→ Under Θ_0,

$H_0 : \Sigma_1 = \Sigma_2 = \dots = \Sigma_r =: \Sigma$

<u>MLE under Θ_0:</u>

$$\tilde{\mu}_i = \bar{X}^{(i)}$$

$$\tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^{r} A_i \quad , \text{ where } A_i = \sum_{j=1}^{n_i} \left(X_j^{(i)} - \bar{X}^{(i)}\right)\left(X_j^{(i)} - \bar{X}^{(i)}\right)^T$$

$$\sup_{\theta \in \Theta_0} L = (2\pi)^{-np/2} \left|n^{-1}A\right|^{-n/2} e^{-np/2}$$

So, the likelihood ratio

$$\Lambda = \frac{n^{np/2} \, |A|^{-n/2}}{\left(\prod_{i=1}^{r} n_i^{n_i p/2}\right)\left(\prod_{i=1}^{r} |A_i|^{-n_i/2}\right)}$$

For large $\min_i n_i$, Under H_0,

$$-2 \log \Lambda \sim \chi^2_{\dim \Theta - \dim \Theta_0}$$

$$\dim \Theta = rp + \frac{rp(p+1)}{2} \quad , \quad \dim \Theta_0 = rp + \frac{p(p+1)}{2}$$

$$\Rightarrow \quad -2 \log \Lambda \overset{\text{approx}}{\sim} \chi^2_{\frac{(r-1)p(p+1)}{2}}$$

$\rightarrow \quad X \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathbb{P}\left(\mu - \sqrt{\sigma^2} \leq X \leq \mu + \sqrt{\sigma^2}\right) \simeq 0.683$$

$$\mathbb{P}\left(\mu - 2\sqrt{\sigma^2} \leq X \leq \mu + 2\sqrt{\sigma^2}\right) \simeq 0.954$$

$$\mathbb{P}\left(\mu - 3\sqrt{\sigma^2} \leq X \leq \mu + 3\sqrt{\sigma^2}\right) \simeq 0.9973$$

Emperical Cummulative Distribution Function Of Sample

$$x_1, \ldots, x_n \qquad \hat{F}(y) = \frac{1}{n} \sum_{i=1}^{n} 1(x_i \leq y)$$

$\downarrow$ emperical cummulative distribution function

Continuity Correction Function

$$\tilde{F}(y) = \begin{cases} \frac{1}{n}\left(\sum_{i=1}^{n} 1(x_i \leq y) - \frac{1}{2}\right) & \\ 0 & \text{if } y < \min(x_i) \\ 1 & \text{if } y > \max(x_i) \end{cases}$$

Recall Φ is cdf of $\mathcal{N}(0,1)$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

Monte Carlo Estimates / Parametric Bootstrap

given $x_1, \ldots, x_n$

$$\bar{x} = \sum_{i=1}^{n} x_i \quad , \quad s^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$$

Sample n observations from $N(\bar{x}, s^2)$

↳ compute "corr coeff" — 1

Repeat this procedure large number of times (say 1000)
$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad$ " B

Corr coef - 1

Corr coef - 2

⋮

Corr coef - B

then sort the corr coeffs

Read the topic around Fig. 4.7