Exam II

Differential Equations
University of Massachusetts Lowell
Summer 2020

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1. Solve the IVP:
Given:

$$y'' - 5y' - 6 = 0 \tag{1}$$
$$y(0) = 4 \wedge y'(0) = 3 \tag{2}$$

Guess that $y = e^{rx}$ for some $r \in \mathbb{R}$, then:

$$y' = re^{rx} \wedge y'' = r^2 e^{rx} \tag{3}$$
$$r^2 e^{rx} - 5re^{rx} - 6e^{rx} = 0 \text{ (apply (3) to (1))} \tag{4}$$
$$r^2 - 5r - 6 = 0 \text{ (divide by } e^{rx}\text{)} \tag{5}$$
$$(r-6)(r-1) = 0 \implies r \in \{-1, 6\} \tag{6}$$
$$(6) \implies y(x) = Ae^{-x} + Be^{6x} \tag{7}$$
$$(2) \wedge (7) \implies A + B = 4 \text{ (apply initial 1)} \tag{8}$$
$$(7) \implies \frac{dy}{dx} = -Ae^{rx} + 6Be^{rx} \tag{9}$$
$$(2) \wedge (9) \implies -A + 6B = 3 \text{ (apply initial 2)} \tag{10}$$
$$\left(\begin{bmatrix} 1 & 1 & 4 \\ -1 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \right) \iff \Big(A = 3 \wedge B = 1 \Big) \tag{11}$$
$$\therefore y(x) = 3e^{-x} + e^{6x} \tag{12}$$

2. Find and categorize the equilibrium solutions of the ODE
Given:

$$\frac{dy}{dx} = y(y-1)^2 \tag{13}$$

Let $f(y) = \frac{dx}{dy}$, then we have table .

y	$f(y)$
-3	-48
-2	-18
-1	-4
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	0
2	2
3	12

Table 1: The rate of change of the dependent variable at different values

We see that f is decreasing at values approaching $y = 0$ from the negative side and increasing at values approacing $y = 0$ from the positive side, so we say that the equilibrium solution $y = 0$ is unstable.

We also see that f is increasing at values approaching $y = 1$ from both sides, so we say that the equilibrium solution $y = 1$ is semi-stable.

3. Find the general solution

Given:

$$y = x^r \text{ for some } r \in \mathbb{R} \tag{14}$$

$$xy'' - 6y' = 0 \tag{15}$$

We can use this information to find the general solution:

$$y' = rx^{r-1} \wedge y'' = (r^2 - r)x^{r-2} \tag{16}$$

$$(15) \wedge (16) \implies (r^2 - r)x^{r-1} - 6rx^{r-1} = 0 \tag{17}$$

$$r^2 - 7r = r(r-7) = 0 \text{ (divide by } x^{r-1}) \tag{18}$$

$$(18) \implies r \in \{0, 7\} \wedge y(x) = Ax^0 + Bx^7 \text{ for any } A, B \in \mathbb{R} \tag{19}$$

$$\therefore y(x) = A + Bx^7 \text{ for any real } A \text{ and } B \tag{20}$$

4. Population over time

Suppose we have a population of hamsters or something described by the following equations, where 2010 is considered to be the begining of time.

$$\frac{dP}{dt} = AP \text{ for some } A \in \mathbb{R} \tag{21}$$

$$P(0) = 100 \wedge P(2) = 400 \tag{22}$$

We see that equation 22 is seperable and so we proceed to solve for the population in 2015, or in other words at $t = 5$.

$$(21) \iff \frac{1}{P}\frac{dP}{dt} = A \iff \int \frac{1}{P} dP = \int A dt \tag{23}$$

$$(23) \iff \ln P = At + B \text{ for some } B \in \mathbb{R} \tag{24}$$

$$(24) \iff P(t) = Ce^{At} \text{ for } C = e^{B} \tag{25}$$

$$P(0) = 100 \iff C = 100 \tag{26}$$

$$P(t) = 100e^{At} \tag{27}$$

$$P(2) = 400 \iff 100e^{2A} = 400 \iff e^{2A} = 4 \iff A = \ln 2 \tag{28}$$

$$\therefore P(t) = 100 \cdot 2^{t} \wedge P(5) = 3200 \text{ hamsters or something} \tag{29}$$

Then, we will have 3200 gerbils in 2015 and the population will double every year thereafter until the entire universe is consumed by the gerbil population. Doomsday...

5. Compute the general solution
Given:

$$y'' - 2y' + 5y = 0 \tag{30}$$

We guess that $y = e^{rx}$ for some $r \in \mathbb{C}$, then:

$$y' = re^{rx} \wedge y'' = r^{2}e^{rx} \tag{31}$$

$$(30) \wedge (31) \iff r^{2}e^{rx} - 2re^{rx} + 5e^{rx} = 0 \tag{32}$$

$$(32) \implies r^{2} - 2r + 5 = 0 \implies r \in \{1 - 2i, 1 + 2i\} \tag{33}$$

$$y(x) = Ae^{1-2i} + Be^{1+2i} \text{ for some } A, B \in \mathbb{C} \tag{34}$$

$$e^{ix} = \cos(x) + i\sin(x) \iff y(x) = e^{x}\Big((A + B)\cos(2x) + i(B - A)\sin(2x)\Big) \tag{35}$$

$$\therefore y(x) = e^{x}\Big(C\cos(2x) + D\sin(2x)\Big) \text{ for } C = A + B \wedge D = i(B - A) \tag{36}$$

6. Determine linear independence of functions
Given:

$$f(x) = xe^{x} \tag{37}$$

$$g(x) = x^{2} \tag{38}$$

We also have the following derivatives:

$$\frac{df}{dx} = e^x + xe^x \tag{39}$$

$$\frac{dg}{dx} = 2x \tag{40}$$

We use the Wronskian determinant to determine linear indepenence. Two functions f and g are linearly independent iff their Wronskian determinant $W(f, g) \neq 0$ where 0 is the zero function in the vector space of continuous functions.

We compute:

$$W(f, g) = \begin{vmatrix} xe^x & x^2 \\ e^x + xe^x & 2x \end{vmatrix} = 2x^2e^x - x^2e^x + x^3x^x \neq 0 \tag{41}$$

We see that $W(f, g) \neq 0$ so we conclude that f and g are linearly independent.

7. Mass attatched to a spring with a dampener

Suppose we have an $m = 2$ Kg mass subject a dampening force with the coefficient $c = 14$ pulled away from a spring $\Delta x = -2$ meters at a force of $F = 48$ Newtons.

By Hooke's law, we have $F = -k\Delta x \iff 48 = 2k \iff k = 24$.

Free spring mass movement is described by the following equation where $x(t)$ is a function of the position of the mass relative the the equilibrium position of the spring with respect to time:

$$mx'' + cx' + kx = 0 \tag{42}$$

We are also given the follwing initial values:

$$x(0) = 2 \tag{43}$$

$$x'(0) = 3 \tag{44}$$

Now, we can plug in the above values to equation 42 and solve by guessing that $x(t) = e^{rt}$ for some real r:

$$2x'' + 14x' + 24x = 0 \tag{45}$$

$$2r^2e^{rt} + 14re^{rt} + 24e^{rt} = 0 \tag{46}$$

$$2r^2 + 14r + 24 = 0 \tag{47}$$

$$r^2 + 7r + 12 = (r+3)(r+4) = 0 \implies r \in \{-3, -4\} \tag{48}$$

$$x(t) = Ae^{-3t} + Be^{-4t} \text{ for some } A, B \in \mathbb{R} \tag{49}$$

$$x(0) = 2 \iff A + B = 2 \tag{50}$$

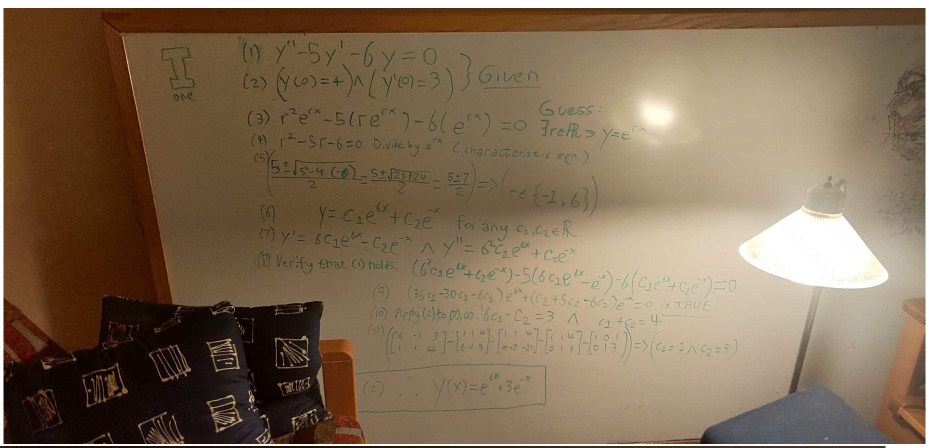
$$x'(t) = -3Ae^{-3t} - 4Be^{-3t} \tag{51}$$

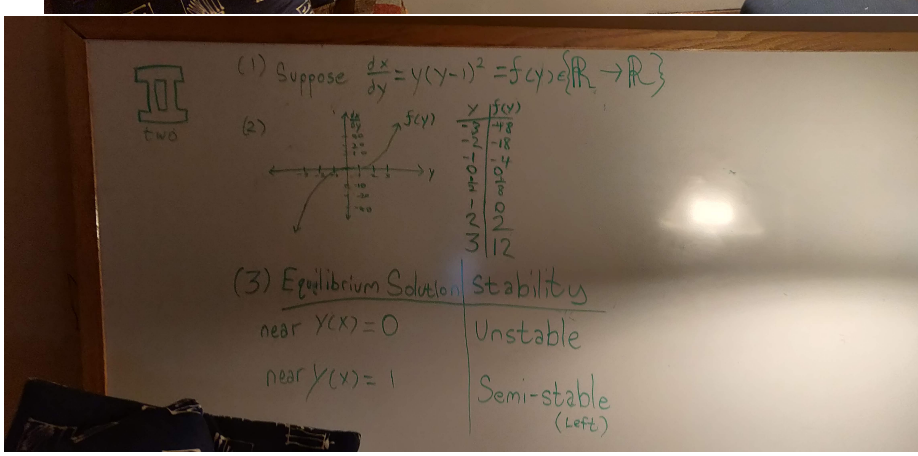
$$x'(0) = 3 \iff -3A - 4B = 3 \tag{52}$$

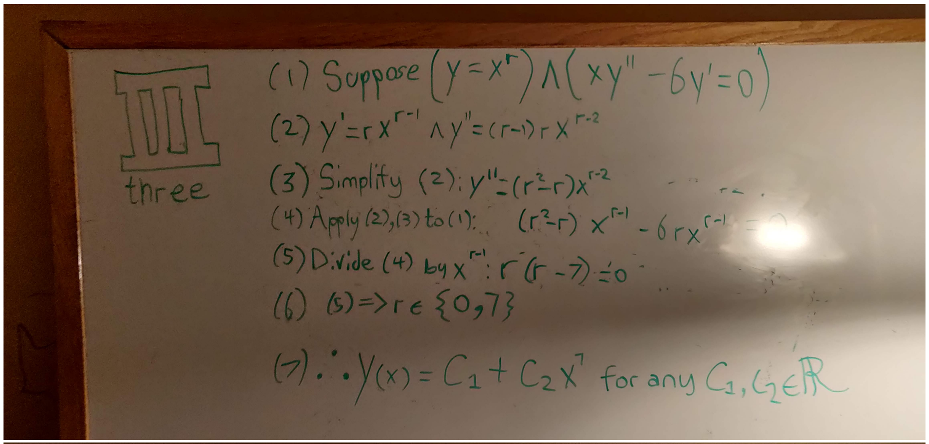
$$\begin{bmatrix} 1 & 1 & 2 \\ -3 & -4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 11 \\ 0 & -1 & 9 \end{bmatrix} \iff A = 11 \wedge B = -9 \tag{53}$$

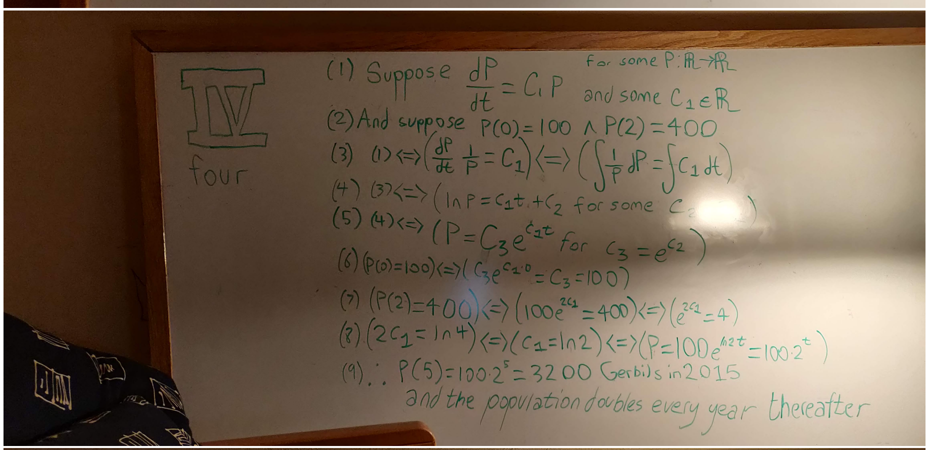
$$\therefore x(x) = 11e^{-3t} - 9e^{-4t} \tag{54}$$

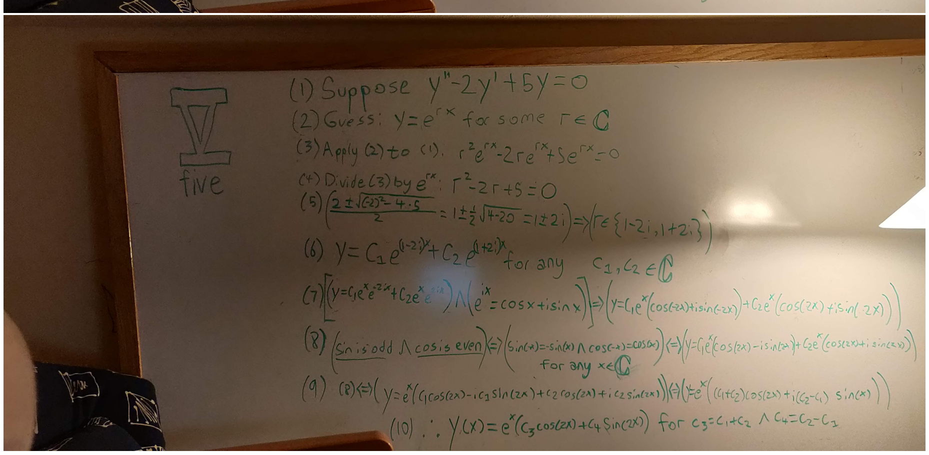
Appendix. Original scratch work

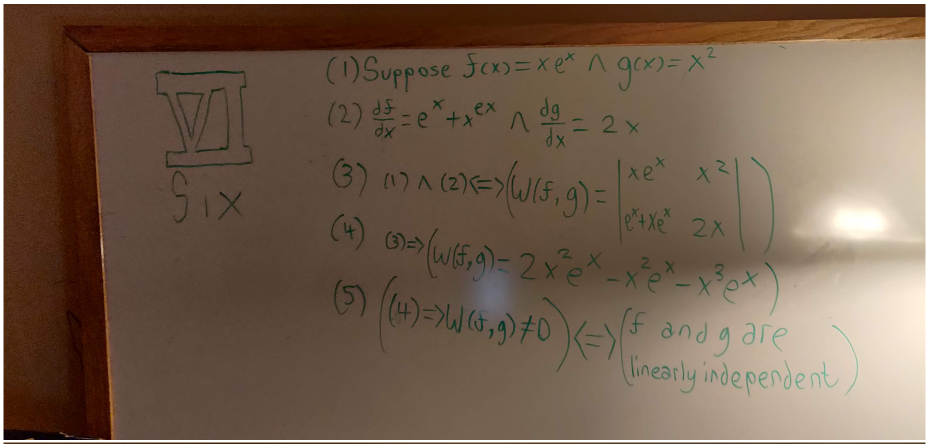


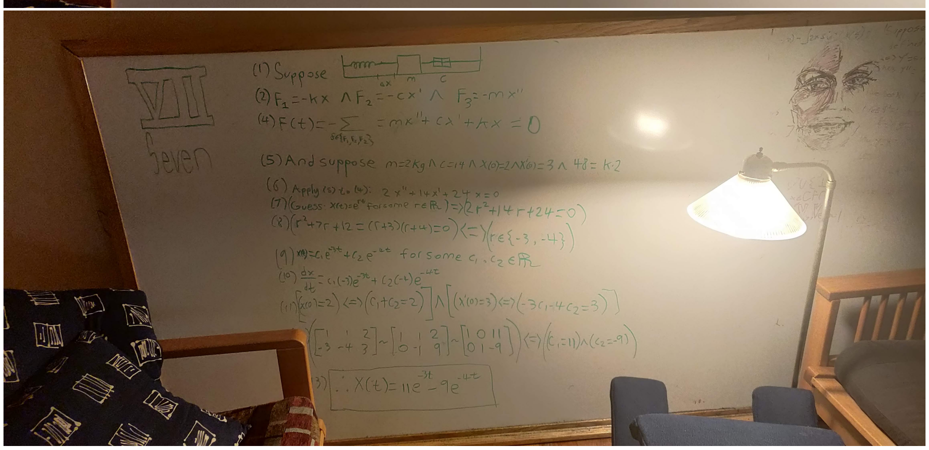



III
three
(1) Suppose $(y = x^r) \wedge (xy'' - 6y' = 0)$
(2) $y' = rx^{r-1} \wedge y'' = (r-1)rx^{r-2}$
(3) Simplify (2): $y'' = (r^2 - r)x^{r-2}$
(4) Apply (2),(3) to (1): $(r^2 - r)x^{r-1} - 6rx^{r-1} = 0$
(5) Divide (4) by x^{r-1}: $r(r-7) = 0$
(6) (5) $\Rightarrow r \in \{0, 7\}$
(7) $\therefore y(x) = C_1 + C_2 x^7$ for any $C_1, C_2 \in \mathbb{R}$


IV
four
(1) Suppose $\frac{dP}{dt} = C_1 P$ for some $P: \mathbb{R} \to \mathbb{R}$ and some $C_1 \in \mathbb{R}$
(2) And suppose $P(0) = 100 \wedge P(2) = 400$
(3) (1) $\Leftrightarrow \left(\frac{dP}{dt}\frac{1}{P} = C_1\right) \Leftrightarrow \left(\int \frac{1}{P}\, dP = \int C_1\, dt\right)$
(4) (3) $\Leftrightarrow (\ln P = C_1 t + C_2$ for some C_2)
(5) (4) $\Leftrightarrow (P = C_3 e^{C_1 t}$ for $C_3 = e^{C_2})$
(6) $(P(0) = 100) \Leftrightarrow (C_3 e^{C_1 \cdot 0} = C_3 = 100)$
(7) $(P(2) = 400) \Leftrightarrow (100e^{2C_1} = 400) \Leftrightarrow (e^{2C_1} = 4)$
(8) $(2C_1 = \ln 4) \Leftrightarrow (C_1 = \ln 2) \Leftrightarrow (P = 100e^{\ln 2 t} = 100 \cdot 2^t)$
(9) $\therefore P(5) = 100 \cdot 2^5 = 3200$ Gerbils in 2015
and the population doubles every year thereafter


V
five
(1) Suppose $y'' - 2y' + 5y = 0$
(2) Guess: $y = e^{rx}$ for some $r \in \mathbb{C}$
(3) Apply (2) to (1): $r^2 e^{rx} - 2re^{rx} + 5e^{rx} = 0$
(4) Divide (3) by e^{rx}: $r^2 - 2r + 5 = 0$
(5) $\left(\frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = 1 \pm \frac{1}{2}\sqrt{4 - 20} = 1 \pm 2i\right) \Rightarrow (r \in \{1-2i, 1+2i\})$
(6) $y = C_1 e^{(1-2i)x} + C_2 e^{(1+2i)x}$ for any $C_1, C_2 \in \mathbb{C}$
(7) $\left[(y = C_1 e^x e^{-2ix} + C_2 e^x e^{2ix}) \wedge (e^{ix} = \cos x + i \sin x)\right] \Rightarrow (y = C_1 e^x(\cos(-2x) + i\sin(-2x)) + C_2 e^x(\cos(2x) + i\sin(2x)))$
(8) $(\underline{\sin \text{ is odd}} \wedge \underline{\cos \text{ is even}}) \Leftrightarrow (\sin(-x) = -\sin(x) \wedge \cos(-x) = \cos(x)$ for any $x \in \mathbb{C}) \Leftrightarrow (y = C_1 e^x(\cos(2x) - i\sin(2x)) + C_2 e^x(\cos(2x) + i\sin(2x)))$
(9) (8) $\Leftrightarrow (y = e^x(C_1\cos(2x) - iC_1\sin(2x) + C_2\cos(2x) + iC_2\sin(2x))) \Leftrightarrow (y = e^x((C_1 + C_2)\cos(2x) + i(C_2 - C_1)\sin(x)))$
(10) $\therefore y(x) = e^x(C_3\cos(2x) + C_4\sin(2x))$ for $C_3 = C_1 + C_2 \wedge C_4 = C_2 - C_1$


VI
Six
(1) Suppose $f(x) = xe^x \wedge g(x) = x^2$
(2) $\frac{df}{dx} = e^x + x^{ex} \wedge \frac{dg}{dx} = 2x$
(3) (1) ∧ (2) ⟺ $\left(W(f,g) = \begin{vmatrix} xe^x & x^2 \\ e^x + xe^x & 2x \end{vmatrix}\right)$
(4) (3) ⟹ $\left(W(f,g) = 2x^2e^x - x^2e^x - x^3e^x\right)$
(5) ((4) ⟹ $W(f,g) \neq 0$) ⟺ (f and g are linearly independent)


VII
Seven
(1) Suppose
Δx m c
(2) $F_1 = -kx \wedge F_2 = -cx' \wedge F_3 = -mx''$
(4) $F(t) = -\sum_{f \in \{F_1, F_2, F_3\}} = mx'' + cx' + kx = 0$
(5) And suppose $m = 2kg \wedge c = 14 \wedge x(0) = 2 \wedge x'(0) = 3 \wedge 48 = k \cdot 2$
(6) Apply (5) to (4): $2x'' + 14x' + 24x = 0$
(7) (Guess: $x(t) = e^{rt}$ for some $r \in \mathbb{R}$) ⟹ $(2r^2 + 14r + 24 = 0)$
(8) $(r^2 + 7r + 12 = (r+3)(r+4) = 0)$ ⟺ $(r \in \{-3, -4\})$
(9) $x(t) = c_1 e^{-3t} + c_2 e^{-4t}$ for some $c_1, c_2 \in \mathbb{R}$
(10) $\frac{dx}{dt} = c_1(-3)e^{-3t} + c_2(-4)e^{-4t}$
(11) $[(x(0) = 2) \Leftrightarrow (c_1 + c_2 = 2)] \wedge [(x'(0) = 3) \Leftrightarrow (-3c_1 - 4c_2 = 3)]$
$\left(\begin{bmatrix} 1 & 1 & 2 \\ -3 & -4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -9 \end{bmatrix}\right) \Leftrightarrow ((c_1 = 11) \wedge (c_2 = -9))$
(13) $\boxed{\therefore x(t) = 11e^{-3t} - 9e^{-4t}}$