CALCULUS

CHAPTER III

Differentiation

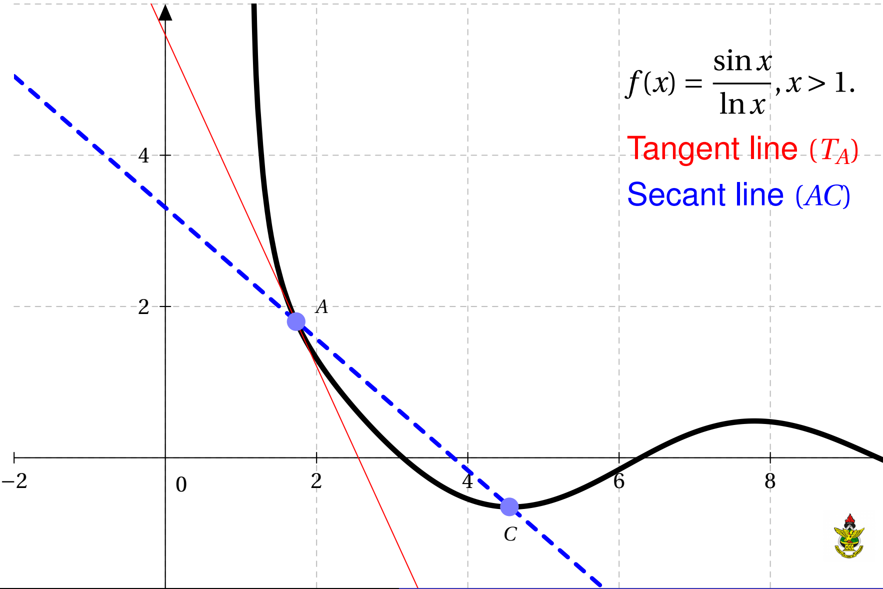
July, 2021.

Outline

1. **Differentiation**
   - Introduction
   - Differentiation rules
   - Implicit differentiation

2. **Applications**

Simulation



$f(x) = \dfrac{\sin x}{\ln x}, x > 1.$

Tangent line (T_A)

Secant line (AC)

Secant and tangent lines

- Consider a moving solid object of coordinate (x, y) in the two dimensional Cartesian plane.
- We assume that its successive positions describe the graph of the function $f(x) = \dfrac{\sin x}{\ln x}$.

☛ The **average rate of change** of the y-coordinate with respect to the x-coordinate in the interval $[x_0, x]$ is $s = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(x_0)}{x - x_0}$.

☛ s is the **slope** of the **secant line** that passes through the points $(x_0, f(x_0))$ and $(x, f(x))$ of the graph.

- Since $\Delta x = x - x_0$, $s = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.

☛ The limit of the slope of the secant line as Δx approaches 0 is the slope of the **tangent line** to the graph of f at the point $(x_0, f(x_0))$.

- The slope of the tangent line is called the derivative of f at x_0.

Derivative

Definition

☛ *The* **derivative** *of the function f with respect to x is the function denoted and defined by*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

provided the limit exists. It's also denoted by $\dfrac{df}{dx}$.

Remark

- $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ *is the derivative of f at x_0.*
- $f'(x_0^-) = \lim_{x \to x_0^-} \dfrac{f(x) - f(x_0)}{x - x_0}$ *is the* **left-derivative** *of f at x_0.*
- $f'(x_0^+) = \lim_{x \to x_0^+} \dfrac{f(x) - f(x_0)}{x - x_0}$ *is the* **right-derivative** *of f at x_0.*

Differentiation

Definition

☛ **Differentiation** *is the process of finding the derivative of a function.*

- f *is said to be* **differentiable** *at* x_0 *when* $f'(x_0)$ *exists.*
- f *is* **differentiable on an interval** *if it is differentiable at any point of the interval.*

Theorem

Constant, polynomial, trigonometric, exponential and logarithmic functions are differentiable on their respective domains. Moreover,

Constant function*:*

$$f(x) = c \Longrightarrow \frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0.$$

Power rule*:*

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}.$$

Trigonometric functions*:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \qquad \textit{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x.$$

Exponential and logarithmic functions*:*

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \qquad \textit{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}.$$

Example

1) $f(x) = x^4$, 2) $f(x) = \dfrac{1}{x^5}$, 3) $f(x) = \sqrt[3]{x}$.

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Solution

1) $f(x) = x^4 \implies \dfrac{\mathrm{d}f}{\mathrm{d}x} = 4x^{4-1} = 4x^3.$

2) $f(x) = x^{-5} \implies \dfrac{\mathrm{d}f}{\mathrm{d}x} = -5x^{-5-1} = -\dfrac{5}{x^6}.$

3) $f(x) = x^{1/3} \implies \dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{1}{3}x^{1/3-1} = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}.$

Let f and g be two differentiable functions at x and c be a constant.

Theorem

Sum Rule: *The sum $f+g$ is differentiable at x and*

$$[f+g]'(x) = f'(x) + g'(x).$$

Product Rule: *$f \cdot g$ is differentiable at x and*

$$[f \cdot g]'(x) = f'(x)g(x) + f(x)g'(x).$$

In particular, $[c \cdot f]'(x) = c \cdot f'(x)$.

Quotient Rule: *$\dfrac{f}{g}$ is differentiable if $g(x) \neq 0$ and*

$$\left[\frac{f}{g}\right]'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Example (Sum and Product Rules)

Find the derivative of $f(x) = 3x^4 - x^3 + 2x - 1$.

$$\begin{aligned} f'(x) &= \left[3x^4 - x^3 + 2x - 1\right]' = \left[3x^4\right]' + \left[-x^3\right]' + [2x]' + [-1]' \\ &= 3\left[x^4\right]' - \left[x^3\right]' + 2\,[x]' \\ &= 12x^3 - 3x^2 + 2. \end{aligned}$$

Exercise

Find the derivative f' of f and the derivative f'' of f' for the following functions.

1) $f(x) = -x^2 + 3$

2) $f(x) = \frac{1}{3}x^3 - x^2$

3) $f(z) = \frac{1}{3z^2} - \frac{5}{2z}$

4) $f(t) = 5\sqrt[3]{t} + \sin t$

5) $f(t) = 2e^t + t^2$

6) $f(s) = \ln s - 3s.$

Example (Product and Quotient Rules)

Find the derivative of $f(x) = x^2 \ln x$ *and* $g(x) = \tan x$.

$$f'(x) = [x^2 \ln x]' = [x^2]' \ln x + x^2 [\ln x]'$$
$$= 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x.$$

$$g'(x) = \left[\frac{\sin x}{\cos x}\right]' = \frac{[\sin x]' \cos x - \sin x [\cos x]'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \frac{1}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x.$$

Find the derivative of

1) $f(x) = (2-3x)(x^2-2x)$

2) $f(x) = (x-4)^2$

3) $f(x) = x^3(1+\sqrt{x})$

4) $f(t) = 3t^2 \sin t$

5) $f(s) = 2s\cos s - 2\sin s$

6) $f(z) = \dfrac{\sin z}{1-\cos z}.$

Let u be a function of x and f a function of u.

Theorem (Chain Rule)

If u is differentiable at x and f is differentiable at $u(x)$, then the function $f \circ u : x \mapsto f(u(x))$ is differentiable at x and

$$\frac{\mathrm{d}}{\mathrm{d}x} f \circ u(x) = \frac{\mathrm{d}f(u)}{\mathrm{d}u} \frac{\mathrm{d}u(x)}{\mathrm{d}x} = u'(x) \left[\frac{\mathrm{d}f}{\mathrm{d}u} \right] (u(x)) .$$

Property (Applications of the chain rule)

Let u be a differentiable function.

1) $\dfrac{\mathrm{d}u^n}{\mathrm{d}x} = \dfrac{\mathrm{d}u^n}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} = nu'u^{n-1}$

2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{u^n}\right] =$

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Property (Applications of the chain rule)

4) $\dfrac{d\sin u}{dx} = \dfrac{d\sin u}{du}\dfrac{du}{dx} = u'\cos u$

5) $\dfrac{d\cos u}{dx} = \dfrac{d\cos u}{du}\dfrac{du}{dx} = -u'\sin u$

6) $\dfrac{d\tan u}{dx} = \dfrac{d\tan u}{du}\dfrac{du}{dx} = u'\sec^2 u$

7) $\dfrac{de^u}{dx} = \dfrac{de^u}{du}\dfrac{du}{dx} = u'e^u$

8) $\dfrac{d\ln u}{dx} = \dfrac{d\ln u}{du}\dfrac{du}{dx} = \dfrac{u'}{u}$

Example

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Let $u(x) = 2x^3 - 3x$.

$h(x) = u^n(x)$ *implies* $h'(x) = nu'(x)\,u^{n-1}(x)$.

Since $u'(x) = 6x^2 - 3$,

$h'(x) = n(6x^2 - 3)\left(2x^3 - 3x\right)^{n-1}$.

Example

- $g(x) = e^{2+\sin x}$.

For $u(x) = 2 + \sin x,\ u'(x) = \cos x$ *and*

$$\begin{aligned} g'(x) &= u'(x)e^{u(x)} \\ &= \cos x e^{2+\sin x}. \end{aligned}$$

Example

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$$\begin{aligned} g'(x) &= u'(x)e^{u(x)} \\ &= \cos x e^{2+\sin x}. \end{aligned}$$

- $f(x) = x\ln(2x+1)$.

$$\begin{aligned} f'(x) &= [x\ln(2x+1)]' \\ &= \ln(2x+1) + x[\ln(2x+1)]' \\ &= \ln(2x+1) + x\frac{2}{2x+1} \\ &= \ln(2x+1) + \frac{2x}{2x+1}. \end{aligned}$$

Exercise

1) $f(x) = x^2\sqrt{1-x^2}$ 2) $f(x) = \sqrt[3]{(x^2-1)^2}$ 3) $f(x) = \left(\dfrac{3x-1}{x^2+3}\right)^2$

4) $f(x) = \cos\left(3x^2\right)$ 5) $f(x) = \sqrt{\cos x}$ 6) $f(t) = \sin^3(4t)$.

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Solution

1) $\dfrac{x(2-3x^2)}{\sqrt{1-x^2}}$ 2) $\dfrac{4x}{3\sqrt[3]{x^2-1}}$ 3) $\dfrac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$

4) $-6x\sin\left(3x^2\right)$ 5) $-\dfrac{\sin x}{2\sqrt{\cos x}}$ 6) $12\sin^2(4t)\cos 4t$.

Discovery (Derivative of inverse functions)

Let f be a function and let $f(x) = y$ for $x \in D_f$.

We call **inverse** *of f the function, denoted by f^{-1}, that maps y to x. That is,*

$$f^{-1}(y) = x.$$

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$$f^{-1}(y) = x.$$

We have

$$\left[f^{-1}\right]'(y) = \frac{1}{f'(x)} = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

Example

- *Let us find the derivative of* $\arccos$, *the inverse of* $\cos$.

$f(x) = \cos x = y$ *and* $f'(x) = -\sin x$ *imply*

$\arccos'(y) = -\dfrac{1}{\sin x} = -\dfrac{1}{\sqrt{1-\cos^2 x}}$ *since* $\sin^2 x = 1 - \cos^2 x$.

Therefore, $\arccos'(y) = -\dfrac{1}{\sqrt{1-y^2}}$.

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Therefore, $\arccos'(y) = -\dfrac{1}{\sqrt{1-y^2}}$.

- *The derivative of* $\arctan$, *the inverse of* $\tan$.

$f(x) = \tan x$ *and* $f'(x) = 1 + \tan^2 x$ *imply*
$\arctan'(y) = \dfrac{1}{\tan' x} = \dfrac{1}{1+\tan^2 x} = \dfrac{1}{1+y^2}$ *since* $\tan x = y$.

Therefore, $\arctan'(y) = \dfrac{1}{1+y^2}$.

Exercise

Find the derivative of the inverse function of the functions

1) $f(x) = \sin x$ 2) $f(x) = \cosh x$ 3) $f(x) = \sinh x$

4) $f(x) = \tanh x$ 5) $f(x) = \ln x$ 6) $f(x) = x^3$.

Example

Differentiate the LHS and RHS of the following equations and express the derivative of y as a function of x and/or y.

1) $x^2 - 2y = 0$ 2) $xy - 2x = 3$ 3) $xy + e^y = 1.$

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Remark

1. *By differentiating the LHS,* $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 - 2y\right]$, *and the RHS,* $\frac{\mathrm{d}0}{\mathrm{d}x}$, *of the equation with respect to* x, *we obtain* $2x - 2y' = 0$ *which yields* $y' = x$.

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2. $\frac{\mathrm{d}}{\mathrm{d}x}\left[xy - 2x\right] = \frac{\mathrm{d}3}{\mathrm{d}x}$ *gives* $(y + xy') - 2 = 0$.
   That is $y' = \dfrac{2-y}{x} = \dfrac{2 - \frac{2x+3}{x}}{x} = -\dfrac{3}{x^2}$.

The equation $xy + e^y = 1$ **implicitly** defines the function y. There is no explicit formula to define y.

However, we can still extract information about $\frac{\mathrm{d}y}{\mathrm{d}x}$.

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Differentiating both sides of $xy + e^y = 1$ with respect to x, we obtain

$(y + xy') + y'e^y = 0$ which implies $(x + e^y)y' = -y$.

Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x + e^y} = -\frac{y}{x + 1 - xy}$.

Exercise

Find the implicit derivative of the function y or θ with respect to x or r:

1) $y^2 - x^2 = \sin(xy)$ 2) $x^3 + y^3 - 9xy = 0$ 3) $x^2y + y^2x = 6$

4) $x + \tan(xy) = 0$ 5) $x\cos(2x + 3y) = y\sin x$ 6) $\cos r + \cot\theta = r$.

Outline

1. Differentiation
   - Introduction
   - Differentiation rules
   - Implicit differentiation

2. Applications

Derivability and continuity

Let f be a differentiable function at x_0. $f'(x_0)$ exists

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$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)\Longrightarrow\frac{\lim\limits_{x\to x_0}f(x)-f(x_0)}{\lim\limits_{x\to x_0}x-x_0}=f'(x_0)\text{ exists}$$

$$\Longrightarrow\lim_{x\to x_0}(f(x)-f(x_0))=\lim_{x\to x_0}f'(x_0)(x-x_0)\text{ exists}$$

$$\Longrightarrow\lim_{x\to x_0}f(x)=\lim_{x\to x_0}\left(f'(x_0)(x-x_0)+f(x_0)\right)\text{ exists}$$

$$\Longrightarrow\lim_{x\to x_0}f(x)=f(x_0)$$

Derivability and continuity

Let f be a differentiable function at x_0. $f'(x_0)$ exists and

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This shows that

Property

☛ *f is continuous at x_0 whenever f is differentiable at x_0.*

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Property

☛ *f is continuous at x_0 whenever f is differentiable at x_0.*

☛ *The equation $(T_{x_0}): L(x)=f'(x_0)(x-x_0)+f(x_0)$ is the equation of the tangent line to the graph of f at $(x_0,f(x_0))$.*

Derivability and continuity

Example

Find the equation of the tangent to the graph of the function defined by $xy + e^y = 1$ *at the point* $(1 - e, 1)$.

Derivability and continuity

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Find the equation of the tangent to the graph of the function defined by $xy + e^y = 1$ *at the point* $(1-e, 1)$.

Solution

The implicit derivative of the function y *is given by*
$y'(x) = -\dfrac{y}{x+e^y}$.
Thus, $y'(1-e) = \dfrac{1}{1-e+e} = -1$ *and*
$L(x) = -(x-(1-e)) + 1 = -x + 2 + e$ *is the equation of the tangent line.*

Linearization

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Definition

The function $L(x) = f(x_0) + f'(x_0)(x - x_0)$ *is also known as the* **linearization** *or the* **linear approximation** *of* f *at* x_0.

Linearization

Example

For $f(x) = 1 - \dfrac{\operatorname{atan} x}{x}$, $L(x) = 0.18x + 0.09$ *at the point* $(2, 0.45)$.

The following shows the values of f *and* L *at some points of the closed interval* $[1,3]$.

x	1	1.4	1.6	1.8	2	2.2	2.4	2.7	3
$f(x)$	0.21	0.32	0.37	0.41	0.45	0.48	0.51	0.55	0.58
$L(x)$	0.27	0.34	0.38	0.41	0.45	0.48	0.52	0.57	0.62

We use $f(x) \approx L(x)$ to say that L is an approximation of f.

Linearization

Exercise

1) Find $L(x)$*, the linearization of* $f(x) = (1+x)^n$ *at* $x_0 = 0$.

2) Find the approximation of $f(x)$ *at* $x_0 = 0.002, 0.009$ *if* $n = 10^3$.

3) Use $L(x)$ *to linearize* $g(x)$.

$a)\ g(x) = (1-x)^6$

$b)\ g(x) = \dfrac{2}{\sqrt{4+3x}}$

$c)\ g(x) = \sqrt[3]{\left(1 - \dfrac{x}{2+x}\right)^2}$

Differential approximation

Definition

☛ *Nearby points* x *of* x_0, *could be denoted by* $x = x_0 + \Delta x$.
We also use the notation $x_0 + \mathrm{d}x$ *where* $\mathrm{d}x$ *is called the* **differential** *of* x.

☛ *Since* $f(x) \approx L(x)$ *then*
$\Delta f = f(x_0 + \mathrm{d}x) - f(x_0) \approx L(x_0 + \mathrm{d}x) - L(x_0) = f'(x_0)\mathrm{d}x = \mathrm{d}f$.

That is, the change in $f, \Delta f$, *at* x_0 *is approximated by* $\mathrm{d}f$ *or*
$\Delta f \approx f'(x_0)\mathrm{d}x$.

Differential approximation

Exercise

1) *The radius of a sphere is increased from* 2 *to* $2.02\ m$.

a) *Estimate the resulting change* dA *in area.*

b) *Express the estimate as a percentage of the sphere's original area.*

2) *Estimate the volume of a material in a cylindrical form with length* $l = 30cm$, *radius* $r = 6cm$ *and shell thickness* $\mathrm{d}r = 0.5cm$.

Monotonic functions

Let f be a differentiable function on an interval I.

Property

- *If* $f'(x) > 0, \forall x \in I$, *then* f *is an increasing function on* I.
- *If* $f'(x) < 0, \forall x \in I$, *then* f *is a decreasing function on* I.
- *If* $f'(x) = 0, \forall x \in I$, *then* f *is a constant function on* I.

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Example

Show that the function $f(x) = (x-4)^2 + 3$ *is decreasing and increasing on the intervals* $I_1 = (-\infty, 4]$ *and* $I_2 = [4, +\infty)$ *respectively.*

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f is differentiable and $f'(x) = 2x - 8$. $f'(x) > 0$ implies that $x > 4$.
That is f increases on the interval I_2.
$f'(x) < 0$ implies that $x \in I_1$. Therefore, f decreases on I_2.

Exercise

Determinate where the graph is increasing and decreasing, and find the critical points.

$$1)\ y = 3x^4 - 4x^3 + 2 \qquad 2)\ y = (x^2+1)^3 \tag{1}$$

$$3)\ y = (x+3)(x^2+6x+6) \qquad 4)\ y = \frac{x-2}{x+1} \tag{2}$$

$$5)\ y = \frac{x}{2} + \cos x, -\pi \le x \le 2\pi \qquad 6)\ y = \sin x + \cos x, -\frac{3}{4}\pi \le x \le \frac{5}{4}\pi. \tag{3}$$

Extrema of a function

Definition

☛ $c \in [a, b]$ *is an interior point if* c *is neither of the endpoints* a *and* b*. That is* $c \in (a, b)$.

☛ *Let* c *be an interior point of* D_f*. If* $f'(c) = 0$ *or* f' *is undefined at* c*, then* c *is called* **critical number** *of* f.

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Remark

Extrema of a function f *could occur at the endpoints of the domain of* f.

Extrema of a function

Finding Extrema On A Closed Interval

Let f be a continuous function on a closed interval $[a, b]$. To find the extrema of f, use the following steps

1. Find the critical numbers of f in $[a, b]$.
2. Evaluate f at all critical numbers and endpoints.
3. The least of these is the minimum and the greatest is the maximum.

Extrema of a function

Example

Find the extrema of $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3$ *on the interval* $I = [-2, 2]$

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Critical points*:* $f'(x) = 0 \implies c = 0$ *and* $c = \pm 1$.

Sign of $f''(c)$: $f''(0) = 0, \quad f''(-1) = -2 < 0, \quad f''(1) = 2 > 0.$

Values of f **at critical points and endpoints:**

$f(0) = 0, \quad f(\pm 1) = \pm 1/5 \mp 1/3 = \mp 2/15,$ *and* $f(\pm 2) = \pm 3.733$

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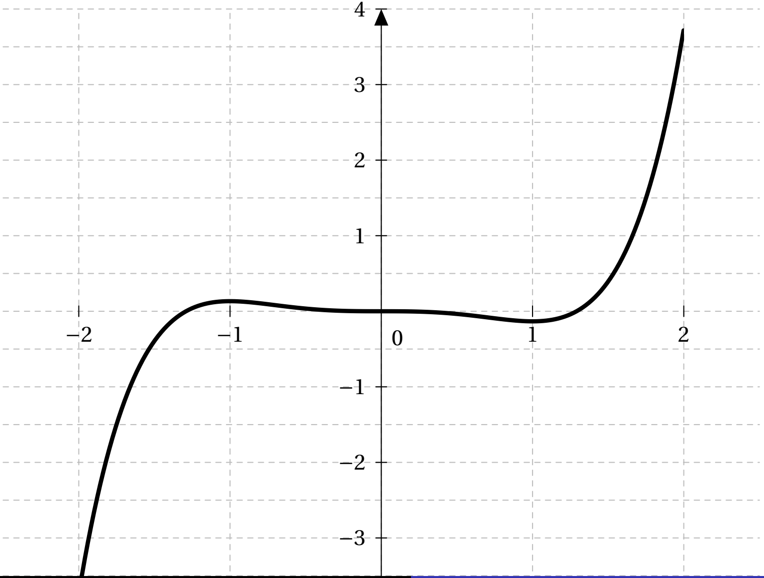
Extrema*:*

- f *has a global minimum, local maximum, local minimum, and global maximum value at* $-2, -1, 1$ *and* 2 *respectively.*
- f *has an inflection point at* $c = 0$.

Extrema of a function



Extrema of a function

Exercise

Find the critical points, endpoints of D_f, the value of the function at each of these points and identify extreme values.

1) $f(x) = x^2 - 6x + 7$

2) $f(x) = (x-1)^2(x-3)^2$

3) $f(x) = \dfrac{x^2}{x-2}$

4) $f(x) = x^{2/3}(x^2 - 4)$

5) $f(x) = x\sqrt{4-x^2}$

6) $f(x) = x^2\sqrt{3-x}$.

7) $f(x) = \begin{cases} 3-x & : x < 0 \\ 3+2x-x^2 & : x \geq 0 \end{cases}$

8) $f(x) = \tan\left(\dfrac{\pi x}{8}\right)$ *with* $D_f = [0,2]$.

Exercise

The height of a body moving vertically is given by $h(t) = -\dfrac{1}{2}gt + v_0 t + h_0$ *where* $g > 0$. *Find the body's maximum height in terms of* h_0, v_0 *and* g.

Mean Value Theorem (MVT)

Theorem

If f is a continuous function on the closed interval $[a,b]$ and differentiable on (a,b), then there is at least one point $c \in (a,b)$ at which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

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Remark

The **MVT** *could be used to show that:*

- *the equation $f'(x) = m$ has a solution in the interval $[a,b]$ if $m = \dfrac{f(b)-f(a)}{b-a}$, that is m is the mean value of f on $[a,b]$.*
- *f has an extremum value at a point of the interval $[a,b]$ if $f(a) = f(b)$.*

Mean Value Theorem (MVT)

Exercise

1) *The position at time t of a moving particle is given by $s(t)$. Find the time at which its instantaneous velocity $s'(t)$ is equal to its average velocity in the interval T.*

a) $s(t) = x^2 + 2x - 1, T = [0, 1]$. *b)* $s(t) = t + \frac{1}{t}, T = [1/2, 2]$.

c) $s(t) = \sqrt{t-1}, T = [1, 3]$.

2) *Does the function f satisfy the hypothesis of the* **MVT** *on the given interval.*

a) $f(x) = x^{4/5}, I = [0, 1]$. *b)* $f(x) = \sqrt{x(1-x)}, I = [0, 1]$.

c) $f(x) = \begin{cases} \dfrac{\sin x}{x} & : -\pi \le x < 0 \\ 0 & : x = 0. \end{cases}$

Hospital's Theorem

Theorem

The functions f and g are differentiable on an open interval I except possibly at $x_0 \in I$. If $g'(x) \neq 0$ and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

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Example

Let $f(x) = x^3 - 3x + 2$ and $g(x) = x^4 - x$. Find $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$.

$\lim_{x\to 1} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ is of indeterminate form. Since f and g are differentiable at $x_0 = 1$, Hospital's Theorem implies that

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{3x^2 - 3}{4x^3 - 1} = \frac{0}{3} = 0.$$

Hospital's Theorem

Remark

The indeterminate form $\dfrac{0}{0}$ *is equivalent to the forms* $0 \times \infty$ *and* $\dfrac{\infty}{\infty}$.

Exercise

Evaluate

1) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$. 2) $\displaystyle\lim_{x\to \infty} \frac{e^x}{x}$

3) $\displaystyle\lim_{x\to \infty} \frac{x+\sin x}{x}$. 4) $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x^2+x}$

5) $\displaystyle\lim_{x\to 0} \frac{2\sin x-\sin(2x)}{x-\sin x}$. 6) $\displaystyle\lim_{x\to \infty} \frac{x^n}{e^x}, n\in\mathbb{N}$.

Leibnitz Theorem

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Express the first, the second and the third derivative of fg *in terms of the nth-derivative of* f *and* g.

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$[fg]' = f'g + fg'$

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Theorem

The nth derivative of the product fg, is given by

$$[fg]^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)}.$$

Exercise

Find the fifth derivative of $e^x \sin x$.