**Homework 7**
Due Wednesday, July 8

1. Chapter 14, Exercise A3 (p. 141)
2. Chapter 14, Exercise B2 (p. 142)
3. Chapter 14, Exercise B4 (p. 142)
4. Chapter 14, Exercise C2 (p. 143)
5. Chapter 14, Exercise C4 (p. 143)
6. Chapter 14, Exercise D1 (p. 143) (Do part (a) only)
7. Chapter 14, Exercise D3 (p. 143) (The center of a group - using the standard notation - is defined as $Z(G) = \{a \in G : ax = xa,\ \forall x \in G\}$; see also Homework 2, Problem 12)
8. Show that $G$ is an Abelian group $\iff$ $f : G \to G$ defined by $f(a) = a^{-1}$ is a homomorphism.
9. Define the mapping $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$, by $\phi(A) = \det(A)$. Show that $\phi$ is a homomorphism, and find $\ker(\phi)$.
10. Show that the set $H = \{X \in GL_2(\mathbb{R}) :\ \det(X) > 0\}$ is a normal subgroup of $GL_2(\mathbb{R})$.
11. Show that the set $H = \{X \in GL_2(\mathbb{R}) :\ X = xI, x \neq 0\}$ is a normal subgroup of $GL_2(\mathbb{R})$.
12. Show that $A_4$ is a normal subgroup of $S_4$.
13. Let $K = \{e, (12)(34)\}$ and $H = \{e, (12)(34), (13)(24), (14)(23)\}$ be subgroups of $S_4$. Show that $K$ is a normal subgroup of $H$, but $K$ is NOT a normal subgroup of $S_4$.
14. Recall the automorphism of a group $G$, which is the set of all isomorphisms from $G$ to $G$,

$$\text{Aut}(G) = \{f \text{ is an isomorphism, } f : G \to G\} = \{f \in S_G :\ f(ab) = f(a)f(b)\}$$

(see Chapter 9, Exercise I, and June 11 lecture notes).

   (a) From Homework 4, Exercise 5, we saw that, for any $a \in G$, the mapping $\pi_a : G \to G$ defined by $\pi_a(x) = axa^{-1}$ is an element of $\text{Aut}(G)$, i.e., $\pi_a \in \text{Aut}(G)$ for all $a \in G$. Show that the mapping $\phi : G \to \text{Aut}(G)$, defined by $\phi(a) = \pi_a$, is a homomorphism. (Remember that $\text{Aut}(G)$ is a group with the composition operation).

   (b) Define $H = \{\pi_a \in \text{Aut}(G) :\ \pi_a(x) = axa^{-1}, a \in G\}$. Show that $H$ is a normal subgroup of $\text{Aut}(G)$.

   (c) Show that $\ker(\phi) = \{a \in G :\ \phi(a) = \epsilon\} = \{a \in G : ax = xa,\ \forall x \in G\}$ (where $\epsilon$ is the identity element of $\text{Aut}(G)$).