$$\hat{\mu}_{MLE} = \bar{\underline{x}}$$

$$\hat{\Sigma}_{MLE} = S_n$$

Lemma: $\bar{\underline{x}} \sim N_p(\mu, \Sigma/n)$

→ We can use characteristic function to prove this

Proof: $\Phi_{\underline{x}_i}(\underset{\sim}{t}) = \exp\left[i \underset{\sim}{t}^T \mu - \frac{1}{2} \underset{\sim}{t}^T \Sigma \underset{\sim}{t}\right]$

$$\bar{\underline{x}} = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i = \sum_{i=1}^{n} \frac{1}{n} x_i = \sum_{i=1}^{n} a_i x_i$$

Let $y = \sum_{j=1}^{n} a_j \underline{x}_j$

$$\Phi_{\underline{y}}(\underset{\sim}{t}) = \prod_{j=1}^{n} \Phi_{a_j x_j}(\underset{\sim}{t})$$

$$= \prod_{j=1}^{n} \exp\left[a_j i t^T \mu - \frac{a_j^2}{2} t^T \Sigma t\right]$$

$$= \exp\left[\sum_{j=1}^{n} \left(a_j i t^T \underline{\mu} - \frac{a_j^2}{2} t^T \Sigma t\right)\right]$$

at $a_j = \frac{1}{n}$

$$= \exp\left[i t^T \mu - \frac{1}{2} t^T (\Sigma/n) t\right]$$

Def^n: Wishart distribution (WD)

Suppose $Y_1 \ldots Y_n$ are i.i.d. $N_p(\underline{0}, \Sigma)$, $\Sigma > 0$

Then $A = \sum_{j=1}^{n} Y_j Y_j^T$ is said to follow wishart distribution with parameters n & Σ, it is denoted by $A \sim W_p(n, \Sigma)$

→ WD is multivariate generalization of χ^2 distribution

Result ①: Suppose $A \sim W_m(n, \Sigma)$ and $C_{q \times m}$ be a non-random matrix, $q \le m$

then $CAC^T \sim W_q(n, C\Sigma C^T)$

Result ②: Let $A_1 \sim W_p(n_1, \Sigma)$ & $A_2 \sim W_p(n_2, \Sigma)$ are independent then

$A_1 + A_2 \sim W_p(n_1+n_2, \Sigma)$

Observe:

It is possible that $x_1 + x_2 \sim N$

but $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is not normal

Eg H/W: $\begin{pmatrix} x \\ y \end{pmatrix} \sim \frac{1}{2} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right) + \frac{1}{2} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}\right)$

Marginal distribution is normal

$Cov(x, y) = 0$

Theorem: $\underline{X}_1, \dots, \underline{X}_n$ iid $N_p(\mu, \Sigma)$

(i) $\bar{\underline{X}} \sim N_p(\underline{\mu}, \Sigma/n)$

(ii) $A = (n-1) S_{n-1} \sim W_p(n-1, \Sigma)$ (next class)

(iii) $\bar{\underline{X}}$ & A are independent

Proof: (i) Can be done using characteristic function

(ii) next class

(iii) $A = \sum_{i=1}^{n} (\underline{x}_i - \bar{\underline{x}})(\underline{x}_i - \bar{\underline{x}})^T$

$= \sum_{i=1}^{n} \underline{Y}_i \underline{Y}_i^T$, where $\underline{Y}_i = \underline{x}_i - \bar{\underline{x}}$

claim: joint dist^n of $(\bar{\underline{x}}, \underline{Y}_i)$ is normal

claim: $Cov(\bar{\underline{x}}, \underline{Y}_i) = \underline{0}$

$$\Rightarrow \quad \underline{\bar{X}} \perp\!\!\!\perp Y_i \quad \forall i$$

$$\Rightarrow \quad \underline{\bar{X}} \perp\!\!\!\perp \sum Y_i Y_i^T = A$$

Consistency:

$$\underline{\bar{X}} \xrightarrow{P} \mu$$

$$S_n \xrightarrow{P} \Sigma$$

$$\bar{X}_i \xrightarrow{P} \mu_i$$