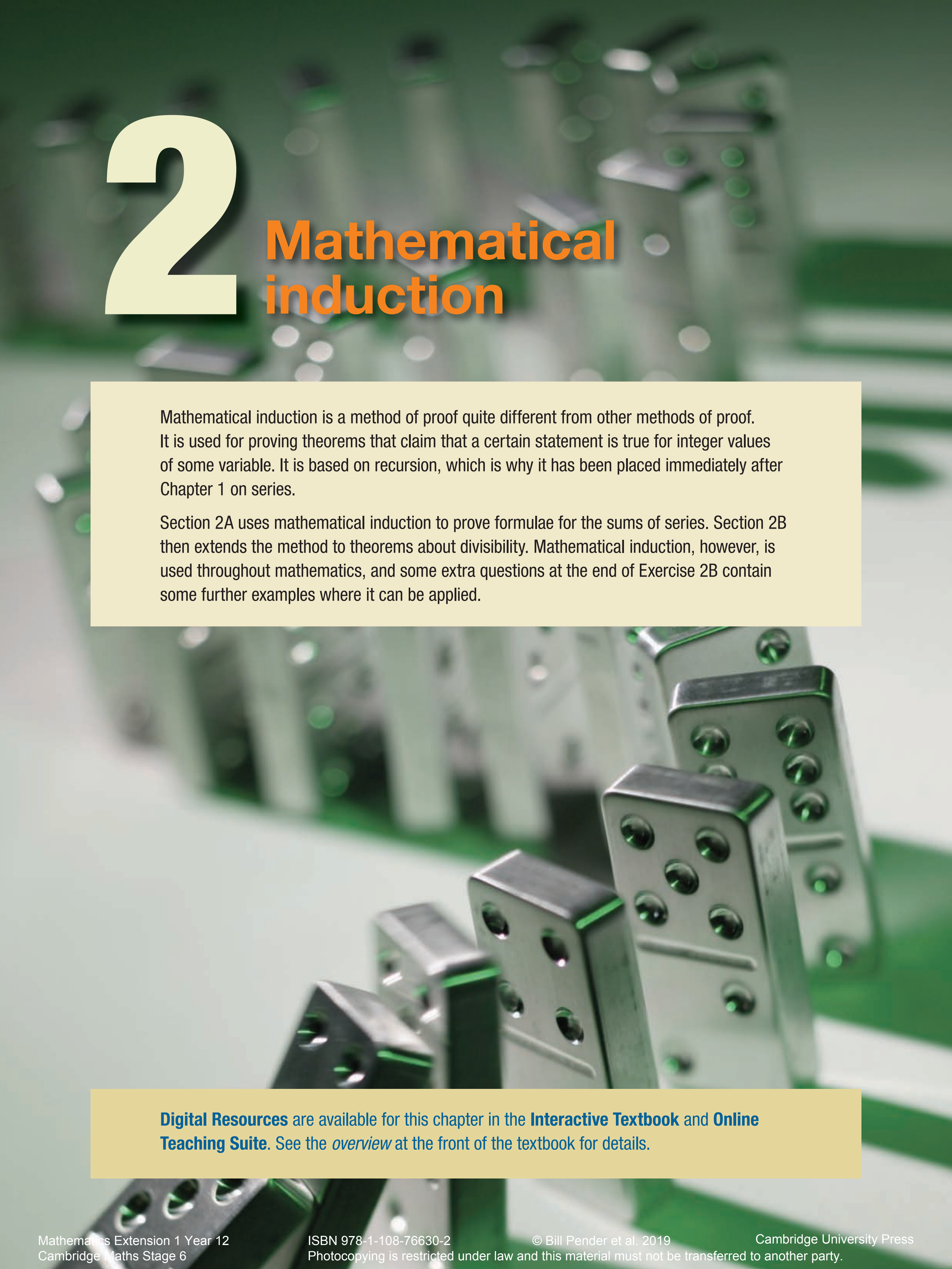

2 Mathematical induction

Mathematical induction is a method of proof quite different from other methods of proof. It is used for proving theorems that claim that a certain statement is true for integer values of some variable. It is based on recursion, which is why it has been placed immediately after Chapter 1 on series.

Section 2A uses mathematical induction to prove formulae for the sums of series. Section 2B then extends the method to theorems about divisibility. Mathematical induction, however, is used throughout mathematics, and some extra questions at the end of Exercise 2B contain some further examples where it can be applied.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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2A Using mathematical induction for series

The logic of mathematical induction is most easily understood when it is applied to sums of series.

In this course, proof by mathematical induction can only be applied after we already have a clear statement of the theorem to be proven. The example below examines a typical situation in which a clear pattern is easily generated, but no obvious explanation emerges for why that pattern occurs.

An example proving a formula for the sum of a series

Find a formula for the sum of the first n cubes, and prove it by mathematical induction.

Some calculations for low values of n: Here is a table of values of the first 10 positive cubes and their partial sums.

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
n^3	1	8	27	64	125	216	343	512	729	1000	$\cdots$
$1^3+2^3+\cdots+n^3$	1	9	36	100	225	441	784	1296	2025	3025	$\cdots$
Form	1^2	3^2	6^2	10^2	15^2	21^2	28^2	36^2	45^2	55^2	$\cdots$

The surprising thing here is that the last row consists of the squares of the *triangular numbers*, where the nth triangular number is the sum of all the positive integers up to n,

$$1+2=3,\quad 1+2+3=6,\quad 1+2+3+4=10,\quad 1+2+3+4+5=15.$$

Using the formula for the sum of an AP (the number of terms times the average of first and last term), the formula for the nth triangular number is $\frac{1}{2}n(n+1)$.

Hence the sum of the first n cubes seems to be $\frac{1}{4}n^2(n+1)^2$.

We have now arrived at a *conjecture*, meaning that we appear to have a true theorem, but we have no clear idea why it is true. We cannot even be sure yet that is true, because showing that a statement is true for the first 10 positive integers is most definitely not a proof that it is true for all integers.

The following worked exercise gives a precise statement and proof of the conjectured result.

Example 1 2A

Prove by mathematical induction that for all integers $n \geq 1$,

$$1^3+2^3+3^3+4^3+\cdots+n^3=\tfrac{1}{4}n^2(n+1)^2.$$

The proof below is a typical proof by mathematical induction. Read it carefully, then read the explanation of the proof in the notes below.

SOLUTION

Proof: By mathematical induction.

A When $n=1$, RHS $= \frac{1}{4}\times 1\times 2^2$
$= 1$
$=$ LHS,

so the statement is true for $n=1$.

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B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose $\quad 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2. \qquad (**)$
We prove the statement for $n = k + 1$.
That is, we prove $1^3 + 2^3 + 3^3 + 4^3 + \cdots + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$.

$$\begin{aligned}
\text{LHS} &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 \\
&= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3, \text{ by the induction hypothesis } (**), \\
&= \tfrac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right) \\
&= \tfrac{1}{4}(k+1)^2(k^2 + 4k + 4) \\
&= \tfrac{1}{4}(k+1)^2(k+2)^2 \\
&= \text{RHS.}
\end{aligned}$$

C It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

Notes on the proof

There are three clear parts.

- Part A proves the statement for the starting value, which in this case is $n = 1$.
- Part B is the most complicated, and proves that whenever the statement is true for some integer $k \geq 1$, then it is also true for the next integer $k + 1$.
- Part C concludes by appealing to the principle of mathematical induction.

Any question on proof by mathematical induction is testing your ability to write a coherent account of the proof — you are advised to follow the structure given here.

The first four lines of Part B are particularly important, and these four sentences should be repeated strictly in all proofs. The first and second sentences of Part B set up what is assumed about k, writing down the specific statement for $n = k$, a statement later referred to as 'the induction hypothesis'. The third and fourth sentences set up specifically what it is that we intend to prove.

Statement of the principle of mathematical induction

With this proof as an example, here is a formal statement of the principle of mathematical induction.

1 MATHEMATICAL INDUCTION

Suppose that a statement is to be proven for all integers n greater than or equal to some starting value n_1. Suppose also that two things have been proven:

A The statement is true for $n = n_1$.

B Whenever the statement is true for some positive integer $k \geq n_1$. then it is also true for the next integer $k + 1$.

Then (part **C**) the statement is true for all integers $n \geq n_1$.

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Mathematical induction is an axiom of mathematics. In formal mathematical logic, even the whole numbers 0, 1, 2, 3, . . . cannot be defined without it, and it allows sentences such as, 'The whole numbers continue for ever' to be made absolutely precise. The final statement **C** in the proof above must never be omitted.

An example proving the formula for the sum of a GP

The next worked example proves the formula for the sum of a GP. We used dots notation . . . when developing the formula, which was perfectly acceptable, but a proof that requires dots depends on mathematical induction for its validity.

Example 2 — 2A

Prove that for all real numbers a and $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, \text{ for all integers } n \geq 1.$$

SOLUTION

The setting-out is almost identical to the previous example.

Proof: By mathematical induction.

A When $n = 1$, $\text{RHS} = \dfrac{a(r^1 - 1)}{r - 1}$

$$= a$$

$$= \text{LHS},$$

so the statement is true for $n = 1$.

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\quad a + ar + ar^2 + \cdots + ar^{k-1} = \dfrac{a(r^k - 1)}{r - 1}. \qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $\quad a + ar + ar^2 + \cdots + ar^k = \dfrac{a(r^{k+1} - 1)}{r - 1}.$

$$\begin{aligned}
\text{LHS} &= a + ar + ar^2 + \cdots + ar^{k-1} + ar^k \\
&= \frac{a(r^k - 1)}{r - 1} + ar^k, \text{ by the induction hypothesis } (**), \\
&= \frac{ar^k - a}{r - 1} + \frac{ar^{k+1} - ar^k}{r - 1} \\
&= \frac{ar^{k+1} - a}{r - 1} \\
&= \frac{a(r^{k+1} - 1)}{r - 1} \\
&= \text{RHS}.
\end{aligned}$$

C It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

Note: The original proof of the formula in Section 1G using the dots . . . was much clearer intuitively. It is very common that the proof by mathematical induction does not display the intuitive idea nearly as well.

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Exercise 2A

FOUNDATION

1 Copy and complete the proof by mathematical induction that for all integers $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

A When $n = 1$, RHS $= \ldots.$
$= \text{LHS},$
so the statement is true for ....

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose ... $(**)$
We prove the statement for $n = k + 1$.
That is, we prove ...
LHS $= \ldots$, by the induction hypothesis $(**)$,
$= \ldots$
$= \text{RHS}.$

C It follows from parts **A** and **B** by mathematical induction that ....

2 Prove by mathematical induction that for all positive integer values of n:

a $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$

b $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

c $1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$

d $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

e $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$

f $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

g $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1)$

h $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

i $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$

3 What are the limiting sums of the series in parts **h** and **i** of the previous question?

DEVELOPMENT

4 Prove by mathematical induction that for all positive integers $n \geq 1$:

a $1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2(n + 1) = \frac{1}{12}n(n + 1)(n + 2)(3n + 1)$

b $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n + 1)^2 = \frac{1}{12}n(n + 1)(n + 2)(3n + 5)$

c $2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (n + 1) \times 2^{n-1} = n \times 2^n$

5 Prove by mathematical induction that for all positive integer values of n:

a $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n + 1)! - 1$

b $2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (n^2 + 1)n! = n(n + 1)!$

c $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n + 1)!} = 1 - \dfrac{1}{(n + 1)!}$

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6 **a** Suppose that the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2 + 2$ is true for the positive integer $n = k$. Prove that it is also true for $n = k + 1$.

b Explain why we cannot conclude that the statement is true for all integers $n \geq 1$.

7 **a** Attempt to prove by mathematical induction that $3 + 6 + 9 + \cdots + 3n = n(n + 1) + 1$ for all positive integer values of n.

b Where does the proof break down?

8 **a** Use the factor theorem to show that $n + 1$ is a factor of $P(n) = 4n^3 + 18n^2 + 23n + 9$, and hence factor $P(n)$.

b Hence prove by mathematical induction that for all integers $n \geq 1$,

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n - 1)(2n + 1) = \tfrac{1}{3}n(4n^2 + 6n - 1).$$

9 Prove by mathematical induction that for all integers $n \geq 1$,

$$1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + n) = \tfrac{1}{6}n(n + 1)(n + 2).$$

(Hint: Use Question **2** part **a**.)

10 Prove by mathematical induction that for all positive integers n,

$$(n + 1)(n + 2)(n + 3) \times \cdots \times 2n = 2^n\big(1 \times 3 \times 5 \times \cdots \times (2n - 1)\big).$$

11 Prove by mathematical induction that for all positive integer values of n:

a $\displaystyle\sum_{r=1}^{n} (r^3 - r) = \tfrac{1}{4}(n - 1)(n)(n + 1)(n + 2)$

b $\displaystyle\sum_{r=1}^{n} \left(3r^5 + r^3\right) = \tfrac{1}{2}n^3(n + 1)^3$

c $\displaystyle\sum_{r=1}^{n} r^2 \times 2^r = (n^2 - 2n + 3) \times 2^{n+1} - 6$

ENRICHMENT

12 Let $H(n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Use mathematical induction to prove that for all positive integers $n \geq 1$,

$$n + H(1) + H(2) + H(3) + \cdots + H(n - 1) = nH(n).$$

13 **a** Prove the trigonometric identity $\dfrac{\cos\alpha - \cos(\alpha + 2\beta)}{2\sin\beta} = \sin(\alpha + \beta)$.

b Hence prove by mathematical induction that for all integers $n \geq 1$,

$$\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n - 1)\theta = \frac{1 - \cos 2n\theta}{2\sin\theta}.$$

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2B Proving divisibility by mathematical induction

Divisibility is another standard situation where mathematical induction can be used to prove a result. The example below again uses low values of n to establish a hypothesis, and then proves that hypothesis using mathematical induction.

An example of proving divisibility

Find the largest integer that is a divisor of $3^{4n} - 1$ for all integers $n \geq 0$. Then prove the result by mathematical induction.

Some calculations for low values of n: Here is a table using just the first four values of n, starting this time at $n = 0$,

n	0	1	2	3	...
$3^{4n} - 1$	0	80	6560	531440	...

It seems likely from this that 80 is a divisor of all the numbers — remember that every number is a divisor of zero. Certainly no number greater than 80 can be a divisor of all of them. So we write down the likely theorem and try to provide a proof. Various proofs are available, but here is the proof by mathematical induction:

Example 3 — 2B

Prove by mathematical induction that for all integers $n \geq 0$,
$3^{4n} - 1$ is divisible by 80.

SOLUTION

The key step in all divisibility proofs is the introduction of an extra pronumeral in the second line of part **B**. In this case we have used the letter m.

Proof: By mathematical induction

A The starting value here is $n = 0$, not $n = 1$ as before.
When $n = 0$, $3^{4n} - 1 = 0$, which is divisible by 80 (and by every number),
so the statement is true for $n = 0$.

B Suppose that $k \geq 0$ is an integer for which the statement is true.
That is, suppose $3^{4k} - 1 = 80m$, for some integer m. $(**)$
We prove the statement for $n = k + 1$.
That is, we prove $3^{4k+4} - 1$ is divisible by 80.

$$\begin{aligned} 3^{4k+4} - 1 &= 3^{4k} \times 3^4 - 1 \\ &= (80m + 1) \times 81 - 1, \text{ by the induction hypothesis } (**), \\ &= 80 \times 81m + 81 - 1 \\ &= 80m \times 81 + 80 \\ &= 80(81m + 1), \text{ which is divisible by 80, as required.} \end{aligned}$$

C It follows from parts **A** and **B** by mathematical induction that the statement is true for all whole numbers n.

Note: In the second sentence of part **B**, the induction hypothesis $(**)$ has interpreted divisibility by 80 as being $80m$ where m is an integer. In the fourth sentence of Part **B**, however, the statement of what is to be proven does not interpret divisibility at all. Proofs of divisibility work more easily this way.

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Further remarks on mathematical induction

Divisibility and summing a series and are classic places where proof by mathematical induction is used. The method, however, is used routinely throughout all branches of mathematics. Some extra questions at the end of Exercise 2B give some applications in geometry, combinatorics and calculus. In each case, the structure and words given in the examples above should be followed.

Exercise 2B

FOUNDATION

1 Copy and complete this proof that $7^n - 1$ is divisible by 6, for all positive integers n.

A When $n = 1$, $7^n - 1 = \ldots$
so the statement is true for $n = 1$.

B Suppose that $k \geq 1$ is a positive integer for which the statement is true.
That is, suppose . . . $(**)$
We prove the statement for $n = k + 1$.
That is, we prove . . .

$$\begin{aligned} 7^{k+1} - 1 &= \ldots \\ &= \ldots, \text{ by the induction hypothesis } (**), \\ &= \ldots, \text{ which is divisible by 6, as required.} \end{aligned}$$

C It follows from parts **A** and **B** by mathematical induction that the statement is true for . . .

2 Prove by mathematical induction that for all integers $n \geq 1$:

a $5^n - 1$ is divisible by 4,
b $9^n + 3$ is divisible by 6,
c $3^{2n} + 7$ is divisible by 8,
d $5^{2n} - 1$ is divisible by 24.

DEVELOPMENT

3 **a** Copy and complete the table of values to the right. Then make a conjecture about the largest number that $11^n - 1$ is divisible by, for all integers $n \geq 0$.

n	0	1	2	3	4
$11^n - 1$					

b Prove your conjecture by mathematical induction.

4 Prove by mathematical induction that for all integers $n \geq 0$:

a $n^3 + 2n$ is divisible by 3,
b $8^n - 7n + 6$ is divisible by 7,
c $9(9^n - 1) - 8n$ is divisible by 64.

5 Prove by mathematical induction that for all integers $n \geq 0$:

a $5^n + 2 \times 11^n$ is divisible by 3,
b $3^{3n} + 2^{n+2}$ is divisible by 5,
c $11^{n+2} + 12^{2n+1}$ is divisible by 133.

6 Prove by mathematical induction that $x - 1$ is a factor of $x^n - 1$ for all integers $n \geq 1$.

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7 Show that for all whole numbers k, if $8k^2 + 14$ is divisible by 4, then $8(k + 1)^2 + 14$ is also divisible by 4. Show, however, that $8n^2 + 14$ is never divisible by 4 if n is a whole number. Which step of proof by induction does this counter-example show is necessary?

8 **a** Show that $f(n) = n^2 - n + 17$ is prime for $n = 0, 1, 2, \ldots, 16$. Show, however, that $f(17)$ is not prime. Which step of proof by induction does this counter-example show is necessary?
b Begin to show that $f(n) = n^2 + n + 41$ is prime for $n = 0, 1, 2, \ldots, 40$ but not for 41.

Note: There is no formula for generating prime numbers — these two quadratics are interesting because of the long unbroken sequences of primes that they produce.

ENRICHMENT

9 Prove by mathematical induction that $3^{2^n} - 1$ is divisible by 2^{n+1} for all integers $n \geq 0$. (Note that 3^{2^n} means 3 to the power of 2^n.)

Further examples of theorems proven using mathematical induction

10 [Geometry]
Prove by mathematical induction that the sum of the angles of a convex polygon with $n \geq 3$ sides is $n - 2$ straight angles.
(Hint: In step B, dissect the $(k + 1)$-gon into a k-gon and a triangle.)

11 [Combinatorics]
Prove by mathematical induction that every n-member set has 2^n subsets.
(Hint: In step B, when a new member is added to a k-member set, every subset of the resulting $(k + 1)$-member set either contains the new member, or does not contain it.)

12 [Calculus]
Use mathematical induction, combined with the product rule, to prove that

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ for all integers } x \geq 1.$$

(Hint: In step B, write x^{k+1} as $x \times x^k$, then apply the product rule.)

13 [The induction step can skip through the integers]
Prove these divisibility results:
a $n^2 + 2n$ is a multiple of 8, for all even integers $n \geq 0$.
b $3^n + 7^n$ is divisible by 10, for all odd integers $n \geq 1$.
(Hint: In step B of each proof, advance from k to $k + 2$.)

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Chapter 2 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 2 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Prove by mathematical induction that for all positive integer values of n:

a $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

b $1 + 7 + 7^2 + \cdots + 7^{n-1} = \frac{1}{6}(7^n - 1)$

c $1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n + 4) = \frac{1}{6}n(n + 1)(2n + 13)$

d $\dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \dfrac{1}{4 \times 5} + \cdots + \dfrac{1}{(n + 1)(n + 2)} = \dfrac{n}{2(n + 2)}$

e $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{n + 2}{2^n}$

2 Prove these results by mathematical induction:

a $7^{2n-1} + 5$ is divisible by 12, for all integers $n \ge 1$,

b $2^{2n} + 6n - 1$ is divisible by 9, for all integers $n \ge 0$,

c $2^{2n+2} + 5^{2n-1}$ is divisible by 21, for all integers $n \ge 1$,

d $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9, for all integers $n \ge 0$.

3 **a** Copy and complete the table of values to the right. Then make a conjecture about the largest number that $2^{3n} - 3^n$ is divisible by, for all whole numbers $n \ge 0$.

n	0	1	2	3
$2^{3n} - 3^n$				

b Prove your conjecture by mathematical induction.

4 Prove by mathematical induction that for all positive integer values of n:

a $\displaystyle\sum_{r=1}^{n} r \times r! = (n + 1)! - 1$ **b** $\displaystyle\sum_{r=1}^{n} \frac{r - 1}{r!} = 1 - \frac{1}{n!}$

What is the limiting sum of the series in part **b**?

5 Prove that $1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$, for all integers $n \ge 1$.
(Hint: Use the factorisation $6k^3 + 15k^2 + 11k + 2 = (k + 1)(6k^2 + 9k + 2)$.)

6 Prove that $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$, for all integers $n \ge 1$.
(Hint: Use the factorisation $2k^4 + 8k^3 + 11k^2 + 6k + 1 = (k + 1)^2(2k^2 + 4k + 1)$.)

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