MATH157: ALGEBRA

CHAPTER 1

Elementary Set Theory

January 18, 2021

Outline

Activity

1. *What is a set?*
2. *Give examples of sets which you are used to?*

Definition

1. *A* **set** S *is a collection of objects.*
2. *Objects are called* **elements**.
3. *The* **cardinality** *or* **size** *of* S *is the number of elements in* S. *It is denoted by* $|S|$.
4. *A set is either empty or not. The* **empty set** *is denoted by* $\emptyset$ *or* $\{\}$ *and* $|\emptyset| = 0$.
5. *A collection* A *of elements of* S *is called a* **subset** *of* S.

Examples of sets

1. $\mathbb{N} := \{1, 2, 3, \cdots\}$, the set of **natural numbers**;
2. $\mathbb{Z} := \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, the set of **integers**;
3. $\mathbb{Z}_- := \{\cdots, -3, -2, -1, 0\}$, the set of **non-positive integers**;
4. $\mathbb{Z}_+ := \{0, 1, 2, 3, \cdots\}$, the set of **non-positive integers**;
5. $\mathbb{R}$, the set of **real numbers**.
6. Students who are attending this lecture form a set which can be named L. The set of female (male) students is a **subset** of L. Let us denote it by F (M).

Exercise

1. *Specify the set A by listing it's elements, where*
   $A = \{\text{whole numbers less than 100 divisible by 16}\}$
2. *Specify the set B by giving a written description of it's elements, where*
   $B = \{0, 1, 4, 9, 16, 25\}$
3. *Does the following sentence specify a set?*
   $C = \{\text{whole numbers close to 50}\}$

Furthermore,

Definition

A set is a collection of elements that verify some properties.

Notation

The **set builder notation***: A set whose elements satisfy some conditions could be formally written as*

$$\mathbb{S} = \left\{ x \middle| \mathit{condition}\ 1; \mathit{condition}\ 2 \right\};$$

meaning that $\mathbb{S}$ *is the set of* $\mathit{objects}\ x$ *such that it satisfies* $\mathit{condition}\ 1$ *and* $\mathit{condition}\ 2$.

Example

1. $\mathbb{Q} := \left\{ \dfrac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N} \right\}$ *is the set of* **rational numbers;**
2. $\mathbb{C} := \left\{ x + iy \mid x, y \in R; i^2 = -1 \right\}$ *is the set of* **complex numbers**.

Exercise (1)

1. *Use dots to help list each set, and state wether it is finite or infinite.*

   (a) $A = \{$*even numbers between* 10000 *and* $20000\}$

   (b) $B = \{$*whole numbers that are multiples of* $3\}$

2. *If the set* I *in each part is finite, write down* $|I|$

   (a) $I = \{$*primes*$\}$.

   (b) $I = \{$*even primes*$\}$.

   (c) $I = \{$*even primes less than* $17\}$.

   (d) $I = \{$*natural numbers less than* $100\}$.

Exercise (2)

1. *Let us consider* $\mathbb{S}_1 = \left\{ n \in \mathbb{N} \mid n < 24; \dfrac{n}{4} \in \mathbb{N} \right\}$.
   (a) List the elements of $\mathbb{S}_1$.
   (b) Is $\mathbb{S}_1$ *a finite set?*
   (c) Find the set $\mathcal{P}(\mathbb{S}_1)$ *of all the subsets of* $\mathbb{S}_1$. *How many are they? Express the result as an integer power of* 2.
   (d) Establish a conjecture that would express the number of subsets, $|\mathcal{P}(\mathbb{S})|$, *of a set* $\mathbb{S}$ *of* n *elements.*
2. *Find an alternative description of the following sets and respond to the questions:*
   (a) $\mathrm{S}_1 = \{ x \in \mathbb{R} \mid x \leq 2; x > -2\}$. *Is* S_1 *a subset of* $\mathbb{Z}$?
   (b) $\mathrm{S}_2 = \{ x \in \mathbb{Z} \mid x \leq 2; x > -2\}$. *Is* S_2 *a subset of* $\mathbb{Z}$?

Definition (Power set)

The **power set** *of a set* S *is the set of all subsets of* S *and it is denoted by* $\mathcal{P}(\mathsf{S})$.
If the cardinality of S *is* n, *then* $|\mathcal{P}(\mathsf{S})| = 2^n$.

Exercise

Find an alternative description of $\mathsf{S} = \{\, x \in \mathbb{N} \mid 3 - 4x \geq -9\}$ *and find* $\mathcal{P}(\mathsf{S})$.

Let A and B be two subset a set S.

We write A $\subset$ B to say that A is a subset of B.

Conversely, we write A $\not\subset$ B to mean that A is not a subset of B.

Exercise

Fill the blanks below with $\subset$ or $\not\subset$.

1. $\mathbb{Z} \cdots \mathbb{N}$.
2. $\mathbb{Z}_+ \cdots \mathbb{N}$.
3. $\mathbb{Q} \cdots \mathbb{R}$.
4. $\{\, x \in \mathbb{Z} \mid 0 \leq x; -3 \leq -x\} \cdots \{0, 1, 2, 3\}$.

Property

1. *Reflexivity*: $A \subset A$.
2. *Antisymmetry*: *if* $A \subset B$ *and* $B \subset A$ *then* $A = B$.
3. *Transitivity*: *if* $A \subset B$ *and* $B \subset C$ *then* $A \subset C$.

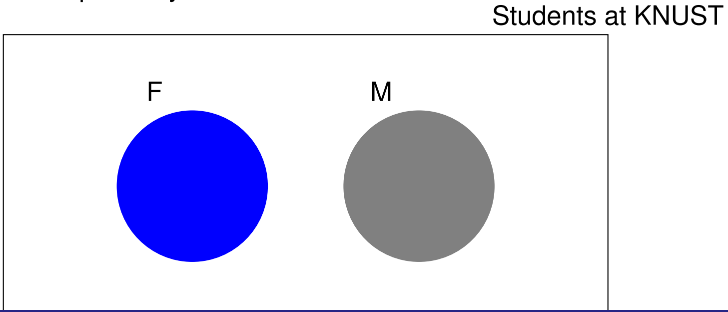
Property 2 is used to prove equality between two sets.

Outline

Definition (Venn diagram)

A set with its subsets and elements can be represented by a diagram that depicts its elements as points in a plane, and its subsets as regions inside closed curves.

We can represent the set of students at KNUST by a rectangle, and the subsets of female and male students by a blue and gray colored circles respectively.



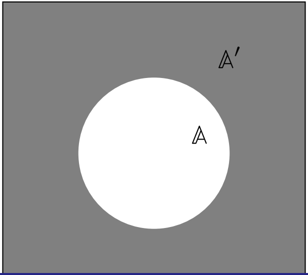
Definition (Complement)

If A *is a subset of* S, *then the* **complement** *of* A *in* S *is*

$$\mathsf{A}' = \{\, x \in \mathsf{S} \mid x \notin \mathsf{A} \} \,.$$

A′ *contains the elements of* S *which are not in* A.

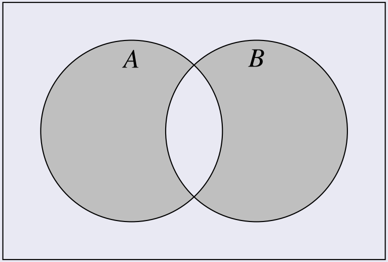
The notations $\mathsf{S} \setminus \mathsf{A}$ and $\mathsf{S} - \mathsf{A}$ are also used. The symbol $\setminus$ or $-$ is called **set difference** and



Definition (Symmetric difference)

The **symmetric difference** *of two sets* A *and* B *is*

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$



Example

Let A $= \{1, 2, 3, 4\}$ *be a subset of* $S = \{0, 1, 2, 3, 4, 5, 6\}$.

•A $- \{3, 4\} = \{1, 2\}$.

•A $- \{3, 4, 5\} = \{1, 2\}$.

•A $- \{1, 2, 3, 4, 5\} = \{\}$.

•A $- \{0, 5\} = \{1, 2, 3, 5\}$.

•A$' =$ S $-$ A $= \{0, 5, 6\}$.

Exercise

Find the complement of

1. $\emptyset$ *and the set* S *in* S.
2. $\mathbb{Z}_-$ *in* $\mathbb{Z}$ *and identify the resulting set.*
3. A$'$ *in* S.

Activity

We consider the sets
$\mathsf{A} = \left\{-\pi, -2, 0, 1.1, \ln 2, \sqrt{5}, e^3\right\}$, $\mathsf{B} = \{-2, 2/3, 1, \ln 2\}$ *and*
$\mathsf{C} = \left\{-5, 2/3, 2\sqrt{3}\right\}$. *Find:*
$\bullet$(A *or* B) $=\left\{-\pi, -2, 0, 2/3, 1, 1.1, \ln 2, \sqrt{5}, e^3\right\}$,
$\bullet$(A *and* B) $=\{-2, \ln 2\}$,
(A *and* C) $=\{\}$,
(B *and* C) $=\{2/3\}$.

Notation

We note
the **union** *of* A *and* B *by* $\mathsf{A} \cup \mathsf{B} = \{\, x \in \mathsf{S} \mid x \in \mathsf{A} \text{ or } x \in \mathsf{B}\}$,
and the **intersection** *of* A *and* B *by*
$\mathsf{A} \cap \mathsf{B} = \{\, x \in \mathsf{S} \mid x \in \mathsf{A} \text{ and } x \in \mathsf{B}\} = \{\, x \in \mathsf{S} \mid x \in \mathsf{A}; x \in \mathsf{B}\}$.

Exercise

1. *State whether the following are true or false:*
   (a) If $\mathsf{A} = \{5, 6, 7\}$ *and* $\mathsf{B} = \{6, 8, 10, 12\}$*; then* $\mathsf{A} \cup \mathsf{B} = \{5, 6, 7, 8, 10, 12\}$.
   (b) If $\mathsf{A} = \{y, x, z\}$ *and* $\mathsf{B} = \{z, y, a\}$*; then* A *intersection* B *is* $\{y, z\}$.
2. *Let* $\mathsf{A} = \{a, b, c, d\}, \mathsf{B} = \{c, d, e, f\}$ *and* $\mathsf{C} = \{b, d, f, g\}$. *Find* $\mathsf{A} \cap \mathsf{B}, \mathsf{A} \cap \mathsf{C}, (\mathsf{A} \cap \mathsf{B}) \cup (\mathsf{A} \cap \mathsf{C}), \mathsf{A} \cap (\mathsf{B} \cup \mathsf{C})$

Solution (2)

$\mathsf{A} \cap \mathsf{B} = \{c, d\}$, $\mathsf{A} \cap \mathsf{C} = \{b, d\}$, $(\mathsf{A} \cap \mathsf{B}) \cup (\mathsf{A} \cap \mathsf{C}) = \{b, c, d\}$.

Exercise

1. *Let the universal set be* $E = \{$*whole numbers less than* $20\}$, *and let,* $A = \{$*squares less than* $20\}, B = \{$*even numbers less than* $20\}$ *and* $C = \{$*odd squares less than* $20\}$.
    1. *Draw* A *and* C *on a Venn diagram, and place the numbers in the correct regions.*
    2. *Draw* B *and* C *on a Venn diagram, and place the numbers in the correct regions.*
    3. *Shade* $A \cap B$ *on a Venn diagram, and place the numbers in the correct regions.*
    4. *Shade* $A \cup B$ *on a Venn diagram, and place the numbers in the correct regions.*
2.
    1. *Draw a Venn diagram of two sets* S *and* T.
    2. *Given that* $|S| = 15, |T| = 20, |S \cup T| = 25$ *and* $|E| = 50$, *insert the number of elements into each of the four regions.*
    3. *Hence find* $|S \cap T|$ *and* $|S \cap T^C|$.

Exercise

A marketing survey of 1000 commuters found that 600 answered listen to the news, 500 listen to music, and 300 listen to both. Let N *be the set of commuters in the sample who listen to news and* M *be the set of commuters in the sample who listen to music. Fill out a two-set Venn diagram and give the number commuters in each of the sets*

$$a)\mathrm{N}\cap\mathrm{M},\qquad b)\mathrm{N}'\cap\mathrm{M},\qquad c)\mathrm{N}\cap\mathrm{M}',\qquad d)\mathrm{N}'\cap\mathrm{M}',\qquad e)\mathrm{N}\cup\mathrm{M}.$$

We now define some properties used to perform calculations with set operators.

Property (Intersection and Union)

For A, B *and* C *three subsets of a set* S,

Property	*Union*	*Intersection*
Identity	$A \cup \emptyset = A$	$A \cap S = A$
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Distributive property	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotent property	$A \cup A = A$	$A \cap A = A$
Morgan's theorem	$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$

Exercise

1. *Let* $A = \{a, b, c, d\}, B = \{c, d, e, f\}$ *and* $C = \{b, d, f, g\}$ *be three subsets of the set* $S = \{a, b, c, d, e, f, g, h\}$. *Find*
   (a) $A \cap B, A \cap C, B \cap C, A \cap (B \cap C), (A \cap B) \cap C$
   (b) $A \cup B, A \cup C, B \cup C, A \cup (B \cup C), (A \cup B) \cup C$
   (c)Compare $(A \cap B) \cup (A \cap C)$ *with* $A \cap (B \cup C)$ *and* $(A \cup B) \cap (A \cup C)$ *with* $A \cup (B \cap C)$
   (d) Compute $(A \cap B)', (A \cup C)'$ *and use Morgan's theorem to find* $A' \cup B', A' \cap C'$.
   (e) Show that $A \Delta C = (A \cup C) - (A \cap C)$.
2. *In general, show that if* S_1, S_2 *and* S_3 *are three sets, then* $(S_1 \cup S_2 \cup S_3)' = S_1' \cap S_2' \cap S_3'$. *Use this result to find* $(A \cup B \cup C)'$.
3. *Find* $A \cup (B \cup C)', A \cup (B \cup C')$. *Prove that* $A \cup (B \cup C)' = (A \cup B') \cap (A \cup C')$.

Solution

(2) $(S_1 \cup S_2 \cup S_3)' = ((S_1)' \cap (S_2 \cup S_3)') = (S_1' \cap S_2' \cap S_3')$.
(3) $A \cup (B \cup C)' = A \cup (B' \cap C') = (A \cup B') \cap (A \cup C')$.