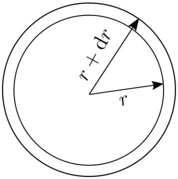
Mass and radius of white dwarfs and neutron stars

Hydrostatic equilibrium

- The inward force of gravity is balanced by the outward force of pressure.



$$4\pi r^2 \mathrm{d}P = -\frac{GM \cdot \mathrm{d}M}{r^2}$$
$$\mathrm{d}M = 4\pi r^2 \rho(r) \mathrm{d}r$$

- This leads to the hydrostatic equations:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2}$$
$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho$$

- The radius of the star R is given by $P(R) = 0$ and the mass is $M(R)$.
- For very dense stars, relativistic effects have to be taken into account.

Tolman-Oppenheimer-Volkoff (TOV) equations

- The TOV-equations are obtained from general relativity when a static and spherically symmetric mass distribution is inserted into the Einstein equations.
- The hydrostatic equations above are modified:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\,[P+\mathcal{E}(r)]\left[M(r)+4\pi r^3 P/c^2\right]}{c^2 r^2[1-2GM(r)/(c^2 r)]}$$

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \frac{\mathcal{E}(r)}{c^2}$$

where $\mathcal{E} = \rho c^2$.

The equation of state (EOS)

- In order to solve the TOV equations, a relation between the energy density $\mathcal{E}$ and the pressure P is needed.
- The function $\mathcal{E}(P)$ is called the equation of state.

White dwarf $\rightarrow$ Degenerate electron gas among inert heavy nuclei (oxygen and carbon).

Neutron star $\rightarrow$ Degenerate neutron gas.

- *White dwarf*:

$$\mathcal{E}(x) = \left(\frac{m_p}{m_e} \frac{(m_e c^2)^4}{(\hbar c)^3} \right) x^3 + \frac{(m_e c^2)^4}{8\pi^2 (\hbar c)^3} F(x)$$

$$P(x) = \frac{(m_e c^2)^4}{8\pi^2 (\hbar c)^3} G(x)$$

- *Neutron star*:

$$\mathcal{E}(x) = \frac{(m_n c^2)^4}{8\pi^2 (\hbar c)^3} F(x)$$

$$P(x) = \frac{(m_n c^2)^4}{8\pi^2 (\hbar c)^3} G(x)$$

where

$$F(x) = x\sqrt{1+x^2}(1+2x^2) - \ln\left(x + \sqrt{1+x^2}\right)$$

$$G(x) = \frac{2}{3}x^2\sqrt{1+x^2} - x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right)$$

Solving the TOV-equations

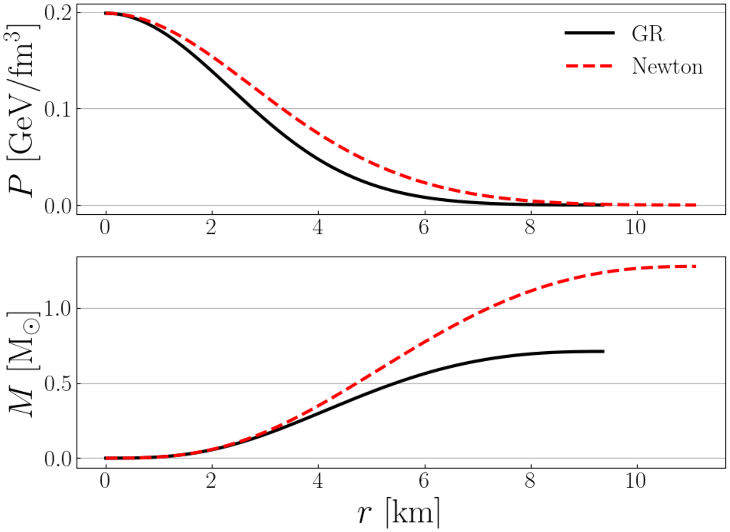
- The initial conditions $P_{j=0} = P(0) = P_c$ and $M_{j=0} = M(0) = 0$ were inserted.
- The numerical solution was then obtained by finding the the energy density $\mathcal{E}_j$ from P_j using the bisection method and then integrating the system of equations using Heun's method:

$$\left(M_j,\, \overset{\text{Root finding}}{P_j \curvearrowright \mathcal{E}_j}\right) \xrightarrow{\text{Integration}} \left(M_{j+1},\, \overset{\text{Root finding}}{P_{j+1} \curvearrowright \mathcal{E}_{j+1}}\right) \xrightarrow{\text{Integration}} \dots$$

- This is repeated until the pressure beomes zero, $P(R) = 0$.

Results

- Example with $P_c = P(0) = 0.2$ GeV/fm^3:



Results

- Examining the mass and radius for many values of P_c: