Real Analysis Assignment 8

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Problem 1(a):

Claim 1.

$$\lim_{x \to 2}(x^2 + 3x) = 10 \tag{1}$$

Proof. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x-2| < \delta_\epsilon, |x^2+3x-10| < \epsilon$. We examine the consequent to find a value of delta that satisifes this implication:

$$\begin{aligned}
|x^2 + 3x - 10| =& |(x+5)(x-2)| && (2)\\
=& |(x-2+7)||(x-2)| && (3)\\
\text{(triangle inequality:)} \quad \leq& (|x-2|+7)|x-2| && (4)\\
\text{(if } |x-2| < \delta) \quad <& (\delta+7)\delta && (5)\\
\text{(if } \delta \leq 1) \quad \leq& (1+7)\delta && (6)\\
=& 8\delta && (7)\\
\text{(if } \delta \leq \frac{\epsilon}{8}) \quad \leq& 8\frac{\epsilon}{8} = \epsilon && (8)
\end{aligned}$$

So, we can choose any $0 < \delta_\epsilon \leq \min(1, \frac{\epsilon}{8})$, and then for an arbitrary $\epsilon > 0$, we have $\forall 0 < |x-2| < \delta_\epsilon, |x^2 + 3x - 10| < \epsilon$. In summary, we have that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x-2| < \delta_\epsilon, |x^2 + 3x - 10| < \epsilon$, which is exactly the definition that $\lim_{x \to 2}(x^2 + 3x) = 10$. □

Problem 1(b):

Claim 2.

$$\lim_{x \to -2}\left(\frac{x^2-5}{x-1}\right) = \frac{1}{3} \tag{9}$$

Proof. We want to show that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x+2| < \delta_\epsilon, |\frac{x^2-5}{x-1} - \frac{1}{3}| < \epsilon$.

We examine the consequent to find a value of delta that satisifes this implication:

$$\begin{aligned}
\left|\frac{x^2-5}{x-1} - \frac{1}{3}\right| =& \left|\frac{3x^2 - x - 14}{3(x-1)}\right| && (10)\\
=& |x+2|\frac{|3x-7|}{3|x-1|} && (11)
\end{aligned}$$

We grow the numerator and shrink the denominator to find a larger fraction we can work with:

$$|3x-7| = |3x-6+1| \leq |3x-6|+1 = |-3||x+2|+1 = 3|x+2|+1 \quad (12)$$
$$3|x-1| = 3|x+2-3| \geq 3(3-|x+2|) \quad (13)$$

Then, we continue with the above examination:

$$|x+2|\frac{|3x-7|}{3|x-1|} \leq |x+2|\frac{3|x+2|+1}{3(3-|x+2|)} \quad (14)$$
$$(\text{if } |x+2|<\delta) \quad < \delta\frac{3\delta+1}{3(3-\delta)} \quad (15)$$
$$(\text{if } \delta \leq 2) \quad \leq \delta\frac{3(2)+1}{3(3-2)} = \delta\frac{7}{3} \quad (16)$$
$$(\text{if } \delta \leq \frac{3\epsilon}{7}) \quad \leq \frac{3\epsilon}{7}\cdot\frac{7}{3} = \epsilon \quad (17)$$

So, we can choose any $0 < \delta_\epsilon \leq \min(2, \frac{3\epsilon}{7})$, and then for an arbitrary $\epsilon > 0$, we have $\forall 0 < |x+2| < \delta_\epsilon, |\frac{x^2-5}{x-1} - \frac{1}{3}| < \epsilon$. In summary, we have that $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x+2| < \delta_\epsilon, |\frac{x^2-5}{x-1} - \frac{1}{3}| < \epsilon$. which is exactly the definition that $\lim_{x\to -2}(\frac{x^2-5}{x-1}) = \frac{1}{3}$. □

Problem 2:

Claim 3. *If* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist, then:*

$$\lim_{x\to a}(3f(x)-4g(x)) = 3\lim_{x\to a} f(x) - 4\lim_{x\to b} g(x) \quad (18)$$

Proof. Let $A = \lim_{x\to a} f(x)$ and $B = \lim_{x\to a} g(x)$. By the definition of the limit of a function, we must have:

$$\forall \epsilon > 0, \exists \delta'_\epsilon > 0, \forall 0 < |x-a| < \delta'_\epsilon, |f(x)-A| < \epsilon \quad (19)$$
$$\forall \epsilon > 0, \exists \delta''_\epsilon > 0, \forall 0 < |x-a| < \delta''_\epsilon, |g(x)-B| < \epsilon \quad (20)$$

We want a value of δ_ϵ where $\forall x \in \mathbb{R}, 0 < |x-a| < \delta_\epsilon \implies |(3f(x)-4g(x))-(3A-4B)$. We can rewrite the consequent as $|3(f(x)-A)-4(g(x)-B)|$, and by the triangle inequality, we have $|3(f(x)-A)-4(g(x)-B)| \leq 3|f(x)-A|+4|g(x)-B|$. By the definitions stated in (19) and (20), we have for $0 < |x-a| < \delta'_{\frac{\epsilon}{6}}$ that $3|f(x)-A| < 3\frac{\epsilon}{6} = \frac{\epsilon}{2}$, and we have for $0 < |x-a| < \delta''_{\frac{\epsilon}{8}}$ that $4|g(x)-B| < 4\frac{\epsilon}{8} = \frac{\epsilon}{2}$, therefore, if we choose any $0 < \delta_\epsilon \leq \min(\delta'_{\frac{\epsilon}{6}}, \delta''_{\frac{\epsilon}{8}})$, we have:

$$3|f(x)-A| + 4|g(x)-B| < 3\frac{\epsilon}{6} + 4\frac{\epsilon}{8} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (21)$$

Therefore, we have $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall 0 < |x-a| < \delta_\epsilon, |(3f(x)-4g(x))-(3A-4B)| < \epsilon$, which is exactly the definition that $\lim_{x\to a}(3f(x)-4g(x)) = 3\lim_{x\to a} f(x) - 4\lim_{x\to b} g(x)$. □

Problem 3(a):

Claim 4. $\lim_{x \to \infty} \frac{\cos^2 x}{x^2} = 0$

Proof. We want to show that $\forall \epsilon > 0, \exists \alpha_\epsilon \in \mathbb{R}, \forall x > \alpha_\epsilon, |\frac{\cos^2 x}{x^2}| < \epsilon$.

We examine $|\frac{\cos^2 x}{x^2}|$ and see that it is simply the square of a real number at every x, so we know it is always positive, thus $|\frac{\cos^2 x}{x^2}| = \frac{\cos^2 x}{x^2}$. Then, since $\forall x, -1 \leq \cos x \leq 1$, we have $\cos^2 x \leq 1$, so $\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$. Since we must have $x > \alpha_\epsilon$, we then see that $\frac{1}{x} < \frac{1}{\alpha_\epsilon}$, thus $\frac{1}{x^2} < \frac{1}{\alpha_\epsilon^2}$. If we choose $\alpha_\epsilon = \sqrt{1/\epsilon}$, then $\frac{1}{x^2} < \frac{1}{\sqrt{1/\epsilon}^2} = \epsilon$. In summary, we have demonstrated that $\forall \epsilon > 0, \exists \alpha_\epsilon \in \mathbb{R}, \forall x > \alpha_\epsilon, |\frac{\cos^2 x}{x^2}| < \epsilon$, which is exactly the definition that $\lim_{x \to \infty} \frac{\cos^2 x}{x^2} = 0$. □

Problem 3(b):

Claim 5. $\lim_{x \to \infty} \cos x \neq \frac{1}{2}$

Proof. Let $\epsilon = \frac{1}{2}$ and let $\alpha \in \mathbb{R}$. By the archimedian property, $\exists n > \alpha$. Then, let $x = 2\pi n > n > \alpha$. Since $\forall n, \cos(2\pi n) = 1$, we see that $|\cos x - \frac{1}{2}| = \frac{1}{2} \geq \epsilon$. We have demonstrated that $\exists \epsilon > 0, \forall \alpha \in \mathbb{R}, \exists x > \alpha, |\cos x - \frac{1}{2}| \geq \epsilon$, and this is exactly the definition that $\lim_{x \to \infty} \cos x \neq \frac{1}{2}$. □