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Stability test for neutral type delay systems: a piecewise linear approximation scheme ⋆

Gerson Portilla * **Irina V. Alexandrova** ** **Sabine Mondié** *

* *Department of Automatic Control, CINVESTAV-IPN, 07360 Mexico D.F., Mexico (e-mail: gportilla,smondie@ctrl.cinvestav.mx).*
** *Department of Control Theory, St. Petersburg State University, 199034 St. Petersburg, Russia (e-mail: i.v.aleksandrova@spbu.ru)*

Abstract: A necessary and sufficient test for the stability of neutral type linear time-delay systems is presented. Our approach combines the Lyapunov functional expressed in terms of the delay Lyapunov matrix for this class of systems with a piecewise linear approximation of the functional argument. As a result, a stability criterion is given in terms of the non-negative definiteness test of a matrix. An example is presented to validate the obtained result.

Keywords: Neutral type delay systems, stability criterion, delay Lyapunov matrix.

1. INTRODUCTION

Neutral type time-delay systems have received significant attention in recent years. Their main feature is that the derivative of the present state depends not only on the past system's states but also on past derivatives of the state. Many contributions on this class of time-delay systems, for example, results on stability analysis Niculescu (2001), robustness analysis (Alexandrova, 2018), and controller design (Palmor, 1980; Yamanaka and Shimemura, 1987) are available among many others. Stability analysis can be tackled through two approaches: time domain and frequency domain analysis. The first one relies on finding Lyapunov functionals, which are positive in some sense and whose time derivative along the solutions of the time-delay system satisfies a negativity condition. The main results on Lyapunov functionals for linear time-delay systems employed in this contribution are those based on prescribed derivative conditions. The second approach, frequency domain analysis, is based on finding stability regions by using the characteristic equation of the time-delay system. It includes the D-subdivision methods and their variations (Vyhlídal and Zítek, 2003; Gu et al., 2005; Michiels and Niculescu, 2007), in particular, the CTCR (cluster treatment of the characteristic roots) paradigm (Sipahi and Olgac, 2006).

Necessary and sufficient stability conditions in a finite number of operations were obtained in the past decade (Mondié et al., 2022) for several classes of time-delay systems, in particular those of neutral type (Gomez et al., 2021). They follow from the substitution of a particular initial condition depending on the fundamental matrix into the Lyapunov-Krasovskii functional with prescribed derivative (Kharitonov, 2013). These results amount to the evaluation of the delay Lyapunov matrix at discrete points of the delay interval. The effective verification of the test is often impractical due to the large number of required evaluation points. In recent works, inspired by Gu (1997) and Seuret and Gouaisbaut (2015), the initial functions were wisely approximated by piecewise linear and Legendre polynomials, allowing a significant reduction of the numerical burden: for the scalar neutral type case, the use of piecewise linear initial conditions approximations belonging to particular sets has led to novel sufficient stability conditions resulting from a minimization problem Alexandrova and Zhabko (2019); for the multivariable single-delay retarded case, necessary and sufficient stability conditions determined by a positivity condition derived from a Legendre polynomials approximation of the initial function is introduced in Bajodek et al. (2021); and a piecewise linear approximation resulting in a minimization problem is given in Alexandrova (2022). It is worth mentioning that in all cases, the resulting conditions require preliminary knowledge of the delay Lyapunov matrix.

Motivated by the ideas introduced in Alexandrova and Zhabko (2019) for the neutral type case and in Alexandrova (2022) for the retarded case, our aim is to use a discretization scheme and obtain efficient necessary and sufficient stability conditions via a finite number of operations for multivariable neutral type systems. These conditions are given in terms of the non-negative definiteness of a matrix depending on the delay Lyapunov matrix.

The present contribution is organized as follows. Section 2 starts with a review of the theoretical preliminaries on neutral linear time-delay systems. It is followed by a reminder of the concepts and results of the Lyapunov-Krasovskii framework. In Section 3, we introduce a piecewise linear discretization scheme for the Lyapunov-Krasovskii functional for neutral linear time-delay systems with prescribed derivative. Then, our main result, necessary and sufficient stability conditions, is achieved through a discretization

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scheme. We present two illustrative examples in Section 4 and conclude with some remarks in Section 5.

Notation: The initial function φ such that $x(\theta) = \varphi(\theta)$, $\theta \in [-h, 0]$, is taken from $PC^1([-h, 0], \mathbb{R}^n)$, which stands for the space of piecewise continuously differentiable functions. This space is endowed with the uniform norm

$$\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|,$$

where $\|\cdot\|$ stands for the Euclidean norm for vectors and the spectral norm for matrices. $C^l([-h, 0], \mathbb{R}^n)$ stands for the space of l time continuously differentiable vector functions; $\Re(s)$ denotes the real part of a complex value; $\lambda_{\min}(W)$ is the smallest eigenvalue of a matrix W; notation $k = \overline{n_1, n_2}$, where $n_1,\ n_2 \in \mathbb{Z}$, $n_1 < n_2$, means that k is an integer between n_1 and n_2; $\lceil\cdot\rceil$ denotes the ceiling function; $\text{vec}(X)$ means the vectorization of the matrix X; $A \otimes B$ stands for the Kronecker product, namely,

$$A \otimes B \overset{\text{def}}{=} \begin{pmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{2n}A & \cdots & b_{nn}A \end{pmatrix},$$

where $B = \{b_{ij}\}_{i,j=1}^n$.

2. PRELIMINARIES

Consider a neutral linear time delay system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - Dx(t-h)] = A_0x(t) + A_1x(t-h), \qquad (1)$$

where $h \geqslant 0$ and A_0, A_1, and D are given real $n \times n$ matrices. It is assumed that the solution $x(t) = x(t, \varphi)$ is a piecewise continuous function which satisfies system (1) almost everywhere. The difference $x(t) - Dx(t-h)$ is continuous for $t \geqslant 0$. The restriction of the solution $x(t, \varphi)$ to the interval $[t-h, t],\ t \geqslant 0$, is denoted by

$$x_t(\varphi) : \theta \mapsto x(t+\theta, \varphi),\ \theta \in [-h, 0].$$

Definition 1. System (1) is exponentially stable if there exist $\gamma > 0$ and $\sigma > 0$ such that for any initial function φ,

$$\|x(t, \varphi)\| \leqslant \gamma e^{-\sigma t}\|\varphi\|_h,\ t \geqslant 0.$$

Assumption 1. Matrix D is a Schur stable matrix, i.e. $|\lambda_j(D)| < 1,\ j = \overline{1, n}$, where $\lambda_j(D)$ are the eigenvalues of D.

Note that Assumption 1 is a necessary condition for the exponential stability of system (1).

Lemma 1. (Kharitonov, 2013) If the matrix D is Schur stable, then there exist values $\rho \in (0, 1)$ and $d \geqslant 1$ such that $\|D^k\| \leqslant d\rho^k,\ k = 0, 1, \ldots$

In particular, in accordance with Kharitonov et al. (2005) one can take $d = \sqrt{\lambda_{\min}(Q)/\lambda_{\max}(Q)}$, where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix solution of the inequality

$$D^TQD - \rho^2Q < 0.$$

Next, we recall important facts on the Lyapunov matrix of system (1) and results on neutral type linear time-delay systems stability analysis.

Definition 2. (Kharitonov, 2013) Let $W \in \mathbb{R}^{n\times n}$ be a positive definite matrix. The delay Lyapunov matrix $U : [-h, h] \to \mathbb{R}^{n\times n}$ is a continuous matrix function, which satisfies the following properties:

(1) Dynamic property
$$U'(\tau) - U'(\tau-h)D = U(\tau)A_0 + U(\tau-h)A_1,\ \tau \in (0, h), \qquad (2)$$
(2) Symmetry property
$$U^T(\tau) = U(-\tau),\ \tau \in [-h, h], \qquad (3)$$
(3) Algebraic property
$$P - D^TPD = -W, \qquad (4)$$
where $P = U'(+0) - U'(-0)$.

Existence and uniqueness conditions for the Lyapunov matrix are reminded in the following result.

Theorem 1. (Kharitonov, 2013) System (1) admits a unique Lyapunov matrix if and only if the system satisfies the Lyapunov condition, i.e., if there exists $\varepsilon > 0$ such that any two points s_1 and s_2 of the spectrum of system (1) satisfy $|s_1 + s_2| > \varepsilon$.

In this paper, we make use of Lyapunov matrix-based functionals with a prescribed quadratic negative definite derivative (Kharitonov, 2013)

$$v_0(\varphi) = (\varphi(0) - D\varphi(-h))^TU(0)(\varphi(0) - D\varphi(-h)) + \sum_{j=2}^{5} I_j, \qquad (5)$$

where,

$$I_2 = 2(\varphi(0) - D\varphi(-h))^T \int_{-h}^{0} \Phi(h+\theta)\varphi(\theta)\mathrm{d}\theta,$$

$$I_3 = \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1)\Psi(\theta_1-\theta_2)\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$I_4 = -\int_{-h}^{0}\int_{-h}^{\theta_1-0} \varphi^T(\theta_1)D^TU''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 - \int_{-h}^{0}\int_{\theta_1+0}^{0} \varphi^T(\theta_1)D^TU''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$I_5 = -\int_{-h}^{0} \varphi^T(\theta)D^TPD\varphi(\theta)\mathrm{d}\theta,$$

$$\Phi(s) = U^T(s)A_1 - U'^T(s)D, \quad s \in (0, h),$$

$$\Psi(s) = A_1^TU(s)A_1 - D^TU'(s)A_1 + A_1^TU'(s)D,$$

and $\Psi(-s) = \Psi^T(s)$, $s \in (0, h)$. The time derivative of the functional (5) along the solutions of system (1) is given by

$$\frac{\mathrm{d}v_0(x_t)}{\mathrm{d}t} = -x^T(t)Wx(t), \quad t \geqslant 0.$$

Now, define $K = \dfrac{(\|A_0\| + \|A_1\|)d}{1-\rho}$, where d and ρ are defined in Lemma 1, and introduce the set

$$S = \Big\{\varphi \in C^2([-h, 0], \mathbb{R}^n) \,\Big|\, \|\varphi(\theta)\| \leqslant \|\varphi(0)\| = 1, \|\varphi'(\theta)\| \leqslant K, \quad \|\varphi''(\theta)\| \leqslant K^2, \quad \theta \in [-h, 0]\Big\}.$$

In recent works, stability and instability results were introduced using the set S.

Theorem 2. (Alexandrova and Zhabko, 2019) System (1) is exponentially stable, if and only if there exists functional (5) and a constant $\mu > 0$ such that

$$v_0(\varphi) \geqslant \mu\|\varphi(0)\|^2 = \mu, \quad \varphi \in S.$$

Lemma 2. (Alexandrova and Zhabko, 2019) If system (1) is unstable, then there exists a function $\varphi \in S$ such that

$$v_0(\varphi) < -a_0 \overset{\text{def}}{=} -\lambda_{\min}(W)/(4\hat{\alpha}).$$

Here, $\hat{\alpha}$ is any bound for $\Re(s)$, where s is an eigenvalue of an unstable system (1). As a rough bound, we can take $\hat{\alpha} = K$.

3. A PIECEWISE LINEAR DISCRETIZATION SCHEME

3.1 Discretization of the functional

Consider a partition of the interval $[-h, 0]$ into N parts of length $\Delta = h/N$ with $\theta_j = -j\Delta$, $j = \overline{0, N}$, and consider the piecewise linear approximation of the function φ (Alexandrova and Zhabko, 2019):

$$l_N(s+\theta_j) = \varphi(\theta_j)\left(1+\frac{s}{\Delta}\right) - \varphi(\theta_{j+1})\frac{s}{\Delta}, \qquad (6)$$

$s \in [-\Delta, 0]$, $j = \overline{0, N-1}$. Let $\eta_N(\theta) = \varphi(\theta) - l_N(\theta)$, $\theta \in [-h, 0]$, be the error of such approximation. Introduce the vector $\widehat{\varphi} = \left(\varphi^T(0), \varphi^T(\theta_1), \ldots, \varphi^T(\theta_N)\right)^T$, and denote $\widehat{\varphi}_j = \varphi(\theta_j)$, $j = \overline{0, N}$.

Let us now compute the value of $v_0(l_N)$, which serves as the approximation of functional (5). We show that $v_0(l_N)$ represents a quadratic form with respect to the vector $\widehat{\varphi}$.

First, we approximate each summand of (5), as follows

$$\begin{aligned} I_1(l_N) &= (\varphi(0) - D\varphi(-h))^T U(0)(\varphi(0) - D\varphi(-h)) \\ &= (\widehat{\varphi}_0 - D\widehat{\varphi}_N)^T U(0)(\widehat{\varphi}_0 - D\widehat{\varphi}_N). \end{aligned}$$

The discretization of $I_2(l_N)$ is given by

$$\begin{aligned} I_2(l_N) =& 2[\varphi(0) - D\varphi(-h)]^T \sum_{k=1}^{N} \int_{-\Delta}^{0} \Phi(s+k\Delta) \\ &\times \left(\varphi(\theta_{N-k})\left(1+\frac{s}{\Delta}\right) - \varphi(\theta_{N-k+1})\frac{s}{\Delta}\right) \mathrm{d}s \\ &= 2[\widehat{\varphi}_0 - D\widehat{\varphi}_N]^T \sum_{k=1}^{N} \left(\mathcal{M}_k \widehat{\varphi}_{N-k} + \mathcal{N}_k \widehat{\varphi}_{N-k+1}\right), \end{aligned}$$

where

$$\mathcal{M}_k = \int_{-\Delta}^{0} \Phi(s+k\Delta)\left(1+\frac{s}{\Delta}\right) \mathrm{d}s,$$
$$\mathcal{N}_k = \int_{-\Delta}^{0} \Phi(s+k\Delta)\left(-\frac{s}{\Delta}\right) \mathrm{d}s.$$

Similarly, we consider the third and fourth summands of the functional:

$$\begin{aligned} I_3(\varphi) + I_4(\varphi) = &\sum_{k=1}^{N}\sum_{j=1}^{N} \int_{-\Delta}^{0}\int_{-\Delta}^{0} \varphi^T(s_1+\theta_{N-k}) \\ &\times \widetilde{\Psi}(s_1 - s_2 + (k-j)\Delta)\varphi(s_2+\theta_{N-j})\mathrm{d}s_2\mathrm{d}s_1, \end{aligned}$$

where $\widetilde{\Psi}(s) = \Psi(s) - D^T U''(s) D$. Substituting approximation (6), we get

$$\begin{aligned} I_3(l_N) + I_4(l_N) = &\sum_{k=1}^{N}\sum_{j=1}^{N} \left(\widehat{\varphi}_{N-k}^T \mathcal{P}_{k-j} \widehat{\varphi}_{N-j}\right. \\ &+ \widehat{\varphi}_{N-k}^T \mathcal{Q}_{k-j} \widehat{\varphi}_{N-j+1} + \widehat{\varphi}_{N-k+1}^T \mathcal{Q}'_{k-j} \widehat{\varphi}_{N-j} \\ &\left.+ \widehat{\varphi}_{N-k+1}^T \mathcal{R}_{k-j} \widehat{\varphi}_{N-j+1}\right). \end{aligned}$$

Here,

$$\mathcal{P}_l = \int_{-\Delta}^{0}\int_{-\Delta}^{0} \widetilde{\Psi}(s_1 - s_2 + l\Delta)\left(1+\frac{s_1}{\Delta}\right)\left(1+\frac{s_2}{\Delta}\right)\mathrm{d}s_2\mathrm{d}s_1,$$

$$\mathcal{Q}_l = \int_{-\Delta}^{0}\int_{-\Delta}^{0} \widetilde{\Psi}(s_1 - s_2 + l\Delta)\left(1+\frac{s_1}{\Delta}\right)\left(-\frac{s_2}{\Delta}\right)\mathrm{d}s_2\mathrm{d}s_1,$$

$$\mathcal{Q}'_l = \int_{-\Delta}^{0}\int_{-\Delta}^{0} \widetilde{\Psi}(s_1 - s_2 + l\Delta)\left(1+\frac{s_2}{\Delta}\right)\left(-\frac{s_1}{\Delta}\right)\mathrm{d}s_2\mathrm{d}s_1,$$

$$\mathcal{R}_l = \int_{-\Delta}^{0}\int_{-\Delta}^{0} \widetilde{\Psi}(s_1 - s_2 + l\Delta)\left(\frac{s_1 s_2}{\Delta^2}\right)\mathrm{d}s_2\mathrm{d}s_1,$$

for $l = \overline{-(N-1), N-1}$ and $l \neq 0$. Otherwise,

$$\begin{aligned} \mathcal{P}_0 = &\int_{-\Delta}^{0}\int_{-\Delta}^{0} \Psi(s_1 - s_2)\left(1+\frac{s_1}{\Delta}\right)\left(1+\frac{s_2}{\Delta}\right)\mathrm{d}s_2\mathrm{d}s_1 \\ &- D^T \int_{-\Delta}^{0} \left(\int_{-\Delta}^{s_1-0}\left(1+\frac{s_1}{\Delta}\right)\left(1+\frac{s_2}{\Delta}\right)U''(s_1-s_2)\mathrm{d}s_2\right. \\ &\left.+ \int_{s_1+0}^{0}\left(1+\frac{s_1}{\Delta}\right)\left(1+\frac{s_2}{\Delta}\right)U''(s_1-s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 D, \end{aligned}$$

$$\begin{aligned} \mathcal{Q}_0 = &\int_{-\Delta}^{0}\int_{-\Delta}^{0} \Psi(s_1 - s_2)\left(1+\frac{s_1}{\Delta}\right)\left(-\frac{s_2}{\Delta}\right)\mathrm{d}s_2\mathrm{d}s_1 \\ &- D^T \int_{-\Delta}^{0} \left(\int_{-\Delta}^{s_1-0}\left(1+\frac{s_1}{\Delta}\right)\left(-\frac{s_2}{\Delta}\right)U''(s_1-s_2)\mathrm{d}s_2\right. \\ &\left.+ \int_{s_1+0}^{0}\left(1+\frac{s_1}{\Delta}\right)\left(-\frac{s_2}{\Delta}\right)U''(s_1-s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 D, \end{aligned}$$

$$\begin{aligned} \mathcal{R}_0 = &\int_{-\Delta}^{0}\int_{-\Delta}^{0} \Psi(s_1 - s_2)\left(\frac{s_1 s_2}{\Delta^2}\right)\mathrm{d}s_2\mathrm{d}s_1 \\ &- D^T \int_{-\Delta}^{0} \left(\int_{-\Delta}^{s_1-0}\frac{s_1 s_2}{\Delta^2}U''(s_1-s_2)\mathrm{d}s_2\right. \\ &\left.+ \int_{s_1+0}^{0}\frac{s_1 s_2}{\Delta^2}U''(s_1-s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 D, \end{aligned}$$

and $\mathcal{Q}'_0 = \mathcal{Q}_0^T$. It is useful to note that

$$\mathcal{P}_l = \mathcal{P}_{-l}^T, \quad \mathcal{Q}'_l = \mathcal{Q}_{-l}^T, \quad \mathcal{R}_l = \mathcal{R}_{-l}^T$$

for all $l = \overline{-(N-1), N-1}$. Finally, discretization of the last summand takes the form

$$\begin{aligned} I_5(l_N) = -\frac{\Delta}{3}\sum_{k=1}^{N}\Big(&\widehat{\varphi}_{N-k}^T D^T P D \widehat{\varphi}_{N-k} \\ &+ \widehat{\varphi}_{N-k}^T D^T P D \widehat{\varphi}_{N-k+1} + \widehat{\varphi}_{N-k+1}^T D^T P D \widehat{\varphi}_{N-k+1}\Big). \end{aligned}$$

Here, we took into account that

$$\int_{-\Delta}^{0} \left(1+\frac{s}{\Delta}\right)^2 \mathrm{d}s = \int_{-\Delta}^{0} \frac{s^2}{\Delta^2}\mathrm{d}s$$
$$= 2\int_{-\Delta}^{0} \left(1+\frac{s}{\Delta}\right)\left(-\frac{s}{\Delta}\right)\mathrm{d}s = \frac{\Delta}{3}.$$

Bringing all five summands together we arrive at the approximation of the form

$$v_0^{\text{approx}}(\varphi) = v_0(l_N) \stackrel{\text{def}}{=} \widehat{\varphi}^T\Lambda_N\widehat{\varphi}, \tag{7}$$

where Λ_N is a block matrix composed of the matrices $\mathcal{M}_k$, $\mathcal{N}_k$, $\mathcal{P}_l$, $\mathcal{Q}_l$, $\mathcal{R}_l$. They depend on the integrals of the Lyapunov matrix and its derivatives multiplied by polynomials. The dimension of Λ_N is $n(N+1)\times n(N+1)$.

Remark 1. Once the Lyapunov matrix is calculated via the semianalytic method, the time-consuming numerical calculation of the integrals in the blocks of Λ_N must be carried out. A recursive method based on Lyapunov matrix properties following the ideas in Alexandrova (2022) allows tackling this task efficiently. Note however that in spite of the availability of explicit recursive formulae, this approach suffers from the accumulation of rounding errors at each step. Due to length restrictions, the recursive method is deferred to a future publication.

3.2 Estimation of the discretization error

The purpose of this section is to determine and bound the error

$$\Upsilon_N = v_0(\varphi) - v_0^{\text{approx}}(\varphi) = v_0(\varphi) - \widehat{\varphi}^T\Lambda_N\widehat{\varphi}.$$

First, the following technical lemma is introduced. Therein, the error of the piecewise linear approximation of functional argument defined in Subsection 3.1 is bounded.

Lemma 3. (Alexandrova, 2022) The piecewise linear approximation error admits the bound

$$\|\eta_N(s+\theta_j)\| \leqslant \frac{1}{2}K^2(-s)(s+\Delta), \quad s\in[-\Delta,0],$$

for all $j=\overline{0,N-1}$.

Having in mind that the Lyapunov condition holds, hence $U(\tau)$ is bounded, denote

$$M_1 = \sup_{\theta\in(0,h)} \|\Phi(\theta)\|,$$
$$M_2 = \sup_{\theta\in(0,h)} \|\Psi(\theta) - D^TU''(\theta)D\|,$$
$$M_3 = |\lambda_{\max}(D^TPD)|.$$

We present the term Υ_N in the form

$$\Upsilon_N = I_2^{\text{err}} + I_3^{\text{err}} + I_4^{\text{err}} + I_5^{\text{err}}$$

with

$$I_2^{\text{err}} = 2[\widehat{\varphi}_0 - D\widehat{\varphi}_N]^T \int_{-h}^{0} \Phi(h+\theta)\eta_N(\theta)\mathrm{d}\theta,$$

$$\begin{aligned} I_3^{\text{err}} + I_4^{\text{err}} = &\int_{-h}^{0}\int_{-h}^{\theta_1-0} \varphi^T(\theta_1)\widetilde{\Psi}(\theta_1-\theta_2)\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &+ \int_{-h}^{0}\int_{\theta_1+0}^{0} \varphi^T(\theta_1)\widetilde{\Psi}(\theta_1-\theta_2)\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0}\int_{-h}^{\theta_1-0} l_N^T(\theta_1)\widetilde{\Psi}(\theta_1-\theta_2)l_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0}\int_{\theta_1+0}^{0} l_N^T(\theta_1)\widetilde{\Psi}(\theta_1-\theta_2)l_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1, \end{aligned}$$

$$\begin{aligned} I_5^{\text{err}} = &-\int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1)D^TPD\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &+ \int_{-h}^{0}\int_{-h}^{0} l_N^T(\theta_1)D^TPDl_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1. \end{aligned}$$

Now, taking account that $\varphi\in S$ we obtain

$$\int_{-h}^{0} \|\eta_N(\theta)\|\mathrm{d}\theta = \sum_{k=1}^{N}\int_{-\Delta}^{0} \|\eta_N(s+\theta_{N-k})\|\mathrm{d}s$$
$$\leqslant \frac{1}{2}K^2N\int_{-\Delta}^{0}(-s)\left(s+\Delta\right)\mathrm{d}s = \frac{1}{12}K^2\frac{h^3}{N^2},$$

$$\int_{-h}^{0} \|l_N(\theta)\|\mathrm{d}\theta = \sum_{k=1}^{N}\int_{-\Delta}^{0} \|l_N(s+\theta_{N-k})\|\mathrm{d}s \leqslant N\Delta = h.$$

Then, bounding each summand of Υ_N we get

$$|I_2^{\text{err}}| \leqslant \frac{1}{6}(1+\|D\|)M_1K^2\frac{h^3}{N^2},$$
$$|I_3^{\text{err}} + I_4^{\text{err}}| \leqslant \frac{1}{6}M_2K^2\frac{h^4}{N^2},$$
$$|I_5^{\text{err}}| \leqslant \frac{1}{6}M_3K^2\frac{h^3}{N^2}.$$

Finally, we prove that the functional approximation error is bounded as

$$|\Upsilon_N| \leqslant \delta_N = \frac{c}{N^2}, \tag{8}$$

where

$$c = \frac{1}{6}K^2h^3\left((1+\|D\|)M_1 + hM_2 + M_3\right).$$

We observe that when the value of N tends to infinity, the error of the functional approximation approaches zero. Hence, the value of quadratic form $\widehat{\varphi}^T\Lambda_N\widehat{\varphi}$ approaches the exact value of the functional. However, the dimension of the quadratic form tends to infinity as well.

3.3 Main result

Finally, we obtain that functional $v_0(\varphi)$ can be presented in the form $v_0(\varphi) = \widehat{\varphi}^T\Lambda_N\widehat{\varphi} + \Upsilon_N$, and admits a quadratic lower bound

$$v_0(\varphi) \geqslant \widehat{\varphi}^T\Lambda_N\widehat{\varphi} - \delta_N, \quad \varphi\in S. \tag{9}$$

The main result of this paper given in Theorem 3 is achieved with the help of the following lemma.

Lemma 4. For any $\varepsilon>0$, if $N\geqslant\sqrt{\frac{c}{\varepsilon}}$, then $|\Upsilon_N|\leqslant\varepsilon$.

Since $N\in\mathbb{N}_+$ in our case, we define the value

$$N^\star = \left\lceil\sqrt{\frac{c}{a_0}}\right\rceil$$

where the constant a_0 is determined in Lemma 2. With such a value of $N^\star$, our sufficient and necessary stability condition is:

Theorem 3. System (1) is exponentially stable, if and only if the Lyapunov condition holds and the matrix $\Lambda_{N^\star}$ is non-negative definite.

Roughly speaking, the necessity part follows from the fact that $v_0(\varphi) \geqslant 0$ for any φ, and in particular for $\varphi = l_N$, if system (1) is exponentially stable. Non-negative definiteness of Λ_N is a necessary condition of exponential stability for any value of N.

The proof of sufficiency part is performed by contradiction and based on Lemma 2. It reveals that tolerance ε in Lemma 4 required to achieve the sufficient condition is equal to the value a_0. Thus, the order of approximation to prove the sufficiency part is given by $N^\star$, which is determined by c. Observe that if c increases then $N^\star$ also does.

We note that if the value N such that bound (9) is strictly positive on S is found then system (1) is exponentially stable. This is an alternative sufficient stability condition.

Remark 2. To carry out the stability test, one has to check the non-negativity of the $n(N+1) \times n(N+1)$ matrix Λ_N. Computing its minimum eigenvalue and checking if it is greater or equal to zero may induce an erroneous conclusion due to the rounding errors accumulated in the recursive computation of the elements of Λ_N. In practice, the Matlab function "cholcov" allows overcoming this issue.

4. ILLUSTRATIVE EXAMPLES

Example 1: The σ-stability analysis of the proportional-integral control of a passive linear system leads to studying a quasipolynomial of neutral type (Castaños et al., 2018). Its time domain representation is of the form (1), with matrices

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{\alpha_2}{\alpha_1} \end{pmatrix},$$

$$A_0 = \frac{1}{\alpha_1}\begin{pmatrix} 0 & \alpha_1 \\ -\sigma^2\alpha_1 + \sigma\beta_1 - \gamma_1 & -\beta_1 + 2\sigma\alpha_1 \end{pmatrix},$$

$$A_1 = \frac{1}{\alpha_1}\begin{pmatrix} 0 & 0 \\ -\sigma^2\alpha_2 + \sigma\beta_2 - \gamma_2 & -\beta_2 + 2\sigma\alpha_2 \end{pmatrix},$$

where

$$\begin{aligned} &\alpha_1 = d + k_p, \quad \gamma_1 = bk_i d + ak_i, \\ &\alpha_2 = (d - k_p)e^{\sigma h}, \quad \gamma_2 = (bk_i d - ak_i)e^{\sigma h}, \\ &\beta_1 = (bk_p + a)d + bd^2 + ak_p + k_i, \\ &\beta_2 = ((bk_p + a)d - bd^2 - ak_p - k_i)e^{\sigma h}. \end{aligned}$$

For the parameter values

$$a = 0.4, \quad b = 50, \quad h = 0.2, \quad d = 0.8, \quad \sigma = 0.3$$

the stability boundaries, obtained by the D-subdivision technique (Neimark, 1949), are depicted by a solid line on Figure 1. The stability of the difference operator imposes in the D-subdivisions the additional condition $|k_p| < 26.67$. The delay Lyapunov matrix is computed for $W = I$ by using the semi-analytic method introduced in Kharitonov (2013). The isolated dots represent points in the space of parameters where the non-negativity test of Λ_N holds. The non-negativity of Λ_N is verified by using the function "cholcov" in Matlab.

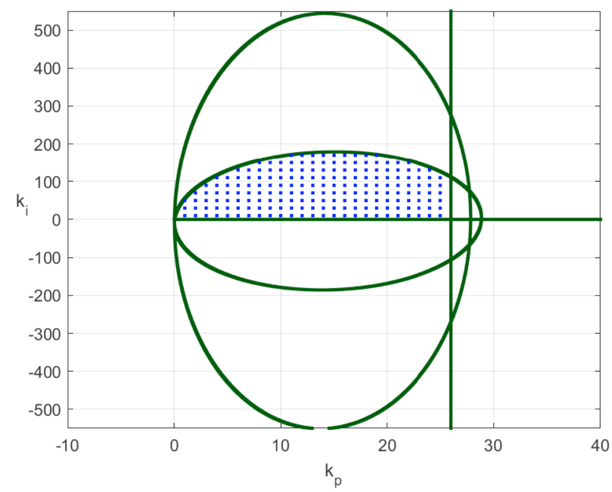

Fig. 1. Non-negative definiteness test of Λ_3

In Figure 1, we observe that the whole stability region is detected for $N = 3$. However, we can only rigorously conclude, from the necessity arguments in the proof of Theorem 3, that points where Λ_3 is negative definite are unstable. For sufficiency of the exponential stability condition in Theorem 3 the semi-definite positivity condition must hold for $N = N^\star$. For the above example, this integer $N^\star$ is evaluated at selected points in the space of parameters, shown in Table 1. The results are compared with those in Gomez et al. (2021) where the stability is tested via a discretization of the delay Lyapunov matrix. The order of approximation $N^\star$, validating the sufficiency of the results in Theorem 3 is substantially smaller than the number $\hat{r}$ in Theorem 4 in Gomez et al. (2021).

Table 1. Stability condition given by Theorem 3 and Theorem 4 in Gomez et al. (2021)

(k_p, k_i)	$N^\star$	Result	$\hat{r}$ (Gomez et al., 2021)	Result
$(1, 1)$	65	Stable	5×10^{22}	—
$(1, -1)$	89	Unstable	3×10^{26}	—

Example 2: The Proportional-Derivative control of the system described by the transfer function

$$H(s) = \frac{15s^2 + 3s - 20}{125s^3 + 70s^2 + 10s + 8}e^{-2s},$$

introduced in Méndez (2011), is described by system (1) with $h = 2$ and matrices

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\dfrac{15}{125}k_d \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{8}{125} & -\dfrac{10}{125} & -\dfrac{70}{125} \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{20}{125}k_p & -\dfrac{1}{125}(3k_p - 20k_d) & -\dfrac{8}{125}(15k_p + 3k_d) \end{bmatrix},$$

where parameters k_p and k_d are the proportional and derivative gains, respectively. The points where the necessary conditions hold in the space of parameters (k_p, k_d) are depicted in Figure 2 for $N = 1$.

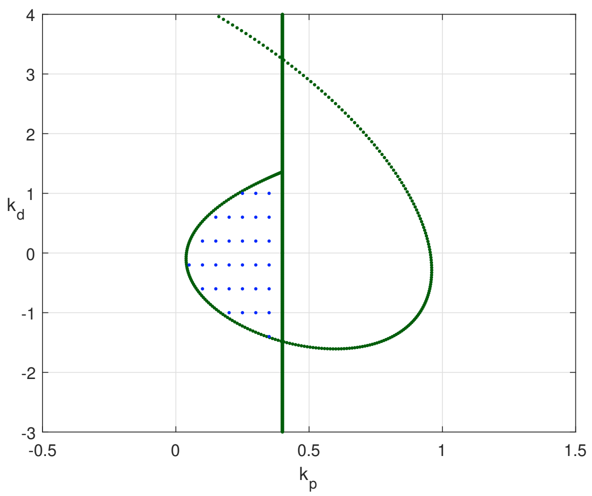

Fig. 2. Non-negative definiteness test of Λ_1

In Table 2, the value $N^\star$ and $\hat{r}$ are likewise compared and evaluated at some selected points of Figure 2. These results confirm the observations in Example 1 on the notable reduction of the value $N^\star$ compared to $\hat{r}$. This makes the test numerically possible when the value r exceeds the computer memory.

Table 2. Stability condition given by Theorem 3 and Theorem 4 in Gomez et al. (2021)

(k_p, k_d)	$N^\star$	Result	$\hat{r}$ (Gomez et al., 2021)	Result
$(0.25, 0)$	32	Stable	2×10^8	—
$(0, 0)$	1	Unstable	1.4×10^9	—

5. CONCLUSION

A stability test in a finite number of operations for neutral type linear time-delay systems is presented. It is based on a piecewise linear approximation of the argument φ in the Lyapunov functionals with prescribed derivative, and recent stability theorems. The resulting finite stability criterion amounts to verify the non-negative definiteness of a matrix. Some examples validating the result are given.

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