Vasudev Gupta

ME18B182

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Part-B

(1) (a) sigmoid

$$f(x) = \text{sigmoid}(x) = \frac{1}{1+e^{-x}}$$

$$f'(x) = -\frac{1}{(1+e^{-x})^2}(0 - e^{-x})$$

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})}\left(\frac{1}{1+e^{-x}}\right)$$

$$f'(x) = [1 - f(x)](f(x))$$

$$f'(x) = [1 - \text{sigmoid}(x)][\text{sigmoid}(x)]$$

(b) hyperbolic tangent

$$f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f'(x) = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$f'(x) = 1 - (f(x))^2$$

$$f'(x) = 1 - [\tanh(x)]^2$$

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(c) $f(x) = relu(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$

$$f'(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \\ 0 & x = 0 \end{cases}$$

Although derivative of relu does not exist at $x = 0$, but we generally assume it to be 0

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Cross-entropy loss

$$loss = \sum_{i=1}^{n} - y_i \log p_i$$

Let 'n' be total no. of categories.

Now $y_i = 1$ where index corresponds to true label
else $y_i = 0$

$$loss = - \log p_i \quad ; \quad p_i = \frac{e^{z_i}}{\sum_{j=1}^{n} e^{z_j}}$$

$$loss = - \log \frac{e^{z_i}}{\sum_j e^{z_j}}$$

$$loss = -z_i + \log \sum_j e^{z_j}$$

$$\frac{\delta loss}{\delta z_i} = -1 + \frac{1}{\sum_j e^{z_i}} e^{z_i}$$

$$\boxed{\frac{\delta loss}{\delta z_i} = -1 + p_i}$$

i corresponds to true label category.

$$\therefore \boxed{\frac{\delta loss}{\delta z_j} = \begin{cases} p_j - 1 & j \equiv \text{true label} \\ p_j & \text{otherwise} \end{cases}}$$

possible values of $j = 1, 2, \cdots n$

~~[illegible]~~

L_2 loss

let last layer has n-neurons
i.e our prediction vector will be n-dimensi-
Let y be target vector of n-dimensions

$$L_2 = (a_i - y_i)^2$$ → assuming reduction to be 'sum'

$$\boxed{\frac{\delta L_2}{\delta a_i} = 2\,|a_i - y_i|}$$ → $i \equiv$ category index

L_1 loss

Let ~~l~~ last layer has n-neurons
i.e prediction vector (a) will be n-dimensional
Let y be target vector of n-dims.

$$L_1 = |a_i - y_i| = \begin{cases} a_i - y_i & a_i \geq y_i \\ y_i - a_i & y_i > a_i \end{cases}$$

$$\frac{\delta L_1}{\delta a_i} = \begin{cases} 1 & a_i > y_i \\ -1 & a_i < y_i \\ 0 & a_i = y_i \end{cases}$$ assuming reduction to be 'sum'

Hinge loss

→ Let 'n' be no of neurons in last layer i.e 'n' categories we want to output.

$$loss = \max(0, 1 - y \cdot a)$$

$$loss = \begin{cases} 0 & ya \geq 1 \\ 1 - y.a & y.a < 1 \end{cases}$$

$$\frac{\delta loss}{\delta a_i} = \begin{cases} 0 & y.a > 1 \\ 0 - y_i & y.a < 1 \end{cases}$$

$$\boxed{\frac{\delta loss}{\delta a_i} = \begin{cases} 0 & y.a \geq 1 \\ -y_i & y.a < 1 \end{cases}}$$

→ assuming reduction to be 'sum'

$i \equiv$ category [possible values → $1, 2. \_\_ . n$]

Huber loss

$$L_\alpha(a, y) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 1/2 \, (y_i - a)^2 & |y_i - a| \leq \alpha \\ \alpha \, (|y_i - a| - \frac{1}{2}\alpha) & \text{otherwise} \end{cases}$$

$$\therefore \frac{\delta L_\alpha(a, y)}{\delta a} = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} -(y_i - a) & |y_i - a| \leq \alpha \\ -\alpha \, \text{sign}(y_i - a) & \text{otherwise} \end{cases}$$

Cosine similarity loss

$$\cos(a, y) = \frac{a \cdot y}{|a||y|}$$

Let a, y be n-dimensional vector

$a \rightarrow$ prediction

$y \rightarrow$ one-hot vector with actual label

$$\cos(a + da, y) = \frac{(a+da)\cdot y}{|a+da||y|}$$

Now

$$|a + da| = \sqrt{(a+da)\cdot(a+da)}$$

$$= \sqrt{|a|^2 + |da|^2 + 2a \cdot da}$$

$$= |a| \sqrt{1 + \frac{|da|^2}{|a|^2} + \frac{2a}{|a|^2} da}$$

$\frac{|da|^2}{|a|^2} \rightarrow 0$

$$\approx |a| \left(1 + \frac{2a}{|a|^2} da\right)^{1/2}$$

Binomial expansion

$$\approx |a| \left(1 + \frac{a \cdot da}{|a|^2}\right)$$

$$\cos(a + da, y) \approx \frac{a \cdot y + da \cdot y}{|a|\left(1 + \frac{a \cdot da}{|a|^2}\right)|y|}$$

$$\approx \frac{a\cdot y + da \cdot y}{|a||y|}\left(1 + \frac{a \cdot da}{|a|^2}\right)^{-1}$$

Binomial expansion

$$\approx \frac{a\cdot y + da \cdot y}{|a||y|}\left(1 - \frac{a \cdot da}{|a|^2}\right)$$

$$\approx \frac{a\cdot y + da\cdot y}{|a||y|} - \frac{a\cdot y}{|a||y|}\frac{a \cdot da}{|a|^2} - \cancel{\frac{(a\cdot da)(da \cdot y)}{|a||y||a|^2}} \rightarrow 0$$

$$\approx \frac{a \cdot y}{|a||y|} + \left(\frac{y}{|a||y|} - \frac{a \cdot y \, a}{|a||y||a|^2}\right)\cdot da$$

$$\cos(a+da, y) = \cos(a,y) + \left(\frac{y}{|a||y|} - \cos(a,y)\frac{a}{|a|^2}\right)\cdot da$$

$$\cos(a+da, y) - \cos(a,y) = \frac{y\cdot da}{|a||y|} - \frac{\cos(a,y)}{a^2} a \cdot da$$

$$\frac{\delta \cos(a,y)}{\delta a_1} = \frac{y_1}{|a||y|} - \frac{\cos(a,y)}{|a|^2} a_1$$

assuming $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ i.e last layer is having n-neurons.

$y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$ 1 is in position which corresponds to label

general term:

$$\boxed{\frac{\delta \cos(a,y)}{\delta a_i} = \frac{y_i}{|a||y|} - \frac{\cos(a,y)}{|a|^2} a_i}$$

$i \equiv$ category index.

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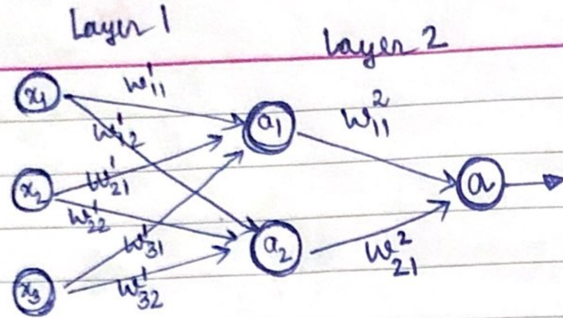

Forward propogation

$$\begin{cases} z_1 = w_{11}x_1 + w_{21}x_2 + w_{31}x_3 + b_1 \\ a_1 = \text{sigmoid}(z_1) \end{cases}$$

$$\begin{cases} z_2 = w_{12}x_1 + w_{22}x_2 + w_{32}x_3 + b_2 \\ a_2 = \text{sigmoid}(z_2) \end{cases}$$

$$\begin{cases} z = w^2_{11}a_1 + w^2_{21}a_2 + b^2_1 \\ a = \text{sigmoid}(z) \end{cases}$$

$$L_2 \equiv \text{Loss} = (a - y)^2$$

$$\frac{\delta L_2}{\delta w^2_{11}} = 2(a-y)\frac{\delta a}{\delta w^2_{11}}$$

$$= 2(a-y)\frac{\delta a}{\delta z}\frac{\delta z}{\delta w^2_{11}}$$

$$\boxed{\frac{\delta L_2}{\delta w^2_{11}} = 2(a-y)\,a(1-a)\,a_1} \quad (10)$$

similarly

$$\boxed{\frac{\delta L_2}{\delta w^2_{21}} = 2(a-y)(a)(1-a)\,a_2} \quad (11)$$

$$\frac{\delta L_2}{\delta w_{11}} = 2(a-y)\frac{\delta a}{\delta w_{11}}$$

$$= 2(a-y)\left(\frac{\delta a}{\delta z}\right)\left(\frac{\delta z}{\delta w_{11}}\right)$$

$$= 2(a-y)(a)(1-a)\left(\frac{\delta z}{\delta a_1}\right)\left(\frac{\delta a_1}{\delta w_{11}}\right)$$

$$= 2(a-y)\left[a(1-a)\right] w^2_{11}\left(\frac{\delta a_1}{\delta z_1}\right)\left(\frac{\delta z_1}{\delta w_{11}}\right)$$

$$\boxed{\frac{\delta L_2}{\delta w^1_{11}} = 2(a-y)\left[a(1-a)\right] w^2_{11}\, a_1(1-a_1)\,x_1} \quad (4)$$

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Backward propogation

$$\frac{\delta L_2}{\delta w_{11}^1} = \boxed{\left(\frac{\delta L_2}{\delta a}\right)\left(\frac{\delta a}{\delta z}\right)}\left(\frac{\delta z}{\delta a_1}\right)\left(\frac{\delta a_1}{\delta z_1}\right)\left(\frac{\delta z_1}{\delta w_{11}^1}\right)$$

$$\frac{\delta L_2}{\delta w_{11}^2} = \boxed{\left(\frac{\delta L_2}{\delta a}\right)\left(\frac{\delta a}{\delta z}\right)}\left(\frac{\delta z}{\delta w_{11}^2}\right)$$

general expression for weights gradient in Layer-1

$$\frac{\delta L_2}{\delta w_{ij}^1} = \frac{\delta L_2}{\delta a}\ \frac{\delta a}{\delta z}\ \frac{\delta z}{\delta a_j}\ \frac{\delta a_j}{\delta z_j}\ \frac{\delta z_j}{\delta w_{ij}^1}$$

$$\frac{\delta L_2}{\delta w_{ij}^1} = 2(a-y)\ a\,(1-a)\ w_{j1}^2\ a_j\,(1-a_j)\ x_i$$

general expression for weights gradient in Layer-2

$$\frac{\delta L_2}{\delta w_{ij}^2} = \frac{\delta L_2}{\delta a}\ \frac{\delta a}{\delta z}\ \frac{\delta z}{\delta w_{ij}^2}$$

$$\frac{\delta L_2}{\delta w_{ij}^2} = 2(a-y)\ a\,(1-a)\ a_i$$

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$$\frac{\delta L_2}{\delta b_1^2} = 2(a-y)\frac{\delta a}{\delta b_1^2}$$

$$= 2(a-y)\frac{\delta a}{\delta z}\frac{\delta z}{\delta b_1^2}$$

$$= 2(a-y)\, a(1-a)\,(1)$$

$$\boxed{\frac{\delta L_2}{\delta b_1^2} = 2(a-y)\, a(1-a)} \quad \text{—(1)}$$

$$\frac{\delta L_2}{\delta b_1^1} = 2(a-y)\frac{\delta a}{\delta b_1^1} = 2(a-y)\frac{\delta a}{\delta z}\frac{\delta z}{\delta b_1^1}$$

$$= 2(a-y)\, a(1-a)\frac{\delta z}{\delta a_1}\frac{\delta a_1}{\delta z_1}\frac{\delta z_1}{\delta b_1^1}$$

$$= 2(a-y)\, a(1-a)\, w_{11}^2\, a_1(1-a_1)\,(1)$$

$$\frac{\delta L_2}{\delta b_1^1} = [2(a-y)]\,[a(1-a)]\,[w_{11}^2]\,[a_1(1-a_1)] \quad \text{—(2)}$$

$$\frac{\delta L_2}{\delta b_2^1} = 2(a-y)\, a(1-a)\, w_{21}^2\, a_2(1-a_2) \quad \text{—(3)}$$

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Parameters

$x_1 = -0.06$
$x_2 = -1.38$
$x_3 = -0.32$
$y = -1.43$

$w^1_{11} = -0.06$
$w^1_{21} = -0.14$
$w^1_{31} = -0.43$
$w^1_{12} = 0.29$
$w^1_{22} = 0$
$w^1_{32} = -0.12$

$b^1_1 = 0.05$
$b^1_2 = -0.22$
$w^2_{11} = -0.59$
$w^2_{21} = -0.14$
$b^2_1 = -0.23$

Forward propogation

$$z_1 = w^1_{11} x_1 + w^1_{21} x_2 + w^1_{31} x_3 + b^1_{11}$$
$$z_1 = (-0.06)(-0.06) + (-0.14)(-1.38) + (-0.43)(-0.32) + 0.05$$
$$z_1 = 0.3844$$

$$z_2 = w^1_{12} x_1 + w^1_{22} x_2 + w^1_{32} x_3 + b^1_2$$
$$z_2 = (0.29)(-0.06) + (0)(-1.38) + (-0.12)(-0.32) - 0.22$$
$$z_2 = -0.199$$

$$a_1 = \frac{1}{1+e^{-z_1}} \qquad a_2 = \frac{1}{1+e^{-z_2}}$$
$$a_1 = \frac{1}{1+e^{-0.3844}} \qquad a_2 = \frac{1}{1+e^{+0.199}}$$
$$a_1 = 0.5949 \qquad a_2 = 0.4504$$

$$z = w^2_{11} a_1 + w^2_{21} a_2 + b^2_1$$
$$z = (-0.59)(0.5949) + (-0.14)(0.4504) - 0.23$$
$$z = -0.644$$

$$a = \frac{1}{1+e^{-z}} = \frac{1}{1+e^{0.644}} = 0.3443$$

$$\hat{y} = 0.3443$$

$$\text{loss} = (\hat{y} - y)^2 = 3.148$$

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similarly

$$\frac{\delta L_2}{\delta w_{21}^1} = 2(a-y)(a(1-a))\, w_{11}^2\, a_1 (1-a_1)\, x_2 \quad \text{— (5)}$$

$$\frac{\delta L_2}{\delta w_{31}^1} = 2(a-y)(a(1-a))\, w_{11}^2\, a_1 (1-a_1)\, x_3 \quad \text{— (6)}$$

$$\frac{\delta L_2}{\delta w_{12}^1} = 2(a-y)(a(1-a))\, w_{21}^2\, a_2 (1-a_2)\, x_1 \quad \text{— (7)}$$

$$\frac{\delta L_2}{\delta w_{22}^1} = 2(a-y)(a(1-a))\, w_{21}^2\, a_2 (1-a_2)\, x_2 \quad \text{— (8)}$$

$$\frac{\delta L_2}{\delta w_{32}^1} = 2(a-y)(a(1-a))\, w_{21}^2\, a_2 (1-a_2)\, x_3 \quad \text{— (9)}$$

Calculating

$$2(a-y)(a(1-a)) = 2(0.3443 + 1.43)(0.3443)(1-0.3443)$$
$$= 0.8011$$

$$2(a-y)\,a(1-a)\, w_{11}^2\, a_1(1-a_1) = 0.8011(-0.59)(0.5949)(1-0.59$$
$$= -0.1139$$

$$2(a-y)(a(1-a))\, w_2^{11}\, a_2(1-a_2) = 0.8011 \times (-0.14)(0.4504)(1-0.4504)$$
$$= -0.02776$$

back propogation

Putting values from forward propogation in 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

$$\frac{\delta L_2}{\delta b_1^2} = 2(0.3443 + 1.43)(0.3443)(1 - 0.3443)$$

$$\boxed{\Delta b_1^2 = 0.8011}$$

$$\frac{\delta L_2}{\delta b_1^1} = \underbrace{2(0.3443 + 1.43)(0.3443)(1 - 0.3443)}_{0.8011}(-0.59)(0.5949)(1 - 0.5949)$$

$$= (0.8011)(-0.59)(0.5949)(1 - 0.5949)$$
$$= -0.1139$$

$$\boxed{\Delta b_1^1 = -0.1139}$$

$$\frac{\delta L_2}{\delta b_2^1} = \underbrace{2(0.343 + 1.43)(0.3443)(1 - 0.3443)}_{0.8011}(-0.14)(0.4504)(1 - 0.4504)$$

$$= (0.8011)(-0.14)(0.4504)(1 - 0.4504)$$
$$= -0.0278$$

$$\boxed{\Delta b_2^1 = -0.0278}$$

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$$\frac{\delta L_2}{\delta w^1_{11}} = (-0.1139)(-0.06)$$

$$\boxed{\Delta w^1_{11} = 0.0068}$$

$$\frac{\delta L_2}{\delta w^1_{21}} = (-0.1139)(-1.38)$$

$$\boxed{\Delta w^1_{21} = 0.15718}$$

$$\frac{\delta L_2}{\delta w^1_{31}} = (-0.1139)(-0.32)$$

$$\boxed{\Delta w^1_{31} = 0.036448}$$

$$\frac{\delta L_2}{\delta w^1_{12}} = (-0.02776)(-0.06)$$

$$\boxed{\Delta w^1_{12} = 0.0016656}$$

$$\frac{\delta L_2}{\delta w^1_{22}} = (-0.02776)(-1.38)$$

$$\boxed{\Delta w^1_{22} = 0.038308}$$

$$\frac{\delta L_2}{\delta w^1_{32}} = (-0.02776)(-0.32)$$

$$\boxed{\Delta w^1_{32} = 0.00888}$$

$$\frac{\delta L_2}{\delta w^2_{11}} = (0.8011)(0.5949)$$

$$\boxed{\Delta w^2_{11} = 0.47657}$$

$$\frac{\delta L_2}{\delta w^2_{21}} = (0.8011)(0.4504)$$

$$\boxed{\Delta w^2_{21} = 0.360815}$$

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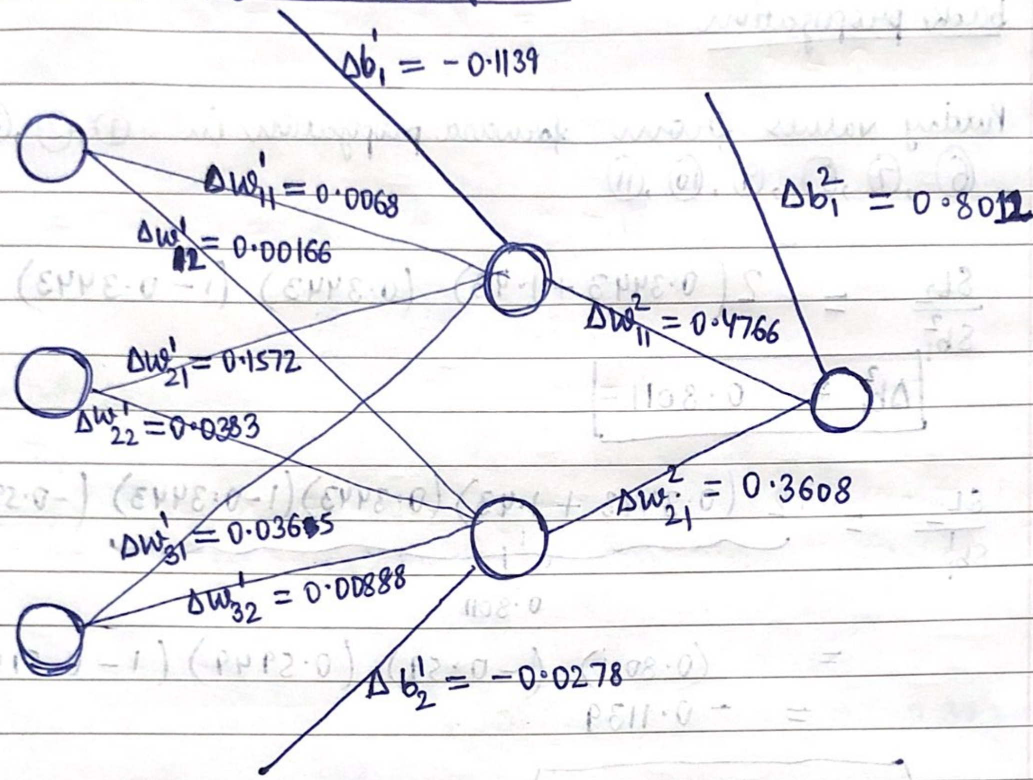
back propogation graph



Submitted by

Vasudev Gupta
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