TOV

Introduction

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{G\left[P+\mathcal{E}(r)\right]\left[M(r)+4\pi r^3 P/c^2\right]}{c^2r^2[1-2GM(r)/(c^2r)]} \\ \frac{\mathrm{d}M}{\mathrm{d}r} &= 4\pi r^2\frac{\mathcal{E}(r)}{c^2}\end{aligned} \tag{1}$$

An interpretation of these equations can be more readily seen by multiplying the first equation by $4\pi r^2\mathcal{E}\mathrm{d}r/c^2 = \mathrm{d}M$ and cancelling $\mathcal{E}$ on both sides:

$$\begin{aligned}4\pi r^2\mathrm{d}P = -\frac{GM\mathrm{d}M}{r^2}\left(1+\frac{P}{\mathcal{E}(r)}\right)\left(1+\frac{4\pi r^3 P}{Mc^2}\right) \\ \times\left(1-\frac{2GM}{c^2r^2}\right)^{-1}\end{aligned} \tag{2}$$

The term on the left hand side is the force exerted on a infinitesimal shell at radius r. The first factor on the right hand side is the newtonian gravitational force from the interior acting on this shell.

$$\begin{aligned}\mathcal{E}(n) &= \frac{(mc^2)^4}{8\pi^2(\hbar c)^3}\Big[x_F\sqrt{1+x_F^2}\left(1+2x_F^2\right) \\ &\qquad - \ln\left(x_F+\sqrt{1+x_F^2}\right)\Big] \\ P(n) &= \frac{(mc^2)^4}{8\pi^2(\hbar c)^3}\Big[\frac{2}{3}x_F^3\sqrt{1+x_F^2} - x_F\sqrt{1+x_F^2} \\ &\qquad + \ln\left(x_F+\sqrt{1+x_F^2}\right)\Big]\end{aligned} \tag{3}$$

where $x_F(n) = \hbar c(3\pi n)^{1/3}/(mc^2)$.

Numerical set-up

The TOV-equations can be made more suitible for numerical calculations by rescaling the variables. Making the substitutions $r = R_0 x$, $P = P_0 p$, $\mathcal{E} = \mathcal{E}_0\varepsilon$ and $M = M_0 m$, the equation for the pressure can be written

$$\begin{aligned}\left(\frac{P_0}{R_0}\right)\frac{\mathrm{d}p}{\mathrm{d}x} = -\left(\frac{GP_0M_0}{c^2R_0^2}\right) \\ \times\frac{\left[p+(\mathcal{E}_0/P_0)\varepsilon\right]\left[m+4\pi R_0^3x^3P_0p/(c^2M_0)\right]}{x^2\left[1-2GM_0m/(c^2R_0x)\right]}.\end{aligned} \tag{4}$$

Cancelling P_0/R_0 on both sides yields

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}x} = -\left(\frac{G_0M_0}{c^2R_0}\right) \\ \times\frac{\left[p+(\mathcal{E}_0/P_0)\varepsilon\right]\left[m+4\pi R_0^3x^3P_0p/(M_0c^2)\right]}{x^2\left[1-2GM_0m/(c^2R_0x)\right]}\end{aligned} \tag{5}$$

Similarily, the equation for the mass becomes

$$\frac{\mathrm{d}m}{\mathrm{d}x} = \left(\frac{4\pi R_0^3\mathcal{E}_0}{M_0c^2}\right)x^2\varepsilon \tag{6}$$

The equations for p and m simplify substantially if the numerical constants are chosen such that

$$1 = \frac{\mathcal{E}_0}{P_0} = \frac{GM_0}{c^2R_0} = \frac{4\pi R_0^3P_0}{M_0c^2}. \tag{7}$$

Then the TOV-equations become

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}x} &= -\frac{(\varepsilon+p)(m+x^3p)}{x(x-2m)} \\ \frac{\mathrm{d}m}{\mathrm{d}x} &= x^2\varepsilon.\end{aligned} \tag{8}$$

A natural choice for $\mathcal{E}_0$ and P_0 is the term in front of the brackets in (3):

$$\mathcal{E}_0 = P_0 = \frac{(mc^2)^4}{8\pi^2(\hbar c)^3} \approx 1.285\ \mathrm{GeV/fm^3}. \tag{9}$$

Equation (7) then fixes M_0 and R_0:

$$\begin{aligned}M_0 &= \sqrt{\frac{c^8}{4\pi\mathcal{E}_0G^3}} \approx 4.63\ \mathrm{M}_\odot \\ R_0 &= \frac{GM_0}{c^2} \approx 6.84\ \mathrm{km}\end{aligned} \tag{10}$$

where $\mathrm{M}_\odot$ is the mass of the sun.

To solve these equations, initial conditions for both p and m are needed. From a physical consideration it is obvious to set $m(0) = 0$. The initial value for the pressure can be either $p(0) = p_0$ or $p(x_0) = 0$ where p_0 is the pressure at the center of the star and x_0 is its radius. When ε depends on the pressure p, it becomes easier to make use of the former condition since m is integrated from the center. The equation for p is however singular for $x = 0$, so the equations have to be integrated from some value Δx close to zero. Since the first two derivatives of m are zero at $x = 0$, the error of setting $m(\Delta x) = 0$ is of the order $(\Delta x)^3$.

Another thing which is needed is the equation of state $\epsilon(p)$ which, for the case of an ideal Fermi gas at zero temperature, can not be found directly but in terms of the number density n:

$$\varepsilon(n) = x_F\sqrt{1+x_F^2}\left(1+2x_F^2\right) - \ln\left(x_F + \sqrt{1+x_F^2}\right)$$
$$p(n) = \frac{2}{3}x_F^3\sqrt{1+x_F^2} - x_F\sqrt{1+x_F^2} + \ln\left(x_F + \sqrt{1+x_F^2}\right) \tag{11}$$

with $x_F(n) = \hbar c(3\pi n)^{1/3}/(mc^2)$. Hence, for a given $p = p_0$ we need a root finding algorithm to find $n = n_0$ such that $p(n_0) = p_0$. Once n_0 has been found, we can plug that into ε to find $\varepsilon(n_0) = \varepsilon(p_0)$.

Having set up the equations which are to be integrated (8) and the equation of state (11), the following steps were taken to solve the TOV-equations:

1. Set the initial conditions by specifying $p(\Delta x) = p_c = p_{j=0}$ and set $m(\Delta x) = 0 = m_{j=0}$.

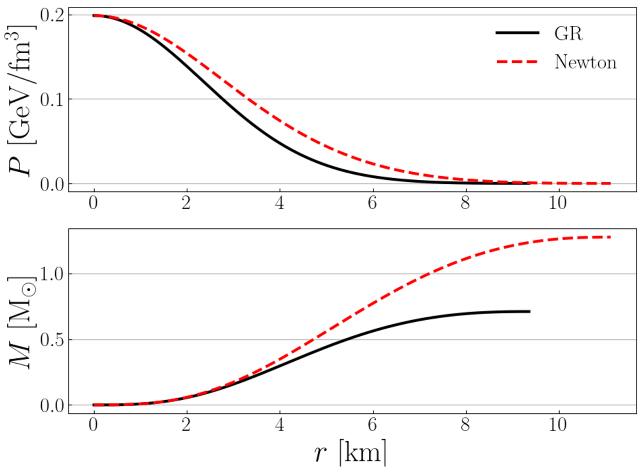
2. Find the energy density by using the bisection method on $p(n) = p_j$ and plugging n into the equation for the energy density to get $\epsilon(n) = \varepsilon_j$.

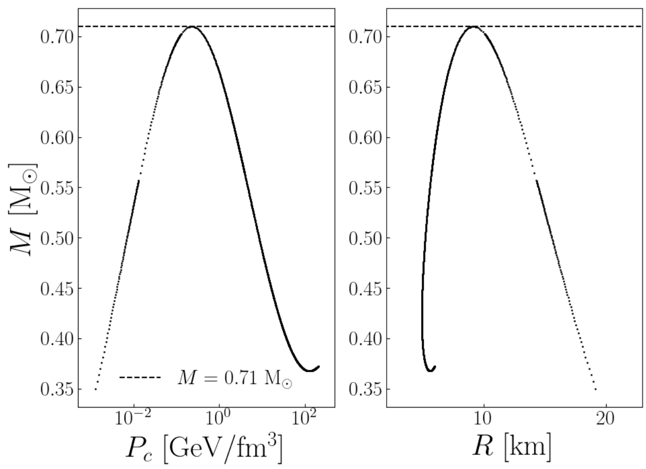
3. Use Heun's integration method on (8) to obtain p_{j+1} and m_{j+1}.

4. Repeat from 2. until the pressure is zero, $p_{j+1} = 0$.

The step size used was $h = 10^{-4}$ and the tolerance for the bisection method was $\Delta = 10^{-8}$.

Results





Notes

- Compare with Newton
- Also do white dwarfs