Sample Mean:

$$\underset{\sim}{\bar{X}} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X}_i$$

① $$E[\underset{\sim}{\bar{x}}] = \frac{1}{n}\sum_{i=1}^{n} E[\underset{\sim}{x}_i] = \underset{\sim}{\mu}$$

$$\Rightarrow E[\underset{\sim}{\bar{x}}] = \mu = E[\underset{\sim}{x}]$$

② $$\mathrm{Cov}(\underset{\sim}{\bar{x}}) = E[\underset{\sim}{\bar{x}}\,\underset{\sim}{\bar{x}}^T] - E[\underset{\sim}{\bar{x}}]\,E[\underset{\sim}{\bar{x}}]^T$$

$$\underset{\sim}{\bar{x}}\,\underset{\sim}{\bar{x}}^T = \left(\frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i^T\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \underset{\sim}{x}_i\,\underset{\sim}{x}_j^T$$

$$\mathrm{Cov}(\underset{\sim}{x}) = E[\underset{\sim}{x}\,\underset{\sim}{x}^T] - \mu\mu^T$$

$$\boxed{E[\underset{\sim}{x}\,\underset{\sim}{x}^T] = \Sigma + \mu\mu^T}$$

Assuming $x_i \perp\!\!\!\perp x_j$, $(i \neq j)$

$$E[\underset{\sim}{x}_i\,\underset{\sim}{x}_j] = E[\underset{\sim}{x}_i]\,E[\underset{\sim}{x}_j]^T = \mu\mu^T$$

when $i = j$, $E[\underset{\sim}{x}_i\,\underset{\sim}{x}_i^T] = \Sigma + \mu\mu^T$

$$\therefore E[\underset{\sim}{\bar{x}}\,\underset{\sim}{\bar{x}}^T] = \frac{1}{n^2}\left((n^2-n)\,\mu\mu^T + n(\Sigma + \mu\mu^T)\right)$$

$$\boxed{E[\underset{\sim}{\bar{x}}\,\underset{\sim}{\bar{x}}^T] = \underset{\sim}{\mu}\,\underset{\sim}{\mu}^T + \frac{\Sigma}{n}} \qquad \therefore \boxed{\mathrm{Cov}(\underset{\sim}{\bar{x}}) = \frac{\Sigma}{n}}$$

→ Next we want to calculate $E[S_n]$

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i \underset{\sim}{x}_i^T - \bar{\underset{\sim}{x}}\bar{\underset{\sim}{x}}^T$$

$$E[S_n] = \frac{1}{n}\sum_{i=1}^{n}\left(\Sigma + \mu\mu^T\right) - \left(\mu\mu^T + \frac{\Sigma}{n}\right)$$

$$= \Sigma + \mu\mu^T - \mu\mu^T - \frac{\Sigma}{n}$$

$$= \left(1 - \frac{1}{n}\right)\Sigma$$

Similarily, $\boxed{E[S_{n-1}] = \Sigma}$ & $\boxed{E[S_n] = \left(\frac{n-1}{n}\right)\Sigma}$

Unbiased Statistic:

A statistic T is unbiased for parameter τ if

$$E[T] = \tau$$

Characteristic function of R.V. :

Characteristic function of a Random Vector $\underset{\sim}{x}$ is defined as

$$\Phi_{\underset{\sim}{x}}(\underset{\sim}{t}) = E\left(e^{i\underset{\sim}{t}^T\underset{\sim}{x}}\right)$$

Cramer Wold Theorem: (Proof will not be discussed)

The distribution function of a random vector $\underset{\sim}{x} \in \mathbb{R}^p$ is known iff for any $\underset{\sim}{\alpha} \in \mathbb{R}^p$, the distribution of $\underset{\sim}{\alpha}^T\underset{\sim}{x}$ is known.

Normal (Gaussian) random variable (univariate):

$X \sim N(\mu, \sigma^2)$, i.e., X is a normal r.v. with mean μ and variance σ^2, if pdf of X is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in \mathbb{R}$$

Multivariate Normal Random Vector: (MVN)

$\underline{X}_{p\times 1}$ is said to follow MVN r.v. with mean $\underset{\sim}{\mu}_{p\times 1}$ & Cov. $\underset{\sim}{\Sigma}_{p\times p}$ if for all $\underset{\sim}{\alpha} \in \mathbb{R}^p$, the random variable $\alpha^T \underline{X} \sim N(\alpha^T \underline{\mu}, \underline{\alpha}^T \Sigma \underline{\alpha})$

Prove:

① $\underline{Y}_{q\times 1} = A_{q\times p} \underset{\sim}{X} + \underset{\sim}{b} \sim N_q(A\underline{\mu} + b, A\Sigma A^T)$

② Components of $\underset{\sim}{X}$ are normal r.v.s.

③ We want to obtain distribution of a subvector

$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}(1)_{q\times 1} \\ \underset{\sim}{X}(2) \end{pmatrix}, \quad q < p \qquad \underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}(1) \\ \underset{\sim}{\mu}(2) \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Take $A = (I_q \;\; 0)$ zero matrix of approp. dim

Using this transformation, try to find mean and variance

Recall (Linear Algebra)

Σ is +ve definite matrix

$\lambda_1, \dots, \lambda_p$ are eigenvalues of Σ

$E_1, \dots, E_p$ are normalized eigenvectors

$P = (E_1, E_2, \dots, E_p)$ (all columns orthogonal to each other)

$\Lambda = \text{Diag}(\lambda_1, \dots, \lambda_p)$

diagonal matrix lambda

then $\Sigma = P \Lambda P^T$ (spectral decomposition)

$\Sigma^{-1} = P \Lambda^{-1} P^T$

$\Sigma^{1/2} = P \Lambda^{1/2} P^T$, where $\Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_p^{1/2})$

<u>pdf of $\underset{\sim}{X}$:</u>

$\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

Let $\underset{\sim}{Y} = \Sigma^{-1/2}(\underset{\sim}{X} - \underset{\sim}{\mu})$

Then $\underset{\sim}{Y} = N_p(\underset{\sim}{0}, I)$ $\quad$ ($Cov(\underset{\sim}{Y}) = A \Sigma A^T = \Sigma^{-1/2} \Sigma \Sigma^{-1/2} = I$

$$\Rightarrow \underset{\sim}{Y} \sim N_p\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix} \right)$$

$\Rightarrow$ If $\underset{\sim}{Y} = (Y_1 \; Y_2 \; .. \; Y_p)^T$ then $Y_1, \dots, Y_p$ are iid univariate $N(0,1)$ r.v.s

$$\Rightarrow f_{\underline{x}}(\underset{\sim}{y}) = \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}} e^{-(y_i - 0)/2}$$

$$= \frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2} y^T y}$$

$$= \frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})}$$

Transformation of r.v.

$$y = \Sigma^{-1/2}(\underline{x} - \underline{\mu})$$

$$\underline{x} = \Sigma^{1/2} y + \mu$$

(We are going for Y to X)

$$J_{x \to y} = |\Sigma^{1/2}|$$

$$J_{y \to x} = \frac{1}{J_{x \to y}} = |\Sigma^{-1/2}|$$

So,

$$\boxed{f_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma^{1/2}|} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}}$$

Probability distribution of MVN variable