

# CALCULUS

## CHAPTER IV

### Integration

July, 2021.



# Outline

## 1 Integration

- Introduction
- Techniques of integration

## 2 Application



# Area Problem

## Discovery

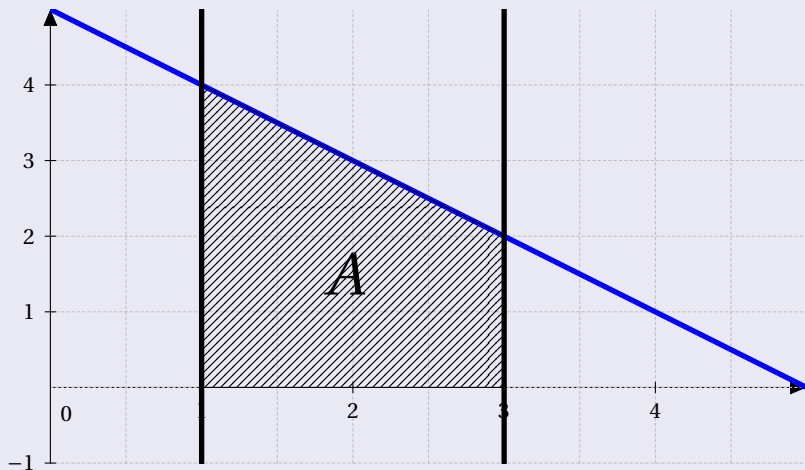
Let  $f(x) = -x + 5$ .

1) Find the area  $A$  of the region bounded by the graph of  $f$ , the  $x$ -axis, and the vertical lines  $x = 1$  and  $x = 3$ .



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*Solution:*  $A = (3 - 1) * 2 + (3 - 1) * 2/2 = 6$

2) Consider the function  $F(x) = -\frac{1}{2}x^2 + 5x + e$ .

a) Express  $\frac{dF(x)}{dx}$  in terms of  $f(x)$ .

b) What does the value  $F(3) - F(1)$  represent in relation to the graph of  $f$ ?



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a) Express  $\frac{dF(x)}{dx}$  in terms of  $f(x)$ .

b) What does the value  $F(3) - F(1)$  represent in relation to the graph of  $f$ ?

*Solution:* (a)  $F'(x) = f(x)$  (b)  $F(3) - F(1) = A$ .

$F$  is called the anti-derivative or indefinite integral of  $f$ .



# Anti-derivative or Indefinite Integral

## Definition

Let  $F$  and  $f$  be two functions such that

$$\frac{dF(x)}{dx} = f(x) \text{ for all } x \in I.$$

☛  $f$  is the derivative of  $F$ .

☛  $F$  is an **anti-derivative** of  $f(x)$  on the interval  $I = [a, b]$ .

We write

$$F(x) = \int f(x) dx$$

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## Theorem (Fundamental Theorem of Calculus (1))

$F(b) - F(a) = F(x) \Big|_a^b = \int_a^b f(x) dx$  is the **definite integral** from  $a$  to  $b$  of  $f(x)$  with respect to  $x$ .



# Anti-derivative or Indefinite Integral

## Exercise

1) Find an anti-derivative  $F(x)$  of  $f(x)$  for

a)  $f(x) = 2$    b)  $f(x) = 2x$    c)  $f(x) = 5x^4$    d)  $f(x) = x^7$    e)  $f(x) = \cos x$   
f)  $f(x) = -\sin x$    g)  $f(x) = -\sec^2 x$    h)  $f(x) = \frac{1}{x}$    e)  $f(x) = e^x$ .

2) Show that  $F(x) = \ln(3x+1)$  and  $G(x) = \ln(3x+1) + 4$  are two anti-derivatives of the same function.



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## Remark

If  $F(x)$  is an anti-derivative of  $f(x)$  then  $G(x) = F(x) + c$ , where  $c$  is a constant, is also an anti-derivative of  $f(x)$

since  $\frac{dG(x)}{dx} = \frac{d}{dx} (F(x) + c) = f(x)$ .



# Anti-derivative or Indefinite Integral

## Summary

$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$
$\int \sin x dx = -\cos x + c$
$\int \cos x dx = \sin x + c$
$\int \cosh x dx = \sinh x + c$
$\int \sinh x dx = \cosh x + c$
$\int (1 + \tan^2 x) dx = \tan x + c$
$\int e^x dx = e^x + c$

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$
$\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + c$
$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arsinh} x + c$
$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arcosh} x + c$
$\int \frac{1}{1+x^2} dx = \arctan x + c$
$\int \frac{1}{x} dx = \ln  x  + c$



# Hyperbolic tangent, $\tanh$ , and cotangent, $\coth$ , functions

☛ Hyperbolic tangent,  $\tanh$  :

$\tanh : \mathbb{R} \rightarrow [-1, 1]$  is defined by  $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ .

$$\frac{d \tanh x}{dx} =$$



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$\frac{d \tanh x}{dx} = 1 - \tanh^2 x$ . This implies  $\frac{d}{dy} \operatorname{atanh} y = \frac{1}{1 - y^2}$  for  $y \in [-1, 1]$ , i.e.  $|y| < 1$ .



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### Theorem (Integrability of continuous function)

*If  $f$  is a continuous function over an interval  $[a, b]$ , or if  $f$  has at most finitely many jump discontinuities there, then the definite integral  $\int_a^x f(u) \, du$ ,  $x \in [a, b]$ , exists and  $f$  is said to be **integrable** over  $[a, b]$ .*

### Remark

*We recall that  $\int_a^x f(u) \, du = F(u) \Big|_a^x = F(x) - F(a)$ .*



## Property (Properties of Definite Integrals)

*If  $f$  and  $g$  are integrable over the interval  $[a, c]$ , then*

$$1) \int_a^a f(x) \, dx = 0$$

$$2) \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx; b \in [a, c].$$

$$3) \int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$$



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$$4) \int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx$$

$$5) \int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$



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$$6) \text{ If } f(x) \leq g(x), \forall x \in [a, b], \text{ then } \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$



## Exercise

*Compute the following definite integrals*

1)  $\int_2^2 x^2 e^x dx$

2)  $\int_1^{-1} \frac{1}{1+x^2} dx$

3)  $\int_0^1 |3x+3| dx$

4)  $\int_0^1 (2x-x^4) dx$

5)  $\int_0^\pi (2\sin x - \cos x) dx$

6)  $\int_0^\pi (1 + \cos t) dt$

7)  $\int_0^{\pi/6} (\sec \theta + \tan \theta)^2 d\theta$

8)  $\int_{\pi/2}^\pi \frac{\sin(2u)}{2\sin u} du$

9)  $\int_{-4}^3 6|6-x-x^2| dx$

10)  $\int_0^\pi |\sin x - \cos x| dx$

11)  $\int_0^{\pi/4} \tan^2 x dx$



# Integration by substitution

## Example

Find  $\int 7(2x-1)(x^2-x+3)^4 dx$ .



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Let  $u = x^2 - x + 3$ .

$u$  is differentiable, and  $\frac{du}{dx} = 2x-1$  implies  $\frac{1}{2x-1} du = dx$ .



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Then,

$$\begin{aligned}\int f(x) dx &= \int 7(2x-1)u^4 \frac{1}{2x-1} du = \int 7u^4 du = 7\left(\frac{1}{4+1} u^{4+1}\right) = \frac{7}{5} u^5 \\ &= \frac{7}{5} (x^2 - x + 3)^5.\end{aligned}$$

## Property

$$\int u' u^n dx = \int \frac{du}{dx} u^n dx = \int u^n du = \frac{1}{n+1} u^{n+1} + c$$



## Example

Find  $\int \sin^2 x \cos x dx$ .



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## Example

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Let  $u = \sin x$ .

$u$  is differentiable, and  $\frac{du}{dx} = \cos x$  implies  $\frac{1}{\cos x} du = dx$ .

Then,

$$\begin{aligned}\int \sin^2 x \cos x dx &= \int u^2 \cos x \frac{1}{\cos x} du = \int u^2 du = \left( \frac{1}{2+1} u^{2+1} \right) = \frac{1}{3} u^3 \\ &= \frac{1}{3} \sin^3 x.\end{aligned}$$

## Property

$$\textcircled{1} \quad \int u' \sin u \, dx = \int \frac{du}{dx} \sin u \, dx = \int \sin u \, du = -\cos u + c$$

$$\textcircled{2} \quad \int u' \cos u \, dx = \sin u + c$$



## Property

$$\textcircled{1} \int \frac{u'}{\sqrt{1-u^2}} dx = \int \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin u + c$$

$$\textcircled{2} \int -\frac{u'}{\sqrt{1-u^2}} dx = \cos u + c$$

## Example

Find  $\int \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}} dx.$



## Property

$$\textcircled{1} \int \frac{u'}{\sqrt{1-u^2}} dx = \int \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} dx = \int \frac{1}{\sqrt{1-u^2}} du = \text{asin } u + c$$

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## Example

Find  $\int \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}} dx$ .

Let  $u = x^3 - x^2$ .

$$\frac{du}{dx} = 3x^2 - 2x \text{ implies } \frac{1}{3x^2 - 2x} du = dx.$$



## Property

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## Example

Find  $\int \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}} dx$ .

Let  $u = x^3 - x^2$ .

$$\frac{du}{dx} = 3x^2 - 2x \text{ implies } \frac{1}{3x^2 - 2x} du = dx.$$

Then,

$$\begin{aligned} \int f(x) dx &= \int \frac{3x^2 - 2x}{\sqrt{1 - u^2}} \frac{1}{3x^2 - 2x} du = \int \frac{1}{\sqrt{1 - u^2}} du = \text{asin}(u) + c \\ &= \text{asin}(x^3 - x^2) + c. \end{aligned}$$



**Theorem (Integration by Substitution)**

$$\int u' \cosh u \, dx = \sinh u + c$$

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$$\int \frac{u'}{\sqrt{u^2 + 1}} \, dx = a \sinh u + c$$

$$\int \frac{u'}{\sqrt{u^2 - 1}} \, dx = a \cosh u + c$$



## Theorem (Integration by Substitution)

$$\int u' \cosh u \, dx = \sinh u + c$$

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$$\int u' (1 + \tan^2 u) \, dx = \tan u + c$$

$$\int u' (1 - \tanh^2 u) \, dx = \tanh u + c$$

$$\int \frac{u'}{\sqrt{u^2 + 1}} \, dx = \operatorname{asinh} u + c$$

$$\int \frac{u'}{\sqrt{u^2 - 1}} \, dx = \operatorname{acosh} u + c$$

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$$\int u' e^u \, dx = e^u + c$$

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$$\int \frac{u'}{1 - u^2} \, dx = \operatorname{atanh} u + c, |u| < 1$$

$$\int \frac{u'}{u} \, dx = \ln |u| + c$$



## Theorem

*In general*

$$\bullet \int f(u(x)) u'(x) dx = \int f(u) du = F(u) \text{ if } \frac{dF(u)}{du} = f(u).$$

$$\bullet \int_a^b f(u(x)) u'(x) dx = \int_{u(a)}^{u(b)} f(u) du = F(u) \Big|_{u(a)}^{u(b)} = F(u(b)) - F(u(a)).$$



## Exercise

Find an anti-derivative of  $f$  if

$$\begin{array}{lll} 1) f(x) = x^2(x^3 - 1)^8 & 2) f(x) = \sin^2 x \cos x & 3) f(u) = 4 \frac{\sin u}{\cos^2 u} \\ 4) f(y) = y \cos y^2 & 5) f(x) = \frac{3x^2 - 2x}{\sqrt{1 + x^3 - x^2}} & 6) f(x) = \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}} \\ 7) f(x) = x 3^{x^2+1} & 8) f(s) = \frac{s^2 - 3s + 1}{s + 1} & 9) f(x) = \frac{4}{(x - 2)(-x + 4)}. \end{array}$$

## Remark

$$\begin{array}{l} 7) 3^{x^2+1} = e^{(x^2+1)\ln 3} \\ 8) s^2 - 3s + 1 = (s + 1)(s - 4) + 5 \text{ and } f(x) = s - 4 + \frac{5}{s + 1}. \\ 9) \frac{4}{(x - 2)(-x + 4)} = \frac{2}{x - 2} + \frac{2}{-x + 4}. \end{array}$$



# Integration by parts

## Theorem (Integration by Parts)

If  $u$  and  $v$  are two continuous functions on the interval  $[a, b]$ , then

$$\int u' v dx = \int (uv)' dx - \int uv' dx = uv - \int uv' dx$$

or

$$\int_a^b u' v dx = \int_a^b (uv)' dx - \int_a^b uv' dx = uv \Big|_a^b - \int_a^b uv' dx.$$

## Proof.

Indeed, the product rule  $(uv)' = u'v + uv'$  implies  $u'v = (uv)' - uv'$ .

That is  $\int_a^b u' v(x) dx = uv \Big|_a^b - \int_a^b uv' dx.$



## Example

Find

(1)  $\int x \cos x dx$  (2)  $\int_0^1 x e^x dx$  (3)  $\int e^x \cos x dx$  (4)  $\int \tan^n x dx$  for  $n \geq 1$ .



## Example

Find

$$(1) \int x \cos x dx \quad (2) \int_0^1 x e^x dx \quad (3) \int e^x \cos x dx \quad (4) \int \tan^n x dx \text{ for } n \geq 1.$$

(1) Let  $u'(x) = \cos x$  and  $v(x) = x$ .

We have  $u(x) = \sin x$  and  $v'(x) = 1$ .

Since  $\int u' v dx = uv - \int u v' dx$ ,

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + c.$$

## Remark

Note that we can also let  $u'(x) = x$  and  $v(x) = \cos x$ ; however, this will make the task more difficult than before.

*The choice depends on which function you can easily integrate or differentiate.*



## Example

Find

$$(1) \int x \cos x dx \quad (2) \int_0^1 x e^x dx \quad (3) \int e^x \cos x dx \quad (4) \int \tan^n x dx \text{ for } n \geq 1.$$

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(2)  $u'(x) = e^x$  and  $v(x) = x$  imply  $u(x) = e^x$ ,  $v'(x) = 1$  and

$$\int_0^1 x e^x dx = x e^x \Big|_0^1 - \int_0^1 e^x dx = (x e^x - e^x + c) \Big|_0^1 = [(x-1)e^x + c]_0^1 = -1$$



## Example

Find

$$(1) \int x \cos x dx \quad (2) \int_0^1 x e^x dx \quad (3) \int e^x \cos x dx \quad (4) \int \tan^n x dx \text{ for } n \geq 1.$$

$$(3) \text{ Let } F(x) = \int e^x \cos x dx.$$

For  $u'(x) = e^x$  and  $v(x) = \cos x$ ,  $u(x) = e^x$ ,  $v'(x) = -\sin x$  and

$$\begin{aligned} F(x) &= e^x \cos x + \int e^x \sin x dx \\ &= e^x \cos x + \left( e^x \sin x - \int e^x \cos x dx \right) \\ &= e^x (\cos x + \sin x) - F(x) + c \end{aligned}$$

which implies

$$2F(x) = e^x (\cos x + \sin x) + c.$$

$$\text{Therefore, } F(x) = \frac{1}{2} e^x (\cos x + \sin x) + c.$$



### Example (Reduction Formulas)

(4)  $\int \tan^n x dx$  for  $n \geq 1$ .



## Example (Reduction Formulas)

$$\begin{aligned}(4) \quad \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx \\&= \int \tan^{n-2} x (\tan^2 x + 1 - 1) dx \\&= \int \tan^{n-2} x \underbrace{(\tan^2 x + 1)}_{\frac{d}{dx} \tan x} dx - \int \tan^{n-2} x dx \\&= \int u^{n-2} u' dx - \int \tan^{n-2} x dx \text{ where } u = \tan x \\&= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.\end{aligned}$$



## Example (Reduction Formulas)

$$\begin{aligned}
 (4) \quad \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx \\
 &= \int \tan^{n-2} x (\tan^2 x + 1 - 1) dx \\
 &= \int \tan^{n-2} x \underbrace{(\tan^2 x + 1)}_{\frac{d}{dx} \tan x} dx - \int \tan^{n-2} x dx \\
 &= \int u^{n-2} u' dx - \int \tan^{n-2} x dx \text{ where } u = \tan x \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.
 \end{aligned}$$

If we let  $u_n = \int \tan^n x dx$ , then

$$u_n = \frac{1}{n-1} \tan^{n-1} x - u_{n-2}$$

is the reduction formula.



## Exercise

1) Find the indefinite integral  $F(x) = \int f(x)dx$  that satisfies  $F(a) = A$  if

(a)  $f(x) = \ln x, a = 1, A = 0$

(b)  $f(x) = x^2 e^x, a = 1, A = e$

(c)  $f(x) = (x^2 - 2x)e^{2x}$

(d)  $f(x) = x^3 \ln x, a = 1, A = -2$

(e)  $f(z) = a \sin z, a = 0, A = 1/2$

2) Evaluate  $\int f$  by using a substitution prior to integration by parts.

(a)  $f(s) = e^{\sqrt{3s+9}}$

(b)  $f(x) = \ln(x + x^2)$

(c)  $f(y) = \sin(\ln y)$

(d)  $f(z) = z(\ln z)^2$



## Exercise

1) Find the reduction formula of  $\int f_n(x) dx$  if

(a)  $f_n(x) = \sin^n x$ ,

(b)  $f_n(x) = \cos^n x$

(c)  $f_n(x) = \ln^n x$

(d)  $f_n(x) = x^n e^x$ .

2) Use the reduction formula to evaluate the anti-derivative of  $f$ .

(a)  $f(x) = \sin^5(2x)$

(b)  $f(t) = 8 \cos^4(2\pi t)$

(c)  $f(\theta) = \sin^2(2\theta) \cos^3(2\theta)$

(d)  $f(x) = 16x^3 \ln^2 x$ .



## Theorem (Fundamental Theorem of Calculus (2))

*If  $f$  is continuous on  $[a, b]$ , then  $F(x) = \int_a^x f(t)dt$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and*

$$\frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

## Remark

*$F$  is the anti-derivative of  $f$  that assigns 0 to  $a$ . That is  $F(a) = 0$ .*



## Example

Find (1)  $\int_1^{x^2} \cos t dt$  and (2)  $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt$ .

$$(1) F(x) = \int_1^{x^2} \cos t dt =$$



## Example

Find (1)  $\int_1^{x^2} \cos t dt$  and (2)  $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt$ .

$$(1) F(x) = \int_1^{x^2} \cos t dt = \sin x^2 - \sin 1.$$

$$(2) \text{ If } F(x) = \int_5^{2x-1} 3t \sin t dt$$



## Example

Find (1)  $\int_1^{x^2} \cos t dt$  and (2)  $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt$ .

$$(1) F(x) = \int_1^{x^2} \cos t dt = \sin x^2 - \sin 1.$$

$$(2) \text{ If } F(x) = \int_5^{2x-1} 3t \sin t dt \text{ then for } u = 2x - 1,$$

$$F(u) = \int_5^u 3t \sin t dt \text{ and}$$

$$\frac{dF(u)}{dx} =$$



## Example

Find (1)  $\int_1^{x^2} \cos t dt$  and (2)  $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt$ .

$$(1) F(x) = \int_1^{x^2} \cos t dt = \sin x^2 - \sin 1.$$

$$(2) \text{ If } F(x) = \int_5^{2x-1} 3t \sin t dt \text{ then for } u = 2x - 1,$$

$$F(u) = \int_5^u 3t \sin t dt \text{ and}$$

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du} \frac{du}{dx}.$$

$$\text{Therefore, } \frac{dF(x)}{dx} = (3u \sin u) (2) = 6(2x - 1) \sin(2x - 1).$$



# Outline

## 1 Integration

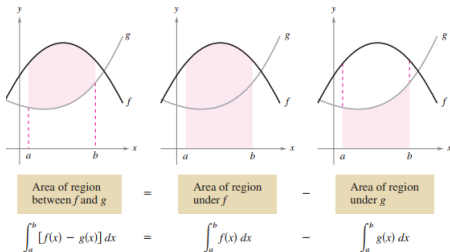
- Introduction
- Techniques of integration

## 2 Application



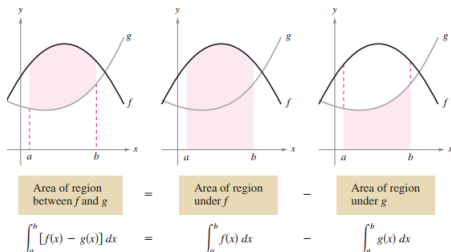
# Area of a Region Between Two Curves

We can extend the application of definite integrals from the area of a region under a curve to the area of a region between two curves.



## Area of a Region Between Two Curves

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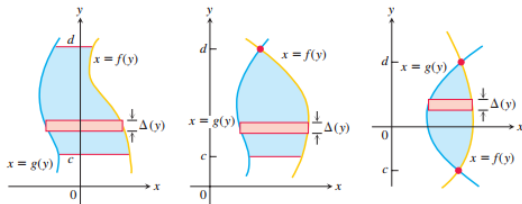


If  $f$  and  $g$  are continuous on  $[a, b]$  and  $g(x) \leq f(x)$  for all  $x$  in  $[a, b]$  then the area of the region bounded by the graphs of  $f$  and  $g$  and the vertical lines  $x = a$  and  $x = b$  is

$$A = \int_a^b [f(x) - g(x)] dx = \int_{a=x_1}^{b=x_2} [\text{top curve} - \text{bottom curve}] dx$$



If a region's bounding curves are described by functions of  $y$ , then



$$A = \int_c^d [f(y) - g(y)] dy = \int_{c=y_1}^{d=y_2} [\text{right curve} - \text{left curve}] dy$$



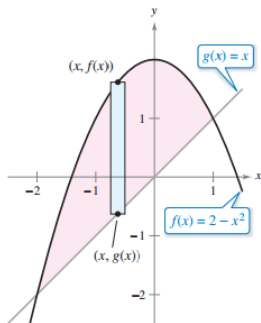
## Example

*Find the area of the region bounded by the graphs of  $f(x) = 2 - x^2$  and  $g(x) = x$*



## Example

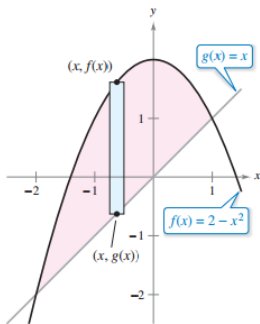
Find the area of the region bounded by the graphs of  $f(x) = 2 - x^2$  and  $g(x) = x$



## Example

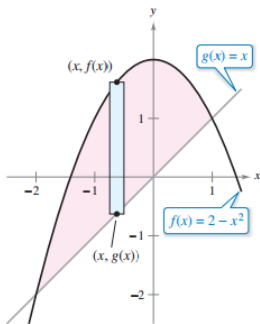
Find the area of the region bounded by the graphs of  $f(x) = 2 - x^2$  and  $g(x) = x$

The plots have two intersections points; that is the limits for integration. First equate  $f(x)$  to  $g(x)$  to find these points.  
 $2 - x^2 = x \Rightarrow x^2 + x - 2 = 0 \Rightarrow x = -2, \text{ or } 1$



## Example

Find the area of the region bounded by the graphs of  $f(x) = 2 - x^2$  and  $g(x) = x$



The plots have two intersections points; that is the limits for integration. First

equate  $f(x)$  to  $g(x)$  to find these points.

$$2 - x^2 = x \implies x^2 + x - 2 = 0 \implies x = -2, \text{ or } 1$$

Then the area is given as

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_{-2}^1 [(2 - x^2) - x] dx \\ &= -\frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-2}^1 = \frac{9}{2} \end{aligned}$$



## Example

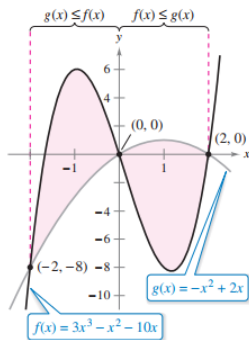
*Find the area of the region between the graphs of  $f(x) = 3x^3 - x^2 - 10x$  and  $g(x) = -x^2 + 2x$ .*



## Example

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$



## Example

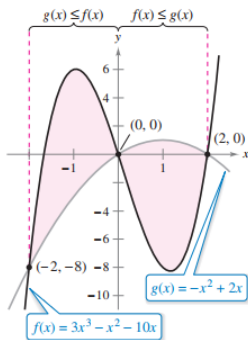
Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$

Again find the intersection points.

$$3x^3 - x^2 - 10x = -x^2 + 2x \implies 3x^3 - 12x = 0 \implies 3x(x-2)(x+2) = 0 \implies x = -2, 0, 2$$

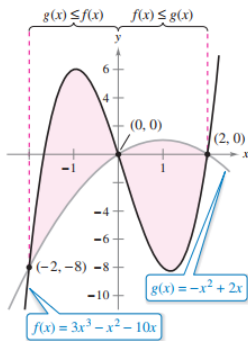
Note  $g(x) \leq f(x)$  when  $x \in [-2, 0]$  and  $f(x) \leq g(x)$  when  $x \in [0, 2]$ .



## Example

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$



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$$3x^3 - x^2 - 10x = -x^2 + 2x \implies 3x^3 - 12x = 0 \implies 3x(x-2)(x+2) = 0 \implies x = -2, 0, 2$$

Note  $g(x) \leq f(x)$  when  $x \in [-2, 0]$  and  $f(x) \leq g(x)$  when  $x \in [0, 2]$ . Then the area is given as

$$\begin{aligned} A &= \int_{-2}^0 [f(x) - g(x)] dx + \int_0^2 [g(x) - f(x)] dx \\ &= \int_{-2}^0 (3x^3 - 12x) dx + \int_0^2 (-3x^3 + 12x) dx \\ &= \left. \frac{3x^4}{4} - 6x^2 \right|_{-2}^0 + \left. \frac{-3x^4}{4} + 6x^2 \right|_0^2 = 24 \end{aligned}$$



## Exercise

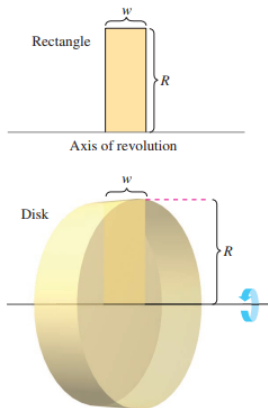
Find the areas of the regions enclosed by the following curves

- 1  $y = x^2 - 2$  and  $y = 2$
- 2  $y = 2x - x^2$  and  $y = -3$
- 3  $y = x^4$  and  $y = 8x$
- 4  $y = 7 - 2x^2$  and  $y = x^2 + 4$
- 5  $x = 2y^2$ ,  $x = 0$ , and  $y = 3$
- 6  $y^2 - 4x = 4$  and  $4x - y = 16$
- 7  $x + y^2 = 0$  and  $x + 3y^2 = 2$

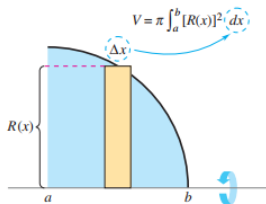


## Volume of A solid: Disk Method

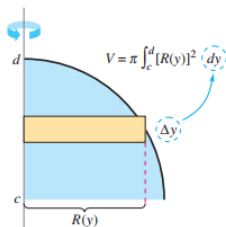
When a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.



# Volume of A solid: Disk Method



Horizontal axis of revolution



Vertical axis of revolution

## Property

*To find the volume of a solid of revolution with the disk method, use one of the formulas below*

$$V = \pi \int_a^b [R(x)]^2 dx \text{ or } V = \pi \int_c^d [R(y)]^2 dy$$



## Volume of a solid

### Example

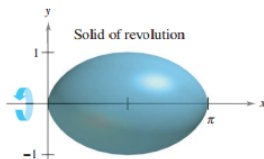
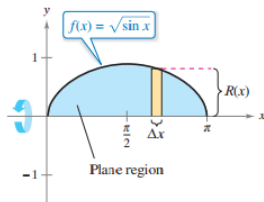
*Find the volume of the solid formed by revolving the region bounded by the graph of  $f(x) = \sqrt{\sin x}$  and the  $x$ -axis  $x \in [0, \pi]$  about the  $x$ -axis.*



# Volume of a solid

## Example

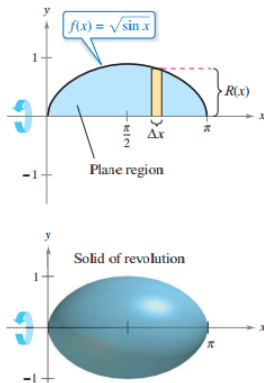
Find the volume of the solid formed by revolving the region bounded by the graph of  $f(x) = \sqrt{\sin x}$  and the  $x$ -axis  $x \in [0, \pi]$  about the  $x$ -axis.



# Volume of a solid

## Example

Find the volume of the solid formed by revolving the region bounded by the graph of  $f(x) = \sqrt{\sin x}$  and the  $x$ -axis  $x \in [0, \pi]$  about the  $x$ -axis.

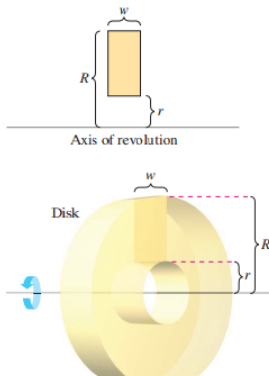


$$\begin{aligned}
 V &= \pi \int_a^b [R(x)]^2 dx \\
 &= \pi \int_0^{\pi} [\sqrt{\sin x}]^2 dx \\
 &= \pi \int_0^{\pi} \sin x \, dx \\
 &= \pi (-\cos x) \Big|_0^{\pi} \\
 &= \pi(1 + 1) \\
 &= 2\pi
 \end{aligned}$$



## Volume of a solid: Washer Method

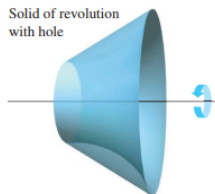
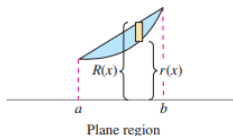
The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer. The washer is formed by revolving a rectangle about an axis with inner  $r(x)$  and outer  $R(x)$  radii.



# Volume of a solid: Washer Method

The volume of a solid of revolution using the washer method is:

$$V = \pi \int_a^b \left[ [R(x)]^2 - [r(x)]^2 \right] dx$$



## Volume of a solid

### Example

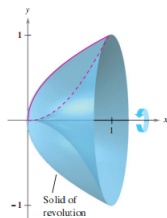
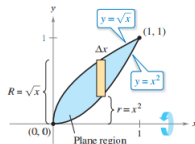
*Find the volume of the solid formed by revolving the region bounded by the graphs of  $y = \sqrt{x}$  and  $y = x^2$  about the  $x$ -axis.*



# Volume of a solid

## Example

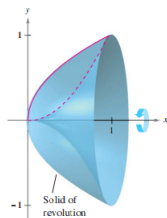
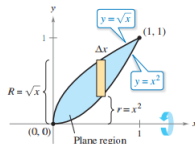
Find the volume of the solid formed by revolving the region bounded by the graphs of  $y = \sqrt{x}$  and  $y = x^2$  about the  $x$ -axis.



# Volume of a solid

## Example

Find the volume of the solid formed by revolving the region bounded by the graphs of  $y = \sqrt{x}$  and  $y = x^2$  about the  $x$ -axis.



$$\begin{aligned}
 V &= \pi \int_a^b [R(x)]^2 - [r(x)]^2 dx \\
 &= \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx \\
 &= \pi \int_0^1 (x - x^4) dx \\
 &= \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right] \bigg|_0^1 \\
 &= \frac{3\pi}{10}
 \end{aligned}$$



## Exercise

- ① Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the  $x$ -axis

①  $y = \frac{1}{x}, y = 0, x = 1, x = 3$

②  $y = e^{-x}, y = 0, x = 0, y = 1$

③  $y = x^2 + 1, y = -x^2 + 2x + 5, x = 0, x = 3$

- ② Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the  $y$ -axis

①  $y = 3(2 - x), y = 0, x = 0$

②  $y = 9 - x^2, y = 0, x = 2, x = 3$

