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# Exam 1

Differential Equations Summer 2020

**1** Suppose: $\frac{dy}{dx} = \frac{y^2+y}{x^2(y+4)}$ (1)

then,

$$\frac{y+4}{y^2+y}\frac{dy}{dy} = \frac{1}{x^2} \iff \int \frac{y+4}{y(y+1)}\,dy = \int \frac{1}{x^2}\,dx$$

$$\int \frac{y+4}{y(y+1)}\,dy = \int \frac{y}{y(y+1)}\,dy + \int \frac{4}{y(y+1)}\,dy = \int \frac{1}{y+1}\,dy + \int \frac{4}{y}\,dy + \int \frac{-4}{y+1}\,dy$$

$$\frac{4}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \iff 4 = A(y+1) + By \quad \begin{array}{l} y=0 \Rightarrow A=4 \\ y=-1 \Rightarrow B=-4 \end{array}$$

$$= \ln|y+1| + 4\ln|y| - 4\ln|y+1| = -\frac{1}{x} + C$$

introduce $C \in \mathbb{R}$

$$\ln|y+1| + 4\left(\ln\left|\frac{y}{y+1}\right|\right) = -x^{-1} + C$$

$$e^{\ln\left|\frac{y^4}{(y+1)^3}\right|} = \ln\left|\frac{(y+1)y^4}{(y+1)^4}\right| = e^{-x^{-1}+C}$$

Let $D = \pm e^C$

$$\boxed{\Rightarrow \frac{y^4}{(y+1)^3} = De^{-x^{-1}}}$$

is an implicit solution to eq. 1

2. Suppose $\frac{dy}{dx} = x^3 e^x$ (1)

Then, $y$ is simply $\int x^3 e^x dx$ by the fundamental theorem of Calculus. (2)

We integrate (2) by parts.

(3) $\int x^3 e^x dx = x^3 e^x - 3\int x^2 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$

$f' = e^x \quad g = x^3 \quad \Big| \quad f' = e^x \quad g = x^2$

$f = e^x \quad g' = 3x^2 \quad \Big| \quad f = e^x \quad g' = 2x$

(4) (3) $= x^3 e^x - 3x^2 e^x + 6\int x e^x dx = x^3 e^x - 3x^2 e^x + 6\left[xe^x - \int e^x dx\right]$

$f' = e^x \quad g = x$

$f = e^x \quad g' = 1$

Introducing $C$

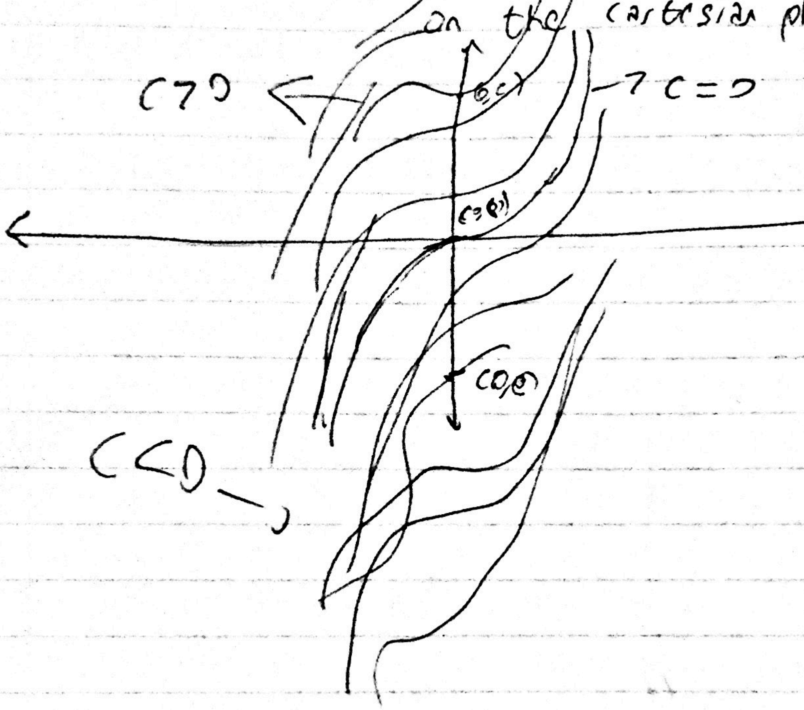
(5) (4) $= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$

Finally we now have:

$$\boxed{\therefore y(x) = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C}$$

3. Let $\frac{dy}{dx} = 3x^2$

then, by simple integration, $y$ must be some $x^3 + C$ for any real $C$.




(simple odd function, a polynomial of degree three)

4 Suppose $\frac{dy}{dx} + \frac{2}{x} y = 3\sqrt{y}$

~~then~~ $\iff \frac{dy}{dx} + \underbrace{\left(\frac{2}{x}\right)}_{P(x)} y = \underbrace{(3)}_{Q(x)} y^{1/2}$

We observe a bernoulli differential equation where $n = \frac{1}{2}$
so we let $v = y^{1-n} = \sqrt{y}$
and then we have: since $1-n = 1-\frac{1}{2} = \frac{1}{2}$

$$\frac{dv}{dx} + (1-n)P(x)y = Q(x)(1-n)$$

$$\implies \frac{dv}{dx} + \left(\frac{1}{2}\right)\left(\frac{2}{x}\right) v = (3)\left(\frac{1}{2}\right)$$

Let $\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$\iff \frac{dv}{dx} + \frac{1}{x} \cdot v = \frac{3}{2}$$

$$\iff x\frac{dv}{dx} + v = x\frac{3}{2}$$

$\frac{3}{4}x^2 + C$

$$\iff xv = \int \frac{3}{2} x \, dx = \frac{3}{4}x^2 + C$$

$\cancel{v = \frac{3}{2} + \frac{C}{x}}$ $\cancel{\iff v = \frac{(3/2)x + C}{x}}$ $\cancel{\iff \sqrt{y} = \frac{3}{2} + \frac{C}{x}}$ $\cancel{\iff y = \left(\frac{3}{2} + \frac{C}{x}\right)^2 = \left(\frac{3}{2}\right)^2 + 3\frac{C}{x} + \left(\frac{C}{x}\right)^2}$

$$xv = \frac{3}{4}x^2 + C$$

$$v = (3/4)x + C/x$$

$$y = ((3/4)x + C/x)^2 \quad \text{since } v = \sqrt{y}$$

$$\boxed{\therefore y(x) = \left[\left(\frac{3}{4}\right)x + \frac{C}{x}\right]^2}$$

5 Suppose $x\frac{dy}{dx} = y - \frac{x^2}{y} \wedge x > 0$

$$\iff \frac{dy}{dx} - \frac{y}{x} = -\frac{x}{y}$$

Let $v = \frac{y}{x}$ $\iff v^{-1} = \frac{x}{y}$

$y = xv$ $\implies \frac{dy}{dx} = v + xv'$

$$\iff v + x\frac{dv}{dx} - v = -\frac{1}{v}$$

$$\iff x\frac{dv}{dx} = \frac{-1}{v} \iff v\frac{dv}{dx} = \frac{-1}{x} \iff \int v\,dv = \int \frac{-1}{x}\,dx$$

$$\implies \frac{1}{2}v^2 = -\ln|x| + C$$

$$v^2 = -2\ln|x| + 2C$$

$$\iff \left(\frac{y}{x}\right)^2 = -2\ln|x| + 2C$$

$$y^2 = -2x^2\ln|x| + 2x^2C$$

$$y^2 = 2x^2(C - \ln|x|)$$

$x > 0 \implies \ln|x| = \ln x$

$$y(x) = \sqrt{2}\,x\sqrt{C - \ln x}$$

let $C_1 = 2C$ (arbitrary anyway) for cleanliness.

$$\boxed{\therefore y(x) = x\sqrt{C_1 - \ln(x^2)}}$$

$\implies$ (I put the 2 back in the radical to get $2\ln x = \ln(x^2)$)

(1) 6 Suppose $(y^2+\sec^2 x)\,dx + (6y^5+2xy)\,dy = 0$

by which we really mean: $\underbrace{(y^2+\sec^2 x)}_{M} + \underbrace{(6y^5+2xy)}_{N}\frac{dy}{dx} = 0$

(1) is an exact equation
meaning that $\exists$ some $f$ where $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

iff: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(2) $\frac{\partial M}{\partial y} = \frac{d}{dy}\left[y^2+\sec^2 x\right] = 2y = \frac{\partial N}{\partial x} = \frac{d}{dx}\left[6y^5+2xy\right] = 2y$

since (2) holds, we can integrate
$M$ to get ~~functions~~ $f(x,y)$ in
terms of some $g(y)$.

(3) $\int M(x,y)\,dx = \int y^2+\sec^2 x\,dx = y^2x + \tan x + g(y)$

(4) so $\frac{\partial f}{\partial y} = (6y^5+2xy) = 2xy + \frac{dg}{dy}$

(5) $6y^5 = \frac{dg}{dy}$

(6) $\Rightarrow \quad y^6 = g(y)$

(7) $\Rightarrow f(x,y) = y^2x + \tan x + y^6$

since $f(x,y) \in \mathbb{R}$ is constant
~~Let D be this constant.~~ Let $D \in \mathbb{R}$ be this constant
Finally, we have our solution to (1):

(8) $\boxed{D = y^2x + \tan x + y^6}$

7 tank capacity = 200 gallons

~~init~~ $x(0) = 20$

$V(0) = 50 = V_0$

$c_i = 1\ \text{lb/gal}$

$r_i = 10\ \text{gal/hour}$ $\qquad c_0 = \dfrac{x(t)}{V_0 + t(r_i - r_0)}$

$r_0 = 5\ \text{gal/hour}$

$x : \mathbb{R} \to \mathbb{R}$ is the amount of lbs salt in the tank at time $t$

$V : \mathbb{R} \to \mathbb{R}$ is the volume of solution in the tank at time $t$

then, $\dfrac{dx}{dt} = r_i c_i - r_0 c_0 = (1\ \text{lb/gal})(10\ \text{gal/h}) - \dfrac{(5\ \text{gal/h})\, x(t)}{\underbrace{50 + (10-5)t}_{V(t)}}$

$$\frac{dx}{dt} = 10\ \text{lb/h} - \frac{5\ \text{gal/h}\ x(t)\ \text{lbs}}{50+5t\ \text{gal}} = 10 - \frac{5x}{50+5t} = 10 - \frac{5\,(x)}{5(10+t)} = 10 - \frac{x}{10+t}$$

$$\frac{dx}{dt} = 10 - \frac{x(t)}{10+t} \iff \frac{dx}{dt} + \frac{1}{10+t}x(t) = 10$$

Let $p(x) = e^{\int \frac{1}{10+t}dt} = e^{\ln(10+t)} = 10+t$ $\qquad \frac{5(t^2+20t+40)}{5t}$

So, $(10+t)\dfrac{dx}{dt} + x(t) = 10(10+t)$

$$(10+t)x(t) = \int 100 + 10t\, dt = 100t + 5t^2 + C$$

$$\Rightarrow x(t) = \frac{100t + 5t^2 + C}{10+t} \wedge x(0) = 20$$

$$\Rightarrow x(0) = 20 = \frac{100\cdot 0 + 5\cdot 0^2 + c}{10+0} = \frac{c}{10}$$

$$\iff c = 200$$

$$\Rightarrow x(t) = \frac{100t + 5t^2 + 200}{10+t}$$

Find fill time:

~~for $V(t) = 200$~~

$V(t) = 200 = 50 + 5t$

$150 = 5t$

$30 = t$

$\therefore$ the tank is full (200 gal) after 30 hours

salt in tank at 30 hrs:

$$x(30) = \frac{100\cdot 30 + 5(30^2) + 200}{10+30}$$

$$= \frac{3000 + 4500 + 200}{40} = \frac{7700}{40}$$

$$\boxed{= \frac{770}{4} = \frac{385}{2} = 192.5\ \text{lbs}}$$