THE HIGHLY COLLAPSED CONFIGURATIONS OF A STELLAR MASS. (SECOND PAPER.)

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1. A study of the equilibrium of degenerate gas spheres has a twofold significance in the analysis of stellar structure, namely, in providing an approach to a proper theory of white dwarfs, and also, we shall see, in providing a certain limiting sequence of configurations to which all stars must tend eventually. A beginning in the study of these configurations was made by the author in a previous communication,* where for convenience the equation of state of degenerate matter was taken to correspond to one or other of the two limiting forms $p=K_1\rho^{5/3}$ or $p=K_2\rho^{4/3}$ according as the density was less than or greater than a certain density ρ' where

$$\rho'=(K_2/K_1)^3,$$

ρ' itself being such that both the equations of state yield the same calculated value for the pressure. Actually in the analysis a certain small temperature gradient was allowed for. Working on the standard model it was assumed that the ratio β of the gas pressure to the total pressure was a constant, but by hypothesis ("highly collapsed") β was taken to be very nearly unity. On these assumptions it followed that stars of mass less than a certain specified $M_{3/2}$ (see I, § 6, page 462) were complete Emden polytropes with index $n=3/2$, and further that configurations of greater mass must be ***composite***, *i.e.* must have inner regions where degeneracy is predominantly relativistic. Lastly, and this was the most important conclusion reached, these composite configurations have a *natural limit*: On the standard model a completely relativistically degenerate configuration has a mass given by (*cf.* I, equation (36))

$$M=-\frac{4}{\pi^{1/2}}\left(\frac{K_2}{G}\right)^{3/2}\left(\xi^2\frac{d\theta_3}{d\xi}\right)_1 . \beta^{-3/2}=M_3\beta^{-3/2}\text{ (say)}, \qquad (1)\dagger$$

where θ_3 is the Emden function with index $n=3$. These configurations have zero radius (*cf.* the remarks in I following the equations (45), (46), page 463).‡

* *M.N.*, **91**, 456, 1931 (referred to as I). See also the earlier papers of the author in *Phil. Mag.*, **11**, 592, 1931, and *Astrophysical Journal*, **64**, 92.

† In I we denoted by M_3 what we have now defined as $M_3\beta^{-3/2}$. It is convenient to separate out the term involving β from the purely "mass factor."

‡ In I this "singularity" was formally avoided by introducing a state of "maximum density" for matter, but now we shall not introduce any such hypothetical states, mainly for the reason that it appears from general considerations that when the central density is high enough for marked deviations from the known gas laws (degenerate or otherwise) to occur the configurations then would have such small radii that they would cease to have any practical importance in astrophysics.

Apart from the above results of a general character, the analysis in I did not lead to any further quantitative results. To obtain by the methods of I anything more exact would have meant very considerable numerical work to "fit" an appropriate solution of Emden's equation with index $n=3/2$ (to describe the outer ordinarily degenerate envelope) with an *Emden function* of index 3 (to describe the inner relativistically degenerate core). It would be very much more satisfactory to take the exact equation describing the degenerate state and treat the whole degenerate parts of a star on the same footing instead of as in I, further subdividing it to correspond to one or other of the two limiting forms of the equation describing the degenerate state. By a very remarkable coincidence the differential equation (governing the structure of a degenerate gas sphere in hydrostatic equilibrium) based on the exact equation of state takes an extremely simple form. We show, in fact, that the structure of the configuration is governed by a solution of the differential equation,

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) = -\left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2}. \tag{2}^*$$

It is to be noticed that there is only one parameter occurring in the equation, and a single system of integrations should suffice to obtain a clear insight into these configurations. Equation (2) has a formal similarity with Emden's equation. Indeed, we shall show that under certain circumstances ϕ can be expressed in terms of the Emden functions with appropriate indices. It is the derivation of the above equation that has led to the developments summarised in this and the following paper. In this paper we shall establish this equation and provide tables of solutions. In the analysis we shall omit all references to radiation pressure, *i.e.* this paper strictly deals with configurations having $\beta=1$. The introduction of radiation in these configurations involves quite delicate considerations, and all these find a proper treatment in the paper following this one.

2. *The Differential Equation governing the Structure of Degenerate Matter in Gravitational Equilibrium.*—The pressure-density relation for a degenerate gas can be written parametrically as follows:—

$$\left.\begin{aligned} p &= \frac{\pi m^4 c^5}{3h^3}[x(2x^2-3)(x^2+1)^{1/2} + 3\sinh^{-1}x], \\ \rho &= \frac{8\pi m^3 c^3 \mu H}{3h^3}x^3, \end{aligned}\right\} \tag{3}$$

where m = mass of the electron, c = velocity of light, h = Planck's constant, H = mass of the proton, μ = molecular weight. Equation (3) is established in Appendix I to this paper, where also $f(x)$ is tabulated. We rewrite (3) as

$$p = A_2 f(x)\,; \qquad \rho = Bx^3, \tag{4}$$

* This equation was given without proof in the author's preliminary note in the *Observatory*, **57**, 373, 1934.

where

$$\left.\begin{aligned} A_2 &= \frac{\pi m^4 c^5}{3h^3}; \qquad B = \frac{8\pi m^3 c^3 \mu H}{3h^3}, \\ f(x) &= x(2x^2-3)(x^2+1)^{1/2} + 3 \sinh^{-1} x. \end{aligned}\right\} \tag{5}$$

The equations of equilibrium are, as usual,

$$\left.\begin{aligned} \frac{dp}{dr} &= -\frac{GM(r)}{r^2}\rho, \\ \frac{dM(r)}{dr} &= 4\pi\rho r^2. \end{aligned}\right\} \tag{6}$$

From (6) we have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho. \tag{7}$$

Substitute for p and ρ from (4). We have

$$\frac{A_2}{B}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi GBx^3. \tag{8}$$

From the definition of $f(x)$ in (5) we easily verify that

$$\frac{df(x)}{dr} = \frac{8x^4}{(x^2+1)^{1/2}}\frac{dx}{dr}, \tag{9}$$

or

$$\frac{1}{x^3}\frac{df(x)}{dr} = \frac{8x}{(x^2+1)^{1/2}}\frac{dx}{dr} = 8\frac{d\sqrt{x^2+1}}{dr}. \tag{10}$$

Hence (8) can be rewritten as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\sqrt{x^2+1}}{dr}\right) = -\frac{\pi GB^2}{2A_2}x^3. \tag{11}$$

Put

$$y^2 = x^2 + 1. \tag{12}$$

Then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy}{dr}\right) = -\frac{\pi GB^2}{2A_2}(y^2-1)^{3/2}. \tag{13}$$

Let x take the value x_0 at the centre.

Further, let y_0 be the corresponding value of y at the centre. Introduce the new variables η and ϕ defined as follows :—

$$r = \alpha\eta; \qquad y = y_0\phi, \tag{14}$$

where

$$\left.\begin{aligned} \alpha &= \left(\frac{2A_2}{\pi G}\right)^{1/2}\frac{1}{By_0}, \\ {y_0}^2 &= {x_0}^2 + 1. \end{aligned}\right\} \tag{15}$$

Our differential equation finally takes the form

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) = -\left(\phi^2 - \frac{1}{{y_0}^2}\right)^{3/2}. \tag{16}$$

By (14) we have to seek a solution of (16) such that ϕ takes the value unity at the origin. Further, from symmetry the derivative of ϕ must

14

vanish at the origin. The *boundary* is defined at the point where the density vanishes, and this by (12) means that if η_1 specifies the boundary

$$\phi(\eta_1) = \frac{1}{y_0}. \tag{17}$$

3. From our definitions of the various quantities we find that

$$\rho = \rho_0 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}, \tag{18}$$

where

$$\rho_0 = Bx_0^3 = B(y_0^2 - 1)^{3/2} \tag{18'}$$

specifies the central density. Also we may notice that the scale of length α introduced in (15) has in terms of the physical quantities the form

$$\alpha = \frac{1}{4\pi m\mu H y_0} \left(\frac{3h^3}{2cG} \right)^{1/2}, \tag{19}$$

or putting in numerical values

$$\alpha = \frac{7 \cdot 720 \times 10^8}{\mu y_0} = l_1 y_0^{-1} \text{ cm. (say)}. \tag{20}$$

4. *The Potential.*—The function ϕ itself has a physical meaning. If V is the inner gravitational potential, then from general theory we have

$$\frac{dV}{dr} = \frac{1}{\rho} \frac{dP}{dr}. \tag{21}$$

From (5) and (10) we see that

$$\frac{dV}{dr} = \frac{8A_2}{B} y_0 \frac{d\phi}{dr}, \tag{22}$$

or integrating

$$V = \frac{8A_2}{B} y_0 \phi + \text{constant}. \tag{23}$$

If we choose the arbitrary zero of the potential on the boundary of the configuration we have by (17) that the "constant" in (23) is $(8A_2/B)$. Hence finally

$$V = \frac{8A_2}{B} y_0 \left(\phi - \frac{1}{y_0} \right). \tag{24}$$

5. *The Mass Relation.*—The mass of the material enclosed up to a point η is clearly

$$M(\eta) = 4\pi \int_0^\eta \rho r^2 dr = 4\pi \alpha^3 \int_0^\eta \rho \eta^2 d\eta. \tag{25}$$

By (18),

$$M(\eta) = 4\pi \rho_0 \frac{\alpha^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2} \eta^2 d\eta, \tag{26}$$

or using our differential equation (16)

$$M(\eta) = -4\pi \rho_0 \frac{\alpha^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) d\eta. \tag{27}$$

Remembering that ρ_0 is given by (18) we have explicitly

$$M(\eta) = -4\pi\left(\frac{2A_2}{\pi G}\right)^{3/2}\frac{1}{B^2}\eta^2\frac{d\phi}{d\eta}. \tag{28}$$

The mass of the whole configuration is therefore

$$M = -4\pi\left(\frac{2A_2}{\pi G}\right)^{3/2}\frac{1}{B^2}\left(\eta^2\frac{d\phi}{d\eta}\right)_{\eta=\eta_1}. \tag{29}$$

We notice that in (28) and (29) y_0 does not *explicitly* occur. It is of course implicitly present inasmuch as in the differential equation defining ϕ, y_0 occurs.

6. *The Relation between the Mean and the Central Density.*—Let $\bar{\rho}(\eta)$ be the mean density of the material inside η. Then

$$M(\eta) = \tfrac{4}{3}\pi\alpha^3\eta^3\bar{\rho}(\eta). \tag{30}$$

Comparing (28) and (30), we have

$$\tfrac{1}{3}\eta^3\bar{\rho}(\eta) = -\rho_0\frac{y_0^3}{(y_0^2-1)^{3/2}}\eta^2\frac{d\phi}{d\eta}, \tag{31}$$

or

$$\frac{\bar{\rho}(\eta)}{\rho_0} = -3\frac{y_0^3}{(y_0^2-1)^{3/2}}\frac{1}{\eta}\frac{d\phi}{d\eta}. \tag{32}$$

From (32) we deduce that *the relation between the mean and the central density of the whole configuration is*

$$\rho_0 = -\bar{\rho}\left(1-\frac{1}{y_0^2}\right)^{3/2}\frac{\eta_1}{3\phi'(\eta_1)} \tag{33}$$

(ϕ' denoting the derivative)—a relation analogous to the corresponding relation in the theory of polytropes.

7. *An Approximation for Configurations with Small Central Densities.*—When the central density is small we should have the law $p = K_1{}^{5/3}$ holding approximately, and the corresponding configurations must have structures which can approximately be represented by an Emden polytrope with index $n=3/2$. We establish this on our differential equation in the following way:—

Now by definition $y_0^2 = x_0^2 + 1$, and we need a first-order approximation when x_0^2 is small. *We shall neglect all quantities of order x_0^4 or higher.* Then

$$y_0 = 1 + \tfrac{1}{2}x_0^2. \tag{34}$$

Put

$$\phi^2 - \frac{1}{y_0^2} = \theta. \tag{35}$$

In our approximation we have

$$\phi = 1 - \tfrac{1}{2}(x_0^2 - \theta). \tag{36}$$

At the origin ϕ takes the value unity. Hence

$$\theta(0) = x_0^2. \tag{37}$$

From (16) we derive the following differential equation for θ :—

$$\frac{1}{2}\frac{d^2\theta}{d\eta^2}+\frac{1}{\eta}\frac{d\theta}{d\eta}=-\theta^{3/2}. \tag{38}$$

Put

$$\xi=2^{1/2}\eta. \tag{39}$$

Then

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^{3/2}, \tag{40}$$

which is Emden's equation with index $n=3/2$, but *the solution we need is not the Emden function in the usual normalisation* * *with* $\theta=1$ *at* $\xi=0$. By (37) our θ takes the value x_0^2 at the origin. Denote by $\theta_{3/2}$ the Emden function. Now it is a property of the differential equation (40) that if θ is any solution then $C^4\theta(C\xi)$ is also a solution where C is any arbitrary constant. Hence if we put

$$C=x_0^{1/2}, \tag{41}$$

and take for θ, $\theta_{3/2}$, we would obtain the solution we need. Hence

$$\theta=x_0^2\theta_{3/2}(x_0^{1/2}\xi)=x_0^2\theta_{3/2}(\sqrt{2x_0}\eta). \tag{42}$$

By (37) then

$$\phi=1-\tfrac{1}{2}x_0^2\{1-\theta_{3/2}(\sqrt{2x_0}\eta)\}+O(x_0^4), \tag{43}$$

which relates ϕ with $\theta_{3/2}$. From (43) we see that for these configurations the boundary η_1 must be such that

$$(\theta_{3/2}\sqrt{2x_0}\eta_1)=0. \tag{44}$$

Let $\xi_1(\theta_{3/2})$ be the boundary of the *Emden function.* Then from (44) we deduce that

$$\eta_1=\frac{\xi_1(\theta_{3/2})}{\sqrt{2x_0}}. \tag{45}$$

From (45) we see that as $y_0\to 1$, $x_0\to 0$, $\eta_1\to\infty$. The radius tends to infinity with the same singularity.

Again from (43) we have

$$\frac{d\phi}{d\eta}=\tfrac{1}{2}x_0^2\sqrt{2x_0}\frac{d\theta_{3/2}(\xi)}{d\xi}. \tag{46}$$

Combining (45) and (46) we have a relation we shall need later :

$$\left(\eta^2\frac{d\phi}{d\eta}\right)_1=\left(\frac{x_0}{2}\right)^{3/2}\left(\xi^2\frac{d\theta_{3/2}}{d\xi}\right)_1. \tag{47}$$

Further,

$$\left(\frac{1}{\eta}\frac{d\phi}{d\eta}\right)_1=x_0^3\left(\frac{1}{\xi}\frac{d\theta_{3/2}}{d\xi}\right)_1. \tag{48}$$

We shall find the above expressions useful when we come to discuss "highly"

* In the sequel by "*Emden function*" *we shall always mean the one which takes the value unity at the origin.* We shall denote the *Emden function* with index n by θ_n.

collapsed configurations ($(1-\beta)$ finite but small), but now we verify that the scheme is consistent. From (48) and (33) we have

$$\rho_0 = -\bar{\rho}\left(\frac{\xi}{3\theta'_{3/2}}\right)_1, \tag{49}$$

which is precisely the formula for an Emden polytrope with index $n=3/2$. Again from (29) and (47)

$$M = -4\pi\left(\frac{2A_2}{\pi G}\right)^{3/2}\frac{1}{B^2}\left(\frac{x_0}{2}\right)^{3/2}\left(\xi^2\frac{d\theta_{3/2}}{d\xi}\right)_1. \tag{50}$$

To compare the above with the formula derived on the law $p=K_1\rho^{5/3}$ we note that the degenerate constant K_1, given by

$$K_1 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m(\mu H)^{5/3}}, \tag{51}$$

is related to our A_1 and B by the relation

$$K_1 = \frac{8}{5}\frac{A_2}{B^{5/3}}. \tag{52}$$

Combining (50) and (52) and setting λ_2 to denote the central density ($=Bx_0^3$) we find that

$$M = -4\pi\left(\frac{5K_1}{8\pi G}\right)^{3/2}\lambda_2^{1/2}\left(\xi^2\frac{d\theta_{3/2}}{d\xi}\right)_1, \tag{53}$$

which is the usual formula since on the law $p=K\rho^{1+\frac{1}{n}}$ the polytropic relation is

$$M = -4\pi\left(\frac{(n+1)K}{4\pi G}\right)^{3/2}\lambda_2^{\frac{3-n}{2n}}\left(\xi^2\frac{d\theta_n}{d\xi}\right)_1. \tag{53'}$$

8. *The Limiting Mass.*—From our differential equation (16) we see that

$$\phi \to \theta_3 \quad \text{as} \quad y_0 \to \infty. \tag{54}$$

But from (20) we see that at the same time the radius tends to zero. From (28) then

$$M \to -4\pi\left(\frac{2A_2}{\pi G}\right)^{3/2}\frac{1}{B^2}\left(\xi^2\frac{d\theta_3}{d\xi}\right)_1. \tag{55}$$

To see that we have now simply recovered our earlier result in I (equation (36)) we have only to notice that the relativistic degenerate constant K_2, defined by

$$K_2 = \left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8(\mu H)^{4/3}}, \tag{56}$$

is related to our A_2 and B by the relation

$$K_2 = \frac{2A_2}{B^{4/3}}. \tag{57}$$

9. As mentioned in § 1, we shall denote by M_3 the mass

$$M_3 = 4\pi\left(\frac{2A_2}{\pi G}\right)^{3/2}\frac{1}{B^2}\omega_3^0, \tag{58}$$

where following Milne we have introduced the quantity $\omega_3{}^0$ defined by

$$\omega_3{}^0 = -\left(\xi^2 \frac{d\theta_3}{d\xi}\right)_1. \tag{59}$$

If we define correspondingly that

$$\Omega(y_0) = -\left(\eta^2 \frac{d\phi}{d\eta}\right)_{\eta=\eta_1} \tag{60}$$

for our function ϕ, then the mass relation can be written as

$$M(y_0)\omega_3{}^0 = M_3\Omega(y_0). \tag{61}$$

As the mass of the configuration increases monotonically with increasing y_0, we have the useful inequality

$$\Omega(y_0) > \omega_3{}^0 \qquad (y_0 \text{ finite}). \tag{62}$$

Finally we may note that the insertion of numerical values in our formula for M_3 yields

$$M_3 = 5{\cdot}728\mu^{-2} \times \odot, \tag{63}$$

where $\odot$ represents the mass of the Sun.

10. *The General Results.*—In the previous sections, §§ 7, 8, 9, we have merely related our present treatment with the results obtained in I on the basis of the polytropic theory. Those results appear as simple limiting cases. However, the exact treatment on the basis of our differential equation

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2 \frac{d\phi}{d\eta}\right) = -\left(\phi^2 - \frac{1}{y_0{}^2}\right)^{3/2} \tag{64}$$

at the same time provides much more quantitative information. The boundary conditions

$$\phi = 1, \qquad \frac{d\phi}{d\eta} = 0 \quad \text{at} \quad \eta = 0, \tag{65}$$

combined with a particular value for y_0, would determine ϕ completely, and therefore the mass of the configuration as well. The equation (64) does not admit of a "homology constant," and hence *each mass has a density distribution characteristic of itself which cannot be inferred from the density distribution in a configuration of a different mass.* This difference between our configurations governed by (64) and polytropes has, as we shall see, an important bearing in the theory of general stellar models considered in the following paper.

Each specified value for y_0 determines uniquely the mass M, the radius R_1 and the ratio of the mean to the central density. We have (collecting together our earlier results) :

$$M/M_3 = \Omega(y_0)/\omega_3{}^0, \tag{66}$$

$$R_1/l_1 = \eta_1/y_0, \tag{67}$$

$$\rho_0/B = (y_0{}^2 - 1)^{3/2}, \tag{68}$$

$$\bar{\rho}/\rho_0 = -\frac{1}{\left(1-\frac{1}{y_0^2}\right)^{3/2}}\frac{3}{\eta_1}\left(\frac{d\phi}{d\eta}\right)_1. \tag{69}$$

In (67) we have introduced a new unit of length ($l_1 = \alpha y_0$),

$$l_1 = \frac{1}{4\pi m\mu H}\left(\frac{3h^3}{2cG}\right) = 7{\cdot}720\mu^{-1} \times 10^8 \text{ cm.}, \tag{67'}$$

and which therefore does not involve factors in y_0. Further, the physical variables determining the structure of the configuration are:

$$\rho = \rho_0 \frac{1}{\left(1-\frac{1}{y_0^2}\right)^{3/2}}\left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2}, \tag{70}$$

$$\bar{\rho} = -\rho_0 \frac{1}{\left(1-\frac{1}{y_0^2}\right)^{3/2}}\frac{3}{\eta}\frac{d\phi}{d\eta}, \tag{71}$$

$$M(\eta) \propto -\eta^2 \frac{d\phi}{d\eta}. \tag{72}$$

11. In § 10 we have reduced the problem of the structure of degenerate gas spheres to a study of our functions ϕ for different initially prescribed values for the parameter y_0. The integration has been numerically effected for the following ten different values of the parameter :—

$$1/y_0^2 = 0{\cdot}8,\ 0{\cdot}6,\ 0{\cdot}5,\ 0{\cdot}4,\ 0{\cdot}3,\ 0{\cdot}2,\ 0{\cdot}1,\ 0{\cdot}05,\ 0{\cdot}02,\ 0{\cdot}01. \tag{73}$$

The following expansion for ϕ near the origin may be noted here for further reference :—

$$\phi = 1 - \frac{q^3}{6}\xi^2 + \frac{q^4}{40}\xi^4 - \frac{q^5(5q^2+14)}{7\,!}\xi^6 + \frac{q^6(339q^2+280)}{3\times 9\,!}\xi^8 - \frac{q^7(1425q^4+11436q^2+4256)}{5\times 11\,!}\xi^{10} + \cdots \tag{74}$$

where

$$q^2 = 1 - \frac{1}{y_0^2}. \tag{75}^*$$

The important quantities of interest are the boundary quantities occurring in equations (66), (67), (69). These are tabulated in Table I for the different values of y_0.

12. From the figures of Table I it is easy to calculate the mass in units of M_3, the radius in units of l_1 and the central density ($= x_0^3$) in units of B

* When $y_0 \to \infty$, $q \to 1$ and the series (74) goes over into the expansion for Emden θ_3 near the origin (cf. *British Association Tables*, **2**, Introduction, equation on top of page v).

($=9\cdot8848 \times 10^5\ \mu$ grams cm.$^{-3}$). These express the chief physical characteristics of these configurations in the "natural system" of units occurring in the theory of these configurations. In Table III they are converted into the more conventional system of units expressing the radius and the density in C.G.S. units and the mass in units of the Sun. The actual figures tabulated are for $\mu=1$. The figures for other values of μ can be obtained by multiplying M by μ^{-2}, R_1 by μ^{-1} and ρ by μ. To see the order of magnitudes involved here it is of interest to point out that the mass $4\cdot852\odot\mu^{-2}$ has a radius only slightly over the radius of the Earth (radius of the Earth 6×10^8 cm. compared to $7\cdot7 \times 10^8$ cm. for the radius of $4\cdot852\odot$). The mass $0\cdot957 M_3$ has a radius considerably less than the radius of the Earth.

TABLE I

$\frac{1}{y_0^2}$	η_1	$-\eta_1^2\phi'(\eta_1)$	$\rho_0/\bar{\rho}$
0	6·8968	2·0182	54·182
·01	5·3571	1·9321	26·203
·02	4·9857	1·8652	21·486
·05	4·4601	1·7096	16·018
·1	4·0690	1·5186	12·626
·2	3·7271	1·2430	9·9348
·3	3·5803	1·0337	8·6673
·4	3·5245	0·8598	7·8886
·5	3·5330	0·7070	7·3505
·6	3·6038	0·5679	6·9504
·8	4·0446	0·3091	6·3814
1	∞	0	5·9907

TABLE II

The Physical Characteristics of Degenerate Spheres in the "Natural" Units

$\frac{1}{y_0^2}$	M/M_3	R_1/l_1	ρ_0/B
0	1	0	∞
·01	0·95733	0·53571	985·038
·02	0·92419	0·70508	343
·05	0·84709	0·99732	82·8191
·1	0·75243	1·28674	27
·2	0·61589	1·66682	8
·3	0·51218	1·96102	3·56423
·4	0·42600	2·22908	1·83711
·5	0·35033	2·49818	1
·6	0·28137	2·79148	0·54433
·8	0·15316	3·61760	0·125
1·0	0	∞	0

TABLE III

The Physical Characteristics of Degenerate Spheres in the Usual Units

(Calculations are for $\mu=1$. For other values μ, M should be multiplied by μ^{-2}, R_1 by μ^{-1}, ρ_c by μ)

$\frac{1}{y_0^2}$	$M/\odot$	ρ_0 in grm./cm.$^{-3}$	ρ_{mean} in grm./cm.$^{-3}$	Radius in cm.
0	5·728	∞	∞	0
·01	5·484	9.737×10^8	4.716×10^7	4.136×10^8
·02	5·294	3.391×10^8	1.578×10^7	5.443×10^8
·05	4·852	8.187×10^7	5.111×10^6	7.699×10^8
·1	4·310	2.669×10^7	2.114×10^6	9.936×10^8
·2	3·528	7.908×10^6	7.960×10^5	1.287×10^9
·3	2·934	3.523×10^6	4.065×10^5	1.514×10^9
·4	2·440	1.816×10^6	2.302×10^5	1.721×10^9
·5	2·007	9.885×10^5	1.345×10^5	1.929×10^9
·6	1·612	5.381×10^5	7.741×10^4	2.155×10^9
·8	0·877	1.236×10^5	1.936×10^4	2.793×10^9
1·0	0	0	0	∞

Now if we define that matter is "relativistically degenerate" for densities greater than $\rho'(=(K_2/K_1)^3)$, then we can from our results easily find the masses which are characterised by central regions of "relativistic degeneracy." The value of x corresponding to ρ' is readily seen to be 1·25. Hence

$$\frac{1}{y_0'^2}=\frac{1}{x'^2+1}=0.39024. \tag{76}$$

From fig. 1 we now see that for $M\leqslant 0.43M_3$ there are no regions which are "relativistically degenerate" on this convention. For $M>0.43M_3$ there are regions in which $x>x'(=1.25)$, and the fraction of the whole radius inside which $x>x'$ rapidly increases to unity. In the mass-radius curve we can therefore draw circles about each point with radii proportional to the actual radii of the corresponding configurations, and draw inside each a concentric circle to represent the "relativistic" region. This has been done in fig. 2 at a few points. We see that even for $M=0.75M_3$ there is barely a "fringe" of ordinarily degenerate regions. This diagram clearly illustrates a general principle that degeneracy never usually sets in without being relativistic.

13. *Comparison with the Results on Emden Polytrope $n=3/2$.*—It is of interest to see in how far the results of the above exact treatment differ from what one would obtain on the law $p=K_1\rho^{5/3}$. We have already shown in § 7 that one gets these Emden configurations as limiting cases for zero density and therefore for small masses (expressed in units of M_3). Our comparison here therefore amounts to a comparison of the results based on an exact treatment of the equation (64) with the limiting form for $y_0\to 1$ extrapolated for all masses. For this purpose it is convenient to rewrite the formulæ for the case of the polytrope $n=3/2$ in the following way.

From (45) and (50) we have now

$$R_1 = \frac{l_1 \xi_1(\theta_{3/2})}{\sqrt{2x_0}}, \tag{77}$$

$$M/M_3 = \left(\frac{x_0}{2}\right)^{3/2} \frac{1}{\omega_3{}^0} \left(\xi^2 \frac{d\theta_{3/2}}{d\xi}\right)_1. \tag{77'}$$

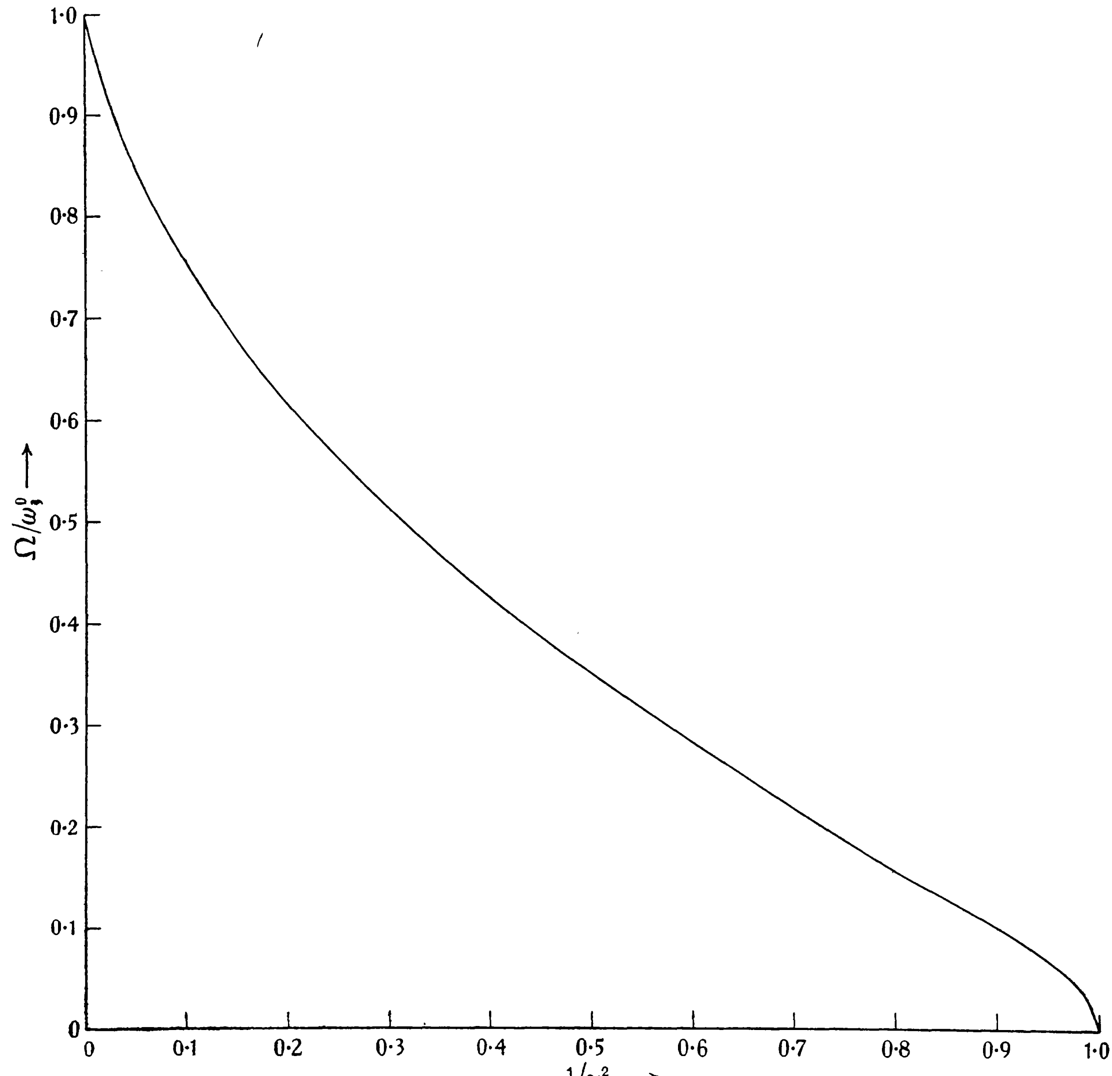

FIG. 1.—$\{(\Omega/\omega_3{}^0),\ 1/y_0{}^2\}$-*relation.*

From (77) and (77′) we have on eliminating x_0

$$2R_1 = \left(\frac{\omega_{3/2}{}^0 M_3}{\omega_3{}^0 M}\right)^{1/3} . \, l_1, \tag{78}$$

where following Milne we have introduced the "invariant" $\omega_{3/2}{}^0$ defined by

$$\omega_{3/2}{}^0 = -\left(\xi^5 \frac{d\theta_{3/2}}{d\xi}\right)_1 = 132{\cdot}3843. \tag{79}$$

It is of interest to notice that the two invariants $\omega_3{}^0$ and $\omega_{3/2}{}^0$ of the Emden equation with the indices $n=3$ and $3/2$ occur in (78) in a "symmetrical way." Numerically (78) is found to be

$$R_1 = 2{\cdot}01647\left(\frac{M_3}{M}\right)^{1/3} . \, l_1. \tag{80}$$

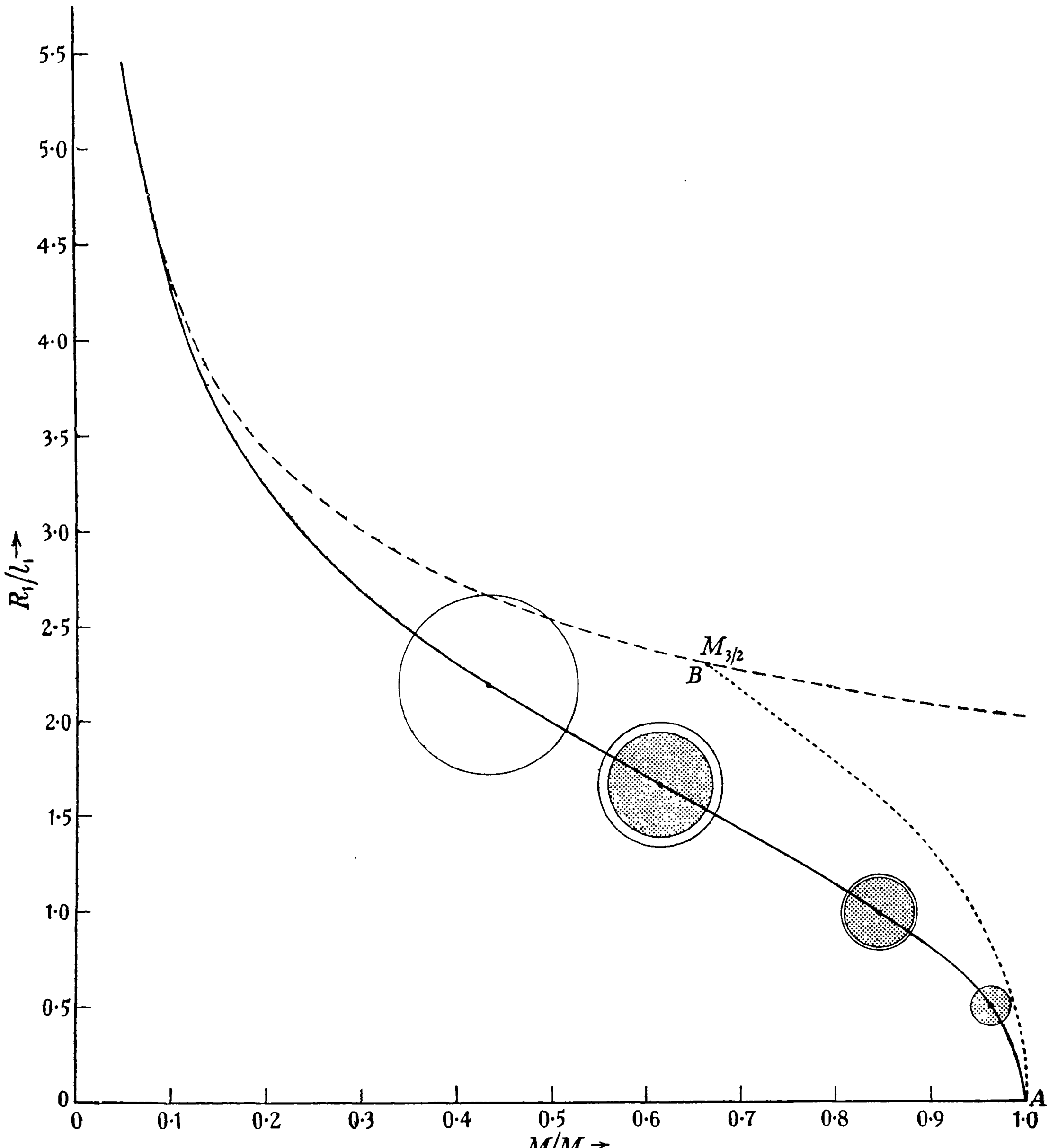

FIG. 2.—*The full line curve represents the exact (mass-radius)-relation for the highly collapsed configurations. This curve tends asymptotically to the - - - - curve as* $M \rightarrow 0$.

(80) expresses the *mass-radius* relation for the polytropic limit, the radius and the mass expressed in the same units as the quantities in Table II. Similarly the mass-central density relation now reads

$$x_0{}^3 = 4{\cdot}42381(M/M_3)^2. \tag{81}$$

The results calculated on the basis of (80) and (81) for the same masses as in Table II are summarised in Table IV. The corresponding curves are shown dotted in figs. 2 and 3.

TABLE IV

M/M_3	R_1/l_1	$x_0{}^3$
1	2·0165	4·4238
0·9573	2·0459	4·0538
0·9242	2·0700	3·7780
0·8471	2·1311	3·1739
0·7524	2·2174	2·5042
0·6159	2·3701	1·6778
0·5122	2·5203	1·1603
0·4260	2·6801	0·8027
0·3503	2·8606	0·5429
0·2814	3·0772	0·3502
0·1532	3·7691	0·1038

One notices clearly from these two curves how marked the deviations from the limiting curves become even for quite small masses. Thus for $M = 0{\cdot}15 M_3$ the central density predicted by our exact treatment is about 25 per cent. greater and the radius about 5 per cent. smaller. The relativistic effects are therefore quite significant even for small masses. They certainly cannot be ignored for masses greater than $0{\cdot}2 M_3$. Of course the extrapolation of the $n = 3/2$ configurations for masses (in units of M_3) approaching unity is quite misleading. These completely collapsed configurations have a natural limit, and our exact treatment now shows how this limit is reached.

It is of interest to compare the full-line curve in fig. 2 representing our exact (mass-radius) curve with what one would obtain by the methods of I, where the degenerate spheres of mass greater than a certain limit $M_{3/2}$ were considered as "composite configurations." The mass $M_{3/2}$ was defined as one in which the Emden polytrope with $n = 3/2$ * would have a central density $\rho' (= (K_2/K_1)^3)$. In our present notation we have by (81)

$$M_{3/2} = \sqrt{\frac{(1{\cdot}25)^3}{4{\cdot}42381}} \,.\, M_3 = 0{\cdot}66446 M_3. \tag{82}$$

This particular point is marked as B in fig. 2 on the - - - - curve. A treatment of the composite configurations by the methods of I would have led to some kind of curve like the dotted one in fig. 2 conjecturally drawn. But fortunately it is now not necessary to go into the very elaborate numerical work that would have been involved to fix the part BA by the methods of I. By a single system of integrations we have now fixed the exact nature of the (mass-radius) curve for these completely collapsed configurations.

* The equation of state being $p = K_1 \rho^{5/3}$.

14. *The Relative Density Distributions in the Different Configurations.*—Our main diagram (fig. 4) now illustrates the relative density distributions in the configurations studied. Here we have plotted (ρ/ρ_0) against (η/η_1) for the different masses for which we have numerical results. The two limiting density distributions specified by Emden, θ_3 and $\theta_{3/2}$, are also shown (dotted) in the same figure. Fig. 4, which is *the* principal outcome of our studies, presents a set of ten out of a continuous family of density distributions

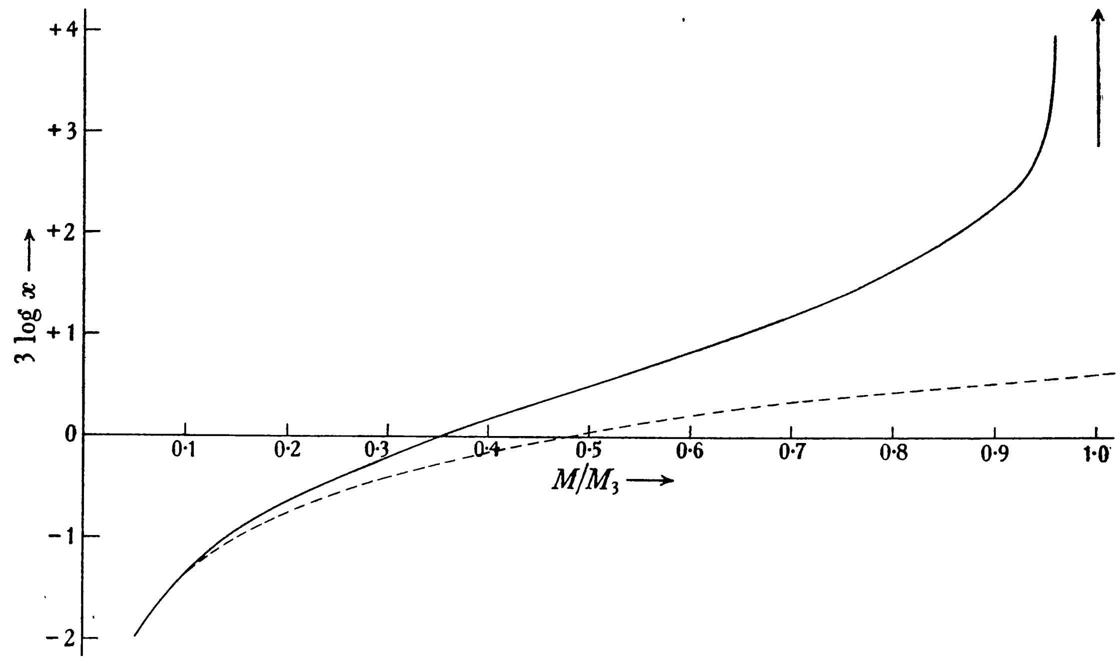

FIG. 3.—*The full line curve represents the exact (mass, log ρ_0)-relation for the highly collapsed configurations. This curve tends asymptotically to the dotted curve as $M \rightarrow 0$.*

covering the range specified by the polytropic distributions of indices 3/2 and 3.

15. *Concluding Remarks.*—In this paper we have strictly confined ourselves to the case "$\beta = 1$." But in stellar problem the radiation pressure (even if small) necessarily plays a deciding rôle, and the question as to in what sense we have to understand the completely degenerate spheres studied here as representing "the limiting sequence of configurations to which all stars must tend eventually" can be answered only by introducing radiation in these configurations. To do this properly we have first to develop adequate methods to treat composite configurations consisting of degenerate cores (of the structures studied here) surrounded by gaseous envelopes. These and related problems are studied in the following paper (p. 226).

16. *Manuscript Copy of Tables.*—The functions ϕ and their derivatives ϕ' (to six and five significant figures respectively) have been computed by the author for the values of $1/y_0^2$ specified in (73). In addition to ϕ and ϕ' the auxiliary functions ρ/ρ_0, $\rho_0/\bar{\rho}$, $-\eta^2\phi'$ and two other functions U and V (defined in equation (91) of the following paper) have also been tabulated. The auxiliary functions were calculated correct to five significant figures. All

the functions were tabulated for steps of 0·1 for the argument η. A manuscript copy of these tables has been deposited in the Library of the Society.*

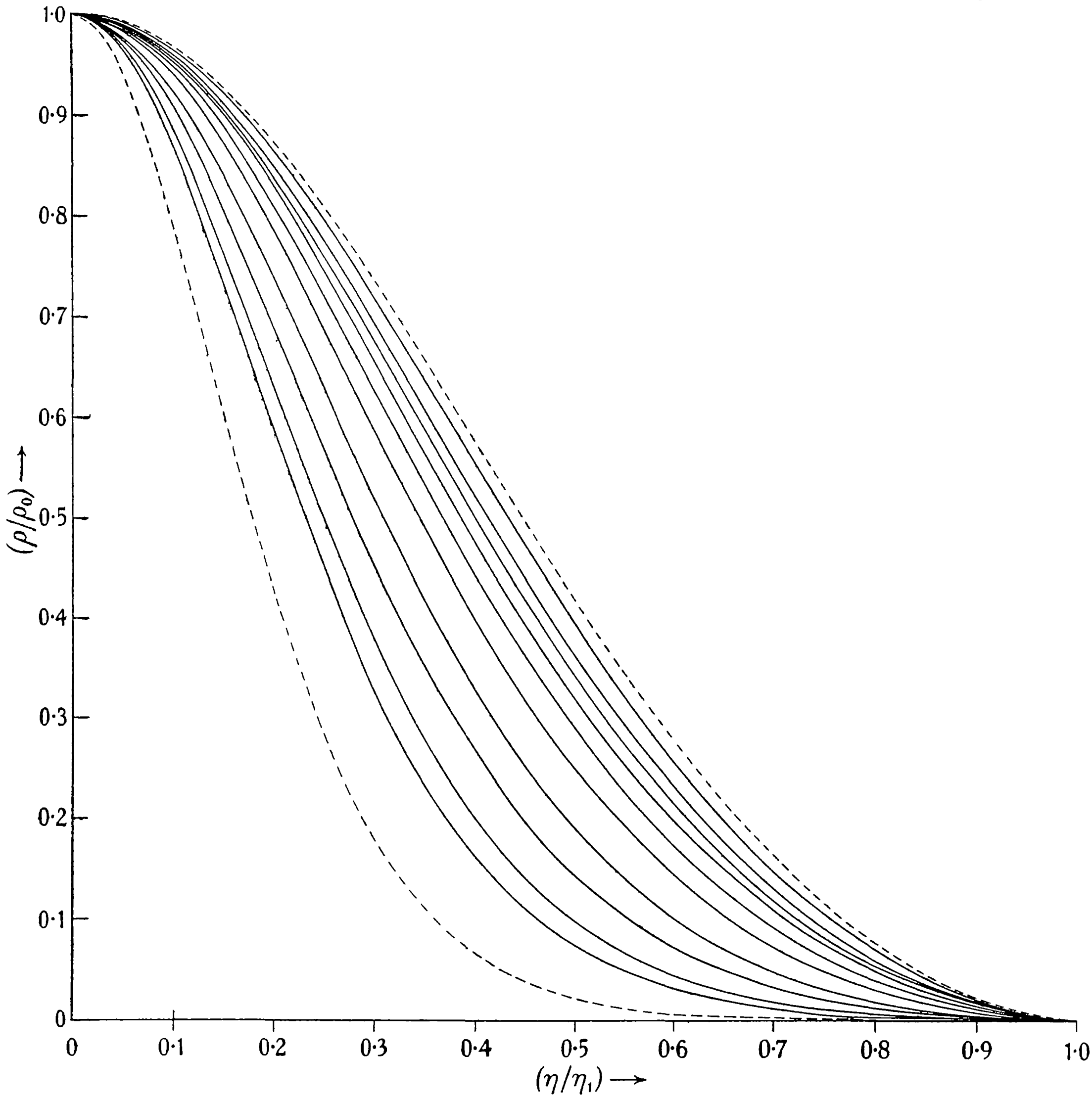

FIG. 4.—*The relative density distributions in the highly collapsed configurations. The upper dotted curve corresponds to the polytropic distribution $n=3/2$ and the lower dotted curve to the polytropic distribution $n=3$. The inner curves represent the density distributions for $1/y_0^2 = 0{\cdot}8, 0{\cdot}6, 0{\cdot}5, 0{\cdot}4, 0{\cdot}3, 0{\cdot}2, 0{\cdot}1, {\cdot}05, {\cdot}02, {\cdot}01$ respectively.*

APPENDIX

The Equation of State for a Degenerate Gas.—The equation has been derived by Stoner (among others),† but we shall give a simpler derivation of the same.

In a completely degenerate electron assembly all the electrons have momenta less than a certain "threshold" value p_0, and in the region of the

* Dr. Chandrasekhar's Tables can be consulted by Fellows on application to the Assistant Secretary (Editors).

† *M.N.*, **92**, 444, 1931.

available phase space of volume $\frac{4}{3}\pi p_0{}^3 V$ every cell of volume h^3 contains just two electrons. Clearly then we have

$$n = \frac{8\pi}{h^3}\int_0^{p_0} p^2 dp, \tag{1*}$$

$$\mathfrak{E} = \frac{8\pi V}{h^3}\int_0^{p_0} E p^2 dp, \tag{2}$$

$$P = \frac{8\pi}{3h^3}\int_0^{p_0} p^3 \frac{dE}{dp} dp, \tag{3}$$

where n is the number of electrons per unit volume in the assembly of volume V, $\mathfrak{E}$ the total energy and E the kinetic energy of a free electron. We have now denoted the pressure by P instead of by "p" as in the text of the paper to avoid confusion with the momentum, which has to be denoted by "p." From (1) and from (2) and (3) we have respectively

$$p_0{}^3 = \frac{3h^3 n}{8\pi}; \qquad P = \frac{8\pi}{3h^3} E(p_0) p_0{}^3 - \frac{\mathfrak{E}}{V}. \tag{4}$$

Equations (1) to (4) are quite general. Now in the relativistic mechanics we have

$$E = mc^2\left\{\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2} - 1\right\}, \tag{5}$$

or

$$p^2 = \frac{E(E + 2mc^2)}{c^2}. \tag{5'}$$

Using (5′) in (3) we have, after some minor transformations, that

$$P = \frac{8\pi m^4 c^5}{3h^3}\int_0^{\theta_0} \sinh^4 \theta d\theta, \tag{6}$$

where

$$\sinh\theta = p/mc; \qquad \sinh\theta_0 = p_0/mc. \tag{7†}$$

(7) yields at once that

$$P = \frac{8\pi m^4 c^5}{3h^3}\left[\frac{\sinh^3\theta\cosh\theta}{4} - \frac{3}{16}\sinh 2\theta + \frac{3}{8}\theta\right]_{\theta=\theta_0}. \tag{8}$$

Writing x for (p_0/mc) we have

$$P = \frac{\pi m^4 c^5}{3h^3}\left[x(2x^2 - 3)(x^2 + 1)^{1/2} + 3\sinh^{-1} x\right], \tag{9}$$

* This equation follows directly from the expression for the number of waves associated with electrons whose energies lie between E and $E + dE$ given by Dirac (*P.R.S.*, **112**, 660, 1926, his unnumbered equation on p. 671). Actually Dirac obtains this result using the Klein-Gordon relativistic wave equation. That the same result would follow from Dirac's relativistic wave equation (on neglecting the states of kinetic energy—which is permissible when no external perturbations are present) is clear from J. von Neumann, *Z. f. Physik*, **48**, 868, 1928.

† θ here introduced will not be confused with the Emden function.

$$\rho = n\mu H = \frac{8\pi m^3 c^3 \mu H}{3h^3} x^3, \tag{10}$$

which are the equations quoted in the text. Our derivation now shows "why" we are able to reduce the differential equation for degenerate gas spheres in gravitational equilibrium to such a simple form. The "reason" is that we have such an elementary integral for P as in (6).

The function $f(x)$ on the right-hand side of (9) has the following asymptotic forms :—

$$f(x) \sim \tfrac{8}{5}x^5 - \tfrac{4}{7}x^7 + \tfrac{1}{3}x^9 - \tfrac{5}{22}x^{11} + \ . \ . \ . \quad x \to 0, \tag{11}$$

$$f(x) \sim 2x^4 - 3x^2 + \ . \ . \ . \quad x \to \infty . \tag{12}$$

Finally we notice that

$$\frac{f(x)}{2x^4} < 1 \quad \text{for all finite } x. \tag{13}$$

The inequality in (13) is a *strict* one. If only the first terms in the expansions (11) and (12) are retained, we can easily eliminate x from (9) and (10) for these limiting cases and obtain, as we should expect, that

$$P = K_1 \rho^{5/3} \quad (x \to 0) ; \qquad P = K_2 \rho^{4/3} \quad (x \to \infty), \tag{14}$$

with

$$K_1 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}} ; \qquad K_2 = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8(\mu H)^{4/3}}. \tag{15*}$$

If we write our "equation of state" (9) and (10) parametrically as (changing to "p" to denote pressure),

$$p = A_2 f(x) ; \qquad \rho = Bx^3, \tag{16}$$

we find, on putting in the numerical values for the constants, that (in C.G.S. units)

$$A_2 = 6{\cdot}0406 \times 10^{22} ; \qquad B = 9{\cdot}8848 \times 10^5 \mu, \tag{17}$$

or

$$\left.\begin{aligned} \log \rho &= \ \ 5{\cdot}9950 + 3 \log x + \log \mu, \\ \log p &= 22{\cdot}7811 + \log f(x) \end{aligned}\right\}. \tag{18}$$

Stoner has previously made some calculations concerning the (p, ρ) relation for a degenerate gas, but for the study in the following paper more accurate tables for $f(x)$ were needed. Accordingly the whole computation was re-

* The law $P = K_2\rho^{4/3}$ was first used by the author in his paper on "Highly Collapsed Configurations," etc. (*M.N.*, **91**, 456, 1931). This law has also been derived by E. C. Stoner (*M.N.*, **92**, 444, 1932), T. E. Sterne (*M.N.*, **93**, 764, 1933), and is also implicitly contained in J. Frenkel (*Z. f. Physik*, **50**, 234, 1928). The law has also been used by L. Landau (*Physik. Zeits. d. Soviet Union*, **1**, 285, 1932). It may also be pointed out that the law $P = K_2\rho^{4/3}$ is implicit in certain equations in a paper by F. Juttner (*Z. f. Physik*, **47**, 542, 1928, equations in §§ 13, 17 ; our equation (6) above is a limiting form of Juttner's integral $Q(\alpha, \gamma ; +1)$). This last work of Juttner is related to his earlier work on the relativistic theory of an ideal classical gas, for a convenient summary of which see W. Pauli, *Relativitätstheorie* (Leipzig, Teubner), § 49.

done and the results are tabulated in Table V. I am indebted to Dr. Comrie and Mr. Sadler for the loan of a manuscript copy of a seven-figure table for $\sinh^{-1} x$, which was valuable in the computations of $f(x)$.

TABLE V

x	$f(x)$	$f(x)/2x^4$
0	0	0
0·2	0·000505	0·15785
0·4	0·015527	·30325
0·6	0·111126	·42873
0·8	0·435865	·53206
1·0	1·229907	·61495
1·2	2·82298	·68070
1·4	5·62991	·73276
1·6	10·14696	·77415
1·8	16·94969	·80731
2·0	26·69159	·83411
2·2	40·10347	·85598
2·4	57·99311	·87398
2·6	81·24509	·88894
2·8	110·8207	·90149
3·0	147·7578	·91209
3·5	279·8113	·93232
4·0	484·5644	·94641
4·5	784·5271	·95659
5·0	1205·2069	·96417
6·0	2525·739	·97444
7·0	4710·192	·98088
8·0	8070·587	·98518
9·0	$1·296694 \times 10^4$	·98818
10·0	$1·980725 \times 10^4$	·99036
20·0	$3·192093 \times 10^5$	·99753
30·0	$1·618212 \times 10^6$	·99890
40·0	$5·116812 \times 10^6$	·99938
50·0	$1·249501 \times 10^7$	·99960
60·0	$2·591280 \times 10^7$	·99972
70·0	$4·801018 \times 10^7$	·99980
80·0	$8·190727 \times 10^7$	·99984
90·0	$13·12039 \times 10^7$	·99988
100·0	$19·9980 \times 10^7$	·99990

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