CALCULUS WITH ANALYSIS

CHAPTER I

Real numbers and points sets

May, 2021.

Outline

1. **Real numbers**
   - Standard subsets
   - Binary operations
2. **Point sets**
   - Absolute value
   - Intervals
3. **Limit points**

Some of the standard subsets of $\mathbb{R}$ are:

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4. $\bar{\mathbb{Q}}$ is the set of irrational numbers or decimals which are not in $\mathbb{Q}$.
5. $\mathbb{R} = \mathbb{Q} \cup \bar{\mathbb{Q}}$ is the set of real numbers.

Let S be an arbitrary set endowed with the binary operation $\star$.

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4. $(S,\star)$ admits an identity element if there exits an unique element $e\in S$ such that $\forall x\in S, e\star x = x\star e = x$.
   An inverse for x is an element x' of S such that $x'\star x = x\star x' = e$.

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$(\mathbb{R}, +)$ (resp. $(\mathbb{R}, \times)$) is closed, associative, commutative and 0 (resp. 1) is the identity element.
Moreover, multiplication is distributive over addition.

Question

We denote by $\mathbb{Z}_-$ *the set of non-positive integers.*

Q1. Fill the table below with True or False

	$(\mathbb{N}, +)$	$(\mathbb{Z}_-, -)$	$(\mathbb{Z}, \times)$	$(\mathbb{Q} - \{0\}, \times)$
Closed				
Commutativity				
Associativity				
Identity exists				
Inverse exists				

Q2. We endow $\mathbb{Z}$ *with the binary operation* $x \star y = 3^x \times 3^y$*. Find the properties of* $\star$*.*

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Absolute value

Definition

The **absolute value** *of a real number x is the non-negative real number denoted by $|x|$ such that*

$$|x| = \begin{cases} x & \textit{if} \quad 0 \leq x \\ -x & \textit{if} \quad x \leq 0. \end{cases}$$

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Example

Let us rewrite the following without the symbol of absolute value.

•$|2| =$

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•$|2| = 2$ *since* 2 *is positive.*

•$|-4| = -(-4) = 4$ *since* -4 *is negative.*

•$|x-2|$ *equals* $x-2$ *if* $x-2 \geq 0$. *That is* $|x-2| = x-2$ *if* $x \geq 2$.
Otherwise, that is for $x \leq 2$, $|x-2| = -(x-2) = -x+2$.

Therefore $|x-2| = \begin{cases} x-2 & \text{if} \quad x \geq 2 \\ -x+2 & \text{if} \quad x \leq 2. \end{cases}$

Absolute value

Exercise

Rewrite the following without the absolute value:

$1)\left|3-\sqrt{2}\right|,$ $2)\left|2\sqrt{2}-\sqrt{5}\right|$

$3)|x+5|$ $4)|2x-3|,$

$5)|-x+a|,$ $6)|-4x-a|$ *for* $a \in \mathbb{R},$

$7)-3|1-2x|$ $8)|1-2x|+3|x-4|-|3x-9|.$

Absolute value

Let solve $|-2x+2| = 4$.

• Rewrite $|-2x+2|$ without the absolute value.

$-2x+2 \geq 0$ implies $-2x \geq -2$. That is $|-2x+2| = -2x+2$ if $x \leq 1$; otherwise $|-2x+2| = -(-2x+2) = 2x-2$.

• Solve the equation for $x \leq 1$ and $x \geq 1$.

⋆ For $x \leq 1$, $|-2x+2| = 4$ becomes $-2x+2 = 4$ which implies that $x = -1$. Since $-1 \leq 1$ then it is a valid solution.

⋆ For $x \geq 1$, $|-2x+2| = 4$ becomes $2x-2 = 4$ which implies that $x = 3$. Since $3 \geq 1$ then it is a valid solution as well.

In summary, the solutions are $\{-1, 3\}$.

Absolute value

Exercise

Rewrite the following without the absolute value:

$1)|x+5| = 10$ $\quad$ $2)|2x-3| = -3,$

$3)|-x+a| = 3a-2,$ $\quad$ $4)|-4x-a| = -a$ *for* $a \geq 0,$

$5)-3|1-2x| = -9$ $\quad$ $6)|1-2x| + 3|x-4| - |3x-9| = 1.$

Open intervals

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Definition

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Example

$(-1, 2)$ *is an open interval and does not include its endpoints* -1 *and* 2.

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Example

$\left[\frac{2}{3}, 2\sqrt{3}\right]$ *is a closed interval and does include its endpoints* $\frac{2}{3}$ *and* $2\sqrt{3}$.

Bounded intervals: Half open or half closed intervals

Definition (Left-closed and right-open)

The set $[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b\}$ *is a* **left-closed and right-open interval.**

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The set $(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$ *is a* **left-open and right-closed interval.**

Remark (Bounded intervals)

The intervals $(a, b), (a, b], [a, b)$ *and* $[a, b]$ *are called* **bounded intervals**.

a *is the* **greatest** **lower bound** *and* b *the* **least** **upper bound**.

Unbounded intervals: Left open or closed intervals

Definition (Left-open)

The set $(a, +\infty) = \{ x \in \mathbb{R} \mid a < x \}$ *is a* **left-open interval.**

Unbounded intervals: Left open or closed intervals

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Unbounded intervals: Right open or closed intervals

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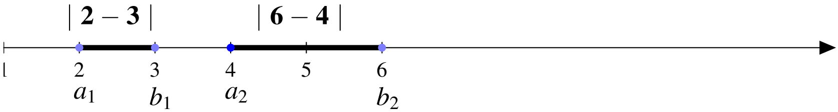
Definition

The set $(-\infty, +\infty)$ *is the set of real numbers,* $\mathbb{R}$. *It's* **unbounded below** *and* **above**.

Intervals and absolute value
Geometric interpretation

The absolute value between two real numbers a and b is the distance between them.



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Any interval I with endpoints a and b, could be represented by a line segment on the real axis.

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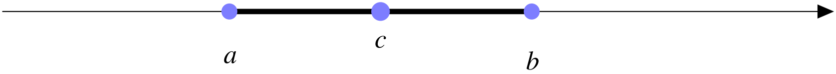

It admits a midpoint and a radius.

Definition

The **midpoint** *of I is the real number $c = \dfrac{1}{2}(b + a)$.*

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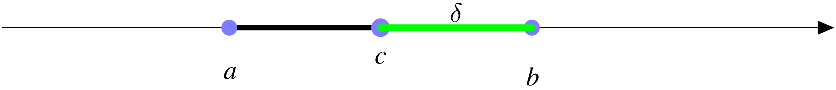

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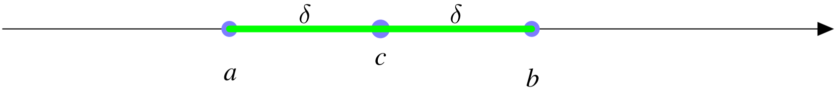
Definition

The **midpoint** *of I is the real number* $c = \frac{1}{2}(b + a)$.

The **radius** *of I is the real number* $\delta = \frac{1}{2}(b - a)$.

Intervals and absolute value

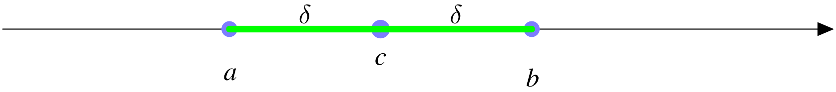
Geometric interpretation



The *distance* from any point x in the interval to the *midpoint* is less than or equal to the *radius.*

Intervals and absolute value

Geometric interpretation



Property

$$(a,b) = \{\, x \in \mathbb{R} \mid a < x < b \} = \{\, x \in \mathbb{R} \mid |x - c| < \delta \}\,.$$
$$[a,b] = \{\, x \in \mathbb{R} \mid a \leq x \leq b \} = \{\, x \in \mathbb{R} \mid |x - c| \leq \delta \}\,.$$
$$(-\infty, a) \cup (b, \infty) = \{\, x \in \mathbb{R} \mid x < a \text{ or } b < x \} = \{\, x \in \mathbb{R} \mid |x - c| > \delta \}\,.$$

Remark

$a = c - \delta$ and $b = c + \delta$.

Inequality with absolute value

Example

Solve $1)|x-8| \leq 2.$

Solution

Inequality with absolute value

Example

Solve $1)|x-8| \leq 2.$

Solution

The midpoint $c = 8$ *and the radius* $\delta = 2$. *Thus,* $a = c - \delta = 6$ *and* $b = c + \delta = 10.$

Inequality with absolute value

Example

Solve $1)|x-8| \leq 2.$

Solution

The midpoint $c=8$ *and the radius* $\delta=2$. *Thus,* $a=c-\delta=6$ *and* $b=c+\delta=10.$

Since the inequality is $\leq$, *the solution set is the interval* $I=[6,10]$.

Inequality with absolute value

Example

Solve $2)|2x+4| < 2.$

Solution

Let rewrite $|2x+4| < 2$ *in the standard form* $|x-c| < \delta.$

$|2x+4| < 2$ *implies* $|2(x+2)| < 2.$

Because $|2(x+2)| = 2|x+2|$, *we have* $|x+2| < 1.$

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The midpoint $c = -2$ *and the radius* $\delta = 1$ *so that* $a = c - \delta = -3$ *and* $b = c + \delta = -1$.

Inequality with absolute value

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Let rewrite $|2x+4| < 2$ *in the standard form* $|x-c| < \delta.$
$|2x+4| < 2$ *implies* $|2(x+2)| < 2.$
Because $|2(x+2)| = 2|x+2|$, *we have* $|x+2| < 1.$
The midpoint $c = -2$ *and the radius* $\delta = 1$ *so that* $a = c - \delta = -3$ *and* $b = c + \delta = -1.$
Since the inequality is $<$ *the solution set is the interval* $I = (-3, -1).$

Inequality with absolute value

Example

Solve $3)|3-2x|>7.$

Solution

$|3-2x|>7$ *implies* $|-2(x-3/2)|>7$ *which implies* $|x-3/2|>7/2.$

Inequality with absolute value

Example

Solve $3)|3-2x|>7$.

Solution

$|3-2x|>7$ *implies* $|-2(x-3/2)|>7$ *which implies* $|x-3/2|>7/2$.
We have $c=3/2,\ \delta=7/2,\ a=c-\delta=-2$ *and* $b=c+\delta=5$.

Inequality with absolute value

Example

Solve $3)|3-2x| > 7$.

Solution

$|3-2x| > 7$ *implies* $|-2(x-3/2)| > 7$ *which implies* $|x-3/2| > 7/2$.
We have $c = 3/2,\ \delta = 7/2,\ a = c-\delta = -2$ *and* $b = c+\delta = 5$.
Since the inequality is $>$ *the solution set is the interval*
$I = (-\infty, -2) \cup (5, +\infty)$.

Exercise

Solve: $1)|4x-3|<4.\quad 2)|-2x-10|>6.\quad 3)|3x+1|<$
$5\quad 3)|3-2x|>1\quad 5)\left|x^2-2x+3\right|<-3.$

Exercise

Find the intersection $I_1 \cap I_2$*, the union* $I_1 \cup I_2$*, and the set difference* $I_1 - I_2$ *of* I_1 *and* I_2*.*

1) $I_1=[-2,4]$ *and* $I_2=[0,1]$. *2)* $I_1=[1,2]$ *and* $I_2=[3,5]$.
3) $I_1=[-2,4)$ *and* $I_2=(0,1)$. *4)* $I_1=(-3,-1)$ *and* $I_2=(-2,0)$.

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Solve: $1)|4x-3|<4.\quad 2)|-2x-10|>6.\quad 3)|3x+1|<5\quad 3)|3-2x|>1\quad 5)\left|x^2-2x+3\right|<-3.$

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Solution

$1)\ I_1\cap I_2=I_2,\ I_1\cup I_2=I_1$ *and* $I_1-I_2=[-2,0)\cup(1,4]$.
$2) I_1\cap I_2=\emptyset,\ I_1\cup I_2=[1,2]\cup[3,5]$ *and* $I_1-I_2=I_1$.
$4) I_1\cap I_2=(-2,-1),\ I_1\cup I_2=(-3,0)$ *and* $I_1-I_2=(-3,-2]$.

δ-neighbourhood

Remark

Given $x_0 \in \mathbb{R}$ and $\delta > 0$,

- *the open interval defined by $|x - x_0| < \delta$ is called a (**open**) δ-**neighbourhood** of x_0.*
- *the closed interval defined by $|x - x_0| \leq \delta$ is called a **closed** δ-**neighbourhood** of x_0.*

Property

Finite union and intersection of neighbourhoods of x_0 is an neighbourhood of x_0.

Outline

Limit point

Let S be a subset of $\mathbb{R}$ and x_0 an arbitrary real number.

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Definition

x_0 *is a* **limit point** *(or* **cluster point** *or* **accumulation point***) of* S *if:*

1. *every* δ*-neighbourhood of* x_0 *contains at least one point of* S *different from* x_0 *itself.*
2. x_0 *does not itself have to be an element of* S.

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2. x_0 *does not itself have to be an element of* S.

Example

1. *Elements of the interval* $[1, 2]$ *are limit points of* $S = \{0\} \cup [1, 2)$.
2. *Only* 0 *is a limit point of* $S = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$.

Let S be a subset of $\mathbb{R}$ and x_0 an arbitrary real number.

Theorem (Bolzano-Weierstrass theorem)

Every bounded infinite subset of $\mathbb{R}$ (or $\mathbb{R}^n$) has at least one limit point.

Let S be a subset of $\mathbb{R}$ and x_0 an arbitrary real number.

Theorem (Bolzano-Weierstrass theorem)

Every bounded infinite subset of $\mathbb{R}$ (or $\mathbb{R}^n$) has at least one limit point.

Example

1) Show that $\{1, 1/2, \ldots, 1/n, \ldots\}$ is bounded and determine its greatest lower bound and least upper bound.
Identity some limit points and how does it illustrate the Bolzano-Weierstrass theorem.
2) Consider the set $\{1, 1.1, 0.9, 1.01, .99, 1.001, .999, \ldots\}$. Is the set bounded?
Does it have a greatest lower bound and least upper bound?
Determine its limits points and determine if it is closed.