Mathematical Framework for Mean Reversion Pairs Trading

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Abstract

This document serves as the technical documentation for the "Mean Reversion Pairs Trader" project. It details the statistical and mathematical foundations implemented in Python, including Time Series Analysis (Stationarity, Cointegration), Regression Analysis (OLS Derivation), and Performance Metrics (Sharpe Ratio, Drawdowns).

Contents

1 Introduction

Pairs trading is a market-neutral statistical arbitrage strategy. It relies on the hypothesis that two assets with the same underlying economic drivers (e.g., Coke and Pepsi) share a long-term equilibrium. While their prices $P_A(t)$ and $P_B(t)$ may diverge in the short term due to noise or liquidity shocks, the "spread" between them is mean-reverting. Basically we find the cointegration b/w two tickers and exploit the relation.

2 Time Series Foundations

2.1 Stationarity [Source: RitvikMaths(TimeSeries Analysis)]

A time series $\{y_t\}$ is defined as *weakly stationary* if its statistical moments are time-invariant. This is the fundamental requirement for any mean-reversion strategy; if the spread is not stationary, there is no guarantee it will return to a mean.

Mathematical Conditions:

1. **Constant Mean:** $\mathbb{E}[y_t] = \mu$ for all $t \in \mathbb{Z}$.
2. **Constant Variance:** $\mathrm{Var}(y_t) = \mathbb{E}[(y_t - \mu)^2] = \sigma^2 < \infty$.
3. **Time-Invariant Covariance:** $\mathrm{Cov}(y_t, y_{t+k}) = \mathbb{E}[(y_t - \mu)(y_{t+k} - \mu)] = \gamma_k$. No seasonality

Crucially, condition (3) implies that the relationship between observations depends only on the lag k, not on the absolute time t.

2.2 Cointegration vs. Correlation

I learned the important difference between correlation and cointegration:

- **Correlation** (ρ): Measures short-term comovement. If $\rho \approx 1$, assets move in the same direction. However, they can drift apart infinitely (e.g., two random walks with drift).
- **Cointegration**: Requires that a linear combination of two non-stationary ($I(1)$) series produces a stationary ($I(0)$) residual.

Let X_t and Y_t be $I(1)$ processes. They are cointegrated if there exists a vector β such that:

$$\epsilon_t = Y_t - \beta X_t \quad \sim \quad I(0) \tag{1}$$

Here, β is the **Hedge Ratio** and ϵ_t is the **Spread**.

3 Phase I: The Engle-Granger Two-Step Method

3.1 Step 1: Derivation of the Hedge Ratio (OLS) [The maths is below and it was a new thing I explored during this project, Python provides statsmodels library for easy calc]

We estimate the hedge ratio β using Ordinary Least Squares (OLS) regression. We posit the model:

$$Y_t = \alpha + \beta X_t + \epsilon_t \tag{2}$$

The goal is to find $\hat{\alpha}$ and $\hat{\beta}$ that minimize the Sum of Squared Residuals (SSR):

$$S(\alpha, \beta) = \sum_{t=1}^{n} (Y_t - \alpha - \beta X_t)^2 \tag{3}$$

Derivation: To find the minimum, we take partial derivatives with respect to α and β and set them to zero:

1. **Derivative w.r.t α:**

$$\frac{\partial S}{\partial \alpha} = -2\sum(Y_t - \alpha - \beta X_t) = 0 \tag{4}$$

$$\implies \sum Y_t = n\alpha + \beta \sum X_t \tag{5}$$

$$\implies \bar{Y} = \alpha + \beta\bar{X} \quad \Rightarrow \quad \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X} \tag{6}$$

2. **Derivative w.r.t β:**

$$\frac{\partial S}{\partial \beta} = -2\sum X_t(Y_t - \alpha - \beta X_t) = 0 \tag{7}$$

$$\sum X_t Y_t = \alpha \sum X_t + \beta \sum X_t^2 \tag{8}$$

Substituting $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$ into the equation:

$$\hat{\beta} = \frac{\sum(X_t - \bar{X})(Y_t - \bar{Y})}{\sum(X_t - \bar{X})^2} = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \tag{9}$$

This $\hat{\beta}$ tells us exactly how many shares of Asset X to short for every share of Asset Y to create a market-neutral portfolio.

3.2 Step 2: The Augmented Dickey-Fuller (ADF) Test [Source: RitvikMaths(TimeSeries Analysis)]

After calculating the spread $\epsilon_t = Y_t - \hat{\beta}X_t$, we must prove it is stationary. The ADF test assumes the spread follows an Autoregressive (AR) process. We fit the model:

$$\Delta\epsilon_t = \gamma\epsilon_{t-1} + \sum_{i=1}^{p} \delta_i \Delta\epsilon_{t-i} + u_t \tag{10}$$

Where $\gamma = \phi - 1$.

- If the series is a Random Walk, $\phi = 1$, so $\gamma = 0$.
- If the series is Mean Reverting, $\phi < 1$, so $\gamma < 0$.

Test Statistic: We calculate the t-statistic for $\hat{\gamma}$:

$$DF_\tau = \frac{\hat{\gamma}}{SE(\hat{\gamma})} \tag{11}$$

This statistic is compared against critical values from the Dickey-Fuller distribution (not the standard Normal distribution). If $DF_\tau <$ Critical Value (e.g., -3.43 for 95% confidence), we reject the null hypothesis of a unit root.

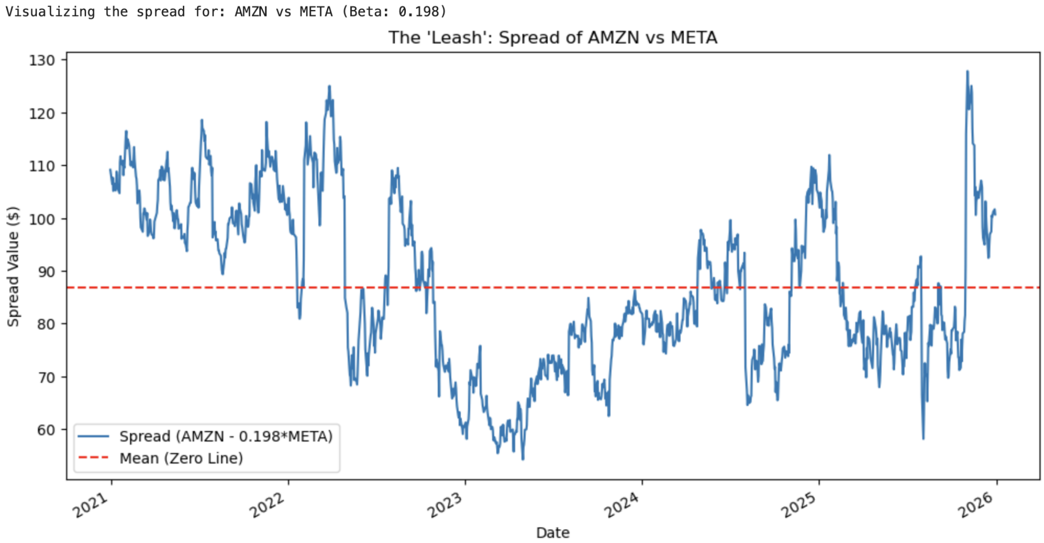

Figure 1: Visualization of the Spread (ϵ_t) for AMZN vs META showing mean-reverting behavior.

4 Phase II: Signal Generation (Z-Score)

To standardize the trading signal across different asset pairs, we transform the dollar spread into a Z-Score. This assumes the spread follows an Ornstein-Uhlenbeck process, which generates a Gaussian distribution of residuals.

Rolling Statistics: To avoid look-ahead bias, we use a rolling window W:

$$Z_t = \frac{\epsilon_t - \mu_{t|t-W}}{\sigma_{t|t-W}} \tag{12}$$

Where:

- $\mu_{t|t-W}$ is the simple moving average (SMA) of the last W days.
- $\sigma_{t|t-W}$ is the rolling standard deviation.

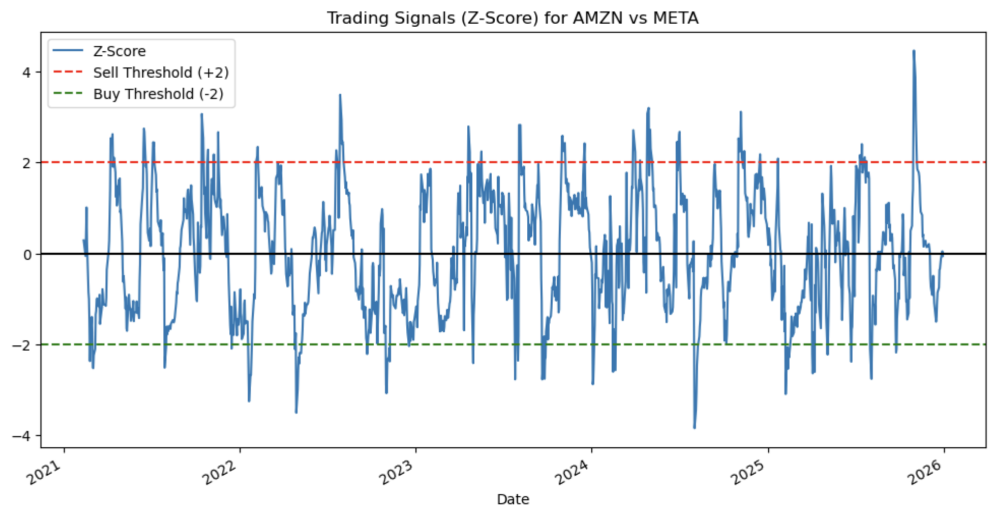

Figure 2: Trading Signals generated by Z-Score normalization. Dotted lines represent Entry (+2/-2) thresholds.

Trading Rules:

- **Entry Short ($Z > \delta_{entry}$):** The spread is statistically expensive. We expect $\epsilon_t \to \mu$.
- **Entry Long ($Z < -\delta_{entry}$):** The spread is statistically cheap.
- **Exit ($Z \to 0$):** The spread has reverted to the mean.
- **Stop Loss ($|Z| > \delta_{stop}$):** The statistical relationship has likely broken (structural break).

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Simulating trades with Safety Net...
Simulation Complete.
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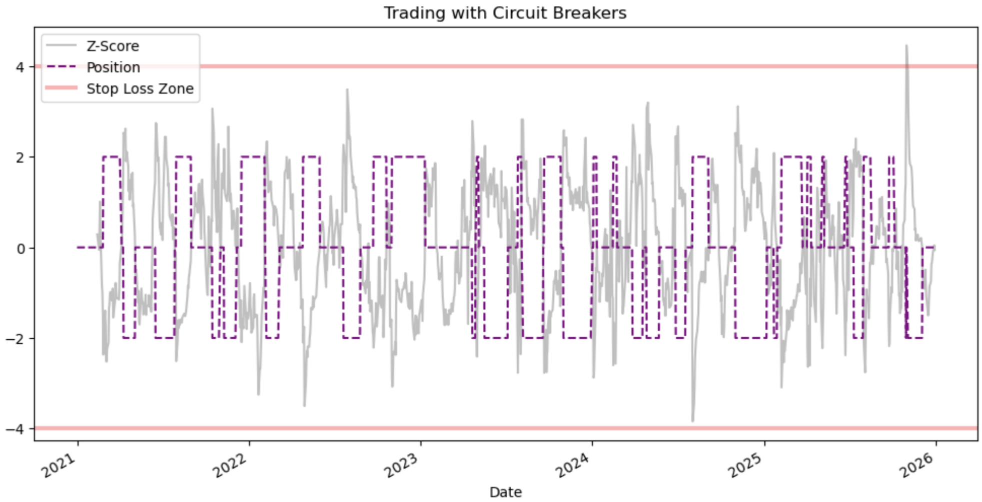

Figure 3: Trading logic visualization including Circuit Breakers (Stop Loss zones at Z ¿ 4).

5 Phase III: Performance Metrics (Very new to me, still trying to understand it better)

5.1 Cumulative Returns with Friction

In a realistic backtest, we must account for transaction costs. Let $P_t \in \{1, 0, -1\}$ be our position vector. The strategy return R_t is:

$$R_t = P_{t-1} \cdot (\epsilon_t - \epsilon_{t-1}) - \underbrace{C \cdot |P_t - P_{t-1}|}_{\text{Transaction Costs}} \tag{13}$$

Where C represents the cost per trade (commission + slippage).

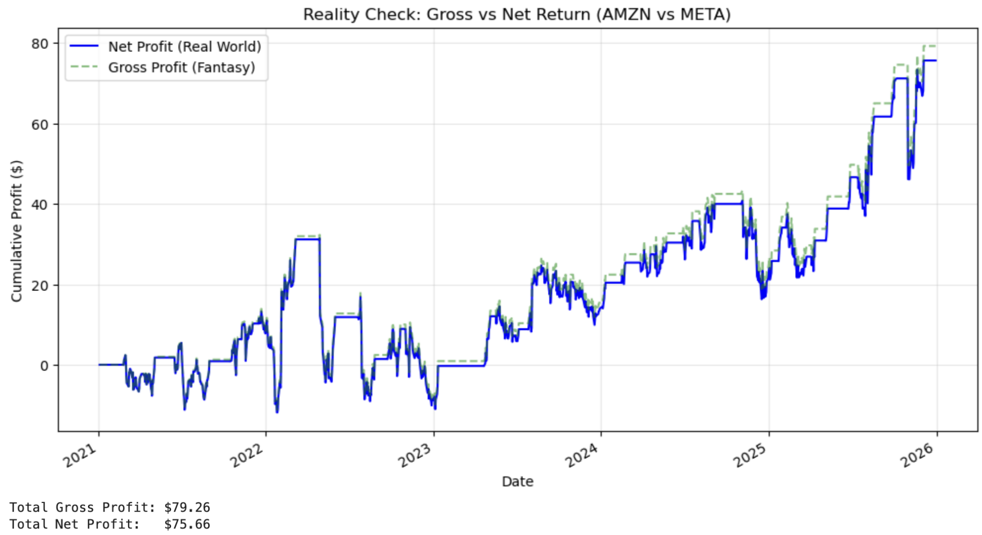

Figure 4: Comparison of Gross Profit (Theoretical) vs. Net Profit (After Friction Costs).

5.2 Sharpe Ratio

The Sharpe Ratio measures excess return per unit of deviation risk.

$$SR = \frac{\bar{R}_p - R_f}{\sigma_p} \times \sqrt{252} \tag{14}$$

Where $\sqrt{252}$ is the annualization factor for daily data. A ratio > 1 indicates the strategy generates returns sufficiently high to justify its volatility.

5.3 Maximum Drawdown (MDD)

The MDD measures the largest peak-to-valley decline in the portfolio equity curve $E(t)$. It assesses the tail risk of the strategy.

$$MDD = \min_t \left(\frac{E(t) - \max_{0 \le \tau \le t} E(\tau)}{\max_{0 \le \tau \le t} E(\tau)} \right) \tag{15}$$

This metric is crucial for determining capital requirements and risk tolerance.

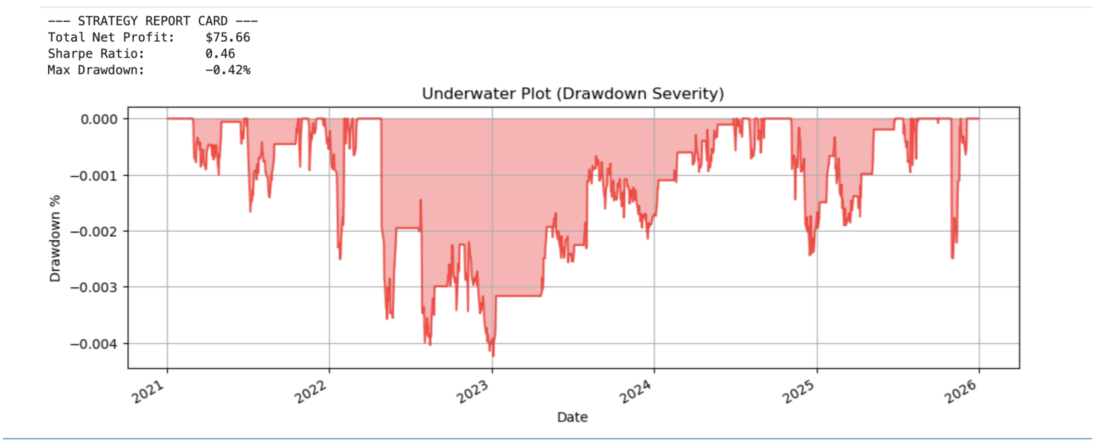

Figure 5: Strategy Report Card and Underwater Plot showing Maximum Drawdown severity.