Real Analysis Assignment 10

Joel Savitz

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Problem 1:
Define $f(x) = \frac{1}{x} + \cos x$ and $A = [1, \infty)$

Claim 1. *$f(x)$ is uniformly continuous on A.*

Proof. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in A, \forall y \in A, |x - y| < \delta_\epsilon \implies |f(x) - f(y)| < \epsilon$. We examine the consequent to find a suitable choice of δ_ϵ:

$$
\begin{aligned}
|f(x) - f(y)| =& |\frac{1}{x} + \cos x - (\frac{1}{y} + \cos y)| && (1) \\
=& |\frac{1}{x} - \frac{1}{y} + \cos x - \cos y| && (2) \\
\text{(triangle inequality)} \quad \leq& |\frac{1}{x} - \frac{1}{y}| + |\cos x - \cos y| && (3)
\end{aligned}
$$

Now we examine each of the last two terms individually. For the first:

$$
\begin{aligned}
|\frac{1}{x} - \frac{1}{y}| =& \frac{|x - y|}{xy} && (4) \\
(xy \geq 1) \quad \leq& |x - y| && (5)
\end{aligned}
$$

For the second:

$$
\begin{aligned}
|\cos x - \cos y| =& |-2\sin(\frac{x-y}{2})\sin(\frac{x-y}{2})| && (6) \\
(\forall x, \sin x \leq 1) \quad \leq& |2\sin(\frac{x-y}{2})| && (7) \\
(\forall x > 0, |\sin x| \leq |x|) \quad \leq& |2\frac{x-y}{2}| = |x - y| && (8)
\end{aligned}
$$

Now, we continue with the full examination:

$$
\begin{aligned}
|\frac{1}{x} - \frac{1}{y}| + |\cos x - \cos y| \leq& 2|x - y| && (9) \\
(\text{if } |x - y| < \delta) \quad <& 2\delta && (10) \\
(\text{if } \delta \leq \frac{\epsilon}{2}) \quad \leq& 2\frac{\epsilon}{2} = \epsilon && (11)
\end{aligned}
$$

Therefore, we can choose $\delta_\epsilon = \frac{\epsilon}{2}$, and we have: $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in A, \forall y \in A, |x - y| < \delta_\epsilon \implies |f(x) - f(y)| < \epsilon$, which is exactly the definition that $f(x)$ is uniformly continuous on $A = [1, \infty)$. $\square$

Problem 2:
Define $f(x) = \frac{1}{x} + \cos x$ and $A = (0, 1]$.

Claim 2. *$f(x)$ is not uniformly continuous on A.*

Proof. We want to show that: $\exists \epsilon > 0, \forall \delta > 0, \exists x \in A, y \in A, |x - y| < \delta \implies |f(x) - f(y)| \geq \epsilon$. We examine the consequent to find a suitable choice of ϵ.

$$\begin{aligned} |f(x) - f(y)| =& |\frac{1}{x} + \cos x - (\frac{1}{y} + \cos y)| && (12)\\ =& |\frac{1}{x} - \frac{1}{y} + \cos x - \cos y| && (13)\\ \text{(rev. triangle inequality)} \quad \geq& |\frac{1}{x} - \frac{1}{y}| - |\cos x - \cos y| && (14)\\ & && (15) \end{aligned}$$

Since $0 < \delta < \delta + 1$, we have $0 < \frac{\delta}{\delta+1} < 1$. Let $x = \frac{\delta}{\delta+1}$ and let $y = \frac{x}{2}$. This is valid since $|x - y| = \frac{\delta}{2(\delta+1)} < \delta$. We look at each term of the last expression to find bounds. For the first:

$$\begin{aligned} |\frac{1}{x} - \frac{1}{y}| =& \frac{|x-y|}{xy} && (16)\\ =& \frac{|x - \frac{x}{2}|}{x\frac{x}{2}} && (17)\\ =& \frac{\frac{x}{2}}{\frac{x^2}{2}} && (18)\\ =& \frac{1}{x} && (19)\\ (x < 1) \quad \geq& 1 && (20) \end{aligned}$$

For the second, we begin with the fact that cosine is monotone decreasing over A:

$$\begin{aligned} &\forall x \in A, \cos 0 = 1 \geq \cos x \geq \cos 1 && (21)\\ &\forall y \in A, -\cos 1 \geq -\cos y \geq -\cos 0 = -1 && (22)\\ &1 - \cos 1 \geq \cos x - \cos y \geq \cos 1 - 1 && (23)\\ &1 - \cos 1 \geq \cos x - \cos y \geq -(1 - \cos 1) && (24)\\ &1 - \cos 1 \geq |\cos x - \cos y| && (25) \end{aligned}$$

We conclude the examination as follows;

$$\begin{aligned} |\frac{1}{x} - \frac{1}{y}| - |\cos x - \cos y| \geq& 1 - (1 - \cos 1) && (26)\\ & \cos 1 = \epsilon && (27) \end{aligned}$$

To conclude, we have that $\exists \epsilon > 0, \forall \delta > 0, \exists x \in A, y \in A, |x - y| < \delta \implies |f(x) - f(y)| \geq \epsilon$. which is exactly the definition that $f(x)$ is not uniformly continuous on A. $\square$

Problem 3:

Suppose $f(x)$ and $g(x)$ are uniformly continuous on some interval E and both functions are bounded. Let $|f(x)| \leq A \wedge |g(x)| \leq B$.

Claim 3. *$f(x) \cdot g(x)$ is uniformly continuous on E.*

Proof. Since $f(x)$ is uniformly continuous on E, we have: $\forall \epsilon > 0, \exists \delta'_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta'_\epsilon \implies |f(x)-f(y)| < \epsilon$. Since $g(x)$ is uniformly continuous on E, we have: $\forall \epsilon > 0, \exists \delta''_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta''_\epsilon \implies |g(x)-g(y)| < \epsilon$. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta_\epsilon \implies |f(x)g(x) - f(y)g(y)| < \epsilon$, so we will examine the consequent to find a suitable value of δ_ϵ.

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| && (28)\\
\text{(triangle inequality)} \quad &\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| && (29)\\
&= |f(x)\big(g(x) - g(y)\big)| + |g(y)\big(f(x) - f(y)\big)| && (30)\\
\text{(use bounds)} \quad &\leq A|g(x) - g(y)| + B|f(x) - f(y)| && (31)
\end{aligned}$$

By the definitions above, we have $\forall x \in E, y \in E, \Big(|x-y| < \delta'_{\frac{\epsilon}{2B}} \implies |f(x) - f(y)| < \frac{\epsilon}{2B}\Big) \wedge \Big(|x-y| < \delta''_{\frac{\epsilon}{2A}} \implies |g(x) = g(y)| < \frac{\epsilon}{2A}\Big)$. Thus, if we choose $\delta_\epsilon = \min(\delta'_{\frac{\epsilon}{2B}} \delta''_{\frac{\epsilon}{2A}})$, we have:

$$\begin{aligned}
A|g(x) - g(y)| + B|f(x) - f(y)| &< A\frac{\epsilon}{2A} + B\frac{\epsilon}{2B} && (32)\\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && (33)
\end{aligned}$$

In conclusion, we have $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall x \in E, y \in E, |x-y| < \delta_\epsilon \implies |f(x)g(x) - f(y)g(y)| < \epsilon$, which is exactly the definition that $f(x) \cdot g(x)$ is uniformly continuous on E. $\square$