Real Analysis Assignment 1

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1(a):
Suppose we have the following equation:

$$|x-1| + |x+2| = 0.5 \tag{1}$$

We have $x \in \mathbb{R}$ if and only if:

$$x \leq -2 \lor -2 \leq x \leq 1 \lor 1 \leq x \tag{2}$$

since $(-\infty, -2] \cup [-2, 1] \cup [1, \infty) = \mathbb{R}$.

Assume $x \leq -2$. Then, $x - 1 < 0 \implies |x-1| = -(x-1)$, and $x + 2 \leq 0 \implies |x+2| = -(x+2)$, so we can solve (1) as follows:

$$\begin{aligned}
& -(x-1) - (x+2) && = 0.5 && (3)\\
= & -x + 1 - x - 2 && = 0.5 && (4)\\
= & -2x - 1 && = 0.5 && (5)\\
\implies & -2x && = 1.5 && (6)\\
\implies & x && = \frac{1.5}{-2} = \frac{-3}{4} && (7)
\end{aligned}$$

However, we have by assumption that $x \leq -2$, and by (7) we have that $x = \frac{-3}{4} > -2$, which is a contradiction. Therefore, $x \leq -2$ is false for any real solution to (1).

Assume $-2 \leq x \leq 1$. Then, $x - 1 \leq 0 \implies |x-1| = -(x-1)$, and $x + 2 \geq 0 \implies |x+2| = x + 2$, so we can solve (1) as follows:

$$\begin{aligned}
& -(x-1) + x + 2 && = 0.5 && (8)\\
= & -x + 1 + x + 2 && = 0.5 && (9)\\
= & 3 && = 0.5 && (10)
\end{aligned}$$

But of course, $3 = 0.5$ is absurd, therefore we reach a contradiction and conclude that $-2 \leq x \leq 1$ is false for any real solution to (1).

Assume $x \geq 1$. Then, $x - 1 \geq 0 \implies |x-1| = x - 1$, and $x + 2 > 0 \implies |x+2| = x + 2$, so we can solve (1) as follows:

$$\begin{aligned} & x - 1 + x + 2 && = 0.5 && (11) \\ =& 2x + 1 && = 0.5 && (12) \\ \implies & 2x && = -0.5 && (13) \\ \implies & x && = \frac{-0.5}{2} = \frac{-1}{4} && (14) \end{aligned}$$

However, we have by assumption that $x \geq 1$, and by (14) we have that $x = \frac{-1}{4} < 1$, which is a contradiction. Therefore, $x \geq 1$ is false for any real solution to (1).

Finally, since we have $x \notin (-\infty, -2] \wedge x \notin [-2, 1] \wedge x \notin [1, \infty)$, we have $x \notin \mathbb{R} = (-\infty, -2] \cup [-2, 1] \cup [1, \infty)$, therefore we have demonstrated that there are no real solutions to (1).

1(b):

Suppose we have the following equation:

$$|x - 1| + |x + 2| = 3.5 \tag{15}$$

We have $x \in \mathbb{R}$ if and only if (2) holds since $(-\infty, -2] \cup [-2, 1] \cup [1, \infty) = \mathbb{R}$.

Assume $x \leq -2$. Then, $x - 1 < 0 \implies |x-1| = -(x-1)$, and $x + 2 \leq 0 \implies |x+2| = -(x+2)$, so we can solve (15) as follows:

$$\begin{aligned} & -(x - 1) - (x + 2) && = 3.5 && (16) \\ =& -x + 1 - x - 2 && = 3.5 && (17) \\ =& -2x - 1 && = 3.5 && (18) \\ \implies & -2x && = 4.5 && (19) \\ \implies & x && = \frac{4.5}{-2} = \frac{-9}{4} && (20) \end{aligned}$$

Indeed, $x = \frac{-9}{4} \leq -2$ so this solution is consistent with our constraints and it must be a member of the solution set for (15).

Assume $-2 \leq x \leq 1$. Then, $x - 1 \leq 0 \implies |x-1| = -(x-1)$, and $x + 2 \geq 0 \implies |x+2| = x + 2$, so we can solve (15) as follows:

$$\begin{aligned} & -(x - 1) + x + 2 && = 3.5 && (21) \\ =& -x + 1 + x + 2 && = 3.5 && (22) \\ =& 3 && = 3.5 && (23) \end{aligned}$$

But of course, $3 = 3.5$ is absurd, therefore we reach a contradiction and conclude that $-2 \leq x \leq 1$ is false for any real solution to (15).

Assume $x \geq 1$. Then, $x-1 \geq 0 \implies |x-1| = x-1$, and $x+2 > 0 \implies |x+2| = x+2$, so we can solve (15) as follows:

$$x - 1 + x + 2 = 3.5 \tag{24}$$
$$=2x + 1 = 3.5 \tag{25}$$
$$\implies 2x = 2.5 \tag{26}$$
$$\implies x = \frac{2.5}{2} = \frac{5}{4} \tag{27}$$

Indeed, $x = \frac{5}{4} \geq 1$ so this solution is consistent with our constraints and it must be a member of the solution set for (15).

Finally, we can describe the real solutions to (15) by $x \in \{\frac{-9}{4}, \frac{5}{4}\}$.

1(c):

Suppose we have the following inequality:

$$|x - 2| < 3 \tag{28}$$

Some $x \in \mathbb{R}$ solves (28) if and only if $x \in (-\infty, 2] \vee x \in [2, \infty)$ since $(-\infty, 2] \cup [2, \infty) = \mathbb{R}$.

Assume $x \leq 2$. Then, $x - 2 \leq 0 \implies |x-2| = -(x-2) = -x+2$, so we can solve (28):

$$-x + 2 < 3 \tag{29}$$
$$\implies -x < 1 \tag{30}$$
$$\implies x > -1 \tag{31}$$

Therefore we have $-1 < x \leq 2$ when $x \leq 2$

Assume $x \geq 2$. Then, $x - 2 \geq 0 \implies |x-2| = x-2$ so we can solve (28):

$$x - 2 < 3 \tag{32}$$
$$\implies x < 5 \tag{33}$$

Therefore we have $2 \leq x < 5$ when $x \geq 2$

Since those two cases describe every case of $x \in \mathbb{R}$, we must have that $-1 < x \leq 2 \vee 2 \leq x < 5$, thus we have demonstrated that $-1 < x < 5$ solves (28).

1(d):

Suppose we have the following inequality:

$$x + \frac{2 - 4x}{x + 1} > 0 \tag{34}$$

Since $x = \frac{x(x+1)}{x+1}$, we can simplify (34) as follows:

$$\frac{x(x+1)+(2-4x)}{x+1} > 0 \tag{35}$$
$$\frac{x^2+x+2-4x}{x+1} > 0 \tag{36}$$
$$\frac{x^2-3x+2}{x+1} > 0 \tag{37}$$
$$\frac{(x-1)(x-2)}{x+1} > 0 \tag{38}$$

Since (38) is undefined when $x = -1$, we must have either $x < -1$ or $x > -1$.

Assume $x > -1$. Then, (38) holds if and only if $(x - 1 > 0 \wedge x - 2 > 0) \vee (x - 1 < 0 \wedge x - 2 < 0)$.

Under this assumption, consider the following two cases:

- Assume $x - 1 > 0 \wedge x - 2 > 0$. Then, $x > 1 \wedge x > 2 \implies x > 2$.

  We conclude that a subset of our solution set is the interval $(2, \infty)$.

- Assume $x - 1 < 0 \wedge x - 2 < 0$. Then, $x < 1 \wedge x < 2 \implies x < 1$.

  We conclude that a subset of our solution set is the interval $(-1, 1)$.

Now, assume $x < -1$. Then, (38) holds if and only if $(x - 1 > 0 \wedge x - 2 < 0) \vee (x - 1 < 0 \wedge x - 2 > 0)$.

Under this assumption, consider the following two cases:

- Assume $x - 1 > 0 \wedge x - 2 < 0$. Then, $x > 1 \wedge x < 2$, but by assumption, $x < -1$, which is a contradiction, so we reject this assumption.

- Assume $x-1 < 0 \wedge x-2 > 0$. Then, $x < 1 \wedge x > 2$, which is a contradiction, so we reject this assumption.

Finally, we conclude that our complete solution set must be the interval $(-1, 1) \cup (2, \infty)$.

2:

Let A and B be two sets such that $A = \{x \in \mathbb{R} : x > 0 \wedge x^2 \geq 3\}$ and $B = \{x \in \mathbb{R} : x^2 < 3\} \cup \{x \in \mathbb{R} : x \leq 0\}$.

The definition of a Dedekind Cut was given in class as follows: (X, Y) forms a Dedekind Cut of $\mathbb{R}$ if:

1. $X \subset \mathbb{R} \wedge Y \subset \mathbb{R}$
2. $X \neq \emptyset \wedge Y \neq \emptyset$
3. $(\forall x \in X)(\forall y \in Y)(x < y)$.

(A, B) does not form a Dedekind Cut since $3 > 0$, and $3 \in A \wedge 0 \in B$, violating the third constraint. Therefore, we instead consider the Dedekind cut (B, A).

By construction (axiom schema of comprehension), we have that $A \subseteq \mathbb{R} \wedge B \subseteq \mathbb{R}$. Since $0 \notin A$, we must have $A \subset \mathbb{R}$, and since $3^2 \geq 3 \implies 3 \notin B$, we have $B \subset \mathbb{R}$. Therefore the first constraint is satisfied.

Suppose $x = 0$. Then $x \in B$ by its definition, so $B \neq \emptyset$. Suppose $x = 3$. Then, $x^2 = 9 \geq 3 \implies x \in A$, so $A \neq \emptyset$. Therefore the second constraint is satisfied.

Let $a \in A$ and let $b \in B$. By definition, we have that $a > 0$. If $b \leq 0$, then $b \leq 0 < a \implies b < a$ is trivial. Assume $b > 0$. By the definition of B, we have that $b^2 \leq 3$, and by the definition of A, we have $a^2 \geq 3$. Then by transitivity, we have $b^2 < 3 \leq a^2 \implies b^2 < a^2$, therefore $b^2 - a^2 = (b + a)(b - a) < 0$. Since $b > 0 \wedge a > 0$, we must have $b + a > 0$, and therefore since $(b + a)(b - a) < 0$ if and only if $(b + a > 0 \wedge b - a < 0) \vee (b + a < 0 \wedge b - a > 0)$, we must have that $b - a < 0$ and thus $b < a$. We conclude that $(\forall b \in B)(\forall a \in A)(b < a)$, therefore the third constraint is satisfied.

Since (B, A) satisfies all of the constraints of the definition of a Dedekind Cut of the reals, we conclude that it indeed must be a Dedekind Cut.