Abstract Algebra: Homework #1

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1 Chapter 3: Excercise A1

Suppose $*$ is defined on $\mathbb{R}$ as $a*b = a+b+k$ for any $a, b \in \mathbb{R}$ for some $k \in \mathbb{R}$.

Theorem 1. *$\langle \mathbb{R}, * \rangle$ is a group.*

Proof. Let a, b, and c be some arbitrary real numbers. Because $a * b = a + b + k \in \mathbb{R}$, we have that the real numbers are closed under $*$. Then, observe that $a * (b * c) = a * (b + c + k) = a + (b + c + k) + k = (a + b + k) + c + k = (a + b + k) * c = (a * b) * c$, so $*$ is associative. We also have $a * -k = a + (-k) + k = -k + a + k = -k * a = a$, so $-k$ is the identity for the real numbers under $*$ Finally, consider the quantity $-(2k + a)$. Since we have $a * -(2k + a) = a + (-2k) + (-a) + k = -(2k + a) * a = -k$, that quantity is the inverse of any a. Since $*$ is closed under the real numbers and $*$ is associative and $-k$ is the identity of $*$ under the real numbers and any real number a has an inverse under $*$ of $-(2k - a)$, $\langle \mathbb{R}, * \rangle$ is a group. $\square$

2 Chapter 3: Excercise A3

Suppose $*$ is defined on $\mathbb{R}$ as $a * b = a + bab$ for any $a, b \in \mathbb{R}$.

Theorem 2. *$\langle \mathbb{R}, * \rangle$ is a group.*

Proof. Let a, b, and c be some arbitrary real numbers. Because $a*b = a+b+ab \in \mathbb{R}$, we have that the real numbers are closed under $*$. Then, observe that $a*(b*c) = a*(b+c+bc) = a+b+c+ab+ac+bc+abc = (a+b+ab)*c = (a*b)*c$, so $*$ is associative. We also have $a * 0 = a + (0) + 0a = 0 * a = a$, so 0 is the identity for the real numbers under $*$ Finally, consider the quantity $\frac{-a}{1+a}$. Since we have $a * \frac{-a}{1+a} = a + \frac{-a}{1+a} + \frac{-a^2}{1+a} = \frac{-a}{1+a} * a = \frac{a^2+a}{1+a} + \frac{-a}{1+a} + \frac{-a^2}{1+a} = 0$, that quantity is the inverse of any a. Since $*$ is closed under the real numbers and $*$ is associative and 0 is the identity of $*$ under the real numbers and any real number a has an inverse under $*$ of $\frac{-a}{1+a}$, $\langle \mathbb{R}, * \rangle$ is a group. $\square$

3 Chapter 3: Excercise B1

Suppose $*$ is defined on $\mathbb{R} \times \mathbb{R}$ as $(a,b) * (c,d) = (ad + bc, bd)$ for any $(a,b), (c,d) \in \mathbb{R} \times \mathbb{R}$.

Theorem 3. *$\langle \mathbb{R}, * \rangle$ is a group.*

Proof. Let $(a,b),(c,d),(e,f) \in \mathbb{R} \times \mathbb{R}$. Consider that $(a,b) * (c,d) = (ad + bc, bd) \in \mathbb{R} \times \mathbb{R}$ since $ad + bc \in \mathbb{R} \wedge bc \in \mathbb{R}$. Then, $\mathbb{R} \times \mathbb{R}$ is closed under $*$ Now observe the following equivalence: $\Big((a,b) * (c,d)\Big) * (e,f) = (ad + bc, bd) * (e,f) = (adf + bcf + bde, bdf) = (a,b) * (cf + de, df) = (a,b) * \Big((c,d) * (e,f)\Big)$ Since $(a,b) * (0,1) = (1a + 0b, 1b) = (a,b) = (0b + 1a, 1b) = (0,1) * (a,b)$, we have that $(0,1)$ is the identity of the real numbers under $*$, Because we have $(a,b) * (\frac{-a}{b^2}, \frac{1}{b}) = (\frac{a}{b} + \frac{-a}{b}, b \cdot \frac{1}{b} = (0,1) = (\frac{-a}{b} + \frac{a}{b}, \frac{1}{b}) = (\frac{-a}{b^2}, \frac{1}{b}) * (a,b)$, that pair $(\frac{-a}{b^2}, \frac{1}{b})$ is the inverse of any real a under 8. Then, since $*$ is a closed associative operation on the real numbers with an identity and an inverse for any element of the real numbers, we have that $\langle \mathbb{R}, * \rangle$ is a group.

$\square$