CALCULUS I (MATH 171)

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DEFINITION

• A sequence is a list of set of numbers that follows a defined partern. For example;

$$1, 2, 3, 4, 5, \cdots \tag{1}$$

• A sequence may be finite called **finite sequence**. For instance;

$$2, 4, 6, 6, 10 \tag{2}$$

• A sequence may be infinite called **infinite sequence**. For instance;

$$3, 6, 9, 12, 15, \cdots \tag{3}$$

• Series is the sum of list of numbers in a defined patter. For instance;

$$1 + 2 + 3 + 4 + 5 + \cdots \tag{4}$$

DEFINITION

• Series may also be finite

$$2 + 4 + 6 + 6 + 10 \tag{5}$$

• or maybe infinite

$$3 + 6 + 9 + 12 + 15 + \cdots \tag{6}$$

• A sequence maybe classified to be an Arithmetic **Progression (AP)** or a **Geometric Progression (GP)**

ARITHMETIC PROGRESSION (AP)

• An Arithmetic Progression or Linear sequence is a sequence of numbers in which each term after the first term is obtained by adding a constant difference to the preceeding term.

• The constant difference is called **common difference**, d.

• Such sequences are characterized by the **first term** and **common difference**.

DEFINITION

- the *nth* term of an AP is given by

$$u_n = a + (n-1)d = l \tag{7}$$

where
$l =$ last term
$a =$ first term
$d =$ common difference
$n =$ number of terms
$u_n =$ nth term

DISCUSSION

- Find the 12th term of the arithmetic progression $2, 7, 12, 17, 22, \cdots$
- Here, $a = 2, d = 5$ and $n = 12$. We therefore compute;

$$\begin{aligned} u_n &= a + (n-1)d \\ u_{12} &= 2 + (12-1)5 \\ u_{12} &= 57 \end{aligned}$$

DISCUSSION

- Find the number of terms of the linear sequence $3, 7, 11, , \cdots , 31$
- Here, $a = 3, d = 4$ and $l = 31$. We therefore compute for n;

$$\begin{aligned} a + (n-1)d &= l \\ 3 + (n-1)4 &= 31 \\ 3 + 4n - 4 &= 31 \\ 4n &= 31 + 1 \\ n &= 8 \end{aligned}$$

there are therefore 8 terms in the given sequence.

DEFINITION

- Find the linear sequence whose 8th term is 38 and 22nd term is 108,
- Given that $u_{n=a+(n-1)d}$, then;

$$
\begin{aligned}
u_8 &= a + 7d = 38..........(1) \\
u_{22} &= a + 21d = 108............(2) \quad (2) - (1) \\
14d &= 70 \\
d &= 5, \quad \text{substitute } d = 5 \text{ into } (1) \\
a + 7(5) &= 38 \\
a &= 3
\end{aligned}
$$

therefore the sequence will be $3, 8, 13, 18, \cdots$

SUM OF AN AP

- The sum of an AP is given by

$$s_n = \frac{n}{2}(2a + (n-1)d) \tag{8}$$

or

$$s_n = \frac{n}{2}(a + l) \tag{9}$$

where
$l =$ last term
$a =$ first term
$d =$ common difference
$n =$ number of terms

DISCUSSION

- Find the sum of the first 10 terms of the linear sequence $2, 7, 12, 17, 22, \cdots$
- Given that $a = 2, d = 5$ and $n = 10$. Using eqn (8), we compute s_{10} as;

$$
\begin{aligned}
s_n &= \frac{n}{2}(2a + (n-1)d) \\
s_{10} &= \frac{10}{2}(2(2) + (10-1)5) \\
s_{10} &= 245
\end{aligned}
$$

DISCUSSION

- Find the sum of the AP $1, 3, 5, 7, \cdots, 101$
- Given that $a = 1, d = 2$ and $l = 101$. Using eqn (9), we compute s_n as;

$$\begin{aligned} l &= a + (n-1)d \\ 101 &= 1 + (n-1)2 \\ 101 &= 1 + 2n - 2 \\ 2n &= 102 \\ n &= 51 \end{aligned}$$

$$\begin{aligned} s_n &= \frac{n}{2}(a + l) \\ s_{51} &= \frac{51}{2}(1 + 101) \\ s_{51} &= 2,601 \end{aligned}$$

DISCUSSION

• Find the sum of the first thirteen terms of a linear sequence whose 8th term is 18 and 11th term is 24.

$$u_8 = a + 7d = 18..........(1)$$
$$u_1 = a + 21d = 24............(2) \quad (2) - (1)$$
$$3d = 6$$
$$d = 2, \quad \text{substitute } d = 2 \text{ into } (1)$$
$$a + 7(2) = 18$$
$$a = 4$$

$$s_n = \frac{n}{2}(2a + (n-1)d)$$
$$s_{13} = \frac{10}{2}(2(4) + (13-1)2$$
$$s_{13} = 208$$

GEOMETRIC PROGRESSION (GP)

- A geometric progression or exponential sequence is a sequence of numbers in which each term after the first is obtained by mulplying the preceeding term by a constant.
- This constant is called the **common ratio**, r.
- GP is characterized by a **common ratio** and a **first term**.

Geomtric Progression, GP

- the *nth* term of a GP is given by

$$u_n = ar^{(n-1)} \tag{10}$$

where
$a =$ first term
$r =$ common ratio
$n =$ number of terms
$u_n =$ nth term

DISCUSSION

- Find the 7th term of the exponential sequence $5, 10, 20, 40, \cdots$
- Given that $a = 5, r = 2$ and $n = 7$. We compute as;

$$u_n = ar^{(n-1)}$$
$$u_7 = 5 \,.\, 2^{(7-1)}$$
$$u_7 = 5 \,.\, 2^6$$
$$u_7 = 320$$

DISCUSSION

- The 3rd term and 6th terms of a GP are $\frac{1}{4}$ and $\frac{1}{32}$ respectively. Find;
- the first term and common ratio
- the 8th term in the sequence
- Here we compute;

$$u_3 = ar^2 = \frac{1}{4}$$
$$u_6 = ar^5 = \frac{1}{32}$$
$$\frac{ar^5}{ar^2} = \frac{\frac{1}{4}}{\frac{1}{32}}$$
$$r = \frac{1}{2} \quad a = 1$$

$$\begin{aligned} u_8 &= ar^7 \\ &= 1\left(\frac{1}{2}\right)^7 \\ &= \frac{1}{128} \end{aligned}$$

SUM OF A GP

- The sum of a GP is given as

$$s_n = \frac{a(1 - r^n)}{1 - r}, \quad r < 1 \tag{11}$$

or

$$s_n = \frac{a(r^n - 1)}{r - 1}, \quad r > 1 \tag{12}$$

or

$$s_n = na, \quad r < 1 \tag{13}$$

DISCUSSIONN

- Find the sum of the first thirteen terms of the GP $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$
- Given that $n = 13, a = \frac{1}{2}$ and $r = \frac{1}{2} < 1$

$$
\begin{aligned}
s_n &= \frac{a(1 - r^n)}{1 - r} \\
s_{13} &= \frac{a(1 - \left(\frac{1}{2}\right)^{13})}{1 - \frac{1}{2}} \\
&= 1 - \left(\frac{1}{2}\right)^{13} \\
&= 0.9998
\end{aligned}
$$

DISCUSSIONN

- • The 3rd term and 6th terms of a GP are 10 and 80 respectively. Find;
- the first term and common ratio
- the sum of the first six terms of the sequence
- Here we compute;

$$u_3 = ar^2 = 10$$
$$u_6 = ar^5 = 80$$
$$\frac{ar^5}{ar^2} = \frac{80}{10}$$
$$r = 2 \quad a = 2.5$$

$$s_n = \frac{a(1-r^n)}{1-r}, \quad r < 1$$
$$s_6 = \frac{2.5(1-2^6)}{1-r}, \quad r < 1$$
$$s_6 = 157$$

INTRODUCTION

A sequence of numbers is simply bunch of numbers in a particular order. For example';

$$\pi, 2\pi, 3\pi, 4\pi, 5\pi, \cdots \tag{14}$$

$$1, 4, 9, 16, 25, \cdots \tag{15}$$

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \tag{16}$$

$$2, -2, 2, -2, 2, \cdots \tag{17}$$

NOTATION

When a sequence increases or decreases without bound, it is said to be **INFINITE**

NOTATION FOR SEQUENCES

The very simplest notation for a sequence is given as a_n; which signifies the "*nth*" term of the sequence.
On the otherhand, the notation "$\{a_n\}$" refers to the entire sequence.
Check the examples below;

$$\{a_n\} = \{2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \cdots\} \tag{18}$$

$$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\} \tag{19}$$

Here, the notations are well defined, where $\{a_n\}$ referred to the entire sequence and 6π is the $3rd$ term in that sequence.

INTRODUCTION

You should also know that; "a_n" in sequences also maybe referred to as the rule that defines the various terms in a particular sequence. For instance,

$$\{a_n\}, \quad \text{where} \quad a_n = n^2 \tag{20}$$

$$\{b_n\} \quad \text{where} \quad b_n = \frac{1}{n} \tag{21}$$

$$\{c_n\} \quad \text{where} \quad c_n = 2(n+1)\pi \tag{22}$$

Most of the times, for clarity sake, they are stated by listing some few earlier terms with the *nth* term. For example;

$$\{a_n\} = \{1, 4, 9, \cdots n^2, \cdots\} \tag{23}$$

$$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\} \tag{24}$$

$$\{c_n\} = \{2\pi, 4\pi, 6\pi, \cdots, 2(n+1)\pi, \cdots\} \tag{25}$$

INTRODUCTION

More so, the notation "a_n" could become more precise by assigning initial and terminating values to the notation in braces.

$$\{a_n\}, = \left\{n^2\right\}_{n=1}^{\infty} \tag{26}$$

$$\{b_n\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \tag{27}$$

$$\{c_n\} = \left\{2(n-1)\pi\right\}_{n=2}^{\infty} \tag{28}$$

INTRODUCTION

Generally, sequences follow this pattern;

$$\begin{aligned} a_1 &= \text{first term} \\ a_2 &= \text{second term} \\ \vdots &= \vdots \\ a_n &= nth \, \text{term} \\ a_{n+1} &= (n+1)th \, \text{term} \\ \vdots &= \vdots \end{aligned}$$

You should also recall from Foundational Mathematics II, where you were taught sequences and series. You were also told that, aside the terms in a sequence, a sequence may also have the common difference (d) and the common ratio (r). Where the first is an "Arithmetic Progression (AP)" and the latter as a "Geometric Sequence (GP)".

GENERATING ELEMENTS OF A SEQUENCE

Here, our discussion is now concentrating on how to derive every single elment of a sequence given its *nth* term. In this case; all we do is to plug the values of "n" into the function-like relation.
You may not be able to list all of the terms in the sequence if the sequence is infinite. Check the illustrations below;

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \cdots\right\} \tag{29}$$

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{\underbrace{-1}_{n=1}, \underbrace{\frac{1}{2}}_{n=1}, \underbrace{-\frac{1}{4}}_{n=2}, \underbrace{\frac{1}{8}}_{n=3}, \underbrace{-\frac{1}{16}}_{n=4}, \cdots\right\} \tag{30}$$

Now you may try your hand on ther examples.

FORMULATING THE NTH TERM

Our discussion here is just opposite to the one we just had in the last slide. We will in this case, try our best to formulate the general rule, given some terms of a sequence. Your knowledge in AP and GP will be very helpful.
Formulate the general rule for the following sequences;

i) $\left\{2, 4, 6, 8, \cdots\right\}$

ii) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right\}$

iii) $\left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \frac{9}{16}, \frac{11}{32} \cdots\right\}$

iv) $\left\{\frac{3}{2}, -\frac{5}{4}, \frac{7}{8}, -\frac{9}{16}, \frac{11}{32} \cdots\right\}$

SOLUTION 1

i) To derive the general rule, here, we simply think of a general formular that will help us generate all elements in the given sequence. So we check for the kind of sequence this example will be. I believe we could all see that the sequence is an AP. This is because it has its first term (a) to be 2 and the common difference (d) as $8-6=6-4=4-2=2$.
From AP, to find an *nth* term we use

$$U_n = a + (n-1)d \tag{31}$$

This becomes;

$$\begin{aligned} U_n &= a + (n-1)d \\ &= 2 + (n-1)2 \\ &= 2n \end{aligned}$$

SOLUTION 1

The *nth* term therefore becomes

$$a_n = \{2n\}_{n=1}^{\infty} \tag{32}$$

SOLUTION 2

ii) This example is in fraction. Here just derive the general form for the numerator alone, and then do same for the denominator. The numerator is of the form $\{1, 1, 1, 1, \cdots\}$
Since 1 is just repeating or the sequence has same elements at the numerator, we simply write the *nth* term as;

$$a_n = \{1\} \tag{33}$$

SOLUTION 2

ii) We now concentrate on the denominator, which is given as; $\{2, 4, 6, 8, \cdots\}$. So we check for the kind of sequence this example will be. I believe we could all see that the sequence is an AP. This is because it has its first term (a) to be 2 and the common difference (d) as $8 - 6 = 6 - 4 = 4 - 2 = 2$.

From AP, to find an *nth* term we use

$$U_n = a + (n-1)d \tag{34}$$

This becomes;

$$\begin{aligned} U_n &= a + (n-1)d \\ &= 2 + (n-1)2 \\ &= 2n \end{aligned}$$

SOLUTION 2

Therefore the general form will be given as;

$$a_n = \left\{ \frac{1}{2n} \right\}_{n=1}^{\infty} \tag{35}$$

SOLUTION 3

iii This example is also in fraction. Here just derive the general form for the numerator alone, and then do same for the denominator. The numerator is of the form $\{3, 5, 7, 9, 11, \cdots\}$

I believe we could all see that the sequence is an AP. This is because it has its first term (a) to be 3 and the common difference (d) as $9 - 7 = 7 - 5 = 5 - 3 = 2$.

SOLUTION 3

iii We then use the *nth* term of an AP to find the general form of the numerator.

$$\begin{aligned} U_n &= a + (n-1)d \\ &= 3 + (n-1)2 \\ &= 2n + 1 \end{aligned}$$

The denominator is also of the form $\{2, 4, 8, 16, 32, \cdots\}$
I believe we could all see that the sequence is a GP. This is because it has its first term (a) to be 2 and the common ratio as (r) as $\frac{16}{8} = \frac{8}{4} = \frac{4}{2} = 2$.

solution 3

iii We then use the *nth* term of a GP to find the general form of the denominator.

$$\begin{aligned} U_n &= ar^{n-1} \\ &= 2(2)^{n-1} && \text{by using indices} \\ &= 2^n \end{aligned}$$

therefore, the general form becomes

$$a_n = \left\{ \frac{2n+1}{2^n} \right\}_{n=1}^{\infty} \tag{36}$$

SOLUTION 4

iv In this example, we can see that the series is alternating, as in their signs. But that should not be a problem. Firstly, ignore the signs, and find the general form for both the numerator and denominator as we did in previous examples. This is given as;

$$a_n = \left\{ \frac{2n+1}{2^n} \right\}_{n=1}^{\infty} \tag{37}$$

refer from example 3.
Now we consider to bring in the alternation in by introducing a restriction $(-1)^{n+1}$.
hence, the general form will be;

$$a_n = \left\{ \frac{(-1)^{n+1}(2n+1)}{2^n} \right\}_{n=1}^{\infty} \tag{38}$$

LIMIT THEOREM ON SEQUENCES

Here, we shall discuss the concept of limit on sequences, and hence use the concept to establish higher mathematical concepts like **Convergence** and **Divergence**.

You should know that sequences could also be treated as functions and its likes.

This implies that sequences can be graphed for very good analysis to be done on them.

To graph the sequence say, $a_n = \left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$, we plot the points (n, a_n) just as we do in functions as;

$$(1, 2), \left(2, \frac{3}{4}\right), \left(3, \frac{4}{9}\right), \left(4, \frac{5}{16}\right), \left(5, \frac{6}{25}\right), \cdots \tag{39}$$

LIMIT THEOREM ON SEQUENCES

The resulting graph when these coordinates are plotted lead us into a very key idea on sequences.

• As the values for "n" is increasing, the values of the sequence "a_n" continually diminish to zero (0).

•This implies that all elements of the sequence will be getting closer and closer to zero as n increases.

• The values of the sequence will neither touch the zero nor go below it. This implies that zero turns to be a limit point to the sequence.

LIMIT THEOREM ON SEQUENCES

In this particular instance, we can refer to zero (0) as the **limit or limiting value** of the sequence a_n.
This mathematically can be written as;

$$\lim_{n \longrightarrow \infty} a_n = \lim_{n \longrightarrow \infty} \left\{ \frac{n+1}{n^2} \right\}_{n=1}^{\infty} = 0 \tag{40}$$

LIMIT DEFINITION

• We say that

$$\lim_{n \longrightarrow \infty} a_n = L \tag{41}$$

if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of a_n approach L as n approaches infinity.

• We say that

$$\lim_{n \longrightarrow \infty} a_n = \infty \tag{42}$$

if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of a_n get larger and larger without bound as n approaches infinity.

• We say that

$$\lim_{n \longrightarrow \infty} a_n = -\infty \tag{43}$$

if we can make a_n as close to L as we want for all sufficiently large n. The value of a_n get smaller and smaller without bound as n approaches infinity.

PRECISE DEFINITION OF LIMIT

- We say that $\lim\limits_{n \longrightarrow \infty} a_n = L, \forall \epsilon > 0$, there is an integer N such that

$$|a_n - L| < \epsilon \qquad \text{whenever} \quad n > N \tag{44}$$

- We say that $\lim\limits_{n \longrightarrow \infty} a_n = \infty, \forall \epsilon > 0$, there is an integer N such that

$$a_n > M \qquad \text{whenever} \quad n > N \tag{45}$$

- We say that $\lim\limits_{n \longrightarrow \infty} a_n = -\infty, \forall \epsilon > 0$, there is an integer N such that

$$a_n < M \qquad \text{whenever} \quad n > N \tag{46}$$

CONVERGENCE AND DIVERGENCE OF SEQUENCES

•Given the sequence a_n, if the limit $\lim_{n \longrightarrow \infty} a_n$ exists and it is finite, we say that the sequence is **Convergent**.

• On the other hand, if the limit $\lim_{n \longrightarrow \infty} a_n$ does not exists and is infinite, we say that the sequence is **Divergent**.

• We therefore mathematically say, that, given the sequence a_n, the sequence converges to L.

• We also mathematically say, that, given the sequence a_n, the sequence diverges to ∞ or $-\infty$.

PROPERTIES OF LIMIT THEOREMS ON SEQUENCES

- $\lim\limits_{n\longrightarrow\infty}(a_n \pm b_n) \quad = \quad \lim\limits_{n\longrightarrow\infty} a_n \pm \lim\limits_{n\longrightarrow\infty} b_n = a \pm b$
- $\lim\limits_{n\longrightarrow\infty} ca_n \quad = \quad c \lim\limits_{n\longrightarrow\infty} a_n = \text{ab}$
- $\lim\limits_{n\longrightarrow\infty}(a_n b_n) \quad = \quad (\lim\limits_{n\longrightarrow\infty} a_n)(\lim\limits_{n\longrightarrow\infty} b_n) = \frac{a}{b}$
- $\lim\limits_{n\longrightarrow\infty} \frac{a_n}{b_n} \quad = \quad \frac{\lim\limits_{n\longrightarrow\infty} a_n}{\lim\limits_{n\longrightarrow\infty} b_n},$ provided that $\lim\limits_{n\longrightarrow\infty} b_n \neq 0$
- $\lim\limits_{n\longrightarrow\infty} a_n^p \quad = \quad \left[\lim\limits_{n\longrightarrow\infty}\right]^p$ if $p > 0$ and $a_n > 0$

EXAMPLES

Determine if the following sequences are convergent or divergent. If convergent, determine its limit.

i. $\left\{ \frac{3n^2-1}{10n+5n^2} \right\}_{n=2}^{\infty}$

ii. $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$

SOLUTION I

For us to establish whether the sequence converges or diverges, we need to determine the limit.

$$
\begin{aligned}
\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty} &= \lim_{n\longrightarrow\infty}\frac{3n^2-1}{10n+5n^2} \\
&= \lim_{n\longrightarrow\infty}\frac{n^2(3-\frac{1}{n^2})}{n^2(\frac{10}{n}+5)} \\
&= \frac{3}{5}
\end{aligned}
$$

Since the limit exists and it is finite, the sequence therefore converges.

SOLUTION II

To determine the limit of this sequence, due to its alternating nature, it would be a bit difficult. We will therefore use theorem 2 to do this.

$$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \lim_{n\longrightarrow\infty}\left|\frac{(-1)^n}{n}\right|$$
$$= \lim_{n\longrightarrow\infty}\frac{1}{n}$$
$$= 0$$

Since the limit exists and it is finite, the sequence converges.

EXAMPLES

Establish the limits of the following using the precise definition of limit of sequences.

i.

$$\frac{2n^2+1}{n^2+1} = 2$$

ii.

$$\frac{1}{n^2+2n-4} = 0$$

SOLUTION 1

Here, we want to prove that $\forall \epsilon > 0, \exists N : n \geq N \implies \left|\frac{2n^2+1}{n^2+1} - 2\right| < \epsilon$

$$\begin{aligned}
\left|\frac{2n^2+1}{n^2+1} - 2\right| &= \left|\frac{-1}{n^2+1}\right| \\
&= \frac{1}{n^2+1} \\
&< \frac{1}{n^2} \\
&< \frac{1}{n}, \qquad \text{if} \quad n > 1 \\
&< \frac{1}{n} \\
&< \epsilon
\end{aligned}$$

SOLUTION 2

Here, we want to prove that $\forall \epsilon > 0, \exists N : n \geq N \implies \left|\frac{1}{n^2+2n-4} - 0\right| < \epsilon$

$$\begin{aligned}
\left|\frac{1}{n^2+2n-4} - 0\right| &= \left|\frac{1}{n^2+2n-4}\right| \\
&= \frac{1}{n^2+2n-4} \\
&< \frac{1}{n^2} \\
&< \frac{1}{n}, \qquad \text{if} \quad n > 1 \\
&< \frac{1}{n} \\
&< \epsilon
\end{aligned}$$

THEOREM I

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim\limits_{n \longrightarrow \infty} a_n = L$

SQUEEZE THEOREM

IF $a_n \leq c_n \leq b_n, \forall n > N$, for some N and $\lim\limits_{n \longrightarrow \infty} a_n = \lim\limits_{n \longrightarrow \infty} b_n = L$, then $\lim\limits_{n \longrightarrow \infty} c_n = L$

MONOTONICITY OF SEQUENCES

Here, we now discuss another essential concept on sequences, that is **Monotonic Sequences**. This concept will be needed in addition with one more concept yet to be established soon, will be used to establish a powerful theorem that redefine how to prove whether a sequence is **Convergent** or **Divergent** as we have already discussed from the limit theorem perspective.

DEFINITION OF MONOTONIC SEQUENCE

Given the sequence, say, $\{a_n\}$, the sequence could be described in two different ways as;

- $\{a_n\}$ is said to be an **increasing sequence** if $a_{n+1} > a_n, \forall n$ values
- $\{a_n\}$ is said to be an **decreasing sequence** if $a_{n+1} < a_n, \forall n$ values
- If $\{a_n\}$ decreases or increases, then it is said to be a **Monotonic Sequence**.

Therefore, to test for monotonicity, we choose to use,

$$a_{n+1} - a_n > 0 \qquad \textbf{for increasing test} \tag{47}$$

and also

$$a_{n+1} - a_n < 0 \qquad \textbf{for decreasing test} \tag{48}$$

EXAMPLES

Discuss if the following sequences are monotonic.

i. $a_n = \left\{ n^2 \right\}_{n=0}^{\infty}$ **ii.** $a_n = \left\{ 2n^2 + 1 \right\}_{n=1}^{\infty}$

iii. $a_n = \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$ **iv.** $a_n = \left\{ -n^2 \right\}_{n=1}^{\infty}$

SOLUTION I

i. $a_n = \left\{ n^2 \right\}_{n=0}^{\infty}$

We then formulate

$$a_n = n^2 \qquad a_{n+1} = (n+1)^2 \tag{49}$$

we then apply the formular

$$\begin{aligned} a_{n+1} - a_n &> 0 \quad \text{we testing for increasing} \\ (n+1)^2 - n^2 &> 0 \\ n^2 + 2n + 1 - n^2 &> 0 \\ 2n + 1 &> 0 \end{aligned}$$

the sequence is hence an **increasing monotonic sequence**

SOLUTION 2

ii. $a_n = \left\{2n^2 + 1\right\}_{n=1}^{\infty}$
We then formulate

$$a_n = 2n^2 + 1 \qquad a_{n+1} = 2(n+1)^2 + 1 \tag{50}$$

we then apply the formular

$$\begin{aligned} a_{n+1} - a_n &> 0 \quad \text{we testing for increasing} \\ (2(n+1)^2 + 1) - (2n^2 + 1) &> 0 \\ n^2 + 2n + 1 - n^2 &> 0 \\ (2n^2 + 4n + 2) - (2n^2 + 1) &> 0 \\ 4n^2 + 1 &> 0 \end{aligned}$$

the sequence is hence an **increasing monotonic sequence**

SOLUTION 3

$$a_n = \frac{n}{n+1} \qquad a_{n+1} = \frac{(n+1)}{n+2} \tag{51}$$

$$\begin{aligned}
a_{n+1} - a_n &> 0 \quad \text{we testing for increasing} \\
\frac{(n+1)}{n+2} - \frac{n}{n+1} &> 0 \\
\frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} &> 0 \\
(n+1)^2 - n(n+2) &> 0 \\
n^2 + 2n + 1 - n^2 - 2n &> 0 \\
1 &> 0
\end{aligned}$$

the sequence is hence an **increasing monotonic sequence**

SOLUTION 4

iv. $a_n = \left\{-n^2\right\}_{n=1}^{\infty}$ We then formulate

$$a_n = -n^2 \qquad\qquad a_{n+1} = -(n+1)^2 \tag{52}$$

we then apply the formular

$$\begin{aligned} a_{n+1} - a_n &> 0 \quad \text{we testing for increasing} \\ -(n+1)^2 - (-n^2) &> 0 \\ -n^2 - 2n - 1 + n^2 &> 0 \\ -2n - 1 &> 0 \end{aligned}$$

the sequence is hence an **decreasing monotonic sequence**

DEFINITION OF BOUNDED SEQUENCE

Given the sequence $\{a_n\}$,

• If $\exists m$ such that $m \leq \{a_n\}, \forall n$, we say that the sequence is **bounded below**. The number m is called the **Lower Bound**. The **Greatest Lower Bound (GLB)** is also called the **Infimum (Inf)**.

• If $\exists m$ such that $M \geq \{a_n\}, \forall n$, we say that the sequence is **bounded above**. The number m is called the **Upper Bound**. The **Greatest Lower Bound (GLB)** is also called the **Supremum**.

• If the sequence a_n has both the **Supremum(Sup)** and the **Infimum**, the sequence a_n is said to be **Bounded**.

EXAMPLES

Discuss if the following sequences are bounded.

i. $a_n = \left\{ n^2 \right\}_{n=0}^{\infty}$

ii. $a_n = \left\{ 2n^2 + 1 \right\}_{n=1}^{\infty}$

iii.$a_n = \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

iv. $a_n = \left\{ -n^2 \right\}_{n=1}^{\infty}$

SOLUTION 1

i. $a_n = \left\{ n^2 \right\}_{n=0}^{\infty}$

Here, what we will do is to list the elements of the sequence.

$$\left\{ n^2 \right\}_{n=0}^{\infty} = \left\{ \underbrace{0}_{n=0}, \underbrace{1}_{n=1}, \underbrace{4}_{n=2}, \underbrace{9}_{n=3}, \underbrace{16}_{n=4}, \cdots \right\}$$

Looking at the elements of the sequence, we realise that

$$\cdots, -5, -4, -3, -2, -1 < a_n, \qquad \textbf{Infimum} = -1$$

On the other hand, the sequence increases without bound which makes it infinite and will therefore have its Supremum as ∞. It has no Supremum. Therefore the bounds of the sequence a_n is given as **Inf = -1** and $Sup = \infty$. The sequence a_n is hence **not bounded**.

SOLUTION 2

ii. $a_n = \left\{2n^2 + 1\right\}_{n=1}^{\infty}$

Here, what we will do is to list the elements of the sequence.

$$\left\{2n^2 + 1\right\}_{n=1}^{\infty} = \left\{\underbrace{3}_{n=1}, \underbrace{9}_{n=2}, \underbrace{19}_{n=3}, \underbrace{33}_{n=4}, \underbrace{51}_{n=5}, \cdots\right\}$$

Looking at the elements of the sequence, we realise that

$$\cdots, -2, -1, 0, 1, 2 < a_n, \qquad \textbf{Infimum} = 2$$

On the other hand, the sequence increases without bound which makes it infinite and will therefore have its Supremum as ∞. It has no Supremum. Therefore the bounds of the sequence a_n is given as $\mathbf{Inf = 2}$ and $Sup = \infty$. The sequence a_n is hence **not bounded**.

SOLUTION 3

iii. $a_n = \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

Here, what we will do is to list the elements of the sequence.

$$\left\{ n^2 \right\}_{n=1}^{\infty} = \left\{ \underbrace{\frac{1}{2}}_{n=1}, \underbrace{\frac{2}{3}}_{n=2}, \underbrace{\frac{3}{4}}_{n=3}, \underbrace{\frac{4}{5}}_{n=4}, \underbrace{\frac{5}{6}}_{n=5}, \dots \right\}$$

Looking at the elements of the sequence, we realise that

$$0 < a_n < 1$$

that is the sequence a_n elements lies in between 0 ans 1
Therefore the bounds of the sequence a_n is given as $\mathbf{Inf = 0}$ and $\mathbf{Sup = 1}$. The sequence a_n is hence **bounded**.

SOLUTION 4

iv. $a_n = \left\{ -n^2 \right\}_{n=1}^{\infty}$

Here, what we will do is to list the elements of the sequence.

$$\left\{ -n^2 \right\}_{n=1}^{\infty} = \left\{ \underbrace{-1}_{n=1}, \underbrace{-4}_{n=}, \underbrace{-9}_{n=3}, \underbrace{-16}_{n=4}, \underbrace{-25}_{n=5}, \cdots \right\}$$

Looking at the elements of the sequence, we realise that

$$\cdots, 4, 3, 2, 1, 0 > a_n, \qquad \textbf{Supremum} = 0$$

On the other hand, the sequence decreases without bound which makes it negative infinite and will therefore have its Infimum as $-\infty$. It has no Infimum. Therefore the bounds of the sequence a_n is given as **Sup = 0** and $Inf = -\infty$. The sequence a_n is hence **not bounded**.

THEOREM 2

Given the sequence a_n, if a_n is bounded and monotonic, then a_n is said to be convergent.

EXAMPLES

Use the theorem above to determine if the following sequences converges or diverges.

i. $a_n = \left\{ n^2 \right\}_{n=0}^{\infty}$

ii. $a_n = \left\{ 2n^2 + 1 \right\}_{n=1}^{\infty}$

iii. $a_n = \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

iv. $a_n = \left\{ -n^2 \right\}_{n=1}^{\infty}$

THANK YOU