Real Analysis Assignment 1

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1(a):
Suppose we have the following equation:

$$|x-1| + |x+2| = 0.5 \tag{1}$$

We have $x \in \mathbb{R}$ if and only if:

$$x \leq -2 \lor -2 \leq x \leq 1 \lor 1 \leq x \tag{2}$$

since $(-\infty, -2] \cup [-2, 1] \cup [1, \infty) = \mathbb{R}$.
Assume $x \leq -2$. Then, $x - 1 < 0 \implies |x-1| = -(x-1)$, and $x + 2 \leq 0 \implies |x+2| = -(x+2)$, so we can solve (1) as follows:

$$
\begin{aligned}
&-(x-1)-(x+2) &&= 0.5 \quad (3)\\
=&-x+1-x-2 &&= 0.5 \quad (4)\\
=&-2x-1 &&= 0.5 \quad (5)\\
\implies &-2x &&= 1.5 \quad (6)\\
\implies &x &&= \frac{1.5}{-2} = \frac{-3}{4} \quad (7)
\end{aligned}
$$

However, we have by assumption that $x \leq -2$, and by (7) we have that $x = \frac{-3}{4} > -2$, which is a contradition. Therefore, $x \leq -2$ is false for any real solution to (1).
Assume $-2 \leq x \leq 1$. Then, $x - 1 \leq 0 \implies |x-1| = -(x-1)$, and $x + 2 \geq 0 \implies |x+2| = x+2$, so we can solve (1) as follows:

$$
\begin{aligned}
&-(x-1)+x+2 &&= 0.5 \quad (8)\\
=&-x+1+x+2 &&= 0.5 \quad (9)\\
=&3 &&= 0.5 \quad (10)
\end{aligned}
$$

But of course, $3 = 0.5$ is absurd, therefore we reach a contradiction and conclude that $-2 \leq x \leq 1$ is false for any real solution to (1).

Assume $x \geq 1$: Then, $x - 1 \geq 0 \implies |x-1| = x-1$, and $x+2 > 0 \implies |x+2| = x+2$, so we can solve (1) as follows:

$$\begin{aligned}
x - 1 + x + 2 &= 0.5 && (11)\\
=2x + 1 &= 0.5 && (12)\\
\implies 2x &= -0.5 && (13)\\
\implies x &= \frac{-0.5}{2} = \frac{-1}{4} && (14)
\end{aligned}$$

However, we have by assumption that $x \geq 1$, and by (14) we have that $x = \frac{-1}{4} < 1$, which is a contradition. Therefore, $x \geq 1$ is false for any real solution to (1).

Finally, since we have $x \notin (-\infty, -2] \wedge x \notin [-2, 1] \wedge x \notin [1, \infty)$, we have $x \notin \mathbb{R} = (-\infty, -2] \cup [-2, 1] \cup [1, \infty)$, therefore we have demonstrated that there are no real solutions to (1).

1(b):

Suppose we have the following equation:

$$|x-1| + |x+2| = 3.5 \tag{15}$$

We have $x \in \mathbb{R}$ if and only if (2) holds since $(-\infty, -2] \cup [-2, 1] \cup [1, \infty) = \mathbb{R}$.

Assume $x \leq -2$. Then, $x - 1 < 0 \implies |x-1| = -(x-1)$, and $x + 2 \leq 0 \implies |x+2| = -(x+2)$, so we can solve (15) as follows:

$$\begin{aligned}
-(x-1) - (x+2) &= 3.5 && (16)\\
= -x + 1 - x - 2 &= 3.5 && (17)\\
= -2x - 1 &= 3.5 && (18)\\
\implies -2x &= 4.5 && (19)\\
\implies x &= \frac{4.5}{-2} = \frac{-9}{4} && (20)
\end{aligned}$$

Indeed, $x = \frac{-9}{4} \leq -2$ so this solution is consistent with our constaints and it must be a member of the solution set for (15).

Assume $-2 \leq x \leq 1$. Then, $x - 1 \leq 0 \implies |x-1| = -(x-1)$, and $x + 2 \geq 0 \implies |x+2| = x+2$, so we can solve (15) as follows:

$$\begin{aligned}
-(x-1) + x + 2 &= 3.5 && (21)\\
= -x + 1 + x + 2 &= 3.5 && (22)\\
=3 &= 3.5 && (23)
\end{aligned}$$

But of course, $3 = 3.5$ is absurd, therefore we reach a contradiction and conclude that $-2 \leq x \leq 1$ is false for any real solution to (15).

Assume $x \geq 1$: Then, $x - 1 \geq 0 \implies |x-1| = x - 1$, and $x + 2 > 0 \implies |x+2| = x+2$, so we can solve (15) as follows:

$$\begin{aligned} x - 1 + x + 2 &= 3.5 \quad (24) \\ =2x + 1 &= 3.5 \quad (25) \\ \implies 2x &= 2.5 \quad (26) \\ \implies x &= \frac{2.5}{2} = \frac{5}{4} \quad (27) \end{aligned}$$

Indeed, $x = \frac{5}{4} \geq 1$ so this solution is consistent with our constaints and it must be a member of the solution set for (15).

Finally, we can describe the real solutions to (15) by $x \in \{\frac{-9}{4}, \frac{5}{4}\}$.