Abstract Algebra: Homework #6

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Wednesday 1 July 2020

1 Chapter 13, Exercise A3

Suppose H is a subgroup of some group G. Furthermore, suppose $\mathsf{G} = \mathbb{Z}_{15} \wedge \mathsf{H} = \langle 5 \rangle$. Then, denoting $+_{15}$ as $+$, the following are the cosets of H:

$$\begin{aligned} H + 0 =& \{0, 5, 10\} \\ H + 1 =& \{1, 6, 11\} \\ H + 2 =& \{2, 7, 12\} \\ H + 3 =& \{3, 8, 13\} \\ H + 4 =& \{4, 9, 14\} \end{aligned}$$

2 Chapter 13, Exercise A4

Denote the elements of D_4 as:

$$R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad R_{\pi/2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad R_{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1, & 3 \end{pmatrix} \quad R_{3\pi/2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \tag{1}$$

$$H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2, & 1 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4, & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \quad D' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \tag{2}$$

The operation table for function composition $\circ$ on D_4 is given in table 1

Suppose H' is a subgroup of some group G. Furthermore, suppose $G = D_4 \wedge H' = \{R_0, D'\}$

$\circ$	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
R_0	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
$R_{\pi/2}$	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	R_0	D'	D	H	V
R_π	R_π	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	V	H	D'	D
$R_{3\pi/2}$	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	R_π	D	D'	V	H
H	H	D	V	D'	R_0	R_π	$R_{\pi/2}$	$R_{3\pi/2}$
V	V	D'	H	D	R_π	R_0	$R_{3\pi/2}$	$R_{\pi/2}$
D	D	V	D'	H	$R_{3\pi/2}$	$R_{\pi/2}$	R_0	R_π
D'	D'	H	D	V	$R_{\pi/2}$	$R_{3\pi/2}$	R_π	R_0

Table 1: Operation table for G under $\circ$

Then, using multiplicative notaiton for $\circ$, we have the following cosets of H'

$$\begin{aligned}
H'R_0 =& \{R_0, D'\}\\
H'R_{\pi/2} =& \{R_{\pi/2}, H\}\\
H'R_\pi =& \{R_\pi, D\}\\
H'R_{3\pi/2} =& \{R_{3\pi/2}, V\}\\
H'H =& \{H, R_{\pi/2}\}\\
H'V =& \{V, R_{3\pi/2}\}\\
H'D =& \{D, R_\pi\}\\
H'D' =& \{D', R_0\}
\end{aligned}$$

3 Chapter 13, Exercise B1

Suppose $H = \langle 3 \rangle$ where $3 \in \mathbb{Z}$.

Then, we can describe the three cosets of H as follows:

$$\begin{aligned}
H + 0 =& \{x \in \mathbb{Z} : \exists k \in \mathbb{Z} \ni x = 3k\}\\
H + 1 =& \{x \in \mathbb{Z} : \exists k \in \mathbb{Z} \ni x = 3k + 1\}\\
H + 2 =& \{x \in \mathbb{Z} : \exists k \in \mathbb{Z} \ni x = 3k + 2\}
\end{aligned}$$

4 Chapter 13, Exercise C2

Suppose G is some group such that $\operatorname{ord}(G) = pq$ for some prime natural p and q.

Theorem 1. G *is not cyclic if any only if every* $x \in G \ni x \neq e \in G$ *satisfies* $\operatorname{ord}(x) = p \vee \operatorname{ord}(x) = q$.

Proof. Suppose G is cyclic. Then, some $x \in G$ satisfies $\langle x \rangle = G \iff \operatorname{ord}(x) = pq$ and we have some $x \in G$ where $\operatorname{ord}(x) = p \vee \operatorname{ord}(x) = q$ does not hold.

Conversely, suppose G is not cyclic. Then, let x be some member of G where $x \neq e \in G$. By Lagrange's theorem, we must have that $\operatorname{ord}(x)$ divides $\operatorname{ord}(G)$, and so we have that $\operatorname{ord}(x)$ divides pq. Then, $\operatorname{ord}(x) \in \{1, p, q, pq\}$. We also have $\big(\operatorname{ord}(x) = 1 \iff x = e\big) \wedge \operatorname{ord}(x) \neq e \implies \operatorname{ord}(x) \neq 1$. And of course $\operatorname{ord}(x) \neq pq$, since otherwise $\langle x \rangle = G$, violating our assumption that G is not cyclic. We have deduced that $\operatorname{ord}(x) = p \vee \operatorname{ord}(x) = q$ holds.

This proves theorem 1. □

5 Chapter 13, Exercise C3

Suppose G is some group where $\operatorname{ord}(G) = 4$.

Theorem 2. G *is not cyclic if and only if every element of* G *is its own inverse.*

Proof. Suppose G is cyclic. Then, we have an $x \in G$ such that $\langle x \rangle = G$. We can write G as $\{e, x, x^2, x^3\}$. By inspection we see that $x^2 \neq e$ and we have an element of G that is not its own inverse, so it is false that every element of G is its own inverse when G is cyclic.

Suppose G is not cyclic. By Lagrange's theorem, the order of every element of G must divide the order of G, so the non identity elements of G must have order 2 or order 4. Since G is not cyclic, no element has order 4, for if it did, that element would generate G and G would not be cyclic. Since every $x \in G$ satisfies $\operatorname{ord}(x) = 2$, we must have $x^2 = e$ for every $x \in G$ and then every element of G is its own inverse.

This proves theorem 3. □

Theorem 3. *Every group of order* 4 *is abelian.*

Proof. Suppose G is not cyclic. Then, by theorem 2, we have that every $x \in G$ satisfies $x^{-1} = x$. Applying this identity, we find that $ab = a^{-1}b^{-1} = (ba)^{-1} = ba$ so any $a, b \in G$ commute and G is abelian.

Instead, suppose G is cyclic. Then, G has a generator x where $G = \{e, x, x^2, x^3\}$. Then, we can write any $y \in G$ as $y = x^i$ for any $i \in \{0, 1, 2, 3\}$. If $a = x^\alpha \in G$ and $b = x^\beta$ are two such sets, we observe that $ab = x^\alpha x^\beta = x^{\alpha+\beta} = x^{\beta+\alpha} = x^\beta x^\alpha = ba$ and see that any two $a, b \in G$ commute and G is abelian.

Since G is abelian if G is cyclic and G is abelian if G is not cyclic, we see by the law of the exluded middle that G is abelian and in general, every group of order 4 is abelian.

This proves theorem 3. $\square$