Real Analysis Assignment 11

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Problem 1:
Define $f_n(x) = \frac{x^2}{2x^2+(nx-3)^2}$.

If $x = 0$, then $\lim\limits_{n\to\infty} f_n(x) = 0$ since the function is a fraction with numerator 0. If $x \neq 0$, then $\lim\limits_{n\to\infty} f_n(x) = 0$ since the expression is a fraction with a nonzero numerator and a denominator with a positive term value that grows arbitrarily as n increases without bound. Thus, $\lim\limits_{n\to\infty} f_n(x) = f(x) = 0$.

Claim 1. *$f_n(x)$ does not converge uniformly to $f(x)$ on $\mathbb{R}$.*

Proof. We want to show that $f_n(x)$ does not converge uniformly to $f(x)$ on $\mathbb{R}$, and we will do so by showing that the negation of the definition of uniform convergence is satisfied by $f_n(x)$. That is to say, $\exists\epsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \exists x \in \mathbb{R}, |f_n(x) - f(x)| \geq \epsilon$. Let $\epsilon = \frac{1}{2}$ and $N \in \mathbb{N}$. Then, $\exists n > N$. Let $x = \frac{n}{3}$. Then, $|f_n(x) - f(x)| = |\frac{(3/n)^2}{2(3/n)^2+(n(3/n)-3)^2}| = \frac{1}{2} \geq \epsilon$, so we have that $f_n(x)$ does not converge uniformly to $f(x)$ on $\mathbb{R}$. □

Problem 2:
Define $f_n(x) = \frac{x^2}{2x^2+(nx-3)^2}$.

Claim 2. *$f_n(x)$ converges uniformly to $f(x)$ on $[\frac{1}{3}, \infty)$.*

Proof. We verify the definition of uniform convergence, that $\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n > N_\epsilon, \forall x \in [\frac{1}{3}, \infty), |f_n(x) - f(x)| < \epsilon$. Let $\epsilon > 0$. We examine the consequent to find a suitable value of N_ϵ. Let $n > N_\epsilon$:

$$
\begin{aligned}
|f_n(x) - f(x)| &= |\frac{x^2}{2x^2+(nx-3)^2}| && (1)\\
\text{(positive num. and denom.)} \quad &= \frac{x^2}{2x^2+(nx-3)^2} && (2)\\
&< \frac{x^2}{(nx-3)^2} && (3)\\
&= \frac{x^2}{n^2x^2-6nx+9} && (4)\\
&< \frac{x^2}{n^2x^2-6nx} && (5)
\end{aligned}
$$

To shrink the denominator further, we will replace $6xn$ with a smaller positive term, and this will introduce a constraint on N_ϵ. We use $\frac{1}{2}n^2x^2$:

$$6nx < \frac{1}{2}n^2x^2 \tag{6}$$
$$6 < \frac{1}{2}nx \tag{7}$$
$$12 < nx \tag{8}$$

But since $x \geq \frac{1}{3}$, we $12 < \frac{n}{3} \implies 36 < n$. We have the constraint $N_\epsilon \geq 36$. Now, we continue with the full examination:

$$\frac{x^2}{n^2x^2 - 6nx} < \frac{x^2}{n^2x^2 - \frac{1}{2}n^2x^2} \tag{9}$$
$$= \frac{x^2}{\frac{1}{2}n^2x^2} \tag{10}$$
$$= \frac{2}{n^2} \tag{11}$$

If $n > \sqrt{2/\epsilon}$, then $\frac{n^2}{2} > \frac{1}{\epsilon}$, so $\frac{2}{n^2} < \epsilon$. By the archimedian propety, we can choose some $N_\epsilon > \max(36, \sqrt{2/\epsilon})$, and we have $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n > N_\epsilon, \forall x \in [\frac{1}{3}, \infty), |f_n(x) - f(x)| < \epsilon$. This is exactly the definition that $f_n(x)$ converges uniformly to $f(x)$ on $[\frac{1}{3}, \infty)$. $\square$

Problem 3:
Define $f_n(x) = \frac{x^{2n}-1}{x^{2n}+1}$.
If $x = 0$, then $f_n(x) = \frac{0-1}{0+1} = -1$, so $\lim_{n\to\infty} f_n(x) = 0$.
If $0 < |x| < 1$, then by the limit laws, $\lim_{n\to\infty} f_n(x) = \frac{\lim_{n\to\infty}(x^2)^n - 1}{\lim_{n\to\infty}(x^2)^n + 1} = \frac{0-1}{0+1} = -1$.
If $|x| = 1$, then $f_n(x) = \frac{1-1}{1+1} = 0$.
If $|x| > 1$, then $\lim_{n\to\infty} f_n(x)$ looks like it will be 1. To show this, we see that $|f_n(x) - 1| = |\frac{x^{2n}-1}{x^{2n}+1} - \frac{x^{2n}+1}{x^{2n}+1}| = \frac{2}{x^{2n}+1} < \frac{2}{x^{2n}}$. Let $\epsilon > 0$. If we set $n > N_\epsilon > \log_x(\sqrt{2/\epsilon})$, then $2n > \log_x(2/\epsilon)$ and $x^{2n} > \frac{2}{\epsilon}$, so $\frac{2}{x^{2n}} < \epsilon$, and we have that $\forall \epsilon > 0, \exists N_\epsilon, \forall n > N_\epsilon, |f_n(x) - 1| < \epsilon$, so indeed $|x| > 1 \implies \lim_{n\to\infty} f_n(x) = f(x) = 1$.

We have the following function: $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} -1 & (\text{if } |x| < 1) \\ 0 & (\text{if } |x| = 1) \\ 1 & (\text{if } |x| > 1) \end{cases}$.

Claim 3. *$f_n(x)$ converges to $f(x)$ uniformly on $A = [\frac{-1}{3}, \frac{1}{3}]$.*

Proof. Since $\forall x \in A, |x| < 1$, we have $f(x) = -1$. We examine $|f_n(x) - f(x)|$:

$$\begin{aligned}
|f_n(x) - f(x)| &= |\frac{x^{2n}-1}{x^{2n}+1} + 1| && (12)\\
&= |\frac{x^{2n}-1}{x^{2n}+1} + \frac{x^{2n}+1}{x^{2n}+1}| && (13)\\
&= |\frac{2x^{2n}}{x^{2n}+1}| && (14)\\
\text{(positive num. and denom.)} \quad &= \frac{2x^{2n}}{x^{2n}+1} && (15)\\
(x^{2n}+1 > 1) \quad &< 2x^{2n} && (16)
\end{aligned}$$

Since $\forall x \in A, |x| \leq \frac{1}{3} < \frac{1}{2}$, we have $2x^{2n} \leq 2\cdot 3^{-2n} < 2\cdot 3^{-2n}$, and $\lim_{n\to\infty} 2\cdot 3^{-2n} = 0$. Thus $\exists c_n = 2\cdot 3^{-2n}$ where $\forall x \in A, |f_n(x) - f(x)| < c_n$, and $\lim_{n\to\infty} c_n = 0$, which is true if and only if $f_n(x)$ converges to $f(x)$ uniformly on $A = [\frac{-1}{3}, \frac{1}{3}]$. □

Claim 4. *$f_n(x)$ does not converge uniformly to $f(x)$ on $A = (0, 1)$.*

Proof. Consider $|f_n(x) - f(x)|$ on A. Since $A = (0, 1)$, we have $|x| < 1$, so $f(x) = -1$. Since $x^{2n} < 1$:

$$\begin{aligned}
|f_n(x) - f(x)| &= |\frac{x^{2n}-1}{x^{2n}+1} + 1| && (17)\\
&= |\frac{2x^{2n}}{x^{2n}+1}| && (18)\\
&< |\frac{2x^{2n}}{x^{2n}+x^{2n}}| && (19)\\
&= |\frac{2x^{2n}}{2x^{2n}}| && (20)\\
&= 1 && (21)
\end{aligned}$$

So 1 is an upper bound of $|f_n(x) - f(x)|$. Let $1 > \epsilon > 0$ and let $x = (1-\epsilon)^{1/2n}$. Then:

$$\begin{aligned}
|f_n(x) - f(x)| &= |\frac{x^{2n}-1}{x^{2n}+1} + 1| && (22)\\
&= \frac{2x^{2n}}{x^{2n}+1} && (23)\\
&= \frac{2(1-\epsilon)}{(1-\epsilon)+1} && (24)\\
&= \frac{2-2\epsilon}{2-\epsilon} && (25)
\end{aligned}$$

We verify that (25) is greater than $1 - \epsilon$:

$$\frac{2 - 2\epsilon}{2 - \epsilon} > 1 - \epsilon \tag{26}$$

$$2 - 2\epsilon > (1 - \epsilon)(2 - \epsilon) = 2 - 3\epsilon + \epsilon^2 \tag{27}$$

$$\epsilon > \epsilon^2 \tag{28}$$

This last inequality is true since $0 < \epsilon < 1$, so $1 - \epsilon$ is not an upper bound of $|f_n(x) - f(x)|$, and we have $\sup\limits_{x \in A} |f_n(x) - f(x)| = 1$. Then, $\lim\limits_{n \to \infty} \left(\sup\limits_{x \in A} |f_n(x) - f(x)| \right) = 1 \neq 0$, which is true if and only if $f_n(x)$ does not converge uniformly to $f(x)$ on $A = (0, 1)$. $\square$