Real Analysis Assignment 6

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Problem 1:

Claim 1. *The minimal subsequential limit of some bounded infinite sequence* x_n *can be computed by the formula* $\sup_k(\inf_{n>k} x_n)$

Proof. Since x_n is bounded, it is bounded above. Denote the upper bound N, then $\forall n, x_n \leq N$. Define a subsequence $y_k = \inf_{n>k} x_n$. Then, y_1 is the greatest lower bound of $\{x_2, x_3, x_4, ...\}$. Since $\{x_3, x_4, x_5, ...\}$ is a subset of $\{x_2, x_3, x_4, ...\}$, we have that y_1 is also a lower bound of the latter set, but since $y_2 = \inf_{n>k} x_n$, we must have $y_1 \leq y_2$. For some y_k, we have y_{k-1} is a lower bound of $\{x_{k+1}, x_{k+2}, ...\}$, since $y_{k-1} = \inf_{n>k} x_n$, but since $y_k = \inf_{n>k} x_n$, we must have $y_k \geq y_{k-1}$. Thus $\forall k, y_k \leq y_{k+1}$, i.e. y_k is monotone increasing. Then since $\forall n > k, y_k \leq x_n$ and $\forall n, x_n \leq N$, we have by transitivity that $y_k \leq N$. Thus y_k is bounded above. Since y_k is a monotone increasing sequence bounded above, we have by the monotone convergence theorem that $\lim y_k$ exists, and it is equal to $\sup y_k$ Let $A = \lim y_k$. Since y_k has a limit A every subsequence of y_k has the same limit A.

Now, pick some $n_1 \in \mathbb{N}$. Then $y_{n_1} = \inf\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, ...\}$. Since y_{n_1} is the greatest lower bound of $\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, ...\}$, we must have that $y_{n_1} + 1$ is not a lower bound of that set. Then, since $y_{n_1} + 1$ is not a lower bound, $\exists n_2 > n_1, y_{n_1} \leq x_{n_2} < y_{n_1} + 1$. Continuing in the same way, we have $y_{n_2} = \inf\{x_{n_2}, x_{n_2+1}, x_{n_2+2}, ...\}$, and since y_{n_2} is the greatest lower bound of the set, we have that $y_{n_2} + \frac{1}{2}$ is not a lower bound of the set, thus $\exists n_2 > n_2, y_{n_2} \leq x_{n_3} < y_{n_2} + \frac{1}{2}$. We can continue this indefinitely, such that for any n_{k-1}, we have that $y_{n_{k-1}} = \inf\{x_{n_k}, x_{n_k+1}, x_{n_k+2}, ...\}$, and so $y_{n_{k-1}} + \frac{1}{k-1}$ is not a lower bound of the set, and thus $\exists n_k > n_{k-1}, y_{n_{k-1}} \leq x_{n_k} < y_{n_{k-1}} + \frac{1}{k-1}$. Applying the squeze theorem and limit laws, we have:

$$\lim(y_{n_{k-1}}) \leq \lim x_{n_k} < \lim(y_{n_{k-1}} + \frac{1}{k-1}) \tag{1}$$

$$\lim(y_{n_{k-1}}) \leq \lim x_{n_k} < \lim(y_{n_{k-1}}) + \lim(\frac{1}{k-1}) \tag{2}$$

$$A \leq \lim x_{n_k} < A + 0 \tag{3}$$

$$\implies \lim x_{n_k} = A \tag{4}$$

Thus A is a subsequential limit of x_n.

Consider some other subsequential limit of x_n, suppose $\lim x_{m_k} = B$. Then:

$$
\begin{aligned}
y_{m_{k-1}} =& \inf\{x_{m_k}, x_{m_k+1}, x_{m_k+2}, ...\} && (5)\\
\forall m_k, y_{m_k-1} \leq& x_{m_k} && (6)\\
\lim y_{m_k-1} \leq& \lim x_{m_k} && (7)\\
A \leq& B && (8)
\end{aligned}
$$

Therefore, A is the minimal subsequential limit of x_n, and it is the value of $\sup_k(\inf_{n>k} x_n)$. □

Problem 2:

Claim 2. *If a_n and b_n are Cauchy sequences, then a_nb_n is a Cauchy sequence.*

Proof. If a sequence is Cauchy, then it is bounded, and if a sequence is bounded, then it is bounded above. Thus, $\exists A, B \in \mathbb{R}$, where $\forall n, a_n \leq A$ and $\forall n, b_n \leq B$. By the definition of a Cauchy sequence, we have that

$$\forall \epsilon > 0, \exists N_\epsilon, \forall n > N_\epsilon, m > N_\epsilon, |a_n - a_m| < \epsilon \tag{9}$$

and likewise

$$\forall \epsilon > 0, \exists M_\epsilon, \forall n > M_\epsilon, m > M_\epsilon, |b_n - b_m| < \epsilon \tag{10}$$

Let $\epsilon > 0$. Thus we have the follwing:

$$
\begin{aligned}
&\exists N_{\frac{\epsilon}{2|B|}}, \forall n > \frac{\epsilon}{2|B|}, m > \frac{\epsilon}{2|B|}, |a_n - a_m| < \frac{\epsilon}{2|B|} && (11)\\
&\exists M_{\frac{\epsilon}{2|A|}}, \forall n > \frac{\epsilon}{2|A|}, m > \frac{\epsilon}{2|A|}, |b_n - b_m| < \frac{\epsilon}{2|A|} && (12)
\end{aligned}
$$

Let $K_\epsilon = \max(N_{\frac{\epsilon}{2|B|}}, M_{\frac{\epsilon}{2|A|}})$. Then, $\forall n > K_\epsilon, m > K_\epsilon$, we have:

$$
\begin{aligned}
&|B||a_n - a_m| < \frac{\epsilon}{2} && (13)\\
&|A||b_n - b_m| < \frac{\epsilon}{2} && (14)\\
\implies &|B||a_n - a_m| + |A||b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && (15)
\end{aligned}
$$

Since $\forall n, a_n \leq A$ and $\forall n, b_n \leq B$, we have:

$$
\begin{aligned}
|B||a_n - a_m| + |A||b_n + b_m| \geq& |b_m||a_n - a_m| + |a_n||b_n - b_m| && (16)\\
|b_m||a_n - a_m| + |a_n||b_n - b_m| =& |a_nb_n - a_nb_m| + |a_nb_m - a_mb_m| && (17)
\end{aligned}
$$

And by the triangle inequality:

$$
\begin{aligned}
|a_nb_n - a_nb_m| + |a_nb_m - a_mb_m| \geq& |a_nb_n - a_nb_m + a_nb_m - a_mb_m| && (18)\\
|a_nb_n - a_nb_m + a_nb_m - a_mb_m| =& |a_nb_n - a_mb_m| && (19)
\end{aligned}
$$

Therefore, by transitivity, $|a_n b_n - a_m b_m| < \epsilon$. Thus $\forall \epsilon > 0, \exists K_\epsilon, \forall n > K_\epsilon, m > K_\epsilon, |a_n b_n - a_m b_m| < \epsilon$, and this is exactly the definition that $a_n b_n$ is a Cauchy sequence. $\square$

Problem 3:
Define a sequence $x_n = \sqrt{n}$.

Part (a):

Claim 3. $\lim(x_{n+2} - x_n) = 0$

Proof. Let $\epsilon > 0$. By the Archimedean property, $\exists N_{\frac{4}{\epsilon^2}} \in \mathbb{N}, N_{\frac{4}{\epsilon^2}} > \frac{4}{\epsilon^2}$. Thus, $\forall n > N_{\frac{4}{\epsilon^2}}, n > \frac{4}{\epsilon^2}$. Therefore:

$$\frac{4}{n} < \epsilon^2 \tag{20}$$

$$\frac{2}{\sqrt{n}} < \epsilon \tag{21}$$

$$\frac{2}{\sqrt{2+n}+\sqrt{n}} < \frac{2}{\sqrt{n}} \tag{22}$$

$$\frac{2}{\sqrt{2+n}+\sqrt{n}} = \frac{n+2-n}{\sqrt{2+n}+\sqrt{n}} \tag{23}$$

$$= \sqrt{n+2} - \sqrt{n} \tag{24}$$

If we assume that for some natural n, $\sqrt{n+2} \leq \sqrt{n}$, we see that $n+2 \leq n$ and thus $2 \leq 0$, which is absurd, therefore $\sqrt{n+2} > \sqrt{n}$, and so $\sqrt{n+2} - \sqrt{n} = |\sqrt{n+2} - \sqrt{n}|$. By the transitivity of the above order relations, we then have $|\sqrt{n+2} - \sqrt{n}| = |\sqrt{n+2} - \sqrt{n} - 0| < \epsilon$. Thus we have demonstrated that $\forall \epsilon > 0, \exists N_\epsilon, \forall n > N_\epsilon, |\sqrt{n+2} - \sqrt{n} - 0| < \epsilon$, which is exactly the definition that $\lim(x_{n+2} - x_n) = 0$. $\square$

Part (b):

Claim 4. $x_n = \sqrt{n}$ *is not a Cauchy sequence.*

Proof. Let $\epsilon = 1$ and let N be any natural number. Define $n = 4(N+1)^2$ and $m = (N+1)^2$. We see that:

$$|x_n - x_m| = |\sqrt{n} - \sqrt{n}| = |\sqrt{4(N+1)^2} - \sqrt{(N+1)^2}| \tag{25}$$

$$|2(N+1) - (N+1)| \tag{26}$$

$$|2N + 2 - N - 1| \tag{27}$$

$$|N+1| \tag{28}$$

Since $N \geq 1$, we have $N + 1 \geq 1 + 1 = 2 \geq 1 = \epsilon$. Thus we have demonstrated that $\exists \epsilon > 0, \forall N, \exists n > N, m > N, |x_n - x_m| \geq \epsilon$, which is the negation of the definition of a Cauchy sequence, therefore $x_n = \sqrt{n}$ is not a Cauchy sequence. $\square$