Unit-3.

Lattices

Partially Ordered Set (POSET) – A relation R defined on a set A is a partial order relation if

1.) R is reflexive i.e., $\forall a \in A$ $(a,a) \in R$.

2.) R is anti-symmetric i.e., if $(a,b) \in R$ and $(b,a) \in R$ then $a=b$.

3.) R is transitive i.e, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

POSET (Partially Ordered Set) – A set A together with some partial order relation is called partially ordered set or poset denoted by (A, R).

A partially ordered relation is denoted by $\preceq$. $a \preceq b$ is read as "a precedes b."

For Example – Let P(S) be the power set under the inclusion relation $\subseteq$. Then, $\subseteq$ is a partial ordering on the power set P(S). Since,

$A \subseteq A \quad \forall A \in P(S)$ (Reflexive).

$A \subseteq B$ and $B \subseteq A$ and $A = B$ (Anti-Symmetric).

$A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ (Transitive).

$\therefore (P(S), \subseteq)$ is a poset.

For Example – The relation / of divisibility is not a ordering relation on set Z of integers. Since, it is not antisymmetric as $7/-7$ and $-7/7$ but $7 \neq -7$.

Comparability – The elements a and b of poset $(A, \preceq)$ are comparable if either $a \preceq b$ or $b \preceq a$. When neither $a \preceq b$ nor $b \preceq a$ then a and b are called incomparable or non-comparable.

For Example – In the poset, $(Z^+, /)$ the integers 3 and 6 are comparable since $3/6$ but 7 and 5 are incomparable because neither $7/5$ or $5/7$.

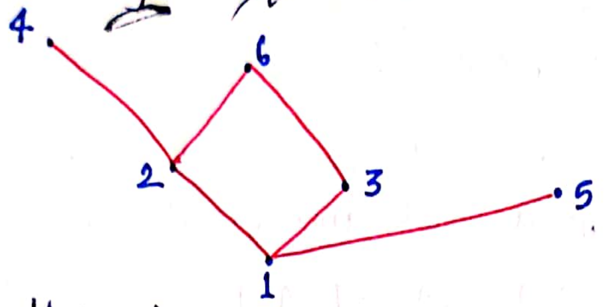
Totally or Linearly Ordered Set – An ordered set A is said to be linearly or totally ordered if every pair of element in A are comparable.

A totally ordered set is also called a chain.

For Example – The poset $(N, \leq)$ is totally ordered set since every two natural numbers are comparable.

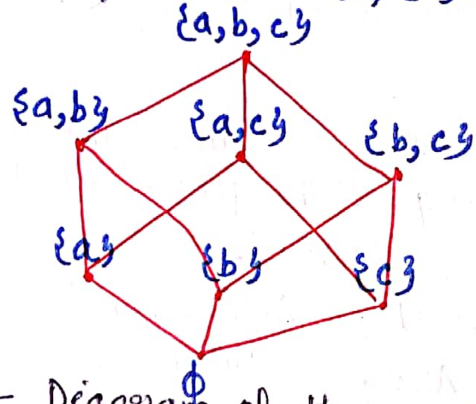
Hasse Diagram – A partial order $\preceq$ on set X can be represented by means of a diagram known as Hasse Diagram.

Ques – Let $X = \{1,2,3,4,5,6\}$, then $/$ is a partial order relation on X. Draw the hasse diagram of $(X, /)$.



Ques – Draw the Hasse Diagram for the partial ordering $\{(A,B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a,b,c\}$.

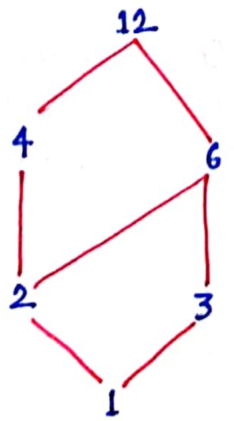
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$



Ques – Draw the Hasse Diagram of the poset $(A, /)$ where $A = \{1, 3, 9, 27, 81\}$.

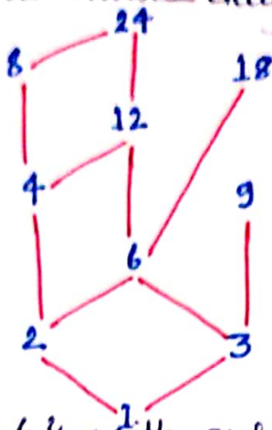
Ques – Let A be the set of factors of 12 and $\leq$ be the relation defined on A such that $x \leq y$ means x divides y.

$A = \{1, 2, 3, 4, 6, 12\}$



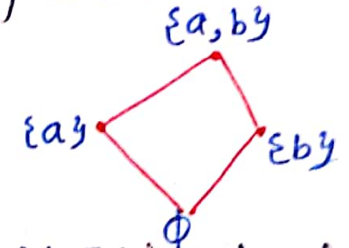
Scanned by CamScanner

3. let $A = \{1,2,3,4,6,8,9,12,18,24\}$ be ordered set with relation "x divides y". Draw its hasse diagram.

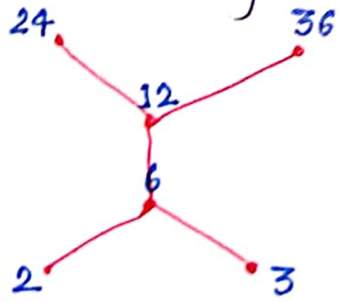


Ques. let $A = \{1,2,3,4,5,6\}$ with relation '$\leq$' defined as "a less than or equal to b". Draw Hasse diagram of $(A, \leq)$.

Ques: let $A = \{a,b\}$ and $P(A) = \{\{a\}\{b\}\{a,b\}\{\phi\}\}$ be power set of A. Draw its hasse diagram.



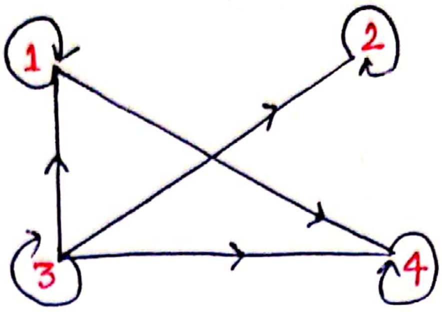
Ques. let $A = \{2,3,6,12,24,36\}$ and relation '$\leq$' be such that "$x \leq y$" if x divides y. Draw the hasse diagram.



Constructing a Hasse Diagram –

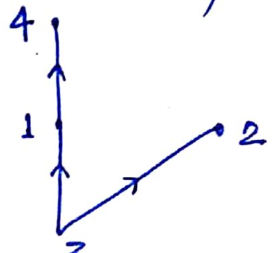
1. Start with a directed graph of the relation.
2. Remove the loops at all the vertices.
3. Remove all edges whose existence is implied by transitive property.
4. Arrange all arrows pointing upwards toward their terminal vertex. Remove all the arrows.

Qus: Draw a Hasse diagram from the directed graph G for a partial order relation on a set $A = \{1, 2, 3, 4\}$.

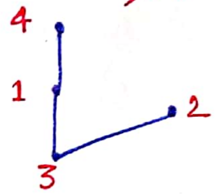


Step 1 – Remove all the self loops i.e.,
(1,1) (2,2) (3,3) (4,4)

Step 2 - Remove transitive edges and arrange all the arrows pointing upward.
(3,4) is transitively implied.

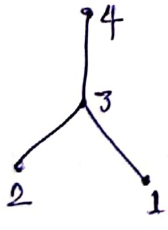


Step 3 – Remove all the arrows.



Qus. Describe the ordered pairs in the relation determined by the Hasse diagram of a poset $(A, \leq)$ on the set $A = \{1, 2, 3, 4\}$.

a.)



b.)

a. Since the relation on A is a partial order, all reflexive pairs (1,1)(2,2)(3,3)(4,4). All edges when converted to upwards towards their vertices give the ordered pairs (1,3)(2,3)(3,4). The transitively implied arcs gives the ordered pairs (1,4)(2,4). Thus the ordered pairs in the relation represented by Hasse Diagram –
{(1,1)(2,2)(3,3)(4,4)(1,3)(2,3)(3,4)(1,4)(2,4)}.

Similarly the ordered pairs in the relation represented by Hasse Diagram as —

{(1,1) (2,2) (3,3) (4,4) (1,2) (1,3) (1,4) (2,3) (2,4) (3,4)}

Greatest Element — let $(P, \leq)$ be a poset. An element $a \in P$ is the greatest element of P if $x \leq a$ for all $x \in P$. i.e., every element in P precedes a. The greatest element, if it exists, is unique.

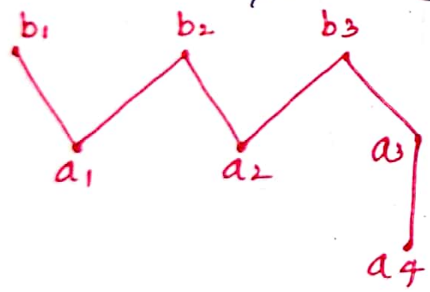
Least Element — let $(P, \leq)$ be a poset. An element $b \in P$ is called the least element if $b \leq x$ for all $x \in P$ i.e., every element in P succeeds b. The least element is unique, if exists.

Minimal Element — let $(A, \leq)$ be a poset. An element is called minimal element if no other element of A strictly precedes a. In Hasse diagram, a is minimal element if no edge enters a from below.

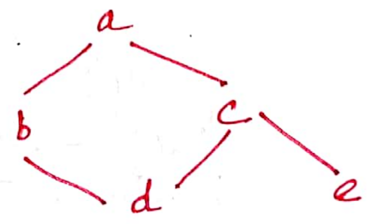
Maximal Element — An element $b \in A$ is called maximal element if no other element of A strictly precedes b. In Hasse diagram, b is maximal element if no edge leaves b in upward direction.

There can be more than one maximal or more than one minimal element.

Ques. Determine all the maximal and minimal element of the POSET from the following Hasse diagrams.



a.)



b.)

c.)

Maximal — b_1, b_2, b_3
Minimal — a_1, a_2, a_4

Maximal — a
Minimal — d, e

Maximal — 4
Minimal — 1.

Ques: Find the least and greatest element in the poset $(Z^+, /)$, if they exists.

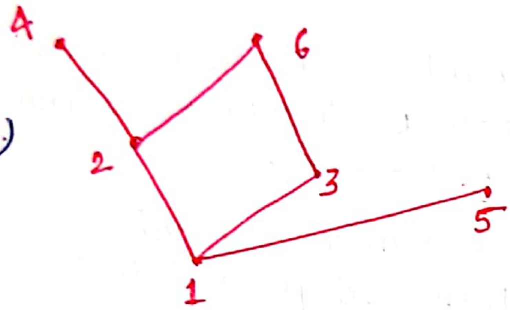
The least element of the poset $(Z^+, /)$ is 1 since $1/n$ whenever n is a positive integer. There is no greatest element since there is no integer which is divisible by all positive integers.

Ques. Determine whether the posets represented by a Hasse diag have a greatest element, least element, minimal and ma elements.

a.)

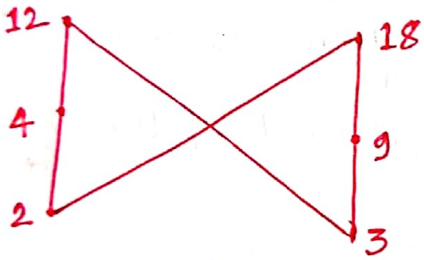
Greatest – 27
least – 1
Maximal – 27
Minimal – 1

b.)



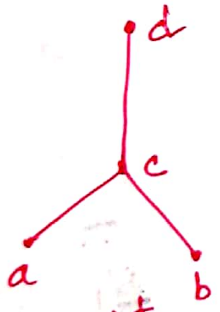
Greatest – No
least – 1
Maximal – 4, 6, 5
Minimal – 1

c.)



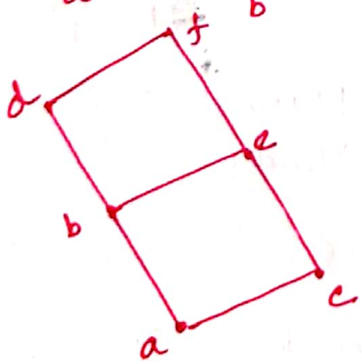
Greatest – No
least – No
Maximal – 12, 18
Minimal – 2, 3

d.)

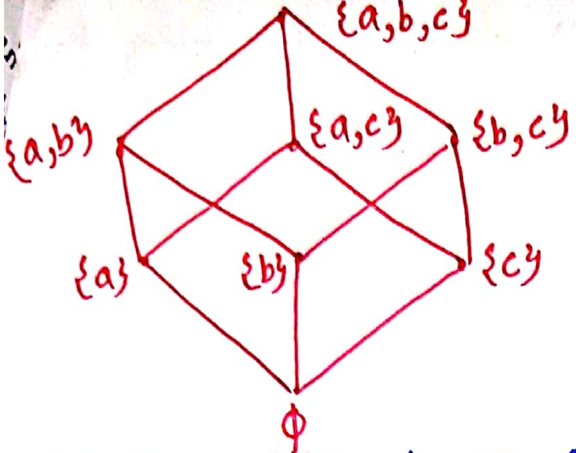


Greatest – d
least – No
Maximal – d
Minimal – a, b

e.)



Greatest – f
Least – a
Maximal – f
Minimal – a



Greatest – {a,b,c}
least – ϕ
Maximal – {a,b,c}
Minimal – ϕ

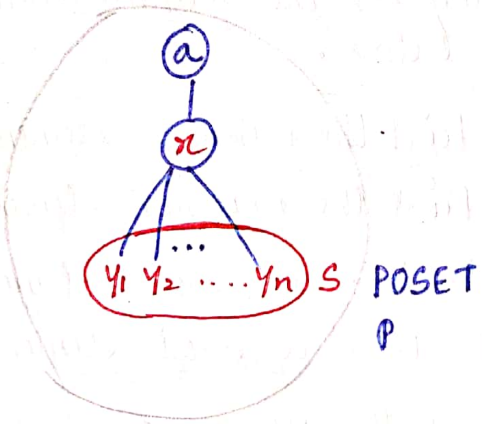
④

Well Ordered Set – A set with an ordering relation is well order if every non-empty subset of the set has a least element. ④

For Example – the set of natural numbers is well ordered. The set Z of integers is not well ordered since the set of negative integers which is a subset of Z, has no least element.

Upper Bound – let $(P, \leq)$ be a poset and S be a subset of P.

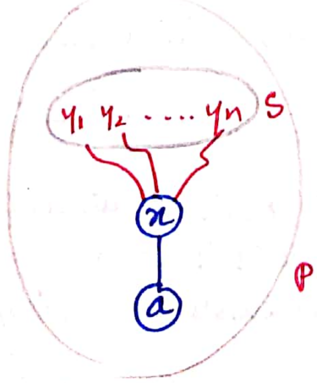
Upperbound(S) :- $x \in P$ is called UB(S) if $\forall y \in S,\ y \leq x$.



then, x and a are the upperbounds of S.

Lower Bound – Let $(P, \leq)$ be a poset and S be a subset of P.

LowerBound(S) :- $x \in P$ is called LowerBound(S) if $\forall y \in S,\ x \leq y$.

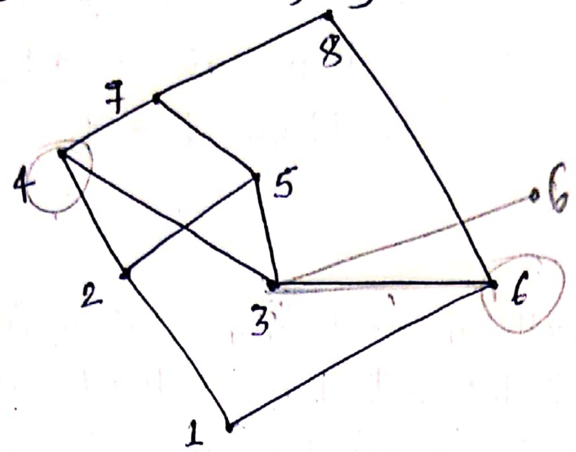


then, x and a are the LowerBounds of S.

Least Upper Bound LUB $(\vee)$:- minimum (UB(S)).

Greatest Lower Bound GLB $(\wedge)$:- maximum (LB(S)).

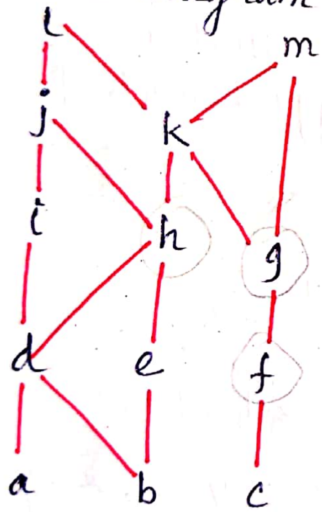
Ques. In the POSET P, find upper bound and least upper bound for $A = \{2, 3\}$ and $B = \{4, 6\}$.



3R4
3R6.

Upper Bound of $\{2,3\} = 4, 5, 7, 8$
Lower Bound of $\{2,3\} = \phi$
Upper Bound of $\{4,6\} = 8$
Lower Bound of $\{4,6\} = 1$

Ques. Determine the following for the partial order represented by Hasse diagram shown Below.



a.) Find the minimal elements. a, b and c
b.) Find the maximal elements. L and m
c.) Is there a greatest element? No
d.) Is there a least element? No
e.) Find the upper bounds of $\{a, b, c\}$? k, l, m
f.) Find all the lower bounds of $\{f, g, h\}$? No
g.) Find the GLB of $\{f, g, h\}$ if it exists? No GLB
h.) Find the LUB of $\{a, b, c\}$ if it exists? No KLM

$a \le x$
$b \le x$
$x \le f$
$x \le g$
$x \le h$
$x \le s$
$y \le x$

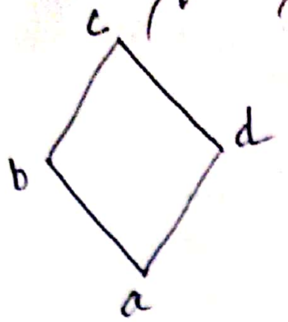
Lattices – A Lattice is a poset $(P, \le)$ in which every 2 element subset of P has both a least upper bound (LUB) and Greatest lower Bound (GLB) i.e., if $(lub(x,y))$ and $glb(x,y)$ exists for every x and y in P. Denoted as,

$x \vee y = lub(x, y)$ (read as join of x and y)
$x \wedge y = glb(x, y)$ (read as meet of x and y.)

Note :– Every Chain is a lattice.

Which of the following Hasse Diagram represent a lattice?

(a)



lub

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

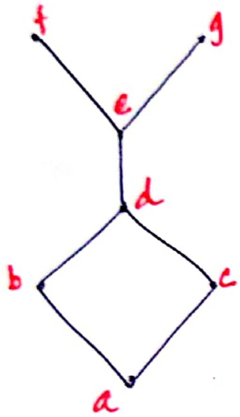
glb

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

5

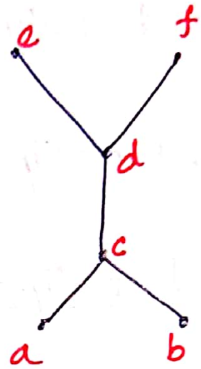
From the table, we can conclude that for every a, b belonging to given poset $a \wedge b$ and $a \vee b$ exists. Therefore (a) is a lattice.

(b)



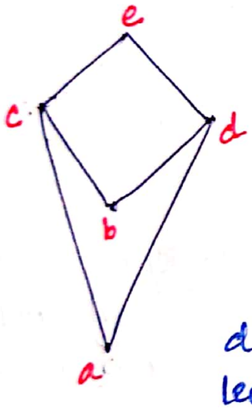
It is not a lattice since $f \vee g$ does not exist. So, there is no upper bound for $\{f, g\}$.

(c)



It is not a lattice as $e \vee f$ and $a \wedge b$ does not exists.

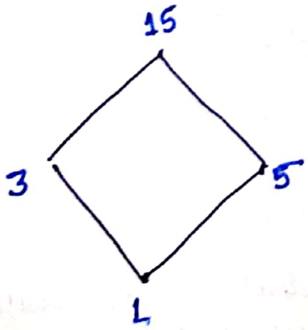
(d)



It is not a lattice, since $a \vee b$ does not exist as there are c, d and e as upper bound and c and d are non-comparable, so least upper bound does not exist.

Ques. Prove that the partially ordered set D_{15} under the relation 'divides' is a lattice.

$D_{15} = \{1, 3, 5, 15\}$.



$\wedge$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

$\vee$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

From the table, we can conclude that meet and join of each pair of elements of D_{15} exists. Therefore, D_{15} is a lattice.

Duality — The dual of any statement in a lattice $(L, \wedge, \vee)$ is defined to be statement that is obtained by interchanging $\wedge$ and $\vee$.

Example - $a \wedge (b \vee c)$; $a \wedge a = a$

$a \vee (b \wedge c)$ [Dual] $\quad$ $a \vee a = a$ [Dual]

Properties of Lattice :-

let L be a lattice and $a, b \in L$ then,

1. Idempotent Property - $a \vee a = a$
   $a \wedge a = a$

2. Commutative Property - $a \vee b = b \vee a$
   $a \wedge b = b \wedge a$

3. Associative Property - $a \vee (b \vee c) = (a \vee b) \vee c$
   $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Property - $a \vee (a \wedge b) = a$
   $a \wedge (a \vee b) = a$

5. $a \vee b = b$ iff $a \leq b$
   $a \wedge b = a$ iff $a \leq b$
   $a \wedge b = a$ iff $a \vee b = b$

6. Distributive Inequality - $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
   $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Lattice as an Algebraic System — Algebraic system is a set together with some rules for combining elements of the set to form other elements of set. A lattice can be treated as an algebraic system, hence one can apply many concepts associated with algebraic system of lattice.

Consider a lattice as an algebraic system $(L, \oplus, *)$ together with two binary operation $\oplus$ and $*$ such that,

$a * b = b * a \qquad a \oplus b = b \oplus a$ (Commutative Law)

$(a * b) * c = a * (b * c)$
$(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (Associative Law)

$a * (a \oplus b) = a \qquad a \oplus (a * b) = a$ (Absorption law)

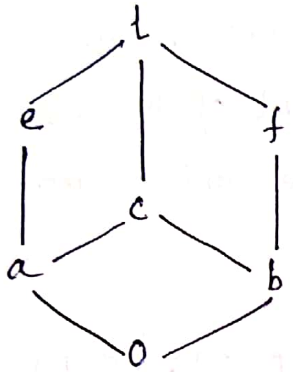
holds for each $a, b, c \in L$. Then, we can define a partial order relation R on L such that (L, R) becomes a lattice with $a \vee b = a \oplus b$ and $a \wedge b = a * b$ $\quad a, b \in L$.

Sub-Lattice — Let $(L, \leq)$ be a lattice. A non-empty subset S of L is called sublattice of L if $a \vee b \in S$ and $a \wedge b \in S$, whenever $a, b \in S$.

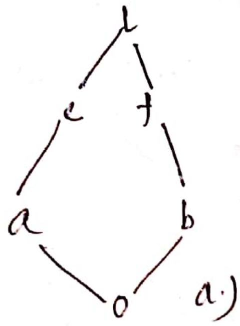
6

Note — Sublattice of a lattice is itself a lattice.
Every singleton set of lattice is sublattice of L.
If m/n (m divides n) then D_m is sublattice of D_n.

Ques. Consider the lattice $(L, \leq)$ as shown below :-

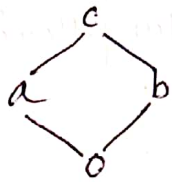


Determine which of the following is sublattice of L.



a.)

L_1



b.)

L_2

a.) L_1 is not sublattice of L since
$a \vee b = c \notin L_1$

b.) LUB

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

GLB

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

For all $a, b \in L_2$
$a \vee b \in L_2$
$a \wedge b \in L_2$
Hence, L_2 is sublattice of L.

Types Of Lattice —

1.) Complete Lattice — A Lattice is called complete if each of its non-empty subsets has a least Upper Bound [LUB] and greatest Lower Bound [GLB].

Every finite lattice is complete because every subset is finite. Also, every complete lattice must have a least element and a greatest element.

2.) Bounded Lattice — A lattice L is said to be bounded if it has a greatest element I or 1 and a least element 0. In such lattice,

$$a \vee I = I \,;\; a \vee 0 = a$$
$$a \wedge I = a \,;\; a \wedge 0 = 0$$

$$\forall a \in L \text{ and } 0 \leq a \leq I$$

Note — Every Finite lattice is bounded.

3. **Complemented lattice** — Let L be a bounded lattice with greatest I or 1 and least element 0. Let $a \in L$ then an element $a' \in L$ is complement of a if,

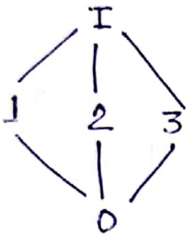
$$a \vee a' = I$$
$$a \wedge a' = 0$$

A lattice L is called complemented if it is bounded and if every element in L has a complement.

1.) 0 and I are complement of each other.
2.) Complement of an element in a lattice is not unique.
3.) Complement is symmetric i.e., If a is complement of b then $b' = a$ also.

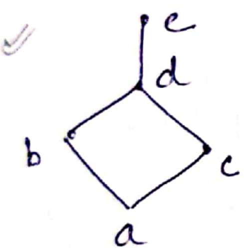
Ques: Consider following lattices. Find complement of each element if given is a lattice.

a.)



$0' = 1$
$1' = 0$
$1' = 2, 3$ as $1 \vee 2 = I$ $1 \vee 3 = I$ $1 \wedge 2 = 0$ $1 \wedge 3 = 0$
$2' = 1, 3$ as $2 \vee 1 = I$ $2 \vee 3 = I$ $2 \wedge 2 = 0$ $2 \wedge 3 = 0$
$3' = 1, 2$ as $3 \vee 1 = I$ $3 \vee 2 = 1$ $3 \wedge 1 = 0$ $3 \wedge 2 = 0$

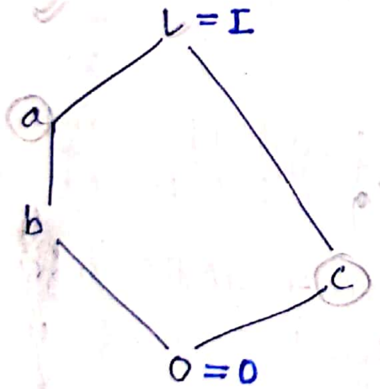
b.)



I = e
0 = a

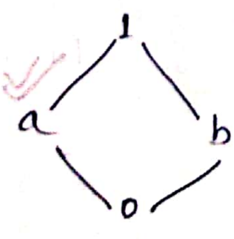
$a' = e$
$e' = a$
b' = No complement
c' = No complement
d' = No complement

c.)



$0' = L$
$L' = 0$
$a' = c$ ($a \vee c = L$, $a \wedge c = 0$).
$b' = c$ ($c \vee b = 1$, and $c \wedge b = 0$).
$c' = b, c$

d.)

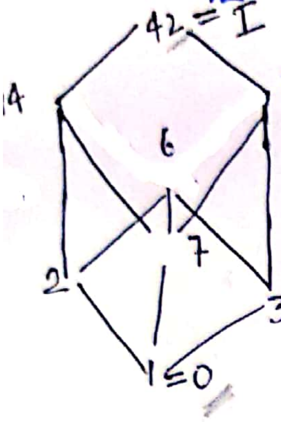


$0' = 1$
$1' = D$
$a' = b$
$b' = a$

Find complement of each element in $(D_{42}, /)$. Is D_{42} a complemented lattice?

7

$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$



$I = 42$
$0 = 1$

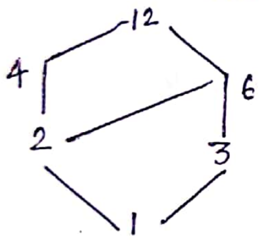
Complements of elements are given by,

$1' = 42$
$42' = 1$
$2' = 21$
$3' = 14$
$6' = 7$
$7' = 6$
$14' = 3$
$21' = 2$
$42' = 1$.

$\therefore$ Hence, D_{42} is a complemented lattice.

Ques. Is D_{12} a complemented Lattice?

$D_{12} = \{1, 2, 3, 4, 6, 12\}$



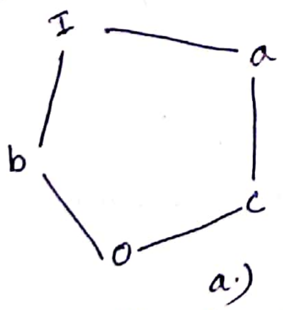
D_{12} is not complemented since 6 has no complement.

4.) Distributive Lattice — A Lattice L is said to be distributive if for any element a, b and c of L following properties are satisfied :-

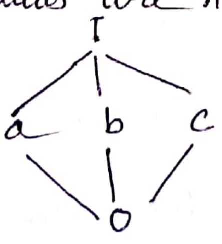
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

otherwise, L is non-distributive Lattice.

Ques Show that the Given lattices are non-distributive.



a.)



b.)

b.) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$a \wedge I = 0 \vee 0$
$a \neq 0$
Hence, it is non-distributive

a) $a \wedge (b \vee c) = a \wedge I = a$
$(a \wedge b) \vee (a \wedge c) = 0 \vee c = c$
$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$
Hence, it is non-distributive

Ques let L be a bounded distributive lattice. Then prove if a com[plement] exists, it is unique.

let a_1 and a_2 be two complements of an element $a \in L$

By definition of complement -

$$\left.\begin{matrix} a \vee a_1 = I \\ a \wedge a_1 = 0 \end{matrix}\right\} - ① \quad \left.\begin{matrix} a \vee a_2 = I \\ a \wedge a_2 = 0 \end{matrix}\right\} - ②$$

Consider, $a_1 = a_1 \vee 0$

$= a_1 \vee (a \wedge a_2)$ from eq (2)

$= (a_1 \vee a) \wedge (a_1 \vee a_2)$ Distributive Property

$= (a \vee a_1) \wedge (a_1 \vee a_2)$ Commutative Property

$= I \wedge (a_1 \vee a_2)$ from eq (1)

$= a_1 \vee a_2$ — (3)

Consider, $a_2 = a_2 \vee 0$

$= a_2 \vee (a \wedge a_1)$ from eq (1)

$= (a_2 \vee a) \wedge (a_2 \vee a_1)$ Distributive Property

$= (a \vee a_2) \wedge (a_1 \vee a_2)$ Commutative Property.

$= I \wedge (a_1 \vee a_2)$ from eq (2)

$= a_1 \vee a_2$ — (4)

Hence, from (3) and (4)

$a_1 = a_2$

So for Bounded distributive lattice, complement is unique.

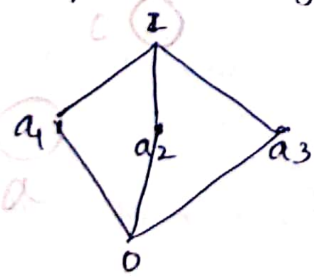
5.) **Modular lattice** – A lattice L is called modular lattice if,

$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

where $a, b, c \in L$.

Note - Every distributive lattice is modular but converse is not true.

Ques. Show that given lattice L is modular.



To show that the lattice is modular, we have to show that

$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

whenever $a, b, c \in L$

Now let $a = a_1$ and $c = I$

$\therefore a_1 \leq I$

and $a \vee (b \wedge c) = a_1 \vee (b \wedge I) = a_1 \vee b$

$(a \vee b) \wedge c = (a_1 \vee b) \wedge I = a_1 \vee b$ for any $b \in L$

com

$\therefore$ If $a_1 \leq I$ then $(a_1 \vee b) \wedge I = a_1 \vee (b \wedge I)$

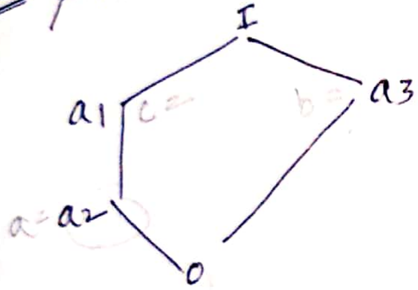
Similarly, we can prove above for $(a = a_2$ and $c = I)$ and $(a = a_3$ and $c = I)$

Now, let $a = 0$ and $c = a_1$

$$a \vee (b \wedge c) = 0 \vee (b \wedge a_1) = b \wedge a_1$$
$$(a \vee b) \wedge c = (0 \vee b) \wedge a_1 = b \wedge a_1$$
$$a \vee (b \wedge c) = (a \vee b) \wedge c$$

Similarly, we can prove (1) for the pairs $(a = 0, c = a_2)$ and $(a = 0$ and $c = a_3)$. Hence, L is a modular Lattice.

Ques. Show that the Lattice L given below is not modular.



let $a, b, c \in L$

$a = a_2$

$b = a_3$ then $a \leq c$

$c = a_1$

Now, $a \vee (b \wedge c) = a_2 \vee (a_3 \wedge a_1)$

$= a_2 \vee 0$

$= a_2$

$(a \vee b) \wedge c = (a_2 \vee a_3) \wedge a_1$

$= I \wedge a_1 = a_1$

$$\boxed{a \vee (b \wedge c) \neq (a \vee b) \wedge c}$$

$\therefore$ L is not modular Lattice.