

Real Analysis Assignment 3

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Problem 1:

Claim 1. $\lim_{n \rightarrow \infty} \frac{6n+5}{4n+7} = \frac{3}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. Then, since \mathbb{R} is closed under the field operations, we have $\frac{1}{8}(\frac{11}{\epsilon} - 14) \in \mathbb{R}$. Since $(1 \in \mathbb{R} \wedge 1 > 0) \wedge \frac{1}{8}(\frac{11}{\epsilon} - 14) \in \mathbb{R}$, by the Archimedean property of \mathbb{R} we have $\exists n \in \mathbb{N} : 1 \cdot n = n > \frac{1}{8}(\frac{11}{\epsilon} - 14)$. This inequality implies that $8n + 14 > \frac{11}{\epsilon} \implies \frac{11}{8n+14} < \epsilon$. Since $x = |-x|$, we have $|\frac{-11}{8n+14}| < \epsilon$. Therefore, $|\frac{12n+10-12n-21}{8n+14}| = |\frac{2(n+5)-3(4n+7)}{8n+14}| = |\frac{6n+5}{4n+7} - \frac{3}{2}| < \epsilon$. Thus we have demonstrated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\frac{6n+5}{4n+7} - \frac{3}{2}| < \epsilon$, which is exactly the definition that $\lim_{n \rightarrow \infty} \frac{6n+5}{4n+7} = \frac{3}{2}$. \square

Problem 2:

Claim 2. $\lim_{n \rightarrow \infty} \sqrt{9 - \frac{(-1)^n}{n}} = 3$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedean property of \mathbb{R} , $\exists n \in \mathbb{N} : n > \frac{1}{3\epsilon}$. Then, $\frac{1}{3n} = \frac{\frac{1}{3}}{n} < \epsilon$. Since $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : \sqrt{y} = x \implies x > 0$, we have $\sqrt{9 - \frac{(-1)^n}{n}} > 0 \implies \sqrt{9 - \frac{(-1)^n}{n}} + 3 > 3 \implies \frac{1}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{1}{3} \implies \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{1}{3}$. Any natural number is either even or odd, i.e. $\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : (n = 2k) \vee (n = 2k - 1)$. Then, $n = 2k \implies \frac{(-1)^n}{n} = \frac{1}{n} \wedge n = 2k - 1 \implies \frac{(-1)^n}{n} = \frac{-1}{n}$, so we have $\forall n \in \mathbb{N}, \frac{(-1)^n}{n} \leq \frac{1}{n}$. Thus, $\frac{\frac{(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \leq \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}$.

NO THIS IS FALSE Since $x = |-x|$, we have $\frac{\frac{(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} = |\frac{-\frac{(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}|$,

and then $|\frac{-\frac{(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| = |\frac{9 - \frac{(-1)^n}{n} - 9}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| = |\frac{(\sqrt{9 - \frac{(-1)^n}{n}})^2 - 3^2}{\sqrt{9 - \frac{(-1)^n}{n}} + 3}| = |\sqrt{9 - \frac{(-1)^n}{n}} - 3|$.

By transitivity, $|\sqrt{9 - \frac{(-1)^n}{n}} - 3| \leq \frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{1}{3n} < \epsilon \implies |\sqrt{9 - \frac{(-1)^n}{n}} - 3| < \epsilon$. Thus we have demonstrated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\sqrt{9 - \frac{(-1)^n}{n}} - 3| < \epsilon$, which is exactly the definition that $\lim_{n \rightarrow \infty} \sqrt{9 - \frac{(-1)^n}{n}} = 3$. \square

Problem 3:

Claim 3. $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{4n^2+9n+1}} = \frac{1}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of \mathbb{R} , $\exists n \in \mathbb{N} : n > \frac{1}{2}(\frac{1}{\epsilon - \frac{9}{2}})$. Then, $\frac{1}{2n} + \frac{9}{2} = \frac{1}{2n} + \frac{9n}{2n} = \frac{9n+1}{2n} < \epsilon$. Since $\forall n \in \mathbb{N}, 1 < 1 + 4n^2 + 9n$, we have $1 < \sqrt{4n^2 + 9n + 1}$. Then, $2n + \sqrt{4n^2 + 9n + 1} > 2n \implies \frac{1}{2n + \sqrt{4n^2 + 9n + 1}} < \frac{1}{2n} \implies \frac{9n+1}{2n + \sqrt{4n^2 + 9n + 1}} < \frac{9n+1}{2n}$. Furthermore, since $\sqrt{4n^2 + 9n + 1} > 1 \implies 2\sqrt{4n^2 + 9n + 1} > 1$, we have $(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1}) > 2n + \sqrt{4n^2 + 9n + 1} \implies \frac{1}{(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1})} < \frac{1}{2n + \sqrt{4n^2 + 9n + 1}} \implies \frac{9n+1}{(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1})} < \frac{9n+1}{2n + \sqrt{4n^2 + 9n + 1}}$. Since $\forall x \in \mathbb{R} : x > 0, |-x| = x$, we have $\frac{9n+1}{(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1})} = \left| \frac{-(9n+1)}{(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1})} \right| = \left| \frac{4n^2 - (4n^2 + 9n + 1)}{(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1})} \right| = \left| \frac{(2n)^2 - (\sqrt{4n^2 + 9n + 1})^2}{(2n + \sqrt{4n^2 + 9n + 1})(2\sqrt{4n^2 + 9n + 1})} \right| = \left| \frac{2n - \sqrt{4n^2 + 9n + 1}}{2\sqrt{4n^2 + 9n + 1}} \right| = \left| \frac{n}{\sqrt{4n^2 + 9n + 1}} - \frac{1}{2} \right|$. Then by transitivity, $\left| \frac{n}{\sqrt{4n^2 + 9n + 1}} - \frac{1}{2} \right| < \frac{9n+1}{2n + \sqrt{4n^2 + 9n + 1}} < \frac{9n+1}{2n} < \epsilon \implies \left| \frac{n}{\sqrt{4n^2 + 9n + 1}} - \frac{1}{2} \right| < \epsilon$. Thus we have demonstrated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, \left| \frac{n}{\sqrt{4n^2 + 9n + 1}} - \frac{1}{2} \right| < \epsilon$ which is exactly the definition that $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{4n^2 + 9n + 1}} = \frac{1}{2}$. \square