Testing Of Covariance Matrix

$X_1 \dots X_n$ is a random sample (iid) from $N_p(\mu, \Sigma)$

$\mu \in \mathbb{R}^p$, $\Sigma > 0$ are unknown

(1) $H_0 : \Sigma = \lambda I$, where I is identity

against H_A: not H_0

Under H_0

$$\bar{\underline{X}} \sim N_p\left(\mu, \frac{\lambda}{n} I\right)$$

The likelihood function is

$$L(\mu, \Sigma) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(\frac{-1}{2} \operatorname{tr}(\Sigma^{-1} A)\right) \exp\left(-\frac{n}{2} (\bar{\underline{x}} - \mu)^T \Sigma^{-1} (\bar{\underline{x}} - \mu)\right)$$

where,

$$A = \sum_{i=1}^{n} (\underline{x}_i - \bar{\underline{x}})(\underline{x}_i - \bar{\underline{x}})^T$$

$$\Theta = \{(\mu, \Sigma) : \mu \in \mathbb{R}^p, \Sigma > 0\}$$

$$\Theta_0 = \{(\mu, \Sigma) : \mu \in \mathbb{R}^p, \Sigma = \lambda I, \lambda > 0\}$$

$$\dim \Theta = p + \frac{p(p+1)}{2}, \quad \dim \Theta_0 = p + 1$$

The likelihood ratio

$$\Lambda = \frac{\sup_{\theta \in \Theta_0} L}{\sup_{\theta \in \Theta} L}$$

Let us compute $\sup_{\theta \in \Theta} L$.

Unrestricted mle

$$\hat{\mu}_{mle} = \bar{x} \quad \& \quad \hat{\Sigma}_{mle} = \frac{A}{n}$$

So,

$$\sup_{\theta \in \Theta} L = (2\pi)^{-np/2} \, |n^{-1}A|^{-n/2} \exp\left(-\tfrac{1}{2} np\right)$$

$$= (2\pi)^{-np/2} \, |A|^{-n/2} \, n^{n/2} \exp\left(-\tfrac{1}{2} np\right)$$

Now,

$$\sup_{\theta \in \Theta_0} L = ?$$

Under Θ_0,

$$\tilde{\mu}_{mle} = \bar{x}$$

$$\tilde{\lambda}_{mle} = \frac{1}{np} \operatorname{tr}(A)$$

So,

$$\sup_{\theta \in \Theta_0} L = (2\pi)^{-np/2} \left(\operatorname{tr}(A)/np\right)^{-np/2} \exp\left(\frac{-np}{2}\right)$$ check this!

→ Therefore,

$$\Lambda = \frac{(np)^{np/2} \, (\operatorname{tr}(A))^{-np/2}}{n^{np/2} \, |A|^{-n/2}}$$

→ Large Sample Test

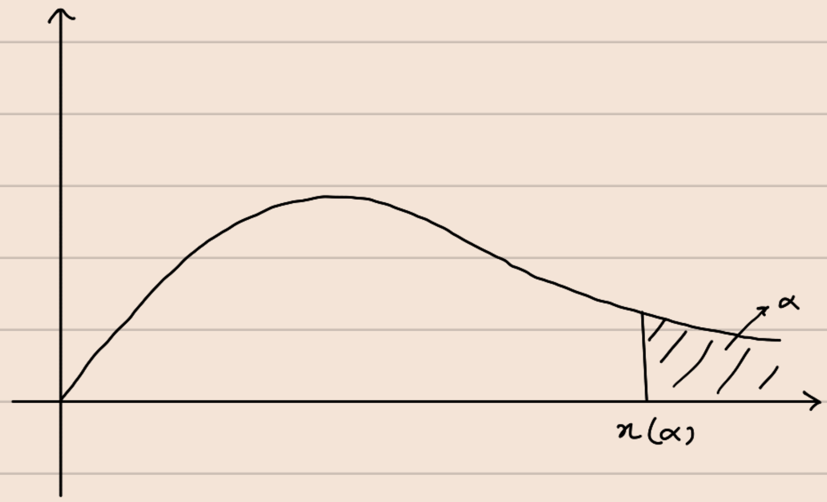
$$-2 \log \Lambda \underset{\text{asymptotic}}{\sim} \chi^2_{\dim \Theta - \dim \Theta_0}, \quad \text{Under } H_0$$

So, under H_0, for large n

$$-2 \log \Lambda \sim \chi^2_{\frac{p(p+1)}{2} - 1}$$

At level α, let $x(\alpha)$ be s.t.

$$\mathbb{P}\left(\chi^2_{\frac{p(p+1)}{2}-1} > x(\alpha)\right) = \alpha$$



we will reject H_0 if

observed $-2\log\Lambda > x(\alpha)$

p-value of $\Lambda = \mathbb{P}\left(\chi^2_{\frac{p(p+1)}{2}-1} > \text{obs}(-2\log\Lambda)\right)$

→ $$\Lambda = \frac{\sup_{\theta\in\Theta_0} L}{\sup_{\theta\in\Theta} L}$$

Eg:

cities:	A	B
infection rate:	20%	5%

$\Theta = \{A, B\}$

$\Theta_0 = \{A\}$

→ The test we discussed is known as test of sphericity pdf

$$\frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(\frac{-1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

If $\Sigma = \lambda I$

$$\cdots\cdots \exp\left(\frac{-1}{2\lambda} (x-\mu)^T (x-\mu)\right)$$ (contours are sphere)

Equality of r cov matrices

Samples

Pop^n I - $N_p(\mu_1, \Sigma_1)$; $X_1^{(1)} \dots X_{n_1}^{(1)}$

Pop^n II - $N_p(\mu_2, \Sigma_2)$;

$\vdots$

Pop^n r - $N_p(\mu_r, \Sigma_r)$; $X_1^{(r)} \dots X_{n_r}^{(r)}$

$H_0 : \Sigma_1 = \Sigma_2 = \dots = \Sigma_r$

against,

H_A : not H_0

we will use LRT (Likelihood ratio test)

→ The likelihood function is

$$L = L(\mu_1, \dots, \mu_r, \Sigma_1, \dots, \Sigma_r)$$

$$= \prod_{i=1}^{r} \left[(2\pi)^{-n_i p/2} |\Sigma_i|^{-n_i/2} \exp\left(\frac{-1}{2} tr\left(\Sigma_i^{-1} A_i\right)\right) \exp\left(\frac{-n_i}{2} \left(\bar{x}^{(i)} - \mu_i\right)^T \Sigma_i^{-1} \left(\bar{x}^{(i)} - \mu_i\right)\right)\right]$$

→ Unrestricted MLE, under Θ, Let $n = \sum_{i=1}^{r} n_i$

$$\hat{\mu}_i = \bar{\underline{X}}^{(i)} \quad \text{and} \quad \hat{\Sigma}_i = \frac{1}{n_i} A_i$$

So,

$$\sup_{\theta \in \Theta} L = (2\pi)^{-np/2} \left(\prod_{i=1}^{r} n_i^{n_i p/2} \right) \left(\prod_{i=1}^{r} |A_i|^{-n_i/2} \right) \exp(-np/2)$$