MATH157: ALGEBRA

CHAPTER 3

Quadratic Equations

February 01, 2021

Outline

Definition

Equations on the form $ax^2 + bx + c = 0$ *are called quadratic equation provided* $a \neq 0$.

Completing the square

$$
\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + bx/a + c/a\right) \\
&= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a^2}\right)
\end{aligned}
$$

Thus,

$$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$$

where $\Delta = b^2 - 4ac$.

Factorizing quadratic polynomials

$$ax^2+bx+c=a\left(x+\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}}\right)\left(x+\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}}\right).$$

Solving quadratic equations

$$ax^2+bx+c=0 \Longleftrightarrow x_1=\frac{-b-\sqrt{\Delta}}{2a} \text{ and } x_2=\frac{-b+\sqrt{\Delta}}{2a}$$

where $\Delta=b^2-4ac$.

Definition

$\Delta=b^2-4ac$ *is called the discriminant.*

If $\Delta>0$, *the equation has two distinct real roots* x_1 *and* x_2.

If $\Delta=0$, *the equation has two identical real roots* $x_1=x_2$.

If $\Delta<0$, *the equation has no real roots (two distinct complex solutions* x_1 *and* x_2*).*

Example

Find and describe the nature of the roots of the following quadratic equations: (1) $2x^2 - x - 6 = 0$. *(2)* $-2x^2 + 5x - 2 = 0$. *(3)* $2x^2 + x - 1 = 0$.
(4) $2x^2 - 12x + 18 = 0$. *(5)* $2x^2 - x + 3 = 0$.

Exercise

(1) Find the real number a *such that* 2 *is a root of the quadratic polynomial* $P(x) = -2x^2 - \frac{a^2}{2}x + 3a + 6$.
(2) Solve in $\mathbb{R}$ *the following a)* $2x^2 - 4x\sqrt{3} + 6 = 0$.
b) $x^2 + (\sqrt{2} + \sqrt{3})x + \sqrt{6} = 0$. *c)* $x^2 - 2(1 - \sqrt{3})x + 2\sqrt{3} = 0$.
3) Assuming that the solutions to the following equations exist, express them in terms of the real number a*:*
a) $ax^2 - (2 + a^2)x + 2a = 0$. *b)* $a^2x^2 - 2x - 2 - a = 0$.
c) $(a - 2)x^2 + 2(a - 2)x - 2 = 0$

Formation of quadratic equation

Let x_1 and x_2 be the roots of the quadratic polynomial $x^2 + bx + c$. We have

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2.$$

Thus $\begin{cases} x_1 + x_2 & = -\dfrac{b}{a} \\ x_1x_2 & = \dfrac{c}{a} \end{cases}$

and $x^2 + bx + c := x^2 - (\text{sum of the roots})x + (\text{product of the roots})$.

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Example

1) Without computing x_1 and x_2, find out if they are the roots of the quadratic polynomial P.

a) $x_1 = 3, x_2 = -1$ and $P(x) = x^2 - 2x - 3$. b) $x_1 = -2, x_2 = 1$ and $P(x) = x^2 + x - 2$. c) $x_1 = 1, x_2 = -1$ and $P(x) = x^2 - 2x - 1$.

Exercise

1) Find the equations with integral coefficients the sum and product of whose roots are respectively

a) -3 *and* 0. *b)* 1.2 *and* 0.8. *c)* $-1/3$ *and* $1/36$.

2) Check if there exist two real numbers x_1 *and* x_2 *such that their sum* S *and product* P *satisfy the following:*

a) $S = -1$ *and* $P = 12$. *b)* $S = -9$ *and* $P = 20$. *c)* $S = 3$ *and* $P = 4$.

d) $S = \sqrt{2}$ *and* $P = 1/2$.

3)Find two reals numbers such that their sum is 1 *and the sum of their inverses is* $-1/6$.

4) A rectangular farm has perimeter $30m$ *and area* $30m^2$. *Find the dimensions of the rectangle.*

Useful identities of the roots in terms of their sum and their product

1) $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2$.
2) $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$.
3) $x_1 - x_2 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$.
4) $x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2)$.

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Example

x_1 and x_2 are the roots of the equation P. Without computing the values of x_1 and x_2,
(a) find $x_1^2 + x_2^2$ and $1/x_1^2 + 1/x_2^2$ if $P(x) = 3x^2 - 5x + 9$.
(b) find $1/x_1 + 1/x_2$ and $x_1 - x_2$ if $P(x) = 2x^2 - x - 2$.
(c) find the polynomial whose roots are i) x_1^2, x_2^2. ii) $x_1 + 1, x_2 + 1$.

Exercise

1) The roots of the equation $2x^2 - 4x + 1 = 0$ *are* x_1 *and* x_2. *Find the equation whose roots are: (a)* $x_1 - 2$ *and* $x_2 - 2$. *(b)* $1/x_1$ *and* $1/x_2$. *(c)* x_1/x_2 *and* x_2/x_1.

2) The equation $4x^2 + 8x - 1 = 0$ *has roots* x_1 *and* x_2. *Find the values of:*

(a) $1/x_1^2 + 1/x_2^2$.

(b) $(x_1 - x_2)^2$.

(c) $x_1^3 x_2 + x_1 x_2^3$.

3) When it is possible, find x *and* y *if:*

(a) $x^2 + y^2 = 25/4$ *and* $xy = -3$.

(b) $xy = -6$ *and* $1/x + 1/y = 1/6$.

(c) $x/y + y/x = -25/12$ *and* $xy = -1/3$.

Find the sign of the following polynomials

a) $P(x) = 2x^2 - 3x + 1$. (b) $Q(x) = -4x^2 + 4x - 1$ (c) $R(x) = 3x^2 + 4x + 3$

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Solution

a) $$\begin{cases} P(x) > 0 & \textit{if } x \in (-\infty, 1/2) \cup (1, +\infty) \\ P(x) < 0 & \textit{if } x \in (1/2, 1) \\ P(x) = 0 & \textit{if } x \in \{1/2, 1\}. \end{cases}$$

b) $$\begin{cases} Q(x) < 0 & \textit{if } x \in (-\infty, 1/2) \cup (1/2, +\infty) \\ Q(x) = 0 & \textit{if } x = 1/2. \end{cases}$$

c) $Q(x) > 0$ *if* $x \in (-\infty, +\infty)$

Sign of $ax^2 + bx + c$

1) Find the roots x_1 and x_2 of $P(x) = ax^2 + bx + c$.
2) Sign of $P(x)$ is the product of the sign of $a, x - x_1$ and $x - x_2$ in each intervals.

Remark

• *If the roots are distinct then the intervals are* $(-\infty, x_1], [x_1, x_2]$ *and* $[x_2, +\infty)$.

$-\infty$ $\quad sign(a) \quad$ x_1 $\quad -sign(a) \quad$ x_2 $\quad sign(a) \quad$ $+\infty$

• *If the roots are identical* $x_1 = x_2$, *the intervals become* $(-\infty, x_1]$, *and* $[x_2, +\infty)$.

$-\infty$ $\quad sign(a) \quad$ $x_1 = x_2$ $\quad sign(a) \quad$ $+\infty$

If there is no roots then the interval is $(-\infty, +\infty)$.

$-\infty$ $\quad sign(a) \quad$ $+\infty$

Exercise

Solve in $\mathbb{R}$, *(a)* $-x^2 + 3x - 2 \geq (x-1)^2$. *(b)* $2x^2 - 4x + 1 \geq (-x+3)^2$.
(c) $(2x^2 - 3x - 1)(2x^2 + 3x + 1) \geq 0$.