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DENSE BARYON MATTER CALCULATIONS WITH REALISTIC POTENTIALS †

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Abstract: The equation of state for dense hyperonic matter is calculated from five potential interaction models. The form of the potential is the same as that of Reid, a sum of Yukawa functions, the coefficients of which are adjusted separately in each partial wave to fit experimental nucleon-nucleon data. The strong short-range central repulsion dominates the equation of state at high density, so special care is taken to make the strengths of the repulsive term consistent with meson theory. A simplified version of the constrained variational technique of Pandharipande is used to calculate the energy versus density. The calculation is performed for neutron matter and a mixture of nucleons and hyperons, $M \leqq 1236$ MeV. The various interaction models produce a spread of about 30% in the equations of state, which are generally stiffer than the equation of state calculated from the Reid potential by Pandharipande. This difference arises because our models have a more realistic repulsion than Reid's potential.

1. Introduction

The neutron star concept dates to the 1930's; it was first discussed in the context of general relativity by Oppenheimer [1]) in 1939. He found that stable, condensed stars can exist, composed of a free neutron gas held together under its gravitational attraction, provided the total mass was not greater than $\frac{2}{3} \odot$ ($\odot \equiv$ mass of the sun). The idea was purely academic until the discovery of the first pulsar in 1967. (Pulsars are objects which emit electromagnetic pulses at regular intervals, of between 0.03 and 4 sec.) Gold [2]) pointed out that the most probable candidate to explain pulsars was a rotating neutron star.

Today most of our understanding of the structure of neutron stars comes from the study of theoretical models. The usual procedure is to calculate separately (i) the equation of state of neutron star matter, which relies heavily on our understanding of low energy nuclear and particle physics, and (ii) the stellar structure, which is a solution of the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium [3]) in general relativity, and requires (i) as input.

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Neutron stars are presumably the residues of supernova implosions. Such remnants cool rapidly by neutrino emission to relatively low temperatures. We shall be interested here in the equilibrium star, and we may consider the matter as being at absolute zero temperature.

The physics of the equation of state for electrically neutral, cold matter divides quite naturally into four density ranges. (a) At the very lowest densities the state of minimum energy [4]) is ^{56}Fe, arranged in a lattice to minimize Coulomb energy. This state persists to densities of the order of 10^7 g/cm^3. (b) The next density region is characterized by neutron-rich nuclei in a lattice [5]); the neutron excess arises because the protons undergo inverse beta decay, in order to lower the electron Fermi energy. (c) At a density of 4.3×10^{11} Baym, Bethe and Pethick [6]) (BBP) (and also earlier authors) found that the nuclei have become so neutron-rich that any further neutrons formed "drip" out of the nuclei. This density region is characterized by a superfluid neutron gas [7]) coexisting with large, neutron-rich nuclei in a lattice. The relation between pressure and density in this neutron-drip region is quite well known [6]) but the magnitude of the charge Z was the subject of some controversy [6,8]). This has been resolved by a better theoretical treatment of the surface energy of nuclei imbedded in a neutron gas [9,10]), with the result that Z remains about 40 throughout this region. The best theoretical approach is the solution of a Hartree-Fock equation [10]) which gives $Z \approx 40$ with some fluctuations. (d) At a density of about 2.4×10^{14}, nuclei disappear, essentially by merging together. We then have a fluid of uniform neutron matter, with a small percentage of protons and electrons "dissolved" in it. Previous work on the equation of state in the last density region is discussed in sect. 2. The aim of the present paper is to take a new look at matter in this density range, which extends from that of normal nuclear matter, $\rho_{\mathrm{NM}} \approx 0.2\ \mathrm{fm}^{-3}$, to the highest densities of relevance to neutron stars, $\approx 3\ \mathrm{fm}^{-3}$. Stars with higher central densities are subject to collapse under the enormous gravitational force.

Qualitatively, the composition of matter is characterized as follows. At densities near ρ_{NM} one finds mostly neutrons. As the density increases, the concentration of protons and electrons increases slowly. At some higher density, somewhat below $1\ \mathrm{fm}^{-3}$, the hyperons $\Lambda, \Sigma, \ldots$ appear, and perhaps the Δ_{33}; at the same time the proton concentration increases sharply. For obvious reasons, we refer to matter at these densities as hyperonic matter (HM). As will become evident, the appearance of hyperons has the desirable effect of maintaining relatively low velocity for all the constituent baryons, so that calculations may be comfortably based on non-relativistic quantum mechanics.

It has been suggested [11]) that high density matter may also contain π^- mesons. We believe this may be correct, but will have little influence on the equation of state (subsect. 7.2).

2. History

Cameron [12,13] was the first researcher to carefully calculate very dense neutron star matter models on the basis of an interparticle interaction. He found that the short-range repulsion played an important role: it had the effect of increasing the maximum mass of neutron stars to about $2\odot$ as compared to $\frac{2}{3}\odot$ obtained by Oppenheimer [1]. Another new feature of his calculation was allowing for the possibility of hyperons $\Sigma, \Lambda, \Delta, \ldots$, which he found to be plentiful even at moderately low densities. For interactions he used those of Levinger-Simmons [13,14] and Skyrme [12,15]. The former is, by now, rather old-fashioned; in particular it suffers because it does not have a Yukawa core. The latter, an empirical effective interaction, was invented for low density systems, and it is therefore not well-justified at high density.

A more recent calculation by Leung and Wang [16] employs a statistical bootstrap baryon level density as a substitute for explicitly including interactions at very high density. Their model predicts a low maximum mass and moment of inertia, which is due to their level density model accounting for some attractive interactions but not for the repulsive ones, which we know from nucleon-nucleon scattering and nuclear saturation to exist at distances of about 0.5 fm.

A similar concept has been used by Wheeler *et al.* [17] who have used a very elegant method to estimate the number of baryon states but have not drawn as explicit conclusions as Leung and Wang [16].

Another school of thought argues that neutron matter should crystalize at very high density. Coldwell [18] uses variational theory with the Reid [19] soft-core potential and finds a lower energy if he introduces a lattice periodicity in the one-body orbits. His work is inconclusive, however, because the trial wave function does not account for the decrease of the wave function when the relative separation of two particles becomes small; his trial wave function is an antisymmetrized product of one-body orbitals.

Anderson [20] and Clark [21] argue that nuclear matter should crystalize at high density, by comparing the Lennard-Jones 6–12 potential to phenomenological nucleon-nucleon potentials. However, Brandow [22] has emphasized that it is the repulsion which is responsible for crystallization and that the arguments of A and C are invalid because they attribute too much importance to the attractive portion of the interaction. As far as the repulsion goes, the soft-core Reid [19] potential has a much gentler slope in the region where repulsion dominates attraction: it goes as r^{-n} with $n = 1+\mu r$. Using $\mu = 5\ \mathrm{fm}^{-1}$ (Reid's $7\mu_\pi$) and $r < 0.6$ fm in the repulsive region, we have $n < 4$ for nuclear interactions whereas $n = 12$ for the LJ potential.

The general arguments of Anderson [20] and Clark [21] do not appear conclusive, and it is necessary to resort to explicit calculations, using the actual nuclear interaction. In this respect, the evidence is contradictory. Canuto and Chitre [23] find crystallization at high density, using the *G*-matrix formulation customary in nuclear matter calculations at normal density (0.16 nucleons/fm^3). Canuto, Lodenquai and

Chitre [24]) tested this method for solid ^{3}He with rather good results. But we are still doubtful whether their method is applicable at high density, especially for the liquid, because many-particle correlations become important. (It is necessary to compare liquid and solid, and therefore to have a method which is good for both.) We have used the method of Guyer and Zane [25]), and obtained a consistently higher energy for solid than for liquid neutron matter. The same result was obtained by Pandharipande [26]) using his variational method.

To compare various methods, a simplified problem was calculated, in which essentially only the repulsive part of the Reid potential was used, and Boltzmann statistics assumed for the neutrons [27]). This time, Pandharipande did find solidification, at a density of about $\rho = 0.8/\text{fm}^3$. However, Cochran and Chester, in a Monte Carlo calculation, did not find any solid up to $\rho = 1.5/\text{fm}^3$. Monte Carlo should be the most reliable; therefore, if anything, Pandharipande's method seems to give solidification too easily. When he adds an attraction, similar to the Reid potential, Pandharipande's solidification point shifts up to $\rho = 2/\text{fm}^3$; when he then uses Fermi statistics, the material remains liquid throughout (up to $3.6/\text{fm}^3$).

It seems to us, therefore, that with realistic, *central* forces between the particles, neutron matter remains liquid up to very high density. In any case, the existence of the controversy shows that the solid energy cannot be *much* lower than the liquid; hence it will be satisfactory to calculate the energy of the liquid for the purpose of obtaining the equation of state.

Turning now to the liquid model of HM, the most thorough published work is a series of papers by Pandharipande, whose method will be described in sect. 5. In the earliest paper [28]) he proposed three models for the nuclear forces. Model A assumed that all particles interact through the same soft-core potential, that of Reid [19]) $T = 1$. He finds for this model a phase transition at the density at which neutron matter rapidly becomes admixed with hyperons, $\rho \approx 1\ \text{fm}^{-3}$; the phase transition is indicated by the characteristic loop of E versus ρ. This force model is, however, unrealistic because, as we know from numerous experimental results, the S-wave central interaction in the $T = 0$ state is much less attractive than in $T = 1$. Model C takes account of this; it is the same as model A except for a 10 % reduction in the attraction for all pairs but like nucleons. This model, which has a more physical basis, results in no phase transition. Model B is the same as model C except for his elimination of the Δ_{33} as a possible constituent. The Δ was found to play only a small role in E versus ρ. By comparing models he concludes that E versus ρ is sensitive to the attraction but not the concentrations.

In a later paper [29]) Pandharipande repeats the calculation but now includes the $T = 0$ Reid [19]) nucleon-nucleon interaction and a theoretical $n\Sigma$ interaction whose derivation is based on the SU(3) model. He finds the somewhat surprising result that now HM is predominantly composed of neutrons, except for the brief appearance of a few protons, electrons, muons and Σ^- near $\rho \approx 0.5\ \text{fm}^{-3}$.

Pandharipande's results [28,29]) have the limitation that they are based rather closely on the Reid [19]) interaction. If we take seriously the meson theory of effective interactions, then there are some evident inconsistences in this potential. To a first approximation one would expect to find the same core in all states because the ω-meson, which couples to all states equally because it is isoscalar, gives the largest repulsive contribution at small distances. One would further expect the range of the repulsion to be given by the mass of the ω-meson and its strength by the ω-nucleon coupling. The core of the Reid [19]) potential is unsatisfactory on all these points; specifically, there is no core in $^3P_1$ and $^1P_1$, the ^3S core is much stronger than the ^1S core and the range is generally too short.

At high density, the core is strongly felt and we therefore believe that these defects in the Reid [19]) interaction should make a difference to the equation of state. We seek to remedy these defects and investigate the effects on the equation of state. We will also make some modifications in the calculational procedure which will permit a detailed analysis of the physics manifested in the equation of state.

3. The repulsive core at short distances

3.1. EXISTENCE OF A CORE

The existence of a repulsive core in the nucleon-nucleon potential is well established experimentally. The most direct evidence is the fact that the ^1S and ^3S phase shifts in nucleon-nucleon scattering become negative [30]) at lab energies above about 300 MeV. If we want to represent the interaction by a static potential, which is certainly the simplest procedure, this potential must be repulsive at short distances.

A less direct, but still stronger argument is the saturation of nuclear forces. This is a very old argument [31]), and leads to the following *necessary* condition

$$\int V(r)r^2\mathrm{d}r > 0, \tag{3.1}$$

where $V(r)$ is the potential, averaged over spin S and angular momentum, i.e., the "Wigner force" in old parlance. Condition (3.1) is by no means sufficient. Both the necessary and the sufficient conditions for saturation have recently been investigated more carefully by Calogero and Simonov [32]). They find, e.g., that the Wigner force must be sufficiently repulsive at small distance r to overcome all possible forces depending on L and S which in some states are attractive, such as a tensor or spin-orbit force.

3.2. VECTOR MESON EXCHANGE

The most reasonable explanation of the repulsive core is the exchange of vector mesons, especially neutral ones. Indeed, a neutral vector meson with strong coupling to the nucleon is known, the ω. The ρ-meson, which exists with charges $+1$, 0 and -1, has a much weaker coupling, at least of the "vector" type,† $\boldsymbol{\phi} \cdot \boldsymbol{j}$: Fom SU(3)

† The ρ also has a tensor type coupling, $\boldsymbol{\sigma} \cdot \nabla \times \boldsymbol{\phi}$, to the nucleon [33]) and this will be discussed in subsect. 3.9.

theory, the ρ-coupling is $\frac{1}{9}$ of the ω-coupling. Fitting of nuclear forces [33]) gives even smaller values, down to about $\frac{1}{20}$ of the ω-coupling. The ϕ-meson apparently has a smaller coupling to the nucleon than the ω, but is coupled strongly to hyperons. We shall neglect both ρ and ϕ in most of our considerations.

One of the great advantages of vector mesons is that they permit a *classical theory,* just like electrodynamics. In first approximation, the interaction between many nucleons m, n is given by the static potential

$$V = \tfrac{1}{2} \sum_{m \neq n} g_m g_n \exp(-\mu_\omega r)/r, \tag{3.2}$$

where μ is determined by the mass of the exchanged meson,

$$\mu_\omega = m_\omega c/\hbar, \tag{3.3}$$

and g_m, g_n are the "charges" of the interacting nucleons. In fact, all nucleons have the same "mesic charge", $g_m = g$. Also, since the ω-meson has isospin $I = 0$, the mesic charge of other baryons, Λ, Σ, Ξ, and that of the 3–3 "resonance particle" Δ should also be g. (This is true in SU(3) theory if the ω is predominantly a unitary singlet vector meson.)

Eq. (3.2) is, of course, the Yukawa potential. It is purely repulsive if caused by vector mesons. To get an attractive potential we may exchange scalar mesons; this gives (3.2) with opposite sign. While scalar mesons (σ) of the mass needed to explain the range of attraction in the nucleon-nucleon potential do not seem to exist, the exchange of a pair of π-mesons has the same effect; this has been treated by Brown, Durso, and others [34]). Exchange of single pions also gives a Yukawa type potential, so that we get altogether for the interaction between a pair of nucleons

$$V = (1/r) \sum C_n \exp(-n\mu_\pi r), \tag{3.4}$$

where μ refers to the pion,

$$\mu_\pi = m_\pi c/\hbar, \tag{3.5}$$

n is the ratio m'/m_π for the exchange of mesons of mass m' and C_n is positive or negative according to the type of meson exchanged.

Reid [19]) has used the form (3.4), with the special choice of integral values of n, to fit experimental nucleon-nucleon phase shifts. We shall use the same form but permit fractional n in some cases.

The interaction by exchange of neutral vector mesons, (3.2), was used for very dense nuclear matter already by Zel'dovich [35]). His interest, however, was the occurrence of sound velocities greater than the velocity of light at high density (see sect. 7 of our paper), not the equation of state as such. Moreover, he did not give any detailed justification for (3.2).

3.3. UNITARY TRANSFORMATION

The best way to derive (3.2) is to start from classical, relativistic field theory and to make a unitary transformation to eliminate the fourth component of the field.

This has been done in a very instructive way by Wentzel [36]). He uses a charged vector meson field, assumes the nucleons to be nearly at rest, and expands the unitary transformation operator in powers of the coupling constant g. We show in appendix A that the transformation is *exact to all orders of the coupling constant,* and we simplify the theory to neutral mesons. The transformed Hamiltonian is then†

$$H = \tfrac{1}{2}\int \mathrm{d}\tau[\mu^2\phi^2+(\nabla\times\phi)^2+c^2\pi^2+c^2\mu^{-2}(\nabla\cdot\pi)^2]+H_{\mathrm{nuc}} \\ +(g'^2/8\pi)\sum_{m\neq n}\mathrm{e}^{-\mu r_{mn}}/r_{mn}-g'\sum_n \boldsymbol{v}_n\cdot\phi(\boldsymbol{r}_n), \quad (3.6)$$

where H_{nuc} is the Hamiltonian for free nucleons, i.e., non-relativistically

$$H_{\mathrm{nuc}} = \sum_n P_n^2/2M, \quad (3.7)$$

$\phi(\boldsymbol{r})$ is the vector wave function of the meson field, π the conjugate momentum, μ is given by (3.3), $\boldsymbol{v}_n$ is the velocity of nucleon n and g' is the "rationalized" coupling constant, $g' = g(4\pi)^{\frac{1}{2}}$. The meson wave function ϕ has *three* components, and may be regarded as a classical vector field.

The next-to-last term in (3.6) is the most important. It does not contain the meson field ϕ but only the nucleon coordinate. It corresponds in all respect to the Coulomb potential in electrodynamics which also can be separated from the interaction with transverse electromagnetic waves. There, as well as here, the separation is not relativistically invariant, but such invariance is irrelevant in our problem because the mass center of the nucleons constitutes a definite framework. The result in (3.6) is the same as the Yukawa formula (3.2) except that in (3.6) the "rationalized" coupling constant is used.

All other terms in (3.6) are small by comparison with the Yukawa term; they will now be discussed.

3.4. VELOCITY-DEPENDENT TERMS

The terms which correspond to the magnetic interaction can be derived in the same way as in electrodynamics, only it is still simpler because the Lorentz condition is automatically satisfied, below W(12.6). It is easiest to obtain the equation for the "vector potential" ϕ from the field Lagrangian (W(12.6)) which gives

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi-\mu^2\phi = -\frac{g'}{c}\sum_n \boldsymbol{v}_n\delta(\boldsymbol{r}-\boldsymbol{r}_n). \quad (3.8)$$

† In electrodynamics there is an A^2 term in the interaction. The analogous term does not occur here because we derived (3.6) on the basis of a non-relativistic nuclear Lagrangian. The essential results of this section would be unchanged by such a term.

If the nucleon velocities are not too large, and independent of time, we may neglect the time dependence and get

$$\phi = \frac{g'}{4\pi} \sum_n \boldsymbol{v}_n \mathrm{e}^{-\mu|\boldsymbol{r}-\boldsymbol{r}_n|}/|\boldsymbol{r}-\boldsymbol{r}_n|. \tag{3.9}$$

The corresponding interaction between nucleons is

$$V_{\mathrm{mag}} = -\frac{g'^2}{8\pi c^2} \sum_{m \neq n} \boldsymbol{v}_m \cdot \boldsymbol{v}_n \mathrm{e}^{-\mu|\boldsymbol{r}_m-\boldsymbol{r}_n|}/|\boldsymbol{r}_m-\boldsymbol{r}_n|, \tag{3.10}$$

which is of order v^2/c^2 relative to the main Yukawa potential (3.6).

In the Fermi sea of nucleons, the velocity $\boldsymbol{v}_m$ occurs as often as $-\boldsymbol{v}_m$ so that V_{mag} averages to zero. This corresponds to pure ω-exchange which gives the potential (3.6) independent of the relative L of the interacting nucleons. In reality, nucleons also have exchange forces which give, without magnetic corrections, an interaction energy (see sect. 5) which may be written, after integrating over the Fermi sphere

$$V_{\mathrm{eff}} = A + Bk_{\mathrm{F}}^2 + Dk_{\mathrm{F}}^4, \tag{3.11}$$

where generally $B > 0$ and $D < 0$. The term Bk_{F}^2 is largest when the *relative* momentum $\boldsymbol{k} = \frac{1}{2}(\boldsymbol{k}_m - \boldsymbol{k}_n)$ is large, i.e., when the two nucleons go in opposite directions. If we assume the magnetic correction to be the same (3.10) for exchange forces (and this is a pure assumption), then for large k, the magnetic interaction has the *same* sign as the Yukawa interaction. We average $\boldsymbol{v}_m \cdot \boldsymbol{v}_n k^2$ over the Fermi sphere, and obtain the corrected interaction

$$V_{\mathrm{eff,\,corr}} = A + Bk_{\mathrm{F}}^2 \left(1 + \frac{k_{\mathrm{F}}^2}{5M^2c^2}\right) + Dk_{\mathrm{F}}^4 \left(1 + \frac{k_{\mathrm{F}}^2}{3M^2c^2}\right). \tag{3.12}$$

In typical situations, $k_{\mathrm{F}} \approx \frac{1}{2}Mc$, so that the correction to the k^2 terms is about 5 %, that to the k^4 term 8 %. Because of the signs of B and D, the two corrections tend to compensate; in any case, they are very small.

3.5. ADVANTAGES OF THE CLASSICAL THEORY

We believe that for the present problem the classical theory is much to be preferred to the customary quantum treatments. These treatments, at some point, always involve an expansion in powers of the coupling constant. Dispersion theory avoids this to some extent, but the trouble comes back in terms of processes involving many mesons, giving rise to unpleasant cuts, etc. The ω-N coupling constant needed to explain the observed NN phase shift seems to be $g^2/\hbar c \geqq 20$ (sect. 4). [This is the unrationalized g, as used in (3.2).] Therefore, the exchange of 20 or more mesons would have to be considered in the quantum theory, clearly an impossible task. The classical theory is valid regardless of the size of $g^2/\hbar c$; in fact, the bigger this quantity is, the better the classical theory becomes.

We are helped in the classical theory by the fact that the velocity of the baryons always remains small compared with c (see above, subsect. 3.4). Our problem is not a high-energy one; in fact, our baryons take the *lowest* states permitted by the Pauli principle and the mutual forces. It is therefore not necessarily appropriate to use the methods designed to treat high-energy phenomena. Our classical method would, of course, be useless for the high energy case, because of the lack of relativistic invariance.

Because our nucleons have fairly low velocity, we have determined the coefficients in the interaction (3.4) the same way as Reid [19]), i.e., by requiring that they fit experimental nucleon-nucleon scattering at *low* energy, <400 MeV. In doing this we find that the main repulsive term corresponds to†

$$g_\omega^2/\hbar c \geqq 22. \tag{3.13}$$

High-energy experiments indicate a much smaller value,

$$g_\omega^2/\hbar c = 8, \tag{3.14}$$

although some direct high-energy arguments have given higher values than 8. We do not know the reason for the discrepancy between 8 and 22.

3.6. RADIATIVE CORRECTIONS

The simplest way to treat radiative corrections is that proposed by Welton [37]) and described even more simply by Weisskopf [38]). According to these authors, the interaction of the particle (nucleon) with the quantized meson field causes fluctuations of the position of the particle, *viz.*

$$\langle r^2 \rangle = \frac{2}{\pi} \frac{g^2}{Mc^2} \frac{\hbar}{Mc} \ln \frac{fMc^2}{\hbar\omega_0}, \tag{3.15}$$

where $\hbar\omega_0$ is a characteristic energy of the particle, e.g., its binding energy, and f a numerical factor of order one, so that the ln term is also of order one. The "vacuum fluctuation" (3.15) is thus the product of the "classical radius" of the particle, g^2/Mc^2, and its Compton wavelength, $\hbar/Mc$.

If the particle is moving in an external potential V, the vacuum fluctuation gives a change of its energy by

$$\delta E = \tfrac{1}{6}\int \nabla^2 V \langle r^2 \rangle |\psi(r)|^2 \mathrm{d}\tau, \tag{3.16}$$

where ψ is the wave function of the particle in potential V. Welton and Weisskopf show that this procedure gives a good account of the Lamb shift.

† In sect. 4 we give the interaction in the form used by Reid,

$$V = (1/x)\Sigma\, a_n \mathrm{e}^{-nx}, \tag{A}$$

with $x = \mu r$, μ as in (3.5). Then for any meson n

$$g_n{}^2/\hbar c = a_n/m_\pi c^2. \tag{B}$$

For our purposes, it is best to use (3.15) directly: Accordingly, the nucleon is not a point particle but is essentially an *extended* source of the size given by (3.15). The "radius" of a nucleon is measured, e.g., in the electromagnetic form factor which indicates

$$\langle r^2 \rangle = 0.7\,\text{fm}^2. \tag{3.17}$$

Could this, perhaps, simply be the "vacuum fluctuation" of the position? The magnitude would agree with (3.15), with $g^2/\hbar c = 22$ and the ln term $= 1$. This interpretation might make it understandable why the nucleon still acts as a point in inelastic scattering at high energy.

Form factor measurements are compatible with the assumption that the charge of the nucleon is distributed about its center with the density

$$\rho(r) = (\alpha^3/8\pi)\mathrm{e}^{-\alpha r}, \tag{3.18}$$

$$\alpha = (12/\langle r^2 \rangle)^{\frac{1}{2}} = 4.1\,\text{fm}^{-1}. \tag{3.19}$$

3.7. EXTENDED SOURCE

It would be quite reasonable to assume that a nucleon is an extended source for the meson field whether this extension is due to radiative corrections or to more subtle causes. We therefore inquire into the NN forces with such an extended source. We take, in particular, the source distribution (3.18), (3.19).

The value of α in (3.19) is very close to

$$\mu_\omega = 3.9\,\text{fm}^{-1}. \tag{3.20}$$

For simplicity, we take them to be the same. Then the fourth component of the meson potential due to the source (3.18) obeys the equation

$$\nabla^2\phi - \mu^2\phi = -4\pi g\rho, \tag{3.21}$$

where $\ddot{\phi}$ has been set to zero. The solution is, for $\alpha = \mu$,

$$\phi(r) = \tfrac{1}{8}g\mu(1+\mu r)\mathrm{e}^{-\mu r}. \tag{3.22}$$

It has, of course, no singularity at the origin. The interaction between two nucleons at distance r is then

$$V = g\int \phi(r_1)\rho(\boldsymbol{r}-\boldsymbol{r}_1)\mathrm{d}\tau_1,$$

which can be evaluated to give

$$V = \tfrac{1}{128}g^2\mu\,\mathrm{e}^{-\mu r}(5+5\mu r+2\mu^2r^2+\tfrac{1}{3}\mu^3r^3). \tag{3.23}$$

While such an interaction is perfectly possible, it is of course very different from the Yukawa potential (3.2). Now when the potential is deduced from experimental phase shifts, the essential features of the repulsive core which are determined are its magnitude V_c and its slope $\mathrm{d}V_c/\mathrm{d}r$ at some distance like 0.6 fm (it does not make much difference if the fit is made at 0.5 or 0.7 fm). Let us postulate that our model IV potential is correct (sect. 4), *viz.*

$$V = \frac{4000\text{ MeV}}{\mu_\pi r} e^{-\mu_\omega r}. \tag{3.24}$$

This corresponds to

$$g^2/\hbar c = 29.6. \tag{3.24a}$$

Then in order to make V of (3.23) agree with (3.24) and its slope at $r = 0.6$ fm, we must choose in (3.23)

$$\mu = 9.1\text{ fm}^{-1}, \tag{3.25}$$

$$g^2/\hbar c = 110. \tag{3.26}$$

Both of these results are very unreasonable: (3.25) says that the meson responsible for the repulsive core would have to have nearly twice the mass of the nucleon. Moreover, since we have assumed $\alpha = \mu$, the rms radius of the extended nucleon would also be reduced by more than a factor of 2, as compared with the measurement (3.17). The very large coupling constant (3.26) is also troublesome. All these changes are undesirable from the physical point of view. Therefore, while mathematically possible, when we turn to numbers, the extended source does not seem very reasonable.

In the application to neutron matter at very high density, the following integral is very important:

$$\int Vr^2 \mathrm{d}r = g^2/\mu^2. \tag{3.27}$$

This is the same result as for a Yukawa potential, as might be expected since $\int \rho(r)\, \mathrm{d}^3 r = 1$. Now the quantity (3.27), when calculated with our extended source parameters, is about 0.71 times its value calculated with the "ω-potential" (3.24). This change is not very large, in spite of the enormous change in μ and g^2. So if the repulsive core is derived from experiment, its effect on ultra-high density matter remains essentially the same, whether or not we are using an extended source.

3.8. CAN THERE BE AN ATTRACTION INSIDE THE CORE

Various authors, e.g. Leung and Wang [16]), have suggested that the repulsive core may, at small r, be followed by a further attractive region. One argument they give is that in addition to vector mesons ω, ρ and ϕ, there are also spin-2 mesons of higher mass, particularly the f, which may interact with nucleons and give rise to a shorter range attraction. At very high density, the energy per baryon is given by

$$E/N = \tfrac{1}{2}\rho \int V(r)\mathrm{d}^3 r, \tag{3.28}$$

where V is the two-body interaction. In our theory, the integral is dominated by the repulsion which automatically makes E/N positive and proportional to ρ. These authors suggest, on the other hand, that an "inside" attraction may be strong enough to make the integrand zero, and then effects other than the potential will dominate. In particular, E/N would then probably not be proportional to ρ, and the equation of state would be much softer.

We shall now try to set an upper limit to the volume integral of the potential over a possible internal attractive well. Clearly, that potential cannot be strong enough to bind *two* nucleons together because then even the deuteron would collapse into this internal well. The barrier provided by the repulsive interaction (ω-meson exchange) is much too low and thin to prevent leakage from the outside to the inside well. Heavy nuclei would collapse even more readily.

The only possibility, then, is that discussed by Bodmer [39]): the attraction (supposed, to be an "ordinary" force) is not strong enough to bind *two* nucleons, but is strong enough to bind a large number, A. Then, in order to form the collapsed nucleus, A nucleons must simultaneously penetrate their mutual potential barrier which is now roughly A times higher. Bodmer estimates the lifetimes for such penetration, and finds that in order to achieve sufficiently long lifetimes, $>10^{31}$ sec, the *minimum* number of nucleons which will form such a collapsed state must be $A \geqq 16$ to 40. At the same time, the radius R of the attractive well must obviously be much smaller than the radius of the repulsive core which we may take to be [19]) 0.6 fm; a reasonable upper limit would be

$$R = 1/\mu_\omega = 0.25\,\text{fm}. \tag{3.29}$$

We would then have, in the collapsed nucleus, A nucleons in a sphere of radius R, giving a density

$$\rho = \frac{3A}{4\pi R^3} = 15A\,\text{nucl/fm}^3, \tag{3.30}$$

and a Fermi momentum

$$k_\text{F} = (\tfrac{3}{2}\pi^2\rho)^{\frac{1}{3}} = 6A^{\frac{1}{3}}\,\text{fm}^{-1}. \tag{3.30a}$$

At such high Fermi momentum we must use relativistic mechanics so that the average kinetic energy of the nucleon is

$$E_\text{kin} = \tfrac{3}{4}\hbar c k_\text{F}. \tag{3.31}$$

The motion of a group of nucleons at such high relativistic energy in close confinement poses of course a lot of problems but we shall disregard these. Instead, we calculate the potential energy per nucleon

$$E_\text{pot} = -\tfrac{1}{2}\rho\int V_\text{att}\text{d}^3r. \tag{3.32}$$

At the limiting value of A (and ρ) the sum $E_\text{kin}+E_\text{pot} = 0$. Inserting, we get

$$\int V_\text{att}\text{d}^3r = \tfrac{3}{2}\hbar c\,\frac{k_\text{F}}{\rho} = \pi(9\pi)^{\frac{1}{3}}\hbar c R^2 A^{-\frac{2}{3}}. \tag{3.33}$$

The repulsive force due to ω-exchange is

$$V_\omega = (g^2/r)\mathrm{e}^{-\mu_\omega r}, \tag{3.34}$$

$$\int V_\omega \mathrm{d}^3 r = 4\pi \frac{g^2}{\mu_\omega^2},$$

therefore the ratio

$$\mathscr{R} = \frac{\int V_{\mathrm{att}} \mathrm{d}^3 r}{\int V_\omega \mathrm{d}^3 r} = \tfrac{1}{4}(9\pi)^{\frac{1}{3}} \frac{\hbar c}{g^2} (\mu_\omega R)^2 A^{-\frac{2}{3}}. \tag{3.35}$$

Using R from (3.29), $g^2/\hbar c = 29.6$ from (3.24a), and $A = 16$, the minimum value permitted by Bodmer,

$$\mathscr{R} \leqq 0.0044. \tag{3.36}$$

The attractive well, if it exists, is therefore completely negligible. Our theory of the repulsive core can then be used without modification.

3.9. THE RHO MESON

We have seen in the preceding discussion that the strength of the short-range repulsion will be the same in all states of relative orbital angular momentum L and spin S, in particular

$$V_\mathrm{c}(^1\mathrm{P}) = V_\mathrm{c}(^3\mathrm{P}) = V_\mathrm{c}(^3\mathrm{S}) = V_\mathrm{c}(^1\mathrm{S}). \tag{3.37}$$

This result presupposes the that ρ-meson makes no contribution to the interaction. However, the tensor coupling of the ρ-meson is not completely negligible and will modify (3.37). If we calculate the contribution of the ρ-meson according to quantum theory, retaining only the single ρ-meson exchange contribution to the potential, then it can be shown [33]) that short-range tensor and spin-spin terms arise. The spin-spin term,

$$+\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2,$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are respectively the Pauli spin matrix and the nuclear isospin operator, modifies (3.37) to read

$$V_\mathrm{c}(^1\mathrm{P}) \geqq V_\mathrm{c}(^3\mathrm{P}) \geqq V_\mathrm{c}(^3\mathrm{S}) \geqq V_\mathrm{c}(^1\mathrm{S}). \tag{3.38}$$

The amount by which (3.38) differs from an equality can be determined by fitting the potential to nucleon-nucleon phase shifts. In the next section we shall describe several models of the potential which allow a range of strengths consistent with (3.38).

4. Interaction models

On the basis of the arguments given in sect. 3, we shall require that our interaction have the following properties:

(i) It must have a repulsive core in all states.

(ii) The core strength must satisfy the inequality

$$V_c(^1\mathrm{P}) \geqq V_c(^3\mathrm{P}) \geqq V_c(^3\mathrm{S}) \geqq V_c(^1\mathrm{S}),$$

in order to be consistent with meson exchange.

(iii) The nucleon-nucleon interaction must give the correct experimental phase shifts for energies up to 350 MeV, and the correct binding energy and quadrupole moment of the deuteron, like Reid's [19]).

(iv) The nucleon-nucleon interaction must saturate nuclear matter at a reasonable density and energy.

(v) Hyperonic interactions must be consistent with the (meager) experimental measurements [40]) in particular measurements on hypernuclei. These indicate that the hyperon-nucleon attraction is less than that of the nucleon-nucleon, but not by very much.

(vi) It is desirable (but not necessary) that the repulsive core have the same range in all states.

(vii) Finally, it is desirable that the repulsive force in (4.1) (see below) have a range corresponding to the exchange of ω-mesons; this means

$$n = 5.5.$$

This condition is satisfied only in our force models IV and V.

We take the potential to be of the same form as Reid [19]), *viz.*

$$V_{LSJ} = \sum_n C_n(LSJ)\mathrm{e}^{-nx}/x, \tag{4.1}$$

$$x = \mu r.$$

Here μ = the reciprocal pion Compton wavelength = $m_\pi c/\hbar$. Following Reid, we let the coefficients depend on the angular momenta L, S and J, and choose them (for $n \neq 1$) to fit experimental data. The coefficient C_1 is taken from the *theoretical* [19]) one-pion exchange potential (OPEP), thus

$$C_1 = 10.463 \times \tfrac{1}{3}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2. \tag{4.2}$$

At high density this is a relatively unimportant component of the potential. We omit it completely from the hyperonic potential and from the nucleon potential for $L \geqq 2$. The potential also includes a tensor force. For one-pion exchange the tensor force $V_\mathrm{T}^\mathrm{OPEP}$ has the form

$$10.463 \times \tfrac{1}{3}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 S_{12}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{\mathrm{e}^{-x}}{x} + \text{counterterm}, \tag{4.3}$$

where

$$S_{12} = (3\boldsymbol{\sigma}_1 \cdot \boldsymbol{r}\boldsymbol{\sigma}_2 \cdot \boldsymbol{r} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 r^2)/r^2. \tag{4.4}$$

The singularity near $r = 0$ is removed by the counterterm [19]), which has the form

$$-10.463 \times \tfrac{1}{3}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 S_{12} \left(\frac{3n}{x} + \frac{3}{x^2}\right) \frac{e^{-nx}}{x}. \tag{4.5}$$

We regard this term as arising from the exchange of a vector meson and will choose the exponent n accordingly (see the discussion of the specific models for our choice).

In practice, condition (v) means that for the hyperon-nucleon and hyperon-hyperon interaction we take a particular average of the nucleon-nucleon interaction for all pairs involving hyperons. Specifically, we define separately an even and odd state potential such that the even state potential is the same as the $^1D_2$ nucleon-nucleon potential and the odd state potential is the spin-isospin average of the P-state nucleon-nucleon potentials, less OPEP. This prescription is admittedly arbitrary but condition (v) is satisfied by virtue of this potential having less attraction than the nucleon-nucleon potential.

For the nucleon-nucleon potential for $L \geqq 2$ we use the same even and odd state potential defined above for hyperons. The particular potentials used in S- and P-states for nucleons will be specified in detail when the models are individually discussed.

For triplet states, especially ^3S and $^3D_1$, there is a tensor force. This must be determined from the phase shifts, but it is not important at high density because of the Pauli principle [see p. 131 of ref. [41])]. The central force for the ^3S state is generally less attractive than that for ^1S. We shall omit the tensor coupling between states of different L but generally include the diagonal terms in our matter calculations.

Model I. The first model is considered for its simplicity, and because it highlights some interesting physics of the HM equation of state. We assume that the potential is the same for all interacting pairs, whether the interacting particles are nucleons or hyperons. Further, the interaction is the same in all odd states, and again the same in all even states (but different from the odd states), so that we have an ordinary plus an exchange potential. The only exception is that we allow like nucleon pairs extra attraction in the relative S-states. We omit OPEP from this force model for simplicity and because it tends to be relatively unimportant at high density.

We take the strength and range of the core from the Reid [19]) ^1S potential, because the ^1S potential is the best determined from nucleon scattering. The Reid ^1D potential has the same core and is used for the even state potential. For like nucleons, the S-wave attraction is the Reid ^1S. We do not use the Reid P-state potential in odd states because the strength and range of the repulsion do not match that of even states. Instead we have refitted the strengths of the repulsive and attractive terms in $^3P_2$, $^3P_1$, $^3P_0$ and $^1P_1$ so that on the average the cores are the same as for the even states. The adjustments were first done by hand, but later Reid [42]) supplied us with more accurate machine fits, with the proper cores. Our odd state potential is the spin and isospin average of the new P-state potentials. It made negligible difference in the resulting HM properties whether we used the hand-fit potentials or the more

TABLE 1
Potentials [a, b, c])

		n	C_n	n	C_n	n	C_n
Pand.	^{1}S	2	0	4	−1650	7	6484
	l even ≠ 0	2	−12.322	4	−1113	7	6484
	l odd	2	0	4	− 933	6	4152
		$g_\omega^2/\hbar c = 47$			$\bar{V} = 2137$ MeV		
Model I	^{1}S	2	0	4	−1650	7	6484
	l even ≠ 0	2	−12.32	4	−1113	7	6484
	l odd	2	−22.79	4	− 370	7	6484
		$g_\omega^2/\hbar c = 47$			$\bar{V} = 2989$ MeV		
Model II	^{1}S	2	0	4	−1650	7	6484
	^{3}S	2	−10	4	−1000	7	6484
	^{1}P	2	0	4	−1426	7	23960
	$^3P_0$	2	27.13	4	− 790.7	7	20660
	$^3P_1$	2	0	4	− 650.9	7	9195
	$^3P_2$	2	0	4	− 714.8	7	2408.2
	l even ≠ 0	2	−12.3	4	−1113	7	6484
	l odd ≠ 1	2	2.71	4	− 774.4	7	8425
		$g_\omega^2/\hbar c = 47.0$			$\bar{V} = 3331$ MeV		
Model III	^{1}S			4	−1650	7	6484
	^{3}S			4	−1230	7	6484
	^{1}P			4	− 200	7	6484
	$^3P_0$			4	− 100	7	14480
	$^3P_1$			4	− 770.0	7	10480
	$^3P_2$			4	− 734.4	7	2484
	l even ≠ 0			4	−1250	7	6484
	l odd ≠ 1			4	− 628.2	7	6484
		$g_\omega^2/\hbar c = 47.0$			$\bar{V} = 2694$ MeV		
Model IV	^{1}S			3.8	−1837.5	5.5	4015.5
	^{3}S			3.8	−1303.5	5.5	4015.5
	^{1}P			3.8	− 856	5.5	4015.5
	$^3P_0$			3.8	− 977.4	5.5	9050.3
	$^3P_1$			3.8	−1408.1	5.5	6538.1
	$^3P_2$			3.8	− 778.71	5.5	1495.0
	l even ≠ 0			3.8	−1441.2	5.5	4015.5
	l odd ≠ 1			3.8	− 995.1	5.5	4015.5
		$g_\omega^2/\hbar c = 29.6$			$\bar{V} = 1773$ MeV		
Model V	^{1}S			3.49	−1208.7	5.5	3047
	^{3}S			3.49	− 812.86	5.5	3047
	^{1}P			3.49	− 972.69	5.5	6000
	$^3P_0$			3.49	− 832.87	5.5	9670.5
	$^3P_1$			3.49	−1075.3	5.5	6910.2
	$^3P_2$			3.49	− 509.5	5.5	1147.7
	l even ≠ 0			3.49	− 902.48	5.5	3047
	l odd ≠ 1			3.49	− 937.9	5.5	4221
		$g_\omega^2/\hbar c = 22.1$			$\bar{V} = 1633$ MeV		

[a]) The quantities n and C_n are defined in (4.1); C_n is given in MeV.

[b]) This table gives the potentials used for calculating the equation of state. In addition, the diagonal projection of OPEP [see note [c])] was used in $l = 0$, 1 nucleon states (see discussion in sect. 4) but is not shown here.

[c]) The coupled ^{3}S-^{3}D and $^3P_2$-$^3F_2$ states were fit to the two-body data by a central (V_c), spin-orbit (V_{LS}) and tensor ($V_T + V_T^{\mathrm{OPEP}}$) potential. Shown here are the diagonal projections of these potentials: $V_c(^3\mathrm{S}\text{-}^3\mathrm{D})$ for the ^{3}S state and $V_c(^3\mathrm{P}\text{-}^3\mathrm{F}) + V_{LS}(^3\mathrm{P}\text{-}^3\mathrm{F}) - {}^2/_5 V_T(^3\mathrm{P}\text{-}^3\mathrm{F})$ for the $^3P_2$ state.

TABLE 2

^3S-^3D potentials [a, b])

Reid:

$E_D = 2.2246$, $Q = 0.2796$, $P_D = 6.470$, $k_F = 1.44$, $-$B.E. $= -11.25$

Potential 6.55

	$n = 1$	$n = 2$	$n = 4$	$n = 6$	$n = 7$
V_c	-10.463	-10	-1000		6484
V_T			-10		-299
V_{LS}			708.91	-2723.1	

$E_D = 2.2569$, $Q = 0.2891$, $P_D = 6.55$, $k_F = 1.345$, $-$B.E. $= -9.37$

E	$\delta(^3S)$	$\delta(^3S$ Reid)	$\delta(^3D)$	$\delta(^3D$ Reid)
0.1	2.95	2.96		
5	2.06	2.07	-0.0028	-0.0029
50	1.07	1.09	-0.12	-0.12
150	0.492	0.497	-0.29	-0.287
250	0.180	0.184	-0.39	-0.37
350	-0.0425	-0.039	-0.47	-0.43

Potential 5.977

	$n = 1$	$n = 2$	$n = 4$	$n = 6$	$n = 7$
V_c	-10.463	-120.56	-399.18		6484
V_T			341.77	-1575.2	
V_{LS}			708.91	-2703.1	

$E_D = 2.2075$, $Q = 0.2908$, $P_D = 5.977$, $k_F = 1.364$, $-$B.E. $= -10.72$

E	$\delta(^3S)$	$\delta(^3D)$
0.1	2.95	
5	2.03	-0.0026
50	1.03	-0.10
150	0.471	-0.24
250	0.176	-0.32
350	-0.034	-0.39

Potential 5.595

	$n = 1$	$n = 2$	$n = 4$	$n = 6$	$n = 7$
V_c	-10.463	-30	-1109.5		6484
V_T			351.77	-976.84	
V_{LS}			708.91	-2723.1	

TABLE 2 (continued)

$E_D = -2.2359$, $Q = 0.2773$, $P_D = 5.595$, $k_F = 1.45$, $-$B.E. $= -11.65$

E	$\delta(^3S)$	$\delta(^3D)$
0.1	2.95	
5	2.05	-0.0027
50	1.07	-0.11
150	0.499	-0.24
250	0.198	-0.30
350	-0.015	-0.35

[a]) In all cases the tensor force includes the OPE term

$$\tfrac{1}{3}\times -10.463\left[\left(1+\frac{3}{x}+\frac{3}{x^2}\right)e^{-x}-\left(\frac{12}{x}+\frac{3}{x^2}\right)e^{-4x}\right]\frac{1}{x}\,\tau_1\cdot\tau_2 .$$

[b]) The potentials are labeled by P_D.

accurate machine fits. Appendix B gives further details concerning the choice of potential. The constants used for even and odd states are given in table 1. Notice also there that the ω-meson coupling to the nucleon is quite large, $g_\omega^2/\hbar c = 47$. It has the same value for models II and III also.

Model II. Here we still take the repulsive core from Reid $^1S_0$. For the np force we calculate a ^3S-^3D potential, requiring the repulsion to be the same as the ^1S. We fit to a few selected phase shifts and the deuteron a central+tensor+$L\cdot S$ force with this core. We show three such refitted potentials in table 2. Sprung [43]) has calculated the saturation properties of these potentials in nuclear matter, using a representative set of odd state potentials refitted by Reid [42]). We see from table 2 that the saturation energy, $-$B.E., and to a lesser extent the saturation k_F are correlated to the D-state probability, P_D, in the coupled ^3S-^3D channel: a larger P_D means a lower nuclear matter saturation density and a lower binding energy. This occurs, qualitatively speaking, because a larger P_D means that more attraction is coming from the tensor coupling between states; thus, the saturation of the tensor force will cost more energy. For our own calculation we pick a ^3S-^3D potential which has about the same P_D as Reid's published potential [19]) but which when used in conjunction with the more repulsive P-state potentials gives somewhat less nuclear matter binding at a somewhat lower density than Reid's. This shift is desirable because (a) the correction terms [three-body correlations and forces, etc., see ref. [41]), sects. 6, 7] generally increase both the density and the binding energy; (b) Reid's potential gives a theoretical density higher than the observed. The Reid [19]) ^1S potential is taken for the $T = 1$ S-wave force. For P-states we use new ^1P, $^3P_2$, $^3P_1$ and $^3P_0$ potentials supplied to us by Reid [42]). The counterterm in OPEP has $n = 4$. The interaction is given in table 1. Note that the chosen ^1P interaction has a larger repulsive term than the ^3P interaction; this seems to be favored by experiments, and it is consistent with subsect. 3.9. For hyperons, as before, we take the ^{1}D Reid [19]) potential for even states and the spin-isospin average of the P-states as our odd state interaction, omitting OPEP.

Model III. This model is similar to model II in that the range of the repulsion is the same. However, we have decided to satisfy the equality in condition (ii) above. For this calculation we use completely potentials supplied by Reid [42]). The phase shift fits are generally quite good, except the ^{1}P potential is not fitted as well as the rest. The central repulsion coefficient is uniformly 6484 and the short range spin-orbit force $-4000\,e^{-7x}/x$. The potential is shown in table 1, and it is not as repulsive as in model II because of the reduced coefficient of e^{-7x}/x in the ^{1}P state. Again the P_D of the ^{3}S-^{3}D potential has about the same value as Reid's [19]). The counterterm in OPEP has $n = 4$.

Model IV. For this model we take into account condition (vii) above, *viz.* that the range of the core is not e^{-7x} if mediated by observed mesons (ω and ρ), but more nearly $e^{-5.5x}$. Our potentials thus have a larger range core than those of models I–III. The strength of the repulsion and the range of the attractive term were determined by fitting the ^{1}S state to the experimental phase shifts, requiring that the volume integral of the repulsive term not differ by much from models I–III. However, now $g_\omega^2/\hbar c = 29.6$. Note that we find the somewhat surprising result that $|C_{3.8}|$ in this model is greater than $|C_4|$ in model III. This is because the 5.5 core extends to larger r than the 7 core, and hence has to be compensated by more attraction of the 3.8 type. The P_D is again about the same as for Reid's [19]) potential. The range of the counterterm ^{3}P states is the same as the range of the core ($n = 5.5$). The average over J of the repulsive terms in the ^{3}P states is the same as the repulsive term in the $^1S_0$ state. As in model III, the fit to the ^{1}P phase shifts in model IV is of poorer quality than to the other states.

Model V. For this model we again take the repulsive term to be $e^{-5.5x}/x$, but we reduce its coefficient substantially in even states. This is consistent with condition (ii), and reduces $g_\omega^2/\hbar c$ (taken from S-states) to 22 which, however, is still higher than the coupling of ω-mesons to nucleons observed in high-energy experiments, which is between 8 and 10. The attractive term is now $e^{-3.49x}/x$, determined just as in model IV. The average ^{3}P repulsion is the same as in model IV but the ^{1}P potential is allowed a somewhat greater repulsive coefficient to improve the phase shift fit. The ^{1}P potential is still the poorest fit to the phases. The P_D is again about the same as for Reid's potential [19]). The potential used is given in table 1. The counterterm has $n = 5.5$.

5. Calculational method

5.1. NUCLEAR MATTER THEORY

Nuclear matter theory [41]), based on the Brueckner-Goldstone theory and the Bethe-Goldstone equation, has up to now been the standard technique for doing nuclear matter calculations. However, this theory does not work at the high densities encountered in the central region of neutron stars. It was invented for moderate density systems, *viz.* for densities on the order of ρ_{NM}. This is obtained only at the lower end of the density range (4) of sect. 1, which is of interest in the present work. In this theory the energy is given as an expansion in linked clusters, with the "small" expansion parameter being $\kappa = \rho \int \xi^2 d\tau$ where ρ is the density and ξ the difference between the correlated and uncorrelated two-particle wave functions; ξ depends only mildly on density. Most of the contribution to nuclear matter energy at ρ_{NM} comes from the two-body terms because κ is relatively small ≈ 0.14. Nuclear matter theory works somewhat better for neutron matter, because κ is smaller due to the absence of the strong correlations induced by the tensor force in the ^3S-^3D channel. Siemens [44]) has solved the BG equation for neutron matter using the Reid [19]) potential for $\rho < 9 \cdot 10^{14}$ g/cm^3 ≈ 0.5 part/fm^3.

The fundamental reason for the breakdown of standard nuclear matter theory is that κ, which is roughly proportional to density, becomes large. Then three- and higher-body clusters need to be considered explicitly and the calculational problem becomes intractable.

5.2. PANDHARIPANDE'S METHOD

Pandharipande's method is not limited in this manner, and fortunately the variational method provides a reliable theory for the study of high-density systems. Although not as esthetically pleasing as standard nuclear matter theory, it is conceptually simpler, and, if implemented cleverly and carefully, it will give an upper bound for the energy.

Pandharipande [45]) has recently developed a variational technique based on the Jastrow trial wave function, which for HM consists of an antisymmetrized product of plane wave states modified by a product of two-body correlation functions

$$\Psi = \mathscr{A} \prod_{kl} f(r_{kl}) \prod_{m} \phi_m(r_m). \tag{5.1}$$

The expectation value of the Hamiltonian,

$$\langle H \rangle = \frac{(\Psi, H\Psi)}{(\Psi, \Psi)}, \tag{5.2}$$

is to be calculated, and made a minimum by variation of f. Pandharipande gives a reasonable prescription for f, *viz.* that it should obey a certain differential equation [(5.7), see below] for $r < d$, and the boundary condition

$$f = 1, \qquad \frac{df}{dr} = 0 \qquad \text{at} \quad r = d. \tag{5.3}$$

For $r > d$, $f = 1$; d may be regarded as a variational parameter.

His work consists of two parts, (a) an essentially exact evaluation [45]) of (5.2), and (b) a simple approximation to it [28]). In (a), which is the more recent paper, he shows that (5.1) can be evaluated for a Bose gas by summing certain diagrams. This in turn can be done by computing a very rapidly converging series of "elementary" diagrams, and then by solving an integral equation due to Van Leeuwen, *et al.* [46]) which can be done essentially exactly. For small d/r_0, the result for $\langle H \rangle$ is found to decrease with increasing d/r_0 where r_0 is related to the density by

$$\tfrac{4}{3}\pi r_0^3 \rho = 1, \tag{5.4}$$

but it settles down to an asymptotic value for $d/r_0 \geqq 2.0$. This value is then regarded as the true result, and is compared for liquid ^{4}He with (i) the experimental value of the binding energy as a function of density, and (ii) the result of Monte Carlo calculations which in principle can be exact for a boson gas, once f is given. Both comparisons are very satisfactory.

For a Fermi gas, exchange diagrams must also be calculated. Pandharipande shows that their contribution decreases rapidly with increasing number of particles exchanged. His theory gives the best known theoretical result for liquid ^{3}He. Hyperon matter is intermediate between a Bose and a Fermi gas.

In part (b), Pandharipande [28]) approximates (5.1) by the simplest two-body diagram which gives

$$\langle H \rangle = \sum_i \frac{p_i^2}{2M} + \tfrac{1}{2} \sum_{ij} \int \psi_{ij}^* \left(v - \frac{\hbar^2}{M} \nabla^2 - \frac{\hbar^2}{M} k_{ij}^2 \right) \psi_{ij} \mathrm{d}^3 r_{ij}, \tag{5.5}$$

where

$$\psi_{ij} = f(r_{ij}) \exp(i\boldsymbol{k}_{ij} \cdot \boldsymbol{r}_{ij}), \qquad \boldsymbol{k}_{ij} = \tfrac{1}{2}(\boldsymbol{k}_i - \boldsymbol{k}_j). \tag{5.6}$$

The second sum goes over all pairs of particles. Minimizing (5.5) with the boundary condition (5.3), and making certain reasonable assumptions, leads to the "Schrödinger equation" for ψ_{ij}

$$\begin{aligned} \left(v - \frac{\hbar^2}{M} \nabla^2 \right) \psi_{ij} &= \left(\lambda_{ij} + \frac{\hbar^2}{M} k_{ij}^2 \right) \psi_{ij}, & r < d, \\ f(r_{ij}) &= 1, & r > d, \end{aligned} \tag{5.7}$$

where the λ_{ij} are Lagrange parameters, to be determined by the boundary condition (5.3). In practice, of course, ψ_{ij} must be resolved in angular momentum components. The same f is used in approximations (a) and (b), but in (b) the energy is simply calculated from (5.5). Further, in (b), Pandharipande demands that d be chosen such that exactly one particle is found (on the average) in the sphere of radius d around particle i, i.e.,

$$4\pi\rho \int_0^d f^2(r) r^2 \mathrm{d}r = 1. \tag{5.8}$$

By this condition he justifies that only the interaction between i and j is taken into account, and all higher terms in both numerator and denominator of (5.2) are neglected. He calls this approximation "constrained variation" (CV).

In his recent paper [45]), Pandharipande shows that CV gives a remarkably good approximation to his more exact method (a). For his assumption about baryon interactions, the agreement is within about 5 %. We shall therefore use his CV method in this paper.

5.3. SIMPLIFICATIONS

Pandharipande's method is entirely satisfactory for numerical calculations. However, we wish to exhibit, as simply as possible, the various terms contributing to the total energy, and to give thereby a feeling for the uncertainty arising from our insufficient knowledge of the interaction. We shall therefore make a number of simplifying assumptions. In the end, the more important ones of these will be checked by direct numerical calculations. The Hamiltonian is

$$H = \sum_i t_i + \tfrac{1}{2}\sum_{ij} v_{ij}, \tag{5.9}$$

and we want to evaluate its expectation value for the ground state.

First of all, we assume the kinetic energy of baryons t_i to be given by the non-relativistic approximation

$$t = m + \frac{p^2}{2m}, \tag{5.10}$$

while we use the relativistic $(p^2+m^2)^{\frac{1}{2}}$ for leptons and pions. Eq. (5.10) is satisfactory because we shall find that in general even the Fermi momentum k_F stays below $\frac{1}{2}m$; moreover, when k_F is large, the kinetic energy is a small fraction of the potential.

To evaluate the energy, we allow f to be an operator which depends on the momentum and angular momentum of the interacting pair. Then, as shown in appendix C, the generalized lowest order expression for the total ground state energy W when particles have different masses is

$$W = \sum_i (\phi_i, t_i\phi_i) + \tfrac{1}{2}\sum_{ij} \left\{ \left(\psi_{ij}, \left[-\frac{\hbar^2}{2\mu_{ij}} k_{ij}^2 - \frac{\hbar^2}{2\mu_{ij}} \nabla_{ij}^2 + v_{ij} \right] \psi_{ij} \right) - \text{exch} \right\}, \tag{5.11}$$

where

$$\boldsymbol{k}_{ij} = \frac{m_i \boldsymbol{k}_j - m_j \boldsymbol{k}_i}{m_i + m_j}, \qquad \mu_{ij} = \frac{m_i m_j}{m_i + m_j},$$

$$\psi_{ij} = f_{ij}\phi_{ij} = f_{ij}(r_{ij}, k_{ij}) e^{i\boldsymbol{k}_{ij}\cdot \boldsymbol{r}_{ij}} \cdot \text{spin} \cdot \text{isospin}. \tag{5.12}$$

The indices (i, j) range over all different species and the unperturbed states of their respective Fermi seas. If we assume for the moment that the concentration $c_i \equiv \rho_i/\rho$ of each species i is known, then following Pandharipande [28]) we determine f by minimizing

$$W - \tfrac{1}{2}\sum_{ij} \{(\psi_{ij}, \lambda_{ij}\psi_{ij}) - \text{exch}\} \tag{5.13}$$

with respect to variations in f for $r < d$, where λ_{ij} is an operator depending on momentum and angular momentum and is chosen so that the boundary condition (5.3) is satisfied in each partial wave and for each value of k. Pandharipande [28]) chooses the Lagrange parameter λ to be a different constant for each partial wave LS, but independent of k. We go beyond this and allow λ to depend also on k. Taking the variation of (5.13) (see appendix C) we find that ψ must satisfy the condition

$$\left(-\frac{\hbar^2\nabla^2}{2\mu}+v\right)\psi = \left(\lambda+\frac{\hbar^2}{2\mu}k^2\right)\psi, \qquad r < d, \tag{5.14}$$
$$\psi = \phi, \qquad r > d.$$

Having solved (5.14), then from (5.11) and (5.14) one immediately obtains

$$W = T+U = T+U_{\rm in}+U_{\rm out}, \tag{5.15}$$

where

$$T = \sum_i (\phi_i, t_i\phi_i), \tag{5.16}$$

$$U_{\rm in} = \tfrac{1}{2}\sum_{ij}\{\psi_{ij}, \lambda_{ij}\psi_{ij})-\text{exch}\}, \qquad r < d, \tag{5.17}$$

$$U_{\rm out} = \tfrac{1}{2}\sum_{ij}\{(\phi_{ij}, v_{ij}\ \phi_{ij})-\text{exch}\}, \qquad r > d, \tag{5.18}$$

or

$$U = \tfrac{1}{2}\sum_{ij}\{(\phi_{ij}, \tilde{v}_{ij}\phi_{ij})-\text{exch}\}, \tag{5.19}$$

with

$$\tilde{v} = \begin{cases} f^{\dagger}\lambda f & r < d \\ v & r > d. \end{cases} \tag{5.20}$$

To determine d, we essentially use Pandharipande's condition (5.8), which for HM yields

$$d \approx 1.2r_0. \tag{5.21}$$

Up to now we have assumed that the concentrations c_α are known. In fact they are not, but to obtain them one imposes the well known conditions for chemical equilibrium: energy is conserved at the Fermi surface for reactions allowed by conservation laws (charge and baryon number; strangeness need not be conserved because there is plenty of time available for the strangeness violating weak processes to occur). This is equivalent to requiring $\delta E/\delta c_\alpha = 0$ (see appendix D) subject to the condition that charge and baryon number are unchanged by the variation. More explicitly, the concentrations are found by solving the set of equations

$$\mu_\alpha+m_\alpha = \mu_{\rm n}+m_{\rm n}-q_\alpha\mu_{\rm e}, \tag{5.22}$$

$$\sum_i q_i c_i = 0 \qquad \text{(charge neutrality)}, \tag{5.23}$$

$$\sum_\alpha c_\alpha = 1 \qquad \text{(baryon conservation)}, \tag{5.24}$$

where q_i is the charge of the ith particle and the index α ranges over baryons. In addition, if there are muons, then

$$\mu_\mu + m_\mu = \mu_e . \tag{5.25}$$

The chemical potential μ is related to the energy per baryon $E = W/(\Omega\rho)$ by

$$\mu_\beta = \frac{\partial}{\partial c_\beta}(E - \sum_\alpha c_\alpha m_\alpha). \tag{5.26}$$

Eqs. (5.22)–(5.24) are a set of non-linear equations in the unknowns c_i. In practice we choose some initial set of concentrations $c_i^{(0)}$ and assume that the solution is $c_i = c_i^{(0)} + \delta c_i$ where δc_i is small. This reduces the set of equations to linear equations, and a matrix inversion is performed to find δc_i. Generally we find stable solutions in this manner after a few (≈ 3) iterations.

Next, we may define an average effective interaction $\bar{v}_l^{\alpha\beta}$ which is a certain average of $\tilde{v}$ over S and J. We then have

$$W = \sum_i (\phi_i, t_i \phi_i) + \frac{4\Omega}{2} \sum_{\alpha,\beta} \sum_l \int_{<k_{F\alpha}} \frac{d^3k_\alpha}{(2\pi)^3} \int_{<k_{F\beta}} \frac{d^3k_\beta}{(2\pi)^3} 4\pi \int_0^\infty r^2 dr\, j_l^2(kr) \bar{v}_l^{\alpha\beta}, \tag{5.27}$$

where the indices $\alpha\beta$ in the second term are the species labels. The factor of 4 is the spin sum (we count species as if they were spin $\frac{1}{2}$; thus the Δ-particle, which in reality has spin $\frac{3}{2}$, counts as two different species), and $k_{F\alpha}$ is the Fermi momentum of species α. Table 3 defines $\bar{v}$ in terms of $\tilde{v}$ and appendix E derives these relationships, assuming for simplicity that the interaction is independent of J. For $r > d$, $\bar{v}$ is just the appropriate average of the basic baryon-baryon *potential* over S and J, and as a result the tensor and spin-orbit potential naturally drop out. Omission of these terms for $r > d$ is thus justified. For $r < d$ we must first solve (5.14) and then average. We found that at high densities, $\rho \gtrsim 5\ \mathrm{fm}^{-3}$, it makes negligible difference whether one averages the potential over J before solving (5.14) or vice versa. For lower densities

TABLE 3

The spin average $\bar{v}_l$ of the effective interaction $\tilde{v}_l(S, T)$ [a])

	Even l	Odd l
(1) Like nucleons	$\frac{1}{2}\tilde{v}_l(S=0\ T=1)$	$\frac{3}{2}\tilde{v}_l(S=1\ T=1)$
(2) Unlike nucleons	$\frac{1}{4}\tilde{v}_l(S=0\ T=1)+\frac{3}{4}\tilde{v}_l(S=1\ T=0)$	$\frac{1}{4}\tilde{v}_l(S=0\ T=0)+\frac{3}{4}\tilde{v}_l(S=1\ T=1)$
(3) Like hyperons	$\frac{1}{2}\tilde{v}_l$	$\frac{3}{2}\tilde{v}_l$
(4) Unlike hyperons	$\tilde{v}_l$	$\tilde{v}_l$

[a]) The $\tilde{v}_l$ is the average over J for given l. The coefficients $(\frac{1}{4}, \frac{3}{4})$ in the case of unlike nucleons arise from the statistical weight average of the spins. The coefficient $\frac{1}{2}$ for even l and $\frac{3}{2}$ for odd l for like particles reflects the Pauli exclusion principle: for odd l the spin must be even under exchange, or $S = 1$; for even l the spin must be odd, or $S = 0$. The coefficient of odd l, like nucleons, is > 1, but if the factor of $\frac{1}{2}$ in (5.27) is taken into account the coefficient is < 1 as it should be.

averaging the ^{3}P potential first can make as much as a 15–20 % difference in the total energy per particle. To average the potential first tends to make the effective interaction more repulsive.

The calculation of

$$4\pi \sum_l (2l+1) \int_0^\infty r^2 \mathrm{d}r\, j_l^2(kr)\bar{v}_l$$

is done separately for $r < d$ and $r > d$. For $r < d$ we retain the lowest two partial waves only. This is permissible because $k_{\mathrm{Fav}} d \lessapprox 1 \approx l_{\max}$ throughout the density range of interest; the appearance of many species of baryons has the effect of keeping k_{Fav} low. For example, for hyperons we have

$$\begin{aligned} U_{\mathrm{in}}^{\alpha\beta} &= 4\pi \left\{ \lambda_0^{\alpha\beta} \int_0^d r^2 \mathrm{d}r (\psi_0^{\alpha\beta})^2 + 3\lambda_1^{\alpha\beta} \int_0^d r^2 \mathrm{d}r (\psi_1^{\alpha\beta})^2 \right\}, \qquad \text{for} \quad \alpha \neq \beta \\ &= 4\pi \left\{ \tfrac{1}{2}\lambda_0^{\alpha\beta} \int_0^d r^2 \mathrm{d}r (\psi_0^{\alpha\beta})^2 + \tfrac{9}{2}\lambda_1^{\alpha\beta} \int_0^d r^2 \mathrm{d}r (\psi_1^{\alpha\beta})^2 \right\}, \qquad \text{for} \quad \alpha = \beta. \end{aligned} \tag{5.28}$$

(Subscripts indicate l. For statistical weights, see table 3.)

The numerical problem of evaluating the integrals over the two Fermi spheres in (5.27) could be a lengthy computational task. To facilitate this integration and to better understand the physics we have made the following numerical approximations. For $r < d$ we expand U in powers of k^2,

$$U_l = 4\pi\lambda_l \int_0^d r^2 \mathrm{d}r \psi_l^2 = A_l + B_l k^2 + D_l k^4. \tag{5.29}$$

Here, A, B and D are determined by solving (5.7) for three values of $0 \leqq k^2 \leqq k_{\mathrm{Fav}}^2$ and fitting a polynomial to these results. We have set

$$k_{\mathrm{Fav}}^3 = 3\pi^2 \rho / N, \tag{5.30}$$

where N is the number of baryon species. Next the coefficients A, B and D are expanded in terms of m_α and m_β, e.g.,

$$A_l(\mu_{\alpha\beta}) = A_l^{(1)} + A_l^{(2)} \left[\frac{1}{2} \frac{m_\alpha + m_\beta}{m_n} - 1 \right], \qquad \mu_{\alpha\beta} = \frac{m_\alpha m_\beta}{m_\alpha + m_\beta}. \tag{5.31}$$

We determine $A_l^{(1)}$ and $A_l^{(2)}$ by calculating A for three values of $\mu_{\alpha\beta}$ and fitting these points by a straight line. The expansions (5.29) and (5.31) are both very accurate. Table 4 a, b shows the coefficients A, B and D for $l = 0, 1$, for several values of the reduced mass and for $d = 0.5$ ($\rho = 3.6$). Note that only A_0 depends appreciably on μ; this is typical at all densities. The $l = 1$ term contributes mostly through the B-coefficient, and even this is small.

Table 5 shows the dependence of A on μ for the interaction model I, sect. 4. Note that $A_0^{(2)}$ is a substantial fraction of $A_0^{(1)}$ and that the μ-dependence favors heavy particles. Physically, this arises because the kinetic energy associated with a given bend in the wave function ψ at small distances is less for greater mass. Thus, more bend is permitted and the massive particles can comfortably sit farther from the repulsive cores of their neighbors, lowering the energy.

TABLE 4

Expansion coefficients of energy in powers of k^2 [a])

Pair	μ	A	B	D
		(a) $4\pi\lambda_l \int_0^d r^2 \mathrm{d}r\psi_l^2$, $l=0$		
nn	μ_0	16.9	−0.88	0.02
nΣ	1.12 μ_0	16.1	−0.86	0.02
$\Sigma\Lambda$	1.30 μ	15.1	−0.83	0.02
		(b) Same, $l=1$		
nn	μ_0	0	0.40	−0.01
nΣ	1.12 μ_0	0	0.40	−0.01
$\Sigma\Lambda$	1.30 μ_0	0	0.35	−0.01
		(c) $4\pi \int_d^\infty r^2 \mathrm{d}r \frac{1}{2}(v_e - v_0) j_0(2kr)$		
	d	A	B	D
	0.315	−11.5	3.02	−0.205
	0.561	− 8.36	4.12	−0.505
	1.13	− 1.82	1.7	−0.306

[a]) To obtain the energy in MeV multiply A, B and D by 41.4 and express k in units of fm^{-1}. The expressions are evaluated for model I, with $d = 0.5$ fm. Here μ_0 is defined as $\frac{1}{2}m_n$.

TABLE 5

Reduced mass dependence of the effective interaction

ρ(fm^{-3})	$A^{(1)}_{l=0}$	$A^{(2)}_{l=0}$	$\frac{1}{2}\rho\lvert A^{(2)}_0\rvert$	k_{Fav}(fm^{-1})
10.0	16.0	−4.69	23.5	2.76
5.90	16.9	−5.21	15.4	2.32
4.50	17.1	−5.43	12.2	2.12
2.33	16.6	−5.75	6.7	1.79
1.50	15.5	−5.79	4.3	1.64
1.00	13.9	−5.69	2.8	1.70

A is in units of 41.4 MeV · fm^3.

For $r > d$, we must of course take into account all values of l, not just $l = 0$ and $l = 1$. On the other hand, for $r > d$ the expansion (5.31) is not needed because the dependence on m_α and m_β arises only because of the terms $k^2 + \nabla^2$ in (5.11) which cancel for $r > d$, by definition of ψ_{ij}. However, it is still of interest to try an expansion in k^2. We shall take model I of sect. 4 as an example. The major part of the interaction is the same for all even l, v_+ and for all odd l, v_-. We will first do the r-integration of (5.27). We use the well known identity

$$\sum_{\substack{l\,\text{even}\\ l\,\text{odd}}} (2l+1)j_l^2(kr) = \tfrac{1}{2}(1 \pm j_0(2kr)),$$

to find

$$4\pi \sum_l (2l+1) \int_d^\infty r^2 \mathrm{d}r\, j_l^2(kr)\bar{v}_l = 4\pi \int_d^\infty \tfrac{1}{2}(\bar{v}_+ + \bar{v}_-) r^2 \mathrm{d}r + 4\pi \int_d^\infty \tfrac{1}{2}(\bar{v}_+ - \bar{v}_-) j_0(2kr) r^2 \mathrm{d}r. \tag{5.32}$$

These integrations are easily done analytically, since v is of the simple form given in eq. (4.1), and therefore

$$4\pi \int_d^\infty r^2 \mathrm{d}r \frac{\mathrm{e}^{-\mu r}}{r} = 4\pi d^2 \mathrm{e}^{-\mu d} \frac{1+\mu d}{(\mu d)^2}, \tag{5.33}$$

$$4\pi \int_d^\infty r^2 \mathrm{d}r \frac{\mathrm{e}^{-\mu r}}{r} j_0(2kr) = \frac{4\pi}{2k} \frac{\mathrm{e}^{-\mu d}}{\mu^2 + 4k^2} (\mu \sin 2kd + 2k \cos 2kd). \tag{5.34}$$

It should be clear from (5.34) that an expansion in k^2 will not in general be good if we stop with k^4. Fig. 1 shows the best fits (dashed) to the k^2 dependence by retaining terms through k^4. We have sacrificed the fit at low k^2 in favor of the region $k^2 \approx k_{\mathrm{Fav}}^2$. Table 4c shows the coefficients of the expansion for several values of d. Notice that the k-dependence occurs mostly through the k^2 term and that the effect is much larger here than in table 4a or b. As we shall see, there are some interesting consequences of

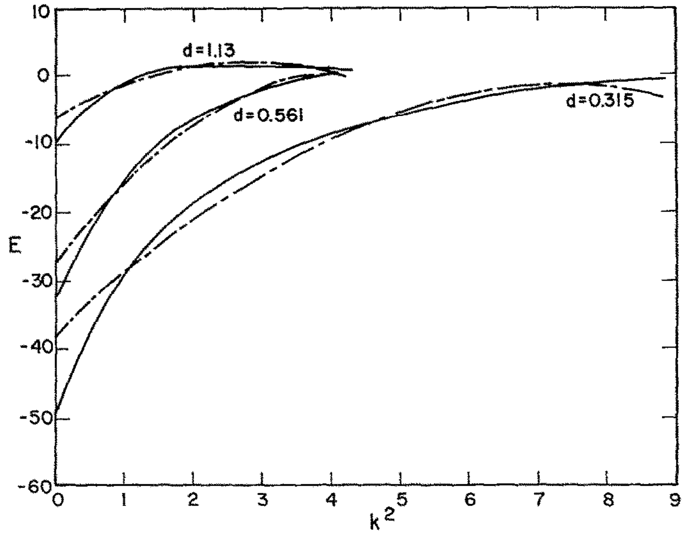

Fig. 1. Comparison of polynomial approximation (dashed curve) to exact expression (solid curve) for the momentum dependent term of (5.32). The curves correspond to three densities with the potential of model I.

this large k^2 term, whose origin is in the exchange character of the interaction: if even and odd state forces were the same, then this effect would be absent (see (5.32)). Although the fits shown in fig. 1 do not look good, we found that the HM results change only slightly when the more explicit expressions, presented next, are used instead.

To get more explicit expressions for $r > d$, we may alternatively do the following. The integration over the two Fermi spheres can be carried out by writing

$$U_{\text{out}}^{\alpha\beta} = (\phi_{\alpha\beta}, \bar{v}\phi_{\alpha\beta}) = \int_{r>d} \mathrm{d}\tau\, \mathrm{e}^{-ik_{\alpha\beta}\cdot r}(P_+ \bar{v}_+(r) + P_- \bar{v}_-(r))\mathrm{e}^{ik_{\alpha\beta}\cdot r},$$

where $P_\pm$ project onto states of $\binom{\text{even}}{\text{odd}}$ l. It immediately follows that

$$U_{\text{out}}^{\alpha\beta} = \int_{r>d} \tfrac{1}{2}(\bar{v}_+ + \bar{v}_-)\mathrm{d}\tau + \int_{r>d} \tfrac{1}{2}(\bar{v}_+ - \bar{v}_-)\mathrm{e}^{2ik_{\alpha\beta}\cdot r}\mathrm{d}\tau. \tag{5.35}$$

Thus,

$$4\int \frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\int \frac{\mathrm{d}^3k_\beta}{(2\pi)^3}\, U_{\text{out}}^{\alpha\beta} = 4\pi\rho_\alpha\rho_\beta \int_d^\infty \tfrac{1}{2}(v_+ + v_-)r^2\mathrm{d}r$$
$$+4\pi\rho_\alpha\rho_\beta \int_d^\infty \tfrac{1}{2}(v_+ - v_-)S(r_\alpha)S(r_\beta)r^2\mathrm{d}r, \tag{5.36}$$

where

$$r_\alpha = \frac{2m_\alpha k_{\mathrm{F}\beta} r}{m_{\alpha\beta}}, \qquad m_{\alpha\beta} = m_\alpha + m_\beta, \qquad S(x) = 3(\sin x - x\cos x)/x^3.$$

The integral over r can easily be done numerically.

The extra attraction for *like nucleons* in S-states for $r > d$ requires evaluating

$$4\pi \int r^2\mathrm{d}r[v(^1\mathrm{S}) - \bar{v}(\text{even})]\, j_0^2(kr). \tag{5.37}$$

This can again be integrated over the Fermi sphere by using the identity

$$\frac{1}{(2\pi)^6}\int_{<k_{\mathrm{F}\alpha}} \mathrm{d}^3k_\alpha \int_{<k_{\mathrm{F}\beta}} \mathrm{d}^3k_\beta\, j_0^2(kr) = \frac{64}{3}\frac{\pi^2}{(2\pi)^6}\int_0^{k_\mathrm{F}} k^2\mathrm{d}k(2k_\mathrm{F}^3 + k^3 - 3k_\mathrm{F}^2 k)\, j_0^2(kr)$$
$$= \frac{k_\mathrm{F}^6}{8\pi^4\xi^6}[\xi^4 - \xi^2 + \xi\sin 2\xi - \sin^2\xi], \qquad \xi = k_\mathrm{F} r. \tag{5.38}$$

We have now reduced the integration over the two Fermi spheres to a tractable form involving at most one one-dimensional integral to be done numerically.

We can easily show that the approximate expression, in terms of k^2 and k^4, can be integrated as follows

$$\frac{1}{2}\sum_{\alpha\beta}\int_{<k_{F\alpha}}\frac{d^3k_\alpha}{(2\pi)^3}\int_{<k_{F\beta}}\frac{d^3k_\beta}{(2\pi)^3}(A^{\alpha\beta}+B^{\alpha\beta}k^2+D^{\alpha\beta}k^4)$$

$$=\frac{1}{2}\rho^2\sum_{\alpha\beta}\Big\{A^{\alpha\beta}c_\alpha c_\beta+\tfrac{3}{10}B^{\alpha\beta}k^2_{\mathrm{Fav}}\left(\frac{2m_\beta}{m_{\alpha\beta}}\right)^2 c_\alpha^{\frac{5}{3}}c_\beta+\tfrac{3}{8}D^{\alpha\beta}k^4_{\mathrm{Fav}}$$

$$\times\left[\frac{1}{7}\left(\frac{2m_\beta}{m_{\alpha\beta}}\right)^2 c_\alpha^{\frac{7}{3}}c_\beta+\frac{1}{5}\left(\frac{2m_\alpha}{m_{\alpha\beta}}\right)^2\left(\frac{2m_\beta}{m_{\alpha\beta}}\right)^2 c_\alpha^{\frac{5}{3}}c_\beta^{\frac{5}{3}}\right]\Big\}. \qquad (5.39)$$

Here, c_α is the concentration of species α. This simplifies for model I of sect. 4. Let the coefficients A, B and D contain both the inner and outer contribution, and therefore except for the reduced mass dependence in A, the coefficients are essentially independent of α, β. In practice we find that replacing $m_\alpha/(m_\alpha+m_\beta)\rightarrow\frac{1}{2}$ changes the energy very little, but has about 10 % effect on the baryon concentrations at the higher densities. As discussed before, the mass terms favor the more massive particles [we see from (5.36) that the mass dependence of $k_{\alpha\beta}$ gives the heavier particle a lower effective Fermi momentum]. In this case (5.39) gives the potential per particle

$$U/N=\tfrac{1}{2}\rho[A+\tfrac{3}{10}Bk^2_{\mathrm{Fav}}\langle c^{\frac{5}{3}}\rangle+\tfrac{3}{8}Dk^4_{\mathrm{Fav}}(\tfrac{1}{7}\langle c^{\frac{7}{3}}\rangle+\tfrac{1}{5}\langle c^{\frac{5}{3}}\rangle^2)]. \qquad (5.40)$$

We must add to this the kinetic energy of the baryons

$$\frac{k^5_{\mathrm{Fav}}}{10\pi^2}\sum_\alpha\frac{c_\alpha^{\frac{5}{3}}}{m_\alpha}, \qquad (5.41)$$

the electron kinetic energy

$$\frac{\lambda^{-5}}{m_e}\frac{1}{32\pi^2}(\sinh\theta-\theta), \qquad (5.42)$$

where

$$e^{\frac{1}{4}\theta}=2\lambda k_{\mathrm{Fe}}, \qquad (5.43)$$

$$\lambda=\frac{\hbar}{m_e c}=386.1\ \mathrm{fm},$$

the extra S-wave attraction for like nucleons (5.37), the rest mass, and $A_0^{(2)}$ terms to get the complete ground state energy.

6. Calculations

We shall now apply the methods of sect. 5 to calculate the energy of hyperon matter with the various interaction models of sect. 4.

Model I. We have already calculated some relevant quantities for this model, as illustrations to the method of subsect. 5.3. In particular, table 4 gives the expansion coefficients A, B and D defined in (5.29). Summing the contributions (a), (b) and (c) of that table, we find that B is large and positive, meaning that the potential energy increases strongly with k^2. The kinetic energy is also positive definite. Thus, these terms favor small kinetic energy, i.e., small concentrations of each species, and therefore many species. The term D is negative and therefore favors few species, but this term is generally smaller than the B term.

The rest mass term [see (5.31)] is

$$22.57 \sum_i m_i \rho_i, \tag{6.1}$$

where all masses m_i are measured in units of the mass of the neutron. Generally in this paper, we have measured energies in units of $\hbar^2/m_n \cdot \mathrm{fm}^2 = 41.4$ MeV, see tables 4 and 5, so that one unit of energy corresponds to $k^2 = 1\ \mathrm{fm}^{-2}$. Therefore we have in (6.1) a factor

$$\frac{m_n c^2}{\hbar^2/m_n} = \left(\frac{m_n c}{\hbar}\right)^2 = 22.57\ \mathrm{fm}^{-2}$$

Eq. (6.1) and the $A_0^{(2)}$ term

$$\tfrac{1}{2} \sum_{ij} \rho_i \rho_j A_0^{(2)} \tfrac{1}{2}(m_i + m_j) \tag{6.2}$$

may be combined to give

$$(22.57 - \tfrac{1}{2}\rho |A_0^{(2)}|) \sum_i m_i \rho_i . \tag{6.3}$$

We see that when $22.57 > \frac{1}{2}\rho|A_0^{(2)}|$, $\sum_i m_i \rho_i$ wants to be minimum and thus the lighter particles are favored. However, if $22.57 < \frac{1}{2}\rho|A_0^{(2)}|$ at some density, then (6.3) will favor a condensation of the system into constituents of rest mass as large as possible. As we see from table 5, such a phase transition would occur at a density of $\rho \approx 10$, which is too high a density to occur ever in a stable star.

An interesting result from (5.40) is that for a given number of species N, the energy should be fairly insensitive to the concentrations. This is because A tends to be larger than B and D and because V depends only on moments, e.g., $\langle c^{\frac{5}{3}} \rangle$, of the (normalized, $\sum_i c_i = 1$) set of concentrations.

Fig. 2 shows the partial densities ρ_i plotted as a function of the total density ρ. At low density, $\rho \lesssim 0.5$ (particles per fm^3), we see that matter consists almost entirely of neutrons. If instead we had, e.g., an equal number of protons and neutrons, we would need an equal number of electrons, and therefore the electron Fermi momentum k_{Fe} would equal the neutron Fermi momentum k_{Fn}. But for given k_{F}, the electron Fermi *energy* is much higher than that of a nucleon because the electron velocity is essentially c (velocity of light) while that of the nucleons is much lower.

At higher density, conditions become relatively less favorable for nucleons because the potential energy terms depending on k^2, i.e., the sum of the terms "k^2" and "k^4" in table 4, are several times the kinetic energy e.g., four times at $\rho = 1.5\ \mathrm{fm}^{-3}$. Then, in absence of hyperons, it becomes more favorable to replace a neutron by a proton plus an electron, because this reduceds k_{Fn}. Thus, the percentage of electrons will increase, and with it the electron chemical potential μ_e. For a narrow range of densities, μ_e becomes larger than the mass of the μ^- meson, and thus some μ^- will appear, but they soon disappear again when the Σ^- enter the picture. The highest electron concentration is 1.8 % at $\rho = 0.475\ \mathrm{fm}^{-3}$.

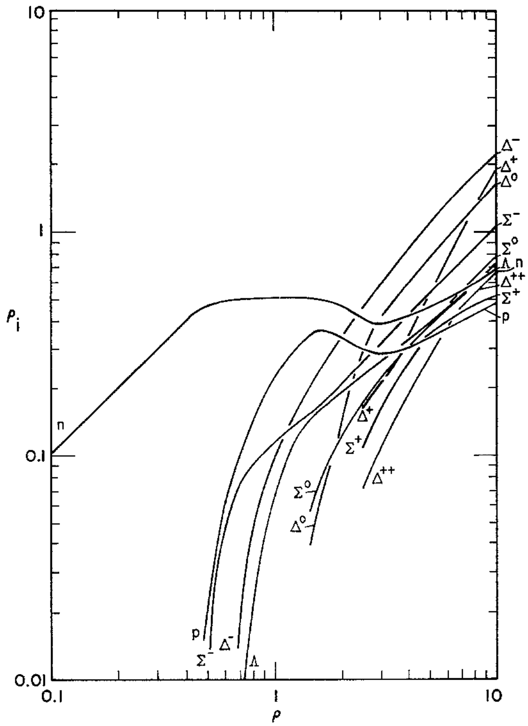

Fig. 2. Partial densities of baryons versus baryon density for model I. Units are baryons per fm³.

At this density, μ_e has risen to 125 MeV, μ_n to 188 MeV. Any further increase in density will cause Σ^- to appear. From (5.22) we see that Σ^- has a chemical potential of 55 MeV at threshold, taking $m_\Sigma - m_n = 258$ MeV. (For a non-interacting baryon gas the chemical potential arises from the kinetic energy term only, and thus the Σ^- would have zero chemical potential at threshold.) This baryon is extremely effective: for one thing, it provides another species (like the proton), and thus serves to keep k_{Fn} down. Another,and even more important function is that it provides negative charge so it can neutralize the proton without requiring "expensive" electrons. These two effects of the Σ^- overcome the counter-effect of the large Σ^- mass. In fact the mass is compensated to such a large extent by the chemical potential of the electron that the Σ^- appears before the Λ, which is the lighter by 75 MeV. As fig. 2 shows the remaining hyperons Δ^-, Δ^0, Σ^0, Σ^+, Δ^+ and Δ^{++} come in strongly as the density rises. The Δ is the largest constituent, having twice the statistical weight (spin $\frac{3}{2}$) of the other baryons.

In table 6a the k_{Fav} independent term of (5.40), $\frac{1}{2}\rho A$, plus the $A_0^{(2)}$ term of (6.2) have been decomposed into an inner and outer contribution. At low density the sum of these terms is negative because the attractive forces dominate; at $\rho = 0.9$ it changes sign because now the correlation function f in (5.12) has sufficiently short range d

TABLE 6

Density dependence of energy: Model I

(a) Momentum-independent terms

ρ	Inner	Outer	Sum
0.1	−8.55	−6.23	−14.8
0.25	−5.69	−26.3	−32.0
0.4	4.74	−48.4	−43.7
0.8	136	−151	−15.0
1.0	212	−192	20
1.5	397	−268	129
2.5	762	−297	459
4.5	1410	31	1440

(b) Momentum-dependent terms

ρ	$k_{\mathrm{Fav}}(\mathrm{fm}^{-1})$	k^0	k^2	k^4	E(kin)	E(total)
0.1	1.14	−15	7	−1	26	17
0.25	1.94	−32	32	−5	47	37
0.4	2.28	−44	67	−14	65	65
0.8	1.58	−15	137	−37	57	177
1.0	1.70	20	163	−44	53	244
1.5	1.64	129	248	−72	50	445
2.5	1.74	459	344	−82	39	944
4.5	2.12	1440	700	−167	49	2240

that a substantial part of the short-range repulsion extends into the region $r > d$. The inner contribution is of course repulsive except at the very lowest densities. By coincidence, $\rho \approx 0.9$ also happens to be about the density at which the hyperons begin to appear, as evidenced in table 6b by the abrupt drop in k_{Fav}, the drop in E_{kin} and the leveling off of the earlier rapid increase in the k-dependent potential terms. In this region of density the system is thus expected to be very sensitive to the strength of the attraction in the two-body interaction involving hyperons. If the hyperon interaction were more attractive than that between nucleons (without compensating for increased repulsion), the *k-independent* terms for a hyperon-nucleon pair would be lower than for the nucleon-nucleon, leading to a drop in the total k-independent term in table 6 when hyperons come in. At the same time the k-dependent terms are lowered because more species are present. The combined effect could easily be a decrease in energy with increasing ρ. Such an effect was in fact observed by Pandharipande [28]) in his model A (see sect. 2). With our model I we find no such behavior.

Fig. 3 shows E versus ρ for model I and compares it to Pandharipande's model A and his neutron matter. Observe that our equation of state is stiffer than Pandhari-

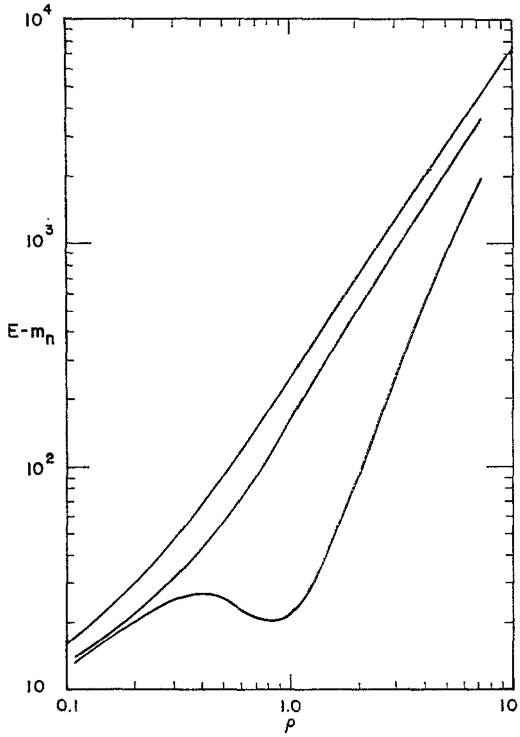

Fig. 3. Energy per particle (MeV) versus baryon density for model I (uppermost curve), Pandharipande's model A (lowermost curve), and his neutron matter.

pande's neutron matter. This is due principally to our modifications of the nucleon-nucleon potential. We have parametrized E versus ρ and find that

$$E = 236\rho^{1.54} + m_n \qquad \text{MeV/part}, \tag{6.4}$$

for $\rho \geqq 0.45$. Since the pressure is given by

$$P = \rho^2 \frac{dE}{d\rho}, \tag{6.5}$$

we find for model I

$$P = 364\rho^{2.54}\ \text{MeV/fm}^3. \tag{6.6}$$

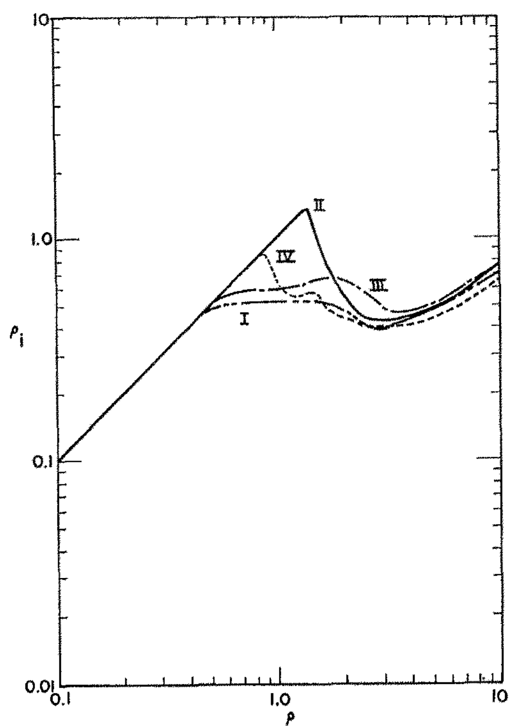

Fig. 4. Partial densities of neutrons for models I, II, III and IV.

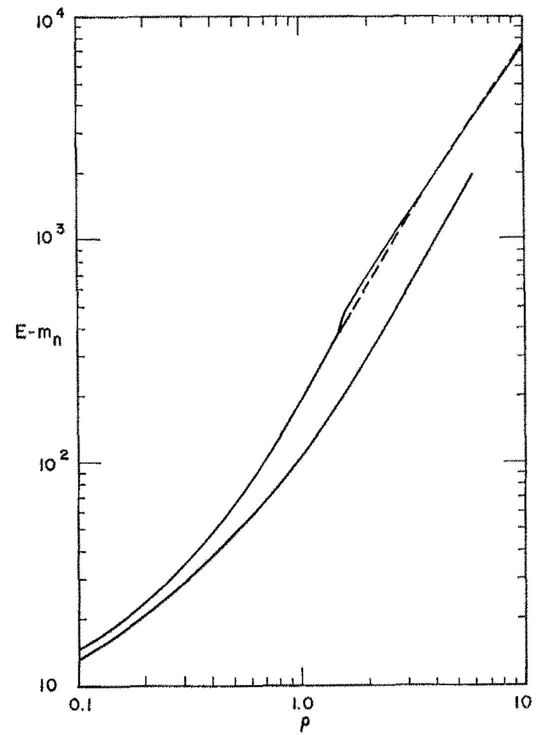

Fig. 5. Energy per particle for model II neutron matter (dashed curve) and hyperon matter (solid curve), and Pandharipande's model C (lowermost curve).

In order to solve the hydrostatic equilibrium equations for the neutron star structure, one needs the relationship $P(\varepsilon)$, where

$$\varepsilon = \rho E. \tag{6.7}$$

This is given implicitly by (6.4)–(6.7).

Model II. This model is solved taking more seriously some of the details of the nucleon-nucleon potential. Whereas in model I we first averaged the potential in odd and even states separately and used this interaction for all baryons, here (and in subsequent models too) we treat separately the singlet and triplet S- and P-states of the nucleons. This means for $r < d$ solving (5.7) separately in each important ($l \leqq 1$) state. For $r > d$ the calculation is done as before, except that we add the differences $v(^3\mathrm{S})-v(\mathrm{even})$, $v(^3\mathrm{P})-v(\mathrm{odd})$ and $v(^1\mathrm{P})-v(\mathrm{odd})$ by fitting a polynomial as follows

$$4\pi \int_d^\infty r^2 \mathrm{d}r (v_l - v) j_l^2(kr) = A + Bk^2 + Dk^4, \tag{6.8}$$

and using (5.39).

TABLE 7
Energy per particle for models I-V [a]

Density	I	II		III		IV		V		P
(fm^{-3})	H	H	N	H	N	H	N	H	N	N
0.5	89	65	65	63	63	61	61	63.5	63.5	57
1.0	245	189	189	189	185	191	181	185	191	156
2.0	680	690	610	560	590	575	560	480	570	490
4.0	1900	1920	1890	1610	1810	1450	1580	1200	1650	1450
10.0	7100	7300	7500	6200	7200	5900	6600	4000	6000	

[a]) Entry H means hyperon matter; N, neutron matter; P, Pandharipande's calculation of neutron matter. The energy (MeV) is measured relative to the neutron mass.

Fig. 4 shows the partial density of neutrons for model II. Compared to model I, fig. 2, there are now more neutrons, therefore fewer hyperons, and the threshold for the appearance of the hyperons lies at a higher density. This difference can be understood on the basis of our choice of interaction in P-states, which for like nucleons is less repulsive and for unlike nucleons more repulsive than for model I, where we took the same interaction in all odd states. This difference energetically favors relatively more neutron pairs for model II, and it raises the density at which the $A_0^{(2)}$ dependence [see (5.31)] of the effective interaction begins to encourage the appearance of many species.

Fig. 5 shows the energy as a function of density. We found that our present procedure of solving (5.7) separately for $^3\mathrm{P}_2$, $^3\mathrm{P}_1$ and $^3\mathrm{P}_0$ gives a significantly lower energy, especially for the lower densities of fig. 5, than the energy obtained by solving (5.7) with an averaged $^3\mathrm{P}$ interaction. This is one reason why the equation of state for model II is less stiff than that for model I for the lower densities. To compare the equations of state refer to table 7. At the higher densities the two models are more similar; this is because the repulsive cores, which have nearly the same strength and range in the important states, dominate. Pandharipande's model C [ref. [28])] is shown for comparison on fig. 5. Our equation of state is uniformly stiffer than his, and this we again attribute to our modifications of the short range behavior of the potential.

A comparison of the HM and neutron matter for model II in fig. 5 shows that the former has lower energy only at sufficiently high density, $\rho \geqq 5\ \mathrm{fm}^{-3}$. For ρ slightly less than $5\ \mathrm{fm}^{-3}$ the state which is realized is therefore neutron matter. For ρ slightly greater than this, hyperon matter is realized. Taken literally, this implies that a phase transition would occur at $\rho = 5\ \mathrm{fm}^{-3}$, such that the concentration would suddenly jump from 100 % neutrons to 10 % neutrons (fig. 4). There will be a sudden change in pressure at this transition density. In the star, of course, there would be a jump of density at a given pressure where the transition occurs. The density at which this effect occurs will depend on the interaction model; the density $5\ \mathrm{fm}^{-3}$ of this model is outside the range of densities that are relevant to neutron stars.

The reason why HM may have a greater energy than neutron matter can be understood on the basis of our selection of d [see (5.3)]. In these calculations we use Pandharipande's [28]) eq. (6) to evaluate the number of particles in the volume $\frac{4}{3}\pi d^3$, and have allowed for the momentum dependence of the correlation function f. As the relative momentum of the interacting pair *increases*, f *decreases*. Thus, for the same value of d, there will be fewer particles in $\frac{4}{3}\pi d^3$ if the substance is neutron matter, than if the substance is HM, where the average Fermi momentum is lower. There is also an effect of the Pauli principle on the unperturbed wave function ϕ: Neutrons are kept apart more effectively by the Pauli principle, again reducing the number of particles in $\frac{4}{3}\pi d^3$ in pure neutron matter. According to Pandharipande's prescription for CV (see subsect. 5.2) d must be adjusted so that on the average only one particle is in $\frac{4}{3}\pi d^3$. Thus, d/r_0 is larger for neutron matter than for hyperonic matter by a few percent. This effect opposes the tendency for the many species to lower the energy. At high density the latter effect wins and HM is lower in energy, but at low density the smaller d/r_0 of hyperonic matter raises its energy more than the smaller k_F lowers it. The effect at largest is about 15 % in energy at $\rho = 1.5\ \mathrm{fm}^{-3}$. Although this exceeds the limit of error of CV, it is not possible to state conclusively whether the phase transition is a physical effect predicted by the theory, or whether it is an effect introduced by the approximation method (CV) used. If the same value of d is used for neutron matter as one finds for HM, then the neutron matter equation of state will of course, be uniformly stiffer than the HM equation of state.

Model III. Model III, like model II, employs a potential (table 1) for nucleons which is more realistic than that of model I. The l-even states are practically identical in models II and III, but the l-odd state central potential, especially the $^1P_1$, is less repulsive in the present case. (Model III was constructed to have the same central repulsion in each partial wave.)

The equations of state for models II and III are substantially alike. Comparing E

versus ρ of model II with model III (see table 7) we see that neutron matter is only slightly softer for model III, and that the effect of the hyperons in softening the equation of state is somewhat greater for model III. Comparing neutron concentrations in fig. 4 shows that the threshold for appearance of the hyperons in model III is shifted back to a substantially lower density from their threshold in model II, and that their appearance is much less sudden. This is due in large part to the ^{1}P interaction being less repulsive now so that protons are allowed to appear at a lower density.

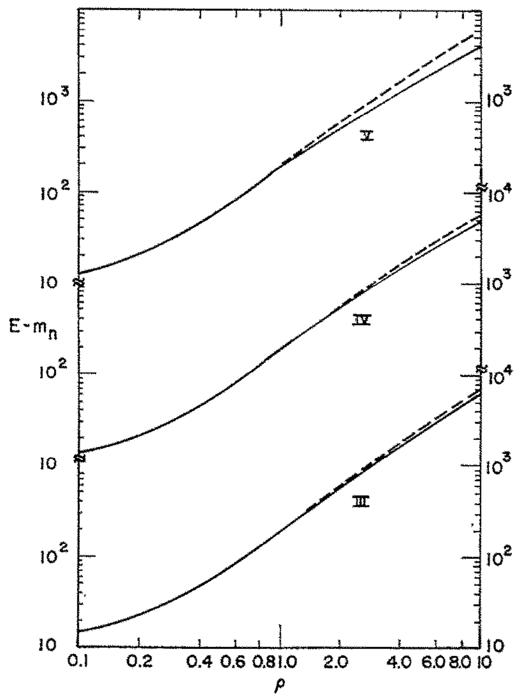

Fig. 6. Energy versus baryon density for neutron matter (dashed curves) and hyperon matter (solid curves). Models III, IV and V are shown.

The comparison of HM and neutron matter in fig. 6 shows that neutron matter is energetically favored for $\rho \lesssim 1.2\ \text{fm}^{-3}$, hyperon matter for greater densities even though according to fig. 6 hyperons enter at $\rho \approx 0.7\ \text{fm}^{-3}$ (see discussion of model II). Now, however, there is much less difference in the energy at the lower densities.

Model IV. As shown in table 1 and discussed in sect. 4, the range of the repulsion has been increased in model IV over models I–III, corresponding now more closely to the mass of the ω-meson. The same central repulsion has been put in all states, and the volume integral of the core term is nearly the same as in model III.

These changes do not make a dramatic difference in the equation of state. Comparing E versus ρ for models III and IV, table 7 and fig. 6, we see that for the longer range

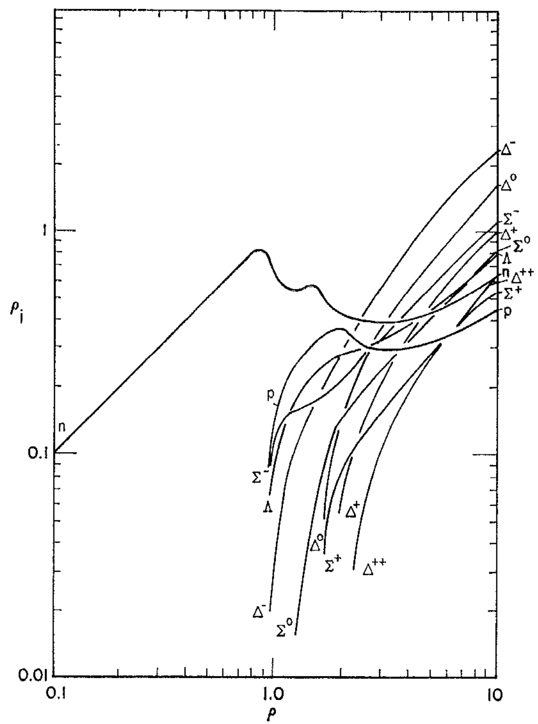

Fig. 7. Partial densities for model V.

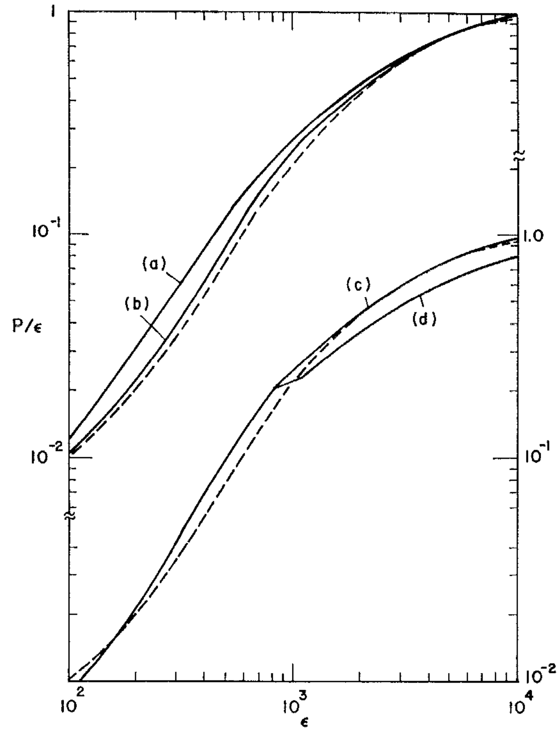

Fig. 8. Equations of state. Curve (a) is model I, hyperon matter; (b) is model III, hyperon matter; (c) is model V, neutron matter; and (d) is model V, hyperon matter. The dashed curve is Pandharipande's neutron matter. Units of P and ε are MeV · fm^{-3}.

potential, E is less at the lower densities, $0.1 < \rho < 0.5\ \mathrm{fm}^{-3}$, less again at the higher densities, $\rho > 2.5\ \mathrm{fm}^{-3}$, and about the same at intermediate densities. This reflects the fact, observed in sect. 4, that the increase in the range of the repulsion holding its total strength fixed is accompanied by an increase in the attraction. Comparison of the neutron partial densities for these two models, fig. 4, shows that generally the two sets of concentrations look quite similar, but that in the present case the threshold for appearance of hyperons is shifted to a somewhat higher density.

Model V. A comparison of the results of this model with model IV shows the effect on the equation of state of decreasing the repulsion in even states and increasing the repulsion in odd states, keeping the range of the repulsion fixed by the ω-meson mass (table 1). A comparison of E versus ρ to model IV, table 7, shows that neutron matter is nearly identical in the two cases. However, there is a relatively large softening of the equation of state when hyperons are included. This is due to our increasing the difference between even and odd state repulsion: it pays energetically to have a lower average Fermi momentum (or more species) because then baryons experience more of the less repulsive S-wave interaction and less of the more repulsive P-wave interaction. Fig. 7 shows the resulting concentrations. The comparison in table 7 shows that for $\rho < 2\ \mathrm{fm}^{-3}$ the energy per particle for the hyperonic matter of model V lies between Pandharipande's neutron matter [28]) and our model I, but for higher densities it lies below both.

The result of Pandharipande's latest calculation [29]) with the Reid [19]) interaction (including tensor terms in the ^{3}S state) for nucleons is that matter throughout the density range relevant to neutron stars is essentially neutron matter, admitting a very small admixture of protons and possibly other hyperons at low density. The main reason he finds fewer hyperons than we do is that the Reid ^{3}S interaction is substantially more repulsive than the ^{3}S interactions we use. Protons are thus discriminated against by the potential at high density; with fewer protons, the Σ^- also will come in less strongly. Other differences also tend to favor fewer hyperons. He uses a hyperonic interaction based on some SU(3) considerations, and this force is somewhat less attractive than our hyperon interaction. Furthermore, he uses relativistic kinetic energies for baryons, whereas we have used the non-relativistic expressions. The latter tend to favor heavier particles. Because he bases this calculation on the Reid [19]) potential, we believe that our present calculations with the modified short range behavior are the more realistic.

Fig. 8 shows P/ε plotted versus ε for models I, III, and V. These curves can be obtained from eqs. (6.5) and (6.7) once $E(\rho)$ is calculated. Pandharipande's [28]) neutron matter curve is plotted for comparison. Relativity requires that $P/\varepsilon < 1$; it is apparent from this figure that the limit is exceeded for ε somewhat greater than 10^4 $\mathrm{MeV} \cdot \mathrm{fm}^{-3}$ (for further discussion of this point see subsect. 7.5). However, for very large ε it is expected [28]) that P/ε approaches unity. If we assume in our model that in the limit there are very many species but that the kinetic energy term (which contains the rest masses of the particles) contributes negligibly, then the energy

(5.27), simplifies to

$$E = \tfrac{1}{2}\rho \overline{V}, \tag{6.9}$$

where

$$\overline{V} \equiv \int \mathrm{d}\tau \tfrac{1}{2}(v(\text{even})+v(\text{odd})). \tag{6.10}$$

This follows because, with many different species present, most of the contribution to the sum arises from unlike particles. We have also used (5.32). In this expression we have let $d = 0$ (correlations are unimportant here at high density) and kept only the first term of (5.32), which is probably not a bad approximation if k_{Fav} is sufficiently great. The $\overline{V}$ for each of our models is shown in table 1. Using (6.5) and (6.7) we find that for large ρ

$$P = \tfrac{1}{2}\rho^2 \overline{V} = \rho E = \varepsilon, \tag{6.11}$$

and P/ε thus a proaches unity in each of our models under these assumptions. As we have remarked the function $P(\varepsilon)$ is the input to calculations of stellar structure, so that the curves shown in fig. 8 are more directly indicative of the sensitivity of neutron star structure to the potential model than the E versus ρ curves. Curves (a), (b) and (c), corresponding respectively to models I, III (hyperonic matter) and V (neutron matter) will yield larger maximum mass stars than Pandharipande's neutron matter equation of state, which is dashed. Curve (d), corresponding to model V, hyperonic matter, may yield a somewhat smaller maximum mass star.

7. Discussion and conclusions

In the calculations described here we have assumed that all particles and resonances are valid as possible constituents of hyperon matter. With this assumption, taking the masses from laboratory experiment and interaction according to the prescription of sect. 4, we found that the light hyperons, $m \leqq 1250$ MeV, may in fact appear at densities of relevance to neutron stars, $\rho \lessapprox 2\ \mathrm{fm}^{-3}$, depending upon the model. Calculations performed but not described here show that at the higher densities heavier hyperons ($m \geqq 1250$ MeV) will appear due to the reduced-mass dependence of the effective interaction.

7.1. SHOULD THE Δ PARTICLE BE INCLUDED?

It is rather clear that the strange particles, Λ and Σ, should be considered as separate particles for purposes of statistical mechanics: their decay is slow, even in the free state, and they have different quantum numbers from the nucleons. However, objections have been raised against the usual assumption that the observed resonances, especially the Δ, may be treated as stable, elementary particles in matter. The case in favor of doing so is the following. In order for any of the resonances to decay, say, $\Delta^- \to \mathrm{n}+\pi^- \to \mathrm{n}+\mathrm{e}^-+\nu$'s, the electron and neutron would have to go to the top of their respective Fermi seas. Thus, remembering the relation between the μ's, the decay is energetically forbidden, unless the Δ itself comes from the top of its Fermi sea, and even then the energy release is zero. Therefore the resonance particle is actually stable in dense matter.

Objections nevertheless can be raised. Kerman has objected that the Δ is a large structure, and there is no room for it in dense nuclear matter. However, Dashen and Rajaraman have recently examined the rate of change of the phase shift in πN scattering, and have concluded that the Δ is a small structure, probably not larger than a nucleon. Rajaraman in turn objected to our assigning the Δ^- a Fermi sea of its own. He says that the Δ is really a neutron and a pion, according to the theory of Chew and Low [47]) and therefore the Δ^- Fermi sea must in some sense be superposed on the neutron sea. This objection is somewhat related to the question of pion condensation which will be treated below.

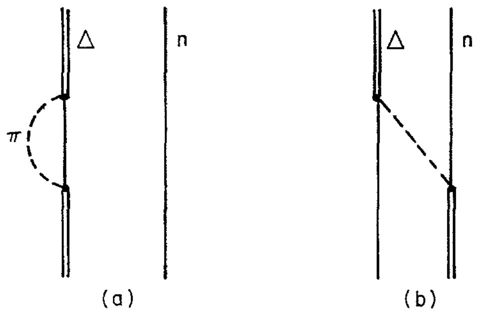

Fig. 9. Feynman diagrams for the 3-3 resonance interacting with pion and nucleon. Diagram (a) is the exchange of diagram (b), and according to Goldstone's theorem if one is included in the calculation, then so must the other.

A more detailed objection has been raised by Sawyer [48]). While the agrees that Δ should exist in dense matter, the argues that a larger mass should be used for them than the observed laboratory mass. This is due to a change of self-energy of the Δ: the Δ^- can transform into $N+\pi^-$ and this transformation possibility gives rise to the self-energy of the Δ. But in dense matter, as already mentioned, many of the neutron states are occupied and the decay into these is forbidden by the Pauli principle. These $n+\pi^-$ states should then also be excluded when calculating the self-energy. Using a model, he calculates the change of the self-energy, and finds this to be 100 to 200 MeV. This will greatly diminish the concentration of Δ at the densities which occur in neutron stars. It will also answer Rajaraman's objections because the neutron and Δ Fermi seas no longer overlap.

The procedure of Sawyer is in apparent contradiction with a well known theorem of many-body theory which says that the Pauli exclusion principle may be ignored in intermediate states [49]) provided that all exchange processes are included. The contradiction arises because we have included a Δ^--neutron potential in the calculation and this may be regarded as having a Δ^-n exchange force as one of its components.

Consider fig. 9a which shows a Δ^- decaying virtually into a π^- and a neutron, in the presence of another neutron in the same state. This graph is identically canceled by fig. 9b, which we may regard as part of the Δ^--neutron exchange potential. Many-body theory tells us that we must include both graphs or neither of them, the net effect being the same (and a consequence of the Pauli exclusion principle). If we regard the process of fig. 9b as being included in the potential and fig. 9a included by virtue of our use of the laboratory Δ-mass, the procedure we have adopted would appear consistent and Sawyer's procedure in contradiction with many-body theory. However, it may make more sense to adopt Sawyer's procedure and regard our Δ^-n potential as *not* including the exchange of fig. 9b. The reason is that the intermediate state may occur with the same energy as the initial state; this means that the "potential" corresponding to this process has some rather strange properties, namely having an oscillatory behavior at large r. It also makes more sense to do the correction to the Δ^- propagator because such calculation will also modify the width of the Δ^-, giving the infinite lifetime one expects.

In summary, we agree in principle with the calculational procedure adopted with Sawyer [48]. Nevertheless, in actual calculations, we have taken the Δ to have its observed laboratory mass. Inasmuch as we guess a value for the Δ-nucleon interaction, it may be academic to worry whether to use the Sawyer mass or the observed mass. A good reason for including the Δ is that there are no pions in the calculations reported here, and that the Δ is an approximate and not too unreasonable procedure for including the attractive π-nucleon p-wave interaction. In any case, we have shown in sect. 6 that the inclusion of hyperons Λ, Σ and Δ makes little difference in the equation of state. A change of the mass of the Δ alone makes even less difference, and therefore the arguments of principle which we have just discussed will have hardly any influence on stellar structure.

7.2. PION CONDENSATION

If we remove the Δ from nuclear matter, this is added reason for considering the pion. Sawyer and Scalapino [11] and Migdal [50] have independently pointed out the possibility that there could be a Bose condensation of pions in dense nuclear matter. Sawyer has emphasized π^- mesons and envisaged a dissociation

$$\mathrm{N} \rightleftarrows \mathrm{P} + \pi^-, \tag{7.1}$$

while Migdal has concentrated on π^0. Without correction terms, both authors predict that the condensate may appear at or slightly above normal nuclear density, but corrections will shift this to higher density.

Barshay, Vagradov and Brown [51] have discussed Migdal's suggestion, and have found that condensation will not occur in normal nuclear matter, a result which is strongly confirmed by direct evidence from ordinary nuclei [52]. But, according to these authors, condensation will occur in neutron matter at density $\rho > 0.38\ \mathrm{fm}^{-3}$, and will then lead to an energy decrease of about 0.7 MeV per nucleon. They have included a correction due to pion-pion interaction which raises the critical density. A similar correction was also included by Sawyer and Scalapino [11]) with a similar result.

None of these authors, however, has included the effect of the condensation on the nucleon-nucleon interaction. If we take, e.g., the process (7.1), and if the pions have momentum k, then we have the correspondence

$$\text{neutron momentum} = \boldsymbol{p}, \qquad \text{proton momentum} = \boldsymbol{p}-\boldsymbol{k}. \tag{7.2}$$

Thus, neutrons and protons will occupy separate Fermi spheres, shifted by k relative to each other. This will increase the mean square relative momentum of protons versus neutrons by k^2. But we have seen in sect. 5 and table 4 that the potential energy increases strongly with relative momentum. We have calculated this effect for the Sawyer-Scalapino case, and have found that it prevents condensation in this case altogether.

However, Baym [53]) has shown that previous calculations of pion condensation are incorrect; he has proved certain conservation laws which must be fulfilled. In comprehensive paper, Baym and Flowers [54]) have discussed various models of pion-nucleon forces, and have found that without nucleon-nucleon forces substantial condensation will indeed occur, but that repulsive nucleon-nucleon forces may prevent it. They conclude that the result is most sensitive to the nucleon-nucleon forces.

Thus the status of pion condensation is unclear. However, in all those calculations where we found condensation of the Sawyer type, the effect on the energy of the system, and hence on the pressure, was very small, usually a few percent. This is in accord with our results on hyperon matter: even large changes in "chemical" composition of the dense matter have very little effect on the equation of state. To get this result for pion condensation it is essential to include the nucleon-nucleon interaction, both the k-independent and k-dependent part; if this is omitted, as in the paper by Sawyer and Scalapino [11]), large reduction of pressure in certain density ranges can result.

7.3. INTERACTION

Perhaps the largest uncertainty in our calculation of the equation of state is the interaction. We have calculated what we consider to be a set of realistic nucleon-nucleon interactions, ranging from relatively weak to relatively strong cores. However, the hyperon interactions are not at all well determined experimentally, and we have used a criterion based only on the evidence that these interactions do not differ by much from the nucleon-nucleon interaction. We have found that under this assumption the equation of state is relatively insensitive to whether or not hyperons are included. If we were to change the balance of attraction and repulsion in the hyperon interaction, we expect that more appreciable changes in the equation of state would result. One could attempt to use theoretical models of the hyperon interactions as an alternative to the arbitrary procedure we have adopted. For example, SU(3) symmetry relates the hyperon to nucleon-nucleon interactions [55]. Such an attempt is premature, however, before a boson exchange theory is verified in the case of the baryons for which we have the best knowledge, the nucleons.

Although a completely satisfactory theory of baryon-baryon interactions does not exist it is interesting to note in passing that one consequence of SU(3) is that the repulsion will look much the same for all members of the SU(3) octet. However, it was noticed in ref. [55] that some cores, in particular the $I = \frac{3}{2}$ ΣN core, are not repulsive. It appears, from scanty experimental data [40], that if there is a strong attraction in this state it is not sufficient to form a bound state. However, these data do not exclude the possibility of a very weak core, or no core.

An even more important change in our results may be caused by the work of Green and Haapakoski [55a] (GH) on the intermediate-range, strong attraction between nucleons, which is usually ascribed to the exchange of two pions. As is well known, to explain the strong attraction in the ^{1}S state, it is necessary to enhance the two-pion force. This is usually done by assuming an attraction between the pions, possibly a σ-meson. GH assume instead that the first pion transforms one of the nucleons into a Δ. They show that this gives just as good a fit to the observed ^{1}S phase shifts as the σ-meson. In fact, it has two great advantages: (i) it moves the minimum in the curve of energy versus density of nuclear matter to lower density, improving the agreement with experiment, (ii) it reduces the required coefficient in the short-range repulsion. Both of these effects are due to the fact that GH's two-pion interaction itself gives some saturation at high relative momenta of the nucleons. Effect (ii) makes it possible to reduce $g_\omega^2/\hbar c$ to 11 (for GH potential 3 which gives the best fit to observed ^{1}S phase shifts at high energy). Then the equation of state of neutron (or hyperon) matter will be made stiffer at moderate densities (0.1 to 0.6 particles $\cdot$ fm^{-3}, see GH table 2) and less stiff at high density.

7.4. SPATIAL EXTENSION

Another consideration at high density is whether the results change if the nucleon is extended spatially, i.e., is not a point particle. This possibility was already discussed in subsect. 3.7, with these results: (a) Very unreasonable numbers have to be assumed for the mass of the meson which is responsible for the interaction, and for its coupling constant, if one wants to reproduce the *observed* repulsive core. (b) In spite of this, the high-density behavior of the equation of state is not changed substantially.

7.5. RELATIVITY

Up to now we have not considered the role of relativity in calculating the equation of state. The theory of special and general relativity are both relevant to the problem. Special relativity enters because the constituents of matter are moving with velocities on the order of $v = \frac{1}{2}c$ and because of the high pressure and energy density. General relativity enters because of the large gravatational fields which are present and create the high pressures.

General relativity may be neglected for the purposes of calculating the equation of state, as long as it is understood that our equation of state is referred to a local inertial frame of reference [56]). General relativity, however, may not be, and is not, ignored in calculating the star properties. This amounts to about a 10 % correction to Newtonian values.

The special relativity correction to the kinetic energy, at a momentum $p = \frac{1}{2}Mc$, amounts to about 7 MeV. The average correction for a Fermi sea of $p_F = \frac{1}{2}Mc$ is only 3 MeV. This is very small compared to the *potential* energy term proportional to k_F^2, see table 6b. We have therefore neglected this relativity correction.

On the other hand, the large potential energy per particle, at high density amounting to the mass of the baryons, may cause some concern that special relativity should be included in the equations of motion. However, we observe that as long as the potential enters in the Klein-Gordon equation as

$$(-\nabla^2+\mu^2)\psi = (E-V)^2\psi, \tag{7.4}$$

the addition of a large constant term to V, and a corresponding constant to E, does not hurt us in going to the non-relativistic limit, the Schroedinger equation.

One rather disturbing feature of our equations of state is that they predict at sufficiently high density that

$$a^2 = \mathrm{d}p/\mathrm{d}\varepsilon > 1, \qquad p/\varepsilon > 1, \tag{7.5}$$

The first of these expressions gives the sound velocity a. It was recognized by Zel'dovich in 1961 [ref. [35])] that such problems may be encountered in high density matter calculations, and several papers have been devoted to trying to understand what they mean. Ruderman has considered this problem in the context of both classical [57, 58]) and quantum [59]) theories. He concludes that in quantum mechanics such behavior is prevented by the formation of baryon-antibaryon pairs (which reduces the energy per net baryon present). Aichelburg, Ecker and Sexl [60]) have considered the quantum mechanical problem and conclude that dynaical reasons, specifically radiation reaction, prevent the velocity of sound from ever exceeding the velocity of light. We thus believe that the results $a/c > 1$ from our calculations at very high density are spurious. A more careful calculation would presumably show the formation of baryon-antibaryon pairs. In any case, at high densities a tends to c, and at the densities actually occurring in neutron stars, the trouble of $a > c$ does not arise.

We believe that the hyperonic matter problem is essentially non-relativistic. One could easily include some of the effects, such as the relativistic form of kinetic energy $(p^2+m^2)^{\frac{1}{2}}$ instead of $p^2/2m+m$. This would have some effect on the concentrations, but very little effect on the energies. Relativistic corrections to eq. (5.7) should also then be systematically considered.

7.6. SUMMARY

We believe that the results of the calculations reported here are the best that can be achieved with our present understanding of baryon interactions. We believe that our use of Pandharipande's VC theory is valid to an accuracy of 5 %, relativity included. The greatest uncertainty still exists in the baryon interactions, although it would require a rather radical departure from current knowledge to obtain an equation of state outside the limits defined by our models I–V. We have found a moderate sensitivity of the equation of state to the assumed repulsion, the variation being roughly 30 % in $\varepsilon(\rho)$, at $\rho = 2\text{fm}^{-3}$.

The application to neutron stars will be reported in a forthcoming paper in Astrophysical Journal. The maximum mass of a neutron star is about two solar masses.

The authors wish to express their appreciation to R. V. Reid for his extensive help in calculating potentials used for models II and III, to D. Sprung for calculating nuclear matter properties for model II and to V. R. Pandharipande for discussions of the material.

Appendix A

UNITARY TRANSFORMATION

The Lagrangian density of a free, neutral vector meson field is

$$L = -\tfrac{1}{2}\sum_{\mu<\nu}\left(\frac{\partial\phi_\nu}{\partial x_\mu} - \frac{\partial\phi_\mu}{\partial x_\nu}\right)^2 - \tfrac{1}{2}\mu^2 \sum \phi_\nu^2. \tag{A.1}$$

This corresponds to Wentzel's [36]) eq. (12.5), except that ϕ is here real, and a factor $\frac{1}{2}$ has been inserted for this reason. (Wentzel's book will be quoted as W in the following.) The $\phi_\nu (\nu = 1, 2, 3, 4)$ are the components of the meson field. The Hamiltonian is formed by setting

$$\mathscr{H}_0 = \sum_\nu \pi_\nu \dot{\phi}_\nu - L - ic\sum_j \frac{\partial}{\partial x_j}(\pi_j \phi_4), \tag{A.2}$$

where j goes from 1 to 3; this corresponds to Wentzel's equations above W(12.11). As usual, the π_ν are the derivatives of L with respect to $\dot{\phi}_\nu$. The $\dot{\phi}_j$ must be eliminated from (A.2), using the relations (cf. W(12.10))

$$\dot{\phi}_j = ic\frac{\partial\phi_4}{\partial x_j} + c^2 \pi_j. \tag{A.3}$$

Just as in electrodynamics, we have (W(12.9))

$$\pi_4 \equiv 0. \tag{A.4}$$

Therefore, ϕ_4 must also be eliminated. This can be done, using the relation analogous to W(12.12),

$$\mu^2\phi_4 = ic\sum \partial\pi_j/\partial x_j. \tag{A.5}$$

Then, after some algebra, we get the analog of W(12.13),

$$\mathscr{H}_0 = \tfrac{1}{2}c^2\sum \pi_j^2+\tfrac{1}{2}\sum_{i<j} f_{ij}^2+\tfrac{1}{2}\mu^2\sum \phi_j^2+\frac{1}{2}\frac{c^2}{\mu^2}\left(\sum\frac{\partial\pi_j}{\partial x_j}\right)^2. \tag{A.6}$$

This can conveniently be written in terms of the vectors $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ and $\boldsymbol{\pi}$, thus:

$$\mathscr{H}_0 = \tfrac{1}{2}c^2\boldsymbol{\pi}^2+\tfrac{1}{2}(\nabla\times\boldsymbol{\phi})^2+\tfrac{1}{2}\mu^2\boldsymbol{\phi}^2+\tfrac{1}{2}c^2\mu^{-2}(\nabla\cdot\boldsymbol{\pi})^2. \tag{A.7}$$

This differs from Wentzel only by the factor $\frac{1}{2}$ and by the fact that $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$ are real.

Now we add to the Lagrangian the interaction with nucleons, see Wentzel sect. 14, *viz.*

$$L' + \sum\phi_\nu\eta_\nu, \tag{A.8}$$

$$\eta_4 = i\eta_0 = i\sum_n g_n\delta(\boldsymbol{r}-\boldsymbol{r}_n), \tag{A.9}$$

$$\eta_j = g_n v_{nj}\delta(\boldsymbol{r}-\boldsymbol{r}_n), \tag{A.10}$$

with v_{nj} the jth component of the velocity of nucleon n, and g_n the coupling constant which, in simple cases, is the same for all nucleons. The Hamiltonian $\mathscr{H}$ is changed by adding to (A.2) the term $-L'$. But (A.5) is changed (cf. W(14.1), (12.8)) into

$$\mu^2\phi_4 = ic\sum\frac{\partial\pi_j}{\partial x_j}+\eta_4. \tag{A.11}$$

Collecting all terms with ϕ_4, and eliminating ϕ_4 by (A.11), the Hamiltonian density becomes

$$\mathscr{H} = \mathscr{H}_0+c\mu^{-2}\eta_0\nabla\cdot\boldsymbol{\pi}+\tfrac{1}{2}\mu^{-2}\eta_0^2-\sum\phi_j\eta_j \equiv \mathscr{H}_0+\mathscr{H}_1, \tag{A.12}$$

where $\mathscr{H}_0$ is given by (A.7)

We now expand $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$ in plane waves, cf. W(5.1), (12.31):

$$\boldsymbol{\phi} = \Omega^{-\frac{1}{2}}\sum \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{q}_k, \tag{A.13a}$$

$$\boldsymbol{\pi} = \Omega^{-\frac{1}{2}}\sum \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{p}_k, \tag{A.13b}$$

$$\boldsymbol{q}_k = \sum_r \boldsymbol{e}_{kr}q_{kr}, \qquad \boldsymbol{p}_k = \sum \boldsymbol{e}_{kr}p_{kr}, \tag{A.14}$$

where the q_{kr}, p_{kr} are quantized wave amplitudes, and the $\boldsymbol{e}_{kr}$ are unit vectors; $r = 1$, 2, 3, with

$$\boldsymbol{e}_{k1} = \boldsymbol{k}/k, \tag{A.14a}$$

in the direction of $\boldsymbol{k}$, and the other two perpendicular. Since $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$ are real, we have

$$\boldsymbol{q}_{-k} = \boldsymbol{q}_k^*, \qquad \boldsymbol{p}_{-k} = \boldsymbol{p}_k^*, \tag{A.15a}$$

and since $e_{-k1} = -e_{k1}$,

$$q_{-k1} = -q^*_{k1}, \qquad p_{-k1} = -p^*_{k1}. \tag{A.15b}$$

The minus sign here is important.

The free Hamiltonian (A.7) may be split into a longitudinal and a transverse part. In analogy with W(12.33), we have for the volume integral of $\mathscr{H}_0$:

$$\mathscr{H}_{0,\,\text{long}} = -\tfrac{1}{2}\mu^2 \sum_k q_{k1}\, q_{-k1} - \tfrac{1}{2}c^2\mu^{-2} \sum p_{k1}\, p_{-k1}\, \omega^2, \tag{A.16}$$

where $\omega^2 = k^2+\mu^2$ and a corresponding expression for the transverse part. Due to (A.15b), (A.16) is positive definite. The only term in (A.12) involving π is

$$\nabla \cdot \boldsymbol{\pi} = -i\Omega^{-\frac{1}{2}} \sum |k| p_{k1}\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.17}$$

Contrary to appearances, this is real, because if we combine the terms $\boldsymbol{k}$ and $-\boldsymbol{k}$, we have

$$|k|(p_{k1}\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + p_{-k1}\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}), \tag{A.17a}$$

which is pure imaginary, using (A.15b).

Similar to W,pp.99,100, we now perform a unitary transformation on H. We replace H by

$$\begin{aligned} \tilde{H} &= \mathrm{e}^{iS} H\, \mathrm{e}^{-iS} \\ &= H + i[S, H] - \tfrac{1}{2}[S, [S, H]] + \ldots . \end{aligned} \tag{A.18}$$

If S is hermitian, the transformation is unitary. We wish the term $[S, H_0]$ to cancel the term with $\nabla \cdot \boldsymbol{\pi}$ in (A.12). Since (A.17) is linear in p_{k1}, and H_0 is quadratic, this can be accomplished by making S linear in q_{k1}. Then S commutes with $H_{0,\,\text{trans}}$ and with the q-term in (A.16). The desired S is

$$S = \frac{i}{\hbar c}\, \Omega^{-\frac{1}{2}} \sum_k \frac{|k|}{\omega^2}\, q_{k1} \sum_n g_n\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}_n}. \tag{A.19}$$

This is again real (hermitian) because of (A.15b). The commutator with H_0 is easily calculated to be

$$[S, H_{0,\,\text{long}}] = c\mu^{-2}\Omega^{-\frac{1}{2}} \sum_k |k| p_{k1} \sum g_n\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}_n}. \tag{A.20}$$

The volume integral of the $\nabla \cdot \boldsymbol{\pi}$ term in (A.12) is

$$H_1 \equiv c\mu^{-2} \int \eta_0 \nabla \cdot \boldsymbol{\pi}\, \mathrm{d}\tau = -ic\mu^{-2}\Omega^{-\frac{1}{2}} \sum |k| p_{k1} \sum g_n\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}_n}. \tag{A.21}$$

This exactly cancels $i[S, H_0]$, as desired.

The only term in $\mathscr{H}_1$ which fails to commute with S is again H_1; we have

$$\begin{aligned} i[S, H_1] &= [S, [S, H_0]] \\ &= \frac{-1}{\mu^2\Omega} \sum_k \frac{k^2}{\omega^2} \sum_m g_m\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}_m} \sum_n g_n\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}_n}. \end{aligned} \tag{A.22}$$

According to (A.18), one-half of this result will enter $\tilde{H}$. Eq. (A.22) does not contain the operators p_{k1} any more, so *any higher commutator brackets are zero,*

$$i[S, [S, H_1]] = [S, [S, [S, H_0]]] = 0, \tag{A.23}$$

etc. The series of commutator brackets in (A.18) therefore breaks off after the second term, and one-half of (A.22) is the *exact* replacement of H_1. Our result is the same as Wentzel's but he had considered it as valid only to second order in g. That the result is exact is well known in electrodynamics.

We now expand the η_0 term in (A.12) in plane waves, using

$$\delta(x-x_n) = \Omega^{-1}\sum_k e^{ik\cdot(x-x_n)},$$

$$\int \eta_0^2 \mathrm{d}\tau = \Omega^{-1}\sum_m\sum_n g_m g_n \sum_k e^{ik\cdot(x_m-x_n)}. \tag{A.24}$$

We multiply this by $1/2\mu^2$, and add one-half of (A.22). Rembering $\omega^2-k^2 = \mu^2$, the result is

$$\int \tilde{\mathscr{H}}\,\mathrm{d}\tau = \int (\mathscr{H}_0 - \sum \eta_i\phi_i)\mathrm{d}\tau + \frac{1}{2\Omega}\sum\sum g_m g_n \sum_k \frac{1}{\omega^2} e^{ik\cdot x_{mn}}. \tag{A.25}$$

Carrying out the summation over k in the usual way, and omitting the self-interaction of the nucleons,

$$\int \tilde{\mathscr{H}}\,\mathrm{d}\tau = \int (\mathscr{H}_0 - \boldsymbol{\eta}\cdot\boldsymbol{\phi})\mathrm{d}\tau + \frac{1}{8\pi}\sum_{m\neq n}\frac{g_m g_n}{r}e^{-\mu r}. \tag{A.26}$$

This proves eq. (3.6) in which we have put the coupling constants of all nucleons equal to g'. The treatment of transverse waves is elementary, subsect. 3.4.

Appendix B

CHOICE OF INTERACTION FOR MODEL I

The following prescription for interaction was chosen for model I. For even states $l \geqq 2$, we took the $^1D_2$ potential of Reid. This is correct for $l = 2$ strictly only when $S = 0$. However, for $S = 1$ it is also reasonable. If one refers to table 3 of ref. [41]) one sees that in nuclear matter, over the range $0.7 \leqq k_F \leqq 1.4$, the sum of the $^3D_1$, $^3D_2$ and $^3D_3$ state contribution to the energy per particle is very nearly equal to the $^1D_2$ contribution. For the higher even partial waves the use of the $^1D_2$ is only a reasonable approximation.

For $l = 0$ we took the ^1D Reid potential as the triplet state and the $^1S_0$ potential of Reid as the singlet state potential. Both have the same core strength and range. The $^1D_2$ potential is shown by later calculations to be very nearly equal to the central potential in the ^3S-^3D states.

For the odd states we insist that

$$\tfrac{1}{10}(5\,^3\bar{P}_2 + 3\,^3P_1 + {}^3P_0 + {}^1P) = {}^1S_0, \tag{B.1}$$

for the coefficient of the core terms. The term $^3\overline{\mathrm{P}}_2$

$$^3\overline{\mathrm{P}}_2 = V_c + V_{LS} - \tfrac{2}{5}V_{\mathrm{T}} \tag{B.2}$$

is the projection of the $^3\mathrm{P}_2$-$^3\mathrm{F}_2$ potential onto the $^3\mathrm{P}_2$ state.

We can not use the Reid P-state potentials because we want to satisfy condition (B.1). Model I therefore required modifying the Reid potential, as follows:

(a) $^3\mathrm{P}_0$: No modification.

(b) $^3\mathrm{P}_2$: We readjusted the central potential by making the repulsion have the range of the $^1\mathrm{S}$ repulsion of Reid (e^{-7x}). Its coefficient was chosen (in conjunction with $^3\mathrm{P}_1$, $^3\mathrm{P}_0$ and $^1\mathrm{P}$) so that (B.1) was satisfied. The attraction was then adjusted by fitting the phase shift for $k = 1.4\ \mathrm{fm}^{-1}$ ($E_{\mathrm{lab}} = 144$ MeV), which should be an important momentum component in nuclear matter. The phase shift was calculated in the Born approximation by first determining the cutoff distance d for which

$$\frac{-2\mu k}{\hbar^2}\int_d^\infty r^2\mathrm{d}r\, j_1^2(kr)v_{\mathrm{R}}(^3\mathrm{P}_2) = \delta(^3\mathrm{P}_2\,\mathrm{obs}),$$

where $v_{\mathrm{R}}(^3\mathrm{P}_2)$ is the "effective" Reid potential (B.2). Then, with the so-determined d we adjusted the attraction so that

$$\int_d^\infty r^2\mathrm{d}r\, j_1^2(kr)(v_{\mathrm{R}}(^3\mathrm{P}_2) - v_{\mathrm{BJ}}(^3\mathrm{P}_2)) = 0,$$

where $v_{\mathrm{BJ}}(^3\mathrm{P}_2)$ is the modified potential.

(c) $^1\mathrm{P}$: We fix the core consistent with (B.1). The attraction was put in terms e^{-2x}/x and e^{-4x}/x. The coefficients were fixed by insisting that $v_{\mathrm{R}}(^1\mathrm{P}) = v_{\mathrm{BJ}}(^1\mathrm{P})$ at the two values of r where $v_{\mathrm{R}} = 0$ and $v_{\mathrm{R}} = 166$ MeV.

(d) $^3\mathrm{P}_1$: We could not put the e^{-7x}/x core in $^3\mathrm{P}_1$ by hand and thereby obtain reasonable looking coefficients. We thus chose a e^{-4x}/x core and adjusted the coefficients of the attraction so that $v_{\mathrm{R}} = v_{\mathrm{BJ}}$ at the points r where $v_{\mathrm{R}} = 0$ and $v_{\mathrm{R}} = 166$ MeV. This constituted some improvement over Reid, who put a core of range e^{-3x}/x in the original $^3\mathrm{P}_1$ potential.

The spin-isospin average of the so-determined P-state potential was used in all odd states for all particles.

Appendix C

LOWEST ORDER CV THEORY FOR PARTICLES OF DIFFERENT MASSES

Let the Hamiltonian H be given by

$$H = \sum_i \frac{p_i^2}{2m_i} + m_i + \tfrac{1}{2}\sum_{ij} v_{ij}.$$

We assume that the wave function Ψ is given as

$$\Psi = \mathscr{A} \prod_{m<n} f_{mn} \prod_i \phi_i,$$

$$\phi_i = \frac{1}{\sqrt{\Omega}} e^{i\boldsymbol{k}_i \cdot \boldsymbol{r}} S_i t_i.$$

Here, f_{mn} is the correlation function operator for the pair m, n, $\mathscr{A}$ is the anti-symmetrization operator ($\mathscr{A}^2 = \mathscr{A}$), S_i is the spin and t_i the isospin wave function, and Ω the total volume. Use of the Van Kampen[61]) cluster expansion leads to the following expression for the energy

$$\langle H \rangle \equiv W = \sum_i C_1(i) + \tfrac{1}{2} \sum_{ij} C_2(ij) + \ldots,$$

where i, j range over all species and the states in their respective Fermi seas. Lowest order CV theory tells that we may truncate the expansion with C_2. Explicit expressions for C_1 and C_2 are

$$C_1(i) = \left(\phi_i, \left(m_i - \frac{\hbar^2}{2m_i} \nabla_i^2\right) \phi_i\right),$$

$$C_2(i,j) = \frac{(\mathscr{A}\Phi_{ij}, H_2 \mathscr{A}\Phi_{ij})}{(\mathscr{A}\Phi_{ij}, \mathscr{A}\Phi_{ij})} - C_1(i) - C_1(j),$$

with $\mathscr{A} = \frac{1}{2}(1-P_{ij})$. The P_{ij} exchanges particles i and j, and

$$H_2 = \frac{p_i^2}{2m_i} + \frac{p_j^2}{2m_j} + m_i + m_j + v_{ij},$$

$$\Phi_{ij} = f_{ij} \phi_i \phi_j.$$

We may rewrite the expression $C_2(ij)$ in terms of the relative and c.m. variables. Define

$$M = m_i + m_j, \qquad \boldsymbol{P} = \boldsymbol{p}_i + \boldsymbol{p}_j, \qquad \boldsymbol{R} = \frac{m_i \boldsymbol{r}_j + m_j \boldsymbol{r}_i}{M},$$

$$\mu = \frac{m_i m_j}{M}, \qquad \boldsymbol{p} = \frac{m_i \boldsymbol{p}_j - m_j \boldsymbol{p}_i}{M}, \qquad \boldsymbol{r} = \boldsymbol{r}_j - \boldsymbol{r}_i,$$

so that

$$H_2 = \frac{\hbar^2 K^2}{2M} + \frac{\boldsymbol{p}^2}{2\mu} + v_{ij} + M,$$

$$\Phi_{ij} = \exp(i\boldsymbol{K} \cdot \boldsymbol{R}) \psi_{ij} / \sqrt{\Omega},$$

where

$$\psi_{ij} = f_{ij} \phi_{ij}, \qquad \phi_{ij} \equiv \exp(i\boldsymbol{k}_{ij} \cdot \boldsymbol{r}_{ij}) S_i S_j t_i t_j / \sqrt{\Omega}.$$

Now,

$$C_2(ij) = \frac{\left(\mathscr{A}\psi_{ij}, \left(\frac{\hbar^2}{2M}K^2 + \frac{p^2}{2\mu} + v_{ij}\right)\mathscr{A}\psi_{ij}\right)}{(\mathscr{A}\psi_{ij}, \mathscr{A}\psi_{ij})} - \frac{\hbar^2}{2\mu}k^2 - \frac{\hbar^2}{2M}K^2$$

$$= [(\psi_{ij}, h_2\psi_{ij}) - \text{exchange}],$$

where

$$h_2 \equiv \frac{p^2}{2\mu} - \frac{\hbar^2}{2\mu}k^2 + v_{ij},$$

$$(\mathscr{A}\psi_{ij}, \mathscr{A}\psi_{ij}) = 1 + O\left(\frac{1}{\Omega}\right) \approx 1.$$

Thus,

$$W \approx \sum_i C_i(i) + \tfrac{1}{2}\sum_{ij} [(\psi_{ij}, h_2\psi_{ij}) - \text{exchange}]. \tag{C.1}$$

Lowest order CV theory stipulates that (C.1) gives the energy only if the correlation function f heals smoothly at $r = d$. Thus, for the purposes of minimizing the energy, the potential is replaced by

$$v_{ij} \to v_{ij} - \lambda_{ij},$$

where λ_{ij} is a "Lagrange parameter operator" which is adjusted to impose the boundary condition on f. Taking the variation of (C.1) with this modification leads to

$$\delta \sum (\psi_{ij}, h_2'\psi_{ij}) = 0, \tag{C.2}$$

where $h_2' = h_2 - \lambda_{ij}$. Taking the variation on (C.2), we find

$$(\delta\psi_{ij}, h_2'\psi_{ij}) + (\psi_{ij}, h_2'\delta\psi_{ij}) = 0,$$

or

$$(\delta\psi_{ij}, h_2'\psi_{ij}) + (\delta\psi_{ij}^*, h_2'\psi_{ij}^*) = 0, \tag{C.3}$$

since h_2' is hermetian. The variations in ψ_{ij} and ψ_{ij}^* are independent and arbitrary, so it follows from (C.3) that

$$h_2'\psi_{ij} = 0,$$

or

$$h_2\psi_{ij} = \lambda_{ij}\psi_{ij}$$

is the condition that W be minimum. The Lagrange parameters λ_{ij} are determined by insisting that f heal smoothly to unity at $r = d$. For $r > d$ it follows that $\psi_{ij} = \phi_{ij}$. In summary,

$$\begin{aligned} h_2\psi_{ij} &= \lambda_{ij}\psi_{ij}, \qquad r < d, \\ h_2\psi_{ij} &= v_{ij}\psi_{ij}, \qquad r > d. \end{aligned} \tag{C.4}$$

Thus from eqs (C.1) and (C.4),

$$W = \sum_i C_1(i) + \tfrac{1}{2}\sum_{ij} \{(\psi_{ij}, \lambda_{ij}\psi_{ij}) - \text{exch}\}_{r<d} + \tfrac{1}{2}\sum_{ij} \{(\psi_{ij}, v_{ij}\psi_{ij}) - \text{exch}\}_{r>d}. \tag{C.5}$$

Appendix D

EQUATION OF CHEMICAL EQUILIBRIUM

Assume that the total energy per baryon, E, has been calculated, given some set of concentrations c_i and the electron chemical potential μ_e

$$E = E(c_1, c_2, \ldots, c_n, \mu_e), \tag{D.1}$$

where $c_i \equiv \rho_i/\rho$, where ρ is the number of baryons per unit volume and ρ_i is the number of particles of type i per unit volume. We want to find the condition for E to be a minimum as a function of its arguments shown in (D.1). We have

$$\delta E = \sum_\alpha \mu_\alpha \delta c_\alpha + \mu_e \frac{\partial c_e}{\partial \mu_e} \delta \mu_e, \tag{D.2}$$

where

$$\mu_i \equiv \frac{\partial E}{\partial c_i} \tag{D.3}$$

are the chemical potentials and $\partial c_e/\partial \mu_e$ may be found from

$$c_e = (3\pi^2\rho)^{-1}(\mu_e^2 - m_e^2c^4)^{\frac{3}{2}}/(\hbar c)^3. \tag{D.4}$$

The variations are to be performed so that the system remains electrically neutral,

$$0 = \sum_\alpha \delta c_\alpha q_\alpha + q_e \delta \mu_e \frac{\partial c_e}{\partial \mu_e}, \tag{D.5}$$

where q_α is the charge on baryon α, and so that the total baryon number is unchanged,

$$0 = \sum_\alpha \delta c_\alpha. \tag{D.6}$$

The index α runs over baryons only.

The constraints (D.5) and (D.6) may be imposed by the use of Lagrange multiplies ϕ and λ,

$$0 = \delta E = \sum_\alpha \mu_\alpha \delta c_\alpha + \mu_e \delta \mu_e \frac{\partial c_e}{\partial \mu_e} + \phi \left(\sum_\alpha q_\alpha \delta c_\alpha + q_e \delta \mu_e \frac{\partial c_e}{\partial \mu_e} \right) + \lambda \sum_\alpha \delta c_\alpha. \tag{D.7}$$

This must be true for independent variations, thus

$$\mu_\alpha + \lambda + q_\alpha \phi = 0, \tag{D.8}$$

$$\mu_e - \phi = 0. \tag{D.9}$$

Eq. (D.9) may be used to eliminate ϕ from (D.8). For neutrons, (D.8) reads

$$\mu_n = -\lambda. \tag{D.10}$$

Using these values of λ and ϕ, we have the following $n+1$ equations

$$\mu_\alpha = \mu_n - q_\alpha \mu_e,$$
$$\sum_\alpha q_\alpha c_\alpha = c_e, \tag{D.11}$$
$$\sum_\alpha c_\alpha = 1,$$

(the first equation is trivial for neutrons) in the $n+1$ unknowns $c_1, \ldots, c_n, \mu_e$. The connection between c_e and μ_e is given by (D.4). If in addition μ^- become possible constituents, it is straightforward to show that (D.11) is supplemented by the condition

$$\mu_\mu = \mu_e. \tag{D.12}$$

Eq. (D.11) may be cast into a more familiar form by removing the baryon mass term from E. Define

$$E' = E - \sum_\alpha c_\alpha m_\alpha. \tag{D.13}$$

Now

$$\mu_\alpha = m_\alpha + \frac{\partial E'}{\partial c_\alpha}. \tag{D.14}$$

If we redefine $\mu_\alpha = \partial E'/\partial c_\alpha$ then (D.11) becomes

$$\mu_\alpha + m_\alpha = \mu_n + m_n - q_\alpha \mu_e, \tag{D.15}$$
$$\sum_\alpha q_\alpha c_\alpha = c_e,$$
$$\sum_\alpha c_\alpha = 1.$$

Appendix E

SPIN AVERAGES FOR LIKE AND UNLIKE NUCLEONS AND HYPERONS

In this appendix, we shall derive expressions for the effective interaction averaged over spin, for the cases of like and unlike particles. Our effective interaction may be represented as

$$v(\boldsymbol{r}', \boldsymbol{r}) = \frac{\delta(r-r')}{r^2} \sum_{lm} \tilde{v}_l(r) Y^*_{lm}(\hat{\boldsymbol{r}}') Y_{lm}(\hat{\boldsymbol{r}}), \tag{E.1}$$

where for nucleons

$$\tilde{v}_l = P(T=0)(P(S=0)\tilde{v}_l(S=0\ T=0) + P(S=1)\tilde{v}_l(S=1\ T=0))$$
$$+P(T=1)(P(S=0)\tilde{v}_l(S=0\ T=1) + P(S=1)\tilde{v}_l(S=1\ T=1)), \tag{E.2}$$

and for other pairs

$$\tilde{v}_l = P(l\ \text{even})\tilde{v}_{l\ \text{even}} + P(l\ \text{odd})\tilde{v}_{l\ \text{odd}}, \tag{E.3}$$

where the P are projection operators; $\tilde{v}$ is given by (5.20) in terms of the potential for $r > d$ and the solution of (5.14) for $r < d$. For $r < d$ $\tilde{v}_l$ depends explicitly on the magnitude of the relative momentum of the interacting pair. We assume for simplicity that J has been averaged out.

In terms of the quantum numbers, $\boldsymbol{k}, s, t$ of the particles, the matrix elements which we need in order to evaluate (5.19) are

$$(\boldsymbol{k}'_a s'_a t'_a;\, \boldsymbol{k}'_b s'_b t'_b|\tilde{v}|\boldsymbol{k}_a s_a t_a;\, \boldsymbol{k}_b s_b t_b)$$

$$= (2\pi)^3\delta(\boldsymbol{K}'-\boldsymbol{K})\int \mathrm{d}^3\boldsymbol{r}'\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-i\boldsymbol{k}'\cdot\boldsymbol{r}'}\langle s'_a t'_a;\, s'_b t'_b|\tilde{v}(\boldsymbol{r},\boldsymbol{r}')|s_a t_a;\, s_b t_b\rangle \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{E.4}$$

where $\boldsymbol{K} = \boldsymbol{k}_a+\boldsymbol{k}_b$, $\boldsymbol{k} = (1/M)(m_b\boldsymbol{k}_a - m_a\boldsymbol{k}_b)$, $\boldsymbol{r} = \boldsymbol{r}_a - \boldsymbol{r}_b$ and $M = m_a+m_b$, and similarly for primed quantities. It is to be understood that the only values of $\boldsymbol{k}$ and $\boldsymbol{k}'$ used are $\boldsymbol{k} = \boldsymbol{k}'$ (for direct matrix element) and $\boldsymbol{k} = -\boldsymbol{k}'$ (exchange matrix element). Using (E.1) in (E.4) and also the partial wave expansion of a plane wave,

$$\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi\sum_{lm} i^l j_l(kr) Y_{lm}(\hat{\boldsymbol{k}})Y^*_{lm}(\hat{\boldsymbol{r}}),$$

we find

$$(\boldsymbol{k}'_a s'_a t'_a;\, \boldsymbol{k}'_b s'_b t'_b|\tilde{v}|\boldsymbol{k}_a s_a t_a;\, \boldsymbol{k}_b s_b t_b)$$

$$= (2\pi)^3\delta(\boldsymbol{K}-\boldsymbol{K}')4\pi\sum_l (2l+1)\int_0^\infty r^2\mathrm{d}r\, j_l(k'r)\langle s'_a t'_a;\, s'_b t'_b|\tilde{v}_l|s_a t_a;\, s_b t_b\rangle j_l(kr)P_l(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}').$$

The direct matrix element is that for which $a' = a$ and $b' = b$. This implies $\boldsymbol{k}'_a = \boldsymbol{k}_a$ and $\boldsymbol{k}'_b = \boldsymbol{k}_b$. Since $\boldsymbol{K}' = \boldsymbol{K}$ already, this condition is guaranteed if we set $\boldsymbol{k} = \boldsymbol{k}'$. Noting $P_l(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}) = 1$, we have for the direct matrix element

$$(\boldsymbol{k}_a s_a t_a;\, \boldsymbol{k}_b s_b t_b|\tilde{v}_{\mathrm{D}}|\boldsymbol{k}_a s_a t_a;\, \boldsymbol{k}_b s_b t_b)$$

$$\equiv 4\pi\sum_l (2l+1)\int_0^\infty r^2\mathrm{d}r\, j_l^2(kr)\langle s_a t_a;\, s_b t_b|\tilde{v}_l|s_a t_a;\, s_b t_b\rangle.$$

The exchange matrix element is that for which $a' = b$ and $b' = a$. This is satisfied for the momenta if $\boldsymbol{k}' = -\boldsymbol{k}$ since $\boldsymbol{K}' = \boldsymbol{K}$ already (this follows only if $m_a = m_b$). Noting that $P_l(-1) = (-)^l$, we have for the exchange matrix element

$$(\boldsymbol{k}_a s_a t_a;\, \boldsymbol{k}_b s_b t_b|\tilde{v}_{\mathrm{E}}|\boldsymbol{k}_b s_b t_b;\, \boldsymbol{k}_a s_a t_a)$$

$$\equiv 4\pi\sum_l (2l+1)\int_0^\infty r^2\mathrm{d}r\, j_l^2(kr)\langle s_a t_a;\, s_b t_b|\tilde{v}_l|s_b t_b;\, s_a t_a\rangle(-)^l.$$

Substituting these in (5.19), we find

$$U = \tfrac{1}{2}\Omega\sum_{\alpha\beta}\int_{<k_{\mathrm{F}\alpha}}\frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\int_{<k_{\mathrm{F}\beta}}\frac{\mathrm{d}^3k_\beta}{(2\pi)^3}\sum_s\sum_t 4\pi\sum_l\int_0^\infty r^2\mathrm{d}r\, j_l^2(kr)$$

$$\times[\langle s_\alpha t_\alpha;\, s_\beta t_\beta|\tilde{v}_l|s_\alpha t_\alpha;\, s_\beta t_\beta\rangle - (-)^l\langle s_\alpha t_\alpha;\, s_\beta t_\beta|\tilde{v}_l|s_\beta t_\beta;\, s_\alpha t_\alpha\rangle]. \tag{E.5}$$

Now we want to average over spin. Consider nucleons first; there are four cases: direct and exchange matrix elements for like and unlike particles.

The spin-average for the direct matrix element is

$$\tfrac{1}{4}\sum_{s}\langle s_\alpha t_\alpha;\, s_\beta t_\beta|\tilde{v}_l|s_\alpha t_\alpha;\, s_\beta t_\beta\rangle = \tfrac{1}{4}\langle t_\alpha t_\beta|\tilde{v}_l(S=0)|t_\alpha t_\beta\rangle+\tfrac{3}{4}\langle t_\alpha t_\beta|\tilde{v}_l(S=1)|t_\alpha t_\beta\rangle,$$

while the spin-average for the exchange matrix element is

$$\tfrac{1}{4}\sum_{s}\langle s_\alpha t_\alpha;\, s_\beta t_\beta|\tilde{v}_l|s_\beta t_\beta;\, s_\alpha t_\alpha\rangle = -\tfrac{1}{4}\langle t_\alpha t_\beta|\tilde{v}_l(S=0)|t_\beta t_\alpha\rangle+\tfrac{3}{4}\langle t_\alpha t_\beta|\tilde{v}_l(S=1)|t_\beta t_\alpha\rangle.$$

We also want to express the matrix elements in terms of the total isospin. For like nucleons $t_\alpha = t_\beta$, and

$$\langle t_\alpha t_\alpha|\tilde{v}_l|t_\alpha t_\alpha\rangle = \tilde{v}_l(T=1),$$

while for unlike nucleons the direct matrix element is

$$\langle t_\alpha t_\beta|\tilde{v}_l|t_\alpha t_\beta\rangle = \tfrac{1}{2}\tilde{v}_l(T=0)+\tfrac{1}{2}\tilde{v}_l(T=1).$$

Thus, in terms of the potentials introduced in (E.2) we have for the direct matrix elements:

(a) like nucleons:

$$\tfrac{1}{4}\tilde{v}_l(S=0\;\; T=1)+\tfrac{3}{4}\tilde{v}_l(S=1\;\; T=1); \tag{E.6}$$

(b) unlike nucleons:

$$\tfrac{1}{2}[\tfrac{1}{4}(\tilde{v}_l(S=0\;\; T=0)+3\tilde{v}_l(S=1\;\; T=0) \\ +\tfrac{1}{4}(\tilde{v}_l(S=0\;\; T=1)+3\tilde{v}_l(S=1\;\; T=1))] \tag{E.7}$$

and for the exchange matrix elements

(a) like nucleons:

$$-\tfrac{1}{4}\tilde{v}_l(S=0\;\; T=1)-\tfrac{3}{4}\tilde{v}_l(S=1\;\; T=1); \tag{E.8}$$

(b) unlike nucleons:

$$-\tfrac{1}{2}[\tfrac{1}{4}(\tilde{v}_l(S=0\;\; T=1)+3\tilde{v}_l(S=1\;\; T=1)) \\ +\tfrac{1}{4}(\tilde{v}_l(S=0\;\; T=0)+3\tilde{v}_l(S=1\;\; T=0))]. \tag{E.9}$$

Finally, using (E.6)–(E.8) with (E.5) we find the results recorded in table 3 for nucleons.

For pairs other than nucleons we have the following cases

Direct matrix elements. (i) Like particles. (ii) Unlike particles.

Exchange matrix elements (like hyperons only).

Because of the assumed form (E.3) the results in this case are easier to obtain: there is only spin to consider. We regard each particle as having spin $\frac{1}{2}$ (some particles will therefore be represented by several different species, such as the Δ which has spin $\frac{3}{2}$)

and the potential as being independent of spin. Then for the direct matrix element

$$\tfrac{1}{4}\sum_{s}\langle s_\alpha s_\beta|\tilde{v}_l|s_\alpha s_\beta\rangle = \tilde{v}_l,$$

and for the exchange matrix element

$$\tfrac{1}{4}\sum_{s}\langle s_\alpha s_\beta|\tilde{v}_l|s_\beta s_\alpha\rangle = \tfrac{1}{2}\tilde{v}_l.$$

For unlike hyperons we take only the direct matrix element, giving row 4 of table 3. For like hyperons, direct+exchange for l odd gives row 3, column 2 and direct −exchange for l even gives row 3, column 2.

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