Homework 6

Due Wednesday, July 1

1. Chapter 13, Exercise A3 (p. 130)
2. Chapter 13, Exercise A4 (p. 130)
3. Chapter 13, Exercise B1 (p. 130)
4. Chapter 13, Exercise C2 (p. 130)
5. Chapter 13, Exercise C3 (p. 131)
6. Chapter 13, Exercise D1 (p. 131)
7. Chapter 13, Exercise E1 (p. 131)
8. Chapter 13, Exercise E3 (p. 131)
9. Recall the matrix representation of the affine group $G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R}) : a \neq 0 \right\}$ (see Homework 4, Problem 15). Let $H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$. Show that H is a subgroup of G, and describe the (right) cosets of H.
10. Consider the group $H = \{e, (12)(34), (13)(24), (14)(23)\}$ from Homework 3, Problem 15, and let $G = A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$ (see also Chapter 13, Exercise A5, p. 130). Write down the cosets of H, and compute $(G : H)$.
11. Let $B_1 = \{1, \dots, k\}$ and $B_2 = \{k+1, \dots, n\}$, where $1 \leq k \leq n-1$. We saw that
$$G_1 = \{f \in S_n : f(x) \in B_1,\ \forall x \in B_1\} \quad \text{and} \quad G_2 = \{g \in S_n : g(x) \in B_2,\ \forall x \in B_2\}$$
are both subgroups of S_n (Homework 3, Problem 6). Let us modify G_1 and G_2, so that $f \in G_1$ is a permutation with all points of B_2 fixed, and likewise $g \in G_2$ is a permutation with all points of B_1 fixed; these G_1 and G_2 still are subgroups of S_n (see Homework 3, Problem 13). With these modified G_1 and G_2, define $H = \{f \circ g :\ f \in G_1,\ g \in G_2\}$.
    (a) For a concrete example, let us consider S_5, $B_1 = \{1, 2\}$, $B_2 = \{3, 4, 5\}$. Write down the elements of G_1, G_2, H, and verify that $(S_5 : H) = 10$.
    (b) Show that H is a subgroup of S_n.
    (c) Show that $(S_n : H) = n!/[k!(n-k)!]$.
12. Show that the following are equivalent: (a) $a \in Hb$, (b) $ab^{-1} \in H$, (c) $Ha = Hb$.
13. Define a **normal subgroup** H of a group G as follows: For any $h \in H$ and $a \in G$, it follows that $aha^{-1} \in H$. Show that if $aH = Ha$ for all $a \in G$, then H is a normal subgroup of G.
14. If H is a subgroup of index 2 in a group G, show that H is a normal subgroup of G. (HINT: You may use Problem 13).