The Structure
of
White Dwarf Stars

Course material for PHYS2200 class

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Abstract

This project applies numerical integration of ordinary differential equations to the prediction of the structure of white dwarf stars. The goal is to determine the dependence of the radius of a white dwarf stars versut its mass. Both the radius and the mass can be determined from the results of astronomical observations and thus the predictions can be verified.

1 Introduction

White dwarfs are thought to be the final evolutionary state of stars whose mass is not high enough to become neutron stars. After the hydrogenfusing lifetime of a star of low or medium mass ends, it will expand to a red giant which fuses helium to carbon and oxygen. If a red giant has insufficient mass to generate the core temperatures required to fuse carbon, which is around 10^9 K, an inert mass of carbon and oxygen will build up at its center. After shedding its outer layers to form a planetary nebula, the star will leave behind the core, which forms the white dwarf.

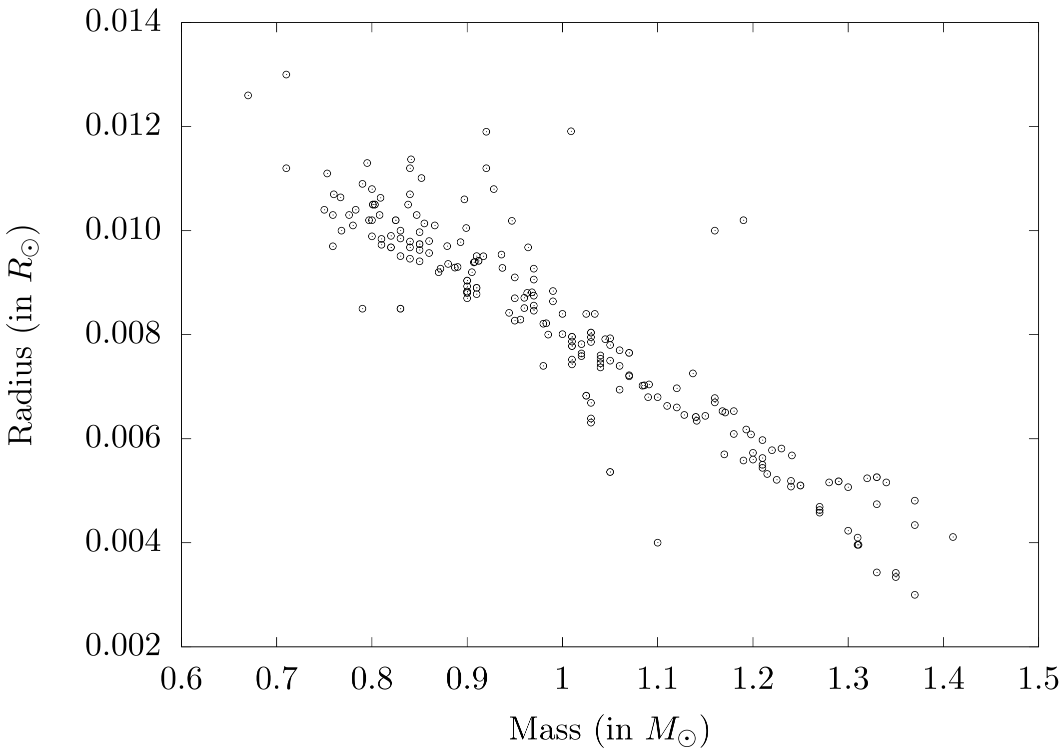

Figure 1: Observational support for the white dwarf mass-radius relation: more massive white dwarfs have smaller radii. The data downloaded from VizieR online database[1]. In the axis labels $M_\odot$ and $R_\odot$ are the mass and the radius of the Sun.

The material in a white dwarf no longer undergoes fusion reactions, so the star is not supported by the heat generated by fusion against gravitational collapse. It is supported only by *electron degeneracy* pressure, causing it to be extremely dense.

To descriibe the stricture of white dwarf star, we chose as a model a system in which the electrons are a gas, similar to the electrons in a metal, and are responsible for the internal pressure of the star. The heavy nuclei random motion and pressure is neglected but they are‘ responsible for the mass of the star and the force of gravity holding the star together. We neglect the relatively small mass of the electrons. We further assume that the star is spherically symmetric.

2 The equations of the mechanical equilibrium

If the star is in mechanical equilibrium, the gravitational force at each point inside is balanced by the force due to the spatial variation of the pressure P. The gravitational force acting on a unit volume of matter at a radius r is

$$F_{\text{grav}} = -G\,\frac{m(r)\,\rho(r)}{r^2}, \tag{1}$$

where G is the gravitational constant, $\rho(r)$ is the mass density of the star, and $m(r)$ is the mass of the star interior to the radius r:

$$m(r) = \int \rho\,\mathrm{d}V = 4\pi \int_0^r \rho(r')\,r'^2\,\mathrm{d}r', \tag{2}$$

where we used the expression for the volume element in spherical coordinates:

$$\mathrm{d}V = 4\pi r^2 \mathrm{d}r.$$

The radial component of the force per unit volume of matter due to the changing pressure is as following:

$$F_{\text{r}} = \frac{\mathrm{d}P}{\mathrm{d}r}. \tag{3}$$

When the star is in equilibrium, we thus have

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\,\frac{m(r)\,\rho(r)}{r^2}. \tag{4}$$

Instead of integral relation Eq. (2), a differential relation between the mass and the density can be obtained by differentiating the Eq. (2) with respect to r:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\,r^2\,\rho(r). \tag{5}$$

The description is completed by specifying *the equation of state*, a relation that gives the pressure, $P(\rho)$, which is required to maintain the matter at a given density, ρ. Using the identity

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{6}$$

Eq.(4) can be written as

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)^{-1} \frac{G\,m(r)}{r^2}\,\rho(r). \tag{7}$$

Equations (5) and (7) are two coupled first-order differential equations for $\rho(r)$ and $m(r)$ that determine the structure of the star for a given equation of state. The values of the dependent variables at $r = 0$ are $\rho = \rho_c$, the (unknown) central density, and $m = 0$. Integration outward in r then gives the density and mass profiles. The radius of the star, R, is being determined by the point at which $\rho = 0$. The total mass of the star is then $M = m(R)$. Since both R and M depend upon ρ_c, variation of this parameter allows to determine the mass-radius relation for white dwarf stars $R(M)$.

3 Background on Quantum Mechanics

3.1 Degenerate fermions

All elementary particles may be classified as either fermion or bosons. Electrons, neutrons, and protons are fermions. The *Pauli exclusion principle* implies that in a many-fermion system each fermion must be in a different quantum state. Thus the lowest energy state of the system results from filling energy levels from the bottom up. *Degenerate* matter corresponds to a many-fermion state in which all the lowest energy levels are filled and all the higher ones are unoccupied.

Degenerate matter plays an important role in a variety of astrophysical applications. For example, in white dwarf stars the electrons are highly degenerate, and in neutron stars the neutrons are highly degenerate.

3.2 Fermi momentum

We use the following quantum mechanivcal result: the number of quantum states for a free particle with the position somewhere in a volume V is $V\,d^3p/(2\pi\hbar)^3$, where d^3p is the volume element in the momentum space. For a spherically symmetric distribution of the electron momenta $\mathrm{d}^3p = 4\pi\,p^2\,\mathrm{d}p$.

Considering a small but macroscopic volume V containing a group of N electrons that occupy the lowest available energy states with magnitude of momentum $0 \le p \le p_f$.

Remembering the two-fold spin degeneracy of each electron state, we have

$$N = 2\,V \int_0^{p_f} \frac{\mathrm{d}^3 p}{(2\,\pi\,\hbar)^3} = 2\,V \int_0^{p_f} \frac{4\,\pi\,p^2\,\mathrm{d}p}{(2\,\pi\,\hbar)^3} = \frac{V}{\pi^2\,\hbar^3}\,\frac{p_f^3}{3}. \tag{8}$$

Eq. (8) determines the value of the local *Fermi momentum*, p_f, i.e. the maximum local magnitude of the electron momentum, in terms of the local electron density:

$$n(r) = \frac{N}{V}, \tag{9}$$

$$p_f(r) = \left(3\,\pi^2\right)^{\frac{1}{3}}\,\hbar\,n(r)^{\frac{1}{3}}. \tag{10}$$

The energy of an electron with the momentum p_f is called *Fermi energy.* The Fermi energy of a nonrelativistic electron is

$$\epsilon_f = \frac{p_f^2}{2m_e} = \left(3\,\pi^2\right)^{\frac{2}{3}} \left(\frac{\hbar^2}{2m_e}\right) n(r)^{\frac{2}{3}}. \tag{11}$$

The Fermi energy of an ultrarelativistic electron is

$$\epsilon_f = c p_f = \left(3\,\pi^2\right)^{\frac{1}{3}}\,(\hbar c)\;n(r)^{\frac{1}{3}}. \tag{12}$$

The condition for the electron gas to be strongly degenerate is that kT, where k is the Boltzmann constant and T is the temperature, should be small in comparison with the energy e_f. We will be interested in such electron densities that $p_f \geq m_e c$. Such momenta correspond to relativistic and ultrarelativistic electrons such that $\epsilon_f \geq m_e c^2 \approx 0.5 \times 10^6$eV. The temperatures, when the approximation of strongly degenerate electron gas is applicable, are

$$T \ll \frac{m_e c^2}{k} \sim 0.6 \times 10^{10} K. \tag{13}$$

3.3 Ideal electron gas

A degenerate electron gas has the peculiar property that it increasingly approaches the *ideal gas* state as its density increases.

Let us consider a gas consisting of electrons and a corresponding number of positively charged nuclei which balance the charge on the electrons. The assumption of ideal-gas properties means that the presence of the nuclei does not affect the thermodynamic quantities for the electron gas. The energy (per electron) of the Coulomb interaction between the electrons and the nuclei, E_c is of the order of

$$E_c \sim \frac{1}{4\pi\epsilon_0}\,\frac{Ze^2}{d}, \tag{14}$$

where Ze is the nuclear charge and d is the mean distance between the electrons and the nuclei,

$$d \sim \left(\frac{ZV}{N}\right)^{\frac{1}{3}}. \tag{15}$$

The condition for an ideal gas is that E_c should be small compared with the mean kinetic energy of the electrons, which is of order of the Fermi energy ϵ_f. For nonrelativistic electrons the inequality $E_c \ll \epsilon_f$, after the substitution of Eq. (15) into Eq. (14) and using Eq. (11) for ϵ_f, gives the condition

$$\frac{N}{V} \gg \left(\frac{e^2 m_e}{4\pi\epsilon_0 \hbar^2}\right)^3 \left(\frac{Z}{3\pi^2}\right)^2 = a_0^{-3}\left(\frac{Z}{3\pi^2}\right)^2 \sim a_0^{-3}, \tag{16}$$

where a_0 is the Bohr radius, $a_0 \approx 5.3 \times 10^{-11}$m.

For ultrarelativistic electrons the inequality $E_c \ll \epsilon_f$ leads to the relation

$$\frac{e^2}{4\pi\epsilon_0 \hbar c} \ll \left(\frac{3\pi^2}{Z^2}\right)^{\frac{1}{3}}. \tag{17}$$

The dimensionless combination of fundamental physical constants on the right is *fine structure constant* α:

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}.$$

Eq. (17) does not impose a restriction on the electron density but rather on the atomic number:

$$Z \ll \left(\frac{3\pi^2}{\alpha^3}\right)^{\frac{1}{2}} \approx 8730. \tag{18}$$

3.4 Pressure Ionization

Let's demonstrate that, as a consequence of quantum mechanics, ionization can be induced by sufficiently high pressure. As a consecuence, most stars are completely ionized over much of their volume.

Suppose the radius of each atom is a and the average spacing between atoms is d. For simplicity, we assume that the stellar material consists only of ions of a single species and the electrons produced by ionizing that species. Electrons in the atoms obey Heisenberg relations of the form $p \cdot a \geq \hbar$. Taking the average volume needed per electron to be V_0, we can write this as $p \geq \hbar / V_0^{1/3}$

The uncertainty principle produces ionization when the effective volume of the atoms becomes too small to confine the electrons. The average volume needed per electron V_0 is related to the average volume needed per nuclei V_i by $Z\,V_0 = V_i$, since there are Z electrons per ion. Thus $p \geq (\hbar Z^{1/3})/V_i^{1/3}$.

If the star is composed entirely of an element with atomic number Z, there are Z electrons in each sphere of diameter d (the average spacing between atoms) and the average number density of electrons n_e is related to d by $n_e \sim Z/d^3$.

This can be solved for d to give $d \sim (Z/n_e)^{1/3}$, which shows that d becomes smaller as n_e becomes larger. If $d < a$ we expect pressure ionization. With increasing density fewer locally bound states are possible until none remain and the electrons are all ionized.

4 Quantum mechanics and the equation of state

We must now determine the equation of state appropriate for a white dwarf. We will assume that the matter consists of a single kind large nuclei (e.g. oxygen) and their electrons. The nuclei, being heavy, contribute nearly all of the mass but make almost no contribution to the pressure since they hardly move at all. The electrons, however, contribute virtually all of the pressure but essentially none of the mass. We will be interested in densities far greater than that of ordinary matter, where the electrons are no longer bound to individual nuclei, but rather move freely through the material. A good model is then a free gas of electrons at zero temperature, treated with relativistic kinematics.

The number of nucleons per unit volume at radius r is approximately $\rho(r)/M_p$, where M_p is the proton mass (we neglect the small difference between the neutron and proton masses). If Y_e is the number of electrons per nucleon, then the concentration of electrons at radius r is

$$n(r) = Y_e \frac{\rho(r)}{M_p}. \tag{19}$$

The total relativistic energy of this group of electrons occupying the lowest possible momentum eigenstates is

$$E = 2\,V \int_0^{p_f} \epsilon(p)\, \frac{4\,\pi\, p^2\, \mathrm{d}p}{(2\,\pi\,\hbar)^3} = \frac{V}{\pi^2\,\hbar^3} \int_0^{p_f} \epsilon(p)\, p^2\, \mathrm{d}p \tag{20}$$

where

$$\epsilon(p) = \sqrt{p^2\,c^2 + m_e^2\,c^4} \tag{21}$$

is the relativistic energy of an electron with mass m_e and momentum p, and p_f is the Fermi momentum given by Eq. (10).

Changing the variable of integration in Eq. (20) to $y = \frac{p}{m_e\,c}$, we arrive to the integral

$$E = \frac{V}{\pi^2\,\hbar^3} \int_0^{p_f} p^2 \sqrt{p^2\,c^2 + m_e^2\,c^4}\, \mathrm{d}p = V\, \frac{m_e^4\,c^5}{\pi^2\,\hbar^3} \int_0^{x} y^2 \sqrt{1+y^2}\, \mathrm{d}y, \tag{22}$$

where the dimensionless parameter x is:

$$x = \frac{p_f}{m_e c}. \tag{23}$$

Using Eq. (10),

$$x = \left\{ \frac{3\pi^2 \hbar^3 n(r)}{m_e^3 c^3} \right\}^{\frac{1}{3}} = \left\{ \frac{n(r)}{n_0} \right\}^{\frac{1}{3}}, \tag{24}$$

where

$$n_0 = \frac{m_e^3 c^3}{3\pi^2 \hbar^3} \tag{25}$$

is the local electron density at which the local Fermi momentum is $p_f(r) = m_e c$. For the reference, $n_0 = 5.87 \times 10^{29}\,\mathrm{cm}^{-3}$.

Another form of Eq. (24):

$$x = \left\{ \frac{\rho(r)}{\rho_0} \right\}^{\frac{1}{3}}, \tag{26}$$

where $\rho(r)$ is the local mass density of matter; from Eq (19)

$$\rho(r) = M_p \frac{n(r)}{Y_e} \tag{27}$$

ρ_0 is the local mass density of matter when the local electron density is n_0:

$$\rho_0 = \frac{M_p n_0}{Y_e} = \frac{M_p m_e^3 c^3}{3\pi^2 \hbar^3 Y_e} = 9.82 \times 10^5\, Y_e^{-1}\,\mathrm{g\,cm}^{-3}. \tag{28}$$

The elementary integral in Eq. (22) is

$$\beta(x) = \int_0^x y^2 \sqrt{1+y^2}\,\mathrm{d}y = \frac{1}{8}\left[x\left(1+2x^2\right)\sqrt{1+x^2} - \ln\left(x + \sqrt{1+x^2}\right) \right]. \tag{29}$$

For the future reference, if $x = x(\alpha)$, where α is a parameter, then

$$\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = x^2 \sqrt{1+x^2}\,\frac{\mathrm{d}x}{\mathrm{d}\alpha}. \tag{30}$$

Eq. (30) is a particular case of the *Leibniz integral rule* that gives a formula for differentiation of a definite integral whose limits are functions of the differential variable.

We finally get the local energy of this group of electrons in the form

$$E = 3V\, n_0\, m_e c^2\, \beta(x). \tag{31}$$

From thermodynamics, the local pressure is related to how the energy of this group of N electrons changes with volume V at fixed N:

$$P(r) = -\frac{\partial E}{\partial V} \tag{32}$$

Taking into account that $x = x(V)$:

$$x = \left\{\frac{n(r)}{n_0}\right\}^{\frac{1}{3}} = \left\{\frac{N}{V\,n_0}\right\}^{\frac{1}{3}}, \tag{33}$$

thus

$$\frac{\mathrm{d}x}{\mathrm{d}V} = -\frac{x}{3\,V}, \tag{34}$$

and using Eq. (30), the local pressure, Eq. (32) has the form:

$$\begin{aligned} P(r) &= -3n_0\,m_e c^2\left\{\beta(x) + V\,x^2\sqrt{1+x^2}\,\frac{\mathrm{d}x}{\mathrm{d}V}\right\} \\ &= -3n_0\,m_e c^2\left\{\beta(x) - \frac{x^3}{3}\sqrt{1+x^2}\right\} \end{aligned} \tag{35}$$

The pressure has the units of energy per unit volume or force per unit area (since energy has the units of force times distance).

We then need the derivative $\frac{\mathrm{d}P}{\mathrm{d}\rho}$ to make use of Eq.(7). Noting that $x = x(\rho)$,

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = n_0\,m_e c^2\,\frac{x^4}{\sqrt{1+x^2}}\,\frac{\mathrm{d}x}{\mathrm{d}\rho}. \tag{36}$$

From Eq. (26),

$$\frac{\mathrm{d}x}{\mathrm{d}\rho} = \frac{1}{3}\,\rho_0^{-\frac{1}{3}}\rho^{-\frac{2}{3}} = \frac{1}{3\,\rho_0}\,\frac{1}{x^2}. \tag{37}$$

Thus,

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = Y_e\,\frac{m_e\,c^2}{M_p}\,\gamma(\rho) \tag{38}$$

where M_p is the mass of the proton, m_e is the mass of of the electron, Y_e is the number of electrons per nucleon, c is the speed of light, and dimensionless function $\gamma(x)$ is

$$\gamma(\rho) = \gamma(x) = \frac{x^2}{3\sqrt{1+x^2}}. \tag{39}$$

Here

$$x = \left(\frac{\rho}{\rho_0}\right)^{1/3} \tag{40}$$

and

$$\rho_0 = \frac{M_p\,m_e^3\,c^3}{3\,\pi^2\,\hbar^3\,Y_e}. \tag{41}$$

Using Eq.(4) in Eq.(7) we get an explicit differential equation governing the evolution of ρ (recall that dimensionless γ is a function of x which is a function of ρ which is a function of r):

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\left(\frac{M_p}{m_e c^2 Y_e}\right) \frac{G\, m(r)}{\gamma(\rho)\, r^2}\, \rho(r). \tag{42}$$

To avoid a fictious numerical divergency in Eq. (42) for small values of r, notice that for sufficiently small r

$$m(r) \approx \frac{4}{3}\pi r^3 \rho_c. \tag{43}$$

Hence, Eq. (42) can be written in the following form:

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{4}{3}\pi\left(\frac{M_p}{m_e c^2 Y_e}\right) \frac{G\, r}{\gamma(\rho_c)}\, \rho_c^2, \tag{44}$$

which avoids diverging factor $1/r^2$.

5 Scaling the Differential Equations

It is always useful to reduce equations describing a physical system to dimensionless form, both for physical insight and for numerical convenience. To do this for the equations of white dwarf structure, we introduce dimensionless radius, density, and mass variables:

$$r = R_0\, \bar{r}, \quad \rho = \rho_0\, \bar{\rho}, \quad m = M_0\, \bar{m} \tag{45}$$

with the radius and mass scales, R_0 and M_0 to be determined for convenience.

Substituting Eq. (45) into Eqs. (5), (42) yields

$$\frac{d\bar{m}}{d\bar{r}} = \left(\frac{4\pi R_0^3 \rho_0}{M_0}\right) \bar{r}^2\, \bar{\rho} \tag{46}$$

and

$$\frac{d\bar{\rho}}{d\bar{r}} = -\left(\frac{G M_p M_0}{m_e c^2 Y_e R_0}\right) \frac{\bar{m}\, \bar{\rho}}{\gamma(\bar{\rho})\, \bar{r}^2}. \tag{47}$$

If we now choose M_0 and R_0 so that the coefficients in parentheses in these two equations are ones, we find

$$R_0 = \left(\frac{m_e c^2 Y_e}{4\pi \rho_0 G M_p}\right) = 7.71 \times 10^8\, Y_e\, \mathrm{cm}, \tag{48}$$

and

$$M_0 = 4\pi R_0^3 \rho_0 = 5.66 \times 10^{33}\, Y_e^2\, \mathrm{gm}, \tag{49}$$

and the dimensionless differential equations are

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{r}} = \bar{r}^2\, \bar{\rho}, \tag{50}$$

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{r}} = -\frac{\bar{m}\, \bar{\rho}}{\gamma(\bar{\rho})\, \bar{r}^2}. \tag{51}$$

These equations are completed by recalling that γ is given by Eq. (39) with $x = \bar{\rho}^{1/3}$.

$$\gamma(\bar{\rho}) = \frac{\bar{\rho}^{2/3}}{3\sqrt{1+\bar{\rho}^{2/3}}} \tag{52}$$

This pair of equations is then integrated from $\bar{r} = 0$, $\bar{\rho} = \bar{\rho}_c$, $\bar{m} = 0$ to the value of $\bar{r}$ at which $\bar{\rho} = 0$, which defines the dimensionless radius of the star $\bar{R}$, and the dimensionless mass of the star is then $\bar{M} = \bar{m}(\bar{R})$. The scaled solution then depends on the dimensionless central mass density $\bar{\rho}_c$.

Eq. (43) in the dimensionless units is

$$\bar{m}(\bar{r}) = \frac{1}{3}\bar{r}^3\,\bar{\rho}_c. \tag{53}$$

Therefore, for $\bar{r} \ll 1$ Eq. (51) has the form,

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{r}} = -\frac{1}{3}\frac{\bar{r}\,\bar{\rho}_c^2}{\gamma(\bar{\rho}_c)}. \tag{54}$$

6 Scaled Dimensionless Differential Equations Summary

We drop the over-bars on the symbols for the scaled dimensionless variables: $\bar{r} \to r$, $\bar{m} \to m$, and $\bar{\rho} \to \rho$, for simplicity. The pair of ordinary differential equations to be integrated are then:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \rho\, r^2 \tag{55}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{m\,\rho}{\gamma(\rho)\, r^2} \tag{56}$$

in which

$$\gamma(\rho) = \frac{\rho^{2/3}}{3\sqrt{1+\rho^{2/3}}} \tag{57}$$

This pair of first order differential equations is then integrated from $r = 0$, $\rho = \rho_c$, $m = 0$ to the value of r at which $\rho = 0$, which defines the dimensionless radius of the star R, and the dimensionless mass of the star is then $M = m(R)$. The scaled solution then depends on the dimensionless central mass density ρ_c.

For small values of r Eq. (56) can be rewritten in the following numerically-preferable form:

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{1}{3}\frac{r\,\rho_c^2}{\gamma(\rho_c)}. \tag{58}$$

References

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