Univariate Variance

$$Var(X) = E\left((X - E(X))^2\right) \geq 0$$

Characterization Of Covariance Matrix

$$Cov(\underline{X}) = E\left[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T\right]$$

$$= E\left[\underline{X}\,\underline{X}^T - E[\underline{X}]\,\underline{X}^T - \underline{X}\,E[\underline{X}]^T + E[\underline{X}]E[\underline{X}]^T\right]$$

$$= E[\underline{X}\underline{X}^T] - E[\underline{X}]\left(E[\underline{X}]\right)^T$$

$\rightarrow$ $\underline{\alpha}^T \underline{x} = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$

say $Y = a_1 X_1 + b X_2$

$$Var(Y) = E[Y^2] - (E[Y])^2$$

$$= a^2 E[X_1^2] + b^2 E[X_2^2] + 2ab\, E[X_1 X_2]$$

$$- a^2 (E[X_1])^2 - b^2 (E[X_2])^2 - 2ab\, E[X_1]\, E[X_2]$$

$$= a^2 Var(X_1) + b^2 Var(X_2) + 2ab\, Cov(X_1, X_2)$$

i.e., $Var(Y) = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

where $Y = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$$\boxed{Var(\underline{\alpha}^T \underline{X}) = \underline{\alpha}^T Cov(\underline{X})\, \underline{\alpha}}$$

Denote $Cov(X) = \Sigma$

$\underline{\alpha}^T \Sigma \alpha \geq 0$ for any $\underline{\alpha} \in \mathbb{R}^P$

→ If Σ is a Cov matrix then $\underline{\alpha}^T \Sigma \underline{\alpha} \geq 0$

If $\underline{\alpha}^T \Sigma \underline{\alpha} \geq 0 \quad \forall\, \underline{\alpha} \in \mathbb{R}^p$ then Σ is a cov matrix of some RV

Proof : Consider a RV X with $E[\underline{X}] = 0$ & $Cov(\underline{X}) = I$

we know,

$$\Sigma_{p \times p} = C^T_{p \times p}\, C_{p \times p} \qquad \text{(from linear algebra)}$$

we define $Y = C^T X$

$$Cov(Y) = Cov(C^T X) = C^T Cov(X)\, C$$
$$= C^T I C = \Sigma$$

Linear Factor Of RV

Let $\underline{X}$ be a RV with $E[\underline{X}] = \underline{\mu}$ & $Cov(\underline{X}) = \Sigma$

Define $\underline{Y} = A_{k \times p}\, \underline{X}_{p \times 1} + \underline{b}_{k \times 1}$

where $\underline{Y} = (Y_1 \ldots Y_n)^T$

$$E[\underline{Y}] = A\underline{\mu} + \underline{b}$$

$$Cov(Y) = \begin{pmatrix} Var(Y_1) & Cov(Y_1, Y_2) & \cdots & Cov(Y_1, Y_p) \\ & Var(Y_2) & & \\ & & \ddots & \\ Cov(Y_p, Y_1) & & & Var(Y_p) \end{pmatrix}$$

Y_i = ith component of $A\underline{X} + \underline{b}$ $\quad (A_{k\times p}\, \underline{X}_{p\times 1} + \underline{b}_{k\times 1})$

$= A_i^T X_p + b_i$

$$\mathrm{Cov}(Y_i, Y_j) = \mathrm{Cov}(A_i^T \underline{x}, A_j^T \underline{x})$$ (as const term doesn't affect covariance)

$$= E\left[A_i^T \underline{x} (A_j^T \underline{x})^T\right] - \left(E[A_i^T \underline{x}]\right)\left(E[A_j^T \underline{x}]\right)^T$$

$$= E\left[A_i^T \underline{x}\, \underline{x}^T A_j\right] - \left(E[A_i^T \underline{x}]\right)\left(E[\underline{x}^T A_j]\right)$$

Finally, $\boxed{\mathrm{Cov}(A\underline{x} + \underline{b}) = A\, \mathrm{Cov}(\underline{x})\, A^T}$ check!

Partition of RV

$$\underline{X} = \begin{pmatrix} X_1 \\ \vdots \\ \cdots \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix}$$

$$E[\underline{x}] = \begin{pmatrix} E[\underline{x}_1] \\ E[\underline{x}_2] \end{pmatrix}$$

$$\mathrm{Cov}(\underline{X}) = \begin{pmatrix} \mathrm{Cov}(\underline{x}_1) & \mathrm{Cov}(\underline{x}_1, \underline{x}_2) \\ \mathrm{Cov}(\underline{x}_2, \underline{x}_1) & \mathrm{Cov}(\underline{x}_2) \end{pmatrix}$$

$$\mathrm{Cov}(\underline{x}_1) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p_1} \\ \\ \sigma_{p_1,1} & \cdots & \cdots & \sigma_{p_1,p_1} \end{pmatrix}, \quad \mathrm{Cov}(\underline{x}_2) = \begin{pmatrix} \sigma_{p_1+1,p_1+1} & \cdots & \sigma_{p_1+1,p} \\ \vdots & & \\ \sigma_{p,p_1+1} & \cdots & \sigma_{p,p} \end{pmatrix}$$

$$\mathrm{Cov}(\underline{x}_1, \underline{x}_2) = E\left[\underline{x}_1 \underline{x}_2^T\right] - E[\underline{x}_1]\, E[\underline{x}_2^T]$$

$$\therefore \quad Cov(\underline{X}) = \begin{pmatrix} Cov(\underline{X}_1) & Cov(\underline{X}_1, \underline{X}_2) \\ Cov(\underline{X}_2, \underline{X}_1) & Cov(\underline{X}_2) \end{pmatrix}$$

where $Cov(\underline{X}_1, \underline{X}_2) = \left(Cov(\underline{X}_2, \underline{X}_1) \right)^T$

Correlation Matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ & \sigma_{22} & & \\ & & \ddots & \\ & & & \sigma_{pp} \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_{11} & & & 0 \\ & \sigma_{22} & & \\ & & \ddots & \\ 0 & & & \sigma_{pp} \end{pmatrix}$$

$$R = D^{-1/2} \Sigma D^{1/2}$$

where
$$D^{-1/2} = \begin{pmatrix} \sigma_{11}^{-1/2} & & \\ & \ddots & \\ & & \sigma_{pp}^{-1/2} \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} & \cdots \\ & \ddots & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}, \quad r_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\,\sigma_{jj}}}$$

$\rightarrow$ correlation b/w X_i & X_j

Sampling From Multivariate Pop^n

Let $X_1, \ldots, X_n$ be random sample from a multivariate pop^n with mean $\underline{\mu}$ and covariance Σ

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots \\ & \ddots & \\ & & \sigma_{pp} \end{pmatrix}$$

$\therefore \frac{p(p+1)}{2}$ unique elements

→ Random $\mathbb{X}$ Sample $= (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$

$$= \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ \vdots & \vdots & & \vdots \\ x_{1p} & x_{2p} & & x_{np} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_p^T \end{pmatrix}, \quad \text{where} \quad \mathbf{y}_j^T = (x_{1j}, x_{2j}, \dots, x_{nj}), \quad 1 \le j \le p$$

Sample mean $\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_2 \end{pmatrix}$

$$\bar{x}_j = \frac{1}{n} \mathbf{y}_j^T 1_n \quad \text{where} \quad 1_n^T = (1, 1, \dots, 1)$$

(Sample Covariance)

$$S_n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{\mathbf{x}})(x_i - \bar{\mathbf{x}})^T$$

(S_{n-1} if divided by $n-1$)

$$= \frac{1}{n} \left[\sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}}) \mathbf{x}_i^T - \sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}}) \bar{\mathbf{x}}^T \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T - \bar{\mathbf{x}} \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i^T$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T - \bar{\mathbf{x}} \bar{\mathbf{x}}^T$$

$$n S_n = \mathbb{X} \mathbb{X}^T - n \bar{\mathbf{x}} \bar{\mathbf{x}}^T$$

$\left(\bar{\mathbf{x}} = \frac{1}{n} \mathbb{X} 1_n \right)$

$$= \mathbb{X} \mathbb{X}^T - n \left(\frac{1}{n} \mathbb{X} 1_n 1_n^T \mathbb{X} \right)$$

$$= \mathbb{X} \left(I_n - \frac{1}{n} 1_n 1_n^T \right) \mathbb{X}^T$$

Sample Correlation Matrix

$$S = \begin{pmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & & \vdots \\ s_{p1} & \cdots & s_{pp} \end{pmatrix}, \quad D = \begin{pmatrix} s_{11} & & & 0 \\ & s_{22} & & \\ & & \ddots & \\ 0 & & & s_{pp} \end{pmatrix}$$

$$\boxed{\begin{aligned} R &= D^{-1/2} S D^{-1/2} \\ &= (r_{ij})_{p \times p} \end{aligned}} \rightarrow \text{Correlation Matrix}$$

$$r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}\, s_{jj}}}$$

Geometry of Sample :- (Geometrical Interpretation)

① Projection of y_j on 1_n

$$\left(\frac{\underset{\sim}{y}_j^T 1_n}{1_n^T 1_n}\right) 1_n = \frac{1}{n}\left(\underset{\sim}{y}_j^T 1_n\right) 1_n = \bar{x}_j 1_n = \begin{pmatrix} \bar{x}_j \\ \vdots \\ \bar{x}_j \end{pmatrix}$$

② Deviation Vector

Note: inner product of deviation vector gives us cov. components

$$d_j = \underset{\sim}{y}_j - \bar{x}_j 1_n \quad \text{(subtracting projection from vector)}$$

$$S_{jj} = \frac{1}{n}\langle d_j, d_j \rangle$$

$$S_{ij} = \frac{1}{n}\langle d_i, d_j \rangle$$

③ Angle between $\underset{\sim}{y}_i$ and $\underset{\sim}{y}_j$ (θ_{ij})

$$\cos\theta_{ij} = \frac{\langle d_i, d_j \rangle}{\sqrt{\langle d_i, d_i \rangle \langle d_j, d_j \rangle}}$$

$$= \frac{s_{ij}}{\sqrt{s_{ii}\ s_{jj}}} = r_{ij}$$

$\therefore$ $\cos\theta_{ij}$ = correlation coefficient

If $\theta_{ij} = 0$, then $r_{ij} = 1$

If $\theta_{ij} = \pi/2$, then $r_{ij} = 0$

If $\theta_{ij} = \pi$, then $r_{ij} = -1$

Independence Of Two Random Variable:

Two random variables X and Y are independent if the joint pdf is the product of marginal pdfs

$$\underbrace{f_{X,Y}(x,y)}_{\text{joint pdf}} = \underbrace{f_X(x)}_{\text{marginal pdf of X}} f_Y(y) \quad \text{—— ①}$$

$$f_X(x) = \int_y f_{X,Y}(x,y)\, dy$$

$\perp\!\!\!\perp \rightarrow$ Independent

Claim: If $X \perp\!\!\!\perp Y$ then $Cov(X,Y) = 0$

Proof: $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$= \int_x\int_y xy\, f_{X,Y}(x,y)\, dx\, dy - \left(\int_x x f_X\, dx\right)\left(\int_y y f_Y\, dy\right)$$

$$= 0 \quad (\because ①)$$

Claim: Uncorrelated $\not\Rightarrow$ independent

X	-1	0	1
$P(X=x)$	$1/3$	$1/3$	$1/3$

$Y = X^2$:

Y	0	1
$P(Y=y)$	$1/3$	$2/3$

$$\text{Cov}(X, Y) = E(XY) = \frac{1}{3} \times 1 + \frac{1}{3} \times (-1) = 0$$

but X and X^2 are related