VICTOR MIRACLE ABOAGYE
TOGETHER WE CAN
T.W.C.21

2. Sequences and Series

A list of number is a sequence or progression..

A sequence can be written as $u_1, u_2, u_3, \ldots$ The nth term of the sequence is u_n.

2.1 Arithmetic Sequences

If the differences between successive term of a sequence are constant, then the sequence is an arithmetic progression (AP).The difference is called the common difference.

$1, 4, 7, 10$ is arithmetic progression with common difference 3.

$1, \frac{1}{2}, 0, -\frac{1}{2}, -1\frac{1}{2}$ is an A.P. with common difference $-\frac{1}{2}$.

If an arithmetic progression has first term a and common difference d, then the nth term is:

$$u_n = a + (n-1)d$$

The sum of first n term is:

$$\sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \cdots + u_n$$

$$= \frac{n}{2}\{2a + (n-1)d\}$$

2.1.1 Examples

2.1.2 Find tenth term and sum to 40 terms of the sequence $3, 5, 7, 9, \ldots$

Solution

The first term is 3 and the common difference is 2.

The tenth term is $3 + (10-1)2 = 21$.

Put $n = 40, a = 3$ and $d = 2$ into the sum of an AP

The sum of 40 terms is $\frac{1}{4}40(2 \times 3 + (40-12)2) = 1680$

2.1.2 The first two term an AP are 5 and 8.How many terms are less than $1,000$?

Solution

Here $a = 5$ and $d = 3$.Term is $5 + (n-1)3 = 2 + 3n$

If $2 + 3n < 1000$, then $n < \frac{998}{3} = 332\frac{2}{3}$.

2.1.2 The fifth of an AP is 13 and the eighth term is 19. Find the first term and nth term .

Solution

The nth term is $a+(n-1)d$ .Use the information above:

$13=a+4d$ and $19=a+7d$

Solve these equations to obtain $d=2$ and $a=5$.

The first term is 5 and the nth term is .

2.1.2 Exercises

1. Find the eleventh term of the following sequence:

   (a) $1,4,7,10$ (b) $13,17,21,25$

   (c) $4.2,4.4,4.6$ (d) $11,8,5,2$
2. Find the sum of the fist 12 terms for the sequences in Question 1.
3. Find expression for the nth term and the sum of the first n terms for the sequences in Question
4. Find the first term and the nth term for the sequences which have:

   (a) 5th term 12 and common difference 3

   (b) 4th term 14 and 9th term 29

   (c) 6th term 2.1 and 10th term 2.9

   (d) Third term 13 and seventh term -3.
5. The nth term of an Arithmetic Progression is $5+3n$. Find the common difference and the first term.
6. Find the common difference and the first term for the sequences which have nth term:

   (a) $2+5n$ (b) $7n-3$

   (c) $2+\frac{n}{5}$ (d) $3-2n$
7. The sum of the fist n terms of an Arithmetic Progression is $2n^2+n$. Find the first 3 terms, and hence find the common difference.
8. Find the first term and the common difference for the APs for which the sum of the first n terms is:

   (a) n^2+n (b) $3n-n^2$

   (c) $2n+3n^2$ (d) $n-\frac{1}{4}n^2$
9. Find the first term of the sequences $1,7,13,19$ which is greater than 100.
10. Which is the first negative term of the sequence $98,93,88,83,\ldots$? How many terms must be taken before their sum is negative?
11. Find the sum of the first n integers.
12. Find the sum of the fist n odd numbers, simplifying your answer as far as possible.
13. Find the sum of the fist ten multiples of 7 which are greater than 100.
14. $3,x,7$ form an Arithmetic Progression. Find x.
15. $53,x,y,25$ form an AP. Find x and y.

2.2 Geometric progressions

If the ratio between successive terms of a sequence is constant, than the sequence is a geometric progression (GP) The ratio is the common ratio.

$2, 6, 18, 54, 162$ is a geometric progression with common ratio 3.

$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$ is a geometric progression with common ratio $-\frac{1}{2}$.

If the first term of a geometric progression is a and the common ratio is r, then the nth term is:

$$u_n = ar^{n-1}$$

The sum of the first n terms of the sequence is:

$$\sum_{i=1}^{n} u_i = u_1 + u_2 + \cdots + u_n$$

$$= \frac{a(r^n - 1)}{r - 1}$$

If $-1 < r < 1$, then the sum to infinity of the sequence is:

$$\sum_{i=1}^{\infty} u_i = u_1 + u_2 + \cdots + u_n + \cdots$$

$$= \frac{a}{1-r}$$

2.2.1 Examples

2.2.2 Find the sixth term and the sum of the first 10 terms of the sequence:

$3, 6, 12, 24, \ldots$

Solution

The first term is 3 and the common ratio is 2.

Hence the sixth term is: $3 \times 2^{6-1} = 96$

The sum to 10 terms is: $\frac{3 \times 9(2^{10} - 1)}{2 - 1} = 3069$

2.2.2 How many terms of the sequence $1, 3, 9, 27, \ldots$ must be taken for the sum to exceed 1,000?

Solution

The sum to n terms is $\frac{1 \times (3^n - 1)}{3 - 1}$.

If this is to be greater than 1,000, then: $3^n > 2001$.

By logs or by trial and error find that $n > 6.9$.

7 terms of the sequence must be taken.

2.2.2 A ball is thrown upwards, and returns to the ground after 5 seconds. After each bounce the time until it returns to each is multiplied by a factor of $\frac{2}{3}$.

Solution

The time of the first bounce is 5 secs. The time for the second bounce is $5\times\frac{2}{3}$. The time for the third bounce is $5\times\left(\frac{2}{3}\right)^2$. And so on.

The total time is the sum to infinity of the GP with first term 5 and common ratio $\frac{2}{3}$.

Total time $=\dfrac{5}{1-\frac{2}{3}}=15$ seconds.

2.2.2 Exercise

1. Find the seventh term and the sum of the first 5 terms for the following geometric progressions:

   (a) $2,4,8,16,\ldots$ (b) $4,2,1,\frac{1}{2},\frac{1}{4},\ldots$

   (c) $2,-6,18,-54,\ldots$ (d) $-2,\frac{1}{2},-\frac{1}{8},\frac{4}{32},\ldots$

2. Find the nth term for each of the sequences in Question 1.
3. Find Expressions for the sum to n terms of each of the sequences in Question1.
4. Find, where possible , the sum to infinity of the sequence in Question 1.
5. Find the first term and the common ratio for the geometric progressions with:
   (a) Second term 3 and fifth term 24
   (b) Third term 8 and sixth term 1.
   (c) Third term 5 and sixth term -40
   (d) Second term 4 and sum to infinity 16.
6. The fourth term of a GP is 4 and the sixth is 100.Find two possible values for the common ratio, and hence find two possible value for the first term.
7. A GP has first term 5 and common ratio 4.How many terms of the sequence are less than $9,000$?
8. A GP has first term 2 and common ratio 1.5.How many term of the sequence most be taken before their sum is greater than 100?
9. A GP has first term 6 and common ratio 0.7.Which is the first term of the sequence less than 1?
10. $3,x,75$ are terms of a GP Find the possible value of x.
11. $8,x,y,-1$ are terms of a GP Find x and y.
12. Find the common ratio and the first term of the sequence whose sum to n terms is :

    (a) $2\times 3^n-2$ (b) $1-\left(\frac{1}{4}\right)^{n-1}$

2.3 Miscellaneous examples

2.4 The first , second and fifth term of an AP form consecutive terms of a GP Find the common ratio.

Solution

Let the first term be a, the common difference be d and the common ratio be r.

Then: $a+d=ar$

And $a+4d=ar^2$

Eliminate d between these equations:

$4(ar-a)=ar^2-a$

$4r-4=r^2-1$

$0=r^2 4r+3$

$(r-3)(r-1)=0$

The common ratio is 3 or 1

2.4 The first terms of GP is two third s of the sum to infinity .Find the common ratio.

Solution

Let the first term be a and the common ratio r.

$$a=\frac{\frac{2}{3}a}{1-r}$$

Hence $1-r=\frac{2}{3}$

The common ratio is $\frac{1}{3}$

2.4 At the beginning of each year a man invests 100 and at 10% compound interest. How much will be have at the beginning of the 13th year ,i.e. just after 13th the investment?

Solution

The 100 which was originally invested to.The 100 which was beginning of second year will have increased to 100×1.1^{11}. And so on .The total sum of money will be the sum of 13 terms of a GP , with first term 100 and common ratio 1.1.

$$\text{Total}=\frac{100\times(1.1^{13}-1)}{1.1-1}=2452$$

2.4 Miscellaneous exercises

1. The third term of an AP is x, and the seventh term is y.Express the common difference and the first term as functions of x and y.
2. $x-3, 2x-3, 6x+1$ are consecutive terms of a GP . Find the possible values of x.
3. Find the sum of all the numbers less than 100 whose last digit is 5.
4. A man invests 50 each year at 9% compound interest. How much will be have in the bank just before his tenth investment ?
5. A job is advertised at a starting salary of $12,750$ with annual increments of 900.How much will the salary be during the 10th years ? How much will be the total sum earned after 10 years?
6. A job is advertised at a starting salary of $12,000$ with annual increase of 5%.What will salary be during the 10th years ?What will the total sum earned after 10 years?

7. A child builds a triangular wall of bricks , with 1 brick on top now , 2 brick on the second , 3 on the third and so on. If the wall is 8 bricks high, how many bricks does it contain? If the child has only 70 bricks ,what is the tallest wall she can build?
8. A large sheet of paper is 0.04 mm thick. How thick is it after it has been folded ten time? (Assuming that this is possible). How many times must it be folded before it is 1 metre thick
9. A king was so pleased with the game of chess that he granted its inventor any prize he desired. The inventor asked for one grain of wheat on the first square, two grain on the second square, four grains on the third square and so on for all 64 square. How many grains is he asking for in total?
10. How many terms of the series $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ must be taken so that the sum is within 0.001 of its sum to infinity?
11. Let x be the recurring decimal $0.12121212\ldots$ Show that x can be written as:
$\frac{12}{100}+\frac{12}{10000}+\frac{12}{1000000}+\cdots$
Write this as the sum of geometric series and hence find x as a fraction in its simplest form.
12. By the method of Question 11 express the following as fraction in their simplest form:
(a) $0.72727272\ldots$ (b) $0.545454\ldots$
(c) $0.037037037\ldots$ (d) $0.135135135\ldots$
(e) $0.063636363\ldots$ (f) $0.00027027027\ldots$
13. Achilles can run at 10 feet per second, and the Tortoise at 1 foot per second. They have a race, and the Tortoise is given a 10 feet start. After 1 second T is 1 foot ahead. When A has run that extra foot, T is still ahead by 0.1 foot. When A has run that extra 0.1 ft, T is 0.01 ft ahead. Find how long it takes for A to catch up with T.

2.5.Examination questions

1. The first term of a geometric series is 1 and the sum of the first 3 terms is $\frac{7}{9}$. Find the two possible values or the common ratio of this series.
2. (i) The sum of an arithmetic progression is -600. The first 3 terms is 12, and the common difference is -3. How many terns are there in series?

(ii) The second and fourth terms of a geometric series are 2 and $\frac{1}{18}$. Calculate the first term and the values of the common ratio. Write down the first four terms of each of the two possible series.
(iii) Calculate the smallest number of terms the geometric series $1+4+16+\cdots$ that will give a total greater than 2×10^6.

3. (i) If a salary is increased during each year by 12% of its value at the start of the year, by what factor is it multiplied each year? By what factor will the original salary have been multiplied after five years and by what percentage has it increased in this time?

(ii) If the cost of repair to a new car is 20 in the first year and this cost increases thereafter by 25 each year, write down and simplify a formula for the cost of repairs in the nth year.

The owner decides to keep the car until the end of the year in which the total repair bill passes $1,000$. How old will the car be when he sells it?

4. The sum S_n of the first n terms of a series is given by $s_n = \frac{1}{4}(5n^2 + 11n)$.Write down an expression for S_{n-1}.Deduce the value of u_n the nth term if the series. Hence show that the series is an arithmetic series , and state the values if the first term and of the common difference.
5. The first term of a geometric series is 10 and thee second term is $10 - x$.Show that the set of values fir which the series is an arithmetic series is convergent is given by $0 < x < 20$. Show also that, for these values, the series convergences to a sum greater than 5.
   Given that the series converges to a sum 100, find the smallest positive integer n such that the sum of the first n terms is greater than 99.

Common errors

1. Be sure whether you are dealing with an arithmetic or a geometric progression. Some sequences are neither of these.
2. Do not confuse the nth term with the sum of n terms.
3. Sometime the term you want to deal with is not the nth but the $(n-1)$th or the $(n+1)$th. Make sure you know the number of the term.
4. Be very careful with problems which involve investment or salaries. Read the question carefully to make sure whether it is the beginning or the end of the year which is being with.
5. Be careful with signs. If an AP is decreasing, then the common difference is negative. If the common ratio of a GP is negative, then the terms are the terms are alternately positive and negative.
6. Be careful when taking square root. Problems involving FPs often require you to solve an equation like $r^2 = 4$.There are two solution: $r = 2$ or $r = -2$.Often the question will help you by requiring the values(plural)of r.

Solution (to exercise)

2.1.2

1.
   (a) 31
   (b) 53
   (c) 6.2
   (d) 19
2.
   (a) 210
   (b) 420

(c) 63.6

(d) -66

3.

(a) $3n-2, \frac{1}{2}n(3n-1)$

(b) $4n+9, \frac{1}{2}n(4n+22)$

(c) $\frac{5}{n}+4, \frac{1}{2}n(8.2+\frac{n}{5})$

(d) $14-3n, \frac{1}{2}n(25-3n)$

4.

(a) $0, 3n-3$

(b) $5, 3n+2$

(c) $1.1, \frac{n}{5}+0.9$

(d) $21, 25-4n$

5. $3, 8$

6.

(a) $5, 7$

(b) $7, 4$

(c) $\frac{1}{5}, \frac{11}{5}$

(d) $-2, 1$

7. $3, 7, 11, 4$

8.

(a) $2, 2$

(b) $2, -2$

(c) $5, 6$

(d) $\frac{3}{4}, -\frac{1}{2}$

9. 103

10. $-2, 41$

11. $\frac{1}{2}n(n+1)$

12. n^2

13. 1365

14. 5

15. $43\frac{2}{3}, 34\frac{1}{3}$

2.2.2

1.

(a) $128, 62$

(b) $\frac{1}{16}, 7\frac{3}{4}$

(c) $1458, 122$

(d) $-\frac{1}{2048}, -1.602$

2.

(a) 2^n

(b) $4\left(\frac{1}{2}\right)^{n-1}$

(c) $2(-3)^{n-1}$

(d) $-2\left(-\frac{1}{4}\right)^{n-1}$

3.

(a) $2\left(2^n - 1\right)$

(b) $8\left(1 - \frac{1}{2}^n\right)$

(c) $\frac{1}{2}\left(1 - (-3)^n\right)$

(d) $-\frac{8}{5}\left(1 - \left(-\frac{1}{4}\right)^n\right)$

4. $b, 8d, -1.6$

5.

(a) $1\frac{1}{2}, 2$

(b) $32, \frac{1}{2}$

(c) $1\frac{1}{4}, -2$

(d) $8, \frac{1}{2}$

6. $\pm 5, \frac{\pm 4}{125}$

7. 6

8. 9

9. 0.706

10. ± 15

11. $-4, 2$

12.

(a) $3,4$

(b) $\frac{1}{2},3$

2.4

1. $d=\frac{1}{4}(y-x), a=\frac{1}{2}(3x-y)$

2. 4 or $-1\frac{1}{2}$

3. 500

4. 709.6

5. $20,850,168,000$

6. $18616,150935$

7. $36,11$ brick high

8. $40.96\,\text{mm}$, 15 time

9. $2^{64}-1$

10. 12

11. $\frac{4}{33}$

12.

(a) $\frac{8}{11}$

(b) $\frac{6}{11}$

(c) $\frac{1}{27}$

(d) $\frac{5}{37}$

13. $1\frac{1}{9}$ seconds.

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References:

Solomon, R.C. (1997), *A Level: Mathematics* (4th Edition) , Great Britain, Hillman Printers(Frome) Ltd.