Real Analysis Assignment 5

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Problem 1:

Claim 1. *Every monotone decreasing bounded below sequence has a limit*

Proof. Let x_n be a monotone decreasing sequence bounded below. Thus by definition, $\forall n, x_{n+1} \leq x_n$. Let M be a lower bound of x_n, i.e $\forall n, x_n \geq M$. Since x_n is a sequence of real numbers bounded below, by the greatest lower bound property of the reals we have that there exists a greatest lower bound, an infimum, of the set x_n. Let $a = \inf x_n$. Let $\epsilon > 0$. Then, since a is the greatest lower bound of x_n, any $a + \epsilon$ is not a lower bound, so there exists some N_ϵ such that $x_{N_\epsilon} < a + \epsilon$. Then, since $\forall n, x_{n+1} \leq x_n$, we must have that $\forall n > N_\epsilon, x_n \leq x_{N_\epsilon}$, and since a is a lower bound of x_n, we have $\forall n, a \leq x_n$, and since $\epsilon > 0$, we of course have $a - \epsilon < a$. Putting this all together, we have that $\forall n > N_\epsilon, a - \epsilon < a \leq x_n \leq x_{N_\epsilon} < a + \epsilon$, thus by transitivity, $\forall n > N_\epsilon, a - \epsilon < x_n < a + \epsilon$, therefore $-\epsilon < x_n - a < \epsilon \iff |x_n - a| < \epsilon$. We have demonstrated that $\forall \epsilon > 0, \exists N_\epsilon : \forall n > N_\epsilon, |x_n - a| < \epsilon$, which is exactly the definition that $\lim x_n = a$, therefore the limit of x_n exists, and every monotone decreasing bounded below sequence has a limit. $\square$

Problem 2:

Claim 2. *The sequence* $a_n = \begin{cases} a_1 = 2 \\ a_{n+1} = \frac{a_n^2+1}{2a_n} & n \geq 1 \end{cases}$ *has a limit.*

Proof. First, note that $a_1 = 2 \geq 1$. Then, let $t \geq 0$. Noting that $\forall x \in \mathbb{R}, x^2 \geq 0$, we have the following:

$$(\sqrt{t} - \frac{1}{\sqrt{t}})^2 \geq 0 \tag{1}$$

$$t - 2 + \frac{1}{t} \geq 0 \tag{2}$$

$$t + \frac{1}{t} \geq 2 \tag{3}$$

Thus $\forall t \geq 0, t + \frac{1}{t} \geq 2$. Knowing this, we can determine a lower bound for a_{n+1}

as follows:

$$a_{n+1} = \frac{a_n^2 + 1}{2a_n} \tag{4}$$
$$= a_n \frac{a_n + \frac{1}{a_n}}{2a_n} \tag{5}$$
$$a_n \frac{2}{2a_n} \leq a_n \frac{a_n + \frac{1}{a_n}}{2a_n} \tag{6}$$
$$1 \leq \frac{a_n^2 + 1}{2a_n} \tag{7}$$
$$1 \leq a_{n+1} \tag{8}$$

Since $a_1 \geq 1 \wedge a_{n+1} \geq 1$, we have $\forall n, a_n \geq 1$, therefore 1 is a lower bound of a_n and a_n is bounded below. Since $a_2 = \frac{a_1^2+1}{2a_1} = \frac{2^2+1}{2 \cdot 2} = \frac{5}{4}$, we see that $a_2 \leq a_1$. Then, since $\forall n, a_n \geq 1 \implies \forall n, a_n^2 \geq 1$:

$$a_{n+1} = \frac{a_n^2 + 1}{2a_n} \tag{9}$$
$$\frac{a_n^2 + a_n^2}{2a_n} \geq \frac{a_n^2 + 1}{2a_n} \tag{10}$$
$$\frac{2a_n^2}{2a_n} \geq \frac{a_n^2 + 1}{2a_n} \tag{11}$$
$$a_n \geq a_{n+1} \tag{12}$$

Therefore, a_n is monotone decreasing. Since a_n is bounded below and monotone decreasing, we have by the monotone convergence theorem that the limit of a_n exists. □

Computation of $\lim a_n$

We know that $\lim a_n$ exists, so it makes sense to demote it by $a = \lim a_n$.

By the limit laws, $\lim a_{n+1} = \frac{\lim a_n^2 + 1}{2 \lim a_n}$

Since they differ by finitely many terms, $\lim a_{n+1} = \lim a_n$, so we can compute as follows:

$$a = \frac{a^2 + 1}{2a} \tag{13}$$
$$2a^2 = a^2 + 1 \tag{14}$$
$$a^2 = 1 \tag{15}$$
$$\implies a = \pm 1 \tag{16}$$

Suppose, for a contradiction, that $\lim a_n = -1$. Then, $\forall \epsilon > 0 \exists N_\epsilon : \forall n > N_\epsilon, a_n \in (-1-\epsilon, -1+\epsilon)$, therefore for any $\epsilon > 0$ this last membership relation must be true for at least one a_n. Let $\epsilon = 1$. Then, some $a_n \in (-2, 0)$, but

$\forall n, a_n \geq 1$, which is a contradiction, thus $\lim a_n \neq -1$ and it must be the case that $\lim a_n = 1$.

Problem 3:

Claim 3. 0 *is not a sub-sequential limit of* $x_n = (-1)^n - \frac{2}{n}$

Proof. Define the neighborhood $B = (\frac{-1}{2}, \frac{1}{2})$. Note that any natural n is either even or odd. If n is even, then $(-1)^n = 1$, but if n is odd, then $(-1)^n = -1$. Now let $n \in \mathbb{N}$. $\forall n \in \mathbb{N}, n > 0$ implies the following:

$$n > 0 \tag{17}$$
$$\implies \frac{1}{n} > 0 \tag{18}$$
$$\implies \frac{-1}{n} < 0 \tag{19}$$
$$\implies \frac{-2}{n} < 0 \tag{20}$$
$$\implies -1 - \frac{2}{n} < -1 \tag{21}$$

If n is odd, then then we have $(-1)^n - \frac{2}{n} < -1$. Thus for all odd n we have $x_n \notin B$. Now let $n \geq 4$ and note the following:

$$n \geq 4 \tag{22}$$
$$\implies -n \leq -4 \tag{23}$$
$$\implies n \leq 2n - 4 \tag{24}$$
$$\implies 1 \leq \frac{2(n-2)}{n} \tag{25}$$
$$\implies \frac{1}{2} \leq \frac{n-2}{n} \tag{26}$$
$$\implies \frac{1}{2} \leq 1 - \frac{2}{n} \tag{27}$$

If n is even, then we have $(-1)^n - \frac{2}{n} \geq \frac{1}{2}$, thus for any even $n \geq 4$, we have $x_n \notin B$. Since $x_n \notin B$ for any odd n, we must also have $x_n \notin N$ for any odd $n > 4$, therefore $\forall n \geq 4, x_n \notin (0 - \frac{1}{2}, 0 + \frac{1}{2})$, and this demonstates that $\forall n \geq 4, x_n$ falls outside of some neighborhood of 0, in particular $B = (\frac{-1}{2}, \frac{1}{2})$, so we conclude that finitely many x_n are in B. This is true if and only if 0 is not a subsequential limit of x_n. $\square$