Abstract Algebra: Homework #7

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Note: I use the term "epimorphism" defined as a surjective homomorphism. This kind of morphism is more relevant to the context of serveral of the following proofs than mere homomorphism. Every epimorphism is a homomorphism so the proofs in question are only stonger and should cover a superset of the relevant constraints.

1 Chapter 14, Exercise A3

Consider the surjection $f : \mathbb{Z}_{15} \to \mathbb{Z}_5$ defined as follows:

$$\begin{array}{ccccc} f : 0 \mapsto 0 & f : 1 \mapsto 1 & f : 2 \mapsto 2 & f : 3 \mapsto 3 & f : 4 \mapsto 4 \\ f : 5 \mapsto 0 & f : 6 \mapsto 1 & f : 7 \mapsto 2 & f : 8 \mapsto 3 & f : 9 \mapsto 4 \\ f : 10 \mapsto 0 & f : 11 \mapsto 1 & f : 12 \mapsto 2 & f : 13 \mapsto 3 & f : 14 \mapsto 4 \end{array}$$

By inspection, we see that $\ker(f) = \{0, 5, 10\}$.

2 Chapter 14, Exercise B2

Suppose we have the function $\phi : \mathcal{D}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ defined by $\phi(f) = \frac{df}{dx}$ where x is the independent variable of f.

Theorem 1. *ϕ is an epimomorphism from $\mathcal{D}(\mathbb{R})$ to $\mathcal{F}(\mathbb{R})$*

Proof. Let $h' \in \mathcal{F}(\mathbb{R})$ be some arbitrary function where the prime is purely formal notation and does not denote an operation. Since h' is continuous on $\mathbb{R}$, it is integrable on the entire interval $(-\infty, \infty)$. By the first part of the fundamental theorem of calculus, we have that h' is the derivative of some h with repsect to the limits of integration implied by the integrability of h'. Then, denoting the independent variable of h by x, we have $\phi(h) = \frac{d}{dx}h = h'$ so we see that ϕ is surjective.

Now suppose $f, g \in \mathcal{D}(\mathbb{R})$ with an independent variable denoted by x. By the linearity of the differentiation operation, we have $\phi(f) + \phi(g) = \frac{d}{dx}f + \frac{d}{dx}g = \frac{d}{dx}(f + g) = \phi(f + g) \in \mathcal{F}(\mathbb{R})$. Since ϕ has this property, it is a homomorphism. Furthermore, since ϕ is surjectve, it is an epimorphism. This proves theorem 1. $\square$

We can describe the kernel of ϕ as $\ker(\phi) = \mathbb{R} \subseteq \mathcal{D}(\mathbb{R})$, since any constant function has a real zero derivative everywhere.

3 Chapter 14, Exercise B4

Suppose we have the function $f : \mathbb{R}^* \to \mathbb{R}^{pos}$ defined by $f(x) = |x|$. Note that the group operation for both the domain and codomain of f is good old real multiplication.

Theorem 2. f *is an epimorphism from* $\mathbb{R}^*$ *to* $\mathbb{R}^{pos}$*.*

Proof. Suppose $x, y \in \mathbb{R}^*$. Then we see$f(x) \cdot f(y) = |x| \cdot |y| = |x \cdot y| = f(xy)$, so f is a homomorphism. Now, consider some $a \in \mathbb{R}^{pos}$. At least one of $f(a) = a$ or $f(-a) = a$ hold since they both hold, so f is surjective. Then, f is a surjective homomorphism if and only if f is an epimorphism. This proves theorem 2 $\square$

We can write the kernel of f as $\ker(f) = \{-1, 1\}$ since $|-1| = |1| = 1$ and $(\forall x \in \mathbb{R}^{pos})(1x = x1 = x)$.

4 Chapter 14, Exercise C2

Suppose $f : G \to H$ is a homomorphism and $\ker(f) = K$.

Theorem 3. f *is injective if and only if* $K = \{e_G\}$

Proof. Suppose f is injective. Then, any $x, y \in G$ satisfy $f(x) = f(y) \implies x = y$. By definition of homomorphism, $f(e_G) = e_H$, so at the very least, $e_G \in K$. Suppose some other $x \in G$ satisfies $f(x) = e$. Then, $x = e_G$ by the reasoning previously elucidated. Therefore the cardinality of K is 1 and $\ker(f)$ is the singleton $K = \{e_G\}$.

Conversely, suppose that f is not injective. This means we must have some $x, y \in G$ where $f(x) = f(y) \implies x = y$ does not hold.

Suppose $x, y \in G$ where $x \neq y$ satisfies $f(x) = f(y)$. Then, $f(x)[f(y)]^{-1} = e_H$, so $f(x)f(y^{-1}) = f(xy^{-1}) = e_H \iff xy^{-1} \in \ker(f)$. We have yet to confirm that the cardinality of K exceeds the singleton K seen above. To determine this property, consider the proposition $xy^{-1} = e_G$ as an assmption. We have $xy^{-1} = e_G \iff x = y$ by multiplication on the right by y, but by assumption, $x \neq y$, so we reach a contradiction and conclude that our assumption must be false, and as such, by the law of the excluded middle, we have $xy^{-1} \neq e_G$, and as such $\ker(f) = K \neq \{e_G\}$ by violation of the axiom of extentionality.

Since both the implication that if f is injective then $K = \{e_G\}$ holds and its converse holds, we must have the bidirective implication specified by theorem 3, which proves the theorem. □

5 Chapter 14, Exercise C4

Suppose $f : G \to H$ is a homomorphism and J is some subgroup of H.

Theorem 4. $f^{-1}(J) = \{x \in G : f(x) \in J\}$ *is a subgroup of* G *and* $\ker(f) \subseteq f^{-1}(J)$

Proof. Suppose some $x, y \in G$ such that $f(x) \in J \wedge f(y) \in J$. Then, $f(x)f(y) = f(xy)$ since f is a homomorphism and $f(xy) \in J$ since the group operation is closed in any subgroup, therefore $xy \in f^{-1}(J)$ by definition. Furthermore, since $e_H \in J$ by definition of subgroup, we must have that any $x \in G$ where $f(x) = e_H \in J$ satisfies $x \in f^{-1}(J)$ by definition, therefore $x \in \ker(f) \implies x \in f^{-1}(J)$, so $\ker(f) \subseteq f^{-1}(J)$ holds and this proves theorem 4. □

6 Chapter 15, Exercise D1

(a) Consider S_3.

We have the following normal subgroups of S_3:

$\{e\}$ and $\{e, (123), (132)\}$ and S_3, the first and last are trivial and the second is by the fact that it has index 2 with respect to S_3.

(b) Consider D_4.
Denote the elements of D_4 as:

$$e = R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad R_{\pi/2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad R_{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad R_{3\pi/2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \tag{1}$$

$$H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \quad D' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \tag{2}$$

We have the following normal subgroups of D_4:

- $\{e\} = \{R_0\}$
- $\{R_0, R_\pi\}$
- $\{R_0, R_{\pi/2}, R_\pi, R_{3\pi/2}\}$
- $\{R_0, R_\pi, V, H\}$
- $\{R_0, R_\pi, D, D'\}$
- D_4

The first and last are trivial, the second is by the fact that it is abelian, and the rest are by the fact of the subgroups being of index 2 with respect to D_4

7 Chapter 14, Exercise D3

Theorem 5. *The center of any group G is a normal subgroup of G, i.e.* $Z(G) \trianglelefteq G$

Proof. By the result of Homework 2 problem 12, we have that $Z(G)$ is a subgroup of G. Then, since $Z(G)$ satisfies $(\forall x \in Z(G))(\forall y \in G)(xy = yx)$, and by multiplication on the right by y^{-1}, we see that $x = yxy^{-1} \in Z(G)$ for any $y \in G$, so $Z(G)$ is a normal subgroup of G. This proves theorem 5. $\square$

8 Abelian group endomorphism

Theorem 6. G *is an Abelian group if and only if* $f : G \to G \ni f(x) = x^{-1}$ *is a homomorphism.*

Proof. Suppose G is an Abelian group. Then, any $x, y \in G$ satisfies $xy = yx$, so we must have $f(xy) = f(yx) = (xy)^{-1} = (yx)^{-1} = x^{-1}y^{-1} = f(x)f(y)$, so G is a homomorphism.

Conversely, suppose f is a homomorphism. Then, any $x, y \in G$ satisfies $f(x)f(y) = f(xy)$, so we must have $f(x)f(y) = x^{-1}y^{-1} = (yx)^{-1} = y^{-1}x^{-1} = (xy)^{-1} = f(xy)$, and of course $(yx)^{-1} = (xy)^{-1} \iff yx = xy$.

Then, G is an abelian group if and only if f is a homomorphism. This proves theorem 6. $\square$

Since f is a homomorphism from G to itself, f is an endomorphism.

9 Homomorphism via determinant

Define $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^* \ni \phi(x) = \det(x)$.

Theorem 7. ϕ *is a homomorphism*

Proof. Since any two $A, B \in GL_2(\mathbb{R})$ satisfy $\phi(x)\phi(y) = \det(x)\det(y) = \det(xy) = \phi(xy)$ by the properties of the determinant of a matrix, we have that ϕ is a homomorphism. This proves theorem 7. $\square$

We can describe the kernel of ϕ by $\ker(\phi) = \{x \in GL_2(\mathbb{R}) : \det(x) = 1\}$, otherwise known as the special linear group commonly denoted $SL_2(\mathbb{R})$.

10 A normal subgroup of $GL_2(\mathbb{R})$

Theorem 8. $H = \{X \in GL_2(\mathbb{R}) : \det(X) > 0\}$ *is a normal subgroup of* $GL_2(\mathbb{R})$*, i.e.* $H \trianglelefteq GL_2(\mathbb{R})$.

Proof. Suppose $A, B \in H$. Consider that $\det(AB) = \det(A)\det(B) > 0$ as well as $\det(B)\det(A) = \det(BA) > 0$, since both $\det(A)$ and $\det(B)$ are positive by definition of H. Then, $AB \in H \iff BA \in H$ since the original choice of matrices was arbitrary, and H is closed under matrix multiplication. Since $\det(A^{-1}) = \det(A)^{-1}$, we have $\det(A) > 0 \iff \det(A^{-1}) > 0$ so every matrix A in H has its inverse A^{-1} in H and we see that H is a subgroup of G. Finally, we have $\Big((H < G) \wedge (AB \in H \iff BA \in H)\Big) \iff H \trianglelefteq GL_2(\mathbb{R})$. This proves theorem 8. □

11 Another normal subgroup of $GL_2(\mathbb{R})$

Theorem 9. $H = \{X \in GL_2(\mathbb{R}) : X = xI \ni x \in \mathbb{R}^*\}$ *is a normal subgroup of* $GL_2(\mathbb{R})$*, i.e.* $H \trianglelefteq GL_2(\mathbb{R})$.

Proof. Suppose we have some $X \in H$ where $X = xI \ni x \in \mathbb{R}^*$. Then consider any $A \in GL_2(\mathbb{R})$. Since any scalar commutes with any matrix, we have $AXA^{-1} = AxIA^{-1} = xAIA^{-1} = xAA^{-1} = xI = X \in H$, so any $X \in H$ has it's congugate with any $A \in G$ in the set H since the conjugate is simply X itself, thus $(X \in H \wedge A \in GL_2(\mathbb{R}) \implies AXA^{-1} \in H) \iff H \trianglelefteq GL_2(\mathbb{R})$. This proves theorem 9. □

12 A demonstration of $A_4 \trianglelefteq S_4$

Theorem 10. $A_4 \trianglelefteq S_4$

Proof. Since every permutation in A_4 is even by definition and an odd permutation composed with an even permuation is odd, the coset of A_4 generated by any odd permutation on S_4 contains only odd permutations. Since every every permutation is either even or odd, we can comfortably conclude that there is only one coset of A_4 other than A_4 itself. Then, we see that A_4 has index 2, so by the result of Homework 6 problem 14, A_4 must be a normal subgroup of S_4, i.e. $A_4 \trianglelefteq S_4$. This proves theorem 10. □

13 A normal and a non normal subgroup

Suppose $K = \{e, (12)(34)\}$ and $H = \{e, (12)(34), (13)(24), (14)(23)\}$ are subgroups of S_4.

Let $a \ntrianglelefteq b$ denote the proposition $\neg(a \trianglelefteq b)$.

Theorem 11. $K \trianglelefteq H \wedge K \ntrianglelefteq S_4$

Proof. By the counterexample $(13) \circ (12)(32) \circ (13) = (14)(23) \notin K$, we clearly must have $K \ntrianglelefteq S_4$. Since every element of K is in H, we have $K \subseteq H$, and since $(12)(34) \circ (12)(34) = e$ and $ee = e$, every element in K has its inverse in K and K is closed under the group operation, so K is a subgroup of H. Furthermore, K is clearly of index 2 with respect to H since it has one non-trivial coset, therefore by the result of Homework 6 problem 14, K is a normal subgroup of H, therefore we have $K \trianglelefteq H \wedge K \ntrianglelefteq S_4$. This proves theorem 11. □

14 Amazing Automorphisms And Analysis

The automorphism of a group G is the set of all isomorphisms from G to itself, foramlly: $\mathrm{Aut}(G) = \{f \in S_G : f(ab) = f(a)f(b)\}$.

(a) Conjugation is a homomorphism

Let G be a group. Define $\pi_a : G \to G$ as $\pi_a(x) = axa^{-1}$ for some $a \in G$.

Theorem 12. $\phi : G \to \mathrm{Aut}(G) \ni \phi(a) = \pi_a$ *is a homomorphism.*

Proof. Suppose $a, b \in G$. Then, consider $f(a)f(b) = \pi_a \circ \pi_b$. For an arbitrary $x \in G$, we see that $(\pi_a \circ \pi_b)(x) = a(bxb^{-1})a^{-1} = (ab)x(ab)^{-1}$. Then, since $f(ab) = \pi_{ab}$, we have $\pi_{ab}(x) = (ab)x(ab)^{-1}$. Thus, $f(ab) = f(a)f(b)$ so f is a homomorphism. □

(b) A normal subgroup of $\mathrm{Aut}(G)$

Define $H = \{\pi_a \in \mathrm{Aut}(G) : \pi_a(x) = ax^{-1} \ni a \in G\}$

Theorem 13. $H \trianglelefteq \mathrm{Aut}(G)$

Proof. Suppose $\pi_a, \pi_b \in H$ and suppose $f \in \mathrm{Aut}(G)$. As demonstrated in the above proof, $\pi_a \circ \pi_b = \pi_{ab} \in H$, so H is closed under the group operation. Of course, we can always find a $\pi_{a^{-1}} \in H$ such that $(\pi_a \circ \pi_{a^{-1}})(x) = aa^{-1}x(aa^{-1})^{-1} = x$, so every element in H has its inverse in H and we see that H is a subgroup of $\mathrm{Aut}(G)$.

Consider that $\pi_a(x) = axa^{-1}$ is simply the composition of permutations on G denoted a, x and a^{-1} — we can consider x a permutation since it is in fact $\epsilon \in \mathrm{Aut}(G)$ — and that f and its inverse f^{-1} are permutations as well since they are bijections from a set to itself. Then, we can multiply our first identity on the left by f and on the right by f^{-1} to get $f\pi_a(x)f^{-1} =$

$f(axa^{-1})f^{-1} = (fa)x(fa)^{-1} = \pi_{fa} \in H$ since (fa) is some bijection from G to itself, an element of Aut(G). Then, $\Big(z \in H \wedge f \in \text{Aut}(G) \implies (fzf^{-1}) \in H\Big) \iff H \trianglelefteq \text{Aut}(G)$. This proves theorem 13. □

(c) The kernel of ϕ

Theorem 14. $\ker(\phi) = \{a \in G : ax = xa \ni a \in G\}$

Denote the identity of Aut(G) by ϵ

Proof. Let $x \in G$ and consider the function ϵ. We have $\epsilon(x) = x$ by definition. Since $\phi(a) = \pi_a$ for some $a \in G$, we can only satisfy $\pi_a(x) = \epsilon(x) = x$ when $axa^{-1} = x$. Then multiplication of this last identity by a on the right yields $ax = xa$. We can generalize that for any $a \in G$, we have $\phi(a) = \epsilon \implies ax = xa$. In fact, we see that if we assume instead for any $a \in G \wedge x \in G$ that $ax = xa$, we see that $x = axa^{-1} = \pi_a(x) = \epsilon(x) = \phi(a)(x)$, so we have the bidirective implication $\phi(a) = \epsilon \iff ax = xa$ for any $x \in G$. We can rewrite this fact as $\ker(\phi) = \{a \in G : ax = xa \ni a \in G\}$ since this indeed satisfies the definition of kernel of a homomorphism. This proves theorem 14. □