Indices & Logarithms Explained with Worked Examples

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INDICES

&

LOGARITHMS

EXPLAINED WITH

WORKED EXAMPLES

By

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PREFACE

After a successful dissemination of the previous books, which are available online, in your hands is another book for potential mathematicians, scientists and engineers. This current work – *Indices and Logarithm Explained with Worked Examples* – offers 250+ worked examples complemented with a comprehensive background on this topic. There are two main parts in this book; one gives a broad explanation of the topic and the other presents the worked examples. The questions used in this work are similar to those in mathematics and engineering textbooks designed for A-level, college and university students. Advanced learners, particularly those returning to study after a break from the academic environment, will also find this helpful. Additionally, it could be used as a reference guide by teachers, tutors, and other teaching staffs during classes and for assessment (home works and examinations).

Finally, many thanks to my colleagues who have offered suggestions and comments, especially Khadijah Olaniyan (Loughborough University, UK), Dr. Abdul Lateef Balogun (Universiti Teknologi PETRONAS, Malaysia), Sakiru Adeleke (IIUM, Malaysia), Teslim Abdul-Kareem (formerly with University of Dundee, UK), Dr. Tijani Abdul-Aziz Apalara (KFUPM, Saudi Arabia), Jimoh JR Ibrahim (University of Brighton, UK) and G. A. Ibraheem (University of Brighton, UK).

Pertinent suggestions, feedbacks and queries are highly welcome and can be directed to the author at the address below.

Coming soon in this series are:

- Worked Examples on Mechanics
- Worked Examples on Circuit Theorems
- Worked Examples on Digital Electronics

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CONTENTS

FUNDAMENTALS OF INDICES AND LOGARITHMS

1. Introduction

Indices (sing. index) together with logarithms[1] are central to many scientific and engineering processes. Most of the equations we encounter, either in simplified or complex forms, are based on them. In fact they are part of our daily activities, e.g. finance, economics and natural phenomenon, although we use them unknowingly. So here we will take a look at how to simplify exponential and logarithmic expressions and how to solve their equations using certain rules that will be discussed shortly. However, graphs of their functions will not be covered here as these will be appropriately dealt with in another book. Readers are reminded that this part of the book focuses on the background regarding the topic and therefore few examples will be provided. For comprehensive and varied examples, do refer to the Worked Examples part (pp. 19 -71).

2. What is Indices?

Well, indices is when a number is expressed in the form x^y where x is called the base and **y** the index. The index, i.e. y, could also be referred to as a power or exponent; they all essentially mean the same. Nonetheless you probably would have found out that one of the terms (index, exponent or power) is used more than the others. In general, power is the most frequently used as such, x^y is read as '**x raised to the power of y**', '**x raised to the power y**' or simply '**x to the power y**'. What does this imply? It means that x has been multiplied by itself y times. For example, if $x = 5$ and $y = 3$ then one can write $\mathbf{5^3}$ read as '**five to the power three**' or '**five cubed**'. This implies $\mathbf{5 \times 5 \times 5}$, which equals $\mathbf{125}$.

So what do you think about $\mathbf{10 \times 10 \times 10 \times 10}$? It is **10 000** in value but it is written in index form as $\mathbf{10^4}$. Now let us work in the reverse order and find out what $\mathbf{10^9}$ is. This can be written as a multiple of tens, i.e. $\mathbf{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}$ and it is the same as **1 000 000 000** (a billion). From this example, it is obvious that we need indices to express numbers such as in the example not only because it saves space and time but more importantly it is sometimes easy to remember and work with. So $\mathbf{10^9}$ is a better way of writing a cumbersome **1 000 000 000**. Calculation involving indices can be carried out on a scientific calculator, commonly via a button marked x^y or something similar.

[1] Precisely, we mean indicial and logarithmic functions.

The next question that might come to mind is whether any number can be written in index form. The answer is yes. Any rational number can be expressed in this form just like any integer can be written in the form $\frac{x}{y}$ where $y = 1$. For instance, 10 can be expressed in both index and fractional forms as 10^1 and $\frac{10}{1}$ respectively, but this 1 is generally omitted. This is the same for any other number.

3. Laws of Indices

Undoubtedly one would need to carry out calculations in index form because it is a useful and compact way to express numbers. There are certain laws[2] that govern these operations and they will be discussed in this section.

3.1. Fundamental laws

There are three fundamental laws

Law (1) Multiplication Law

This law can be written as

$$\boxed{x^n \times x^m = x^{n+m}}$$

For example, $x^3 \times x^2 = x^{3+2} = x^5$. This can alternatively be expressed as

$$\begin{aligned} x^3 \times x^2 &= (x \times x \times x) \times (x \times x) \\ &= x \times x \times x \times x \times x \\ &= x^5 \end{aligned}$$

The following should be noted about this rule:

i) The terms must have the same base otherwise the law cannot be used. For example $5^3 \times 6^2 \neq 5^{3+2} \neq 6^{3+2}$ because 5 and 6 are not like base.

ii) The sign between the terms must be a multiplication and not an addition. In other words, $2^3 \times 2^2 = 2^{3+2} = 2^5$ but $2^3 + 2^2 \neq 2^{3+2}$. This can easily be proven because $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ and $2^3 + 2^2 = 8 + 4 = 12$.

iii) The power can either be same or different.

iv) The terms can be two or more.

Law (2) Quotient (or Division) Law

This law can be written as

[2] They are also referred to as theorems, laws and properties. I will use them interchangeably in this work.

$$\boxed{x^n \div x^m = x^{n-m}} \qquad \text{or} \qquad \boxed{\frac{x^n}{x^m} = x^{n-m}}$$

For example, $x^7 \div x^4 = x^{7-4} = x^3$. Again, this can alternatively be expressed as

$$\begin{aligned}\frac{x^7}{x^4} &= \frac{x \times x \times x \times x \times x \times x \times x}{x \times x \times x \times x} \\ &= x \times x \times x \\ &= x^3\end{aligned}$$

The following should be noted about this rule:

i) The terms must have the same base otherwise the law cannot be used. For example $6^5 \div 5^3 \neq 6^{5-3} \neq 5^{5-3}$.
ii) The sign between the terms must be a division and not a subtraction. In other words, $2^5 \div 2^3 = 2^{5-3} = 2^2$ but $2^5 - 2^3 \neq 2^{5-3}$. Again as previously performed $2^5 \div 2^3 = 32 \div 8 = 4$ and $2^5 - 2^3 = 32 - 8 = 24$ and $2^5 = 2 \times 2 = 4$.
iii) The power can either be the same or different.
iv) The base is any real number excluding zero. i.e. $x \in \mathbb{R} - \{0\}$.
v) The terms can be two or more.

Law (3) Power Law

This law states that

$$\boxed{(x^m)^n = x^{mn}}$$

For example, $(x^4)^2 = x^{4\times 2} = x^8$. Alternatively, this can be shown using Law 1 as

$$\begin{aligned}(x^4)^2 &= (x \times x \times x \times x)^2 \\ &= [x \times x \times x \times x] \times [x \times x \times x \times x] \\ &= x \times x \times x \times x \times x \times x \times x \times x \\ &= x^8\end{aligned}$$

3.2. Special or derived laws

Other laws of indices include

Law (4) Zero Power Law

This law can be written as

$$\boxed{x^0 = 1}$$

It is simply put as '**anything**'[3] to the power of zero is 1. Although this rule might look odd, it is logical and well established. This can be derived from Law 2. Given that

$$x^n \div x^m = x^{n-m}$$

If ***m*=*n***, then

$$x^m \div x^m = x^{m-m}$$
$$= x^0$$

But

$$x^m \div x^m = \frac{x^m}{x^m}$$
$$= 1$$
$$\therefore x^0 = 1$$

Law (5) Negative Power Law

This law can be written as

$$x^{-n} = \frac{1}{x^n}$$

Provided that $x \neq 0$, it implies that a number to the power of a negative index is the reciprocal of that number to the same but with a positive index. Sometimes one might need to carry out some operations that require a change from a positive index to a negative index. Let us account for this and re-state the rule as '**whenever the reciprocal of a number with an index is taken, the sign of the index changes**'. For instance,

(a) $$x^{-3} = \frac{1}{x^3}$$

or

(b) $$\frac{1}{x^3} = x^{-3}$$

or

(c) $$\frac{1}{x^{-3}} = x^3$$

[3] This excludes zero.

In each of the above three cases, we have taken the reciprocal of the left-hand side to obtain the corresponding right-hand side. So in (a) and (c) the sign of the index changed from negative to positive whilst in (b) it changed from positive three (+3) to negative three (-3).

This negative index rule can be derived from Law 2 (division rule) and Law 4 (power of zero law)[4] as follows:

$$x^{-n} = 1 \times x^{-n}$$

Using Laws 4,

$$= x^{0} \times x^{-n}$$

Using Laws 1,

$$\therefore x^{-n} = x^{0-n} \qquad ----(\text{i})$$

Using Law 2, the right-hand side of equation (i) can be written as

$$x^{0-n} = x^{0} \div x^{n}$$

this implies that

$$x^{0-n} = \frac{x^{0}}{x^{n}}$$

Using Laws 4,

$$\therefore x^{-n} = \frac{1}{x^{n}} \qquad ----(\text{ii})$$

Equating equations (i) and (ii), therefore

$$x^{-n} = \frac{1}{x^{n}}$$

Law (6) Fractional Power (or Root) Law

This law can be written as

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

This simply means that '**anything**' raised to the power of **one – nth** is equal to the **nth** root of '**the same thing**'[5]. If it is not clear, this will shortly be put this into context. But

[4] Or simply from the division rule since Law 4 is also a special case of Law 2.

remember that a power is the inverse of a root just like a whole number is the opposite of a fraction. For example, $\mathbf{9^{\frac{1}{2}}} = \sqrt[2]{\mathbf{9}}$ and $\mathbf{8^{\frac{1}{3}}} = \sqrt[3]{\mathbf{8}}$. We know that the answers are **3** and **2** respectively. One can however obtain the same answers using Laws 2 and 3. How? Here we go:

$$9^{\frac{1}{2}} = (9)^{\frac{1}{2}}$$

using Law 1

$$= (3 \times 3)^{\frac{1}{2}}$$
$$= (3^2)^{\frac{1}{2}}$$

using Law 3

$$= 3^{\left(2\times\frac{1}{2}\right)}$$
$$= 3$$

Similarly,

$$8^{\frac{1}{3}} = (8)^{\frac{1}{3}}$$

using Law 1

$$= (2 \times 2 \times 2)^{\frac{1}{3}}$$
$$= (2^3)^{\frac{1}{3}}$$

using Law 3

$$= 2^{\left(3\times\frac{1}{3}\right)}$$
$$= 2$$

What about a situation when the numerator of the fractional index is not 1? Yes, there is a rule for this case which states that

$$\boldsymbol{x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^{m}}$$

or

[5] Note that $\sqrt{x} = \sqrt[2]{x}$ but 2 is generally omitted. It is read as '**square root of x**'. Similarly $\sqrt[3]{x}$ is the '**cube root of x**'. While $\sqrt[4]{x}$ and $\sqrt[5]{x}$ the '**fourth root of x**' and '**fifth root of x**' respectively. So what are $\sqrt[6]{x}$ and $\sqrt[100]{x}$?

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

This is a combination of Laws 3 and 4 because

$$\begin{aligned} x^{\frac{m}{n}} &= x^{\left(\frac{1}{n}\times m\right)} \\ &= \left(x^{\frac{1}{n}}\right)^m \\ &= \left(\sqrt[n]{x}\right)^m \end{aligned}$$

For example, $32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^2 = 2^2 = 4$. You can double-check this with a calculator or move to the Worked Examples section for much more.

Law (7) Same Power Law

So far we have introduced rules for dealing with terms that share a common base. We will now look at rules to be used when the bases of the terms are not the same. In general, the operations are carried out as you would normally do with one exception. This exclusion is that when the index of the terms is the same then the following rules should be applied:

$$\begin{aligned} x^n \times y^n &= (x \times y)^n \\ &= (xy)^n \end{aligned}$$

or

$$x^n \div y^n = (x \div y)^n \quad \text{i.e.} \quad \frac{x^n}{y^n} = \left[\frac{x}{y}\right]^n$$

Also,

$$\left(x^a y^b\right)^n = x^{an} y^{bn} \quad \text{and} \quad \left(\frac{x^a}{y^b}\right)^n = \frac{x^{an}}{y^{bn}}$$

The rules above are very simple and can easily be verified. Let us illustrate this using these two examples:

$$\begin{aligned} 4^2 \times 3^2 &= (4 \times 3)^2 \\ &= (12)^2 \\ &= 144 \end{aligned}$$

$$\boxed{\mathbf{4^2 \times 3^2 = 16 \times 9 = 144}}$$

and

$$15^2 \div 5^2 = (15 \div 5)^2$$
$$= (3)^2$$
$$= 9$$
$$\boxed{\mathbf{15^2 \div 5^2 = 225 \div 25 = 9}}$$

Conversely, it can be said that if there are terms in a bracket which is raised to the power of $\boldsymbol{n}$, then the power applies to each term in the bracket provided the operation between them is either multiplication or division. Consequently,

$$\boldsymbol{(x+y)^n \neq x^n + y^n}$$

and

$$\boldsymbol{(x-y)^n \neq x^n - y^n}$$

Finally, have you wondered if there is a rule relating to powers of one (1), i.e. $\boldsymbol{x^1}$? I guess not. But whatever the case, there is no special rule for numbers or expressions raised to power of **1** because this is the default state of all numbers. To save time and space, it is an unspoken rule as previously mentioned. Remember that $\boldsymbol{x}$, $\boldsymbol{y}$ means $\boldsymbol{1x}$ and $\boldsymbol{1y}$ respectively but **1** is always left unwritten.

Before proceeding to logarithms, you may wish to go through the Worked Examples on indices first. These are contained in sections 1- 4 (pp. 19 – 38).

4. What is Logarithm?

Logarithm is a derived term from two Greek words, namely: ***logos*** (expression) and ***arithmos*** (number) (Singh, 2011). Thus, logarithm is a technique of expressing numbers. In fact, it is a system of evaluating multiplication, division, powers and roots by appropriately converting them to addition and subtraction. This concept is primarily attributed to a Scotsman from Edinburgh, John Napier (1550 – 1617) (Bird, 2010) who developed it.

Technically, the logarithm[6] of a number to a given base is the value of the power to which the base must be raised in order to produce the number. Let x (for $x > 0$) and y be the number[7] and its associated base respectively and c equals to '**the logarithm to base y of x**' or '**the logarithm of x to base y**', then we can write this as

$$\boldsymbol{log_y\, x = c}$$

[6] This is commonly shortened to or written as **log** and sometimes as **lg.**

[7] $\boldsymbol{y}$ must be a positive real number excluding **1**, i.e. $\boldsymbol{y} \in \mathbb{R}^+ - \{\mathbf{1}\}$. This is because **1** raised to the power of anything is **1** and so $\mathbf{log_1}\, \boldsymbol{x}$ will only be valid for $\boldsymbol{x} = \mathbf{1}$ since $\mathbf{1}^{\boldsymbol{c}} = \mathbf{1}$ for $\boldsymbol{c} \in \mathbb{R}$.

For example, the logarithm of 100 to base 10 is 2 because, if the base 10 is raised to the power of 2 we will get the number (100). In other words, $\mathbf{100 = 10^2}$. It is evident that indices and logarithm are related and one can change from one form to the other, i.e. one is the inverse of the other. This relationship can be written as

$$\boldsymbol{log_y x = c \quad \Leftrightarrow \quad y^c = x}$$

Logarithms are classified according to the value of their base. Essentially, there are two (or three) types.

a) **Common Logarithm**: This is a logarithm to the base of 10, i.e. $\mathbf{log_{10}}\, \boldsymbol{N}$. In general, when the base is 10, it is usually omitted. In other words, $\log_{10} N$ is simply written as $\mathbf{log}\, \boldsymbol{N}$. Common logarithm is also called Briggsian named after Henry Briggs.
b) **Natural Logarithm**: This is logarithm to the base of an irrational number denoted as $\boldsymbol{e}$, where $e = 2.718\,281 \ldots$ or $e = 2.7183$ to 4 decimal places. Why '**natural**'? It is possible that this logarithm is classified as natural due to the behaviour of certain natural phenomena (e.g. radioactive decay, charging-discharging a capacitor, frequency response, biological functions, interest rate, etc.) being dependent on the functions of e. Natural logarithm is also called hyperbolic or Napierian logarithm, named after its inventor John Napier. Furthermore, the logarithmic ratio unit, ***neper*** (abbreviated as ***nep***), is also used in his honour[8]. Usually, natural logarithm is written as $\mathbf{ln}\, \boldsymbol{N}$ (read as 'lin N') instead of $\mathbf{log}_{\boldsymbol{e}}\, \boldsymbol{N}$ as one might have expected.
c) The third category (if we may say) is any other logarithm with a base other than **10** and $\boldsymbol{e}$.

5. Laws of Logarithm

Like indices, there are certain laws governing the operation of logarithms and these will be discussed under the following headings.

5.1. Fundamental laws

Essentially, there are three main laws of logarithms.

Law (1) Addition-Product Law

This rule can be written as

$$\boldsymbol{log_x M + log_x N = log_x (MN)}$$

[8] This is when the base is $\boldsymbol{e}$, often with current ratio. However, when the base is 10, usually for power ratio (attenuation or amplification), the unit of logarithm ratio is decibels ($\boldsymbol{dB}$).

In other words, the sum of logs of numbers to the same base is equal to the log of their products and vice versa. The following should be noted about the rule:

i) The logs must have the same base otherwise the law cannot be used. For example $\log_5 2 + \log_3 8 \neq \log_5 16 \neq \log_3 16$.
ii) The sign between the terms must be addition and not multiplication. In other words, $\log_2 5 + \log_2 10 = \log_2 50$ but $\log_2 5 \times \log_2 10 \neq \log_2 50$. This can simply be proven as with the indices.

Law (2) Subtraction-Quotient Law

This rule can be written as

$$\boxed{\log_x M - \log_x N = \log_x \left(\frac{M}{N}\right)}$$

The following should be noted about the rule:

i) The logs must have the same base otherwise the law cannot be used. For example $\log_5 32 - \log_3 8 \neq \log_5 4 \neq \log_3 4$.
ii) The coefficient of the log must be 1.
iii) The sign between the terms must be subtraction and not division. In other words, $\log_2 50 - \log_2 10 = \log_2 \left(\frac{50}{10}\right) = \log_2 5$ but $\log_2 50 \div \log_2 10 \neq \log_2 5$. Again, this can be proven.

Law (3) Power Law

This rule can be written as

$$\boxed{\log_x M^N = N\log_x M}$$

There are three special formulae or properties resulting from the above power law, namely:

$$\boxed{\log_M M^N = N} \; , \; \boxed{\log_{(M)^N} M = \frac{1}{N}} \text{ and } \boxed{\log_{(M)^x} N^x = \log_M N}$$

5.2. Special or derived laws

Other laws of logarithms include

Law (4) Unity Law (or Log of Unity Law)

This rule can be written as

$$\log_x 1 = 0$$

This rule states that the logarithm of unity (1) to any base is zero. Yes, it is true because the power that any number can be raised to produce one is actually a zero. This is in accordance with the power of zero law in indices. From this it can be said that

$$log_y\, x = \begin{cases} +ve & for\ x > 1 \\ -ve & for\ 0 < x < 1 \end{cases}$$

One may be prompted to ask about the logarithm of zero. Well, it has been said that logarithms are only defined (or valid) for numbers greater than zero. This is because given that

$$log_y\, x = c \quad \Leftrightarrow \quad y^c = x$$

a) When $\boldsymbol{x} = \mathbf{0}$, we have

$$log_y\, 0 = c \quad \Leftrightarrow \quad y^c = 0$$

There is no value for $\boldsymbol{c}$ to satisfy the above condition. Let us put this in practice by taking $\boldsymbol{y} = \mathbf{2}$ and ask ourselves if there is a power which 2 can be raised to produce zero? Could this be $\mathbf{1}$? No, because $\mathbf{2^1 = 2}$. Did you say power zero? No, we just talked about a rule of power zero, so $2^0 = 1$. You may start wondering whether a lower power would work. What is less than a zero? The answer is a negative number. In this case, let us take -0.5 and evaluate,

$$\begin{aligned} 2^{-0.5} &= 2^{-\frac{1}{2}} \\ &= \frac{1}{2^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

At least by now we know the answer is not zero. Actually the answer is 0.7071. What else do you have in mind, try it and let us know if you have obtained a zero.

This said, you may want to try negative infinity, i.e. $-\infty$. So let us work this out.

$$\begin{aligned} 2^{-\infty} &= \frac{1}{2^{\infty}} \\ &\cong 0 \end{aligned}$$

The above is simply an approximation! Therefore, it can be said that the limiting value of logarithm of zero to any base is negative infinity. That is

$$\lim_{x \to 0^+} \log_y x = -\infty$$

b) When $x < 0$, i.e. x is a negative, we have

$$log_y(-ve) = c \quad \leftrightarrow \quad y^c = -ve$$

Remember y is a positive number; now what is the positive number[9] that when it is raised to a certain power (positive, negative, whole number or fraction) the answer will be a negative number. Again, there is none.

Law (5) (Logarithm to the) Same Base Law

This rule can be written as

$$\log_x x = 1$$

The logarithm of any number to the same base is 1. This is because $x^1 = x$.

Law (6) Change of Base Law

This law can be written as

$$\log_N M = \frac{\log_x M}{\log_x N}$$

The logarithm of a number to a certain base is the same as the logarithm of the number divided by the logarithm of the base such that both are given a new but same base. For example,

$$\log_5 125 = \frac{\log_{10} 125}{\log_{10} 5}$$
$$= \frac{\log_8 125}{\log_8 5}$$
$$= \frac{\log_9 125}{\log_9 5}$$

You can check this out with a calculator; they will all be equal.

There is a special application of this rule when one need to multiply two or more logs together. This states that

[9] Here we mean only real numbers because a complex number such as $2j$ when squared gives a negative number. In other words $(2j)^2 = -4$. For further details on complex numbers, refer to 'Complex Numbers Explained with Worked Examples' by the same author, which is available online.

$$(\log_x M)(\log_y N) = (\log_x N)(\log_y M)$$

This is because

$$(\log_x M)(\log_y N) = \frac{\log M}{\log x} \times \frac{\log N}{\log y}$$
$$= \frac{\log N}{\log x} \times \frac{\log M}{\log y}$$
$$= (\log_x N)\left(\log_y M\right)$$

For instance

$$\begin{aligned}\log_5 100 \times \log_{10} 25 &= \log_5 25 \times \log_{10} 100\\ &= 2 \times 2\\ &= 4\end{aligned}$$

Can one imagine evaluating the above logs separately before multiplying? Without a calculating aid, this will of course prove difficult if not impossible. Using a calculator, the following was obtained.

$$\begin{aligned}\log_5 100 \times \log_{10} 25 &= 2.8614 \times 1.3979\\ &= 4\end{aligned}$$

Using this rule, we can also show that

$$\log_N M = \frac{1}{\log_M N}$$

6. Indicial-Logarithmic Equation

Indicial equations are equations involving powers, where either the base or the exponent is the unknown variable to be found. Indicial equations can generally be solved using any known method of solving polynomials – linear or otherwise – in conjunction with the laws of indices and logarithms discussed above.

The method of solving an indicial equation will be determined by the nature of the equation, which can be broadly grouped into two.

a) **When the unknown is the base**: this is often solved by simplifying the expressions, and a suitable method of solving polynomials is subsequently applied. Kindly refer to section 10 (pp 47 – 51) for examples.

b) **When the unknown is the index**: in this case, if the base of the two sides of the equation can be made the same then equate their index otherwise the log of both sides must be taken (sections 11 and 12, pp 51 – 58). In the latter case, one need to apply the one-to-one property of logarithm which states that if

$$\log_x M = \log_x N\,, \quad then \quad M = N$$

In both aforementioned instances, there must be only a single term on each side of the equation before the index or base can be equated. Refer to the Worked Examples part of this book for further clarification.

7. Scientific Notation

Scientific notation (also known as '**standard form**') is a way of representing a number $\boldsymbol{N}$ such that

$$N = M \times 10^c \qquad where\ 1 \leq M < 10$$

8. Logarithm-Antilogarithm Table

8.1. Mathematical tables

Mathematical tables[10] are used to find the square (or square root), cube (or cube root), $\boldsymbol{n}$ power (or $\boldsymbol{nth}$ root) and logarithm (or antilogarithm[11]) of numbers. Sine, cosine and tangent of angles are some other functions that can be obtained using a mathematical table.

Since we are dealing with logarithm here, we will show how a log (logarithm) table can be used to perform calculations involving multiplications, divisions, powers and roots. It should be mentioned that a log table is used in conjunction with another table, called antilogarithm table. Antilog table, as it is commonly called, is the opposite of log table. The tables are meant for the common log, i.e. to base 10, but the procedure and principle is applicable to any base.

How is it possible to represent logs of numbers in a single table? Yes, it is doable correct to four significant figures. One of the techniques used to achieve this is explained below. Supposing we know that

[10] With the proliferation of calculating devices, it is not very common nowadays but it still remains a useful invention. Addition and subtraction of numbers is possible but could one imagine multiplication and division without a calculating device such as a calculator?

[11] Anti-logarithm is the opposite of logarithm to a certain base. For instance, if $\log_N M$ is x then the antilog of x to base N is M. In other words, the antilog of x is N^x. Therefore, the antilog (2) and antilog (3) to base 10 are $\mathbf{10^2 = 100}$ and $\mathbf{10^3 = 1\,000}$ respectively.

$$\log_{10} 2 = 0.3010$$

we can therefore find the value of $\log_{10} 20$ for example. This is accomplished using relevant laws of logarithms and the scientific notation. Hence, the above log can be resolved as

$$\begin{aligned}\log_{10} 20 &= \log_{10}(2 \times 10)\\ &= \log_{10} 2 + \log_{10} 10\\ &= 0.3010 + 1\\ &= 1.3010\end{aligned}$$

What about $\log_{10} 200$? Well, the same principle is applicable.

$$\begin{aligned}\log_{10} 200 &= \log_{10}(2 \times 100)\\ &= \log_{10} 2 + \log_{10} 10^2\\ &= 0.3010 + 2\\ &= 2.3010\end{aligned}$$

It can therefore be said that $\boldsymbol{2 \times 10^n,\ for\ n \in \mathbb{Z}}$ have the same log value except that they differ in their leading integer or digit. This also applies to any number. The above illustration shows that a cell in the table can be used to represent a group of numbers.

8.2. Components of a number

In general, given that $\boldsymbol{N}$ can be expressed in scientific form as $x \times 10^n$ and that $log_{10} x = a$, then $\boldsymbol{log\ N = log_{10}(x \times 10^n) = a + n}$. In this case, the fractional part $\boldsymbol{a}$ or $\boldsymbol{\log_{10} x}$ is called the **mantissa** and $\boldsymbol{n}$ the integer part is referred to as the **characteristic** of $log\ N$. Whilst mantissa is always positive, the characteristic can either be positive or negative. When the characteristic is negative it is written as $\boldsymbol{\bar{n}}$ (read as '**bar n**'). For example, if a number has a mantissa and characteristic of 0.3010 and -3 respectively, it will be written as $\bar{3}.3010$ and read as '**bar three point three, zero, one, zero**'. You however need to remember that $\bar{3}.3010$ is equivalent to $\boldsymbol{-3 + 0.3010 = -2.6990}$. It should be further noted that the values in the log tables are for mantissa and the characteristics are simply multipliers and are obtained from the number itself. This is easy when the latter is expressed in scientific form.

8.3. When to use the log-antilog table

The log tables are typically used for multiplication and divisions rather than for addition and subtraction of numbers. With multiplication, the logs of the multiplier and multiplicand are added and the antilog of the result is taken to obtain the answer. On the other hand, when dividing numbers, the log of the divisor is subtracted from the log of the dividend and followed by taking the antilog of the resulting value.

We can also evaluate powers and roots of numbers using log-antilog tables. Remember that power is the number of times a number is multiplied so in this case, we multiple the log of that number by the index. Similarly, a root which is the inverse of power is computed by division. In each case one need to take the antilog of the result to obtain the solution.

8.4. How to use the log-antilog table

Looking at log-antilog tables, there are three distinct parts:

(a) **The first column**: This is numbered from 10 to 99 and 0.00 to 0.99 for log-table and antilog-table respectively. They are used for the first two digits of the number whose log or antilog[12] is to be found.

(b) **A set of 10-column**: These ten columns are numbered from 0 to 9 and are used for the third digit. For example, the log of 234 is placed on the row numbered '23' but under the fifth column, i.e. the column numbered '4'.

(c) **Mean difference**: This is a set of 9-column, known as the '**mean difference**', which can be found written above the columns. They are numbered from 1 to 9 and used for the fourth number. However, the number found here is added to that obtained in (b) above. What about a fifth number? Since the table is a 'four-figure' table, it implies that one can only evaluate up to four digits. We therefore need to round off numbers to four significant figures before finding their logs in the log tables. Also, if the number whose log is to be determined is a two-digit or three-digit number, assume there is / are one and two zeros respectively to the right of the number. 2-digit or 3-digit numbers have no value in the mean difference section.

In summary, to use the log table do the following:

1) Determine the **characteristic** of the number as described above.
2) Locate the row corresponding to the first two digits in the log table and move across the table until you get to the column number corresponding to the third digit of the number whose log you are looking for. Note down the value in this cell of the table.

[12] In the case of antilog table, the first two digits are those immediate to the left of the decimal point. The integer to the right of the decimal is a multiplier.

3) Stay on the same row and move until you get to the section under the **mean difference**. Now read the value in the cell of the column number (in the mean difference section) corresponding to the fourth digit of the number whose log is to be found.
4) Add the numbers obtained in 2 and 3 together. This gives the mantissa of the log of the number you are looking for.
5) The log of the number is now the **characteristic** obtained in 1 dot(.) the **mantissa** obtained in 4. For instance, given a number whose characteristic and mantissa are 3 and 4567 respectively, then the log of the number is '**3.4567**'.
6) Repeat 1 to 5 for each number.
7) If two numbers are multiplied, add the logs but subtract the logs if there is a division between them. For $\boldsymbol{nth}$ root and $\boldsymbol{nth}$ power, divide and multiply the log respectively by $\boldsymbol{n}$.
8) The resulting value from 7 above should contain two parts: the integer and decimal part.
9) Repeat steps 2 to 5 on the decimal part (mantissa) obtained in 7. You should however use antilog-table in this case.
10) Know that the integer part from 7 is simply a multiplier. Thus, provided that the number obtained from the antilog table in 9 is written with a dot after the first digit, we can therefore write the final answer as $\boldsymbol{x} \times \mathbf{10}^{\boldsymbol{n}}$, where $\boldsymbol{x}$ and $\boldsymbol{n}$ are the answer from the antilog table and the integer from 7 respectively.

8.5. Illustration of the use of log-antilog table

To further clarify this, the process of using log-antilog tables will be illustrated by multiplying $\mathbf{401.5}$ by $\mathbf{2.67}$.

$$\mathbf{401.5 \times 2.67}$$

	Number	Log	What to do
Step 1.	401.5	2.6036	Remember that $\mathbf{401.5}$ can be written in standard form as $\mathbf{4.015} \times \mathbf{10^2}$ so its **characteristic** is 2. Write this down and insert a dot after it. In the log table, we now need to locate 40 in the first column. Since the third digit is 1 (one), move across this row until you reach the column labeled '1' and read off the number in the cell formed by the intersection of 'row 40' and 'column 1'. This is 6031. The next step is to stay on the same row and move

			until you get to the column labeled '5' in the '**mean difference**' column section. This is because the fourth digit in the number whose log we are looking for is 5. Coincidentally, the number in the mean difference section is 5, though this is not always the case. Finally, add the two results as $\mathbf{6031 + 5 = 6036}$. This implies that the mantissa is 0.6036. Therefore, the log of 401.5 is 2.6036, i.e. **characteristic** dot (.) **characteristic**.
Step 2.	2.67	0.4265	Do the same as described above for 2.67. However, since there is no fourth number here, the value in the '**mean difference**' section of the table is taken to be zero. Thus, the log of 2.67 is 0.4265. Remember also that $\mathbf{2.67 = 2.67 \times 10^0}$ and that is why its characteristic is zero.
Step 3.	401.5 $\times$ 2.67	2.6036 $+$ 0.4265 $=$ 3.0301	Since the two numbers have been multiplied, one should add their logs.
Step 4.	Antilog	1072	We want the actual result and not log so the antilog of the number in step 3 should be obtained. This number contains two parts: the integer 3 and the decimal .0301. Find the antilog of .0301 in the antilog table the same way we found the logs of the two numbers. This should give you 1072.
Step 5.	Answer	1072	The final answer is $\mathbf{1.072 \times 10^3 = 1072}$. Note that although a calculator would give a more accurate answer as 1072.005, which is 1072 correct to 4 significant figures.

END OF FUNDAMENTALS OF INDICES AND LOGARITHMS

AND

BEGINNING OF WORKED EXAMPLES

WORKED EXAMPLES

Section 1: Basic Applications of Laws of Indices (Numbers Only)

INTRODUCTION

In this section of the Worked Examples, questions are based on the laws of indices involving numbers and are structured such that one, two or more laws can be used at a time. The law(s) to be used is stated in the **Hint box**. Sometimes different combination of the laws can be employed to solve a single question but only one approach is shown. Nonetheless, the author provides hints and/or a prevailing alternative method to a question. No calculators will be required in this and many of the subsequent sections. However, one can be used to confirm answers.

1) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 1 of indices exclusively.

$$\boxed{\boldsymbol{x^m \times x^n = x^{m+n}}}$$

(a) $10^2 \times 10^3$

Solution

$$\begin{aligned} 10^2 \times 10^3 &= 10^{2+3} \\ &= 10^5 \\ &= \mathbf{100\,000} \end{aligned}$$

(b) $2^6 \times 2^{-3}$

Solution

$$\begin{aligned} 2^6 \times 2^{-3} &= 2^{6+(-3)} \\ &= 2^3 \\ &= \mathbf{8} \end{aligned}$$

(c) $7^{1/2} \times 7^{3/2}$

Solution

$$\begin{aligned} 7^{1/2} \times 7^{3/2} &= 7^{(1/2 + 3/2)} \\ &= 7^2 \\ &= \mathbf{49} \end{aligned}$$

(d) $5^{3\frac{1}{3}} \times 5^{-\frac{1}{3}}$

Solution

$$\begin{aligned} 5^{3\frac{1}{3}} \times 5^{-\frac{1}{3}} &= 5^{3\frac{1}{3}+\left(-\frac{1}{3}\right)} \\ &= 5^3 \\ &= \mathbf{125} \end{aligned}$$

2) Without using a calculator, simplify the following.

Hint

In the following questions, we will be applying Law 2 of indices exclusively.

$$\boxed{\boldsymbol{x^m \div x^n = x^{m-n}}}$$

(a) $6^5 \div 6^3$

Solution

$$\begin{aligned} 6^5 \div 6^3 &= 6^{5-3} \\ &= 6^2 \\ &= \mathbf{36} \end{aligned}$$

(b) $11^{\frac{7}{2}} \div 11^{\frac{3}{2}}$

Solution

$$\begin{aligned} 11^{\frac{7}{2}} \div 11^{\frac{3}{2}} &= 11^{\left(\frac{7-3}{2}\right)} \\ &= 11^2 \\ &= \mathbf{121} \end{aligned}$$

(c) $4^{-2} \div 4^{-5}$

Solution

$$4^{-2} \div 4^{-5} = 4^{-2-(-5)}$$
$$= 4^{-2+5} = 4^3$$
$$= \mathbf{64}$$

(d) $13^{1/5} \div 13^{-4/5}$

Solution

$$13^{1/5} \div 13^{-4/5} = 13^{\frac{1}{5}-\left(-\frac{4}{5}\right)}$$
$$= 13^{\frac{1}{5}+\frac{4}{5}} = 13^{\frac{1+4}{5}}$$
$$= 13^1$$
$$= \mathbf{13}$$

3) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 3 of indices exclusively.

$$\boxed{(x^m)^n = x^{mn}}$$

(a) $(10^2)^3$

Solution

$$(10^2)^3 = 10^{2\times 3}$$
$$= 10^6$$
$$= \mathbf{1\,000\,000}$$

(b) $(3^{-2})^{-2}$

Solution

$$(3^{-2})^{-2} = 3^{-2\times -2}$$
$$= 3^4$$
$$= \mathbf{81}$$

(c) $(11^4)^{\frac{1}{2}}$

Solution

$$(11^4)^{\frac{1}{2}} = 11^{4\times\frac{1}{2}}$$
$$= 11^2$$
$$= \mathbf{121}$$

(d) $216^{\frac{2}{3}}$

Solution

$$216^{\frac{2}{3}} = (6^3)^{\frac{2}{3}}$$
$$= 6^2$$
$$= \mathbf{36}$$

NOTE

Initially it does not look like Law 3 can be applied as presented in the Hint box but we have managed to change **216** so that the question looks like the format presented. Good. This technique will be utilized in most cases.

4) Without using a calculator, simplify the following.

Hint

In the following questions, we will be applying Law 4 of indices exclusively.

$$\boxed{x^0 = 1}$$

(a) 5^0

Solution

$$5^0 = \mathbf{1}$$

(b) 14^0

Solution

$$14^0 = \mathbf{1}$$

(c) 0.003^0

Solution

$$0.003^0 = \mathbf{1}$$

(d) $(-41)^0$

Solution

$$(-41)^0 = \mathbf{1}$$

(e) $\left(\frac{1}{234}\right)^0$

Solution

$$\left(\frac{1}{234}\right)^0 = \mathbf{1}$$

(f) $\left(\frac{7}{9}\right)^0$

Solution

$$\left(\frac{7}{9}\right)^0 = \mathbf{1}$$

(g) $\left(-\frac{3}{29}\right)^0$

Solution

$$\left(-\frac{3}{29}\right)^0 = \mathbf{1}$$

5) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 5 of indices exclusively.

$$\boxed{x^{-m} = \frac{1}{x^m}}$$

(a) 1^{-3}

Solution

$$1^{-3} = \frac{1}{1^3}$$
$$= \mathbf{1}$$

(b) 6^{-1}

Solution

$$6^{-1} = \frac{1}{6^1}$$
$$= \frac{\mathbf{1}}{\mathbf{6}}$$

(c) $(-7)^{-1}$

Solution

$$(-7)^{-1} = \frac{1}{(-7)^1}$$
$$= -\frac{\mathbf{1}}{\mathbf{7}}$$

(d) 5^{-3}

Solution

$$5^{-3} = \frac{1}{5^3}$$
$$= \frac{\mathbf{1}}{\mathbf{125}}$$

(e) $\frac{1}{9^{-2}}$

Solution

$$\frac{1}{9^{-2}} = \frac{1}{\frac{1}{9^2}}$$
$$= \frac{1}{\frac{1}{81}}$$
$$= \mathbf{81}$$

NOTE

In general,

$$\frac{1}{x^{-m}} = x^m$$

so

$$\frac{1}{9^{-2}} = 9^2$$
$$= 81$$

as before.

(f) $\left(\frac{1}{5}\right)^{-1}$

Solution

$$\left(\frac{1}{5}\right)^{-1} = 5^1$$
$$= \mathbf{5}$$

(g) $\left(\frac{1}{3}\right)^{-3}$

Solution

$$\left(\frac{1}{3}\right)^{-3} = 3^3$$
$$= \mathbf{27}$$

(h) $\frac{7^{-1}}{4^{-1}}$

Solution

$$\frac{7^{-1}}{4^{-1}} = \frac{4^1}{7^1}$$
$$= \mathbf{\frac{4}{7}}$$

(i) $\left(\frac{15}{13}\right)^{-1}$

Solution

$$\left(\frac{15}{13}\right)^{-1} = \mathbf{\frac{13}{15}}$$

6) Without using a calculator, simplify the following.

Hint

In the following questions, we will be applying Law 6(a) of indices exclusively.

$$\boxed{x^{\frac{1}{n}} = \sqrt[n]{x}}$$

(a) $9^{\frac{1}{2}}$

Solution

$$9^{1/2} = \sqrt{9}$$
$$= \sqrt{3^2}$$
$$\mathbf{= 3}$$

NOTE

In general,

$$\sqrt[n]{a^n} = a$$

Hence, for ease of evaluation one can express the radicand (the number under the root symbol) in index form when determining roots of numbers.

(b) $27^{\frac{1}{3}}$

Solution

$$27^{1/3} = \sqrt[3]{27}$$
$$= \sqrt[3]{3^3}$$
$$\mathbf{= 3}$$

(c) $256^{\frac{1}{4}}$

Solution

$$256^{\frac{1}{4}} = \sqrt[4]{256}$$
$$= \sqrt[4]{4^4}$$
$$\mathbf{= 4}$$

(d) $1^{\frac{1}{5}}$

Solution

$$1^{1/5} = \sqrt[5]{1}$$
$$= \sqrt[5]{1^5}$$
$$\mathbf{= 1}$$

(e) $\left(\frac{1}{64}\right)^{\frac{1}{6}}$

Solution

$$\left(\frac{1}{64}\right)^{1/6} = \sqrt[6]{\frac{1}{64}}$$
$$= \sqrt[6]{\left(\frac{1}{2}\right)^6}$$
$$= \mathbf{\frac{1}{2}}$$

(f) $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

Solution

$$\left(\frac{16}{81}\right)^{1/4} = \sqrt[4]{\frac{16}{81}}$$
$$= \sqrt[4]{\frac{2^4}{3^4}}$$
$$= \sqrt[4]{\left(\frac{2}{3}\right)^4}$$
$$= \mathbf{\frac{2}{3}}$$

(g) $(-32)^{\frac{1}{5}}$

Solution

$$(-32)^{\frac{1}{5}} = \sqrt[5]{(-32)}$$

$$= \sqrt[5]{(-2)^5}$$

$$\boxed{= -2}$$

(h) $0.04^{1/2}$

Solution

$$0.04^{1/2} = \left(\frac{4}{100}\right)^{1/2}$$

$$= \sqrt{\frac{4}{100}} = \frac{\sqrt{4}}{\sqrt{100}}$$

$$= \frac{\sqrt{2^2}}{\sqrt{10^2}} = \frac{2}{10}$$

$$= \mathbf{\frac{1}{5}}$$

NOTE

In general,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

and

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt{ab}$$

7) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 6(b) of indices exclusively.

$$\boxed{x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m}$$

(a) $8^{\frac{2}{3}}$

Solution

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2$$

$$= \left(\sqrt[3]{2^3}\right)^2$$

$$= 2^2$$

$$= \mathbf{4}$$

(b) $27^{\frac{4}{3}}$

Solution

$$27^{\frac{4}{3}} = \left(\sqrt[3]{27}\right)^4$$

$$= \left(\sqrt[3]{3^3}\right)^4$$

$$= 3^4$$

$$= \mathbf{81}$$

(c) $4^{2\frac{1}{2}}$

Solution

$$4^{2\frac{1}{2}} = 4^{\frac{5}{2}}$$

$$= \left(\sqrt{4}\right)^5 = \left(\sqrt{2^2}\right)^5$$

$$= 2^5$$

$$= \mathbf{32}$$

(i) $\left(\frac{4}{9}\right)^{\frac{3}{2}}$

Solution

$$\left(\frac{4}{9}\right)^{3/2} = \left(\sqrt{\frac{4}{9}}\right)^3$$

$$= \left(\sqrt{\frac{2^2}{3^2}}\right)^3$$

$$= \left(\frac{2}{3}\right)^3$$

$$= \mathbf{\frac{8}{27}}$$

(d) $\left(3\frac{3}{8}\right)^{\frac{2}{3}}$

Solution

$$\left(3\frac{3}{8}\right)^{\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}}$$

$$= \left(\sqrt[3]{\frac{27}{8}}\right)^2 = \left(\sqrt[3]{\frac{3^3}{2^3}}\right)^2$$

$$= \frac{3^2}{2^2}$$

$$= \mathbf{\frac{9}{4}}$$

(e) $32^{0.4}$

Solution

$$32^{0.4} = 32^{\frac{4}{10}}$$
$$= 32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^2$$
$$= \left(\sqrt[5]{2^5}\right)^2$$
$$= 2^2$$
$$\mathbf{= 4}$$

(f) $0.027^{2/3}$

Solution

$$0.027^{2/3} = \left(\frac{27}{1000}\right)^{2/3}$$
$$= \left(\sqrt[3]{\frac{27}{1000}}\right)^2 = \left(\sqrt[3]{\frac{3^3}{10^3}}\right)^2$$
$$= \frac{3^2}{10^2}$$
$$= \mathbf{\frac{9}{100}}$$

8) Without using a calculator, simplify the following.

Hint

In the following questions, we will be applying Laws 3 and 5 of indices exclusively.

$$\boxed{(x^m)^n = x^{mn}} \text{ and } \boxed{x^{-m} = \frac{1}{x^m}}$$

(a) $(5^2)^{-1}$

Solution

$$(5^2)^{-1} = 5^{2\times -1}$$
$$= 5^{-2}$$
$$= \frac{1}{5^2}$$
$$= \mathbf{\frac{1}{25}}$$

(b) $(2^{-3})^2$

Solution

$$(2^{-3})^2 = 2^{-3\times 2}$$
$$= 2^{-6}$$
$$= \frac{1}{2^6}$$
$$= \mathbf{\frac{1}{64}}$$

(c) $(0.01)^2$

Solution

$$(0.01)^2 = (10^{-2})^2$$
$$= 10^{-4}$$
$$= \frac{1}{10^4}$$
$$= \mathbf{\frac{1}{10\,000}}$$

(d) $\left[\left(\frac{1}{8}\right)^{\frac{1}{6}}\right]^2$

Solution

$$\left[\left(\frac{1}{8}\right)^{\frac{1}{6}}\right]^2 = \left(\frac{1}{8}\right)^{\frac{1}{3}}$$
$$= (8^{-1})^{\frac{1}{3}}$$
$$= [(2^3)^{-1}]^{\frac{1}{3}}$$
$$= (2^{-3})^{\frac{1}{3}}$$
$$= 2^{-1}$$
$$= \mathbf{\frac{1}{2}}$$

9) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Laws 5 and 6 of indices exclusively.

$$\boxed{x^{-m} = \frac{1}{x^m}} \text{ and } \boxed{x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m}$$

(a) $\left(\frac{1}{64}\right)^{-2/3}$

Solution

$$\left(\frac{1}{64}\right)^{-2/3} = (64)^{2/3}$$
$$= \left(\sqrt[3]{64}\right)^2 = \left(\sqrt[3]{4^3}\right)^2$$
$$= 4^2$$
$$= \mathbf{16}$$

(b) $\left(2\frac{1}{4}\right)^{-\frac{1}{2}}$

Solution

$$\left(2\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\frac{9}{4}\right)^{-\frac{1}{2}}$$
$$= \left(\frac{4}{9}\right)^{\frac{1}{2}}$$
$$= \sqrt{\frac{4}{9}} = \sqrt{\frac{2^2}{3^2}}$$
$$= \mathbf{\frac{2}{3}}$$

(c) $0.008^{-\frac{1}{3}}$

Solution

$$0.008^{-\frac{1}{3}} = \left(\frac{8}{1000}\right)^{-\frac{1}{3}}$$
$$= \left(\frac{1000}{8}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1000}{8}}$$
$$= \sqrt[3]{\frac{10^3}{2^3}} = \frac{10}{2}$$
$$= \mathbf{5}$$

NOTE

For this type of questions, one does not simplify the fraction, i.e. $\frac{8}{1000}$ unless it is known that the simplified form can be expressed in the power that would eliminate the root.

(d) $\left(\frac{36}{64}\right)^{-\frac{3}{2}}$

Solution

$$\left(\frac{36}{64}\right)^{-\frac{3}{2}} = \left(\frac{9}{16}\right)^{-\frac{3}{2}}$$
$$= \left(\frac{16}{9}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{16}{9}}\right)^3$$
$$= \left(\sqrt{\frac{4^2}{3^2}}\right)^3 = \frac{4^3}{3^3}$$
$$= \mathbf{\frac{64}{27}}$$

NOTE

Unlike with the previous question, the fraction here needs to be simplified first before proceeding with the root operation.

10) Without using a calculator, evaluate the following.

Hint

Since laws 1- 6 have been tried independently (for numbers) and with commonly occurring combinations, I hope we have developed necessary confidence. In the following questions, we will be applying the relevant rule(s) of indices including Law 7.

(a) $16^{-\frac{1}{4}}$

Solution

$$16^{-\frac{1}{4}} = (2^4)^{-\frac{1}{4}}$$
$$= 2^{-1}$$
$$= \mathbf{\frac{1}{2}}$$

NOTE

Here, we have expressed the base in an index form and applied Law 3. This is an alternative way to using the fractional power law and it is a very useful way of dealing with situations where roots might prove difficult to determine. We will employ this extensively in subsequent questions.

(b) $\left(1\frac{7}{9}\right)^{1\frac{1}{2}}$

Solution

$$\left(1\frac{7}{9}\right)^{1\frac{1}{2}} = \left(\frac{16}{9}\right)^{\frac{3}{2}}$$
$$= \left[\frac{4^2}{3^2}\right]^{\frac{3}{2}} = \left[\frac{4}{3}\right]^{2\times\frac{3}{2}}$$
$$= \left(\frac{4}{3}\right)^3 = \frac{4^3}{3^3}$$
$$= \mathbf{\frac{64}{27}}$$

(c) $\left(1\frac{1}{3}\right)^{-3}$

Solution

$$\left(1\frac{1}{3}\right)^{-3} = \left(\frac{4}{3}\right)^{-3}$$
$$= \left(\frac{3}{4}\right)^3$$
$$= \mathbf{\frac{27}{64}}$$

(d) $(10\ 000)^{-\frac{3}{4}}$

Solution

$$(10\ 000)^{-\frac{3}{4}} = (10^4)^{-\frac{3}{4}}$$
$$= 10^{-3}$$
$$= \frac{1}{10^3}$$
$$= \mathbf{\frac{1}{1\ 000}}$$

(e) $\sqrt{27^{-\frac{2}{3}}}$

Solution

$$\sqrt{27^{-\frac{2}{3}}} = \left(27^{-\frac{2}{3}}\right)^{\frac{1}{2}}$$
$$= 27^{\left(-\frac{2}{3}\times\frac{1}{2}\right)} = 27^{-\frac{1}{3}}$$
$$= \frac{1}{27^{\frac{1}{3}}} = \frac{1}{(3^3)^{\frac{1}{3}}}$$
$$= \mathbf{\frac{1}{3}}$$

NOTE

This last question can obviously be approached in various ways using different law combinations.

(f) $\sqrt[5]{36^{2.5}}$

Solution

$$\sqrt[5]{36^{2.5}} = (36^{2.5})^{1/5}$$
$$= \left(36^{\frac{5}{2}}\right)^{\frac{1}{5}} = 36^{\frac{1}{2}}$$
$$= (6^2)^{\frac{1}{2}}$$
$$= \mathbf{6}$$

(g) $\frac{21\times10^6}{7\times10^3}$

Solution

$$\frac{21\times10^6}{7\times10^3} = \frac{21}{7}\times10^{6-3}$$
$$= 3\times10^3$$
$$= \mathbf{3\ 000}$$

(h) $33\times10^3 \div (11\times10^{-2})$

Solution

$$33\times10^3 \div (11\times10^{-2}) = \frac{33\times10^3}{11\times10^{-2}}$$
$$= \frac{33}{11}\times\frac{10^3}{10^{-2}}$$
$$= 3\times10^{3+2}$$
$$= 3\times10^5$$
$$= \mathbf{300\ 000}$$

(i) $(-8)^{\frac{1}{3}}$

Solution

$$(-8)^{\frac{1}{3}} = [(-2)^3]^{\frac{1}{3}}$$
$$= \mathbf{-2}$$

(j) $(-64)^{-\frac{4}{3}}$

Solution

$$(-64)^{-\frac{4}{3}} = [(-4)^3]^{-\frac{4}{3}}$$
$$= (-4)^{-4} = \frac{1}{(-4)^4}$$
$$= \mathbf{\frac{1}{256}}$$

NOTE

The key rule to know here is that

$$(-x)^m = \begin{cases} x^m, & \text{if } m \text{ is an even number} \\ -x^m, & \text{if } m \text{ is an odd number} \end{cases}$$

(k) $\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{2}\right)^{-3}$

Solution

$$\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{2}\right)^{-3} = 2 + 2^2 + 2^3$$
$$= 2 + 4 + 8$$
$$= \mathbf{14}$$

NOTE

Remember that one cannot apply Law 1 when there is addition (+), so the above question should be simplified numerically.

Section 2: Basic Applications of Laws of Indices (Numbers and Letters)

INTRODUCTION

This section is similar in structure and approach to section 1 except that numbers and letters will be used here. Again no calculators are required.

11) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 1 of indices exclusively.

$$\boxed{x^m \times x^n = x^{m+n}}$$

(a) $a^2 \times a^3$

Solution

$$a^2 \times a^3 = a^{2+3}$$
$$= \mathbf{a^5}$$

(b) $a^4 \times a^{-3} \times a^5$

Solution

$$a^4 \times a^{-3} \times a^5 = a^{4+(-3)+5}$$
$$= \mathbf{a^6}$$

(c) $b^{\frac{1}{3}} \times b^{\frac{5}{3}}$

Solution

$$b^{\frac{1}{3}} \times b^{\frac{5}{3}} = b^{\frac{1}{3}+\frac{5}{3}}$$
$$= b^{\frac{1+5}{3}} = b^{\frac{6}{3}}$$
$$= \mathbf{b^2}$$

(d) $3c^7 \times 5c^0 \times 2c^{-6}$

Solution

$$3c^7 \times 5c^0 \times 2c^{-6}$$
$$= (3 \times 5 \times 2)$$
$$\times (c^7 \times c^0 \times c^{-6})$$
$$= 30\,(c^{7+0-6})$$
$$= \mathbf{30c}$$

(e) $6^{d-3} \times 6^{3-d}$

Solution

$$6^{d-3} \times 6^{3-d} = 6^{(d-3+3-d)}$$
$$= 6^0$$
$$= \mathbf{1}$$

NOTE

As shown above, it is evident that the index can be exclusively numbers, wholly letters or a combination of both and the laws will still be valid.

12) Without using a calculator, simplify the following.

Hint

In the following questions, we will be applying Law 2 of indices exclusively.

$$\boxed{x^m \div x^n = x^{m-n}}$$

(a) $e^8 \div e^5$

Solution

$$e^8 \div e^5 = e^{8-5}$$
$$= \boldsymbol{e^3}$$

NOTE

Remember that,

$$e^8 \div e^5 = \frac{e^8}{e^5}$$
$$= \frac{e \times e \times e \times e \times e \times e \times e \times e}{e \times e \times e \times e \times e}$$
$$= e \times e \times e$$
$$= \boldsymbol{e^3}$$

(b) $f^{-3} \div f^{-5}$

Solution

$$f^{-3} \div f^{-5} = f^{-3-(-5)}$$
$$= f^{-3+5}$$
$$= \boldsymbol{f^2}$$

(c) $\frac{18g^7}{6g^4}$

Solution

$$\frac{18g^7}{6g^4} = \frac{18}{6} \times g^{7-4}$$
$$= \boldsymbol{3g^3}$$

(d) $32h^{11} \div 8h^4$

Solution

$$32h^{11} \div 8h^4 = (32 \times h^{11}) \div (8 \times h^4)$$
$$= (32 \div 8)(h^{11} \div h^4)$$
$$= (4)(h^{11-4})$$
$$= \boldsymbol{4h^7}$$

NOTE

Remember that,

$$32h^{11} \div 8h^4 = \frac{32h^{11}}{8h^4} = \frac{32}{8}(h^{11-4})$$
$$= \boldsymbol{4h^7}$$

(e) $\frac{j^{13}}{(-j)^6}$

Solution

$$\frac{j^{13}}{(-j)^6} = \frac{j^{13}}{j^6}$$
$$= j^{13-6}$$
$$= \boldsymbol{j^7}$$

13) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 3 of indices exclusively.

$$\boxed{(x^m)^n = x^{mn}}$$

(a) $(k^3)^7$

Solution

$$(k^3)^7 = k^{3\times 7}$$
$$= \boldsymbol{k^{21}}$$

(b) $(-l^5)^2$

Solution

$$(-l^5)^2 = l^{5\times 2}$$
$$= \boldsymbol{l^{10}}$$

NOTE

Remember that the square of a negative number or expression gives a positive number or expression. Alternatively, we can write

$$(-l^5)^2 = (-1 \times l^5)^2$$
$$= (-1)^2 \times (l^5)^2$$
$$= 1 \times \ l^{5\times 2}$$
$$= \boldsymbol{l^{10}}$$

(c) $(-m^5)^4$

Solution

$$(-m^5)^4 = (-1\ \times\ m^5)^4$$
$$= (-1)^4\ \times\ m^{5\times 4}$$
$$= 1\ \times m^{20}$$
$$= \boldsymbol{m^{20}}$$

(d) $(-n^6)^5$

Solution

$$(-n^6)^5 = (-1\ \times n^6)^5$$
$$= (-1)^5\ \times n^{6\times 5}$$
$$= -1\ \times\ n^{30}$$
$$= \boldsymbol{-n^{30}}$$

(e) $(-2p^5q)^3$

Solution

$$(-2p^5q)^3 = (-2)^3 \times p^{5\times 3}\ \times\ q^{\ 3}$$
$$= \boldsymbol{-8p^{15}q^3}$$

(f) $(27r^6s^{12})^{\frac{1}{3}}$

Solution

$$(27r^6s^{12})^{\frac{1}{3}} = (3^3r^6s^{12})^{\frac{1}{3}}$$
$$= \left(3^{3\times\frac{1}{3}}\right)\left(r^{6\times\frac{1}{3}}\right)\left(s^{12\times\frac{1}{3}}\right)$$
$$= \boldsymbol{3r^2s^4}$$

14) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Law 5 of indices exclusively.

$$\boxed{\boldsymbol{x^{-m} = \frac{1}{x^m}}}$$

(a) t^{-3}

Solution

$$t^{-3} = \boldsymbol{\frac{1}{t^3}}$$

(b) $\frac{1}{u^{-5}}$

Solution

$$\frac{1}{u^{-5}} = \boldsymbol{u^5}$$

(c) $7v^{-\frac{1}{7}}$

Solution

$$7v^{-\frac{1}{7}} = \boldsymbol{\frac{7}{v^{\frac{1}{7}}}}$$

(d) $\left(\frac{1}{x}\right)^{-\frac{3}{5}}$

Solution

$$\left(\frac{1}{x}\right)^{-\frac{3}{5}} = \boldsymbol{x^{\frac{3}{5}}}$$

(e) $11y^{-6}$

Solution

$$11y^6 = 11 \times \frac{1}{y^6}$$
$$= \boldsymbol{\frac{11}{y^6}}$$

(f) $(4z)^{-3}$

Solution

$$(4z)^{-3} = \left(\frac{1}{4z}\right)^3$$

$$= \frac{1}{4^3z^3}$$

$$= \boldsymbol{\frac{1}{64z^3}}$$

15) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Laws 1 and 3 of indices exclusively.

$\boxed{x^m \times x^n = x^{m+n}}$ **and** $\boxed{(x^m)^n = x^{mn}}$

(a) $3a^{-1} \times (2a)^2$

Solution

$$3a^{-1} \times (2a)^2 = 3 \times 2^2 \times a^{-1} \times a^2$$
$$= 3 \times 4 \times a^{-1+2}$$
$$= \boldsymbol{12a}$$

(b) $(2b^2)^3 \times (3b)^2$

Solution

$$(2b^2)^3 \times (3b)^2 = (2^3b^{2\times3}) \times (3^2b^2)$$
$$= 8b^6 \times 9b^2$$
$$= 8 \times 9 \times b^{6+2}$$
$$= \boldsymbol{72b^8}$$

(c) $(x^2y^3)^4 \times (xy^{-3})^3$

Solution

$$(x^2y^3)^4 \times (xy^{-3})^3 = (x^{2\times4}y^{3\times4}) \times (x^3y^{-3\times3})$$
$$= x^8y^{12} \times x^3y^{-9}$$
$$= x^{8+3}y^{12-9}$$
$$= \boldsymbol{x^{11}y^3}$$

(d) $(2p^4q^2)^3 \times (3pq^3)^2$

Solution

$$(2p^4q^2)^3 \times (3pq^2)^2$$
$$= (2^3p^{4\times3}q^{2\times3}) \times (3^2p^2q^{2\times2})$$
$$= 8p^{12}q^6 \times 9p^2q^4$$
$$= 8 \times 9 \times p^{12+2}q^{6+4}$$
$$= 72 \times p^{14}q^{10}$$
$$= \boldsymbol{72p^{14}q^{12}}$$

(e) $(3d^{-1})^{\frac{1}{2}} \times \left(3d^5\right)^{\frac{3}{2}}$

Solution

$$(3d^{-1})^{\frac{1}{2}} \times (3d^5)^{\frac{3}{2}} = \left(3^{\frac{1}{2}} \times d^{-\frac{1}{2}}\right) \times \left(3^{\frac{3}{2}} \times d^{\frac{15}{2}}\right)$$
$$= 3^{\frac{1}{2}+\frac{3}{2}} \times d^{-\frac{1}{2}+\frac{15}{2}}$$
$$= 3^{\frac{1+3}{2}} \times d^{-\frac{1+15}{2}}$$
$$= 3^2 \times d^7$$
$$= \boldsymbol{9d^7}$$

16) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Laws 1 and 4 of indices exclusively.

$\boxed{x^m \times x^n = x^{m+n}}$ **and** $\boxed{x^{-m} = \frac{1}{x^{-m}}}$

(a) $a^{-7} \times a^{-6}$

Solution

$$a^{-7} \times a^{-6} = a^{-7-6}$$
$$= a^{-13}$$
$$= \boldsymbol{\frac{1}{a^{13}}}$$

(b) $7b \times 3b^{-2}$

Solution

$$7b \times 3b^{-2} = 7 \times 3 \times b^{1-2}$$
$$= 21b^{-1}$$
$$= \boldsymbol{\frac{21}{b}}$$

(c) $5\,c^{\frac{3}{2}} \times 2\,c^{-\frac{7}{2}}$

Solution

$$5\,c^{\frac{3}{2}} \times 2\,c^{-\frac{7}{2}} = 5 \times 2 \times c^{\frac{3}{2}-\frac{7}{2}}$$
$$= 10 \times c^{\frac{3-7}{2}}$$

$$= 10\,c^{-2}$$

$$\boxed{= \mathbf{\frac{10}{c^2}}}$$

(d) $11d^3e \times 4d^2e^{-3}$

Solution

$$11d^3e \times 4d^2e^{-3} = 11 \times 4 \times d^{3+2}e^{1-3}$$

$$= 44d^5e^{-2}$$

$$\boxed{= \mathbf{\frac{44d^5}{e^2}}}$$

(e) $\frac{f^{-3}}{f^5}$

Solution

$$\frac{f^{-3}}{f^5} = \frac{1}{f^5 \times f^3}$$

$$= \frac{1}{f^{5+3}}$$

$$\boxed{= \mathbf{\frac{1}{f^8}}}$$

17) Without using a calculator, evaluate the following.

Hint

In the following questions, we will be applying Laws 3 and 5 of indices exclusively.

$\boxed{(x^m)^n = x^{mn}}$ **and** $\boxed{x^{-m} = \frac{1}{x^m}}$

(a) $(p^{-3})^4$

Solution

$$(p^{-3})^4 = p^{-3\times4}$$

$$= p^{-12}$$

$$\boxed{= \mathbf{\frac{1}{p^{12}}}}$$

(b) $(4q^2)^{-3}$

Solution

$$(4q^2)^{-3} = \frac{1}{(4q^2)^3}$$

$$= \frac{1}{4^3q^6}$$

$$\boxed{= \mathbf{\frac{1}{64q^6}}}$$

(c) $(2p^2q^4r^3)^{-3}$

Solution

$$(2p^2q^4r^3)^{-3} = \frac{1}{(2p^2q^4r^3)^3}$$

$$= \frac{1}{2^3p^{2\times3}q^{4\times3}r^{3\times3}}$$

$$\boxed{= \mathbf{\frac{1}{8p^6q^{12}r^9}}}$$

(d) $\left(\frac{1}{3}r^{-2}\right)^{-3}$

Solution

$$\left(\frac{1}{3}r^{-2}\right)^{-3} = (3^{-1}r^{-2})^{-3}$$

$$= 3^{-1\times-3}r^{-2\times-3}$$

$$= 3^3r^6$$

$$\boxed{= \mathbf{27r^6}}$$

(e) $\left(\frac{r}{3s}\right)^{-2}$

Solution

$$\left(\frac{r}{3s}\right)^{-2} = \left(\frac{3s}{r}\right)^2$$

$$= \frac{3^2s^2}{r^2}$$

$$\boxed{= \mathbf{\frac{9s^2}{r^2}}}$$

18) Without using a calculator, evaluate the following.

Hint

Since the laws (1- 6) have been used independently (for numbers and letters) and with common combinations, I hope we have developed necessary confidence. In the following questions, we will be applying the relevant rule(s) of indices including Law

7.

(a) $16t^3 \times (2t^2)^{-3}$

Solution

$$\begin{aligned}16t^3 \times (2t^2)^{-3} &= 16t^3 \times 2^{-3}t^{-6}\\ &= 16 \times 2^{-3} \times t^{3-6}\\ &= \frac{16}{2^3} \times t^{-3}\\ &= \frac{16}{8} \times \frac{1}{t^3}\\ &= \boldsymbol{\frac{4}{t^3}}\end{aligned}$$

(b) $(3u^{-3})^3 \times (9u)^{-1}$

Solution

$$\begin{aligned}(3u^{-3})^3 \times (9u)^{-1} &= 3^3u^{-9} \times \frac{1}{9u}\\ &= \frac{3^3}{u^9} \times \frac{1}{9u}\\ &= \frac{3^3}{3^2 \times u \times u^9}\\ &= \frac{3^{3-2}}{u^{1+9}}\\ &= \boldsymbol{\frac{3}{u^{10}}}\end{aligned}$$

(c) $(3u^{-2}v)^2 \div (2uv)^{-3}$

Solution

$$\begin{aligned}(3u^{-2}v)^2 \div (2uv)^{-3} &= 3^2u^{-4}v^2 \div \frac{1}{2^3u^3v^3}\\ &= 9u^{-4}v^2 \times 8u^3v^3\\ &= 9 \times 8(u^{-4+3}v^{2+3})\\ &= 72u^{-1}v^5\\ &= \boldsymbol{\frac{72v^5}{u}}\end{aligned}$$

(d) $(3xy^3z^5)^2 \div (3xy^3z^5)$

Solution

$$\begin{aligned}&(3xy^3z^5)^2 \div (3xy^3z^5)\\ &= 3^2x^2y^{3\times2}z^{5\times2} \div 3xy^3z^5\\ &= 9x^2y^6z^{10} \div 3xy^3z^5\\ &= (9 \div 3) \times (x^2y^6z^{10} \div xy^3z^5)\\ &= 3 \times (x^{2-1}y^{6-3}z^{10-5})\\ &= 3 \times xy^3z^5\\ &= \boldsymbol{3xy^3z^5}\end{aligned}$$

NOTE

We can consider the expression in the brackets as a unit and simply evaluate it as:

$$(3xy^3z^5)^2 \div (3xy^3z^5) = (3xy^3z^5)^{2-1}$$
$$= \boldsymbol{3xy^3z^5}$$

as before.

(e) $\frac{10a^5 \times 4a^3}{8a^{-2}}$

Solution

$$\begin{aligned}\frac{10a^5 \times 4a^3}{8a^{-2}} &= \frac{10 \times 4 \times a^5 \times a^3}{8 \times a^{-2}}\\ &= \left\{\frac{10 \times 4}{8}\right\} \times \left\{\frac{a^5 \times a^3}{a^{-2}}\right\}\\ &= \left\{\frac{40}{8}\right\} \times \left\{a^{5+3-(-2)}\right\}\\ &= 5 \times a^{10}\\ &= \boldsymbol{5a^{10}}\end{aligned}$$

(f) $\frac{-(b^2)^4}{b^6 \times (-b)}$

Solution

$$\begin{aligned}\frac{-(b^2)^4}{b^6 \times (-b)} &= \frac{-(b^{2\times4})}{-b \times b^6}\\ &= \frac{-1 \times b^8}{-1 \times b \times b^6}\\ &= \frac{b^8}{b^7}\\ &= b^{8-7}\\ &= \boldsymbol{b}\end{aligned}$$

(g) $\frac{(-c)^2 \times c^6}{-c^7}$

Solution

$$\frac{(-c)^2 \times c^6}{-c^7} = \frac{c^2 \times c^6}{-1 \times c^7}$$

$$= \frac{c^8}{-1 \times c^7}$$
$$= -1 \times c^{8-7}$$
$$= \boldsymbol{-c}$$

(h) $\sqrt[3]{27d^{-12}}$

Solution

$$\sqrt[3]{27d^{-12}} = (27d^{-12})^{1/3}$$
$$= (3^3 d^{-12})^{\frac{1}{3}}$$
$$= 3^{\left(3 \times \frac{1}{3}\right)} \times d^{\left(-12 \times \frac{1}{3}\right)}$$
$$= 3d^{-4}$$
$$= \boldsymbol{\frac{3}{d^4}}$$

(i) $(3p^6q^8)^{\frac{1}{4}} \times (27p^{-2})^{\frac{1}{4}}$

Solution

$$(3p^6q^8)^{\frac{1}{4}} \times (27p^{-2})^{\frac{1}{4}} = [(3p^6q^8)(27p^{-2})]^{\frac{1}{4}}$$
$$= [(3 \times 27)(p^{6-2})(q^8)]^{\frac{1}{4}}$$
$$= [(3^4)(p^4)(q^8)]^{\frac{1}{4}}$$
$$= [(3pq^2)^4]^{\frac{1}{4}}$$
$$= \boldsymbol{3pq^2}$$

(j) $(24x^4y^2)^{\frac{1}{3}} \div (3xy^2)^{\frac{1}{3}}$

Solution

$$(24x^4y^2)^{\frac{1}{3}} \div (3xy^2)^{\frac{1}{3}} = (24x^4y^2 \div 3xy^2)^{\frac{1}{3}}$$
$$= \left(\frac{24x^4y^2}{3xy^2}\right)^{\frac{1}{3}} = (8x^3)^{\frac{1}{3}}$$
$$= (2^3x^3)^{\frac{1}{3}}$$
$$= \boldsymbol{2x}$$

NOTE

Until now we have only indirectly applied Law 7 (Same Power Law). The last two questions clearly underscore the relevance of this rule (though one can still do without it in these questions).

Section 3: Advanced Applications of Laws of Indices

INTRODUCTION

This section provides advanced questions on indices. As a result, readers should be familiar with laws of indices, and should in addition be able to handle complex algebraic expressions. If you have not done so, sections 1 and 2 are good starting points for beginners.

19) Simplify the following.

(a) $(\theta^2 - \varphi^2)^{-\frac{1}{3}} \times (\theta - \varphi)^{\frac{4}{3}} \times (\theta + \varphi)^{\frac{1}{3}}$

Solution

$$(\theta^2 - \varphi^2)^{-\frac{1}{3}} \times (\theta - \varphi)^{\frac{4}{3}} \times (\theta + \varphi)^{\frac{1}{3}}$$
$$= [(\theta - \varphi)(\theta + \varphi)]^{-\frac{1}{3}} \times (\theta - \varphi)^{\frac{4}{3}} \times (\theta + \varphi)^{\frac{1}{3}}$$

We have employed difference of two squares to obtain the expression above. Now we can carry on with the application of laws of indices.

$$= (\theta - \varphi)^{-\frac{1}{3}} \times (\theta + \varphi)^{-\frac{1}{3}} \times (\theta - \varphi)^{\frac{4}{3}} \times (\theta + \varphi)^{\frac{1}{3}}$$
$$= (\theta - \varphi)^{-\frac{1}{3}+\frac{4}{3}} \times (\theta + \varphi)^{-\frac{1}{3}+\frac{1}{3}}$$
$$= (\theta - \varphi)^1 \times (\theta + \varphi)^0$$
$$= \boldsymbol{\theta - \varphi}$$

(b) $\left(p^{\frac{1}{5}}q^{\frac{1}{3}}\right)^{-4} \times \left(p^{\frac{4}{5}}q^{\frac{1}{2}}\right)^3 \times (p^{-6}q)^{\frac{1}{3}}$

Solution

$$\left(p^{\frac{1}{5}}q^{\frac{1}{3}}\right)^{-4} \times \left(p^{\frac{4}{5}}q^{\frac{1}{2}}\right)^3 \times (p^{-6}q)^{\frac{1}{3}}$$
$$= \left(p^{-\frac{4}{5}}q^{-\frac{4}{3}}\right) \times \left(p^{\frac{12}{5}}q^{\frac{3}{2}}\right) \times \left(p^{-2}q^{\frac{1}{3}}\right)$$
$$= \left(p^{-\frac{4}{5}} \times p^{\frac{12}{5}} \times p^{-2}\right)\left(q^{-\frac{4}{3}} \times q^{\frac{3}{2}} \times q^{\frac{1}{3}}\right)$$
$$= \left(p^{-\frac{4}{5}+\frac{12}{5}-2}\right)\left(q^{-\frac{4}{3}+\frac{3}{2}+\frac{1}{3}}\right)$$
$$= \left(p^{-\frac{2}{5}}\right)\left(q^{\frac{1}{2}}\right)$$
$$= \boldsymbol{\frac{q^{\frac{1}{2}}}{p^{\frac{2}{5}}}}$$

(c) $\sqrt[3]{x^9 y^{\frac{1}{3}} z^{\frac{1}{2}}} \times y^{\frac{8}{9}} \times \left(3^6 x^6 y^2 z^{\frac{1}{3}}\right)^{-\frac{1}{2}}$

Solution

$$\sqrt[3]{x^9 y^{\frac{1}{3}} z^{\frac{1}{2}}} \times y^{\frac{8}{9}} \times \left(3^6 x^6 y^2 z^{\frac{1}{3}}\right)^{-\frac{1}{2}}$$

$$= \left(x^9 y^{\frac{1}{3}} z^{\frac{1}{2}}\right)^{\frac{1}{3}} \times y^{\frac{8}{9}} \times \left(3^6 x^6 y^2 z^{\frac{1}{3}}\right)^{-\frac{1}{2}}$$

$$= x^3 y^{\frac{1}{9}} z^{\frac{1}{6}} \times y^{\frac{8}{9}} \times 3^{-3} x^{-3} y^{-1} z^{-\frac{1}{6}}$$

$$= 3^{-3} \times x^{(3-3)} y^{\left(\frac{1}{9}+\frac{8}{9}-1\right)} z^{\left(\frac{1}{6}-\frac{1}{6}\right)}$$

$$= 3^{-3} x^0 y^0 z^0$$

$$= 3^{-3} = \frac{1}{3^3}$$

$$= \mathbf{\frac{1}{27}}$$

(d) $\sqrt[4]{\alpha^{\frac{7}{2}} \beta^5 \gamma^{-8}} \div \sqrt{\alpha^{-\frac{1}{4}} \beta^{\frac{5}{2}} \gamma^{-10}}$

Solution

$$\sqrt[4]{\alpha^{\frac{7}{2}} \beta^5 \gamma^{-8}} \div \sqrt{\alpha^{-\frac{1}{4}} \beta^{\frac{5}{2}} \gamma^{-10}}$$

$$= \left(\alpha^{\frac{7}{2}} \beta^5 \gamma^{-8}\right)^{\frac{1}{4}} \div \left(\alpha^{-\frac{1}{4}} \beta^{\frac{5}{2}} \gamma^{-10}\right)^{\frac{1}{2}}$$

$$= \alpha^{\frac{7}{8}} \beta^{\frac{5}{4}} \gamma^{-2} \div \alpha^{-\frac{1}{8}} \beta^{\frac{5}{4}} \gamma^{-5}$$

$$= \alpha^{\left(\frac{7}{8}+\frac{1}{8}\right)} \beta^{\left(\frac{5}{4}-\frac{5}{4}\right)} \gamma^{(-2+5)}$$

$$= \alpha^1 \beta^0 \gamma^3$$

$$= \boldsymbol{\alpha\gamma^3}$$

(e) $(\omega^2 - 1)^3 \times \sqrt[3]{\omega + 1} \div (\omega - 1)^{-\frac{1}{3}}$

Solution

$$(\omega^2 - 1)^3 \times \sqrt[3]{\omega + 1} \div (\omega - 1)^{-\frac{1}{3}}$$

$$= [(\omega - 1)(\omega + 1)]^3 \times (\omega + 1)^{\frac{1}{3}} \times (\omega - 1)^{\frac{1}{3}}$$

$$= (\omega - 1)^3 \times (\omega + 1)^3 \times (\omega + 1)^{\frac{1}{3}} \times (\omega - 1)^{\frac{1}{3}}$$

$$= (\omega - 1)^{3+\frac{1}{3}} \times (\omega + 1)^{3+\frac{1}{3}}$$

$$= (\omega - 1)^{\frac{10}{3}} \times (\omega + 1)^{\frac{10}{3}}$$

$$= [(\omega - 1)(\omega + 1)]^{\frac{10}{3}}$$

$$= (\omega^2 - 1)^{\frac{10}{3}}$$

$$= \boldsymbol{\sqrt[3]{(\omega^2 - 1)^{10}}}$$

(f) $(2x^2y^{-3}z)^3 \times (x^{-3}y^6z^9)^{\frac{1}{3}} \div 16(x^4y^6z^2)^{-\frac{1}{2}}$

Solution

$$(2x^2y^{-3}z)^3 \times (x^{-3}y^6z^9)^{\frac{1}{3}} \div 16(x^4y^6z^2)^{-\frac{1}{2}}$$

$$= (2^3x^6y^{-9}z^3) \times \{(x^{-1}y^2z^3) \div 16(x^{-2}y^{-3}z^{-1})\}$$

$$= (8x^6y^{-9}z^3) \times \left\{\frac{1}{16}(x^{-1+2}y^{2+3}z^{3+1})\right\}$$

$$= (8x^6y^{-9}z^3) \times \frac{1}{16}(xy^5z^4)$$

$$= \frac{8}{16}(x^{6-1}y^{-9+5}z^{3+4})$$

$$= \frac{1}{2}(x^5y^{-4}z^7)$$

$$= \boldsymbol{\frac{x^5z^7}{2y^4}}$$

Section 4: Applications of Laws of Indices (Simplification)

INTRODUCTION

In this section, as the title suggests, laws of indices will be used to re-write an expression in its simplified form, either prescribed in the question or otherwise. Sometimes, the process will involve substituting the value of a variable.

20) Simplify the following, giving each answer in the form 3^n.

Hint

Questions 20 and 21 are similar except while multiplication and division are used between the terms in the former, the latter has addition and subtraction between its terms. In both cases the final expression must be expressed in $\boldsymbol{nth}$ power of 3.

(a) $81^3 \div 9^6$

Solution

$$81^3 \div 9^6 = (3^4)^3 \div (3^2)^6$$
$$= 3^{12} \div 3^{12} = 3^{12-12}$$
$$\mathbf{= 3^0}$$

(b) $(\frac{1}{3^2})^{-8} \times (3^4)^{-3}$

Solution

$$\left(\frac{1}{3^2}\right)^{-8} \times (3^4)^{-3} = (3^{-2})^{-8} \times (3^4)^{-3}$$
$$= 3^{-2\times-8} \times 3^{4\times-3}$$
$$= 3^{16} \times 3^{-12}$$
$$= 3^{16-12}$$
$$\mathbf{= 3^4}$$

(c) $(3^4)^{-5} \div (3^{-5})^4$

Solution

$$(3^4)^{-5} \div (3^{-5})^4 = 3^{-20} \div 3^{-20}$$
$$= 3^{-20-(-20)}$$
$$= 3^{-20+20}$$
$$\mathbf{= 3^0}$$

NOTE

Alternatively, one can re-arrange the expression in the brackets such that

$$(3^4)^{-5} \div (3^{-5})^4 = (3^{-5})^4 \div (3^{-5})^4$$
$$= (3^{-5})^{4-4}$$
$$= (3^{-5})^0$$
$$\mathbf{= 3^0}$$

as before.

(d) $3^2 \div 27 \times 9^3$

Solution

$$3^2 \div 27 \times 9^3 = 3^2 \div 3^3 \times (3^2)^3$$
$$= 3^{2-3} \times 3^6$$
$$= 3^{-1} \times 3^6 = 3^{-1+6}$$
$$\mathbf{= 3^5}$$

(e) $\frac{(3^{-2})^3 \times 3^9}{3^{11}}$

Solution

$$\frac{(3^{-2})^3 \times 3^9}{3^{11}} = \frac{3^{-6} \times 3^9}{3^{11}}$$
$$= \frac{3^{-6+9}}{3^{11}}$$
$$= \frac{3^3}{3^{11}}$$
$$= 3^{3-11}$$
$$\mathbf{= 3^{-8}}$$

(f) $\frac{\sqrt{48}}{\sqrt[3]{192}}$

Solution

$$\frac{\sqrt{48}}{\sqrt[3]{192}} = \frac{(48)^{\frac{1}{2}}}{(192)^{\frac{1}{3}}}$$
$$= \frac{(3\times 16)^{\frac{1}{2}}}{(3\times 64)^{\frac{1}{3}}} = \frac{(3)^{\frac{1}{2}} \times (16)^{\frac{1}{2}}}{(3)^{\frac{1}{3}} \times (64)^{\frac{1}{3}}}$$
$$= \frac{(3)^{\frac{1}{2}} \times (4^2)^{\frac{1}{2}}}{(3)^{\frac{1}{3}} \times (4^3)^{\frac{1}{3}}} = \frac{(3)^{\frac{1}{2}} \times 4}{(3)^{\frac{1}{3}} \times 4}$$
$$= 3^{\frac{1}{2}} \div 3^{\frac{1}{3}} = 3^{\frac{1}{2}-\frac{1}{3}}$$
$$\mathbf{= 3^{\frac{1}{6}}}$$

21) Express each of the following in the form 3^n.

(a) $3^{50} + 3^{50} + 3^{50}$

Solution

$$3^{50} + 3^{50} + 3^{50} = 3^{50}(1+1+1)$$
$$= 3^{50}(3)$$
$$= 3^{50+1}$$
$$\mathbf{= 3^{51}}$$

(b) $3^{-100} + 3^{-100} + 3^{-100}$

Solution

$$3^{-100} + 3^{-100} + 3^{-100} = 3^{-100}(1+1+1)$$
$$= 3^{-100}(3)$$
$$= 3^{-100+1}$$
$$\mathbf{= 3^{-99}}$$

(c) $27^{11} - 9^{16} - 3^{32}$

Solution

$$\begin{aligned} 27^{11} - 9^{16} - 3^{32} &= (3^3)^{11} - (3^2)^{16} - 3^{32} \\ &= 3^{33} - 3^{32} - 3^{32} \\ &= 3(3^{32}) - 3^{32} - 3^{32} \\ &= 3^{32}(3 - 1 - 1) = 3^{32}(1) \\ &= \mathbf{3^{32}} \end{aligned}$$

(d) $81^{0.5} + 81^{0.5} + 81^{0.5}$

Solution

$$\begin{aligned} 81^{0.5} + 81^{0.5} + 81^{0.5} &= 81^{0.5}(1 + 1 + 1) \\ &= 81^{0.5}(3) \\ &= (3^4)^{0.5}(3) \\ &= (3^2)(3) \\ &= 3^{2+1} \\ &= \mathbf{3^3} \end{aligned}$$

22) Without using a calculator, evaluate the following, leaving the answers in the form $a\sqrt[n]{b}$, where a, b and n are real numbers. State the value of a, b and n.

Hint

Note that the first two questions can also be solved using surd but I guess it will be easier to use indices. Law 7 should be applied here.

(a) $\sqrt[5]{81} \times \sqrt[5]{6}$

Solution

$$\begin{aligned} \sqrt[5]{81} \times \sqrt[5]{6} &= (81)^{\frac{1}{5}} \times (6)^{\frac{1}{5}} \\ &= (3^4)^{\frac{1}{5}} \times (3 \times 2)^{\frac{1}{5}} \\ &= 3^{\frac{4}{5}} \times 3^{\frac{1}{5}} \times 2^{\frac{1}{5}} \\ &= 3^{\frac{4}{5}+\frac{1}{5}} \times 2^{\frac{1}{5}} = 3 \times 2^{\frac{1}{5}} \\ &= 3 \times \sqrt[5]{2} \\ &= \mathbf{3\sqrt[5]{2}} \end{aligned}$$

$$\therefore \boldsymbol{a = 3, b = 2 \text{ and } n = 5}$$

(b) $\sqrt[3]{32} \times \sqrt[3]{14}$

Solution

$$\begin{aligned} \sqrt[3]{32} \times \sqrt[3]{14} &= (32)^{\frac{1}{3}} \times (14)^{\frac{1}{3}} \\ &= (2^5)^{\frac{1}{3}} \times (2 \times 7)^{\frac{1}{3}} \\ &= 2^{\frac{5}{3}} \times 2^{\frac{1}{3}} \times 7^{\frac{1}{3}} \\ &= 2^{\frac{5}{3}+\frac{1}{3}} \times 7^{\frac{1}{3}} = 2^2 \times 7^{\frac{1}{3}} \\ &= 4 \times \sqrt[3]{7} \\ &= \mathbf{4\sqrt[3]{7}} \end{aligned}$$

$$\therefore \boldsymbol{a = 4, b = 7 \text{ and } n = 3}$$

NOTE

This can also be dealt with using rules or properties of surds. See '***Surds Explained with Worked Examples***' by Shefiu S. Zakariyah.

(c) $5^{-\frac{3}{2}} + 5^{-\frac{1}{2}} + 5^{\frac{1}{2}}$

Solution

$$\begin{aligned} 5^{-\frac{3}{2}} + 5^{-\frac{1}{2}} + 5^{\frac{1}{2}} &= 5^{\frac{1}{2}}(5^{-2} + 5^{-1} + 5^0) \\ &= 5^{\frac{1}{2}}\left(\frac{1}{25} + \frac{1}{5} + 1\right) \\ &= 5^{\frac{1}{2}}\left(\frac{1 + 5 + 25}{25}\right) \\ &= 5^{\frac{1}{2}}\left(\frac{31}{25}\right) \\ &= \frac{31}{25}\sqrt{5} \end{aligned}$$

$$\therefore \boldsymbol{a = \frac{31}{25}, b = 5 \text{ and } n = 2}$$

(d) $7^{\frac{7}{3}} + 7^{\frac{1}{3}} - 7^{\frac{4}{3}}$

Solution

$$\begin{aligned} 7^{\frac{7}{3}} + 7^{\frac{1}{3}} - 7^{\frac{4}{3}} &= 7^{\frac{1}{3}}(7^2 + 7^0 - 7^1) \\ &= 7^{\frac{1}{3}}(49 + 1 - 7) \\ &= 7^{\frac{1}{3}}(43) \\ &= 43\sqrt[3]{7} \end{aligned}$$

$$\therefore \boldsymbol{a = 43, b = 7 \text{ and } n = 3}$$

23) If $x = 3$, without using a calculator find the value of each of the following.

Hint

You may first simplify the expression and then substitute for $\boldsymbol{x}$ or vice versa.

(a) $(5x)^{-1}$

Solution

$$(5x)^{-1} = (5 \times 3)^{-1}$$
$$= 15^{-1}$$
$$= \frac{1}{15^1}$$
$$= \boldsymbol{\frac{1}{15}}$$

(b) $\frac{1}{3}x^{-2}$

Solution

$$\frac{1}{3}x^{-2} = \frac{1}{3x^2}$$
$$= \frac{1}{3 \times 3^2}$$
$$= \boldsymbol{\frac{1}{27}}$$

(c) $\frac{2}{(2x)^{-2}}$

Solution

$$\frac{2}{(2x)^{-2}} = 2(2x)^2$$
$$= 2(2 \times 3)^2 = 2(6)^2$$
$$= 2 \times 36$$
$$= \boldsymbol{72}$$

(d) $\sqrt[3]{\frac{8}{(x^3)^{-1}}}$

Solution

$$\sqrt[3]{\frac{8}{(x^3)^{-1}}} = \sqrt[3]{8(x^3)}$$
$$= \sqrt[3]{2^3x^3} = \sqrt[3]{(2x)^3}$$
$$= 2x$$
$$= 2 \times 3$$
$$= \boldsymbol{6}$$

24) If $y = -27$ without using a calculator find the value of each of the following.

Hint

This is similar to the previous question except that the substituting value is negative and this would need to be diligently dealt with. Refer to the rule on page 27.

(a) $9y^{-\frac{1}{3}}$

Solution

$$9y^{-\frac{1}{3}} = \frac{9}{y^{\frac{1}{3}}}$$
$$= \frac{9}{(-27)^{\frac{1}{3}}} = \frac{9}{[(-3)^3]^{\frac{1}{3}}}$$
$$= \frac{9}{(-3)}$$
$$= \boldsymbol{-3}$$

(b) $\frac{1}{3}y^{\frac{2}{3}}$

Solution

$$\frac{1}{3}y^{\frac{2}{3}} = 3^{-1}(-27)^{\frac{2}{3}}$$
$$= 3^{-1}(-27)^{2\times\frac{1}{3}} = 3^{-1}[(-27)^2]^{\frac{1}{3}}$$
$$= 3^{-1}[(27)^2]^{\frac{1}{3}} = 3^{-1}[(3^3)^2]^{\frac{1}{3}}$$
$$= 3^{-1}\left[3^{\left(3\times2\times\frac{1}{3}\right)}\right] = 3^{-1} \times 3^2$$
$$= \boldsymbol{3}$$

(c) $2y^{\frac{1}{6}}$

Solution

$$2y^{\frac{1}{6}} = 2(-27)^{\frac{1}{6}}$$
$$= 2[(-3)^3]^{\frac{1}{6}}$$
$$= 2(-3)^{\frac{1}{2}}$$
$$= 2\sqrt{-3}$$

No solution

(d) $\left(\frac{1}{9}y\right)^{\frac{3}{4}}$

Solution

$$\left(\frac{1}{9}y\right)^{\frac{3}{4}} = \left(\frac{1}{9}\times -27\right)^{\frac{3}{4}}$$
$$= (-3)^{\frac{3}{4}} = \left(\sqrt[4]{-3}\right)^3$$
$$= \sqrt[4]{-27}$$

No solution

NOTE

Although it is said that there are no solutions to questions (c) and (d), what is meant is that the answers are not real values. In reality, the two questions have answers, which are complex numbers. For example, the solution to (c) are $2(j3) = j6$ and $2(-j3) = -j6$, i.e. $\pm j6$. For further details on complex numbers, kindly refer to '***Complex Numbers Explained with Worked Examples***' by Shefiu S. Zakariyah. This and others are freely available online.

25) Rewrite the following expressions using index notation without fractions.

(a) $\sqrt[5]{32x^2}$

Solution

$$\sqrt[5]{32x^2} = (32x^2)^{\frac{1}{5}}$$
$$= (32)^{\frac{1}{5}}(x^2)^{\frac{1}{5}}$$
$$= (2^5)^{\frac{1}{5}}x^{\frac{2}{5}}$$
$$= \mathbf{2x^{\frac{2}{5}}}$$

(b) $3x^2\left(\sqrt[3]{x}\right)$

Solution

$$3x^2\left(\sqrt[3]{x}\right) = 3x^2 \times x^{\frac{1}{3}}$$
$$= 3x^{\left(2+\frac{1}{3}\right)}$$
$$= \mathbf{3x^{\frac{7}{3}}}$$

(c) $\frac{1}{\sqrt[3]{x^2}}$

Solution

$$\frac{1}{\sqrt[3]{x^2}} = \frac{1}{(x^2)^{\frac{1}{3}}}$$
$$= \frac{1}{x^{\frac{2}{3}}}$$
$$= \mathbf{x^{-\frac{2}{3}}}$$

(d) $\frac{6\sqrt{x}}{x\left(\sqrt[3]{x}\right)}$

Solution

$$\frac{6\sqrt{x}}{x\left(\sqrt[3]{x}\right)} = \frac{6x^{\frac{1}{2}}}{x \times x^{\frac{1}{3}}}$$
$$= \frac{6x^{\frac{1}{2}}}{x^{\left(1+\frac{1}{3}\right)}} = \frac{6x^{\frac{1}{2}}}{x^{\frac{4}{3}}}$$
$$= 6x^{\frac{1}{2}-\frac{4}{3}} = 6x^{\frac{3-8}{6}}$$
$$= \mathbf{6x^{-\frac{5}{6}}}$$

26) Without using a calculator simplify $\left(\sqrt{5}\right)^{-2} + \left(\sqrt{5}\right)^{-1} + \left(\sqrt{5}\right)^{0} + \left(\sqrt{5}\right)^{1} + \left(\sqrt{5}\right)^{2}$, giving the answer in surd form.

Solution

$$\left(\sqrt{5}\right)^{-2} + \left(\sqrt{5}\right)^{-1} + \left(\sqrt{5}\right)^{0} + \left(\sqrt{5}\right)^{1} + \left(\sqrt{5}\right)^{2}$$
$$= \sqrt{5}\left\{\left(\sqrt{5}\right)^{-3} + \left(\sqrt{5}\right)^{-2} + \left(\sqrt{5}\right)^{-1} + \left(\sqrt{5}\right)^{0} + \left(\sqrt{5}\right)\right\}$$
$$= \sqrt{5}\left\{\frac{1}{\left(\sqrt{5}\right)^3} + \frac{1}{\left(\sqrt{5}\right)^2} + \frac{1}{\sqrt{5}} + 1 + \sqrt{5}\right\}$$

$$= \sqrt{5}\left\{\frac{1}{5\sqrt{5}} + \frac{1}{5} + \frac{1}{\sqrt{5}} + 1 + \sqrt{5}\right\}$$

$$= \sqrt{5}\left\{\frac{1 + \sqrt{5} + 5 + 5\sqrt{5} + 25}{5\sqrt{5}}\right\}$$

$$= \sqrt{5}\left\{\frac{31 + 6\sqrt{5}}{5\sqrt{5}}\right\} = \frac{31 + 6\sqrt{5}}{5}$$

$$= \frac{1}{5}\left(31 + 6\sqrt{5}\right)$$

Section 5: Conversions between Indices and Logarithms (Basic)

INTRODUCTION

In this section, we want to practise the conversion from an index notation to an equivalent logarithm form and vice versa. This is exclusively to show conversion between indices and logarithm and no final answer is therefore required.

27) Change each of the following index forms into their equivalent logarithmic forms.

(a) $7^3 = 343$

Solution

$$7^3 = 343$$
$$\Rightarrow$$
$$\log_7 343 = 3$$

(b) $3^6 = 729$

Solution

$$3^6 = 729$$
$$\Rightarrow$$
$$\log_3 729 = 6$$

(c) $125^{\frac{1}{3}} = 5$

Solution

$$125^{\frac{1}{3}} = 5$$
$$\Rightarrow$$
$$\log_{125} 5 = \frac{1}{3}$$

(d) $(0.01)^{2.5} = 0.00001$

Solution

$$(0.01)^{2.5} = 0.00001$$
$$\Rightarrow$$
$$\log_{0.01}(0.00001) = 2.5$$

(e) $2^{-3} = \frac{1}{8}$

Solution

$$2^{-3} = \frac{1}{8}$$
$$\Rightarrow$$
$$\log_2\left(\frac{1}{8}\right) = -3$$

(f) $\sqrt[3]{64} = 4$

Solution

$$\sqrt[3]{64} = 4$$
$$\Rightarrow$$
$$64^{\frac{1}{3}} = 4$$
$$\Rightarrow$$
$$\log_{64} 4 = \frac{1}{3}$$

(g) $3^x = 0.12$

Solution

$$3^x = 0.12$$
$$\Rightarrow$$
$$\log_3 0.12 = x$$

(h) $5^{\frac{t}{2}} = R$

Solution

$$5^{\frac{t}{2}} = R$$
$$\Rightarrow$$
$$\log_5 R = \frac{t}{2} \quad \text{or} \quad 2\log_5 R = t$$

(i) $e^{x+1} = 4$

Solution

$$e^{x+1} = 4$$
$$\Rightarrow$$
$$\log_e 4 = x + 1 \quad \text{or} \quad \ln 4 = x + 1$$

(j) $e^5 = 3V + 2$

Solution

$$e^5 = 3V + 2$$

$$\Rightarrow$$

$$\boldsymbol{\log_e(3V+2) = 5 \quad \text{or} \quad \ln(3V+2) = 5}$$

(k) $2.5e^{-Rt/L} = 5$

Solution

$$2.5e^{-Rt/L} = 5$$

Divide both sides by 2.5 and then convert as

$$e^{-Rt/L} = 2$$

$$\boldsymbol{\log_e 2 = -Rt/L \quad \text{or} \quad \ln 2 = -Rt/L}$$

(l) $6^{2t} - 4 = 2x$

Solution

$$6^{2t} - 4 = 2x$$

Re-arrange the above equation and then convert as

$$6^{2t} = 2x + 4$$

$$\Rightarrow$$

$$\boldsymbol{\log_6(3x+4) = 2t}$$

28) Express the following logarithmic forms in their corresponding index notation.

(a) $\log_{13} 1 = 0$

Solution

$$\log_{13} 1 = 0$$

$$\Rightarrow$$

$$\boldsymbol{13^0 = 1}$$

(b) $\log_{39} 39 = 1$

Solution

$$\log_{39} 39 = 1$$

$$\Rightarrow$$

$$\boldsymbol{39^1 = 39}$$

(c) $\log_{2.6} 6.76 = 2$

Solution

$$\log_{2.6} 6.76 = 2$$

$$\Rightarrow$$

$$\boldsymbol{2.6^2 = 6.76}$$

(d) $\log_4\left(\frac{1}{64}\right) = -3$

Solution

$$\log_4\left(\frac{1}{64}\right) = -3$$

$$\Rightarrow$$

$$\boldsymbol{4^{-3} = \frac{1}{64}}$$

(e) $\log_{\frac{1}{3}}\left(\frac{1}{27}\right) = 3$

Solution

$$\log_{\frac{1}{3}}\left(\frac{1}{27}\right) = 3$$

$$\Rightarrow$$

$$\boldsymbol{\left(\frac{1}{3}\right)^3 = \frac{1}{27}}$$

(f) $\log 2y = 0.1$

Solution

$$\log 2y = 0.1$$

$$\Rightarrow$$

$$\boldsymbol{10^{0.1} = 2y}$$

(g) $\log_e 0.3Q = -t/RC$

Solution

$$\log_e 0.3Q = -t/RC$$

$$\Rightarrow$$

$$\boldsymbol{e^{-t/RC} = 0.3Q}$$

(h) $\log_2(2a+b) = 3.6$

Solution

$$\log_2(2a+b) = 3.6$$

$$\Rightarrow$$

$$\boldsymbol{2^{3.6} = 2a + b}$$

Section 6: Applications of Laws of Logarithms

INTRODUCTION

In this section of the Worked Examples, questions are based on the laws of logarithms and are structured such that one out of the three main laws or their combinations can be used to solve a question as stated in the Hint box. The derived laws will be employed in subsequent sections.

29) Without using a calculator, simplify the following leaving the final answer in log form.

Hint

In the following questions, we will be applying Law 1 of logarithms exclusively.

$$\boxed{\log_x M + \log_x N = \log_x(MN)}$$

(a) $\log_5 11.5 + \log_5 2$

Solution

$$\log_5 11.5 + \log_5 2 = \log_5(11.5 \times 2)$$
$$= \mathbf{\log_5 23}$$

(b) $\log_7 \sqrt{2} + \log_7 \sqrt{18}$

Solution

$$\log_7 \sqrt{2} + \log_7 \sqrt{18} = \log_7(\sqrt{2} \times \sqrt{18})$$
$$= \log_7(\sqrt{2 \times 18})$$
$$= \log_7(\sqrt{36})$$
$$= \mathbf{\log_7 6}$$

(c) $\log_{11} 3 + \log_{11} 4 + \log_{11} 5$

Solution

$$\log_{11} 3 + \log_{11} 4 + \log_{11} 5 = \log_{11}(3 \times 4 \times 5)$$
$$= \mathbf{\log_{11} 60}$$

30) Without using a calculator, simplify the following leaving the final answer in log form.

Hint

In the following questions, we will be applying Law 2 of logarithms exclusively.

$$\boxed{\log_x M - \log_x N = \log_x\left(\frac{M}{N}\right)}$$

(a) $\log_{10} 35 - \log_{10} 17.5$

Solution

$$\log_{10} 35 - \log_{10} 17.5 = \log_{10}\left(\frac{35}{17.5}\right)$$
$$= \mathbf{\log_{10} 2}$$

(b) $\log_6 \sqrt[3]{250} - \log_6 \sqrt[3]{2}$

Solution

$$\log_6 \sqrt[3]{250} - \log_6 \sqrt[3]{2} = \log_6\left(\frac{\sqrt[3]{250}}{\sqrt[3]{2}}\right)$$
$$= \log_6\left(\sqrt[3]{\frac{250}{2}}\right)$$
$$= \log_6(\sqrt[3]{125})$$
$$= \log_6\left(\sqrt[3]{5^3}\right)$$
$$= \mathbf{\log_6 5}$$

(c) $\log_9 5^{\frac{5}{2}} - \log_9 25$

Solution

$$\log_9 5^{\frac{5}{2}} - \log_9 25 = \log_9\left(\frac{5^{\frac{5}{2}}}{25}\right)$$
$$= \log_9\left(\frac{5^{\frac{5}{2}}}{5^2}\right)$$
$$= \log_9\left(5^{\frac{5}{2}-2}\right)$$
$$= \log_9 5^{\frac{1}{2}}$$
$$= \mathbf{\log_9(\sqrt{5})}$$

31) Without using a calculator, simplify the following leaving the final answer in log form.

Hint

In the following questions, we will be applying Law 3 of logarithms exclusively.

$$\boxed{\log_x M^N = N \log_x M}$$

(a) $\log_3 16$

Solution

$$\log_3 16 = \log_3 16$$
$$= \log_3 2^4$$
$$= \mathbf{4 \log_3 2}$$

(b) $\log_2 \sqrt{125}$

Solution

$$\log_2 \sqrt{125} = \log_2 125^{\frac{1}{2}}$$
$$= \log_2 (5^3)^{\frac{1}{2}}$$
$$= \log_2 5^{\frac{3}{2}}$$
$$= \mathbf{\frac{3}{2} \log_2 5}$$

(c) $\log_5 \sqrt[3]{81}$

Solution

$$\log_5 \sqrt[3]{81} = \log_5 81^{\frac{1}{3}}$$
$$= \log_5 (3^4)^{\frac{1}{3}}$$
$$= \log_5 3^{\frac{4}{3}}$$
$$= \mathbf{\frac{4}{3} \log_5 3}$$

32) Without using a calculator, simplify the following.

Hint

In the following questions, we will be applying the relevant rule(s) of logarithms.

(a) $\log_5 125 + 3 \log_5 25 - \log_5 250$

Solution

$$\log_5 125 + 3 \log_5 25 - \log_5 250$$
$$= \log_5 5^3 + 3 \log_5 5^2 - \log_5 (125 \times 2)$$
$$= 3\log_5 5 + 2 \times 3 \log_5 5 - (\log_5 125 + \log_5 2)$$
$$= 3\log_5 5 + 2 \times 3 \log_5 5 - (\log_5 5^3 + \log_5 2)$$
$$= 3\log_5 5 + 6 \log_5 5 - 3\log_5 5 - \log_5 2$$
$$= 6 \log_5 5 - \log_5 2$$
$$= 6(1) - \log_5 2$$
$$= \mathbf{6 - \log_5 2}$$

(b) $\log_4 27^{\frac{3}{4}} + \log_4 9^{\frac{3}{2}} - \log_4 81$

Solution

$$\log_4 27^{\frac{3}{4}} + \log_4 9^{\frac{3}{2}} - \log_4 81$$
$$= \log_4 (3^3)^{\frac{3}{4}} + \log_4 (3^2)^{\frac{3}{2}} - \log_4 3^4$$
$$= \log_4 3^{\frac{9}{4}} + \log_4 3^3 - \log_4 3^4$$
$$= \log_4 \left(\frac{3^{\frac{9}{4}} \times 3^3}{3^4} \right)$$
$$= \log_4 \left[3^{\left(\frac{9}{4} + 3 - 4\right)} \right]$$
$$= \log_4 \left(3^{\frac{5}{4}} \right)$$
$$= \mathbf{\frac{5}{4} \log_4 3}$$

(c) $\log_2 \sqrt{2} + \log_3 \sqrt[3]{3} + \log_4 \sqrt[4]{4}$

Solution

$$\log_2 \sqrt{2} + \log_3 \sqrt[3]{3} + \log_4 \sqrt[4]{4}$$
$$= \log_2 2^{\frac{1}{2}} + \log_3 3^{\frac{1}{3}} + \log_4 4^{\frac{1}{4}}$$
$$= \frac{1}{2} \log_2 2 + \frac{1}{3} \log_3 3 + \frac{1}{4} \log_4 4$$
$$= \frac{1}{2}(1) + \frac{1}{3}(1) + \frac{1}{4}(1)$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$
$$= \frac{6 + 4 + 3}{12}$$
$$= \mathbf{\frac{13}{12}}$$

> **NOTE**
>
> Although we have only used numbers to show the three fundamental laws of logarithm, the same applies when variables (or letters) are used as would be shown later.

Section 7: Logarithms (simplification)

INTRODUCTION

The first part of the current section requires that we express a given logarithmic expression in terms of a log whose value is given. The second part is to prove the equality of a given problem.

33) Given that $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 5 = 0.6990$ and $\log_{10} 7 = 0.8451$, find the value of the following correct to 4 significant figures.

(a) $\log_{10} 14$

Solution

$$\begin{aligned}\log_{10} 14 &= \log_{10}(2 \times 7)\\ &= \log_{10} 2 + \log_{10} 7\\ &= 0.3010 + 0.8451\\ &= \mathbf{1.146}\end{aligned}$$

$$\therefore \mathbf{\log 14 = 1.146}$$

(b) $\log_{10} 40$

Solution

$$\begin{aligned}\log_{10} 40 &= \log_{10}(8 \times 5)\\ &= \log_{10}(2^3 \times 5)\\ &= \log_{10} 2^3 + \log_{10} 5\\ &= 3\log_{10} 2 + \log_{10} 5\\ &= 3(0.3010) + 0.6990\\ &= 0.903 + 0.6990\\ &= \mathbf{1.602}\end{aligned}$$

$$\therefore \mathbf{\log 40 = 1.602}$$

Alternatively,

$$\begin{aligned}\log_{10} 40 &= \log_{10}(4 \times 10)\\ &= \log_{10}(2^2 \times 10)\\ &= \log_{10} 2^2 + \log_{10} 10\\ &= 2\log_{10} 2 + 1\\ &= 2(0.3010) + 1\\ &= 0.6020 + 1\\ &= \mathbf{1.602}\end{aligned}$$

$$\therefore \mathbf{\log 40 = 1.602}$$

as before.

(c) $\log_{10}\left(\frac{1}{6}\right)$

Solution

$$\begin{aligned}\log_{10}\left(\frac{1}{6}\right) &= \log_{10} 6^{-1}\\ &= -\log_{10} 6\\ &= -(\log_{10} 2 \times 3)\\ &= -(\log_{10} 2 + \log_{10} 3)\\ &= -(0.3010 + 0.4771)\\ &= -(0.7781)\\ &= \mathbf{-0.7781}\end{aligned}$$

$$\therefore \mathbf{\log\left(\frac{1}{6}\right) = -0.7781}$$

(d) $\log 0.00042$

Solution

$$\begin{aligned}\log 0.00042 &= \log(42 \times 10^{-5})\\ &= \log 42 + \log 10^{-5}\\ &= \log 2 \times 3 \times 7 - 5\log 10\\ &= \log 2 + \log 3 + \log 7 - 5\\ &= 0.3010 + 0.4771 + 0.8451\\ &\quad - 5\\ &= 1.6232 - 5\\ &= \mathbf{-3.377}\end{aligned}$$

$$\therefore \mathbf{\log 0.00042 = -3.377}$$

(e) $\log_{10} 3.75 + \log_{10} 4$

Solution

$$\begin{aligned}\log_{10} 3.75 + \log_{10} 4 &= \log_{10}(3.75 \times 4)\\ &= \log_{10} 15\\ &= \log_{10}(3 \times 5)\\ &= \log_{10} 3 + \log_{10} 5\end{aligned}$$

$$= 0.4771 + 0.6990$$

$$\mathbf{= 1.176}$$

$$\therefore \mathbf{\log_{10} 3.75 + \log_{10} 4 = 1.176}$$

(f) $\log_2 25$

Solution

$$\log_2 25 = \frac{\log 25}{\log 2}$$

$$= \frac{\log 5^2}{\log 2}$$

$$= \frac{2\log 5}{\log 2}$$

$$= \frac{2(0.6990)}{0.3010}$$

$$\mathbf{= 4.645}$$

$$\therefore \mathbf{\log_2 25 = 4.645}$$

(g) $\log_7 0.5$

Solution

$$\log_7 0.5 = \frac{\log 0.5}{\log 7}$$

$$= \frac{\log\left(5 \times \frac{1}{10}\right)}{\log 7}$$

$$= \frac{\log 5 + \log 10^{-1}}{\log 7}$$

$$= \frac{\log 5 - \log 10}{\log 7}$$

$$= \frac{0.6990 - 1}{0.8451}$$

$$\mathbf{= -0.3562}$$

$$\therefore \mathbf{\log_7 0.5 = -0.3562}$$

34) Show that

(a) $\log_P Q \times \log_Q P = 1$

Solution

LHS

$$\log_P Q \times \log_Q P = \frac{\log Q}{\log P} \times \frac{\log P}{\log Q}$$

$$\mathbf{= 1\ (RHS)}$$

NOTE

Alternatively (though longer),

Let

$$\log_P Q = y$$

$$\Rightarrow$$

$$Q = P^y \quad - - - - - - - - \text{(i)}$$

and

$$\log_Q P = x$$

$$\Rightarrow$$

$$P = Q^x \quad - - - - - - - - \text{(ii)}$$

Substitute for b in the left-hand side of the original equation

$$\log_P Q \times \log_Q P = \log_P (P^y) \times \log_{(P^y)} P$$

$$= (y)\ \log_P \mathrm{P} \times \left(\frac{1}{y}\right) \log_Q Q$$

$$= \mathrm{y} \times \frac{1}{\mathrm{y}}$$

$$\mathbf{= 1\ (RHS)}$$

(b) $\log_b Q^{x-n} - \log_b Q^{y+n} = (x - y) \log_b Q$

Solution

LHS

$$\log_b Q^{x-n} - \log_b Q^{y+n} = \log_b \left\{\frac{(Q^{x-n})}{(Q^{y+n})}\right\}$$

$$= \log_b [Q^{(x-n)-(y+n)}]$$

$$= \log_b Q^{(x-y)}$$

$$= (x - y) \log_b Q$$

$$\mathbf{= RHS\ (proved)}$$

Section 8: Finding Anti-logarithms of Numbers

INTRODUCTION

Typically, logarithm of a number is required. However, the inverse might be needed sometimes. This is called anti-logarithm and it is the focus of this section. This requires a calculating device. Good, isn't it?

35) Determine the antilog to the given base of the following, giving the answer correct to 4 significant figures.

(a) $antilog\ 1.2545\ (to\ base\ 10)$

Solution

Let $\boldsymbol{a}$ equals 1.2545 $to\ base$ 10, thus

$$1.2545 = \log_{10} a$$
$$\Rightarrow$$
$$a = 10^{1.2545}$$
$$= \mathbf{17.97}$$

(b) $antilog\ 3.2467\ (to\ base\ e)$

Solution

Let $\boldsymbol{b}$ equals 3.2467 $to\ base$ e, thus

$$3.2467 = \log_e b = \ln b$$
$$\Rightarrow$$
$$b = e^{3.2467}$$
$$= \mathbf{25.71}$$

(c) $antilog\ 4.2730\ (to\ base\ 5)$

Solution

Let $\boldsymbol{c}$ equals 4.2730 $to\ base$ 5, thus

$$4.2730 = \log_5 c$$
$$\Rightarrow$$
$$c = 5^{4.2730}$$
$$= 969.8$$
$$= \mathbf{9.698 \times 10^2}$$

(d) $antilog\ \bar{3}.4786\ (to\ base\ 10)$

Solution

Let $\boldsymbol{d}$ equals $\bar{3}.4786$ $to\ base$ 10, thus

$$\bar{3}.4786 = \log_{10} d$$
$$\Rightarrow$$
$$d = 10^{\bar{3}.4786}$$
$$= 10^{-3+0.4786} = 10^{-2.5214}$$
$$= 0.003\ 010\ 232$$
$$= \mathbf{3.010 \times 10^{-3}}$$

(e) $antilog\ 2.0506\ (to\ base\ 6)$

Solution

Let $\boldsymbol{f}$ equals 2.0506 $to\ base$ 6, thus

$$2.0506 = \log_6 f$$
$$\Rightarrow$$
$$f = 6^{2.0506}$$
$$= \mathbf{39.42}$$

(f) $antilog\ \bar{2}.9675\ (to\ base\ 20)$

Solution

Let $\boldsymbol{g}$ equals $\bar{2}.9675$ $to\ base$ 20, thus

$$\bar{2}.9675 = \log_{20} g$$
$$\Rightarrow$$
$$g = 20^{\bar{2}.9675}$$
$$= 20^{-2+0.9675} = 20^{-1.0325}$$
$$= 0.045\ 361$$
$$= \mathbf{4.536 \times 10^{-2}}$$

NOTE

As would be shown later, the antilog of numbers to base 10 can be found from anti-log tables.

Section 9: Conversions between Indices and Logarithms (Advanced)

INTRODUCTION

This section provides advanced questions on logarithms and as a result, readers need to be familiar with laws of logarithms. If you have not done so, previous sections (5 - 8) are recommended starting points for beginners.

36) Express y in terms of x if

(a) $\log_5(y + 2) = x$

Solution

$$\log_5(y + 2) = x$$
$$\Rightarrow$$
$$y + 2 = 5^x$$

$$\therefore \boldsymbol{y = 5^x - 2}$$

(b) $\frac{1}{2}\log_3(y-5) = x$

Solution

$$\frac{1}{2}\log_3(y-5) = x$$

$$\Rightarrow$$

$$\log_3(y-5) = 2x$$

$$y - 5 = 3^{2x}$$

$$\therefore \mathbf{y = 3^{2x} + 5}$$

37) Rewrite the following in their simplest log form.

(a) $E = \frac{x^5}{y^{0.6}z^{0.3}}$

Solution

$$E = \frac{x^5}{y^{0.6}z^{0.3}}$$

$$\Rightarrow$$

$$\log E = \log\left[\frac{x^5}{y^{0.6}z^{0.3}}\right]$$

$$= \log x^5 - \log y^{0.6}z^{0.3}$$

$$= \log x^5 - [\log y^{0.6} + \log z^{0.3}]$$

$$= 5\log x - [0.6\log y + 0.3\log z]$$

$$\therefore \boldsymbol{\log E = 5\log x - 0.6\log y - 0.3\log z}$$

(b) $V = \frac{\pi h}{3}(l+h)(l-h)$

Solution

$$V = \frac{\pi h}{3}(l+h)(l-h)$$

$$\Rightarrow$$

$$\log V = \log\left[\frac{\pi h}{3}(l+h)(l-h)\right]$$

$$= \log\frac{\pi h}{3} + \log(l+h) + \log(l-h)$$

$$= \log \pi h - \log 3 + \log(l+h) + \log(l-h)$$

$$\therefore \boldsymbol{\log V = \log \pi + \log h - \log 3 + \log(l+h) + \log(l-h)}$$

(c) $Q = \frac{1}{\beta}(2\alpha - 1)^2(\sqrt[3]{\gamma})$

Solution

$$Q = \frac{1}{\beta}(2\alpha - 1)^2(\sqrt[3]{\gamma})$$

$$\Rightarrow$$

$$\ln Q = \ln\left[\frac{1}{\beta}(2\alpha - 1)^2(\sqrt[3]{\gamma})\right]$$

$$= \ln\left(\frac{1}{\beta}\right) + \ln(2\alpha-1)^2 + \ln\sqrt[3]{\gamma}$$

$$= \ln 1 - \ln\beta + 2\ln(2\alpha - 1) + \ln\gamma^{\frac{1}{3}}$$

$$= 0 - \ln\beta + 2\ln(2\alpha-1) + \frac{1}{3}\ln\gamma$$

$$\therefore \boldsymbol{\ln Q = 2\ln(2\alpha - 1) + \frac{1}{3}\ln\gamma - \ln\beta}$$

NOTE

Here we have taken the natural log of both sides of the equation rather than the common log. Apart from satisfying the instruction in the question, it is correct to take log to any base when dealing with this type of problem. We will employ this when solving equations later in this book.

(d) $f_c = \frac{1}{2\pi\sqrt{LC}}$

Solution

$$f_c = \frac{1}{2\pi\sqrt{LC}}$$

$$\Rightarrow$$

$$\log f_c = \log\left[\frac{1}{2\pi\sqrt{LC}}\right]$$

$$= \log 1 - \log 2\pi\sqrt{L \times C}$$

$$= 0 - \left[\log 2 + \log\pi + \log\sqrt{L} + \log\sqrt{C}\right]$$

$$= -\left[\log 2 + \log\pi + \log L^{\frac{1}{2}} + \log C^{\frac{1}{2}}\right]$$

$$= -\left[\log 2 + \log\pi + \frac{1}{2}\log L + \frac{1}{2}\log C\right]$$

$$\therefore \log f_c = -\left[\log 2 + \log \pi + \frac{1}{2}\log L + \frac{1}{2}\log C\right]$$

38) Express the following without logarithms.

(a) $\log y = \log a + \log b - \log c - 1$

Solution

$$\log y = \log a + \log b - \log c - 1$$
$$= (\log a + \log b) - (\log c + \log 10)$$
$$= (\log ab) - (\log 10c)$$
$$= \log\left[\frac{ab}{10c}\right]$$

Therefore

$$\log y = \log\left[\frac{ab}{10c}\right]$$

$\Rightarrow$

$$y = \frac{ab}{10c}$$

(b) $\log_2 P = 2 + \frac{1}{5}\log_2 R + 2\log_2 V$

Solution

$$\log_2 P = 2 + \frac{1}{5}\log_2 R + 2\log_2 V$$
$$= 2\log_2 2 + \log_2 R^{\frac{1}{5}} + \log_2 V^2$$
$$= \log_2 2^2 + \log_2 R^{\frac{1}{5}} + \log_2 V^2$$
$$= \log_2 4R^{\frac{1}{5}}V^2$$

Therefore

$$\log_2 P = \log_2 4R^{\frac{1}{5}}V^2$$

$\Rightarrow$

$$P = 4R^{\frac{1}{5}}V^2$$

(c) $\ln P = \frac{1}{2}\ln(V^2 + 3) - 3\ln R + 1$

Solution

$$\ln P = \frac{1}{2}\ln(V^2 + 3) - 3\ln R + 1$$
$$= \ln(V^2 + 3)^{\frac{1}{2}} - \ln R^3 + \ln e$$
$$= \ln\left[\frac{(V^2 + 3)^{\frac{1}{2}}e}{R^3}\right]$$

Therefore

$$\ln P = \ln\left[\frac{(V^2 + 3)^{\frac{1}{2}}e}{R^3}\right]$$

$\Rightarrow$

$$P = \frac{(V^2 + 3)^{\frac{1}{2}}e}{R^3}$$

(d) $\log_3 S = 2(\log_3 L + \log_3 2) - (\log_3 25 + 2\log_3 \pi + 2\log_3 r)$

Solution

$$\log_3 S = 2(\log_3 L + \log_3 2) - (\log_3 25 + 2\log_3 \pi + 2\log_3 r)$$
$$= 2(\log_3 2L) - (\log_3 25\pi^2 r^2)$$
$$= 2\log_3 2L - \log_3 25\pi^2 r^2$$
$$= \log_3 2^2L^2 - \log_3 25\pi^2 r^2$$
$$= \log_3\left\{\frac{4L^2}{25\pi^2 r^2}\right\}$$

Therefore

$$\log_3 S = \log_3\left\{\frac{4L^2}{25\pi^2 r^2}\right\}$$

$\Rightarrow$

$$S = \frac{4L^2}{25\pi^2 r^2}$$

Section 10: Indicial Equations (I) (Unknown as the Base)

INTRODUCTION

The next three sections deal with equations involving indices. In this current section, the base is the variable to solve for. Notes will be added on the key steps as deemed necessary particularly for leading questions.

39) Solve the following indicial equations given that $x > 0$.

(a) $x^{\frac{1}{4}} = 2$

Solution

$$x^{\frac{1}{4}} = 2$$

Raise both sides to the power of 4

$$\left(x^{\frac{1}{4}}\right)^4 = 2^4$$

$$x = 2^4$$

$$\therefore \boldsymbol{x = 16}$$

(b) $y^{\frac{2}{3}} = 4$

Solution

$$y^{\frac{2}{3}} = 4$$

Raise both sides to the power of $\frac{3}{2}$

$$\left(y^{\frac{2}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}}$$

$$y = (2^2)^{\frac{3}{2}}$$

$$y = 2^3$$

$$\therefore \boldsymbol{y = 8}$$

(c) $6z^3 = 162$

Solution

$$6z^3 = 162$$

Divide both sides by 6

$$z^3 = 27$$

Express 27 as a power of 3 so the powers on both the LHS and RHS cancel each other

$$z^3 = 3^3$$

$$\therefore \boldsymbol{z = 3}$$

(d) $2\alpha^{-3} = -250$

Solution

$$2\alpha^{-3} = -250$$

$$\alpha^{-3} = -125$$

$$(\alpha^{-3})^{-\frac{1}{3}} = (-125)^{-\frac{1}{3}}$$

$$\alpha = \frac{1}{(-125)^{\frac{1}{3}}}$$

$$\alpha = \frac{1}{[(-5)^3]^{\frac{1}{3}}}$$

$$\alpha = -\frac{1}{5}$$

$$\therefore \boldsymbol{\alpha = -\frac{1}{5}}$$

(e) $\beta^{\frac{5}{2}} = \beta^2\sqrt{3}$

Solution

$$\beta^{\frac{5}{2}} = \beta^2\sqrt{3}$$

Square both sides

$$\left(\beta^{\frac{5}{2}}\right)^2 = \left(\beta^2\sqrt{3}\right)^2$$

$$\beta^5 = 3\beta^4$$

$$\beta^5 - 3\beta^4 = 0$$

$$\beta^4(\beta - 3) = 0$$

Therefore, either

$$\beta^4 = 0$$

or

$$\beta - 3 = 0$$

$$\beta = 3$$

$$\therefore \boldsymbol{\beta = 0 \ or \ \beta = 3}$$

NOTE

Note that the approach shown below will give only one value $\boldsymbol{\beta}$.

$$\beta^{\frac{5}{2}} = \beta^2\sqrt{3}$$

$$\Longrightarrow$$

$$\beta^{\frac{5}{2}} = \beta^2\left(3^{\frac{1}{2}}\right)$$

Divide both sides by β^2

$$\frac{\beta^{\frac{5}{2}}}{\beta^2} = 3^{\frac{1}{2}}$$

$$\beta^{\frac{5}{2}-2} = 3^{\frac{1}{2}}$$

$$\beta^{\frac{1}{2}} = 3^{\frac{1}{2}}$$

Now square both sides

$$\left(\beta^{\frac{1}{2}}\right)^2 = \left(3^{\frac{1}{2}}\right)^2$$

$$\therefore \boldsymbol{\beta = 3}$$

(f) $4\gamma = \frac{1}{108\sqrt{\gamma}}$

Solution

$$4\gamma = \frac{1}{108\sqrt{\gamma}}$$

$$4\gamma = 108\gamma^{-\frac{1}{2}}$$

$$\gamma = 27\gamma^{-\frac{1}{2}}$$

Multiply both sides by $\gamma^{\frac{1}{2}}$ (one could also divide both sides by $\gamma^{-\frac{1}{2}}$)

$$\gamma \times \gamma^{\frac{1}{2}} = 27\gamma^{-\frac{1}{2}} \times \gamma^{\frac{1}{2}}$$

$$\gamma^{\frac{3}{2}} = 27\gamma^{-\frac{1}{2}+\frac{1}{2}}$$

$$\gamma^{1+\frac{1}{2}} = 27$$

$$\gamma^{\frac{3}{2}} = 27$$

Raise both sides to the power of $\frac{2}{3}$

$$\left(\gamma^{\frac{3}{2}}\right)^{\frac{2}{3}} = (27)^{\frac{2}{3}}$$

$$\gamma = (3^3)^{\frac{2}{3}}$$

$$\gamma = 3^2$$

$$\therefore \boldsymbol{\gamma = 9}$$

(g) $\omega^{\frac{2}{3}} = \sqrt[3]{4}\omega^{\frac{1}{3}}$

Solution

$$\omega^{\frac{2}{3}} = \sqrt[3]{4}\omega^{\frac{1}{3}}$$

$$\left(\omega^{\frac{2}{3}}\right)^3 = \left(\sqrt[3]{4}\omega^{\frac{1}{3}}\right)^3$$

$$\omega^2 = 4\omega$$

$$\omega^2 - 4\omega = 0$$

$$\omega(\omega - 4) = 0$$

Therefore, either

$$\omega = 0$$

or

$$\omega - 4 = 0$$

$$\omega = 4$$

$$\therefore \boldsymbol{\omega = 0 \text{ or } \omega = 4}$$

40) Solve the following equations.

Hint

Unlike most of the questions we have encountered so far, the indicial questions here will result in quadratic equations. So there will always be two values to the unknown variable. For clarity, I will provide relevant notes on the steps taken in solving each problem.

(a) $2 + x = \sqrt{8}x^{\frac{1}{2}}$

Solution

$$2 + x = \sqrt{8}x^{\frac{1}{2}}$$

Square both sides

$$(2 + x)^2 = \left(8^{\frac{1}{2}}x^{\frac{1}{2}}\right)^2$$

Open the bracket

$$4 + 4x + x^2 = 8x$$

Collect the like terms

$$x^2 - 4x + 4 = 0$$

Factorise the above equation

$$(x - 2)(x - 2) = 0$$

This implies that

$$x - 2 = 0$$

$$x = 2$$

$$\therefore \boldsymbol{x = 2 \text{ (twice)}}$$

(b) $v^{\frac{1}{2}} + 3v^{-\frac{1}{2}} = 4$

Solution

$$v^{\frac{1}{2}} + 3v^{-\frac{1}{2}} = 4$$

Multiply through by $v^{\frac{1}{2}}$

$$v^{\frac{1}{2}}\left(v^{\frac{1}{2}} + 3v^{-\frac{1}{2}}\right) = 4v^{\frac{1}{2}}$$

$$\Rightarrow$$

$$v^{\frac{1}{2}} \times v^{\frac{1}{2}} + 3v^{-\frac{1}{2}} \times v^{\frac{1}{2}} = 4v^{\frac{1}{2}}$$

This equals

$$v^{\frac{1}{2}+\frac{1}{2}} + 3v^{-\frac{1}{2}+\frac{1}{2}} = 4v^{\frac{1}{2}}$$

$$\Rightarrow$$

$$v + 3v^0 = 4v^{\frac{1}{2}}$$

$$v + 3 = 4v^{\frac{1}{2}}$$

Now square both sides

$$(v+3)^2 = \left(4v^{\frac{1}{2}}\right)^2$$

Open the bracket

$$v^2 + 6v + 9 = 16v$$

Collect the like terms

$$v^2 - 10v + 9 = 0$$

Factorise the above equation

$$(v-1)(v-9) = 0$$

Therefore, either

$$v - 1 = 0$$

$$v = 1$$

or

$$v - 9 = 0$$

$$v = 9$$

$$\therefore \boldsymbol{v = 1 \text{ or } v = 9}$$

(c) $y^{\frac{1}{5}} + y^{\frac{2}{5}} = 2$

Solution

$$y^{\frac{1}{5}} + y^{\frac{2}{5}} = 2$$

Re-arrange this to have

$$y^{\frac{2}{5}} + y^{\frac{1}{5}} - 2 = 0$$

$$\left(y^{\frac{1}{5}}\right)^2 + y^{\frac{1}{5}} - 2 = 0$$

$$\left(y^{\frac{1}{5}} + 2\right)\left(y^{\frac{1}{5}} - 1\right) = 0$$

Therefore, either

$$y^{\frac{1}{5}} + 2 = 0$$

$$y^{\frac{1}{5}} = -2$$

Raise both sides to power 5

$$y = (-2)^5$$

$$y = -32$$

or

$$y^{\frac{1}{5}} - 1 = 0$$

$$y^{\frac{1}{5}} = 1$$

Raise both sides to power 5

$$y = (1)^5$$

$$y = 1$$

$$\therefore \boldsymbol{y = 1 \text{ or } -32}$$

NOTE

In this question we could have simplified the equation as we did in the preceding ones until we have a quadratic equation of the form

$$\boldsymbol{ax^2 + bx + c = 0}$$

and then solve for the unknown variable ($\boldsymbol{y}$ in this case). However, this is not always easy and/or necessary. Once the power of one term is twice the other then it is a quadratic equation and we can use any appropriate technique of solving a quadratic equation. To simplify it, we sometimes substitute for the terms with fractional power as will be shown in the next question.

41) By letting $b = a^{\frac{1}{3}}$, or otherwise, find the values of a for which $a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} = 1$.

Solution

$$a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} = 1$$

$$a^{\frac{1}{3}} - \frac{2}{a^{\frac{1}{3}}} = 1$$

Now let $b = a^{\frac{1}{3}}$ and substitute this in the above equation, which gives

$$b - \frac{2}{b} = 1$$

Multiply through by b and collect the like terms

$$b^2 - 2 = b$$

$$b^2 - b - 2 = 0$$

$$(b - 2)(b + 1) = 0$$

Therefore, either

$$b - 2 = 0$$

$$b = 2$$

Now substitute for b

$$a^{\frac{1}{3}} = 2$$

$$a = 2^3$$

$$a = 8$$

or

$$b + 1 = 0$$

$$b = -1$$

Again, substitute for b

$$a^{\frac{1}{3}} = -1$$

$$a = (-1)^3$$

$$a = -1$$

$$\therefore \boldsymbol{a = 8 \text{ or } a = -1}$$

NOTE

Alternatively,

$$a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} = 1$$

$$a^{\frac{1}{3}} - \frac{2}{a^{\frac{1}{3}}} = 1$$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} - 2 = a^{\frac{1}{3}}$$

$$\left(a^{\frac{1}{3}}\right)^2 - a^{\frac{1}{3}} - 2 = 0$$

$$\left(a^{\frac{1}{3}} - 2\right)\left(a^{\frac{1}{3}} + 1\right) = 0$$

Therefore, either

$$a^{\frac{1}{3}} - 2 = 0$$

$$a^{\frac{1}{3}} = 2$$

$$a = 8$$

or

$$a^{\frac{1}{3}} - 1 = 0$$

$$a^{\frac{1}{3}} = -1$$

$$a = (-1)^3$$

$$a = -1$$

$$\therefore \boldsymbol{a = 8 \text{ or } a = -1}$$

as before.

Section 11: Indicial Equations (II) (Unknown as the Index)

INTRODUCTION

We are still dealing with indicial equations and in this current section, the exponent (or index) is the variable to solve for. Again, notes will accompany the solution when appropriate.

42) Solve the following indicial equations.

Hint

In the following questions, if there can be only one term on each side of the equation such that they are both in the same base to a certain power then equate the index - the job is done! It is as easy as that, so let's get started.

(a) $2^x = 1$

Solution

$$2^x = 1$$
$$2^x = 2^0$$
$$\therefore \boldsymbol{x = 0}$$

(b) $100^x = 1000$

Solution

$$100^x = 1000$$
$$(10^2)^x = 10^3$$
$$10^{2x} = 10^3$$

Therefore

$$2x = 3$$
$$\therefore \boldsymbol{x = \frac{3}{2}}$$

(c) $25^x = 125$

Solution

$$25^x = 125$$
$$5^{2x} = 5^3$$

Therefore

$$2x = 3$$
$$\therefore \boldsymbol{x = \frac{3}{2}}$$

(d) $6^{-x} = 216$

Solution

$$6^{-x} = 216$$
$$6^{-x} = 6^3$$

Therefore

$$-x = 3$$
$$\therefore \boldsymbol{x = -3}$$

(e) $49^y = \frac{1}{343}$

Solution

$$49^y = \frac{1}{343}$$
$$(7^2)^y = \frac{1}{7^3}$$
$$(7^2)^y = 7^{-3}$$
$$7^{2y} = 7^{-3}$$

Therefore

$$2y = -3$$
$$\therefore \boldsymbol{y = -\frac{3}{2}}$$

(f) $4^{y-3} = 64$

Solution

$$4^{y-3} = 64$$
$$4^{y-3} = 4^3$$

Therefore

$$y - 3 = 3$$
$$\therefore \boldsymbol{y = 6}$$

(g) $2^{y+2} = 0.125$

Solution

$$2^{y+2} = 0.125$$
$$2^{y+2} = \frac{125}{1000}$$
$$= \frac{1}{8} = \frac{1}{2^3}$$
$$= 2^{-3}$$

Therefore, if

$$2^{y+2} = 2^{-3}$$

then

$$y + 2 = -3$$
$$\therefore \boldsymbol{y = -5}$$

(h) $\frac{8}{2^{-y}} = 32^y$

Solution

$$\frac{8}{2^{-y}} = 32^y$$
$$8(2^y) = (2^5)^y$$
$$(2^3)(2^y) = (2^5)^y$$

$$2^{3+y} = 2^{5y}$$

Therefore

$$3 + y = 5y$$
$$3 = 4y$$
$$\therefore \boldsymbol{y = \frac{3}{4}}$$

(i) $3^{4m} \div 3^{3-m} = \frac{1}{243}$

Solution

$$3^{4m} \div 3^{3-m} = \frac{1}{243}$$
$$3^{4m-(3-m)} = \frac{1}{3^5}$$
$$3^{4m-3+m} = 3^{-5}$$
$$3^{5m-3} = 3^{-5}$$

Therefore

$$5m - 3 = -5$$
$$5m = -2$$
$$\therefore \boldsymbol{m = -\frac{2}{5}}$$

(j) $2^{m+5} \div 4^{-m} = 32$

Solution

$$2^{m+5} \div 4^{-m} = 32$$
$$2^{m+5} \div (2^2)^{-m} = 2^5$$
$$2^{m+5} \div 2^{-2m} = 2^5$$
$$2^{m+5+2m} = 2^5$$
$$2^{3m+5} = 2^5$$

Therefore

$$3m + 5 = 5$$
$$3m = 5 - 5$$
$$3m = 0$$
$$\therefore \boldsymbol{m = 0}$$

(k) $(3^n)^3 \times 3^{3-n} = \frac{1}{27}$

Solution

$$(3^n)^3 \times 3^{3-n} = \frac{1}{27}$$
$$3^{3n} \times 3^{3-n} = 3^{-3}$$
$$3^{3n+3-n} = 3^{-3}$$
$$3^{2n+3} = 3^{-3}$$

Therefore

$$2n + 3 = -3$$
$$2n = -6$$
$$\therefore \boldsymbol{n = -3}$$

(l) $2^{2n+3} - 2^{2n+1} = \frac{3}{8}$

Solution

$$2^{2n+3} - 2^{2n+1} = \frac{3}{8}$$
$$2^{2n}(2^3) - 2^{2n}(2) = \frac{3}{8}$$

Factorise

$$2^{2n}(2^3 - 2) = \frac{3}{8}$$
$$2^{2n}(6) = \frac{3}{8}$$

Divide both sides by 6

$$2^{2n} = 3/8 \div 6$$
$$2^{2n} = \frac{3}{8} \times \frac{1}{6}$$
$$2^{2n} = \frac{1}{16}$$
$$2^{2n} = 2^{-4}$$

Therefore

$$2n = -4$$
$$\therefore \boldsymbol{n = -2}$$

NOTE

Alternatively, we could have solved this problem by substitution as shown below. This approach will be used for questions in **No 43**.

$$2^{2n+3} - 2^{2n+1} = \frac{3}{8}$$

Let $\boldsymbol{x = 2^{2n}}$ thus

$$x(2^3) - x(2) = \frac{3}{8}$$
$$8x - 2x = \frac{3}{8}$$
$$6x = \frac{3}{8}$$

Divide both sides by 6

$$x = \frac{1}{16}$$

$$= \frac{1}{2^4}$$

$$= 2^{-4}$$

Substituting back, we have

$$2^{2n} = 2^{-4}$$

Therefore

$$2n = -4$$

$$\therefore \boldsymbol{n = -2}$$

as before.

(m) $6^{1-2n} \times 36^{2n-1} = 216$

Solution

$$6^{1-2n} \times 36^{2n-1} = 216$$

$$6^{1-2n} \times (6^2)^{2n-1} = 6^3$$

$$6^{1-2n} \times 6^{4n-2} = 6^3$$

$$6^{1-2n+4n-2} = 6^3$$

$$6^{2n-1} = 6^3$$

Therefore

$$2n - 1 = 3$$

$$2n = 4$$

$$\therefore \boldsymbol{n = 2}$$

(n) $\frac{5^{2x-3}}{25^{3x-2}} = 125$

Solution

$$\frac{5^{2x-3}}{25^{3x-2}} = 125$$

$$\frac{5^{2x-3}}{(5^2)^{3x-2}} = 5^3$$

$$\frac{5^{2x-3}}{5^{6x-4}} = 5^3$$

$$5^{2x-3-(6x-4)} = 5^3$$

$$5^{2x-3-6x+4} = 5^3$$

$$5^{-4x+1} = 5^3$$

Therefore

$$-4x + 1 = 3$$

$$-4x = 2$$

$$\therefore \boldsymbol{x = -\frac{1}{2}}$$

(o) $\frac{27^{3x}}{3^{x+4}} = \frac{243}{9^{x-2}}$

Solution

$$\frac{27^{3x}}{3^{x+4}} = \frac{243}{9^{x-2}}$$

$$\frac{(3^3)^{3x}}{3^{x+4}} = \frac{3^5}{(3^2)^{x-2}}$$

$$\frac{3^{9x}}{3^{x+4}} = \frac{3^5}{3^{2x-4}}$$

$$3^{9x-x-4} = 3^{5-2x+4}$$

$$3^{8x-4} = 3^{9-2x}$$

Therefore

$$8x - 4 = 9 - 2x$$

$$10x = 13$$

$$\therefore \boldsymbol{x = \frac{13}{10}}$$

43) Solve the following equations.

Hint

In the following questions, the simplification will lead to a quadratic equation and therefore two solutions to the unknown variable.

(a) $2^{2x} - 10(2^x) + 16 = 0$

Solution

$$2^{2x} - 10(2^x) + 16 = 0$$

$$(2^x)^2 - 10(2^x) + 16 = 0$$

Let $\boldsymbol{y = 2^x}$ thus

$$y^2 - 10y + 16 = 0$$

$$(y - 8)(y - 2) = 0$$

Therefore, either

$$y - 8 = 0$$

$$y = 8$$

$$2^x = 8$$

$$2^x = 2^3$$
$$x = 3$$

or

$$y - 2 = 0$$
$$y = 2$$
$$2^x = 2$$
$$2^x = 2^1$$
$$x = 1$$
$$\therefore \boldsymbol{x = 1 \text{ or } x = 3}$$

(b) $2^{2x+1} - 3(2^x) + 1 = 0$

Solution

$$2^{2x+1} - 3(2^x) + 1 = 0$$
$$2(2^{2x}) - 3(2^x) + 1 = 0$$
$$2(2^x)^2 - 3(2^x) + 1 = 0$$

Let $\boldsymbol{w = 2^x}$ thus

$$2w^2 - 3w + 1 = 0$$
$$(2w - 1)(w - 1) = 0$$

Therefore, either

$$2w - 1 = 0$$
$$w = \frac{1}{2}$$
$$2^x = \frac{1}{2}$$
$$2^x = 2^{-1}$$
$$x = -1$$

or

$$w - 1 = 0$$
$$w = 1$$
$$2^x = 2^0$$
$$x = 0$$
$$\therefore \boldsymbol{x = -1 \text{ or } x = 0}$$

(c) $3^{2x+1} - 4(3^{x+1}) + 9 = 0$

Solution

$$3^{2x+1} - 4(3^{x+1}) + 9 = 0$$
$$3(3^{2x}) - 4(3^x \times 3) + 9 = 0$$
$$3(3^x)^2 - 12(3^x) + 9 = 0$$

Let $\boldsymbol{v = 3^x}$ thus

$$3v^2 - 12v + 9 = 0$$
$$(3v - 3)(v - 3) = 0$$

Therefore, either

$$3v - 3 = 0$$
$$3v = 3$$
$$v = 1$$
$$3^x = 3^0$$
$$x = 0$$

or

$$v - 3 = 0$$
$$v = 3$$
$$3^x = 3$$
$$3^x = 3^1$$
$$x = 1$$
$$\therefore \boldsymbol{x = 0 \text{ or } x = 1}$$

(d) $5^{2x} - 20(5^{x-1}) - 5 = 0$

Solution

$$5^{2x} - 20(5^{x-1}) - 5 = 0$$
$$5^{2x} - 20(5^x \times 5^{-1}) - 5 = 0$$
$$5^{2x} - 20 \times \frac{1}{5}(5^x) - 5 = 0$$
$$(5^x)^2 - 4(5^x) - 5 = 0$$

Let $\boldsymbol{u = 5^x}$ thus

$$u^2 - 4u - 5 = 0$$
$$(u - 5)(u + 1) = 0$$

Therefore, either

$$u - 5 = 0$$
$$u = 5$$
$$5^x = 5$$
$$5^x = 5^1$$
$$x = 1$$

or

$$u + 1 = 0$$
$$u = -1$$
$$5^x = -1$$

No solution for x in this case

$$\therefore \boldsymbol{x = 1}$$

(e) $6^{2x} - 9(6^x) + 18 = 0$

Solution

$$6^{2x} - 9(6^x) + 18 = 0$$
$$(6^x)^2 - 9(6^x) + 18 = 0$$

Let $\boldsymbol{m = 6^x}$ thus

$$m^2 - 9m + 18 = 0$$
$$(m - 6)(m - 3) = 0$$

Therefore, either

$$m - 6 = 0$$
$$m = 6$$
$$4^x = 6$$
$$4^x = 4^1$$
$$x = 1$$

or

$$m - 3 = 0$$
$$m = 3$$
$$6^x = 3$$
$$x = \log_6 3$$
$$x = 0.6131$$
$$\therefore \boldsymbol{x = 1 \text{ or } x = 0.6131}$$

(f) $10^{2x} - 20(10^x) + 100 = 0$

Solution

$$10^{2x} - 20(10^x) + 100 = 0$$
$$(10^x)^2 - 20(10^x) + 100 = 0$$

Let $\boldsymbol{n = 10^x}$ thus

$$n^2 - 20n + 100 = 0$$
$$(n - 10)(n - 10) = 0$$

Therefore

$$n - 10 = 0$$
$$n = 10$$
$$10^x = 10$$
$$10^x = 10^1$$
$$x = 1$$
$$\therefore \boldsymbol{x = 1 \text{ (twice)}}$$

Section 12: Indicial Equations (III) (Involving Logarithms)

INTRODUCTION

In this final category of indicial equations, we will be dealing with equations where it will not be practical to express all terms in the same base. The way out in this case is to employ some rules (or properties) of logarithms as will be shown shortly. Whilst the examples use both natural and common logs, logarithms to any base can be employed.

44) Solve the following equations, giving the results to 4 significant figures.

(a) $3^x = 7$

Solution

$$3^x = 7$$

We will need to take the logarithm to base 10 of both sides.

$$\log 3^x = \log 7$$
$$x \log 3 = \log 7$$
$$x = \frac{\log 7}{\log 3}$$
$$x = \frac{0.84510}{0.47712}$$
$$\therefore \boldsymbol{x = 1.771}$$

NOTE

Alternatively, we could have solved this problem by converting the index form to logarithmic form as

$$3^x = 7$$
$$\Rightarrow$$
$$x = \log_3 7$$
$$\boldsymbol{= 1.771}$$

as before.

It also worth noting that since 7 cannot be expressed in the form of the 3^n where n is any real number ($\mathbf{3^n}, \boldsymbol{n} \in \mathbb{R}$) the only viable means of solving the problem is by using logarithmic relationship.

(b) $5^{2x-1} = 11$

Solution

$$5^{2x-1} = 11$$

$$\Rightarrow$$

$$\ln 5^{2x-1} = \ln 11$$

$$(2x-1)\ln 5 = \ln 11$$

$$2x - 1 = \frac{\ln 11}{\ln 5}$$

$$2x - 1 = 1.4899$$

$$2x = 1 + 1.4899$$

$$x = \frac{2.4899}{2}$$

$$\therefore \boldsymbol{x = 1.245}$$

(c) $2^{2x} = 5^{x-3}$

Solution

$$2^{2x} = 5^{x-3}$$

$$\Rightarrow$$

$$\log 2^{2x} = \log 5^{x-3}$$

$$2x \log 2 = (x-3)\log 5$$

$$2x \log 2 = x \log 5 - 3 \log 5$$

$$2x \log 2 - x \log 5 = -3 \log 5$$

$$x(2\log 2 - \log 5) = -3 \log 5$$

$$x = \frac{-3\log 5}{2\log 2 - \log 5}$$

$$= \frac{-3(0.69897)}{2(0.30103) - (0.69897)}$$

$$\therefore \boldsymbol{x = 21.64}$$

(d) $4 \times 7^{3x+1} = 6^{4x}$

Solution

$$4 \times 7^{3x+1} = 6^{4x}$$

This implies

$$\ln(4 \times 7^{3x+1}) = \ln(6^{4x})$$

$$\ln 4 + \ln 7^{3x+1} = \ln 6^{4x}$$

$$\ln 4 + (3x+1)\ln 7 = 4x \ln 6$$

$$\ln 4 + 3x \ln 7 + \ln 7 = 4x \ln 6$$

$$\ln 4 + \ln 7 = 4x \ln 6 - 3x \ln 7$$

$$\ln(4 \times 7) = x\,(4\ln 6 - 3\ln 7)$$

$$\ln 28 = x\,(\ln 6^4 - \ln 7^3)$$

$$x = \frac{\ln 28}{\ln 6^4 - \ln 7^3}$$

$$x = \frac{3.33220}{7.16704 - 5.83773}$$

$$\therefore \boldsymbol{x = 2.5067}$$

(e) $6(8^{3x+2}) = 5^{2x-7}$

Solution

$$6(8^{3x+2}) = 5^{2x-7}$$

This implies

$$\ln 6(8^{3x+2}) = \ln 5^{2x-7}$$

$$\ln 6 + \ln 8^{3x+2} = (2x-7)\ln 5$$

$$\ln 6 + (3x+2)\ln 8 = 2x \ln 5 - 7\ln 5$$

$$\ln 6 + 3x\ln 8 + 2\ln 8 = 2x\ln 5 - 7\ln 5$$

Collect the like terms

$$\ln 6 + 2\ln 8 + 7\ln 5 = 2x\ln 5 - 3x\ln 8$$

$$\ln 6 + \ln 8^2 + \ln 5^7 = x\,(\ln 5^2 - \ln 8^3)$$

$$\ln(6 \times 8^2 \times 5^7) = x\left[\ln\left(\frac{5^2}{8^3}\right)\right]$$

$$x = \frac{\ln(6 \times 8^2 \times 5^7)}{\ln\left(\frac{5^2}{8^3}\right)}$$

$$x = \frac{17.217}{-3.0194}$$

$$\therefore \boldsymbol{x = -5.702}$$

(f) $5.5^{2-x} \times 9^{2x+3} = 10.5^{7x}$

Solution

$$5.5^{2-x} \times 9^{2x+3} = 10.5^{7x}$$

$$\log 5.5^{2-x} \times 9^{2x+3} = \log 10.5^{7x}$$

$$\log 5.5^{2-x} + \log 9^{2x+3} = 7x \log 10.5$$

$$(2-x)\log 5.5 + (2x+3)\log 9 = 7x\log 10.5$$

Open the brackets

$$2\log 5.5 - x\log 5.5 + 2x\log 9 + 3\log 9 = 7x\log 10.5$$

Collect the like terms

$$2\log 5.5 + 3\log 9 = 7x\log 10.5 + x\log 5.5 - 2x\log 9$$

$$\log 5.5^2 + \log 9^3 = x(\log 10.5^7 + \log 5.5 - \log 9^2)$$

Simplify and make x the subject

$$\log(5.5^2 \times 9^3) = x\log\left(\frac{10.5^7 \times 5.5}{9^2}\right)$$

$$x = \frac{\log(5.5^2 \times 9^3)}{\log\left(\frac{10.5^7 \times 5.5}{9^2}\right)}$$

$$\therefore \boldsymbol{x = 0.7263}$$

NOTE

Alternatively (for ease of evaluation using a calculator),

$$\log 5.5^2 + \log 9^3 = x(\log 10.5^7 + \log 5.5 - \log 9^2)$$

$$x = \frac{\log 5.5^2 + \log 9^3}{\log 10.5^7 + \log 5.5 - \log 9^2}$$

$$= \frac{1.48073 + 2.86273}{7.14833 + 0.74036 - 1.90849}$$

$$= \frac{4.34346}{5.9802}$$

$$\therefore \boldsymbol{x = 0.7263}$$

as before.

(g) $3(13.6^{x-3}) \times 2(4^{3x}) = 3.8^{x-1}$

Solution

$$3(13.6^{x-3}) \times 2(4^{3x}) = 3.8^{x-1}$$

This implies

$$\log 3(13.6^{x-3}) \times 2(4^{3x}) = \log 3.8^{x-1}$$

$$\log 3(13.6^{x-3}) + \log 2(4^{3x}) = (x-1)\log 3.8$$

$$\log 3 + (x-3)\log 13.6 + \log 2 + 3x\log 4 = x\log 3.8 - \log 3.8$$

Open the brackets

$$\log 3 + x\log 13.6 - 3\log 13.6 + \log 2 + 3x\log 4 = x\log 3.8 - \log 3.8$$

Collect the like terms

$$\log 3 + \log 2 + \log 3.8 - \log 13.6^3 = x\log 3.8 - x\log 13.6 - 3x\log 4$$

Simplify and make x the subject

$$\log 3 + \log 2 + \log 3.8 - \log 13.6^3 = x(\log 3.8 - \log 13.6 - \log 4^3)$$

$$\log\left(\frac{3 \times 2 \times 3.8}{13.6^3}\right) = x\left[\log\left(\frac{3.8}{13.6 \times 4^3}\right)\right]$$

$$x = \frac{\log\left(\frac{22.8}{13.6^3}\right)}{\log\left(\frac{3.8}{13.6 \times 4^3}\right)}$$

$$x = \frac{-2.04268}{-2.35994}$$

$$\therefore \boldsymbol{x = 0.8656}$$

Section 13: Logarithmic Equations

INTRODUCTION

Another class of equations, known as logarithmic equations, will be dealt with in this section. In simple cases, we can convert a logarithmic equation into an indicial equivalent. In a more advanced and complex case, the problem is simplified such that there is only a term on both sides of the equation each expressed in log form to the same base. Once this is done, the following property of logarithm is applied to determine the variable and the job is done.

If

$$\boldsymbol{\log_x M = \log_x N,}$$

$$\textbf{then} \quad \boldsymbol{M = N}$$

45) Solve the following logarithmic equations leaving the answer correct to 3 significant figures.

(a) $\log_2(3x-1)=2$

Solution

$$\log_2(3x-1)=2$$

Write this in index form

$$(3x-1)=2^2$$
$$3x-1=4$$
$$3x=5$$
$$x=\frac{5}{3}$$
$$\therefore\ \boldsymbol{x=1.60}$$

(b) $\ln(x+5)=3$

Solution

$$\ln(x+5)=3$$

Write this in index form

$$x+5=e^3$$
$$x+5=20.086$$
$$x=15.086$$
$$\therefore\ \boldsymbol{x=15.1}$$

(c) $\log_{11}(x^2-6)=\frac{1}{3}$

Solution

$$\log_{11}(x^2-6)=\frac{1}{3}$$

Write this in index form

$$(x^2-6)=11^{\frac{1}{3}}$$
$$x^2=6+\sqrt[3]{11}$$
$$x^2=8.2240$$
$$x=\pm\sqrt{8.2240}$$
$$\therefore\ \boldsymbol{x=\pm 2.87}$$

(d) $\ln(x^2-2)=1$

Solution

$$\ln(x^2-2)=1$$

Write this in index form

$$x^2-2=e^1$$
$$x^2=2+e$$
$$x=\sqrt{2+e}$$
$$\therefore\ \boldsymbol{x=\pm 2.17}$$

(e) $\log_2 x-\log_4 3=2$

Solution

$$\log_2 x-\log_4 3=2$$

Write this in index form

$$\log_2 x-\frac{\log_2 3}{\log_2 4}=2$$
$$\log_2 x-\frac{\log_2 3}{\log_2 2^2}=2$$
$$\log_2 x-\frac{\log_2 3}{2}=2$$
$$2\log_2 x-\log_2 3=4$$
$$\log_2 x^2-\log_2 3=4$$
$$\log_2\left(\frac{x^2}{3}\right)=4$$
$$\frac{x^2}{3}=2^4$$
$$\frac{x^2}{3}=16$$
$$x^2=3\times 16=48$$
$$x=\pm\sqrt{48}=\pm 4\sqrt{3}$$
$$\therefore\ \boldsymbol{x=\pm 6.923}$$

NOTE

Whilst the values of x are 6.923 and −6.923, the only one that satisfy the original logarithmic equation $\log_2 x-\log_4 3=2$ is $x=6.923$.

However both values satisfy the modified version of the equation, i.e. $\log_2 x^2 - \log_2 3 = 4$.

Alternatively, we can solve the equation as follows

$$\log_2 x - \log_4 3 = 2$$

$$\log_2 x - \log_{2^2} 3 = 2\log_2 2$$

$$\log_2 x - \frac{1}{2}\log_2 3 = \log_2 2^2$$

$$\log_2\left(\frac{x}{\sqrt{3}}\right) = \log_2 4$$

Therefore,

$$\frac{x}{\sqrt{3}} = 4$$

$$\therefore \boldsymbol{x = 4\sqrt{3}}$$

46) Solve the following logarithmic equations.

(a) $\log_9(5x - 4) = \log_3 4$

Solution

$$\log_9(5x - 4) = \log_3 4$$

$$\Longrightarrow$$

$$\frac{\log_3(5x - 4)}{\log_3 9} = \log_3 4$$

$$\frac{\log_3(5x - 4)}{\log_3 3^2} = \log_3 4$$

$$\frac{\log_3(5x - 4)}{2} = \log_3 4$$

$$\log_3(5x - 4) = 2\log_3 4$$

$$\log_3(5x - 4) = \log_3 4^2$$

Thus

$$5\text{x} - 4 = 4^2$$

$$5\text{x} - 4 = 16$$

$$5x = 20$$

$$x = \frac{20}{5}$$

$$\therefore \boldsymbol{x = 4}$$

NOTE

Alternatively,

$$\log_9(5x - 4) = \log_3 4$$

change the base of the LHS to 3, we then have

$$\frac{1}{2}\log_3(5x - 4) = \log_3 4$$

$$\log_3(5x - 4) = 2\log_3 4$$

$$\log_3(5x - 4) = \log_3 4^2$$

Thus

$$5\text{x} - 4 = 4^2$$

$$5\text{x} - 4 = 16$$

$$5x = 20$$

$$x = \frac{20}{5}$$

$$\therefore \boldsymbol{x = 4}$$

as before.

(b) $\log_6 x + \log_6(x - 5) = 2$

Solution

$$\log_6 x + \log_6(x - 5) = 2$$

$$\Longrightarrow$$

$$\log_6 x(x - 5) = \log_6 36$$

Therefore

$$x(x - 5) = 36$$

$$x^2 - 5x - 36 = 0$$

$$(x + 4)(x - 9) = 0$$

Either

$$x + 4 = 0$$

$$x = -4$$

or

$$x - 9 = 0$$

$$x = 9$$

$$\therefore \boldsymbol{x = -4 \text{ or } x = 9}$$

NOTE

Although the values of x are -4 and 9, the only one that satisfy the original logarithmic equation is $x = 9$ because log is not valid for negative numbers. Therefore $x = -4$ is not a solution to this equation.

(c) $\log_3(x+2) + \log_3(x+4) = 1$

Solution

$$\log_3(x+2) + \log_3(x+4) = 1$$

$\Rightarrow$

$$\log_3(x+2)(x+4) = \log_3 3$$

Therefore

$$(x+2)(x+4) = 3$$
$$x^2 + 6x + 8 = 3$$
$$x^2 + 6x + 5 = 0$$
$$(x+5)(x+1) = 0$$

Either

$$x + 5 = 0$$
$$x = -5$$

or

$$x + 1 = 0$$
$$x = -1$$
$$\therefore\ \boldsymbol{x = -5 \text{ or } x = -1}$$

NOTE

Again, there is no solution for $\log_3(x+2) + \log_3(x+4) = 1$ as $x = -1$ and $x = -5$ do not satisfy the above logarithmic equation.

(d) $\log_7(x^2 - 2x + 25) - \log_7(x+1) = 1$

Solution

$$\log_7(x^2 - 2x + 25) - \log_7(x+1) = 1$$

$\Rightarrow$

$$\log_7\left[\frac{(x^2 - 2x + 25)}{(x+1)}\right] = \log_7 7$$

Therefore

$$\frac{x^2 - 2x + 25}{x+1} = 7$$
$$x^2 - 2x + 25 = 7\,(x+1)$$
$$x^2 - 2x + 25 = 7x + 7$$
$$x^2 - 9x + 18 = 0$$
$$(x-3)(x-6) = 0$$

Either

$$x - 3 = 0$$
$$x = 3$$

or

$$x - 6 = 0$$
$$x = 6$$
$$\therefore\ \boldsymbol{x = 3 \text{ or } x = 6}$$

(e) $\log_{10}(x^2 - 4x + 7) = \log_{10} 100$

Solution

$$\log_{10}(x^2 - 4x + 7) = \log_{10} 100$$

$\Rightarrow$

$$x^2 - 4x + 7 = 100$$
$$x^2 - 4x - 93 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{4 \pm \sqrt{388}}{2}$$
$$= \frac{4 \pm 19.7}{2}$$

Either

$$x = \frac{4 + 19.7}{2}$$
$$= 11.9$$

or

$$x = \frac{4 - 19.7}{2} = -7.85$$
$$\therefore\ \boldsymbol{x = 11.9 \text{ or } x = -7.85}$$

(f) $\log_{10}(x^2 - 3x + 5) = 1$

Solution

$$\log_{10}(x^2 - 3x + 5) = 1$$

$$\Rightarrow$$

$$\log_{10}(x^2 - 3x + 5) = \log_{10} 10$$

Therefore

$$x^2 - 3x + 5 = 10$$

$$x^2 - 3x - 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2a}$$

$$= \frac{3 \pm \sqrt{29}}{2}$$

$$= \frac{3 \pm 5.385}{2}$$

$$x = \frac{3 + 5.385}{2}$$

$$\mathbf{= 4.19}$$

or

$$x = \frac{3 - 5.385}{2}$$

$$\mathbf{= -1.19}$$

$$\therefore\ \boldsymbol{x = 4.19 \text{ or } x = -1.19}$$

(g) $\log_8(2x^2 - 13x + 84) = 2$

Solution

$$\log_8(2x^2 - 13x + 84) = 2$$

$$\Rightarrow$$

$$\log_8(2x^2 - 13x + 84) = \log_8 64$$

Therefore

$$2x^2 - 13x + 84 = 64$$

$$2x^2 - 13x + 20 = 0$$

$$(2x - 5)(x - 4) = 0$$

Either

$$2x - 5 = 0$$

$$2x = 5$$

$$x = \frac{5}{2} = 2.5$$

or

$$x - 4 = 0$$

$$x = 4$$

$$\therefore\ \boldsymbol{x = 2.5 \text{ or } x = 4}$$

(h) $\log_6(x^2 - 23) = 0$

Solution

$$\log_6(x^2 - 23) = 0$$

$$\Rightarrow$$

$$\log_6(x^2 - 23) = \log_6 1$$

Therefore

$$x^2 - 23 = 1$$

$$x^2 - 24 = 0$$

$$x^2 - \left(\sqrt{24}\right)^2 = 0$$

$$\left(x + \sqrt{24}\right)\left(x - \sqrt{24}\right) = 0$$

Either

$$x + \sqrt{24} = 0$$

$$x = -2\sqrt{6}$$

$$= -4.90$$

or

$$x - \sqrt{24} = 0$$

$$x = 2\sqrt{6}$$

$$\mathbf{= 4.90}$$

$$\therefore\ \boldsymbol{x = \pm\sqrt{24} = \pm 4.90}$$

NOTE

Alternatively, we can simply evaluate as

$$x^2 - 24 = 0$$

$$x^2 = 24$$

$$x = \pm\sqrt{24}$$

$$\mathbf{= \pm 4.90}$$

$$\therefore\ \boldsymbol{x = \pm\sqrt{24} = \pm 4.90}$$

as before.

(i) $\log_2(x^2 - 4x + 12) = 3$

Solution

$$\log_2(x^2 - 4x + 12) = 3$$

$$\Rightarrow$$

$$\log_2(\mathrm{x}^2 - 4\mathrm{x} + 12) = \log_2 8$$

Therefore

$$x^2 - 4x + 12 = 8$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

Either

$$x - 2 = 0$$

$$x = 2$$

or

$$x - 2 = 0$$

$$x = 2$$

$$\therefore\ \boldsymbol{x = 2}\ \textbf{(twice)}$$

(j) $\log_9(x^2 + 7x + 15) = \frac{1}{2}$

Solution

$$\log_9(\mathrm{x}^2 + 7\mathrm{x} + 15) = \frac{1}{2}$$

$$\Longrightarrow$$

$$\log_9(\mathrm{x}^2 + 7\mathrm{x} + 15) = \log_9 3$$

Therefore

$$x^2 + 7x + 15 = 3$$

$$x^2 + 7x + 12 = 0$$

$$(x+4)(x+3) = 0$$

Either

$$x + 4 = 0$$

$$x = -4$$

or

$$x + 3 = 0$$

$$x = -3$$

$$\therefore\ \boldsymbol{x = -3}\ \textbf{or}\ \boldsymbol{x = -4}$$

Section 14: Scientific Notation

INTRODUCTION

We are now through with indices and logarithms - not absolutely though as we have one last part of logarithm to deal with. Before then however, we need to touch on expressing numbers in standard form and this is the focus of this section.

47) Express the following numbers in standard form.

(a) 97.246

Solution

$$97.246 = 9.724\,6 \times 10$$

$$= \mathbf{9.724\,6 \times 10}$$

(b) 0.1032

Solution

$$0.1032 = 1.032 \times \frac{1}{10}$$

$$= \mathbf{3.64 \times 10^{-1}}$$

(c) 875.4

Solution

$$875.4 = 8.754 \times 100$$

$$= \mathbf{8.754 \times 10^{2}}$$

(d) 0.0135 43

Solution

$$0.0135\,43 = 1.354\,3 \times \frac{1}{100}$$

$$= \mathbf{1.354\,3 \times 10^{-2}}$$

(e) 7 840.021

Solution

$$7\,840.021 = 7.840\,021 \times 1000$$

$$= \mathbf{7.840\,021 \times 10^{3}}$$

(f) 0.005 566

Solution

$$0.005\,566 = 5.566 \times \frac{1}{1000}$$

$$= \mathbf{5.566 \times 10^{-3}}$$

(g) 67503

Solution

$$67503 = 6.7503 \times 10\,000$$

$$= \mathbf{6.750\,3 \times 10^{4}}$$

(h) 0.000 673

Solution

$$0.000\,673 = 6.73 \times \frac{1}{10\,000}$$

$= \mathbf{6.73 \times 10^{-4}}$

(i) 620 350

Solution

$$620\ 350 = 6.203\ 50 \times 100\ 000$$
$$= \mathbf{6.203\ 50 \times 10^5}$$

(j) 0.000 085 04

Solution

$$0.000\ 085\ 04 = 8.504 \times \frac{1}{100\ 000}$$
$$= \mathbf{8.504 \times 10^{-5}}$$

48) Evaluate the following, giving the result in standard form.

(a) $23\ 500 \times 450\ 000$

Solution

$$23\ 500 \times 450\ 000 = (2.35 \times 10^4)(4.5 \times 10^5)$$
$$= (2.35 \times 4.5)(10^4 \times 10^5)$$
$$= (10.575)(10^9)$$
$$= 10.575 \times 10^9$$
$$= \mathbf{1.057\ 5 \times 10^{10}}$$

(b) $0.000\ 0043 \times 0.0020$

Solution

$$0.000\ 0043 \times 0.0020$$
$$= (4.3 \times 10^{-6})(2.0 \times 10^{-3})$$
$$= (4.3 \times 2.0)(10^{-6} \times 10^{-3})$$
$$= (8.6)(10^{-9})$$
$$= 8.6 \times 10^{-9}$$
$$= \mathbf{8.6 \times 10^{-9}}$$

(c) $(7.65 \times 10^4)(2.06 \times 10^3)$

Solution

$$(7.65 \times 10^4)(2.06 \times 10^3)$$
$$= (7.65 \times 2.06)(10^4 \times 10^3)$$
$$= (15.759)(10^7)$$
$$= 15.759 \times 10^7$$
$$= \mathbf{1.575\ 9 \times 10^8}$$

(d) $\frac{(3.78\times10^3)(1.92\times10^5)}{8.30\times10^{12}}$

Solution

$$\frac{(3.78 \times 10^3)(1.92 \times 10^5)}{8.30 \times 10^{12}}$$
$$= \frac{3.78 \times 1.92 \times 10^3 \times 10^5}{8.30 \times 10^{12}}$$
$$= \left(\frac{3.78 \times 1.92}{8.30}\right)\left(\frac{10^3 \times 10^5}{10^{12}}\right)$$
$$= \left(\frac{3.78 \times 1.92}{8.30}\right)\left(\frac{10^3 \times 10^5}{10^{12}}\right)$$
$$= 0.8744 \times 10^{-4}$$
$$= \mathbf{8.74 \times 10^{-5}}$$

Section 15: Solving Common Logarithms using Log-Antilog Tables

INTRODUCTION

In this last section of this book, the primary attention is on the use of log and antilog tables to solve problems. It should be noted that answers obtained using these tables are accurate only to four significant figures; calculators would give more accurate results. Kindly refer to pages 14 to 18 for a detailed description on their usage as notes will be kept at minimal.

49) Express the following numbers in standard form and write down the integer parts of their logarithms.

Solution

	Number	Standard form	Integer part
a)	4.7	4.7×10^0	0
b)	12.93	1.293×10	1
c)	795	7.95×10^2	2
d)	7 004	7.004×10^3	3
e)	25 500	2.55×10^4	4
f)	783 000	7.83×10^5	5
g)	2 000 000	2.0×10^6	6

50) Given that $\log 2.75 = 0.4393$, without using the log table, find the logarithm of the following numbers.

Hint

Although we will not be using log-antilog tables in the following questions, it is important to mention that the tables were constructed based on the approach used here. This allows several numbers to be placed in a single cell of the tables.

(a) 27.5

Solution

$$27.5 = 2.75 \times 10$$
$$\begin{aligned}\log 27.5 &= \log(2.75 \times 10)\\ &= \log 2.75 + \log 10\\ &= 0.4393 + 1\\ &= 1.4393\end{aligned}$$
$$\therefore \boldsymbol{\log 27.5 = 1.4393}$$

(b) 275

Solution

$$\begin{aligned}\log 275 &= \log(2.75 \times 100)\\ &= \log 2.75 + \log 100\\ &= 0.4393 + 2\\ &= 2.4393\end{aligned}$$
$$\therefore \boldsymbol{\log 275 = 2.4393}$$

(c) 2750

Solution

$$\begin{aligned}\log 2750 &= \log(2.75 \times 1000)\\ &= \log 2.75 + \log 1000\\ &= 0.4393 + 3\\ &= 3.4393\end{aligned}$$
$$\therefore \boldsymbol{\log 2750 = 3.4393}$$

(d) 27 500

Solution

$$\begin{aligned}\log 27500 &= \log(2.75 \times 10000)\\ &= \log 2.75 + \log 10000\\ &= 0.4393 + 4\\ &= 4.4393\end{aligned}$$
$$\therefore \boldsymbol{\log 27500 = 4.4393}$$

(e) 0.275

Solution

$$\begin{aligned}\log 0.275 &= \log(2.75 \times 10^{-1})\\ &= \log 2.75 + \log 10^{-1}\\ &= 0.4393 - 1\\ &= -0.5607\end{aligned}$$
$$\therefore \boldsymbol{\log 0.275 = -0.5507 \text{ or } \bar{1}.4393}$$

NOTE

You may want to refer to section 8.2 on how to convert a negative number into the **characteristic** and **mantissa** and vice versa.

(f) 0.027 5

Solution

$$\begin{aligned}\log 0.027\,5 &= \log(2.75 \times 10^{-2})\\ &= \log 2.75 + \log 10^{-2}\\ &= 0.4393 - 2\\ &= -1.5607\end{aligned}$$
$$\therefore \boldsymbol{\log 0.027\,5 = -1.5507 \text{ or } \bar{2}.4393}$$

(g) 0.002 75

Solution

$$\begin{aligned}\log 0.002\,75 &= \log(2.75 \times 10^{-3})\\ &= \log 2.75 + \log 10^{-3}\\ &= 0.4393 - 3\\ &= -2.5607\end{aligned}$$
$$\therefore \boldsymbol{\log 0.002\,75 = -2.5507 \text{ or } \bar{3}.4393}$$

(h) 0.000 275

Solution

$$\begin{aligned}\log 0.000\,275 &= \log(2.75 \times 10^{-4})\\ &= \log 2.75 + \log 10^{-4}\end{aligned}$$

$$= 0.4393 - 4$$
$$= -3.5607$$
$$\therefore \log 0.000\,275 = -3.5507 \text{ or } \bar{4}.4393$$

51) Use a log table to express the following as powers of 10.

Hint

In the following questions, we need to use our log table to find the logarithm of the number and subsequently express it in the power of 10 using the relationship below.

$$\boxed{log_{10}\, y = x \quad \Leftrightarrow \quad y = 10^x}$$

(a) 2.014

Solution

$$\log 2.014 = 0.3040$$
$$\therefore 2.014 = 10^{0.3040}$$

(b) 19.20

Solution

$$\log 19.20 = 1.2833$$
$$\therefore 19.20 = 10^{1.2833}$$

(c) 742.3

Solution

$$\log 742.3 = 2.8706$$
$$\therefore 742.3 = 10^{2.8706}$$

(d) 1879

Solution

$$\log 1879 = 3.2739$$
$$\therefore 1879 = 10^{3.2739}$$

52) Write down the logarithm of the following numbers.

Hint

The solutions to the questions to follow are presented in two-column tables. The left column represents the number while the right column is the corresponding logarithm. Log tables will be used in this case.

(a) 80, 81, 82, 83, 84 and 85.

Solution

Number	Logarithm
80	1.9031
81	1.9085
82	1.9138
83	1.9191
84	1.9243
85	1.9294

(b) 23.0, 23.1, 23.2, 23.3, 23.4 and 23.5.

Solution

Number	Logarithm
23.0	1.3617
23.1	1.3636
23.2	1.3655
23.3	1.3674
23.4	1.3692
23.5	1.3711

(c) 191.0, 191.1, 191.2, 191.3, 191.4 and 191.5.

Solution

Number	Logarithm
191.0	2.2810
191.1	2.2812
191.2	2.2814
191.3	2.2817
191.4	2.2819

191.5	2.2821

(d) 7.58, 758, 758 000 and 75.8

Solution

Number	Logarithm
7.58	0.8797
75.8	1.8797
758	2.8797
7 580	3.8797
75 800	4.8797
758 000	5.8797

53) Write down the numbers whose logarithms are given below.

Hint

The solutions to the questions to follow are presented in two-column tables. The left column represents the log of a number while the right column is the number whose log is given in the left column. In other words, the right column is the antilog of the corresponding number in the left column; therefore antilog tables should be used.

(a) 0.8900, 0.8910, 0.8920, 0.8930 and 0.8940

Solution

Logarithm	Number
0.8900	7.762
0.8910	7.780
0.8920	7.798
0.8930	7.816
0.8940	7.834

(b) 2.6215, 1.6215, 0.6215, $\bar{1}.6215$ and $\bar{2}.6215$.

Solution

Logarithm	Number
2.6215	418.3
1.6215	41.83
0.6215	4.183
$\bar{1}.6215$	0.4 183
$\bar{2}.6215$	0.041 83

54) Calculate the following. Check your answers with a calculator wherever possible.

Hint

Now it is time to show how the log-antilog tables can be used to perform calculations which, without a calculating device, would have been difficult or impractical to undertake. Again, we will set this out in a tabular form but an alternative method will be shown at the very end. Kindly refer to a detailed description on how this is carried out on pages 14 -18 particularly the table on pages 17 and18.

(a) 3.46×72.41

Solution

$$3.46 \times 72.41$$

No	Log
3.46	0.5391
	plus
72.41	1.8598
$\mathbf{3.46 \times 72.41}$	= 2.3989

Antilog = 2505

$$\therefore \mathbf{3.46 \times 72.41 = 250.5}$$

(b) $576 \div 25.8$

Solution

$$576 \div 25.8$$

No	Log
576	2.7604
	minus
25.8	1.4116
$\mathbf{576 \div 25.8}$	= 1.3488

Antilog = 2232

$\therefore \mathbf{576 \div 25.9 = 22.32}$

(c) $5.503 \times 14.64 \times 202$

Solution

$$5.503 \times 14.64 \times 202$$

No	Log
5.503	0.7406
	plus
14.64	1.1656
	plus
202	2.3054
$\mathbf{5.503 \times 14.64 \times 202}$	= 4.2116

Antilog = 1628

$\therefore \mathbf{5.503 \times 14.64 \times 202 = 16\ 280}$

(d) $\frac{105.7 \times 63.44}{913.6}$

Solution

$$\frac{105.7 \times 63.44}{913.6}$$

No	Log	
105.7	2.0241	
	plus	
63.44	1.8024	
86.23 x 4058	= 3.8265	3.8265
		minus
913.6		2.9608
$\mathbf{\frac{105.7 \times 63.44}{913.6}}$		= 0.8657

Antilog = 7340

$$\therefore \mathbf{\frac{105.7 \times 63.44}{913.6} = 7.340}$$

(e) $3.926^2 \times 85.43$

Solution

$$3.926^2 \times 85.43$$

No	Log	
3.926	0.5940	
$\mathbf{3.926^2}$	0.5940 x 2	=1.1880
		plus
85.43		1.9317
$\mathbf{3.926^2 \times 85.43}$		= 3.1197

Antilog =1317

$\therefore \mathbf{3.926^2 \times 85.43 = 1\ 317}$

(f) $\left(\frac{8.245}{0.8971}\right)^3$

Solution

$$\left(\frac{8.245}{0.8971}\right)^3$$

No	Log	
8.245	0.9162	
	minus	
0.8971	$\bar{1}.9528$	
$\mathbf{\frac{8.245}{0.8971}}$	0.9634	=0.9634
$\mathbf{\left(\frac{8.245}{0.8971}\right)^3}$		0.9634 x 3
		= 2.8902

Antilog = 7766

$$\therefore \left(\frac{8.245}{0.8971}\right)^3 = \mathbf{776.6}$$

(g) $\sqrt[4]{0.3174 \times 0.1986}$

Solution

$$\sqrt[4]{0.3174 \times 0.1986}$$

No	Log	
0.3174	$\bar{1}.5017$	

	plus	
0.1986	$\bar{1}.2980$	
$\mathbf{0.3174 \times 0.1986}$	$= \bar{2}.7997$	$\bar{2}.7997$
$\mathbf{\sqrt[4]{0.3174 \times 0.1986}}$		$\bar{2}.7997 \div 4$
		$\bar{1}.6999$

Antilog = 5010

$$\therefore \sqrt[4]{0.3174 \times 0.1986} = 0.5010$$

(h) $\sqrt[5]{19.2 \times 12.3 \times 75.4}$

Solution

$$\sqrt[5]{19.2 \times 12.3 \times 75.4}$$

No	Log	
19.2	1.2833	
	plus	
12.3	1.0899	
	plus	
75.4	1.8774	
$\mathbf{19.2 \times 12.3 \times 75.4}$	= 4.2506	4.2506
$\mathbf{\sqrt[5]{19.2 \times 12.3 \times 75.4}}$		$4.2506 \div 5$
		= 0.8501

Antilog = 7081

$$\therefore \sqrt[5]{19.2 \times 12.3 \times 75.4} = 7.081$$

(i) $\sqrt[6]{0.2511^5}$

Solution

$$\sqrt[6]{0.2511^5}$$

No	Log	
0.2511	$\bar{1}.3999$	
$\mathbf{0.2511^5}$	$\bar{1}.3999 \times 5$	
	$= \bar{4}.9995$	$\bar{4}.9995$
$\mathbf{\sqrt[6]{6.838^5}}$		$\bar{4}.9995 \div 6$
		$= \bar{1}.4999$

Antilog = 3161

$$\therefore \sqrt[6]{0.2511^5} = 0.3161$$

(j) $\sqrt[3]{\frac{1989}{2014}}$

Solution

$$\sqrt[3]{\frac{1989}{2014}}$$

No	Log	
1989	3.2987	
	minus	
2014	3.3040	
$\mathbf{\frac{1989}{2014}}$	$= \bar{1}.9947$	$\bar{1}.9947$
$\mathbf{\sqrt[3]{\frac{1989}{2014}}}$		$\bar{1}.9947 \div 3$
		$= \bar{1}.9982$

Antilog = 9959

$$\therefore \sqrt[3]{\frac{1989}{2014}} = 0.9959$$

(k) $1.840^2 \times 1.435^3$

Solution

$$1.840^2 \times 1.435^3$$

No	Log	
1.840	0.2648	
$\mathbf{1.840^2}$	0.2648 x 2	
	= 0.5296	0.5296
1.435	0.1568	plus
$\mathbf{1.435^3}$	0.1568 x 3	0.4704
$\mathbf{5.836^2 \times 1.283^3}$		=1.0000

Antilog = 10

$$\therefore 1.840^2 \times 1.435^3 = 10.00$$

(l) $\sqrt[3]{\frac{313 \times 99}{95.43}}$

Solution

$$\sqrt[3]{\frac{313 \times 99}{95.43}}$$

No	Log	
313	2.4955	
	plus	
99	1.9956	
	=4.4911	4.4911
		minus
95.43		1.9796
$\frac{313 \times 99}{95.43}$		=2.5115
$\sqrt[3]{\frac{313 \times 99}{95.43}}$		2.5115÷ 3
		=0.8372

Antilog = 6874

$$\therefore \sqrt[3]{\frac{313 \times 99}{95.43}} = 6.874$$

(m) $\frac{41.53 \times 70.03}{648.2 \times 2.547}$

Solution

$$\frac{41.53 \times 70.03}{648.2 \times 2.547}$$

No	Log	
41.53	1.6183	
	plus	
70.03	1.8453	
41.53 x 70.03	=3.4636	
Numerator		3.4636
648.2	2.8117	
	plus	minus
2.547	0.4060	
648.2 x 2.547	=3.2177	
Denominator		3.2177
$\frac{41.53 \times 70.03}{648.2 \times 2.547}$	(Numerator) – (Denominator)	=0.2459

Antilog = 1762

$$\therefore \frac{41.53 \times 70.03}{648.2 \times 2.547} = 1.762$$

(n) $\frac{(57.5)^2 \times 7.88}{\sqrt[5]{679.8}}$

Solution

$$\frac{(57.5)^2 \times 7.88}{\sqrt[5]{679.8}}$$

No	Log	
57.5	1.7597	
57.5^2	1.7597 x 2 = 3.5194	
	plus	
7.88	0.8965	
57.5^2 x 7.88	= 4.4159	
Numerator		4.4159
679.8	2.8324	
$\sqrt[5]{679.8}$	2.8324 ÷ 5	minus
Denominator	= 0.5665	0.5665
$\frac{(57.5)^2 \times 7.88}{\sqrt[5]{679.8}}$	(Numerator) – (Denominator)	=3.8494

Antilog = 7069

$$\therefore \frac{(57.5)^2 \times 7.88}{\sqrt[5]{679.8}} = 7069$$

(o) $\sqrt[3]{\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^2}$

Solution

$$\sqrt[3]{\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^2}$$

No	Log	
48.11	1.6822	
	plus	
3.753	0.5743	
48.11 × 3.753	= 2.2565	

Numerator		2.2565
7.003	0.8453	
	plus	minus
6.099	0.7852	
$\mathbf{8.687 \times 6.285}$	=1.6305	
Denominator		1.6305
$\dfrac{\mathbf{38.32 \times 2.964}}{\mathbf{8.687 \times 6.285}}$	**(Numerator) – (Denominator)**	=0.6260
$\left(\dfrac{\mathbf{48.11 \times 3.753}}{\mathbf{7.003 \times 6.099}}\right)^2$		0.6260 x 2
		=1.2520
$\sqrt[3]{\left(\dfrac{\mathbf{48.11 \times 3.753}}{\mathbf{7.003 \times 6.099}}\right)^2}$		=.2520 ÷ 3
		= 0.4173

Antilog = 2614

$$\therefore \sqrt[3]{\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^2} = \mathbf{2.614}$$

NOTE

There is an alternative way to solve the problems presented so far and the immediate last question has been chosen to show this. This could be applied to all.

$$\sqrt[3]{\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^2} = \left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^{\frac{2}{3}}$$

Take the log of

$$\log\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^{\frac{2}{3}} = \frac{2}{3}\log\frac{48.11 \times 3.753}{7.003 \times 6.099}$$

$$= \tfrac{2}{3}[\log 48.11 + \log 3.753 - \log 7.003 - \log 6.099]$$

Now, look in the log table for the logs of the numbers above, which gives

$$= \tfrac{2}{3}[1.6822 + 0.5743 - 0.8453 - 0.7852]$$

Simplify this to obtain

$$= \tfrac{2}{3}[0.6260] = 0.4173$$

In other word

$$\log\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^{\frac{2}{3}} = 0.4173$$

Now look up for 0.4173 in the antilog table. This gives 2614.

$$\therefore \sqrt[3]{\left(\frac{48.11 \times 3.753}{7.003 \times 6.099}\right)^2} = \mathbf{2.614}$$

55) Using the log and antilog tables, evaluate the following correct to 3 significant figures. Take the value of $\log\pi$ to be 0.4971.

(a) The area of a rectangle which is 40.5 m long and 3.4 m wide.

Solution

$$Area = length \times width = 40.5\,m \times 3.4\,m$$

No	Log
40.5	1.6075
	plus
3.4	0.5315
	= 2.1390

Antilog = 1377

Thus,

$$40.5 \times 3.4 = 137.7$$

$$\therefore \boldsymbol{Area = 138\,m^2\ (3\,s.f.)}$$

(b) The volume of a cube of edge 3.56 mm.

Solution

$$Volume = x^3 = 3.56^3\,mm^3$$

No	Log
3.56	0.5514

3.56³	0.5514 x 3
	= 1.6542

Antilog = 4510

Thus,

$$3.56^3 = 45.10$$

$$\therefore \boldsymbol{Volume = 45.1\ mm^3}\ (\boldsymbol{3\ s.f.})$$

(c) The area of a circular disc 11.86 cm in diameter.

Solution

$$Area\ of\ circular\ disc, A = \pi r^2$$

Where

$$r = \frac{d}{2} = \frac{11.86}{2}$$

$$= 5.93\ cm$$

Therefore,

$$A = \pi \times 5.93^2$$

No	Log
$\boldsymbol{\pi}$	0.4971
5.93	0.7731
5.93²	0.7731 x 2
	= 1.5462
$\boldsymbol{\pi \times 5.93^2}$	0.4971+1.5462
	=2.0433

Antilog = 1105

Thus,

$$\pi \times 5.93^2 = 110.5$$

$$\therefore \boldsymbol{Area = 111\ cm^2}\ (\boldsymbol{3\ s.f.})$$

(d) The volume of a sphere with a radius of 3.5 mm.

Solution

$$Volume\ of\ a\ sphere = \frac{4}{3}\pi r^3$$

$$= \frac{4 \times \pi \times 3.5^3}{3}$$

No	Log	
4	0.6021	
	plus	
$\boldsymbol{\pi}$	0.4971	
3.5	0.5441	
3.5³	0.5441 x 3 = 1.6323	
	plus	
$\boldsymbol{4 \times \pi \times 3.5^3}$	=2.7315	
Numerator		2.7315
3	0.4771	minus
Denominator		0.4771
$\boldsymbol{\frac{4 \times \pi \times 3.5^3}{3}}$	**(Numerator) – (Denominator)**	=2.2544

Antilog = 1797

Thus,

$$\frac{4}{3}\pi \times 3.5^3 = 179.7$$

$$\therefore \boldsymbol{Area = 180\ mm^2}\ (\boldsymbol{3\ s.f.})$$

END OF WORKED EXAMPLES

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