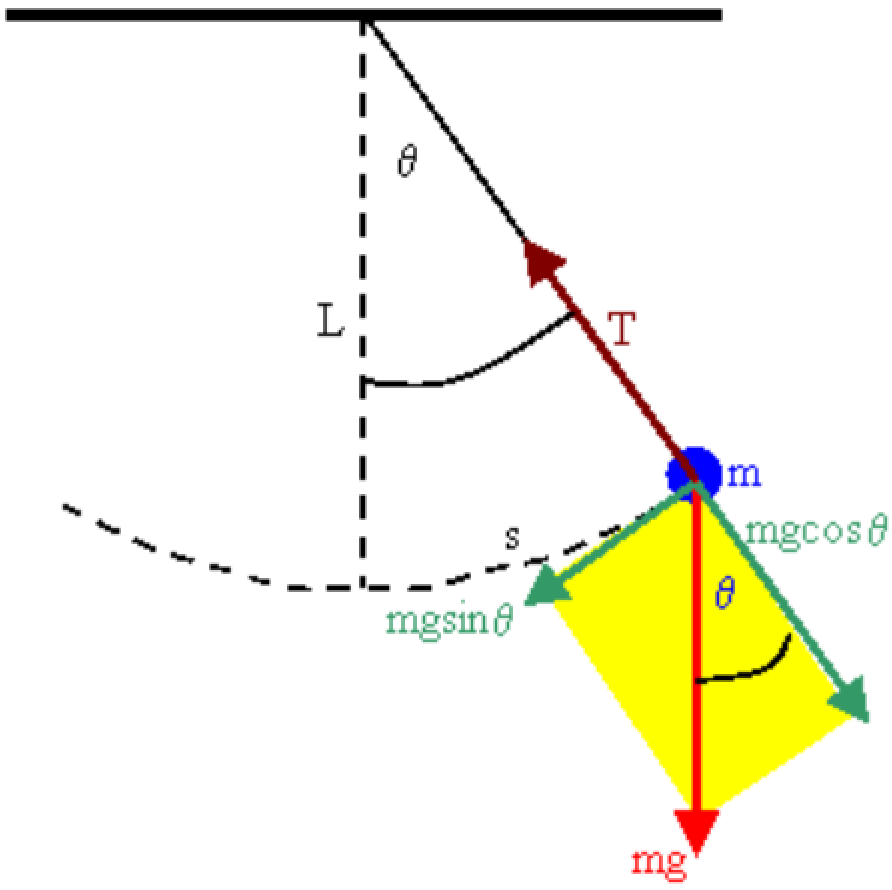
APPLICATION OF RUNGE KUTTA

SINGLE & DOUBLE PENDULUM

Deriving Motion Equation



$$T = mg\cos(\theta)$$

$$ma = -mg\sin(\theta)$$

$$0 = a + g\sin(\theta)$$

$$0 = \frac{d^2 s}{dt} + g\sin(\theta)$$

$$s = L\theta$$

$$0 = L\frac{d^2\theta}{dt^2} + g\sin(\theta)$$

- Motion Equation $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$

Single Pendulum Modelling (1)

- Position and velocity of CM

$$x = \frac{l}{2}\sin\theta;\ y = \frac{l}{2}\cos\theta$$

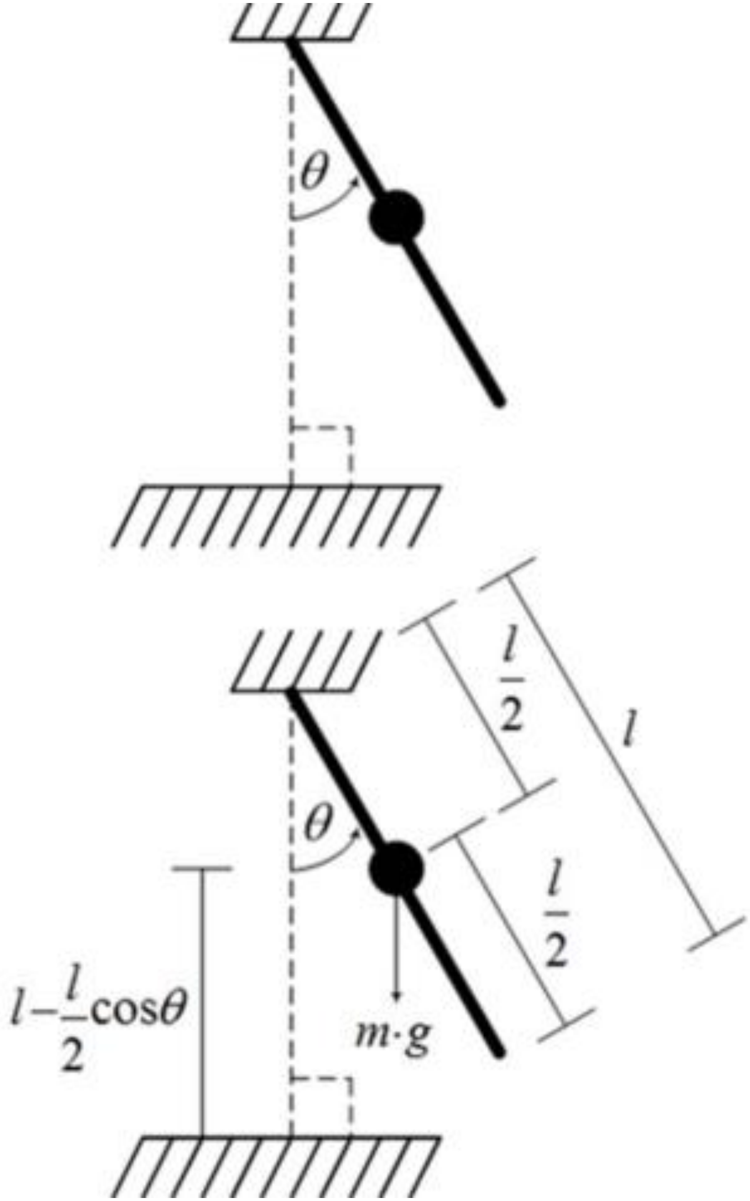
$$\dot{x} = \frac{l}{2}\cos\theta\dot{\theta};\ \dot{y} = -\frac{l}{2}\sin\theta\dot{\theta}$$

$$v^2 = \frac{l^2}{4}\cos^2\theta\dot{\theta}^2 + \frac{l^2}{4}\sin^2\theta\dot{\theta}^2$$

$$= \frac{l^2}{4}\dot{\theta}^2$$

- Kinetic Energy $E_K = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2$
- Potential Energy $E_P = mgh$

$$= mg\left(l - \frac{l}{2}\cos\theta\right)$$



Single Pendulum Modelling (2)

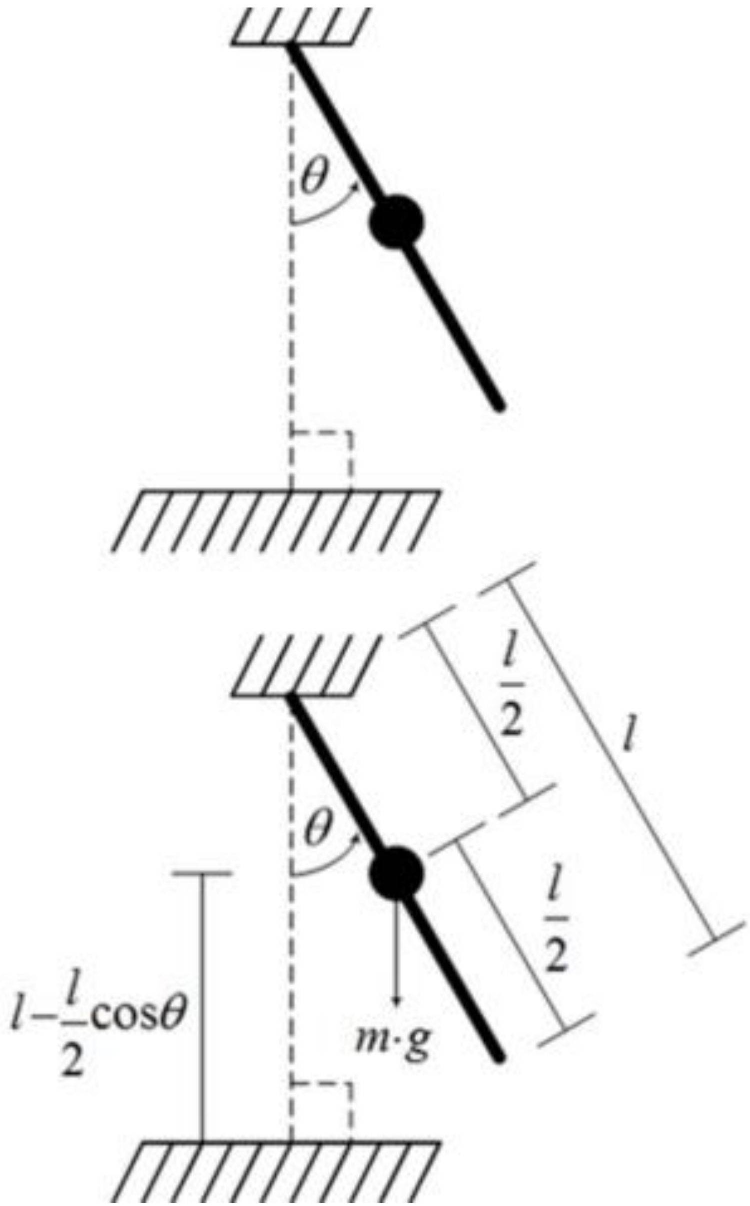
- Lagrange function

$$L = \sum E_K - \sum E_P$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2 - mg\left(l - \frac{l}{2}\cos\theta\right)$$
$$= \frac{1}{8}ml^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - mgl\left(1 - \frac{1}{2}\cos\theta\right)$$

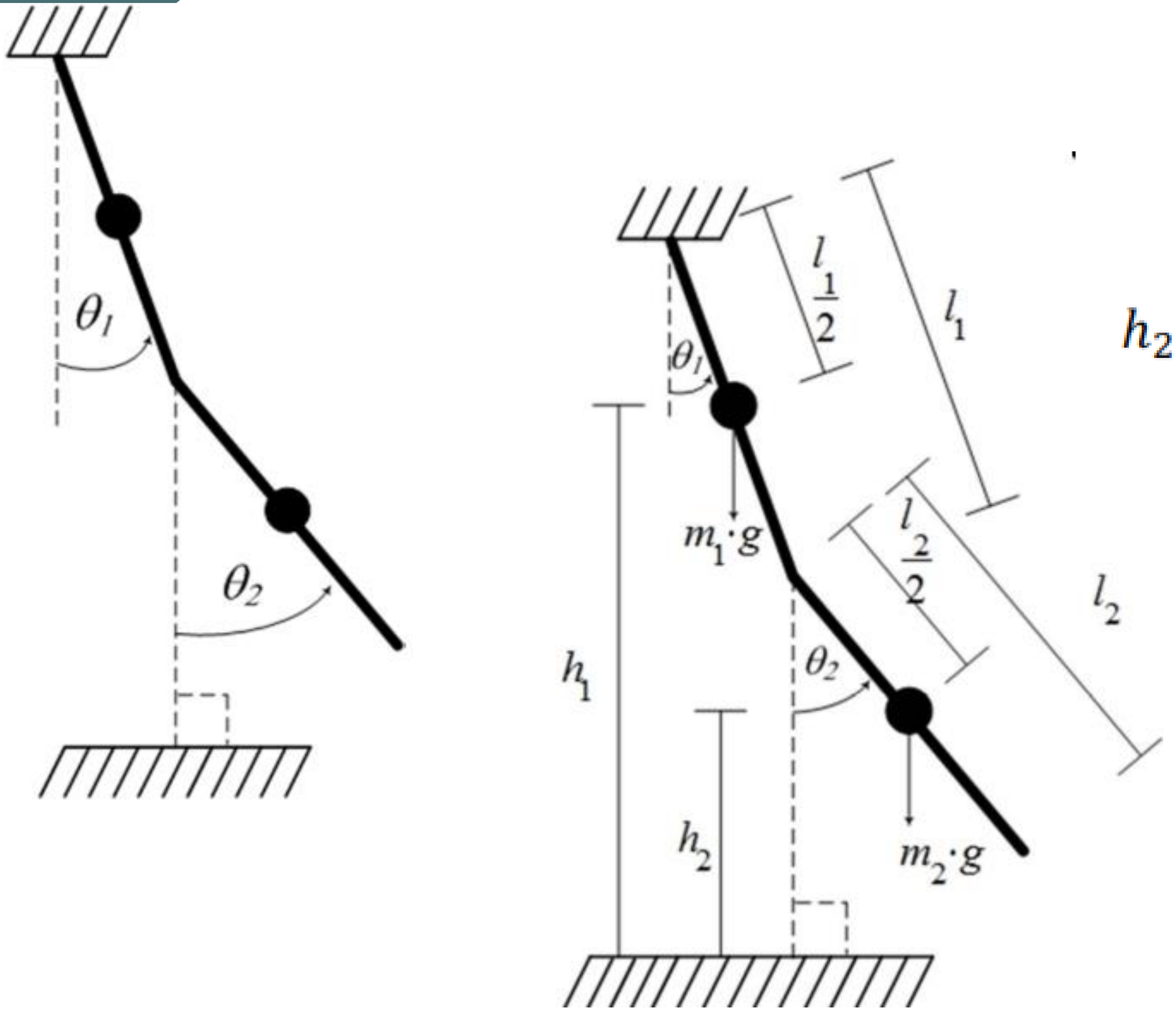
- Lagrange equation $\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\theta}} - \dfrac{\partial L}{\partial\theta} = \tau$

$$\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{4}ml^2\dot{\theta} + I\dot{\theta} \qquad \frac{\partial L}{\partial\theta} = -mg\frac{l}{2}\sin\theta$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{4}ml^2\ddot{\theta} + I\ddot{\theta}$$

- Motion equation $\left(\dfrac{1}{4}ml^2 + I\right)\ddot{\theta} + mg\dfrac{l}{2}\sin\theta = \tau$



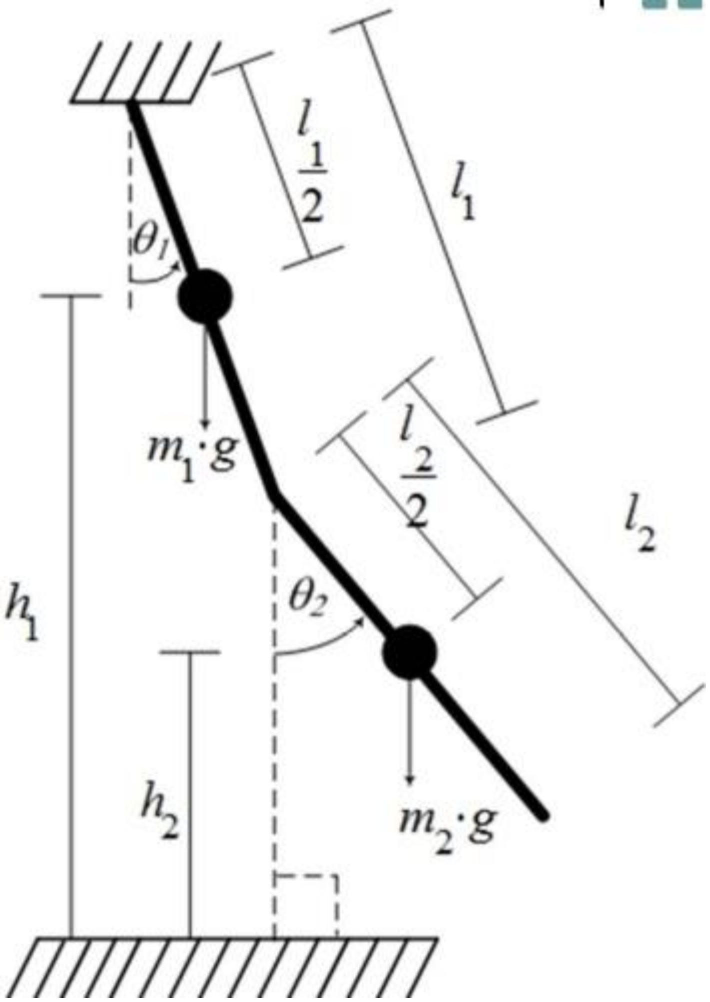
Double Pendulum System (1)



$$h_1 = L_1 + L_2 - \frac{1}{2} L_1 \cos\theta_1$$

$$h_2 = L_1 + L_2 - [L_1 \cos\theta_1 + \frac{1}{2} L_2 \cos\theta_2]$$

Double Pendulum System (2)



$$X_1 = \frac{1}{2} L_1 \sin\,\theta_1$$

$$Y_1 = \frac{1}{2} L_1 \cos\,\theta_1$$

$$X_2 = L_1 \sin\,\theta_1 + \frac{1}{2} L_2 \sin\,\theta_2$$

$$Y_2 = L_1 \cos\,\theta_1 + \frac{1}{2} L_2 \cos\,\theta_2$$

$$\dot{X}_1 = \frac{1}{2} L_1 \cos\theta_1\ \dot{\theta}_1$$

$$\dot{Y}_1 = -\frac{1}{2} L_1 \sin\theta_1\ \dot{\theta}_1$$

$$\dot{X}_2 = L_1 \cos\theta_1\ \dot{\theta}_1 + \frac{1}{2} L_2 \cos\theta_2\ \dot{\theta}_2$$

$$\dot{Y}_2 = -L_1 \sin\theta_1\ \dot{\theta}_1 - \frac{1}{2} L_2 \sin\theta_2\ \dot{\theta}_2$$

Double Pendulum System (3)

$$[\dot{X_1}^2 + \dot{Y_1}^2] = \left(\frac{1}{2}L_1\cos\theta_1\ \dot{\theta}_1\right)^2 + \left(-\frac{1}{2}L_1\sin\theta_1\ \dot{\theta}_1\right)^2$$

$$= \frac{1}{4}L_1{}^2\cos^2\theta_1\ \dot{\theta_1}^2 + \frac{1}{4}L_1{}^2\sin^2\theta_1\ \dot{\theta_1}^2$$

$$= \frac{1}{4}L_1{}^2\ \dot{\theta_1}^2$$

$$[\dot{X_2}^2 + \dot{Y_2}^2] = \left(L_1\cos\theta_1\ \dot{\theta}_1 + \frac{1}{2}L_2\cos\theta_2\ \dot{\theta}_2\right)^2 + \left(-L_1\sin\theta_1\ \dot{\theta}_1 - \frac{1}{2}L_2\sin\theta_2\ \dot{\theta}_2\right)^2$$

$$= L_1{}^2\cos^2\theta_1\ \dot{\theta_1}^2 + \frac{1}{4}L_2{}^2\cos^2\theta_2\ \dot{\theta_2}^2 + L_1L_2\cos\theta_1\cos\theta_2\ \dot{\theta}_1\dot{\theta}_2$$

$$+L_1{}^2\sin^2\theta_1\ \dot{\theta_1}^2 + \frac{1}{4}L_2{}^2\sin^2\theta_2\ \dot{\theta_2}^2 + L_1L_2\sin\theta_1\sin\theta_2\ \dot{\theta}_1\dot{\theta}_2$$

$$= L_1{}^2\ \dot{\theta_1}^2 + \frac{1}{4}L_2{}^2\ \dot{\theta_2}^2 + L_1L_2\ \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

Double Pendulum System (4)

$$[\dot{X_1}^2 + \dot{Y_1}^2] = \left(\frac{1}{2}L_1 \cos\theta_1 \ \dot{\theta}_1\right)^2 + \left(-\frac{1}{2}L_1 \sin\theta_1 \ \dot{\theta}_1\right)^2$$

$$= \frac{1}{4}L_1{}^2 \cos^2\theta_1 \ \dot{\theta_1}^2 + \frac{1}{4}L_1{}^2 \sin^2\theta_1 \ \dot{\theta_1}^2$$

$$= \frac{1}{4}L_1{}^2 \ \dot{\theta_1}^2$$

$$[\dot{X_2}^2 + \dot{Y_2}^2] = \left(L_1 \cos\theta_1 \ \dot{\theta}_1 + \frac{1}{2}L_2 \cos\theta_2 \ \dot{\theta}_2\right)^2 + \left(-L_1 \sin\theta_1 \ \dot{\theta}_1 - \frac{1}{2}L_2 \sin\theta_2 \ \dot{\theta}_2\right)^2$$

$$= L_1{}^2 \cos^2\theta_1 \ \dot{\theta_1}^2 + \frac{1}{4}L_2{}^2 \cos^2\theta_2 \ \dot{\theta_2}^2 + L_1 L_2 \cos\theta_1 \cos\theta_2 \ \dot{\theta}_1\dot{\theta}_2$$

$$+L_1{}^2 \sin^2\theta_1 \ \dot{\theta_1}^2 + \frac{1}{4}L_2{}^2 \sin^2\theta_2 \ \dot{\theta_2}^2 + L_1 L_2 \sin\theta_1 \sin\theta_2 \ \dot{\theta}_1\dot{\theta}_2$$

$$= L_1{}^2 \ \dot{\theta_1}^2 + \frac{1}{4}L_2{}^2 \ \dot{\theta_2}^2 + L_1 L_2 \ \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

Double Pendulum System (5)

$$E_k = \sum \frac{1}{2} m v^2 + \sum \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2} m_1 [\dot{X_1}^2 + \dot{Y_1}^2] + \frac{1}{2} m_2 [\dot{X_2}^2 + \dot{Y_2}^2] + \frac{1}{2} I_1 \dot{\theta}_1^2 + + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$= \frac{1}{2} m_1 \left[\frac{1}{4} {L_1}^2 \dot{\theta_1}^2\right] + \frac{1}{2} m_1 [{L_1}^2 \dot{\theta_1}^2 + \frac{1}{4} {L_2}^2 \dot{\theta_2}^2 + L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$E_p = \sum mgh$$

$$= m_1 g h_1 + m_2 g h_2$$

$$= m_1 g L_1 + m_1 g L_2 - \frac{1}{2} m_1 g L_1 \cos\theta_1 + m_2 g L_1 + m_2 g L_2 - m_2 g L_1 \cos\theta_1 - \frac{1}{2} m_2 g L_2 \cos\theta_2$$

Double Pendulum System (6)

Lagrange Equation.

$L = E_k - E_p$

$$= \frac{1}{2}m_1\left(\frac{1}{4}L_1^2\dot{\theta}_1^2\right) + \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + \frac{1}{4}L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\dot{\theta}_1 - \theta_2)\right) + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_1^2$$
$$- m_1gL_1 - m_2gL_2 + \frac{1}{2}m_1gL_1cos\theta_1 - m_2gL_1 - m_2gL_2$$
$$+ m_2gL_1cos\theta_1 + \frac{1}{2}m_2gL_2cos\theta_2$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}m_1 + m_2\right)L_1^2\dot{\theta}_1^2 + \frac{1}{4}m_2L_2^2\dot{\theta}_2^2 + m_1L_1L_2\dot{\theta}_1\dot{\theta}_1\cos(\theta_1 - \theta_2) + I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2\right]$$
$$- (m_1 + m_2)gL_1 - (m_1 + m_2)gL_2 + \left(\frac{1}{2}m_1 + m_2\right)gL_1cos\theta_1$$
$$+ \frac{1}{2}m_2gL_2cos\theta_2$$

Motion Equation

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$$

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

Double Pendulum System (7)

Komponen τ_1

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) - \left(\frac{1}{2} m_1 + m_2\right) g L_1 sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \left(\frac{1}{4} m_1 + m_2\right) L_1^2 \dot{\theta}_1 + \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + I_1 \dot{\theta}_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \left(\frac{1}{4} m_1 + m_2\right) L_1^2 \ddot{\theta}_1 + \frac{1}{2} m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\,(\dot{\theta}_1 - \dot{\theta}_2) + I_1\, \ddot{\theta}_1$$

Maka

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$= \left(\frac{1}{4} m_1 + m_2\right) L_1^2 \ddot{\theta}_1 + \frac{1}{2} m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\,(\dot{\theta}_1 - \dot{\theta}_2) + I_1\, \ddot{\theta}_1 - \left[-\frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) - \left(\frac{1}{2} m_1 + m_2\right) g L_1 sin\theta_1\right]$$

$$\tau_1 = \left[\left(\frac{1}{4} m_1 + m_2\right) L_1^2 + I_1^2\right] \ddot{\theta}_1 + \frac{1}{2} m_2 L_1 L_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \sin(\theta_1 - \theta_2)\,(\dot{\theta}_1 - \dot{\theta}_2) + \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2)\right] + \left(\frac{1}{2} m_1 + m_2\right) g L_1 sin\theta_1$$

Double Pendulum System (8)

Komponen τ_2

$$\frac{\partial L}{\partial \theta_2} = (-1) - \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{1}{2} m_2 g L_2 \sin \theta_2$$

$$= \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{1}{2} m_2 g L_2 \sin \theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{4} m_2 {L_2}^2 \dot{\theta}_2 + \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) + I_2 \dot{\theta}_2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{4} m_2 {L_2}^2 \ddot{\theta}_2 + \frac{1}{2} m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) \left(\dot{\theta}_1 - \dot{\theta}_2\right) + I_2 \ddot{\theta}_2$$

Sehingga,

$$\tau_2 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

$$\tau_2 = \frac{1}{4} m_2 {L_2}^2 \ddot{\theta}_2 + I_2 \ddot{\theta}_2 + \frac{1}{2} m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) \left(\dot{\theta}_1 - \dot{\theta}_2\right) - \frac{1}{2} m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \frac{1}{2} m_2 g L_2 \sin \theta_2$$

Double Pendulum System (9)

$$\tau_1 = \left[\left(\frac{1}{4}m_1 + m_2\right)L_1^2 + I_1^2\right]\ddot{\theta}_1 + \frac{1}{2}m_2 L_1 L_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \frac{1}{2}m_2 L_1 L_2\left[-\dot{\theta}_2 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right) + \dot{\theta}_1 \dot{\theta}_2\, sin(\theta_1 - \theta_2)\right] + \left(\frac{1}{2}m_1 + m_2\right)gL_1 sin\theta_1$$

$$\tau_2 = \frac{1}{2}m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + \left[\frac{1}{4}m_2 L_2^2 + I_2\right]\ddot{\theta}_2 - \frac{1}{2}m_2 L_1 L_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right) - \frac{1}{2}m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \frac{1}{2}m_2 g L_2 \sin\theta_2$$

Berdasarkan Persamaan τ_1 dan τ_2 dapat dibentuk matriks untuk mencari nilai θ_1 dan θ_2 :

$$\begin{pmatrix} \left(\frac{1}{4}m_1 + m_2\right)L_1^2 + I_1 & \frac{1}{2}m_2 L_1 L_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2}m_2 L_1 L_2 \cos(\theta_1 - \theta_2) & \frac{1}{4}m_2 L_2^2 + I_2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}m_2 L_1 L_2 sin(\theta_1 - \theta_2)\dot{\theta}_2 & -\frac{1}{2}m_2 L_1 L_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ -\frac{1}{2}m_2 L_1 L_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) & -\frac{1}{2}m_2 L_1 L_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) \end{pmatrix}\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} \left(\frac{1}{2}m_1 + m_2\right)gL_1 sin\theta_1 \\ \frac{1}{2}m_2 g L_2 \sin\theta_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} &M\ddot{\theta} + C\dot{\theta} + G = \tau \\ &\ddot{\theta} = M^{-1}\tau - M^{-1}C - M^{-1}G \end{aligned}}$$

4th Order Runge Kutta

$$\frac{\partial^2 y}{\partial x^2} = f\left(x, y, y'\right)$$

$$k_1 = \frac{h}{2} f\left(x, y, y'\right)$$

$$k_2 = \frac{h}{2} f\left(x + \frac{h}{2}, y + \frac{h}{2}\left(y' + \frac{k_1}{2}\right), y' + k_1\right)$$

$$k_3 = \frac{h}{2} f\left(x + \frac{h}{2}, y + \frac{h}{2}\left(y' + \frac{k_1}{2}\right), y' + k_2\right)$$

$$k_4 = \frac{h}{2} f\left(x + h, y + h\left(y' + k_3\right), y' + 2k_3\right)$$

$$y_n = y_{n-1} + h\left(y'_{n-1} + \frac{k_1 + k_2 + k_3}{3}\right)$$

$$y'_n = y'_{n-1} + \frac{k_1 + 2k_2 + 2k_3 + k_4}{3}$$

THANK YOU