Real Analysis Assignment 3

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Problem 1:

Claim 1. $\lim \frac{6n+5}{4n+7} = \frac{3}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. Then, since $\mathbb{R}$ is closed under the field operations, we have $\frac{1}{8}(\frac{11}{\epsilon} - 14) \in \mathbb{R}$. Since $(1 \in \mathbb{R} \wedge 1 > 0) \wedge \frac{1}{8}(\frac{11}{\epsilon} - 14) \in \mathbb{R}$, by the Archimedian property of $\mathbb{R}$ we have $\exists n \in \mathbb{N} : 1 \cdot n = n > \frac{1}{8}(\frac{11}{\epsilon} - 14)$. This inequality implies that $8n + 14 > \frac{11}{\epsilon} \implies \frac{11}{8n+14} < \epsilon$. Since $x = |-x|$, we have $|\frac{-11}{8n+14}| < \epsilon$. Therefore, $|\frac{12n+10-12n-21}{8n+14}| = |\frac{2(n+5)-3(4n+7)}{8n+14}| = |\frac{6n+5}{4n+7} - \frac{3}{2}| < \epsilon$. Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\frac{6n+5}{4n+7} - \frac{3}{2}| < \epsilon$, which is exactly the definition that $\lim \frac{6n+5}{4n+7} = \frac{3}{2}$. □

Problem 2:

Claim 2. $\lim \sqrt{9 - \frac{(-1)^n}{n}} = 3$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of $\mathbb{R}$, $\exists n \in \mathbb{N} : n > \frac{1}{3\epsilon}$. Then, $\frac{1}{3n} = \frac{\frac{1}{n}}{3} < \epsilon$. Since $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : \sqrt{y} = x \implies x > 0$, we have $\sqrt{9 - \frac{(-1)^n}{n}} > 0 \implies \sqrt{9 - \frac{(-1)^n}{n}} + 3 > 3 \implies \frac{1}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{1}{3} \implies \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{\frac{1}{n}}{3}$.

Consider the quantity $\frac{-(-1)^n}{n}$. Any natural number is either even or odd, i.e. $\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : (n = 2k) \vee (n = 2k-1)$. Then, $n = 2k \implies \frac{-(-1)^n}{n} = \frac{-1}{n}$ and $\frac{-1}{n} \leq \frac{-(-1)^n}{n} \leq \frac{1}{n}$ holds. Alternatively, $n = 2k-1 \implies \frac{-(-1)^n}{n} = \frac{1}{n}$ and $\frac{-1}{n} \leq \frac{-(-1)^n}{n} \leq \frac{1}{n}$ holds, so we have $\forall n \in \mathbb{N}, \frac{-1}{n} \leq \frac{-(-1)^n}{n} \leq \frac{1}{n}$. Thus, $\frac{\frac{-1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \leq \frac{\frac{-(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \leq \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}$, and this is true if and only if $|\frac{\frac{-(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| \leq \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}$.

Then, $|\frac{-\frac{(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| = |\frac{9 - \frac{(-1)^n}{n} - 9}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| = |\frac{(\sqrt{9 - \frac{(-1)^n}{n}})^2 - 3^2}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| = |\sqrt{9 - \frac{(-1)^n}{n}} - 3|$. By transitivity, $|\sqrt{9 - \frac{(-1)^n}{n}} - 3| \leq \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{1}{3n} < \epsilon \implies |\sqrt{9 - \frac{(-1)^n}{n}} -$

$3| < \epsilon$. Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\sqrt{9 - \frac{(-1)^n}{n}} - 3| < \epsilon$. which is exactly the definition that $\lim \sqrt{9 - \frac{(-1)^n}{n}} = 3$. ☐

Problem 3:

Claim 3. $\lim \frac{n}{\sqrt{4n^2+9n+1}} = \frac{1}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of $\mathbb{R}$, $\exists n \in \mathbb{N} : n > \frac{1}{2}(\frac{1}{\epsilon - \frac{9}{2}})$. Then, $\frac{1}{2n} + \frac{9}{2} = \frac{1}{2n} + \frac{9n}{2n} = \frac{9n+1}{2n} < \epsilon$. Since $\forall n \in \mathbb{N}, 1 < 1 + 4n^2 + 9n$, we have $1 < \sqrt{4n^2 + 9n + 1}$. Then, $2n + \sqrt{4n^2 + 9n + 1} > 2n \implies \frac{1}{2n+\sqrt{4n^2+9n+1}} < \frac{1}{2n} \implies \frac{9n+1}{2n+\sqrt{4n^2+9n+1}} < \frac{9n+1}{2n}$. Furthermore, since $\sqrt{4n^2 + 9n + 1} > 1 \implies 2\sqrt{4n^2 + 9n + 1} > 1$, we have $(2n+\sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1}) > 2n + \sqrt{4n^2 + 9n + 1} \implies \frac{1}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} < \frac{1}{2n+\sqrt{4n^2+9n+1}} \implies \frac{9n+1}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} < \frac{9n+1}{2n+\sqrt{4n^2+9n+1}}$. Since $\forall x \in \mathbb{R} : x > 0, |-x| = x$, we have $\frac{9n+1}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} = |\frac{-(9n+1)}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})}| = |\frac{4n^2-(4n^2+9n+1)}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})}| = |\frac{(2n)^2-(\sqrt{4n^2+9n+1})^2}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})}| = |\frac{2n-\sqrt{4n^2+9n+1}}{2\sqrt{4n^2+9n+1}}| = |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}|$. Then by transitivity, $|\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \frac{9n+1}{2n+\sqrt{4n^2+9n+1}} < \frac{9n+1}{2n} < \epsilon \implies |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \epsilon$ Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \epsilon$ which is exactly the definition that $\lim \frac{n}{\sqrt{4n^2+9n+1}} = \frac{1}{2}$. ☐

Problem 4:

Claim 4. $\lim(\sqrt{n^2 + 5n} - n) = \frac{5}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of $\mathbb{R}$, $\exists n \in \mathbb{N} : n > \frac{2\epsilon+5}{2\epsilon-5}$. This inequality implies $\frac{1}{n} < \frac{2\epsilon-5}{2\epsilon+5}$, so $\frac{n}{n^2+5n} < \frac{n}{n^2} = \frac{1}{n} < \frac{2\epsilon-5}{2\epsilon+5}$.

Since $n > 0 \wedge n^2 + 5n > 0$, we have $\frac{n}{n^2+5n} > 0$, therefore $\frac{-n}{n^2+5n} < 0$, and of course $\frac{-n}{\sqrt{n^2+5n}} < 0$ since $\forall x \in \mathbb{R}, (\exists y \in \mathbb{R} : x = \sqrt{y}) \implies x > 0$, so $\frac{-n}{\sqrt{n^2+5n}} < \frac{2\epsilon-5}{2\epsilon+5}$. Thus, $-n(2\epsilon + 5) = -2\epsilon n - 5n < \sqrt{n^2 + 5n}(2\epsilon - 5) = 2\epsilon\sqrt{n^2 + 5n} - 5\sqrt{n^2 + 5n} \implies -5n + 5\sqrt{n^2 + 5n} = -5(n - \sqrt{n^2 + 5n}) < 2\epsilon\sqrt{n^2 + 5n} + 2\epsilon n = 2\epsilon(n + \sqrt{n^2 + 5n}) \implies \frac{-5(n-\sqrt{n^2+5n})}{2(n+\sqrt{n^2+5n})} < \epsilon$.

Assume that $n \geq \sqrt{n^2 + 5n}$. Then, $n^2 \geq n^2 + 5n \implies 0 \geq 5n \implies 0 \geq n$, but $n > 0$ since $n \in \mathbb{N}$, which is a contradiction, therefore $n < \sqrt{n^2 + 5n}$, so $n - \sqrt{n^2 + 5n} < 0$ and therefore $\frac{5(n-\sqrt{n^2+5n})}{2(n+\sqrt{n^2+5n})} < 0$. Thus $|\frac{5(n-\sqrt{n^2+5n})}{2(n+\sqrt{n^2+5n})}| = \frac{-5(n-\sqrt{n^2+5n})}{2(n+\sqrt{n^2+5n})} < \epsilon$.

Then we see that $|\frac{5(n-\sqrt{n^2+5n})}{2(n+\sqrt{n^2+5n})}| = |\frac{2(5n)-5(n+\sqrt{n^2+5n})}{2(n+\sqrt{n^2+5n})}| = |\frac{5n}{\sqrt{n^2+5n}+n} - \frac{5}{2}| = |\frac{(\sqrt{n^2+5n})^2-n^2}{\sqrt{n^2+5n}+n} - \frac{5}{2}| = |\sqrt{n^2 + 5n} - n - \frac{5}{2}| < \epsilon$. Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\sqrt{n^2 + 5n} - n - \frac{5}{2}| < \epsilon$, which is exactly the definition that $\lim(\sqrt{n^2 + 5n} - n) = \frac{5}{2}$. ☐

Problem 5:

Claim 5. $\lim(3 + 2(-1)^n) \neq 5$

Proof. Let $\epsilon = 1$, let $n \in \mathbb{N}$, and let $m = 2n + 1$. Then, $|3 - 2(-1)^m - 5| = |3 - 2(-1)^2(-1) - 5| = |-4| = 4 \geq 1 = \epsilon$. Thus we have demonstrated that $\exists \epsilon \in \mathbb{R} : \epsilon > 0 : \forall N_\epsilon \in \mathbb{N}, \exists n \in \mathbb{N} : n > N_\epsilon : |3 - 2(-1)^n - 5| \geq \epsilon \iff \neg(\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |3 - 2(-1)^n - 5| < \epsilon)$, which is the negation of $\lim(3 + 2(-1)^n) = 5$, and $\neg(\lim(3 + 2(-1)^n) = 5) \iff \lim(3 + 2(-1)^n) \neq 5$. $\square$