Neutron Star Matter Equation of State

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Abstract

Neutron stars are highly compact objects with masses comparable to that of our Sun but radii of only about 10 km. The structure of neutron stars is encapsulated in the Tolman-Oppenheimer-Volkoff (TOV) equations, which represent the generalization of Newtonian gravity to the domain of general relativity. Remarkably, the only input required to solve the TOV equations is the equation of state of cold, neutron-rich matter in chemical equilibrium. In this contribution we derive analytic expressions for the equation of state of an electrically neutral, relativistic free Fermi gas of neutrons, protons, and electrons in chemical equilibrium. Then, we introduce simple "scaling" concepts to rewrite the TOV equations in a form amenable to standard numerical algorithms. Finally, we highlight the ongoing synergy between astrophysics and nuclear physics that will need to be maintained, and indeed enhanced, to elucidate some of the most fascinating and challenging problems associated with the structure, dynamics, and composition of neutron stars.

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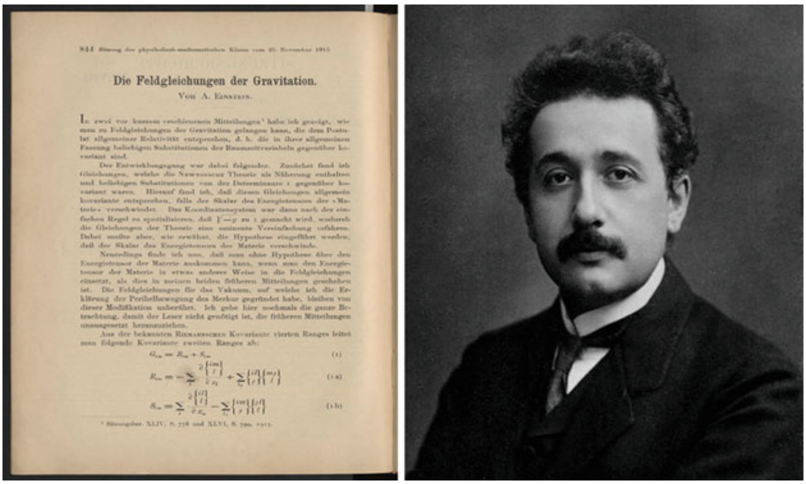
1 Introduction

One century ago, on November 25, 1915, Albert Einstein published his landmark paper on *Die Feldgleichungen der Gravitation* (Einstein 1915); see Fig. 1. Almost two decades later in an experiment with no connection with the new laws of gravitation, Chadwick discovered the neutron (Chadwick 1932). Very soon after Chadwick's announcement, the term *neutron star* appears in writing for the first time in the 1933 proceedings of the American Physical Society by Baade and Zwicky who wrote: *With all reserve we advance the view that supernovae represent the transition from ordinary stars into "neutron stars", which in their final stages consist of extremely closed packed neutrons* (Baade and Zwicky 1934). It appears, however, that a couple of years earlier, Landau speculated on the existence of dense stars that look like giant atomic nuclei (Yakovlev et al. 2013). Ultimately in 1939, it would fall on the able hands of Oppenheimer and Volkoff to perform the first calculation of the structure of neutron stars by employing the full power of Einstein's theory of general relativity (Oppenheimer and Volkoff 1939). Using what it is now commonly referred to as the Tolman-Volkoff-Oppenheimer (TOV) equations (Oppenheimer and Volkoff 1939; Tolman 1939), Oppenheimer and Volkoff demonstrated that a neutron star supported exclusively by the quantum mechanical pressure from its degenerate neutrons will collapse into a black hole once its mass exceeds seven tenths of a solar mass.

It would take almost 30 years after the seminal work by Oppenheimer and Volkoff for a young graduate student by the name of Jocelyn Bell to discover "pulsars" which—after a period of confusion in which they were mistaken as potential beacons from an extraterrestrial civilization—were finally identified as


Die Feldgleichungen der Gravitation.

Von A. Einstein.

Fig. 1 A copy of Einstein's historical article on "The Field Equations of Gravitation" that revolutionized our understanding of the laws of gravitation next to a picture of a young Einstein during his 1905–1915 "decade mirabilis" rapidly rotating neutron stars (Hewish et al. 1968). Since then, the field has evolved enormously and to date a few thousands neutron stars have been observed (Lattimer and Prakash 2004). Recently, the existence of neutron stars with masses as large as two solar masses has been firmly established (Antoniadis et al. 2013; Demorest et al. 2010). This represents a maximum mass limit nearly three times larger than the original prediction by Oppenheimer and Volkoff. Given that the only physics accounted for by Oppenheimer and Volkoff is rooted in the Pauli exclusion principle, the mere existence of massive neutron stars highlights the vital role that nuclear interactions play in explaining the structure and composition of neutron stars.

But how do nuclear interactions leave their imprint on the structure of neutron stars? It is a remarkable fact—and one responsible for creating a unique synergy between astrophysics and nuclear physics—that the only input required to solve the TOV equations is the *equation of state* (EOS) of neutron-rich matter. The EOS encodes a fundamental relation between the pressure P and two other (intensive) thermodynamic quantities, such as the density n and the temperature T. The best known equation of state is that of a classical ideal gas: $P = n k_{\rm B} T$, where $k_{\rm B}$ is Boltzmann's constant. However, unlike a classical ideal gas, neutron stars are highly degenerate objects where quantum effects play a predominant role. That is, the thermodynamic regime of relevance to neutron stars involves high densities and low temperatures, such that the interparticle separation is small relative to the thermal de Broglie wavelength of the particle. These conditions are well satisfied in a neutron star despite that its core temperature of $T \lesssim 10^9\,{\rm K}$ is enormous for normal standards. Yet, this temperature is small relative to the "Fermi temperature" $T_{\rm F}$, a quantity that provides a convenient proxy for the density. In the case of neutron stars, the dimensionless ratio of the physical temperature of the stellar core to its Fermi temperature is very small indeed, i.e., $T/T_{\rm F} \lesssim 10^{-3}$. This suggests an important approximation: *Neutron stars may be effectively treated at zero-temperature systems.*

Although in this chapter we will focus on the equation of state of a uniform system of neutrons, protons, and electrons at zero temperature and in chemical equilibrium, the structure and composition of a neutron star are much more interesting and significantly more complex. Thus, aided by Fig. 2, we now embark on a very brief journey through a neutron star (Piekarewicz 2014). The outermost surface of the neutron star contains a very thin atmosphere of only a few centimeters thick that is believed to be composed of hydrogen, but may also contain heavier elements such as helium and carbon. The electromagnetic radiation that reaches both terrestrial and space-based telescopes is often used to constrain critical parameters of the neutron star. For example, assume blackbody emission from a stellar surface at a temperature T provides a determination of the neutron star radius R through the Stefan-Boltzmann law: $L = 4\pi\sigma R^2 T^4$, where L is the stellar luminosity and σ the Stefan-Boltzmann constant. Just below the atmosphere lies the $\sim$100 m thick envelope that acts as "blanket" between the hot interior and the "cold" surface. Further, below lies the nonuniform crust which is speculated to consist of exotic structures, such as Coulomb crystals of very neutron-rich nuclei as well as nuclear pasta phases. The nonuniform crust sits above a uniform liquid core that consists

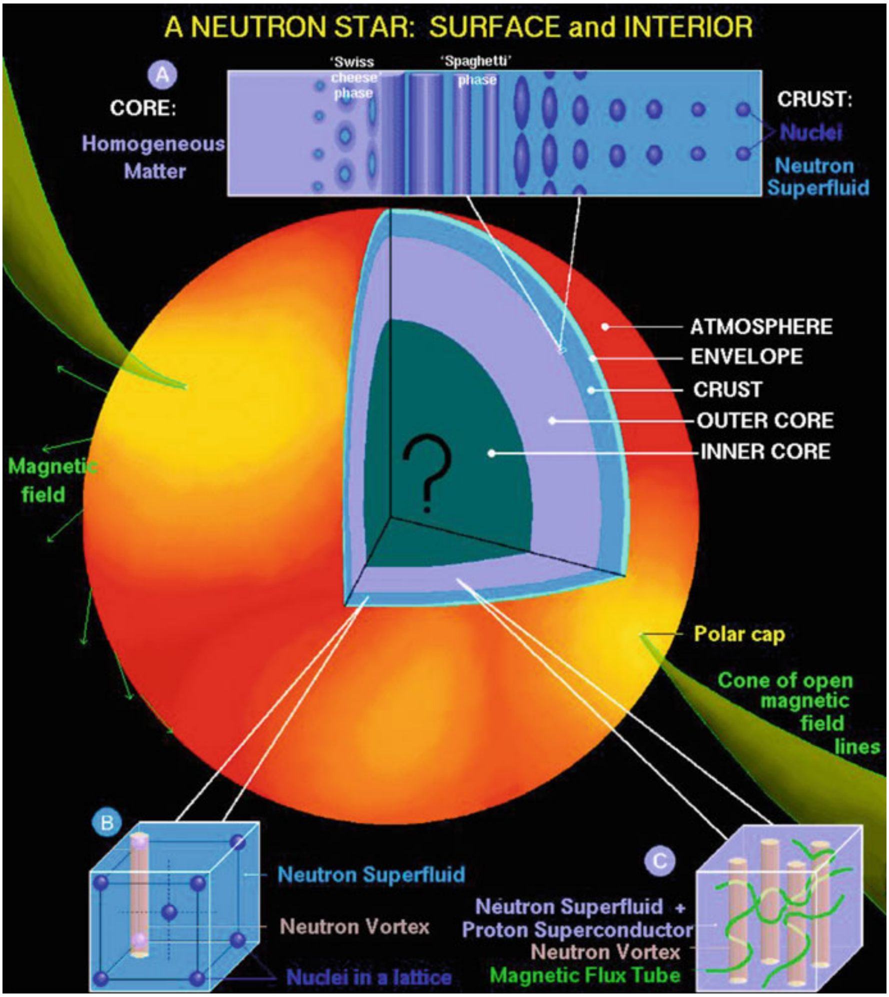


Fig. 2 An accurate rendition of the fascinating structure and exotic phases in the interior of a neutron star—courtesy of Dany Page

of neutrons, protons, and electrons. Finally, there is also the fascinating possibility, marked with a question mark in Fig. 2, of an inner core made of strange quark matter or some other exotic state of matter. However, as alluded earlier in this paragraph, we will limit our discussion to the uniform stellar core that accounts for practically all the mass and for nearly 90 % of the size of a neutron star.

We have organized this chapter as follows. In Sect. 2, we discuss the two main ingredients required to understand the structure of neutron stars: (a) the equation of state of an electrically neutral, relativistic free Fermi gas of neutrons, protons, and electrons in chemical equilibrium and (b) the solution of the TOV equations.

We then proceed in Sect. 3 to present results for the structure of neutron stars using both the simple EOS derived in Sect. 2 and more sophisticated ones whose derivation is beyond the scope of this contribution. Finally, we conclude in Sect. 4 with a summary of our main results and with a brief discussion of some of the most interesting and challenging open questions that remain to date.

2 Formalism

The two topics developed in this section lie at the heart of the physics of neutron stars. The first topic centers around the equation of state of neutron star matter. Here, we start by deriving analytic expressions for the equation of state of a relativistic free Fermi gas of neutrons. This derivation serves as the cornerstone for addressing the equation of state of a multicomponent system consisting of neutrons, protons, and electrons in chemical equilibrium. The second topic introduces the TOV equations and develops a scaling transformation that is vital in treating systems with enormous disparity in scales, such as neutron stars. For example, with masses comparable to that of our Sun but largely supported by the pressure of its neutrons, neutron stars involve a mass disparity of 57 orders of magnitude!

It is important to underscore that the topics addressed in this section should be treated as two independent modules. Given the pedagogical nature of this volume as well as the inherit complexity of the equation of state, we treat neutrons, protons, and electrons as noninteracting Fermi gases with correlations limited to those induced by the Pauli exclusion principle. Yet, fundamental concepts that are critical to our understanding of neutron stars, such as the nuclear symmetry energy and chemical equilibrium, are discussed in sufficient detail. On the other hand, the TOV module is general, at least for the widely used case of spherically symmetric neutrons stars in hydrostatic equilibrium. As developed, all that is required as input for the TOV equations is the equation of state: specifically, a relation between the pressure and the energy density.

2.1 Neutron Star Matter Equation of State

The main objective of this section is to obtain the equation of state of neutron star matter, simulated as an electrically neutral system of neutrons, protons, and electrons in chemical equilibrium. However, for pedagogical reasons we find it convenient to start by developing the EOS of a one-component relativistic Fermi gas (e.g., of neutrons) followed by a discussion of a two-component Fermi gas (of neutrons and protons) where the critical concept of the nuclear symmetry energy is introduced.

2.1.1 A One-Component Relativistic Fermi Gas

Although the pioneering calculation by Oppenheimer and Volkoff represents the first application of general relativity to the structure of neutron stars, the assumption for the underlying EOS was simple, namely, that of a degenerate gas of noninteracting neutrons (Oppenheimer and Volkoff 1939). The zero-temperature EOS of a free Fermi gas of neutrons (or in general any fermion) of mass m satisfying the relativistic dispersion relation $\epsilon(p) = \sqrt{(pc)^2 + (mc^2)^2}$ is given by the following expression for the energy density ($\mathcal{E} \equiv E/V$) in terms of the number density ($n \equiv N/V$):

$$\begin{aligned}\mathcal{E}(n) &= \frac{(mc^2)^4}{(\hbar c)^3}\frac{1}{\pi^2}\int_0^{x_{\rm F}} x^2\sqrt{1+x^2}\,dx \\ &= \mathcal{E}_0\Big[x_{\rm F}y_{\rm F}(x_{\rm F}^2+y_{\rm F}^2) - \ln(x_{\rm F}+y_{\rm F})\Big] \rightarrow nmc^2\begin{cases}1+\dfrac{3}{10}x_{\rm F}^2 & \text{if } x_{\rm F} \ll 1\,, \\ \dfrac{3}{4}x_{\rm F} & \text{if } x_{\rm F} \gg 1\,,\end{cases}\end{aligned} \tag{1}$$

where the dimensionless Fermi momentum and Fermi energy are given by

$$x_{\rm F} \equiv \frac{p_{\rm F}\,c}{mc^2} \quad \text{and} \quad y_{\rm F} \equiv \frac{\varepsilon_{\rm F}}{mc^2} = \sqrt{1+x_{\rm F}^2}\,, \tag{2}$$

with the former related to the number density as follows:

$$n = \frac{k_{\rm F}^3}{3\pi^2} \quad (p_{\rm F} \equiv \hbar k_{\rm F})\,. \tag{3}$$

Although the above expression for the energy density is exact, we find it instructive to provide its leading behavior in both the nonrelativistic ($x_{\rm F} \ll 1$) and ultra-relativistic ($x_{\rm F} \gg 1$) limits. Note that Eq. (1) defines the natural scale for the energy density in terms of the mass of the fermion and fundamental constants:

$$\mathcal{E}_0 = \frac{1}{8\pi^2}\frac{(mc^2)^4}{(\hbar c)^3}. \tag{4}$$

In the zero-temperature limit of interest here, the pressure is obtained from the derivative of the energy density with respective to the density. That is,

$$P = n\frac{\partial\mathcal{E}}{\partial n} - \mathcal{E} \quad \text{or} \quad P + \mathcal{E} = n\varepsilon_{\rm F} = \frac{k_{\rm F}^3\varepsilon_{\rm F}}{3\pi^2}\,. \tag{5}$$

Using the expression obtained in Eq. (1) for the energy density, one obtains

$$P(n) = \mathcal{E}_0\left[\frac{2}{3}x_{\rm F}^3 y_{\rm F} - x_{\rm F}y_{\rm F} + \ln(x_{\rm F}+y_{\rm F})\right] \rightarrow nmc^2\begin{cases}\dfrac{x_{\rm F}^2}{5} & \text{if } x_{\rm F} \ll 1\,, \\ \dfrac{x_{\rm F}}{4} & \text{if } x_{\rm F} \gg 1\,.\end{cases} \tag{6}$$

Again, as in the case of the energy density, we provide expressions for the pressure in both the nonrelativistic and ultra-relativistic limits. This expression is the hallmark of quantum degeneracy pressure. As a consequence of the Pauli exclusion principle, a fermionic system generates a nonzero pressure even at zero temperature. Note that as the system becomes ultra-relativistic, namely, the density increases to a point that the fermion mass becomes negligible, the pressure "loses" one power of the Fermi momentum. This weakening of the pressure has profound consequences, as it is ultimately responsible for the gravitational collapse of white dwarf stars with masses in excess of $M_{\rm Ch} \approx 1.4\,M_{\odot}$: the so-called Chandrasekhar mass (Chandrasekhar 1931). In essence, it is quantum mechanics, through the Pauli exclusion principle, that provides the pressure support against the gravitational collapse of low-mass white dwarf stars—but it is special relativity, through the modification of the nonrelativistic dispersion relation, that ultimately leads to their collapse.

2.1.2 A Two-Component Fermi Gas: The Nuclear Symmetry Energy

Although the most common perception of a neutron star is that of a conglomerate of neutrons, it is clearly energetically advantageous for some of the neutrons on top of the Fermi surface to beta decay into protons, electrons, and antineutrinos. In anticipation to this more general case, we want to compute the energy and pressure of a system of neutrons and protons by assuming that only the total baryon density of the system is conserved; the individual neutron (n_n) and proton (n_p) densities can fluctuate subject to the constraint that $n \equiv n_n + n_p =$ constant. To quantify such a *neutron-proton asymmetry*, we introduce a dimensionless *asymmetry* parameter α that is given by

$$\alpha \equiv \frac{n_n - n_p}{n_n + n_p} \tag{7}$$

In turn, we introduce individual neutron $k_{\rm F}^{(\rm n)}$ and proton $k_{\rm F}^{(\rm p)}$ Fermi momenta through the following definitions:

$$n_n = \frac{\left(k_{\rm F}^{(\rm n)}\right)^3}{3\pi^2} = \left(\frac{1+\alpha}{2}\right) n \equiv \left(\frac{1+\alpha}{2}\right)\left(\frac{2k_{\rm F}^3}{3\pi^2}\right), \tag{8a}$$

$$n_p = \frac{\left(k_{\rm F}^{(\rm p)}\right)^3}{3\pi^2} = \left(\frac{1-\alpha}{2}\right) n \equiv \left(\frac{1-\alpha}{2}\right)\left(\frac{2k_{\rm F}^3}{3\pi^2}\right), \tag{8b}$$

where have defined the Fermi momentum $k_{\rm F}$ as a proxy for the *total* density of the system:

$$n_n + n_p = n = \frac{2k_{\rm F}^3}{3\pi^2}. \tag{9}$$

In the symmetric ($\alpha = 0$) limit with equal densities for neutrons and protons, one obtains $k_{\rm F}^{\rm (n)} = k_{\rm F}^{\rm (p)} = k_{\rm F}$. However, in general the individual Fermi momenta are not equal and are given by

$$k_{\rm F}^{\rm (n)} = (1+\alpha)^{1/3} k_{\rm F}\,, \tag{10a}$$

$$k_{\rm F}^{\rm (p)} = (1-\alpha)^{1/3} k_{\rm F}\,. \tag{10b}$$

We are now in a position to calculate how the energy of the system *increases* from the symmetric limit as the system develops a neutron-proton asymmetry. Note that the energy *must* increase as a consequence of the Pauli exclusion principle. In the symmetric case, the Fermi momentum of both species is equal, but as soon as the system develops a neutron excess, protons with momenta at or *below* the Fermi surface must be transferred *above* the neutron Fermi surface.

For historical reasons, it is customary to address the role of a neutron-proton asymmetry on the equation of state in terms of the energy per nucleon E/A rather than the energy per volume $\mathcal{E}$; of course, both are simply related to each other: $E/A \equiv \mathcal{E}/n$. In this way, the total energy per nucleon at a fixed density n and neutron-proton asymmetry α may be written as follows:

$$\frac{E}{A}(n,\alpha) = \left(\frac{1+\alpha}{2}\right)\frac{E_n}{N}(n_n) + \left(\frac{1-\alpha}{2}\right)\frac{E_p}{Z}(n_p)\,. \tag{11}$$

Note that the total baryon (or nucleon) number is given by $A = N + Z$. This expression indicates how to compute the energy per nucleon as a function of both n and α. Once the total density n and the neutron-proton asymmetry α are specified, both the neutron and proton Fermi momenta are obtained, as per Eqs. (9) and (10). And given that the Fermi momentum is all that is needed to evaluate the energy of a free Fermi gas [see Eq. (1)], the evaluation of E/A can now be completed.

Although strictly correct, the implementation suggested above fails to provide an intuitive picture on the energy cost of inducing a neutron-proton asymmetry. To gain some insights into such a cost, it is instructive to provide an expansion of the energy per nucleon in powers of α. That is,

$$\frac{E}{A}(n,\alpha) = \varepsilon_{\rm SNM}(n) + \alpha^2 S(n) + \alpha^4 S_4(n) + \ldots \tag{12}$$

Note that only even powers of α appear in the expansion since in the absence of nuclear and electromagnetic interactions, it is equally costly to turn protons into neutrons than neutrons into protons; throughout this contribution we neglect the very small neutron-proton mass difference. The first term in the expansion represents the energy per particle of *symmetric* nuclear matter and is identical to the one provided in Eq. (1) (recall the simple relation between the energy per particle and the energy density). The leading-order correction $S(n)$ is known as the nuclear *symmetry energy* and plays a critical role in the structure, dynamics, and composition of neutron stars (Brown 2000; Ducoin et al. 2010; Fattoyev and Piekarewicz 2010; Gandolfi et al. 2014; Hebeler et al. 2013; Horowitz and Piekarewicz 2001a,b, 2002; Lattimer and Prakash 2007; Steiner et al. 2005, 2010); see also Bertulani and Piekarewicz (2012) and references contained therein.

Assuming, as we have done so far, that the only important correlations are those induced by the Pauli exclusion principle, one obtains a remarkably simple expression for the symmetry energy:

$$\frac{S(n)}{mc^2} = \frac{x_{\rm F}^2}{6\,y_{\rm F}} \to \frac{1}{6}\begin{cases} x_{\rm F}^2 & \text{if } x_{\rm F} \ll 1\,, \\ x_{\rm F} & \text{if } x_{\rm F} \gg 1\,. \end{cases} \tag{13}$$

That is, the energy cost of converting protons into neutrons (or vice versa) increases quadratically with the Fermi momentum at low density and eventually becomes linear at very high density. Given that neutron stars are systems with a large neutron excess ($\alpha \simeq 0.8$), it is pertinent to examine the next correction to the energy of asymmetric nuclear matter, namely, $S_4(n)$. This term—indeed all terms—may also be evaluated analytically; we obtain

$$\frac{S_4(n)}{mc^2} = \frac{x_{\rm F}^2}{648 y_{\rm F}^5}\left(3x_{\rm F}^4 + 3x_{\rm F}^2 y_{\rm F}^2 + 4y_{\rm F}^4\right) \to \frac{1}{648}\begin{cases} 4x_{\rm F}^2 & \text{if } x_{\rm F} \ll 1\,, \\ 10x_{\rm F} & \text{if } x_{\rm F} \gg 1\,. \end{cases} \tag{14}$$

That is, S_4 is heavily suppressed relative to the leading symmetry energy term S. Moreover, we have verified that this trend persists to all orders, namely, subsequent higher-order terms in the expansion, $S_6(n)$, $S_8(n)$, ..., are highly suppressed relative to the previous term in the expansion. As a result, we conclude—at least in the case of a free Fermi gas—that the so-called parabolic approximation i.e.,

$$\frac{E}{A}(n,\alpha) \approx \varepsilon_{\rm SNM}(n) + \alpha^2 S(n)\,, \tag{15}$$

provides an excellent approximation to the energy of asymmetric matter. In particular, it is possible to establish the following important connection between the energy of pure neutron matter, the energy of symmetric nuclear matter, and the symmetry energy:

$$\frac{E}{A}(n,\alpha \equiv 1) \equiv \varepsilon_{\rm PNM}(n) \approx \varepsilon_{\rm SNM}(n) + S(n)\,. \tag{16}$$

In this context, the symmetry energy may be regarded as the energy cost required to convert symmetric nuclear matter into pure neutron matter. This intuitive picture remains valid in the case of more realistic models.

2.1.3 A Multicomponent Fermi Gas: Chemical Equilibrium

Although most of the pressure support against the gravitational collapse of a neutron star is provided by the neutrons, the timescale for the formation of a neutron star is long enough for the system to settle into its absolute ground state. Chemical or "beta" equilibrium is established between two weak-interaction processes: neutron beta decay ($n \to p + e^- + \bar{\nu}_e$) and electron capture ($p + e^- \to n + \nu_e$). That is,

$$n \rightleftharpoons p + e^- \,. \tag{17}$$

Note that both the neutrino and the antineutrino are missing from the above weak-interaction reactions. This is because neutrinos and antineutrinos are the main cooling agents of the nascent proto-neutron star, so they escape the star and thus may be effectively regarded as having zero chemical potential. Besides chemical equilibrium, the long-range nature of the Coulomb force dictates that charge neutrality must also be enforced; macroscopic systems with a net electric charge are unstable against fission. This implies equal proton and electron densities (i.e, $n_p = n_e$).

So what is then the equation of state of cold, *fully catalyzed* matter? Adding the electronic contribution to the energy per nucleon given in Eq. (11) results in the following equation:

$$\frac{E}{A}(n,\alpha) = \left(\frac{1+\alpha}{2}\right)\frac{E_n}{N}(n_n) + \left(\frac{1-\alpha}{2}\right)\left(\frac{E_p}{Z}(n_p) + \frac{E_e}{Z}(n_e)\right) . \tag{18}$$

Note that since the system is in β-equilibrium, only the total baryon density is conserved; neutrons can turn into protons (and electrons) and vice versa. This implies that in β-equilibrium, the neutron-proton asymmetry is not fixed but rather emerges from minimizing the energy per particle at every given density. That is, $\alpha(n)$ is obtained by minimizing Eq. (18) with respect to α at fixed n. To carry out the minimization, it is convenient to resort back to the expression for the energy density of a free Fermi gas in terms of the underlying momentum integral; see Eq. (1). That is,

$$\mathcal{E}(n,\alpha) = \frac{(mc^2)^4}{(\hbar c)^3}\frac{1}{\pi^2}\left(\int_0^{x_{\rm F}^{\rm (n)}} x^2\sqrt{1+x^2}\,dx + \int_0^{x_{\rm F}^{\rm (p)}} x^2\sqrt{1+x^2}\,dx + \int_0^{x_{\rm F}^{\rm (e)}} x^3\,dx\right), \tag{19}$$

where we have assumed ultra-relativistic electrons and equal masses for neutrons and protons ($m_n = m_p \equiv m$). Note at the densities of relevance to the neutron star core, treating the electrons in the ultra-relativistic limit constitutes an excellent approximation. The advantage of writing the above expression in integral form is that all the dependence on α is contained in the Fermi momentum of the individual constituents [see Eq. (10)] so the derivative with respect to α is simple. That is,

$$\frac{\partial}{\partial\alpha}\mathcal{E}(n,\alpha) = 0 \implies \mu_n = \mu_p + \mu_e \,. \tag{20}$$

Note that at zero temperature, the chemical potential μ equals the Fermi energy $\varepsilon_{\rm F}$; the equality of the chemical potential is a general result that also holds even at finite temperature. One can gain valuable insights into the composition of a neutron star by examining the consequences of enforcing chemical equilibrium. We obtain the following simple expression:

$$x_{\rm F}^{\rm (p)} = \frac{\left(x_{\rm F}^{\rm (n)}\right)^2}{2y_{\rm F}^{\rm (n)}} \to \frac{1}{2}\begin{cases} \left(x_{\rm F}^{\rm (n)}\right)^2 & \text{if } x_{\rm F}^{\rm (n)} \ll 1\,, \\ x_{\rm F}^{\rm (n)} & \text{if } x_{\rm F}^{\rm (n)} \gg 1\,. \end{cases} \tag{21}$$

This last relation encapsulates the critical role that electrons play in driving hadronic matter to be neutron rich. As indicated earlier, in the absence of electrons (i.e., if charge neutrality is not enforced), then the ground state of the system consists of an equal mixture of neutrons and protons. However, as soon as charge neutrality is demanded, the system becomes very neutron rich. Indeed, one obtains

$$\alpha \geq \frac{7}{9} \iff Y_p \equiv \frac{1-\alpha}{2} \leq \frac{1}{9}, \tag{22}$$

where $Y_p = Z/A$ denotes the proton fraction of the system. That is, the proton fraction attains its largest value in the ultra-relativistic limit when the neutron density is still eight times larger than the proton density.

2.2 The Tolman-Oppenheimer-Volkoff Equations

Given that an equation of state is all that is required to solve the TOV equations, we are now in a position to study the structure and composition of neutron stars. The TOV equations represent the generalization of Newtonian gravity to the domain of general relativity. For static, spherically symmetric stars in hydrostatic equilibrium, the TOV equations may be written as a pair of first-order differential equations. That is,

$$\frac{dP(r)}{dr} = -\frac{G}{c^2}\frac{\Big(\mathcal{E}(r) + P(r)\Big)\Big(M(r) + 4\pi r^3 \frac{P(r)}{c^2}\Big)}{r^2\Big(1 - 2GM(r)/c^2 r\Big)}\,, \tag{23a}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \frac{\mathcal{E}(r)}{c^2}\,, \tag{23b}$$

where $M(r)$, $\mathcal{E}(r)$, and $P(r)$ are the respective profiles for the mass, energy density, and pressure. Given the boundary conditions in terms of a central pressure $P(0) = P_c$ and enclosed mass at the origin $M(0) = 0$, the TOV equations may be solved using any reasonable solver, such as the Runge-Kutta method—yet, even a spreadsheet such as Excel may be used (Jackson et al. 2005). Note that the stellar radius R and mass M are determined from the following two conditions: $P(R) = 0$ and $M = M(R)$. As already mentioned repeatedly, also essential to the solution of the problem is the specification of an equation of state $P = P(\mathcal{E})$, namely, a relation connecting the pressure to the energy density.

With the exception of a few choices for the equation of state (mostly academic in nature), the TOV equations must be solved numerically. This, however, presents some nontrivial challenges associated with the diversity of scales involved in the problem. Indeed, a Fermi gas composed mostly of subatomic scale neutrons, each with a mass of $m = 1.675 \times 10^{-27}$ kg, provides most of the pressure support against the gravitational collapse of neutron stars with masses comparable to that of our own Sun, namely, $M_{\odot} = 1.989 \times 10^{30}$ kg. This represents a mismatch in scales of 57 orders of magnitude! That is, there are approximately 10^{57} neutrons in a neutron star (Avogadro's number $\sim 6{\times}10^{23}$ pales in comparison!). Thus, in order to avoid numerical overflows—and to gain critical insights into the problem—one must first properly *rescale* the TOV equations.

2.2.1 Rescaling the Tolman-Oppenheimer-Volkoff Equations

The rescaling of the TOV equations involves introducing various, as yet undetermined, dimensionful scales that characterize the natural units in the system. That is,

$$r = R_0\, x, \tag{24a}$$

$$M = M_0\, m, \tag{24b}$$

$$P = P_0\, p; \tag{24c}$$

$$\mathcal{E} = \mathcal{E}_0\, \epsilon. \tag{24d}$$

where R_0, M_0, P_0, and $\mathcal{E}_0$ represent the natural units of length, mass, pressure, and energy density in the system. In turn, x, m, p, and ϵ are the dimensionless quantities that enter into the scaled TOV equations. At the end of the computation, one can restore "physical" units by multiplying the appropriate physical observable by the relevant dimensionful parameter. In this manner the TOV equations may be written as follows:

$$\frac{dp(x)}{dx} = -\left[\frac{GM_0\mathcal{E}_0}{c^2 P_0 R_0}\right] \frac{\left(\epsilon(x) + \left[\frac{P_0}{\mathcal{E}_0}\right] p(x)\right)\left(m(x) + \left[\frac{4\pi R_0^3 P_0}{M_0 c^2}\right] x^3 p(x)\right)}{x^2\left(1 - \left[\frac{2GM_0}{c^2 R_0}\right]\frac{m(x)}{x}\right)}\,, \tag{25a}$$

$$\frac{dm(x)}{dx} = \left[\frac{4\pi R_0^3 P_0}{M_0 c^2}\right] x^2 \epsilon(x)\ . \tag{25b}$$

At first glance, it seems difficult to appreciate the advantage of rescaling the equations. However, one must realize that all quantities appearing within the square brackets remain at our disposal. In the particular case of the energy density and pressure, these scales have already been defined in the previous section; see Eq. (4). That is,

$$\mathcal{E}_0 = P_0 \equiv \frac{1}{8\pi^2}\frac{(m_n c^2)^4}{(\hbar c)^3} \approx 1.285\,\mathrm{GeV/fm^3}. \tag{26}$$

This value is slightly larger than the rest mass energy of a single neutron occupying a volume of 1 fm^3; this value is also equivalent to a pressure of approximately 10^{30} atmospheres! To fix the two remaining scales, namely, R_0 and M_0, we adopt the following natural choice:

$$\left[\frac{2GM_0}{c^2 R_0}\right] = \left[\frac{4\pi R_0^3 \mathcal{E}_0}{3M_0 c^2}\right] = 1 \tag{27}$$

Setting the above two quantities to unity, i.e., to a number that is neither too small nor too large, provides "natural" length and mass scales for the problem. Indeed, one obtains

$$R_0 = \sqrt{\frac{3\pi}{\alpha_G}}\,\lambda_n \approx 8.378\,\mathrm{km}\,, \tag{28a}$$

$$M_0 = \left(\frac{R_0}{R_s^{\odot}}\right) M_{\odot} \approx 2.837 M_{\odot}\,. \tag{28b}$$

where α_G is the (minute) dimensionless strength of the gravitational coupling between two neutrons (or the ratio of the neutron mass to the Planck mass), λ_n is the Compton wavelength of the neutron, and $R_s^{\odot}$ is the Schwarzschild radius of the Sun. Numerically, these quantities take the following values:

$$\alpha_G = \frac{G m_n^2}{\hbar c} \approx 5.922 \times 10^{-39} \tag{29}$$

$$\lambda_n = \frac{\hbar c}{m_n c^2} \approx 0.210 \times 10^{-18}\,\mathrm{km}\,, \tag{30}$$

$$R_s^{\odot} = \frac{2GM_{\odot}}{c^2} \approx 2.953\,\mathrm{km}\,. \tag{31}$$

The estimates obtained in Eq. (28) suggest—without the need for any detailed calculation—that neutron stars are compact objects of a few solar masses with typical radii of about 10 km. That is, by adopting the choice illustrated in Eq. (27), quantities that differ by many orders of magnitude conspire in just the right way to provide natural length and mass scales for the problem. Moreover, adopting these choices yields the following simplified form of the scaled TOV equations:

$$\frac{dp}{dx} = -\frac{1}{2}\frac{\Big[(\epsilon(x) + p(x)\Big]\Big[m(x) + 3x^3 p(x)\Big]}{x^2\Big[1 - m(x)/x\Big]} \,, \tag{32a}$$

$$\frac{dm}{dx} = 3x^2\epsilon(x) \,. \tag{32b}$$

Note that no approximations were made in reducing the TOV equations to this highly simplified form.

3 Results

After laying down the groundwork in the previous section, we are now in a position to compute the structure of neutron stars. We start by displaying in Fig. 3 the pressure $P(r)$ and mass $M(r)$ profiles for the heaviest neutron star that can be supported against gravitational collapse by the pressure of a degenerate Fermi gas of neutrons. Such calculation, originally carried out in 1939 by Oppenheimer and

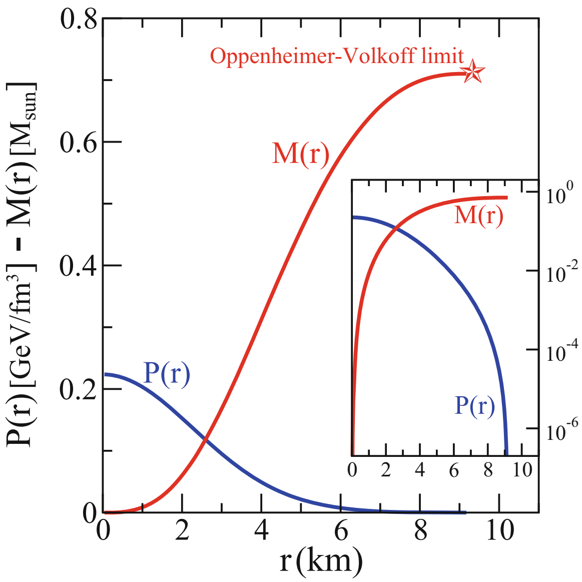


Fig. 3 Pressure and mass profiles for the heaviest neutron star that can be supported exclusively by a degenerate Fermi gas of neutrons, as first predicted by Oppenheimer and Volkoff (1939). The *inset* uses a logarithmic scale to accentuate the rapid variation of both profiles

Volkoff (1939), predicts that a neutron star with a value that exceeds $M \approx 0.71\, M_{\odot}$ will collapse into a black hole. Thus, as in the case of white dwarf stars, neutron stars exceeding certain mass limit will also collapse under their own weight. However, unlike white dwarf stars, neutron stars attain this mass limit at a finite radius; $R \approx 9.2\,\text{km}$ in the case of a degenerate Fermi gas of neutrons. Such is a unique and novel effect that is deeply rooted in general relativity; that is, neutron stars with masses that exceed its maximum limit are unstable against small density fluctuations and will collapse into a black hole.

It is important to underscore that the ingredients required to generate Fig. 3 in its entirety have all been presented in the previous section. That is, one starts by computing the underlying equation of state of a relativistic free Fermi gas as is given in parametric form in Eqs. (1) and (6). One then inputs such an EOS, preferably in the form of the energy density as a function of the pressure $\varepsilon = \varepsilon(p)$, into the scaled TOV equations displayed in Eq. (32). The set of coupled TOV equations is solved using a standard solver such as the Runge-Kutta method. Note, however, that care must be exercised in selecting a small enough step size $x = r/R_0$ to account for the rapid changes of both the pressure and mass profiles, as illustrated using a logarithmic scale in the inset to Fig. 3. Finally, one then directly compares the theoretical predictions against observation by rescaling the various dimensionless quantities according to the pressure, length, and mass scales defined in Eqs. (26) and (28).

Although the pioneering approach by Oppenheimer and Volkoff (1939) continues to be used to date, evidence now suggests that the assumed Fermi gas equation of state is not sufficient to explain observations. Indeed, within the last five years, two neutron stars with masses in the vicinity of two solar masses have been reported (Antoniadis et al. 2013; Demorest et al. 2010); this is almost three times larger than the Oppenheimer-Volkoff limit. This mere fact indicates that dynamical effects above and beyond simple fermionic correlations must be at play, thereby highlighting the critical role that nuclear interactions play in accounting for the structure of neutron stars. This has created a powerful synergy between nuclear physics and astrophysics that has established neutron stars as unique cosmic laboratories for the study of matter under extreme conditions of density and neutron-proton asymmetry. Indeed, the search for the underlying equation of state of neutron star matter is at the heart of one of the overarching questions that animates nuclear physics today: *How does subatomic matter organize itself and what phenomena emerge?*

Whereas a detailed description of some of the major ongoing efforts to elucidate the nature of the equation of state goes beyond the scope of this work, it is nevertheless instructive to briefly address some of the current debate. To do so, we display in Fig. 4 the range of neutron star masses predicted by some realistic equations of states as a function of the central pressure—which is an input to the TOV equations. We refer to the EOS as “realistic” as in all three cases, NL3, FSUGold, and FSUGarnet, the underlying theoretical model provides a quantitatively accurate description of various nuclear properties. However, given that the structure of neutron stars is sensitive to densities and neutron-proton

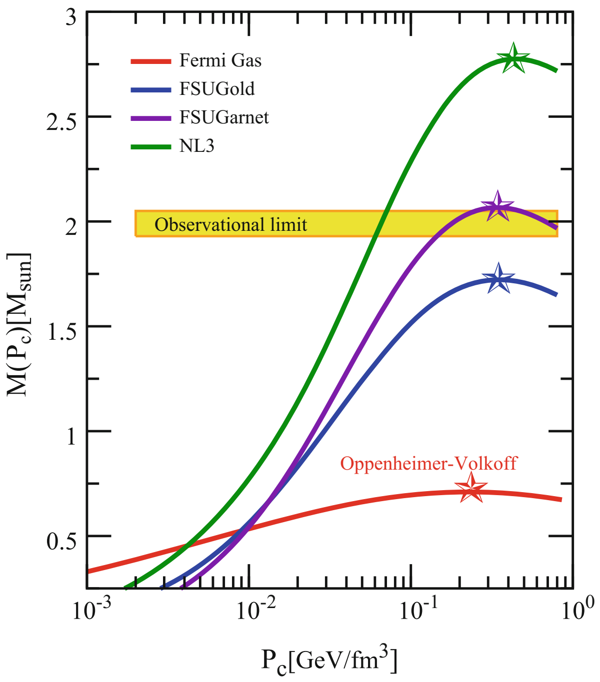


Fig. 4 Neutron star mass as a function of the central pressure as predicted by a simple Fermi gas model (Oppenheimer and Volkoff 1939) and three realistic equations of state (Chen and Piekarewicz 2015). The *horizontal band* indicates the observational constraints from the mass measurements reported in Demorest et al. (2010) and Antoniadis et al. (2013)

asymmetries not accessible in terrestrial laboratories, extrapolations are unavoidable and this leads to the differences displayed in the figure. Note that the "stars" in Fig. 4 indicate the maximum mass predicted by the given model; to the left the star is stable to small density fluctuations, but to the right it is not. Also indicated in the figure with a yellow band is the current observational limit on the maximum neutron star mass. Particularly notable is the vital role played by nuclear interactions in going from the Oppenheimer-Volkoff prediction to the present observational limit. In the common lingo, we say that the simple Fermi gas equation of state used by Oppenheimer and Volkoff must be "stiffen" significantly in order to account for massive neutron stars. Stiffening implies that the pressure required to support the neutron star against gravitational collapse must increase rapidly with density. Conversely, the pressure increases slowly with density for a "soft" equation of state. From the various EOS depicted in Fig. 4, NL3 is the stiffest.

Figure 4 suggests that models such as FSUGarnet and NL3 satisfy (quite comfortably in the latter case) the current observational constraint. However, focusing on a single observable can be misleading, as the equation of state leaves its imprint in many neutron star properties above and beyond the stellar mass; notably the stellar

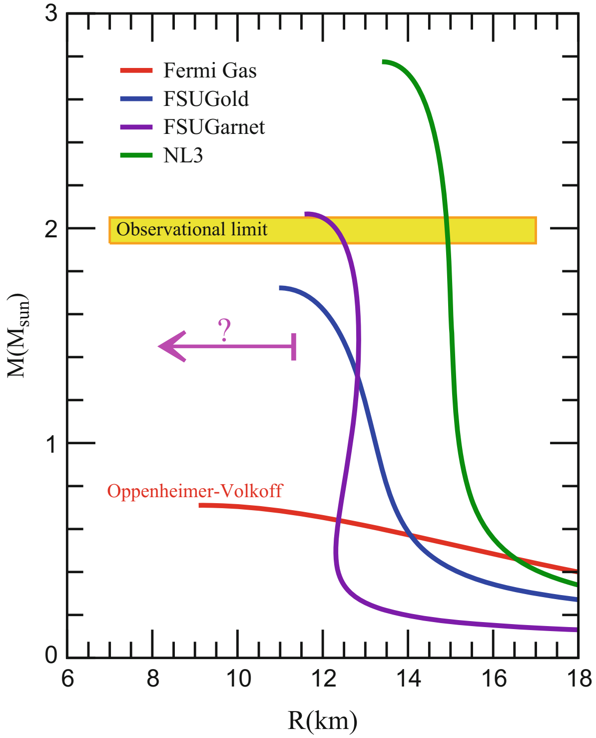


Fig. 5 Mass-Radius relation as predicted by a simple Fermi gas model (Oppenheimer and Volkoff 1939) and three realistic equations of state (Chen and Piekarewicz 2015). The *horizontal band* indicates the observational constraints from the mass measurements reported in Demorest et al. (2010) and Antoniadis et al. (2013). The *arrow* and *question mark* are motivated by some recent analyses that seem to suggest fairly small stellar radii (Guillot and Rutledge 2014; Ozel et al. 2016)

radius. Indeed, the *mass-radius* (MR) relation displayed in Fig. 5 is often regarded as the "holy grail" of neutron star structure. This is because knowledge of the MR relation *uniquely* determines the underlying equation of state (Lindblom 1992). As illustrated in Fig. 5, NL3 predicts that neutron stars with masses $M \approx 1 - 2.4\,M_{\odot}$ share a common radius of about 15 km. That is, the stiffness of NL3 is reflected in the prediction of massive neutron stars with large stellar radii. For a neutron star with a mass of $M = 1.4\,M_{\odot}$—the so-called "canonical" neutron star—this represents a discrepancy of about 3 km relative to the predictions of FSUGarnet, which, as NL3, accounts for the $2\,M_{\odot}$-limit and also reproduces a myriad of nuclear properties. Although stellar radii are notoriously difficult to measure, two recent studies seem to favor relatively small stellar radii (Guillot and Rutledge 2014; Guillot et al. 2013; Ozel et al. 2016); however, a consensus has yet to be reached (Lattimer and Steiner 2014). This situation is depicted by an arrow and a question mark in Fig. 5. One should underscore that accounting simultaneously for large stellar masses and small radii represents an enormous challenge for theory. But with every challenge, new opportunities emerge. And in this case, the solution to this puzzle may bring us closer than ever to answer one of the most fundamental questions in nuclear astrophysics: *what is the nature of dense stellar matter*?

4 Conclusions

Neutron stars are unique cosmic laboratories for the study of neutron-rich matter under extreme conditions of density. Given that a neutron star is bound by gravity and not by the strong force, most of the exotic states of matter predicted to exist in its interior cannot be realized under normal laboratory conditions. Thus, neutron stars provide a powerful intellectual bridge between nuclear physics and astrophysics.

In hydrostatic equilibrium, spherically symmetric neutron stars are described by the Tolman-Oppenheimer-Volkoff equations, which represent the generalization of Newtonian gravity to the domain of general relativity. And just as in the Newtonian case, their solution requires the specification of an *equation of state*, namely, a functional relation between the energy density that is the source of gravity, and the pressure that provides the support against collapse. What makes neutron stars unique celestial laboratories is that assuming that the pressure support is provided in its entirety by the degenerate neutrons—as Oppenheimer and Volkoff assumed in their 1939 paper—is in stark contradiction with observation. That is, above and beyond the fermionic correlations induced by the Pauli principle, the complicated nuclear dynamics plays an essential role in describing the structure, dynamics, and composition of these fascinating objects.

Although delving into the many modern approaches to the equation of state of neutron star matter is well beyond the scope of this chapter, we introduced fundamental concepts associated to multicomponent Fermi systems, such as the symmetry energy and beta equilibrium. Once we have discussed the impact of beta equilibrium—and charge neutrality—on the EOS, we proceeded to discuss the important role of *scaling* in solving the TOV equations. Indeed, by insisting on using "natural" units, we discovered—without the need for any detailed calculation—that neutron stars are compact objects of a few solar masses with typical radii of about 10 km. However, given that "the devil is in the detail," we solved the TOV equations to explore their sensitivity to the underlying equation of state. To do so, we incorporated some modern equations of state that have been calibrated to reproduce both the properties of finite nuclei and neutron stars. We underscore that the whole approach can be implemented in terms of two separate modules: one that solves the (scaled) TOV equations in general and without regard to a specific equation of state and the other that generates the desired equation of state that can then be used as input into the TOV equations.

Neutron star physics is a fruitful and stimulating area of investigation because many challenges remain. In this chapter, we highlighted one particular puzzle facing the field. On the one hand, we mentioned how enormous advances in observational astronomy have been used to identify the two most massive neutron stars to date.

This requires the equation of state to be "stiff," namely, the pressure must increase rapidly with density to be able to support such stars against gravitational collapse. Other observations—although not nearly as precise as those used to determine masses—seem to suggest that stellar radii are relatively small, thereby suggesting a "soft" equation of state. If such small stellar radii are confirmed, then perhaps the EOS is soft at intermediate densities in order to account for the small stellar radii and will then stiffen at high densities to account for massive neutron stars. Such unique behavior may be indicative of an exotic phase transition inside these fascinating objects.

5 Cross-References

► Neutron Stars as Probes for General Relativity and Gravitational Waves
► Nuclear Matter in Neutron Stars
► Strange Quark Matter Inside Neutron Stars
► Supernovae and Supernova Remnants: The Big Picture in Low Resolution
► The Masses of Neutron Stars

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