# Chemical Potentials Updated Notes

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**Abstract**

Hopefully found an expression for

$$\frac{\partial \mu_i}{\partial x_j} \tag{1}$$

where $\mu_i$ is the chemical potential for baryon $i$ and $x_j$ is a dependent variable (in our case: $n_B, x_e, x_\Lambda$, baryon density, electron fraction, and Lambda fraction respectively). Expression is given in Section 3.5 and in summary. Calculation with details is performed in Section 3.

## Contents

# 1 NPE-$\Lambda$ Sound Speed Difference

First, we follow the same main steps outlined in the PJ-PRD Paper to arrive at the sound speed difference expression.

1. We begin with the speed-of-sound difference

$$c_s^2 - c_e^2 = \frac{1}{\mu_{avg}} \frac{\partial P}{\partial n_B}\bigg|_{x,y,z} - \frac{1}{\mu_n} \frac{dP}{dn_B} \tag{2}$$

   where $\mu_{\text{avg}}$ is given by

$$\mu_{\text{avg}} = \frac{\sum_i \mu_i n_i}{n_B} = \sum_i \mu_i x_i \tag{3}$$

2. First, we need to choose our independent variables in this case. Below we will refer to these independent variables as $x_j$.

   - Baryon density: $n_B$
   - Electron Fraction: $x_e = n_e/n_B$
   - Lambda Fraction: $x_\Lambda = n_\Lambda/n_B$

   Then the dependent variables are

   - Proton fraction: $x_e$ which in this simple case is equal to $x_p$
   - Neutron fraction: $x_n$ which in this case is equal to $x_n = 1 - x_p - x_\Lambda$

   These are found by enforcing baryon number conservation and electric charge neutrality. Note: we need the neutron fraction to be a dependent variable for the $\mu_n - \mu_{\text{avg}}$ step to yield the correct $\tilde{\mu}_i$. This also comes from the step where we consider $\varepsilon + p$ which yields the generalized sound speed difference expression in terms of the neutron chemical potential $1/\mu_n$

$$\varepsilon + p = \sum_i \mu_i n_i = \mu_n n_n + \mu_p n_p + \mu_e n_e + \mu_\Lambda n_\Lambda \tag{4}$$

$$= \mu_n(n_B - n_p - n_\Lambda) + \mu_p n_p + \mu_e n_e + \mu_\Lambda n_\Lambda \tag{5}$$

$$= \mu_n n_B + (\mu_p + \mu_e - \mu_n) n_p + (\mu_\Lambda - \mu_n) n_\Lambda \tag{6}$$

$$= \mu_n n_B \qquad \text{in beta equilibrium} \tag{7}$$

   (Similar to eqn. 35 in PJ-PRD)

3. Having chosen our independent and dependent variables, we can consider the expression for the total derivative of the pressure $P$ with respect to baryon density $n_B$ (this appears in the last term in the expression for the sound speed difference expression).

$$\frac{dP}{dn_B} = \frac{\partial P}{\partial n_B}\bigg|_{x_e,x_\Lambda} + \frac{\partial P}{\partial x_n}\bigg|_{n_B,x_\Lambda} \frac{dx_e}{dn_B} + \frac{\partial P}{\partial x_\Lambda}\bigg|_{n_B,x_e} \frac{dx_\Lambda}{dn_B} \tag{8}$$

   and insert this into the last term of the speed-of-sound difference expression.

$$c_s^2 - c_e^2 = \frac{1}{\mu_{avg}} \frac{\partial P}{\partial n_B}\bigg|_{x_e,x_\Lambda} - \frac{1}{\mu_n} \left[ \frac{\partial P}{\partial n_B}\bigg|_{x_e,x_\Lambda} + \frac{\partial P}{\partial x_e}\bigg|_{n_B,x_\Lambda} \frac{dx_e}{dn_B} + \frac{\partial P}{\partial x_\Lambda}\bigg|_{n_B,x_e} \frac{dx_\Lambda}{dn_B} \right] \tag{9}$$

4. Next, we can collect into the first term the $\partial P/\partial n_B$ terms which yields a $\mu_n - \mu_{\text{avg}}$ expression.

$$c_s^2 - c_e^2 = \left( \frac{\mu_n - \mu_{avg}}{\mu_{avg}\mu_n} \right) \frac{\partial P}{\partial n_B}\bigg|_{x_e,x_\Lambda} - \frac{1}{\mu_n} \left[ \frac{\partial P}{\partial x_e}\bigg|_{n_B,x_\Lambda} \frac{dx_e}{dn_B} + \frac{\partial P}{\partial x_\Lambda}\bigg|_{n_B,x_e} \frac{dx_\Lambda}{dn_B} \right] \tag{10}$$

5. We can now deal with the $\mu_n - \mu_{avg}$ term. In this case, we have in the first line the definition. In the second line, we re-write the dependent fractions in terms of the independent fractions.

$$\mu_{\text{avg}} = \mu_n x_n + \mu_p x_p + \mu_\Lambda x_\Lambda + \mu_e x_e \tag{11}$$
$$= \mu_n(1 - x_e - x_\Lambda) + \mu_p x_e + \mu_\Lambda x_\Lambda + \mu_e x_e \tag{12}$$

Next, we can calculate $\mu_n - \mu_{\text{avg}}$ and collect terms in like fractions which yields combinations of chemical potentials.

$$\mu_n - \mu_{\text{avg}} = \mu_n - \mu_n + x_e(\mu_p + \mu_e - \mu_n) + x_\Lambda(\mu_\Lambda - \mu_n) \tag{13}$$
$$= -x_e \tilde{\mu}_{x_e} - x_\Lambda \tilde{\mu}_{x_\Lambda} \tag{14}$$

with

$$\tilde{\mu}_{x_e} = \mu_n - \mu_p - \mu_e \tag{15}$$
$$\tilde{\mu}_{x_\Lambda} = \mu_n - \mu_\Lambda \tag{16}$$

We see that in this case, the $\tilde{\mu}_i$ are combinations of chemical potentials that vanish in $\beta$ equilibrium. We also see that for each independent fraction $x_j$ (that is, the independent variables minus the baryon density $n_B$) we can associate a $\tilde{\mu}_j$ so that we can write this as

$$\mu_n - \mu_{\text{avg}} = -\sum_i x_i \tilde{\mu}_i \qquad i \in \text{ind. fractions} \tag{17}$$

6. Next, thinking ahead to $\beta$-equlibrium, we can ignore the first term and then just deal with the total derivative terms.

$$c_s^2 - c_e^2 = -\frac{1}{\mu_n}\left[\left.\frac{\partial p}{\partial x}\right|_{n_B,y}\frac{dx}{dn_B} + \left.\frac{\partial p}{\partial y}\right|_{n_B,x}\frac{dy}{dn_B}\right] \tag{18}$$

Then using $P = n_B^2 \frac{\partial E}{\partial n_B}|_{x_e,x_\Lambda}$, we can re-write this sound speed difference as

$$\boxed{c_s^2 - c_e^2 = -\frac{n_B^2}{\mu_n}\left[\left.\frac{\partial \tilde{\mu}_{x_e}}{\partial n_B}\right|_{x_e,x_\Lambda}\frac{dx_e}{dn_B} + \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial n_B}\right|_{x_e,x_\Lambda}\frac{dx_\Lambda}{dn_B}\right]} \tag{19}$$

7. We then can use the fact that in $\beta$-equilibrium the total derivative of the $\tilde{\mu}_i$ chemical potentials vanishes.

$$d\tilde{\mu}_{x_e} = \left.\frac{\partial \tilde{\mu}_{x_e}}{\partial n_B}\right|_{x_e,x_\Lambda} dn_B + \left.\frac{\partial \tilde{\mu}_{x_e}}{\partial x_e}\right|_{n_B,x_\Lambda} dx_e + \left.\frac{\partial \tilde{\mu}_{x_e}}{\partial x_\Lambda}\right|_{n_B,x_e} dx_\Lambda = 0 \tag{20}$$
$$d\tilde{\mu}_{x_\Lambda} = \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial n_B}\right|_{x_e,x_\Lambda} dn_B + \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial x_e}\right|_{n_B,x_\Lambda} dx_e + \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial x_\Lambda}\right|_{n_B,x_e} dx_\Lambda = 0 \tag{21}$$

We can divide each equation by $dn_B$ to get expressions in terms of $dx_e/dn_B$, $dx_\Lambda/dn_B$.

$$0 = \left.\frac{\partial \tilde{\mu}_{x_e}}{\partial n_B}\right|_{x_e,x_\Lambda} + \left.\frac{\partial \tilde{\mu}_{x_e}}{\partial x_e}\right|_{n_B,x_\Lambda}\frac{dx_e}{dn_B} + \left.\frac{\partial \tilde{\mu}_{x_e}}{\partial x_\Lambda}\right|_{n_B,x_e}\frac{dx_\Lambda}{dn_B} \tag{22}$$
$$0 = \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial n_B}\right|_{x_e,x_\Lambda} + \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial x_e}\right|_{n_B,x_\Lambda}\frac{dx_e}{dn_B} + \left.\frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial x_\Lambda}\right|_{n_B,x_e}\frac{dx_\Lambda}{dn_B} \tag{23}$$

This is a system of linear equations where the *ith* equation takes the form

$$\left.\frac{\partial \tilde{\mu}_i}{\partial n_B}\right|_{x,y,\ldots} + \nabla\tilde{\mu}_i \cdot \frac{dx_i}{dn_B} = 0 \tag{24}$$

for $x_i$ being an independent fraction and where $\nabla = [\partial/\partial x_i, \ldots]$ is a sort of gradient.

8. For now, taking these two equations we can solve them in Mathematica which gives

$$\frac{dx_e}{dn_B} = \frac{\frac{\partial\tilde{\mu}_{x_e}}{\partial x_\Lambda}\frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial n_B} - \frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial x_\Lambda}\frac{\partial\tilde{\mu}_{x_e}}{\partial n_B}}{\frac{\partial\tilde{\mu}_{x_e}}{\partial x_\Lambda}\frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial x_e} - \frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial x_\Lambda}\frac{\partial\tilde{\mu}_{x_e}}{\partial x_e}} \tag{25}$$

$$\frac{dx_\Lambda}{dn_B} = \frac{\frac{\partial\tilde{\mu}_{x_e}}{\partial x_e}\frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial n_B} - \frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial x_e}\frac{\partial\tilde{\mu}_{x_e}}{\partial n_B}}{\frac{\partial\tilde{\mu}_{x_e}}{\partial n_B}\frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial x_e} - \frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial x_e}\frac{\partial\tilde{\mu}_{x_\Lambda}}{\partial n_B}} \tag{26}$$

To get an analytic expression the form above quickly gets messy as we add more particles. However, to do this numerically should be easy as these are systems of linear equations.

# 2 Calculation of Partial Derivatives of Chemical Potentials

In an RMF-Walecka type model, we have the chemical potential for a Baryon $i$ given by

$$\mu_i = E^*_{F_i} + \text{Other Meson Contributions} \tag{27}$$

In these sort of Lagrangians, the mesons only interact with baryons but do not mix among themselves. For a concrete example, we have the chemical potential for the $i$th Baryon from our given Lagrangian

$$\begin{aligned}
\mathcal{L} = \sum_j \bar{\psi}_j (i\gamma^\mu\partial_\mu - &\underbrace{(m_j - g_{\sigma j}\sigma)}_{m_j^*} + \gamma_0 \underbrace{(\mu_j - g_{\omega j}\omega_0 - g_{\phi j}\phi_0 - g_{\rho j}\rho_0^a\tau_a)}_{\mu_j^*})\psi_j && (28)\\
&- \frac{1}{2}m_\sigma^2\sigma^2 - \frac{b}{3}m_N(g_\sigma\sigma)^3 - \frac{c}{4}(g_\sigma\sigma)^4 && (29)\\
&+ \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu && (30)\\
&+ \frac{1}{2}m_\phi^2\phi^\mu\phi_\mu && (31)\\
&+ \frac{1}{2}m_\rho^2\rho_a^\mu\rho_\mu^a && (32)
\end{aligned}$$

given by

$$\boxed{\mu_i = \mu_i^* + g_{\omega i}\omega_0 + g_{\phi i}\phi_0 + I_{3B}g_{\rho i}\rho_0 \qquad \mu_i^* = \sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{33}$$

We then want to calculate the partial derivatives of these chemical potentials with respect to the independent variables.

$$\frac{\partial\mu_i}{\partial x_j} \qquad i \in \text{Baryons} \qquad j \in \text{Independent Variables} \tag{34}$$

When we apply this to the above chemical potential, we find

$$\frac{\partial\mu_i}{\partial x_j} = \frac{\partial}{\partial x_j}\sqrt{k_{F_i}^2 + m_i{*}^2} + g_{\omega i}\frac{\partial\omega_0}{\partial x_j} + g_{\phi i}\frac{\partial\phi_0}{\partial x_j} + I_{3B}g_{\rho i}\frac{\partial\rho_0}{\partial x_j} \tag{35}$$

For the $\omega_0, \phi_0, \rho_0$ fields, we have the mean field equation of motions which take a particularly simple form so these partial derivatives are somewhat "trivial". The issue is dealing with the first term which involves the $\sigma$ field which self-couples among other non-trivial couplings.

## 2.1 RMF Equations of Motion

Applying the procedure outlined in Glenndenning compact stars, we can re-write these equations of motion in the mean field approximation in terms of the scalar and vector densities $n_i^s$ and $n_i$ (the vector density is the usual density associated with a Fermion). First, we have the equations of motion for the non-$\sigma$ fields given by

$$m_\omega^2 \omega_0 = \sum_i g_{\omega i} n_i \qquad\qquad n_i = \frac{k_{F_i}^3}{3\pi^2} \tag{36}$$

$$m_\rho^2 \rho_{03} = \sum_i g_\rho I_{3i} n_i \tag{37}$$

$$m_\phi^2 \phi_0 = \sum_i g_\phi n_i \tag{38}$$

and the $\sigma$ field given by

$$m_\sigma^2 \sigma + \frac{\partial U}{\partial \sigma} = \sum_i g_{\sigma i} n_i^s \tag{39}$$

with the vector density given by

$$n_i^s = \frac{g_{sB}}{2\pi^2} \int_0^{k_{F_i}} \frac{m_i^*\, k^2}{\sqrt{k^2 + m_i^{*2}}}\, dk = \frac{m_i^*}{2\pi^2} \left[ k_{F_i} \sqrt{k_{F_i}^2 + m_i^{*2}} - m_i^{*2} \ln \frac{k_{F_i} + \sqrt{k_{F_i}^2 + m_i^{*2}}}{m_i^*} \right] \tag{40}$$

$$= \frac{m_i^*}{2\pi^2} \left[ k_{F_i} E_{F_i}^* - m_i^{*2} \ln \frac{k_{F_i} + E_{F_i}^*}{m_i^*} \right] \tag{41}$$

This last expression is highly coupled.

## 2.2 "Trivial" Partial Derivatives

For the $\omega_0$, $\rho_0$, $\rho_0$ fields, we can get the partial derivatives in the following manner. We solve for the fields from the equation of motion, re-write the sums in terms of the independent fractions, and then take the partial derivatives. For example, we have for $\omega_0$

$$\omega_0 = \frac{1}{m_\omega^2} \sum_i g_{\omega i} n_i \tag{42}$$

$$= \frac{1}{m_\omega^2} \left[ g_{\omega N}(n_n + n_p) + g_{\omega H} n_\Lambda \right] \tag{43}$$

$$= \frac{1}{m_\omega^2} \left[ g_{\omega N} n_B (1 - x_\Lambda) + g_{\omega H} n_B x_\Lambda \right] \tag{44}$$

where $g_{\omega N}$ is the $\omega$-nucleon coupling and $g_{\omega H}$ is the $\omega$-Hyperon coupling. Having written this in terms of the independent variables, taking the partial derivatives is straightfoward. Repeating this for the other mesons gives us

$$m_\rho^2 \rho_{03} = \sum_i g_\rho I_{3i} n_i = g_\rho \left( \frac{1}{2} n_p - \frac{1}{2} n_n \right) \tag{45}$$

$$= g_\rho \left( \frac{1}{2} n_e - \frac{1}{2}(n_B - n_e - n_\Lambda) \right) \tag{46}$$

$$= g_\rho \left( n_e - \frac{1}{2} n_B + \frac{1}{2} n_\Lambda \right) \tag{47}$$

$$= g_\rho n_B \left( x_e + \frac{1}{2} x_\Lambda - \frac{1}{2} \right) \tag{48}$$

and

$$m_\phi^2 \phi_0 = \sum_i g_\phi n_i = g_{\phi N}(n_n + n_p) + g_{\phi H} n_\Lambda \tag{49}$$

$$= g_{\phi N} n_B (1 - x_\Lambda) + g_{\phi H} n_\Lambda \tag{50}$$

For clarity this is given here

$$\boxed{\omega_0 = \frac{n_B}{m_\omega^2}\left[g_{\omega N}(1-x_\Lambda) + g_\omega x_\Lambda\right]} \tag{51}$$

$$\boxed{\rho_{03} = \frac{g_{\rho N} n_B}{2m_\rho^2}\left(2x_e + x_\Lambda - 1\right)} \tag{52}$$

$$\boxed{\phi_0 = \frac{n_B}{m_\phi^2}\left[g_{\phi N}(1-x_\Lambda) + g_{\phi H} x_\Lambda\right]} \tag{53}$$

For the chemical potential derivatives to be calculated in the following sections, this gives us the following for the partial derivatives of the $\omega_0$ and $\rho_{03}$ fields with respect to the independent variables $x_e, x_\Lambda, n_B$.

1. $\omega$ meson

$$\frac{\partial \omega_0}{\partial n_B} = \frac{1}{m_\omega^2}\left(g_{\omega N}(1-x_\Lambda) + g_{\omega H} x_\Lambda\right) \tag{54}$$

$$\frac{\partial \omega_0}{\partial x_e} = 0 \tag{55}$$

$$\frac{\partial \omega_0}{\partial x_\Lambda} = \frac{n_B}{m_\omega^2}\left(g_{\omega H} - g_{\omega N}\right) \tag{56}$$

2. $\rho_{03}$ meson field

$$\frac{\partial \rho_{03}}{\partial n_B} = \frac{g_{\rho N}}{2m_\rho^2}(2x_e + x_\Lambda - 1) \tag{57}$$

$$\frac{\partial \rho_{03}}{\partial x_e} = \frac{g_{\rho N} n_B}{m_\rho^2} \tag{58}$$

$$\frac{\partial \rho_{03}}{\partial x_\Lambda} = \frac{g_{\rho N} n_B}{2m_\rho^2} \tag{59}$$

3. $\phi_0$ meson field

$$\frac{\partial \phi_0}{\partial n_B} = \frac{1}{m_\phi^2}\left[g_{\phi N}(1-x_\Lambda) + g_{\phi H} x_\Lambda\right] \tag{60}$$

$$\frac{\partial \phi_0}{\partial x_e} = 0 \tag{61}$$

$$\frac{\partial \phi_0}{\partial x_\Lambda} = \frac{n_B}{m_\phi^2}\left(g_{\phi H} - g_{\phi N}\right) \tag{62}$$

### 2.3 Non-Trivial Partial Derivative

Let $\mu_i' = \sqrt{k_{F_i}^2 + m_i^{*2}}$. Then the partial derivative of $\mu_i'$ with respect to independent variable $x_j$ is given by

$$\frac{\partial \mu_i'}{\partial x_j} = \frac{\partial}{\partial x_j}\sqrt{k_{F_i}^2 + m_i^{*2}} \tag{63}$$

$$= \frac{1}{2}\left[k_{F_i}^2 + m_i^{*2}\right]^{-1/2}\left[2k_{F_i}\frac{\partial k_{F_i}}{\partial x_j} + 2m_i^*\frac{\partial m_i^*}{\partial x_j}\right] \tag{64}$$

$$= \frac{k_{F_i}\dfrac{\partial k_{F_i}}{\partial x_j} - g_{\sigma i} m_i^* \dfrac{\partial \sigma}{\partial x_j}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{65}$$

using $m_i^* = m_i - g_{\sigma i}\sigma$. At this stage, we have two unknowns

$$\frac{\partial k_{F_i}}{\partial x_j} \qquad \frac{\partial \sigma}{\partial x_j} \tag{66}$$

The first can be dealt with by using

$$k_{F_i} = (3\pi^2 n_i)^{1/3} \tag{67}$$

and relating $n_i$ to $x_j := n_j/n_B$ via baryon number conservation, charge conservation. The second one we talk about below.

# 3 Calculation of $\sigma$ Field Partial Derivatives

So we have seen that to calculate partial derivatives of the chemical potentials with respect to the independent variables, we need the partial derivatives of the $\sigma$-field:

$$\frac{\partial \sigma}{\partial x_j} \qquad j \in \text{Ind. Vars} \tag{68}$$

This section outlines how to do this.

## 3.1 Sigma Field Equation of Motion

The $\sigma$ field obeys the following equation of motion

$$m_\sigma^2 \sigma + \frac{\partial U}{\partial \sigma} = \sum_i g_{\sigma i} n_i^s \tag{69}$$

where

$$U(\sigma) = \frac{1}{3} b m_N (g_\sigma \sigma)^3 + \frac{1}{4} c (g_\sigma \sigma)^4 \tag{70}$$

and $m_N, b, c$ are the nucleon mass, and sigma self coupling constants respectively. Note: $g_{\sigma i}$ is the coupling constant between the sigma field and the $i$th Baryon.

Importantly: $n_i^s$ is the scalar density which at zero temperature is given by

$$n_i^s = \frac{m_i^*}{2\pi^2} \left[ k_{F_i} E_{F_i} - m_i^{*2} \ln \frac{k_{F_i} + E_{F_i}}{m_i^*} \right] \tag{71}$$

where $k_{F_i} = (3\pi^2 n_i)^{1/3}$ is the Fermi momentum for the $i$th Baryon and $E_{F_i} = \sqrt{k_{F_i}^2 + m_i^{*2}}$ is the effective Fermi energy with $m_i^* = m_i - g_{\sigma i}\sigma$ the effective mass.

## 3.2 Sigma Partial Derivatives

The strategy here is to take the partial derivative of both sides with respect to our desired variable $x_j$ and solve/isolate for $\partial\sigma/\partial x_j$.

$$\frac{\partial \sigma}{\partial x_j} \left( m_\sigma^2 + \frac{\partial^2 U}{\partial \sigma^2} \right) = \sum_i g_{\sigma i} \frac{\partial n_i^s}{\partial x_j} \tag{72}$$

This then reduces to a problem of determining $\partial n_i^s/\partial x_j$ in terms of $\partial\sigma/\partial x_j$. One immediate thing to note however is that $n_i^s$ is given primarily in terms of $k_{F_i}$ so for ease of calculation it pays to re-write these partial derivatives in terms of $k_{F_i}$. That is, we perform a chain rule of the form

$$\frac{\partial \sigma}{\partial x_j} \left( m_\sigma^2 + \frac{\partial^2 U}{\partial \sigma^2} \right) = \sum_i g_{\sigma i} \frac{\partial n_i^s}{\partial k_{F_i}} \frac{\partial k_{F_i}}{\partial x_j} \tag{73}$$

to instead calculate $\partial n_i^s/\partial k_{F_i}$ which is easier.

### 3.3 Finding $\partial n_i^s/\partial k_{F_i}$

In this section, we find $\partial n_i^s/\partial k_{F_i}$. This is done as follows. We just straight forwardly take the derivative.

$$\frac{\partial n_i^s}{\partial k_{F_i}} = \underbrace{\frac{1}{2\pi^2}\frac{\partial m_i^*}{\partial k_{F_i}}\left[k_{F_i}E_{F_i} - m_i^{*2}\ln\frac{k_{F_i}+E_{F_i}}{m_i^*}\right]}_{\text{Term 1}} + \underbrace{\frac{m_i^*}{2\pi^2}\frac{\partial}{\partial k_{F_i}}\left[k_{F_i}E_{F_i} - m_i^{*2}\ln\frac{k_{F_i}+E_{F_i}}{m_i^*}\right]}_{\text{Term 2}} \tag{74}$$

In the following, we will use

$$\frac{\partial m_i^*}{\partial k_{F_i}} = \frac{\partial}{\partial k_{F_i}}(m_i - g_{\sigma i}\sigma) = -g_{\sigma i}\frac{\partial \sigma}{\partial k_{F_i}} \tag{75}$$

and

$$\frac{\partial E_{F_i}}{\partial k_{F_i}} = \frac{\partial}{\partial k_{F_i}}\sqrt{k_{F_i}^2 + m_i^{*2}} \tag{76}$$

$$= \frac{1}{2}\left[k_{F_i}^2 + m_i^{*2}\right]^{-1/2}\left[2k_{F_i} + 2m_i^*\left(-g_{\sigma i}\frac{\partial\sigma}{\partial k_{F_i}}\right)\right] \tag{77}$$

$$= \frac{k_{F_i} - g_{\sigma i}m_i^*\frac{\partial\sigma}{\partial k_{F_i}}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{78}$$

and

$$\frac{\partial}{\partial k_{F_i}}\ln\frac{k_{F_i} + \sqrt{k_{F_i}^2 + m^{*2}}}{m_i^*} = \frac{m_i^* + g_{\sigma i}k_{F_i}\frac{\partial\sigma}{\partial k_{F_i}}}{m_i^*\sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{79}$$

where the last one comes from Mathematica.

For now, focusing on Term 2, we can calculate the derivatives of the two terms inside of the bracket. First, we have

$$\frac{\partial}{\partial k_{F_i}}(k_{F_i}E_{F_i}) = \frac{\partial}{\partial k_{F_i}}\left(k_{F_i}\sqrt{k_{F_i}^2 + m_i^{*2}}\right) \tag{80}$$

$$= \sqrt{k_{F_i}^2 + m_i^{*2}} + k_{F_i}\frac{\partial}{\partial k_{F_i}}\sqrt{k_{F_i}^2 + m_i^{*2}} \tag{81}$$

$$= \sqrt{k_{F_i}^2 + m_i^{*2}} + k_{F_i}\frac{1}{2}\left[k_{F_i}^2 + m_i^{*2}\right]^{-1/2}\left[2k_{F_i} + 2_i^*\left(-g_{\sigma i}\frac{\partial\sigma}{\partial k_{F_i}}\right)\right] \tag{82}$$

$$= \sqrt{k_{F_i}^2 + m_i^{*2}} + \frac{k_{F_i}(k_{F_i} - g_{\sigma i}m_i^*\frac{\partial\sigma}{\partial k_{F_i}})}{\sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{83}$$

$$= \frac{k_{F_i}^2 + m_i^{*2}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} + \frac{k_{F_i}^2 - g_{\sigma i}m_i^*k_{F_i}\frac{\partial\sigma}{\partial k_{F_i}}}{k_{F_i}^2 + m_i^{*2}} \tag{84}$$

$$= \frac{2k_{F_i}^2 + m_i^{*2} - g_\sigma m_i^*k_{F_i}\frac{\partial\sigma}{\partial k_{F_i}}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{85}$$

Second, we have

$$\frac{\partial}{\partial k_{F_i}}m_i^{*2}\ln\frac{k_{F_i} + \sqrt{k_{F_i}^2 + m_i^{*2}}}{m_i^*} = 2m_*\left(-g_\sigma\frac{\partial\sigma}{\partial k_{F_i}}\right)\ln\frac{k_{F_i} + \sqrt{k_{F_i}^2 + m_i^{*2}}}{m_i^*} + m_i^{*2}\left(\frac{m_i^* + k_{F_i}g_\sigma\frac{\partial\sigma}{\partial k_{F_i}}}{m_i^*\sqrt{k_{F_i}^2 + m_i^{*2}}}\right) \tag{86}$$

Putting these two expressions back together gives us:

$$\frac{2k_{F_i}^2 + m_i^{*2} - g_\sigma m_* k_{F_i}\dfrac{\partial\sigma}{\partial k_{F_i}}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} + 2m_* g_\sigma \frac{\partial\sigma}{\partial k_{F_i}} \ln\frac{k_{F_i} + \sqrt{k_{F_i}^2 + m_i^{*2}}}{m_*} - \frac{m_i^{*2}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} - \frac{m_* k_{F_i} g_\sigma \dfrac{\partial\sigma}{\partial k_{F_i}}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} \tag{87}$$

This then simplifies greatly to

$$\frac{2k_{F_i}^2}{\sqrt{k_{F_i}^2 + m_i^{*2}}} - 2g_\sigma m_i^* \frac{\partial\sigma}{\partial k_{F_i}}\left[\frac{k_{F_i}}{\sqrt{k_{F_i}^2 + m_i^{*2}}} - \ln\frac{k_{F_i} + \sqrt{k_{F_i}^2 + m_i^{*2}}}{m_i^*}\right] \tag{88}$$

Now, we want to add back to this the first term which gives us

$$\frac{\partial n_i^s}{\partial k_{F_i}} = \frac{-g_{\sigma i}}{2\pi^2}\frac{\partial\sigma}{\partial k_{F_i}}\left[k_{F_i}E_{F_i} - m_i^{*2}\ln\frac{k_{F_i}+E_{F_i}}{m_i^*}\right] + \frac{m_i^* k_{F_i}^2}{\pi^2 E_{F_i}} - \frac{g_{\sigma i} m_i^{*2}}{\pi^2}\frac{\partial\sigma}{\partial k_{F_i}}\left[\frac{k_{F_i}}{E_{F_i}} - \ln\frac{k_{F_i}+E_{F_i}}{m_i^*}\right] \tag{89}$$

or condensed together

$$\boxed{\frac{\partial n_i^s}{\partial k_{F_i}} = \left[\frac{3}{2}\frac{g_{\sigma i} m_i^{*2}}{\pi^2}\ln\frac{k_{F_i}+E_{F_i}}{m_i^*} - \frac{g_{\sigma i}}{\pi^2}\left(\frac{1}{2}k_{F_i}E_{F_i} + m_i^{*2}\frac{k_{F_i}}{E_{F_i}}\right)\right]\frac{\partial\sigma}{\partial k_{F_i}} + \frac{m_i^*}{\pi^2}\frac{k_{F_i}^2}{E_{F_i}}} \tag{90}$$

So we see that this is of the form

$$\frac{\partial n_i^s}{\partial k_{F_i}} = \alpha_i \frac{\partial\sigma}{\partial k_{F_i}} + \beta_i \tag{91}$$

with

$$\alpha_i = \left[\frac{3}{2}\frac{g_{\sigma i} m_i^{*2}}{\pi^2}\ln\frac{k_{F_i}+E_{F_i}}{m_i^*} - \frac{g_{\sigma i}}{\pi^2}\left(\frac{1}{2}k_{F_i}E_{F_i} + m_i^{*2}\frac{k_{F_i}}{E_{F_i}}\right)\right] \tag{92}$$

$$\beta_i = \frac{m_i^*}{\pi^2}\frac{k_{F_i}^2}{E_{F_i}} \tag{93}$$

## 3.4 Returning to Sigma Partials

In section 3.2, we had the following

$$\frac{\partial\sigma}{\partial x_j}\left(m_\sigma^2 + \frac{\partial^2 U}{\partial\sigma^2}\right) = \sum_i g_{\sigma i}\frac{\partial n_i^s}{\partial k_{F_i}}\frac{\partial k_{F_i}}{\partial x_j} \tag{94}$$

where in section 3.3, we solved for $\partial n_i^s/\partial k_{F_i}$. Inserting this in gives us

$$\frac{\partial\sigma}{\partial x_j}\left(m_\sigma^2 + \frac{\partial^2 U}{\partial\sigma^2}\right) = \sum_i g_{\sigma i}\left(\alpha_i\frac{\partial\sigma}{\partial k_{F_i}} + \beta_i\right)\frac{\partial k_{F_i}}{\partial x_j} \tag{95}$$

$$= \sum_i g_{\sigma i}\left(\alpha_i\frac{\partial\sigma}{\partial x_j} + \beta_i\frac{\partial k_{F_i}}{\partial x_j}\right) \tag{96}$$

From this expression, we can solve for $\partial\sigma/\partial x_j$ and get

$$\frac{\partial\sigma}{\partial x_j}\left[m_\sigma^2 + \frac{\partial^2 U}{\partial\sigma^2} - \sum_i g_{\sigma i}\alpha_i\right] = \sum_i g_{\sigma i}\beta_i\frac{\partial k_{F_i}}{\partial x_j} \tag{97}$$

$$\Longrightarrow \boxed{\frac{\partial\sigma}{\partial x_j} = \frac{\sum_i g_{\sigma i}\beta_i\dfrac{\partial k_{F_i}}{\partial x_j}}{\left[m_\sigma^2 + \dfrac{\partial^2 U}{\partial\sigma^2} - \sum_i g_{\sigma i}\alpha_i\right]}} \tag{98}$$

for $\alpha_i, \beta_i$ given by

$$\alpha_i = \left[\frac{3}{2}\frac{g_{\sigma i} m_i^{*2}}{\pi^2} \ln \frac{k_{F_i} + E^*_{F_i}}{m_i^*} - \frac{g_{\sigma i}}{\pi^2}\left(\frac{1}{2}k_{F_i}E^*_{F_i} + m_i^{*2}\frac{k_{F_i}}{E^*_{F_i}}\right)\right] \tag{99}$$

$$\beta_i = \frac{m_i^*}{\pi^2}\frac{k_{F_i}^2}{E^*_{F_i}} \tag{100}$$

and $i$ sums over the Baryons present in the system.

### 3.5 Putting everything back together

Then, we have found our two unknowns. So $\partial \mu_i'/\partial x_j$ is given by

$$\frac{\partial \mu_i'}{\partial x_j} = \frac{k_{F_i}}{E^*_{F_i}}\frac{\partial k_{F_i}}{\partial x_j} - \frac{g_{\sigma i} m_i^*}{E^*_{F_i}}\left[\frac{\sum_i g_{\sigma i}\beta_i \dfrac{\partial k_{F_i}}{\partial x_j}}{\left[m_\sigma^2 + \dfrac{\partial^2 U}{\partial \sigma^2} - \sum_i g_{\sigma i}\alpha_i\right]}\right] \tag{101}$$

and the entire chemical potential derivative for a baryon $i$ with respect to independent variable $x_j$ is given by

$$\boxed{\frac{\partial \mu_i}{\partial x_j} = \frac{k_{F_i}}{E^*_{F_i}}\frac{\partial k_{F_i}}{\partial x_j} - \frac{g_{\sigma i} m_i^*}{E^*_{F_i}}\left[\frac{\sum_i g_{\sigma i}\beta_i \dfrac{\partial k_{F_i}}{\partial x_j}}{\left[m_\sigma^2 + \dfrac{\partial^2 U}{\partial \sigma^2} - \sum_i g_{\sigma i}\alpha_i\right]}\right] + g_{\omega i}\frac{\partial \omega_0}{\partial x_j} + g_{\phi i}\frac{\partial \phi_0}{\partial x_j} + I_{3B} g_{\rho i}\frac{\partial \rho_0}{\partial x_j}} \tag{102}$$

# 4 Electron Chemical Potential Partial Derivatives

First, we can consider the electron chemical potential which takes the simple form below

$$\mu_e = \sqrt{k_{F_e}^2 + m_e^2} \qquad k_{F_e} = \left(3\pi^2 n_e\right)^{1/3} \tag{103}$$

Then using $x_e = n_e/n_B \implies n_e = x n_B$ we have $k_{F_e} = (3\pi^2 x n_B)^{1/3}$. This gives us the partial derivatives of the electron chemical potential with respect to the independent variables as

$$\frac{\partial \mu_e}{\partial n_B} = \frac{\partial}{\partial n_B}\sqrt{(3\pi^2 x n_B)^{2/3} + m_e^2} = \left(\frac{\pi^4 x_e^2}{3 n_B}\right)^{1/3}\left[(3\pi^2 x_e n_B)^{2/3} + m_e^2\right]^{-1/2} \tag{104}$$

$$\frac{\partial \mu_e}{\partial x_e} = \frac{\partial}{\partial x_e}\sqrt{(3\pi^2 x_e n_B)^{2/3} + m_e^2} = \left(\frac{\pi^4 n_B^2}{3 x_e}\right)^{1/3}\left[(3\pi^2 x_e n_B)^{2/3} + m_e^2\right]^{-1/2} \tag{105}$$

$$\frac{\partial \mu_e}{\partial x_\Lambda} = \frac{\partial}{\partial y}\sqrt{(3\pi^2 x_e n_B)^{2/3} + m_e^2} = 0 \tag{106}$$

Though this is easy to compute symbolically.

# 5 Summary of key steps

1. We have the speed of sound difference as

$$c_s^2 - c_e^2 = -\frac{n_B^2}{\mu_n}\left[\frac{\partial \tilde{\mu}_{x_e}}{\partial n_B}\bigg|_{x_e, x_\Lambda}\frac{dx_e}{dn_B} + \frac{\partial \tilde{\mu}_{x_\Lambda}}{\partial n_B}\bigg|_{x_e, x_\Lambda}\frac{dx_\Lambda}{dn_B}\right] \tag{107}$$

   To get this, we need to solve for the fractions $dx_e/dn_B$ and $dx_\Lambda/dn_B$ in terms of partial derivatives of $\tilde{\mu}_i$ which reduces to solving a system of linear equations.

2. Next, we need to determine $\partial\tilde{\mu}_i/\partial x_j$ with $i \in \{x_e, x_\Lambda\}$ and $x_j \in \{n_B, x_e, x_\Lambda\}$.
   - We break this down into determining $\partial\mu_i/\partial x_j$ where $i \in$ baryons, leptons and $j \in \{n_B, x_e, x_\Lambda\}$.
   - The baryon chemical potential in an RMF model is given by

$$\mu_i = \sqrt{k_{F_i}^2 + m_i*^2} + g_{\omega i}\omega_0 + g_{\phi i}\phi_0 + I_{3B} g_{\rho i}\rho_0 \tag{108}$$

     To determine the partial derivative, we need derivatives of the meson fields. The derivatives of the $\omega_0, \rho_0, \phi_0$ fields are straight forward but the derivative of $E_{F_i}^* = \sqrt{k_{F_i}^* + m_i^{*2}}$ is more tricky as it requires the partial derivative of $\sigma$ and its equation of motion is highly nonlinear.
   - We start to take the derivative of $E_{F_i}^*$ with respect to independent variable $x_j$ and find that it depends on two unknowns

$$\frac{\partial k_{F_i}^*}{\partial x_j} \qquad \frac{\partial \sigma}{\partial x_j} \tag{109}$$

     the first being straight forward.
   - From the $\sigma$ equation of motion, we differentiate it with respect to $x_j$ to solve for $\partial\sigma/\partial x_j$. Then plugging this result back in then gives us the chemical potential partial derivative for the baryon. We arrive at the following expression

$$\boxed{\frac{\partial \mu_i}{\partial x_j} = \frac{k_{F_i}}{E_{F_i}^*}\frac{\partial k_{F_i}}{\partial x_j} - \frac{g_{\sigma i} m_i^*}{E_{F_i}^*}\left[\frac{\sum_k g_{\sigma k}\beta_k \frac{\partial k_{F_k}}{\partial x_j}}{\left[m_\sigma^2 + \frac{\partial^2 U}{\partial \sigma^2} - \sum_k g_{\sigma k}\alpha_k\right]}\right] + g_{\omega i}\frac{\partial \omega_0}{\partial x_j} + g_{\phi i}\frac{\partial \phi_0}{\partial x_j} + I_{3B} g_{\rho i}\frac{\partial \rho_0}{\partial x_j}} \tag{110}$$

     with $\alpha_i, \beta_i$ defined above. The sums of $k$ is a sum over all baryons in the system.
   - We then would evaluate this for the different baryons and take the appropriate linear combinations for $\tilde{\mu}_i$.
3. Next, once we have $\partial\tilde{\mu}_i/\partial x_j$ we can return to the system of linear equations for $dx_e/dn_B$ and $dx_\Lambda/dn_B$ and solve for them numerically.
4. Lastly, we can calculate the speed of sound difference by plugging in. At this point we should have all necessary values.