

# MATH157: ALGEBRA

## CHAPTER 3

### Quadratic Equations

February 01, 2021



# Outline

## 1 Quadratic equations

- Solving quadratic equations
- Relationships between the roots and the coefficients
- Quadratic inequality

## 2 Cubic equations



## Definition

*Equations on the form  $ax^2 + bx + c = 0$  are called quadratic equation provided  $a \neq 0$ .*



## Completing the square

$$\begin{aligned} ax^2 + bx + c &= a \left( x^2 + bx/a + c/a \right) \\ &= a \left( x^2 + 2 \frac{b}{2a} x + \left( \frac{b}{2a} \right)^2 - \left( \frac{b}{2a} \right)^2 + \frac{c}{a} \right) \\ &= a \left( \left( x + \frac{b}{2a} \right)^2 + \frac{-b^2 + 4ac}{4a^2} \right) \end{aligned}$$

Thus,

$$ax^2 + bx + c = a \left( \left( x + \frac{b}{2a} \right)^2 - \frac{\Delta}{4a^2} \right)$$

where  $\Delta = b^2 - 4ac$ .



## Factorizing quadratic polynomials

$$ax^2 + bx + c = a \left( x + \frac{b}{2a} + \sqrt{\frac{\Delta}{4a^2}} \right) \left( x + \frac{b}{2a} - \sqrt{\frac{\Delta}{4a^2}} \right).$$

## Solving quadratic equations

$$ax^2 + bx + c = 0 \iff x_1 = \frac{-b - \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

where  $\Delta = b^2 - 4ac$ .

## Definition

$\Delta = b^2 - 4ac$  is called the *discriminant*.

If  $\Delta > 0$ , the equation has *two distinct real roots*  $x_1$  and  $x_2$ .

If  $\Delta = 0$ , the equation has *two identical real roots*  $x_1 = x_2$ .

If  $\Delta < 0$ , the equation has *no real roots* (two distinct complex solutions  $x_1$  and  $x_2$ ).



## Example

*Find and describe the nature of the roots of the following quadratic equations: (1)  $2x^2 - x - 6 = 0$ . (2)  $-2x^2 + 5x - 2 = 0$ . (3)*

$$2x^2 + x - 1 = 0.$$

$$(4) 2x^2 - 12x + 18 = 0. (5) 2x^2 - x + 3 = 0.$$

## Exercise

(1) Find the real number  $a$  such that 2 is a root of the quadratic polynomial  $P(x) = -2x^2 - \frac{a^2}{2}x + 3a + 6$ .

(2) Solve in  $\mathbb{R}$  the following a)  $2x^2 - 4x\sqrt{3} + 6 = 0$ .

b)  $x^2 + (\sqrt{2} + \sqrt{3})x + \sqrt{6} = 0$ . c)  $x^2 - 2(1 - \sqrt{3})x + 2\sqrt{3} = 0$ .

3) Assuming that the solutions to the following equations exist, express them in terms of the real number  $a$ :

a)  $ax^2 - (2 + a^2)x + 2a = 0$ . b)  $a^2x^2 - 2x - 2 - a = 0$ .

c)  $(a - 2)x^2 + 2(a - 2)x - 2 = 0$



## Formation of quadratic equation

Let  $x_1$  and  $x_2$  be the roots of the quadratic polynomial  $x^2 + bx + c$ . We have

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2.$$

$$\text{Thus } \begin{cases} x_1 + x_2 &= -\frac{b}{a} \\ x_1x_2 &= \frac{c}{a} \end{cases}$$

and  $x^2 + bx + c := x^2 - (\text{sum of the roots})x + (\text{product of the roots}).$



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## Example

1) Without computing  $x_1$  and  $x_2$ , find out if they are the roots of the quadratic polynomial  $P$ .

a)  $x_1 = 3, x_2 = -1$  and  $P(x) = x^2 - 2x - 3$ . b)  $x_1 = -2, x_2 = 1$  and  $P(x) = x^2 + x - 2$ . c)  $x_1 = 1, x_2 = -1$  and  $P(x) = x^2 - 2x - 1$ .





## Exercise

1) Find the equations with integral coefficients the sum and product of whose roots are respectively

a)  $-3$  and  $0$ . b)  $1.2$  and  $0.8$ . c)  $-1/3$  and  $1/36$ .

2) Check if there exist two real numbers  $x_1$  and  $x_2$  such that their sum  $S$  and product  $P$  satisfy the following:

a)  $S = -1$  and  $P = 12$ . b)  $S = -9$  and  $P = 20$ . c)  $S = 3$  and  $P = 4$ .

d)  $S = \sqrt{2}$  and  $P = 1/2$ .

3) Find two real numbers such that their sum is  $1$  and the sum of their inverses is  $-1/6$ .

4) A rectangular farm has perimeter  $30m$  and area  $30m^2$ . Find the dimensions of the rectangle.



## Useful identities of the roots in terms of their sum and their product

$$1) x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2.$$

$$2) (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2.$$

$$3) x_1 - x_2 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}.$$

$$4) x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2).$$



## Useful identities of the roots in terms of their sum and their product

- 1)  $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2.$
- 2)  $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2.$
- 3)  $x_1 - x_2 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}.$
- 4)  $x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2).$

### Example

$x_1$  and  $x_2$  are the roots of the equation  $P$ . Without computing the values of  $x_1$  and  $x_2$ ,

(a) find  $x_1^2 + x_2^2$  and  $1/x_1^2 + 1/x_2^2$  if  $P(x) = 3x^2 - 5x + 9$ .

(b) find  $1/x_1 + 1/x_2$  and  $x_1 - x_2$  if  $P(x) = 2x^2 - x - 2$ .

(c) find the polynomial whose roots are      i)  $x_1^2, x_2^2$ . ii)  $x_1 + 1, x_2 + 1$ .



## Exercise

1) The roots of the equation  $2x^2 - 4x + 1 = 0$  are  $x_1$  and  $x_2$ . Find the equation whose roots are: (a)  $x_1 - 2$  and  $x_2 - 2$ . (b)  $1/x_1$  and  $1/x_2$ . (c)  $x_1/x_2$  and  $x_2/x_1$ .

2) The equation  $4x^2 + 8x - 1 = 0$  has roots  $x_1$  and  $x_2$ . Find the values of:

(a)  $1/x_1^2 + 1/x_2^2$ .

(b)  $(x_1 - x_2)^2$ .

(c)  $x_1^3 x_2 + x_1 x_2^3$ .

3) When it is possible, find  $x$  and  $y$  if:

(a)  $x^2 + y^2 = 25/4$  and  $xy = -3$ .

(b)  $xy = -6$  and  $1/x + 1/y = 1/6$ .

(c)  $x/y + y/x = -25/12$  and  $xy = -1/3$ .



Find the sign of the following polynomials

a)  $P(x) = 2x^2 - 3x + 1$ . (b)  $Q(x) = -4x^2 + 4x - 1$  (c)  $R(x) = 3x^2 + 4x + 3$



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### Solution

$$\begin{aligned} \text{a)} \quad & \begin{cases} P(x) > 0 & \text{if } x \in (-\infty, 1/2) \cup (1, +\infty) \\ P(x) < 0 & \text{if } x \in (1/2, 1) \\ P(x) = 0 & \text{if } x \in \{1/2, 1\}. \end{cases} \\ \text{b)} \quad & \begin{cases} Q(x) < 0 & \text{if } x \in (-\infty, 1/2) \cup (1/2, +\infty) \\ Q(x) = 0 & \text{if } x = 1/2. \end{cases} \\ \text{c)} \quad & Q(x) > 0 \text{ if } x \in (-\infty, +\infty) \end{aligned}$$

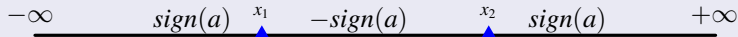


## Sign of $ax^2 + bx + c$

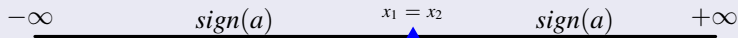
- 1) Find the roots  $x_1$  and  $x_2$  of  $P(x) = ax^2 + bx + c$ .
- 2) Sign of  $P(x)$  is the product of the sign of  $a$ ,  $x - x_1$  and  $x - x_2$  in each intervals.

### Remark

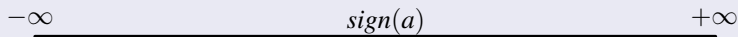
- If the roots are distinct then the intervals are  $(-\infty, x_1]$ ,  $[x_1, x_2]$  and  $[x_2, +\infty)$ .



- If the roots are identical  $x_1 = x_2$ , the intervals become  $(-\infty, x_1]$ , and  $[x_2, +\infty)$ .



If there is no roots then the interval is  $(-\infty, +\infty)$ .



## Exercise

*Solve in  $\mathbb{R}$ , (a)  $-x^2 + 3x - 2 \geq (x - 1)^2$ . (b)  $2x^2 - 4x + 1 \geq (-x + 3)^2$ .  
(c)  $(2x^2 - 3x - 1)(2x^2 + 3x + 1) \geq 0$ .*

