

A sufficient condition for k -contraction in Lurie systems

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Abstract: We consider a Lurie system obtained via a connection of a linear time-invariant system and a nonlinear feedback function. Such systems often have more than a single equilibrium and are thus not contractive with respect to any norm. We derive a new sufficient condition for k -contraction of a Lurie system. For $k = 1$, our sufficient condition reduces to the standard stability condition based on the bounded real lemma and a small gain condition. For $k = 2$, our condition guarantees well-ordered asymptotic behaviour of the closed-loop system: every bounded solution converges to an equilibrium, which is not necessarily unique. We apply our results to derive a sufficient condition for k -contractivity of a networked system.

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1. INTRODUCTION

Consider a nonlinear system obtained by connecting a linear time-invariant (LTI) system with state vector $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^q$:

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx + Du,\end{aligned}\quad (1)$$

with a time-varying nonlinear feedback control $u = -\Phi(t, y)$. The resulting closed-loop system is known as the Lurie system after the Russian mathematician A. I. Lurie. Such systems play an important role in systems and control theory, as many nonlinear real-world systems can be represented as a Lurie system. A non-trivial and well-studied topic is proving that the closed-loop system is asymptotically stable for Φ belonging to certain classes of nonlinear functions, e.g., the class of sector-bounded functions (Khalil, 2002, Ch. 7).

When $D = 0$, the closed-loop system is

$$\dot{x} = Ax - B\Phi(t, Cx). \quad (2)$$

In the 1940s and 1950s, Aizerman and Kalman conjectured that for certain classes of nonlinear functions the stability analysis of the closed-loop system (2) can be reduced to the stability analysis of certain classes of linear systems. These conjecture are now known to be false.

Several authors studied (2) using contraction theory. Smith (1986) reformulated a bound on the Hausdorff dimension of compact invariant sets of mappings (Douady and Oesterle, 1980) to dynamical systems, applied it to

a Lurie system, and demonstrated the results by bounding the Hausdorff dimension of attractors of the Lorentz equation. However, his results are highly conservative, especially for large-scale systems. Andrieu and Tarbouriech (2019) provide a Linear Matrix Inequality (LMI) sufficient condition for contraction w.r.t. Euclidean norms under differential sector bound or monotonicity assumptions on the nonlinearity (see also (Bullo, 2022, Theorem 3.24) for a similar condition under different assumptions), and use it to design controllers which guarantee contraction of the closed-loop system. This design method was then revisited by Giaccagli et al. (2022) where it was shown that the designed controllers yield a closed-loop system with infinite gain margin. Proskurnikov et al. (2022) provide a sufficient condition for contraction w.r.t. non-Euclidean norms (see also Davydov et al. (2022) where this question was studied in the context of recurrent neural networks). However, a Lurie system may have more than a single equilibrium point (Miranda-Villatoro et al., 2018), and then it is clearly not contractive w.r.t. any norm.

Following the seminal work of Muldowney (1990), Wu et al. (2022a) recently introduced the notion of k -contractive systems. Classical contractivity implies that under the phase flow of the system the tangent vectors to the phase space contract exponentially fast. k -contractivity implies that the same property holds for elements of k th exterior powers of the tangent spaces. Roughly speaking, this is equivalent to the fact that the flow of the variational equation contracts k -dimensional parallelotopes at an exponential rate. In particular, a 1-contractive system is just a contractive system. However, a k -contractive system, with $k > 1$, may not be contractive in the standard sense.

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For example, every bounded solution of a time-invariant 2-contractive system converges to an equilibrium point, which may not be unique (Li and Muldowney, 1995). Thus, 2-contraction may be useful for analyzing multi-stable systems that cannot be analyzed using standard contraction theory.

The basic tools required to define and study k -contractivity are the k th multiplicative and k th additive compounds of a matrix. The reason for this is simple: k th multiplicative compounds provide information on the volume of parallelotopes generated by k vertices, and k th additive compounds describe the dynamics of k th multiplicative compounds when the vertices follow a linear dynamics.

Here, we derive and prove a novel sufficient condition for k -contractivity of a Lurie system. A unique feature of this condition is that it combines an algebraic Riccati inequality (ARI) that includes the k th additive compounds of the matrices of the LTI, and a kind of gain condition on the Jacobian of the nonlinear function Φ , and is therefore less conservative than the sufficient condition in (Smith, 1986).

In the special case $k = 1$, our condition reduces to a small-gain sufficient condition for standard contraction. However, for $k > 1$ our results may be used to analyze systems that are not contractive. We demonstrate this by deriving a sufficient condition for k -contractivity of a Hopfield network. These networks are often used as associative memories with every stored pattern corresponding to an equilibrium of the dynamics. Thus, they are typically multi-stable, and cannot be analyzed using standard contraction theory.

We use standard notation. For a square matrix A , A^T is the transpose of A , and $\text{tr}(A)$ [$\det(A)$] is the trace [determinant] of A . A symmetric matrix $P \in \mathbb{R}^{n \times n}$ is called positive definite [positive semi-definite] if $x^T P x > 0$ [$x^T P x \geq 0$] for all $x \in \mathbb{R}^n \setminus \{0\}$. Such matrices are denoted by $P \succ 0$ and $P \succeq 0$, respectively. Given a matrix $A \in \mathbb{R}^{n \times m}$, we use $\sigma_1(A) \geq \dots \geq \sigma_{\min\{n,m\}}(A) \geq 0$ to denote the ordered singular values of A . The $n \times n$ identity matrix is denoted by I_n . The L_2 norm of a vector x is $|x|_2 := (x^T x)^{1/2}$. For two integers i, j , with $i \leq j$, let $[i, j] := \{i, i+1, \dots, j\}$.

The next section reviews known definitions and results that are used later on. Section 3 includes our main results.

2. PRELIMINARIES

We review several definitions and results on matrix compounds and matrix measures (also known as logarithmic norms (Ström, 1975)) that will be used in Section 3.

2.1 Matrix compounds

Let $Q_{k,n}$ denote the set of increasing sequences of k numbers from $[1, n]$ ordered lexicographically. For example, $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$. For $A \in \mathbb{R}^{n \times m}$ and $k \in [1, \min\{n, m\}]$, a *minor of order k* of A is the determinant of some $k \times k$ submatrix of A . Consider the $\binom{n}{k} \times \binom{m}{k}$ minors of order k of A . Each such minor is defined by a set of row indices $\kappa^i \in Q_{k,n}$ and column indices $\kappa^j \in Q_{k,m}$. This mi-

nor is denoted by $A(\kappa^i | \kappa^j)$. For example, for $A = \begin{bmatrix} 4 & 5 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$,

we have $A(\{1,3\} | \{1,2\}) = \det \begin{bmatrix} 4 & 5 \\ 0 & 3 \end{bmatrix} = 12$.

Definition 1. The k th *multiplicative compound matrix* of $A \in \mathbb{R}^{n \times m}$, denoted $A^{(k)}$, is the $\binom{n}{k} \times \binom{m}{k}$ matrix that includes all the minors of order k ordered lexicographically.

For example, for $n = m = 3$ and $k = 2$, we have

$$A^{(2)} = \begin{bmatrix} A(\{1,2\} | \{1,2\}) & A(\{1,2\} | \{1,3\}) & A(\{1,2\} | \{2,3\}) \\ A(\{1,3\} | \{1,2\}) & A(\{1,3\} | \{1,3\}) & A(\{1,3\} | \{2,3\}) \\ A(\{2,3\} | \{1,2\}) & A(\{2,3\} | \{1,3\}) & A(\{2,3\} | \{2,3\}) \end{bmatrix}.$$

Definition 1 has several implications. First, if A is square then $(A^T)^{(k)} = (A^{(k)})^T$, and in particular if A is symmetric then so is $A^{(k)}$. Also, $A^{(1)} = A$ and if $A \in \mathbb{R}^{n \times n}$ then $A^{(n)} = \det(A)$. If D is an $n \times n$ diagonal matrix, i.e. $D = \text{diag}(d_1, \dots, d_n)$ then $D^{(k)} = \text{diag}(d_1 \dots d_k, d_1 \dots d_{k-1} d_{k+1}, \dots, d_{n-k+1} \dots d_n)$. In particular, every eigenvalue of $D^{(k)}$ is the product of k eigenvalues of D . In the special case $D = pI_n$, with $p \in \mathbb{R}$, we have that $(pI_n)^{(k)} = p^k I_r$, with $r := \binom{n}{k}$.

The *Cauchy-Binet formula* (see, e.g., (Fallat and Johnson, 2011, Thm. 1.1.1)) asserts that

$$(AB)^{(k)} = A^{(k)} B^{(k)} \quad (3)$$

for any $A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{p \times m}$, $k \in [1, \min\{n, p, m\}]$. This justifies the term *multiplicative compound*.

When $n = p = m = k$, Eq. (3) becomes the familiar formula $\det(AB) = \det(A) \det(B)$. If A is $n \times n$ and non-singular then (3) implies that $I_n^{(k)} = (AA^{-1})^{(k)} = A^{(k)} (A^{-1})^{(k)}$, so $A^{(k)}$ is also non-singular with $(A^{(k)})^{-1} = (A^{-1})^{(k)}$. Another implication of (3) is that if $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ then the eigenvalues of $A^{(k)}$ are all the $\binom{n}{k}$ products:

$$\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}, \text{ with } 1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

The usefulness of the k th multiplicative compound in analyzing k -contraction follows from the following fact. Fix k vectors $x^1, \dots, x^k \in \mathbb{R}^n$. The parallelotope generated by these vectors (and the zero vertex) is

$$\mathcal{P}(x^1, \dots, x^k) := \left\{ \sum_{i=1}^k r_i x^i \mid r_i \in [0, 1] \text{ for all } i \right\}.$$

Let $X := [x^1 \dots x^k] \in \mathbb{R}^{n \times k}$. The volume of $\mathcal{P}(x^1, \dots, x^k)$ satisfies (Gantmacher, 1960, Chapter IX):

$$\text{volume}(\mathcal{P}(x^1, \dots, x^k)) = |X^{(k)}|_2. \quad (4)$$

Note that since $X \in \mathbb{R}^{n \times k}$, the dimensions of $X^{(k)}$ are $\binom{n}{k} \times 1$, that is, $X^{(k)}$ is a column vector.

In the special case $k = n$, Eq. (4) becomes

$$\text{volume}(\mathcal{P}(x^1, \dots, x^n)) = |X^{(n)}|_2 = |\det(X)|.$$

When the vertices of the parallelotope follow a linear time-varying dynamics, the evolution of the k th multiplicative compound depends on another algebraic construction called the k th additive compound.

Definition 2. The k th *additive compound matrix* of a square matrix $A \in \mathbb{R}^{n \times n}$ is defined by

$$A^{[k]} := \frac{d}{d\varepsilon} (I_n + \varepsilon A)^{(k)}|_{\varepsilon=0} = \frac{d}{d\varepsilon} (\exp(A\varepsilon))^{(k)}|_{\varepsilon=0}. \quad (5)$$

Example 1. Suppose that $A = pI_n$, with $p \in \mathbb{R}$. Then

$$\begin{aligned} (I_n + \varepsilon A)^{(k)} &= ((1 + \varepsilon p)I_n)^{(k)} \\ &= (1 + \varepsilon p)^k I_r, \end{aligned}$$

where $r := \binom{n}{k}$, so

$$\begin{aligned} (pI_n)^{[k]} &= \frac{d}{d\varepsilon} (1 + \varepsilon p)^k I_r|_{\varepsilon=0} \\ &= kpI_r. \end{aligned}$$

Definition 2 implies that $A^{[1]} = A$, $A^{[n]} = \text{tr}(A)$, and that

$$(I_n + \varepsilon A)^{(k)} = I_r + \varepsilon A^{[k]} + o(\varepsilon). \quad (6)$$

Thus, $\varepsilon A^{[k]}$ is the first-order term in the Taylor series of $(I + \varepsilon A)^{(k)}$. Also, $(A^T)^{[k]} = (A^{[k]})^T$, and in particular if A is symmetric then so is $A^{[k]}$.

Example 2. If $D = \text{diag}(d_1, \dots, d_n)$ then $(I + \varepsilon D)^{(k)} = \text{diag}(\prod_{i=1}^k (1 + \varepsilon d_i), \dots, \prod_{i=n-k+1}^n (1 + \varepsilon d_i))$, so (6) gives $D^{[k]} = \text{diag}(\sum_{i=1}^k d_i, \dots, \sum_{i=n-k+1}^n d_i)$. In particular, every eigenvalue of $D^{[k]}$ is the sum of k eigenvalues of D .

More generally, if $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ then the eigenvalues of $A^{[k]}$ are all the $\binom{n}{k}$ sums:

$$\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}, \text{ with } 1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

It follows from (6) and the properties of the multiplicative compound that $(A + B)^{[k]} = A^{[k]} + B^{[k]}$ for any $A, B \in \mathbb{R}^{n \times n}$, thus justifying the term *additive compound*. In fact, the mapping $A \rightarrow A^{[k]}$ is linear (Schwarz, 1970).

Below we will use the following relations. Let $A \in \mathbb{R}^{n \times n}$, and $k \in [1, n]$. If $T \in \mathbb{R}^{n \times n}$ is invertible, then

$$(TAT^{-1})^{(k)} = T^{(k)} A^{(k)} (T^{(k)})^{-1}, \quad (7)$$

and combining this with Definition 2 gives

$$(TAT^{-1})^{[k]} = T^{(k)} A^{[k]} (T^{(k)})^{-1} \quad (8)$$

(see, e.g., Bar-Shalom et al. (2023)).

For more on the applications of compound matrices to systems and control theory, see e.g. (Wu and Margaliot, 2022; Margaliot and Sontag, 2019; Ofir et al., 2022; Ofir and Margaliot, 2021; Grussler and Sepulchre, 2022; Li et al., 1999), and the recent tutorial by Bar-Shalom et al. (2023).

2.2 Matrix measures

A norm $|\cdot| : \mathbb{R}^n \rightarrow \mathbb{R}_+$ induces a matrix norm $\|\cdot\| : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}_+$ defined by $\|A\| := \max_{|x|=1} |Ax|$, and a matrix measure $\mu(\cdot) : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by $\mu(A) := \lim_{\varepsilon \downarrow 0} \frac{\|I + \varepsilon A\| - 1}{\varepsilon}$. The matrix measure is sub-additive, i.e. $\mu(A+B) \leq \mu(A) + \mu(B)$, and $\mu(cI_n) = c$ for any $c \in \mathbb{R}$.

Denote the L_2 norm by $|x|_2 := (x^T x)^{1/2}$. The corresponding matrix norm is $\|A\|_2 = (\lambda_{\max}(A^T A))^{1/2}$, and the corresponding matrix measure is (Vidyasagar, 2002):

$$\mu_2(A) = (1/2)\lambda_{\max}(A + A^T), \quad (9)$$

where $\lambda_{\max}(S)$ denotes the largest eigenvalue of the symmetric matrix S .

For an invertible matrix $H \in \mathbb{R}^{n \times n}$ the scaled L_2 norm is defined by $|x|_{2,H} := |Hx|_2$, and the induced matrix measure is

$$\begin{aligned} \mu_{2,H}(A) &= \mu_2(HAH^{-1}) \\ &= (1/2)\lambda_{\max}(HAH^{-1} + (HAH^{-1})^T). \end{aligned} \quad (10)$$

As shown in (Wu et al., 2022a), a sufficient condition for the system $\dot{x} = f(t, x)$ to be k -contractive is that $\mu((J(t, x))^{[k]}) \leq -\eta < 0$ for all t, x , where $J := \frac{\partial f}{\partial x}$. For $k = 1$, this reduces to the standard sufficient condition for contraction, namely, $\mu(J(t, x)) \leq -\eta < 0$ for all t, x . Note that if $A, T \in \mathbb{R}^{n \times n}$, with T non-singular, then

$$\begin{aligned} \mu_{2,T^{(k)}}(A^{[k]}) &= \mu_2(T^{(k)} A^{[k]} (T^{(k)})^{-1}) \\ &= \mu_2((TAT^{-1})^{[k]}). \end{aligned} \quad (11)$$

3. MAIN RESULTS

In this section, we derive a sufficient condition for k -contraction of the closed-loop system (2). We assume that the nonlinearity Φ is continuously differentiable and denote its Jacobian by $J_\Phi(t, y) := \frac{\partial \Phi}{\partial y}(t, y)$. The Jacobian of (2) is then

$$J(t, x) := A - BJ_\Phi(t, Cx)C. \quad (12)$$

We can now state our main result. For a symmetric matrix $S \in \mathbb{R}^{n \times n}$, we denote its ordered eigenvalues as $\lambda_1(S) \geq \dots \geq \lambda_n(S)$.

Theorem 1. Consider the Lurie system (2). Fix $k \in [1, n]$. Suppose that there exist $\eta_1, \eta_2 \in \mathbb{R}$ and $P \in \mathbb{R}^{n \times n}$, where $P = QQ$ with $Q \succ 0$, such that

$$\begin{aligned} P^{(k)} A^{[k]} + (A^{[k]})^T P^{(k)} + \eta_1 P^{(k)} \\ + Q^{(k)} \left((QBB^T Q)^{[k]} + (Q^{-1} C^T C Q^{-1})^{[k]} \right) Q^{(k)} \preceq 0, \end{aligned} \quad (13)$$

and

$$\sum_{i=1}^k \lambda_i(Q^{-1} C^T ((J_\Phi^T(t, y) J_\Phi(t, y) - I_q) C Q^{-1}) \leq -\eta_2, \quad (14)$$

for all $t \geq 0, y \in \mathbb{R}^q$. Then the Jacobian of the closed-loop system (2) satisfies

$$\mu_{2,Q^{(k)}}(J^{[k]}(t, x)) \leq -(\eta_1 + \eta_2)/2 \text{ for all } t \geq 0, x \in \mathbb{R}^n.$$

In particular, if $\eta_1 + \eta_2 > 0$, then the closed-loop system (2) is k -contractive with rate $(\eta_1 + \eta_2)/2$ w.r.t. the scaled L_2 norm $|z|_{2,Q^{(k)}} = |Q^{(k)} z|_2$.

Remark 1. Note that when $k = 1$, Eq. (13) holds for some $\eta_1 > 0$ if and only if the familiar ARI

$$PA + A^T P + PBB^T P + C^T C \prec 0 \quad (15)$$

holds. Assuming the LTI subsystem is minimal, (15) holds if and only if A is Hurwitz and the H_∞ norm of the LTI subsystem is less than 1. Similarly, (14) holds for any $\eta_2 > 0$ if $\|J_\Phi\|_2 \leq 1$, so in this case Thm. 1 becomes a small-gain sufficient condition for standard contraction.

The rest of this section is devoted to the proof of Thm. 1. We require the following result.

Lemma 1. Fix $M \in \mathbb{R}^{n \times m}$, $N \in \mathbb{R}^{m \times n}$, and $k \in \{1, \dots, n\}$. Then

$$(-MN - N^T M^T - N^T N)^{[k]} \preceq (MM^T)^{[k]}.$$

Proof. The identity

$MN + N^T M^T = (M^T + N)^T (M^T + N) - MM^T - N^T N$ gives

$$Z := -MM^T - MN - N^T M^T - N^T N \preceq 0.$$

Thus, Z is symmetric with all (real) eigenvalues smaller or equal to zero. Hence, the same properties hold for $Z^{[k]}$, so

$$Z^{[k]} = (-MM^T - MN - N^T M^T - N^T N)^{[k]} \preceq 0,$$

and this completes the proof.

We can now prove Thm. 1.

Proof. Let $R := QJQ^{-1} + Q^{-1}J^T Q$, with J defined in (12). Then

$$\begin{aligned} R^{[k]} &= (Q(A - BJ_\phi C)Q^{-1} + Q^{-1}(A - BJ_\phi C)^T Q)^{[k]} \\ &= (QAA^{-1} + Q^{-1}A^T Q)^{[k]} \\ &\quad - (QBJ_\phi CQ^{-1} + Q^{-1}C^T J_\phi^T B^T Q)^{[k]}. \end{aligned}$$

Multiplying (13) on the left- and on the right-hand side by $(Q^{(k)})^{-1}$, and using (8) gives

$$(QAA^{-1} + Q^{-1}A^T Q)^{[k]} \preceq -\eta_1 I_r - (QBB^T Q + Q^{-1}C^T CQ^{-1})^{[k]}, \quad (16)$$

so

$$\begin{aligned} R^{[k]} &\preceq -\eta_1 I_r - (QBB^T Q + Q^{-1}C^T CQ^{-1})^{[k]} \\ &\quad - (QBJ_\phi CQ^{-1} + Q^{-1}C^T J_\phi^T B^T Q)^{[k]}. \end{aligned} \quad (17)$$

It follows from Lemma 1 with $M = QB$ and $N = J_\phi CQ^{-1}$ that

$$\begin{aligned} &(-QBJ_\phi CQ^{-1} - Q^{-1}C^T J_\phi^T B^T Q - Q^{-1}C^T J_\phi^T J_\phi CQ^{-1})^{[k]} \\ &\preceq (QBB^T Q)^{[k]}, \end{aligned}$$

and combining this with (17) gives

$$R^{[k]} \preceq -\eta_1 I_r + (Q^{-1}C^T (J_\phi^T J_\phi - I_q) CQ^{-1})^{[k]}. \quad (18)$$

Using (14), we get

$$\mu_2(R^{[k]}) \leq -\eta_1 - \eta_2,$$

and the definition of R gives $2\mu_{2,Q^{(k)}}(J^{[k]}) = \mu_2(R^{[k]}) \leq -\eta_1 - \eta_2$. In particular, if $\eta_1 + \eta_2 > 0$ then the closed-loop system is k -contractive with rate $(\eta_1 + \eta_2)/2$ w.r.t. the scaled L_2 norm $|z|_{2,Q^{(k)}} = |Q^{(k)}z|_2$. This completes the proof of Thm. 1. \square

Remark 2. Define the symmetric matrix

$$\begin{aligned} S &:= QAA^{-1} + Q^{-1}A^T Q + \eta_1 k^{-1} I_n + QBB^T Q \\ &\quad + Q^{-1}C^T CQ^{-1}. \end{aligned}$$

Then condition (13) can be written more succinctly as $S^{[k]} \preceq 0$, that is, $\sum_{i=1}^k \lambda_i(S) \leq 0$.

Remark 3. Note that for the particular case $P = pI_n$, with $p > 0$, and $C = I_n$ (i.e., the LTI output is $y = x$), we have $Q = p^{1/2}I$, and Eq. (18) gives

$$2\mu_{2,Q^{(k)}}(J^{[k]}) \leq -\eta_1 - p^{-1}k + p^{-1} \sum_{i=1}^k \sigma_i^2(J_\Phi).$$

Thus, in this case a sufficient condition for k -contraction is

$$\sum_{i=1}^k \sigma_i^2(J_\Phi(t, y)) < k + \eta_1 p \text{ for all } t \geq 0, y \in \mathbb{R}^n. \quad (19)$$

4. AN APPLICATION: k -CONTRACTION IN A NETWORKED SYSTEM

Consider the nonlinear networked dynamical system

$$\dot{x} = -\alpha x + Wf(x), \quad (20)$$

where $x \in \Omega \subseteq \mathbb{R}^n$, $\alpha > 0$, $W \in \mathbb{R}^{n \times n}$ is a matrix of interconnection weights, and $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$. In the context of neural network models, the f_i s are the neuron activation functions. We assume that the state space Ω is convex and that f is continuously differentiable. Let $J_f(z) := \frac{\partial f}{\partial z}(z)$.

Intuitively speaking, it is clear that a larger α makes the system “more stable”. The next result formalizes this by providing a sufficient condition for k -contraction.

Corollary 2. Consider (20) with $\alpha > 0$. Fix $k \in [1, n]$. If

$$\|J_f(x)\|_2^2 \sum_{i=1}^k \sigma_i^2(W) < \alpha^2 k \text{ for all } x \in \Omega, \quad (21)$$

then (20) is k -contractive. Furthermore, if f is uniformly bounded and (21) holds with $k = 2$ then every trajectory of (20) converges to an equilibrium point (which is not necessarily unique).

Remark 4. Note that if (21) holds for some $k \in [1, n]$ then it holds for any $\ell \geq k$. Thus, if the sufficient condition for k -contraction holds then the system is also ℓ -contractive for any $\ell \geq k$. This agrees with the results in Wu et al. (2022a,b). Note also that if $W = 0$ or $f(x) = 0$ then (21) holds for $k = 1$ (and thus for any $k \in [1, n]$). This is reasonable, as in this case we have $\dot{x} = -\alpha x$, and this is indeed k -contractive for any $k \geq 1$.

Proof. The proof is based on Thm. 1. We first represent the networked system (20) as a Lurie system. By (21), there exists γ such that

$$0 < \gamma < \alpha \text{ and } \|J_f(y)\|_2^2 \sum_{i=1}^k \sigma_i^2(W) < \gamma^2 k. \quad (22)$$

We can represent (20) as the interconnection of the LTI system with $(A, B, C) = (-\alpha I_n, \gamma I_n, I_n)$ and the nonlinearity $\Phi(y) := -\gamma^{-1}Wf(y)$, that is,

$$\begin{aligned} \dot{x} &= -\alpha x + \gamma u, \\ u &= \gamma^{-1}Wf(x). \end{aligned} \quad (23)$$

For this Lurie system, there exist $Q \succ 0$ with $P = QQ$ and $\eta_1 > 0$ such that (13) holds if and only if

$$-2k\alpha P^{(k)} + Q^{(k)}(\gamma^2 P + P^{-1})^{[k]} Q^{(k)} \prec 0. \quad (24)$$

Taking $P = pI_n$, with $p > 0$, Eq. (24) simplifies to

$$(-2\alpha + \gamma^2 p + p^{-1})kp^k < 0, \quad (25)$$

which indeed admits a solution $p > 0$ since $\alpha > 0$ and $\gamma < \alpha$. We conclude that there exists a matrix $P = pI_n$, with $p > 0$, and a scalar $\eta_1 > 0$ for which (13) holds.

We now show that (21) implies that (14) holds for some $\eta_2 > 0$. Since $P = pI_n$ and $C = I_n$, we may apply the result in Remark 3. Recall that for any $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, we have

$$\sum_{i=1}^k \sigma_i^s(AB) \leq \sum_{i=1}^k (\sigma_i(A)\sigma_i(B))^s \quad (26)$$

for any $k \in [1, \min\{m, p, n\}]$, $s > 0$ (Horn and Johnson, 1991, Thm. 3.3.14). Consider

$$\begin{aligned} \sum_{i=1}^k \sigma_i^2(J_\Phi) &= \sum_{i=1}^k \sigma_i^2(-\gamma^{-1}WJ_f) \\ &\leq \gamma^{-2} \sum_{i=1}^k \sigma_i^2(W)\sigma_i^2(J_f) \\ &< k, \end{aligned}$$

where the first inequality follows from (26), and the second from (22). We conclude that the sufficient condition (19) holds, and Thm. 1 implies that (20) is k -contractive.

Suppose now that (21) holds with $k = 2$. Then (20) is 2-contractive. If in addition f is uniformly bounded, then all the trajectories of (20) are bounded, and by known results on time-invariant 2-contractive systems (Li and Muldowney, 1995) we then have that all trajectories converge to an equilibrium point. This completes the proof of Corollary 2. \square

A particular example of a system in the form (20) is the famous Hopfield network (Hopfield, 1982). This network has been used as an associative memory, where each equilibrium corresponds to a stored pattern (see, e.g., Krotov and Hopfield (2016)). Hence, the network is multi-stable and thus not contractive w.r.t. any norm. Corollary 2 may be used to prove that a Hopfield network is k -contractive for $k > 1$. To show this, consider (20) with $n = 10$, $\alpha = 1/2$, $W = 1_n 1_n^T$, where $1_n \in \mathbb{R}^n$ is a column vector of ones, and $f(x) = 0.07 \tanh(x)$. In this case, (20) has at least 3 equilibrium points: $e^1 = 0$, $e^2 \approx 1.1403 \cdot 1_n$, and $e^3 = -e^2$, so it is certainly not 1-contractive. However, condition (21) holds for $k = 2$ since $\|J_f(x)\|_2^2 \leq 0.07^2$, $\sum_{i=1}^2 \sigma_i^2(W) = 100$, and $\alpha^2 k = 1/2$. Thus, the system is 2-contractive. Since f is also uniformly bounded, we conclude that all trajectories converge to an equilibrium point. Note that when using the network as an associative memory, such a property is very useful.

5. CONCLUSION

We derived a sufficient condition for k -contraction of Lurie systems. For $k = 1$, this reduces to the standard sufficient condition for contraction. However, often Lurie systems admit more than a single equilibrium point, and are thus not contractive (that is, not 1-contractive) w.r.t. any norm. Our condition may still be used to guarantee a well-ordered behaviour of the closed-loop system. For example in the time-invariant case, establishing that the system is 2-contractive implies that any bounded solution converges to an equilibrium, that is not necessarily unique. Such a property is important, for example, in dynamical models of associative memories, where every equilibrium corresponds to a stored memory.

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