

Real Analysis Assignment 8

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Problem 1(a):

Claim 1.

$$\lim_{x \rightarrow 2} (x^2 + 3x) = 10 \quad (1)$$

Proof. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x - 2| < \delta_\epsilon, |x^2 + 3x - 10| < \epsilon$. We examine the consequent to find a value of delta that satisfies this implication:

$$|x^2 + 3x - 10| = |(x + 5)(x - 2)| \quad (2)$$

$$= |(x - 2 + 7)|(x - 2)| \quad (3)$$

$$\text{(triangle inequality:)} \leq (|x - 2| + 7)|x - 2| \quad (4)$$

$$\text{(if } |x - 2| < \delta) < (\delta + 7)\delta \quad (5)$$

$$\text{(if } \delta \leq 1) \leq (1 + 7)\delta \quad (6)$$

$$= 8\delta \quad (7)$$

$$\text{(if } \delta \leq \frac{\epsilon}{8}) \leq 8\frac{\epsilon}{8} = \epsilon \quad (8)$$

So, we can choose any $0 \leq \delta_\epsilon \leq \min(1, \frac{\epsilon}{8})$, and then for an arbitrary $\epsilon > 0$, we have $\forall 0 < |x - 2| < \delta_\epsilon, |x^2 + 3x - 10| < \epsilon$. In summary, we have that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x - 2| < \delta_\epsilon, |x^2 + 3x - 10| < \epsilon$, which is exactly the definition that $\lim_{x \rightarrow 2} (x^2 + 3x) = 10$. \square

Problem 1(b):

Claim 2.

$$\lim_{x \rightarrow -2} \left(\frac{x^2 - 5}{x - 1} \right) = \frac{1}{3} \quad (9)$$

Proof. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x + 2| < \delta_\epsilon, \left| \frac{x^2 - 5}{x - 1} - \frac{1}{3} \right| < \epsilon$.

We examine the consequent to find a value of delta that satisfies this implication:

$$\left| \frac{x^2 - 5}{x - 1} - \frac{1}{3} \right| = \left| \frac{3x^2 - x - 14}{3(x - 1)} \right| \quad (10)$$

$$= |x + 2| \frac{|3x - 7|}{3|x - 1|} \quad (11)$$

We grow the numerator and shrink the denominator to find a larger fraction we can work with:

$$|3x - 7| = |3x - 6 + 1| \leq |3x - 6| + 1 = |-3||x + 2| + 1 = 3|x + 2| + 1 \quad (12)$$

$$3|x - 1| = 3|x + 2 - 3| \geq 3(3 - |x + 2|) \quad (13)$$

Then, we continue with the above examination:

$$|x + 2| \frac{|3x - 7|}{3|x - 1|} \leq |x + 2| \frac{3|x + 2| + 1}{3(3 - |x + 2|)} \quad (14)$$

$$(\text{if } |x + 2| < \delta) < \delta \frac{3\delta + 1}{3(3 - \delta)} \quad (15)$$

$$(\text{if } \delta \leq 2) \leq \delta \frac{3(2) + 1}{3(3 - 2)} = \delta \frac{7}{3} \quad (16)$$

$$(\text{if } \delta \leq \frac{3\epsilon}{7}) \leq \frac{3\epsilon}{7} \cdot \frac{7}{3} = \epsilon \quad (17)$$

So, we can choose any $0 \leq \delta_\epsilon \leq \min(2, \frac{3\epsilon}{7})$, and then for an arbitrary $\epsilon > 0$, we have $\forall 0 < |x + 2| < \delta_\epsilon, |\frac{x^2 - 5}{x - 1} - \frac{1}{3}| < \epsilon$. In summary, we have that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x + 2| < \delta_\epsilon, |\frac{x^2 - 5}{x - 1} - \frac{1}{3}| < \epsilon$. which is exactly the definition that $\lim_{x \rightarrow -2} (\frac{x^2 - 5}{x - 1}) = \frac{1}{3}$. \square

Problem 2:

Claim 3. If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist, then:

$$\lim_{x \rightarrow a} (3f(x) - 4g(x)) = 3\lim_{x \rightarrow a} f(x) - 4\lim_{x \rightarrow b} g(x) \quad (18)$$

Proof. Let $A = \lim_{x \rightarrow a} f(x)$ and $B = \lim_{x \rightarrow a} g(x)$. By the definition of the limit of a function, we must have:

$$\forall \epsilon > 0, \exists \delta'_\epsilon > 0, \forall 0 < |x - a| < \delta'_\epsilon, |f(x) - A| < \epsilon \quad (19)$$

$$\forall \epsilon > 0, \exists \delta''_\epsilon > 0, \forall 0 < |x - a| < \delta''_\epsilon, |g(x) - B| < \epsilon \quad (20)$$

We want a value of δ_ϵ where $\forall x \in \mathbb{R}, 0 < |x - a| < \delta_\epsilon \implies |(3f(x) - 4g(x)) - (3A - 4B)| < \epsilon$. We can rewrite the consequent as $|3(f(x) - A) - 4(g(x) - B)|$, and by the triangle inequality, we have $|3(f(x) - A) - 4(g(x) - B)| \leq 3|f(x) - A| + 4|g(x) - B|$. By the definitions stated in (19) and (20), we have for δ'_ϵ that $3|f(x) - A| < 3\frac{\epsilon}{6} = \frac{\epsilon}{2}$, and we have for δ''_ϵ $4|g(x) - B| < 4\frac{\epsilon}{8} = \frac{\epsilon}{2}$, therefore, if we choose any $0 < \delta_\epsilon \leq \min(\delta'_\epsilon, \delta''_\epsilon)$, we have:

$$3|f(x) - A| + 4|g(x) - B| < 3\frac{\epsilon}{6} + 4\frac{\epsilon}{8} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (21)$$

Therefore, we have $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x - a| < \delta_\epsilon, |(3f(x) - 4g(x)) - (3A - 4B)| < \epsilon$, which is exactly the definition that $\lim_{x \rightarrow a} (3f(x) - 4g(x)) = 3\lim_{x \rightarrow a} f(x) - 4\lim_{x \rightarrow b} g(x)$. \square

Problem 3(a):

Claim 4. $\lim_{x \rightarrow \infty} \frac{\cos^2 x}{x^2} = 0$

Proof. We want to show that $\forall \epsilon > 0, \exists \alpha_\epsilon \in \mathbb{R}, \forall x > \alpha_\epsilon, \left| \frac{\cos^2 x}{x^2} \right| < \epsilon$.

We examine $\left| \frac{\cos^2 x}{x^2} \right|$ and see that it is simply the square of a real number at every x , so we know it is always positive, thus $\left| \frac{\cos^2 x}{x^2} \right| = \frac{\cos^2 x}{x^2}$. Then, since $\forall x, -1 \leq \cos x \leq 1$, we have $\cos^2 x \leq 1$, so $\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$. Since we must have $x > \alpha_\epsilon$, we then see that $\frac{1}{x} < \frac{1}{\alpha_\epsilon}$, thus $\frac{1}{x^2} < \frac{1}{\alpha_\epsilon^2}$. If we choose $\alpha_\epsilon = \sqrt{1/\epsilon}$, then $\frac{1}{x^2} < \frac{1}{\sqrt{1/\epsilon}^2} = \epsilon$. In summary, we have demonstrated that $\forall \epsilon > 0, \exists \alpha_\epsilon \in \mathbb{R}, \forall x > \alpha_\epsilon, \left| \frac{\cos^2 x}{x^2} \right| < \epsilon$, which is exactly the definition that $\lim_{x \rightarrow \infty} \frac{\cos^2 x}{x^2} = 0$. \square

Problem 3(b):

Claim 5. $\lim_{x \rightarrow \infty} \cos x \neq \frac{1}{2}$

Proof. Let $\epsilon = \frac{1}{2}$ and let $\alpha \in \mathbb{R}$. By the archimedian property, $\exists n > \alpha$. Then, let $x = 2\pi n > n > \alpha$. Since $\forall n, \cos(2\pi n) = 1$, we see that $\left| \cos x - \frac{1}{2} \right| = \frac{1}{2} \geq \epsilon$. We have demonstrated that $\exists \epsilon > 0, \forall \alpha \in \mathbb{R}, \exists x > \alpha, \left| \cos x - \frac{1}{2} \right| \geq \epsilon$, and this is exactly the definition that $\lim_{x \rightarrow \infty} \cos x \neq \frac{1}{2}$. \square