MATH157: ALGEBRA

CHAPTER 2

Indices - Logarithm and Exponential Functions

February 01, 2021

Outline

$8 = 2 \times 2 \times 2$ can be rewritten as 2^3. This is read as '2 (raised) to the power 3.

Definition

Let x be a real number. If there exist two real numbers a and n such that $x = a^n$, then

- *the expression a^n is the* **index form** *of x;*
- *a is called the* **base***;*
- *n is called the* **index** *(or the power/exponent).*

Property (Index laws)

Let n and m be two integers, a and b two real numbers.

(1) $a^n \times a^m = a^{n+m}$.

(2) $\dfrac{a^n}{a^m} = a^{n-m}$ *provided* $a \neq 0$

(3) $(a^n)^m = a^{nm}$.

(4) $(ab)^m = a^m b^m$.

(5) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m} = a^m b^{-m}$ *provided* $b \neq 0$.

(6) $a^0 = 1$ *provided* $a \neq 0$.

Application of (1) $a^n \times a^m = a^{n+m}$.

Example

$3^4 \times 3^5 =$

Application of $(1)\ a^n \times a^m = a^{n+m}$.

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Application of (2) $\dfrac{a^n}{a^m} = a^{n-m}$.

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$\dfrac{4^{-2}}{4^{-1}} = 4^{-2+1} = 4^{-1}$.

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$(11^{-4})^5 = 11^{-20}$.
$(4^{-2})^{-1} = 4^2$.

Property

(7) $a^{\frac{1}{n}} = \sqrt[n]{a}$ *and* $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ *provided* $a \geq 0$.

Example

$9^{\frac{3}{2}} =$

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$9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 27$ (or $(3^2)^{\frac{3}{2}} = 3^3 = 27$).

$32^{\frac{3}{5}} =$

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$32^{\frac{3}{5}} = (2^5)^{\frac{3}{5}} = 8.$

Exercise

- Find $\frac{3^3 \times 3^7}{3^2}$, $\frac{2^{-3} \times 2^7}{4^2}$ and $(27 \times 16)^{\frac{2}{3}}$
- Show that $\frac{8a^2b^2}{3a^3b} \div \frac{4ab^2}{9a^3b^5} = 6ab^4$ and $36 \times 2^{m-2} + 2^{m+3} - 2^m = 2^{m+4}$.
- Express in the simplest product of index forms: $(a)\, \frac{8^3 \times 5^4 \times 7^3}{5^3 \times 7^5 \times 2^6}$,

$(b)\ \frac{(0.6)^2 \times 12^5 \times 54^3}{9^2 \times 5^3 \times (0.8)^3 \times (0.4)^4}$, $(c)\ \frac{(3^4 \times 2^{-3})^3}{(9^{-1} \times 2^2)^4}$, $\frac{10^2 \times 3^2}{8 \times 5^2} \div \sqrt{\frac{2^5 \times 3^9}{6}}$,

Outline

Definition

Equations of the form $a^x = b$ *are called* **exponential equations**.

If $a > 0$ and the index form of b is a^n, then $a^x = a^n$ implies that $x = n$.

Example

Solve $(1)\ 2^x = 8$ *and* $(2)\ 3^{-4x} - 1 = 8$.

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Solve $(1)\ 2^x = 8$ *and* $(2)\ 3^{-4x} - 1 = 8$.

(1) *Since* $8 = 2^3,\ 2^x = 8 \implies 2^x = 2^3 \implies x = 3.$

$(2) 3^{-4x} - 1 = 8 \implies 3^{-4x} = 3^2 \implies -4x = 2 \implies x = -1/2$

Exercise

Find x *for which* $(a)\ 3^{-2x} = 27^{x+1}$, $(b)\ 5^{3x} = 5^{7x-2}$, $(c)\ 5^{5-9x} = \dfrac{1}{25^{x-2}}$,
$(d)\ 21^{-1+2x} - 21^{x^2} = 0$, $(e)\ 4^x \times 3^{2x} = 6$, $(f)\ 4^{x+1} + 2^{2x} = 20$.
Find x *and* y *solutions of* $2^{5x+1} = 16^{y+1}$ *and* $2^{x-13} = 32^{1-y}$.

Activity

Use the calculator to find out the values of 2^x *for* $x \in \{-4, -3, -1, 0, 1, 2, 3\}$. *Show the points of coordinates* $(x, 2^x)$ *on a graph.*

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Solution

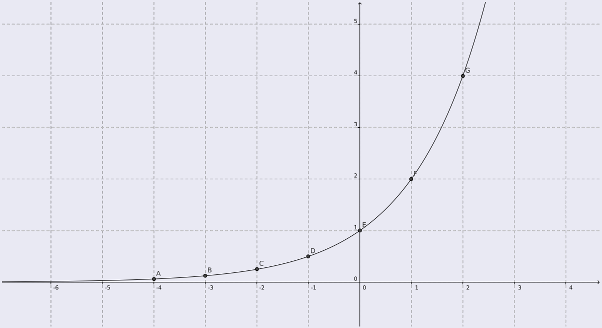

Figure: Exponential graph

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- *The real number* $e = 2.718281828459045235360287471352 7\ldots$ *is called* **Euler's number**.

The function that associates to any real number x, *the real number* $y = e^x$ *is called* **exponential function**.

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Property

$e^0 = 1, e^1 = e$ and $e^x > 0, x \in \mathbb{R}$.

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Exercise

Solve $(a)\ e^{\frac{1}{2}x^2+1} = e^{x\sqrt{2}}$ and $(b)\ e^{2x} - 3e^x + 2 = 0$.

Outline

Definition (Natural logarithm function)

Natural logarithm function $\ln$ *is the inverse-function of the exponential function* e.
That is, if $y > 0$,

Exponential form		*Natural logarithm form*
$e^x = y$	$\Longleftrightarrow$	$x = \ln y$

For any two positive real numbers x and y, we have
$(a)\ \ln(xy) = \ln x + \ln y$, $(b)\ \ln(x^n) = n \ln x$, and $(c) \ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
For instance

$$e^{x-1} = 3 \implies (\ln\left[e^{x-1}\right] = \ln 3 \implies)x - 1 = \ln 3 \implies x = 1 + \ln 3.$$

Solve in $\mathbb{R}$: $e^{x^2 - 4x + 1 + \ln 6} = 6$.

Definition (Logarithm function)

If b and x are two positive real numbers, then **base b logarithm** *of x is denoted by $\log_b(x)$ and is defined by* $\log_b(x) = \dfrac{\ln x}{\ln b}$.

- $b = e \implies \log_e x = \dfrac{\ln x}{\ln e} = \ln x.$
- $b = 10 \implies \log_{10} x = \dfrac{\ln x}{\ln 10} = \log x.$
- If $b \neq e$ and $b \neq 10$ we use $\log_b$.

Note that $\log_b 1 = 0$ and $\log_b b = 1$.

Property (Conversion)

For $b > 0$ and $y > 0$,

Exponential form		*Logarithmic form*
$b^x = y$	$\iff$	$x = \log_b y.$

Example

The logarithmic form of :

$5^2 = 25$ *is* $2 = \log_5 25$

$11^x = 21$ *is* $x = \log_{11} 21.$

Property

For $b > 0, x > 0$ *and* $y > 0$

$(a)\ \log_b 1 = 0, \log_b b = 1$ *and* $\log_b x = \dfrac{\ln x}{\ln b}.$

$(b)\ \log_b(xy) = \log_b x + \log_b y,$

$(c)\ \log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y.$

$(d)\ \log_b(x^n) = n \log_b x.$

Example

We want to simplify $(1)\ \log_4 10 + \log_4 2$,

$(2)\ \log_2(\sqrt{3}+1) + \log_2(\sqrt{3}-1)$ *and expand* $(3)\ \log\left(\dfrac{10x^4}{3y^2}\right)$.

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(1) $\log_4 10 + \log_4 2 = \log_4(10 \times 2) = \log_4(20)$.

(2) $\log_2(\sqrt{3}+1)+\log_2(\sqrt{3}-1) = \log_2\left[(\sqrt{3}+1)(\sqrt{3}-1)\right] = \log_2(2) = 1$.

(3) $\log\left(\dfrac{10x^4}{3\sqrt{y}}\right) = \log\left(10x^4\right) - \log(3\sqrt{y}) = \log 10 + 4\log x - \log 3 - \dfrac{1}{2}\log y$

$= 1 - \log 3 + 4\log x - \dfrac{1}{2}\log y$

Exercise

Solve

(1) $(x-2)\log_4(x-2)=0$,

(2) $\log\left(x^2+x-1\right)=0$,

(3)$x-y=2-b$ *and* $\log_b(x+y)=1$ *provided* $b>0$.

Given $\log_7 2=\alpha, \log_7 3=\beta$ *and* $\log_7 5=\gamma$, *express in terms of* α, β *and* γ

(4) $\log_7 6$.

(5) $\log_7 \dfrac{15}{2}$.