MATH 157:

ALGEBRA

Semester 1

Lecture Notes

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DEPARTMENT OF MATHEMATICS

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SECTION 1

SIGMA NOTATION

The symbol Σ (capital sigma) is often used as shorthand notation to indicate the sum of a number of similar terms. For example,

Suppose we weigh five children. We will denote their weights by x_1, x_2, x_3, x_4 and x_5 . The sum of their weights $x_1 + x_2 + x_3 + x_4 + x_5$ is written more compactly as

$$\sum_{j=1}^{5} x_j$$

The symbol Σ means 'add up' . Underneath Σ we see j = 1 and on top of it 5. This means that j is replaced by whole numbers starting at the bottom number, 1, until the top number, 5, is reached. Thus,

$$\sum_{j=2}^{5} x_j = x_2 + x_3 + x_4 + x_5$$

and

$$\sum_{j=2}^{4} x_j = x_2 + x_3 + x_4$$

So the notation $\sum_{j=1}^{n} x_j$ tells us:

- To add the scores x_j,
- Where to start: x_1,
- Where to stop: x_n (where n is some number).

Now take the weight of the children to be $x_1 = 10$kg, $x_2 = 12$kg, $x_3 = 14,$ kg $x_4 = 8$kg and $x_5 = 11$kg.

Then the total weight (in kilograms) is

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$= 10 + 12 + 14 + 8 + 11$$

$$= 55$$

Notice that we have used i instead of j in the formula above. The j is what we call a dummy variable - any letter can be used, i.e.,

$$\sum_{j=1}^{n} x_j = \sum_{i=1}^{n} x_i$$

Now let us find $\sum_{i=1}^{4} 2x_1$ where $x_1 = 2$, $x_2 = 3$, $x_3 = -2$ and $x_4 = 1$.

Again, starting with $i = 1$ we replace the expression $2x_i$ with its value and add up the terms until $i = 4$ is reached. So,

$$\sum_{i=1}^{4} 2x_i = 2x_1 + 2x_2 + 2x_3 + 2x_4$$

$$= 2(2) + 2(3) + 2(4) + 2(1)$$

$$= 4 + 6 - 4 + 2$$

$$= 8$$

Similarly, let us find $\sum_{k=1}^{3}(x_k - 4)$ where $x_1 = 7$, $x_2 = 4$ and $x_3 = 1$.

Here,

$$\sum_{k=1}^{3}(x_k - 4) = (x_1 - 4) + (x_2 - 4) + (x_3 - 4)$$

$$= (7 - 4) + (4 - 4) + (1 - 4)$$

$$= 3 + 0 + (-3) = 0$$

Notice that this is different from $\sum_{k=1}^{3} x_k - 4$ where $x_1 = 7,\ x_2 = 4,\ x_3 = 1$.

In this case, we have,

$$\sum_{k=1}^{3} x_k - 4 = x_1 + x_2 + x_3 - 4$$

$$= 7 + 4 + 1 - 4$$

$$= 8$$

We use brackets to indicate what should be included in the sum. In the previous example, there were no brackets, so the '4' was not included in the sum.

Rules of Summation

We will prove three rules of summation. These rules will allow us to evaluate formula containing sigma notation more easily and allow us to derive equivalent formula.

Rule 1: If c is a constant, then

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

To see why Rule 1 is true, let's start with the left hand side of this equation,

$$\sum_{i=1}^{n} cx_i = cx_1 + cx_2 + cx_3 + \cdots + cx_n$$

$$= c(x_1 + x_2 + x_3 + \cdots + x_n)$$

$= c \sum_{i=1}^{n} x_i$

As required.

Rule 2: If c is a constant, then

$$\sum_{i=1}^{n} c = nc$$

This rule looks a bit strange as there is no $'x_i'$. The left hand side of this formula means 'sum c, n times'. That is,

$$\sum_{i=1}^{n} c = c + c + \cdots + c$$

$$= n \times c$$

$$= nc$$

Rule 3:

$$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

To prove this rule, let's start with the left hand side,

$$\sum_{i=1}^{n} (x_i + y_i) = (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + \cdots + (x_n + y_n)$$

$$= \left(x_1 + x_2 + x_3 + \cdots + x_n\right) + (y_1 + y_2 + y_3 + \cdots + y_n)$$

$= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$

SECTION 2

SET THEORY

Definition:

A SET is a collection of well-defied objects of the same kind. Objects that form a set are called ELEMENTS or MEMBERS of the set. Examples:

a. A set of days of the week.
b. A set of regions in Ghana.
c. A set of counting numbers less than 10.

SET NOTATION

A set is usually denoted by a capital letter – A, B, C, Q, Z, etc., and the members are enclosed in parenthesis {}. Example

A = {2,4,6,8}

The elements of the set A are 2, 4, 6, 8.

OPERATION ON SET

- **Intersect of sets – symbol:** ∩

  Two or more sets are said to intersect if there are elements common to the sets in question. For example,

  If A = {2,3,5,7,11} and B = {3,5,7,9,11,13}, then A∩B = {3,5,7,11}

Examples:

M = {multiples of 3 between 10 and 20}

N = {even numbers between 10 and 20}

i. Find $M \cap N$

M = {12,15,18}

N = {12,14,16,18}

ii. P = {0,2,4,6} and Q = {1,2,4,5} Find $P \cap Q$

iii. If = {7,9,13} and Q = {1,7,13} Find $P \cap Q$

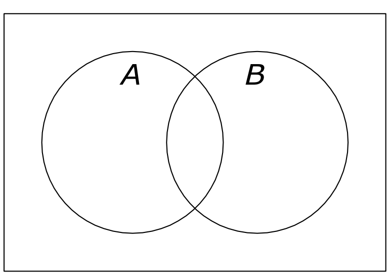
iv. If Q = {1,3,5,7,9,11,13,15} and R = {1,2,3,5,6,7,10,11,12} Find $Q \cap R$

v. M = {1,2,3,4,5, $\cdots$,20},Q = {3,4,5,6,7,8} and R = {2,3,5,7}

If Q and R are subsets of M, find $Q \cap R$

vi. If P = {2,3,5,7} and Q = {2,4,6,8} find $P \cap Q$

vii. Which of the following describes the relationship between the sets A and B in the Venn diagram below?



a. $A \subset B$,

b. $A \cap B = 5$,

c $A \cap B = \emptyset$,

d. $A \cup B = \{1,2,3,4,5,6,7\}$

- **Union of Sets – symbol:** ∪

  When two or more sets are put together to form one set then union of the set is formed. For example

  If Ab = {2,4,6,8} and B = {2,3,5,7}, then A∪B = {2,3,5,6,7,8}.

- **Complement of a set**

  The complement of a set A is the set of all elements in a universal set U, which are not elements of set A. The complement of A is written as A'

PROPERTIES OF SET OPERATIONS

Let A, B, and C be three intersecting subsets of the universal set U. We can make the following operational deductions relating the sets A, B, and C.

1. **Commutative Property**

   $A \cap B = B \cap A$ and A ∪ B= B ∪ A. therefore intersection and union are commutative. For example

   If A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10} then A ∩ B = B ∩ A = {2, 4}.

   Also A ∪ B = B ∪A = {1, 2, 3, 4, 5, 6, 8, 10}

2. **Associative Property**

   $(A \cap B) \cap C = (A \cap B) \cap C$ and

   $(A \cup B) \cup C = A \cup (B \cup C)$

   Therefore, intersection (∩) and union (∪) of sets are both associative. For example:

   If A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6} then

$(A \cap B) \cap C = (A \cap B) \cap C$ = {4} and (A ∪ B) ∪ C = A ∪ (B ∪ C) =

{1, 2, 3, 4, 5, 6, 8}

3. Distributive Property

A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C). Therefore, union of sets (∪) is distributive over intersection of sets (∩).

Also A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C). Therefore, intersection of sets (∩) is distributive over intersection of sets (∪).

For example:

If A = {1,2,3,4}, B = {1,3,5,7} and C = {2,3,5,7} then A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).

Proof:

B ∩ C = {3,5,7}

A ∪ (B ∩ C) = {1,2,3,4,5,7} --- (1)

A ∪ B = {1,2,3,4,5,7}

A ∪ C = {1,2,3,4,5,7}

(A ∪ B) ∩ (A ∪ C) = {1,2,3,4,5,7} --- (2)

(1) = (2)

Therefore, A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C), the proof.

PRACTICAL PROBLEM USING THE VENN DIAGRAM

- **2 - SET PROBLEM**

Example 1

There are 20 students in Pharmacy I, 16 of them are fluent in French and 10 of them are fluent I English. Each student is fluent in at least one of the two languages.

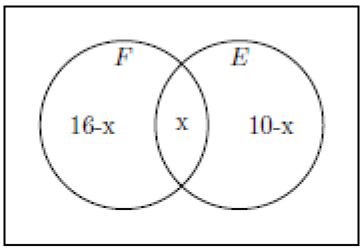
i. Illustrate this information on a Venn diagram

ii. How many students are fluent in both English and French?

iii. How many students are fluent in only one language?

✓ **Solution**

(i)



(ii) $16 - x + x + 10 - x = 20$

$-x = 20 - 26$

$-x = -6$

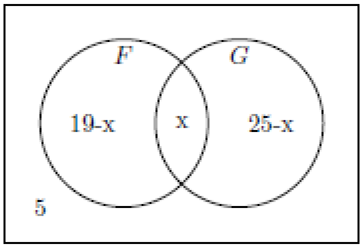
$x = 6$

(iii) $(16 - x) + (10 - X) = 16 - 6 + 10 - 6 = 14$

Example 2

In a class of 39 students, 19 offer French and 25 offer Ga, 5 students do not offer any of the two languages. How many students offer only French?

✓ **Solution**



$n(\cup) = 39, n(F) = 19, n(G) = 25$

x = number of students who offer both subjects

(i) $19 - x + x + 25 - x + 5 = 39$

$x = 10$

(ii) Only French

$\Rightarrow 19 - x = 19 - 10 = 9$

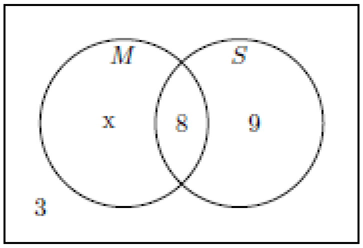
Example 3.

25 student in a class took an examination in mathematics and science and 8 passed in both mathematics and science. 3 students did not pass in any of the subjects.

(i) How many passed in mathematics.

(ii) Find the probability meeting a student who passed in one subject only.

✓ **Solution:**



(i) $x - 8 + 9 + 3 = 25$

$x = 25 - 20$

$x = 5$

Therefore, $5 + 8 = 13$

(ii) $\frac{5+9}{25} = \frac{14}{25}$

❖ 3 - SET PROBLEMS

Example 1

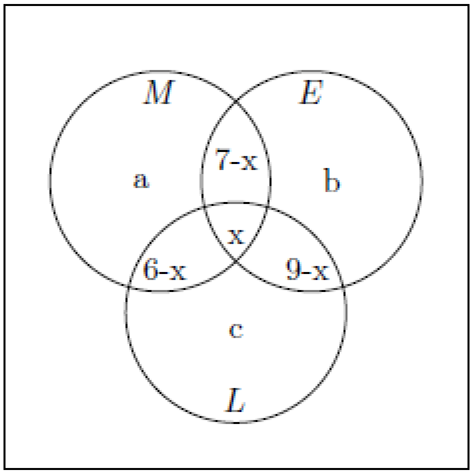
Mathematics, English and Life Skills books were distributed to 50 students in a class. 22 had Mathematics books, 21 English books and 25 Life Skills books, 7 had Mathematics and English books, 6 Mathematics and Life Skills books and 9 English and Life Skills books. Find the number of students who had

(i) All the three books

(ii) Exactly 2 of the books

(iii) Only Life Skills books

✓ Solution:



$n(U) = 50, n(M) = 22, \; n(E) = 21,$

$n(L) = 25, \quad n(M \cap E) = 7, \quad n(M \cap L) = 6, \quad n(E \cap L) = 9, \quad n(M \cap E \cap L) = x$

Here, we need to determine the values of a, b, c.

$a + 7 - x + x + 6 - x = 22$

$a = 22 - 13 + x$

$a = 9 + x$

$b + 7 - x + x + 9 - x = 21$

$b = 21 - 16 + x$

$b = 5 + x$

$c + 6 - x + x + 9 - x = 25$

$c = 25 - 15 + x$

$c = 10 + x$

Therefore,

$(9 + x) + (7 - x) + x + (6 - x) + (5 + x) + (9 - x) + (10 + x) = 50$

$46 + x = 50$

$x = 4$

(i) 4

(ii) Exactly 2 of the books = 10

(iii) 14

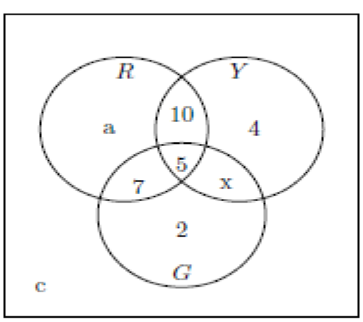
Example 2

A group of 50 girls were asked which of the three colours; Red, Yellow and Green they liked. 5 of them said they liked all three colours, 25 liked Red and 22 liked Green. 15 liked Red and Yellow, 12 liked Red and Green, 4 liked only Yellow and 2 liked only Green.

(i) Illustrate the information in a Venn diagram.

(ii) How many girls did not like any of the 3 colours

<u>**Solution:**</u>



$n(\xi) = 50, \quad n(R) = 25, \quad n(G) = 22, \quad n(R \cap Y) = 15, \quad n(R \cap G) = 12$

$\text{n(Y only)} = 4, \qquad \text{n(G only)} = 2, \qquad \text{n}(\text{R} \cap \text{G} \cap \text{Y}) = 5$

$x + 5 + 7 + 2 = 22$

$x + 14 = 22$

$x = 22 - 14$

$x = 8$

$a + 7 + 5 + 10 = 25$

$a = 25 - 22$

$a = 3$

$3 + 10 + 5 + 7 + 4 + 8 + 2 + c = 50$

$c = 50 - 39$

$$c = 11$$

SECTION 3

INDICES

Any expression written as a^n is defined as the variable a raised to the power of the number n.

n is called a power, an index or an exponent of a.

For example, the number 2^4 is in the index form. The number "4" is the index and the "2" is called the base. An index tells you how many of the base number to multiply together. In general:

a^3 means a x a x a ... {3 factor}

a^n means a x a x a ... a ... {n factor}

LAWS OF INDICES

1. $\boldsymbol{a^m \times a^n = a^{m+n}}$

   Examples: $a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5 = a^{3+2}$

2. $\boldsymbol{a^m \div a^n = a^{m-n}}$ **or** $\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$

   Example: $\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^2 = a^{5-3}$

3. $\boldsymbol{(a^m)^n = a^{mn}}$ **or** $\boldsymbol{(a^x b^y)^n = a^{xn} b^{yn}}$

   Example: $(a^3)^2 = (a^3 \times a^3) = (a \times a \times a) \times (a \times a \times a) = a^6$

PROPERTIES OF INDICES

The following properties of indices should be noted carefully in addition to the basic three laws above. They are used to simplify expressions.

Property 1:

$$\boldsymbol{a^0 = 1}$$

i .e. any non-zero number raised to the power zero is equal to 1. For example,

$2^0 = 1;\ 4^0 = 1$

Property 2:

$$\boldsymbol{(ab)^n = a^n b^n}$$

For example, $(2 \times 3)^2 = 2^2 \times 3^2 = 4 \times 9 = 36$

Property 3:

$$\boldsymbol{a^{-n} = \frac{1}{a^n}, \qquad provided\ a \neq 0}$$

i.e. any number raised to a negative index (power) is equal to the reciprocal of the number raised to the same but positive index.

Note: $\frac{1}{a^{-n}} = a^n$

for example, $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

note: any number raised to the power -1 is equal to the reciprocal of the number. For example, $a^{-1} = \frac{1}{a}$

Property 4:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \qquad \textit{provided } b \neq 0$$

For example, $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$

Property 5:

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n, \qquad \textit{provided } a \neq 0 \textit{ and } b \neq 0$$

For example, $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

Property 6:

a) $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ i.e. any number raised to the power $\frac{1}{2}$ is equal to the square root of the number. For example, $16^{\frac{1}{2}} = \sqrt{16} = 4$.

b) $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$ i.e. any number raised to the power $\frac{1}{3}$ is equal to the cube root of the number. For example, $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$.

c) $\boldsymbol{a^{\frac{1}{4}} = \sqrt[4]{a}}$ i.e. any number raised to the power $\frac{1}{4}$ is equal to the fourth root of the number. For example, $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. any number raised to the power $\frac{1}{n}$ is equal to the nth root of the number.

Also, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

For example, $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

Note: any number raised to the power 1 is equal to the number.

i.e. $3^1 = 3$

SECTION 4

LOGARITHMS

The logarithm of any number b to the base a (written $\log_a b$) is the index or power to which the base must be raised to give the number. From the definition,

$$\log_a b = c \quad \Rightarrow \quad b = a^c \quad provided\ a \neq 1$$

For example, $\log_2 16 = 4 \quad \Rightarrow \quad 16 = 2^4$

Note: the base of logarithm can be written in any positive number except 1. The logarithm of a negative number does not exist. When the base of a logarithm is not written, it means it is in base 10.

LAWS OF LOGARITHMS

1. $\boldsymbol{\log_a x + \log_a y = \log_a(xy)}$

   e.g. $\log_2 3 + \log_2 4 = \log_2 12$

2. $\boldsymbol{\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)}$

   E.g. $\log_2 21 - \log_2 7 = \log_2 \left(\frac{21}{7}\right) = \log_2 3$

3. $\boldsymbol{\log_a x^n = n \log_a x}$

E.g. $\log_2 5^2 = 2 \log_2 5$

4. $\mathbf{\log_a x = \frac{\log_b x}{\log_b a}}$

E.g. $\log_2 8 = \frac{\log_3 8}{\log_3 2}$

Note: x and y are any positive numbers.

PROPERTIES OF LOGARITHMS

The following two special properties of logarithms should be noted.

Property 1:

$$\mathbf{\log_a a = 1,}$$

i.e. the logarithm of any number to the base of the same number is 1.

For example, $\log_2 2 = 1$

Property 2:

$$\mathbf{\log_a 1 = 0,}$$

i.e. the logarithm of one (1) to the base is zero (0).

For example, $\log_3 1 = 1$

SECTION 5

QUADRATIC FUNCTIONS AND EQUATIONS

A quadratic equation is an equation of the form

$$ax^2 + bx + c = 0$$

Where a, b and c are real numbers and $a \neq 0$.

SQUARE ROOT PROPERTY

If b is a real number and $a^2 = b$, then

$$a^2 = \pm\sqrt{b}$$

Example:

1. Solve $x^2 = 49$

   $x = \pm\sqrt{49} = \pm 7$

2. Solve $2x^2 = 4$

   $x^2 = 2$

   $x = \pm\sqrt{2}$

3. Solve $(y-3)^2 = 4$

   $(y-3) = \pm\sqrt{4} = \pm 2$

   $y = 3 \pm 2$

$y = 1$ or 5

4. Solve $(x + 2)^2 = 25$

   $x + 2 = \pm\sqrt{25} = \pm 5$

   $x = -2 \pm 5$

   $x = -2 + 5$ or $x = -2 - 5$

   $x = 3$ or $x = -7$

5. Solve $(3x - 17)^2 = 28$

   $3x - 17 = \pm 28 = \pm 2\sqrt{7}$

   $3x = 17 \pm 2\sqrt{7}$

   $$x = \frac{17 \pm 2\sqrt{7}}{3}$$

SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

What constant term should be added to the following expressions to create a perfect square trinomial?

$x^2 - 10$ add $5^2 = 25$

$x^2 + 16$ Add $8^2 = 64$

$x^2 - 7x$ Add $\left(\frac{7}{2}\right)^2 = \frac{49}{2}$

We now look at a method for solving quadratics that involves a technique called **completing the square**.

It involves creating a trinomial that is a perfect square, setting the factored trinomial equal to a constant, then using the square root property from the previous section.

Solving a Quadratic Equation by Completing a Square

1) If the coefficient of x^2 is NOT 1, divide both sides of the equation by the coefficient.
2) Isolate all variable terms on one side of the equation.
3) Complete the square (half the coefficient of the x term squared, added to both sides of the equation).
4) Factor the resulting trinomial.
5) Use the square root property.

Examples

1. Solve by completing the square.

$$y^2 + 6y = -8$$

$$y^2 + 6y + 9 = -8 + 9$$

$$(y + 3)^2 = 1$$

$$y + 3 = \pm\sqrt{1} = \pm 1$$

$$y = -3 \pm 1$$

$$y = -4 \text{ or } -2$$

2. Solve by completing the square.

$$y^2 + y - 7 = 0$$

$$y^2 + y = 7$$

$$y^2 + y + {}^{1}/_{4} = 7 + {}^{1}/_{4}$$

$$(y + {}^{1}/_{2})^2 = {}^{29}/_{4}$$

$$y + \frac{1}{2} = \pm\sqrt{\frac{29}{4}} = \pm\frac{\sqrt{29}}{2}$$

$$y = \frac{1}{2} \pm \frac{\sqrt{29}}{2} = \frac{-1 \pm \sqrt{29}}{2}$$

3. Solve by completing the square.

$$2x^2 + 14x - 1 = 0$$

$$2x^2 + 14x = 1$$

$$x^2 + 7x = {}^{1}/_{2}$$

$$x^2 + 7x + \frac{49}{4} = {}^{1}/_{2} + \frac{49}{4} = \frac{51}{4}$$

$$\left(x + \frac{7}{2}\right)^2 = \frac{51}{4}$$

$$x + \frac{7}{2} = \pm\sqrt{\frac{51}{4}} = \pm\frac{\sqrt{51}}{2}$$

$$x = -\frac{7}{2} \pm \frac{\sqrt{51}}{2} = \frac{-7 \pm \sqrt{51}}{2}$$

SOLVING QUADRATIC EQUATIONS BY THE QUADRATIC FORMULA

Another technique for solving quadratic equations is to use the **quadratic formula**.

The formula is derived from completing the square of a general quadratic equation.

A quadratic equation written in standard form,

$$ax^2 + bx + c = 0$$

has the solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve $11n^2 - 9n = 1$ by the quadratic formula.

$11n^2 - 9n - 1 = 0,$

Therefore $a = 11,\ \ b = -9,\ \ c = -1$

$$n = \frac{9 \pm \sqrt{(-9)^2 - 4(11)(-1)}}{2(11)} = \frac{9 \pm \sqrt{81 + 44}}{22}$$

$$n = \frac{9 \pm \sqrt{125}}{22} = \frac{9 \pm 5\sqrt{5}}{22}$$

Solve $\frac{1}{8}x^2 + x - \frac{5}{2} = 0$ by the quadratic formula.

$x^2 + 8x - 20 = 0$ (multiply both sides by 8)

$a = 1,\ \ b = 8,\ \ c = -20$

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-20)}}{2(1)} = \frac{-8 \pm \sqrt{64 + 80}}{2}$$

$$x = \frac{-8 \pm \sqrt{144}}{2} == \frac{-8 \pm 12}{2} = \frac{-20}{2}\ or\ \frac{4}{2}$$

$$x = -10\ \ or\ 2$$

Solve $x(x + 6) = -30$ by the quadratic formula.

$x^2 + 6x + 30 = 0$

$$a = 1, \qquad b = 6, \qquad c = 30$$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(30)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 120}}{2}$$

$$x = \frac{-6 \pm \sqrt{-84}}{2}$$

There is no real solution.

THE DISCRIMINANT

The expression under the radical sign in the formula

$$b^2 - 4ac$$

is called the ***discriminant***. The discriminant will take on a value that is positive, 0, or negative. The value of the discriminant indicates two distinct real solutions, one real solution, or no real solutions, respectively.

Example:

Use the discriminant to determine the number and type of solutions for the following equation.

$$5 - 4x + 12x^2 = 0$$

$$a = 12, \qquad b = -4, \qquad and\ c = 5$$

$$b^2 - 4ac = (-4)^2 - 4(12)(5)$$

$$= 16 - 240$$

$$= -224$$

There are no real solutions.

STEPS IN SOLVING QUADRATIC EQUATIONS

1) If the equation is in the form$(ax + b)^2 + c$, use the square root property to solve.

2) If not solved in step 1, write the equation in standard form.

3) Try to solve by factoring.

4) If you haven't solved it yet, use the quadratic formula.

Example:

Solve $12x = 4x^2 + 4$

$$0 = 4x^2 - 12x + 4$$

$$0 = 4(x^2 - 3x + 1)$$

Let $a = 1$, $b = -3$, $c = 1$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

Example 2:

Solve the following quadratic equation.

$$\frac{5}{8}m^2 + m - \frac{1}{2} = 0$$

$$5m^2 + 8m - 4 = 0$$

$$(5m - 2)(m + 2) = 0$$

$5m - 2 = 0$ or $m + 2 = 0$

$m = \frac{2}{5}$ or $m = -2$

GRAPHING QUADRATIC EQUATIONS IN TWO VARIABLES

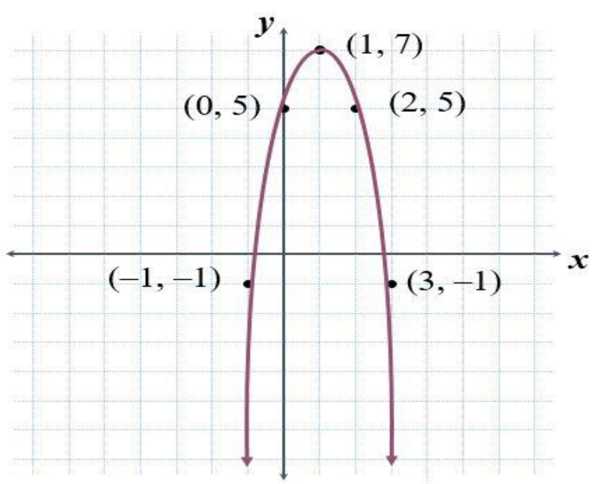
The graph of a quadratic equation is a **parabola**. The highest point or lowest point on the parabola is the **vertex**.

Axis of symmetry is the line that runs through the vertex and through the middle of the parabola.

Example:

Graph $y = 2x^2 - 4$

x	3	2	1	0	-1
y	-1	5	7	5	-1



Since a = –2 and b = 4, the graph opens down and the x-coordinate of the vertex is

$$-\frac{4}{2(-2)} = 1$$

SECTION 6

SERIES AND SEQUENCE

Definitions:

- A sequence is a set of numbers (terms) written in a defined order with a rule or formula for obtaining the terms. For example,

  1, 3, 5, 7, 9, 11, and 2, 4, 6, 8, 10, are sequences. Each term is obtained by adding 2 to the preceding term.

- A Series is formed when the term of a sequence are added. For example,

  1 + 3 + 5 + 7 + 9 + … is a series. If the series stops after a finite number of the terms, it is called a **finite series.**

  For example, 1+ 3+ 5+ 7+ 9 is a finite series of five terms. However, if the series does not stop but continues indefinitely it is called an **infinite series.**

  Thus, 2 + 4 + 6 + 8 + 10 + … is an infinite series. Note that the same definitions are given to finite and infinite sequences.

TYPES OF SEQUENCE

Two types of sequences shall be considered, namely:

1. Arithmetical Progression (AP), sometimes called linear sequence.
2. Geometrical Progression (GP), also called Exponential sequence.

1. Arithmetical Progression (A.P.) or Linear sequence

An **arithmetical progression** (A.P.) or a **linear sequence** is a sequence in which any term differs from the preceding term by a constant called the **common difference** which may be positive or negative. For example, 2, 5, 8, 11 is an Arithmetical Progression (AP) with common difference 3.

For a linear sequence, the first term is denoted by $'a'$ and the common difference by$'d'$. The nth term is denoted byU_n, where U_1 means the first term, U_2 means the second term etc. the first four term of an AP are:

$U_1 = a;\ \ U_2 = a + d;\ \ U_3 = a + 2d$ and$U_4 = a + 3d$.

i.e. $a, (a + d), (a + 2d), (a + 3d)$.

The general term or the formula for obtaining the term of any linear sequence is given by:

$$U_n = l = a + (n - 1)d$$

where n is the number of terms and$l$, the last term. This formula can be used to find any term of an AP if the first term a and the common difference d are known.

Note that $d = U_2 - U_1 = U_3 - U_2$ etc.

i.e. the difference between consecutive terms. For example, the common difference of the AP $2, 5, 8, 11$ is

$d = 5 - 2 = 8 - 5 = 11 - 8 = 3$

Examples

Find the 11^{th} term of a linear sequence of the term $4, 9, 14, 19, \ldots$

✓ **Solution**

For the sequence, the first term, $a = 4$ and the common difference, $d = 9 - 4 = 5$

The general formula or term of an AP is:

$$U_n = a + (n-1)d$$

We want to find the 11^{th} term, therefore $n = 11$

Substituting $a = 4$, $d = 5$ and $n = 11$ into U_n, we have

$$U_{11} = 4 + (11-1)5$$

$$= 4 + 10(5)$$

$$= 4 + 50$$

$$= 54$$

Find the 8^{th} term of the linear sequence of the form $47, 42, 37, 32, \ldots$

✓ Solution

Form the sequence, the first term, $a = 47$ and the common difference, $d = 42 - 47 = -5$

The general formula or term of an AP is

$$U_n = a + (n-1)d$$

We want to find the 8^{th} term, therefore $n = 8$ into the formula, we have

$$U_8 = 47 + (8-1)(-5)$$

$$= 47 + 7(-5)$$

$$= 47 - 35$$

$$= 12$$

Find the 12^{th} term of an AP of the form $7, 6\frac{1}{4}, 5\frac{1}{2}, \ldots$

✓ Solution

From the sequence, the first term, $a = 7$ and the common difference, $d = 6\frac{1}{4} - 7 = -\frac{3}{4}$

The general formula or term of an AP is

$$U_n = a + (n-1)d$$

We want to find the 12^{th} term, therefore $n = 12$ into the formula, we have

$$U_{12} = 7 + (12-1)\left(-\frac{3}{4}\right)$$

$$= 7 + (11)\left(-\frac{3}{4}\right)$$

$$= 7 - \frac{33}{4} = -\frac{5}{4}$$

Find the formula for the nth term of the linear sequence $10, 6, 2, -2, \ldots$

✓ Solution

Form the sequence, the first term, $a = 10$ and the common difference, $d = 6 - 10 = -4$

The nth term of the linear sequence is given by

$$U_n = a + (n-1)d$$

$$U_n = 10 + (n-1)(-4)$$

$$U_n = 10 - 4n + 4$$

$$U_n = 14 - 4n$$

Sum of an AP (Linear Sequence)

There are two important formula for finding the sum of an A.P. or a linear sequence. The **sum** of the first n terms of an A.P. or linear sequence denoted by S_n is given by:

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \ldots (1)$$

$$= \frac{n}{2}(a + l) \qquad\qquad \ldots (2)$$

All symbols denoted their usual meaning. The first formula is used when a and d are known.

The second formula is also used when the first term a and the last term l are known but not the common difference. For any arithmetic sequence we can write the following two properties:

1. $d = U_n - U_{n-1}$
   where $d = U_2 - U_1 = U_3 - U_2 \quad etc.$

2. $U_n = S_n - S_{n-1}$

   Where $U_2 = S_2 - S_1$

   $U_3 = S_3 - S_2 \quad etc.$

Example:

Find the sum of the first six terms of the AP $3, 5, 7, 9, \ldots$

✓ Solution:

From the sequence, the first term, $a = 3$ and the common difference, $d = 5 - 3 = 2$

The sum of the nth term of an AP,

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Hence the sum of the first six terms,

$$S_6 = \frac{6}{2}[2(3) + (6-1)12] = 3[6+10]$$

$$= 3(16) = 48$$

Example

Find the sum of the AP $1,3,5, \dots, 101$.

✓ Solution

To find the sum we have to find the number of terms n first.

From the sequence, $a = 1$, $d = 3 - 1 = 2$ and the last term is $l = 101$

$$l = a + (n-1)d$$

$$101 = 1 + (n-1)2$$

$$101 = 1 + 2n - 2$$

$$2n = 102$$

$$n = 51$$

The sum of the nth term of an AP, $S_n = \frac{n}{2}(a + l)$. Here the first term and the last term are known.

Hence the sum of the AP,

$$S_n = \frac{51}{2}(1 + 101) = \frac{51}{2}(102) = 2601$$

Alternatively, using $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{51} = \frac{51}{2}[2(1) + (51-1)2]$$

$$S_{51} = \frac{51}{2}[2 + 100] = \frac{51}{2}(102) = 2601$$

The fifth term of an arithmetic progression is -1 and the sum of the first twenty terms is-240. Find the third term.

✓ Solution

The 5^{th} term,

$U_5 = a + 4d = -1$ ... (1)

The sum of the first twenty terms,

$$S_{20} = \frac{20}{2}[2a + (20 - 1)d] = -240$$

$2a + 19d = -24$ ... (2)

Solving (1) and (2) simultaneously gives:

$a = 7$ and $d = -2$

The third term,

$U_3 = a + 2d = 7 + 2(-2) = 7 - 4 = 3$

Arithmetic Mean

If three numbers a, b and c are in arithmetical progression, then b is called the **arithmetic mean** of a and c. The common difference of the progression is given by

$b - a$ or $c - b$.

Therefore $b - a = c - b$

$\Rightarrow 2b = a + c$

$b = \dfrac{a + c}{2}$

Therefore the arithmetic mean of a and c is $\frac{1}{2} = (a + c)$. This is the ordinary 'average' of a and c.

Find the arithmetic mean of 8 and 60 is $\frac{1}{2}(8 + 60)$.

✓ Solution

$\frac{1}{2}(8 + 60) = 34$

2. Geometrical Progression (GP) or Exponential Sequence.

A **Geometrical Progression (GP)** or **exponential sequence** is a sequence where each term is obtained from the preceding term by multiplying by a constant factor. This constant factor is called the **common ratio**, denoted by r.

For example, the sequence 6, 12, 24, 48 is a G.P. with common ratio 2.

The general term or formula of a GP with first term a and common ratio r is given by:

$$U_n = ar^{n-1}$$

Where n is the number of terms. The first four terms of a GP are:

$U_1 = a$; $U_2 = ar$; $U_3 = ar^2$ and $U_4 = ar^3$

i.e. $a,\ ar,\ ar^2,\ ar^3$ ...

Note that the common ratio, r is given by:

$$r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \frac{U_n}{U_{n-1}}$$

For the sequence, $12, 24, 48$, the common ratio $r = \frac{12}{6} = \frac{24}{12} = 2$

Example:

Find the 7^{th} term of an exponential sequence of the form $5, 10, 20, 30, \ldots$

Solution:

From the sequence, the first term, $a = 5$ and the common ratio, $r = \frac{10}{5} = 2$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

We want to find the 7^{th} term, therefore $n = 7$

Substituting $a = 5,\ r = 2$ and $n = 7$ into the formula, the 7^{th} term,

$$U_n = 5 \times (2)^{7-1} = 5 \times (2)^6$$

$$= 5 \times 64 = 320$$

Example:

Find the 11^{th} term of the exponential sequence of the form $324, 108, 36, 12, \ldots$

Solution:

Form the sequence, the first term, $a = 324$ and the common ratio, $r = \frac{108}{324} = \frac{36}{108} = \frac{1}{3}$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

We want to find the 11^{th} term, therefore $n = 11$.

Substituting $a = 324, r = \frac{1}{3}$ and $n = 11$ into the formula, the 11^{th} term,

$$U_{11} = 324 \times \left(\frac{1}{3}\right)^{11-1} = 324 \times \left(\frac{1}{3}\right)^{10}$$

Find the formula in terms of n of the G.P $2, 4, 8, 16, \ldots$

Solution:

From the sequence, the first term, $a = 2$ and the common ratio, $r = \frac{4}{2} = 2$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

Substituting $a = 2$ and $r = 2$ into the formula, the formula in terms of n is

$$U_n = 2 \times (2)^{n-1} = 2^n$$

Note: the general term is a formula in terms of n

Sum of a GP or exponential sequence

The general formula of the sum of the first n terms of an exponential sequence is given by:

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r}}$$

This formula is used when for $r < 1$ i.e when the common ratio is less than unity.

When $r > 1$ i.e when the common ratio is greater than 1, the sum of the first n terms is given by:

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$$

For any geometric sequence, we can write the following two properties:

1. $\boldsymbol{r = \frac{U_n}{U_{n-1}}}$
2. $\boldsymbol{U_n = S_n - S_{n-1}}$

Example:

Find the sum of the first eight terms of the exponential sequence $\frac{1}{2}$, 1, 2, 4, ...

Solution:

The first term $a = \frac{1}{2}$ and the common ratio $r = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2$. Therefore the sum of the first eight terms is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Since $r > 1$ i.e $r = 2 > 1$. Therefore sum of the first 8 terms,

$$S_8 = \frac{\frac{1}{2}(2^8 - 1)}{2 - 1} = \frac{1}{2}(256 - 1) = 127.5$$

Sum to infinity of a geometrical progression

When the common ratio, r of a GP is less than unity, the sum to n terms is given by of a G.P is given by:

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a - ar^n}{1 - r}$$

Which may be written as

$$S_n = \frac{a}{1 - r} = \frac{ar^n}{1 - r}$$

Since $r < 1$e, as $n \to \infty$ (i.e as n becomes large), $r^n \to 0$.

Hence $\frac{ar^n}{1-r} \to 0$ as $n \to \infty$

Thus $S_n = \frac{a}{1-r}$ as $n \to \infty$

The quantity $\frac{a}{1-r}$ is called the **sum to infinity**, denoted by S_∞.

Hence the sum to infinity of a GP is:

$$S_\infty = \frac{a}{1-r}$$

which is valid when $-1 < r < 1$.

Example:

Find the sum to infinity of the following exponential sequence

i. $1, \frac{4}{5}, \left(\frac{4}{5}\right)^2, \ldots$

ii. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots$

Solution:

i. $a = 1$ and $r = \frac{4}{5} \div 1 = \frac{4}{5}$

Therefore $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{4}{5}} = \frac{1}{\frac{1}{5}} = 5$

ii. $a = \frac{5}{2}$ and $r = \frac{5}{4} \div \frac{5}{2} = \frac{1}{2}$

Therefore $S_\infty = \frac{a}{1-r} = \frac{\frac{5}{2}}{1-\frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}} = 5$

<u>Applications of Geometric Progression</u>

Example:

Ali bought a new car for Gh¢ 50,000.00. It is reckoned that the value of the depreciation per annum is 20% of its value at the beginning of the year. Calculate the number of years after which the car is worth Gh¢ 13,110.00.

Solution:

This is an application of a GP. If the car depreciates by 20% each year, then the values of the car for successive years form an exponential series, $a,\ ar,\ ar^2, \dots, ar^n$

The first term, $a = \text{Gh¢ } 50{,}000.00$ and the common ratio, $r = 80\% = 0.8$

Note: value decreases, therefore $r = (100 - 20)\% = 80\%$

Hence the second term is

$ar = (50{,}000)(0.8) = \text{Gh¢}40{,}000$ which is the value after 1 year

The third term is $ar^2 = (50{,}000)(0.8)^2 = \text{Gh¢}32{,}000$

Which is the value after 2 years, and so on. When Gh¢13,110 has been reached,

$$13{,}000 = ar^n$$

$$13{,}110 = (50{,}000)(0.8)^n$$

$$\frac{13{,}110}{50{,}000} = (0.8)^n$$

$$0.2622 = (0.8)^n$$

Taking logarithms to base 10 on both sides

$$\log 0.2622 = \log(0.8)^n$$

$$\log 0.2622 = n \log(0.8)$$

$$n = \frac{\log 0.2622}{\log 0.8} = 5.999$$

Hence it will take 6 years to reach more than Gh¢13,110.00

COMBINED AP AND GP

Example:

The first, third and the ninth terms of a linear sequence are the first three terms of an exponential sequence. If the seventh term of the linear sequence is 14, calculate

i) The 20^{th} term of the linear sequence

ii) The sum of the first twelve terms of the exponential sequence.

Solution:

Let $d =$ common difference of the linear sequence and $a =$ first term of both the A. P and G. P.

For the linear sequence, $U_n = a + (n-1)d$

$U_1 = a,\ \ U_3 = a + 2d$ and $U_9 = a + 8d$

The exponential sequence is $a,\ (a + 2d),\ (a + 8d)$.

The common ratio $= \frac{a+2d}{a} = \frac{a+8d}{a+2d}$

$$(a + 2d)(a + 2d) = a(a + 8d)$$

$$a^2 + 4ad + 4d^2 = a^2 + 8ad$$

$$4d^2 = 4ad$$

$$d^2 = ad$$

$$d = a$$

Now the seventh term of linear sequence is 14. Thus

$$a + 6d = 14 \quad \Longrightarrow 7d = 14 \quad \Longrightarrow d = 2$$

But $a = d \quad \Rightarrow a = 2$

i) For the A.P, $a = 2$ and $d = 2$. Hence the 20^{th} term,

$$U_{20} = a + 19d = 2 + 19(2) = 40$$

ii) The exponential sequence

$$= a, a + 2d, a + 8d$$

$$= 2, (2 + 4), (2 + 16) = 2, 6, 18$$

The first term, $a = 2$ and the common ratio, $r = \frac{6}{2} = 3$

The sum of the first n terms of a GP is given by

$S_n = \frac{a(r^n - 1)}{r - 1}$ since $r < 1$

Therefore $S_{12} = \frac{2(3^{12} - 1)}{3 - 1} = 3^{12} - 1$

SECTION 7

PERMUTATION OF OBJECTS

An ordered arrangement of objects is called a permutation. For example, the possible permutations of the letters A, B and C are as follows: ABC, ACB, BAC, BCA, CAB, CBA.

Definitions:

(a) The number of permutations of n distinct objects, taken all together is:

$$n! = n(n-1)(n-2)(n-3)\cdots\cdots\cdots 3*2*1 \quad \left(or\ {}^{n}P_{n}\right)$$

(b) The number of permutations of n distinct objects taken k at a time is:

$${}^{n}P_{k}\ or\ P(n,k) = \frac{n!}{(n-k)!},\ where\ k < n$$

(c) The number of permutations of n objects consisting of groups of which n_1 of the first group are alike, n_2 of the second group are alike and so on for the kth group with n_k objects which are alike is:

$$\frac{n!}{n_1!n_2!n_3!\cdots\cdots\cdots n_k!} \text{ where } n_1 + n_2 + n_3 + \cdots\cdots\cdots + n_k = n$$

(d) *Circular Permutations:* Permutations that occur when objects are arranged in a circle are called circular permutations. The number of ways of arranging n different objects in a circle is given by:

$$\frac{n!}{n} = (n-1)!$$

Examples

1. (i) The number of permutations of 10 distinct digits taken two at a time is

$$^{10}P_2 = \frac{10!}{(10-2)!} = 10*9 = 90$$

(ii) A company codes its customers by giving each customer an eight character code. The first 3 characters are the letters A, B and C in any order and the remaining 5 are the digits 1, 2, 3, 4 and 5 also in any order. If each letter and digit can appear only once then the number of customers the company can code is obtained as follows:

The first 3 letters can be filled in 3!

The next 5 digits can be filled in 5!

Then the required number $= 3! * 5! = 720$

2.(a) The number of permutations of the letters of the word, **POSSIBILITY**, which contains 3I's and 2S's is

$$\frac{11!}{3!*2!*1!*1!*1!*1!*1!*1!} = 11*10*9*8*7*6*5*2$$

$$= 3,326,400$$

(b) The number of arrangements of the letters of the word, **ADDING**, if

the two letters D and D are together (**ADDING**)

$$= 5! = 120$$

(c) The number of circular permutations of 6 persons sitting around a circular

table $= \frac{n!}{n} = (n-1)! = 5! = 120$

SECTION 8

COMBINATION OF OBJECTS

A Combination is a selection of objects in which the order of selection does not matter.

Definition:

The number of ways in which k objects can be selected from n distinct objects, irrespective of their order is defined by:

$$^{n}C_{k} \ or \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!\,k!} = \frac{^{n}P_{k}}{k!}, \ where \, k < n$$

<u>Example</u>

1.(a) (i) The number of ways a committee of 5 people can be chosen out of 9 is

$$^{9}C_{5} = \frac{9!}{4!5!} = 126$$

(ii) The number of combinations of the letters a, b, c, d and e, taken three at a time is $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$ which are listed below:

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde abc,

(b) Find the number of ways in which a committee of 4 can be chosen from 6 boys and 5 girls if it must

(i) consist of 2 boys and 2 girls

✓ **Solution:**

(i) The number of ways of choosing 2 boys from 6 and 2 girls from 5 is as follows:

$$\binom{6}{2} * \binom{5}{2} = \frac{6!}{4!\,2!} * \frac{5!}{3!\,2!}$$

$$= 15 * 10 = 150$$

Examples:

1) In how many ways can 4 students be selected from a class of 20 to represent the class in an essay competition?

   $^{20}C_4$

2) In how many ways can 2 drugs be selected from 13 different drugs?

   $^{13}C_2$

3) In how many ways can a committee consisting of 4 pharmacists and 3 biologist be selected from 9 pharmacist and 8 biologist, if
   i) there are no restrictions ?
   ii) a particular pharmacist must be selected ?
   iii) a particular biologist cannot serve on the committee?
   iv) a particular pharmacist and biologist must serve on the committee.

✓ **Solution**

$9\ phar \qquad 8\ bio$

$4\ \ phar \qquad 3\ bio$

i. $^9C_4 \times {}^8C_3$

ii. 1P
$^8C_3 \times {}^8C_3$

iii. $^{9}C_{4} \times {}^{7}C_{3}$

iv. $^{8}C_{3} \times {}^{7}C_{2}$

Example:

A committee of 3 is to be selected from 5 pharmacists and 4 biologist. In how many ways can this be done, if

i) there are no restrictions
ii) the committee should include at least one pharmacist.
iii) at least one from each discipline.
iv) the committee should include more pharmacists than biologists.

✓ **Solutions:**

$Phar = 5$ $Bio = 4$

i) $^{9}C_{3}$
ii)

possibility	No. of ways
1P 2B	$^{5}C_{1} \times {}^{4}C_{2} = 5 \times 6 = 30$
2P 1B	$^{5}C_{2} \times {}^{4}C_{1} = 10 \times 4 = 40$
3P 0B	$^{5}C_{3} \times {}^{4}C_{0} = 10 \times 1 = 10$

iii) 30 + 40 = 70 ways
iv) 2P 1B = $^{5}C_{2} \times {}^{4}C_{1} = 10 \times 1 = 40$

Example:

A student is to answer 8 out of 13 questions in an examination. In how many ways can this be done if

i) there are no restrictions ?
ii) she must answer the first 2 questions
iii) she must answer at least 3 of the first four questions.

✓ **Solution:**

i) $^{13}C_8$

ii) $\cancel{1}$ $\cancel{2}$ 3 4 5 6 7 8 9 10 11 12 13

$^{11}C_6$

iii) 1 2 $\boxed{3\ 4}$ 5 6 7 8 9 10 11 12 13

Answering 3 questions of the first 4	
$^4C_3 \times {}^9C_5$	Or $^4C_4 \times {}^9C_4$

SECTION 9

THE BINOMIAL THEOREM

The factorial operation is the exclamation sign, !, and it is placed after an integer variable or number to indicate factorial function, which is defined as: $n! = n(n-1)(n-2)(n-3)\ldots 2\times 1$

From this definition, it follows that $n! = n[(n-1)!] = n(n-1)(n-2)!$ etc.

For example, 4! = 4 x 3 x 2 x 1 = 24

3! = 3 x 3 x 1 = 6

2! = 2 x 1 = 2

1! = 1

and by definition 0! =1

The binomial theorem or the binomial formula is a formula to expand a power of the binomial expression $a+b$, that is, it is a formula that enables us to expand $(a+b)^n$. Descriptions of the binomial theorem frequently use a shorthand notation for the coefficients of each term consisting of numbers over each other enclosed in parenthesis e.g. $\binom{n}{k}$

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$. The notation $\binom{n}{k}$ is often called the binomial coefficient "n choose k"

In general, the binomial theorem is as follows:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{3}a^{n-r}b^r + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^o b^n$$

Applying the binomial theorem to $(a + b)^3$, we obtain

$$(a + b)^3 = \binom{3}{0} a^3 b^0 + \binom{3}{1}a^2 b^1 + \binom{3}{2}a^1 b^2 + \binom{3}{3}a^0 b^3$$

$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

Example

Expand $(x+2y)^5$

Solution

Apply the binomial theorem. If n is a positive integer, then

$$(x+2y)^5 = \binom{5}{0}x^5(2y)^0 + \binom{5}{1}x^4(2y)^1 + \binom{5}{2}x^3(2y) + \binom{5}{3}x^2(2y)^3 + \binom{5}{4}x^1(2y)^4 + \binom{5}{5}x^0(2y)^5$$

$$= \frac{5!}{5!0!}x^5 + \frac{5!}{4!1!}x^4 2y + \frac{5!}{3!2!}x^3(4y^2) + \frac{5!}{2!3!}x^2(8y^3) + \frac{5!}{1!4!}x(16y^4) + \frac{5!}{0!5!}(32y^5)$$

$$= x^5 + 10x^4 y + 40x^3 y^2 + 80x^2 y^3 + 80xy^4 + 32y^5$$

Example

Find the fifth term of $(a+2x^3)^{17}$

Solution

The fifth term of $(a+2x^3)^{17} = \binom{17}{4} a^{13}(2x^3)^4$

$= 38080a^{13}x^{12}$

Example

Find the term involving y^5 in the expansion of $(2x^2+y)^{10}$

Solution

Thus the sixth term, $= \binom{10}{5}(2x^2)^5 y^5$

$= 8064x^{10}y^5$

Example

Find the constant term in the expansion of $(2x^2+\frac{1}{x})^9$

The 7th term of this expansion is $\binom{9}{6}(2x^2)^3(\frac{1}{x})^6 = 84(2^3)x^6(\frac{1}{x^6})$

$= \underline{672}$

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+...+x^n$$

Example

Expand $(1+x)^4$

Solution

$$(1+x)^4 = 1+4x+\frac{4.3x^2}{2!}+\frac{4.3.2x^3}{3!}+x^4$$

$$= 1+4x+6x^2+4x^3+x^4$$

Example

Expand $(1+2x)^3$ in ascending power of x

Solution

$$(1+2x)^3 = 1+3(2x)+\frac{3.2(2x)^2}{2!}+(2x)^3$$

$$= 1+6x+12x^2+8x^3$$

Example

Expand $(1-2x)^5$ in ascending power of x.

Solution

$$(1-2x)^5 = 1+5(-2x)+\frac{5.4(-2x)^2}{2!}+\frac{5.4.3(-2x)^3}{3!}+\frac{5.4.3.2(-2x)^4}{4!}+(-2x)^5$$

$$= 1-10x+40x^2-80x^3+80x^4-32x^5$$

Example

Expand $(1+\tfrac{x}{2})^4$ in ascending powers of x.

Solution

$$(1+\tfrac{x}{2})^4 = 1+4(\tfrac{x}{2})+\frac{4.3(\tfrac{x}{2})^2}{2!}+\frac{4.3.2(\tfrac{x}{2})^3}{3!}+(\tfrac{x}{2})^4$$

$$= 1+2x+\frac{4.3x^2}{4.2.1}+\frac{4.3.2x^3}{8.3.2.1}+\frac{x^4}{16}$$

$$= 1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}$$

Example

Expand $(2+x)^4$ in ascending powers of x.

Solution

$$(2+x)^4 = 2^4(1+\tfrac{x}{2})^4$$

$$2^4(1+\tfrac{x}{2})^4 = 2^4\{1+4(\tfrac{x}{2})+\frac{4.3(\tfrac{x}{2})^2}{2!}+\frac{4.3.2(\tfrac{x}{2})^3}{3!}+(\tfrac{x}{2})^4\}$$

$$= 2^4\{1+2x+\frac{4.3x^2}{4.2.1}+\frac{4.3.2x^3}{8.3.2.1}+\frac{x^4}{16}\}$$

$$= 16+32x+24x^2+8x^3+x^4$$

Example

The first three terms of the binomial expansion $(1+px)^n$ are 1, $-10x$ and $40x^2$ respectively.

Find the values of p and n.

Solution

$$(1+px)^n = 1+npx+\frac{n(n-1)(px)^2}{2!}$$

Comparing coefficients,

$$np = -10\text{.......... ....}(1)$$

$$\frac{n(n-1)p^2}{2!} = 40$$

$$n^2p^2 - np^2 = 80\text{.......}(2)$$

From (1) $n = \frac{-10}{p}$

Substituting the value of n in (2) we have

$$\left(\frac{-10}{p}\right)^2 p^2 - \left(\frac{-10}{p}\right) p^2 = 80$$

$$100+10p=80.$$

$$10p=80-100$$

$$10p=-20$$

$$p=-2$$

Substitute the value of $p=-2$ in (1)

$$-2n=-10$$

$$n=\frac{-10}{-2}$$

$$n=5$$

PASCAL TRIANGLE

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1

This is $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

SECTION 10

MATRICES

DEFINITION AND NOTATIONS

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A*,*B*, *C* etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{mn}$$

A is a matrix of order $m\,n$. ith row jth column element of the matrix denoted by a_{ij}.

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

SPECIAL TYPES OF MATRICES

1. **Square matrix**

A matrix in which numbers of rows are equal to number of columns is called a square matrix. For example A=$\begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$ B=$\begin{bmatrix} 2 & 4 & 7 \\ 1 & 3 & 5 \\ 7 & 11 & 13 \end{bmatrix}$ A and B are square matrices.

2. **Diagonal matrix**

A square matrix A = $(a_{ij})_{mn}$is called a diagonal matrix if each of its non-diagonal element is zero.

That is $a_{ij} = 0\ if\ i \neq j$ and at least one element $a_{ii} \neq 0$.

Examples

$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 13 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. **IDENTITY MATRIX**

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n

That is $a_{ij} = \begin{cases} 0 \; if \; i \neq j \\ 1 \; if \; i = j \end{cases}$

Examples

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. **Symmetric Matrix**

A square matrix A =$(a_{ij})_{mn}$ said to be a symmetric if $a_{ij} = a_{ji}$for all i and j.

Example

A=$\begin{pmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ \boldsymbol{a_{12}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ \boldsymbol{a_{13}} & \boldsymbol{a_{23}} & \boldsymbol{a_{33}} \end{pmatrix}$ B=$\begin{pmatrix} \mathbf{8} & \mathbf{-2} & \mathbf{7} \\ \mathbf{-2} & \mathbf{-9} & \mathbf{3} \\ \mathbf{7} & \mathbf{3} & \mathbf{5} \end{pmatrix}$

5. **Column matrix**

A column matrix has only one column of elements. For example, A=$\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is a 2 x 1 matrix or a column matrix.

6. **A row matrix**

A row matrix has only one row of elements for example, B=(3 2) is a 1x 2 matrix or a row matrix .

7. The null or Zero matrix

The null or zero matrix is a square matrix in which every element is zero. For example, O=$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

8. Rectangular matrix

A matrix in which the number of rows is not equal to the number of columns is known as rectangular matrix. For example, A=$\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 8 \end{bmatrix}$ A is rectangular matrix of order 3 x 2

B=$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 1 & 3 \end{bmatrix}$ B is a rectangular matrix of order 2 x 3

9. Equal matrices

Two matrices are said to be equal if

i. They are of the same order

ii. Their corresponding elements are equal.

That is if A=$(a_{ij})_{m\times n}$ $and\ B = (b_{ij})_{m\times n}$ then $a_{ij} = b_{ij}$ for all $i\ and\ j$

That is their corresponding entries are the same. For example if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then a=e, b=f, c=g, d=h. If $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$=$\begin{pmatrix} 2 & 4 \\ -3 & 9 \end{pmatrix}$ then e=2, f=4, g=-3 and h=9

TRANSPOSE OF A MATRIX

The transpose of matrix $A = (a_{ij})_{m\times n}$, written A^t is the matrix obtained by writing the rows of A in order as columns. That is, by the transpose of an $m \times n$ matrix A, we mean a matrix of order $n \times m$ having the rows of A as its columns and the columns of A as its rows.

For example, A=$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 2 \end{pmatrix}$

PROPERTIES OF TRANSPOSE

1. $(A+B)^t = A^t + B^t$
2. $(A^t)^t = A$
3. $(AB)^t = B^t A^t$

ADDITION AND SUBTRACTION OF MATRICES

ADDITION OF MATRICES

Let A=$(a_{ij})_{m\times n}$ and B=$(b_{ij})_{m\times n}$ are two matrices with the same order then the sum of the two matrices is given by $A + B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = \left(a_{ij} + b_{ij}\right)_{m\times n}$

Two matrices can be added if they are of the same order. Thus we can add 2×2 matrix to another 2 x 2 matrix. To add two or more matrices, simply add the corresponding elements. This gives a matrix of the same order. For example, for these 2 x 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

Note the following

1. The zero matrix is the identity matrix under addition i.e. **0+A=A**
2. Matrix addition is commutative and associative i.e. A+B=B +A,
   (A+B)+C=A + (B+C)

Example one

Two matrices A and B are given by A=$\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$.

a. Find A+B
b. 5B+A

Solution

a. **A+B=**$\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}+\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}=\begin{pmatrix} 5+2 & 10+4 \\ 15+6 & 20+8 \end{pmatrix}=\begin{pmatrix} 7 & 14 \\ 21 & 28 \end{pmatrix}$
b. **5B+A=5** $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}+\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}=\begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}+\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}=\begin{pmatrix} 15 & 30 \\ 45 & 60 \end{pmatrix}$

SUBTRACTION OF MATRICES

Two matrices can be subtracted if they are of the same order. Thus we can subtract a 2 x 2 matrix from another 2 x 2 matrix. To subtract matrices, simply subtract corresponding elements. This gives a matrix of the same order. For example, for these 2 x 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}-\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$.

Note : A-B≠B-A Matrix subtraction is not commutative.

Example

The matrices A,B and and C are given by A=$\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$ B=$\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ and C=$\begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$. Find

1. A-B
2. B-C

Solution

1. A-B=$\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}-\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}=\begin{pmatrix} 1-2 & 3-4 \\ 5-6 & 7-8 \end{pmatrix}=\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

2. B-C=$\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$= $\begin{pmatrix} 2-3 & 4--5 \\ 6--7 & 8-9 \end{pmatrix}$=$\begin{pmatrix} -1 & 9 \\ 13 & -1 \end{pmatrix}$

MULTIPLICATION OF MATRICES

Multiplication by a scalar

To multiply a matrix by a number (scalar), multiply each element of the matrix by the number. For example given the number or scalar k and the matrix A=$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then

kA=$\begin{pmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{pmatrix}$

Examples

1. 3$\begin{pmatrix} 1 & 2 & 2 \\ 5 & 6 & 4 \\ 0 & 3 & 1 \end{pmatrix}$=$\begin{pmatrix} 3 & 6 & 6 \\ 15 & 18 & 12 \\ 0 & 9 & 1 \end{pmatrix}$
2. 4$\begin{pmatrix} -1 & 9 \\ 13 & -1 \end{pmatrix}$=$\begin{pmatrix} -4 & 36 \\ 52 & -4 \end{pmatrix}$

Multiplication of two (2 x 2) matrices

Two matrices *A* and *B* are said to be confirmable for product *AB* if number of columns in *A* equals to the number of rows in matrix *B*. Let *A* = $(a_{ij})_{m\times n}$ anb $(b_{ij})_{n\times r}$ be two matrices the product matrix C= *AB*, is matrix of order $m\times$ r

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_i b_{2j} + \cdots + a_{in}b_{nj}$.

Two matrices A and B can only be multiplied together if the number of columns in the first matrix A equals the number of the rows in the second matrix B. When this happens the matrix is said to be compatible. To find the product of two matrices, multiply every row of the first matrix by every column of the second matrix. In general when multiplying a matrix of order (m x n) by a matrix of order (n x p), the resulting matrix has order (m x p)

i.e. (m x n) x (n x p)= (m x p).

Given the matrices A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B=$\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, the product of

AB=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$=$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

Example

B=$\begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$ C=$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

Find BC and CB

BC=$\begin{pmatrix} 6 & 9 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ =$\begin{pmatrix} (6\times 1)+(9\times -1) & (6\times 2)+(9\times 0) \\ (-4\times 1)+(-6\times -1) & (-4\times 2)+(-6\times 0) \end{pmatrix}$= $\begin{pmatrix} -3 & 12 \\ 2 & -8 \end{pmatrix}$

CB=$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$ = $\begin{pmatrix} (1\times 6)+(2\times -4) & (1\times 9)+(2\times -6) \\ (-1\times 6)+(-6\times -1) & (-1\times -9)+(0\times -6) \end{pmatrix}$=$\begin{pmatrix} -2 & -3 \\ -6 & -9 \end{pmatrix}$

Note BC≠CB

Multiplication of (3 x 3) matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}=\begin{pmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+em+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{pmatrix}$$

Examples

A=$\begin{pmatrix} 1 & 2 & 2 \\ 5 & 6 & 4 \\ 0 & 3 & 1 \end{pmatrix}$ B=$\begin{pmatrix} 3 & 6 & 6 \\ 15 & 18 & 12 \\ 1 & 9 & 1 \end{pmatrix}$ C=$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 3 & 1 \end{pmatrix}$

Find

1. AB
2. BC
3. AC

SOLUTION

1. **AB=**$\begin{pmatrix}1 & 2 & 2\\ 5 & 6 & 4\\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix}3 & 6 & 6\\ 15 & 18 & 12\\ 1 & 9 & 1\end{pmatrix} = \begin{pmatrix}35 & 60 & 32\\ 109 & 174 & 106\\ 46 & 63 & 37\end{pmatrix}$
2. **BC=**$\begin{pmatrix}3 & 6 & 6\\ 15 & 18 & 12\\ 1 & 9 & 1\end{pmatrix}\begin{pmatrix}2 & 2 & 2\\ 1 & 3 & 4\\ 3 & 3 & 1\end{pmatrix}=\begin{pmatrix}30 & 42 & 36\\ 84 & 120 & 114\\ 14 & 32 & 39\end{pmatrix}$
3. **AC=**$\begin{pmatrix}1 & 2 & 2\\ 5 & 6 & 4\\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 2 & 2\\ 1 & 3 & 4\\ 3 & 3 & 1\end{pmatrix}=\begin{pmatrix}10 & 14 & 12\\ 28 & 40 & 38\\ 6 & 12 & 13\end{pmatrix}$

DETERMINANTS AND INVERSION OF MATRICES

Let $A = (a_{ij})_{m\times n}$ be a square matrix of order n , then the number $|A|$called determinant of the matrix A.

(i) Determinant of 2 x 2 matrix

Let A=$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ then $|A| = \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$=$a_{11}a_{22} - a_{12}a_{21}$

Example

Find the determinant of B=$\begin{pmatrix}3 & -5\\ 1 & 6\end{pmatrix}$

Solution

$|B| = (3 \ \times 6) - (-4 \ \times \ 1) = 18 - (-4) = 22$

Hence the determinant of a matrix can be expressed as a single numerical value

Examples

Find the determinants of the following matrices

1. A=$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$
2. B=$\begin{pmatrix} 4 & -2 \\ 1 & 7 \end{pmatrix}$
3. C=$\begin{pmatrix} 10 & -5 \\ 1 & 3 \end{pmatrix}$
4. D=$\begin{pmatrix} 7 & 5 \\ 3 & 3 \end{pmatrix}$

ii. Determinant of 3× 3 matrix

Let B=$\begin{pmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ \boldsymbol{a_{12}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ \boldsymbol{a_{13}} & \boldsymbol{a_{23}} & \boldsymbol{a_{33}} \end{pmatrix}$

Then $|B| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Calculate the determinant of the following matrices

1. **A=**$\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{6} & \mathbf{8} \\ \mathbf{1} & \mathbf{9} & \mathbf{5} \end{pmatrix}$
2. **B=**$\begin{pmatrix} \mathbf{2} & \mathbf{-3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{8} & \mathbf{9} & \mathbf{1} \end{pmatrix}$

Solution

1. $|A|$=0
2. $|B|$=-279

Some Properties of the determinant

1. The determinant of a matrix A and its transpose A^t are equal. ie $|A| = |A^t|$
2. Let A be a square matrix

(i) If A has a row (column) of zeros then $|A|=0$

(ii) If A has two identical rows (or columns) then $|A|=0$

3. If A is triangular matrix then $|A|$ is product of the diagonal elements.

THE INVERSE OF A 2 X 2 MATRIX

The inverse of a matrix A is the matrix A^{-1} such that AA^{-1}=A^{-1}A=I, where I is the unit matrix. Any matrix multiply by its inverse matrix the result is the unit matrix. It is only possible to calculate the inverses of square matrices. If the determinant of a square matrix is **zero** then the matrix is called a singular matrix and has no inverse. If the determinant of a square matrix is **not zero** then the matrix is called **non singular matrix** and has an inverse.

The inverse of a (2 x 2) matrix A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where det A =ad-bc. Thus to find the inverse of any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. Interchange the positions of a and d i.e. the elements in the leading diagonals
2. Change the signs of b and c i.e. the elements of the other diagonals
3. Multiply this new matrix by $\frac{1}{ad-bc}$ i.e. the reciprocal of the determinants of the matrix.

Examples

Given the matrix A=$\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ find the inverse matrix A^{-1} of A.

Approach one

det A= 12-15=-3

A^{-1}=$\frac{1}{-3}\begin{pmatrix} 6 & -3 \\ -5 & 2 \end{pmatrix}$ =$\begin{pmatrix} \frac{6}{-3} & \frac{-3}{-3} \\ \frac{-5}{-3} & \frac{2}{-3} \end{pmatrix}$ =$\begin{pmatrix} -2 & 1 \\ \frac{5}{3} & -\frac{2}{3} \end{pmatrix}$

Approach two

Let the inverse of matrix A^{-1}=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ since AA^{-1}=I

$\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2a+3c & 2b+3d \\ 5a+6c & 5b+6d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ equating corresponding elements

2b+3d=0 i.e. 2b=-3d, b=$-\frac{3}{2}$ d

5a+6c=0 i.e. 5a=-6c, a=$-\frac{6}{5}$ c

2a+3c=1 5b+6d=1

Substituting for a and b gives $\begin{pmatrix} 2\left(-\frac{6}{5}c\right)+3c & 2(-\frac{3}{2}\text{ d })+3d \\ 5\left(-\frac{6}{5}c\right)+6c & 5\left(-\frac{3}{2}d\right)+6d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} \frac{3}{5}c & 0 \\ 0 & -\frac{3}{2}d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Equating corresponding elements $\frac{3}{5}c=1 \rightarrow c=\frac{5}{3}$, $-\frac{3}{2}d = 1 \rightarrow d=-\frac{2}{3}$

Since b=$-\frac{3}{2}$ d $\rightarrow$b=1 and a=$-\frac{6}{5}$ c $\rightarrow$ a= -2

Therefore the inverse matrix $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{5}{3} & -\frac{2}{3} \end{pmatrix}$.

Note that AA^{-1}=I i.e. $\begin{pmatrix} -2 & 1 \\ \frac{5}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Example two

Find the determinants and inverses of the following matrices using the two approaches

1. A=$\begin{pmatrix} 6 & 3 \\ 5 & 2 \end{pmatrix}$

2. A=$\begin{pmatrix} 2 & 5 \\ 5 & 6 \end{pmatrix}$

USING THE INVERSE TO SOLVE MATRIX EQUATION

Given any matrix equation Ax=b, we can use the identities AA^{-1} = A^{-1}A=I and AI=I to solve the equation.

i.e.Ax=b premultiply both sides by A^{-1}

$A^{-1}AX = A^{-1}b$

IX= $A^{-1}b$ since $A^{-1}A = I$

X= $A^{-1}b$ since IX=XI where A^{-1} is the inverse of A and I is the unit matrix.

Example one

Solve the simultaneous equations;

$5x - 3y = 28$

$-2x + 4y = 14$

Solution

Writing the equation in the matrix form we have

$\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 28 \\ 14 \end{pmatrix}$

i.e. AX=b $\rightarrow A^{-1}$AX= $A^{-1}b$,

IX= A^{-1}b since A^{-1}A= b

X= $A^{-1}b$ since IX=XI

Where A=$\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}$ and b=$\begin{pmatrix} 28 \\ 14 \end{pmatrix}$ but A^{-1}=$\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$

X=$\frac{1}{14}\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 28 \\ 14 \end{pmatrix}$ =$\begin{pmatrix} 11 \\ 9 \end{pmatrix}$

$\boldsymbol{x = 11 \; and \; y = 9}$

Example two

Given the matrix A$=\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ find a 2 x 2 matrix X such that AX=XA=I. Where I is the 2 x 2 unit matrix.

Solution

If AX=XA=I, then X is the inverse of A. To find the inverse of A, first find the determinant of A i.e. det A=4-(-6)=10

X=inverse of A

$$= \frac{\mathbf{1}}{\mathbf{10}}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2/5 & -1/5 \\ 3/10 & 1/10 \end{pmatrix}$$

SECTION 11

COORDINATE GEOMETRY

EQUATION OF A STRAIGHT LINE

The equation of a straight line is satisfied by the coordinates of all the points on the line.

1. The general equation of a straight line is of the form $ax + by + c = 0$ where a,b,c are constants.
2. The slope equation is of the form $y = mx + c$, where m is the gradient of the line and c is the intercept on the y-axis.
3. The intercept equation is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the intercepts on the x and y axes respectively. Thus any equation of this form cuts the x-axis at (a,0) and the y-axis at (0,b)

TO FIND THE GRADIENT OF THE LINE WHEN THE EQUATION IS GIVEN

To find the gradient, express the given equation in the form $y = mx + c$ and take the coeffient of x ($ie\ m$) as the gradient

$y = mx + c$, where m is the gradient and c is the intercept on the y-axis.

Example one

Find the gradient of the line with equation $4x + 8y = 6$

Solution

Making y the subject of the equation we have $8y = -4x + 6 \rightarrow y = -\frac{1}{2}x + \frac{3}{4}$

The coefficient of x is $-\frac{1}{2}$, the gradient of the line is $-\frac{1}{2}$

Example two

If the equation of a straight line is $10x - 40y + 15 = 0$. Find its gradient

Solution

$40y = 10x + 15$

$y = \frac{10}{40}x + \frac{15}{40}$

$y = \frac{1}{4}x + \frac{3}{8}$

The coefficient of x is $\frac{1}{4}$, therefore the gradient is of the line is $\frac{1}{4}$

Example three

Find the angle of the straight line whose equation is $6x + 4y - 2 = 0$ makes with the positive $x - axis$

Solution

$4y = -6x + 2$

$y = -\frac{6}{4}x + \frac{2}{4}$

$y = -\frac{3}{2}x + \frac{1}{2}$

The coefficient of x is $-\frac{3}{2}$, therefore the gradient is of the line is $-\frac{3}{2}$

Hence if the angle is θ, then $\tan\theta = -\frac{3}{2} = -1.5$

$\theta = tan^{-1}(-1.5) = 123.7^0$

Note that the gradient of a line is equal to the tangent of the angle the line makes with the positive $x - axis$

Example four (trial)

State the gradient and hence find the angles the straight line whose equations are given make the positive $x - axis$

1. $5x + 5y - 10 = 0$
2. $3x - y = 1$

To find the intercept on the $x - axis$

To find the intercept of a straight line on the x-axis, put $y = 0$ in the equation and solve for x

To find the intercept on the $y - axis$

To find the intercept of a straight line on the y-axis, put $x = 0$ in the equation and solve for y

Example five

Find the gradient of the straight line with the equation $14x + 8y + 4 = 0$ and the intercepts on the $x \ and \ y - axis$

Solution

To find the gradient of the line, make y the subject of the equation of the line

$14x + 8y + 4 = 0$

$8y = -14x - 4$

$y = -\frac{14}{8}x - \frac{4}{8}$

$y = -\frac{7}{4}x - \frac{2}{4}$

The gradient of the line is $-\frac{7}{4}$

To find the intercept on the $y - axis$ put $x = 0$ in the equation and solve for y

$14(0) + 8y + 4 = 0$

$8y = -4$

$y = -\frac{4}{8}$

$y = -\frac{1}{2}$

The intercept on the $y - axis$ is $(0, -\frac{1}{2})$

To find the intercept on the $x - axis$ put $y = 0$ in the given equation and solve for x

$14y + 8(0) + 4 = 0$

$14y = -4$

$y = -\frac{4}{14}$

$y = -\frac{2}{7}$

The intercept on the $x - axis$ is $(-\frac{2}{7}, 0)$

FINDING EQUATION OF A LINE GIVEN THE GRADIENT AND A POINT ON THE LINE

Let m be the gradient of the line and (x_1, y_1) be the given point, then the equation of the line is given by $\boldsymbol{y - y_1 = m(x - x_1)}$

Example one

Find the equation of the line whose gradient is 6, passing through the point (2,4)

Solution

Here $m = 6\ and\ (x_1, y_1) = (2,4)$

The equation of the line is $y - 4 = 6(x - 2)$

$y - 4 = 6x - 12$

$y - 6x + 8 = 0$

Example two

Find the equation of the line with gradient -2 and passing through (0,6)

Solution

$y - 6 = -3(x - 0)$

$y - 6 = -3x$

$y + 3x - 6 = 0$

FINDING THE EQUATION OF A LINE GIVEN TWO POINTS ON THE LINE

Let the two given points be $A(x_1, y_1)$ and $B(x_2, y_2)$. First find the gradient of the line using these two points. Thus the gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$. With the gradient and any of the points on the line form the equation.

Example one

Find the equation of the line which passes through the points $A(2,1)and\ B(\ 3,6)$

Solution

First find the gradient of the line

$m = \frac{6-1}{3-1} = \frac{5}{2}$

Using the point (2,1), the equation of the line is $y - 1 = \frac{5}{2}(x - 2)$

$y - 1 = \frac{5}{2}x - 5$

$y - \frac{5}{2}x + 4 = 0$

$2y - 5x + 8 = 0$

Example two

Find the equation of the line passing through A(-10,4) and B(6,8)

Solution

The gradient of the line AB= $\frac{\mathbf{8-4}}{\mathbf{6--10}} = \frac{\mathbf{4}}{\mathbf{16}} = \frac{\mathbf{1}}{\mathbf{4}}$

Using the point (-10,4), the equation of the straight line becomes

$y - 4 = \frac{1}{4}(x - -10)$

$\boldsymbol{y - 4 = \frac{1}{4}x + \frac{10}{4}}$

$\boldsymbol{4y - 16 = x + 10}$

$\boldsymbol{4y - x - 26 = 0}$

INTERSECTION OF LINES

If two lines intersect, the coordinates of the point of intersection must satisfy both equations. Hence the coordinates of the points of intersection can be obtained by solving the two equations simultaneously.

Example one

What is the point of intersection of the two lines $6x - 9y = 24$ *and* $9x + 12y + 15 = 0$

Solution

$\boldsymbol{6x - 9y = 24}$

$\boldsymbol{9x + 12y = -15}$

Solving the two equations simultaneously

We have $x = 1$ *and* $y = -2$

The point of intersection is$(1, -2)$

Example two

Two lines with equation $5x + 5y - 10 = 0$ *and* $10x - 5y + 25 = 0$ meet at a point A, find the coordinates of A

Solution

$5x + 5y = 10$

$10x - 5y = -25$

Solving the two equations simultaneously, we have $x = -1$ *and* $y = 3$

The coordinates of A are $(-1,3)$

CONDITIONS FOR PARALLEL AND PERPENDICULAR LINES

Parallel lines

If two lines l_1 *and* l_2 are parallel, they are equally inclined to the positive direction of the $x - axis$ (corresponding angles). Therefore $\tan\theta$ is the gradient of both l_1 *and* l_2. Hence two lines are said to be parallel if they have equal gradient

Example

Find the equation of the line which passes through the (3,-2) and parallel to

1. $10x - 2y + 6 = 0$
2. $10x - 5y + 25 = 0$

Solution

1. $2y = 10x + 6$

   $y = 5x + 3$

   Gradient of the line is 5.

   Since the line is parallel to the equation they must have the same gradient. Hence the gradient of the new is 5

   The new equation is

   $y - y_1 = m(x - x_1)$

   $y - -2 = 5(x - 3)$

   $y + 2 = 5x - 15$

   $y - 5x + 17 = 0$

2. $10x - 5y + 25 = 0$

   $5y = -10x - 25$

   $y = -2x - 5$

   $m = -2$

   The new equation is

   $y - y_1 = m(x - x_1)$

   $y - -2 = -2(x - 3)$

   $y + 2 = -2x + 6$

   $y - 2x - 4 = 0$

Perpendicular lines

If the gradient of one line is equal to the negative reciprocal of the other, then the two lines are said to meet at right angles or are perpendicular. Thus if the gradient of one line is m the other line is$-\frac{1}{m}$ then the lines are perpendicular.

$note\ i.e\ m_1 = -\frac{1}{m_2} \quad or\ \ m_1 m_2 = -1$

Example one

Find the equation of the line which passes through the point (2, 6) and perpendicular to the line $4y - 6x + 2 = 0$

Solution

The gradient of the line is $4y = 6x - 2$

$y = \frac{3}{2}x - \frac{1}{4}$

$y = \frac{3}{2}$

Therefore the gradient of the line which passes through (2,6) is $-\frac{2}{3}$ since the two lines are perpendicular.

The equation of the new line is $y - 6 = -\frac{2}{3}(x - 2)$

$3y - 18 = -2(x - 2)$

$3y - 18 = -2x + 4$

$3y + 2x - 22 = 0$

Example two

The lines $3x + 6y - 9 = 0 \; and \; 12x + 15y - 27 = 0$ intersect at M. Find

a. The coordinates of M
b. The equation of the line through M, perpendicular to the line $6x - 4y + 1 = 0$

Solution

a. To find the point of intersection M, solve the two equations simultaneously

$3x + 6y - 9 = 0$

$12x + 15y - 27 = 0$

$x = 1, y = 1$

The coordinates of M is (1,1)

b. The gradient of line M is equal to the negative reciprocal of the gradient of the line
   $6x - 4y + 1 = 0$
   $4y = -6x - 1$
   $y = -\frac{3}{2}x - \frac{1}{4}$
   The gradient of M is $\frac{2}{3}$
   Hence the equation of the line is $y - 1 = \frac{2}{3}(x - 1)$
   $3y - 3 = 2x - 2$
   $3y - 2x - 1 = 0$

PERPENDICULAR DISTANCE FROM A POINT TO A LINE

If $ax + by + c = 0$ is the general equation of any line, then the perpendicular distance from any point $P(h, k)$ to the line is given by

$\pm \frac{ah+bk+c}{\sqrt{a^2+b^2}}$

Example one

Find the perpendicular distance from a point (1,2) to the line $6x + 8y = 10$

Solution

Put the equation in the general form first i.e. $6x + 8y - 10 = 0$

Perpendicular distance from (1,2) to $6x + 8y - 10 = 0$ is $\frac{6(1)+8(2)-10}{\sqrt{1^2+2^2}} = \frac{12}{\sqrt{5}}$ =5.3665

ANGLE BETWEEN TWO INTERSECTING STRAIGHT LINES

In general, the angle θ between two straight lines with gradient m_1 *and* m_2 is given by

$\tan\theta = \frac{m-m}{1+m_1m_2}$ where $m_1 > m_2$

The acute angle θ between two straight lines with gradients m_1 *and* m_2 is given by

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$ in this case it is irrelevant which gradient is taken m_1 *and* m_2

Example one

Find the angle between the lines $4x - 2y = 8$ *and* $x - 2y = 8$

Solution

$4x - 2x = 8$

$2y = 4x - 8$

$x = 2y - 4$

$m_1 = 2$

$x - 2y = 8$

$2y = x - 8$

$y = \frac{1}{2}x - 4$

$m_2 = \frac{1}{2}$

$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2} = \frac{2-1/2}{1+2(\frac{1}{2})} = \frac{1.5}{2} = 0.74$

$\theta = 36.50^0$

Example two

Find the acute angle between the lines $4x + 2y = 8 \text{ } and \text{ } 2y - 6x + 14 = 0$

Solution

$4x + 2y = 8$

$\mathbf{2y = -4x + 8}$

$\boldsymbol{y = -2x + 4}$

$\boldsymbol{m_1 = -2}$

$2y - 6x + 14 = 0$

$2y = 6x - 14$

$y = 3x - 7$

$m_2 = 3$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{-2-3}{1+-2(3)}\right| = \left|\frac{-5}{1-6}\right| = \left|\frac{-5}{-5}\right| = 1$

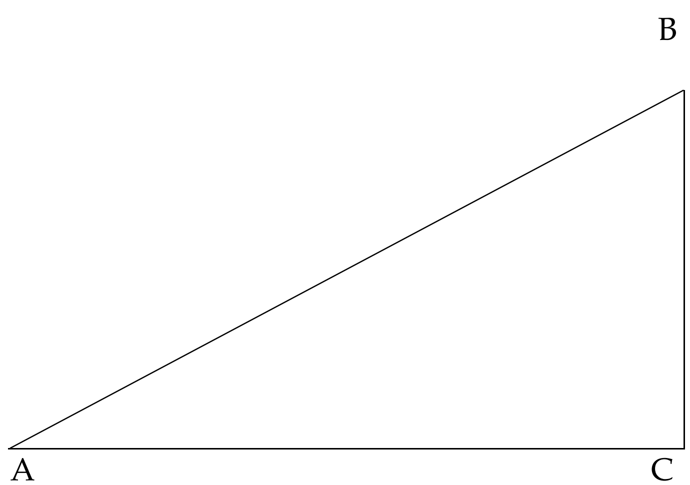
$\theta = 45^0$

SECTION 12

TRIGONOMETRY

The names of the six trigonometric functions, along with their abbreviations

Name	**Abbreviation**
Sine	Sin
Cosine	Cos
Tangent	Tan
Cosecant	Csc
Secant	sec
Cotangent	cot



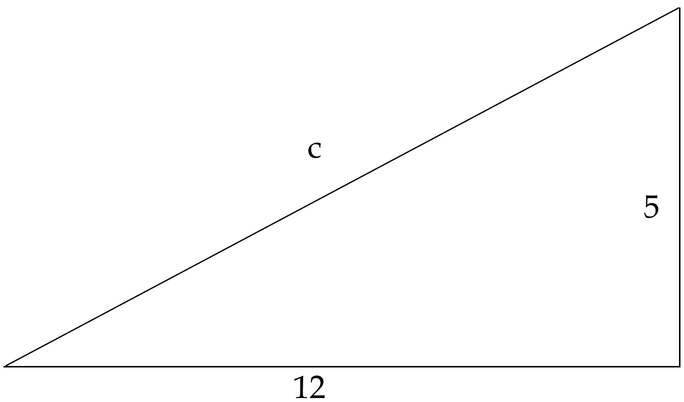
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

$\csc\theta = \frac{\text{hypotenuse}}{opposite}$

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\sec\theta = \frac{\text{hypotenuse}}{adjacent}$

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ $\cot\theta = \frac{\text{adjacent}}{opposite}$

Find the value of the six trigonometric functions of θ in the figure below



Solution

$c^2 = a^2 + b^2 = 5^2 + 12^2 = 169$

$c = \sqrt{169}$

$c = 13$

1. $\sin\theta = \frac{opposite}{hypotenus} = \frac{5}{13}$
2. $cos\theta = \frac{adjacent}{hypoenus} = \frac{12}{13}$
3. $\tan\theta = \frac{opposite}{adjacent} = \frac{5}{12}$
4. $\csc\theta = \frac{hypoyenus}{opposite} = \frac{13}{5}$
5. $\sec\theta = \frac{hypotenus}{opposite} = \frac{13}{12}$
6. $\cot\theta = \frac{adjacent}{opposite} = \frac{12}{5}$

Trigonometric functions of special angles

$\boldsymbol{\theta}$	$\mathbf{30^0 = \frac{\pi}{6}}$	$\mathbf{45^0 = \frac{\pi}{4}}$	$\mathbf{60^0 = \frac{\pi}{3}}$
$\boldsymbol{sin\ \theta}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{\sqrt{2}}{2}}$	$\mathbf{\frac{\sqrt{3}}{2}}$
$\boldsymbol{cos\ \theta}$	$\mathbf{\frac{\sqrt{3}}{2}}$	$\mathbf{\frac{\sqrt{2}}{2}}$	$\mathbf{\frac{1}{2}}$
$\boldsymbol{tan\theta}$	$\mathbf{\frac{\sqrt{3}}{3}}$	$\mathbf{1}$	$\mathbf{\sqrt{3}}$

Many relationships exist among the six trigonometric functions. These relationships are described using trigonometric identities.

SOME BASIC TRIGONOMETRY IDENTITIES

1. $\csc\theta = \frac{1}{\sin\theta}\ where \sin\theta \neq 0$
2. $\sec\theta = \frac{1}{\cos\theta}$ where $\cos\theta \neq 0$
3. $\cot\theta = \frac{1}{\tan\theta}\ where \tan\theta\ is\ defined\ and\ not\ 0$
4. $\cot\theta = \frac{\cos\theta}{\sin\ \theta}$
5. $\sin\theta = \frac{1}{\csc\theta}$
6. $\cos\theta = \frac{1}{\sec\theta}$
7. $\tan\theta = \frac{1}{\cot\theta}$
8. $\tan\theta = \frac{\sin\theta}{\cos\ \theta}$
9. $\cos^2\theta + \sin^2\theta = 1$

10.$\sin^2\theta = 1 - \cos^2\theta$

11.$\cos^2\theta = 1 - \sin^2\theta$

12.$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

13.$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

14.$1 + \tan^2\theta = \sec^2\theta$

15.$\cot^2\theta + 1 = \csc^2\theta$

Using the identities to solve questions

Example one

Simplify $5\sin^2\theta + 4\cos^2\theta$

Solution

$$5\sin^2\theta + 4\cos^2\theta = 5\sin^2\theta + 4(1 - \sin^2\theta)$$
$$= 5\sin^2\theta + 4 - 4\sin^2\theta$$
$$= \boldsymbol{\sin^2\theta + 4}$$

Example two

Prove that $\tan\theta + \cot\theta = \sec\theta csc\theta$

Solution

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin\theta}{\cos\theta}\frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\frac{\cos\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta sin\theta}$$
$$= \frac{1}{\cos\theta\sin\theta}$$
$$= \frac{1}{\cos\theta} \times \frac{1}{\sin\theta}$$

$$= \sec\theta\csc\theta$$

SUM AND DIFFERENCE FORMULAS

We will now derive identities for the trigonometric functions of the sum and difference of two angles. For the sum of any two angles A and B, we have the *addition formulas*

$$sin\ (A + B) = sin\ Acos\ B + cos\ A\ sin\ B$$

$$cos\ (A + B) = cos\ A\ cosB - sin\ A\ sin\ B$$

$$sin\ (A - B) = sin\ Acos\ B - cos\ A\ sin\ B$$

$$cos\ (A - B) = cos\ A\ cosB + sin\ A\ sin\ B$$

Using the identity $tan\ \theta = \frac{sin\ \theta}{cos\ \theta}$ and the addition formulas for Sine and Cosine, we can derive the addition formula for tangent

$$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{sin\ Acos\ B + cos\ A\ sin\ B}{cos\ A\ cosB - sin\ A\ sin\ B}$$

$$= \frac{\frac{sinA\cos B}{\cos Acos\ B} + \frac{cosA\sin B}{\cos Acos\ B}}{\frac{\cos Acos\ B}{\cos Acos\ B} - \frac{\sin A\sin B}{\cos Acos\ B}}$$

$$= \frac{\frac{\sin Acos\ B}{\cos Acos\ B} + \frac{cosAsinB}{\cos A\cos B}}{1 - \frac{\sin A}{\cos A} \times \frac{sinB}{\cos B}} = \frac{\tan A + \tan B}{1 - tanAtanB}$$

$$\tan(A + B) = \frac{\tan A + tanB}{1 - \tan A\tan B}$$

$$\tan(A - B) = \frac{\tan A - tanB}{1 + \tan A\tan B}$$

Example

Given angles A and B such that $sin\ A\ = \frac{4}{5}$, $cos\ A\ = \frac{3}{5}$, $sin\ B\ = \frac{12}{13}$, and $s\ B\ = \frac{5}{13}$, find the exact values of sin $(A + B)$, cos $(A + B)$, and tan $(A + B)$.

Solution

$\boldsymbol{sin}\ (\boldsymbol{A}+\boldsymbol{B}) = \ sin\ Acos\ B + cos\ A\ sin\ B$

$= \frac{\mathbf{4}}{\mathbf{5}} \times \frac{\mathbf{5}}{\mathbf{13}} + \frac{\mathbf{3}}{\mathbf{5}} \times \frac{\mathbf{12}}{\mathbf{13}} = \frac{\mathbf{56}}{\mathbf{12}}$

$\boldsymbol{cos}(\boldsymbol{A}+\boldsymbol{B}) = \ cos\ A\ cosB - sin\ A\ sin\ B$

$= \frac{\mathbf{3}}{\mathbf{5}} \times \frac{\mathbf{5}}{\mathbf{13}} - \frac{\mathbf{4}}{\mathbf{5}} \times \frac{\mathbf{12}}{\mathbf{13}} = -\frac{\mathbf{33}}{\mathbf{65}}$

tan(A+B)=$\frac{\tan A + tanB}{1 - \tan A \tan B}$

$= \frac{\boldsymbol{sin}(\boldsymbol{A}+\boldsymbol{B})}{\boldsymbol{Cos}\ (\boldsymbol{A}+\boldsymbol{B})} = \frac{\frac{\mathbf{56}}{\mathbf{65}}}{-\frac{\mathbf{13}}{\mathbf{65}}} = -\frac{\mathbf{56}}{\mathbf{33}}$

DOUBLE ANGLE AND HALF ANGLE

$sin\ 2\theta = 2sin\ \theta cos\ \theta$

$cos2\theta = cos^2\theta - sin^2\theta$

$tan2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Note $cos2\theta\ = 2\cos^2\theta - 1\ \ or\ 1 - 2\sin^2\theta$

$using\ the\ identities\ \cos^2\theta = 1 - \sin^2\theta\ and\ \sin^2\theta = 1 - \cos^2\theta$

Example

Prove that $sin\ 3\theta\ = 3sin\theta - 4sin^3\theta$

Solution

Using $3\theta = sin(2\theta + \theta)$, the addition formula for sine and double angles formula

$sin3\theta = \sin(2\theta + \theta)$

$sin3\theta = (2\theta + \theta)$

$= sin2\theta cos\theta + cos2\theta sin\theta$

$= (2sin\theta cos\theta)cos\theta + (1 - 2\sin^2\theta)sin\theta$

$= 2sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

OTHER IDENTITIES

$sinAcos\ B = \frac{1}{2}(\sin(A + B) + Sin(A - B))$

$cosAsinB = \frac{1}{2}(\sin(A + B) - \sin(A - B))$

$cosAcos\ B = \frac{1}{2}(\cos(A + B) + cos(A - B))$

$sinAsin\ B = -\frac{1}{2}(\cos(A + B) + cos(A - B))$

$\sin A + \sin B = 2\sin\frac{1}{2}(A + B)cos\frac{1}{2}(A - B)$

$\sin A - \sin B = 2\cos\frac{1}{2}(A + B)sin\frac{1}{2}(A - B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A + B)cos\frac{1}{2}(A - B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A + B)sin\frac{1}{2}(A - B)$

COFUNCTION THEOREM

If A and B are the complementary acute angles in a right triangle $\triangle ABC$, then the following relations hold

$sin\ A = cos\ B$ $\quad$ $sec\ A = csc\ B$ $\quad$ $tan\ A = cot\ B$

$sin\ B = cos\ A$ $\quad$ $sec\ B = csc\ A$ $\quad$ $tan\ B = cot\ A$

So sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. That is how the functions cosine, cosecant, and cotangent got the "co" in their names. The Cofunction Theorem says that any trigonometric function of an acute angle is equal to its cofunction of the complementary angle.

Example

Write each of the following numbers as trigonometric functions of an angle less than 45^0

(a) $sin\ 65^0$
(b) $cos\ 78^0$
(c) $tan\ 59^0$

Solution

(a) The complement of $65^0$is $90^0 - 65^0$ = 25^0 and the cofunction of sin is cos, so by the Cofunction Theorem we know that sin 65^0= $cos 25^0$
(b) The complement of 78^0 is $90^0 - 78^0$ = 12^0 and the cofunction of cos is sin, so cos78^0 = sin12^0
(c) The complement of 59^0 is $90^0 - 59^0$ = 31^0 and the cofunction of tan is cot, so tan 59^0 = cot 31^0

RADIANS AND DEGREES

Definition: A *radian* is a unit for measuring angles. All the way around a circle is 360^0, and is defined also to be $2\pi\, radians$.

This is to do with how radians are defined: the length of circle "subtended" by an angle, divided by the radius.

So $360^0 \;=\; 2\pi\, radians.$

Or $180^0 \;=\; \pi\, radians.$

Degrees to radians: x degrees = $\frac{\pi}{180}\; x$ radians.

Radians to degrees: x radians = $\frac{180}{\pi}\; x$ degree

Radians are used a lot because in calculus, $d(sin\, x)/dx \;=\; cos\, x$, *but only if* x *is measured in radians!* So if no notation is used, we mean radians. So cos 60 means the cosine of 60 radians! But $cos\; 60^0$ is the cosine of 60 degrees (which is $\frac{1}{2}$)

Some useful conversions:

Recall that $2\pi\, radians = \; 360^0$. So

$\frac{\pi}{6} radians = 30^0$

$\frac{\pi}{4} radians = 45^0$

$\frac{\pi}{3} radians = 60^0$

$\frac{\pi}{2} radians = 90^0$

Example

Convert the angle in degree to radians

a. -135^0

Solution

$$-135^0 = -135 \times \frac{\pi radians}{180^0} = -\frac{135\pi}{180} radians = -\frac{3\pi}{4} \text{ radians}$$

Example

Convert each angle in radians to degree

a. $-\frac{5\pi}{3} radians$

b. $1 radian$

Solution

To convert radians to degrees, multiply by $\frac{180^0}{\pi radians}$

a. $-\frac{5\pi}{3} radians = -\frac{5\pi}{3} radians \times \frac{180^0}{\pi radians} = -300^0$

b. $1 radian = 1 radian \times \frac{180^0}{\pi radians} = \frac{180^0}{\pi} \cong 57^0$

The cos and sin of certain angles should be known exactly (without need for a calculator)

Angle degrees	θradians	$\cos\theta$	$\sin\theta$
0	0	1	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90	$\frac{\pi}{2}$	0	1