# Real Analysis Assignment 5

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**Problem 1:**

**Claim 1.** *Every monotone decreasing bounded below sequence has a limit*

*Proof.* Let $x_n$ be a monotone decreasing sequence bounded below. Thus by definition, $\forall n, x_{n+1} \leq x_n$. Let $M$ be the lower bound of $x_n$, i.e.e $\forall n, x_n \geq M$. Since $x_n$ is a sequence of real numbers bounded below, by the completeness of the reals we have that there exists a greatest lower bound, an infimum, of the set $\{x_n : n \in \mathbb{N}\}$. Let $a = \inf\{x_n : n \in \mathbb{N}\}$. Let $\epsilon > 0$. Then, since $a$ is the greatest lower bound of $x_n$, there exists some $N$ such that $x_N < a + \epsilon$. Then, since $\forall n, x_{n+1} \leq x_n$, we must have that $\forall n > N, x_n \leq x_N$, and since $a$ is a lower bound of $x_n$, we have $\forall n, a \leq x_n$, and since $\epsilon > 0$, we of course have $a - \epsilon < a$. Putting this all together, we have that $\forall n > N, a - \epsilon < a \leq x_n \leq x_N < a + \epsilon$, thus by transitivity, $\forall n > N, a - \epsilon < x_n < a + \epsilon$, therefore $-\epsilon < x_n - a < \epsilon \iff |x_n - a| < \epsilon$. We have demonstrated that $\forall \epsilon > 0, \exists N : \forall n > N, |x_n - a| < \epsilon$, which is exactly the definition that $\lim x_n = a$, therefore the limit of $x_n$ exists, and every monotone decreasing bounded belows sequence has a limit. $\square$

**Problem 2:**

**Claim 2.** *The sequence* $a_n = \begin{cases} a_1 = 2 \\ a_{n+1} = \frac{a_n^2+1}{2a_n} & n \geq 1 \end{cases}$ *has a limit.*

*Proof.* First, note that $a_1 = 2 \geq 1$. Then, let $t \geq 0$. Noting that $\forall x \in \mathbb{R}, x^2 \geq 0$, we have the following:

$$(\sqrt{t} - \frac{1}{\sqrt{t}})^2 \geq 0 \tag{1}$$

$$t - 2 + \frac{1}{t} \geq 0 \tag{2}$$

$$t + \frac{1}{t} \geq 2 \tag{3}$$

Thus $\forall t \geq 0, t + \frac{1}{t} \geq 2$. Knowing this, we can determine a bound for $a_{n+1}$ as follows:

$$
\begin{aligned}
a_{n+1} =& \frac{a_n^2+1}{2a_n} && (4)\\
=& a_n \frac{a_n + \frac{1}{a_n}}{2a_n} && (5)\\
a_n \frac{2}{2a_n} \leq & a_n \frac{a_n + \frac{1}{a_n}}{2a_n} && (6)\\
1 \leq & \frac{a_n^2+1}{2a_n} && (7)\\
1 \leq & a_{n+1} && (8)
\end{aligned}
$$

Since $a_1 \geq 1 \wedge a_{n+1} \geq 1$, we have $\forall n, a_n \geq 1$, therefore 1 is a lower bound of $a_n$ and $a_n$ is bounded. Since $a_2 = \frac{a_1^2+1}{2a_1} = \frac{2^2+1}{2 \cdot 2} = \frac{5}{4}$, we see that $a_2 \leq a_1$. Then, since $\forall n, a_n \geq 1 \implies \forall n, a_n^2 \geq 1$:

$$
\begin{aligned}
a_{n+1} =& \frac{a_n^2+1}{2a_n} && (9)\\
\frac{a_n^2+a_n^2}{2a_n} \geq & \frac{a_n^2+1}{2a_n} && (10)\\
\frac{2a_n^2}{2a_n} \geq & \frac{a_n^2+1}{2a_n} && (11)\\
a_n \geq & a_{n+1} && (12)
\end{aligned}
$$

Therefore, $a_n$ is monotone decreasing. Since $a_n$ is bounded below and monotone decreasing, By the monotone conversion theorem, the limit of $a_n$ exists. □

Computation of $\lim a_n$

We know that $\lim a_n$ exists, so it makes sense to demote it by $a = \lim a_n$.

By the limit laws, $\lim a_{n+1} = \frac{\lim a_n^2+1}{2 \lim a_n}$

Since they differ by finitely many terms, $\lim a_{n+1} = \lim a_n$, so we can compute as follows:

$$
\begin{aligned}
a =& \frac{a^2+1}{2a} && (13)\\
2a^2 =& a^2+1 && (14)\\
a^2 =& 1 && (15)\\
\implies & a \in \{-1, 1\} && (16)
\end{aligned}
$$

Suppose, for a contradiction, that $\lim a_n = -1$. Then, $\forall \epsilon > 0 \exists N_\epsilon : \forall n > N_\epsilon, a_n \in (-1-\epsilon, -1+\epsilon)$, therefore for any $\epsilon > 0$ this last membership relation must be true for at least one $a_n$. Let $\epsilon = 1$. Then, some $a_n \in (-2, 0)$, but $\forall n, a_n \geq 1$, which is a contradiction, thus $\lim a_n \neq -1$ and it must be the case that $\lim a_n = 1$.

**Problem 3:**

**Claim 3.** 0 *is not a sub-sequential limit of* $x_n = (-1)^n - \frac{2}{n}$

*Proof.* Define the neighborhood $N = (\frac{-1}{2}, \frac{1}{2})$. Note that any $n$ is either even or odd, and if $n$ is even, then $(-1)^n = 1$, and if $n$ is odd, then $(-1)^n = -1$. Now let $n \in \mathbb{N}$. $\forall n \in \mathbb{N}, n > 0$ implies the following:

$$
\begin{aligned}
& n > 0 && (17)\\
\implies & \frac{1}{n} > 0 && (18)\\
\implies & \frac{-1}{n} < 0 && (19)\\
\implies & \frac{-2}{n} < 0 && (20)\\
\implies & -1 - \frac{2}{n} < -1 && (21)
\end{aligned}
$$

If $n$ is odd, then then we have $(-1)^n - \frac{2}{n} < -1$. Thus for all odd $n$ we have $x_n \notin N$. Now let $n \geq 4$ and note the following:

$$
\begin{aligned}
& n \geq 4 && (22)\\
\implies & -n \leq -4 && (23)\\
\implies & n \leq 2n - 4 && (24)\\
\implies & 1 \leq 2\frac{n-2}{n} && (25)\\
\implies & \frac{1}{2} \leq \frac{n-2}{n} && (26)\\
\implies & \frac{1}{2} \leq 1 - \frac{2}{n} && (27)
\end{aligned}
$$

If $n$ is even, then we have $(-1)^n - \frac{2}{n} \geq \frac{1}{2}$, thus for any even $n \geq 4$, we have $x_n \notin N$. Since $x_n \notin N$ for any odd $n$, we must also have $x_n \notin N$ for any odd $n > 4$, therefore $\forall n \geq 4, x_n \notin (0 - \frac{1}{2}, 0 + \frac{1}{2})$, and this demonstates that infinitely many $x_n$ fall outside a neighborhood of 0, so we conclude that finitely many $x_n$ are in $N$. This is true if and only if 0 is not a subsequential limit of $x_n$. □