Abstract Algebra: Homework #5

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1 Chapter 11, Exercise A2

Suppose $f = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 4 & 5 & 4 \end{smallmatrix}\right)$.

Then, f generates a subgroup of S_6 — denoted $\langle f \rangle$ — like so:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 2 & 5 & 4 \end{pmatrix} \tag{1}$$

$$f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 5 & 2 \end{pmatrix} \tag{2}$$

$$f^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 6 & 5 & 1 \end{pmatrix} \tag{3}$$

$$f^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = e \in S_6 \tag{4}$$

2 Chapter 11, Exercise B1

Suppose we have some group G such that $\operatorname{ord}(G) = n$.

Theorem 1. *G is cyclic* $\iff \exists x \in G \ni \operatorname{ord}(x) = n$

Proof. Since G is cyclic, it must be generated by some element $a \in G$, and since $\operatorname{ord}(G) = n$, we can rewrite G as $\{e, a, a^2, a^3, ..., a^{n-1}\}$. Then, since $a^n \in G$ and all other powers of a are already present, we deduce that $a^n = e$, and therefore we have $\operatorname{ord}(a) = n$ and we have the implication G is cyclic $\implies \exists x \in G \ni \operatorname{ord}(x) = n$.

Conversely, suppose we have some $a \in G$ where $\mathrm{ord}(a) = n$. Then, by definition we have $a^n = e$. Define the following set H: $H = \{x \in G : a^{n-i} \ni 0 \leq i < n\}$. First, note that $x \in H \implies x \in G$ by construction. We can see that $|H| = n$ since every a^{n-i} must be distinct due to the fact that $\mathrm{ord}(a)$ is defined to be the smallest positive integer where $a^n = e$. Then, since $|H| = n$ and every element is a power of a under an operation closed on G, we must have that $x \in G \implies x \in H$ and thus by the Axiom of Extentionality we have $H = G$. Then, G is generated by some element $a \in G$ so G is cyclic. Thus, we have the implication that $\exists x \in G \ni \mathrm{ord}(x) = n \implies G$ is cyclic.

Finally, since we have the implication in both directions, we must have the bidirectional implication that: G is cyclic $\iff \exists x \in G \ni \mathrm{ord}(x) = n$. This proves theorem 1. $\square$

3 Chapter 11, Exercise B3

Suppose G is some group generated by $a \in G$ and suppose $b \in G$.

Theorem 2. $\exists k \in \mathbb{Z} \ni k\,\mathrm{ord}(b) = \mathrm{ord}(a)$ *i.e. the order of* b *is a factor of the order of* a.

Proof. Let $n = \mathrm{ord}(a)$ and let $m = \mathrm{ord}(b)$. We have $m = \mathrm{ord}(b) \iff b^m = e$. Since $G = \langle a \rangle$, we have $G = \{e, a, a^2, a^3, ..., a^{n-1}\}$. Since $b \in G$, we must have that $\exists z \in \mathbb{Z} \ni 0 < z \leq n$ such that $a^z = b$. Then if we take both sides to the power of m we have $a^{mz} = b^m = e = a^n$. Then clearly m is a factor of n, since $a^n = a^{mz}$ so we conclude that the order of b is a factor of the order of a. This proves theorem 2. $\square$

4 Chapter 11, Exercise D1

Suppose G is a group and $a, b \in G$.

Theorem 3. $\left(\exists k \in \mathbb{Z} \ni a = b^k\right) \implies \langle a \rangle \subseteq \langle b \rangle$

Proof. Suppose that some integer k exists where $a = b^k$. Then, suppose some $x \in \langle a \rangle$. We must have some integer n where $a^n = x$, so we subsitute the identity defined in our initial andecedent for a to get $b^{nk} = x$, and we see that neccessarily $x \in \langle b \rangle$ since it is some power of b. Thus $x \in \langle a \rangle \implies x \in$

$\langle b \rangle \iff \langle a \rangle \subseteq \langle b \rangle$. We arrive at the implication $\left(\exists k \in \mathbb{Z} \ni a = b^k\right) \implies \langle a \rangle \subseteq \langle b \rangle$ and this proves theorem 3. $\square$

5 Chapter 11, Exercise D2

Suppose $a \in \langle b \rangle$ for some $a, b \in G$ where there exists some integer k such that $a = b^k$

Theorem 4. $\exists q \in \mathbb{Z} \ni a^q = b \iff \langle a \rangle = \langle b \rangle$

Proof. Suppose we have some $q \in \mathbb{Z}$ where $a^q = b$. We can rewrite $\langle b \rangle$ as the equivalent set $\{e, b, b^2 b^3, ..., b^{n-1}\}$. If we substitute a^q for every b, we have $\langle b \rangle = \{e, a^q, a^{2q}, a^{3q}, ..., a^{(n-1)q}\}$ and therefore a generates $\langle b \rangle$, or $\langle a \rangle = \langle b \rangle$.

Conversely, suppose $\langle a \rangle = \langle b \rangle$. Then, we can rewrite this equation as $\{e, a, a^2, a^3, ..., a^{n-1}\} = \{e, b, b^2, b^3, ..., b^{n-1}\}$. Since any $a^q \in \langle a \rangle = \langle b \rangle$, we must have an integer q where $a^q = b \in \langle b \rangle$.

Then, since we have the individual implication in both directions, we have the biconditional implication $\exists q \in \mathbb{Z} \ni a^q = b \iff \langle a \rangle = \langle b \rangle$. This proves theorem 4. $\square$

6 Chapter 12, Exercise B1

Suppose some $m, n \in \mathbb{Z}$.

Define a relation $\sim$ as $m \sim n \iff |m| = |n|$

Theorem 5. $\sim$ *is an equivalence relation.*

Proof. Suppose some $a, b, c \in \mathbb{Z}$.

Triviallly $|a| = |a| \iff a \sim a$, i.e. $\sim$ is reflexive.

Suppose $a \sim b$. Then, $|a| = |b| \iff |b| = |a|$ since equality is symmetric, so $a \sim b$ and we have $a \sim b \implies b \sim a$, i.e. $\sim$ is symmetric.

Suppose $a \sim b \wedge b \sim c$. Then, $|a| = |b| \wedge |b| = |c| \iff |a| = |c|$ since equality is transitive, so $a \sim c$ and we have $a \sim b \wedge b \sim c \implies a \sim c$, i.e. $\sim$ is transitive.

Then, $\sim$ is reflexive, symmetric, and transitive, if and only if $\sim$ is an equivalence relation. This proves theorem 5. $\square$

We can describe the eqivalence class of some $a \in \mathbb{Z}$ by $[a] = \{a, -a\}$.

7 Chapter 12, Exercise B5

Suppose some $m, n \in \mathbb{R}$.

Define a relation $\sim$ as $m \sim n \iff a - b \in \mathbb{Q}$.

Theorem 6. $\sim$ *is an equivalence relation.*

Proof. Suppose $a, b, c \in \mathbb{R}$.

Obviously $a - a = 0 \in \mathbb{Q}$, so $a \sim a$ and $\sim$ is reflexive.

Suppose that $a \sim b$. Then, $a - b \in \mathbb{Q} \iff b - a \in \mathbb{Q}$, so $b \sim a$. This demonstrates that $a \sim b \implies b \sim a$ so we see that $\sim$ is symmetric.

Suppose that $a \sim b \wedge b \sim c$. Then, $a - b \in \mathbb{Q} \wedge b - c \in \mathbb{Q} \iff a - b + b - c = a - c \in \mathbb{Q}$, so $a \sim c$. This demonstrates that $a \sim b \wedge b \sim c \implies a \sim c$ so we see that $\sim$ is transitive.

Then, $\sim$ is reflexive, symmetric, and transitive if and only if $\sim$ is an equivalence relation. This proves theorem 6. $\square$

If we let floor(x) denote the $n \in \mathbb{Q} \ni n \leq x \wedge (\forall m \in \mathbb{Z})(m \geq n \implies m = n)$, we can describe the equivalence class of an $x \in \mathbb{R}$ with respect to $\sim$ by $[x] = \{y : y = n + \text{floor}(x) \ni n \in \mathbb{Q}\}$. We see from this description of $[x]$ that $[0] = \mathbb{Q}$.

8 Chapter 12, Exercise D3

Suppose G is some group with elements $a, b \in G$.

Define a relation $\sim$ as $a \sim b \iff \exists x \in G \ni a = xbx^{-1}$.

Theorem 7. $\sim$ *is an equivalence relation.*

Proof. Let $a \in G$. We observe that $eae^{-1} = a \iff a \sim a$, so we see that $\sim$ is reflexive.

Let $a, b, x \in G \ni a \sim b$ such that $a = xbx^{-1}$. Then, we must have $b = x^{-1}ax$, by multiplication of the left side by x^{-1} and the right side by x. We see that $b = yay^{-1}$ for $y = x^{-1} \in G$, so $b \sim a$ and $a \sim b \implies b \sim a$, therefore $\sim$ is symmetric.

Let $a, b, c, x, y \in G \ni a \sim b \wedge b \sim c$ such that $a = xbx^{-1} \wedge b = ycy^{-1}$. By substitution, we find that $a = x(ycy^{-1})x^{-1} = xyc(xy)^{-1}$, and since $xy \in G$, it is evident that $a \sim c$ and $a \sim b \wedge b \sim c \implies a \sim c$, therefore $\sim$ is transitive.

Then, $\sim$ is reflexive, symmetric, and transitive if and only if $\sim$ is an equivalence relation. This proves theorem 7. $\square$

9 The cyclic subgroups of Z_{12}

The following are the cyclic subgroups of Z_{12}.

$$\langle 0 \rangle = \{0\} \text{ (trivial)} \tag{5}$$
$$\langle 1 \rangle = Z_{12} \text{ (trivial)} \tag{6}$$
$$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\} \tag{7}$$
$$\langle 3 \rangle = \{0, 3, 6, 9\} \tag{8}$$
$$\langle 4 \rangle = \{0, 4, 8\} \tag{9}$$
$$\langle 6 \rangle = \{0, 6\} \tag{10}$$
$$\tag{11}$$

Then, the following are the orders of the generator of each cyclic subgroup:

$$\mathrm{ord}(0) = 1 \tag{12}$$
$$\mathrm{ord}(1) = 12 \tag{13}$$
$$\mathrm{ord}(2) = 6 \tag{14}$$
$$\mathrm{ord}(3) = 4 \tag{15}$$
$$\mathrm{ord}(4) = 3 \tag{16}$$
$$\mathrm{ord}(6) = 2 \tag{17}$$

By inspection, we see that the orders of each generator element divide 12, where $12 = \mathrm{ord}(Z_{12})$.

10 The permutation known as $x + 1$

Suppose that $f(x) = x + 1 \in S_{\mathbb{R}}$.

Semi-formally, we can describe the elements of $\langle f \rangle$ as $\{..., x-2, x-1, x, x+1, x+2, x+3, x+4, ...\}$. In general, $\forall x \in \mathbb{Z}$ we have $f^n(x) = x + n$, the proof of which is trivial and hence omitted.

Theorem 8. $\mathbb{Z} \cong \langle f \rangle$

Proof. Define $\phi : \mathbb{Z} \to \langle f \rangle$ by $\phi(n) = f^n$. Suppose some $x, y \in \mathbb{Z}$.

Furthermore, suppose that $\phi(x) = \phi(y)$. Then since in general $f^n(x) = x + n$, we have in particular that $f^x(z) = x + z = y + z = f^y(z) \iff x = y$, so ϕ is injective.

Suppose that we have some $f^n \in \langle f \rangle$, where $n \in \mathbb{Z}$. Since $f^n(x) = x + n$, we see that $\phi(n) = f^n$, so ϕ is surjective.

At this point, we observe that ϕ is injecive and ϕ is surjective if and only if ϕ is bijecive.

Removing the previously imposed constraint that $\phi(x) = \phi(y)$, we see that $\phi(x + y) = f^{x+y} = f^x f^y = \phi(x)\phi(y)$, so we conclude that the bijection ϕ is an isomorphism from $\mathbb{Z}$ to $\langle f \rangle$, therefore $\mathbb{Z} \cong \langle f \rangle$. This proves theorem 8. □

11 The real-valued function known as $x + 1$

Suppose that $f(x) = x + 1 \in \mathcal{F}(\mathbb{R})$.

We can describe the elements of $\langle f \rangle$ as $\langle f \rangle = \{n(x+1) : n \in \mathbb{Z}\}$, the proof of which is trivial and hence omitted.

Theorem 9. $\mathbb{Z} \cong \langle f \rangle$

Note: I use multiplicative notation to denote vector addition of real-valued functions.

Proof. Define $\phi : \mathbb{Z} \to \langle f \rangle$ by $\phi(n) = f^n$. Suppose some $x, y \in \mathbb{Z}$.

Furthermore, suppose that $\phi(x) = \phi(y)$. Then since in general $f^n(x) = n(x + 1)$, we have $f^x(z) = x(z + 1) = y(z + 1) = f^y(z) \iff x = y$, so ϕ is injective.

Suppose that we have some $f^n \in \langle f \rangle$, where $n \in \mathbb{Z}$. Since in general we have $f^n(x) = n(x + 1)$, we see that $\phi(n) = f^n$, so ϕ is surjective.

At this point, we observe that ϕ is injecive and ϕ is surjective if and only if ϕ is bijecive.

Removing the previously imposed constraint that $\phi(x) = \phi(y)$, we see that $\phi(x + y) = f^{x+y} = f^x f^y = \phi(x)\phi(y)$, so we conclude that the bijection ϕ is an isomorphism from $\mathbb{Z}$ to $\langle f \rangle$, therefore $\mathbb{Z} \cong \langle f \rangle$. This proves theorem 9. □

12 A further look at $GL_2(\mathbb{R})$

Let $A \in GL_2(\mathbb{R})$ such that $A = \begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix} \wedge b \neq 0 \neq a$

Theorem 10. *For any integer* n*, we have* $\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix}$

Proof. First, note that $A^1 = A \iff \begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix}$.
If we assume that theorem 10 holds for some $n \in \mathbb{Z} \ni n > 0$, we find that $\left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix} \right) \implies \left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^{n+1} = \begin{bmatrix} a^{n+1} & (b^{n+1} - a^{n+1}) \\ 0 & b^{n+1} \end{bmatrix} \right)$ since $\begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix} \begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^{n+1} & (b^{n+1} - a^{n+1}) \\ 0 & b^{n+1} \end{bmatrix}$, so we see that theorem 10 holds when n is restricted to be greater than 0.

Then since A is a two by two matrix, we have $A^{-1} = \frac{1}{ab-(b-a)0} \begin{bmatrix} b & -(b-a) \\ -0 & a \end{bmatrix} = \begin{bmatrix} a^{-1} & (b^{-1} - a^{-1}) \\ 0 & b^{-1} \end{bmatrix}$ If we assume that If we assume that theorem 10 holds for some $n \in \mathbb{Z} \ni n < 0$, we find that $\left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix} \right) \implies \left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^{n-1} = \begin{bmatrix} a^{n-1} & (b^{n-1} - a^{n-1}) \\ 0 & b^{n-1} \end{bmatrix} \right)$ since $\begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix} \begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} a^{n-1} & (b^{n-1} - a^{n-1}) \\ 0 & b^{n-1} \end{bmatrix}$, so we see that theorem 10 holds when n is restricted to be greater than 0.

Thus we have covered every integer but the additive identity, $0 \in \mathbb{Z}$. We see that $A^0 = AA^{-1} = \begin{bmatrix} a^n & (b^n - a^n) \\ 0 & b^n \end{bmatrix} \begin{bmatrix} a^{-1} & (b^{-1} - a^{-1}) \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^0 & (b^0 - a^0) \\ 0 & b^0 \end{bmatrix}$ and conclude that 10 holds for any integer n. $\square$

By theorem 10 we have the following:

$$A^{-3} = \begin{bmatrix} a^{-3} & (b^{-3} - a^{-3}) \\ 0 & b^{-3} \end{bmatrix} \tag{18}$$

$$A^{-2} = \begin{bmatrix} a^{-2} & (b^{-2} - a^{-2}) \\ 0 & b^{-2} \end{bmatrix} \tag{19}$$

$$A^{-1} = \begin{bmatrix} a^{-1} & (b^{-1} - a^{-1}) \\ 0 & b^{-1} \end{bmatrix} \tag{20}$$

$$A^{0} = \begin{bmatrix} a^{0} & (b^{0} - a^{0}) \\ 0 & b^{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \tag{21}$$

$$A^{1} = \begin{bmatrix} a & (b - a) \\ 0 & b \end{bmatrix} \tag{22}$$

$$A^{2} = \begin{bmatrix} a^{2} & (b^{2} - a^{2}) \\ 0 & b^{2} \end{bmatrix} \tag{23}$$

$$A^{3} = \begin{bmatrix} a^{3} & (b^{3} - a^{3}) \\ 0 & b^{3} \end{bmatrix} \tag{24}$$

13 Proof of an equivalence relation

Suppose $f, g, h \in \mathcal{F}(\mathbb{R})$.

Define the relation $\sim$ as:

$$f \sim g \iff (\forall x \in \mathbb{R})(f(x) - g(x) = c \text{ for some } c \in \mathbb{R}) \tag{25}$$

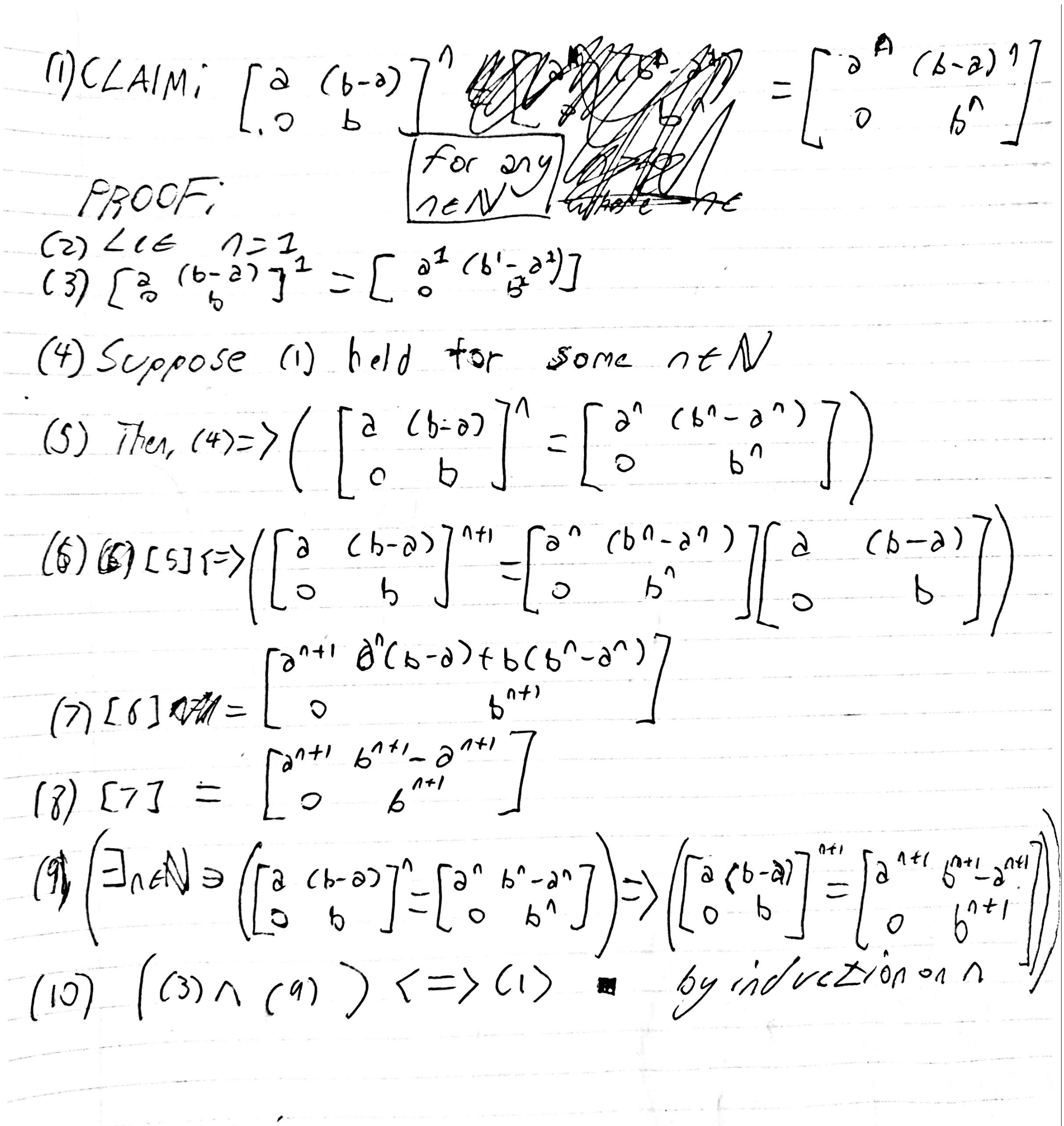
Theorem 11. *$\sim$, as described in (25), is an equivalence relation*

Proof. Obviously $f(x) - f(x) = 0 \in \mathbb{R}$ for all real x, so we have that $f \sim f$ i.e. $\sim$ is reflexive.

Then, if we suppose that $f(x) - g(x) = c \in \mathbb{R}$ for all real x, we must then have that $g(x) - f(x) = -c \in \mathbb{R}$, so $f \sim g \implies g \sim f$ i.e. $\sim$ is symmetric.

Alternatively, if we suppose that both $f(x) - g(x) = c \in \mathbb{R} \land g(x) - h(x) = d \in \mathbb{R}$ for all real x, we can add the two equations together to get $f(x) - h(x) = c + d \in \mathbb{R}$, so we must have $f \sim g \land g \sim h \implies f \sim h$, i.e. $\sim$ is transitive.

Finally, $\sim$ is reflexive, symmetric, and transitive if and only if $\sim$ is an equivalence relation. This proves theorem 11. $\square$



(1) CLAIM: $\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & (b-a)^n \\ 0 & b^n \end{bmatrix}$ $\boxed{\text{for any } n \in N}$

PROOF:

(2) Let $n = 1$

(3) $\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^1 = \begin{bmatrix} a^1 & (b^1-a^1) \\ 0 & b^1 \end{bmatrix}$

(4) Suppose (1) held for some $n \in N$

(5) Then, (4) $\Rightarrow \left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & (b^n-a^n) \\ 0 & b^n \end{bmatrix} \right)$

(6) [5] $\Leftrightarrow \left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^{n+1} = \begin{bmatrix} a^n & (b^n-a^n) \\ 0 & b^n \end{bmatrix} \begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix} \right)$

(7) [6] $= \begin{bmatrix} a^{n+1} & a^n(b-a)+b(b^n-a^n) \\ 0 & b^{n+1} \end{bmatrix}$

(8) [7] $= \begin{bmatrix} a^{n+1} & b^{n+1}-a^{n+1} \\ 0 & b^{n+1} \end{bmatrix}$

(9) $\left(\exists n \in N \ni \left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & b^n-a^n \\ 0 & b^n \end{bmatrix} \right) \Rightarrow \left(\begin{bmatrix} a & (b-a) \\ 0 & b \end{bmatrix}^{n+1} = \begin{bmatrix} a^{n+1} & b^{n+1}-a^{n+1} \\ 0 & b^{n+1} \end{bmatrix} \right) \right)$

(10) $((3) \wedge (9)) \Leftrightarrow (1)$ ■ by induction on n

Figure 1: An early sketch for the proof of theorem 10

14 Proof of another equivalence relation

Suppose X is a set, and define the bijection $f : X \to X$.

Define the following relation for some $a, b \in X$:

$$a \sim b \iff \exists n \in \mathbb{Z} \ni f^n(a) = b \tag{26}$$

Theorem 12. *$\sim$, as described in (26), is an equivalence relation*

Proof. Suppose $a, b, c \in X$.

Obviously $\exists n \in \mathbb{Z} \ni f^n(a) = a$ since $f^0(a) = a$ so we have that $a \sim a$ i.e. $\sim$ is reflexive.

Then, if we suppose that $a \sim b$, we have that some integer n must exist such that $f^n(a) = b$, but since f is bijective, we can apply f^{-1} to both sides of the last equation n times to get $f^{-n}(b) = a$ and since $-n \in \mathbb{Z}$ we then have $b \sim a$ and so we have the implication that $a \sim b \implies b \sim a$, i.e. $\sim$ is symmetric.

Alternatively, if we suppose that $a \sim b \wedge b \sim c$ hold, we have integers n and m such that $f^n(a) = b \wedge f^m(b) = c$, so we can construct the composition $(f^n \circ f^m)(a) = f^{n+m}(a) = c$, and since $n + m \in \mathbb{Z}$, we have $a \sim c$ and so we have the implication that $a \sim b \wedge b \sim c \implies a \sim c$, i.e. $\sim$ is symmetric.

Finally, $\sim$ is reflexive, symmetric, and transitive if and only if $\sim$ is an equivalence relation. This proves theorem 12. $\square$