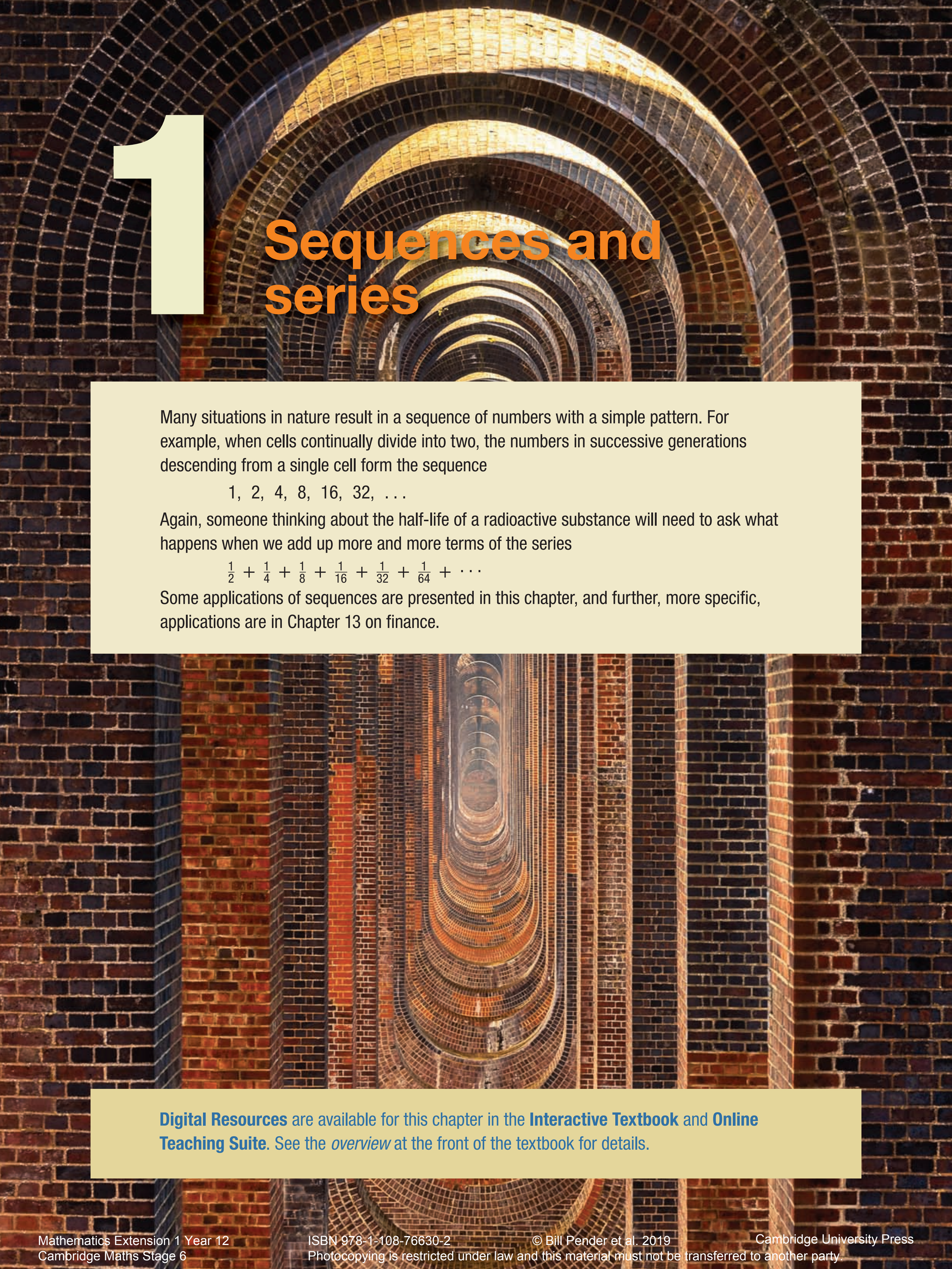


1 Sequences and series

Many situations in nature result in a sequence of numbers with a simple pattern. For example, when cells continually divide into two, the numbers in successive generations descending from a single cell form the sequence

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \ldots$$

Again, someone thinking about the half-life of a radioactive substance will need to ask what happens when we add up more and more terms of the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \cdots$$

Some applications of sequences are presented in this chapter, and further, more specific, applications are in Chapter 13 on finance.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press

1A Sequences and how to specify them

A typical *infinite sequence* is formed by arranging the positive odd integers in increasing order:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \dots$$

The three dots . . . indicate that the sequence goes on forever, with no last term. The sequence starts with the first term 1, then has second term 3, third term 5, and so on. The symbol T_n will usually be used to stand for the nth term, thus

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 5, \quad T_4 = 7, \quad T_5 = 9, \quad \dots$$

The two-digit odd numbers less than 100 form a *finite sequence*:

$$1,\ 3,\ 5,\ 7,\ \dots,\ 99$$

where the dots . . . stand for the 45 terms that have been omitted.

There are three different ways to specify a sequence, and it is important to be able to display a given sequence in each of these different ways.

Write out the first few terms

The easiest way is to write out the first few terms until the pattern is clear to the reader. Continuing with our example of the positive odd integers, we could write the sequence as

$$1,\ 3,\ 5,\ 7,\ 9,\ \dots$$

This sequence clearly continues as $11,\ 13,\ 15,\ 17,\ 19,\ \dots$, and with a few more calculations, it becomes clear that $T_{11} = 21$, $T_{14} = 27$, and $T_{16} = 31$.

Give a formula for the *n*th term

The formula for the nth term of this sequence is

$$T_n = 2n - 1,$$

because the nth term is always 1 less than $2n$. Giving the formula does not rely on the reader recognising a pattern, and any particular term of the sequence can now be calculated quickly:

$$\begin{aligned} T_{30} &= 60 - 1 \\ &= 59 \end{aligned} \qquad \begin{aligned} T_{100} &= 200 - 1 \\ &= 199 \end{aligned} \qquad \begin{aligned} T_{244} &= 488 - 1 \\ &= 487 \end{aligned}$$

Say where to start and how to proceed

The sequence of odd positive integers starts with 1, then each term is 2 more than the previous one. Thus the sequence is completely specified by writing down these two statements:

$$\begin{aligned} &T_1 = 1, && \text{(start the sequence with 1)} \\ &T_n = T_{n-1} + 2, \quad \text{for } n \ge 2. && \text{(every term is 2 more than the previous term)} \end{aligned}$$

Such a specification is called a *recursive* formula of a sequence. Most of the sequences studied in this chapter are based on this idea.

ISBN 978-1-108-76630-2

Example 1 1A

a Write down the first five terms of the sequence given by $T_n = 7n - 3$.
b Describe how each term T_n can be obtained from the previous term T_{n-1}.

SOLUTION

a $T_1 = 7 - 3 = 4$ $\quad T_2 = 14 - 3 = 11$ $\quad T_3 = 21 - 3 = 18$ $\quad T_4 = 28 - 3 = 25$ $\quad T_5 = 35 - 3 = 32$

b Each term is 7 more than the previous term. That is, $T_n = T_{n-1} + 7$.

Example 2 1A

a Find the first five terms of the sequence given by $T_1 = 14$ and $T_n = T_{n-1} + 10$.
b Write down a formula for the nth term T_n.

SOLUTION

a $T_1 = 14$ $\quad T_2 = T_1 + 10 = 24$ $\quad T_3 = T_2 + 10 = 34$ $\quad T_4 = T_3 + 10 = 44$ $\quad T_5 = T_4 + 10 = 54$

b From this pattern, the formula for the nth term is clearly $T_n = 10n + 4$.

1 THREE WAYS TO SPECIFY A SEQUENCE

- Write out the first few terms until the pattern is clear to the reader.
- Give a formula for the nth term T_n.
- Say where to start and how to proceed. That is:
  – Say what the value of T_1 is.
  – Then for $n \geq 2$, give a formula for T_n in terms of the preceding terms.

Using the formula for T_n to solve problems

Many problems about sequences can be solved by forming an equation using the formula for T_n.

Example 3 1A

Find whether 300 and 400 are terms of the sequence $T_n = 7n + 20$.

SOLUTION

Put $T_n = 300$.
Then $7n + 20 = 300$
$7n = 280$
$n = 40$.
Hence 300 is the 40th term.

Put $T_n = 400$.
Then $7n + 20 = 400$
$7n = 380$
$n = 54\frac{2}{7}$.
Hence 400 is not a term of the sequence.

ISBN 978-1-108-76630-2

Example 4 1A

a Find how many negative terms there are in the sequence $T_n = 12n - 100$.
b Find the first positive term of the sequence $T_n = 7n - 60$.

SOLUTION

a Put $T_n < 0$.
Then $12n - 100 < 0$
$n < 8\frac{1}{3}$,
so there are eight negative terms.

b Put $T_n > 0$.
Then $7n - 60 > 0$
$7n > 60$
$n > 8\frac{4}{7}$.
Thus the first positive term is $T_9 = 3$.

Note: The question, 'Find the first positive term' requires two answers:
- Which number term is it?
- What is its value?

Thus the correct answer is, 'The first positive term is $T_9 = 3$'.

Exercise 1A FOUNDATION

1 Alex collects stamps. He found a collection of 700 stamps in the attic a few years ago, and every month since then he has been buying 150 interesting stamps to add to his collection. Thus the numbers of stamps at the end of each month after his discovery form a sequence

$850,\ 1000,\ \ldots$

a Copy and continue the sequence to at least 12 terms followed by dots . . . .
b After how many months did his collection first exceed 2000 stamps?

2 Write down the next four terms of each sequence.
a $6,\ 16,\ 26,\ \ldots$ **b** $3,\ 6,\ 12,\ \ldots$ **c** $38,\ 34,\ 30,\ \ldots$ **d** $24,\ 12,\ 6,\ \ldots$
e $-1,\ 1,\ -1,\ \ldots$ **f** $1,\ 4,\ 9,\ \ldots$ **g** $\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots$ **h** $16,\ -8,\ 4,\ \ldots$

3 Find the first four terms of the sequence whose nth term T_n is:
a $T_n = 5n - 2$ **b** $T_n = 5^n$ **c** $T_n = 6 - 2n$ **d** $T_n = 7 \times 10^n$
e $T_n = n^3$ **f** $T_n = n(n + 1)$ **g** $T_n = (-1)^n$ **h** $T_n = (-3)^n$

4 Write down the first four terms of each sequence described below.
a The first term is 11, and every term after that is 50 more than the previous term.
b The first term is 15, and every term after that is 3 less than the previous term.
c The first term is 5, and every term after that is twice the previous term.
d The first term is -100, and every term after that is one fifth of the previous term.

5 Write out the first twelve terms of the sequence $7,\ 12,\ 17,\ 22,\ \ldots$
a How many terms are less than 30?
b How many terms lie between 20 and 40?
c What is the 10th term?
d What number term is 37?
e Is 87 a term in the sequence?
f Is 201 a term in the sequence?

ISBN 978-1-108-76630-2

Cambridge University Press

6 Write out the first twelve terms of the sequence $\frac{3}{4},\ 1\frac{1}{2},\ 3,\ 6,\ \dots$

a How many terms are less than 400?
b How many terms lie between 20 and 100?
c What is the 10th term?
d What number term is 192?
e Is 96 a term in the sequence?
f Is 100 a term in the sequence?

DEVELOPMENT

7 For each sequence, write out the first five terms. Then explain how each term is obtained from the previous term.

a $T_n = 12 + n$
b $T_n = 4 + 5n$
c $T_n = 15 - 5n$
d $T_n = 3 \times 2^n$
e $T_n = 7 \times (-1)^n$
f $T_n = 80 \times \left(\frac{1}{2}\right)^n$

8 The nth term of a sequence is given by $T_n = 3n + 1$.

a Put $T_n = 40$, and hence find which term of the sequence 40 is.
b Put $T_n = 30$, and hence show that 30 is not a term of the sequence.
c Similarly, find whether 100, 200 and 1000 are terms of the sequence.

9 Answer each question by forming an equation and solving it.

a Find whether 44, 200 and 306 are terms of the sequence $T_n = 10n - 6$.
b Find whether 40, 72 and 200 are terms of the sequence $T_n = 2n^2$.
c Find whether 8, 96 and 128 are terms of the sequence $T_n = 2^n$.

10 The nth term of a sequence is given by $T_n = 10n + 4$.

a Put $T_n < 100$, and hence find how many terms are less than 100.
b Put $T_n > 56$, and find the first term greater than 56. State its number and its value.

11 Answer each question by forming an inequation and solving it.

a How many terms of the sequence $T_n = 2n - 5$ are less than 100?
b What is the first term of the sequence $T_n = 7n - 44$ greater than 100?

12 In each part, the two lines define a sequence T_n. The first line gives the first term T_1. The second line defines how each subsequent term T_n is obtained from the previous term T_{n-1}. Write down the first four terms of each sequence.

a $T_1 = 5$,
$T_n = T_{n-1} + 12$, for $n \ge 2$.

b $T_1 = 12$,
$T_n = T_{n-1} - 10$, for $n \ge 2$.

c $T_1 = 20$,
$T_n = \frac{1}{2}T_{n-1}$, for $n \ge 2$.

d $T_1 = 1$,
$T_n = -T_{n-1}$, for $n \ge 2$.

13 Give a recursive formula for the nth term T_n of each sequence in terms of the $(n - 1)$th term T_{n-1}.

a $16,\ 21,\ 26, \dots$
b $7,\ 14,\ 28, \dots$
c $9,\ 2,\ -5,\ \dots$
d $4,\ -4,\ 4,\ \dots$

14 Write down the first four terms of each sequence. Then state which terms of the whole sequence are zero.

a $T_n = \sin 90n°$
b $T_n = \cos 90n°$
c $T_n = \cos 180n°$
d $T_n = \sin 180n°$

15 **a** Which terms of the sequence $T_n = n^2 - 3n$ are 28 and 70?
b How many terms of this sequence are less than 18?

16 **a** Which terms of the sequence $T_n = \frac{3}{32} \times 2^n$ are $1\frac{1}{2}$ and 96?
b Find the first term in this sequence which is greater than 10.

ISBN 978-1-108-76630-2

17 The correct definition of a sequence is: 'A *sequence* is a function whose domain is the set of positive integers'. Graph the sequences in question 2, with n on the horizontal axis and T_n on the vertical axis. If a simple curve joins the points, draw it and give its equation.

18 A sequence is defined by $T_n = \dfrac{1}{n} - \dfrac{1}{n+1}$.

a Find $T_1 + T_2 + T_3 + T_4$, and give a formula for $T_1 + T_2 + \cdots + T_n$.

b Show that $T_n = \dfrac{1}{n(n+1)}$, and find which term of the sequence $\frac{1}{30}$ is.

19 **a** Which terms of the sequence $T_n = \dfrac{n-1}{n}$ are 0.9 and 0.99?

b Find $T_{n+1} : T_n$, and prove that $\dfrac{T_n}{T_{n+1}} + \dfrac{1}{n^2} = 1$.

c Find $T_2 \times T_3 \times \cdots \times T_n$.

d Prove that $T_{n+1} - T_{n-1} = \dfrac{2}{n^2 - 1}$.

ENRICHMENT

20 **a** Write out the first 12 terms of the *Fibonacci sequence*, which is defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \ge 3.$$

b Write out the first 12 terms of the *Lucas sequence*, which is defined by

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2}, \quad \text{for } n \ge 3.$$

c Explain why every third term of each sequence is even and the rest are odd.

d Write out the first twelve terms of the sequences

$$L_1 + F_1,\ L_2 + F_2,\ L_3 + F_3,\ \ldots \qquad \text{and} \qquad L_1 - F_1,\ L_2 - F_2,\ L_3 - F_3,\ \ldots$$

How are these two new sequences related to the Fibonacci sequence, and why?

A possible project:

21 This open-ended investigation could be developed into a project.

a Generate the Fibonacci and Lucas sequences on a spreadsheet such as Excel.

b Use the spreadsheet to generate the successive ratios $\dfrac{F_n}{F_{n-1}}$ and $\dfrac{L_n}{L_{n-1}}$.

c Investigate the golden mean and its relationship with these sequences.

d Investigate how and why these sequences occur in the natural world.

e Investigate other sequences generated in similar ways.

ISBN 978-1-108-76630-2

An introduction to countably infinite sets:

22 This difficult question introduces some of the extraordinary ideas about infinity associated with Georg Cantor in the late 19th century. Sequences are all about listing, and as discussed in Chapter 13 (Year 11), an infinite set S is called *countably infinite* if its members can be *listed*, meaning that they can be written in a sequence $T_1, T_2, T_3, \ldots$. Cantor, using a Hebrew letter, assigned the symbol $\aleph_0$, 'aleph nul', to this infinity.

a The set of integers is countably infinite because it can be listed as

$0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4,\ \ldots.$

What is the 20th number on the list, and what is the position of the integer -20?

b The table to the right contains all the positive rational numbers. In fact, every positive rational number appears infinitely many times in the table. By taking successive diagonals, show that the positive rational numbers can be listed.

1	2	3	4	5	6	7	8	
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	$\ldots$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\ldots$
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\ldots$
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\ldots$
$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\ldots$
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	$\ldots$
$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

c Copy and complete the following proof by contradiction that *the set of real numbers cannot be listed*. This means that set of real numbers is not countably infinite, and that the infinity of real numbers is 'greater' that the infinity of whole numbers.

Suppose that there were a listing of all the real numbers in the interval $0 \le x < 1$,

$T_1,\ T_2,\ T_3,\ T_4,\ T_5,\ T_6,\ \ldots$

Imagine that each real number T_n in the sequence is written as an infinite decimal string of digits 0.dddddd . . . (where each d represents a digit).

Define a real number x in the interval $(0, 1)$ by specifying each decimal place in turn,

$$n\text{th decimal place of } x = \begin{cases} 1, & \text{if the } nth \text{ decimal place of } T_n \text{ is zero,} \\ 0, & \text{otherwise.} \end{cases}$$

Then x is not on the list because . . .

(A minor qualification is needed — see Question **8** of Exercise 1I.)

ISBN 978-1-108-76630-2

1B Arithmetic sequences

A simple type of sequence is an *arithmetic sequence*. This is a sequence such as

$$3,\ 13,\ 23,\ 33,\ 43,\ 53,\ 63,\ 73,\ 83,\ 93,\ \dots,$$

in which the difference between successive terms is constant — in this example each term is 10 more than the previous term. Notice that all the terms can be generated from the *first term* 3 by repeated addition of this *common difference* 10.

In the context of successive terms of sequences, the word *difference* will always mean some term minus the previous term.

Definition of an arithmetic sequence

Arithmetic sequences are called APs for short. The initials stand for 'arithmetic progression' — an old name for the same thing.

2 ARITHMETIC SEQUENCES

- The *difference* between successive terms in a sequence T_n always means some term minus the previous term, that is,

  $$\text{difference} = T_n - T_{n-1}, \quad \text{where } n \ge 2.$$

- A sequence T_n is called an *arithmetic sequence* or AP if

  $$T_n - T_{n-1} = d, \quad \text{for all } n \ge 2,$$

  where d is a constant, called the *common difference*.
- The terms of an arithmetic sequence can be generated from the first term by repeated addition of this common difference:

  $$T_n = T_{n-1} + d, \quad \text{for all } n \ge 2.$$

Example 5 1B

Test whether each sequence is an AP. If the sequence is an AP, find its first term a and its common difference d.

a $46,\ 43,\ 40,\ 37,\ \dots$ **b** $1,\ 4,\ 9,\ 16,\ \dots$ **c** $\log_e 6,\ \log_e 12,\ \log_e 24,\ \log_e 48,\ \dots$

SOLUTION

a $T_2 - T_1 = 43 - 46 = -3$ $\quad T_3 - T_2 = 40 - 43 = -3$ $\quad T_4 - T_3 = 37 - 40 = -3$

Hence the sequence is an AP with $a = 46$ and $d = -3$.

b $T_2 - T_1 = 4 - 1 = 3$ $\quad T_3 - T_2 = 9 - 4 = 5$ $\quad T_4 - T_3 = 16 - 9 = 7$

The differences are not all the same, so the sequence is not an AP.

c $T_1 = \log_e 6 = \log_e 2 + \log_e 3$ $\quad T_2 = \log_e 12 = 2\log_e 2 + \log_e 3$ $\quad T_3 = \log_e 24 = 3\log_e 2 + \log_e 3$ $\quad T_4 = \log_e 48 = 4\log_e 2 + \log_e 3$

Hence the sequence is an AP with $a = \log_e 6$ and $d = \log_e 2$.

ISBN 978-1-108-76630-2

A formula for the *n*th term of an AP

Let the first term of an AP be a and the common difference be d. Then the first few terms of the sequence are

$$T_1 = a, \qquad T_2 = a + d, \qquad T_3 = a + 2d, \qquad T_4 = a + 3d, \qquad \dots$$

From this pattern, the general formula for the nth term is clear:

3 THE *n*TH TERM OF AN AP

$$T_n = a + (n - 1)d$$

where a is the first term and d is the common difference.

Example 6 — 1B

a Write out the first five terms of the AP with $a = 130$ and $d = -3$.
b Find the 20th term and a formula for the nth term.
c Find the first negative term.

SOLUTION

a $130,\ 127,\ 124,\ 121,\ 118,\ \dots$

b
$$\begin{aligned} T_{20} &= a + 19d \\ &= 130 + 19 \times (-3) \\ &= 73 \\ T_n &= 130 - 3(n - 1) \\ &= 133 - 3n \end{aligned}$$

c Put
$$\begin{aligned} T_n &< 0 \\ 133 - 3n &< 0 \\ 3n &> 133 \\ n &> 44\tfrac{1}{3}, \end{aligned}$$
so the first negative term is $T_{45} = -2$.

Example 7 — 1B

a Find a formula for the nth term of the sequence 26, 35, 44, 53, ...
b How many terms are there in the sequence 26, 35, 44, 53, . . . , 917?

SOLUTION

a This is an AP with $a = 26$ and $d = 9$.

Hence
$$\begin{aligned} T_n &= a + (n - 1)d \\ &= 26 + 9(n - 1) \\ &= 26 + 9n - 9 \\ &= 17 + 9n. \end{aligned}$$

b Put $T_n = 917$.

Then
$$\begin{aligned} 17 + 9n &= 917 \\ 9n &= 900 \\ n &= 100, \end{aligned}$$
so there are 100 terms in the sequence.

ISBN 978-1-108-76630-2

Example 8 1B

The first term of an AP is 105 and the 10th term is 6. Find the common difference and write out the first five terms.

SOLUTION

First, we know that $T_1 = 105$,
that is, $a = 105$. (1)
Secondly, we know that $T_{10} = 6$,
so using the formula for the 10th term, $a + 9d = 6$. (2)
Substituting (1) into (2), $105 + 9d = 6$
$9d = -99$
$d = -11$,
so the common difference is $d = -11$ and the sequence is 105, 94, 83, 72, 61, . . .

Arithmetic sequences and linear functions

Take a linear function such as $f(x) = 30 - 8x$, and substitute the positive integers. The result is an arithmetic sequence

$$22,\ 14,\ 6,\ -2,\ -10,\ \ldots .$$

x	1	2	3	4	5
$f(x)$	22	14	6	-2	-10

The formula for the nth term of this AP is $T_n = 22 - 8(n - 1) = 30 - 8n$. This is a function whose domain is the set of positive integers, and its equation is the same as the linear function above, with only a change of pronumeral from x to n.

Every arithmetic sequence can be generated in this way.

Exercise 1B

FOUNDATION

1 Write out the next three terms of these sequences. They are all APs.
a 3, 8, 13, . . . **b** 35, 25, 15, . . . **c** $4\frac{1}{2}$, 6, $7\frac{1}{2}$, . . .

2 Write out the first four terms of the APs whose first terms and common differences are:
a $a = 3$ and $d = 2$ **b** $a = 7$ and $d = -4$ **c** $a = 30$ and $d = -11$
d $a = -9$ and $d = 4$ **e** $a = 3\frac{1}{2}$ and $d = -2$ **f** $a = 0.9$ and $d = 0.7$

3 Find the differences $T_2 - T_1$ and $T_3 - T_2$ for each sequence to test whether it is an AP. If the sequence is an AP, state the values of the first term a and the common difference d.
a 3, 7, 11, . . . **b** 11, 7, 3, . . . **c** 23, 34, 45, . . .
d -12, -7, -2, . . . **e** -40, 20, -10, . . . **f** 1, 11, 111, . . .
g 8, -2, -12, . . . **h** -17, 0, 17, . . . **i** 10, $7\frac{1}{2}$, 5, . . .

4 Use the formula $T_n = a + (n - 1)d$ to find the 11th term T_{11} of the APs in which:
a $a = 7$ and $d = 6$ **b** $a = 15$ and $d = -7$ **c** $a = 10\frac{1}{2}$ and $d = 4$

5 Use the formula $T_n = a + (n - 1)d$ to find the nth term T_n of the APs in which:
a $a = 1$ and $d = 4$ **b** $a = 100$ and $d = -7$ **c** $a = -13$ and $d = 6$

ISBN 978-1-108-76630-2

6 **a** Find the first term a and the common difference d of the AP $6,\ 16,\ 26,\ \ldots$

b Find the ninth term T_9, the 21st term T_{21} and the 100th term T_{100}.

c Use the formula $T_n = a + (n-1)d$ to find a formula for the nth term T_n.

7 Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If the sequence is an AP, use the formula $T_n = a + (n-1)d$ to find a formula for the nth term T_n.

a $8,\ 11,\ 14,\ \ldots$ **b** $21,\ 15,\ 9,\ \ldots$ **c** $8,\ 4,\ 2,\ \ldots$

d $-3,\ 1,\ 5,\ \ldots$ **e** $1\frac{3}{4},\ 3,\ 4\frac{1}{4},\ \ldots$ **f** $12, -5, -22, \ldots$

g $\sqrt{2},\ 2\sqrt{2},\ 3\sqrt{2},\ \ldots$ **h** $1,\ 4,\ 9,\ 16,\ \ldots$ **i** $-2\frac{1}{2},\ 1,\ 4\frac{1}{2},\ \ldots$

DEVELOPMENT

8 **a** Use the formula $T_n = a + (n-1)d$ to find the nth term T_n of $165,\ 160,\ 155,\ \ldots$

b Solve $T_n = 40$ to find the number of terms in the finite sequence $165,\ 160,\ 155,\ \ldots,\ 40$.

c Solve $T_n < 0$ to find the first negative term of the sequence $165,\ 160,\ 155,\ \ldots$

9 Find T_n for each AP. Then solve $T_n < 0$ to find the first negative term.

a $20,\ 17,\ 14,\ \ldots$ **b** $82,\ 79,\ 76,\ \ldots$ **c** $24\frac{1}{2},\ 24,\ 23\frac{1}{2},\ \ldots$

10 Find the number of terms in each finite sequence.

a $10,\ 12,\ 14,\ \ldots,\ 30$ **b** $1,\ 4,\ 7,\ \ldots,\ 100$ **c** $105,\ 100,\ 95,\ \ldots,\ 30$

d $100,\ 92,\ 84,\ \ldots,\ 4$ **e** $-12,\ -10\frac{1}{2},\ -9,\ \ldots,\ 0$ **f** $2,\ 5,\ 8,\ \ldots,\ 2000$

11 The nth term of an arithmetic sequence is $T_n = 7 + 4n$.

a Write out the first four terms, and hence find the values of a and d.

b Find the sum and the difference of the 50th and the 25th terms.

c Prove that $5T_1 + 4T_2 = T_{27}$.

d Which term of the sequence is 815?

e Find the last term less than 1000 and the first term greater than 1000.

f Find which terms are between 200 and 300, and how many of them there are.

12 **a** Let T_n be the sequence $8,\ 16,\ 24,\ \ldots$ of positive multiples of 8.

i Show that the sequence is an AP, and find a formula for T_n.

ii Find the first term of the sequence greater than 500 and the last term less than 850.

iii Hence find the number of multiples of 8 between 500 and 850.

b Use the same steps to find the number of multiples of 11 between 1000 and 2000.

c Use the same steps to find the number of multiples of 7 between 800 and 2000.

13 **a** The first term of an AP is $a = 7$ and the fourth term is $T_4 = 16$. Use the formula $T_n = a + (n-1)d$ to find the common difference d. Then write down the first four terms.

b Find the 20th term of an AP with first term 28 and 11th term 108.

c Find the 100th term of an AP with first term 32 and 20th term -6.

14 Ionian Windows charges \$500 for the first window, then \$300 for each additional window.

a Write down the cost of 1 window, 2 windows, 3 windows, 4 windows, . . .

b Use the formula $T_n = a + (n-1)d$ to find the cost of 15 windows.

c Find the cost of n windows.

d Find the maximum number of windows whose total cost is less than \$10 000.

ISBN 978-1-108-76630-2

Cambridge University Press

15 Many years ago, 160 km of a railway line from Nevermore to Gindarinda was built. On 1st January 2001, work was resumed, with 20 km of new track completed each month.

a Write down the lengths of track 1 month later, 2 months later, 3 months later, ...

b Use the formula $T_n = a + (n - 1)d$ to find how much track there was after 12 months.

c Find a formula for the length after n months.

d The distance from Nevermore to Gindarinda is 540 km. Form an equation and solve it to find how many months it took to complete the track.

16 [Simple interest and APs]
A principal of \$2000 is invested at 6% per annum simple interest. Let $\$A_n$ be the total amount (principal plus interest) at the end of n years.

a Write out the values of A_1, A_2, A_3 and A_4.

b Use the formula $T_n = a + (n - 1)d$ to find a formula for A_n, and evaluate A_{12}.

c How many years will it take before the total amount exceeds \$6000?

17 a Write down the first few terms of the AP generated by substituting the positive integers into the linear function $f(x) = 12 - 3x$. Then write down a formula for the nth term.

b i Find the formula of the nth term T_n of the AP $-3,\ -1,\ 1,\ 3,\ 5\ldots$. Then write down the linear function $f(x)$ that generates this AP when the positive integers are substituted into it.

ii Graph the function and mark the points $(1, -3)$, $(2, -1)$, $(3, 1)$, $(4, 3)$, $(5, 5)$.

18 Find the common difference of each AP. Then find x if $T_{11} = 36$.

a $5x - 9,\ 5x - 5,\ 5x - 1,\ \ldots$

b $16,\ 16 + 6x,\ 16 + 12x,\ \ldots$

19 Find the common difference of each AP. Then find a formula for the nth term T_n.

a $\log_3 2,\ \log_3 4,\ \log_3 8,\ \ldots$

b $\log_a 54,\ \log_a 18,\ \log_a 6,\ \ldots$

c $x - 3y,\ 2x + y,\ 3x + 5y,\ \ldots$

d $5 - 6\sqrt{5},\ 1 + \sqrt{5},\ -3 + 8\sqrt{5},\ \ldots$

e $1.36,\ -0.52,\ -2.4,\ \ldots$

f $\log_a 3x^2,\ \log_a 3x,\ \log_a 3,\ \ldots$

20 How many terms of the sequence $100,\ 97,\ 94,\ \ldots$ have squares less than 400?

ENRICHMENT

21 a What are the first term and difference of the AP generated by substituting the positive integers into the linear function with gradient m and y-intercept b?

b What are the gradient and y-intercept of the linear function that generates an AP with first term a and difference d when the positive integers are substituted into it?

22 [The set of all APs forms a two-dimensional space.]
Let $\mathcal{A}(a, d)$ represent the AP whose first term is a and difference is d.

a The *sum* of two sequences T_n and U_n is defined to be the sequence whose nth term is $T_n + U_n$. Show that for all constants λ and μ, and for all values of a_1, a_2, d_1 and d_2, the sequence $\lambda\mathcal{A}(a_1, d_1) + \mu\mathcal{A}(a_2, d_2)$ is an AP, and find its first term and common difference.

b Write out the sequences $\mathcal{A}(1, 0)$ and $\mathcal{A}(0, 1)$. Show that any AP $\mathcal{A}(a, d)$ with first term a and difference d can be written in the form $\lambda\mathcal{A}(1, 0) + \mu\mathcal{A}(0, 1)$, and find λ and μ.

ISBN 978-1-108-76630-2

Cambridge University Press

1C Geometric sequences

A *geometric sequence* is a sequence like this:

$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ 1458,\ \ldots$$

in which the *ratio* of successive terms is constant — in this example, each term is 3 times the previous term. Because the ratio is constant, all the terms can be generated from the *first term* 2 by repeated multiplication by this *common ratio* 3.

In the context of successive terms of sequences, the word *ratio* will always mean some term divided by the previous term.

Definition of a geometric sequence

The old name was 'geometric progression' and geometric sequences are called GPs for short.

4 GEOMETRIC SEQUENCES

- The *ratio* of successive terms in a sequence T_n always means some term divided by the previous term, that is,
  $$\text{ratio} = \frac{T_n}{T_{n-1}}, \text{ where } n \ge 2.$$
- A sequence T_n is called a *geometric sequence* if
  $$\frac{T_n}{T_{n-1}} = r, \text{ for all } n \ge 2,$$
  where r is a non-zero constant, called the *common ratio*.
- The terms of a geometric sequence can be generated from the first term by repeated multiplication by this common ratio:
  $$T_n = T_{n-1} \times r, \text{ for all } n \ge 2.$$

Thus arithmetic sequences have a common difference and geometric sequences have a common ratio, so the methods of dealing with them are quite similar.

Example 9 1C

Test whether each sequence is a GP. If the sequence is a GP, find its first term a and its ratio r.

a $40,\ 20,\ 10,\ 5,\ \ldots$ **b** $5,\ 10,\ 100,\ 200,\ \ldots$ **c** $e^2, e^5, e^8, e^{11}, \ldots$

SOLUTION

a Here $\frac{T_2}{T_1} = \frac{20}{40} = \frac{1}{2}$ and $\frac{T_3}{T_2} = \frac{10}{20} = \frac{1}{2}$ and $\frac{T_4}{T_3} = \frac{5}{10} = \frac{1}{2}$,

so the sequence is a GP with $a = 40$ and $r = \frac{1}{2}$.

ISBN 978-1-108-76630-2

b Here $\dfrac{T_2}{T_1} = \dfrac{10}{5} = 2$ and $\dfrac{T_3}{T_2} = \dfrac{100}{10} = 10$ and $\dfrac{T_4}{T_3} = \dfrac{200}{100} = 2.$

The ratios are not all the same, so the sequence is not a GP.

c Here $\dfrac{T_2}{T_1} = \dfrac{e^5}{e^2} = e^3$ and $\dfrac{T_3}{T_2} = \dfrac{e^8}{e^5} = e^3$ and $\dfrac{T_4}{T_3} = \dfrac{e^{11}}{e^8} = e^3,$

so the sequence is a GP with $a = e^2$ and $r = e^3$.

A formula for the *n*th term of a GP

Let the first term of a GP be a and the common ratio be r. Then the first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad T_5 = ar^4, \quad \dots$$

From this pattern, the general formula for the nth term is clear:

5 THE *n*TH TERM OF A GP

$$T_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

Example 10 1C

Write out the first five terms, and calculate the 10th term, of the GP with:

a $a = 3$ and $r = 2$,

b $a = 45$ and $r = \frac{1}{3}$.

SOLUTION

a $3,\ 6,\ 12,\ 24,\ 48,\ \dots$

$$\begin{aligned} T_{10} &= ar^9 \\ &= 3 \times 2^9 \\ &= 1536 \end{aligned}$$

b $45,\ 15,\ 5,\ 1\tfrac{2}{3},\ \tfrac{5}{9},\ \dots$

$$\begin{aligned} T_{10} &= a \times r^9 \\ &= 45 \times \left(\tfrac{1}{3}\right)^9 \\ &= 5 \times 3^{-7} \end{aligned}$$

Zeroes and GPs don't mix

No term of a GP can be zero. For example, if $T_2 = 0$, then $\dfrac{T_3}{T_2}$ would be undefined, contradicting the definition that $\dfrac{T_3}{T_2} = r$.

Similarly, the ratio of a GP cannot be zero. Otherwise $T_2 = ar$ would be zero, which is impossible, as we have explained above.

Negative ratios and alternating signs

The sequence $2,\ -6,\ 18,\ -54,\ \dots$ is an important type of GP. Its ratio is $r = -3$, which is negative, so the terms are alternately positive and negative.

ISBN 978-1-108-76630-2

Example 11 1C

a Show that $2,\ -6,\ 18,\ -54,\ \ldots$ is a GP, and find its first term a and ratio r.
b Find a formula for the nth term, and hence find T_6 and T_{15}.

SOLUTION

a Here $\dfrac{T_2}{T_1} = \dfrac{-6}{2} = -3$ and $\dfrac{T_3}{T_2} = \dfrac{18}{-6} = -3$ and $\dfrac{T_4}{T_3} = \dfrac{-54}{18} = -3$,

so the sequence is a GP with $a = 2$ and $r = -3$.

b Using the formula for the nth term, $T_n = ar^{n-1}$
$= 2 \times (-3)^{n-1}$.

Hence $T_6 = 2 \times (-3)^5$
$= -486$, because 5 is odd.

and $T_{15} = 2 \times (-3)^{14}$
$= 2 \times 3^{14}$, because 14 is even.

Using a switch to alternate the sign

Here are two classic GPs with ratio -1:

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \ldots \qquad \text{and} \qquad 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots$$

The first has formula $T_n = (-1)^n$, and the second has formula $T_n = (-1)^{n-1}$.

These sequences provide a way of writing any GP that alternates in sign using a *switch*. For example, the sequence $2,\ -6,\ 18,\ -54,\ \ldots$ in the previous worked example has formula $T_n = 2 \times (-3)^{n-1}$, which can also be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^{n-1}$$

to emphasise the alternating sign, and $-2,\ 6,\ -18,\ 54,\ \ldots$ can be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^n.$$

Solving problems involving GPs

As with APs, the formula for the nth term allows many problems to be solved by forming an equation and solving it.

Example 12 1C

a Find a formula for the nth term of the geometric sequence $5,\ 10,\ 20,\ \ldots$.
b Hence find whether 320 and 720 are terms of this sequence.

SOLUTION

a The sequence is a GP with $a = 5$ and $r = 2$.
Hence $T_n = ar^{n-1}$
$= 5 \times 2^{n-1}$.

b Put $T_n = 320$.
Then $5 \times 2^{n-1} = 320$
$2^{n-1} = 64$
$n - 1 = 6$
$n = 7$,
so 320 is the seventh term T_7.

c Put $T_n = 720$.
Then $5 \times 2^{n-1} = 720$
$2^{n-1} = 144$.
But 144 is not a power of 2,
so 720 is not a term of the sequence.

ISBN 978-1-108-76630-2

Example 13 1C

The first term of a GP is 448 and the seventh term is 7. Find the common ratio and write out the first seven terms.

SOLUTION

First, we know that $T_1 = 448$

that is, $a = 448.$ (1)

Secondly, we know that $T_7 = 7$

so using the formula for the 7th term, $ar^6 = 7.$ (2)

Substituting (1) into (2),

$$448r^6 = 7$$
$$r^6 = \tfrac{1}{64}$$
$$r = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}.$$

Thus either the ratio is $r = \frac{1}{2}$, and the sequence is

$$448,\ 224,\ 112,\ 56,\ 28,\ 14,\ 7,\ \ldots$$

or the ratio is $r = -\frac{1}{2}$, and the sequence is

$$448,\ -224,\ 112,\ -56,\ 28,\ -14, 7,\ \ldots$$

Geometric sequences and exponential functions

Take the exponential function $f(x) = 54 \times 3^{-x}$, and substitute the positive integers. The result is a geometric sequence

$$18,\ 6,\ 2,\ \tfrac{2}{3},\ \tfrac{2}{9},\ \ldots$$

x	1	2	3	4	5
$f(x)$	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$

The formula for the nth term of this GP is $T_n = 18 \times \left(\frac{1}{3}\right)^{n-1} = 54 \times 3^{-n}$. This is a function whose domain is the set of positive integers, and its equation is the same as the exponential function, with only a change of pronumeral from x to n.

Thus the graph of an arithmetic sequence is the positive integer points on the graph of a linear function, and the graph of a geometric sequence is the positive integer points on the graph of an exponential function.

Exercise 1C

FOUNDATION

1 Write out the next three terms of each sequence. They are all GPs.

a $1,\ 2,\ 4,\ \ldots$ **b** $81,\ 27,\ 9,\ \ldots$ **c** $-7,\ -14,\ -28,\ \ldots$

d $-2500,\ -500,\ -100,\ \ldots$ **e** $3,\ -6,\ 12,\ \ldots$ **f** $-25,\ 50,\ -100,\ \ldots$

g $5,\ -5,\ 5,\ \ldots$ **h** $-1000,\ 100,\ -10,\ \ldots$ **i** $0.04,\ 0.4,\ 4,\ \ldots$

2 Write out the first four terms of the GPs whose first terms and common ratios are:

a $a = 12$ and $r = 2$ **b** $a = 5$ and $r = -2$ **c** $a = 18$ and $r = \frac{1}{3}$

d $a = 18$ and $r = -\frac{1}{3}$ **e** $a = 6$ and $r = -\frac{1}{2}$ **f** $a = -7$ and $r = -1$

ISBN 978-1-108-76630-2

3 Find the ratios $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ for each sequence to test whether it is a GP. If the sequence is a GP, write down the first term a and the common ratio r.

a $4,\ 8,\ 16,\ \dots$
b $16,\ 8,\ 4,\ \dots$
c $2,\ 4,\ 6,\ \dots$
d $-1000,\ -100,\ -10,\ \dots$
e $-80,\ 40,\ -20,\ \dots$
f $29,\ 29,\ 29,\ \dots$
g $1,\ 4,\ 9,\ \dots$
h $-14,\ 14,\ -14,\ \dots$
i $6,\ 1,\ \frac{1}{6},\ \dots$

4 Use the formula $T_n = ar^{n-1}$ to find the fourth term of the GP with:

a $a = 5$ and $r = 2$
b $a = 300$ and $r = \frac{1}{10}$
c $a = -7$ and $r = 2$
d $a = -64$ and $r = \frac{1}{2}$
e $a = 11$ and $r = -2$
f $a = -15$ and $r = -2$

5 Use the formula $T_n = ar^{n-1}$ to find the nth term T_n of the GP with:

a $a = 1$ and $r = 3$
b $a = 5$ and $r = 7$
c $a = 8$ and $r = -\frac{1}{3}$

6 a Find the first term a and the common ratio r of the GP $7,\ 14,\ 28,\ \dots$
b Find the sixth term T_6 and an expression for the 50th term T_{50}.
c Find a formula for the nth term T_n.

7 a Find the first term a and the common ratio r of the GP $10,\ -30,\ 90,\ \dots$
b Find the sixth term T_6 and an expression for the 25th term T_{25}.
c Find a formula for the nth term T_n.

8 Find $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ to test whether each sequence is a GP. If the sequence is a GP, use the formula $T_n = ar^{n-1}$ to find a formula for the nth term, then find T_6.

a $10,\ 20,\ 40,\ \dots$
b $180,\ 60,\ 20,\ \dots$
c $64,\ 81,\ 100,\ \dots$
d $35,\ 50,\ 65,\ \dots$
e $\frac{3}{4},\ 3,\ 12,\ \dots$
f $-48,\ -24,\ -12,\ \dots$

9 Find the common ratio of each GP, find a formula for T_n, and find T_6.

a $1,\ -1,\ 1,\ \dots$
b $-2,\ 4,\ -8,\ \dots$
c $-8,\ 24,\ -72,\ \dots$
d $60,\ -30,\ 15,\ \dots$
e $-1024,\ 512,\ -256,\ \dots$
f $\frac{1}{16},\ -\frac{3}{8},\ \frac{9}{4},\ \dots$

DEVELOPMENT

10 Use the formula $T_n = ar^{n-1}$ to find how many terms there are in each finite sequence.

a $1,\ 2,\ 4,\ \dots,\ 64$
b $-1,\ -3,\ -9,\ \dots,\ -81$
c $8,\ 40,\ 200,\ \dots,\ 125\,000$
d $7,\ 14,\ 28,\ \dots,\ 224$
e $2,\ 14,\ 98,\ \dots,\ 4802$
f $\frac{1}{25},\ \frac{1}{5},\ 1,\ \dots,\ 625$

11 a The first term of a GP is $a = 25$ and the fourth term is $T_4 = 200$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.
b Find the common ratio r of a GP for which:
i $a = 3$ and $T_6 = 96$
ii $a = 1000$ and $T_7 = 0.001$
iii $a = 32$ and $T_6 = -243$
iv $a = 5$ and $T_7 = 40$

ISBN 978-1-108-76630-2

Cambridge University Press

12 The nth term of a geometric sequence is $T_n = 25 \times 2^n$.

a Write out the first six terms and hence find the values of a and r.

b Which term of the sequence is 6400?

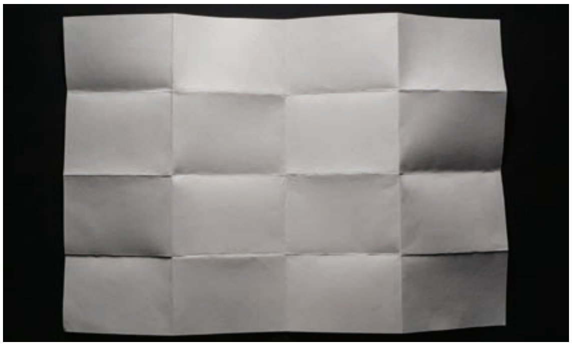
c Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.

d Prove that $T_9 \times T_{11} = 25 \times T_{20}$.

e Write out the terms between 1000 and 100 000. How many of them are there?

f Verify by calculations that $T_{11} = 51\,200$ is the last term less than 100 000 and that $T_{12} = 102\,400$ is the first term greater than 100 000.

13 A piece of paper 0.1 mm thick is folded successively 100 times. How thick is it now?



14 [Compound interest and GPs]
A principal \$$P$ is invested at 7% per annum compound interest. Let A_n be the total amount at the end of n years.

a Write down A_1, A_2 and A_3.

b Show that the total amount at the end of n years forms a GP with first term $1.07 \times P$ and ratio 1.07, and find the nth term A_n.

c Use trial-and-error on the calculator to find how many full years it will take for the amount to double, and how many years it will take for it to become ten times the original principal.

15 [Depreciation and GPs]
A car originally costs \$20 000, then at the end of every year, it is worth only 80% of what it was worth a year before. Let W_n be its worth at the end of n years.

a Write down expressions for W_1, W_2 and W_3, and find a formula for W_n.

b Use trial-and-error on the calculator to find how many complete years it takes for the value to fall below \$2000.

ISBN 978-1-108-76630-2

16 Find the nth term of each GP.

a $\sqrt{6}, 2\sqrt{3}, 2\sqrt{6}, \ldots$ **b** $ax, a^2x^3, a^3x^5, \ldots$ **c** $-\frac{x}{y}, -1, -\frac{y}{x}, \ldots$

17 a Find a formula for T_n in $2x, 2x^2, 2x^3, \ldots$ Then find x if $T_6 = 2$.

b Find a formula for T_n in $x^4, x^2, 1, \ldots$ Then find x if $T_6 = 3^6$.

c Find a formula for T_n in $2^{-16}x, 2^{-12}x, 2^{-8}x, \ldots$ Then find x if $T_6 = 96$.

18 a Show that $2^5, 2^2, 2^{-1}, 2^{-4}, \ldots$ is a GP, and find its nth term.

b Show that $\log_2 96,\ \log_2 24,\ \log_2 6, \ldots$ is an AP, and show that $T_n = 7 - 2n + \log_2 3$.

19 a Write down the first few terms of the GP generated by substituting the positive integers into the exponential function $f(x) = \frac{4}{25} \times 5^x$. Then write down a formula for the nth term.

b i Find the formula of the nth term T_n of the GP 5, 10, 20, 40, 80, ... Then write down the exponential function $f(x)$ that generates this GP when the positive integers are substituted into it.

ii Graph the function (without the same scale on both axes) and mark the points $(1, 5)$, $(2, 10)$, $(3, 20)$, $(4, 40)$, $(5, 80)$.

20 [GPs are essentially exponential functions.]

a Show that if $f(x) = kb^x$ is any exponential function, then the sequence $T_n = kb^n$ is a GP, and find its first term and common ratio.

b Conversely, if T_n is a GP with first term a and ratio r, find the exponential function $f(x)$ such that $T_n = f(n)$.

c Plot on the same axes the points of the GP $T_n = 2^{4-n}$ and the graph of the continuous function $y = 2^{4-x}$.

ENRICHMENT

21 a What are the first term and common ratio of the GP generated by substituting the positive integers into the exponential function $f(x) = cb^x$?

b What is the equation of the exponential function that generates a GP with first term a and ratio r when the positive integers are substituted into it?

22 [Products and sums of GPs]

Suppose that $T_n = ar^{n-1}$ and $U_n = AR^{n-1}$ are two GPs.

a Show that the sequence $V_n = T_nU_n$ is a GP, and find its first term and common ratio.

b Show that the sequence $W_n = T_n + U_n$ is a GP if and only if $r = R$ and $a + A \neq 0$, and find the formula for W_n in this case. (Hint: if W_n is a GP, then $W_nW_{n+2} = {W_{n+1}}^2$. Substitute into this condition, and deduce that $(R - r)^2 = 0$.)

ISBN 978-1-108-76630-2

Cambridge University Press

1D Solving problems involving APs and GPs

This section deals with APs and GPs together and presents some further approaches to problems about the terms of APs and GPs.

A condition for three numbers to be in AP or GP

The three numbers $10,\ 25,\ 40$ form an AP because the differences $25 - 10 = 15$ and $40 - 25 = 15$ are equal.

Similarly, $10,\ 20,\ 40$ form a GP because the ratios $\frac{20}{10} = 2$ and $\frac{40}{20} = 2$ are equal.

These situations occur quite often and a formal statement is worthwhile:

6 THREE NUMBERS IN AP OR GP

- Three numbers a, m and b form an AP if
  $m - a = b - m$, that is, $m = \frac{1}{2}(a + b)$.
- Three non-zero numbers a, g and b form a GP if
  $\dfrac{g}{a} = \dfrac{b}{g}$, that is, $g^2 = ab$.

The number m is already familiar because it is the mean of a and b. In the context of sequences, the numbers m and g are called *the arithmetic mean* and *a geometric mean* of a and b, but these terms are not required in the course.

Example 14 1D

a Find the value of m if $3,\ m,\ 12$ form an AP.

b Find the value of g if $3,\ g,\ 12$ form a GP.

SOLUTION

a Because $3,\ m,\ 12$ form an AP,

$$\begin{aligned} m - 3 &= 12 - m \\ 2m &= 15 \\ m &= 7\tfrac{1}{2}. \end{aligned}$$

b Because $3,\ g,\ 12$ form a GP,

$$\begin{aligned} \frac{g}{3} &= \frac{12}{g} \\ g^2 &= 36 \\ g &= 6 \text{ or } -6. \end{aligned}$$

Solving problems leading to simultaneous equations

Many problems about APs and GPs lead to simultaneous equations. These are best solved by elimination.

7 PROBLEMS ON APS AND GPS LEADING TO SIMULTANEOUS EQUATIONS

- With APs, eliminate a by subtracting one equation from the other.
- With GPs, eliminate a by dividing one equation by the other.

ISBN 978-1-108-76630-2

Cambridge University Press

Example 15 — 1D

The third term of an AP is 16 and the 12th term is 79. Find the 41st term.

SOLUTION

Let the first term be a and the common difference be d.

Because $T_3 = 16$, $\quad a + 2d = 16, \quad (1)$

and because $T_{12} = 79$, $\quad a + 11d = 79. \quad (2)$

Subtracting (1) from (2), $\quad 9d = 63$ (the key step that eliminates a)

$d = 7.$

Substituting into (1), $\quad a + 14 = 16$

$a = 2.$

Hence $\quad T_{41} = a + 40d$

$= 282.$

Example 16 — 1D

Find the first term a and the common ratio r of a GP in which the fourth term is 6 and the seventh term is 162.

SOLUTION

Because $T_4 = 6$, $\quad ar^3 = 6, \quad (1)$

and because $T_7 = 162$, $\quad ar^6 = 162. \quad (2)$

Dividing (2) by (1), $\quad r^3 = 27$ (the key step that eliminates a)

$r = 3.$

Substituting into (1), $\quad a \times 27 = 6$

$a = \frac{2}{9}.$

Solving GP problems involving trial-and-error or logarithms

Equations and inequations involving the terms of a GP are index equations, so logarithms are needed for a systematic approach.

Trial-and-error, however, is quite satisfactory for simpler problems, and the reader may prefer to leave the application of logarithms until Chapter 13.

Example 17 — 1D

a Find a formula for the nth term of the geometric sequence $2,\ 6,\ 18,\ \ldots$.

b Use trial-and-error to find the first term greater than 1 000 000.

c Use logarithms to find the first term greater than 1 000 000.

ISBN 978-1-108-76630-2

SOLUTION

a This is a GP with $a = 2$ and $r = 3$,
so $T_n = ar^{n-1}$
$= 2 \times 3^{n-1}$.

b Put $T_n > 1000000$.
Using the calculator, $T_{12} = 354294$
and $T_{13} = 1062882$.
Hence the first term over 1000000 is $T_{13} = 1062882$.

c Put $T_n > 1000000$.

$$\begin{aligned}
\text{Then } 2 \times 3^{n-1} &> 1000000 \\
3^{n-1} &> 500000 \\
n-1 &> \log_3 500000 \quad \text{(remembering that } 2^3 = 8 \text{ means } 3 = \log_2 8\text{)} \\
n-1 &> \frac{\log_{10} 500000}{\log_{10} 3} \quad \text{(the change-of-base formula)} \\
n-1 &> 11.94\ldots \\
n &> 12.94\ldots
\end{aligned}$$

Hence the first term over 1000000 is $T_{13} = 1062882$.

Exercise 1D

FOUNDATION

1 Find the value of m if each set of numbers below forms an arithmetic sequence.
(Hint: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find m.)

a $5, m, 17$ **b** $32, m, 14$ **c** $-12, m, -50$
d $-23, m, 7$ **e** $m, 22, 32$ **f** $-20, -5, m$

2 Each triple of number forms a geometric sequence. Find the value of g. (Hint: Form an equation using the identity $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$, then solve it to find g.)

a $2, g, 18$ **b** $48, g, 3$ **c** $-10, g, -90$
d $-98, g, -2$ **e** $g, 20, 80$ **f** $-1, 4, g$

3 Find x if each triple of three numbers forms: **i** an AP, **ii** a GP.

a $4,\ x,\ 16$ **b** $1,\ x,\ 49$ **c** $16,\ x,\ 25$ **d** $-5,\ x,\ -20$
e $x,\ 10,\ 50$ **f** $x,\ 12,\ 24$ **g** $x,\ -1,\ 1$ **h** $x,\ 6,\ -12$
i $20,\ 30,\ x$ **j** $-36,\ 24,\ x$ **k** $-\frac{1}{4},\ -3,\ x$ **l** $7,\ -7,\ x$

DEVELOPMENT

4 In these questions, substitute into $T_n = a + (n-1)d$ or $T_n = ar^{n-1}$.

a Find the first six terms of the AP with first term $a = 7$ and sixth term $T_6 = 42$.
b Find the first four terms of the GP with first term $a = 27$ and fourth term $T_4 = 8$.
c Find the first five terms of the AP with $a = 48$ and $T_5 = 3$.
d Find the first five terms of the GP with $a = 48$ and $T_5 = 3$.

ISBN 978-1-108-76630-2

5 Use simultaneous equations and the formula $T_n = a + (n - 1)d$ to solve these problems.
a Find the first term and common difference of the AP with $T_{10} = 18$ and $T_{20} = 48$.
b Find the first term and common difference of the AP with $T_5 = 24$ and $T_9 = -12$.
c Find the first term and common difference of the AP with $T_4 = 6$ and $T_{12} = 34$.

6 Use simultaneous equations and the formula $T_n = ar^{n-1}$ to solve these problems.
a Find the first term and common ratio of the GP with $T_3 = 16$ and $T_6 = 128$.
b Find the first term and common ratio of the GP with $T_2 = \frac{1}{3}$ and $T_6 = 27$.
c Find the first term and common ratio of the GP with $T_5 = 6$ and $T_9 = 24$.

7 a The third term of an AP is 7 and the seventh term is 31. Find the eighth term.
b The common difference of an AP is -7 and the 10th term is 3. Find the second term.
c The common ratio of a GP is 2 and the sixth term is 6. Find the second term.

8 Use either trial-and-error or logarithms to solve these problems.
a Find the smallest value of n such that $3^n > 1\,000\,000$.
b Find the largest value of n such that $5^n < 1\,000\,000$.
c Find the smallest value of n such that $7^n > 1\,000\,000\,000$.
d Find the largest value of n such that $12^n < 1\,000\,000\,000$.

9 Let T_n be the sequence $2, 4, 8, \ldots$ of powers of 2.
a Show that the sequence is a GP, and show that the nth term is $T_n = 2^n$.
b Find how many terms are less than $1\,000\,000$. (You will need to solve the inequation $T_n < 1\,000\,000$ using trial-and-error or logarithms.)
c Use the same method to find how many terms are less than $1\,000\,000\,000$.
d Use the same method to find how many terms are less than 10^{20}.
e How many terms are between $1\,000\,000$ and $1\,000\,000\,000$?
f How many terms are between $1\,000\,000\,000$ and 10^{20}?

10 Find a formula for T_n for these GPs. Then find how many terms exceed 10^{-6}. (You will need to solve the inequation $T_n > 10^{-6}$ using trial-and-error or logarithms.)
a $98, 14, 2, \ldots$
b $25, 5, 1, \ldots$
c $1, 0.9, 0.81, \ldots$

11 When light passes through one sheet of very thin glass, its intensity is reduced by 3%. (Hint: 97% of the light gets though each sheet.)
a If the light passes through 50 sheets of this glass, find by what percentage (correct to the nearest 1%) the intensity will be reduced.
b What is the minimum number of sheets that will reduce the intensity below 1%?

12 a Find a and d for the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.
b Find a and r for the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.
c The fourth, sixth and eighth terms of an AP add to -6. Find the sixth term.

13 Each set of three numbers forms an AP. Find x and write out the numbers.
a $x - 1,\ 17,\ x + 15$
b $2x + 2,\ x - 4,\ 5x$
c $x - 3,\ 5,\ 2x + 7$
d $3x - 2,\ x,\ x + 10$

14 Each set of three numbers forms a GP. Find x and write out the numbers.
a $x,\ x + 1,\ x$
b $2 - x,\ 2,\ 5 - x$

ISBN 978-1-108-76630-2

15 Find x and write out the three numbers if they form: **i** an AP, **ii** a GP.

a $x,\ 24,\ 96$ **b** $0.2,\ x,\ 0.00002.$ **c** $x,\ 0.2,\ 0.002.$
d $x - 4,\ x + 1,\ x + 11$ **e** $x - 2,\ x + 2,\ 5x - 2$ **f** $\sqrt{5} + 1,\ x,\ \sqrt{5} - 1$
g $\sqrt{2},\ x,\ \sqrt{8}$ **h** $2^4,\ x,\ 2^6$ **i** $7,\ x,\ -7$

16 **a** Find a and b if a, b, 1 forms a GP, and b, a, 10 forms an AP.
b Find a and b if $a,\ 1,\ a + b$ forms a GP, and $b,\ \frac{1}{2},\ a - b$ forms an AP.

17 **a** Three non-zero numbers form both an AP and a GP. Prove that they are all equal. (Hint: Let the numbers be $x - d$, x and $x + d$, and prove that $d = 0$.)
b Show that in an AP, the first, fourth and seventh terms form another AP.
c Show that in a GP, the first, fourth and seventh terms form another GP.

18 **a** Show that if the first, second and fourth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the positive integer multiples of the first term.
b Show that if the first, second and fifth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the odd positive integer multiples of the first term.
c Find the common ratio of the GP in which the first, third and fourth terms form an arithmetic sequence. $\left(\text{Hint: } r^3 - 2r^2 + 1 = (r - 1)(r^2 - r - 1)\right)$
d Find the GP in which each term is one more than the sum of all the previous terms.

ENRICHMENT

19 Let a and b be positive real numbers with $a \le b$. Let $m = \frac{1}{2}(a + b)$ and $g = \sqrt{ab}$, so that the sequence a, m, b forms an AP and the sequence a, g, b forms a GP.
a Explain why $(a - b)^2 \ge 0$.
b Expand this to prove that $(a + b)^2 \ge 4ab$, and hence show that $g \le m$.
c Find a trivial, and a non-trivial, example of numbers a and b so that $g = \frac{1}{2}(a + m)$.

20 [Geometric sequences in musical instruments]
The pipe lengths in a rank of organ pipes decrease from left to right. The lengths form a GP, and the 13th pipe along is exactly half the length of the first pipe (making an interval called an *octave*).
a Show that the ratio of the GP is $r = \left(\frac{1}{2}\right)^{\frac{1}{12}}$.
b Show that the eighth pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).
c Show that the fifth pipe along is just under four-fifths the length of the first pipe (a *major third*).
d Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.
e What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

21 Construction of the positive geometric mean of two positive numbers a and b:
a Construct a number line OAB with O at zero, $OA = a$, and $OB = b$.
b Construct the midpoint M of AB, and construct the circle with diameter AB.
c Construct the midpoint N of OM, and construct the circle with diameter OM.
d Let the circles meet at S and T.
e Prove that $OS = OT$ is the positive geometric mean of a and b.

ISBN 978-1-108-76630-2

1E Adding up the terms of a sequence

Adding the terms of a sequence is often important. For example, a boulder falling from the top of a high cliff falls 5 metres in the first second, 15 metres in the second second, 25 metres in the third second, and so on. The distance that it falls in the first 10 seconds is the sum of the 10 numbers

$$5 + 15 + 25 + 35 + \cdots + 95 = 500.$$

A notation for the sums of terms of a sequence

For any sequence $T_1, T_2, T_3, \ldots$, define S_n to be the sum of the first n terms of the sequence.

8 THE SUM OF THE FIRST *n* TERMS OF A SEQUENCE

Given a sequence $T_1, T_2, T_3, \ldots$, define

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

The sum S_n is variously called:

- the *sum of the first n terms* of the sequence,
- the *sum to n terms* of the sequence,
- the *nth partial sum* of the sequence ('partial' meaning 'part of the sequence').

For example, the sum of the first 10 terms of the sequence $5, 15, 25, 35, \ldots$ is

$$\begin{aligned} S_{10} &= 5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85 + 95 \\ &= 500 \end{aligned}$$

which is also called the 10th partial sum of the sequence.

The sequence $S_1, S_2, S_3, S_4, \ldots$ of sums

The partial sums $S_1, S_2, S_3, S_4, \ldots$ form another sequence. For example, with the sequence $5, 15, 25, 35, \ldots$,

$$S_1 = 5 \qquad \begin{aligned} S_2 &= 5 + 15 \\ &= 20 \end{aligned} \qquad \begin{aligned} S_3 &= 5 + 15 + 25 \\ &= 45 \end{aligned} \qquad \begin{aligned} S_4 &= 5 + 15 + 25 + 35 \\ &= 80 \end{aligned}$$

Example 18 1E

Copy and complete this table for the successive sums of a sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n										

SOLUTION

Each entry for S_n is the sum of all the terms T_n up to that point.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n	5	20	45	80	125	180	245	320	405	500

ISBN 978-1-108-76630-2

Recovering the sequence from the partial sums

Suppose we know that the partial sums S_n of a sequence are the successive squares,

$$S_n\colon 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ \ldots$$

and we want to recover the terms T_n. The first term is $T_1 = S_1 = 1$, and then we can take successive differences, giving the sequence

$$T_n\colon 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ \ldots$$

9 RECOVERING THE TERMS FROM THE PARTIAL SUMS

The original sequence T_n can be recovered from the sequence S_n of partial sums by taking successive differences,

$$\begin{aligned} T_1 &= S_1 \\ T_n &= S_n - S_{n-1}, \text{ for } n \ge 2. \end{aligned}$$

Example 19 1E

By taking successive differences, list the terms of the original sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n										
S_n	1	5	12	22	35	51	70	92	117	145

SOLUTION

Each entry for T_n is the difference between two successive sums S_n.

n	1	2	3	4	5	6	7	8	9	10
T_n	1	4	7	10	13	16	19	22	25	28
S_n	1	5	12	22	35	51	70	92	117	145

Example 20 1E

Confirm the example given above by proving algebraically that if the partial sums S_n of a sequence are the successive squares, then the sequence T_n is the sequence of odd numbers.

SOLUTION

We are given that $S_n = n^2$.

Hence $T_1 = S_1$

$= 1$, which is the first odd number,

and for $n \ge 2$, $T_n = S_n - S_{n-1}$

$= n^2 - (n-1)^2$

$= 2n - 1$, which is the nth odd number.

Note: Taking successive differences in a sequence is analogous to differentiation in calculus, and the results have many similarities to differentiation. For example, in the worked example above, taking finite differences of a quadratic function yields a linear function. Questions 13–14 in Exercise 1E have further analogies, which are not pursued in this course.

ISBN 978-1-108-76630-2

Sigma notation

This is a concise notation for the sums of a sequence. For example:

$$\begin{aligned}\sum_{n=2}^{5} n^2 &= 2^2 + 3^2 + 4^2 + 5^2 \\ &= 4 + 9 + 16 + 25 \\ &= 54\end{aligned}$$

$$\begin{aligned}\sum_{n=6}^{10} n^2 &= 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\ &= 36 + 49 + 64 + 81 + 100 \\ &= 330\end{aligned}$$

The first sum says 'evaluate the function n^2 for all the integers from $n = 2$ to $n = 5$, then add up the resulting values'. There are 4 terms, and their sum is 54.

10 SIGMA NOTATION

Suppose that $T_1,\ T_2,\ T_3,\ \dots$ is a sequence. Then

$$\sum_{n=5}^{20} T_n = T_5 + T_6 + T_7 + T_8 + \cdots + T_{20}$$

(Any two integers can obviously be substituted for the numbers 5 and 20.)

We used the symbol Σ before in Chapter 13 of the Year 11 book. It stands for the word 'sum', and is a large version of the Greek capital letter Σ called 'sigma' and pronounced 's'. The superscripts and subscripts on the sigma sign, however, are used for the first time in this chapter.

Example 21 1E

Evaluate these sums.

a $\displaystyle\sum_{n=4}^{7} (5n + 1)$

b $\displaystyle\sum_{n=1}^{5} (-2)^n$

SOLUTION

a $$\begin{aligned}\sum_{n=4}^{7} (5n + 1) &= 21 + 26 + 31 + 36 \\ &= 114\end{aligned}$$

b $$\begin{aligned}\sum_{n=1}^{5} (-2)^n &= -2 + 4 - 8 + 16 - 32 \\ &= -22\end{aligned}$$

Series

The word *series* is often used imprecisely, but it always refers to the activity of adding up terms of a sequence. For example, the phrase

'the series $1 + 4 + 9 + \cdots$'

means that one is considering the successive partial sums $S_1 = 1,\ S_2 = 1 + 4,\ S_3 = 1 + 4 + 9,\ \dots$ of the sequence of positive squares.

The precise definition is that a *series* is the sequence of partial sums of a sequence. That is, given a sequence $T_1,\ T_2,\ T_3,\ \dots$, the corresponding series is the sequence

$$S_1 = T_1,\quad S_2 = T_1 + T_2,\quad S_3 = T_1 + T_2 + T_3,\quad S_4 = T_1 + T_2 + T_3 + T_4,\quad \dots$$

ISBN 978-1-108-76630-2

Exercise 1E

FOUNDATION

1 Find the sum S_4 of the first four terms of each sequence.

a $3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots$

b $2,\ 6,\ 18,\ 54,\ 162,\ 486,\ \ldots$

c $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$

d $32 - 16 + 8 - 4 + 2 - 1 + \cdots$

2 Find the partial sums S_4, S_5 and S_6 for each series (which you will need to continue).

a $1 - 2 + 3 - 4 + \cdots$

b $81 + 27 + 9 + 3 + \cdots$

c $30 + 20 + 10 + \cdots$

d $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

3 Copy and complete these tables of a sequence and its partial sums.

a

T_n	2	5	8	11	14	17	20
S_n							

b

T_n	40	38	36	34	32	30	28
S_n							

c

T_n	2	−4	6	−8	10	−12	14
S_n							

d

T_n	7	−7	7	−7	7	−7	7
S_n							

4 Rewrite each partial sum without sigma notation, then evaluate it.

a $\displaystyle\sum_{n=1}^{6} 2n$

b $\displaystyle\sum_{n=1}^{6} (3n + 2)$

c $\displaystyle\sum_{k=3}^{7} (18 - 3n)$

d $\displaystyle\sum_{n=5}^{8} n^2$

e $\displaystyle\sum_{n=1}^{4} n^3$

f $\displaystyle\sum_{n=0}^{5} 2^n$

g $\displaystyle\sum_{n=2}^{4} 3^n$

h $\displaystyle\sum_{\ell=1}^{31} (-1)^{\ell}$

i $\displaystyle\sum_{\ell=1}^{40} (-1)^{\ell-1}$

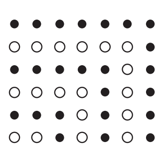
j $\displaystyle\sum_{n=5}^{105} 4$

k $\displaystyle\sum_{n=0}^{4} (-1)^n(n + 5)$

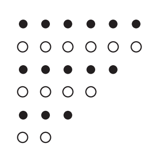
l $\displaystyle\sum_{n=0}^{4} (-1)^{n+1}(n + 5)$

DEVELOPMENT

5 **a** Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2.



b Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$.



c Part **b** shows why the sums $1,\ \ 1 + 2 = 3,\ \ 1 + 2 + 3 = 6,\ \ 1 + 2 + 3 + 4 = 10,\ \ldots$ are called the *triangular numbers*. Write out the first 15 triangular numbers.

6 Each table below gives the successive sums $S_1,\ S_2,\ S_3,\ \ldots$ of a sequence. By taking successive differences, write out the terms of the original sequence.

a

T_n							
S_n	1	4	9	16	25	36	49

b

T_n							
S_n	2	6	14	30	62	126	254

c

T_n							
S_n	−3	−8	−15	−24	−35	−48	−63

d

T_n							
S_n	8	0	8	0	8	0	8

ISBN 978-1-108-76630-2

7 [The Fibonacci and Lucas sequences] Each table below gives the successive sums S_n of a sequence. By taking successive differences, write out the terms of the original sequence.

a

T_n								
S_n	1	2	3	5	8	13	21	34

b

T_n								
S_n	3	4	7	11	18	29	47	76

8 The nth partial sum of a series is $S_n = 3^n - 1$.
- a Write out the first five partial sums.
- b Take differences to find the first five terms of the original sequence.
- c Write down S_{n-1}, then use the result $T_n = S_n - S_{n-1}$ to find a formula for T_n. (Hint: This will need the factorisation $3^n - 3^{n-1} = 3^{n-1}(3-1) = 2 \times 3^{n-1}$.)

9 Repeat the steps of the previous question for the sequence whose nth partial sum is:
- a $S_n = 10(2^n - 1)$
- b $S_n = 4(5^n - 1)$
- c $S_n = \frac{1}{4}(4^n - 1)$

(Hint: You will need factorisations such as $2^n - 2^{n-1} = 2^{n-1}(2-1)$.)

10 Find the nth term and the first three terms of the sequence for which S_n is:
- a $S_n = 3n(n+1)$
- b $S_n = 5n - n^2$
- c $S_n = 4n$
- d $S_n = n^3$
- e $S_n = 1 - 3^{-n}$
- f $S_n = \left(\frac{1}{7}\right)^n - 1$

11 Rewrite each sum in sigma notation, starting each sum at $n = 1$. Do not evaluate it.
- a $1^3 + 2^3 + 3^3 + \cdots + 40^3$
- b $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40}$
- c $3 + 4 + 5 + \cdots + 22$
- d $2 + 2^2 + 2^3 + \cdots + 2^{12}$
- e $-1 + 2 - 3 + \cdots + 10$
- f $1 - 2 + 3 - \cdots - 10$

ENRICHMENT

12 In these sequences, the first term will not necessarily obey the same rule as the succeeding terms, in which case the formula for the sequence will need to be given piecewise:
- a $S_n = n^2 + 4n + 3$
- b $S_n = 7(3^n - 4)$
- c $S_n = \dfrac{1}{n}$
- d $S_n = n^3 + n^2 + n$

Find T_1 and a formula for T_n for each sequence. How could you have predicted whether or not the general formula would hold for T_1?

13
- a The partial sums of a sequence T_n are given by $S_n = 2^n$. Use the formula in Box 9 to find a formula for T_n.
- b Confirm your answer by writing out the calculation in table form, as in Question **6**.
- c In Chapter 11 of the Year 11 book, you differentiated $y = e^x$. What is the analogy to this result?

14
- a Prove that $n^3 - (n-1)^3 = 3n^2 - 3n + 1$.
- b The partial sums of a sequence T_n are given by $S_n = n^3$. Use the formula in Box 9 to find a formula for T_n.
- c The terms of the sequence T_n are the partial sums of a third sequence U_n. Use the formula in Box 9 to find a formula for T_n.
- d Confirm your answer by writing out in table form the successive taking of differences in parts **b** and **c**.
- e In Chapter 8 of the Year 11 volume, you differentiated powers of x. What is the analogy to these results?

15
- a Write out the terms of $\displaystyle\sum_{r=1}^{10}\left(\frac{1}{r} - \frac{1}{r+1}\right)$ and hence show that the sum is $\frac{10}{11}$.
- b Rationalise the denominator of $\dfrac{1}{\sqrt{k+1}+\sqrt{k}}$ and hence evaluate $\displaystyle\sum_{n=1}^{15}\frac{1}{\sqrt{k+1}+\sqrt{k}}$.
- c Evaluate $\displaystyle\sum_{r=1}^{4}\left(\sum_{s=1}^{4}\left(\sum_{t=1}^{4} rst\right)\right)$.

ISBN 978-1-108-76630-2

1F Summing an arithmetic series

There are two formulae for adding up the first n terms of an AP.

Adding the terms of an AP

Consider adding the first six terms of the AP

$$5 + 15 + 25 + 35 + 45 + 55 + \cdots$$

$$\begin{aligned}
&\text{Writing out the sum,} & S_6 &= 5 + 15 + 25 + 35 + 45 + 55.\\
&\text{Reversing the sum,} & S_6 &= 55 + 45 + 35 + 25 + 15 + 5,\\
&\text{and adding the two,} & 2S_6 &= 60 + 60 + 60 + 60 + 60 + 60\\
& & &= 6 \times 60, \text{ because there are 6 terms in the series.}\\
&\text{Dividing by 2,} & S_6 &= \tfrac{1}{2} \times 6 \times 60\\
& & &= 180.
\end{aligned}$$

Notice that 60 is the sum of the first term $T_1 = 5$ and the last term $T_6 = 55$.

In general, let $\ell = T_n$ be the last term of an AP with first term a and difference d.

$$\begin{aligned}
&\text{Then} & S_n &= a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.\\
&\text{Reversing the sum,} & S_n &= \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a,\\
&\text{and adding,} & 2S_n &= (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)\\
& & &= n \times (a + \ell), \text{ because there are } n \text{ terms in the series.}\\
&\text{Dividing by 2,} & S_n &= \tfrac{1}{2}n(a + \ell).
\end{aligned}$$

Example 22 1F

Add up all the integers from 100 to 200 inclusive.

SOLUTION

The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms.

The first term is $a = 100$ and the last term is $\ell = 200$.

$$\begin{aligned}
\text{Using} \quad S_n &= \tfrac{1}{2}n(a + \ell),\\
S_{101} &= \tfrac{1}{2} \times 101 \times (100 + 200)\\
&= \tfrac{1}{2} \times 101 \times 300\\
&= 15\,150.
\end{aligned}$$

An alternative formula for summing an AP

This alternative form is equally important.

$$\begin{aligned}
&\text{The previous formula is} & S_n &= \tfrac{1}{2}n(a + \ell), \text{ where } \ell = T_n = a + (n - 1)d.\\
&\text{Substituting } \ell = a + (n - 1)d, & S_n &= \tfrac{1}{2}n\big(a + a + (n - 1)d\big)\\
&\text{so} & S_n &= \tfrac{1}{2}n\big(2a + (n - 1)d\big).
\end{aligned}$$

ISBN 978-1-108-76630-2

11 TWO FORMULAE FOR SUMMING AN AP

Suppose that the first term a of an AP, and the number n of terms, are known.

- When the last term $\ell = T_n$ is known, use $S_n = \frac{1}{2}n(a + \ell)$.
- When the difference d is known, use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$.

If you have a choice, use the first because it is simpler.

Example 23 1F

Consider the arithmetic series $100 + 94 + 88 + 82 + \cdots$.

a Find S_{10}.

b Find S_{41}.

SOLUTION

The series is an AP with $a = 100$ and $d = -6$.

a Using $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$,

$$\begin{aligned} S_{10} &= \tfrac{1}{2} \times 10 \times (2a + 9d) \\ &= 5 \times (200 - 54) \\ &= 730. \end{aligned}$$

b Similarly,

$$\begin{aligned} S_{41} &= \tfrac{1}{2} \times 41 \times (2a + 40d) \\ &= \tfrac{1}{2} \times 41 \times (200 - 240) \\ &= \tfrac{1}{2} \times 41 \times (-40) \\ &= -820. \end{aligned}$$

Example 24 1F

a Find how many terms are in the sum $41 + 45 + 49 + \cdots + 401$.

b Hence evaluate the sum $41 + 45 + 49 + \cdots + 401$.

SOLUTION

a The series is an AP with first term $a = 41$ and difference $d = 4$.

To find the numbers of terms, put $T_n = 401$

$$\begin{aligned} a + (n - 1)d &= 401 \\ 41 + 4(n - 1) &= 401 \\ 4(n - 1) &= 360 \\ n - 1 &= 90 \\ n &= 91. \end{aligned}$$

Thus there are 91 terms in the series.

b Because we now know both the difference d and the last term $\ell = T_{91}$, either formula can be used. It's always easier to use $S_n = \frac{1}{2}n(a + \ell)$ if you can.

Using $S_n = \frac{1}{2}n(a + \ell)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (41 + 401) \\ &= \tfrac{1}{2} \times 91 \times 442 \\ &= 20\,111. \end{aligned}$$

OR

Using $S_n = \frac{1}{2}n(2a + (n - 1)d)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (2a + 90d) \\ &= \tfrac{1}{2} \times 91 \times (82 + 360) \\ &= 20\,111. \end{aligned}$$

ISBN 978-1-108-76630-2

Solving problems involving the sums of APs

Problems involving sums of APs are solved using the formulae developed for the nth term T_n and the sum S_n of the first n terms.

Example 25 1F

a Find an expression for the sum S_n of n terms of the series $40 + 37 + 34 + \cdots$.

b Hence find the least value of n for which the partial sum S_n is negative.

SOLUTION

The sequence is an AP with $a = 40$ and $d = -3$.

a
$$\begin{aligned} S_n &= \tfrac{1}{2}n(2a + (n-1)d) \\ &= \tfrac{1}{2} \times n \times \big(80 - 3(n-1)\big) \\ &= \tfrac{1}{2} \times n \times (80 - 3n + 3) \\ &= \frac{n(83 - 3n)}{2} \end{aligned}$$

b
$$\begin{aligned} &\text{Put} & S_n &< 0. \\ &\text{Then} & \frac{n(83-3n)}{2} &< 0 \\ &\boxed{\times\ 2} & n(83 - 3n) &< 0 \\ &\boxed{\div\ n} & 83 - 3n &< 0, \quad \text{because } n \text{ is positive,} \\ && 83 &< 3n \\ && n &> 27\tfrac{2}{3}. \end{aligned}$$

Hence S_{28} is the first sum that is negative.

Example 26 1F

The sum of the first 10 terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION

$$\begin{aligned} &\text{The first piece of information given is} & S_{10} &= 0 \\ && 5(2a + 9d) &= 0 \\ &\boxed{\div\ 5} & 2a + 9d &= 0. & (1) \\ &\text{The second piece of information given is} & T_1 + T_2 &= 24 \\ && a + (a + d) &= 24 \\ && 2a + d &= 24. & (2) \\ &\text{Subtracting (2) from (1),} & 8d &= -24 \\ && d &= -3, \\ &\text{and substituting this into (2),} & 2a - 3 &= 24 \\ && a &= 13\tfrac{1}{2}. \end{aligned}$$

Hence the AP is $13\frac{1}{2} + 10\frac{1}{2} + 7\frac{1}{2} + \cdots$.

ISBN 978-1-108-76630-2

Exercise 1F

FOUNDATION

1 Let $S_7 = 2 + 5 + 8 + 11 + 14 + 17 + 20$. By reversing the sum and adding in columns, evaluate S_7.

2 State how many terms each sum has, then find the sum using $S_n = \frac{1}{2}n(a + \ell)$.

a $1 + 2 + 3 + 4 + \cdots + 100$

b $1 + 3 + 5 + 7 + \cdots + 99$

c $2 + 4 + 6 + 8 + \cdots + 100$

d $3 + 6 + 9 + 12 + \cdots + 300$

e $101 + 103 + 105 + \cdots + 199$

f $1001 + 1002 + 1003 + \cdots + 10\,000$

3 Use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_6 of the first 6 terms of the series with:

a $a = 5$ and $d = 10$

b $a = 8$ and $d = 2$

c $a = -3$ and $d = -9$

d $a = -7$ and $d = -12$

4 Use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum of the stated number of terms.

a $2 + 5 + 8 + \cdots$ (12 terms)

b $40 + 33 + 26 + \cdots$ (21 terms)

c $-6 - 2 + 2 + \cdots$ (200 terms)

d $33 + 30 + 27 + \cdots$ (23 terms)

e $-10 - 7\frac{1}{2} - 5 + \cdots$ (13 terms)

f $10\frac{1}{2} + 10 + 9\frac{1}{2} + \cdots$ (40 terms)

5 First use the formula $T_n = a + (n - 1)d$ to find how many terms there are in each sum. Then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum, where ℓ is the last term T_n.

a $50 + 51 + 52 + \cdots + 150$

b $8 + 15 + 22 + \cdots + 92$

c $-10 - 3 + 4 + \cdots + 60$

d $4 + 7 + 10 + \cdots + 301$

e $6\frac{1}{2} + 11 + 15\frac{1}{2} + \cdots + 51\frac{1}{2}$

f $-1\frac{1}{3} + \frac{1}{3} + 2 + \cdots + 13\frac{2}{3}$

6 Find these sums by any appropriate method.

a $2 + 4 + 6 + \cdots + 1000$

b $1000 + 1001 + \cdots + 3000$

c $1 + 5 + 9 + \cdots$ (40 terms)

d $10 + 30 + 50 + \cdots$ (12 terms)

7 Find and simplify the sum of the first n terms of each series.

a $5 + 10 + 15 + \cdots$

b $10 + 13 + 16 + \cdots$

c $3 + 7 + 11 + \cdots$

d $-9 - 4 + 1 + \cdots$

e $5 + 4\frac{1}{2} + 4 + \cdots$

f $\big(1 - \sqrt{2}\big) + 1 + \big(1 + \sqrt{2}\big) + \cdots$

8 Use either standard formula for S_n to find a formula for the sum of the first n:

a positive integers,

b odd positive integers,

c positive integers divisible by 3,

d odd positive multiples of 100.

DEVELOPMENT

9 a How many legs are there on 15 fish, 15 ducks, 15 dogs, 15 beetles, 15 spiders, and 15 ten-legged grubs? How many of these creatures have the mean number of legs?

b Matthew Flinders High School has 1200 pupils, with equal numbers of each age from 6 to 17 years inclusive. It also has 100 teachers and ancillary staff, all aged 30 years, and one Principal aged 60 years. What is the total of the ages of everyone in the school?

c An advertising graduate earns \$28 000 per annum in her first year, then each successive year her salary rises by \$1600. What are her total earnings over 10 years?

ISBN 978-1-108-76630-2

10 By substituting appropriate values of k, find the first term a and last term ℓ of each sum. Then evaluate the sum using $S_n = \frac{1}{2}n(a + \ell)$. (Note that all four series are APs.)

a $\sum_{k=1}^{200} (600 - 2k)$ **b** $\sum_{k=1}^{61} (93 - 3k)$

c $\sum_{k=1}^{40} (3k - 50)$ **d** $\sum_{k=10}^{30} (5k + 3)$

11 Find the sums of these APs, whose terms are logarithms.

a $\log_a 2 + \log_a 4 + \log_a 8 + \cdots + \log_a 1024$

b $\log_5 243 + \log_5 81 + \log_5 27 + \cdots + \log_5 \frac{1}{243}$

c $\log_b 36 + \log_b 18 + \log_b 9 + \cdots + \log_b \frac{9}{8}$

d $\log_x \frac{27}{8} + \log_x \frac{9}{4} + \log_x \frac{3}{2} + \cdots$ (10 terms)

12 Solve these questions using the formula $S_n = \frac{1}{2}n(a + \ell)$ whenever possible — otherwise use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$.

a Find the last term if a series with 10 terms and first term -23 has sum -5.

b Find the first term if a series with 40 terms and last term $8\frac{1}{2}$ has sum 28.

c Find the common difference if a series with 8 terms and first term 5 has sum 348.

d Find the first term if a series with 15 terms and difference $\frac{2}{7}$ has sum -15.

13 **a** Show that the sum to n terms of the AP $60 + 52 + 44 + 36 + \cdots$ is $S_n = 4n(16 - n)$.

b Hence find how many terms must be taken to make the sum: **i** zero, **ii** negative.

c Find the two values of n for which the sum S_n is 220.

d Show that $S_n = -144$ has two integer solutions, but that only one has meaning.

e For what values of n does the sum S_n exceed 156?

f Prove that no sum S_n can exceed 256.

14 First use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_n for each arithmetic series. Then use quadratic equations to find the number of terms if S_n has the given value.

a $42 + 40 + 38 + \cdots$ where $S_n = 0$ **b** $60 + 57 + 54 + \cdots$ where $S_n = 0$

c $45 + 51 + 57 + \cdots$ where $S_n = 153$ **d** $2\frac{1}{2} + 3 + 3\frac{1}{2} + \cdots$ where $S_n = 22\frac{1}{2}$

15 Find the first term and the number of terms if a series has:

a $d = 4, \ell = 32$ and $S_n = 0$ **b** $d = -3, \ell = -10$ and $S_n = 55$

16 **a** Logs of wood are stacked with 10 on the top row, 11 on the next, and so on. If there are 390 logs, find the number of rows, and the number of logs on the bottom row.

b A stone dropped from the top of a 245-metre cliff falls 5 metres in the first second, 15 metres in the second second, and so on in arithmetic sequence. Find a formula for the distance after n seconds, and find how long the stone takes to fall to the ground.

c A truck spends several days depositing truckloads of gravel from a quarry at equally spaced intervals along a straight road. The first load is deposited 20 km from the quarry, the last is 10 km further along the road. If the truck travels 550 km during these deliveries, including its return to the quarry after the last delivery, how many trips does it make, and how far apart are the deposits?

ISBN 978-1-108-76630-2

17 a The sum of the first and fourth terms of an AP is 16, and the sum of the third and eighth terms is 4. Find the sum of the first 10 terms.

b The sum of the first 10 terms of an AP is zero, and the 10th term is -9. Find the first and second terms.

c The sum to 16 terms of an AP is 96, and the sum of the second and fourth terms is 45. Find the fourth term, and show that the sum to four terms is also 96.

ENRICHMENT

18 a Find $1 + 2 + \cdots + 24$.

b Show that $\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n} = \frac{n+1}{2}$.

c Hence find the sum of the first 300 terms of
$\frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \cdots$.

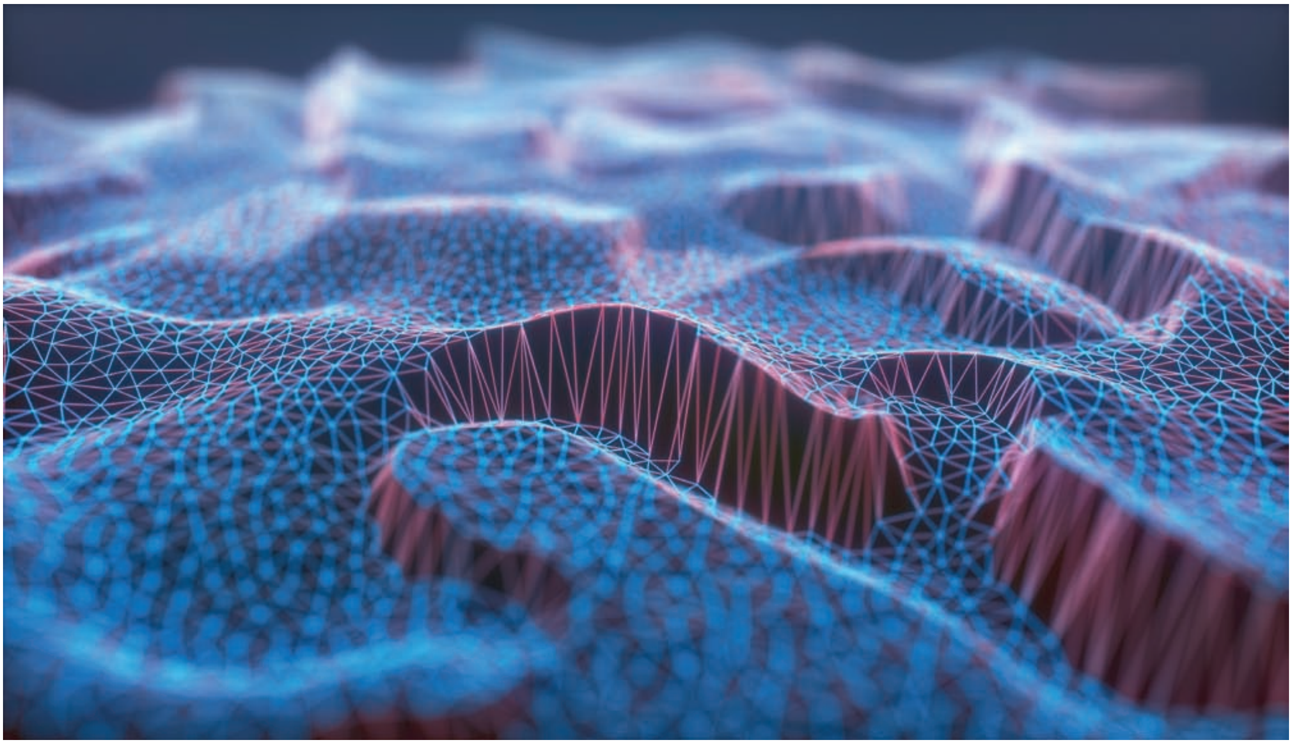
19 a Find a formula for the nth triangular number $S_n = 1 + 2 + 3 + \cdots + n$.

b For what values of n is S_n: i divisible by 5, ii even?

c What is the smallest value of n for which S_n is divisible by:

i 29, ii 35, iii 26, iv 38,

v two distinct primes, vi three distinct primes, vii four distinct primes?



ISBN 978-1-108-76630-2

1G Summing a geometric series

There is also a simple formula for finding the sum of the first n terms of a GP. The approach, however, is quite different from the approach used for APs.

Adding up the terms of a GP

This method is easier to understand with a general GP. Let us find the sum S_n of the first n terms of the GP $a + ar + ar^2 + \cdots$

Writing out the sum, $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \quad (1)$

Multiplying both sides by r, $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$

Subtracting (1) from (2), $\quad (r - 1)\,S_n = ar^n - a.$

Then provided that $r \neq 1$, $\quad S_n = \dfrac{a(r^n - 1)}{r - 1}.$

If $r < 1$, there is a more convenient form. Taking opposites of top and bottom,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Method for summing a GP

Thus again there are two forms to remember.

12 TWO FORMULAE FOR SUMMING A GP

Suppose that the first term a, the ratio r, and the number n of terms are known.

- When $r > 1$, use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$.
- When $r < 1$, use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

Example 27 **1G**

a Find the sum of all the powers of 5 from 5^0 to 5^7.

b Find the sum of the first six terms of the geometric series $2 - 6 + 18 - \cdots$.

SOLUTION

a The sum $5^0 + 5^1 + \cdots + 5^7$ is a GP with $a = 1$ and $r = 5$.

Using $S_n = \dfrac{a(r^n - 1)}{r - 1}$, $\quad$ (in this case $r > 1$)

$$S_8 = \frac{a(r^8 - 1)}{r - 1} \quad \text{(there are 8 terms)}$$

$$= \frac{1 \times (5^8 - 1)}{5 - 1}$$

$$= 97\,656.$$

ISBN 978-1-108-76630-2

b The series $2 - 6 + 18 - \cdots$ is a GP with $a = 2$ and $r = -3$.

$$\text{Using } S_n = \frac{a(1 - r^n)}{1 - r}, \qquad \text{(in this case } r < 1)$$

$$\begin{aligned} S_6 &= \frac{a(1 - r^6)}{1 - r} \\ &= \frac{2 \times (1 - (-3)^6)}{1 + 3} \\ &= -364. \end{aligned}$$

Solving problems about the sums of GPs

As always, read the question very carefully and write down all the information in symbolic form.

Example 28 1G

The sum of the first four terms of a GP with ratio 3 is 200. Find the four terms.

SOLUTION

It is known that $S_4 = 200.$

$$\begin{aligned} \text{Using the formula, } \quad \frac{a(3^4 - 1)}{3 - 1} &= 200 \\ \frac{80a}{2} &= 200 \\ 40a &= 200 \\ a &= 5. \end{aligned}$$

So the series is $5 + 15 + 45 + 135 + \cdots$

Solving problems involving trial-and-error or logarithms

As remarked already in Section 1D, logarithms are needed for solving GP problems systematically, but trial-and-error is quite satisfactory for simpler problems.

Example 29 1G

a Find a formula for the sum of the first n terms of the GP $2 + 6 + 18 + \cdots$.

b How many terms of this GP must be taken for the sum to exceed one billion?

SOLUTION

a The sequence is a GP with $a = 2$ and $r = 3$,

$$\begin{aligned} \text{so } S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{2(3^n - 1)}{3 - 1} \\ &= 3^n - 1. \end{aligned}$$

ISBN 978-1-108-76630-2

b Put $S_n > 1\,000\,000\,000$.
Then $3^n - 1 > 1\,000\,000\,000$
$3^n > 1\,000\,000\,001$.
Using trial-and-error on the calculator,
$3^{18} = 387\,420\,489$
and $3^{19} = 1\,162\,261\,467$,
so S_{19} is the first sum over one billion.

OR

Put $S_n > 1\,000\,000\,000$.
Then $3^n - 1 > 1\,000\,000\,000$
$3^n > 1\,000\,000\,001$
$n > \dfrac{\log_{10} 1\,000\,000\,001}{\log_{10} 3}$
$n > 18.86\ldots$,
so S_{19} is the first sum over one billion.

An exceptional case

If the ratio of a GP is 1, then the formula for S_n doesn't work, because the denominator $r - 1$ would be zero. All the terms, however, are equal to the first term a, so the formula for the partial sum S_n is just

$$S_n = an.$$

This series is also an AP with first term a and difference 0. The last term is also a, so

$$S_n = \tfrac{1}{2}n(a + \ell) = \tfrac{1}{2}n(a + a) = an.$$

Exercise 1G

FOUNDATION

1 Let $S_6 = 2 + 6 + 18 + 54 + 162 + 486$. By taking $3S_6$ and subtracting S_6 in columns, evaluate S_6.

2 'As I was going to St Ives, I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits. Kits, cats, sacks and wives, how many were going to St Ives?'

Only the speaker was going *to* St Ives, but how many were going the other way?

3 **a** Use the formula $S_7 = \dfrac{a(r^7 - 1)}{r - 1}$ to find $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6$.

b Use the formula $S_7 = \dfrac{a(1 - r^7)}{1 - r}$ to find $1 - 3 + 3^2 - 3^3 + 3^4 - 3^5 + 3^6$.

4 Find these sums using $S_n = \dfrac{a(r^n - 1)}{r - 1}$ when $r > 1$, or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ when $r < 1$.
Then find a formula for the sum S_n of the first n terms of each series.

a $1 + 2 + 4 + 8 + \cdots$ (10 terms)
b $2 + 6 + 18 + \cdots$ (5 terms)
c $-1 - 10 - 100 - \cdots$ (5 terms)
d $-1 - 5 - 25 - \cdots$ (5 terms)
e $1 - 2 + 4 - 8 + \cdots$ (10 terms)
f $2 - 6 + 18 - \cdots$ (5 terms)
g $-1 + 10 - 100 + \cdots$ (5 terms)
h $-1 + 5 - 25 + \cdots$ (5 terms)

5 Find these sums. Then find a formula for the sum S_n of the first n terms of each series. Be careful when dividing by $1 - r$, because $1 - r$ is a fraction in each case.

a $8 + 4 + 2 + \cdots$ (10 terms)
b $9 + 3 + 1 + \cdots$ (6 terms)
c $45 + 15 + 5 + \cdots$ (5 terms)
d $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$
e $8 - 4 + 2 - \cdots$ (10 terms)
f $9 - 3 + 1 - \cdots$ (6 terms)
g $-45 + 15 - 5 + \cdots$ (5 terms)
h $\frac{2}{3} - 1 + \frac{3}{2} - \frac{9}{4} + \frac{27}{8}$

6 Find an expression for S_n. Hence approximate S_{10} correct to four significant figures.

a $1 + 1.2 + (1.2)^2 + \cdots$
b $1 + 0.95 + (0.95)^2 + \cdots$
c $1 + 1.01 + (1.01)^2 + \cdots$
d $1 + 0.99 + (0.99)^2 + \cdots$

ISBN 978-1-108-76630-2

Cambridge University Press

DEVELOPMENT

7 The King takes a chessboard of 64 squares, and places 1 grain of wheat on the first square, 2 grains on the next square, 4 grains on the next square, and so on.

a How many grains are on: i the last square, ii the whole chessboard?

b Given that 1 litre of wheat contains about 30 000 grains, how many cubic kilometres of wheat are there on the chessboard?

8 Find S_n and S_{10} for each series, rationalising the denominators in your answers.

a $1 + \sqrt{2} + 2 + \cdots$ b $2 - 2\sqrt{5} + 10 - \cdots$

9 Find these sums. First write out some terms and identify a and r.

a $\sum_{n=1}^{7} 3 \times 2^n$ b $\sum_{n=3}^{8} 3^{n-1}$ c $\sum_{n=1}^{8} 3 \times 2^{3-n}$

10 a The first term of a GP is $\frac{1}{8}$ and the fifth term is 162. Find the first five terms of the GP, then find their sum.

b The first term of a GP is $-\frac{3}{4}$ and the fourth term is 6. Find the sum of the first six terms.

c The second term of GP is 0.08 and the third term is 0.4. Find the sum to eight terms.

d The ratio of a GP is $r = 2$ and the sum to eight terms is 1785. Find the first term.

e A GP has ratio $r = -\frac{1}{2}$ and the sum to eight terms is 425. Find the first term.

11 a Each year when the sunflower paddock is weeded, only half the previous weight of weed is dug out. In the first year, 6 tonnes of weed is dug out.

i How much is dug out in the 10th year?

ii What is the total dug out over 10 years (correct to four significant figures)?

b Every two hours, half of a particular medical isotope decays. If there was originally 20 grams, how much remains after a day (correct to two significant figures)?

c The price of Victoria shoes is increasing over a 10-year period by 10% per annum, so that the price in each of those 10 years is P, $1.1 \times P$, $(1.1)^2 \times P$, $\ldots$ I buy one pair of these shoes each year.

i Find an expression for the total paid over 10 years (correct to the nearest cent).

ii Hence find the initial price P if the total paid is \$900.

12 The number of people attending the yearly Abletown Show is rising by 5% per annum, and the number attending the yearly Bush Creek Show is falling by 5% per annum. In the first year under consideration, 5000 people attended both shows.

a Find the total number attending each show during the first six years.

b Show that the number attending the Abletown Show first exceeds ten times the number attending the Bush Creek Show in the 25th year.

c What is the ratio (correct to three significant figures) of the total number attending the Abletown Show over these 25 years to the total attending the Bush Creek Show?

13 a Show that the sum S_n of the first n terms of $7 + 14 + 28 + \cdots$ is $S_n = 7(2^n - 1)$.

b For what value of n is S_n equal to 1785?

c Show that $T_n = 7 \times 2^{n-1}$, and find how many terms are less than 70 000.

d Use trial-and-error to find the first sum S_n that is greater than 70 000.

e Prove that the sum S_n of the first n terms is always 7 less than the $(n + 1)$th term.

ISBN 978-1-108-76630-2

Cambridge University Press

14 The powers of 3 that are greater than 1 form a GP 3, 9, 27, ...

a Find using logarithms how many powers of 3 there are between 2 and 10^{20}.

b Show that $S_n = \frac{3}{2}(3^n - 1)$, and find the smallest value of n for which $S_n > 10^{20}$.

15 Find a formula for S_n, and hence find n for the given value of S_n.

a $5 + 10 + 20 + \cdots$ where $S_n = 315$

b $5 - 10 + 20 - \cdots$ where $S_n = -425$

c $18 + 6 + 2 + \cdots$ where $S_n = 26\frac{8}{9}$

d $48 - 24 + 12 - \cdots$ where $S_n = 32\frac{1}{4}$

16 Find the nth terms of the sequences:

a $\dfrac{2}{1}, \dfrac{2+4}{1+3}, \dfrac{2+4+6}{1+3+5}, \ldots$

b $\dfrac{1}{1}, \dfrac{1+2}{1+4}, \dfrac{1+2+4}{1+4+16}, \ldots$

17 **a** Show that in any GP, $S_{2n} : S_n = (r^n + 1) : 1$. Hence find the common ratio of the GP if $S_{12} : S_6 = 65 : 1$.

b Show that if S_n and Σ_n are the sums to n terms of GPs with ratios r and r^2 respectively, but the same first term, then $\Sigma_n : S_n = (r^n + 1) : (r + 1)$.

c In any GP, let $R_n = T_{n+1} + T_{n+2} + \cdots + T_{2n}$. Show that $R_n : S_n = r^n : 1$, and hence find r if $R_8 : S_8 = 1 : 81$.

ENRICHMENT

18 Given a GP in which $T_1 + T_2 + \cdots + T_{10} = 2$ and $T_{11} + T_{12} + \cdots + T_{30} = 12$, find $T_{31} + T_{32} + \cdots + T_{60}$.

19 Show that the formula for the nth partial sum of a GP can also be written independently of n, in terms only of a, r and the last term $\ell = T_n = ar^{n-1}$, as

$$S_n = \frac{r\ell - a}{r - 1} \quad \text{or} \quad S_n = \frac{a - r\ell}{1 - r}.$$

a Hence find:

i $1 + 2 + 4 + \cdots + 1\,048\,576$

ii $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{2187}$

b Find n and r if $a = 1$, $\ell = 64$, $S_n = 85$.

c Find ℓ and n if $a = 5$, $r = -3$, $S_n = -910$.

20 **a** The sequence $T_n = 2 \times 3^n + 3 \times 2^n$ is the sum of two GPs. Find S_n.

b The sequence $T_n = 2n + 3 + 2^n$ is the sum of an AP and a GP. Use a combination of AP and GP formulae to find S_n.

c It is given that the sequence $10, 19, 34, 61, \ldots$ has the form $T_n = a + nd + b2^n$, for some values of a, d and b. Find these values, and hence find S_n.

ISBN 978-1-108-76630-2

1H The limiting sum of a geometric series

There is a sad story of a perishing frog, dying of thirst only 8 metres from the edge of a waterhole. He first jumps 4 metres towards it, his second jump is 2 metres, then each successive jump is half the previous jump. Does the frog perish?

The jumps form a GP, whose terms T_n and sums S_n are as follows:

T_n	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...
S_n	4	6	7	$7\frac{1}{2}$	$7\frac{3}{4}$	$7\frac{7}{8}$	$7\frac{15}{16}$	...

The successive jumps $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots$ have limit zero, because they are halving each time. It seems too that the successive sums S_n have limit 8, meaning that the frog's total distance gets 'as close as we like' to 8 metres. So provided that the frog can stick his tongue out even the merest fraction of a millimetre, eventually he will get some water to drink and be saved.

The limiting sum of a GP

We can describe all this more precisely by looking at the sum S_n of the first n terms and examining what happens as $n \to \infty$.

The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ is a GP with $a = 4$ and $r = \frac{1}{2}$.
Using the formula for the sum to n terms of the series,

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \qquad \text{(using this formula because } r < 1\text{)} \\ &= \frac{4\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} \\ &= 4 \times \left(1-\left(\tfrac{1}{2}\right)^n\right) \div \tfrac{1}{2} \\ &= 8\left(1-\left(\tfrac{1}{2}\right)^n\right). \end{aligned}$$

As n increases, the term $\left(\frac{1}{2}\right)^n$ gets progressively closer to zero:

$$\left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4},\quad \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8},\quad \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16},\quad \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32},\quad \left(\tfrac{1}{2}\right)^6 = \tfrac{1}{64},\quad \ldots$$

so that $\left(\frac{1}{2}\right)^n$ has limit zero as $n \to \infty$.

Hence S_n does indeed have limit $8(1-0) = 8$, as the table of values suggested.

There are several different common notations and words for this situation:

13 NOTATIONS FOR THE LIMITING SUM

Take as an example the series $4 + 2 + 1 + \frac{1}{2} + \cdots$.

- $S_n \to 8$ as $n \to \infty$. ('S_n has limit 8 as n increases without bound.')
- $\lim\limits_{n\to\infty} S_n = 8$ ('The limit of S_n, as n increases without bound, is 8.')
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ has *limiting sum* $S_\infty = 8$.
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ *converges to the limit* $S_\infty = 8$.
- $4 + 2 + 1 + \frac{1}{2} + \cdots = 8$.

The symbols S_∞ and S are both commonly used for the limiting sum.

ISBN 978-1-108-76630-2

The general case

Suppose now that T_n is a GP with first term a and ratio r, so that

$$T_n = ar^{n-1} \qquad \text{and} \qquad S_n = \frac{a(1 - r^n)}{1 - r}.$$

Suppose also that the ratio r lies in the interval $-1 < r < 1$.

Then as $n \to \infty$, the successive powers $r^1, r^2, r^3, r^4, \ldots$ get smaller and smaller,

that is, $\quad r^n \to 0 \quad$ and $\quad 1 - r^n \to 1$.

Thus both the nth term T_n and the sum S_n *converge* to a limit,

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} ar^{n-1} = 0, \qquad \text{and} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}.$$

14 THE LIMITING SUM OF A GEOMETRIC SERIES

- Suppose that $|r| < 1$, that is, $-1 < r < 1$.
  Then $\quad r^n \to 0$ as $n \to \infty$,
  so the terms and the partial sums of the GP both *converge to a limit*,
  $$\lim_{n\to\infty} T_n = 0 \qquad \text{and} \qquad S_\infty = \lim_{n\to\infty} S_n = \frac{a}{1 - r}.$$
- If $|r| \geq 1$, then the partial sums S_n do not converge to a limit.

Example 30 1H

Explain why these series have limiting sums, and find them.

a $18 + 6 + 2 + \cdots$

b $18 - 6 + 2 - \cdots$

SOLUTION

a Here $a = 18$ and $r = \frac{1}{3}$.
Because $-1 < r < 1$, the series converges.

$$S_\infty = \frac{18}{1 - \frac{1}{3}} = 18 \times \frac{3}{2} = 27$$

b Here $a = 18$ and $r = -\frac{1}{3}$.
Because $-1 < r < 1$, the series converges.

$$S_\infty = \frac{18}{1 + \frac{1}{3}} = 18 \times \frac{3}{4} = 13\tfrac{1}{2}$$

ISBN 978-1-108-76630-2

Example 31 1H

a For what values of x does the series $1 + (x - 2) + (x - 2)^2 + \cdots$ converge?

b When the series does converge, what is its limiting sum?

SOLUTION

The sequence is a GP with first term $a = 1$ and ratio $r = x - 2$.

a The GP converges when

$$\begin{aligned} -1 &< r < 1 \\ -1 &< x - 2 < 1 \\ \boxed{+2} \qquad 1 &< x < 3. \end{aligned}$$

b The limiting sum is then

$$\begin{aligned} S_\infty &= \frac{1}{1 - (x - 2)} \\ &= \frac{1}{3 - x}. \end{aligned}$$

Solving problems involving limiting sums

As always, the first step is to write down in symbolic form everything that is given in the question.

Example 32 1H

Find the ratio of a GP whose first term is 10 and whose limiting sum is 40.

SOLUTION

We know that $S_\infty = 40.$

Using the formula, $\dfrac{a}{1 - r} = 40,$

and substituting $a = 10$ gives

$$\begin{aligned} \frac{10}{1 - r} &= 40 \\ 10 &= 40(1 - r) \\ 1 &= 4 - 4r \\ 4r &= 3 \\ r &= \tfrac{3}{4}. \end{aligned}$$

Sigma notation for infinite sums

When $-1 < r < 1$ and the GP converges, the limiting sum S_∞ can also be written as an infinite sum, either using sigma notation or using dots, so that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad \text{or} \qquad a + ar + ar^2 + \cdots = \frac{a}{1 - r},$$

and we say that 'the series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\dfrac{a}{1 - r}$'.

ISBN 978-1-108-76630-2

Exercise 1H

FOUNDATION

1 a Copy and complete the table of values opposite for the GP with $a = 18$ and $r = \frac{1}{3}$.

n	1	2	3	4	5	6
T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n						

b Find the limiting sum using $S_\infty = \dfrac{a}{1 - r}$.

c Find the difference $S_\infty - S_6$.

2 a Copy and complete the table of values opposite for the GP with $a = 24$ and $r = -\frac{1}{2}$.

n	1	2	3	4	5	6
T_n	24	-12	6	-3	$1\frac{1}{2}$	$-\frac{3}{4}$
S_n						

b Find the limiting sum using $S_\infty = \dfrac{a}{1 - r}$.

c Find the difference $S_\infty - S_6$.

3 Identify the first term a and ratio r of each GP and hence find S_∞.

a $8 + 4 + 2 + \cdots$
b $-4 - 2 - 1 - \cdots$
c $1 - \frac{1}{3} + \frac{1}{9} - \cdots$
d $36 - 12 + 4 - \cdots$
e $60 - 30 + 15 - \cdots$
f $60 - 12 + 2\frac{2}{5} - \cdots$

4 Find each ratio r to test whether there is a limiting sum. Find the limiting sum if it exists.

a $1 - \frac{1}{2} + \frac{1}{4} - \cdots$
b $4 - 6 + 9 - \cdots$
c $12 + 4 + \frac{4}{3} + \cdots$
d $1000 + 100 + 10 + \cdots$
e $-2 + \frac{2}{5} - \frac{2}{25} + \cdots$
f $-\frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \cdots$

5 Bevin dropped the Nelson Bros Bouncy Ball from a height of 8 metres. It bounced continually, each successive height being half of the previous height.

a Show that the first distance travelled down-and-up is 12 metres, and explain why the successive down-and-up distances form a GP with $r = \frac{1}{2}$.

b Through what distance did the ball 'eventually' travel?

6 These examples will show that a GP does not have a limiting sum when $r \geq 1$ or $r \leq -1$. Copy and complete the tables for these GPs, then describe the behaviour of S_n as $n \to \infty$.

a $r = 1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

b $r = -1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

c $r = 2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

d $r = -2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

7 For each series, find S_∞ and S_4, then find the difference $S_\infty - S_4$.

a $80 + 40 + 20 + \cdots$
b $100 + 10 + 1 + \cdots$
c $100 - 80 + 64 - \cdots$

ISBN 978-1-108-76630-2

DEVELOPMENT

8 When Brownleigh Council began offering free reflective house numbers to its 10000 home owners, 20% installed them in the first month. The number installing them in the second month was only 20% of those in the first month, and so on.

- **a** Show that the numbers installing them each month form a GP.
- **b** How many home owners will 'eventually' install them? ('Eventually' means take S_∞.)
- **c** How many eventual installations were not done in the first four months?

9 The Wellington Widget Factory has been advertising its unbreakable widgets every month. The first advertisement brought in 1000 sales, but every successive advertisement is only bringing in 90% of the previous month's sales.

- **a** How many widget sales will the advertisements 'eventually' bring in?
- **b** About how many eventual sales were not brought in by the first 10 advertisements?

10 Find the limiting sums if they exist, rationalising denominators where necessary.

- **a** $1 + (1.01) + (1.01)^2 + \cdots$
- **b** $1 + (1.01)^{-1} + (1.01)^{-2} + \cdots$
- **c** $16\sqrt{5} + 4\sqrt{5} + \sqrt{5} + \cdots$
- **d** $7 + \sqrt{7} + 1 + \cdots$
- **e** $4 - 2\sqrt{2} + 2 - \cdots$
- **f** $5 - 2\sqrt{5} + 4 - \cdots$
- **g** $9 + 3\sqrt{10} + 10 + \cdots$
- **h** $1 + (1 - \sqrt{3}) + (1 - \sqrt{3})^2 + \cdots$

11 Expand each series for a few terms. Then write down a and r, and find the limiting sum.

- **a** $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{1}{3}\right)^n$
- **b** $\displaystyle\sum_{n=1}^{\infty} 7 \times \left(\tfrac{1}{2}\right)^n$
- **c** $\displaystyle\sum_{n=1}^{\infty} 40 \times \left(-\tfrac{3}{5}\right)^n$

12 Find, in terms of x, an expression for the limiting sum of the series on the LHS of each equation. Then solve the equation to find x.

- **a** $5 + 5x + 5x^2 + \cdots = 10$
- **b** $5 - 5x + 5x^2 - \cdots = 15$
- **c** $x + \dfrac{x}{3} + \dfrac{x}{9} + \cdots = 2$
- **d** $x - \dfrac{x}{3} + \dfrac{x}{9} - \cdots = 2$

13 **a** Suppose that $a + ar + ar^2 + \cdots$ is a GP with limiting sum. Show that the four sequences
$a + ar + ar^2 + \cdots,\ a - ar + ar^2 + \cdots,\ a + ar^2 + ar^4 + \cdots,\ ar + ar^3 + ar^5 + \cdots,$
are all GPs, and that their limiting sums are in the ratio $1 + r : 1 - r : 1 : r$.

b Find the limiting sums of these four GPs, and verify the ratio proven above:

- **i** $48 + 24 + 12 + \cdots$
- **ii** $48 - 24 + 12 + \cdots$
- **iii** $48 + 12 + \cdots$
- **iv** $24 + 6 + \cdots$

14 Find the condition for each GP to have a limiting sum, then find that limiting sum.

- **a** $7 + 7x + 7x^2 + \cdots$
- **b** $2x + 6x^2 + 18x^3 + \cdots$
- **c** $1 + (x - 1) + (x - 1)^2 + \cdots$
- **d** $1 + (1 + x) + (1 + x)^2 + \cdots$

15 Find the condition for each GP to have a limiting sum, then find that limiting sum.

- **a** $1 + (x^2 - 1) + (x^2 - 1)^2 + \cdots$
- **b** $1 + \dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots$

16 **a** Show that a GP has a limiting sum if $0 < 1 - r < 2$.

b By calculating the common ratio, show that there is no GP with first term 8 and limiting sum 2.

c A GP has positive first term a, and has a limiting sum S_∞. Show that $S_\infty > \tfrac{1}{2}a$.

d Find the range of values of the limiting sum of a GP with:

- **i** $a = 6$
- **ii** $a = -8$
- **iii** $a > 0$
- **iv** $a < 0$

ISBN 978-1-108-76630-2

17 The series $v + v^2 + v^3 + \cdots$ has a limiting sum w.

a Write w in terms of v.

b Find v in terms of w.

c Hence find the limiting sum of the series $w - w^2 + w^3 - \cdots$, assuming that $|w| < 1$.

d Test your results with $v = \frac{1}{3}$.

ENRICHMENT

18 Suppose that $T_n = ar^{n-1}$ is a GP with a limiting sum.

a Find the ratio r if the limiting sum equals 5 times the first term.

b Find the first three terms if the second term is 6 and the limiting sum is 27.

c Find the ratio if the sum of all terms except the first equals 5 times the first term.

d Show that the sum S of all terms from the third on is $\dfrac{ar^2}{1 - r}$.

i Hence find r if S equals the first term.

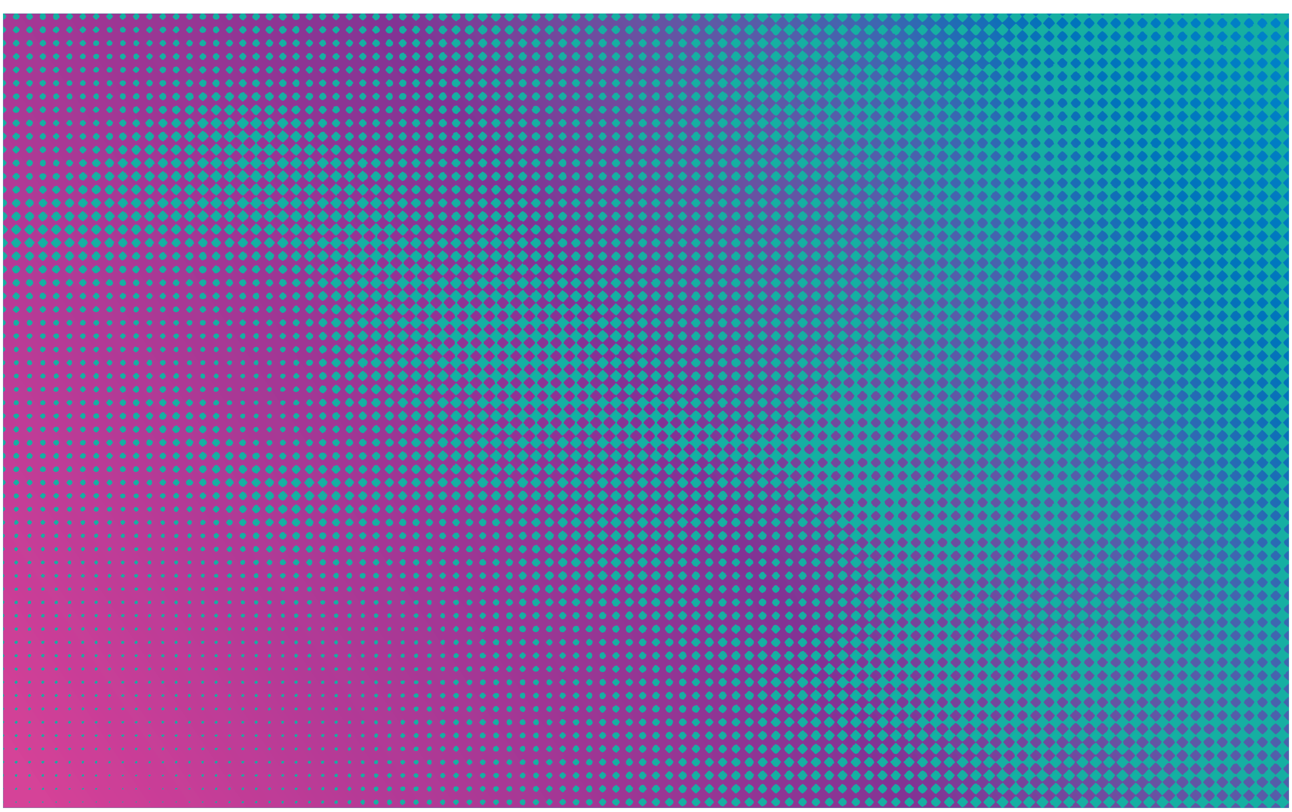
ii Find r if S equals the second term.

iii Find r if S equals the sum of the first and second terms.

19 The series $4 + 12 + 36 + \cdots$ has no limiting sum because $r > 1$. Nevertheless, substitution into the formula for the limiting sum gives

$$S_\infty = \frac{4}{1 - 3} = -2.$$

Can any meaning be given to this calculation and its result? (Hint: Look at the extension of the series to the *left* of the first term.)



ISBN 978-1-108-76630-2

1I Recurring decimals and geometric series

It would not have been easy in Chapter 2 of the Year 11 book to convert a recurring decimal back to a fraction. Now, however, we can express a recurring decimal as an infinite GP — its value is the limiting sum of that GP, which is easily expressed as a fraction.

Example 33 1I

Express these recurring decimals as infinite GPs. Then use the formula for the limiting sum to find their values as fractions reduced to lowest terms.

a $0.\dot{2}\dot{7}$ **b** $2.6\dot{4}\dot{5}$

SOLUTION

a Expanding the decimal, $0.\dot{2}\dot{7} = 0.272727\ldots$
$= 0.27 + 0.0027 + 0.000027 + \cdots$

This is an infinite GP with first term $a = 0.27$ and ratio $r = 0.01$.

Hence
$$\begin{aligned} 0.\dot{2}\dot{7} &= \frac{a}{1 - r} \\ &= \tfrac{0.27}{0.99} \\ &= \tfrac{27}{99} \\ &= \tfrac{3}{11}. \end{aligned}$$

b This example is a little more complicated, because the first part is not recurring.
Expanding the decimal, $2.6\dot{4}\dot{5} = 2.645454545\ldots$
$= 2.6 + (0.045 + 0.00045 + \cdots)$
This is 2.6 plus an infinite GP with first term $a = 0.045$ and ratio $r = 0.01$.
Hence
$$\begin{aligned} 2.6\dot{4}\dot{5} &= 2.6 + \tfrac{0.045}{0.99} \\ &= \tfrac{26}{10} + \tfrac{45}{990} \\ &= \tfrac{286}{110} + \tfrac{5}{110} \\ &= \tfrac{291}{110}. \end{aligned}$$

15 EXPRESSING A RECURRING DECIMAL AS A FRACTION

- To convert a recurring decimal as a fraction, write the recurring part as a GP.
- The ratio will be between 0 and 1, so the series will have an infinite sum.

ISBN 978-1-108-76630-2

Exercise 1I

FOUNDATION

Note: These prime factorisations will be useful in this exercise:

$9 = 3^2$ $\quad$ $999 = 3^3 \times 37$ $\quad$ $99\,999 = 3^2 \times 41 \times 271$

$99 = 3^2 \times 11$ $\quad$ $9999 = 3^2 \times 11 \times 101$ $\quad$ $999\,999 = 3^3 \times 7 \times 11 \times 13 \times 37$

1 Write each recurring decimal as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

a $0.\dot{3}$ b $0.\dot{1}$ c $0.\dot{7}$ d $0.\dot{6}$

2 Write each recurring decimal as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

a $0.\dot{2}\dot{7}$ b $0.\dot{8}\dot{1}$ c $0.\dot{0}\dot{9}$ d $0.\dot{1}\dot{2}$

e $0.\dot{7}\dot{8}$ f $0.\dot{0}2\dot{7}$ g $0.\dot{1}3\dot{5}$ h $0.\dot{1}8\dot{5}$

DEVELOPMENT

3 Write each recurring decimal as the sum of an integer or terminating decimal and an infinite GP. Then express it as a fraction in lowest terms.

a $12.\dot{4}$ b $7.\dot{8}\dot{1}$ c $8.4\dot{6}$ d $0.2\dot{3}\dot{6}$

4 a Express the repeating decimal $0.\dot{9}$ as an infinite GP, and hence show that it equals 1.

b Express $2.7\dot{9}$ as 2.7 plus an infinite GP, and hence show that it equals 2.8.

5 Use GPs to express these as fractions in lowest terms.

a $0.\dot{0}95\dot{7}$ b $0.\dot{2}47\dot{5}$ c $0.\dot{2}3076\dot{9}$ d $0.\dot{4}2857\dot{1}$

e $0.25\dot{5}\dot{7}$ f $1.1\dot{0}3\dot{7}$ g $0.0\dot{0}027\dot{1}$ h $7.7\dot{7}1428\dot{5}$

ENRICHMENT

6 Last year we proved in Section 2B of the Year 11 book that $\sqrt{2}$ is irrational. Why can we now conclude that when $\sqrt{2}$ is written as a decimal, it is not a recurring decimal?

7 a [The periods of recurring decimals]

Let p be any prime other than 2 or 5. Explain why the cycle length of the recurring decimal equal to $1/p$ is n digits, where n is the least power of 10 that has remainder 1 when divided by p.

b Use the factorisations of $10^k - 1$ given at the start of this exercise to predict the periods of the decimal representations of $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{27}$, $\frac{1}{37}$, $\frac{1}{41}$, $\frac{1}{101}$ and $\frac{1}{271}$, then write each as a recurring decimal.

8 a Use limiting sums of GPs to prove that $0.46\dot{9} = 0.47$.

b Explain how every recurring decimal with an infinite string of 9s can be written as a terminating decimal.

c Explain how every terminating decimal can be written as a recurring decimal with an infinite string of 9s.

d What qualification was needed in Question **22c** of Exercise 1A?

ISBN 978-1-108-76630-2

Two techniques in mental arithmetic: You would have used quite a bit of mental arithmetic in this chapter. Here are some techniques that are well worth knowing and practising to make life easier (and this course does not require them).

9 **Doubling and halving** are easy. This means that when multiplying and dividing with even numbers, we can break down the calculation into smaller pieces that can be done mentally.

a To multiply by an even number, take out the factors of 2, then multiply the resulting odd numbers together, then use doubling to get the final answer.

$$14 \times 24 = 2^4 \times (7 \times 3) = 2^4 \times 21 = 2^3 \times 42 = 2^2 \times 84 = 2 \times 168 = 336$$

b To multiply by a multiple of 5, combine each 5 with a 2 using doubling and halving

$$15 \times 26 = 30 \times 13 = 390$$

$$125 \times 108 = 250 \times 54 = 500 \times 27 = 1000 \times 13\tfrac{1}{2} = 13\,500$$

c To divide by 5 or a multiple of 5, double top and bottom.

$$\frac{62}{5} = \frac{124}{10} = 12.4 \qquad \frac{48}{15} = \frac{96}{30} = 3.2$$

d Some practice — make up your own, and use a calculator to check.

$$11 \times 44,\ 12 \times 77,\ 18 \times 14,\ 14 \times 35,\ 15 \times 21,\ 75 \times 16,\ \frac{85}{5},\ \frac{42}{5},\ \frac{36}{15}$$

10 **The difference of squares** makes multiplying two odd numbers straightforward as long as you know your squares.

$$13 \times 17 = (15 - 2)(15 + 2) = 15^2 - 2^2 = 225 - 4 = 221$$

a To find a square, add and subtract to give a product that can be done easily, then use the difference of squares in reverse. In this example, multiplying by 20 is simple.

$$23^2 = (20 \times 26) + 3^2 = (10 \times 52) + 3^2 = 520 + 9 = 529$$

b Half-integers can easily be squared in this way.

$$\left(8\tfrac{1}{2}\right)^2 = (8 \times 9) + \left(\tfrac{1}{2}\right)^2 = 72\tfrac{1}{4}$$

c Some practice — it is worth learning by heart the squares up to 20^2.

$$9 \times 13,\ 17 \times 23,\ 23 \times 37,\ 17^2,\ 13 \times 21,\ 18^2,\ 17 \times 19,\ 19^2,\ 17 \times 21,\ 41^2,\ 28^2$$

ISBN 978-1-108-76630-2

Chapter 1 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 1 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Write out the first 12 terms of the sequence $50,\ 41,\ 32,\ 23,\ \ldots$

- **a** How many positive terms are there?
- **b** How many terms lie between 0 and 40?
- **c** What is the 10th term?
- **d** What number term is -13?
- **e** Is -100 a term in the sequence?
- **f** What is the first term less than -35?

2 The nth term of a sequence is given by $T_n = 58 - 6n$.

- **a** Find the first, 20th, 100th and the 1000000th terms.
- **b** Find whether 20, 10, -56 and -100 are terms of the sequence.
- **c** Find the first term less than -200, giving its number and its value.
- **d** Find the last term greater than -600, giving its number and its value.

3 Find the original sequence T_n if its partial sums S_n are:

- **a** the sequence $4,\ 11,\ 18,\ 25,\ 32,\ 39,\ \ldots,$
- **b** the sequence $0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots,$
- **c** given by $S_n = n^2 + 5,$
- **d** given by $S_n = 3^n,$

4 Evaluate these expressions:

- **a** $\displaystyle\sum_{n=3}^{6} (n^2 - 1)$
- **b** $\displaystyle\sum_{n=-2}^{2} (5n - 3)$
- **c** $\displaystyle\sum_{n=0}^{6} (-1)^n$
- **d** $\displaystyle\sum_{n=1}^{6} \left(\tfrac{1}{2}\right)^n$

5

- **a** Write out the first eight terms of the sequence $T_n = 5 \times (-1)^n$.
- **b** Find the sum of the first seven terms and the sum of the first eight terms.
- **c** How is each term obtained from the previous term?
- **d** What are the 20th, 75th and 111th terms?

6 Test each sequence to see whether it is an AP, a GP or neither. State the common difference of any AP and the common ratio of any GP.

- **a** $76,\ 83,\ 90,\ \ldots$
- **b** $100,\ -21,\ -142,\ \ldots$
- **c** $1,\ 4,\ 9,\ \ldots$
- **d** $6,\ 18,\ 54,\ \ldots$
- **e** $6,\ 10,\ 15,\ \ldots$
- **f** $48,\ -24,\ 12,\ \ldots$

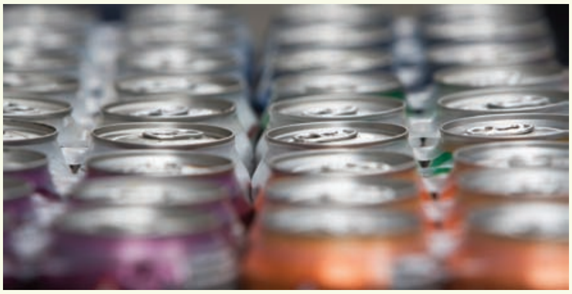
7

- **a** State the first term and common difference of the AP $23,\ 35,\ 47,\ \ldots$
- **b** Use the formula $T_n = a + (n - 1)d$ to find the 20th term and the 600th term.
- **c** Show that the formula for the nth term is $T_n = 11 + 12n$.

Mathematics Extension 1 Year 12 Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

d Hence find whether 143 and 173 are terms of the sequence.
e Hence find the first term greater than 1000 and the last term less than 2000.
f Hence find how many terms there are between 1000 and 2000.

8 A shop charges \$20 for one case of soft drink and \$16 for every subsequent case.
a Show that the costs of 1 case, 2 cases, 3 cases, . . . form an AP and state its first term and common difference.
b Hence find a formula for the cost of n cases.
c What is the largest number of cases that I can buy with \$200, and what is my change?
d My neighbour paid \$292 for some cases. How many did he buy?



9 **a** Find the first term and common ratio of the GP $50, 100, 200, \dots$
b Use the formula $T_n = ar^{n-1}$ to find a formula for the nth term.
c Hence find the eighth term and the twelfth term.
d Find whether 1600 and 4800 are terms of the sequence.
e Find the product of the fourth and fifth terms.
f Use logarithms, or trial-and-error on the calculator, to find how many terms are less than 10 000 000.

10 On the first day that Barry exhibited his paintings, there were 486 visitors. On each subsequent day, there were only a third as many visitors as on the previous day.
a Show that the number of visitors on successive days forms a GP and state the first term and common ratio.
b Write out the terms of the GP until the numbers become absurd.
c For how many days were there at least 10 visitors?
d What was the total number of visitors while the formula was still valid?
e Use the formula $S_\infty = \dfrac{a}{1-r}$ to find the 'eventual' number of visitors if the absurdity of fractional numbers of people were ignored.

ISBN 978-1-108-76630-2

Review

11 Find the second term x of the sequence $15,\ x,\ 135$:

a if the sequence is an AP, **b** if the sequence is a GP.

12 Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum of the first 41 terms of each AP.

a $51 + 62 + 73 + \cdots$ **b** $100 + 75 + 50 + \cdots$ **c** $-35 - 32 - 29 - \cdots$

13 Use the formula $T_n = a + (n-1)d$ to find the number of terms in each AP, then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum of the series.

a $23 + 27 + 31 + \cdots + 199$

b $200 + 197 + 194 + \cdots - 100$

c $12 + 12\frac{1}{2} + 13 + \cdots + 50$

14 Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ to find the sum of the first 6 terms of each GP.

a $3 + 6 + 12 + \cdots$ **b** $6 - 18 + 54 - \cdots$ **c** $-80 - 40 - 20 - \cdots$

15 Find the limiting sum of each GP, if it exists.

a $240 + 48 + 9\frac{3}{5} + \cdots$ **b** $-6 + 9 - 13\frac{1}{2} + \cdots$ **c** $-405 + 135 - 45 + \cdots$

16 **a** For what values of x does the GP $(2 + x) + (2 + x)^2 + (2 + x)^3 + \cdots$ have a limiting sum?

b Find a formula for the value of this limiting sum when it does exist.

17 Use the formula for the limiting sum of a GP to express as a fraction:

a $0.\dot{3}\dot{9}$ **b** $0.\dot{4}6\dot{8}$ **c** $12.30\dot{4}\dot{5}$

18 **a** The second term of an AP is 21 and the ninth term is 56. Find the 100th term.

b Find the sum of the first 20 terms of an AP with third term 10 and 12th term -89.

c The third term of a GP is 3 and the eighth term is -96. Find the sixth term.

d Find the difference of the AP with first term 1 if the sum of the first 10 terms is -215.

e Find how many terms there are in an AP with first term $4\frac{1}{2}$ and difference -1 if the sum of all the terms is 8.

f Find the common ratio of a GP with first term 60 and limiting sum 45.

g The sum of the first 10 terms of a GP with ratio -2 is 682. Find the fourth term.

ISBN 978-1-108-76630-2