CALCULUS WITH ANALYSIS

CHAPTER II

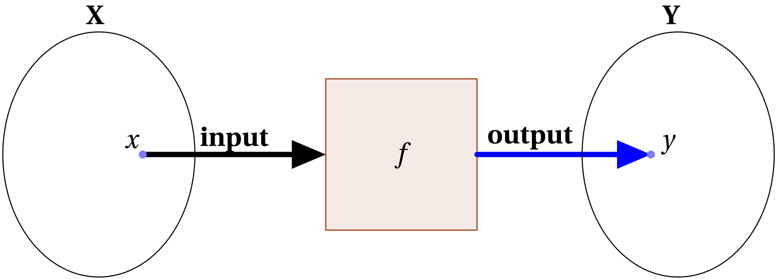
Real Valued Functions

May, 2021.

Outline

Definition

In general, a function is a map between two sets that assigns to each element in the first set a unique element in the second set.



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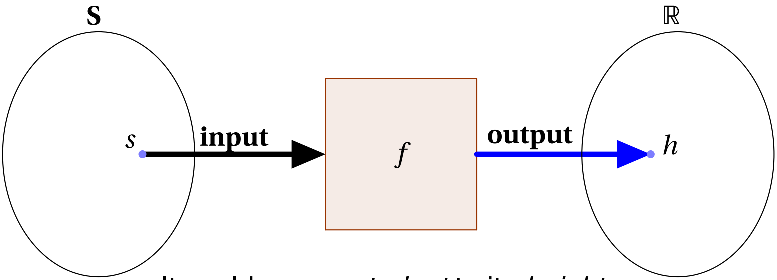

It could map a *student* to its *height*.

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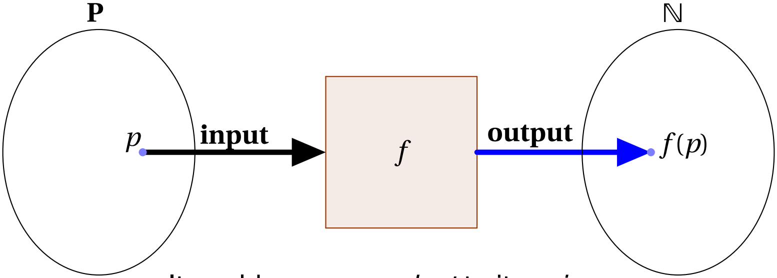

It could map a *product* to its *price*.

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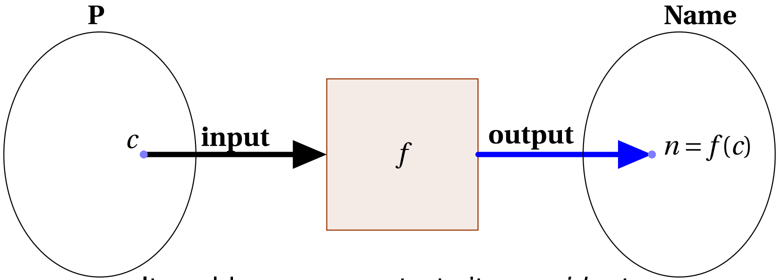

It could map a *country* to its *president*.

Definition

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A **real-valued function** f *assigns a unique real number* y *to each input* x.

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If the function f *is defined from a set* X *to* Y, *then we write*

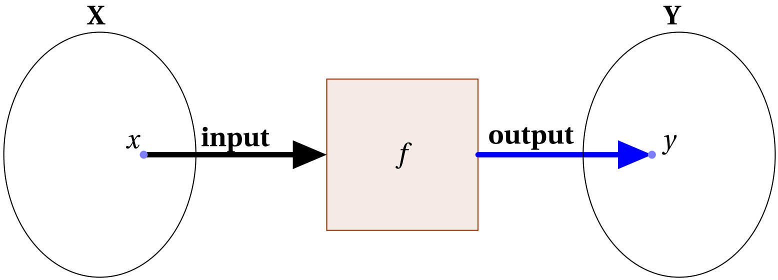
$$\begin{aligned} f: \quad X &\to Y \\ x &\mapsto y = f(x) \end{aligned}$$

Remark

Uniqueness here means an input cannot yield more than one output i.e. $x \mapsto y_1, y_2$ *is not allowed.*

However, two different inputs x_1 *and* x_2 *can be assigned to the same output* y.

Definition



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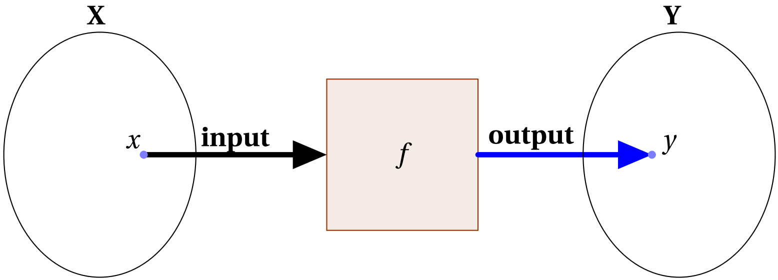

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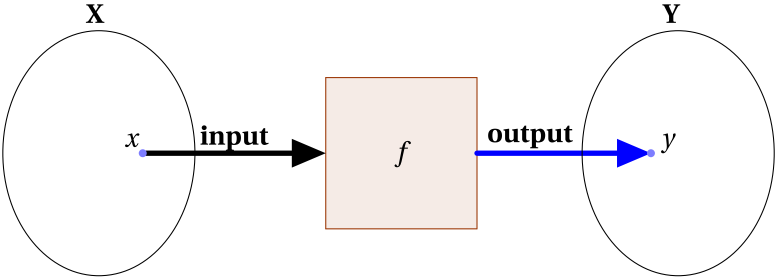

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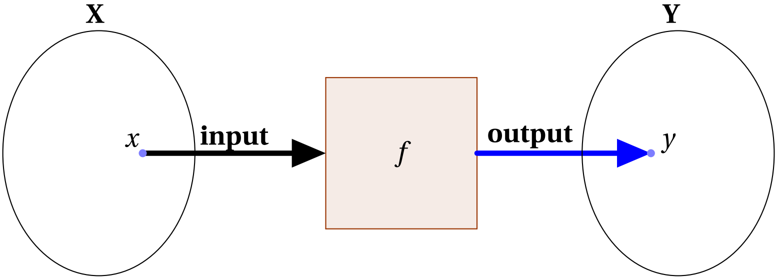
Definition



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☛ The set of all assigned outputs, $f(X) = \{ f(x) \ : \ x \in X \}$, is called the **range/image** of f.

☛ The **graph** of f is the collection of the points $(x, f(x))$. It represents a curve in the Cartesian plane.

Which of the following are functions or represent the graph of a function?

1) $f(x) = 2$ 2) $f(x) = x^2 - x + 1$ 3) $f(x) = \dfrac{2x-1}{x^2+1}$

4) $f(x) = \sin x$ 5) $f(x) = x - \sin x$ 6) $f(x) = e^x - 1.$

7) $f(x) = \begin{cases} x+1 & : x \le 2 \\ -x & : 2 < x \end{cases}$ 8) $f(x) = \begin{cases} x^2+1 & : x \le -1 \\ x+5 & : -1 \le x \end{cases}$

9) $G = \{(0,1), (-2,1), (1,0)\}.$

10) $G = \{(1,1), (0,3), (1,0)\}.$

11) $G = \left\{ (x,y) \ : \ x^2 + y = 1 \right\}.$

12) $G = \left\{ (x,y) \ : \ x^2 + y^2 = 1 \right\}.$

Domain and range

Type	$f(x)$	Domain	Range
Constant function	a	$\mathbb{R}$	$\{a\}$
Absolute value function	$\|x\|$	$\mathbb{R}$	$\mathbb{R}_+$
Even power function	x^{2n}	$\mathbb{R}$	$\mathbb{R}_+$
Odd power function	x^{2n+1}	$\mathbb{R}$	$\mathbb{R}$
Rational function	$\frac{1}{x}$	$\mathbb{R}-\{0\}$	$\mathbb{R}-\{0\}$
Radical function	$\sqrt{x}$	$\mathbb{R}_+$	$\mathbb{R}_+$
Trigonometric function	$\sin(x)$	$\mathbb{R}$	$[-1,1]$
Trigonometric function	$\cos(x)$	$\mathbb{R}$	$[-1,1]$
Exponential function	e^x	$\mathbb{R}$	$(0,+\infty)$
Logarithmic function	$\ln x$	$(0,+\infty)$	$\mathbb{R}$

$$n \in \mathbb{Z}_+, \quad \mathbb{R}-\{0\} = (-\infty,0)\cup(0,+\infty), \quad \mathbb{R}_+ = [0,+\infty).$$

Polynomial functions

If f is a polynomial, then $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

- n is a non-negative integer called the degree;
- a_n is a non-zero real number;
- a_i's are called the coefficients of the polynomial f.

Property

The domain of a polynomial function is $\mathbb{R}$.

Polynomial functions

Exercise

1. *Which of the following are not polynomial functions?*

a. $f(x) = 1$

b. $f(x) = x^2 + x^{-1} + 1$

c. $f(x) = -2x^3 + x^{1/2} - 1$

d. $f(x) = x^4\sqrt{5} - \pi.$

Rational functions

Definition

A **rational function** *is a ratio* $f(x) = \dfrac{P(x)}{Q(x)}$*, where* P *and* Q *are polynomials.*

Property

The domain of f *is all real numbers except the roots of the polynomial* $Q(x)$.

That is $D_f = \{ x \in \mathbb{R} : Q(x) \neq 0 \}$.

Exercise (Constant and step functions)

Find the domain and the range of the following functions:

1) $f(x) = -4$

2) $f(x) = e$

3) $f(x) = \begin{cases} \sqrt{3} & : x < 1 \\ -1 & : x \geq 1 \end{cases}$

4) $f(x) = \begin{cases} 5 & : -4 \leq x \leq 3 \\ 0 & : 3 < x \end{cases}$

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Solution

1)$D_f = \mathbb{R}$. 2) $D_f = \mathbb{R}$. 3) $D_f = \mathbb{R}$. 4) $D_f = [-4, +\infty)$.

Exercise (Polynomial functions)

Find the domain of

$a)\ f(x) = 2x+1 \quad b)\ f(x) = -x^3 + \sqrt{2} \quad c)\ f(x) = -x^5 + x^2 \ln 2 - 4e.$

Exercise (Rational functions)

Find the domain of

$1)\ f(x) = \dfrac{1}{2x} \quad 2)\ f(x) = \dfrac{1-x}{1+x} \quad 3)\ f(x) = \dfrac{x^3-2x}{x(-x-6)}$

$4)\ f(x) = 3x-1-\dfrac{1}{2x-6} \quad 5)\ f(x) = \dfrac{x}{1-2x+x^2} \quad 6)\ f(x) = \dfrac{x^2-2x}{(x-3)(1-x^2)}$

Solution

$a-c)$ $D_f = \mathbb{R}$

1) $D_f = \mathbb{R} - \{0\}$

2) $D_f = \mathbb{R} - \{-1\}$

3) $D_f = \mathbb{R} - \{-6, 0\}$

4) $D_f = \mathbb{R} - \{3\}$

5) $D_f = \mathbb{R} - \{1\}$

Solution

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4) $D_f = \mathbb{R} - \{3\}$

5) $D_f = \mathbb{R} - \{1\}$

6) $f(x) = \dfrac{x^2 - 2x}{(x-3)(1-x^2)}$

f is defined for x which satisfy $(x-3)(1-x^2) \neq 0$.

However, $(x-3)(1-x^2) = 0 \implies x-3 = 0$ *or* $x^2 - 1 = 0$.

That is $x = 3, -1, 1$.

Thus, $D_f = \mathbb{R} - \{-1, 1, 3\}$.

Exercise (Radical functions)

Find the domain of

1) $f(x) = \sqrt{x}$

2) $f(x) = 2 - \sqrt{x-1}$

3) $f(x) = x^2 - x\sqrt{2-x}$

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Solution

$1) x \geq 0 \implies D_f = [0, +\infty)$

$2) x - 1 \geq 0 \implies x \geq 1.$ *Then* $D_f = [1, +\infty).$

$3) 2 - x \geq 0 \implies x \leq 2 \implies D_f = (-\infty, 2].$

Exercise (Algebraic functions)

Find the domain of the following functions:

1) $f(x) = \sqrt{5-2x}$

2) $f(x) = \sqrt{x} - \dfrac{1}{x-1}$

3) $f(x) = \dfrac{2-x}{\sqrt{x-1}-2}$

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Solution

1)$5-2x \geq 0 \implies D_f = (-\infty, 5/2]$

2) $f(x)$ *is defined if* $x \geq 0$ *and* $x-1 \neq 0$. *This implies*

$D_f = [0,+\infty) - \{1\} = [0,1) \cup (1,+\infty)$.

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3) $f(x)$ *is defined if* $x-1 \geq 0$ *and* $\sqrt{x-1}-2 \neq 0$.

$x-1 \geq 0 \implies x \geq 1 \implies x \in [1,+\infty)$.

$\sqrt{x-1}-2 = 0 \implies \sqrt{x-1} = 2 \implies x-1 = 4 \implies x = 5$.

Therefore, $D_f = [1,+\infty) - \{5\} = [1,5) \cup (5,+\infty)$.

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$x-1 \geq 0 \implies x \geq 1 \implies x \in [1, +\infty)$.

$\sqrt{x-1}-2 = 0 \implies \sqrt{x-1} = 2 \implies x-1 = 4 \implies x = 5$.

Therefore, $D_f = [1, +\infty) - \{5\} = [1,5) \cup (5, +\infty)$.

4) $f(x)$ *is defined if* $3-2x \geq 0$ *and* $x-1 \neq 0$. *That is*

$D_f = (-\infty, 3/2) - \{1\}$.

Definition

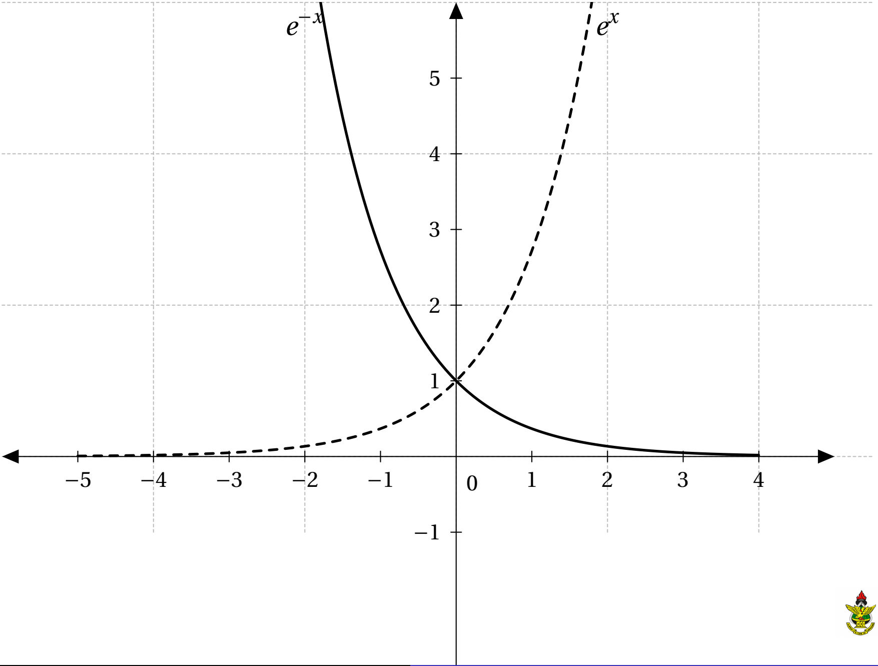
The function $f(x) = a^x$, *where* $a > 0$ *and* $a \neq 1$, *is called* **exponential function** *with base* a.

Property

The domain of an exponential function is $\mathbb{R}$ *and the range is* $(0, +\infty)$

Example

1) $\left(\frac{2}{3}\right)^x$ 2) 2^x 3) 3^{-x} 4) $\sqrt{7}^x$ 5) e^x 6) e^{-x}.


e^{-x}
e^{x}
5
4
3
2
1
0
−1
−5 −4 −3 −2 −1 1 2 3 4

Definition

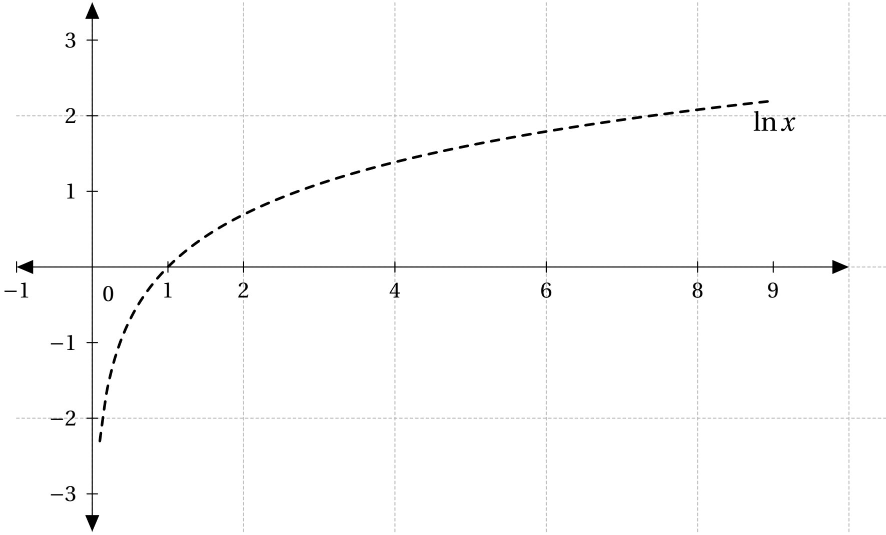
The function $f(x) = \log_a(x)$, *where* $a > 0$ *and* $a \neq 1$, *is called* **logarithmic function** *with base* a.

Property

The domain of a logarithmic function is $(0, +\infty)$ *and the range is* $\mathbb{R}$.

Example

1) $\log_{\frac{2}{3}} x$ 2) $\log_2 x$ 3) $\log_{1/3} x$ 4) $\log_{\sqrt{7}} x$ 5) $\log_e x$ 6) $\log_{1/e} x$.


3
2
1
−1
−2
−3
−1
0
1
2
4
6
8
9
$\ln x$

Exercise (Logarithmic and exponential functions)

Find the domain of:

1) $f(x) = e^x$, 2) $f(x) = 2^{x^2-1}$, 3) $f(x) = xe^{\sqrt{x}-1}$,

4) $f(x) = \ln x$, 5) $f(x) = \log_5(1-3x)$, 6) $f(x) = \ln\left(\dfrac{1}{x-1}\right)$,

7) $f(x) = e^{\frac{1}{x+1}-x}$, 8) $f(x) = e^{x^2-1} + \ln(|x|+1)$.

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Solution

1) $D_f = \mathbb{R}$.

2) $D_f = \mathbb{R}$.

3) $x \geq 0 \implies D_f = \mathbb{R}_+$.

4) $x > 0 \implies D_f = (0, +\infty)$.

5) $1 - 3x > 0 \implies D_f = (-\infty, 1/3)$.

6) $x - 1 \neq 0$ *and* $x - 1 > 0 \implies D_f = (1, +\infty)$.

Transcendental functions: Trigonometric functions

Definition

If x is an acute angle in a right triangle, then:

1. *The functions sine, cosine and tangent are defined by*
$$\sin(x) = \frac{opposite}{hypotenuse},\ \cos(x) = \frac{adjacent}{hypotenuse}, \tan(x) = \frac{opposite}{adjacent}$$

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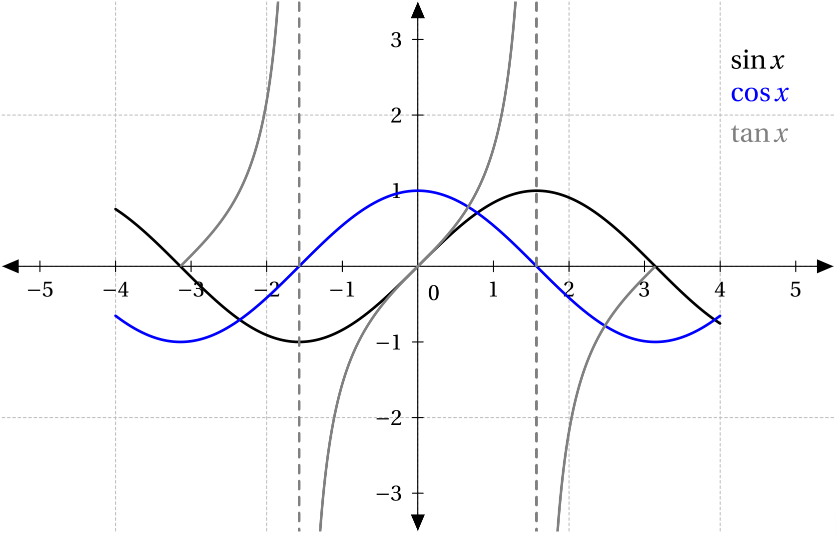
2. *Their reciprocal functions are*

secant*:* $\sec(x) = \dfrac{1}{\cos x}$,

cosecant*:* $\csc(x) = \dfrac{1}{\sin x}$,

cotangent*:* $\cot(x) = \dfrac{1}{\tan x}$.

Transcendental functions: Trigonometric functions



Transcendental functions: Trigonometric functions

Remark

$f(x)$	*Domain*	*Range*
sin	$\mathbb{R}$	$[-1,1]$
cos	$\mathbb{R}$	$[-1,1]$
tan	$\mathbb{R}-\left\{\dots,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\dots\right\}$	$\mathbb{R}$
sec	$\mathbb{R}-\left\{\dots,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\dots\right\}$	$(-\infty,-1)\cup(1,+\infty)$
csc	$\mathbb{R}-\{\dots,-2\pi,-\pi,0,\pi,2\pi,\dots\}$	$(-\infty,-1)\cup(1,+\infty)$
cot	$\mathbb{R}-\{\dots,-2\pi,-\pi,0,\pi,2\pi,\dots\}$	$\mathbb{R}$

$D_{\tan} = D_{\sec} = \mathbb{R}-\{\,\pi/2+k\pi \;:\; k\in\mathbb{Z}\}$ *and*

$D_{\cot} = D_{\csc} = \mathbb{R}-\{\,k\pi \;:\; k\in\mathbb{Z}\}$.

Let f be a function and D_f its domain. We assume that if $x \in D_f$ then $-x \in D_f$.

Definition

1. *f is an* **even** *function if* $f(-x) = f(x)$.

Let f be a function and D_f its domain. We assume that if $x \in D_f$ then $-x \in D_f$.

Definition

1. *f is an* **even** *function if* $f(-x) = f(x)$.
2. *f is an* **odd** *function if* $f(-x) = -f(x)$.

Let f be a function and D_f its domain. We assume that if $x \in D_f$ then $-x \in D_f$.

Definition

1. f *is an* **even** *function if* $f(-x) = f(x)$.
2. f *is an* **odd** *function if* $f(-x) = -f(x)$.

Example

- *The functions* $x \mapsto x^2,\ x \mapsto |x|,\ x \mapsto \cos x$ *are even functions since*

$(-x)^2 = x^2,$

$|-x| = |x|$ *and*

$\cos(-x) = \cos x.$

- *The functions* $x \mapsto x,\ x \mapsto -x^3 + 2x,\ x \mapsto \sin(x)$ *are odd functions.*

Exercise

Show that the functions $g(x) = -x^4 + 2x^2 - 1,\ h(x) = \cos(x) + x^2,\ i(x) = x\sin x$ *are even functions and* $j(x) = x^3 - x, k(x) = x\cos x - x^3$ *are odd functions.*

Exercise

Show that the functions

$g(x) = -x^4 + 2x^2 - 1,\ h(x) = \cos(x) + x^2,\ i(x) = x\sin x$ *are even functions and* $j(x) = x^3 - x, k(x) = x\cos x - x^3$ *are odd functions.*

Solution

$h(-x) = \cos(-x) + (-x)^2 = h(x).$

$i(-x) = (-x)\sin(-x) = -x(-\sin x) = x\sin x = i(x).$

$k(-x) = -(x)\cos(-x) - (-x)^3 = -x\cos x + x^3 = -k(x).$

Exercise

Determine whether the functions below are even, odd or neither.

1) $f(x) = e^{x^2-1} + \ln(|x|+1)$

2) $f(x) = \dfrac{x^2-2}{x(1-x^2)}$

3) $f(x) = x^2\sin(x)$

4) $f(x) = x\sqrt{|x|-1}$

5) $f(x) = \ln\left(\tan x - e^{|x|}\right)$

6) $f(x) = x-1$

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5) $f(x) = \ln\left(\tan x - e^{|x|}\right)$

6) $f(x) = x - 1$

Solution

1) Even *2) Odd* *3) Odd*

4) Odd *5) Neither* *6) Neither*

Let f be a function, and D_f its domain.

Definition

f is a **periodic function** *if there exists a positive real number t such that $f(x+t) = f(x)$ for all $x \in D_f$.*

The minimum of such t's, T, is called the **period** *of f.*

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Example

The trigonometric functions are periodic functions.

$\sin(x+2k\pi) = \sin(x+2\pi) = \sin(x)$ *for $k \in \mathbb{Z}$, however,* $T = 2\pi$.

$\cos(x+2k\pi) = \cos(x+2\pi) = \cos(x), T = 2\pi$.

$\tan(x+(2k+1)\pi) = \tan(x+\pi) = \tan(x), T = \pi$.

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Example

The trigonometric functions are periodic functions.

$\sin(x+2k\pi)=\sin(x+2\pi)=\sin(x)$ *for $k \in \mathbb{Z}$, however,* $T=2\pi$.

$\cos(x+2k\pi)=\cos(x+2\pi)=\cos(x), T=2\pi$.

$\tan(x+(2k+1)\pi)=\tan(x+\pi)=\tan(x), T=\pi$.

Exercise

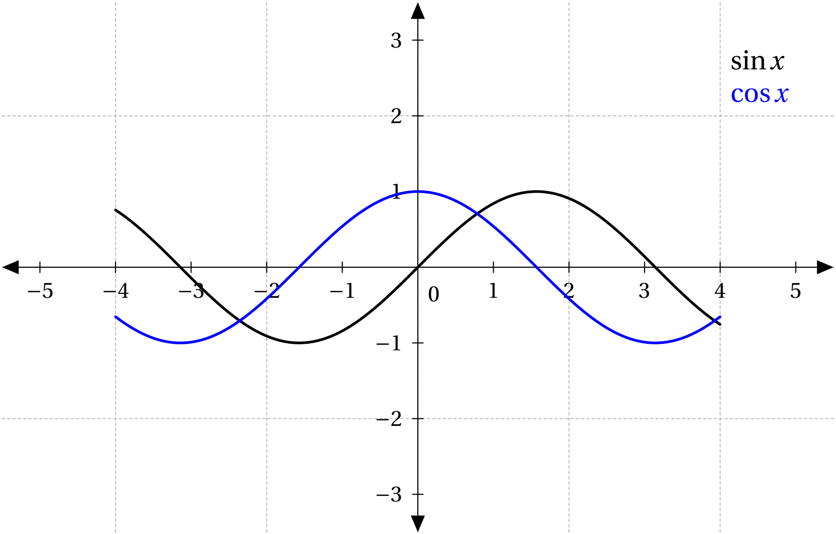
Find the period of the following functions

1) $f(x)=\sin(2x)$, 2) $f(x)=\cos(-2x+\pi/3)$, 3) $f(x)=x-\sin(x)$.

Remark

The graph of an even function is symmetric about the y-axis.

The graph of an odd function is symmetric about the origin.


sin x
cos x
3
2
1
−1
−2
−3
−5
−4
−3
−2
−1
0
1
2
3
4
5

Monotonic functions

Let I be an open interval. x_1 and x_2 are two elements of I such that $x_1 < x_2$.

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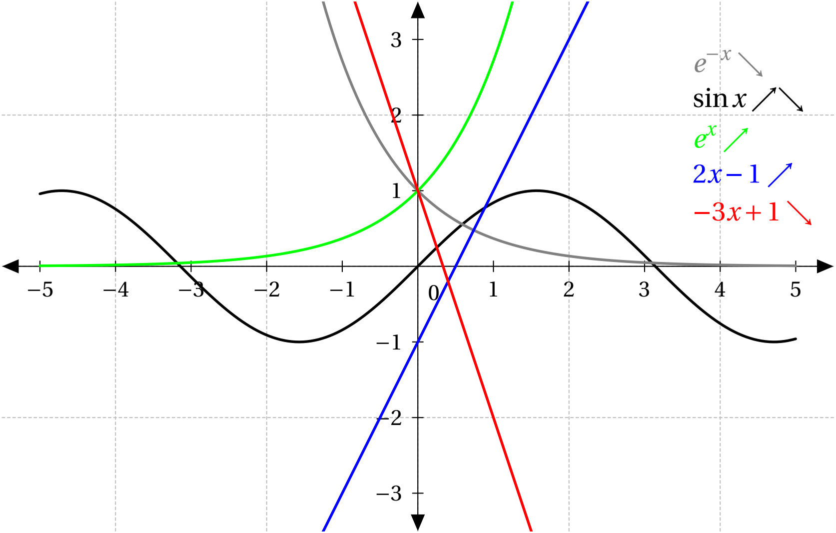
Example

- *The functions* e^x, $\tan(x)$ *and* $ax+b$, *where* $a>0$, *are increasing on their respective domains.*
- *The functions* e^{-x}, $\cot(x)$ *and* $ax+b$, *where* $a<0$, *are decreasing on their respective domains.*

Monotonic functions



Example

Show that the function $f(x) = \sqrt{x-2}$ is an increasing function on its domain.

$D_f = [2, +\infty)$.

For $x_1, x_2 \in D_f$ *and* $x_1 < x_2$,

$$\begin{aligned} 2 < x_1 < x_2 &\implies 0 < x_1 - 2 < x_2 - 2 \\ &\implies 0 < \sqrt{x_1 - 2} < \sqrt{x_2 - 2} \\ &\implies f(x_1) < f(x_2). \end{aligned}$$

Thus, f *is an increasing function on its domain.*

We could also look for the sign of $f(x_2) - f(x_1)$. *Indeed,*

$$\begin{aligned} f(x_2) - f(x_1) = \sqrt{x_2 - 2} - \sqrt{x_1 - 2} &= \frac{(x_2 - 2) - (x_1 - 2)}{\sqrt{x_2 - 2} + \sqrt{x_1 - 2}} \\ &= \frac{x_2 - x_1}{\sqrt{x_2 - 2} + \sqrt{x_1 - 2}} > 0 \end{aligned}$$

since $x_2 - x_1 > 0$ *and* $\sqrt{x_2 - 2} + \sqrt{x_1 - 2} \geq 0$.

Example

Show that $f(x) = (2-x)^2 + 1$ *decreases on* $(-\infty, 2]$ *and increases on* $[2, +\infty)$.

$D_f = \mathbb{R}$.

For $x_1, x_2 \in (-\infty, 2]$,

$$\begin{aligned} x_1 < x_2 \leq 2 &\implies -x_1 > -x_2 > -2 \\ &\implies 2 - x_1 > 2 - x_2 > 0 \\ &\implies (2-x_1)^2 > (2-x_2)^2 > 0 \\ &\implies (2-x_1)^2 + 1 > (2-x_2)^2 + 1 > 1 \\ &\implies f(x_1) > f(x_2). \end{aligned}$$

f *is decreasing on* $(-\infty, 2]$.

Example

Show that $f(x) = (2-x)^2 + 1$ *decreases on* $(-\infty, 2]$ *and increases on* $[2, +\infty)$.

For $x_1, x_2 \in [2, +\infty)$,

$$
\begin{aligned}
2 \le x_1 < x_2 &\implies -2 > -x_1 > -x_2 \\
&\implies 0 > 2 - x_1 > 2 - x_2 \\
&\implies 0 < (2-x_1)^2 < (2-x_2)^2 \\
&\implies 1 < (2-x_1)^2 + 1 < (2-x_2)^2 + 1 \\
&\implies f(x_1) < f(x_2).
\end{aligned}
$$

f *is an increasing function on* $[2, +\infty)$.

Maxima and minima

Definition (Minimum)

☛ *f is said to be* **bounded below** *if its range is bounded below. There exists a real number m such that $m \leq f(x), \forall x \in D_f$.*

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☛ f *is said to have a* **global minimum value** *at the point* x_0 *if* $f(x_0) \leq f(x)$ *for all* x *in the domain of* f.
In particular, f *is bounded below.*

Remark

Show that m *is a minimum of* f *is equivalent to show that* $f(x) - m \geq 0$.

Maxima and minima

Definition (Maximum)

☛*If if its range is bounded above then f is said to be* **bounded above**.

There exists a real number M such that $f(x) \leq M, \forall x \in D_f$.

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Maxima and minima

Definition (Maximum)

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There exists a real number M such that $f(x) \leq M, \forall x \in D_f$.

☛ *if $f(x) \leq f(x_0)$ for all x in a neighbourhood of x_0, then f has a* **local maximum value** *at the point x_0.*

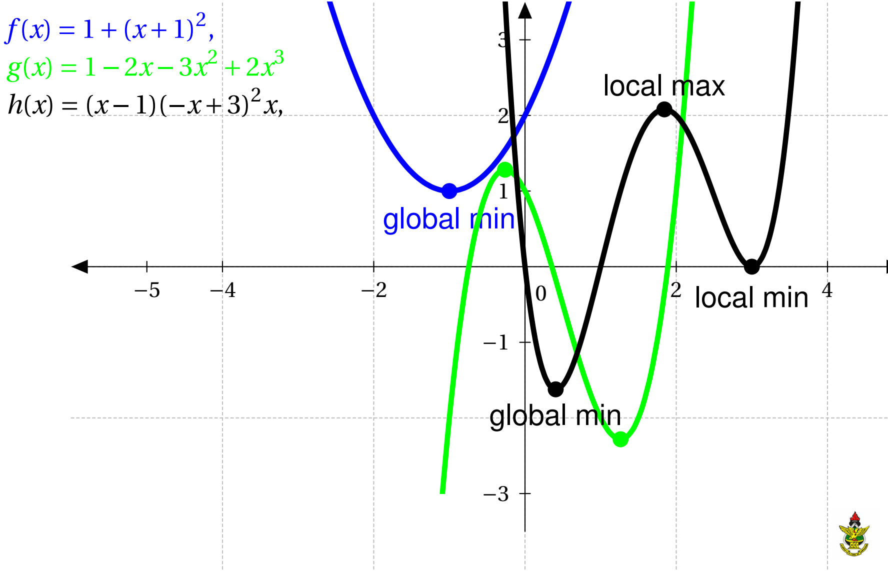
☛ *The maximum is* **global** *if $f(x) \leq f(x_0)$ for all x in the domain of f.*

f is bounded above.

Remark

Show that M is a maximum of f is equivalent to show that $f(x) - M \leq 0$.

Maxima and minima



Maxima and minima

Example

Show that 4 *is a minimum of the function* $f(x) = 2x^2 - 4x + 6$.

•4 *is a lower bound.*
For all $x \in D_f = \mathbb{R}$, $f(x) - 4 = 2x^2 - 4x + 2 = 2(x-1)^2 \geq 0$.

That is $f(x) \geq 4$.

•4 *is a minimum*
Since 4 *is a lower bound of* f *and* $f(1) = 4$, 4 *is a minimum of* f.

Maxima and minima

Example

Show that 2 *is an upper bound of the function* $f(x) = \dfrac{1}{x} + 2$ *on the interval* $(-\infty, 0)$

•2 *is an upper bound.*

For all $x \in (-\infty, 0),\ f(x) - 2 = 2 - \dfrac{1}{x} - 2 = -\dfrac{1}{x} < 0$

since $x \in (-\infty, 0) \implies x < 0 \implies -x > 0.$

That is $f(x) \leq 2$ *for all* $x \in (-\infty, 0).$

• *Is* 2 *a maximum?*

Let us assume that 2 *is in the range of* f *and solve the equation* $f(x) = 2.$

$f(x) = 2 \implies \dfrac{1}{x} = 0 \implies 1 = 0,$ *which is a contradiction.*

Thus, 2 *is not a maximum.*