MATH157: ALGEBRA

CHAPTER 4

Sequences And Series

March 01, 2021

Outline

I-1) Basic definitions

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- *Size:* *A sequence has either finite or infinite terms.*

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- *Formula: The terms may be related by some rules or a formula.*
- *Size: A sequence has either finite or infinite terms.*
- *Notation: The nth term of a sequence is denoted by a_n, reads a subscript (or sub) n. Note that a in the expression a_n could be replaced by any other letter.*

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2. *The list of all positive even integers* $2, 4, 6, 8, \ldots$ *is an infinite sequence. Its* n*th term is* $a_n = 2n$ *since*
   $a_1 = 2 = 2 \times 1,$
   $a_2 = 4 = 2 \times 2,$
   $a_3 = 6 = 2 \times 3, \ldots$

Exercise

1) Find the next two terms of the following infinite sequences and express the nth term a_n as a function of n.

a) $0, 1, 4, 9, \ldots$ *b)* $3, 6, 9, 12, \ldots$ *c)* $7, 11, 15, 19, \ldots$

d) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

2) List the first three terms of the following sequences.

a) $a_n = \dfrac{1}{n^2 + n}$ *for* $1 \leq n.$ *b)* $b_n = 2^{-n}\sqrt{n}$ *for* $n \geq 0.$

c) $c_n = (-1)^n(n-4)$ *for* $3 \leq n.$

Exercise

Let $u_n, n = 1, 2, 3, \ldots$ *be a sequence of real numbers. If* $u_1 = -2$ *and* $u_{n+1} = 2u_n + 3$, *find* u_2, u_3 *and* u_4.

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Remark (Recursive definition)

The formula that allows any term of a sequence, except the first, to be computed from the previous term is called a **recursive formula**.

A **recursive definition** *of a sequence* a_n *defines the first term of the sequence,* a_p, *and a relationship between* a_{n+1} *and* a_n.

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A **recursive definition** *of a sequence a_n defines the first term of the sequence, a_p, and a relationship between a_{n+1} and a_n.*

Exercise

2) Write the recursive formula of the following sequences.
a) $3, 6, 9, 12, \ldots$ *b)* $7, 11, 15, 19, \ldots$ *c)* $a_n = 3 \cdot 2^n$ *for* $n \geq 0$.
d) $a_n = \ln(5^n)$ *for* $n \geq 1$.

I-2) Arithmetic sequence

Definition

An **arithmetic sequence** *is defined by the recursive formula* $a_{n+1} = a_n + d$ or $a_{n+1} - a_n = d$ *where* d *is a constant number called the* **common difference**.

Example

The sequence $7, 11, 15, 19, \ldots$ *is an arithmetic sequence with first term* $a_1 = 7$ *and common difference* $d = 4$.

Exercise

Find the common difference of the following arithmetic sequences.

a) $-3, -1, 1, 3, \ldots$ *b)* $a_n = 2n + 6$.

Theorem

- *If the first term is a_1 and the common difference is d, then the general expression or explicit formula is*

$$a_n = a_1 + d(n-1).$$

- *If the sequence is defined from a_p and the common difference is d, then the explicit formula is*

$$a_n = a_p + d(n-p).$$

Exercise

Find the explicit formula of the following arithmetic sequences.

a) $-3, -1, 1, 3, \ldots$ *for* $n \geq 1$. *b)* $5, 9, 13, 17, \ldots$ *for* $n \geq 1$.

c) $1, 3/2, 2, 5/2, \ldots$ *for* $n \geq 0$.

I-3) Geometric sequence

Definition

The sequence $a_n, n = 1, 2, 3, \ldots$ *is a geometric sequence with common ratio* r *if and only if*

- *Recursive formula:* $\dfrac{a_{n+1}}{a_n} = r$ or $a_{n+1} = ra_n$.
- *Explicit formula:* $a_n = a_1 r^{n-1}$ or $a_n = a_p r^{n-p}$.

Example

- *The sequence* $a_n = 3 \cdot 2^n, n \geq 1$ *has common ratio* $r = a_2/a_1 = a_3/a_2 = \cdots = a_{n+1}/a_n = 2$ *and first term* $a_1 = 6$.

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• *The sequence* $81, 27, 9, 3, \ldots$ *is a geometric sequence with common ratio* $r = 3/9 = 1/3$ *and first term* $b_1 = 81$. *Therefore, its recursive formula is* $b_{n+1} = \dfrac{b_n}{3}$ *and explicit formula is*

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Exercise

1) Find the ratio and the explicit formula of the geometric sequence
$1, 1/3, 1/9, \ldots$
2) Write the first 3 terms of the geometric sequence
(a) $a_1 = 1, r = \sqrt{2}$.
(b) $a_1 = 1/4, r = -2$.
(c) $a_4 = 1$ *and* $a_7 = -8$.

Outline

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Definition

A series is the indicated sum of the terms of a sequence. That is $\sum_{i=1}^{n} a_i$ *is the series of terms* $a_i, 1 \leq i \leq n$.

Example

- *The sigma notation of the series of terms* $a_n = 1/n, 1 \leq n \leq 5$, *is* $\displaystyle\sum_{n=1}^{5} \frac{1}{n}$.

Exercise

Find the value of the series

(a) $\displaystyle\sum_{i=1}^{3} (-1)^i (2i+1)$. *(b)* $\displaystyle\sum_{i=0}^{2} i^2 c_i$ *for* $c_n = 2^n$.

Property

(1) $\sum_{i=3}^{n} a_{i-2} = \sum_{j=1}^{n-2} a_j$. *We have used* $j = i - 2$

(2) $\sum (ka_n) = k \sum a_n$.

(3) $\sum (a_n + b_n) = \sum a_n + \sum b_n$.

However,

(4) $\sum (a_n b_n) \neq \sum a_n \cdot \sum b_n$ *in general.*

II-2) Arithmetic series

Definition

The series $\sum_{i=1}^{n} a_i$ *is an* **arithmetic series** *if* $a_n, n \geq 1$ *is an arithmetic sequence.*

The m*th partial sum* $S_m = a_1 + a_2 + \cdots + a_m$ *is given by*

$S_m = \frac{m}{2}(a_1 + a_m)$.

In general,

$$\sum_{i=p}^{m} a_i = \frac{m-p+1}{2}(a_p + a_m).$$

Exercise

1) Is $1+1+2+3+4+5$ *an arithmetic series? Justify your answer.*

2) Write the sum of the first 15 terms of the arithmetic series $1+4+7+\cdots$ *in the sigma notation. Find the sum.*

II-3) Geometric series

Definition

The series $\sum_{i=1}^{n} a_i$ *is a* **geometric series** *if* $a_n, n \geq 1$ *is an geometric sequence.*

$$S_m = a_1 + a_2 + \cdots + a_m = \frac{1 - r^m}{1 - r} a_1;$$

and

$$S_m = \sum_{i=p}^{m} a_i = \frac{1 - r^{m-p+1}}{1 - r} a_p.$$

Exercise

1) Find the sum of the n terms of the geometric series

a) $a_1 = 1$ *and* $r = 1/2$. *Evaluate* S_{10} *and* S_∞.

b) $2 - 8 + 32 - \cdots + a_7$.

2) $1 + \sqrt{3} + 3 + 3\sqrt{3} + \ldots$ *Evaluate* S_6.

In summary, for $n \geq p$, we have

	Arithmetic ($d \neq 0$)	Geometric ($r \neq 0, 1$)
Sequence	$u_{n+1} = u_n + d$ or $u_n = u_p + d(n-p)$	$u_{n+1} = ru_n$ or $u_n = u_p r^{n-p}$
Series	$S_n = \sum_{i=p}^{n} u_i$	
	$S_n = \dfrac{n-p+1}{2}(u_p + u_n)$	$S_n = \dfrac{1-r^{n-p+1}}{1-r} u_p.$

Outline

Definition (Factorial)

- $0! = 1$
- $1! = 1$
- $2! = 2 \times 1$
- $3! = 3 \times 2! = 3 \times 2 \times 1$
- $6! = 6 \times 5! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
- $n! = n \times (n-1)! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

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Exercise

Find the explicit formula of the sequence defined by the recursive formula $u_{n+1} = (n+1)u_n$ *with* $u_0 = 1$.

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Remark

$n!$ *is the number of permutations of* n *distinguishable objects taken* n *at a time.*

Permutations without replacement

The number of ways of drawing r objects from a collection of n objects without replacement is

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Exercise

1) Find the number of different arrangements that are possible for the letters of the word 'RANDOMIZE'.

2) In how many ways can 6 students be assigned to 10 empty seats?

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Solution

1) $P(\{R, A, N, D, O, M, I, Z, E\}) = {}^{9}P_{9} = 9!.$

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Solution

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2) ${}^{10}P_6 = \frac{10!}{4!}$.

Permutations with repetition

The number of permutations of n objects taken r at a time when

p_1 of them are of one kind,

p_2 of them are of a second kind,

$\vdots$

p_m of them are of an mth kind is

$$\frac{n!}{p_1!p_2!\cdots p_m!}.$$

Exercise

Find the number of arrangements that are possible for the letters of the words 'ELEVEN' and 'STATISTICS'

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Solution

$$P(E,L,E,V,E,N) = \frac{6!}{3!} = 120.$$

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Solution

$$P(E,L,E,V,E,N) = \frac{6!}{3!} = 120. \qquad P(S,T,A,T,I,S,T,I,C,S) = \frac{10!}{3!3!2!}.$$

Exercise

Solve for x

1) ${}^{6}P_2 = x \cdot {}^{5}P_3$ *2)* ${}^{x}P_6 = 30 \cdot {}^{x}P_4$ ${}^{13}P_5 - 1287 \cdot {}^{x}P_x = 0.$

Permutations with replacement

The number of ways of drawing r objects from a collection of n with replacement is

$$n^r.$$

Combinations

The number of combinations of n objects taken r at a time is

$$^{n}C_r = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \binom{n}{p}.$$

Example

How many different combinations of 8 letters can be selected from an alphabet of 10 letters. *Ans:* $^{10}C_8 = \dfrac{10!}{(10-8)!8!} = 45.$

Exercise

Show that $1)\ {}^{n}C_r = {}^{n}C_{n-r}$,
$2)\ {}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ *(Pascal's rule)*

Definition

- *A binomial is a polynomial with two terms.*
- *Any binomial can be expressed as $x+y$ where x and y are the first and the second term respectively.*
- *The nth power of the binomial $x+y$ is defined and given by*

$$(x+y)^n = x^n + {}^nC_{n-1}x^{n-1}y + {}^nC_{n-2}x^{n-2}y^2 + \cdots + {}^nC_{n-r}x^{n-r}y^r + \cdots + y^n,$$

$$= \sum_{r=0}^{n} {}^nC_{n-r}x^{n-r}y^r = \sum_{r=0}^{n} \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r.$$

Example

For instance
$$(1+y)^5 = 1^5 + \frac{5}{1!}y + \frac{5(4)}{2!}y^2 + \frac{5(4)(3)}{3!}y^3 + \frac{5(4)(3)(2)}{4!}y^4 + y^5.$$
That is, $(1+y)^5 = 1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5.$

Exercises

1) Expand the following products in ascending powers of x up to and including the term x^3.

a) $(1+x+x^2)(1-x)^5$ b) $\dfrac{1}{1+3x}$

c) $\dfrac{1-2x+x^2}{1+3x}$ d) $\sqrt[3]{1+x}$.

2) Write the 6th term of the binomial expansion $(3a+2b)^7$.

3) For $(\frac{x}{2}-2y)^{11}$ identify the term $\dfrac{165}{4}x^7y^4$ and write the next term.

4) Use the binomial expansions to evaluate (a) $\sqrt[4]{1.08}$ and (b) $\sqrt[4]{8.72}$ correct to five places of decimals.