# Real Analysis Assignment 9

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**Problem 1:**

**Claim 1.** $f(x) = 3x^2 + 4x + 5$ *is continuous at every point* $x_0$

*Proof.* We want to prove that at any $x_0 \in \mathbb{R}$, $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x - x_0| < \delta_\epsilon, |(3x^2 + 4x + 5) - (3x_0^2 + 4x_0 + 5)| < \epsilon$.

We examine as follows:

$$\begin{aligned}
|(3x^2 + 4x + 5) - (3x_0^2 + 4x_0 + 5)| &= |3(x^2 - x_0^2) + 4(x - x_0)| && (1)\\
&= |3(x - x_0)(x + x_0) + 4(x - x_0)| && (2)\\
\text{(triangle inequality)} \quad &\leq 3|x - x_0| \cdot |x + x_0| + 4|x - x_0| && (3)\\
&= 3|x - x_0| \cdot |x - x_0 + 2x_0| + 4|x - x_0| && (4)\\
\text{(triangle inequality)} \quad &\leq 3|x - x_0| \cdot (|x - x_0| + 2|x_0|) + 4|x - x_0| && (5)\\
\text{(if } |x - x_0| < \delta) \quad &< 3\delta(\delta + 2|x_0|) + 4\delta && (6)\\
&= \delta(3\delta + 6|x_0| + 4) && (7)\\
\text{(if } \delta \leq 1) \quad &\leq \delta(7 + 6|x_0|) && (8)\\
\text{(if } \delta \leq \frac{\epsilon}{7 + 6|x_0|}) \quad &\leq \frac{\epsilon \cdot (7 + 6|x_0|)}{7 + 6|x_0|)} = \epsilon && (9)
\end{aligned}$$

Thus we can choose any $0 < \delta_\epsilon \leq \min(1, \frac{\epsilon}{7+6|x_0|)})$ and then we have for our arbitrary $x_0$ that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x - x_0| < \delta_\epsilon, |(3x^2+4x+5)-(3x_0^2+4x_0+5)| < \epsilon$, which is exactly the definition that $f(x)$ is continuous at any point $x_0$. $\square$

**Problem 2:**

Suppose $f(x)$ is continuous at some $c \in \mathbb{R}$ and assume $f(c) < 2$.

**Claim 2.** *There exists a neighborhood* $A$ *of* $c$ *such that* $\forall x \in A, f(x) < 2$

*Proof.* $f$ is continuous at $c$ if and only if $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x - c| < \delta_\epsilon, |f(x) - f(c)| < \epsilon$. In particular, $\exists \delta_{2-f(c)}, \forall |x - c| < \delta_{2-f(c)}, |f(x) - f(x)| < 2 - f(c)$. Let $A = (c - \delta_{2-f(c)}, c + \delta_{2-f(c)})$. We see that $x \in A$ which is a neighborhood of $c$. Then for any $x \in A$, $|f(x) - f(c)| < 2 - f(c) \iff -(2 - f(c)) < f(x) - f(c) < 2 - f(c) \implies f(x) < 2$. Thus there exists a neighborhood $A$ of $c$ such that $\forall x \in A, f(x) < 2$. $\square$

**Problem 3:**

Supose $f(x)$ is continuous on the closed interval $[a,b]$ and assume $\forall x \in [a,b], f(x) > 0$.

**Claim 3.** $\inf\limits_{x\in[a,b]} f(x) > 0.$

*Proof.* By the Extreme Value Theorem, $f$ attains maximum and minimum values in $[a,b]$. In particular, $\exists x_m \in [a,b], f(x_m) = \inf f(x), x \in [a,b]$, and since $\forall x \in [a,b], f(x) > 0$, we have that $f(x_m) > 0$, so $\inf\limits_{x\in[a,b]} f(x) > 0$. □

Define the following function on the closed interval $[-1,1]$:

$g(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$g(x)$ is discontinuous at 0 since $0 = \lim\limits_{x\to 0} g(x) \neq g(x) = 1$. We also have that $\forall x \in [-1,1], g(x) > 0$, since $|x| = 0 \iff x = 0$, but $g(0) = 1 > 0$. However, it is clear that $\inf\limits_{x\in[-1,1]} g(x) = 0$.

**Problem 4:**

Define $f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 2x-1 & \text{if } x \text{ is rational} \end{cases}$

**Claim 4.** *$f(x)$ is discontinuous at every real $x_0 \neq 1$.*

*Proof.* Suppose $x_0 \in \mathbb{R}, x_0 \neq 1$ We will show that $f(x)$ satisfies the negation of the definition of continuity for real $x_0$ under this constraint, that is to say, $\exists \epsilon > 0, \forall \delta > 0, \exists x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| \geq \epsilon$. Let $\epsilon = (x_0 - 1)^2 > 0$ and $\delta > 0$. We consider three cases:

- $x \in \mathbb{Q} \wedge x_0 \geq 0$
- $x \in \mathbb{Q} \wedge x_0 < 0$
- $x \in \mathbb{R}\backslash\mathbb{Q}$

If $\exists x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| \geq \epsilon$ holds for each of the three cases, then $f(x)$ is discontinuous at all $x_0 \neq 1$.

Suppose $x_0 \in \mathbb{Q} \wedge x_0 \geq 0$. Then since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{Q}$, $\exists x_0 < x < x_0 + \delta$. Then, $0 \leq x_0 < x \implies x_0^2 < x^2$, so:

$$x_0^2 - 2x_0 + 1 < x^2 - 2x_0 + 1 \tag{10}$$

$$(x_0 - 1)^2 < |x^2 - (2x_0 - 1)| \tag{11}$$

$$\epsilon = (x_0 - 1)^2 \leq |f(x) - f(x_0)| \tag{12}$$

Suppose $x_0 \in \mathbb{Q} \wedge x_0 < 0$. Then since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{Q}$, $\exists x_0 - \delta < x < x_0$. Then, $0 > x_0 > x \implies x_0^2 < x^2$, so:

$$x_0^2 - 2x_0 + 1 < x^2 - 2x_0 + 1 \tag{13}$$

$$(x_0 - 1)^2 < |x^2 - (2x_0 - 1)| \tag{14}$$

$$\epsilon = (x_0 - 1)^2 \leq |f(x) - f(x_0)| \tag{15}$$

Suppose $x_0 \in \mathbb{R}\backslash\mathbb{Q}$. Then since $\mathbb{Q}$ is dense in $\mathbb{R}\backslash\mathbb{Q}$, $\exists x_0 - \delta < x < x_0$. Then:

$$\begin{align}
-2x >& -2x_0 \tag{16}\\
x_0^2 - 2x + 1 >& x_0^2 - 2x_0 + 1 \tag{17}\\
|x_0^2 - (2x-1)| >& (x_0-1)^2 \tag{18}\\
|2x - 1 - x_0^2| >& (x_0-1)^2 \tag{19}\\
|f(x) - f(x_0)| \geq& (x_0-1)^2 = \epsilon \tag{20}\\
& \tag{21}
\end{align}$$

Since $\forall x_0 \neq 1, \exists \epsilon > 0, \forall \delta > 0, \exists x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| \geq \epsilon$, we have that $f(x)$ is discontinuous at every $x_0 \neq 1$.

We will now show that $f(x)$ is continuous at $x_0 = 1$ by finding for any $\epsilon > 0$ a $\delta_\epsilon > 0$ where $|x-1| < \delta_\epsilon$ implies that $|f(x) - 1| < \epsilon$.

Suppose that $x \in \mathbb{Q}$. Then:

$$\begin{align}
|f(x) - 1| =& |2x - 1 - 1| \tag{22}\\
=& 2|x-1| \tag{23}\\
(\text{if } |x-1| < \delta) \quad <& 2\delta \tag{24}\\
(\text{if } \delta \leq \frac{\epsilon}{2}) \quad \leq& 2\frac{\epsilon}{2} = \epsilon \tag{25}
\end{align}$$

Suppose that $x \in \mathbb{R}\backslash\mathbb{Q}$. Then:

$$\begin{align}
|f(x) - 1| =& |x^2 - 1| \tag{26}\\
=& |x-1| \cdot |x+1| \tag{27}\\
=& |x-1| \cdot |x - 1 + 2| \tag{28}\\
(\text{triangle inequality}) \quad \leq& |x-1| \cdot (|x-1| + 2) \tag{29}\\
(\text{if } |x-1| < \delta) \quad <& \delta(\delta + 2) \tag{30}\\
(\text{if } \delta \leq 1) \quad \leq& \delta(1+2) = 3\delta \tag{31}\\
(\text{if } \delta \leq \frac{\epsilon}{3}) \quad \leq& 3\frac{\epsilon}{3} = \epsilon \tag{32}
\end{align}$$

So, we can choose any $0 < \delta_\epsilon \leq \min(1, \frac{\epsilon}{3})$, and we have that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x-1| < \delta_\epsilon, |f(x) - 1| < \epsilon$, which is exactly the definition that $f(x)$ is continuous at $x_0 = 1$. Therefore, $f(x)$ is discontinuous at every point $x_0 \neq 1$. $\square$