Tolman-Oppenheimer-Volkov (TOV) equations

- The equations are obtained from general relativity when a static and spherically symmetric mass distribution of a perfect fluid is inserted into the Einstein equations.
- The equations relate the pressure $P(r)$ at radial coordinate r with the total gravitational mass $M(r)$ contained inside r and the energy density $\mathcal{E}(r) = \rho(r)/c^2$ at r:

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{G\,[P+\mathcal{E}(r)]\left[M(r)+4\pi r^3 P/c^2\right]}{c^2 r^2[1-2GM(r)/(c^2 r)]} \\
\frac{\mathrm{d}M}{\mathrm{d}r} &= 4\pi r^2 \frac{\mathcal{E}(r)}{c^2}
\end{aligned} \tag{1}$$

- They can be rewritten in a form which make them easier to interpret.

Interpretation of TOV-equations

First equation:

$$4\pi r^2 \mathrm{d}P = -\frac{GM \cdot \mathrm{d}M}{r^2}\left(1+\frac{P}{\mathcal{E}}\right)\left(1+\frac{4\pi r^3 P}{Mc^2}\right)\left(1-\frac{2GM}{c^2 r}\right)^{-1}$$

- The left hand side is the net force acting on a shell lying between radii r and $r+\mathrm{d}r$.
- First factor on right hand side is the attractive Newtonian gravity acting on the shell.
- The other factors are general relativistic corrections to the Newtonian gravity.

Second equation:

$$\mathrm{d}M = 4\pi r^2 \mathcal{E} \mathrm{d}r / c^2$$

- $\mathrm{d}M$ is the gravitational mass of the shell.

The equation of state (EOS)

- In order to solve the TOV equations, a relation between the energy density $\mathcal{E}$ and the pressure P is needed.
- The function $\mathcal{E}(P)$ is called the equation of state.
- For a degenerate gas of noninteracting neutrons of mass m_n, the pressure and energy density can be found as function of number density n:

$$\mathcal{E}(n) = \mathcal{E}_0 \left(x_F \sqrt{1 + x_F^2} \left(1 + 2x_F^2\right) - \ln \left(x_F + \sqrt{1 + x_F^2} \right) \right)$$

$$P(n) = \mathcal{E}_0 \left(\frac{2}{3} x_F^3 \sqrt{1 + x_F^2} - x_F \sqrt{1 + x_F^2} + \ln \left(x_F + \sqrt{1 + x_F^2} \right) \right)$$

where

$$x_F(n) = \frac{\hbar c (3\pi^2 n)^{1/3}}{m_n c^2} \quad \text{and} \quad \mathcal{E}_0 = \frac{(m_n c^2)^4}{8\pi^2 (\hbar c)^3}.$$

Solving the TOV-equations

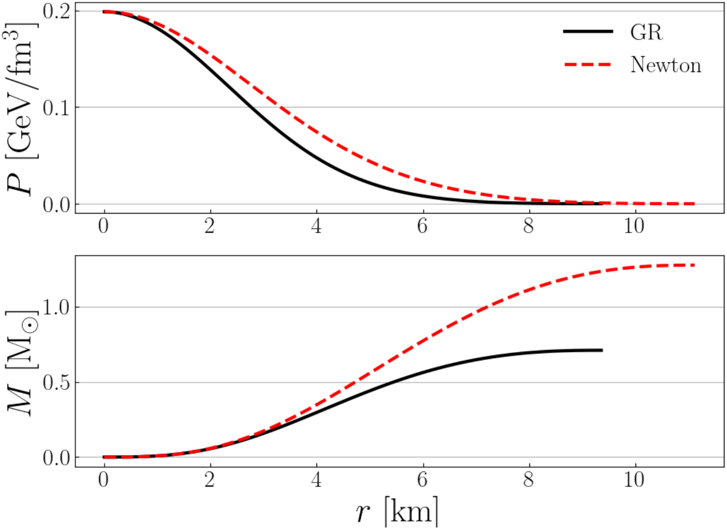
- The initial conditions $P_{j=0} = P(0) = P_c$ and $M_{j=0} = M(0) = 0$ were inserted.
- The numerical solution was then obtained by finding the the energy density $\mathcal{E}_j$ from P_j using the bisection method and then integrating the system of equations using Heun's method:

$$
\begin{tikzcd}[column sep=small]
(M_j, & P_j \arrow[r, bend left, "\text{Root finding}"] & \mathcal{E}_j) \\
\xrightarrow{\text{Integration}} (M_{j+1}, & P_{j+1} \arrow[r, bend left, "\text{Root finding}"] & \mathcal{E}_{j+1}) \\
\xrightarrow{\text{Integration}} \dots
\end{tikzcd}
$$

- This is repeated until the pressure beomes zero, $P(R) = 0$.

Results

- Example with $P_c = P(0) = 0.2\ \text{GeV}/\text{fm}^3$:



Results

- Examining the mass and radius for many values of P_c: