Real Analysis Assignment 3

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October 14, 2020

Problem 1:

Claim 1. $\lim \frac{6n+5}{4n+7} = \frac{3}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of $\mathbb{R}$ we have $\exists N_\epsilon \in \mathbb{N} : N_\epsilon > \frac{1}{8}(\frac{11}{\epsilon} - 14)$. Then, $\forall n > N_\epsilon, n > \frac{1}{8}(\frac{11}{\epsilon})$. This inequality implies the following:

$$\frac{11}{8n+14} < \epsilon \tag{1}$$

$$\forall x < 0, |x| = -x \implies \frac{11}{8n+14} = |\frac{-11}{8n+14}| < \epsilon \tag{2}$$

$$|\frac{-11}{8n+14}| = |\frac{12n+10-12n-21}{8n+14}| = |\frac{2(6n+5)-3(4n+7)}{8n+14}| \tag{3}$$

$$|\frac{2(6n+5)-3(4n+7)}{8n+14}| = |\frac{6n+5}{4n+7} - \frac{3}{2}| < \epsilon \tag{4}$$

Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\frac{6n+5}{4n+7} - \frac{3}{2}| < \epsilon$, which is exactly the definition that $\lim \frac{6n+5}{4n+7} = \frac{3}{2}$. □

Problem 2:

Claim 2. $\lim \sqrt{9 - \frac{(-1)^n}{n}} = 3$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of $\mathbb{R}$, $\exists N_\epsilon \in \mathbb{N} : N_\epsilon > \frac{1}{3\epsilon}$, so $\forall n > N_\epsilon, n > \frac{1}{3\epsilon}$.

Assume for contradiction that $\frac{(-1)^n}{n} > 1$. Then, $(\frac{(-1)^n}{n})^2 = \frac{1}{n^2} > 1$, so $n^2 < 1$. Since $n > 0$, this implies that $n < 1$, but of course $n \geq 1$, which is a contradiciton, therefore $\frac{(-1)^n}{n} < 1 < 9$, so $9 - \frac{(-1)^n}{n} > 0$, and $\sqrt{9 - \frac{(-1)^n}{n}}$ exists.

Then, since $\sqrt{9 - \frac{(-1)^n}{n}} > 0$, we have:

$$\sqrt{9-\frac{(-1)^n}{n}}+3>3 \tag{5}$$

$$\frac{1}{\sqrt{9-\frac{(-1)^n}{n}}+3}<\frac{1}{3} \tag{6}$$

$$\frac{\frac{1}{n}}{\sqrt{9-\frac{(-1)^n}{n}}+3}=|\frac{\frac{-(-1)^n}{n}}{\sqrt{9-\frac{(-1)^n}{n}}+3}|<\frac{1}{3n} \tag{7}$$

$$|\frac{-\frac{(-1)^n}{n}}{\sqrt{9-\frac{(-1)^n}{n}}+3}|=|\frac{9-\frac{(-1)^n}{n}-9}{\sqrt{9-\frac{(-1)^n}{n}}+3}| \tag{8}$$

$$|\frac{9-\frac{(-1)^n}{n}-9}{\sqrt{9-\frac{(-1)^n}{n}}+3}|=|\frac{(\sqrt{9-\frac{(-1)^n}{n}})^2-3^2}{\sqrt{9-\frac{(-1)^n}{n}}+3}|=|\sqrt{9-\frac{(-1)^n}{n}}-3|. \tag{9}$$

By transitivity, $|\sqrt{9-\frac{(-1)^n}{n}}-3|<\frac{1}{3n}<\epsilon \implies |\sqrt{9-\frac{(-1)^n}{n}}-3|<\epsilon$. Thus we have demonstated that $\forall\epsilon\in\mathbb{R}:\epsilon>0,\exists N_\epsilon\in\mathbb{N}:\forall n\in\mathbb{N}:n>N_\epsilon,|\sqrt{9-\frac{(-1)^n}{n}}-3|<\epsilon$. which is exactly the definition that $\lim\sqrt{9-\frac{(-1)^n}{n}}=3$. □

Problem 3:

Claim 3. $\lim\frac{n}{\sqrt{4n^2+9n+1}}=\frac{1}{2}$

Proof. Let $\epsilon\in\mathbb{R}:\epsilon>0$. By the Archimedian property of $\mathbb{R}$, $\exists N_\epsilon:N_\epsilon>\frac{18}{8\epsilon}$. Then, $\forall n>N_\epsilon, n>\frac{18}{8\epsilon}$. We can then derive the following inequalities and equations:

$$n > \frac{18}{8\epsilon} \implies \frac{18}{8n} < \epsilon \tag{10}$$

$$\frac{9n+9n}{8n^2+18n} = \frac{18}{8n+18} < \frac{18}{8n} \tag{11}$$

$$\frac{9n+1}{2(4n^2+9n+1)} = \frac{9n+1}{8n^2+18n+2} < \frac{9n+9n}{8n^2+18n+2} < \frac{9n+9n}{8n^2+18n} \tag{12}$$

$$\frac{9n+1}{(\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} = \frac{9n+1}{2(4n^2+9n+1)} \tag{13}$$

$$\frac{(4n^2+9n+1)-4n^2}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} < \frac{9n+1}{(\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} \tag{14}$$

$$\frac{(4n^2+9n+1)-4n^2}{(2n+\sqrt{4n^2+9n+1})(2\sqrt{4n^2+9n+1})} = \frac{\sqrt{4n^2+9n+1}-2n}{(2\sqrt{4n^2+9n+1})} \tag{15}$$

$$\frac{\sqrt{4n^2+9n+1}-2n}{(2\sqrt{4n^2+9n+1})} = \frac{-(2n-\sqrt{4n^2+9n+1})}{2\sqrt{4n^2+9n+1}} \tag{16}$$

Now assume for a contradiction that $\sqrt{4n^2+9n+1} \leq 2n$. This inequality implies:

$$4n^2+9n+1 \leq 4n^2 \tag{17}$$

$$9n+1 \leq 0 \tag{18}$$

$$n \leq \frac{-1}{9} < 0 \tag{19}$$

But $n > 0$ since $n \in \mathbb{N}$, which is a contradiction, therefore $\sqrt{4n^2+9n+1} > 2n \implies 0 > 2n - \sqrt{4n^2+9n+1}$, so $|\frac{2n-\sqrt{4n^2+9n+1}}{2\sqrt{4n^2+9n+1}}| = \frac{-(2n-\sqrt{4n^2+9n+1})}{2\sqrt{4n^2+9n+1}}$. Finally, $|\frac{2n-\sqrt{4n^2+9n+1}}{2\sqrt{4n^2+9n+1}}| = |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}|$, and by the transitivity of the above order relations, we have $|\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \epsilon$. Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\frac{n}{\sqrt{4n^2+9n+1}} - \frac{1}{2}| < \epsilon$ which is exactly the definition that $\lim \frac{n}{\sqrt{4n^2+9n+1}} = \frac{1}{2}$. □

Problem 4:

Claim 4. $\lim(\sqrt{n^2+5n} - n) = \frac{5}{2}$

Proof. Let $\epsilon \in \mathbb{R} : \epsilon > 0$. By the Archimedian property of $\mathbb{R}$, $\exists N_\epsilon \in \mathbb{N} : N_\epsilon > \frac{25}{4\epsilon}$. Then, $\forall n > N_\epsilon, n > \frac{25}{4\epsilon}$. We have the following inequalities and equations:

$$n > \frac{25}{4\epsilon} \implies \frac{25}{4n} < \epsilon \tag{20}$$

$$\frac{5}{2} + \sqrt{n^2+5n} > 0 \implies \frac{\frac{25}{4}}{n+\frac{5}{2}+\sqrt{n^2+5n}} < \frac{25}{4n} \tag{21}$$

$$\frac{\frac{25}{4}}{n+\frac{5}{2}+\sqrt{n^2+5n}} = \frac{(n^2+5n)+\frac{25}{4}-(n^2+5n)}{n+\frac{5}{2}+\sqrt{n^2+5n}} \tag{22}$$

$$(a-b) = \frac{a^2-b^2}{a+b} \implies \frac{(n^2+5n)+\frac{25}{4}-(n^2+5n)}{n+\frac{5}{2}+\sqrt{n^2+5n}} = n+\frac{5}{2}-\sqrt{n^2+5n} \tag{23}$$

$$n+\frac{5}{2}-\sqrt{n^2+5n} = -\,(\sqrt{n^2+5n}-(n+\frac{5}{2}) \tag{24}$$

Now, assume for the purpose of contradiction that $n+\frac{5}{2} \leq \sqrt{n^2+5n}$. Then, we have:

$$n^2+5n+\frac{25}{4} \leq n^2+5n \tag{25}$$

$$\frac{25}{4} \leq 0 \tag{26}$$

But of course $\frac{25}{4} > 0$, which is a contradiction. Thus, $n+\frac{5}{2} > \sqrt{n^2+5n}$. So $-(\sqrt{n^2+5n}-(n+\frac{5}{2})) = |\sqrt{n^2+5n}-(n+\frac{5}{2})| = |\sqrt{n^2+5n}-n-\frac{5}{2})|$, and by transitivity of the above order relations, we have $|\sqrt{n^2+5n}-n-\frac{5}{2})| < \epsilon$. Thus we have demonstated that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |\sqrt{n^2+5n}-n-\frac{5}{2}| < \epsilon$, which is exactly the definition that $\lim(\sqrt{n^2+5n}-n) = \frac{5}{2}$. □

Problem 5:

Claim 5. $\lim(3+2(-1)^n) \neq 5$

Proof. Let $\epsilon = 1$, let $N \in \mathbb{N}$, and let $m = 2N+1$. Then, $|3-2(-1)^m-5| = |3-2(-1)^2(-1)-5| = |-4| = 4 \geq 1 = \epsilon$. Thus we have demonstrated that $\exists \epsilon \in \mathbb{R} : \epsilon > 0 : \forall N_\epsilon \in \mathbb{N}, \exists n \in \mathbb{N} : n > N_\epsilon : |3-2(-1)^n-5| \geq \epsilon \iff \neg(\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > N_\epsilon, |3-2(-1)^n-5| < \epsilon)$, which is the negation of $\lim(3+2(-1)^n) = 5$, and $\neg(\lim(3+2(-1)^n) = 5) \iff \lim(3+2(-1)^n) \neq 5$. □