

# Real Analysis Assignment 3

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## Problem 1:

**Claim 1.**  $\lim_{n \rightarrow \infty} \frac{6n+5}{4n+7} = \frac{3}{2}$

*Proof.* Let  $\epsilon \in \mathbb{R} : \epsilon > 0$ . By the Archimedian property of  $\mathbb{R}$  we have  $\exists N_\epsilon \in \mathbb{N} : N_\epsilon > \frac{11}{8\epsilon}$ . Then,  $\forall n > N_\epsilon, n > \frac{11}{8\epsilon}$ . This inequality implies the following:

$$n > \frac{11}{8\epsilon} \implies \frac{11}{8n} < \epsilon \quad (1)$$

$$\frac{11}{8n+14} < \frac{11}{8n} < \epsilon \quad (2)$$

$$\forall x < 0, |x| = -x \implies \frac{11}{8n+14} = \left| \frac{-11}{8n+14} \right| < \epsilon \quad (3)$$

$$\left| \frac{-11}{8n+14} \right| = \left| \frac{12n+10-12n-21}{8n+14} \right| = \left| \frac{2(6n+5)-3(4n+7)}{8n+14} \right| \quad (4)$$

$$\left| \frac{2(6n+5)-3(4n+7)}{8n+14} \right| = \left| \frac{6n+5}{4n+7} - \frac{3}{2} \right| < \epsilon \quad (5)$$

Thus we have demonstrated that  $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, \left| \frac{6n+5}{4n+7} - \frac{3}{2} \right| < \epsilon$ , which is exactly the definition that  $\lim_{n \rightarrow \infty} \frac{6n+5}{4n+7} = \frac{3}{2}$ .  $\square$

## Problem 2:

**Claim 2.**  $\lim_{n \rightarrow \infty} \sqrt{9 - \frac{(-1)^n}{n}} = 3$

*Proof.* Let  $\epsilon \in \mathbb{R} : \epsilon > 0$ . By the Archimedian property of  $\mathbb{R}$ ,  $\exists N_\epsilon \in \mathbb{N} : N_\epsilon > \frac{1}{3\epsilon}$ , so  $\forall n > N_\epsilon, n > \frac{1}{3\epsilon}$ .

Assume for contradiction that  $\frac{(-1)^n}{n} > 1$ . Then,  $(\frac{(-1)^n}{n})^2 = \frac{1}{n^2} > 1$ , so  $n^2 < 1$ . Since  $n > 0$ , this implies that  $n < 1$ , but of course  $n \geq 1$ , which is a contradiction, therefore  $\frac{(-1)^n}{n} < 1 < 9$ , so  $9 - \frac{(-1)^n}{n} > 0$ , and  $\sqrt{9 - \frac{(-1)^n}{n}}$  exists.

Then, since  $\sqrt{9 - \frac{(-1)^n}{n}} > 0$ , we have:

$$\sqrt{9 - \frac{(-1)^n}{n}} + 3 > 3 \quad (6)$$

$$\frac{1}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} < \frac{1}{3} \quad (7)$$

$$\frac{\frac{1}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} = \left| \frac{\frac{-(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \right| < \frac{1}{3n} \quad (8)$$

$$\left| \frac{-\frac{(-1)^n}{n}}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \right| = \left| \frac{9 - \frac{(-1)^n}{n} - 9}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \right| \quad (9)$$

$$\left| \frac{9 - \frac{(-1)^n}{n} - 9}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \right| = \left| \frac{(\sqrt{9 - \frac{(-1)^n}{n}})^2 - 3^2}{\sqrt{9 - \frac{(-1)^n}{n}} + 3} \right| = \left| \sqrt{9 - \frac{(-1)^n}{n}} - 3 \right|. \quad (10)$$

By transitivity,  $|\sqrt{9 - \frac{(-1)^n}{n}} - 3| < \frac{1}{3n} < \epsilon \implies |\sqrt{9 - \frac{(-1)^n}{n}} - 3| < \epsilon$ . Thus we have demonstrated that  $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\sqrt{9 - \frac{(-1)^n}{n}} - 3| < \epsilon$ . which is exactly the definition that  $\lim \sqrt{9 - \frac{(-1)^n}{n}} = 3$ .  $\square$

### Problem 3:

**Claim 3.**  $\lim \frac{n}{\sqrt{4n^2 + 9n + 1}} = \frac{1}{2}$

*Proof.* Let  $\epsilon \in \mathbb{R} : \epsilon > 0$ . By the Archimedian property of  $\mathbb{R}$ ,  $\exists N_\epsilon : N_\epsilon > \frac{18}{8\epsilon}$ . Then,  $\forall n > N_\epsilon, n > \frac{18}{8\epsilon}$ . We can then derive the following inequalities and

equations:

$$n > \frac{18}{8\epsilon} \implies \frac{18}{8n} < \epsilon \quad (11)$$

$$\frac{9n+9n}{8n^2+18n} = \frac{18}{8n+18} < \frac{18}{8n} \quad (12)$$

$$\frac{9n+1}{2(4n^2+9n+1)} = \frac{9n+1}{8n^2+18n+2} < \frac{9n+9n}{8n^2+18n+2} < \frac{9n+9n}{8n^2+18n} \quad (13)$$

$$\frac{9n+1}{(\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} = \frac{9n+1}{2(4n^2+9n+1)} \quad (14)$$

$$\frac{(4n^2+9n+1)-4n^2}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} < \frac{9n+1}{(\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} \quad (15)$$

$$\frac{(4n^2+9n+1)-4n^2}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} = \frac{\sqrt{4n^2+9n+1}-2n}{(2\sqrt{4n^2+9n+1})} \quad (16)$$

$$\frac{\sqrt{4n^2+9n+1}-2n}{(2\sqrt{4n^2+9n+1})} = \frac{-(2n-\sqrt{4n^2+9n+1})}{2\sqrt{4n^2+9n+1}} \quad (17)$$

Now assume for a contradiction that  $\sqrt{4n^2+9n+1} \leq 2n$ . This inequality implies:

$$4n^2+9n+1 \leq 4n^2 \quad (18)$$

$$9n+1 \leq 0 \quad (19)$$

$$n \leq \frac{-1}{9} < 0 \quad (20)$$

But  $n > 0$  since  $n \in \mathbb{N}$ , which is a contradiction, therefore  $\sqrt{4n^2+9n+1} > 2n \implies 0 > 2n - \sqrt{4n^2+9n+1}$ , so  $|\frac{2n-\sqrt{4n^2+9n+1}}{2\sqrt{4n^2+9n+1}}| = \frac{-(2n-\sqrt{4n^2+9n+1})}{2\sqrt{4n^2+9n+1}}$ . Finally,  $|\frac{2n-\sqrt{4n^2+9n+1}}{2\sqrt{4n^2+9n+1}}| = |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}|$ , and by the transitivity of the above order relations, we have  $|\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \epsilon$ . Thus we have demonstrated that  $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \epsilon$  which is exactly the definition that  $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{4n^2+9n+1}} = \frac{1}{2}$ .  $\square$

#### Problem 4:

**Claim 4.**  $\lim(\sqrt{n^2+5n}-n) = \frac{5}{2}$

*Proof.* Let  $\epsilon \in \mathbb{R} : \epsilon > 0$ . By the Archimedian property of  $\mathbb{R}$ ,  $\exists N_\epsilon \in \mathbb{N} : N_\epsilon > \frac{25}{4\epsilon}$ . Then,  $\forall n > N_\epsilon, n > \frac{25}{4\epsilon}$ . We have the following inequalities and equations:

$$n > \frac{25}{4\epsilon} \implies \frac{25}{4n} < \epsilon \quad (21)$$

$$\frac{5}{2} + \sqrt{n^2 + 5n} > 0 \implies \frac{\frac{25}{4}}{n + \frac{5}{2} + \sqrt{n^2 + 5n}} < \frac{25}{4n} \quad (22)$$

$$\frac{\frac{25}{4}}{n + \frac{5}{2} + \sqrt{n^2 + 5n}} = \frac{(n^2 + 5n) + \frac{25}{4} - (n^2 + 5n)}{n + \frac{5}{2} + \sqrt{n^2 + 5n}} \quad (23)$$

$$(a - b) = \frac{a^2 - b^2}{a + b} \implies \frac{(n^2 + 5n) + \frac{25}{4} - (n^2 + 5n)}{n + \frac{5}{2} + \sqrt{n^2 + 5n}} = n + \frac{5}{2} - \sqrt{n^2 + 5n} \quad (24)$$

$$n + \frac{5}{2} - \sqrt{n^2 + 5n} = -(\sqrt{n^2 + 5n} - (n + \frac{5}{2})) \quad (25)$$

Now, assume for the purpose of contradiction that  $n + \frac{5}{2} \leq \sqrt{n^2 + 5n}$ . Then, we have:

$$n^2 + 5n + \frac{25}{4} \leq n^2 + 5n \quad (26)$$

$$\frac{25}{4} \leq 0 \quad (27)$$

But of course  $\frac{25}{4} > 0$ , which is a contradiction. Thus,  $n + \frac{5}{2} > \sqrt{n^2 + 5n}$ . So  $-(\sqrt{n^2 + 5n} - (n + \frac{5}{2})) = |\sqrt{n^2 + 5n} - (n + \frac{5}{2})| = |\sqrt{n^2 + 5n} - n - \frac{5}{2}|$ , and by transitivity of the above order relations, we have  $|\sqrt{n^2 + 5n} - n - \frac{5}{2}| < \epsilon$ . Thus we have demonstrated that  $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\sqrt{n^2 + 5n} - n - \frac{5}{2}| < \epsilon$ , which is exactly the definition that  $\lim(\sqrt{n^2 + 5n} - n) = \frac{5}{2}$ .  $\square$

### Problem 5:

**Claim 5.**  $\lim(3 + 2(-1)^n) \neq 5$

*Proof.* Let  $\epsilon = 1$ , let  $N \in \mathbb{N}$ , and let  $m = 2N + 1$ . Then,  $|3 - 2(-1)^m - 5| = |3 - 2(-1)^2(-1) - 5| = |-4| = 4 \geq 1 = \epsilon$ . Thus we have demonstrated that  $\exists \epsilon \in \mathbb{R} : \epsilon > 0 : \forall N \in \mathbb{N}, \exists n \in \mathbb{N} : n > N, |3 - 2(-1)^n - 5| \geq \epsilon \iff \neg(\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |3 - 2(-1)^n - 5| < \epsilon)$ , which is the negation of  $\lim(3 + 2(-1)^n) = 5$ , and  $\neg(\lim(3 + 2(-1)^n) = 5) \iff \lim(3 + 2(-1)^n) \neq 5$ .  $\square$