Abstract Algebra: Final Exam

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Thursday 6 August 2020

1. Elements and intersection of groups
Suppose $G = \{g \in S_4 : g(1) = 3\}$ and $H = \{h \in S_4 : h(2) = 2\}$.
The following are the elements of G:

$$\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}\quad\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\quad\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix} \tag{1}$$

The following are the elements of H:

$$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}\quad\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\quad\begin{pmatrix}1&2&3&4\\4&2&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&2&2&1\end{pmatrix} \tag{2}$$

Then, the following are the elements of $G \cap H$

$$\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix} \tag{3}$$

We have:

$$\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\circ\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix} \notin G \cap H \tag{4}$$

Since $G \cap H$ is not closed under function compsition, we have that $G \cap H$ is not a group.

2. Nonsingular rational matrices transformed by scalar multiples of the identity
Suppose $H = \{X \in GL_2(\mathbb{R}) : X = xI \in x \in \mathbb{R}^*$ and $K = GL_2(\mathbb{Q})$.

Theorem 1. *The set* $HK = \{XY : X \in H, Y \in K\}$ *is a subgroup of* $GL_2(\mathbb{R})$.

Proof. Suppose $A, B \in HK$. Since each element of HK is some scalar mutliple of I times some matrix with rational entries, we can write $A = \begin{bmatrix} xa & xb \\ xc & xd \end{bmatrix}$ for some $x \in \mathbb{R}^*$ and $a, b, c, d \in \mathbb{Q}$ as well as $B = \begin{bmatrix} ye & yf \\ yg & yh \end{bmatrix}$ for some $y \in \mathbb{R}^*$ and $e, f, g, h \in \mathbb{Q}$. Then, $AB = \begin{bmatrix} xaye + xbyg & xayf + xbyh \\ xcye + xdyg & xcyf + xdyh \end{bmatrix}$, which is just $AB = xy \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$, and since this is none other than a scalar multiple of the identity matrix multiplied by a matrix with rational entries, we have that $AB \in HK$, and HK is closed under multiplication. Since both 0 and 1 are rational, we have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in HK$, so the identity of $GL_2(\mathbb{R})$ is in HK. Then, we can invert A since $\det(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}) \neq 0$ and $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$ by their respective definitions, so by the properties of the determinant, we must have that their matrix product has nonzero determinent, i.e. $\det(A) \neq 0$, therefore A is invertible and it's inverse is $A^{-1} = \frac{1}{x(ad-bd)} \begin{bmatrix} xd & -xb \\ -xc & xa \end{bmatrix}$, and since this is the product of a scalar multiple of the identity matrix and a matrix with rational entries, we have that $A^{-1} \in HK$, therefore every element of HK has its inverse in HK. Finally, since HK is closed under matrix multiplication, since the identity of $GL_2(\mathbb{R})$, the identity matrix, is in HK, and since every element of HK has its inverse in HK, we conclude that HK is a subgroup of $GL_2(\mathbb{R})$. □

3. Normal subgroups

Suppose H and K are normal subgroups of a group G, where $H \subseteq K$.

Theorem 2. *If* G/H *is an Abelian group, then* G/K *is an abelian group.*

Proof. Suppose $Ha, Hb \in G/H$. Then, since G/H is Abelian, we have $HaHb = HbHa$, thus $H(ab) = H(ba)$ by definition. Therefore we have some $h_1, h_2 \in H$ where we have $h_1ab = h_2ba$, but since $H \subseteq K$, we have just shown that for an arbitrary $h_1, h_2 \in K$, it is also the case that $h_1ab = h_2ba$, and this is exactly the definition that $K(ab) = K(ba)$, i.e. $KaKb = KbKa$, and this demonstrates that G/K is an abelian group. □

4. The non-field $\mathbb{R}^2$.

Consider $\mathbb{R}^2$.

A field has no divisors of zero.

That is, if there exists some $a, b \in F$ where F is some set with a well-defined multiplication operation such that $ab = 0 \in F$ and $a \neq 0 \wedge b \neq 0$, then F is not a field.

The zero element of $\mathbb{R}^2$ is $(0, 0)$

We have that $(1, 0) \neq (0, 0) \wedge (0, 1) \neq (0, 0)$, however it is indeed the case that $(1, 0)(0, 1) = (1 \cdot 0, 0 \cdot 1) = (0, 0)$, therefore $\mathbb{R}^2$ has divisors of zero and so we conclude it is not a field.