Real Analysis Assignment 4

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Problem 1:

Claim 1. $\lim x_n = a > 0 \implies \lim \sqrt{x_n} = \sqrt{a}$

Proof. Assume that $\lim x_n = 0 > 0$. We have by the definition of $\lim x_n = a$ that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n > N_\epsilon, |x_n - a| < \epsilon$. Let $\epsilon > 0$. Then, there exists an $N_{\sqrt{a}\epsilon}$ such that $\forall n \in \mathbb{N} : n > N_{\sqrt{a}\epsilon}, |x_n - a| < \epsilon\sqrt{a}$. Thus, $\frac{|x_n - a|}{\sqrt{a}} < \epsilon$. Since $\sqrt{x_n} > 0$, we have $\frac{|x_n - a|}{\sqrt{x_n}+\sqrt{a}} < \frac{|x_n - a|}{\sqrt{a}}$. Then, since $\forall x > 0$ we have $|x| = x$, and since $\forall a, b \in \mathbb{R}, |a||b| = |ab|$, we see that $\frac{|x_n - a|}{\sqrt{x_n}+\sqrt{a}} = |\frac{x_n - a}{\sqrt{x_n}+\sqrt{a}}|$, and since $\forall a, b \in \mathbb{R}, a - b = \frac{a^2 - b^2}{a+b}$, we have $|\frac{x_n - a}{\sqrt{x_n}+\sqrt{a}}| = |\sqrt{x_n} - \sqrt{a}|$. Finally, since $|\sqrt{x_n} - \sqrt{a}| < \frac{|x_n - a|}{\sqrt{a}} < \epsilon$, we have by transitivity that $|\sqrt{x_n} - \sqrt{a}| < \epsilon$. Thus we have demonstrated that for $K_\epsilon = N_{\sqrt{a}\epsilon}, \forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists K_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > K_\epsilon, |\sqrt{x_n} - \sqrt{a}| < \epsilon$, which is exactly the definition that $\lim \sqrt{x_n} = \sqrt{a}$. Therefore, we have the implication $\lim x_n = a > 0 \implies \lim \sqrt{x_n} = \sqrt{a}$ $\square$

Problem 2:

Claim 2. $\lim x_n = 2 \implies \lim \frac{1}{x_n} = \frac{1}{2}$.

Proof. Assume $\lim x_n = 2$. We have by the definition of $\lim x_n = 2$ that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n > N_\epsilon, |x_n - 2| < \epsilon$. Let $\epsilon = 1$. Then, $\exists N_1 \in \mathbb{N} : \forall n \in \mathbb{N} : n < N_1, |x_n - 2| < 1$. By the reverse triangle inequality, $||x_n| - |2|| < |x_n - 2| < 1$, but $||x_n| - |2|| = |2 - |x_n||$, and by transitivity $|2 - |x_n|| < 1 \iff -1 < 2 - |x_n| < 1 \implies 1 < |x_n|$. Also note that $1 < |x_n| \implies 1 > \frac{1}{|x_n|}$. Let $\epsilon > 0$ and let $K_\epsilon = \max(\{N_1, N_{2\epsilon}\})$. Then, $|\frac{1}{x_n} - \frac{1}{2}| = \frac{|x_n - 2|}{2|x_n|}$, and since $K_\epsilon \geq N_1$, we have for all $n > K_\epsilon$ that $\frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2}$, and since $K_\epsilon \geq N_{2\epsilon}$, we have for all $n > K_\epsilon$ that $\frac{|x_n - 2|}{2} < \frac{2\epsilon}{2} = \epsilon$, and by transitivity $|\frac{1}{x_n} - \frac{1}{2}| < \frac{|x_n - 2|}{2} < \epsilon \implies |\frac{1}{x_n} - \frac{1}{2}| < \epsilon$. Thus we have demonstrated that for $K_\epsilon = \max(\{N_{2\epsilon}, N_1\})$, $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists K_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > K_\epsilon, |\frac{1}{x_n} - \frac{1}{2}| < \epsilon$. which is exactly the definition that $\lim \frac{1}{x_n} = \frac{1}{2}$. Therefore, we have the implication $\lim x_n = 2 \implies \lim \frac{1}{x_n} = \frac{1}{2}$. $\square$

Problem 3:

Claim 3. $\lim x_n = -1 \implies \lim |5x_n + 3| = 2$

Proof. Assume that $\lim x_n = -1$. We have by the definition of $\lim x_n = -1$ that $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : \forall n > N_\epsilon, |x_n + 1| < \epsilon$. Then, there exists an $N_{\frac{\epsilon}{5}}$ such that $\forall n \in \mathbb{N} : n > N_{\frac{\epsilon}{5}}, |x_n + 1| < \frac{\epsilon}{5}$. Thus, $|5x_n + 5| + |-2| - |2| = |5x_n + 5| = |5||x_n + 1| = 5|x_n + 1| < \epsilon$. By the triangle inequality, $|5x_n + 5 + (-2)| - |2| \leq |5x_n + 5| + |-2| - |2| < \epsilon$, and by the reverse triangle inequality, $||5x_n + 3| - 2| = ||5x_n + 3| - |2|| \leq |5x_n + 5 + (-2)| - |2|$, so by transitivity, $||5x_n + 3| - 2| < \epsilon$. Thus we have demonstrated that for $K_\epsilon = N_{\frac{\epsilon}{5}}$, $\forall \epsilon \in \mathbb{R} : \epsilon > 0, \exists K_\epsilon \in \mathbb{N} : \forall n \in \mathbb{N} : n > K_\epsilon, ||5x_n + 3| - 2| < \epsilon$, which is exactly the definition that $\lim |5x_n + 3| = 2$. Therefore, we have the implication that $\lim x_n = -1 \implies \lim |5x_n + 3| = 2$ □

It is not the case that $\lim |5x_n + 3| = 2 \implies \lim x_n = -1$.
As a counterexample, consider the sequence $x_n = \frac{-1}{5}$.
Then, $\lim |5x_n + 3| = \lim |5 \cdot \frac{-1}{5} + 3| = \lim |-1 + 3| = \lim |2| = \lim 2 = 2$.