Abstract Algebra: Homework #8

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1 Chapter 15, Exercise A1

Suppose $G = \mathbb{Z}_{10} \wedge H = \{0, 5\}$.

Then, table 1 describes the operation table for G/H with respect to coset multiplication defined for cosets of an abelian group, denoted $*$.

I exclusively use multiplicative notation here because I like it better, but $aH \ni a \in G$ denotes the coset $a +_{10} H$. Since G is abelian, I use left and right cosets interchangably.

The following are the elements of G/H:

$$
\begin{aligned}
H0 &= \{0, 5\} \\
H1 &= \{1, 6\} \\
H2 &= \{2, 7\} \\
H3 &= \{3, 8\} \\
H4 &= \{4, 9\}
\end{aligned}
$$

$*$	H0	H1	H2	H3	H4
H0	H0	H1	H2	H3	H4
H1	H1	H2	H3	H4	H0
H2	H2	H3	H4	H0	H1
H3	H3	H4	H0	H1	H2
H4	H4	H0	H1	H2	H3

Table 1: Operation table for G/H under $*$

If we replace each HX in the table with an $\mathsf{f}(\mathsf{HX})$ where $\mathsf{f} : \mathsf{G}/\mathsf{H} \to \mathbb{Z}_5 \ni \mathsf{f}(\mathsf{HX}) = \mathsf{X}$ and replace $*$ by $+_5$, we construct the operation table for $\mathbb{Z}_5$. By table inspection, this f is an isomorphism from G/H to $\mathbb{Z}_5$, so clearly $\mathsf{G}/\mathsf{H} \cong \mathbb{Z}_5$.

2 Chapter 15, Exercise A4

Denote the elements of D_4 as:

$$R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad R_{\pi/2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad R_{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad R_{3\pi/2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \tag{1}$$

$$H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \quad D' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \tag{2}$$

The operation table for function composition $\circ$ on D_4 is given in table 1

$\circ$	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
R_0	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
$R_{\pi/2}$	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	R_0	D'	D	H	V
R_π	R_π	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	V	H	D'	D
$R_{3\pi/2}$	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	R_π	D	D'	V	H
H	H	D	V	D'	R_0	R_π	$R_{\pi/2}$	$R_{3\pi/2}$
V	V	D'	H	D	R_π	R_0	$R_{3\pi/2}$	$R_{\pi/2}$
D	D	V	D'	H	$R_{3\pi/2}$	$R_{\pi/2}$	R_0	R_π
D'	D'	H	D	V	$R_{\pi/2}$	$R_{3\pi/2}$	R_π	R_0

Table 2: Operation table for D_4 under $\circ$

Now that we have better notation than Pinter, let $G = D_4 \wedge H \leq G \ni H = \{R_0, R_\pi, H, V\}$

Note that other than H, G/H contains only one other element since $(G : H) = 2$.

We can then fully describe $G/H = \Big\{\{R_0, R_\pi, H, V\}, \{R_{\pi/2}, R_{3\pi/2}, D, D'\}\Big\}$.

Table 3 give the operation table for G/H under coset multiplication:

$*$	H	HD
H	H	HD
HD	HD	H

Table 3: Operation table for G/H under coset multiplication

3 Chapter 15, Exercise C1

Suppose $H \trianglelefteq G$ where G is a group.

Theorem 1. $(\forall x \in G)(x^2 \in H) \iff (\forall X \in G/H)(X^2 = H)$

Proof. Suppose that $(\forall x \in G)(x^2 \in H)$. Let $X \in G/H$. Then, $X = Hx \ni x \in G$. Therefore, $XX = (Hx)(Hx) = H(x^2) = H$ since $x^2 \in H \implies h(x^2) \in H$ for any $h \in H$ since H is closed under the group operation. Then, $(\forall X \in G/H)(X^2 = H)$.

Conversely, suppose $(\forall X \in G/H)(X^2 = H)$. Let $x \in G$ and let $X = Hx$. Then, $X^2 == (Hx)(Hx) = Hx^2$. By assumption, $X^2 = Hx^2 = H$, and by Pinter chapter 15 theorem 5 part 2, we have $Hx^2 = H \iff x^2 \in H$.

The first implication and its converse thus proved demostrates bidirecitonal implication. This proves theorem 1. □

4 Chapter 15, Exercise D1

Suppose $H \trianglelefteq G$ where G is a group.

Theorem 2. $|H| \in \mathbb{N} \land |G/H| \in \mathbb{N} \implies |G| \in \mathbb{N}$

Proof. Let $n = |H| \in \mathbb{N}$ and let $m = |G/H|$. Since G/H is the set of all left cosets of H with respect to G, we can write $|G/H|$ as $(G : H)$, the index of H with respect to G. By Lagrange's theorem, we have $|G| = (G : H) \cdot |H| = mn$. Since $m, n \in \mathbb{N}$ and the naturals are closed under multiplication, we must have $|G| \in \mathbb{N}$. This proves theorem 2. □

5 Chapter 15, Exercise E2

Suppose $H \trianglelefteq G$ where G is a group.

Theorem 3. $m = (G : H) \implies (\forall x \in G/H)(\text{ord}(x)|m)$

Proof. Suppose $\mathfrak{m} = (\mathsf{G} : \mathsf{H})$. Then, since $(\mathsf{G} : \mathsf{H}) = |\mathsf{G}/\mathsf{H}|$, we have as a consequence of Lagrange's theorem that $(\forall x \in \mathsf{G}/\mathsf{H})(\operatorname{ord}(x) \mid |\mathsf{G}/\mathsf{H}|)$, therefore $(\forall x \in \mathsf{G}/\mathsf{H})(\operatorname{ord}(x)|\mathfrak{m})$. This proves theorem 3. ☐

6 Chapter 15, Exercise E5

Suppose $\mathsf{H} \trianglelefteq \mathsf{G}$ where G is a group.

Theorem 4. $\mathfrak{m} = (\mathsf{G} : \mathsf{H}) \implies a^{\mathfrak{m}} \in \mathsf{G}$ *for any* $a \in \mathsf{G}$

Proof. Suppose $\mathfrak{m} = (\mathsf{G} : \mathsf{H})$ and let $a \in \mathsf{G}$. Then, $a^{\mathfrak{m}} \in \mathsf{G}$ since G is closed under multiplication. This proves theorem 4 for some reason. ☐

7 Chapter 15, Exercise E6

Suppose $\mathsf{H} \trianglelefteq \mathsf{G}$ where G is a group.

Theorem 5. $(\forall x \in \mathbb{Q}/\mathbb{Z})(\operatorname{ord}(x) \in \mathbb{N})$

Proof. Suppose $x \in \mathbb{Q}/\mathbb{Z}$. Then x can be written as some unique $y + \mathbb{Z}$, where $y \in \mathbb{Q}$. Then, we can write $y = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ by the definition of $\mathbb{Q}$. We want to find an $n \in \mathbb{N}$ such that $n(y+\mathbb{Z}) = \mathbb{Z}$. Any $n(y+\mathbb{Z})$ is just $(ny+\mathbb{Z})$, however this n is no aribtrary integer, it is in fact the same n we see in the denominator of $y = \frac{m}{n}$, for then $(ny+\mathbb{Z}) = (n\frac{m}{n} = \mathbb{Z}) = (m+\mathbb{Z})$, and since $m \in \mathbb{Z}$, we have $m+\mathbb{Z} = \mathbb{Z}$, and every $x \in \mathbb{Q}/\mathbb{Z}$ satisfies $n(y+\mathbb{Z}) = \mathbb{Z}$ for any $y = \frac{m}{n} \in \mathbb{Q}$, therefore $\operatorname{ord}(x) = n \in \mathbb{N}$. This proves theorem 5. ☐