Remark: If Σ is not +ve definite, then pdf of $\underset{\sim}{X}$ does not exist

In such a scenario $\underset{\sim}{X}$ is known as singular normal random variable

Random Variable

- Discrete (Support set countable)
- Continuous (Support set uncountable)
  - Absolutely cont. (Prob. measure is abs. cont wrt Lebesgue measure)
  - Singular (The lebesgue measure of support set is 0)

Example of singular RV:

$X_1 \sim N(0,1)$

$X_2 = X_1 + 1$

$\therefore X_2 \sim N(1,1)$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

$$\text{Support} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1 + 1\}$$

Lebesgue measure (support set) = 0
over $\mathbb{R}^2$
(as dimension of support = 1)

→ $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}^{(1)}_{q\times 1} \\ \underset{\sim}{X}^{(2)}_{(p-q)\times 1} \end{pmatrix} \; ; \; \underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{\mu}^{(2)} \end{pmatrix} \; ; \; \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Claim: If $\Sigma_{12} = 0$, then $X^{(1)} \perp\!\!\!\perp X^{(2)}$

Proof:

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(\frac{-1}{2} (\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \right)$$

Observe that

$$(\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) = \begin{pmatrix} \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{pmatrix}^T \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{pmatrix}^T \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{pmatrix}$$

$$= (\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})^T \Sigma_{11}^{-1} (\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}) + (\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})^T \Sigma_{22}^{-1} (\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})$$

&

$$|\Sigma| = |\Sigma_{11}| \, |\Sigma_{22}|$$

$$\therefore \; f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{1}{(2\pi)^{q/2}\,|\Sigma_{11}|} \exp\left((\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})^T \Sigma_{11}^{-1} (\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}) \right)$$

$$\times \frac{1}{(2\pi)^{(p-q)/2}\,|\Sigma_{22}|} \exp\left((\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})^T \Sigma_{22}^{-1} (\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)}) \right)$$

$$= f_{\underset{\sim}{X}^{(1)}}(\underset{\sim}{x}^{(1)}) \times f_{\underset{\sim}{X}^{(2)}}(\underset{\sim}{x}^{(2)})$$

$$\Rightarrow \; X_{(1)} \perp\!\!\!\perp X_{(2)}$$

Conditional Density:

Conditional pdf of $X \mid Y = y$ is defined as $f_{X|Y=y} = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

If $X \perp\!\!\!\perp Y$, $f_{X|Y=y}(x) = f_X(x)$

$\rightarrow$ $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}^{(1)} \\ \underset{\sim}{X}^{(2)} \end{pmatrix}$$

we want to obtain conditional pdf of $\underset{\sim}{X}^{(1)} \mid \underset{\sim}{X}^{(2)} = \underset{\sim}{x}^{(2)}$

we know,

$$\underset{\sim}{A}\underset{\sim}{X} + \underset{\sim}{b} \sim N_p(A\underset{\sim}{\mu} + \underset{\sim}{b}, \; A\Sigma A^T)$$

Take $A = \begin{pmatrix} I_q & \Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-q} \end{pmatrix}$

$$Y = A(\underset{\sim}{x} - \underset{\sim}{\mu})$$

$$\Rightarrow Y = \begin{pmatrix} \underset{\sim}{X}^{(1)} - \underset{\sim}{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{X}^{(2)} - \underset{\sim}{\mu}^{(2)}) \\ \underset{\sim}{X}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{Y}^{(1)} \\ \underset{\sim}{Y}^{(2)} \end{pmatrix}$$

$$E[Y] = 0$$

1 $\operatorname{Cov}(\underset{\sim}{y}^{(1)}) = \operatorname{Cov}(\underset{\sim}{x}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underset{\sim}{x}^{(2)})$

$= \operatorname{Cov}(\underset{\sim}{x}^{(1)}) + \Sigma_{12}\Sigma_{22}^{-1}\operatorname{Cov}(\underset{\sim}{x}^{(2)})(\Sigma_{12}\Sigma_{22}^{-1})^T$

$-2\operatorname{Cov}(\underset{\sim}{x}^{(1)}, \underset{\sim}{x}^{(2)})(\Sigma_{12}\Sigma_{22}^{-1})^T$

$= \Sigma_{11} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$\operatorname{Cov}(\underset{\sim}{y}^{(1)}) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \Sigma_{11\cdot 2} \rightarrow ?$

$\operatorname{Cov}(y^{(2)}) = \Sigma_{22}$

$\operatorname{Cov}(y^{(1)}, y^{(2)}) = \operatorname{Cov}(x^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}x^{(2)}, x^{(2)})$

$= \operatorname{Cov}(x^{(1)}, x^{(2)}) - \Sigma_{12}\Sigma_{22}^{-1}\operatorname{Cov}(x^{(2)})$

$= \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}$

$= 0$

Thus $y^{(1)} \perp\!\!\!\perp y^{(2)}$

$\therefore \boxed{\underset{\sim}{f}_{\underline{y}^{(1)} \mid \underline{y}^{(2)} = y^{(2)}} = f_{\underline{y}^{(1)}}}$

$\Rightarrow X^{(1)} - \mu^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(X^{(2)} - \mu^{(2)}) \mid X^{(2)} = x^{(2)} \sim N_q(0, \Sigma_{11\cdot 2})$

$\Rightarrow \underset{\sim}{X}^{(1)} \mid \underset{\sim}{X}^{(2)} = \underset{\sim}{x}^{(2)} \sim N_q(\underset{\sim}{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)}), \Sigma_{11\cdot 2})$