Real Analysis Assignment 9

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Problem 1:

Claim 1. $f(x) = 3x^2 + 4x + 5$ *is continuous at every point* x_0

Proof. We want to prove that at any $x_0 \in \mathbb{R}$, $\forall \epsilon > 0 \exists \delta_\epsilon > 0 \forall |x - x_0| < \delta_\epsilon, |(3x^2 + 4x + 5) - (3x_0^2 + 4x_0 + 5)| < \epsilon$.

We examine as follows:

$$\begin{aligned}
|(3x^2 + 4x + 5) - (3x_0^2 + 4x_0 + 5)| =& |3(x^2 - x_0^2) + 4(x - x_0)| && (1)\\
=& |3(x - x_0)(x + x_0) + 4(x - x_0) && (2)\\
\text{(triangle inequality)} \quad \leq& 3|x - x_0| \cdot |x + x_0| + 4|x - x_0| && (3)\\
=& 3|x - x_0| \cdot |x - x_0 + 2x_0| + 4|x - x_0| && (4)\\
\text{(triangle inequality)} \quad \leq& 3|x - x_0| \cdot (|x - x_0| + 2|x_0|) + 4|x - x_0| && (5)\\
\text{(if } |x - x_0| < \delta) \quad <& 3\delta(\delta + 2|x_0|) + 4\delta && (6)\\
=& 3\delta(\delta + 2|x_0| + 4) && (7)\\
\text{(if } \delta \leq 1) \quad \leq& 3\delta(5 + 2|x_0|) && (8)\\
\left(\text{if } \delta \leq \frac{\epsilon}{3(5 + 2|x_0|)}\right) \quad \leq& \frac{\epsilon \cdot 3(5 + 2|x_0|)}{3(5 + 2|x_0|)} = \epsilon && (9)
\end{aligned}$$

Thus we can choose any $0 < \delta_\epsilon \leq \min(1, \frac{\epsilon}{3(5+2|x_0|)})$ and then we have for our arbitrary x_0 that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x - x_0| < \delta_\epsilon, |(3x^2 + 4x + 5) - (3x_0^2 + 4x_0 + 5)| < \epsilon$, which is exactly the definition that $f(x)$ is continuous at any point x_0. $\square$

Problem 2:

Suppose $f(x)$ is continuous at some $c \in \mathbb{R}$ and assume $f(c) < 2$.

Claim 2. *There exists a neighborhood* A *of* c *such that* $\forall x \in A, f(x) < 2$

Proof. f is continuous at c if and only if $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x - c| < \delta_\epsilon, |f(x) - f(x)| < \epsilon$. In particular, $\exists \delta_{2-f(c)}, \forall |x - c| < \delta_{2-f(c)}, |f(x) - f(x)| < 2 - f(c)$. Let $A = (c - \delta_{2-f(c)}, c + \delta_{2-f(c)})$. We see that $x \in A$ which is a neighborhood of c. Then for any $x \in A$, $|f(x) - f(c)| < 2 - f(c) \iff -(2 - f(c)) < f(x) - f(c) < 2 - f(c) \implies f(x) < 2$. Thus there exists a neighborhood A of c such that $\forall x \in A, f(x) < 2$. $\square$

Problem 3:

Supose $f(x)$ is continuous on the closed interval $[a, b]$ and assume $\forall x \in [a, b], f(x) > 0$.

Claim 3. $\inf\limits_{x\in[a,b]} f(x) > 0$.

Proof. By the Extreme Value Theorem, f attains maximum and minimum values in $[a, b]$. In particular, $\exists x_m \in [a, b], f(x_m) = \inf f(x), x \in [a, b]$, and since $\forall x \in [a, b], f(x) > 0$, we have that $f(x_m) > 0$, so $\inf\limits_{x\in[a,b]} f(x) > 0$. $\square$

Define the following function on the closed interval $[-1, 1]$:

$$g(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$g(x)$ is discontinuous at 0 since $0 = \lim\limits_{x\to 0} g(x) \neq g(x) = 1$. We also have that $\forall x \in [-1, 1], g(x) > 0$, since $|x| = 0 \iff x = 0$, but $g(0) = 1 > 0$. However, it is clear that $\inf\limits_{x\in[-1,1]} g(x) = 0$.

Problem 4:

Define $f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 2x - 1 & \text{if } x \text{ is rational} \end{cases}$

Claim 4. *$f(x)$ is discontinuous at every real $x_0 \neq 1$.*

Proof. Suppose $x_0 \in \mathbb{R}, x_0 \neq 1$ We will show that $f(x)$ satisfies the negation of the definition of continuity for real x_0 under this constraint, that is to say, $\exists\epsilon > 0, \forall\delta > 0, \exists x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| \geq \epsilon$. Let $\epsilon = (x_0 - 1)^2 > 0$ and $\delta > 0$. We consider three cases:

- $x \in \mathbb{Q} \wedge x_0 \geq 0$
- $x \in \mathbb{Q} \wedge x_0 < 0$
- $x \in \mathbb{R}\backslash\mathbb{Q}$

If $\exists x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| \geq \epsilon$ holds for each of the three cases, then $f(x)$ is discontinuous at all $x_0 \neq 1$.

Suppose $x_0 \in \mathbb{Q} \wedge x_0 \geq 0$. Then since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{Q}$, $\exists x_0 < x < x_0 + \delta$. Then, $0 \leq x_0 < x \implies x_0^2 < x^2$, so:

$$x_0^2 - 2x_0 + 1 < x^2 - 2x_0 + 1 \tag{10}$$
$$(x_0 - 1)^2 < |x^2 - (2x_0 - 1)| \tag{11}$$
$$\epsilon = (x_0 - 1)^2 \leq |f(x) - f(x_0)| \tag{12}$$

Suppose $x_0 \in \mathbb{Q} \wedge x_0 < 0$. Then since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{Q}$, $\exists x_0 - \delta < x < x_0$. Then, $0 > x_0 > x \implies x_0^2 < x^2$, so:

$$x_0^2 - 2x_0 + 1 < x^2 - 2x_0 + 1 \tag{13}$$
$$(x_0 - 1)^2 < |x^2 - (2x_0 - 1)| \tag{14}$$
$$\epsilon = (x_0 - 1)^2 \leq |f(x) - f(x_0)| \tag{15}$$

Suppose $x_0 \in \mathbb{R}\backslash\mathbb{Q}$. Then since $\mathbb{Q}$ is dense in $\mathbb{R}\backslash\mathbb{Q}$, $\exists x_0 - \delta < x < x_0$. Then:

$$\begin{aligned}
-2x >& -2x_0 && (16)\\
x_0^2 - 2x + 1 >& x_0^2 - 2x_0 + 1 && (17)\\
|x_0^2 - (2x-1)| >& (x_0 - 1)^2 && (18)\\
|2x - 1 - x_0^2| >& (x_0 - 1)^2 && (19)\\
|f(x) - f(x_0)| \geq& (x_0 - 1)^2 = \epsilon && (20)\\
& && (21)
\end{aligned}$$

Since $\forall x_0 \neq 1, \exists \epsilon > 0, \forall \delta > 0, \exists x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| \geq \epsilon$, we have that $f(x)$ is discontinuous at every $x_0 \neq 1$.

We will now show that $f(x)$ is continuous at $x_0 = 1$ by finding for any $\epsilon > 0$ a $\delta_\epsilon > 0$ where $|x - 1| < \delta_\epsilon$ implies that $|f(x) - 1| < \epsilon$. Suppose that $x \in \mathbb{Q}$. Then:

$$\begin{aligned}
|f(x) - 1| =& |2x - 1 - 1| && (22)\\
=& 2|x - 1| && (23)\\
(\text{if } |x-1| < \delta) \quad <& 2\delta && (24)\\
(\text{if } \delta \leq \frac{\epsilon}{2}) \quad \leq& 2\frac{\epsilon}{2} = \epsilon && (25)
\end{aligned}$$

Suppose that $x \in \mathbb{R}\backslash\mathbb{Q}$. Then:

$$\begin{aligned}
|f(x) - 1| =& |x^2 - 1| && (26)\\
=& 2|x - 1| \cdot |x + 1| && (27)\\
=& 2|x - 1| \cdot |x - 1 + 2| && (28)\\
(\text{triangle inequality}) \quad \leq& 2|x - 1| \cdot (|x - 1| + 2) && (29)\\
(\text{if } |x-1| < \delta) \quad <& \delta(\delta + 2) && (30)\\
(\text{if } \delta \leq 1) \quad \leq& \delta(1 + 2) = 3\delta && (31)\\
(\text{if } \delta \leq \frac{\epsilon}{3}) \quad \leq& 3\frac{\epsilon}{3} = \epsilon && (32)
\end{aligned}$$

So, we can choose any $0 < \delta_\epsilon < \min(1, \frac{\epsilon}{3})$, and we have that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall |x - 1| < \delta_\epsilon, |f(x) - 1| < \epsilon$, which is exactly the definition that $f(x)$ is continuous at $x_0 = 1$. Therefore, $f(x)$ is discontinuous at every point $x_0 \neq 1$. $\square$