Real Analysis Assignment 2

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Note: For the scope of this document, let $\ni$ denote "such that"

1:

Suppose E is a non-empty subset of $\mathbb{R}$ that is bounded below, i.e. $E \subseteq \mathbb{R} \ni (E \neq \emptyset) \wedge (\exists x \in \mathbb{R} \ni \forall y \in E, x \leq y)$.

Theorem 1. $\inf E$ *exists.*

Proof. Let A be the set of all lower bounds of E, and let B be the set of all non-lower bounds of E. More precisely, let $A = \{x \in \mathbb{R} : \forall y \in E, x \leq y\}$, and let $B = \{x \in \mathbb{R} : \exists y \in E : x > y\}$.

Suppose $x \in \mathbb{R}$. Then, we have either $\exists y \in \mathbb{R} \ni x > y$ or we have $\neg(\exists y \in \mathbb{R} \ni x > y) = \forall y \in \mathbb{R} x \leq y$, i.e. either x is not a lower bound of E or it is. If the former, then by definition $x \in B$, and if the latter, then $x \in A$. Since this covers every possible element of $\mathbb{R}$, we have that $A \cup B = \mathbb{R}$.

Suppose $x \in E$. Then, since $x < x + 1$, we have that $\exists y \in E \ni y < x + 1$, and this y is our previously introduced x, therefore $x + 1 \in B$ since it is not a lower bound of E, and so $B \neq \emptyset$.

Since E is bounded below, we have that $\exists x \in \mathbb{R} \ni \forall y \in E, x \leq y$ by the definition of a lower bound, and this is exactly the definition of membership in A, so $x \in A$, and therefore $A \neq \emptyset$.

Now let $a \in A$ and $b \in B$. $b \in B \iff \exists y \in E \ni y < b$, so let $\beta \in E$ be such that $\beta < b$. $a \in A \iff \forall x \in E, a \leq x$, and since $\beta \in E$, we have $a \leq \beta$. Putting these together, we have $a \leq \beta < b$, and by transitivity of order we have $a < b$, therefore in general we have that $\forall a \in A, \forall b \in B, a < b$.

Since $A \cup B = \mathbb{R} \wedge A \neq \emptyset \wedge B \neq \emptyset \wedge \forall a \in A, b \in B, a < b$, we have that (A, B) is a Dedekind cut of $\mathbb{R}$.

By Dedekind's axiom, we have that for any Dedekind cut (X, Y) of $\mathbb{R}$, there exists some $z \in \mathbb{R} \ni \forall x \in X, \forall y \in Y, x \leq z \leq y$.

Therefore, there exists some $c \in \mathbb{R}$ such that $\forall a \in A, \forall b \in B, a \leq c \leq b$.

Assume that c is not a lower bound of E. Then, $\exists x \in E \ni x < c$. Let $y = \frac{x+c}{2}$. Then, $x < y < c$. Since $\exists x \in E \ni x < y$, we have $y \in B$ by definition, so $\exists b \in B \ni b < c$, however we have by Dedekind's axiom that $\forall b \in B, c \leq B$, which is a contradiction. Therefore c must be a lower bound of E.

Let x be some lower bound of E. Then, $\forall y \in E, x \leq y$, so $x \in A$. By Dedekind's axiom, we have that $\forall a \in A, a \leq c$, therefore $x \leq c$, so c must be the greatest lower bound of E, or in other words, $c = \inf E$.

Since we have constructed $\inf E$, we have demonstrated that $\inf E$ exists, and this proves theorem 1.

$\square$

2:

Suppose E_1, E_2 are nonempty subsets of $\mathbb{R}$ bounded above, where E_1 is a subset of E_2, i.e. $E_1 \subseteq \mathbb{R}, E_2 \subseteq \mathbb{R} \ni E_1 \subseteq E_2 \wedge E_1 \neq \emptyset \neq E_2 \wedge (\exists x \in \mathbb{R} \ni \forall y \in E_1, y \leq x) \wedge (\exists x \in \mathbb{R} \ni \forall y \in E_2, y \leq x)$.

Theorem 2. $\sup E_1 \leq \sup E_2$.

Proof. By the Dedekind property of $\mathbb{R}$, $\sup E_2$ exists. Let $\beta = \sup E_2$. Then, $\forall x \in E_2, x \leq \beta$.

Let $x \in E_1$. Since $E_1 \subseteq E_2 \iff y \in E_1 \implies y \in E_2$, we have $x \in E_2$, therefore $x \leq \beta$, so $\forall x \in E_1, x \leq \beta$, and we have that β is an upper bound of E_1.

By the Dedekind property of $\mathbb{R}$, $\sup E_1$ exists. Let $\alpha = \sup E_1$. By definition, α is the least upper bound of E_1, i.e. $(\forall x \in \mathbb{R} \ni \forall y \in E_1, x \geq y)(\alpha \leq x)$. Since β an upper bound of E_1, i.e. $\forall x \in E_1, \beta \geq x$, we must then have that $\alpha \leq \beta$, or in other words $\sup E_1 \leq \sup E_2$. This proves theorem 2. $\square$

3:

Let $A = \{x \in \mathbb{R} : x = 2 + \frac{1}{n} \text{ for some } n \in \mathbb{N}\}$.

Theorem 3. $\inf A = 2$

Proof. Let $n \in \mathbb{N}$. Then, $n > 0 \implies \frac{1}{n} > 0$, thus $2 + \frac{1}{n} > 2 + 0 = 2$. so $\forall n \in \mathbb{N}, 2 + \frac{1}{n} > 2$, therefore 2 must be a lower bound of A.

Let $\epsilon \in \mathbb{R} \ni \epsilon > 0$. By the archimedian property of $\mathbb{R}$, $\exists n \in \mathbb{N} \ni \epsilon > \frac{1}{n} > 0$, therefore $2+\epsilon > 2+\frac{1}{n} > 2$. $2+\frac{1}{n} \in A$ by definition, therefore $\exists x \in A \ni x < 2+\epsilon$ for all $\epsilon > 0$, where $x = 2 + \frac{1}{n}$. Since any arbitrarily small amount greater than 2 is not a lower bound of A, and since 2 is a lower bound of A, 2 must therefore be the greatest lower bound of A, or in other words, $2 = \inf A$. This proves theorem 3.

$\square$