Abstract Algebra: Homework #3

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Note: for the scope of this document, let $\ni$ denote "such that".

1 Chapter 7, Exercise A1

Suppose we have some $f, g, h \in S_6$ such that:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 5 & 4 & 2 \end{pmatrix} \tag{1}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix} \tag{2}$$

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 5 & 2 \end{pmatrix} \tag{3}$$

Then, we have the following inverse functions and compositions of functions:

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 3 & 5 & 4 & 1 \end{pmatrix} \tag{4}$$

$$g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix} \tag{5}$$

$$h^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 4 & 5 & 3 \end{pmatrix} \tag{6}$$

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \end{pmatrix} \tag{7}$$

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 5 & 6 & 3 \end{pmatrix} \tag{8}$$

2 Chapter 7, Exercise B2

Suppose we have some $f \in S_6$ such that:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix} \tag{9}$$

Then, we generate a cyclic subgroup of S_6 containing the following elements.

$$f^2 = ff = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 5 & 6 \end{pmatrix} \tag{10}$$

$$f^3 = fff = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 3 & 6 & 5 \end{pmatrix} \tag{11}$$

$$f^4 = ffff = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = e \tag{12}$$

$$f^5 = ff^4 = fe = f \tag{13}$$

Let $A = \{f, f^2, f^3, f^4\}$.

It is readily checked that A is a 4^{th} order cyclic subgroup of S_6 under the operation of function composition.

3 Chapter 7, Exercise B3

Define a subset of S_5 as $A = \{e, f, g, h\}$ with elements defines as follows:

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{14}$$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \tag{15}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} \tag{16}$$

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \tag{17}$$

Table 1 describes the behavior of A under function composition.

We see by inspection of table 1 that A is closed under $\circ$, contains the identity of S_6, contains an inverse for each element, and contains absolutely zero non-commutative elements.

Then, we conclude that A is a subgroup of S_6, and furthermore we conclude that A is an Abelian subgroup of S_6 since any element of A commutes with any other element of A.

$\circ$	e	f	g	h
e	e	f	g	h
f	f	e	h	g
g	g	h	e	f
h	h	g	f	e

Table 1: Operation table for A under $\circ$

4 Chapter 7, Exercise C2

Suppose $A = \{x \in \mathbb{R} \ni x \neq 0\}$ and $G = \{e, f, g, h\}$ where:

$$e(x) = x \wedge f(x) = \frac{1}{x} \wedge g(x) = -x \wedge h(x) = \frac{-1}{x} \tag{18}$$

Table 2 describes the behavior of A under function composition.

$\circ$	e	f	g	h
e	e	f	g	h
f	f	e	h	g
g	g	h	e	f
h	h	g	f	e

Table 2: Operation table for A under $\circ$

By inspection of table 2, we see that G is closed under $\circ$, that the identity of S_A is in G, and that every element in G has its inverse under $\circ$ in G, so we conclude that G is a subgroup of S_A.

Remark 1: Tables 1 and 2 are identical, though the elements of G are defined in terms of an enirely different A.

In fact, if we continue to let G denote the group in this problem and if we let G' denote the G from the previous problem, and let e', f', g', h' denote the e, f, g, h from the previous problem and if we define a $\phi : G \to G'$ such that $\phi(e) = e', \phi(f) = f', \phi(g) = g', \phi(h) = h'$, we observe the linearity of ϕ, that $(\forall x, y \in G)(\phi(x \circ y) = \phi(x) \circ \phi(y))$ and that ϕ is a bijection, so we have group isomorphism between G and G', or $G \cong G'$.

5 Chapter 7, Exercise F2

Suppose H is the group of symmetries of the rectangle.

Denote the full 2π radian rotation of the rectangle as r_e, since it is the identity of G

Denote the π radian rotation of the rectangle as r_π.

Denote the horizontal and vertical flips of the rectangle as f_h and f_v respectively.

More precisely, we define the following four permutations:

$$r_e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \wedge r_\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \wedge f_h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \wedge f_v = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \tag{19}$$

Then, table 3 describes the behavior of H under function compositon

$\circ$	r_e	r_π	f_h	f_v
r_e	r_e	r_π	f_h	f_v
r_π	r_π	r_e	f_v	f_h
f_h	f_h	f_v	r_e	r_π
f_v	f_v	f_h	r_π	r_e

Table 3: Operation table for H under $\circ$

Remark 2: Define a function $\psi : H \to G$, where G is defined as in the previous exercise, such that $\psi(r_e) = e$, $\psi(r_\pi) = f$, $\psi(f_h) = g$, and $\psi(f_v) = h$. We see by inspection that ψ is bijective and it is readily checked via tables 2 and 3 that $(\forall x, y \in H)(\psi(x) \circ \psi(y) = \psi(x \circ y))$ holds. Then, we have $H \cong G$, and by transitivity of isomorphism we have that tables 1, 2, and 3 describe isomorphic groups.

6 Chapter 7, Exercise H3

Suppose some set B is a subset of a finite set A.

Let G be the subset of S_A where $(\forall x \in B)(f(x) \in B)$.

Theorem 1. *G is a subgroup of A.*

Proof. We have by the definition of G that for any $f \in G$, we have $f(x) \in B$ for any $x \in B$, so for any two $f, g \in B$, we must also have $(f \circ g)(x) = f(g(x)) \in B$, so G is closed under $\circ$.

Then, for some $f \in G$, we have $f(x) \in B$ for any $x \in B$, and we have that any permutation is a one-to-one correspondence, so if we simply reverse the direction of the mappings for f, we have a g where $g(f(x)) = x$ and since $x \in B$ and $f(x) \in B$ we must also have $g \in B$ and we see that $g = f^{-1}$. So, every element of G has its inverse in G and then it follows immediately that the identity of S_A is in G.

Since G is closed under function composition and every element has an inverse, we conclude that G is a subgroup of S_A. This proves theorem 1. $\square$

7 Chapter 8, Exercise A1(f)

Suppose $f = (6\ 1\ 4\ 8)(2\ 3\ 4\ 5)(1\ 2\ 4\ 9\ 3) \in S_9$.

We can rewrite f as $\binom{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9}{3\ 5\ 4\ 9\ 2\ 1\ 7\ 6\ 8}$.

8 Chapter 8, Exercise A2(d)

Suppose $f = \binom{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9}{9\ 8\ 7\ 4\ 3\ 6\ 5\ 1\ 2} \in S_9$.

We can rewrite f as $(1\ 9\ 2\ 8)(3\ 7\ 5)$.

9 Chapter 8, Exercise A3(b)

Suppose $f = (4\ 1\ 6)(8\ 2\ 3\ 5)$

We can rewrite f as $(6\ 1)(6\ 4)(5\ 3)(5\ 2)(5\ 8)$.

10 Chapter 8, Exercise B1(b)

Let $\alpha = (1\ 2\ 3\ 4)$.

Then:

$$\alpha^{-1} = (1\ 4\ 3\ 2) \tag{20}$$
$$\alpha^{2} = (1\ 3)(2\ 4) \tag{21}$$
$$\alpha^{3} = (1\ 4\ 3\ 2) \tag{22}$$
$$\alpha^{4} = (1)(2)(3)(4) \tag{23}$$
$$\alpha^{5} = (1\ 2\ 3\ 4) = \alpha \tag{24}$$

11 Chapter 8, Exercise C1(c)

Let $f = (1\ 2)(7\ 6)(3\ 4\ 5)$.

If we rewrite f as $(1\ 2)(7\ 6)(5\ 4)(5\ 3)$ we can see by inspection that f is an even permutation since it is the product of an even number of transpositions.

12 Symmetries of the equilateral triangle

Let $G \subseteq S_3$ be the symmetries of the equilateral triangle.

Denote the rotations by r_1, r_2, r_3 for a rotation r_n of $\frac{2n\pi}{3}$ radians. Denote the flips about each axis of symmetry by f_1, f_2, f_3.

More precisely, we define the following permutations:

$$r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad r_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \tag{25}$$
$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tag{26}$$

Table 4 describes the behavior of G under function composition.

Remark 3: We see $G = S_3$ by the observation that $|G| = 3! = 6 = |S_3|$.

13 A subgroup of S_4

Suppose $G = \{ \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ x & y & z & 4 \end{smallmatrix}\right) \in S_4 \ni x, y, z \in \{1, 2, 3\} \}$.

Then, we “arbitrarily” denote the elements of G as follows:

$\circ$	r_3	r_1	r_2	f_1	f_2	f_3
r_3	r_3	r_1	r_2	f_1	f_2	f_3
r_1	r_1	r_2	r_3	f_3	f_1	f_2
r_2	r_2	r_3	r_1	f_2	f_3	f_1
f_1	f_1	f_3	f_2	r_3	r_2	r_1
f_2	f_2	f_1	f_3	r_2	r_3	r_2
f_3	f_3	f_2	f_1	r_1	r_2	r_3

Table 4: Operation table for G under $\circ$

$$r_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \qquad r_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \tag{27}$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \tag{28}$$

By sheer coincidence, table 4 describes the behavior of G under function composition.

14 Chapter 8, Exercises C2 and C3

Theorem 2. *If two permutations are both even or both odd, then their product is even.*

Proof. Suppose we have some $\pi \in S_n$ that can be factored into m transpositions. We note that $(-1)^m = 1$ iff m is even and $(-1)^m = -1$ iff m is odd.

We know that for any two permutations $a, b \in S_n$ composed of q and r transpositions each respectively, if we let s denote the number of transpositions of ab, we have $(-1)^s = (-1)^q \cdot (-1)^r$.

Then, suppose we have two even permutations x and y composed of i and j transpositions respectively. We have $(-1)^i \cdot (-1)^j = 1 \cdot 1 = 1$ which implies that xy must be an even permutation.

Alternatively, suppose we have two odd permutations x and y composed of i and j transpositions respectively. We have $(-1)^i \cdot (-1)^j = -1 \cdot -1 = 1$ which implies that xy must be an even permutation.

Finally, we conclude that if any two permutations are either both even or both odd, we must have that their product is even.

This proves theorem 2. $\square$

15 A subset of S_4 consisting of transpositions

Suppose $G = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subseteq S_4$.

Denote the elements of G as follows:

$$e = e \tag{29}$$
$$f = (1\ 2)(3\ 4) \tag{30}$$
$$g = (1\ 3)(2\ 4) \tag{31}$$
$$h = (1\ 4)(2\ 3) \tag{32}$$

$\circ$	e	f	g	h
e	e	f	g	h
f	f	e	h	g
g	g	h	e	f
h	h	g	f	e

Table 5: Operation table for G under $\circ$

Hmm, this looks suspicously similar to table 2.

Anyway, we confidently conclude that due to the fact that G is closed under $\circ$ and has an inverse under $\circ$ for every element in the set, including the identity of S_4, G is a subgroup of S_4.