Real Analysis Assignment 12

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Problem 1:

Define $f(x) = \begin{cases} -3, & x = 4 \\ 4, & x = 5 \\ 0, & x \in [3,6], x \neq 4, x \neq 5 \end{cases}$ on the interval $[3, 6]$.

Claim 1. *f is Riemann integrable over $[3, 6]$ and $\int\limits_{[3,6]} f = 0$.*

Proof. Let $\delta > 0$ be small, and define the partition $P = \{3, 4-\delta, 4+\delta, 5-\delta, 5+\delta, 6\}$. We calculate the lower sum:

$$\begin{aligned} L(f, P) &= \sum_{j=1}^{5} \inf_{x \in I_j} f(x) \cdot \Delta x_j && (1) \\ &= 0(4-\delta-3) - 3(4+\delta-4+\delta) + 0(5-\delta-4-\delta) \\ &+ 0(5+\delta-5+\delta) + 0(6-5-\delta) && (2) \\ &= -6\delta && (3) \end{aligned}$$

And the upper sum:

$$\begin{aligned} U(f, P) &= \sum_{j=1}^{5} \sup_{x \in I_j} f(x) \cdot \Delta x_j && (4) \\ &= 0(4-\delta-3) + 0(4+\delta-4+\delta) + 0(5-\delta-4-\delta) \\ &+ 4(5+\delta-5+\delta) + 0(6-5-\delta) && (5) \\ &= 8\delta && (6) \end{aligned}$$

Let $\epsilon > 0$ and impose the constraint that $\delta < \frac{\epsilon}{14}$. Then, $U(f, P) - L(f, P) = 8\delta - (-6\delta) = 14\delta < 14\frac{\epsilon}{14} = \epsilon$. Therefore, $\forall \epsilon > 0, \exists P_\epsilon = P \in \mathcal{P}([3, 6]), U(f, P_\epsilon) - L(P, \epsilon) < \epsilon$, so f satisfies Riemann's condition on the interval $[3, 6]$. This is true if and only if f is Riemann integrable over $[3, 6]$. Then, since $L(f, P) \leq L(f) \leq U(f) \leq U(f, P)$, we have for a small $\delta > 0$ that $-6\delta \leq L(f) \leq U(f) \leq 8\delta$, and since when δ is arbitrarily small, -6δ and 8δ will be arbitrarily close to 0, so $0 \leq L(f) \leq U(f) \leq 0$, implying that $L(f) = U(f) = 0$ and equivalently that $\int\limits_{[3,6]} f = 0$. $\square$

Problem 2:

Define $f(x) = \begin{cases} 3, & 1 \leq x < 2 \\ 2, & 2 \leq x \leq 4 \\ 4, & 4 < x \leq 5 \end{cases}$ on the interval $[1, 5]$.

Claim 2. *f is Riemann integrable on $[1, 5]$ and* $\int\limits_{[1,5]} f = 11$.

Proof. By inspection, $\forall x \in [1, 5], |f(x)| \leq 4$, therefore f is bounded on $[1, 5]$. Define the finite sequence of intervals $a_1 = (2 - \frac{\epsilon}{8}, 2 + \frac{\epsilon}{8})$ and $a_2 = (4 - \frac{\epsilon}{8}, 4 + \frac{\epsilon}{8})$. Then, the set of discontinuities of f in $[1, 5]$, i.e. $\{2, 4\}$, is a subset of $a_1 \cup a_2$. Furthermore, $(2 + \frac{\epsilon}{8} - (2 - \frac{\epsilon}{8})) + (4 + \frac{\epsilon}{8} - (4 - \frac{\epsilon}{8})) = \frac{\epsilon}{2} < \epsilon$, therefore the set of discontinuities of f on $[1, 5]$ has Lebesgue measure 0 and is bounded on that same interval. This is true if and only if f is Riemann integrable on $[1, 5]$.

Let $\delta > 0$ be small and define the partition $P = \{1, 2-\delta, 2+\delta, 4-\delta, 4+\delta, 5\}$. We calculate the lower sum:

$$\begin{aligned} L(f, P) &= \sum_{j=1}^{5} \inf_{x \in I_j} f(x) \cdot \Delta x_j & (7) \\ &= 3(2 - \delta - 1) + 2(2 + \delta - 2 + \delta) + 2(4 - \delta - 2 - \delta) \\ &\quad + 2(4 + \delta - 4 + \delta) + 4(5 - 4 - \delta) & (8) \\ &= 11 - 3\delta & (9) \end{aligned}$$

And the upper sum:

$$\begin{aligned} U(f, P) &= \sum_{j=1}^{5} \sup_{x \in I_j} f(x) \cdot \Delta x_j & (10) \\ &= 3(2 - \delta - 1) + 3(2 + \delta - 2 + \delta) + 2(4 - \delta - 2 - \delta) \\ &\quad + 4(4 + \delta - 4 + \delta) + 4(5 - 4 - \delta) & (11) \\ &= 11 + 3\delta & (12) \end{aligned}$$

As in the previous problem, we have $L(f, P) \leq L(f) \leq U(f) \leq U(f, P)$, so for an arbitrarily small $\delta > 0$, we have $11 - 3\delta \leq L(f) \leq U(f) \leq 11 + 3\delta$ therefore $11 = L(f) = U(f)$, or equivalently $\int\limits_{[1,5]} f = 11$. $\square$