CALCULUS

CHAPTER V

Differential Equation

July, 2021.

Outline

Discovery

Let $f(x) = 2x\ln x$*. Use integration by part to solve the equation* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ *for* y *with* $y(1) = \dfrac{1}{2}$.

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$dy = f(x)dx \implies \displaystyle\int dy = \int 2x\ln x dx \implies y = \int 2x\ln x dx.$

Let $u' = 2x$ *and* $v = \ln x$. *Then,* $u = x^2, v' = \dfrac{1}{x}$ *and*

$y = x^2\ln x - \displaystyle\int x^2\frac{1}{x}dx = x^2\ln x - \int x dx = x^2\ln x - \frac{1}{2}x^2 + c.$

$y(1) = 0 \implies 1\ln 1 - \dfrac{1}{2} + c = \dfrac{1}{2} \implies c = 1$. *That is*

$$y = x^2\left(\ln x - \frac{1}{2}\right) + 1.$$

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Definition

An equation that involves derivatives, $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}, n = 1, 2\ldots$, *or differentials,* $\mathrm{d}x, \mathrm{d}y, \ldots$, *is called an* **ordinary differential equation** *(ODE).*

Remark

- *The solution* $y = x^2(\ln x - 1/2) + c$ *contains an arbitrary constant* c; *it is called the* **general solution** *of the ODE.*

Whereas, the solution $y = x^2(\ln x - 1/2) + 1$, *that satisfies the initial condition* $y(1) = 1/2$ *is called a* **particular solution**.

Example

Verify whether the function y *is a solution of the ODE* $y'' - y = 0$.

1) $y = \sin x$ 2) $y = 4e^{-x}$ 3) $y = ce^x$.

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Solution: 1) *No.* *2) Yes* *3) Yes.*

Show that $y = cx^3$ *is the general solution of the ODE* $xy' - 3y = 0$. *Find the particular solution that satisfies the initial condition* $y(-3) = 2$.

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Show that $y = cx^3$ *is the general solution of the ODE* $xy' - 3y = 0$. *Find the particular solution that satisfies the initial condition* $y(-3) = 2$.

Solution: $y' = 3cx^2 \implies xy' = 3cx^3 = 3y \implies xy' - 3y = 0$;

$y(-3) = 2 \implies c = -2/27$; *the particular solution is* $y = -\dfrac{2}{27}x^3$.

Exercise

1) *Determine whether the function is a solution of the ODE* $xy' - 2y = x^3 e^x$.

$a.\ y = x^2 \quad b.\ y = x^2(2+e^x) \quad c.\ y = \ln x \quad d.\ y = x^2 e^x - 5x^2.$

2) *Verify the solution of the ODE.*

$a.\ ODE:\ 3y' + 5y = -e^{-2x},\ y = e^{-2x}$

$b.\ ODE:\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{xy}{y^2-1},\ y^2 - 2\ln y = x^2$

$c.\ ODE:\ y'' + 2y' + 2y = 0,\ y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$

3) *Use integration to find a general solution of the ODE.*

$a.\ y' = 10x^4 - 2x^3 \quad b.\ y' = \dfrac{e^x}{4+e^x} \quad c.\ \dfrac{\mathrm{d}y}{2x} = \mathrm{d}x\sqrt{4x^2+1} \quad d.\ y' = \tan^2 x.$

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The order of an ODE is determined by the highest-order derivative in the equation.

Example

*Which of the equations are a first-order ODE (***1ODE***)? Find $f(x,y)$.*

1) $y' = x^2 + y$, 2) $y'y = 2xy' - xy$, 3) $\dfrac{\mathrm{d}y}{x} = \mathrm{d}x \sin x$ 4) $(xy')' = 3$.

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Solution:

1) $Yes, f = x^2 + y$ 2) $Yes, f = \dfrac{xy}{y-2x}$ 3) $Yes, f = x\sin x,$ 4) $No.$

Definition (Separable Equation)

An ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

is called **separable** *because its variables can be separated,*

$$g(y)\mathrm{d}y = f(x)\mathrm{d}x.$$

Remark

The separable equation can be solved by a pair of integrations:

$$\int g(y)\mathrm{d}y = \int f(x)\mathrm{d}x.$$

Example

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Solution:

1) $\Longrightarrow \dfrac{\mathrm{d}y}{y} = \dfrac{x\mathrm{d}x}{x^2+4} \Longrightarrow \displaystyle\int \dfrac{\mathrm{d}y}{y} = \int \dfrac{x\mathrm{d}x}{x^2+4} \Longrightarrow \ln|y| = \dfrac{1}{2}\ln|x^2+4| + c$

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$\Longrightarrow \ln|y| = \dfrac{1}{2}\ln\left(x^2+4\right) + c$ *since* x^2+4 *is positive,*

$\Longrightarrow |y| = e^c\sqrt{x^2+4}$

$\Longrightarrow y = \pm C\sqrt{x^2+4}$ *where* $C = e^c$.

Example

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2) *Given the initial condition* $y(0)=1$, *find the particular solution of* $xy\,\mathrm{d}x + e^{-x^2}(y^2-1)\,\mathrm{d}y = 0$.

$$(2) \implies \frac{(y^2-1)\,\mathrm{d}y}{y} = -xe^{x^2}\,\mathrm{d}x \implies \int\left(y-\frac{1}{y}\right)\mathrm{d}y = \int -xe^{x^2}\,\mathrm{d}x$$

$$\implies \frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + c.$$

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$$\implies \frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + c.$$

From the initial condition $y(0) = 1$, $1/2 = -1/2 + c$ *which implies* $c = 1$.

Thus, the solution is defined by the implict formula

$$\frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + 1.$$

Exercise

1) *Find the equation of the curve that passes through the point* $(1,3)$ *and ha a slope of* y/x^2 *at any point* (x,y).

2) *The rate of change of the number of coyotes* $N(t)$ *in a population is directly proportional to* $650 - N(t)$, *where* t *is the time in years. When* $t = 0$, *the population is* 300, *and* $N(2) = 500$. *Find the population when* $t = 3$.

3) *Solve the logistic ODE* $y' = ky\left(1 - \frac{y}{L}\right)$.

4) *Find the general solution of the ODE.*

$a.\ y\ln x - xy' = 0 \quad b.\ \sqrt{x} + y'\sqrt{y} = 0 \quad c.\ \frac{\mathrm{u}}{\mathrm{v}} = uv\sin v^2.$

5) *Find the orthogonal trajectories of the family.*

$a.\ x^2 + y^2 = c \quad b.\ y^2 = 2cx \quad c. y = ce^x$

Definition

Equations of the form

$$y' + p(x)y = q(x)$$

are called **first-order linear ODEs**.

The general solution of this differential equation is

$$yu = \int qu\,\mathrm{d}x + c$$

where $u = e^{\int p(x)\mathrm{d}x}$ *is called the* **integrating factor** *since its presence makes the equation integrable.*

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Remark

The equation $y' + p(x)y = q(x)$ *is linear in* y *because* y *and its derivative* y' *occur only to the first power, and they are not multiplied together.*

Example

1) *Determine whether the differential equation is linear. Explain your reasoning. If YES then write the equation in the standard form* $y' + py = q$.

$a.\ x^3y' + xy = e^x + 1 \quad b.\ 2xy - y'\ln x = y$

$c.\ y' - y\sin x = xy^2 \quad d.\ \dfrac{2 - y'}{y} = 5x.$

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Solution:

$a.$ *YES*, $y' + \dfrac{1}{x^2} y = \dfrac{e^x+1}{x^3}$

$b.$ *YES*, $y' - \dfrac{2x}{\ln x} y = -\dfrac{y}{\ln x}$

$c.$ *NO, because of* y^2.

$d.$ *YES*, $y' + 5xy = 2$

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Solve the equation

1) $y' - \frac{3}{x}y = x, x > 0$ 2) $y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}$

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Solution:

Since $p(x) = -3/x$, *the integrating factor is*

$u(x) = e^{\int p(x)\mathrm{d}x} = e^{\int \frac{-3}{x}\mathrm{d}x} = e^{-3\ln|x|} = e^{-3\ln x} = x^{-3}$ *since* $x > 0$.

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$yu = \int qu\mathrm{d}x \Longrightarrow yx^{-3} = \int x\dot{x}^{-3}\mathrm{d}x = \int x^{-2}\mathrm{d}x = -x^{-1} + c.$

Therefore, $y = -x^2 + cx^3, x > 0.$

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Integration by parts of the RHS, $\frac{1}{3}\int x^{-4/3}(1+\ln x)\mathrm{d}x$ *(by setting* $w' = x^{-4/3}$ *and* $v = 1 + \ln x$ *and considering* $\int w'v = wv - \int wv'$*), leads to the solution*

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leads to the solution

$$y = x^{1/3}\left(-x^{-1/3}(\ln x + 1) + \int x^{-4/3}\mathrm{d}x + c\right)$$
$$= x^{1/3}\left(-x^{-1/3}(\ln x + 1) - 3x^{-1/3} + c\right)$$
$$= cx^{1/3} - \ln x - 4.$$

Exercise

Solve the following differential equations.

1) $xy' + y = e^x, x > 0$

2) $e^x y' + 2e^x y = 1$

3) $xy' + 3y = \dfrac{\sin x}{x^2}, x > 0$

4) $y' + y\tan x = \cos^2 x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

5) $xy' - y = 2x\ln x$

6) $\theta\dfrac{\mathrm{d}y}{\theta} - 2y = \theta^3\sec\theta\tan\theta, \theta > 0, y(\pi/3) = 2$

7) $(x+1)y' - 2(x^2 + x)y = \dfrac{e^x}{x+1}, x > -1, y(0) = 5.$

Outline

So far, all the equations have been first order ODEs. Now we will turn to higher-order equations; in particular, second-order equation with constant coefficients.

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$$ay'' + by' + cy = d(x)$$

where $a \neq 0, b$ *and* c *are constant real numbers.*

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- *The equation is said to be* **homogeneous** *if* $d(x) = 0$. *That is*

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Solving a second-order homogeneous ODE with constant coefficients

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Assuming that the solution is on the form $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$ so that the equation becomes
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$ay'' + by' + c = ar^2 e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}$.
In order words, $ay'' + by' + c = 0$ yields $ar^2 + br + c = 0$.

Definition

$$ar^2 + br + c = 0$$

is called the **characteristic polynomial** *of the ODE* $ay'' + by' + c = 0$.

The roots of the characteristic polynomial determine the solutions of the homogeneous ODE.

Property

☛ **Real and distinct roots**.

If we denote the roots by r_1 *and* r_2, *then the general solution of the ODE is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \text{ where } c_1, c_2 \in \mathbb{R}.$$

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- **Complex and distinct roots**.

If we denote the roots by $r_1 = \alpha + i\beta$ *and* $r_2 = r_1 = \alpha - i\beta$, *then the general solution of the ODE is*

$$y = \left(c_1 \cos\beta + c_2 \sin\beta\right) e^{\alpha x} \text{ where } c_1, c_2 \in \mathbb{R}.$$

Example

Solve (1) $2y'' + 7y' = 4y$, *and* (2) $y'' + 2y' + y = 0$.

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Solution

(1) *The characteristic polynomial of* $2y'' + 7y' - 4y = 0$ *is* $2r^2 + 7r - 4 = 0$. *The discriminant* $\Delta = b^2 - 4ac = 49 + 32 = 81 > 0$ *yields two distinct real roots* $r_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = -4, \dfrac{1}{2}$.

Thus, the general solution is $y = c_1 e^{-4x} + c_2 e^{\frac{x}{2}}$.

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Thus, the general solution is $y = c_1 e^{-4x} + c_2 e^{\frac{x}{2}}$.

(2) *The characteristic polynomial is* $r^2 + 2r + 1 = (r+1)^2$. *We get a double real root* $r_1 = -1$.

Thus $y = (c_1 + c_2 x)e^{-x}$.