

CALCULUS

CHAPTER IV

Integration

July, 2021.



Outline

1 Integration

- Introduction
- Techniques of integration

2 Application



Area Problem

Discovery

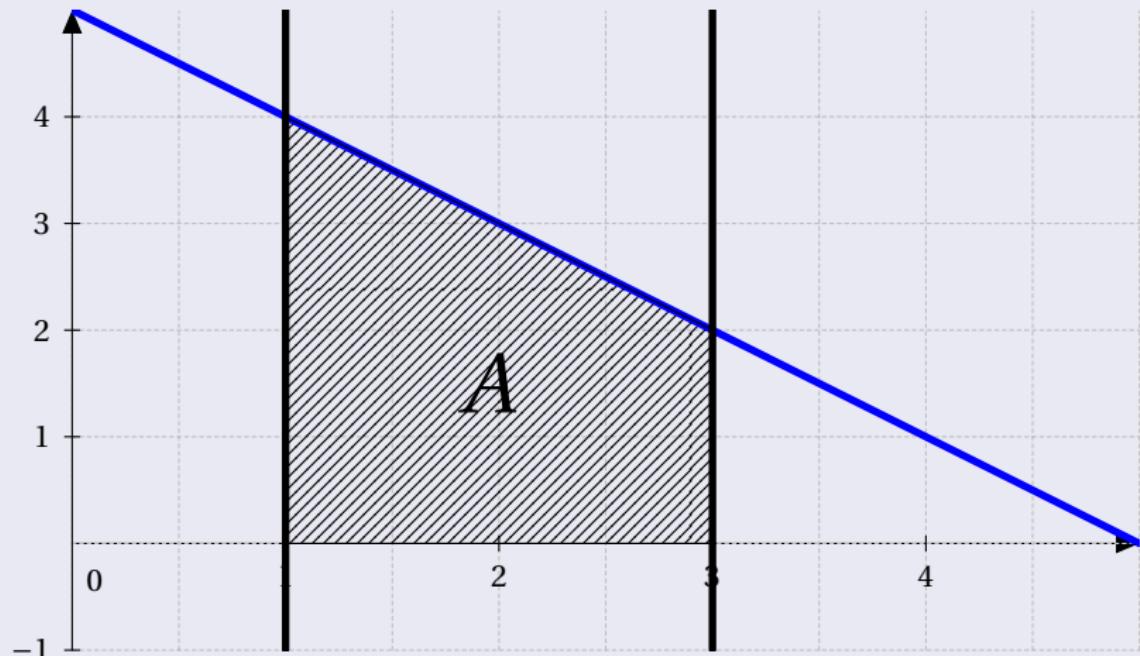
Let $f(x) = -x + 5$.

- 1) Find the area A of the region bounded by the graph of f , the x -axis, and the vertical lines $x = 1$ and $x = 3$.



Area Problem

Discovery



Area Problem

Discovery

Let $f(x) = -x + 5$.

- 1) Find the area A of the region bounded by the graph of f , the x -axis, and the vertical lines $x = 1$ and $x = 3$.

Solution: $A = (3 - 1) * 2 + (3 - 1) * 2/2 = 6$

- 2) Consider the function $F(x) = -\frac{1}{2}x^2 + 5x + e$.

a) Express $\frac{dF(x)}{dx}$ in terms of $f(x)$.

b) What does the value $F(3) - F(1)$ represent in relation to the graph of f ?



Area Problem

Discovery

Let $f(x) = -x + 5$.

1) Find the area A of the region bounded by the graph of f , the x -axis, and the vertical lines $x = 1$ and $x = 3$.

Solution: $A = (3 - 1) * 2 + (3 - 1) * 2/2 = 6$

2) Consider the function $F(x) = -\frac{1}{2}x^2 + 5x + e$.

a) Express $\frac{dF(x)}{dx}$ in terms of $f(x)$.

b) What does the value $F(3) - F(1)$ represent in relation to the graph of f ?

Solution: (a) $F'(x) = f(x)$ (b) $F(3) - F(1) = A$.

F is called the anti-derivative or indefinite integral of f .



Anti-derivative or Indefinite Integral

Definition

Let F and f be two functions such that

$$\frac{dF(x)}{dx} = f(x) \text{ for all } x \in I.$$

- f is the derivative of F .
- F is an **anti-derivative** of $f(x)$ on the interval $I = [a, b]$.

We write

$$F(x) = \int f(x) dx$$

and read **indefinite integral** of f .



Anti-derivative or Indefinite Integral

Definition

Let F and f be two functions such that

$$\frac{dF(x)}{dx} = f(x) \text{ for all } x \in I.$$

- f is the derivative of F .
- F is an **anti-derivative** of $f(x)$ on the interval $I = [a, b]$.

We write

$$F(x) = \int f(x) dx$$

and read **indefinite integral** of f .

Theorem (Fundamental Theorem of Calculus (1))

$F(b) - F(a) = F(x) \Big|_a^b = \int_a^b f(x) dx$ is the **definite integral** from a to b of $f(x)$ with respect to x .



Anti-derivative or Indefinite Integral

Exercise

1) Find an anti-derivative $F(x)$ of $f(x)$ for

- a) $f(x) = 2$ b) $f(x) = 2x$ c) $f(x) = 5x^4$ d) $f(x) = x^7$ e) $f(x) = \cos x$
f) $f(x) = -\sin x$ g) $f(x) = -\sec^2 x$ h) $f(x) = \frac{1}{x}$ e) $f(x) = e^x$.

2) Show that $F(x) = \ln(3x+1)$ and $G(x) = \ln(3x+1) + 4$ are two anti-derivatives of the same function.



Anti-derivative or Indefinite Integral

Exercise

1) Find an anti-derivative $F(x)$ of $f(x)$ for

- a) $f(x) = 2$ b) $f(x) = 2x$ c) $f(x) = 5x^4$ d) $f(x) = x^7$ e) $f(x) = \cos x$
f) $f(x) = -\sin x$ g) $f(x) = -\sec^2 x$ h) $f(x) = \frac{1}{x}$ e) $f(x) = e^x$.

2) Show that $F(x) = \ln(3x+1)$ and $G(x) = \ln(3x+1) + 4$ are two anti-derivatives of the same function.

Remark

If $F(x)$ is an anti-derivative of $f(x)$ then $G(x) = F(x) + c$, where c is a constant, is also an anti-derivative of $f(x)$ since $\frac{dG(x)}{dx} = \frac{d}{dx}(F(x) + c) = f(x)$.



Anti-derivative or Indefinite Integral

Summary

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \cosh x \, dx = \sinh x + c$$

$$\int \sinh x \, dx = \cosh x + c$$

$$\int (1 + \tan^2 x) \, dx = \tan x + c$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$$

$$\int \frac{-1}{\sqrt{1-x^2}} \, dx = \arccos x + c$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \operatorname{asinh} x + c$$

$$\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{acosh} x + c$$

$$\int \frac{1}{1+x^2} \, dx = \operatorname{atan} x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$



Hyperbolic tangent, \tanh , and cotangent, \coth , functions

☛ Hyperbolic tangent, \tanh :

$$\tanh : \mathbb{R} \rightarrow [-1, 1] \text{ is defined by } \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$
$$\frac{d \tanh x}{dx} =$$



Hyperbolic tangent, \tanh , and cotangent, \coth , functions

• Hyperbolic tangent, \tanh :

$\tanh : \mathbb{R} \rightarrow [-1, 1]$ is defined by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$\frac{d \tanh x}{dx} = 1 - \tanh^2 x$. This implies $\frac{d}{dy} \operatorname{atanh} y = \frac{1}{1 - y^2}$ for $y \in [-1, 1]$, i.e. $|y| < 1$.



Hyperbolic tangent, tanh, and cotangent, coth, functions

• Hyperbolic tangent, tanh :

$\tanh : \mathbb{R} \rightarrow [-1, 1]$ is defined by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$\frac{d \tanh x}{dx} = 1 - \tanh^2 x$. This implies $\frac{d}{dy} \operatorname{atanh} y = \frac{1}{1-y^2}$ for $y \in [-1, 1]$, i.e. $|y| < 1$.

This means, $\int \frac{1}{1-y^2} dy = \operatorname{atanh} y + C$ if $|y| < 1$.



Hyperbolic tangent, tanh, and cotangent, coth, functions

• Hyperbolic tangent, tanh :

$\tanh : \mathbb{R} \rightarrow [-1, 1]$ is defined by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$\frac{d \tanh x}{dx} = 1 - \tanh^2 x$. This implies $\frac{d}{dy} \operatorname{atanh} y = \frac{1}{1-y^2}$ for $y \in [-1, 1]$, i.e. $|y| < 1$.

This means, $\int \frac{1}{1-y^2} dy = \operatorname{atanh} y + C$ if $|y| < 1$.

• Hyperbolic cotangent, coth :

$\coth : \mathbb{R} \rightarrow (-\infty, -1) \cup (1, +\infty)$, $x \mapsto \frac{\cosh x}{\sinh x}$.

$\frac{d \coth x}{dx} =$



Hyperbolic tangent, tanh, and cotangent, coth, functions

Hyperbolic tangent, tanh :

$\tanh : \mathbb{R} \rightarrow [-1, 1]$ is defined by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$\frac{d \tanh x}{dx} = 1 - \tanh^2 x$. This implies $\frac{d}{dy} \operatorname{atanh} y = \frac{1}{1-y^2}$ for $y \in [-1, 1]$, i.e. $|y| < 1$.

This means, $\int \frac{1}{1-y^2} dy = \operatorname{atanh} y + C$ if $|y| < 1$.

Hyperbolic cotangent, coth :

$\coth : \mathbb{R} \rightarrow (-\infty, -1) \cup (1, +\infty)$, $x \mapsto \frac{\cosh x}{\sinh x}$.

$\frac{d \coth x}{dx} = 1 - \coth^2 x$. This implies that $\frac{d}{dy} \operatorname{acoth} y = \frac{1}{1-y^2}$ for $y \in (-\infty, -1) \cup (1, +\infty)$, i.e. $|y| > 1$.



Hyperbolic tangent, tanh, and cotangent, coth, functions

☛ Hyperbolic tangent, tanh :

$\tanh : \mathbb{R} \rightarrow [-1, 1]$ is defined by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$\frac{d \tanh x}{dx} = 1 - \tanh^2 x$. This implies $\frac{d}{dy} \operatorname{atanh} y = \frac{1}{1-y^2}$ for $y \in [-1, 1]$, i.e. $|y| < 1$.

This means, $\int \frac{1}{1-y^2} dy = \operatorname{atanh} y + c$ if $|y| < 1$.

☛ Hyperbolic cotangent, coth :

$\coth : \mathbb{R} \rightarrow (-\infty, -1) \cup (1, +\infty)$, $x \mapsto \frac{\cosh x}{\sinh x}$.

$\frac{d \coth x}{dx} = 1 - \coth^2 x$. This implies that $\frac{d}{dy} \operatorname{acoth} y = \frac{1}{1-y^2}$ for $y \in (-\infty, -1) \cup (1, +\infty)$, i.e. $|y| > 1$.

This means, $\int \frac{1}{1-y^2} dy = \operatorname{acoth} y + c$ if $|y| > 1$.



Theorem (Integrability of continuous function)

If f is a continuous function over an interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^x f(u) du$, $x \in [a, b]$, exists and f is said to be **integrable** over $[a, b]$.

Remark

We recall that $\int_a^x f(u)du = F(u) \Big|_a^x = F(x) - F(a).$



Property (Properties of Definite Integrals)

If f and g are integrable over the interval $[a, c]$, then

$$1) \int_a^a f(x) \, dx = 0$$

$$2) \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx; b \in [a, c].$$

$$3) \int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$$



Property (Properties of Definite Integrals)

If f and g are integrable over the interval $[a, c]$, then

$$1) \int_a^{\textcolor{blue}{a}} f(x) \, dx = 0$$

$$2) \int_a^{\textcolor{blue}{b}} f(x) \, dx + \int_{\textcolor{blue}{b}}^c f(x) \, dx = \int_a^c f(x) \, dx; b \in [a, c].$$

$$3) \int_{\textcolor{blue}{b}}^{\textcolor{red}{a}} f(x) \, dx = - \int_{\textcolor{red}{a}}^{\textcolor{blue}{b}} f(x) \, dx$$

$$4) \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$$

$$5) \int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$



Property (Properties of Definite Integrals)

If f and g are integrable over the interval $[a, c]$, then

$$1) \int_a^{\textcolor{blue}{a}} f(x) \, dx = 0$$

$$2) \int_a^{\textcolor{blue}{b}} f(x) \, dx + \int_{\textcolor{blue}{b}}^c f(x) \, dx = \int_a^c f(x) \, dx; b \in [a, c].$$

$$3) \int_{\textcolor{blue}{b}}^{\textcolor{red}{a}} f(x) \, dx = - \int_{\textcolor{red}{a}}^{\textcolor{blue}{b}} f(x) \, dx$$

$$4) \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$$

$$5) \int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

$$6) \text{ If } f(x) \leq g(x), \forall x \in [a, b], \text{ then } \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$



Exercise

Compute the following definite integrals

$$1) \int_2^2 x^2 e^x dx$$

$$2) \int_1^{-1} \frac{1}{1+x^2} dx$$

$$3) \int_0^1 |3x+3| dx$$

$$4) \int_0^1 (2x - x^4) dx$$

$$5) \int_0^\pi (2\sin x - \cos x) dx$$

$$6) \int_0^\pi (1 + \cos t) dt$$

$$7) \int_0^{\pi/6} (\sec \theta + \tan \theta)^2 d\theta$$

$$8) \int_{\pi/2}^{\pi} \frac{\sin(2u)}{2\sin u} du$$

$$9) \int_{-4}^3 6|6-x-x^2| dx$$

$$10) \int_0^\pi |\sin x - \cos x| dx$$

$$11) \int_0^{\pi/4} \tan^2 x dx$$



Integration by substitution

Example

Find $\int 7(2x - 1)(x^2 - x + 3)^4 dx$.



Integration by substitution

Example

Find $\int 7(2x-1)(x^2-x+3)^4 dx$.

Let $u = x^2 - x + 3$.

u is differentiable, and $\frac{du}{dx} = 2x-1$ implies $\frac{1}{2x-1} du = dx$.



Integration by substitution

Example

Find $\int 7(2x-1)(x^2-x+3)^4 dx$.

Let $u = x^2 - x + 3$.

u is differentiable, and $\frac{du}{dx} = 2x-1$ implies $\frac{1}{2x-1} du = dx$.

Then,

$$\begin{aligned}\int f(x) dx &= \int 7(2x-1)u^4 \frac{1}{2x-1} du = \int 7u^4 du = 7\left(\frac{1}{4+1}u^{4+1}\right) = \frac{7}{5}u^5 \\ &= \frac{7}{5}(x^2 - x + 3)^5.\end{aligned}$$

Property

$$\int u' u^n dx = \int \frac{du}{dx} u^n dx = \int u^n du = \frac{1}{n+1} u^{n+1} + c$$



Example

Find $\int \sin^2 x \cos x dx$.



Example

Find $\int \sin^2 x \cos x dx$.

Let $u = \sin x$.

u is differentiable, and $\frac{du}{dx} = \cos x$ implies $\frac{1}{\cos x} du = dx$.



Example

Find $\int \sin^2 x \cos x dx$.

Let $u = \sin x$.

u is differentiable, and $\frac{du}{dx} = \cos x$ implies $\frac{1}{\cos x} du = dx$.

Then,

$$\begin{aligned}\int \sin^2 x \cos x dx &= \int u^2 \cos x \frac{1}{\cos x} du = \int u^2 du = \left(\frac{1}{2+1} u^{2+1} \right) = \frac{1}{3} u^3 \\ &= \frac{1}{3} \sin^3 x.\end{aligned}$$

Property

$$\textcircled{1} \quad \int u' \sin u \, dx = \int \frac{du}{dx} \sin u \, dx = \int \sin u \, du = -\cos u + c$$

$$\textcircled{2} \quad \int u' \cos u \, dx = \sin u + c$$



Property

$$\textcircled{1} \quad \int \frac{u'}{\sqrt{1-u^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \int \frac{1}{\sqrt{1-u^2}} du = \arcsin u + c$$

$$\textcircled{2} \quad \int -\frac{u'}{\sqrt{1-u^2}} dx = \arccos u + c$$

Example

Find $\int \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}} dx.$



Property

- ① $\int \frac{u'}{\sqrt{1-u^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \int \frac{1}{\sqrt{1-u^2}} du = \arcsin u + c$
- ② $\int -\frac{u'}{\sqrt{1-u^2}} dx = \arccos u + c$

Example

Find $\int \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}} dx.$

Let $u = x^3 - x^2.$

$$\frac{du}{dx} = 3x^2 - 2x \text{ implies } \frac{1}{3x^2 - 2x} du = dx.$$



Property

- ① $\int \frac{u'}{\sqrt{1-u^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \int \frac{1}{\sqrt{1-u^2}} du = \arcsin u + c$
- ② $\int -\frac{u'}{\sqrt{1-u^2}} dx = \arccos u + c$

Example

Find $\int \frac{3x^2 - 2x}{\sqrt{1-(x^3 - x^2)^2}} dx.$

Let $u = x^3 - x^2.$

$$\frac{du}{dx} = 3x^2 - 2x \text{ implies } \frac{1}{3x^2 - 2x} du = dx.$$

Then,

$$\begin{aligned} \int f(x) dx &= \int \frac{3x^2 - 2x}{\sqrt{1-u^2}} \frac{1}{3x^2 - 2x} du = \int \frac{1}{\sqrt{1-u^2}} du = \arcsin(u) + c \\ &= \arcsin(x^3 - x^2) + c. \end{aligned}$$



Theorem (Integration by Substitution)

$$\int u' \cosh u \, dx = \sinh u + c$$

$$\int u' \sinh u \, dx = \cosh u + c$$

$$\int \frac{u'}{\sqrt{u^2 + 1}} \, dx = \operatorname{asinh} u + c$$

$$\int \frac{u'}{\sqrt{u^2 - 1}} \, dx = \operatorname{acosh} u + c$$



Theorem (Integration by Substitution)

$$\int u' \cosh u \, dx = \sinh u + c$$

$$\int u' \sinh u \, dx = \cosh u + c$$

$$\int u' (1 + \tan^2 u) x \, dx = \tan u + c$$

$$\int u' (1 - \tanh^2 u) \, dx = \tanh u + c$$

$$\int \frac{u'}{\sqrt{u^2 + 1}} \, dx = \operatorname{asinh} u + c$$

$$\int \frac{u'}{\sqrt{u^2 - 1}} \, dx = \operatorname{acosh} u + c$$

$$\int \frac{u'}{1 + u^2} \, dx = \operatorname{atan} u + c$$

$$\int \frac{u'}{1 - u^2} \, dx = \operatorname{atanh} u + c, |u| < 1$$



Theorem (Integration by Substitution)

$$\int u' \cosh u \, dx = \sinh u + c$$

$$\int u' \sinh u \, dx = \cosh u + c$$

$$\int u'(1 + \tan^2 u)x \, dx = \tan u + c$$

$$\int u'(1 - \tanh^2 u) \, dx = \tanh u + c$$

$$\int u' e^u \, dx = e^u + c$$

$$\int \frac{u'}{\sqrt{u^2 + 1}} \, dx = \operatorname{asinh} u + c$$

$$\int \frac{u'}{\sqrt{u^2 - 1}} \, dx = \operatorname{acosh} u + c$$

$$\int \frac{u'}{1 + u^2} \, dx = \operatorname{atan} u + c$$

$$\int \frac{u'}{1 - u^2} \, dx = \operatorname{atanh} u + c, |u| < 1$$

$$\int \frac{u'}{u} \, dx = \ln|u| + c$$



Theorem

In general

- $\int f(u(x))u'(x)dx = \int f(u)du = F(u)$ if $\frac{dF(u)}{du} = f(u).$
- $\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du = F(u)\Big|_{u(a)}^{u(b)} = F(u(b)) - F(u(a)).$



Exercise

Find an anti-derivative of f if

- 1) $f(x) = x^2(x^3 - 1)^8$ 2) $f(x) = \sin^2 x \cos x$ 3) $f(u) = 4 \frac{\sin u}{\cos^2 u}$
- 4) $f(y) = y \cos y^2$ 5) $f(x) = \frac{3x^2 - 2x}{\sqrt{1 + x^3 - x^2}}$ 6) $f(x) = \frac{3x^2 - 2x}{\sqrt{1 - (x^3 - x^2)^2}}$
- 7) $f(x) = x 3^{x^2+1}$ 8) $f(s) = \frac{s^2 - 3s + 1}{s + 1}$ 9) $f(x) = \frac{4}{(x-2)(-x+4)}.$

Remark

7) $3^{x^2+1} = e^{(x^2+1)\ln 3}$

8) $s^2 - 3s + 1 = (s+1)(s-4) + 5$ and $f(x) = s-4 + \frac{5}{s+1}.$

9) $\frac{4}{(x-2)(-x+4)} = \frac{2}{x-2} + \frac{2}{-x+4}.$



Integration by parts

Theorem (Integration by Parts)

If u and v are two continuous functions on the interval $[a, b]$, then

$$\int u'v \, dx = \int (uv)' \, dx - \int uv' \, dx = uv - \int uv' \, dx$$

or

$$\int_a^b u'v \, dx = \int_a^b (uv)' \, dx - \int_a^b uv' \, dx = uv \Big|_a^b - \int_a^b uv' \, dx.$$

Proof.

Indeed, the product rule $(uv)' = u'v + uv'$ implies $u'v = (uv)' - uv'$.

That is $\int_a^b u'v(x) \, dx = uv \Big|_a^b - \int_a^b uv' \, dx$.



Example

Find

- (1) $\int x \cos x dx$ (2) $\int_0^1 xe^x dx$ (3) $\int e^x \cos x dx$ (4) $\int \tan^n x dx$ for
 $n \geq 1$.



Example

Find

(1) $\int x \cos x dx$ (2) $\int_0^1 x e^x dx$ (3) $\int e^x \cos x dx$ (4) $\int \tan^n x dx$ for
 $n \geq 1$.

(1) Let $u'(x) = \cos x$ and $v(x) = x$.

We have $u(x) = \sin x$ and $v'(x) = 1$.

Since $\int u' v dx = uv - \int uv' dx$,

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + c.$$

Remark

Note that we can also let $u'(x) = x$ and $v(x) = \cos x$; however, this will make the task more difficult than before.

The choice depends on which function you can easily integrate or differentiate.



Example

Find

$$(1) \int x \cos x dx \quad (2) \int_0^1 x e^x dx \quad (3) \int e^x \cos x dx \quad (4) \int \tan^n x dx \text{ for } n \geq 1.$$

(1) Let $u'(x) = \cos x$ and $v(x) = x$.

We have $u(x) = \sin x$ and $v'(x) = 1$.

Since $\int u' v dx = uv - \int uv' dx$,

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + c.$$

(2) $u'(x) = e^x$ and $v(x) = x$ imply $u(x) = e^x$, $v'(x) = 1$ and

$$\int_0^1 x e^x dx = x e^x \Big|_0^1 - \int_0^1 e^x dx = (x e^x - e^x + c) \Big|_0^1 = [(x-1)e^x + c]_0^1 = -1$$



Example

Find

$$(1) \int x \cos x dx \quad (2) \int_0^1 x e^x dx \quad (3) \int e^x \cos x dx \quad (4) \int \tan^n x dx \text{ for } n \geq 1.$$

$$(3) \text{ Let } F(x) = \int e^x \cos x dx.$$

For $u'(x) = e^x$ and $v(x) = \cos x$, $u(x) = e^x$, $v'(x) = -\sin x$ and

$$\begin{aligned} F(x) &= e^x \cos x + \int e^x \sin x dx \\ &= e^x \cos x + \left(e^x \sin x - \int e^x \cos x dx \right) \\ &= e^x (\cos x + \sin x) - F(x) + c \end{aligned}$$

which implies

$$2F(x) = e^x (\cos x + \sin x) + c.$$

$$\text{Therefore, } F(x) = \frac{1}{2} e^x (\cos x + \sin x) + c.$$



Example (Reduction Formulas)

(4) $\int \tan^n x dx$ for $n \geq 1$.



Example (Reduction Formulas)

$$\begin{aligned}(4) \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx \\&= \int \tan^{n-2} x (\tan^2 x + 1 - 1) dx \\&= \int \tan^{n-2} x (\underbrace{\tan^2 x + 1}_{\frac{d}{dx} \tan x}) dx - \int \tan^{n-2} x dx \\&= \int u^{n-2} u' dx - \int \tan^{n-2} x dx \text{ where } u = \tan x \\&= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.\end{aligned}$$



Example (Reduction Formulas)

$$\begin{aligned}
 (4) \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx \\
 &= \int \tan^{n-2} x (\tan^2 x + 1 - 1) dx \\
 &= \int \tan^{n-2} x (\underbrace{\tan^2 x + 1}_{\frac{d}{dx} \tan x}) dx - \int \tan^{n-2} x dx \\
 &= \int u^{n-2} u' dx - \int \tan^{n-2} x dx \text{ where } u = \tan x \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.
 \end{aligned}$$

If we let $u_n = \int \tan^n x dx$, then

$$u_n = \frac{1}{n-1} \tan^{n-1} x - u_{n-2}$$

is the reduction formula.



Exercise

1) Find the indefinite integral $F(x) = \int f(x)dx$ that satisfies

$F(a) = A$ if

- (a) $f(x) = \ln x, a = 1, A = 0$
- (b) $f(x) = x^2 e^x, a = 1, A = e$
- (c) $f(x) = (x^2 - 2x)e^{2x}$
- (d) $f(x) = x^3 \ln x, a = 1, A = -2$
- (e) $f(z) = a \sin z, a = 0, A = 1/2$

2) Evaluate $\int f$ by using a substitution prior to integration by parts.

- (a) $f(s) = e^{\sqrt{3s+9}}$
- (b) $f(x) = \ln(x + x^2)$
- (c) $f(y) = \sin(\ln y)$
- (d) $f(z) = z(\ln z)^2$



Exercise

1) Find the reduction formula of $\int f_n(x)dx$ if

- (a) $f_n(x) = \sin^n x,$
- (b) $f_n(x) = \cos^n x$
- (c) $f_n(x) = \ln^n x$
- (d) $f_n(x) = x^n e^x.$

2) Use the reduction formula to evaluate the anti-derivative of $f.$

- (a) $f(x) = \sin^5(2x)$
- (b) $f(t) = 8 \cos^4(2\pi t)$
- (c) $f(\theta) = \sin^2(2\theta) \cos^3(2\theta)$
- (d) $f(x) = 16x^3 \ln^2 x.$



Theorem (Fundamental Theorem of Calculus (2))

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$, differentiable on (a, b) , and

$$\frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Remark

F is the anti-derivative of f that assigns 0 to a . That is $F(a) = 0$.



Example

Find (1) $\int_1^{x^2} \cos t dt$ and (2) $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt.$

$$(1) F(x) = \int_1^{x^2} \cos t dt =$$



Example

Find (1) $\int_1^{x^2} \cos t dt$ and (2) $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt.$

$$(1) F(x) = \int_1^{x^2} \cos t dt = \sin x^2 - \sin 1.$$

$$(2) \text{ If } F(x) = \int_5^{2x-1} 3t \sin t dt$$



Example

Find (1) $\int_1^{x^2} \cos t dt$ and (2) $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt.$

$$(1) F(x) = \int_1^{x^2} \cos t dt = \sin x^2 - \sin 1.$$

(2) If $F(x) = \int_5^{2x-1} 3t \sin t dt$ then for $u = 2x - 1,$

$$F(u) = \int_5^u 3t \sin t dt \text{ and}$$

$$\frac{dF(u)}{dx} =$$



Example

Find (1) $\int_1^{x^2} \cos t dt$ and (2) $\frac{d}{dx} \int_5^{2x-1} 3t \sin t dt.$

$$(1) F(x) = \int_1^{x^2} \cos t dt = \sin x^2 - \sin 1.$$

(2) If $F(x) = \int_5^{2x-1} 3t \sin t dt$ then for $u = 2x - 1,$

$$F(u) = \int_5^u 3t \sin t dt \text{ and}$$

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du} \frac{du}{dx}.$$

$$\text{Therefore, } \frac{dF(x)}{dx} = (3u \sin u)(2) = 6(2x-1) \sin(2x-1).$$



Outline

1 Integration

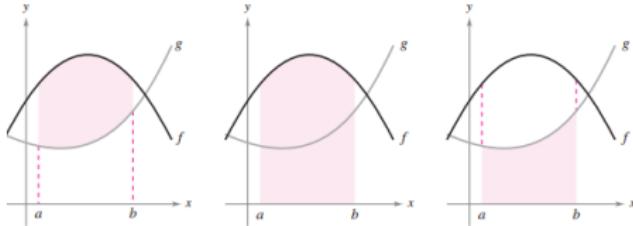
- Introduction
- Techniques of integration

2 Application



Area of a Region Between Two Curves

We can extend the application of definite integrals from the area of a region under a curve to the area of a region between two curves.



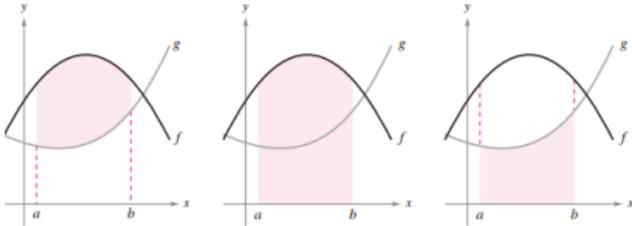
$$\begin{array}{ccc} \text{Area of region} & = & \text{Area of region} \\ \text{between } f \text{ and } g & & \text{under } f \\ \hline \end{array}$$

$$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$



Area of a Region Between Two Curves

We can extend the application of definite integrals from the area of a region under a curve to the area of a region between two curves.



$$\begin{array}{c} \text{Area of region} \\ \text{between } f \text{ and } g \end{array} = \begin{array}{c} \text{Area of region} \\ \text{under } f \end{array} - \begin{array}{c} \text{Area of region} \\ \text{under } g \end{array}$$

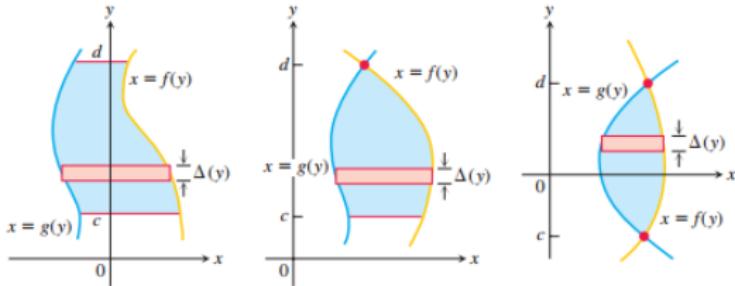
$$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$ then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx = \int_{a=x_1}^{b=x_2} [\text{top curve} - \text{bottom curve}] dx$$



If a region's bounding curves are described by functions of y , then



$$A = \int_c^d [f(y) - g(y)] dy = \int_{c=y_1}^{d=y_2} [\text{right curve} - \text{left curve}] dy$$



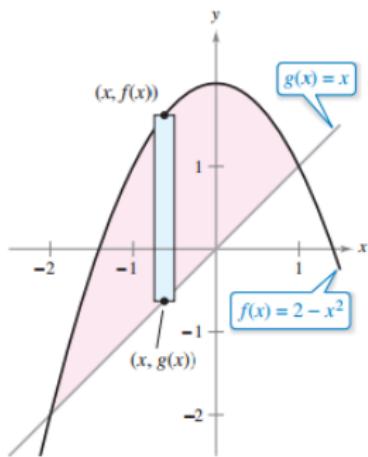
Example

*Find the area of the region bounded by the graphs of
 $f(x) = 2 - x^2$ and $g(x) = x$*



Example

*Find the area of the region bounded by the graphs of
 $f(x) = 2 - x^2$ and $g(x) = x$*



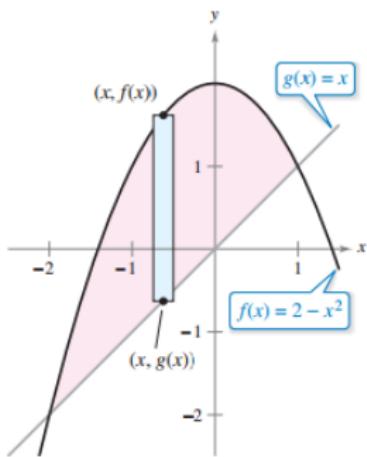
Example

Find the area of the region bounded by the graphs of

$$f(x) = 2 - x^2 \text{ and } g(x) = x$$

The plots have two intersection points; that is the limits for integration. First equate $f(x)$ to $g(x)$ to find these points.

$$2 - x^2 = x \implies x^2 + x - 2 = 0 \implies x = -2, \text{ or } 1$$



Example

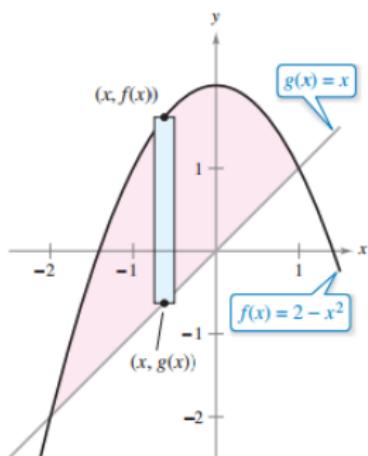
Find the area of the region bounded by the graphs of

$$f(x) = 2 - x^2 \text{ and } g(x) = x$$

The plots have two intersection points; that is the limits for integration. First equate $f(x)$ to $g(x)$ to find these points.

$$2 - x^2 = x \implies x^2 + x - 2 = 0 \implies x = -2, \text{ or } 1$$

Then the area is given as



$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_{-2}^1 [(2 - x^2) - x] dx \\ &= \left. -\frac{x^3}{3} - \frac{x^2}{2} + 2x \right|_{-2}^1 = \frac{9}{2} \end{aligned}$$



Example

Find the area of the region between the graphs of

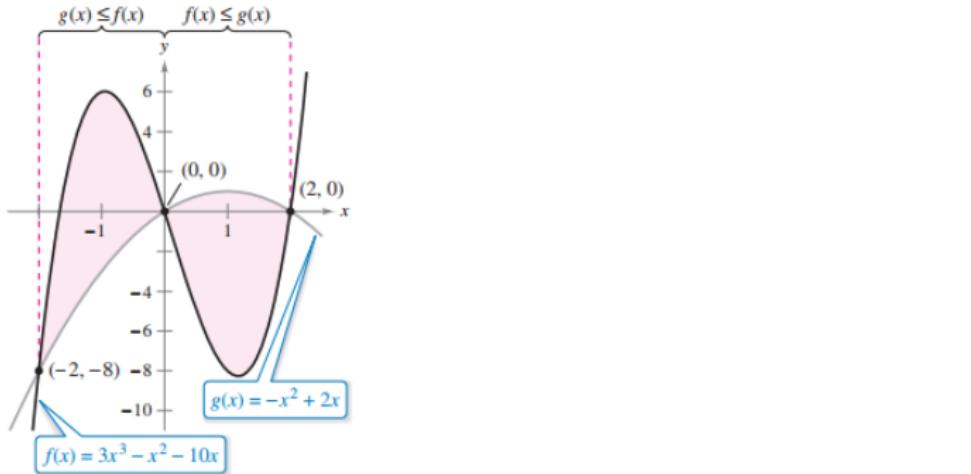
$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$



Example

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$



Example

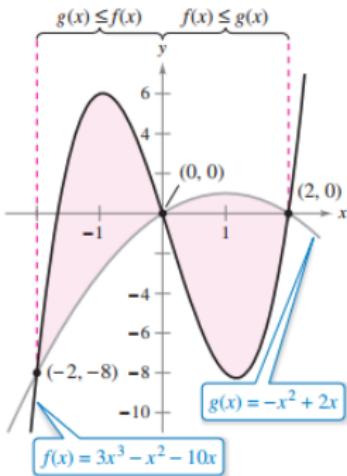
Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$

Again find the intersection points.

$$3x^3 - x^2 - 10x = -x^2 + 2x \implies 3x^3 - 12x = 0 \implies 3x(x-2)(x+2) = 0 \implies x = -2, 0, 2$$

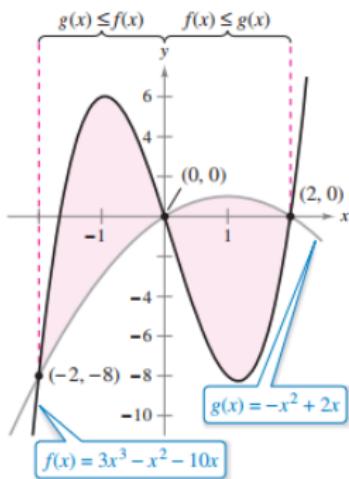
Note $g(x) \leq f(x)$ when $x \in [-2, 0]$ and
 $f(x) \leq g(x)$ when $x \in [0, 2]$.



Example

Find the area of the region between the graphs of

$$f(x) = 3x^3 - x^2 - 10x \text{ and } g(x) = -x^2 + 2x.$$



Again find the intersection points.

$$\begin{aligned} 3x^3 - x^2 - 10x &= -x^2 + 2x \implies 3x^3 - 12x = \\ 0 &\implies 3x(x-2)(x+2) = 0 \implies x = -2, 0, 2 \end{aligned}$$

Note $g(x) \leq f(x)$ when $x \in [-2, 0]$ and $f(x) \leq g(x)$ when $x \in [0, 2]$. Then the area is given as

$$\begin{aligned} A &= \int_{-2}^0 [f(x) - g(x)] dx + \int_0^2 [g(x) - f(x)] dx \\ &= \int_{-2}^0 (3x^3 - 12x) dx + \int_0^2 (-3x^3 + 12x) dx \\ &= \left. \frac{3x^4}{4} - 6x^2 \right|_{-2}^0 + \left. \frac{-3x^4}{4} + 6x^2 \right|_0^2 = 24 \end{aligned}$$



Exercise

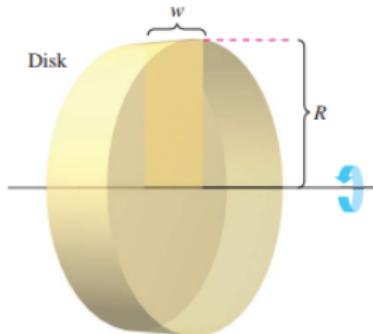
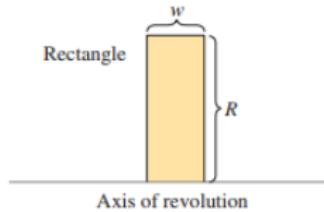
Find the areas of the regions enclosed by the following curves

- ① $y = x^2 - 2$ and $y = 2$
- ② $y = 2x - x^2$ and $y = -3$
- ③ $y = x^4$ and $y = 8x$
- ④ $y = 7 - 2x^2$ and $y = x^2 + 4$
- ⑤ $x = 2y^2$, $x = 0$, and $y = 3$
- ⑥ $y^2 - 4x = 4$ and $4x - y = 16$
- ⑦ $x + y^2 = 0$ and $x + 3y^2 = 2$

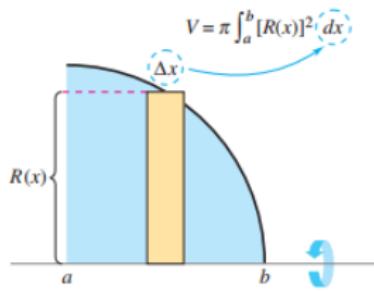


Volume of A solid: Disk Method

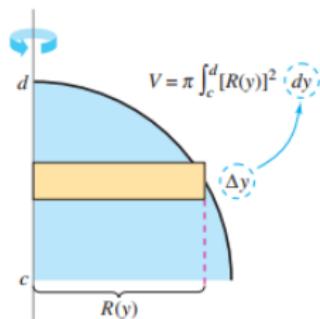
When a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.



Volume of A solid: Disk Method



Horizontal axis of revolution



Vertical axis of revolution

Property

To find the volume of a solid of revolution with the disk method, use one of the formulas below

$$V = \pi \int_a^b [R(x)]^2 dx \text{ or } V = \pi \int_c^d [R(y)]^2 dy$$



Volume of a solid

Example

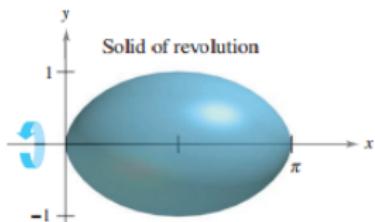
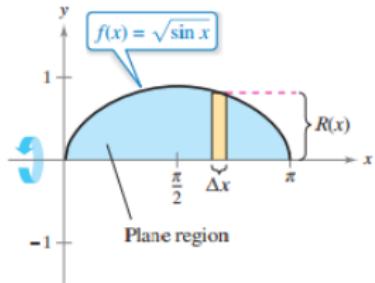
Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $x \in [0, \pi]$ about the x-axis.



Volume of a solid

Example

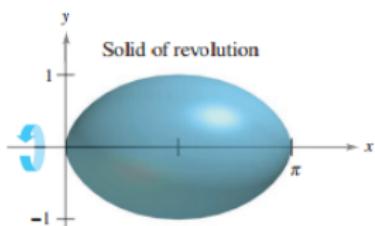
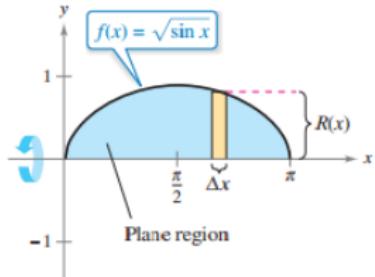
Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $x \in [0, \pi]$ about the x-axis.



Volume of a solid

Example

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $x \in [0, \pi]$ about the x-axis.

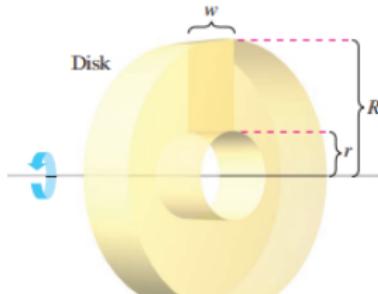
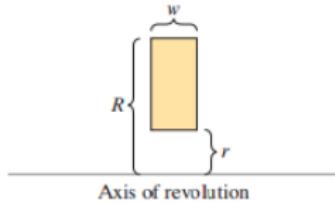


$$\begin{aligned}
 V &= \pi \int_a^b [R(x)]^2 dx \\
 &= \pi \int_0^\pi [\sqrt{\sin x}]^2 dx \\
 &= \pi \int_0^\pi \sin x \, dx \\
 &= \pi (-\cos x) \Big|_0^\pi \\
 &= \pi(1 + 1) \\
 &= 2\pi
 \end{aligned}$$



Volume of a solid: Washer Method

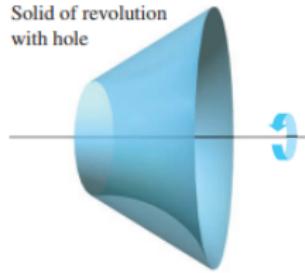
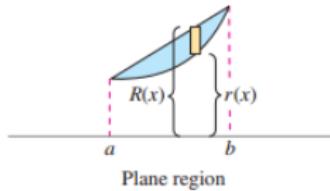
The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer. The washer is formed by revolving a rectangle about an axis with inner $r(x)$ and outer $R(x)$ radii.



Volume of a solid: Washer Method

The volume of a solid of revolution using the washer method is:

$$V = \pi \int_a^b [R(x)]^2 - [r(x)]^2 dx$$



Volume of a solid

Example

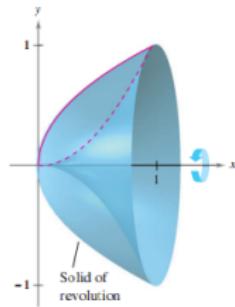
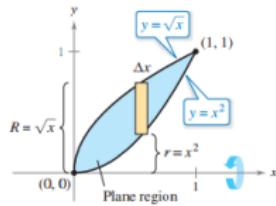
Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.



Volume of a solid

Example

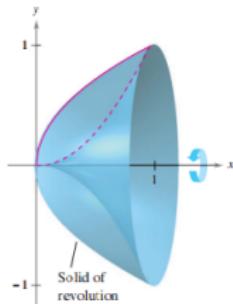
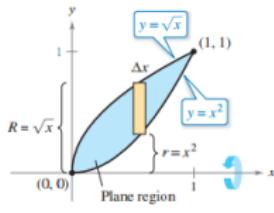
Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.



Volume of a solid

Example

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.



$$\begin{aligned}
 V &= \pi \int_a^b [R(x)]^2 - [r(x)]^2 dx \\
 &= \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx \\
 &= \pi \int_0^1 (x - x^4) dx \\
 &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right] \Big|_0^1 \\
 &= \frac{3\pi}{10}
 \end{aligned}$$



Exercise

1 Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x -axis

① $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 3$

② $y = e^{-x}$, $y = 0$, $x = 0$, $y = 1$

③ $y = x^2 + 1$, $y = -x^2 + 2x + 5$, $x = 0$, $x = 3$

2 Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y -axis

① $y = 3(2 - x)$, $y = 0$, $x = 0$

② $y = 9 - x^2$, $y = 0$, $x = 2$, $x = 3$

