Real Analysis Assignment 7

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Problem 1:
Define $x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... \frac{1}{\sqrt{n}} = \sum_{j=1}^{n} \frac{1}{\sqrt{j}}$.

Claim 1. *x_n is not a Cauchy sequence.*

Proof. Let $\epsilon = \frac{3}{2}$ and let $N \in \mathbb{N}$. Suppose $m > N$ and $n = 4m > m$. Then, we see that:

$$|x_n - x_m| = |\sum_{j=1}^{n} \frac{1}{\sqrt{j}} - \sum_{j=1}^{m} \frac{1}{\sqrt{j}}| \tag{1}$$

$$= |\sum_{j=1}^{4m} \frac{1}{\sqrt{j}} - \sum_{j=1}^{m} \frac{1}{\sqrt{j}}| \tag{2}$$

$$\sum_{j=m+1}^{4m} \frac{1}{\sqrt{j}} = |\sum_{j=m+1}^{4m} \frac{1}{\sqrt{j}}| \text{ (positive terms)} \tag{3}$$

$$\sum_{j=m+1}^{4m} \frac{1}{\sqrt{j}} \geq \sum_{j=m+1}^{4m} \frac{1}{\sqrt{4m}} = \frac{3m}{2\sqrt{m}} = \frac{3\sqrt{m}}{2} \tag{4}$$

$$m > N \geq 1 \implies \sqrt{m} > 1 \implies \frac{3\sqrt{m}}{2} \geq \frac{3}{2} \tag{5}$$

Thus, $\exists \epsilon > 0, \forall N, \exists n > m > N, |x_n - x_m| \geq \epsilon$, which is exactly the definition that x_n is not a Cauchy sequence. ☐

Problem 2(a):

Claim 2. $\lim a_n = -\infty \implies \lim \frac{1}{a_n} = 0$.

Proof. Since $\lim a_n = -\infty$, we have by definition that $\forall \beta < 0, \exists N_\beta, \forall n > N_\beta, a_n < \beta$. Let $\beta < 0$. Then, $\forall n > N_\beta$, we have $a_n < \beta$. Since $\beta < 0$, we have:

$$a_n < \beta < 0 \tag{6}$$
$$-a_n > -\beta > 0 \tag{7}$$
$$\frac{-1}{\beta} > \frac{-1}{a_n} > 0 \tag{8}$$
$$\frac{-1}{\beta} > |\frac{1}{a_n}| > 0 \tag{9}$$

We see that $M_\epsilon = N_{\frac{-1}{\epsilon}}$ will work. Thus, let $\epsilon > 0$, and let $\beta = \frac{-1}{\epsilon} < 0$, then define $M_\epsilon = N_{\frac{-1}{\epsilon}}$. Then, $\forall n > M_\epsilon$, we have by the above order relations, that $|\frac{1}{a_n}| < \frac{-1}{\beta} = \frac{-1}{-1/\epsilon} = \epsilon$, therefore $\forall \epsilon > 0, \exists M_\epsilon, \forall n > M_\epsilon, |\frac{1}{a_n}| < \epsilon$, which is exactly the definition that $\lim \frac{1}{a_n} = 0$. □

Problem 2(b):

Claim 3. $(\lim a_n = 0 \wedge a_n < 0) \implies \lim \frac{1}{a_n} = -\infty$

Proof. Since $\lim a_n = 0$, we have by definition that $\forall \epsilon > 0, \exists N_\epsilon, \forall n > N_\epsilon, |a_n| < \epsilon$. Let $\epsilon > 0$. Then for some N_ϵ, we have $\forall n > N_\epsilon$, that $|a_n| < \epsilon$, which is definitionally equivalent to $-\epsilon < a_n < \epsilon$, and since by assumption $a_n < 0$, we have $-\epsilon < a_n < 0$, therefore $\frac{1}{a_n} < \frac{-1}{\epsilon} < 0$, so a value of $M_\beta = N_{\frac{-1}{\beta}}$ will work. Let $\beta < 0$ and let $\epsilon = \frac{-1}{\beta} > 0$, then define $M_\beta = N_{\frac{-1}{\beta}}$. By the above reasoning, $\forall n > M_\beta$, we have $\frac{1}{a_n} < \frac{-1}{\epsilon} = \frac{-1}{-1/\beta} = \beta$. Then, $\forall \beta < 0, \exists M_\beta, \forall n > M_\beta, \frac{1}{a_n} < \beta$, and this is exactly the definition that $\lim \frac{1}{a_n} = -\infty$. □

Problem 3:

Claim 4. *If* $\lim x_n \neq \infty$*, then there exists an infinite subsequence of* x_n *that is bounded above.*

Proof. If $\lim x_n \neq \infty$, then by definition $\exists \alpha \in \mathbb{R}, \forall N \in \mathbb{N}, \exists n > N, x_n \leq \alpha$. Let $\alpha > 0$ and let $N \in \mathbb{N}$. Then, $\exists n_1 > N, x_{n_1} \leq \alpha$. Since $n_1 \in \mathbb{N}$, we must have $\exists n_2 > n_1, x_{n_2} \leq \alpha$, and since $n_2 \in \mathbb{N}$, we must have $\exists n_3 > n_2, x_{n_3} \leq \alpha$. We can continue this indefinitely, so for some $n_{k-1} \in \mathbb{N}$, we must have $\exists n_k > n_{k-1}, x_{n_k} \leq \alpha$. Thus we have a strictly increasing set of indices $n_1 < n_2 < ... < n_{k-1} < n_k < ...$ where $\forall k, x_{n_k} \leq \alpha$, so α is an upper bound of x_{n_k}, and there exists an infinite subsequence of x_n that is bounded above. □