MATH157: ALGEBRA

CHAPTER 5

Matrices And Determinants

March 22, 2021

Outline

Definition

- *A* **matrix** A *is an array with a finite number of rows and columns. Its elements can be real or complex numbers.*
- *We will let* a_{ij} *represents the element in the* i*th row and* j*th column.*
- *If* n *is the numbers of rows and* m *the number of columns, then* A *is an* $n \times m$ *(reads* n *by* m*) matrix. We also write* $A = (a_{ij})_{n\times m} = (a_{ij})$.
- (n, m) *is the* **size** *of* A.

$$n \text{ rows} \left\{ \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}}_{m \text{ columns}} \right. = A$$

For instance $E = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \sqrt{3} & -1 \end{pmatrix}$ is a 2×3 matrix, $e_{13} = 3$ and $e_{22} = \sqrt{3}$.

Definition

- *A matrix whose entries are all zeros is called a* **zero matrix**.
- *An* $n \times 1$ *matrix or a matrix with only one row is called a* **row matrix** *and a* $1 \times m$ *matrix is a* **column matrix**.
- $A_{i:}$ *is the* i*th row vector and* $A_{:j}$ *is the* j*th column vector of the rectangular matrix* A.
- $I = (I_{ij})_{n \times n}$ *with* $I_{ii} = 1$ *and* $I_{ij} = 0$ *if* $i \neq j$ *is called the* **identity matrix**.

Property (Equality)

$A = B$ *if and only if* $a_{ij} = b_{ij}$ *for all* i *and* j.

Example

Find x *and* y *such that* $\begin{pmatrix} 2(x-y) & 2 & -1 \\ 0 & xy-1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Outline

Definition

If $A = (a_{ij})_{n\times m}$ *and* $B = (b_{ij})_{n\times m}$ *then*
$A + B = (a_{ij} + b_{ij})_{n\times m}$;
$kA = (ka_{ij})_{n\times m}$ *for any scalar* k.

Remark

The addition of matrices with different sizes is not defined.

Example

Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & -2 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.
The **linear combination** $A - 2B$ *of* A *and* B *is given by*
$A - 2B =$

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The **linear combination** $A - 2B$ *of* A *and* B *is given by*
$A - 2B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -4 & 4 & 2 \\ -2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 5 \\ -2 & 1 & 1 \end{pmatrix}$

Property

Consider any matrices A, B and C with the same sizes and any scalars k and k'. The following results hold

(1) $A + B = B + A$,

(2) $(A + B) + C = A + (B + C)$,

(3) $A + 0 = A$,

(4) $k(A + B) = kA + kB$,

(5) $(k + k')A = kA + k'A$,

(6) $1A = A$.

Matrix multiplication

Criterion

The product of two matrices of sizes $(n, \mathbf{p})$ and $(\mathbf{q}, m)$ is defined if and only if $p = q$.

Definition

Let A be an $n \times p$ matrix and B a $p \times m$ matrix. The product $AB = (c_{ij})$ is a $n \times m$ matrix defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

Row vector times column vector

If the sizes are $1 \times p$ and $p \times 1$ then

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1p}b_{p1}$$

is a 1×1 matrix or a scalar.

Example

$A = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *yields* $AB = 2$

Exercise

Find AB *if (a)* $A = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, *(b)* $A = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *(c)* $A = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Row vector times a rectangular matrix

If the sizes are $1 \times p$ and $p \times m$ then

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{pmatrix} = (AB_{:1}, AB_{:2}, \cdots, AB_{:m})$$ is a $1 \times m$ matrix.

Example

$A = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}$ yields $AB = \begin{pmatrix} 2 & 3 \end{pmatrix}$

Exercise

Find AB if (a) $A = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, (b) $A = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & -5 \\ 0 & 2 \end{pmatrix}$ (c) $A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 7 \\ 4 & -5 \\ 2 & 3 \end{pmatrix}$

Product of rectangular matrices

Example

$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$ *yields* $AB = \begin{pmatrix} 7 & 13 \\ -3 & 0 \end{pmatrix}$.

Exercise

Find AB *if (a)* $A = \begin{pmatrix} -2 & 2 \\ 0 & -1 \\ 1 & -4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -3 & -2 \\ 2 & 0 & -1 \end{pmatrix}$,
(b) $A = \begin{pmatrix} -2 & 0 & 1 \\ 2 & -1 & -4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

Column vector times row vector

If the sizes are $n \times 1$ and $1 \times m$ then

$$AB = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1m} \\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{1m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1}b_{11} & a_{n1}b_{12} & \cdots & a_{n1}b_{1m} \end{pmatrix}$$

is a $n \times m$ matrix.

Example

$A = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ yields $AB = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix}$

Exercise

Find AB if (a) $A = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$. (b) $A = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.

Property

Let A, B, C be three matrices. Whenever the product and sums are defined,
(1) $(AB)C = A(BC)$,
(2) $A(B + C) = AB + AC$;
(3) $(A + B)C = AC + BC$;
(4) $k(AB) = (kAB) = A(kB)$, *where* k *is a scalar;*
(5) $0 \times A = 0$;
(6) $I \times A = A$, *where* I *is the identity matrix;*
(7) The product is not commutative i.e $AB = BA$ *is not always true.*

Exercise

1) Show that $AB \neq BA$ *for* $A = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$.
2) Show that $AB = 0$ *for* $A = \begin{pmatrix} 0 & -1 \\ 0 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}$.
3) Show that $AB = AC$ *for* $A = \begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -1 \\ 5 & 4 \end{pmatrix}$ *and* $C = \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$.
2) Show that $I_2M = MI_2 = M$ *for* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Outline

Definition (Square matrix)

• *Matrices of size* (n, n) *are called* **square** *matrices or* n**-square** *matrices of* **order** n.

• *The* $a_{ii}, 1 \leq n \leq n,$ *entries of a square matrix are called the* **diagonal elements**. *If the nondiagonal elements are all zero, then the matrix is called a* **diagonal matrix**. *It is denoted by* $\mathbf{A} = \mathbf{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

Definition (Matrix Function)

A function f which maps a matrix A of size (n,m) to another matrix $A' = f(A)$ of size (n', m') is called a matrix function.

In particular

Definition (Transpose)

The matrix function that interchanges rows and columns of a matrix A is called the **Transpose** *of A.*
We denote the transpose of $A = (a_{ij})_{n\times m}$ by $\mathbf{A}^{\mathbf{T}} = (a_{ji})_{m\times n}$

Thus, the ith row of A becomes the ith column of A^T. For instance
$\begin{pmatrix} 1 & -3 & 4 \\ 5 & 6 & -2 \end{pmatrix}^T =$

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Thus, the ith row of A becomes the ith column of A^T. For instance $\begin{pmatrix} 1 & -3 & 4 \\ 5 & 6 & -2 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 \\ -3 & 6 \\ 4 & -2 \end{pmatrix}.$

Property

Let A and B be two matrices and k be a scalar. Provided that the following operations are defined, we have

$$\text{(1)} \quad (A+B)^T = A^T + B^T, \qquad \text{(3)} \quad (kA)^T = kA^T,$$
$$\text{(2)} \quad (A^T)^T = A, \qquad \text{(4)} \quad (AB)^T = B^T A^T.$$

Exercise

Find $(2A - B^T)^T$ *where* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} -4 & 2 \\ 2 & 2 \\ 5 & 0 \end{pmatrix}$.

Exercise

If $A = \begin{pmatrix} 2 & -3 & 4 \\ -3 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$, *show that* $A^T = A$ *and* $B^T = -B$.

Exercise

If $A = \begin{pmatrix} 2 & -3 & 4 \\ -3 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$, *show that* $A^T = A$ *and* $B^T = -B$.

Definition

• *An* n*-square matrix* A *is said to be* **symmetric** *if* $\mathbf{A^T} = \mathbf{A}$. *That is* $a_{ji} = a_{ij}$ *for* $1 \leq i, j \leq n$.

• *An* n*-square matrix for which* $\mathbf{A^T} = -\mathbf{A}$ *is said to be* **skew-symmetric**. *That is* $a_{ji} = -a_{ij}$ *for* $1 \leq i, j \leq n$. *In particular, the diagonal elements* a_{ii} *must all be zero.*

Let A be a n-square real matrix.

Definition (Power of Matrices)

Powers of A are defined as follows:

$$A^0 = I_n, \qquad A^2 = AA, \qquad A^3 = A^2A, \qquad \ldots, \qquad A^{n+1} = A^nA.$$

Definition (Matrix polynomials)

A matrix polynomial is a polynomial with square matrices as variables.

The standard form of a matrix polynomial P is $P(A) = \sum_{i=0}^{d} k_i A^i$.

Exercise

If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ *and* $P(X) = -2X + X^2$ *is a matrix polynomial, find* $P(A)$.

Definition (Inverse)

An n-square matrix is said to be **invertible** *or* **nonsingular** *if there exists a unique n-square matrix B such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I_n}$.*
B is the **inverse** *of A and is denoted by $\mathbf{A^{-1}}$.*

Theorem

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible then $\mathbf{A^{-1}} = \dfrac{\mathbf{1}}{\det(\mathbf{A})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $\det(A) = ad - bc$ is the determinant of A and is nonzero.
In general, if $\det(A) = 0$ then A^{-1} does not exist or A is not invertible or A is a **singular matrix**.

Example

Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$. Show that A is nonsingular and $B = \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$ is the inverse of A.

Outline

Example

Let $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$ *be a* 3*-square real matrix. Evaluate the determinants* d_3 *and* d_2' *by expanding by the* 3*rd row and the* 2*nd column respectively. Show that* $d_3 = d_2'$.

Example

Let $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$ *be a* 3*-square real matrix. Evaluate the determinants* d_3 *and* d_2' *by expanding by the* 3*rd row and the* 2*nd column respectively. Show that* $d_3 = d_2'$.

Solution

We have

						d_i
	36	+	12	+	16	=64
	+		+		+	
$(a_{ij}c_{ij})$	4	+	12	+	48	= 64
	+		+		+	
	24	+	40	+	0	=64
d_j'	= 64		= 64		= 64	

Let $A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a 3-square matrix.

Definition

• $\mathbf{A_{ij}}$ *is the submatrix of A obtained by removing/deleting the ith row and the jth column of A.*

That is $A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$, $A_{22} = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$ and $A_{31} = \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$.

Definition

The **determinant** *of the* 3*-square matrix* $A = (a_{ij})$ *is the real number obtained by summing* $(-1)^{i+j} a_{ij} \det(A_{ij})$ *along the* i*th row or the* j*th column.*

The expansion by the i*th row is*

$$\det A = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + a_{i3}(-1)^{i+3} \det(A_{i3}).$$

The expansion by the j*th column is*

$$\det A = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + (-1)^{3+j} a_{3j} \det(A_{3j}).$$

Definition

The real number defined and denoted by $c_{ij} = (-1)^{i+j} \det(A_{ij})$ *is called a co-factor of* a_{ij}*.*

Exercise

1) Compute the determinant of A *if (a)* $A = \begin{pmatrix} 3 & 0 & 4 \\ -11 & 1 & 10 \\ -2 & 0 & -1 \end{pmatrix}$. *(b)* $A = \begin{pmatrix} -6 & 2 & 2 \\ -8 & 2 & 3 \\ -3 & 0 & 0 \end{pmatrix}$.

2) Evaluate the determinant of $A = \begin{pmatrix} 1 & -\frac{k^2}{10} & \frac{k}{7} \\ 2 & -1 & 3 \\ -1 & 4 & -2 \end{pmatrix}$ *for any real scalar* k. *Could* A *be a singular matrix?*

Definition

The **adjoint** *of a n-square matrix A is the transpose of the matrix of co-factors and is denoted by $adj(A) = (c_{ij})^T$.*

Theorem

The inverse of an invertible n-square matrix A is given by

$$A^{-1} = \frac{1}{\det A} adj(A).$$

Example

The inverse of $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ exists and is given by

$$A^{-1} = -\frac{1}{4} \begin{pmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

Let A, B and C be three square matrices.

Property

(1) $\det A^T = \det A.$

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & -1 & -2 \end{pmatrix}$. $\det A = 4$ *and* $\det A^T =$

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Property

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Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & -1 & -2 \end{pmatrix}$. $\det A = 4$ *and* $\det A^T$ =4.

Property

(2) If $A = (A_{:1}, A_{:2}, \dots, A_{:n})$ *and*
$B = (A_{:1}, \dots, A_{:(j-1)}, kA_{:j}, A_{:(j+1)}, \dots, A_{:n})$ *then*
$\det B = k \det A$.
Note that $A_{:j}$ *is the* j*th column vector of* A.

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ *and* $B = (A_{:1}, A_{:2}, -2A_{:3}) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}$,
$\det A = 4$ *and* $\det B =$

Property

(2) If $A = (A_{:1}, A_{:2}, \ldots, A_{:n})$ *and*
$B = (A_{:1}, \ldots, A_{:(j-1)}, kA_{:j}, A_{:(j+1)}, \ldots, A_{:n})$ *then*
$\det B = k \det A$.
Note that $A_{:j}$ *is the* j*th column vector of* A.

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ *and* $B = (A_{:1}, A_{:2}, -2A_{:3}) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}$,
$\det A = 4$ *and* $\det B = -8 = -2 \det A$.

Property

(3) $\det(kA) = k^n \det A$ *where* n *is the order of* A.

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ *and* $B = 2A = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 2 & -2 \\ 2 & -2 & -4 \end{pmatrix}$,
$\det A = 2^2$ *and* $\det B =$

Property

(3) $\det(kA) = k^n \det A$ *where* n *is the order of* A.

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ *and* $B = 2A = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 2 & -2 \\ 2 & -2 & -4 \end{pmatrix}$,
$\det A = 2^2$ *and* $\det B = 2^5 = 2^3(2^2)$.

Property

(4) If B is obtained from A by interchanging two rows or columns then $\det B = -\det A$.

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ *and* $B = (A_{:1}, A_{:3}, A_{:2}) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & -1 \end{pmatrix}$,
$\det A = 4$ *and* $\det B =$

Property

(4) If B is obtained from A by interchanging two rows or columns then $\det B = -\det A$.

Example

For $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$ *and* $B = (A_{:1}, A_{:3}, A_{:2}) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & -1 \end{pmatrix}$,
$\det A = 4$ *and* $\det B = -4 = -\det A$.

Property

(5) If two rows (or columns) vectors of A *are parallel, then* $\det A = 0$.

Example

For $C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & 6 & -3 \end{pmatrix}$, $\det C =$

Property

(5) If two rows (or columns) vectors of A are parallel, then $\det A = 0$.

Example

For $C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & 6 & -3 \end{pmatrix}$, $\det C = 0$.

Property

(6) $\det A = a_{11}a_{22}\cdots a_{nn}$ *if* A *is a triangular matrix.*

Example

For $C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$, $\det C =$

Property

(6) $\det A = a_{11}a_{22}\cdots a_{nn}$ *if* A *is a triangular matrix.*

Example

For $C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$, $\det C = -2 = 1 \times 1 \times (-2)$.

Property

(7) $\det(AB) = \det A \det B$.

Example

If $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix}$, *then* $\det A = 2$, $\det B = -3$ *and* $\det AB = \begin{vmatrix} -2 & 3 \\ 2 & 0 \end{vmatrix} = -6 = 2 \times (-3) = \det A \det B$.

Outline

$$\begin{cases} x-2y+z &= 1,\\ 3x-5y+5z &= 13,\\ x-3y &= -5. \end{cases} \iff \underbrace{\begin{pmatrix}1 & -2 & 1\\ 3 & -5 & 5\\ 1 & -3 & 0\end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix}x\\ y\\ z\end{pmatrix}}_{\mathbf{X}} = \underbrace{\begin{pmatrix}1\\ 13\\ -5\end{pmatrix}}_{\mathbf{b}}$$

$$AX = b$$

Definition

The matrix A is called **coefficient matrix** *and* $\begin{pmatrix}1 & -2 & 1 & 1\\ 3 & -5 & 5 & 13\\ 1 & -3 & 0 & -5\end{pmatrix}$ *is called* **augmented matrix**.

Exercise

1) Find the coefficient matrix A of the system of equation

$$\begin{cases} x_1 + 3x_4 & = 0 \\ -x_2 + x_3 & = 3 \\ 2x_1 + 11x_2 - x_4 + 3x_3 & = 1 \end{cases}$$

2) Find the system of equations that corresponds to the augmented matrix $A' = \begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ -2 & 3 & -1 & 0 & -3 \\ 1 & 4 & -2 & 9 & 4 \end{pmatrix}$.

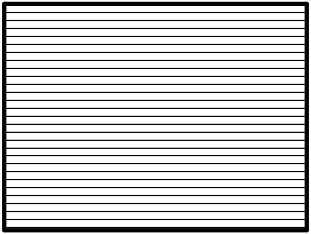
Conditions for echelon matrices

(C_1) If there are any rows of all zeros then they are at the bottom of the matrix.
(C_2) If a row does not consist of all zeros then its first non-zero entry is 1. This is called a leading 1.
(C_3) In any two successive rows, neither of which consists of all zeros, the leading 1 of the lower row is to the right of the leading 1 of the higher row.
(C_4) If a column contains a leading 1 then all the other entries of that column are zero.

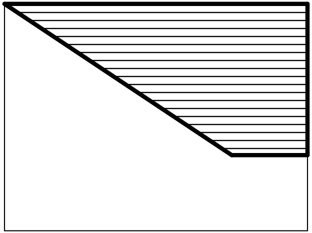
Definition

A matrix is on the **row echelon form** *if it satisfies conditions* C_1, C_2 *and* C_3.
A matrix is on the **reduce row echelon form** *if it satisfies all the four conditions.*

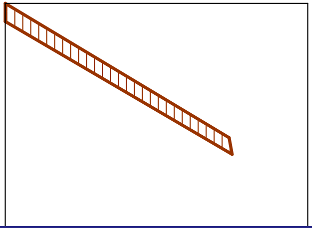
Standard form



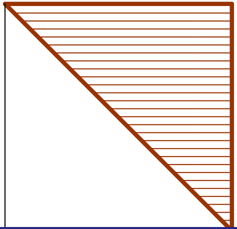
Row echelon form



Reduced row echelon form



Triangular form



Definition

Gaussian elimination *method consists of reducing augmented matrices into their row echelon form.*
Gauss-Jordan elimination *method reduces augmented matrices into reduced row echelon matrices.*

Example

Use the Gaussian and Gauss-Jordan eliminations to solve the following system of linear equations

$$\begin{aligned} R_1 &: & 2x - 4y - 3z &= 8 \\ R_2 &: & x - 3y - 2z &= 6 \\ R_3 &: & -3x + 6y + 8z &= -5 \end{aligned}$$

Transformation of the augmented matrix $E = \begin{pmatrix} 2 & -4 & -3 & 8 \\ 1 & -3 & -2 & 6 \\ -3 & 6 & 8 & -5 \end{pmatrix}$

$Step1 : R_2 \to R_1, R_2 \to R_1$

$$\begin{pmatrix} 1 & -3 & -2 & 6 \\ 2 & -4 & -3 & 8 \\ -3 & 6 & 8 & -5 \end{pmatrix}$$

$Step2 : R_2 - 2R_1 \to R_2, R_3 + 3R_1 \to R_3$

$$\begin{pmatrix} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & -3 & 2 & 13 \end{pmatrix}$$

$Step3 : R_3 + 3R_2/2 \to R_3$

$$\begin{pmatrix} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 7/2 & 7 \end{pmatrix}$$

$Step4 : R_2/2 \to R_2, 2R_3/7 \to R_3$

$$\begin{pmatrix} 1 & -3 & -2 & 6 \\ 0 & 1 & 1/2 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

row echelon form

$Step5 : R_1 + 3R_2 \to R_1$

$$\begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$Step6 : R_1 + R_3/2 \to R_1, R_2 - R_3/2 \to R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Reduced row echelon form

Activity

Let $A = \begin{pmatrix} -4 & -2 \\ 5 & 5 \end{pmatrix}$ *and let* $E = (A|I_2)$ *be an augmented matrix. Transform* E *into its reduced row-echelon form* $E' = (I_2|A')$. *Show that* A' *is the inverse of* A.

Activity

Let $A = \begin{pmatrix} -4 & -2 \\ 5 & 5 \end{pmatrix}$ *and let* $E = (A|I_2)$ *be an augmented matrix. Transform* E *into its reduced row-echelon form* $E' = (I_2|A')$. *Show that* A' *is the inverse of* A.

Indeed, $A' = \left(\begin{array}{cc|cc} -4 & -2 & 1 & 0 \\ 5 & 5 & 0 & 1 \end{array}\right)$ *and* $E' = \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & -1/5 \\ 0 & 1 & 1/2 & 2/5 \end{array}\right)$.

We have $A' = \begin{pmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{pmatrix}$ *and* $A'A = AA' = I_2$.

Exercise

Use Gauss-Jordan Elimination method to find the inverse of

$A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 2 \\ -2 & 0 & 0 \end{pmatrix}$.

Theorem (Cramer's rule)

If A is an n-square invertible matrix, then the solution to the system of linear equations $Ax = b$ is given by

$$x_1 = \frac{det(A_1)}{det(A)}, x_2 = \frac{det(A_2)}{det(A)}, \ldots, x_n = \frac{det(A_n)}{det(A)}$$

where A_i is the matrix found by replacing the ith column of A with b.

Example

For $\begin{cases} 2x + y & = 1 \\ 3x + 2y & = 1 \end{cases}$, *we have* $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ *with* $\det(A) = 1, \det(A_1) = 1$ *and* $\det(A_2) = -1$.

Therefore, $x_1 = \dfrac{det(A_1)}{det(A)} = 1$ *and* $x_2 = \dfrac{det(A_2)}{det(A)} = -1$.

Exercise

Use Cramer's rule to determine the solution to the system

$$\begin{cases} 3x - y + 5z & = -2 \\ -4x + y + 7z & = 10 \\ 2x + 4y - z & = 3. \end{cases}$$

Exercise

Use Cramer's rule to determine the solution to the system

$$\begin{cases} 3x - y + 5z & = -2 \\ -4x + y + 7z & = 10 \\ 2x + 4y - z & = 3. \end{cases}$$

Solution

$(212/187, 273/187, 107/187)$.