TOV

Introduction

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\left[P+\mathcal{E}(r)\right]\left[M(r)+4\pi r^3 P/c^2\right]}{c^2 r^2[1-2GM(r)/(c^2 r)]}$$
$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \frac{\mathcal{E}(r)}{c^2} \tag{1}$$

An interpretation of these equations can be more readily seen by multiplying the first equation by $4\pi r^2 \mathcal{E}\mathrm{d}r/c^2 = \mathrm{d}M$ and cancelling $\mathcal{E}$ on both sides:

$$4\pi r^2 \mathrm{d}P = -\frac{GM\mathrm{d}M}{r^2}\left(1+\frac{P}{\mathcal{E}(r)}\right)\left(1+\frac{4\pi r^3 P}{Mc^2}\right) \times \left(1-\frac{2GM}{c^2 r^2}\right)^{-1} \tag{2}$$

The term on the left hand side is the force exerted on a infinitesimal shell at radius r. The first factor on the right hand side is the newtonian gravitational force from the interior acting on this shell.

Equation of state

Equation of state for a white dwarf:

In this model for a white dwarf, it is assumed that its matter content is composed of inert heavy nuclei (e.g. oxygen or carbon nuclei) in a sea of electrons. Due to the high density, the electrons are not bound to the nuclei and can move freely. Furthermore, it is assumed that the electrons form a zero temperature Fermi gas. From Fermi statistics, the number density of electrons is then given by

$$n = \frac{p_F^3}{3\pi^2\hbar^3} \tag{3}$$

where p_F is the Fermi momentum. The energy density is also given by

$$\mathcal{E}_e = 2\int_0^{p_F} \sqrt{c^2p^2+m_e^2c^4}\,\frac{4\pi p^2}{(2\pi\hbar)^3}\mathrm{d}p \tag{4}$$

where $\sqrt{c^2p^2+m_e^2c^4}$ is the energy of an electron with momentum p and mass m_e. By calculating this integral, one finds that

$$\mathcal{E}_e = \frac{(m_ec^2)^4}{8\pi^2(\hbar c)^3}\left[x_F\sqrt{1+x_F^2}\left(1+2x_F^2\right) - \ln\left(x_F+\sqrt{1+x_F^2}\right)\right] \tag{5}$$

where $x_F = cp_F/m_ec^2$. The total energy density is then the sum of the rest masses of the nuclei and $\mathcal{E}_e$,

$$\mathcal{E} = Ynm_Nc^2 + \mathcal{E}_e, \tag{6}$$

where Y is the number of nuclei per electron ($Y = 2$ for carbon and oxygen for example) and m_N is the mass of the nuclei which is taken to be equal to the proton mass. The mass difference between the neutron and proton is neglected. The pressure can be found by utilizing the first law of thermodynamics and taking the derivative of the energy with respect to volume. By transforming the derivative of the energy with respect to volume to a derivative of the energy density with respect to the number density, we find that

$$P = n\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}n} - \mathcal{E} = \frac{(m_ec^2)^4}{8\pi^2(\hbar c)^3}\left[\frac{2}{3}x_F^3\sqrt{1+x_F^2} - x_F\sqrt{1+x_F^2} + \ln\left(x_F+\sqrt{1+x_F^2}\right)\right]. \tag{7}$$

Now we know both $\mathcal{E}$ and P as functions of x_F (which in turn is a function of the number density n) but we don't have an explicit expression for the function $\mathcal{E}(P)$. Given a pressure $P = P_0$ we can however use a root finding algorithm to find the x_F such that $P(x_F) - P_0 = 0$. Having found this value of x_F, we can plug it into the equation for the energy density to find $\mathcal{E}(P_0)$.

Equation of state for a neutron star

Numerical set-up

The TOV-equations can be made more suitible for numerical calculations by rescaling the variables. Making the substitutions $r = R_0x$, $P = P_0p$, $\mathcal{E} = \mathcal{E}_0\varepsilon$ and $M = M_0m$, the equation for the pressure can be written

$$\left(\frac{P_0}{R_0}\right)\frac{\mathrm{d}p}{\mathrm{d}x} = -\left(\frac{GP_0M_0}{c^2R_0^2}\right) \times \frac{\left[p+(\mathcal{E}_0/P_0)\varepsilon\right]\left[m+4\pi R_0^3x^3P_0p/(c^2M_0)\right]}{x^2\left[1-2GM_0m/(c^2R_0x)\right]}. \tag{8}$$

Cancelling P_0/R_0 on both sides yields

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\left(\frac{G_0M_0}{c^2R_0}\right) \times \frac{\left[p+(\mathcal{E}_0/P_0)\varepsilon\right]\left[m+4\pi R_0^3x^3P_0p/(M_0c^2)\right]}{x^2\left[1-2GM_0m/(c^2R_0x)\right]} \tag{9}$$

Similarily, the equation for the mass becomes

$$\frac{\mathrm{d}m}{\mathrm{d}x} = \left(\frac{4\pi R_0^3 \mathcal{E}_0}{M_0 c^2}\right) x^2 \varepsilon \tag{10}$$

The equations for p and m simplify substantially if the numerical constants are chosen such that

$$1 = \frac{\mathcal{E}_0}{P_0} = \frac{GM_0}{c^2 R_0} = \frac{4\pi R_0^3 P_0}{M_0 c^2}. \tag{11}$$

Then the TOV-equations become

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} &= -\frac{(\varepsilon + p)(m + x^3 p)}{x(x-2m)} \\ \frac{\mathrm{d}m}{\mathrm{d}x} &= x^2 \varepsilon. \end{aligned} \tag{12}$$

A natural choice for $\mathcal{E}_0$ and P_0 is the term in front of the brackets in (**??**):

$$\mathcal{E}_0 = P_0 = \frac{(mc^2)^4}{8\pi^2 (\hbar c)^3} \approx 1.285 \text{ GeV/fm}^3. \tag{13}$$

Equation (11) then fixes M_0 and R_0:

$$\begin{aligned} M_0 &= \sqrt{\frac{c^8}{4\pi \mathcal{E}_0 G^3}} \approx 4.63 \text{ M}_\odot \\ R_0 &= \frac{GM_0}{c^2} \approx 6.84 \text{ km} \end{aligned} \tag{14}$$

where $\text{M}_\odot$ is the mass of the sun.

To solve these equations, initial conditions for both p and m are needed. From a physical consideration it is obvious to set $m(0) = 0$. The initial value for the pressure can be either $p(0) = p_0$ or $p(x_0) = 0$ where p_0 is the pressure at the center of the star and x_0 is its radius. When ε depends on the pressure p, it becomes easier to make use of the former condition since m is integrated from the center. The equation for p is however singular for $x = 0$, so the equations have to be integrated from some value Δx close to zero. Since the first two derivatives of m are zero at $x = 0$, the error of setting $m(\Delta x) = 0$ is of the order $(\Delta x)^3$.

Another thing which is needed is the equation of state $\epsilon(p)$ which, for the case of an ideal Fermi gas at zero temperature, can not be found directly but in terms of the number density n:

$$\begin{aligned} \varepsilon(n) &= x_F \sqrt{1 + x_F^2}\left(1 + 2x_F^2\right) - \ln\left(x_F + \sqrt{1 + x_F^2}\right) \\ p(n) &= \frac{2}{3} x_F^3 \sqrt{1 + x_F^2} - x_F \sqrt{1 + x_F^2} \\ &\qquad + \ln\left(x_F + \sqrt{1 + x_F^2}\right) \end{aligned} \tag{15}$$

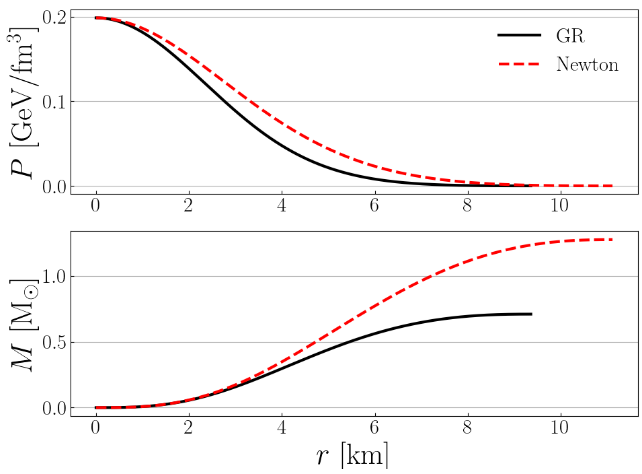
with $x_F(n) = \hbar c(3\pi n)^{1/3}/(mc^2)$. Hence, for a given $p = p_0$ we need a root finding algorithm to find $n = n_0$ such that $p(n_0) = p_0$. Once n_0 has been found, we can plug that into ε to find $\varepsilon(n_0) = \varepsilon(p_0)$.

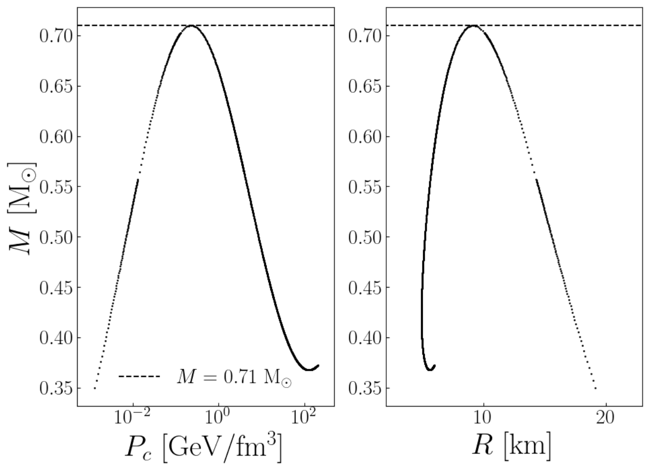
Having set up the equations which are to be integrated (12) and the equation of state (15), the following steps were taken to solve the TOV-equations:

1. Set the initial conditions by specifying $p(\Delta x) = p_c = p_{j=0}$ and set $m(\Delta x) = 0 = m_{j=0}$.
2. Find the energy density by using the bisection method on $p(n) = p_j$ and plugging n into the equation for the energy density to get $\epsilon(n) = \varepsilon_j$.
3. Use Heun's integration method on (12) to obtain p_{j+1} and m_{j+1}.
4. Repeat from 2. until the pressure is zero, $p_{j+1} = 0$.

The step size used was $h = 10^{-4}$ and the tolerance for the bisection method was $\Delta = 10^{-8}$.

Results





Notes

- Compare with Newton
- Also do white dwarfs