χ^2 Random Variable:

1. Central χ^2 RV

Let $x_1, x_2, \ldots, x_k$ are iid $N(0,1)$
Then $Y = \sum_{i=1}^{k} x_i^2$ said to follow central χ^2 distribution with k degrees of freedom

<u>Notation</u>: $Y \sim \chi^2_k$

2. <u>Non - central χ^2</u>:

Let $X_1, \ldots, X_k$ are independent normal RV's with $X_i \sim N(\mu_i, 1)$
then,
$Y = \sum_{i=1}^{k} X_i^2$ is said to follow non - central chi-squared distribution with k degrees of freedom & non- centrality parameter

$$\delta = \sum_{i=1}^{k} \mu_i^2$$

<u>Notation</u>: $Y \sim \chi^2_k(\delta)$

Now,
1. $\underset{\sim}{x} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$
Then $(\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \sim \chi^2_{(p)}$

<u>Proof</u>: $x \longrightarrow Y = \Sigma^{-1/2}(\underset{\sim}{x} - \underset{\sim}{\mu}) \sim N_p(0, I_p)$
Now, $(\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) = Y^T Y = \sum_{i=1}^{k} Y_i^2 \sim \chi^2_{(p)}$

(2) $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

Then $\quad \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} \sim \chi^2_{(p)} \, (\underset{\sim}{\mu}^T \Sigma^{-1} \underset{\sim}{\mu})$

Proof: $\quad X \longrightarrow Y = \Sigma^{-1/2} \underset{\sim}{X} \sim N(\Sigma^{-1/2}\mu, I)$

Now, $\quad \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} = y^T y = \Sigma y_i^2$

$$\Rightarrow \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} \sim \chi^2_{(p)} \, (\underset{\sim}{\mu}^T \Sigma^{-1} \underset{\sim}{\mu})$$

Characteristic Eq^n:

$$\Phi_{\underset{\sim}{x}}(t) = E_{\underset{\sim}{x}}\left[e^{i \underset{\sim}{t}^T \underset{\sim}{x}}\right]$$

→ $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$

$$\Phi_{\underset{\sim}{x}}(\underset{\sim}{t}) = E\left[e^{i \underset{\sim}{t}^T \underset{\sim}{x}}\right]$$

$$Y = \underline{t}^T \underline{x} \sim N(\underline{t}^T \underline{\mu}, \underline{t}^T \Sigma \underline{t})$$

Observe: $\quad \Phi_{\underline{x}}(\underline{t}) = E\left[e^{i \underline{t}^T \underline{x}}\right] = \Phi_Y(1)$, where $Y = \underline{t}^T \underline{x}$

$$= e^{i \underline{t}^T \underline{\mu} - \frac{1}{2} \underline{t}^T \Sigma \underline{t}}$$

$$\boxed{\Phi_{\underset{\sim}{x}}(\underset{\sim}{t}) = e^{i \underset{\sim}{t}^T \underset{\sim}{\mu} - \frac{1}{2} \underset{\sim}{t}^T \Sigma \underset{\sim}{t}}}$$

→ Let $\underset{\sim}{X} \sim N_p(\underline{\mu}, I_p)$

$B_{p \times p}$ is real, symmetric matrix

$\text{rank}(B) = k$

Then $\quad \underline{x}^T B \underline{x} \sim \chi^2_{(k)} (\underline{\mu}^T B \underline{\mu}) \iff B$ is idempotent i.e., $B^2 = B$

Proof: $(\Leftarrow)$

Let B be important

$\exists$ an orthogonal matrix H s.t.,

$$H^T B H = \begin{pmatrix} I_k & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix}$$

Transform $X \to \underline{Y} = H^T \underline{x} \sim N_p(H^T \underset{\sim}{\mu}, I)$

$$\Rightarrow \quad \underline{X} = H \underline{Y}$$

$$X^T B X = \underline{Y}^T H^T B H Y$$

$$= \sum_{i=1}^{k} Y_i^2 \sim \chi^2_{(k)}(\delta)$$

where $\delta = \sum_{i=1}^{k} \mu_i^2$

$$= (H^T \underline{\mu})^T (H^T BH)(H^T \underline{\mu})$$

$$= \mu^T B \mu$$

$(\Rightarrow)$ Let $\underline{X}^T B \underset{\sim}{X} \sim \chi^2_{(k)}(\underset{\sim}{\mu}^T B \underset{\sim}{\mu})$

$\exists$ an orthogonal matrix st $H^T B H = \begin{pmatrix} \lambda_1 & 0 & & & & \\ 0 & \lambda_2 & & & & 0 \\ & & \ddots & & & \\ & & & \lambda_n & & \\ & 0 & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix}$

$$Y = H^T X$$

$$\underline{X}^T B \underset{\sim}{X} = \sum_{i=1}^{k} \lambda x_i^2 \quad \text{check!}$$

$$Y_i^2 \sim \chi^2_{(i)}(\delta_i^2)$$