Abstract Algebra: Homework #9

Joel Savitz

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1 A consequence of the fundamental homomorphism theorem

Suppose $G = \mathbb{Z}_{15}$ and $H = \mathbb{Z}_5$.

Let $f : G \to H \ni f = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{smallmatrix}\right)$ be a hommorphism from G to H.

We have $\ker(f) = \{x \in G : f(x) = 0 \in H\} = \{0, 5, 10\} \subseteq G$, therefore $G/H = \{\{0, 5, 10\}, \{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 13\}, \{4, 9, 14\}\}$.

By the fundamental homomorphism theorem, we have $H \cong G/K$ since K is the kernel of a homomorphism from G to H.

2 Chapter 16, Exercise A3

Suppose $G = S_3$ and $H = \mathbb{Z}_2$.

Denote the elements of S_3 as follows:

$$r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad r_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \tag{1}$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tag{2}$$

Then, table 1 describes the behavior of G under function composition.

Define $K = \{r_1, r_3, r_3\}$

Theorem 1. $H \cong G/K$

$\circ$	r_3	r_1	r_2	f_1	f_2	f_3
r_3	r_3	r_1	r_2	f_1	f_2	f_3
r_1	r_1	r_2	r_3	f_3	f_1	f_2
r_2	r_2	r_3	r_1	f_2	f_3	f_1
f_1	f_1	f_3	f_2	r_3	r_2	r_1
f_2	f_2	f_1	f_3	r_1	r_3	r_2
f_3	f_3	f_2	f_1	r_2	r_1	r_3

Table 1: Operation table for G under $\circ$

Proof. Define $\phi : G \to H$ by $f = \{(r_1, 0), (r_2, 0), (r_3, 0), (f_1, 1), (f_2, 1), (f_3, 1)\}$. By inspection, f is surjective, since by definition it maps some element of G to both $0 \in H$ and $1 \in H$ at least once, so it is surjective. Let $x, y \in S_3$ and consider $f(x) + f(y)$. We will use the term rotation to refer any $x \in G$ where $f(x) = 0$ and flip to refer to any $x \in G$ where $f(x) = 1$. Since f maps every element of G to an element of H, and $H = \{0, 1\}$, every element of G is either a flip or a rotation. Suppose x is a flip and y is a flip. Then, xy is a rotation by table 1 so $f(xy) = 0$ and $f(x)+f(y) = 1+1 = 0$, thus $f(xy) = f(x)+f(y)$. Suppose x is a rotation and y is a rotation. Then, xy is a rotation by table 1 so $f(xy) = 0$ and $f(x) + f(y) = 0 + 0 = 0$, thus $f(xy) = f(x) + f(y)$. Suppose x is a flip and y is a rotation. Then, by table 1, xy is a flip. Alternatively, if x is a rotation and y is a flip, table 1 still constrains us such that xy must be a flip, thus we will consider the two cases together. Then, we have either $f(x) = 0 \wedge f(y) = 1$ or $f(x) = 1 \wedge f(y) = 0$, but in both cases we have $f(x)+f(y) = 1 = f(xy)$. Since the proposition holds for all possible cases, we have that for any $x, y \in G$, it holds that $f(x) + f(y) = f(xy)$, so we conclude that f is a homomorphism onto H.

By inspection, $\ker(f) = K = \{r_1, r_2, r_3\}$ since this is exactly the subset of G containing every element that is mapped to $0 \in H$ by f. As demonstrated above, every rotation composed with a rotation is simply another rotation, and $r_2 \circ r_1 = r_1 \circ r_2 = r_3 \circ r_3 = r_3$, so K is a subgroup of G, and since it is the kernel of a homomorphism, K is furthermore a normal subgroup of G, so G/K is well defined. By the fundamental homomorphism theorem, the quotient group formed by the kernel of a homomorphism onto the codomain is isomorphic to the range of the homomorphism, so $H \cong G/K$. This proves theorem 1. $\square$

Table 2 gives the operation table for H under $\circ$.

$\circ$	r_3	r_1	r_2
r_3	r_3	r_1	r_2
r_1	r_1	r_2	r_3
r_2	r_2	r_3	r_1

Table 2: Operation table for H under $\circ$

Table 3 gives the operation table for G/K under coset composition.

$\circ$	K	Kr_1	Kr_2
K	K	Kr_1	Kr_2
Kr_1	Kr_1	Kr_2	K
Kr_2	Kr_2	K	Kr_1

Table 3: Operation table for G/K under coset $\circ$

3 Chapter 16, Exercise A4

Suppose a, b, and c are some sets.

Let $G = \mathcal{P}(\{a, b, c\})$, $H = \mathcal{P}(\{a, b\})$, and $K = \{\emptyset, \{c\}\}$.

Theorem 2. $H \cong G/K$

Proof. Define $f : G \to H$ by $f(X) = X \cap \{a, b\} \in H \ni X \in G$ and assume it is a homomorphism from G to H. Let $z \in H$. Clearly $f(z) = z$, so f is surjective. Then, suppose $x \in G \ni f(x) = \emptyset$. Since $f(x) = x \cap \{a, b\}$, we have that $f(x) = \emptyset$ only if $a \notin x \wedge b \notin x$ by definition of set intersection. Then, we construct this set by the axiom schema of comprehension over this last predicate, i.e. we have $\ker(f) = \{x \in \mathcal{P}(G) : a \notin x \wedge b \notin x\}$. Suppose $y \in K$. By inspection $a \notin y \wedge b \notin y$. Thus, $K \subseteq \ker(f)$. Alternatively take some $y \in \ker(f)$. By the definition of this set, $a \notin y \wedge b \notin y$, and since there are no other elements in $\mathcal{P}(G)$ besides $\{c\}$ and $\emptyset$, we must have $y \in K$, so $\ker(f) \subseteq K$, and by the axiom of extentionality, we have $K \subseteq \ker(f) \wedge \ker(f) \subseteq K \iff K = \ker(f)$, and by the fundamental homomorphism theorem, the quotient group formed by the kernel of a homomorphism onto the codmain is isomorphic to the range of that homormorphism, so $H \cong G/K$ and this proves theorem 2. $\square$

4 Chapter 16, Exercise A5

Suppose $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $H = \mathbb{Z}_3$ and $K = \{(0,0),(1,1),(2,2)\}$.

Theorem 3. $H \cong G/K$

Proof. Define $f : G \to H$ by $f(a,b) = a-b$ and assume f is a homomorphism. Since for all $a,b \in \mathbb{Z}_3$, we have $a-b=0 \iff a=b$, we must have that $\ker(f) = K$ since these are all the elements of G of the form $(a,a) \ni a \in \mathbb{Z}_3$. To verify this, inspect K. Suppose $x \in \mathbb{Z}_3$. Then at the very least, we have $f((x,0)) = x \ni (x,0) \in \mathbb{Z}_3 \times \mathbb{Z}_3$, so f is surjective. Then, since by the fundamental homomorphism theorem the group formed by the kernel of a homomorphism onto the codomain is isomorphic to the range of the homomorphism, we have $H \cong G/K$ and this proves theorem 3. $\square$

5 The FHT applied to $\mathcal{F}(\mathbb{R})$

Define $\alpha : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ by $\alpha(f) = f(1) \ni f \in \mathcal{F}(\mathbb{R})$, and define $\beta : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ by $\beta(f) = f(2) \ni f \in \mathcal{F}(\mathbb{R})$.

Instead of using the normal, boring terminology of "surjective homomorphism", I will use an equivalent and much cooler term from category theory, the epic morphism, or epimorphism. To be precise for this context, an epic morphism is a surjective homomorphism.

Theorem 4. *α and β are epic morphisms from $\mathcal{F}(\mathbb{R}) \to \mathbb{R}$.*

Proof. Suppose $f, g \in \mathcal{F}(\mathbb{R})$. Then, $\alpha(f)+\alpha(g) = f(1)+g(1) = (f+g)(1) = \alpha(f+g)$ and $\beta(f)+\beta(g) = f(2)+g(2) = (f+g)(2) = \beta(f+g)$ since the vector addition of real valued functions is linear, so α and β are homomorphisms. To verify the epic property of these morphisms, take $a \in \mathbb{R}$, and define $f \in \mathcal{F}(\mathbb{R}) \ni f(x) = (x-1)+a$ and define $g \in \mathcal{F}(\mathbb{R}) \ni g(x) = (x-2)+a$. By inspection, we see that $\alpha(f) = a$ and $\beta(g) = a$, so α and β are surjetive, i.e. α and β are epic morphisms from $\mathcal{F}(\mathbb{R}) \to \mathbb{R}$ and this proves theorem 4. $\square$

Now suppose $J = \{f \in \mathcal{F}(\mathbb{R}) : (1,0) \in f\}$ and $K = \{f \in \mathcal{F}(\mathbb{R}) : (2,0) \in f\}$.

Theorem 5. $\mathbb{R} \cong \mathcal{F}(\mathbb{R})/J \wedge \mathbb{R} \cong \mathcal{F}(\mathbb{R})/K$

Proof. Suppose $f \in \mathcal{F}(\mathbb{R})$. Since $\alpha(f) = f(1) = 0$ if only if $(1, 0) \in f$, and this is exactly the predicate of the set comprehension used to construct J, we have $\ker(\alpha) = J$. Since $\beta(f) = f(2) = 0$ if only if $(2, 0) \in f$, and this is exactly the predicate of the set comprehension used to construct K, we have $\ker(\beta) = K$. By theorem 4, we have that α and β are epic morphisms, so by the fundamental homomorphism theorem, the quotient groups formed by the kernel of some epic morphism are isomorphic to the range of that morphism, that is to say, $\mathbb{R} \cong \mathcal{F}(\mathbb{R})/J \land \mathbb{R} \cong G/K$ and this proves theorem 5. □

Theorem 6. $\mathcal{F}(\mathbb{R})/J \cong \mathcal{F}(\mathbb{R})/K$.

Proof. $\mathbb{R} \cong \mathcal{F}(\mathbb{R})/J \land \mathbb{R} \cong \mathcal{F}(\mathbb{R})/K \iff \mathcal{F}(\mathbb{R})/J \cong \mathcal{F}(\mathbb{R})/K$ by the transitivity of the equivalence relation that is group isomorphism. This proves theorem 6. lol. □

6 Chapter 16, Exercise C1

Suppose G is some abelian group, and let $H = \{x \in G : x = y^2 \ni y \in G\}$.

Theorem 7. *The function* $f : G \to H \ni f(x) = x^2 \ni x \in G$ *is an epic morphism from* G *to* H.

Proof. Suppose $x, y \in G$. Since G is abelian, we have $f(x)f(y) = xxyy = xyxy = f(xy)$, so the status of f as a homomorphism is established. Then, consider any $y \in H$. By definition of H, there must exist some $x \in G$ such that $x^2 = G$, so $f(x) = x^2 = y \in H$ and f is an epic morphism. This proves theorem 7. □

7 Chapter 16, Exercise F

For any two subgroups A and B of some group G, define $AB = \{x \in G : x = yz \ni y \in A \land z \in B\}$.

Theorem 8. *For any group* G *with a normal subgroup* H *and a subgroup* K*, we have* $K/(H \cap K) \cong HK/H$.

Proof. Since $H \trianglelefteq G$, we must have that for any $x \in H$, and any $a \in G$ we have $axa^{-1} \in H$. If we take some $b \in K$, we must also then have $bxb^{-1} \in H$ since $K \subseteq G$. Every element of $H \cap K$ has it's inverse in $H \cap K$ if and only if this is true for both H and K individually, which it is by definition of a subgroup, and since $H \cap K \subseteq K$ by definition of set intersection, we have that $(H \cap K) \trianglelefteq K$.

Now we consider the set HK. Trivially, $HK \subseteq G$ since groups are closed under their operation. Let $a, b \in HK$ then, $a = rs \ni r \in H \wedge s \in K$ and $b = jk \ni j \in H \wedge k \in K$. Therefore $ab = (rs)(jk)$, and since H is normal with respect to G, it commutes with every element of G and the elements of K are among them due to subset status. So, we can rearange $ab = (rj)(sk) \in HK$ since $rj \in H$ and $sk \in K$. Therefore HK is closed under the group operation. Suppose $x = yz \in HK \ni y \in H \wedge z \in K$. Then, $y^{-1} \in H \wedge z^{-1} \in K$, and since $e \in H \wedge e \in K$, we have $y^{-1}e \in HK \wedge ez^{-1} \in K$, and of course $z^{-1}y^{-1} = (yz)^{-1} \in HK$, so every element of HK has its inverse in HK, so HK is a subgroup of G. Now since every element x of H can be written as $xe \ni e \in K$, it is clear that $x \in HK$, so $H \subseteq HK$. Since H is a subgroup of G, every $a, b \in H$ satisfies $ab \in H$, so this also applies with respect to K and we additionally conclude from the fact that H is already a subgroup that every element of H has its inverse in H, so $H \leq K$. Since $HK \subseteq G$, it is trivial that for any $x \in H$, we have $axa^{-1} \in H$ for any $a \in HK$, so in fact we have $H \trianglelefteq HK$. With this last fact established, we have that HK/H must be well-defined.

Take some $x \in HK/H$. By definition, this x can be written as $x = Ha \ni a \in HK$, and then any $y \in Ha$ can be written as $y = ha \ni a \in HK$, and then this y can be written as some $pq \ni p \in H \wedge q \in K$, therefore $y = hpq$. Since $h \in H \wedge p \in H$, we must have $hp \in H$. We also have $q \in K$. Since $y = (hp)q \in Ha$ we have that any Ha can be written as Hq for some $q \in K$.

Now define the function $f : K \to HK/H \ni f(k) = Hk \ni k \in K$. Let $a, b \in K$. Then, we see that $f(a)f(b) = (Ka)(Kb) = K(ab) = f(ab)$ so f is a homomorphism. Consider any $y \in HK/H$. Since y can be written as some $Hk \ni k \in K$, we must have a $k \in K$ where $f(k) = Ka$ therefore f is surjective and is an epic morphism from K to HK/H.

Suppose $x \in H \cap K$. Then, $f(x) = Hx = H$ since $x \in H$. Therefore $x \in \ker(f)$. Alternatively, suppose $x \in \ker(f)$. Then, $f(x) = Hx = H$, so $x \in H \cap K$ or else this would not hold, and by the axiom of extentionality, we have $H \cap K \subseteq \ker(f) \wedge \ker(f) \subseteq H \cap K \iff \ker(f) = H \cap K$. By the fundamental homomorphism theorem, we have that the quotient group formed by the kernel of an epic morphism is isomorphic to the range of the morphism, i.e. $H \cap K \cong HK/H$. This proves theorem 8. $\square$

8 Chapter 16, Exercise I

Theorem 9. *For any two normal subgroups* H *and* K *of some group* G *where* $\mathsf{H} \subseteq \mathsf{K}$ *and some function* $\phi : \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K} \ni \phi(\mathsf{H}a) = \mathsf{K}a \ni \mathsf{H}a \in \mathsf{G}/\mathsf{H}$*, we have that* $(\mathsf{G}/\mathsf{H})/(\mathsf{K}/\mathsf{H}) \cong \mathsf{G}/\mathsf{K}$*.*

Proof. Suppose $\mathsf{H}a, \mathsf{H}b \in \mathsf{G}/\mathsf{H} \ni \mathsf{H}a = \mathsf{H}b$. Then $\phi(\mathsf{H}a) = \mathsf{K}a$ and $\phi(\mathsf{H}b) = \mathsf{K}b$, but since $\mathsf{H}a = \mathsf{H}b \iff ab^{-1} \in \mathsf{H}$, we must have $\mathsf{K}a = \mathsf{K}b \iff ab^{-1} \in \mathsf{K}$ since $\mathsf{H} \subseteq \mathsf{K}$. Therefore ϕ is well-defined. We see that $\phi(\mathsf{H}a)\phi(\mathsf{H}b) = (\mathsf{K}a)(\mathsf{K}b) = \mathsf{K}(ab) = \phi(\mathsf{H}(ab)) = \phi((\mathsf{H}a)(\mathsf{H}b))$, so ϕ must be a homomorphism. Furthermore, for any $\mathsf{K}a \in \mathsf{G}/\mathsf{K}$, it is obvious from the definition that $\phi(\mathsf{H}a) = \mathsf{K}a$, so ϕ maps G/H onto G/K, so ϕ is an epic morphism. Consider an element $x \in \mathsf{K}/\mathsf{H} = \{x \in \mathsf{G}/\mathsf{H} : x = \mathsf{H}a \ni a \in \mathsf{K}\}$. Since the identity element of G/K is nothing but K itself, we see that $\phi(x) = \mathsf{K}$ since $x = \mathsf{H}a \ni a \in \mathsf{K}$ and $\mathsf{K} = \mathsf{K}a$ for any $a \in \mathsf{K}$. therefore, $\mathsf{K}/\mathsf{H} \subseteq \ker(\phi)$. Alternatively, consider some $y \in \ker(\phi)$. Since $\phi(y) = \mathsf{K} = \mathsf{K}a \ni a \in \mathsf{K}$, we must have that y is of the form $y = \mathsf{H}a \ni a \in \mathsf{K}$, therefore $y \in \mathsf{K}/\mathsf{H}$ by the definition of quotient group and $\ker(\phi) \subseteq \mathsf{K}/\mathsf{H}$ and then by the axiom of extentionality, $\ker(\phi) = \mathsf{K}/\mathsf{H}$. Finally, since K/H is the kernel of an epic morphism from G/H to G/K, we must have $(\mathsf{G}/\mathsf{H})/(\mathsf{K}/\mathsf{H}) \cong \mathsf{G}/\mathsf{K}$ by the fundamental homomorphism theorem. This proves theorem 9. $\square$