Real Analysis Assignment 10

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Problem 1:
Define $f(x) = \frac{1}{x} + \cos x$ and $A = [1, \infty)$

Claim 1. *$f(x)$ is uniformly continuous on A.*

Proof. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in A, \forall y \in A, |x-y| < \delta_\epsilon \implies |f(x) - f(y)| < \epsilon$. We examine the consequent to find a suitable choice of δ_ϵ:

$$|f(x) - f(y)| = |\frac{1}{x} + \cos x - (\frac{1}{y} + \cos y)| \tag{1}$$

$$= |\frac{1}{x} - \frac{1}{y} + \cos x - \cos y| \tag{2}$$

$$\text{(trianlge inequality)} \quad \leq |\frac{1}{x} - \frac{1}{y}| + |\cos x - \cos y| \tag{3}$$

Now we examine each of the last two terms individually. For the first:

$$|\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{xy} \tag{4}$$

$$(xy \geq 1) \quad \leq |x - y| \tag{5}$$

For the second:

$$|\cos x - \cos y| = |-2\sin(\frac{x-y}{2})\sin(\frac{x-y}{2})| \tag{6}$$

$$(\forall x, \sin x \leq 1) \quad \leq |2\sin(\frac{x-y}{2})| \tag{7}$$

$$(\forall x > 0, |\sin x| \leq |x|) \quad \leq |2\frac{x-y}{2}| = |x-y| \tag{8}$$

Now, we continue with the full examination:

$$|\frac{1}{x} - \frac{1}{y}| + |\cos x - \cos y| \leq 2|x-y| \tag{9}$$

$$(\text{if } |x-y| < \delta) \quad < 2\delta \tag{10}$$

$$(\text{if } \delta \leq \frac{\epsilon}{2}) \quad \leq 2\frac{\epsilon}{2} = \epsilon \tag{11}$$

Therefore, we can choose $\delta_\epsilon = \frac{\epsilon}{2}$, and we have: $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in A, \forall y \in A, |x-y| < \delta_\epsilon \implies |f(x) - f(y)| < \epsilon$, which is exactly the definition that $f(x)$ is uniformly continuous on $A = [1, \infty)$. $\square$

Problem 2:

Define $f(x) = \frac{1}{x} + \cos x$ and $A = (0, 1]$.

Claim 2. *$f(x)$ is not uniformly continuous on A.*

Proof. We want to show that: $\exists \epsilon > 0, \forall \delta > 0, \exists x \in A, y \in A, |x - y| < \delta \implies |f(x) - f(y)| \geq \epsilon$. We examine the consequent to find a suitable choice of ϵ.

$$\begin{aligned}
|f(x) - f(y)| =& |\frac{1}{x} + \cos x - (\frac{1}{y} + \cos y)| && (12)\\
=& |\frac{1}{x} - \frac{1}{y} + \cos x - \cos y| && (13)\\
\text{(rev. triangle inequality)} \quad \geq& |\frac{1}{x} - \frac{1}{y}| - |\cos x - \cos y| && (14)\\
& && (15)
\end{aligned}$$

Since $0 < \delta < \delta + 1$, we have $0 < \frac{\delta}{\delta+1} < 1$. Let $x = \frac{\delta}{\delta+1}$ and let $y = \frac{x}{2}$. This is valid since $|x - y| = \frac{\delta}{2(\delta+1)} < \delta$. We look at each term of the last expression to find bounds. For the first:

$$\begin{aligned}
|\frac{1}{x} - \frac{1}{y}| =& \frac{|x - y|}{xy} && (16)\\
=& \frac{|x - \frac{x}{2}|}{x\frac{x}{2}} && (17)\\
=& \frac{\frac{x}{2}}{\frac{x^2}{2}} && (18)\\
=& \frac{1}{x} && (19)\\
(x < 1) \quad \geq& 1 && (20)
\end{aligned}$$

For the second, we begin with the fact that cosine is monotone decreasing over A:

$$\begin{aligned}
&\forall x \in A, \cos 0 = 1 \geq \cos x \geq \cos 1 && (21)\\
&\forall y \in A, -\cos 1 \geq -\cos y \geq -\cos 0 = -1 && (22)\\
&1 - \cos 1 \geq \cos x - \cos y \geq \cos 1 - 1 && (23)\\
&1 - \cos 1 \geq \cos x - \cos y \geq -(1 - \cos 1) && (24)\\
&1 - \cos 1 \geq |\cos x - \cos y| && (25)
\end{aligned}$$

We conclude the examination as follows;

$$\begin{aligned}
|\frac{1}{x} - \frac{1}{y}| - |\cos x - \cos y| \geq& 1 - (1 - \cos 1) && (26)\\
\cos 1 =& \epsilon && (27)
\end{aligned}$$

To conclude, we have that $\exists \epsilon > 0, \forall \delta > 0, \exists x \in A, y \in A, |x - y| < \delta \implies |f(x) - f(y)| \geq \epsilon$. which is exactly the definition that $f(x)$ is not uniformly continuous on A. □

Problem 3:

Suppose $f(x)$ and $g(x)$ are uniformly continuous on some interval E and both functions are bounded. Let $|f(x)| \leq A \wedge |g(x)| \leq B$.

Claim 3. *$f(x) \cdot g(x)$ is uniformly continuous on E.*

Proof. Since $f(x)$ is uniformly continuous on E, we have: $\forall \epsilon > 0, \exists \delta'_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta'_\epsilon \implies |f(x) - f(y)| < \epsilon$. Since $g(x)$ is uniformly continuous on E, we have: $\forall \epsilon > 0, \exists \delta''_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta''_\epsilon \implies |g(x) - g(y)| < \epsilon$. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta_\epsilon \implies |f(x)g(x) - f(y)g(y)| < \epsilon$, so we will examine the consequent to find a suitable value of δ_ϵ.

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| && (28)\\
\text{(triangle inequality)} &\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| && (29)\\
&= |f(x)\big(g(x) - g(y)\big)| + |g(y)\big(f(x) - f(y)\big)| && (30)\\
\text{(use bounds)} &\leq A|g(x) - g(y)| + B|f(x) - f(y)| && (31)
\end{aligned}$$

By the definitions above, we have $\forall x \in E, y \in E, \left(|x-y| < \delta'_{\frac{\epsilon}{2B}} \implies |f(x) - f(y)| < \frac{\epsilon}{2B}\right) \wedge \left(|x-y| < \delta''_{\frac{\epsilon}{2A}} \implies |g(x) = g(y)| < \frac{\epsilon}{2A}\right)$. Thus, if we choose $\delta_\epsilon = \min(\delta'_{\frac{\epsilon}{2B}} \delta''_{\frac{\epsilon}{2A}})$, we have:

$$\begin{aligned}
A|g(x) - g(y)| + B|f(x) - f(y)| &< A\frac{\epsilon}{2A} + B\frac{\epsilon}{2B} && (32)\\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && (33)
\end{aligned}$$

In conclusion, we have $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta_\epsilon \implies |f(x)g(x) - f(y)g(y)| < \epsilon$, which is exactly the definition that $f(x) \cdot g(x)$ is uniformly continuous on E. $\square$