$X_1 \ldots X_n$ are i.i.d. $N_p(\mu, \Sigma)$

What about the situation when normality is not there? Or underlying distribution is not known.

Asymptotic:

$X_1 \ldots X_n$ are i.i.d. with mean μ & Variance σ^2

We have,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow[n \to \infty]{\text{in dist}} N(0,1) \quad , \text{ by CLT}$$

$$S^2 \xrightarrow[n \to \infty]{\text{in } P} \sigma^2 \quad , \text{ WLLN}$$

$$\text{i.e., } \quad S/\sigma \xrightarrow[n \to \infty]{\text{in } P} 1 \quad , \text{ (Continuous Mapping Theorem)}$$

Using Slutsky's Theorem

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \xrightarrow[n \to \infty]{\text{in dist}} N(0,1)$$

Therefore for large n,

$\frac{\bar{X} - \mu}{S/\sqrt{n}}$ is approximately $N(0,1)$ distributed

→ Applications:

1. To test $H_0 : \mu = \mu_0$
2. For constructing confidence set for μ

Multivariate

$\underline{X}_1, \dots, \underline{X}_n$ are iid with mean $\underline{\mu}$ & covariance matrix Σ, $\Sigma > 0$

Then,

$$(\underline{\bar{x}} - \mu) \xrightarrow[n \to \infty]{\text{in dist}} N(\underline{0}, \Sigma)$$

$$S \xrightarrow[n \to \infty]{\text{in } p} \Sigma$$

Then,

$$n(\bar{x} - \mu)^T S^{-1} (\bar{x} - \mu) \xrightarrow[n \to \infty]{\text{in dist}} \chi^2_p$$

Application:

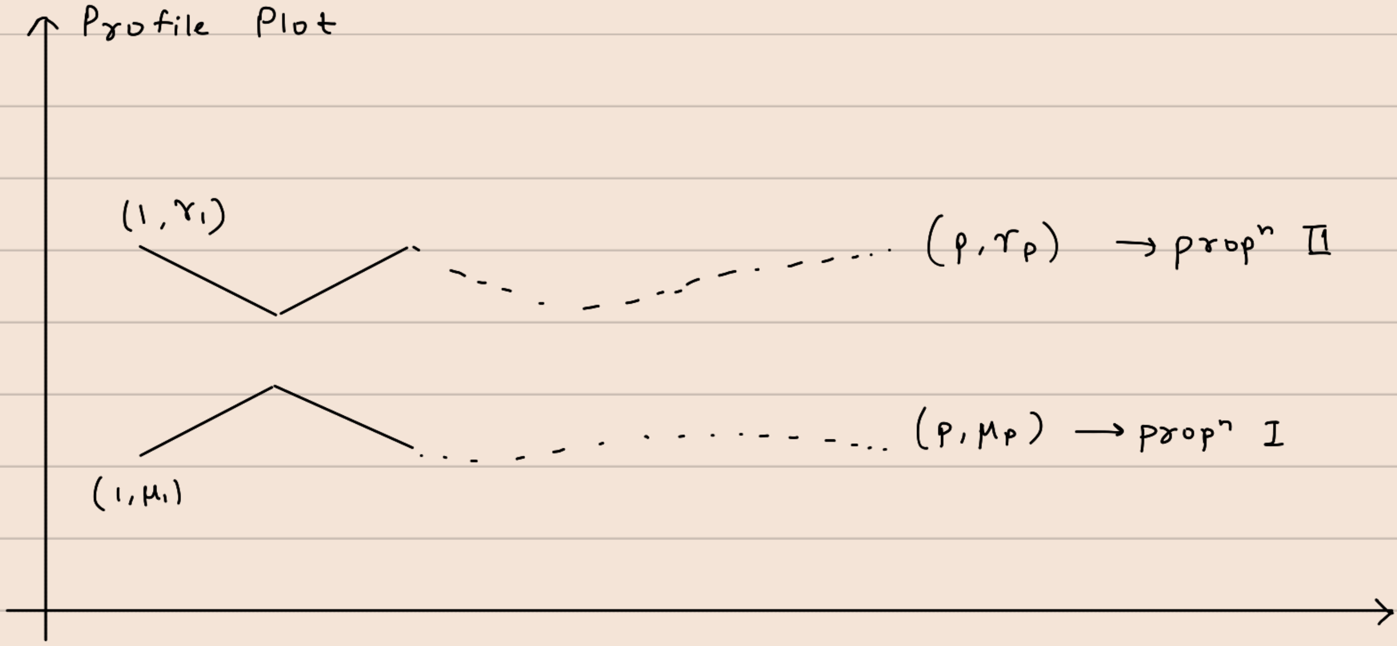
1. Testing $H_0 : \underline{\mu} = \underline{\mu}_0$
2. Finding confidence region for $\underline{\mu}$

Profile Analysis: (Chapter 6)

Prop^n - I : $N_p(\underline{\mu}, \Sigma)$

Prop^n - II : $N_p(\underline{\gamma}, \Sigma)$

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}, \quad \underline{\gamma} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{pmatrix}$$



Samples:

$X_1 \dots X_n$ from propⁿ I $\xrightarrow[\text{mean}]{\text{sample}}$ $\underline{\bar{X}}$

$Y_1 \dots Y_n$ from propⁿ II $\xrightarrow[\text{mean}]{\text{sample}}$ $\underline{\bar{Y}}$

Goals of Profile Analysis:

(i) To test $H_{01} : \mu_k - \gamma_k = \mu_{k-1} - \gamma_{k-1}$ for $k = 2, \dots, p$

against $H_{A1} : H_{01}$ is not true

(ii) If H_{01} is accepted

$H_{02} : \mu_k = \gamma_k \quad , \quad k = 1, 2, \dots, p$

against $H_{A2} : H_{02}$ is not true

equivalently : $H'_{02} : \sum_{k=1}^{p} \mu_k = \sum_{k=1}^{p} \gamma_k$

against $H'_{A2} : H'_{02}$ is not true

(iii) If H_{01} & H_{02} are accepted

$H_{03} : \mu_1 = \mu_2 = \dots = \mu_p = \gamma_1 = \gamma_2 = \dots = \gamma_p$

$H_{A3} : H_{03}$ is not true

$$\left.\begin{array}{ll} \underline{\bar{X}} & \sim N_p(\mu, \Sigma/n) \\ \underline{\bar{Y}} & \sim N_p(\gamma, \Sigma/m) \\ (n-1) S_x & \sim W_p(n-1, \Sigma) \\ (m-1) S_y & \sim W_p(m-1, \Sigma) \end{array}\right\} \text{indep}$$

$$(n+m-2) S = (n-1) S_x + (m-1) S_y \sim W_p(n+m-2, \Sigma)$$

$$\underline{\bar{X}} - \underline{\bar{y}} \sim N_p\left(\mu - \gamma, \left(\frac{1}{n} + \frac{1}{m}\right) \Sigma\right)$$

Take

$$\underset{p-1 \times p}{A} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \\ 0 & \cdots & \cdots & 1 & -1 \end{pmatrix}$$

Note that

$$H_{01} : A(\underline{\mu} - \underline{\gamma}) = 0$$

and

$$A(\underline{\bar{x}} - \underline{\bar{y}}) \sim N_{p-1}\left(A(\underline{\mu} - \underline{\gamma}), \left(\frac{1}{n} + \frac{1}{m}\right) A \Sigma A^T\right)$$

$$(n+m-2) A S A^T \sim W_{p-1}(n+m-2, A \Sigma A^T)$$

Use Hotelling's T^2 to test $H_{01} : A(\underline{\mu} - \underline{\gamma}) = \underline{0}$

(Homework)

Profile Analysis Pg. 323

(ii) To test $H'_{02} : \sum_{k=1}^{p} \mu_k = \sum_{k=1}^{p} \gamma_k$, i.e., $1^T(\mu - \gamma) = 0$

against $H'_{A2} : H'_{02}$ is not true

Similar to previous one,

instead of A, Take

$$1^T(\bar{x} - \bar{y}) \sim N\left(1^T(\underline{\mu} - \underline{\gamma}), \left(\frac{1}{n} + \frac{1}{m}\right) 1^T \Sigma 1\right)$$

Fill details : Homework (Use χ^2 or T^2 test)

$\rightarrow$ $(n+m-2)\, \underline{1}^T S\, \underline{1} \sim W(n+m-2,\ 1^T \Sigma 1)$

(iii) $H_{03} : \mu_1 = \mu_2 = \ldots = \mu_p = \gamma_1 = \gamma_2 = \ldots = \gamma_p$

$H_{A3} : H_{03}$ is not true

Under H_{03},

$$\bar{\bar{\underset{\sim}{Z}}} = \frac{n\bar{x} + m\bar{y}}{n+m} \sim N_p\left(\mu, \left(\frac{1}{m+n}\right)\Sigma\right)$$

$H'_{03} : \mu_1 = \ldots = \mu_p$

$$A = \begin{pmatrix} 1 & -1 & 0 & \ldots & & 0 \\ 0 & 1 & -1 & \ldots & & 0 \\ & & & & & \\ 0 & \ldots & & 0 & 1 & -1 \end{pmatrix}$$

$H'_{03} : A\mu = 0$

fill details – homework