13 Differential equations

A *differential equation* is an equation that involves a derivative.

For example, we saw in Question 4 of the Chapter 16 Review in the Year 11 book that the radioactive decay of a mass M of caesium-137, with a half-life of about 30.2 years, obeys the differential equation

$$\frac{dM}{dt} = -kM, \text{ where } k \doteqdot 0.023,$$

which says that the rate at which caesium-137 is decreasing is proportional to the mass M present at that time t.

When calculus is used in science or engineering or elsewhere, it is very common that the behaviour being studied is modelled by a differential equation, as in the example with caesium-137. This chapter is a first introduction to an extremely important branch of calculus.

Changes of pronumeral can be confusing when learning methods. The first four sections are all explained using only the standard pronumerals x and y. Examples from science and elsewhere require other pronumerals such as t for time and M for mass, and will not be discussed until Section 13E.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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13A Differential equations

We begin by explaining what a differential equation is and what a solution of a differential equation is, and by introducing some basic terminology.

In this chapter, all logarithms have base e, and we will usually write $\log x$ or $\ln x$ instead of $\log_e x$.

Differential equations and their order

A *differential equation*, or DE for short, is an equation that involves a derivative. Here are some examples of DEs.

$$y' = -7y \qquad y' = 1 + y^2 \qquad y'' + 49y = 0$$

$$yy' + x = 0 \qquad y' = y(y-1) \qquad x^2y'' - xy' + y = 0$$

The *order of a differential equation* is the order of the highest derivative that occurs within it. For example, the DEs in the last column above have order 2 because they involve the second derivative, whereas the other four have order 1.

The chapter is mostly concerned with first-order DEs, with a few simple higher-order examples.

Example 1 — 13A

State whether these equations are differential equations, and if they are, state their order.

a $x^2y'' - xy' + y = 0$

b $x^2y^2 - xy + 1 = 0$

c $y' = x^2 + 1$

d $y = y''''$

SOLUTION

Parts **a**, **c** and **d** are DEs of orders 2, 1 and 4 respectively.
Part **b** is not a DE because it does not involve a derivative.

What is a solution of a differential equation?

DEs are equations, and they have solutions. The solutions are not numbers, but functions of x, or more generally relations in x and y. We can test whether a function is a solution by substituting it into the DE, as in the next worked example.

Example 2 — 13A

Test whether each function is a solution of the differential equation $y' = -7y$.

a $y = 20e^{7x}$

b $y = 20e^{-7x}$

SOLUTION

a Differentiating, $y' = 140e^{7x}$, so substituting into the DE $y' = -7y$,

$\text{LHS} = 140e^{7x}$ and $\text{RHS} = -7 \times 20e^{7x}$.

Because LHS $\neq$ RHS, the function is not a solution.

b Differentiating, $y' = -140e^{-7x}$, so substituting into the DE $y' = -7y$,

$\text{LHS} = -140e^{-7x}$ and $\text{RHS} = -7 \times 20e^{-7x}$.

Because LHS $=$ RHS, the function is a solution.

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Note: The differential equation $y' = -7y$ in part **b** above only involves algebraic operations, yet the solution $y = 20e^{-7x}$ involves an exponential function. The next worked example below is also algebraic, but its solution involves a trigonometric function. This behaviour is typical for DEs, just as we have already seen with integrals such as $\int \frac{dx}{1 + x^2} = \tan^{-1} x + C$.

Example 3 13A

Show that $y = \tan x$ is a solution of the differential equation $y' = 1 + y^2$.

SOLUTION

Substituting $y = \tan x$ into the DE,

$$\begin{aligned}\text{LHS} &= \frac{d}{dx}\tan x \\ &= \sec^2 x,\end{aligned} \qquad \begin{aligned}\text{RHS} &= 1 + \tan^2 x \\ &= \sec^2 x.\end{aligned}$$

Hence $y = \tan x$ is a solution of the DE.

Indefinite integrals are equivalent to DEs

An indefinite integral such as $\int 2x\, dx$ says, 'Find a function whose derivative is $2x$'. It is thus equivalent to the DE

$$y' = 2x,$$

and the primitive $y = x^2$ is a solution of this DE because on substitution,

$$\text{LHS} = y' = \frac{d}{dx}(x^2) = 2x = \text{RHS}.$$

In general, the indefinite integral $\int f(x)\, dx$ is equivalent to the DE $y' = f(x)$.

1 DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

- A *differential equation* or *DE* is an equation that involves a derivative. Its solutions are functions, or in general, relations.
- The *order of a differential equation* is the order of the highest derivative that occurs within it.
- To test if a function is a solution of a DE, substitute it into the DE.
- An indefinite integral is equivalent to a differential equation because finding $\int f(x)\, dx$ means finding a solution of the DE $y' = f(x)$.

Indefinite integrals and the family of solutions

We saw in Section 4*I* that the solutions of the DE

$$y' = 2x$$

involve a single arbitrary constant,

$$y = x^2 + C, \text{ for some constant } C,$$

producing the family of solution curves sketched on the next page.

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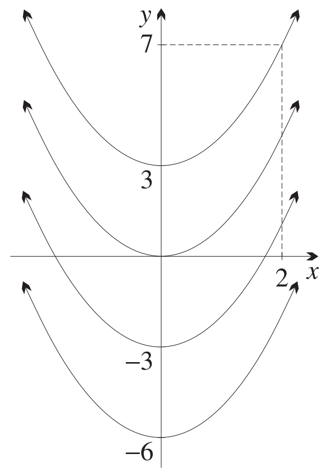
When working with motion in Chapter 9, we often started with acceleration and integrated twice, which needed two arbitrary constants. Our most common such differential equation was the second-order DE

$$\ddot{x} = -g,$$

whose solution is $x = -\frac{1}{2}gt^2 + At + B$, for constants A and B. This is a two-parameter family of solutions, not nearly so easy to represent graphically.



General solution of a differential equation

The general solution of any differential equation normally involves arbitrary constants, and so forms a family of curves. The next two examples demonstrate this. They show a solution family with one arbitrary constant, and a solution family with two arbitrary constants.

Example 4 13A

Show that $y = Ae^x - x - 1$ is a solution of $y' = x + y$, for all values of A.

SOLUTION

Substituting $y = Ae^x - x - 1$ into the DE $y' = x + y$,

$$\begin{aligned} \text{LHS} &= y' \\ &= Ae^x - 1, \end{aligned} \quad \text{and} \quad \begin{aligned} \text{RHS} &= x + (Ae^x - x - 1) \\ &= Ae^x - 1. \end{aligned}$$

Hence $y = Ae^x - x - 1$ is a solution for all values of A.

Note: Three curves in the family of solutions are drawn in worked Example 7.

Example 5 13A

a Show that $y = \sin 7x$ and $y = \cos 7x$ satisfy $y'' = -49y$.

b Show that $y = A\sin 7x + B\cos 7x$ satisfies $y'' = -49y$, for all A and B.

c Find the value of the second derivative y'' when $y = 5$.

d What transformation maps each solution of the DE to its second derivative?

SOLUTION

a
$$\begin{aligned} &\frac{d^2}{dx^2}(\sin 7x) \\ &= \frac{d}{dx}(7\cos 7x) \\ &= -49\sin 7x \\ &= -49y \end{aligned} \qquad \begin{aligned} &\frac{d^2}{dx^2}(\cos 7x) \\ &= \frac{d}{dx}(-7\sin 7x) \\ &= -49\cos 7x \\ &= -49y \end{aligned}$$

b
$$\begin{aligned} &\frac{d^2}{dx^2}(A\sin 7x + B\cos 7x) \\ &= \frac{d}{dx}(7A\cos 7x - 7B\sin 7x) \\ &= -49A\sin 7x - 49B\cos 7x \\ &= -49(A\sin 7x - B\cos 7x) \\ &= -49y \end{aligned}$$

c Because $y'' = -49y$, if $y = 5$, then $y'' = -49 \times 5 = -245$.

d Because $y'' = -49y$, every solution is mapped to its second derivative by a reflection in the x-axis followed by a vertical dilation with factor 49.

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How many arbitrary constants are there in the general solution?

Normally, the most general solution of an nth-order differential equation has n arbitrary constants, but we cannot prove this in the present course.

We can only sometimes prove that even a solution with the expected number of arbitrary constants is the most general solution of the DE, in the sense that every solution can be obtained from it by substituting suitable arbitrary constants.

Initial value problems and particular solutions

We saw, especially in motion and rates, that arbitrary constants in the general solution can be evaluated provided that we have suitable *initial conditions* (or *boundary conditions*). The resulting curve is a *particular solution*, and the problem of finding a particular solution given a DE and suitable initial conditions is called an *initial value problem* or *IVP*. There may also be restrictions on the variables.

A particular solution may or may not be a connected curve. Many familiar functions such as $y = \frac{1}{x}$ and $y = \tan x$ have two or more branches, and a solution curve (or *integral curve*) may be just one branch of such a function.

The next two worked examples pick out particular family members that satisfy the initial conditions. The first is equivalent to the familiar indefinite integral $\int 2x\, dx$, whose solution family is sketched above.

Example 6 — 13A

Solve the differential equation $y' = 2x$, given that $P(2, 7)$ lies on the curve.

SOLUTION

Integrating, $\quad y = x^2 + C$, for some constant C.

When $x = 2,\ y = 7$, so $\quad 7 = 4 + C$,

so $C = 3$ and $\quad y = x^2 + 3$, as on the diagram drawn below Box 1.

Now we can use the same substitution method to pick out three particular solutions of the family in worked Example 4, given three different initial conditions.

Example 7 — 13A

We showed in worked Example **4** that $y = Ae^x - x - 1$, where A is a constant, is the general solution of $y' = x + y$.

a Find the particular solution passing through the origin.

b Find A if $y = -2$ when $x = 0$, and write down this particular solution.

c Find the particular solution through $(0, -1)$.

d Analyse the gradient and concavities of the solutions in parts **a**–**c**, identifying any stationary points and inflections.

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e By substituting $y' = 0$ into the DE, identify the set of points where the solution curves have gradient zero.

f Sketch the three solutions of the DE, and draw dashed the answer to part **e**.

g Do all solution curves to this DE have a stationary point?

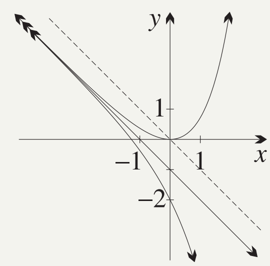
SOLUTION

a Substituting $(0, 0)$, $\quad 0 = A - 0 - 1$,
so $A = 1$, giving $\quad y = e^x - x - 1$.

b Substituting $(0, -2)$, $\quad -2 = A - 0 - 1$,
so $A = -1$, giving $\quad y = -e^x - x - 1$.

c Substituting $(0, -1)$, $\quad -1 = A - 0 - 1$,
so $A = 0$, giving $\quad y = -x - 1$.

f



d For part **a**, $\quad y' = e^x - 1 \quad$ and $\quad y'' = e^x$,
so there is a minimum turning point at $(0, 1)$, and the curve is always concave up.
For part **b**, $\quad y' = -e^x - 1 \quad$ and $\quad y'' = -e^x$,
so the curve is always decreasing and always concave down.
Part **c** is the line $y = -x - 1$.

e Substituting $y' = 0$ into the DE $y' = x + y$ gives the line $x + y = 0$.

g Clearly, no. The solution curve in part **a** crosses $x + y = 0$, and has a stationary point there. But the other two solution curves do not cross $x + y = 0$, so they do not have any stationary points.

Note: Further detail about this DE is given in worked Example 13 in Section 13B.

Initial values with higher-order differential equations

An initial condition is usually a point on the curve. For higher-order DEs, however, the initial conditions may involve also a point on the graph of the derivative, that is, a value of y' for some particular value of x. For two arbitrary constants, we need two initial conditions, which usually form two simultaneous equations.

Example 8 — 13A

We showed in worked Example **5b** that $y = A \sin 7x + B \cos 7x$ is a solution of $y'' = -49y$ for all A and B. Find the solution for which $y(0) = 1$ and $y'(0) = 14$.

SOLUTION

The solution is $\quad y = A \sin 7x + B \cos 7x$,
and the derivative is $\quad y' = 7A \cos 7x - 7B \sin 7x$.
When $x = 0$, $y = 1$, so $\quad 1 = 0 + B$, $\quad$ (1)
and when $x = 0$, $y' = 14$, so $\quad 14 = 7A + 0$. $\quad$ (2)
Hence $A = 2$ and $B = 1$, and $\quad y = 2 \sin 7x + \cos 7x$.

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2 A FAMILY OF SOLUTIONS, AND PARTICULAR SOLUTIONS

- The general solution of a DE normally involves one or more arbitrary constants, giving a *family* of solutions.
- The number of arbitrary constants is normally equal to the degree of the DE.
- These constants can be evaluated to give *particular solutions* if one or more *initial conditions* (or *boundary conditions*) are known.
- These conditions are usually points on the curve, but if higher derivatives occur, they may also involve the values of derivatives for particular values of x.
- A *solution curve* (or *integral curve*) may be just one of the connected branches of a function.
- The problem of finding a particular solution, given the DE and suitable initial conditions, is called an *initial value problem* or *IVP*.

A relation can be a solution of a differential equation

A relation that is not a function can also be a solution of the° differential equation. We will not be dealing with such situations in any detail, and some simple examples will be sufficient. Such examples will normally need the chain rule (and possibly alsothe product and quotient rules) to differentiate expressions involving y. For example, the next worked example requires differentiating y^2 and xy with respect to x,

$$\begin{aligned}\frac{d}{dx}(y^2) &= \frac{d}{dy}(y^2) \times \frac{dy}{dx}\\ &= 2yy',\end{aligned} \qquad \begin{aligned}\frac{d}{dx}(xy) &= y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\\ &= y + xy'.\end{aligned}$$

Example 9 13A

a Differentiate the equation $x^2 + y^2 = a^2$ of the circle with respect to x, where $a > 0$, and show that each equation in the family of curves satisfies the differential equation $y' = -\frac{x}{y}$. What geometrical significance does this have?

b Differentiate the equation $xy = a^2$ of the rectangular hyperbola with respect to x, where $a > 0$, and show that the equation satisfies the differential equation $y' = -\frac{y}{x}$. What geometrical significance does this have?

SOLUTION

a We differentiate $\quad x^2 + y^2 = a^2.$

Using the working above, $2x + \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 0$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}, \text{ as required.}$$

The radius OP has gradient $\frac{y}{x}$, and from the DE, the tangent has gradient $y' = -\frac{x}{y}$.

Thus the radius and tangent are perpendicular.

(If $y = 0$, then the the division by y above was invalid — this corresponds to the vertical tangents to the circle at the x-intercepts, when y' is undefined.)

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b We differentiate $xy = a^2$.

Using the working above, $y + xy' = 0$

$$y' = -\frac{y}{x}, \quad \text{as required.}$$

(There is no division by 0 because x is a solution of $xy = a^2$, so cannot be zero.)

The interval OP has gradient $\frac{y}{x}$, and from the DE, the tangent has gradient $-\frac{y}{x}$, which is the opposite of this. Thus the interval OP and the tangent form an isosceles triangle with the x-axis.

These two examples — one a function and the other a relation that is not a function — show that relations form families of solutions of a DE, with arbitrary constants, in the same way that functions do.

3 RELATIONS CAN BE SOLUTIONS OF DIFFERENTIAL EQUATIONS

- Relations that are not functions can be solutions of differential equations, with the usual arbitrary constants and families of curves.
- Forming a differential equation from a relation that is not a function will involve the chain rule.

First-order linear differential equations

A first-order differential equation is called *linear* if it can be put into the form

$y' + f(x)y = g(x)$, where $f(x)$ and $g(x)$ are functions of x.

For example, here are some first-order linear differential equations:

$$y' = 5x^3 \qquad y' - 6y = 0 \qquad y' - 4xy = 3x^2$$
$$x^2y' + xy = 0 \qquad y - e^xy' = 1 \qquad y'\sin x - y\cos x = 1$$

The first row are all in the given form already. Either of the terms $f(x)y$ or $g(x)$ can be missing, but y' must be present for it to be a differential equation.

The first DE is equivalent to an indefinite integral. The second is the exponential growth DE — we will discuss these equations later in Sections 13D–13E.

The second row can all be put into the standard form. Careful readers will notice that division by x^2 or $\sin x$ involves division by zero, but in this introductory course, we will mostly not bother very much with such difficulties during the solution, but instead sort out any difficulties afterwards. Compare the remarks about division by zero in worked Example 9 above.

An alternative definition of a first-order linear differential equation uses the form

$f_1(x)y' + f(x)y = g(x)$, where $f_1(x)$, $f(x)$ and $g(x)$ are functions of x.

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Example 10 13A

Classify these DEs in terms of their order and whether they are linear.

a $y' = \dfrac{y}{x}$

b $y' = \dfrac{x}{y}$

SOLUTION

a $y' = \dfrac{y}{x}$ is first-order linear because it can be put into the form $y' - \dfrac{y}{x} = 0$.

b $y' = \dfrac{x}{y}$ is first-order, but it is not linear, because it becomes $yy' = x$.

4 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

- A *first-order linear DE* is a DE that can be put into the form
$$y' + f(x)y = g(x).$$
- An indefinite integral is equivalent to a first-order linear DE with $f(x) = 0$,
$$y' = g(x).$$
- An exponential growth DE is a first-order linear DE with $g(x) = 0$ and $f(x)$ a constant, that is
$$y' - ky = 0, \quad \text{where } k \text{ is a constant.}$$

Exercise 13A

FOUNDATION

Note: In this chapter, all logarithms have base e, and we will usually write $\log x$ or $\ln x$ instead of $\log_e x$.

1 In each case, state the order of the differential equation

a $y' - y = x$
b $y'y = 3x$
c $y' + 4y' - y = \sin x$
d $y' + y\cos x = e^x$
e $y'' - \frac{1}{2}(y')^2 = 0$
f $y' + y^2 = 1$
g $y' + xy = 0$
h $xy'' + y' = x^2$
i $y'' - xy' + e^x y = 0$

2 For each of the first-order differential equations in Question **1**, state whether it is linear or non-linear.

3 For each of the differential equations in Question **1**, state how many arbitrary constants will appear in the general solution.

4 Show by substitution that the given function is a particular solution of the differential equation.

a $y = 5x^3$; $\quad xy' - 3y = 0$
b $y = x^2 - 1$; $\quad xy' - 2y = 2$
c $y = 3e^{-x}$; $\quad y' + y = 0$
d $y = \sqrt{x^2 + 4}$; $\quad y'y = x$

5 Find the general solution of the differential equation by integration.

a $y' = 2x - 3$
b $y' = 12e^{-2x} + 4$
c $y' = \sec^2 x$
d $y' = 6\cos 2x + 9\sin 3x$
e $y' = \sqrt{1 - 5x}$
f $y' = 4x\cos x^2$

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DEVELOPMENT

6 Show by substitution that the given function with arbitrary constant C is the general solution of the differential equation.

a $y = Ce^x - x - 1; \quad y' = x + y$

b $y = Cxe^{-x}; \quad xy' = y(1 - x)$

c $y = \sin(x + C); \quad (y')^2 = 1 - y^2$

d $y = \dfrac{C}{x} + 2; \quad \dfrac{dy}{dx} = \dfrac{2 - y}{x}$

7 Verify that the given function is a particular solution of the differential equation.

a $x^2y'' - 2xy' + 2y = 6; \quad y = x^2 - 2x + 3$

b $y'' - 6y' + 5y = 0; \quad y = 2e^x + e^{5x}$

c $y'' + \pi^2 y = 0; \quad y = \cos \pi x - 3 \sin \pi x$

d $y'' + 4y' + 5y = 0; \quad y = e^{-2x} \sin x$

e $x^2y'' + xy' + y = 0; \quad y = \cos(\log x)$

8 Solve these second-order differential equations by integrating twice.

a $y'' = 2$

b $y'' = \cos 2x$

c $y'' = e^{\frac{1}{2}x}$

d $y'' = \sec^2 x$

9 **a** Show that $y = e^{-x}$ and $y = e^{3x}$ are each solutions of the equation $y'' - 2y' - 3y = 0$.

b Now show that $y = Ae^{-x} + Be^{3x}$ is also a solution of this equation for any values of the constants A and B.

10 Verify by substitution that the given function is the general solution of the differential equation for all values of the constants A, B and C.

a $y''' = 6; \quad y = x^3 + Ax^2 + Bx + C$

b $y'' + 3y' + 2y = 4x; \quad y = Ae^{-x} + Be^{-2x} + 2x - 3$

c $y'' + 4y = 0; \quad y = A \cos 2x + B \sin 2x$

d $y'' + 2y' + 2y = 0; \quad y = Ae^{-x} \cos x$

e $y'' = y(x^2 - 1); \quad y = Ae^{-\frac{1}{2}x^2}$

11 Solve these initial value problems. In each case, use integration to find the general solution, then use the initial condition to evaluate the constant.

a $y' = 1; \quad y(2) = 1$

b $y' = 2x - 3; \quad y(0) = 2$

c $y' = 3x^2 + 6x - 9; \quad y(1) = 2$

d $y' = \sin x; \quad y(\pi) = 3$

e $y' = 6e^{2x}; \quad y(0) = 0$

f $y' = 3\sqrt{x} - 2; \quad y(4) = 7$

12 When an indefinite integral involves the reciprocal of a quadratic, the expression needs to be separated into the sum of two fractions, as in the following examples.

a Consider the initial value problem $y' = \dfrac{1}{x(1 - x)}$, where $y\left(\frac{1}{2}\right) = 0$.

i Show that $\dfrac{1}{x(1 - x)} = \dfrac{1}{x} + \dfrac{1}{1 - x}$.

ii Find the general solution of the DE.

iii Hence solve the IVP.

b Consider the initial value problem $y' = \dfrac{4}{(2 - x)(2 + x)}$, where $y(0) = 1$.

i Show that $\dfrac{4}{(2 - x)(2 + x)} = \dfrac{1}{2 - x} + \dfrac{1}{2 + x}$.

ii Find the general solution of the DE.

iii Hence solve the IVP.

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13 **a** **i** Differentiate both sides of the equation $x^2 + y^2 = 9$ with respect to x, using the chain rule where necessary.

ii Hence show that $\frac{dy}{dx} = -\frac{x}{y}$ at each point on the circle.

iii Part **ii** needs qualification. Where is the differential equation undefined and why is this expected?

b Likewise, find $\frac{dy}{dx}$ for these curves:

i the parabola $y^2 = x + 4$

ii the hyperbola $xy = c^2$

iii the ellipse $9x^2 + 16y^2 = 144$

iv the hyperbola $x^2 - 4y^2 = 4$

v the hyperbola $xy - y^2 = 1$

vi the folium of Descartes $x^3 + y^3 = 3xy$

14 **a** Show that $y = \sin x$ is a solution of $y'' + y = 0$.

b Find the value of the second derivative y'' when $y = 12$.

c Suppose that $y = f(x)$ is a solution of this differential equation. What does the equation say about $f''(x)$? Answer in terms of a transformation of $f(x)$.

15 Use integration and the initial conditions to find the solution of $y'' = \sec^2 x$ with $y(0) = 1$ and $y'(0) = 1$.

16 Show that the function $y = \tan x$ is a solution of the initial value problem $y'' = 2yy'$, with $y\left(\frac{\pi}{4}\right) = 1$ and $y'\left(\frac{\pi}{4}\right) = 2$.

ENRICHMENT

17 For each DE, use substitution to find the values of λ, if any, that make the function $y = Ae^{\lambda x}$ a solution for all values of A.

a $y'' - 4y' + 3y = 0$

b $y'' + 2y' + y = 0$

c $y'' - 3y' + 4y = 0$

18 Consider the function $y = \sec x$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

a Find a linear first-order initial value problem that has this solution.

b Find a non-linear first-order IVP that has this solution.

19 Consider the initial value problem $y' = -2xy$ with $y(0) = 1$. Suppose that the function $y = f(x)$ is a solution.

a Write down the y-intercept of the graph of $y = f(x)$.

b Calculate the gradient of this graph at the y-intercept.

c **i** Differentiate the differential equation to show that $y'' = (4x^2 - 2)y$.

ii Hence determine the concavity of $y = f(x)$ at the y-intercept.

d Determine the value of $f'''(0)$.

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20 Consider the differential equation $(y')^2 - xy' + y = 0$

a Show that $y = cx - c^2$ is a general solution of this equation.

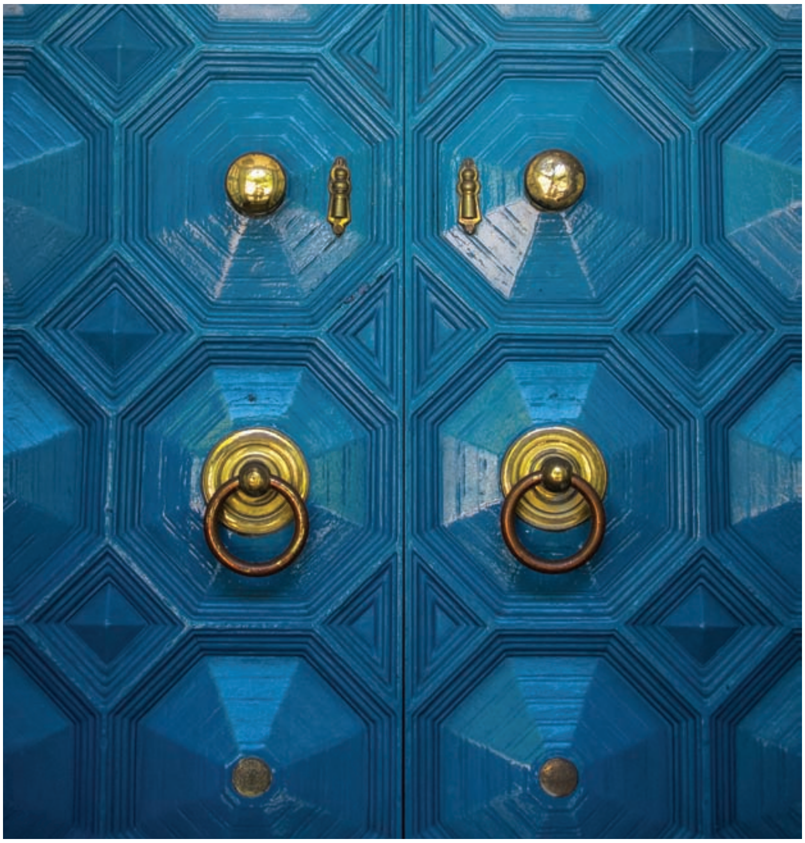
b Draw the particular solutions corresponding to $c = -2,\ -1\frac{1}{2},\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2},\ 2,$ for the domain $-4 \le x \le 4$. What do you notice?

c Find the coordinates of the point where the line corresponding to $c = p$ intersects the line corresponding to $c = p + h$.

d Show that in the limit as $h \to 0$, the coordinates of this point are $x = 2p$ and $y = p^2$.

e Eliminate p from these two equations and show that the resulting curve is a solution of the original differential equation. Try to explain what has happened.

21 Find the general solution of $y^{(n)} = 1$. How many arbitrary constants did you use?



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13B Slope fields

Before we embark on any more algebra, this section takes a visual approach to solving a DE. A *slope field* is a visual display that provides an impression of possible solution curves. The method only applies to a first-order differential equation of a special type.

Slope fields (or gradient fields or direction fields)

A slope field can only be drawn for a first-order differential equation that can be written with y' as the subject,

$$y' = G(x, y), \quad \text{where } G(x, y) \text{ is an expression in } x \text{ and } y. \qquad (*)$$

At each point P on a solution curve, the tangent to the curve at P has gradient given by equation $(*)$. This allows us to draw a *line element* at P, of a fixed short length, inclined at the same angle as the tangent at that point.

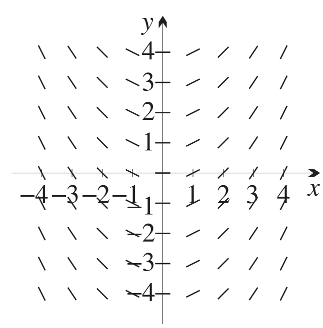
- Choose a suitable grid of points $P(x, y)$ in the plane.
- Use the equation $(*)$ to construct a table of values of the gradients at these grid points.
- At each point, draw a short line element of fixed length with that gradient.
- Centre each line element on the grid point whose tangent it represents.

Our first example will be equivalent to an indefinite integral, where we have already seen how to construct families of curves. Consider the differential equation

$$y' = \tfrac{1}{2}x.$$

All the rows in the table of slopes are the same because y is not involved.

y \ x	-4	-3	-2	-1	0	1	2	3	4
4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
0	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-1	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2



The slope field suggests a family of parabolas, all with the x-axis as axis of symmetry, and all vertical translations of each other. We know that this is true because by integration, the general solution is

$$y = \tfrac{1}{4}x^2 + C.$$

Constructing the table of gradients and the slope field is a time-consuming procedure. Refer to Question 4 in Exercise 13B for instructions how to use the free WolframAlpha.com website to draw a slope field of a differential equation.

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Terminology — slope fields or direction fields or gradient fields

Each short line element displays the gradient of the solution curve at that point. It does not indicate direction, and its length does not have any significance, so it must not be confused with a vector. The terms *slope field* or *gradient field* therefore seem preferable to another commonly used term *direction field*. Some authors distinguish all three terms.

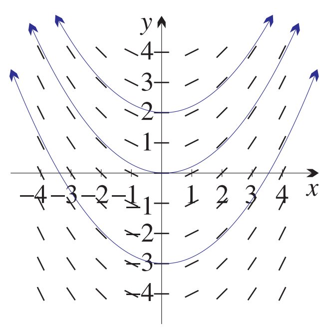
5 SLOPE FIELDS

Suppose that a DE can be written with y' as the subject, that is, as

$$y' = G(x, y)\text{ , where } G(x, y) \text{ is an expression in } x \text{ and } y.$$

- At each point in the coordinate plane where $G(x, y)$ is defined, the solution curve has the gradient given by this equation.
- Choose a suitable grid of points in the plane.
- Draw up a table showing the gradients at these grid points.
- Display these gradients by short tilted *line elements* of a fixed length, centred on these grid points in the coordinate plane.
- The resulting diagram is called a *slope field* (or *gradient field* or *direction field*).

Sketching a solution curve (or integral curve)



Place a pencil on the diagram and follow the gradients. Keep in mind that the line elements represent gradients. On the right, three solution curves have been drawn on the slope field we drew.

This is not an accurate procedure, and it it is quite unlike joining up the dots when sketching a curve from plotted points. For example, the curves do not pass through the centres of neighbouring line elements, and drawing a solution curve means threading the curve through the slope field. The purpose is to get a global view of what is happening, usually before any detailed calculations.

- Apart from some strange singularities, two solution curves can never cross, because the gradient at each point can only have one value.
- An *isocline* is a curve passing through points where the tangents have equal gradient. In the slope field above, every vertical line is an isocline — this is because the DE is equivalent to an indefinite integral, so all the solution curves are vertical translations of each other, and the gradient is therefore independent of y.
- In the slope field above, the y-axis is a special isocline because it joins all the points where the gradient is zero. Every solution curve has a stationary point where it crosses the y-axis.
- If you are given an initial condition, such as the origin, start there and draw the approximate curve, in both directions, as far as any restrictions allow. If there is no initial condition, draw at least three representative curves in different places on the plane.

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Example 11 13B

a Construct a slope table for the differential equation $y'y + x = 0$.
b From the table, draw the slope field.
c What family of curves look as if they are solution curves?
d Check your conjecture by differentiation.

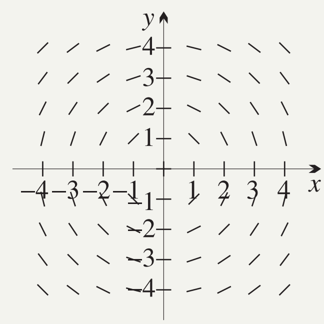
SOLUTION

a Solving $y'y + x = 0$ for y' gives $y' = -\frac{x}{y}$, provided that $x \neq 0$.

Construct a table of grid points for y' as follows.

$y \backslash x$	-4	-3	-2	-1	0	1	2	3	4
4	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$	-1
3	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1	$-\frac{4}{3}$
2	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
1	4	3	2	1	0	-1	-2	-3	-4
0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
-1	-4	-3	-2	-1	0	1	2	3	4
-2	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
-3	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$
-4	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

b



c It looks as if the solution curves are circles with centre the origin.

d Test whether $x^2 + y^2 = r^2$ is a solution, for any constant r.
Differentiating using the chain rule, $2x + 2yy' = 0$, as required.

Vertical line element

The slope field above has vertical line elements on the x-axis. Why was this done when vertical lines do not have gradient?

- The line elements on the x-axis are drawn vertical because they arise from division of a non-zero number x by y, which is zero. (You could even enter ∞ instead of $*$ into the table of grid points.)
- This does not include the origin, where the central element $\frac{0}{0}$ in the table has no meaning at all.

Some things to look for in a slope field

The diagram looks as if it is self-interpreting, but some advice about what to look for is useful. Here are four points to think about when looking at the diagram:

- Look at where the slope is zero, that is, where the line elements are horizontal. These are the places where a solution curve has a stationary point.
- Look at where the line elements slope upwards (positive gradient), and where they slope downwards (negative gradient). This tells you where a solution curve is increasing and where it is decreasing.

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For example, looking at the slope field drawn in the previous worked Example:

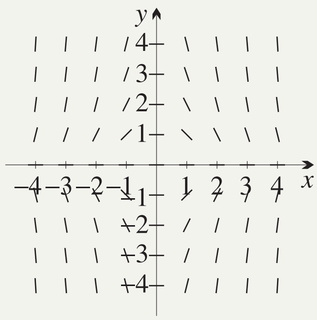
- The points with gradient zero are the y-axis (excluding the origin).
- The gradients are negative in quadrants 1 and 3, and positive in 2 and 4.
- The isoclines — curves joining points where the gradients are equal — are all the lines through the origin. That is, they are the radii of the circles.

Example 12 — 13B

Which DEs are possibly graphed in the diagram to the right?

A $y' = -xy$ **B** $y' = -x^2y$

C $y' = -xy^2$ **D** $y' = -x - y$



SOLUTION

- The slopes are zero on the y-axis. This excludes option D.
- The slopes are positive in the second quadrant. This excludes option B.
- The slopes are negative in the third quadrant. This excludes option C.
- Option A has the right sign in each quadrant. Notice also that the gradients become steeper away from the axes, as they should in option A.

6 SLOPE FIELDS AND THE SOLUTION CURVES

- Two solution curves can never cross (apart from some strange singularities).
- This is not an accurate procedure. In particular, asymptotic behaviour may not be clear.
- Precede the sketch by looking for *isoclines*, which are curves through points of equal gradient in the slope field.
- If there is an initial condition, construct the *solution curve* (or *integral curve*) beginning at a given initial point. Otherwise draw about three solution curves.

An isocline that is an asymptote

The next worked example extends worked Example 7 in the previous section by looking at the slope field of the differential equation. It also gives some further insight into the importance of isoclines.

Example 13 — 13B

a Sketch the slope field of the differential equation $y' = x + y$.

b Sketch representative solution curves, including the solution curve through the origin.

c Identify all the isoclines of the slope field.

d Explain the significance of the isocline $y = -x$.

e Explain the significance of the isocline $y = -x - 1$.

f Show that $y = Ae^x - x - 1$ is a solution, for all constants A.

g Explain these equations in terms of the slope field, including isoclines.

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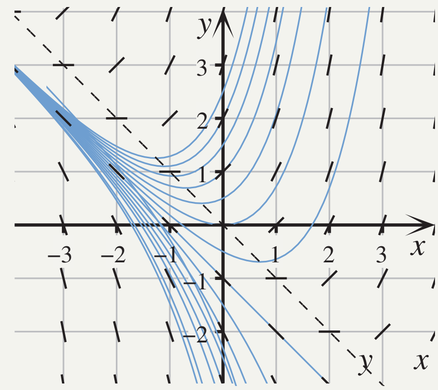
SOLUTION

a

y \ x	−4	−3	−2	−1	0	1	2	3	4
4	0	1	2	3	4	5	6	7	8
3	−1	0	1	2	3	4	5	6	7
2	−2	−1	0	1	2	3	4	5	6
1	−3	−2	−1	0	1	2	3	4	5
0	−4	−3	−2	−1	0	1	2	3	4
−1	−5	−4	−3	−2	−1	0	1	2	3
−2	−6	−5	−4	−3	−2	−1	0	1	2
−3	−7	−6	−5	−4	−3	−2	−1	0	1
−4	−8	−7	−6	−5	−4	−3	−2	−1	0



The most obvious solution curve is the line $y = -x - 1$.

b The solution through the origin is drawn here. As we saw in worked Example 7, the line $y = -x - 1$ is also a solution, and there are also solutions below $y = -x - 1$. Turn back now to the diagram for worked Example 7 to see how the three solution curves drawn there fit onto this slope field.

c The table of values above makes it clear that every line with gradient -1 is an isocline.

d Every solution curve that crosses the isocline $y = -x$ has a stationary point at the intersection, and given the slopes on both sides, it will be a minimum turning point. The curves below $y = -x - 1$, however, do not cross this line.

e The isocline $y = -x - 1$ is exceptional in many ways.

— It is a solution of the DE, because it satisfies $y' = x + y$. When the line is substituted into the DE, both sides equal -1.

— This isocline is a line, and its gradient equals the gradient of the slope field at each point on it.

— No other solution curve crosses this curve, and the other solution curves fall into two groups on each side of this line.

— The other solution curves are asymptotic to this line in the second quadrant. Look at the diagram, and look back again to worked Example 7.

f Substituting into $y' = x + y$,

$$\begin{aligned} \text{LHS} &= \frac{d}{dx}(Ae^x - x - 1) \\ &= Ae^x - 1 \\ \text{RHS} &= x + (Ae^x - x - 1) \\ &= \text{LHS}, \end{aligned}$$

so $y = Ae^x - x - 1$ is a solution, for all constants A.

g The isocline $y = -x - 1$ is a solution curve — it corresponds to $A = 0$.
The solution curves above $y = -x - 1$ curl upwards, corresponding to $A > 0$.
The solution curves below $y = -x - 1$ curl downwards, corresponding to $A < 0$.
The fact that every solution curve is asymptotic to the isocline $y = -x - 1$ corresponds to the limit $\lim\limits_{x \to -\infty} (Ae^{kx} - 1) = -1$.

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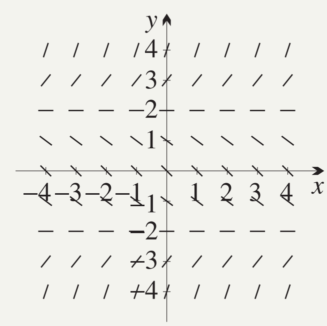
Constant solutions are horizontal asymptotes

The derivative of a constant function is zero. Hence any constant solutions $y = k$ of a DE stands out on the slope field. Look at the diagrams below — they are the two lines consisting of horizontal line elements. A constant solution is an isocline, and it divides the other solutions into those above and those below. Usually many of the solution curves in the family have this line as a horizontal asymptote on the left or right.

Constant solutions are particular examples of the more general phenomenon of *equilibrium solutions*, for reasons that the next worked example will make clear.

Example 14 13B



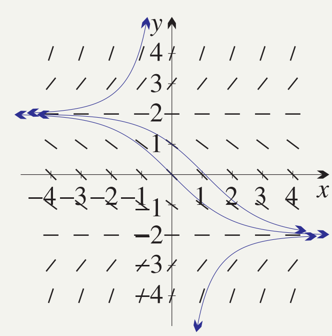
a Sketch possible solution curves to the slope field in the upper diagram to the right. It is the slope field of the differential equation $y' = \frac{1}{4}(y - 2)(y + 2)$.

b From the slope field, identity the constant solutions, that is, the equilibrium solutions.

c Substitute into the DE to show that they are solutions.

d If the horizontal axis is time, describe the behaviour of the solution curves near those constant solutions, and distinguish between them.

SOLUTION



a Four different connected solution curves are drawn in the lower diagram to the right. (Whether or not they have vertical asymptotes will become clear when the logistic equation is solved in Section 13D.)

b The horizontal isoclines $y = 2$ and $y = -2$ stand out because both consist of points where the gradient is zero.

c They are solution curves because substituting $y = 2$ or $y = -2$ into the DE gives LHS $= y' = 0$ and RHS $= 0$.

d The time x moves forwards, so the two constant solutions have different meanings, because $y = -2$ is an asymptote on the right, and $y = 2$ is an asymptote on the left.

— For $y = -2$: As time increases, the middle and bottom solution curves have the limit $y = -2$. This means that over time, the system moves towards *equilibrium* at $y = -2$.

— For $y = 2$: If the curve starts at the point $P(-5, 2.001)$, it will go up without bound. If the curve starts at the point $P(-5, 1.999)$, it will go down, then move towards the horizontal asymptote $y = -2$. Thus a minutely small change in initial situation — even a quantum fluctuation — may produce a huge change in the final situation.

Thus $y = 2$ is an equilibrium solution in the sense that with this solution, y does not change over time. But the equilibrium is an *unstable equilibrium* because the slightest fluctuation will send it permanently away from $y = 2$.

The equilibrium at $y = -2$, however, is a *stable equilibrium* because any slight change will see the system return to where it was.

Note: The differential equation in this example is a *logistic equation*. Such equations will be discussed further in Sections 13D–13E.

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Isoclines and the slopefield

The worked examples in the previous discussion should have made it clear how useful isoclines can be in examining the solution curves of a DE. Here is a summary of some uses that we have made of them.

- Isoclines consisting of horizontal line elements show where solution curves that cross them have stationary points.
- Some isoclines have a particular property that makes them stand out — they are lines consisting of line elements with the same gradient as the line.
  - Such an isocline is a solution curve, so no other solution curve crosses it.
  - It therefore divides the other solutions into two groups on either side.
  - Normally many of the solution curves have this line as an asymptote.
- Constant solutions have this property, and are the most important isoclines of all. They are horizontal lines consisting of horizontal line elements, and are seen immediately on the slope field.
- The presence of an isocline can often be used to help identify a correct DE from a list of options. Any option that does not have the requisite isocline can immediately be eliminated from consideration.

Using technology to deal with slope fields

Everyone needs to plot by hand a few slope fields after first preparing the tables and doing the calculations. But this is laborious, and technology is a great assistance. There is software available, there are online resources, and there are some calculators with screens that will do the job. Here are some suggestions:

- The WolframAlpha.com website is free for the tasks required in this chapter. Question 4 gives some initial instructions about commands, but use the website's help functions.
- A link to a Desmos calculator set up to plot slope fields has been provided with Question 4 in the interactive textbook.
- If your calculator has a screen, check if it can handle slope fields.

Exercise 13B

FOUNDATION

1 In each case, find the value of y' at the given point.

a $y' = 2x - 3$ at $(1, 1)$
b $y' = 2\cos x - 1$ at $(0, 0)$
c $y' = 4 - y^2$ at $(0, 1)$
d $y' = \dfrac{1}{1 + y}$ at $(3, 1)$
e $y' = \dfrac{y}{x} + 1$ at $(-2, 1)$
f $y' = xy - x$ at $(1, -2)$

2 Answer these questions for the differential equation $y' = \frac{1}{2}x - 1$.

a Copy and complete the table of values for the slope field.

b Draw a number plane with a scale of 1 cm = 1 unit with domain $[-1, 5]$ and range $[-1, 5]$.

c Through each grid point in the table, draw a line element $\frac{1}{2}$ cm long, centred on the point and with gradient as given in the table.

d Notice that the vertical line $x = 1$ is an isocline. Why is this expected from the table?

e Check for any other isoclines evident in the table or graph.

y \ x	-1	0	1	2	3	4	5
3	$-\frac{3}{2}$	-1	$-\frac{1}{2}$				
2	$-\frac{3}{2}$						
1							
0							
-1							
-2							
-3							

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f The slope field indicates a positive gradient to the right of $x = 2$ and a negative gradient to the left of $x = 2$. What will be the concavity of a solution curve?

g Starting at the origin, draw an integral curve (solution curve) to the right and to the left. What type of curve might this be?

h Draw two more integral curves, starting at $(0, 2)$ and $(0, 3)$, making sure that none of the curves cross.

3 Consider the differential equation $y' + y = x$.

a Make y' the subject.

b Copy and complete the table of values for the slope field.

y \ x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-2	$-\frac{3}{2}$	-1				
$\frac{1}{2}$	$-\frac{3}{2}$	-1					
0							
$-\frac{1}{2}$							
-1							
$-\frac{3}{2}$							
-2							

c Draw a number plane with a scale of 2 cm = 1 unit with domain $[-1, 2]$ and range $[-2, 1]$.

d Through each grid point in the table, draw a line element $\frac{1}{2}$ cm long, centred on the point and with gradient as given in the table.

e Look carefully at the table for matching entries. Check that these agree with the isoclines in your graph.

f Look carefully at your graph. What concavity would you expect for the solution curve passing through the origin?

g Draw this solution curve.

h Now add solution curves that pass through $(-1, -1)$ and $(1, -1)$, making sure that none of the curves cross.

i Which line do all your solution curves appear to have as an asymptote?

j Is this line a solution of the differential equation?

4 [Technology]

Various mathematical applications can be used to save time plotting slope fields. A Desmos slope field plotter is provided in the interactive textbook. In addition, the slope field for $y' = x + y$ can be plotted in the free internet application WolframAlpha.com by using the command

```
slope field x + y, {x, −4, 4}, {y, −4, 4}.
```

The two terms in braces are optional, and are used to indicate the domain $-4 \le x \le 4$ and range $-4 \le y \le 4$.

Use WolframAlpha or Desmos to plot the following slope fields. In each case:

i identify any points or isoclines where $y' = 0$,

ii identify any other obvious isoclines,

iii state how the gradients of the line elements change along the line $x = 1$, from bottom to top, and

iv state how the gradients of the line elements change along the line $y = 2$, from left to right.

a $y' = -y^2$ **b** $y' = -\dfrac{1}{x^2}$ **c** $y' = \cos\left(\frac{\pi}{4}x\right)$

d $y' = 1 - x + y$ **e** $y' = \dfrac{2y}{x} - y$ **f** $y' = \dfrac{2x}{y + 1} - x$

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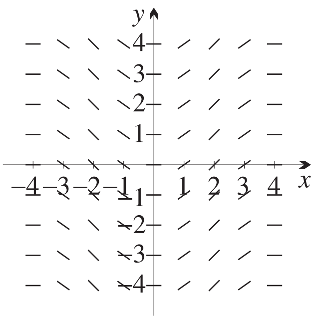
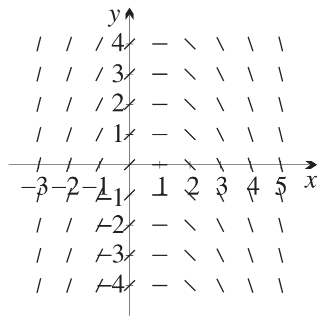
5 For each slope field below, draw the solution curves that pass through the two given points. Ensure that at each point on each curve, the gradient is roughly the average of the slopes indicated at nearby points.

a



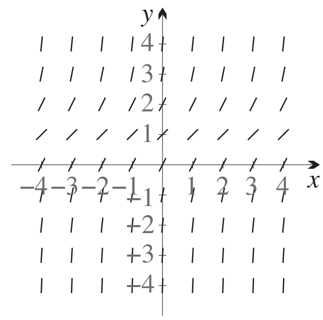
i $(0, 0)$ **ii** $(2, 0)$

b



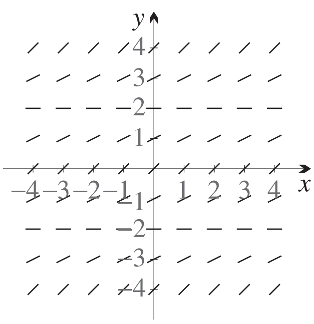
i $(0, 0)$ **ii** $(1, 3)$

c



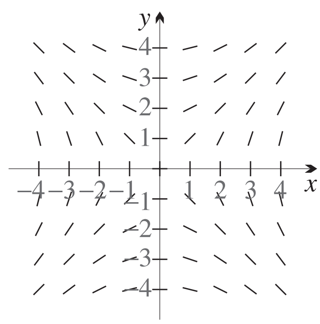
i $(0, 1)$ **ii** $(1, 0)$

d



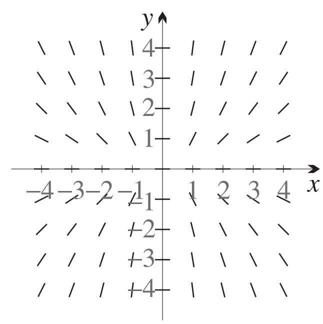
i $(-2, 0)$ **ii** $(2, 0)$

e



i $(0, 2)$ **ii** $(-2, 0)$

f

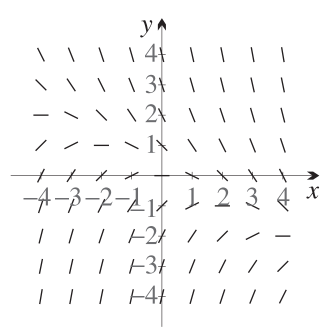


i $(1, 1)$ **ii** $(-2, 1)$

6 For each slope field in the previous question, use the isoclines to determine whether y' is a function of x alone, a function of y alone, or a combination of both.

DEVELOPMENT

7 The slope field for the differential equation $y' = -\frac{1}{2}x - y$ is drawn to the right. Make a copy of the slope field and answer the following questions.



a On your copy of the slope field, draw the solution curves through the points $(-2, 0)$ and $(2, 0)$.

b Look carefully at the slope of the line elements on the vertical line $x = 1$, from bottom to top.

i Do the gradients increase or decrease?

ii Do your solution curves converge (get closer) or diverge (further apart) as they cross $x = 1$, from left to right?

c Explain why the line $y = \frac{1}{2} - \frac{1}{2}x$ is an isocline.

d Show that the equation of this isocline is also a solution of the DE, then add the solution to your copy.

e What do you notice about the isocline and the two solution curves you have drawn?

8 Consider the differential equation $y' = \dfrac{9 - y^2}{9}$.

a Draw the slope field for this DE.

b What are the constant solutions for this DE?

c Are these constant solutions isoclines?

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d Consider the slope of the line elements on the vertical line $x = 1$.

i If $y > 0$, will the solution curves converge or diverge as they cross $x = 1$ from left to right?

ii What happens to the solution curves if $y < 0$?

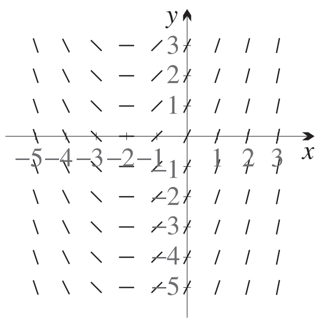
iii Is the same true as the solution curves cross other vertical lines from left to right?

iv What do you conclude about the constant solutions?

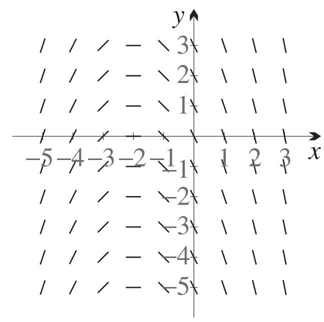
e Confirm all your answers by drawing the solution curve through $(0, 0)$.

9 By considering isoclines and constant solutions, determine which of the slope fields shown below corresponds to the differential equation $y' = -1 - y$.

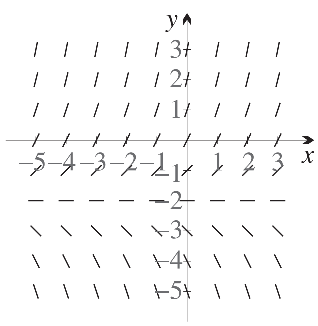
A



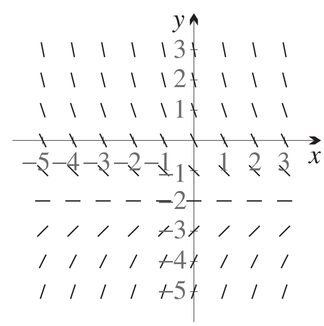
B



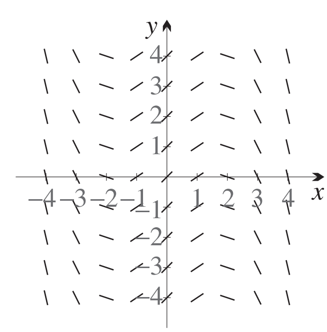
C



D



10 Consider the slope field shown to the right. Look for any of the important features of the slope field: constant solutions, points where $y' = 0$, isoclines, converging or diverging solution curves. Hence determine which of the following DEs corresponds to the slope field.

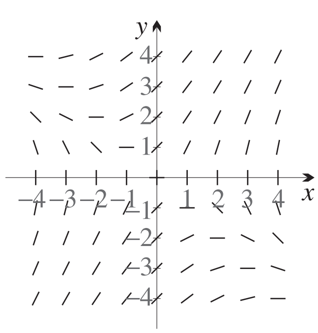


A $y' = \frac{1}{3}(x^2 - 3)$ **B** $y' = \frac{1}{3}(y^2 - 3)$

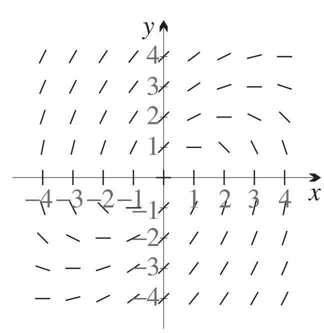
C $y' = \frac{1}{3}(3 - x^2)$ **D** $y' = \frac{1}{3}(3 - y^2)$

11 Which slope field below corresponds to the DE $y' = 1 - \dfrac{x}{y}$?

A



B



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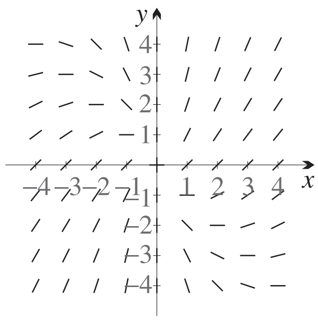
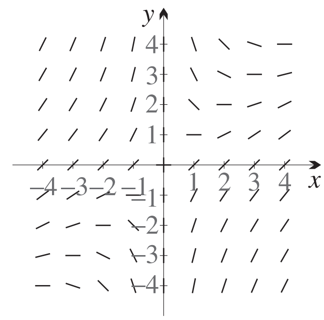
C



D



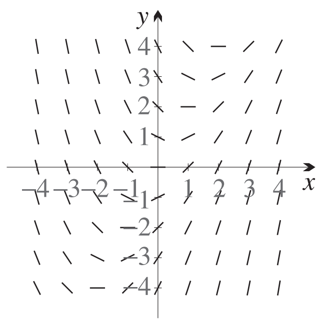
12 In each part, determine which DE corresponds to the slope field shown.

a **A** $y' = x + \frac{1}{2}y$

B $y' = x - \frac{1}{2}y$

C $y' = \frac{1}{2}x + y$

D $y' = \frac{1}{2}x - y$

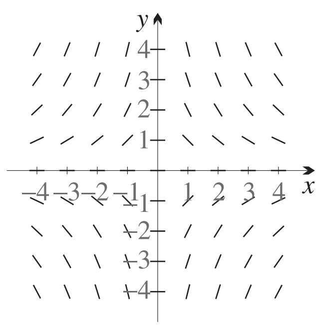


b **A** $y' = \dfrac{2xy}{1 + y^2}$

B $y' = \dfrac{2xy}{1 + x^2}$

C $y' = \dfrac{-2xy}{1 + y^2}$

D $y' = \dfrac{-2xy}{1 + x^2}$



13 **a** **i** Draw the slope field for the differential equation $y' = -\dfrac{y}{x}$.

ii Add the solution curves through $(2, 2)$ and $(-2, -2)$ to your graph.

b **i** On a separate number plane draw the slope field for $y' = \dfrac{x}{y}$.

ii Add the solution curves through $(2, 0)$ and $(-2, 0)$.

c **i** Show that the hyperbola $xy = 4$ is the solution of the IVP in part **a**.

ii Show that the hyperbola $x^2 - y^2 = 4$ is the solution of the IVP in part **b**.

iii Graph these two hyperbolas (without their slope fields) on a third number plane.

d If you have drawn your graph in part **c** carefully enough, then the hyperbolas will be perpendicular where they intersect. Why?

14 **a** **i** Draw the slope field for the differential equation $y' = \dfrac{2y}{x}$.

ii Add the solution curve through $(2, 2)$ to your graph.

b **i** On a separate number plane draw the slope field for $y' = -\dfrac{x}{2y}$.

ii Add the solution curve through $(2, 0)$ $\big($it also passes through $(-2, 0)\big)$.

c **i** Show that the parabola $y = \frac{1}{2}x^2$ is the solution of the IVP in part **a**.

ii Show that the ellipse $x^2 + 2y^2 = 4$ is the solution of the IVP in part **b**.

iii Graph these two (without their slope fields) on a third number plane.

d If you have drawn your graph in part **c** carefully enough, then the parabola and ellipse will be perpendicular where they intersect. Why?

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15 [Shifting]

a Show that the equation of the circle with radius 4 and centre the origin satisfies the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

b What is the equation of this circle if it is shifted 3 units right and 1 units up?

c Show that this new circle satisfies the differential equation $\dfrac{dy}{dx} = -\dfrac{x-3}{y-1}$.

d It should be clear from this that if a curve is translated h units right and k units up, then the new DE is obtained by replacing x by $(x-h)$ and y by $(y-k)$.

The hyperbola $x^2 - y^2 = 1$ satisfies the DE $y' = \dfrac{x}{y}$, and the slope field is graphed in Question **5e**.

i Write down the DE for the shifted hyperbola $(x-1)^2 - (y+2)^2 = 1$.

ii Sketch its slope field by shifting the one in Question **5e**.

16 Slope fields can be used to draw the solution curves for differential equations that cannot be solved algebraically. For example, the function $\Phi(x)$ used in statistics is defined by the integral formula

$$\Phi(x) = \int_{-\infty}^{x} \tfrac{1}{\sqrt{2\pi}} e^{-t^2}\, dt\,.$$

Differentiating both sides gives the differential equation:

$$\Phi'(x) = \tfrac{1}{\sqrt{2\pi}} e^{-x^2}.$$

a Plot the slope field for $y = \Phi(x)$, that is, for $y' = \tfrac{1}{\sqrt{2\pi}} e^{-x^2}$.

b It is known that $y = \Phi(x)$ has two asymptotes, $y = 1$ and $y = 0$. Using these asymptotes and the slope field, sketch the integral curve through $\left(0, \tfrac{1}{2}\right)$.

c Use your graph to estimate $\Phi(1)$ correct to one decimal place.

17 [An alternative method]

Here is an alternative method, based on isoclines, for drawing slope fields and solution curves. Consider the differential equation $y' = xy$.

a What two isoclines correspond to $y' = 0$? Add horizontal line elements to these.

b Write down the equation of the isoclines for the constant gradient $y' = C$, where $C \neq 0$, and name the type of curve this is.

c Graph the isoclines for each value $C = 1, -1, 2, -2, 3, -3,$ for $-4 \le x \le 4$ and $-4 \le y \le 4$.

d On the isocline for $C = 1$, add line elements at 45° to the horizontal, roughly equally spaced along the isocline.

e Likewise add the line elements for $C = 2$ at angle 63° (note that $\tan 63° \doteqdot 2$).

f Add the line elements for $C = 3$ at angle 72°.

g Similarly add line elements for the isoclines where $C = -1, -2, -3$.

h Draw the integral curve (solution curve) through $(0, 1)$. Ensure that your curve crosses each isocline at the correct angle.

i Likewise draw the integral curves through $\left(0, \tfrac{1}{4}\right)$ and $\left(0, -\tfrac{1}{8}\right)$.

j **i** What is an advantage of this technique?

ii What is a disadvantage of this technique?

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18 [A comparison of methods]
The alternative method for drawing the slope field presented in the previous question is impractical when the expression for y' is a complicated or an unknown curve. In some instances, however, it is much preferable. Consider the differential equation $y' = -\frac{x}{y}$.

a **i** What can be said about the line elements of the slope field for points on the y-axis other than the origin?
ii The value of y' at points on the x-axis is undefined. By considering $\lim\limits_{y \to 0^+} y'$ and $\lim\limits_{y \to 0^-} y'$ when $x \neq 0$, what can be said about the line elements of the slope field for points on the x-axis?
iii Draw the slope field using a grid of points as in Questions **2** and **3** or by using software such as the free internet application WolframAlpha.
iv What shape seems to be suggested?

b **i** What is the equation of the isocline corresponding to $y' = C$?
ii On a new number plane, draw by hand the isoclines for $C = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \sqrt{3}$ and $\sqrt{3}$. (Hint: What angles of inclination correspond to these gradients?)
iii Explain why each line element is perpendicular to its isocline.
iv Add line elements to each isocline at equally spaced intervals.
v Is the shape of part **c** clearer now?

ENRICHMENT

19 Prove the following three statements about isoclines.
a If $y' = f(x)$, then the isoclines $y' = c$ are the vertical lines.
b If $y' = g(y)$, then the isoclines $y' = c$ are the horizontal lines.
c If $y = cx + b$ is a solution of a first-order DE for a specific value of the constant c, then this line is also an isocline. (Compare this result with Question **7c**.)

20 None of the integral curves in previous questions crosses itself. In this question you will investigate what happens when a curve crosses itself.

The folium of Descartes has equation $x^3 + y^3 = 3axy$ for different values of the constant a. In this question put $a = 1$, so that $x^3 + y^3 = 3xy$.

a Show that $y' = \dfrac{y - x^2}{y^2 - x}$.
b Draw the slope field, with $-2 \le x \le 2$ and $-2 \le y \le 2$.
c The integral curve is horizontal where $y' = 0$. What curve do these points lie on?
d The integral curve is vertical where $\dfrac{1}{y'} = 0$. What curve do these points lie on?
e Use the slope field to plot the integral curve that passes through $(\sqrt[3]{2}, \sqrt[3]{4})$. You may assume that this curve also passes through $(\sqrt[3]{4}, \sqrt[3]{2})$ and the origin.
f What happens at the origin?
g The integral curve appears to have an asymptote that is also an isocline. Write down its equation, show that it is indeed an isocline, and show that this is also a solution of the DE in part **a**.
h In which line does the integral curve have symmetry? Explain this in terms of:
i the equation of the folium $x^3 + y^3 = 3axy$,
ii the differential equation $y' = \dfrac{y - x^2}{y^2 - x}$.

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13C Separable differentiable equations

Many particular types of differential equations can be solved by systematic approaches. *Separable DEs*, after a suitable rearrangement, can be solved just by integration.

Separable differential equations

A first-order differential equation is called *separable* if y' can be written as the product of a function of x and a function of y,

$$y' = f(x)\, g(y).$$

There are three types of separable DEs.

- The general case is $y' = f(x)\, g(y)$, where neither function is a constant.
- The case $y' = f(x)$, where $g(y) = 1$, is equivalent to an indefinite integral.
- The case $y' = g(y)$, where $f(x) = 1$, will be discussed in Section 13D.

Solving a separable differential equation

The key step is to separate the dx and dy in the derivative, exactly as we were doing in the last chapter when integrating by substitution.

We write the DE as $\dfrac{1}{g(y)}\, dy = f(x)\, dx,$

then integrate both sides, $\displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$

Note: The chain rule is the justification of this. First, it allows us to separate the dx and the dy, as is the first line above. Secondly, it allows us to cancel dx.

The more complete argument is $\dfrac{dy}{dx} = f(x)\, g(y)$

$\boxed{\div g(y)}$ $\dfrac{1}{g(y)}\dfrac{dy}{dx} = f(x),$

integrating with respect to x, $\displaystyle\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)\, dx$

and cancelling dx, $\displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$

Example 15 13C

a Solve $y' = -2xe^{y}$. **b** Find the solution curve through $(0, 0)$.

SOLUTION

a Arrange the equation as $-e^{-y}\, dy = 2x\, dx$ (Why was this convenient?)

and integrate, $\displaystyle\int -e^{-y} dy = \int 2x\, dx$

$$\begin{aligned} e^{-y} &= x^2 + C, \text{ for some constant } C \\ -y &= \log(x^2 + C) \\ y &= -\log(x^2 + C). \end{aligned}$$

b Substituting $(0, 0)$ gives $0 = -\log C,$

so $C = 1$ and $y = -\log(x^2 + 1).$

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Example 16 13C

a Solve the DE $y' = \frac{x}{y}$.

b Find the solution passing through $P(4, 5)$.

SOLUTION

a The DE is $\qquad \frac{dy}{dx} = \frac{x}{y}$,

and separating dy and dx, $\qquad y\,dy = x\,dx$.

Now we can integrate, $\qquad \int y\,dy = \int x\,dx$,

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$, for some constant C,

and putting $D = 2C$, $\qquad y^2 - x^2 = D$, for some constant D.

b Substituting the point $P(4, 5)$, $\qquad 25 - 16 = D$,

so $D = 9$, and the solution is $\qquad y^2 - x^2 = 9$.

7 SEPARABLE DIFFERENTIAL EQUATIONS

- A first-order DE is called *separable* if it can be put into the form
  $$y' = f(x)\,g(y).$$
- The general case is $y' = f(x)\,g(y)$, where neither function is a constant.
  — The case $y' = f(x)$, where $g(y) = 1$, is equivalent to an indefinite integral.
  — The case $y' = g(y)$, where $f(x) = 1$, will be discussed in Section 13D.
- To solve the general case, integrate after putting it into the form
  $$\frac{1}{g(y)}\,dy = f(x)\,dx.$$

Look for constant solutions before solving

It is easy to check whether a constant function $y = k$ is a solution of a DE because the derivative y' is zero. This step is important because when the DE is solved by the methods above, constant solutions are often missing because of a division by zero or other issue.

Constant solutions were discussed in Section 13B — they correspond on the slope field to horizontal isoclines consisting of horizontal line elements, and they are usually horizontal asymptotes of nearby solution curves.

The first step in solving any DE is therefore, ‘Look for constant solutions.’

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Example 17 13C

a Solve $y' = -xy^2$.

b Find the particular solution given that:

i $y(1) = \frac{1}{2}$, **ii** $y(2) = 0$.

SOLUTION

a First, the constant function $y = 0$ is trivially a solution of the DE.

Now we can write the DE as $-y^{-2}\,dy = x\,dx$, because $y \neq 0$,

and integrating, $$\int -y^{-2}\,dy = \int x\,dx,$$

so for some constant C, $$y^{-1} = \tfrac{1}{2}x^2 + C. \qquad (*)$$

b i Substituting $y(1) = \frac{1}{2}$ into $(*)$, $2 = \frac{1}{2} + C$,

so $C = \frac{3}{2}$ and $$y^{-1} = \tfrac{1}{2}x^2 + \tfrac{3}{2}$$

$$y = \frac{2}{x^2 + 3}.$$

ii Substituting $y(2) = 0$ into $(*)$ is impossible because of division by zero, but the first solution $y = 0$ satisfies $y(2) = 0$, so it is the required solution.

8 CONSTANT SOLUTIONS

- Always check first whether any constant functions are solutions of the DE.
- These are horizontal isoclines consisting of horizontal line elements, and they are usually horizontal asymptotes of nearby solution curves.

We remarked in worked Example 14 that a constant solution corresponds to *stable equilibrium* when it is an asymptote on the right to nearby solution curves, and corresponds to *unstable equilibrium* when it is an asymptote on the left.

Dealing with absolute values in the solution

Many solutions obtained by the methods used with separable DEs result in absolute values because $\dfrac{1}{x}$ has primitive $\log|x|$. This requires some care because although an arbitrary constant C may take any value, the power e^C is never negative and never zero.

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Example 18 13C

a Solve the DE $y' = x(1 - y)$.

b Find the particular solution passing through:

i the origin, **ii** $(1, 1)$.

SOLUTION

a First, the constant function $y = 1$ is trivially a solution.

Now we can write the DE as $\dfrac{dy}{1 - y} = x\,dx$, because $y \neq 1$,

$$\text{and integrating,} \quad \int \frac{dy}{y - 1} = -\int x\,dx$$

$$\log|y - 1| = -\tfrac{1}{2}x^2 + C, \text{ for some constant } C$$

$$|y - 1| = e^{-\frac{1}{2}x^2 + C},$$

and putting $A = e^C$, $\quad |y - 1| = Ae^{-\frac{1}{2}x^2}$, where A is positive.

Hence $\quad y - 1 = Ae^{-\frac{1}{2}x^2}$, where A can be positive or negative.

We have already remarked that $y - 1 = 0$ is a solution of the original DE,

so the general solution is $y - 1 = Ae^{-\frac{1}{2}x^2}$, for any constant A,

$$y = 1 + Ae^{-\frac{1}{2}x^2}. \qquad (*)$$

b **i** Substituting $(0, 0)$ into $(*)$ gives $0 = 1 + A \times 1$,

so $A = -1$, and the particular solution is $y = 1 - e^{-\frac{1}{2}x^2}$.

ii Substituting $(1, 1)$ into $(*)$ gives $1 = 1 + Ae^{-\frac{1}{2}}$

so $A = 0$, giving the constant function $y = 1$ identified on the first line.

Example 19 13C

a Solve the DE $y' = xy$.

b Find the solution for which:

i $y(2) = 3$, **ii** $y(3) = 0$.

SOLUTION

a First, the constant function $y = 0$ is trivially a solution.

Now we can write the DE as $\dfrac{dy}{y} = x\,dx$, because $y \neq 0$,

$$\text{and integrating,} \quad \int \frac{dy}{y} = \int x\,dx$$

$$\log|y| = \tfrac{1}{2}x^2 + C, \text{ for some constant } C,$$

so $\quad |y| = e^C e^{\frac{1}{2}x^2}$

$$|y| = A\,e^{\frac{1}{2}x^2}, \text{ where } A > 0.$$

Hence $\quad y = A\,e^{\frac{1}{2}x^2}$, where A can be positive or negative.

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We have already remarked that $y = 0$ is a solution of the original DE,

so the general solution is $y = A\,e^{\frac{1}{2}x^2}$, for any constant A. $(*)$

b **i** Substituting $y(2) = 3$ gives $3 = Ae^2$,

so $A = 3e^{-2}$ and $y = 3e^{\frac{1}{2}x^2 - 2}$.

ii This solution is the constant solution $y = 0$.

9 DEALING WITH ABSOLUTE VALUE IN SOLUTIONS

When interpreting absolute value signs in a solution

- Acknowledge that e^C is always positive.
- Modify the solution when removing the absolute value signs.
- Modify the solution again if any constant function is a solution.

Exercise 13C

FOUNDATION

1 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x - 1}{y + 1}$.

a Multiply through by $y + 1$ and then by dx, so that the variables are separated.

b Use the result $\displaystyle\int (x + a)^n\,dx = \frac{1}{n + 1}(x + a)^{n+1} + C$ to find the general solution of this differential equation. Write your answer without using fractions.

2 Likewise find the general solutions of these separable equations. Make y the subject of the solution in each case.

a $\dfrac{dy}{dx} = xe^{-y}$

b $\dfrac{dy}{dx} = 4x^3(1 + y^2)$

3 **a** Explain why $y = 0$, where $x \neq 0$, is a solution of $\dfrac{dy}{dx} = -\dfrac{y^2}{x}$. (Always look first for constant solutions.)

b Use the method of separable DEs to find the other solutions.

4 Suppose that the solution curve for $\dfrac{dy}{dx} = -\dfrac{x}{y}$ passes through $(1, \sqrt{3})$.

a Separate the variables and hence write down the corresponding equation of integrals.

b The general solution of this DE is a relation, not a function. Find the general solution, writing your answer without fractions.

c Hence determine the equation of the solution curve through the given point.

5 Likewise, for each DE, find the solution curve passing through the given point.

a $\dfrac{dy}{dx} = \dfrac{x}{y}$, through $(0, 1)$

b $\dfrac{dy}{dx} = (1 + x)(1 + y^2)$, with $y(-1) = 0$

c $\dfrac{dy}{dx} = -2y^2x$, with $y(1) = \frac{1}{2}$

d $\dfrac{dy}{dx} = e^{-y}\sec^2 x$, through $(\frac{\pi}{4}, \log 2)$

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DEVELOPMENT

6 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2y + 4}{x}$.

a Find the constant solution, substituting to show that it is a solution of the DE.
b Use separation of variables to find the other solutions of the DE.
c How can the solutions in parts **a** and **b** be combined?

7 Consider the differential equation $y' = -xy$.

a Find the constant solution, substituting to show it is a solution of the DE.
b Use separation of variables to find the other solutions of the DE.
c How can the solutions in parts **a** and **b** be combined?

8 Use a similar approach to Questions **5** and **6** to solve these DEs.

a $\dfrac{dy}{dx} = \dfrac{2 - y}{x}$ **b** $\dfrac{dy}{dx} = \dfrac{xy}{1 + x^2}$ **c** $\dfrac{dy}{dx} = \dfrac{-2y}{x}$

d $\dfrac{dy}{dx} = y \sin x$ **e** $\dfrac{dy}{dx} = \dfrac{3y}{x^2}$ **f** $\dfrac{dy}{dx} = \dfrac{y(1 - x)}{x}$

9 **a** Find all the constant solutions of $\dfrac{dy}{dx} = 3x^2 \cos^2 y$ in the interval $-2\pi \le y \le 2\pi$.

b Find all the non-constant solutions.

10 Consider the initial value problem $\dfrac{dy}{dx} = \dfrac{2y}{x - 1}$ with $y(2) = 1$.

a Show that the constant solution of the DE is not a solution of the IVP.
b Use separation of variables to find the general solution of the DE.
c Hence solve the IVP.

11 Consider the initial value problem $\dfrac{dy}{dx} = (y - 1)\tan x$ with $y(\frac{\pi}{4}) = 3$.

a Show that the constant solution of the DE is not a solution of the IVP.
b Use separation of variables to find the general solution of the DE.
c Hence solve the IVP.

12 Use a similar approach to Questions **8** and **9** to solve these IVPs.

a $\dfrac{dy}{dx} = \dfrac{y}{x}$ with $y(2) = 1$ **b** $\dfrac{dy}{dx} = \dfrac{y}{2x}$ with $y(1) = 2$

c $\dfrac{dy}{dx} = \dfrac{-2xy}{1 + x^2}$ with $y(1) = 2$ **d** $\dfrac{dy}{dx} = -\dfrac{y}{x}$ with $y(2) = 1$

e $\dfrac{dy}{dx} = y \cos x$ with $y(\frac{\pi}{2}) = 1$ **f** $\dfrac{dy}{dx} = \dfrac{y(2 - x)}{x^2}$ with $y(2) = \frac{1}{2}$

13 **a** Differentiate $\log(\log x)$.

b Hence find the general solution of $(x \log x)y' = y$.

14 **a** Show that $\dfrac{x}{x + 2} = 1 - \dfrac{2}{x + 2}$.

b Hence solve the initial value problem $(x + 2)y' - xy = 0$ with $y(0) = 1$.

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15 **a** Use the double-angle formulae to rewrite $2\cos^2 x$ in terms of $\cos 2x$

b The solution of the DE $\dfrac{dy}{dx} = \dfrac{2\cos^2 x}{y}$ is a relation and not a function. Use part **a** to find its equation, given that it passes through $(0, \sqrt{2})$.

16 **a** Let $y = x \times u$, where u is an unknown function of x. Use the product rule to find an expression for y'.

b Consider the differential equation $xy' = 2x + 2y$.

i Use the result of part **a** to write a corresponding differential equation for u that is separable.

ii Solve this DE for u.

iii Hence write down the general solution of $xy' = 2x + 2y$.

ENRICHMENT

17 Suppose that $(x^2 + 1)y' + (y^2 + 1) = 0$ with $y(0) = 1$.

a Find a general solution of this DE.

b Show that the general solution is equivalent to $\dfrac{y + x}{1 - xy} = D$.

c Hence find the solution of the IVP. Make y the subject of your answer.

18 For the unwary mathematician, the initial value problem

$$\frac{dy}{dx} = xy^{\frac{1}{2}} \quad \text{with} \quad y(2) = 1$$

appears to have two solutions:

$$y_1 = \tfrac{1}{16}x^4 \quad \text{and} \quad y_2 = \tfrac{1}{16}(x^2 - 8)^2.$$

a Show that both y_1 and y_2 satisfy the initial condition $y(2) = 1$.

b Show that both y_1 and y_2 satisfy the modified DE $(y')^2 = x^2 y$.

c Draw the slope field for the original DE, $y' = xy^{\frac{1}{2}}$. Then add both y_1 and y_2 to the graph and observe that y_2 clearly does not follow the slope field.

d Explain algebraically why y_2 is not a solution, and then correctly derive y_1.

19 [Picard Iterations]

Suppose that $y' = f(x, y)$ with $y(x_0) = y_0$ has solution $y(x)$.

a Explain why $y(x) = y_0 + \displaystyle\int_{x_0}^{x} f\big(t, y(t)\big)\, dt$.

b Now consider the sequence $y_0, y_1(x), y_2(x) \ldots$, where y_0 is constant, and where

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$\text{with } y_{n+1}(x) = y_0 + \int_{x_0}^{x} f\big(t, y_n(t)\big)\, dt.$$

Suppose that $y' = -xy$ with $y(0) = 1$. That is, $f(x, y) = -xy$ and $y_0 = 1$.

i Find the solution of this IVP using separation of variables.

ii Use the formulae above to find $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$.

iii It can be shown that the function $y_n(x)$ is a series approximation that converges to the solution of the IVP as $n \to \infty$. Use $y_4(x)$ with the value $x = \frac{1}{2}$ to approximate e correct to 4 decimal places.

iv Investigate better approximations by using higher values of n.

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13D $y' = g(y)$ and the logistic equation

We now consider equations of the form $y' = g(y)$. The exponential growth DE has this form, and so does the logistic equation, which develops the natural growth DE by modelling amongst other things populations restricted by predators or lack of food.

This section, however, is still mostly about the equations rather than the situations that they are modelling, so we continue to use the pronumerals x and y. Section 13E will explain some models and use a variety of pronumerals.

Solving $y' = g(y)$

There are two equivalent approaches to solving a DE of the form $\dfrac{dy}{dx} = g(y)$.

- Regard it as a separable DE, write it as $\dfrac{1}{g(y)}\,dy = dx$, and integrate.
- Take reciprocals, write it as $\dfrac{dx}{dy} = \dfrac{1}{g(y)}$, and integrate.

Example 20 — 13D

Solve $y' = e^y$ using both approaches.

SOLUTION

EITHER regard it as a separable DE, $\quad e^{-y}\,dy = dx$

and integrate,
$$-\int e^{-y}\,dy = -\int dx$$
$$e^{-y} = -x + C, \quad \text{for some constant } C$$
$$x = -e^{-y} + C.$$

OR take reciprocals, $\quad \dfrac{dx}{dy} = e^{-y}$

and integrate, $\quad x = -e^{-y} + C$, for some constant C.

After either approach, it may be appropriate to solve for y, giving
$$y = -\log\,(-x + C).$$

10 SOLVING $y' = g(y)$

- Solve $y' = g(y)$ by integration using either of the forms
$$\frac{1}{g(y)}\,dy = dx \qquad \text{OR} \qquad \frac{dx}{dy} = \frac{1}{g(y)}.$$

The exponential growth DE

Exponential growth has a very simple DE
$$y' = ky, \quad \text{where} \quad k \neq 0,$$
which says in words that the rate of change of the quantity (such as population or mass of a radioactive isotope) is proportional to the quantity. Growth occurs when k is positive, and decay when k is negative.

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Chapters 11 and 16 of the Year 11 book presented exponential growth without discussing this DE in any detail. We will now examine this DE — think of x as time.

These DEs are a special case of a linear DE, because they can be put into the form $y' + f(x)y = g(x)$. In this case $f(x)$ is a constant and $g(x) = 0$.

Example 21 13D

Solve $y' = ky$, where $k \neq 0$, using both approaches.

SOLUTION

First, the constant function $y = 0$ is trivially a solution. Otherwise divide by y.

Rearranging, $\frac{1}{y}\,dy = k\,dx$,

and integrating, $\int \frac{1}{y}\,dy = \int k\,dx$,

so for some constant C, $\log|y| = kx + C$,

Hence $|y| = e^{kx+C}$,

and putting $A = e^C$, $|y| = Ae^{kx}$.

OR

Taking reciprocals, $\frac{dx}{dy} = \frac{1}{ky}$,

and integrating, $x = \int \frac{1}{ky}\,dx$,

so for some constant C, $x = \frac{1}{k}\log|y| + C$.

Hence $kx - kC = \log|y|$

$|y| = e^{kx-kC}$,

and putting $A = e^{-kC}$, $|y| = Ae^{kx}$.

With either working, $|y| = Ae^{kx}$ where A is positive.

Hence $y = Ae^{kx}$ where A can be positive or negative,

and because the constant function $y = 0$ is trivially a solution,

$y = Ae^{kx}$, where A can be any real number.

Example 22 13D

Solve the differential equation $y' = -3y$ given the initial value $y(2) = 50$.

SOLUTION

First, the constant function $y = 0$ is trivially a solution.

Rearranging, $\frac{1}{y}\,dy = -3\,dx$,

and integrating, $\log|y| = -3x + C$, for some constant C,

so $|y| = e^{-3x+C}$

$|y| = Ae^{-3x}$, where $A = e^C$.

Hence $y = Ae^{-3x}$, where A can be positive or negative,

and because the constant function $y = 0$ is trivially a solution,

the general solution is $y = Ae^{-3x}$, where A can be any real number.

Substituting $y(2) = 50$ gives $50 = Ae^{-6}$,

so $A = 50e^6$, and $y = 50e^{-3(x-2)}$.

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Autonomous DEs

A differential equation that does not involve the independent variable x is called *autonomous*. All the DEs in this section are autonomous, and the title of this could have been, *'First-order autonomous differential equations'*.

Think of x as time. The exponential growth DE above is independent of time, which means that the differential equation describing such phenomena — populations, radioactive decay, the cooling of a kettle of hot water taken off the stove — are true for all times. These things are laws of physics, and laws of physics are usually independent of time because the laws hold at all times. The most general physical laws also hold in all places, provided that we factor into the DE any gravitational forces, so these general laws are independent of both space and time — think of Newton's second law of motion $F = m\ddot{x}$, and of the motion of a mass oscillating on a spring in accordance with $\ddot{x} = -n^2x$. Thus the absence here of a variable in an equation has amazing significance in the physical world.

The logistic equation — solving the differential equation

A *logistic differential equation* has a very general definition, but in this course it is a DE of the form

$y' = ky(P - y)$, where P and k are non-zero constants.

The example below is the simplest logistic equation — both constants P and k are set equal to 1.

Solving the logistic equation directly involves converting a single fraction into the sum of two fractions using a procedure known as *partial fractions*. Partial fractions are not in the course, so such a decomposition will be given in each question where it is needed.

Example 23 13D

a Show that $\dfrac{1}{y(1-y)} = \dfrac{1}{y} + \dfrac{1}{1-y}$. (This is called a *partial fractions decomposition*.)

b Hence solve $y' = y(1-y)$, writing the solution with y as the subject.

SOLUTION

a RHS $= \dfrac{(1-y) + y}{y(1-y)} =$ LHS.

b First, the constant functions $y = 0$ and $y = 1$ are trivially solutions.

Otherwise, rearranging, $\dfrac{1}{y(1-y)}\,dy = dx$,

and using part **a**, $\left(\dfrac{1}{y} + \dfrac{1}{1-y}\right) dy = dx$.

$$\begin{aligned}
\text{Integrating,}\quad \log|y| - \log|1-y| &= x + C, \text{ for some constant } C\\
\log\left|\frac{y}{1-y}\right| &= x + C\\
\left|\frac{y}{1-y}\right| &= e^{x+C}\\
\left|\frac{y}{1-y}\right| &= Ae^x, \text{ where } A = e^C \text{ is positive.}
\end{aligned}$$

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Hence $\dfrac{y}{1-y} = Ae^x$, where A can be positive or negative.

Making y the subject,
$$y = Ae^x - Ae^x y$$
$$Ae^x y + y = Ae^x$$
$$y = \frac{Ae^x}{Ae^x + 1}.$$

This is not a good form because the arbitrary constant occurs twice.

Dividing top and bottom by Ae^x, $y = \dfrac{1}{1 + Be^{-x}}$, where $B = A^{-1}$.

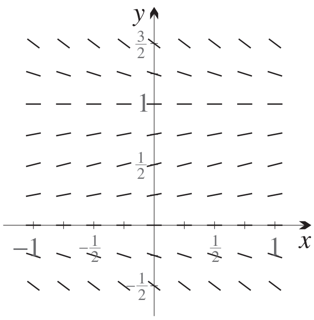
Hence $y = 0$, or $y = 1$, or $y = \dfrac{1}{1 + Be^{-x}}$, for some non-zero constant B.

Note: The solution $y = 1$ corresponds to $B = 0$, and the solution $y = 0$, corresponds to $B \to \infty$. It is probably better to consider these two solutions as special cases, and add to the general solution the condition, 'where B is non-zero'.

The logistic equation — the slope field and three types of solution

To the right is the slope field of this DE $y' = y(1 - y)$. Go back now to worked Example 14 in Section 13B to see some solution curves drawn on a very similar slope field. The slope field makes clear some things that are difficult to make out from the algebra above.

- The two constant solutions $y = 0$ and $y = 1$ are clear from the slope field — in fact they are seen first.
- No solution curve ever crosses the two horizontal lines $y = 0$ and $y = 1$. This means that in practice the other connected solution curves fall into three distinct groups:
  - — those with range $y > 1$,
  - — those with range $0 < y < 1$,
  - — those with range $y < 0$.



The next worked example picks out a solution curve in each of the three regions.

Example 24 13D

In worked Example 23, we obtained the general solution $y = \dfrac{1}{1 + Be^{-x}}$ of the differential equation $y' = y(1 - y)$.

a Use this to find connected solution curves passing through:

i $(0, \tfrac{1}{2})$, **ii** $(0, 2)$, **iii** $(0, -1)$.

In each case, identify any asymptotes and any symmetries, state the domain and range of the connected curve, and briefly describe the situation if x is time and y is population.

b How are the solution curves in parts **a ii** and **a iii** related?

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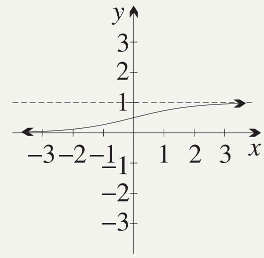
SOLUTION



a **i** Substituting $(0, \tfrac{1}{2})$, $\dfrac{1}{2} = \dfrac{1}{1 + B}$,

so $B = 1$, and the solution curve is $y = \dfrac{1}{1 + e^{-x}}$.

The domain is all real x, and the range is $0 < y < 1$.

Taking limits, $\lim_{x \to \infty} y = 1$ and $\lim_{x \to -\infty} y = 0$,

so $y = 0$ and $y = 1$ are horizontal asymptotes.

The solution curve has point symmetry in $(0, \tfrac{1}{2})$, because

$$\begin{aligned} y(-x) &= \frac{1}{1 + e^{x}} \times \frac{e^{-x}}{e^{-x}} \\ &= \frac{e^{-x}}{e^{-x} + 1} \\ &= \frac{e^{-x} + 1}{e^{-x} + 1} - \frac{1}{e^{-x} + 1} \\ &= 1 - \frac{1}{1 + e^{-x}} \\ &= 1 - y(x). \end{aligned}$$

On the left of the x-axis, the population begins very small and increases, first at an increasing rate. Then on the right of the x-axis, the population increases at a decreasing rate, approaching a limiting value, which we take as the stable population.

ii Substituting $(0, 2)$ $2 = \dfrac{1}{1 + B}$,

so $B = -\tfrac{1}{2}$, and the solution curve is $y = \dfrac{1}{1 - \frac{1}{2}e^{-x}}$.

There is a vertical asymptote when $e^{-x} = 2$,

that is, $x = -\log 2$,

so that $y \to \infty$ as $x \to (-\log 2)^{+}$.

Our connected solution curve does not cross this asymptote,

so because the initial value is at $x = 0$, the domain is $x > -\log 2$.

Taking limits, $\lim_{x \to \infty} y = 1$,

giving a horizontal asymptote at $y = 1$, and range $y > 1$.

The population is originally greater than the stable population, and then decreases at a decreasing rate, approaching the same stable population as in part **i**.

iii Substituting $(0, -1)$, $-1 = \dfrac{1}{1 + B}$,

so $B = -2$, and the solution curve is $y = \dfrac{1}{1 - 2e^{-x}}$.

There is a vertical asymptote when $e^{-x} = \tfrac{1}{2}$

that is, $x = \log 2$

so that $y \to -\infty$ as $x \to (\log 2)^{-}$.

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Our connected solution curve does not cross this asymptote,
so because the initial value is at $x = 0$, the domain is $x < \log 2$.
Taking limits, $\lim\limits_{x \to -\infty} y = 0$,
giving a horizontal asymptote at $y = 0$, and range $y < 0$.

The negative values of y mean that this solution curve has no meaning for populations.

b The function $y = \dfrac{1}{1 - \frac{1}{2}e^{-x}}$ that we found in part **a ii** has another disconnected branch to the left of $y = -\log 2$, and this branch is below $y = 0$. Similarly, the function $y = \dfrac{1}{1 - 2e^{-x}}$ that we found in part **a iii** has another disconnected branch to the right of $y = -\log 2$, and this branch is above $y = 1$. In general, each solution curve of the DE above $y = 1$ is paired with a solution curve below $y = 0$, and vice versa.

In applications, however, it is rare for more than one of these branches to have any meaning, or for there to be any physical relationship between them.

Using the differential equation to find the second derivative

The middle solution curve that we found in part **a i** looks as if it has an inflection at $\left(0, \frac{1}{2}\right)$, and the symmetry that we established proves this. But our usual methods of examining the first and second derivatives look complicated because of the need to differentiate $y = \dfrac{1}{1 + e^{-x}}$ twice. The next worked example show how we can use the differential equation itself to make this process far quicker.

Example 25 — 13D

In worked Example 24, we were examining solution curves of $y' = y(1 - y)$.

a Prove that $y'' = y'(1 - 2y) = y(1 - y)(1 - 2y)$.

b Use this result and the DE itself to analyse the gradient and concavity of the solution curves in parts **a i**–**a iii** of worked Example 24.

SOLUTION

a Expanding, $y' = y - y^2$.

Using the chain rule,
$$y'' = \frac{d}{dy}(y - y^2) \times \frac{dy}{dx}$$
$$y'' = (1 - 2y)y'$$
$$= y(1 - y)(1 - 2y).$$

b First, the DE $y' = y(1 - y)$ tells us that y' is positive for $0 < y < 1$ and negative for $y < 0$ or $y > 1$. Hence the solution curves in the middle region are always increasing, and the solution curves in the top and bottom region are always decreasing.

Secondly, $y'' = y(1 - y)(1 - 2y)$ tells us that y'' is positive for $0 < y < \frac{1}{2}$ or $y > 1$, and negative for $\frac{1}{2} < y < 1$ or $y < 0$. Hence the solution curves in the top region are always concave up, and the solution curves in the bottom region are always concave down, The solution curves in the middle region change concavity from up to down at $y = \frac{1}{2}$, giving a point of inflection there.

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The summary below uses a more general form of the logistic equation.

11 THE LOGISTIC DIFFERENTIAL EQUATION

- The logistic equation is a first-order DE of the form
  $y' = ky(P - y)$, where k and P are constants,
  where the following dotpoints assume that P and k are positive.
- The constant functions $y = 0$ and $y = P$ are solutions of the DE.
- The general solution consists otherwise of three groups of solution curves:
  — Solution curves between $y = 0$ and $y = P$, with domain all real numbers.
  — Solution curves above $y = P$.
  — Solution curves below $y = 0$.
- The solution curves in the second group are the upper branches of two-branch functions whose lower branch is a solution in the third group, and vice versa.
- In applications, one group, or even two groups, may have no significance.

But look at Question 15 in Exercise 13E, where the third group does have significance — it describes the way in which a population moves to extinction if its numbers ever fall below a certain threshold.

Note: Go back to the slope field above worked Example 24 (and see also worked Example 14 in Section 13B). The slope field displays the solution curves in the middle group very nicely. But it fails to identify that every solution curve in the upper or lower groups has a vertical asymptote. As is often the case with slope fields, some aspects of a problem are very clearly displayed, but other aspects may be deceptive.

Exercise 13D

FOUNDATION

1 Consider the differential equation $\dfrac{dy}{dx} = -y$.

a What is the constant solution of this equation?
b Use separation of variables to solve the DE.
c Make y the subject of this solution, simplifying the constant part of the expression.
d Check that the constant solution is included in your answer to part **c**.
e Find the solution for the initial condition $y(0) = 2$.

2 Consider the differential equation $\dfrac{dy}{dx} = 3y$.

a What is the constant solution of this equation?
b Write down the DE obtained by taking the reciprocal of both sides.
c Use direct integration to obtain x as a function of y.
d Make y the subject of this solution, simplifying the constant part of the expression.
e Check that the constant solution is included in your answer to part **d**.
f Find the solution for the initial condition $y(0) = -1$.

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3 Find the solutions of these autonomous IVPs $\big($*autonomous* means that ithas the form $y' = g(y)\big)$. Use either of the methods given in Questions **1** and **2**.

a $y' - y = 0$, with $y(0) = -3$

b $y' + 2y = 0$, with $y(0) = 1$

c $y' = -3y$, with $y(0) = 2$

d $y' = 2y$, with $y(0) = -1$

4 Consider the differential equation $\dfrac{dy}{dx} = 2 - y$.

a What is the constant solution of this equation?

b Use separation of variables or take reciprocals to solve this DE.

c Make y the subject, simplifying the constant part of the expression.

d Check that the constant solution is included in your answer to part **c**.

e Find the solution for the initial condition $y(0) = 3$.

5 Follow the procedures in Question **4** to solve these IVPs.

a $y' = 1 - y$, with $y(0) = 3$

b $y' = y - 1$, with $y(0) = 0$

c $y' = \frac{1}{2}(y + 1)$, with $y(0) = 1$

d $y' = 2(3 - y)$, with $y(0) = 4$

6 Solve these initial value problems.

a $y' = 2y^2$, with $y(0) = 3$

b $y' = -y^2$, with $y(0) = 1$

c $y' = 1 + y^2$, with $y(\frac{\pi}{4}) = 1$

d $y' = -e^y$, with $y(0) = 0$

e $y' = e^{-y}$, with $y(3) = 0$

f $y' = y^{\frac{2}{3}}$, with $y(0) = 1$

DEVELOPMENT

7 Consider the differential equation $y' = ky$, where k is an unknown constant.

a Find the general solution of this DE.

b Evaluate the arbitrary constant, given the initial condition $y(0) = 20$.

c Finally, evaluate k given the condition $y(2) = 5$.

d Simplify your solution, and hence evaluate $y(3)$.

8 Once again consider the differential equation $y' = ky$, where k is an unknown constant.

a Find the general solution of this DE.

b Evaluate the arbitrary constant, given the initial condition $y(0) = 8$.

c Finally, evaluate k given the condition $y(2) = 18$.

d Simplify your solution, and hence evaluate $y(4)$.

9 In solving certain problems involving support beams in engineering, the fourth-order differential equation $y'''' = \lambda^4 y$ is encountered, which is sometimes written as $y^{(4)} = \lambda^4 y$.

a Show that $y = Ae^{\lambda x} + Be^{-\lambda x} + C\cos \lambda x + D\sin \lambda x$ is a solution of the DE.

b A beam rests on a support at a point O. At a horizontal distance x along the beam, the downwards deflection is y. Thus $y(0) = 0$, and we may also assume that $y''(0) = 0$. Find the value of C.

c If the beam is also resting on a support at $x = 10$, then both $y(10) = 0$ and $y''(10) = 0$. From this it can be shown that $\lambda = \frac{n\pi}{10}$.

i Use these results to show that $A = B = 0$.

ii Write down the solution of the beam IVP.

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10 **a** Find the general solution of the DE $y' = e^{-y}$

b Describe the family of curves that you have found.

c Draw the slope field for this DE.

i Draw the solution curve that passes through the origin.

ii Draw two other solution curves.

iii Describe how your two solution curves can be obtained by simple transformations.

iv What feature of the slope field makes this possible?

d Evaluate the arbitrary constant, given that the curve passes through $(0, 1)$.

11 The solution of $y'y = 2$ is a relation that is not a function.

a Find the general solution of the DE.

b Describe the family of curves you have found.

c Draw the slope field for this DE.

i Draw the solution curve that passes through the origin.

ii Draw two other solution curves.

iii Describe how your two solution curves can be obtained by simple transformations of the curve in part **i**.

iv What feature of the slope field makes this possible.

d Evaluate the arbitrary constant, given that the curve passes through $(0, 1)$.

12 Let $L(x) = \dfrac{1}{1 + e^{-x}}$ and consider the curve $y = L(x)$

a What is the y-intercept?

b Explain why the function is always positive.

c Determine $\lim\limits_{x \to \infty} L(x)$ and $\lim\limits_{x \to -\infty} L(x)$.

d Find L' and hence show that the curve has no stationary points.

e **i** Show that $L' = \dfrac{1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$.

ii Use the result in part **i** to find L''.

iii Hence find the point of inflection of $y = L(x)$.

f Sketch $y = L(x)$.

g **i** Show by substitution that $y = L(x)$ is a solution of the logistic DE $y' = y(1 - y)$.

ii Use this DE to prove the formula for $L'(x)$ given in part **e i**.

iii Use this DE again to prove the formula for $L''(x)$ that you found in part **e ii**.

13 **a** Show that $\dfrac{1}{y(1 - y)} = \dfrac{1}{y} + \dfrac{1}{(1 - y)}$.

b Consider the logistic differential equation $y' = y(1 - y)$.

i What are the constant solutions of this equation?

ii Use part **a** to find the general solution of the logistic DE.

c Show that the solution is the result of shifting the function in Question **12**, and determine the shift.

d The constant solutions cannot be obtained in the usual way from the general solution. Use the general solution $y = \dfrac{1}{1 + Be^{-x}}$ to answer the following.

i Find $\lim\limits_{B \to \infty} y$. Is this one of the constant solutions?

ii Find $\lim\limits_{B \to 0^+} y$. Is this one of the constant solutions?

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14 A more generalised version of the logistic equation is $y' = ry(1 - y)$, for some constant r.

a Find the constant solutions.

b Use part **a** of Question **13** to find the general solution.

c Suppose that the initial condition is $y(0) = y_0$. Determine the value of the arbitrary constant B given in the answer to part **a**.

d Show that the solution given in the answer to part **a** is the result of shifting the function $y = \dfrac{1}{1 + e^{-rx}}$ right by $\frac{1}{r}\log B$.

e By using the answers to Question **13d**, or otherwise, what happens as:

i $y_0 \to 0^+$, **ii** $y_0 \to 1^-$.

15 Draw the slope field for the logistic differential equation $y' = y(1 - y)$.

a Add the two constant solutions $y = 0$ and $y = 1$ to the graph.

b Add the solution curve $y = \dfrac{1}{1 + e^{-x}}$ to the slope field. Notice that this curve occupies the part of the number plane between the two constant solutions.

c Show by substitution that $y = \dfrac{1}{1 - e^{-x}}$ is also a solution of the DE.

d Follow the curve-sketching menu to add this second function to the graph. Notice that this curve has two branches that both lie outside the two constant solutions.

16 **a** Show that $\dfrac{1}{(y - 1)(y - 3)} = \dfrac{1}{2}\left(\dfrac{1}{y - 3} - \dfrac{1}{y - 1}\right)$.

b Consider the modified logistic equation $y' = -(1 - y)(3 - y)$.

i What are the constant solutions?

ii Use part **a** to find the general solution of the given DE.

c Using the general solution given in the answers:

i Which of the constant solutions is found by taking $B \to 0^+$?

ii Which of the constant solutions is found by taking $B \to \infty$?

d Draw the slope field, the two constant solutions, and the solutions with $B = -1$ and $B = 1$.

e **i** Find an expression for y'' in terms of y alone.

ii Hence determine the location of the inflection point for the solution curve with $B = 1$.

17 Some first-order differential equations can be made much simpler by using a substitution. Here is a very important example.

Once again, consider the logistic equation, $y' = ry(1 - y)$.

a Make the substitution $v = \dfrac{1}{y}$ and show that $v' = r(1 - v)$.

b Solve the differential equation for v by any appropriate means.

c Hence find the general solution of the logistic equation.

18 Some second-order differential equations can be turned into first-order equations with a substitution. Here is a simple example.

Consider the second-order initial value problem $y'' = 2(1 - y')$, with $y(0) = 1$ and $y'(0) = 0$.

a Put $v = y'$. Write down the corresponding differential equation for v.

b Write down the initial condition for v.

c Solve the initial value problem for v.

d Hence write down y' as a function of x.

e Finally integrate to find y as a function of x.

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ENRICHMENT

19 **a** Prove that if $y = f(x)$ is a solution of the autonomous differential equation $y' = g(y)$, then $y = f(x - C)$ is also a solution.

b Describe the graph of $y = f(x - C)$ as a transformation of $y = f(x)$.

c Describe the isoclines of $y' = g(y)$.

d How are parts **b** and **c** related?

e Review your solutions of any of the first-order differential equations in this exercise in light of this result.

20 This question combines several techniques from earlier in the exercise to solve the simple harmonic motion differential equation. That equation is $\frac{d^2y}{dx^2} + y = 0$, which is autonomous.

a Begin by putting $v = \frac{dy}{dx}$.

i Use the chain rule to show that $\frac{d^2y}{dx^2} = v\frac{dv}{dy}$.

ii Hence write down a separable differential equation in terms of v and y alone.

b The general solution of the DE in part **a ii** is a relation, not a function. Find it.

c Explain the situation geometrically for different cases of arbitrary constant.

d Assuming that $C = r^2$, what are the standard parametric equations?

e Confirm that using these parametric equations gives $y' = v$.

f Use the results of Question **19** to write down the general solution of the original DE.

g Hence show that $y = A\cos x + B\sin x$ is the general solution of the simple harmonic motion differential equation, $y'' + y = 0$:

i first by expanding the result of the previous part,

ii then by direct substitution into the DE.

21 Care must be taken when the solution of a DE involves inverse functions. Solutions may be inadvertently lost or added, as in the following example. It demonstrates that any solution of a DE should be thoroughly checked before it is accepted as correct.

Consider the initial value problem $\frac{dy}{dx} = -\sqrt{1 - y^2}$, with $y(0) = 1$.

a Find the implicit general solution of the DE.

b Evaluate the unknown constant by applying the initial condition.

c Making y the subject without taking care, it would seem that the solution of the IVP is $y = \cos x$. Explain why this solution is not valid for all values of x. It may help to substitute this solution into each side of the DE.

d What is the correct solution of the IVP in which y is the subject?

22 It is possible for an IVP to have multiple solutions. Consider the initial value problem

$$\frac{dy}{dx} = 3y^{\frac{2}{3}}, \text{ with } y(0) = 0.$$

Show that each of the following functions is a solution of this IVP.

a $y = x^3$

b $y = 0$

c $y = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } x \geq 0 \end{cases}$

d $y = \begin{cases} x^3 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$

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13E Applications of differential equations

In this final section, we apply differential equations to some situations in science where they occur naturally. In particular, we shall examine the use of the exponential growth DE and the logistic equation to model physical situations such as population growth. Mostly our variables will no longer be x and y, and in particular, our independent variable will typically be t for time.

First, however, we discuss another approach to solving DEs when we know the form of the solution, but not the values of the constants in the formula.

Evaluating unknown constants when the form of the solution is known

People who work with DEs learn to recognise these equations. They may not know the solution, but often they do know what the solution looks like. The next worked example demonstrates how to find the actual solution in two such situations.

Example 26 **13E**

a Find the values of b and c so that $y = e^{3x} + bx + c$ is a solution of $y' = 3y - 18x$.
b Find n so that $y = 10\cos nx$ is a solution of the second-order DE $y'' = -49y$.

SOLUTION

a Substituting $y = e^{3x} + bx + c$ into the DE,

$$\text{LHS} = 3e^{3x} + b, \qquad \text{RHS} = 3(e^{3x} + bx + c) - 18x$$
$$= 3e^{3x} + (3b - 18)x + 3c.$$

Equating coefficients of x, $\quad 3b - 18 = 0 \quad (1)$
and equating constants, $\quad 3c = b, \quad (2)$
so $b = 6$ and $c = 2$, giving the solution $\quad y = 3e^{3x} + 6x + 2.$

b Substituting $y = 10\cos nx$ into the DE,

$$\text{LHS} = \frac{d}{dx}(-10n\sin nx) \qquad \text{RHS} = -49 \times 10\cos nx$$
$$= -10n^2\cos nx, \qquad = -490\cos nx.$$

Hence $-10n^2 = -490$
$n = 7$ or $n = -7$,
giving the solutions $y = 10\cos 7x$ and $y = 10\cos(-7x)$,
which are the same function because cosine is an even function.

Solving the natural growth DE in a practical situation

This example demonstrates the formation of the differential equation and the procedure to solve it.

Example 27 **13E**

The rabbits on Bandicoot Island are increasing. Fifty years ago, 100 rabbits were released, and now there are 10 000 rabbits. Assume in this example that the rate of increase of rabbits is proportional to the number of rabbits.

a Find the equation for the population N at time t years after they were introduced.
b Find how many rabbits there will be in another:
i 25 years, **ii** 50 years.

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SOLUTION

a Writing the assumption about the rate of increase of rabbits as a DE,

$$\frac{dN}{dt} = kN, \text{ for some constant of proportionality } k.$$

First, the constant function $N = 0$ is trivially a solution of the DE.

$$\begin{aligned}
&\text{Otherwise, dividing by } N, & \frac{1}{N}\,dN &= k\,dt,\\
&\text{and integrating,} & \log|N| &= kt + C, \text{ for some constant } C,\\
& & |N| &= Ae^{kt}, \text{ where } A = e^{C} > 0,\\
& & N &= Ae^{kt}, \text{ where } A \text{ can be positive or negative.}\\
&\text{Because } N = 0 \text{ is a solution,} & N &= Ae^{kt}, \text{ for any constant } A.\\
&\text{When } t = 0,\ N = 100, \text{ so} & 100 &= A \times 1\\
&\text{so} & N &= 100e^{kt}.\\
&\text{When } t = 50,\ N = 10\,000, \text{ so} & 10\,000 &= 100e^{50k}\\
& & k &= \tfrac{1}{50}\log 100.\\
&\text{Hence} & N &= 100e^{kt}, \text{ where } k = \tfrac{1}{50}\log 100.
\end{aligned}$$

b

$$\begin{aligned}
&\text{Substituting } t = 75, & kt &= \frac{1}{50} \times \log 100 \times 75\\
& & &= 1.5\log 100\\
&\text{so population after another 25 years} & &= 100 \times e^{1.5\log 100}\\
& & &= 1\,000\,000 \text{ rabbits.}\\
&\text{Substituting } t = 100, & \text{kt} &= 2\log 100\\
&\text{so population after another 50 years} & &= 100 \times \mathrm{e}^{2\log 100}\\
& & &= 1\,000\,000 \text{ rabbits.}
\end{aligned}$$

The logistic equation can model a limit to exponential growth

The rabbits on Bandicoot Island can't increase forever because there is a limit to the amount of food on the island, and there may be predators. The most straightforward model to limit that growth is the logistic equation discussed in the previous section.

Example 28 13E

It is estimated that the maximum carrying capacity of the Bandicoot Island is 20000 rabbits. Use the population values in the previous worked example, but model the population growth by the logistic equation

$$\frac{dN}{dt} = kN(P - N), \text{ where } P = 20000 \text{ is the maximum population.}$$

a Prove the partial fractions decomposition $\dfrac{1}{N(P - N)} = \dfrac{1}{P}\left(\dfrac{1}{N} + \dfrac{1}{P - N}\right)$.

b Find the equation for the population N at time t years after introduction.

c Find how many rabbits there will be in another:

i 25 years, **ii** 50 years.

d Comment on these results in comparison with the previous worked example.

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SOLUTION

a RHS $= \dfrac{1}{P} \times \dfrac{(P-N)+N}{N(P-N)} =$ LHS.

b First, the constant functions $N = 0$ and $N = P$ are trivially solutions.

Rearranging the DE, $\dfrac{1}{N(P-N)}\,dN = k\,dt$

and by part **a**, $\left(\dfrac{1}{N} + \dfrac{1}{P-N}\right)dN = Pk\,dt.$

Integrating, $\log|N| - \log|P-N| = Pkt + C$, for some constant C,

$$\log\left|\frac{N}{P-N}\right| = Pkt + C$$

$$\left|\frac{N}{P-N}\right| = Ae^{Pkt}, \text{ where } A = e^C > 0,$$

$$\frac{N}{P-N} = Ae^{Pkt}, \text{ where } A \text{ can be positive or negative.}$$

Making N the subject, $N = APe^{Pkt} - ANe^{Pkt}$

$$ANe^{Pkt} + N = APe^{Pkt}$$

$$N = \frac{APe^{Pkt}}{Ae^{Pkt}+1}.$$

Dividing through by Ae^{Pkt} and replacing A^{-1} by B,

$$N = \frac{P}{1+Be^{-Pkt}}.$$

Now we substitute $P = 20000$ and apply the initial conditions.

When $t = 0$, $N = 100$, so $100 = \dfrac{20\,000}{1+B}$

$$1 + B = 200$$

so $B = 199$, and $N = \dfrac{P}{1+199e^{-Pkt}}$

When $t = 50$, $N = 10000$, so $10000 = \dfrac{20000}{1+199e^{-20000\times k\times 50}}$

$$1 + 199e^{-1\,000\,000k} = 2$$

$$e^{-1\,000\,000k} = \frac{1}{199}$$

$$1000000k = \log 199$$

$$k = \frac{\log 199}{1000000}.$$

Hence $N = \dfrac{P}{1+Be^{-Pkt}}$, where $P = 20\,000$, $B = 199$ and $k = \dfrac{\log 199}{1\,000000}$.

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c Substituting $t = 75$, $\quad -Pkt = -20000 \times \dfrac{\log 199}{1000000} \times 75$

$$= -1.5 \log 199$$

so population after another 25 years $= \dfrac{20\,000}{1 + 199e^{-1.5\log 199}}$

$$\doteqdot 18676 \text{ rabbits.}$$

Substituting $t = 100$, $\quad -Pkt = -20000 \times \dfrac{\log 199}{1000000} \times 100$

$$= -2 \log 199$$

so population after another 50 years $= \dfrac{20000}{1 + 199e^{-2\log 199}}$

$$\doteqdot 19900 \text{ rabbits.}$$

d The reduction in the predicted population using the logistic model is dramatic. In the previous worked example, the population increased as a GP, whereas in this worked example the population rises rapidly to approach the limit of 20000.

Notice that in this worked example, the value 50 years ago was 100, and the value in 50 years time is $19000 = 20000 - 100$. The function $N = \dfrac{P}{1 + Be^{-Pkt}}$ has point symmetry in the point $(50, 10\,000)$. See worked Example 24 in Section 13D.

Exercise 13E

FOUNDATION

1 **a** In each case find the values of a and b given that:

i $y = ax^2 + bx$ is a solution of $y' = 1 - 4x$,

ii $y = e^{-x}(a\cos x + b\sin x)$ is a solution of $y' = 2e^{-x}\sin x$,

iii $y = ax + b + 3e^{-x}$ is a solution of $y' = x - y$.

b In each case find the values of a, b and c given that:

i $y = ax^2 + bx + c + 4e^{-2x}$ is a solution of $y' + 2y = x^2 - 3x - 4$,

ii $y = ax^2 + bx + c - \sin 2x$ is a solution of $y'' + 4y = x^2 + 5x$.

c Find the possible values of λ given that $y = 5e^{\lambda x}$ is a solution of $y'' + 5y' + 6y = 0$.

2 In a laboratory, a scientist has a sample of radioactive material. The material decays at a rate proportional to the amount present. That is,

$$\frac{dR}{dt} = kR,$$

where R is the amount present at time t days, and k is an unknown constant.

a Find the general solution of this DE.

b Initially there is 100 grams of the material. Determine the arbitrary constant in your solution.

c After 4 days only 20 grams of the substance remains radioactive. Determine the value of k.

d Hence determine the amount present after 12 days.

e Show that $R = 100 \times \left(\frac{1}{5}\right)^{\frac{1}{4}t}$, then use this formula to check part **d** mentally.

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3 A metal ingot is put in a fridge until its temperature is 5°C. The ingot is then taken out of the fridge and left in a room maintained at 25°C. Let H be the temperature of the ingot after t minutes. Experiments show that the rate of change of temperature over time is proportional to the difference in temperature between the ingot and the room. That is,

$$\frac{dH}{dt} = k(H - 25), \quad \text{for some constant } k.$$

a Find the general solution of this DE.
b Use the initial condition to determine the arbitrary constant.
c After 10 minutes the ingot is at 15°C. Determine the value of k.
d Find how long it takes, correct to the nearest minute, for the temperature to reach 24°C.

4 A conical tank with height 12 m and radius 4 m is filled with water. The water in the tank evaporates at a rate proportional to the circular surface area exposed to the air. Let r metres be the radius of the surface at time t hours. (Recall that the volume of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)
a Write down a differential equation for the evaporation in terms of the radius.
b Use part **a** and the chain rule to find a DE for the rate of change of the radius.
c Use the given information to solve this DE.
d After 6 hours the depth of the water in the tank is $10\frac{1}{2}$ m. Determine the value of k.
e Hence give a formula for the volume of water at time t. Note any restrictions on t.

5 A certain tank is in the shape of a cylinder. It is filled with water to a height of 400 cm. A tap at the bottom of the tank is opened and, as the water empties, the rate of change of height of water in the tank is proportional to the square root of the height. That is:

$$\frac{dh}{dt} = k\sqrt{h}\,.$$

a State the initial condition, and explain why the constant k must be negative.
b Solve the IVP. You may assume that the arbitrary constant of integration is positive.
c After 20 minutes the height of water in the tank is 100 cm. Find the value of k.
d How long does the tank take to drain?
e Is the function you found in part **b** valid for larger values of t?

6 A certain curve has the property that the tangent at any point passes through the origin.
a Write down the gradient of the line from $(0, 0)$ to the point (x, y).
b Now suppose that (x, y) is on this curve. Write down a differential equation for this curve.
c Hence determine the general equation of this curve.
d Which special case is a solution of the problem, but not of the DE?

7 A certain curve has the property that the normal at any point passes through the origin.
a Write down the gradient of the line from $(0, 0)$ to the point (x, y).
b Now suppose that (x, y) is on this curve. Write down a differential equation for this curve.
c Hence determine the general equation of this curve, which is a relation and not a function.

8 A tangent is drawn to a curve, and it is found that its x-intercept is 1 less than the x-coordinate of the point of contact.
a Write down the gradient of the line from $(x - 1, 0)$ to the point (x, y).
b Now suppose that (x, y) is on this curve. Write down a differential equation for this curve.
c Hence determine the general equation of this curve.
d Find such a curve passing through $(0, 1)$.

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DEVELOPMENT

9 The atmospheric pressure P on the planet Nebula changes with altitude h at a rate proportional to P. That is,

$$\frac{dP}{dh} = kP, \quad \text{for some constant } k.$$

Let the atmospheric pressure at ground level be $P = P_0$.

a Find the general solution of this DE.

b Measurements from satellites orbiting the planet show that the pressure at 10 000 m is 40 kPa, and the pressure at 6000 m is 80 kPa. Find the value of k.

c Determine the pressure at ground level correct to the nearest kPa.

10 A tangent to a curve intersects the coordinate axes at A and B. It is found that the point of contact of the tangent is also the mid-point of AB.

a Let the point of contact with the curve be (x, y). Find the coordinates of A and B in terms of x and y.

b Use the gradient of AB to determine a differential equation for this curve.

c Hence determine the general equation of this curve.

11 According to Fick's law, diffusion across a cell membrane is governed by a differential equation. If $C(t)$ is the concentration of a solute in a cell, and S is the concentration of the solute in the surrounding medium, then

$$\frac{dC}{dt} = k(S - C), \quad \text{for some constant } k.$$

a Find the general solution of this DE.

b Suppose that initially $C(0) = C_0$, where $C_0 < S$. Solve the IVP.

12 In order to save an endangered species, it has been decided to release 40 animals on an island where there are no predators. The maximum number that can survive on the island is called the *carrying capacity*, which is 1000. It is assumed that the population N of these animals at time t years after release fits the logistic growth equation

$$\frac{dN}{dt} = kN(1000 - N), \quad \text{for some constant } k.$$

a Show that $\dfrac{1000}{N(1000 - N)} = \dfrac{1}{N} + \dfrac{1}{1000 - N}$.

b Use the result of part **a** to find the general solution of the logistic growth equation.

c Determine the arbitrary constant by applying the initial condition $N(0) = 40$, then simplify the function.

d Given that the population of animals after 1 year was 80, find the value of k correct to four significant figures.

e What will the population be after 5 years, correct to the nearest whole number?

13 In the mid-1800s, Verhulst estimated the population growth of the United States of America using the logistic differential equation

$$\dot{N} = kN(P - N), \quad \text{for some positive constant } k,$$

where N is the population in millions, and t is the number of years after 1850.

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a Show that $\dfrac{P}{N(P-N)} = \dfrac{1}{N} + \dfrac{1}{P-N}$.

b Use the result of part **a** to find the general solution of the logistic growth equation.

c It was estimated at the time that the carrying capacity of the USA was $P = 187.5$, and the population in 1850 was recorded as $N(0) = 23.2$, both in units of millions. Determine the arbitrary constant and simplify the function.

d The estimate used for k was $k = 1.6 \times 10^{-4}$, which predicted a population of 59.8 million in 1890. The actual population of the USA in 1890 was 63.0 million. Calculate a new value for k.

e Using the revised value of k, compare the predicted population for 1930 using this model with the actual population, which was 123.2 million.

f The population of the USA in 2018 was approximately 327 million. Comment on this value.

14 Once again consider the logistic initial value problem,

$$\frac{dN}{dt} = kN(P - N), \text{ with } N(0) = N_0.$$

a Use the result of Question **12a** to find the general solution of the logistic DE.

b Apply the initial condition, and hence show that $N = \dfrac{N_0 P}{N_0 + (P - N_0)e^{-kPt}}$.

c Suppose that the population is N_1 when $t = t_1$. Find a formula for k.

d Now suppose that $t_1 = 1$ and that $N(2) = N_2$. Find a quadratic equation for P with coefficients that only involve the values N_0, N_1 and N_2.

15 Biologists are modelling the population of an endangered species of fish in a river system. The indigenous population of the area are permitted to harvest 200 fish once a year in January as part of their culture. Data collected on a recent field trip in December indicate that there are 500 fish in the river system. The data also suggest that the following mathematical model should be used,

$$\frac{dy}{dt} = -2 + \tfrac{1}{24}y(16 - y),$$

where y is the population of fish in the river measured in hundreds at time t years after the next harvest.

a The term $\frac{1}{24}y(16 - y)$ on the right-hand side represents the familiar logistic growth model. What is the significance of the -2 in this equation?

b Show that the DE can be re-written as $\dfrac{dy}{dt} = -\tfrac{1}{24}(y - 4)(y - 12)$, and write down the initial condition assuming that the harvest goes ahead.

c Show that $\dfrac{24}{(y-4)(y-12)} = \dfrac{3}{y-12} - \dfrac{3}{y-4}$.

d Hence solve the IVP in part **b**.

e According to this model, the fish in the river will die out. When will that be?

f **i** If the most recent harvest had not occurred, what would the initial condition change to?

ii It can be shown that the solution of the DE for this initial condition is

$$y = \frac{4\left(3 + 7e^{-\frac{1}{3}t}\right)}{1 + 7e^{-\frac{1}{3}t}}.$$

Investigate what happens for this solution over time. Comment on the result.

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16 **a** Consider the integral $I = \int \frac{dx}{x \log x}$.

i Use the substitution $u = \log x$ to simplify this integral.

ii Determine the integral for u and then back-substitute to find the original integral.

b In studying the survival of a population after an epidemic, Gompertz proposed the following alternative differential equation for population growth

$$\frac{dN}{dt} = kN \log N,$$

where N is the population at time t. Use the results of part **a** to solve this differential equation.

ENRICHMENT

17 A tank initially holds 100 litres of a solution of water and 200 g of a radioactive substance. Water flows into one end of the tank through a pipe at a rate of 5 litres per minute and mixes with the solution. A pipe at the other end of the tank allows the mixed solution to flow out at the same rate. Each minute the radioactive substance decays at a rate of 0.1 times the amount present. Form and solve a differential equation for the situation and hence find a formula for M, the amount of radioactive material in the tank at time t.

18 An object is heated to 100°C and then placed in a room at 20°C. Let h be the temperature of the object after t minutes. The rate of change of temperature is proportional to its difference from the room temperature. That is,

$$\frac{dh}{dt} = k(h - 20), \text{ for some constant } k.$$

After 10 minutes the temperature of the object is 80°C.

a Solve the initial value problem and find the value of the constant k.

b At the 10-minute mark, refrigeration equipment is turned on, which lowers the temperature in the room by 1°C/min. You may assume that the rate of change of temperature continues to be proportional to the difference, and that the value of k is unchanged. Let $H(t)$ be the temperature of the object t minutes after the refrigeration is turned on.

i Write down the initial value problem for $H(t)$ in terms of k.

ii Let $y = H - 20 + t$. What is the corresponding IVP for y?

iii Find y, and hence determine $H(t)$.

19 Often mathematicians try to simplify a problem by removing constants and parameters from a differential equation. Here the logistic equation in Question **12** will be simplified to one such as those investigated in Section 13D. Let the IVP be

$$\frac{dN}{dt} = kN(P - N), \text{ with } N(0) = N_0.$$

a Put $N = P_y$, where y is an unknown function of t. Also put $r = kP$, and hence determine the corresponding IVP in terms of y, t and r.

b Now put $x = rt$ to obtain a differential equation in terms of y and x alone. What is the new initial condition?

c Next make the substitution $v = \frac{1}{y}$ to get a differential equation in v.

d Find the general solution for v, and hence write down the corresponding solution for y.

e Apply the initial condition to evaluate the arbitrary constant, then simplify y.

f Hence determine the solution of the original IVP.

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20 You have shown several times that the DE $y' = ry(1 - y)$, with initial condition $y(0) = \frac{1}{2}$, has solution $y = \dfrac{1}{1 + e^{-rx}}$.

a Draw the graphs for $r = 1,\ 2,\ 4$ and 8.

b For each fixed value of x, find $\lim\limits_{r \to \infty} y$ for:

i $x > 0$, **ii** $x = 0$, **iii** $x < 0$.

c Sketch the resulting function $u(x)$. This is called the *Heaviside step function*, and it has many applications. Electrical engineers use it as an ideal switch, and it models the quantum steps that occur in quantum mechanics.

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Chapter 13 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 13 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 In each part, state the order of the DE, whether or not it is linear, and how many arbitrary constants will appear in the general solution.

a $y' + xy = \cos x$ **b** $y'' + x^2y' - 3y = e^x \sin x$ **c** $y''' + y''y' = x - 2$

2 Consider the DE $xy' = y(1 - x^2)$.

a Make y' the subject of the equation.

b Copy and complete the table of values for the slope field.

$y \backslash x$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	3	$\frac{4}{3}$	0	-3	$*$				
$\frac{3}{2}$	$\frac{9}{4}$	$\frac{5}{4}$	0						
1									
$\frac{1}{2}$									
0									
$-\frac{1}{2}$									
-1									
$-\frac{3}{2}$									
-2									

c Draw a number plane with a scale of 2 cm = 1 unit, with domain $[-2, 2]$ and range $[-2, 2]$.

d Through each grid point corresponding to the table, draw a line element $\frac{1}{2}$ cm long, centred on the point and with gradient as given in the table.

e Look carefully at the table for matching entries. Check that these agree with the isoclines in your graph.

f Look carefully at your graph. What asymptote seems to be suggested? Does this agree with any constant solutions?

g What symmetry seems to be suggested by the slope field? How is this evident in the equation for y' and in the table?

h In this instance, every solution curve passes through the origin. Add the integral curves (solution curves) that pass through $\left(1, -\frac{1}{2}\right)$, $(1, 1)$ and $(1, 2)$.

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Review

3 Show by substitution that the given function with arbitrary constant C is a general solution of the accompanying differential equation.

a $y = Cx^2e^x$,
$xy' = y(2 + x)$

b $y = \sqrt{x^2 + C}$,
$y'y = x$

c $y = \dfrac{1}{x^2 + C}$,
$y' = -2xy^2$

4 Consider the differential equation $\dfrac{dy}{dx} = -\frac{1}{2}y$.

a What is the constant solution of this equation?

b Write down the DE obtained by taking the reciprocal of both sides.

c Use direct integration to obtain x as a function of y.

d Make y the subject of this solution, simplifying the constant part of the expression.

e Check that the constant solution is included in your answer to part **d**.

f Find the solution for the initial condition $y(0) = 3$.

5 In each case draw the slope field for the given DE using appropriate technology. Then **i** identify any points or isoclines where $y' = 0$, **ii** state how the gradients of the line elements change along the line $x = 1$, from top to bottom, and **iii** state how the gradients of the line elements change along the line $y = 1$, from left to right. **iv** Finally draw three appropriate solution curves.

a $y' = \frac{1}{4}(x^2 - 4)$

b $y' = \frac{1}{8}(y^2 - 4)$

c $y' = \frac{1}{2}(x + y)$

6 Verify that the given function is a general solution of the differential equation for all values of the constants A, B, C and D.

a $y = Ae^{-x} + Be^{-2x}$,
$y'' + 2y' + y = 0$

b $y = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$,
$y'' + 2y' + 5y = 0$

c $y = A\cos x + B\sin x + Ce^{2x}$,
$y''' - 2y'' + y' - 2y = 0$

d $y = Ae^{2x} + Be^{-2x} + C\cos 2x + D\cos 2x$
$y''' - 16y = 0$

DEVELOPMENT

7 Use separation of variables to solve each differential equation. Note that the solution of part **b** is a relation that is not a function.

a $\dfrac{dy}{dx} = \dfrac{-2xy}{1 + x^2}$

b $\dfrac{dy}{dx} = \dfrac{1 - x}{2 + y}$

c $\dfrac{dy}{dx} = \dfrac{y(x - 1)}{x}$

8 Solve each IVP by taking the reciprocal and using direct integration.

a $y' = \frac{1}{2}(1 - y)$, with $y(0) = 2$

b $y' = \frac{1}{5}(5 - y)$, with $y(0) = 2$

9 **a** Show that $\dfrac{1}{1 - x^2} = \dfrac{1}{2}\left(\dfrac{1}{1 + x} + \dfrac{1}{1 - x}\right)$.

b Hence solve $\dfrac{dy}{dx} = \dfrac{y}{1 - x^2}$ by separation of variables.

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Review

10 Let $L(x) = \dfrac{1}{1 - e^{-x}}$ and consider the curve $y = L(x)$.

a Determine the domain and any intercepts.

b Determine $\lim\limits_{x \to \infty} L(x)$ and $\lim\limits_{x \to -\infty} L(x)$.

c Explain why there is a vertical asymptote at $x = 0$ and investigate the behaviour of function on either side.

d Evaluate $y = L(x)$ at $x = \log 2$ and at $x = -\log 2$.

e **i** Show that $L'(x) = \dfrac{-1}{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2}$.

ii Use part **i** to determine $L''(x)$.

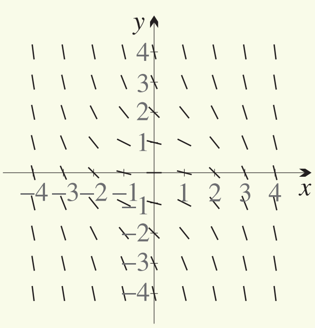
f Hence determine the concavity of the graph of $y = L(x)$.

g Sketch $y = L(x)$, showing all these features.

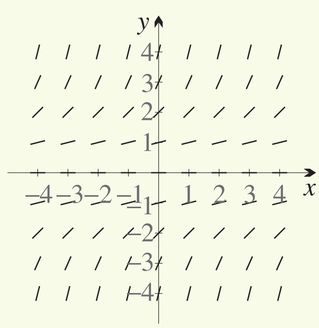
h Show by substitution that $y = L(x)$ is a solution of the logistic DE $y' = y(1 - y)$.

11 Which of the slope fields shown below corresponds to the DE $y' = \frac{1}{4}(x^2 + y^2)$?

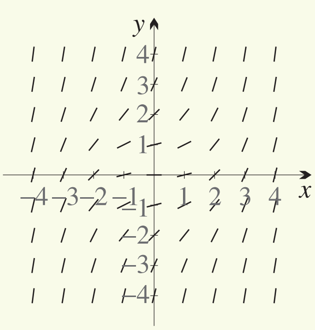
A



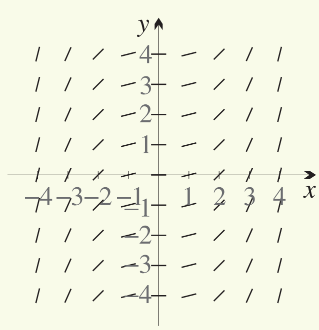
B



C



D



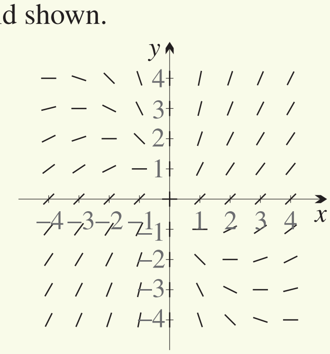
12 Determine which of the DEs corresponds to the slope field shown.

A $y' = 1 + \dfrac{x}{y}$

B $y' = 1 + \dfrac{y}{x}$

C $y' = 1 - \dfrac{x}{y}$

D $y' = 1 - \dfrac{y}{x}$



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Review

13 Use separation of variables to solve initial value problem.

a $\dfrac{dy}{dx} = -\dfrac{y}{x}$, with $y(2) = 1$

b $\dfrac{dy}{dx} = (1 + 2x)e^{-y}$, with $y(1) = 0$

c $\dfrac{dy}{dx} = \dfrac{y^2}{\sqrt{x}}$, with $y(0) = -1$

14 a Show that $\dfrac{1}{y(1 - y)} = \dfrac{1}{y} + \dfrac{1}{(1 - y)}$.

b Consider the logistic differential equation $y' = y(1 - y)$.

i What are the constant solutions of this equation?

ii Use part **a** to find the general solution of the logistic DE.

iii Hence find the solution that passes through $\left(0, \frac{1}{4}\right)$.

15 a Show that $\dfrac{1}{(2 - y)(3 - y)} = \dfrac{1}{y - 3} - \dfrac{1}{y - 2}$.

b Consider the harvest logistic differential equation $y' = -\frac{1}{5}(3 - y)(2 - y)$.

i What are the constant solutions of this equation?

ii Use part **a** to find the general solution of the harvest logistic DE with initial condition $y(0) = 1$.

iii For what value of x does $y = 0$?

16 A company selling mobile phones has been using the logistic differential equation to model sales over the last two years. That is

$$\frac{dN}{dt} = kN(5 - N)$$

where N is the number of people in millions who bought a mobile phone t years after the company began tracking sales with this mathematical model. The value of the constant k is unknown.

a Show that $\dfrac{5}{N(5 - N)} = \dfrac{1}{N} + \dfrac{1}{5 - N}$.

b Use the result of part **a** to find the general solution of the logistic growth equation.

c When the company started using this model, they had already sold 1 million phones. Determine the arbitrary constant, and simplify your solution.

d It is now 2 years since the company started tracking phone sales in this way, and the company has sold 400 000 phones in that time. Find the value of k correct to four significant figures.

e According to this model, what will the projected sales in the next year be, correct to the nearest thousand?

17 Let the function $y = f(x)$ be a solution of IVP $y' = x(1 - 2y)$, with $y(0) = 1$.

a Differentiate the given DE, and hence find a formula for y''.

b Hence show that $y = f(x)$ has a maximum turning point at $x = 0$.

c Now solve the IVP by separation of variables to confirm your answers.

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Review

18 a A differential equation of the form $y' = f(x)$ is solved. Explain how the solutions are related by a simple transformation.

b Likewise, a differential equation of the form $y' = g(y)$ is solved. You may assume that the general solution of the DE is a function and not a relation. Explain how the solutions are related by a simple transformation.

ENRICHMENT

19 The DE $(y')^2 = 1 - y^2$ appears to have two distinct general solutions, $y_1 = \cos(x + A)$ and $y_2 = \sin(x + B)$, as well as constant solutions.

a Determine the constant solutions of this DE.

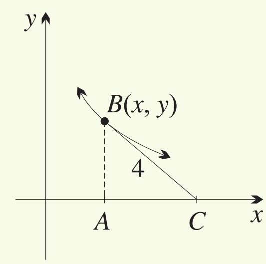
b Show that both y_1 and y_2 satisfy the DE.

c Expand the trigonometric functions in y_1 and y_2, and hence show that these two solutions are in fact identical apart from the values of the constants A and B.

d Determine the relationship between A and B.

20 [Tractrix]

A child pulls a box of toys over a smooth floor using a string of length 4 m. When the obvious coordinate system is applied, the child is initially at the origin, and the box is at $(0, 4)$. It is noted that as the child moves along the x-axis, the string is always tangent to the unknown path followed by the box of toys. Thus when the child is at C and the box is at $B(x, y)$, the line BC is tangent to the curve and $|BC| = 4$. The situation is shown in the diagram to the right.



a Use $\triangle ABC$ to show that $\dfrac{dy}{dx} = \dfrac{-y}{\sqrt{16 - y^2}}$.

b Note any restrictions on y and state the initial condition.

c State the constant solution of the DE. Is it a solution of the initial value problem?

d This IVP cannot be solved using the techniques in this course. Nevertheless a solution curve, called a *tractrix*, can be drawn as follows.

i Draw the slope field for this DE.

ii Add the solution curve that corresponds to the initial condition given.

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