12 Further calculus

This chapter is mostly concerned with three different ways in which integration can be extended to a greater range of functions. Much of the material concerns the use of trigonometry in integration, and readers should be aware of the structured progression of trigonometry in the two books.

- Trigonometric identities have been developing through Chapters 6, 11 and 17 of the Year 11 book.
- Trigonometric equations also began in those chapters, and were the main subject of the previous chapter.
- Trigonometric integrals were introduced in Chapter 7 of this book and are developed further in this chapter.

Trigonometry has many purposes and applications, but its principal significance in calculus is its role in integration, and more generally in the differential equations that are the subject of the next chapter.

Sections 12A–12B develop the calculus of the inverse trigonometric functions, with the surprising result that inverse trigonometric functions are required for the integration of two purely algebraic functions. The standard forms are:

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x + C \qquad \text{and} \qquad \int \frac{1}{1 + x^2}\,dx = \tan^{-1} x + C.$$

Section 12C uses the $\cos 2\theta$ formula to integrate $\sin^2 x$ and $\cos^2 x$ and reviews the reverse chain rule for trigonometric integrals. Then Sections 12D–12E generalise the reverse chain rule to a more general method of integration by substitution, which applies to all integrals, whether trigonometric or not.

Section 12F moves in a different direction, using calculus to find the volumes of solids generated by rotating a curve about the x-axis or the y-axis.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

12A Inverse trigonometric functions — differentiating

The inverse trigonometric functions were defined in Chapter 17 of the Year 11 book. We can now apply the normal processes of calculus to them. This section develops their derivatives and applies them to curve-sketching and maximisation.

Differentiating $\sin^{-1} x$ and $\cos^{-1} x$

To differentiate $y = \sin^{-1} x$ and $y = \cos^{-1} x$, we change to the inverse functions and use the known derivatives of the sine and cosine functions. We need to keep track of the restrictions to the domain so that the choice can be made later between positive and negative square roots.

A Let $y = \sin^{-1} x$.

Then $x = \sin y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,

so $\dfrac{dx}{dy} = \cos y$.

Because y is in the first or fourth quadrant, $\cos y$ is positive,

$$\text{so} \quad \cos y = +\sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

$$\text{Thus} \quad \frac{dx}{dy} = \sqrt{1 - x^2},$$

$$\text{so} \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

$$\text{Hence} \quad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.$$

B Let $y = \cos^{-1} x$.

Then $x = \cos y$, where $0 \le y \le \pi$,

so $\dfrac{dx}{dy} = -\sin y$.

Because y is in the first or second quadrant, $\sin y$ is positive,

$$\text{so} \quad \sin y = +\sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}.$$

$$\text{Thus} \quad \frac{dx}{dy} = -\sqrt{1 - x^2},$$

$$\text{so} \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

$$\text{Hence} \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

Differentiating $\tan^{-1} x$

The problem of which square root to choose does not arise when differentiating $y = \tan^{-1} x$.

Let $y = \tan^{-1} x$.

Then $x = \tan y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$\text{so} \quad \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y.$$

$$\text{Hence} \quad \frac{dx}{dy} = 1 + x^2$$

and $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, giving the standard form $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1 + x^2}$.

1 STANDARD FORMS FOR DIFFERENTIATION

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$$

ISBN 978-1-108-76630-2

Cambridge University Press

Example 1 12A

Differentiate these functions.

a $y = x\tan^{-1}x$

b $y = \sin^{-1}(ax + b)$

SOLUTION

a
$$\begin{aligned} y &= x\tan^{-1}x \\ y' &= vu' + uv' \\ &= \tan^{-1}x \times 1 + x \times \frac{1}{1+x^2} \\ &= \tan^{-1}x + \frac{x}{1+x^2} \end{aligned}$$

Let $u = x$
and $v = \tan^{-1}x$.
Then $u' = 1$
and $v' = \dfrac{1}{1+x^2}$.

b
$$\begin{aligned} y &= \sin^{-1}(ax + b) \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{\sqrt{1-(ax+b)^2}} \times a \\ &= \frac{a}{\sqrt{1-(ax+b)^2}} \end{aligned}$$

Let $u = ax + b$,
then $y = \sin^{-1}u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1-u^2}}$.

Linear extensions

The method used in part **b** above can be applied to all three inverse trigonometric functions, giving a further set of standard forms.

2 FURTHER STANDARD FORMS FOR DIFFERENTIATION

$$\frac{d}{dx}\sin^{-1}(ax + b) = \frac{a}{\sqrt{1-(ax+b)^2}}$$

$$\frac{d}{dx}\cos^{-1}(ax + b) = -\frac{a}{\sqrt{1-(ax+b)^2}}$$

$$\frac{d}{dx}\tan^{-1}(ax + b) = \frac{a}{1+(ax+b)^2}$$

Example 2 12A

a Find the points A and B on the curve $y = \cos^{-1}(x - 1)$ where the tangent has gradient -2.

b Sketch the curve, showing these points.

ISBN 978-1-108-76630-2

SOLUTION

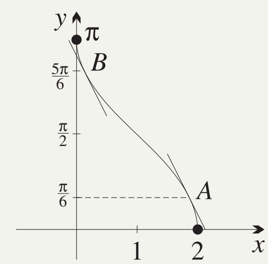
a Differentiating, $y' = -\dfrac{1}{\sqrt{1-(x-1)^2}}$.

Put $y' = -2$.

Then
$$-\frac{1}{\sqrt{1-(x-1)^2}} = -2$$
$$1-(x-1)^2 = \tfrac{1}{4}$$
$$(x-1)^2 = \tfrac{3}{4}$$
$$x-1 = \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3}$$
$$x = 1+\tfrac{1}{2}\sqrt{3} \text{ or } 1-\tfrac{1}{2}\sqrt{3},$$

so the points are $A\left(1+\frac{1}{2}\sqrt{3}, \frac{\pi}{6}\right)$ and $B\left(1-\frac{1}{2}\sqrt{3}, \frac{5\pi}{6}\right)$.

b



Functions whose derivatives are zero are constants

Several identities involving inverse trigonometric functions can be obtained by showing that some derivative is zero, and hence that the original function is a constant. The following identity is the clearest example — it has been proven already in Section 17C of the Year 11 book using symmetry arguments.

Example 3 — 12A

a Differentiate $\sin^{-1}x + \cos^{-1}x$.

b Hence prove the identity $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

SOLUTION

a
$$\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$$
$$= 0$$

b Hence $\sin^{-1}x + \cos^{-1}x = C$, for some constant C.

Substitute $x = 0$, then $0 + \frac{\pi}{2} = C$,

so $C = \frac{\pi}{2}$, and $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, as required.

Curve sketching using calculus

The usual methods of curve sketching can now be extended to curves whose equations involve the inverse trigonometric functions. The next worked example applies calculus to sketching the curve $y = \cos^{-1}\cos x$, which was sketched without calculus in Section 17C of the Year 11 book.

Example 4 — 12A

Use calculus to sketch $y = \cos^{-1}\cos x$.

SOLUTION

The function is periodic with the same period 2π as $\cos x$.
A simple table of test values gives some key points.
The shape of the curve joining these points can be obtained by calculus.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	...
y	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	...

ISBN 978-1-108-76630-2

Cambridge University Press

Differentiating using the chain rule,

$$\frac{dy}{dx} = \frac{\sin x}{\sqrt{1 - \cos^2 x}} = \frac{\sin x}{\sqrt{\sin^2 x}}.$$

Let $u = \cos x$,
then $y = \cos^{-1} u$.

Hence $\dfrac{du}{dx} = -\sin x$

and $\dfrac{dy}{du} = -\dfrac{1}{\sqrt{1 - u^2}}$.

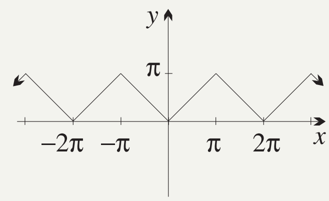
When $\sin x$ is positive, $\sqrt{\sin^2 x} = \sin x$,

so $\dfrac{dy}{dx} = 1$.

When $\sin x$ is negative, $\sqrt{\sin^2 x} = -\sin x$,

so $\dfrac{dy}{dx} = -1$.

Hence $\dfrac{dy}{dx} = \begin{cases} 1, & \text{for } x \text{ in quadrants 1 and 2,} \\ -1, & \text{for } x \text{ in quadrants 3 and 4,} \end{cases}$



This means that the graph consists of a series of intervals, each with gradient 1 or −1.

Chain-rule extensions to the standard forms

The usual chain-rule extensions to the standard forms can be used with the inverse trigonometric forms. They provide an alternative to the chain-rule setting out.

3 CHAIN-RULE EXTENSIONS TO THE STANDARD FORMS

$$\frac{d}{dx}\sin^{-1} u = \frac{u'}{\sqrt{1 - u^2}} \quad \text{OR} \quad \frac{d}{dx}\sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1 - \big(f(x)\big)^2}}$$

$$\frac{d}{dx}\cos^{-1} u = -\frac{u'}{\sqrt{1 - u^2}} \qquad \frac{d}{dx}\cos^{-1} f(x) = -\frac{f'(x)}{\sqrt{1 - \big(f(x)\big)^2}}$$

$$\frac{d}{dx}\tan^{-1} u = \frac{u'}{1 + u^2} \qquad \frac{d}{dx}\tan^{-1} f(x) = \frac{f'(x)}{1 + \big(f(x)\big)^2}$$

Example 5 — 12A

Use the chain-rule extension formulae to differentiate:

a $\tan^{-1} e^{-5x}$

b $\cos^{-1}(3x^2 + 2)$

SOLUTION

a Here $u = e^{-5x}$ and $u' = -5e^{-5x}$,

$$\text{so } \frac{d}{dx}(\tan^{-1} e^{-5x}) = \frac{u'}{1 + u^2} = \frac{-5e^{-5x}}{1 + e^{-10x}}.$$

OR

Here $f(x) = e^{-5x}$ and $f'(x) = -5e^{-5x}$,

$$\text{so } \frac{d}{dx}(\tan^{-1} e^{-5x}) = \frac{f'(x)}{1 + \big(f'(x)\big)^2} = \frac{-5e^{-5x}}{1 + e^{-10x}}.$$

b Here $u = 3x^2 + 2$ and $u' = 6x$,

$$\text{so } \frac{d}{dx}\big(\cos^{-1}(3x^2 + 2)\big) = \frac{u'}{1 + u^2} = \frac{6x}{1 + (3x^2 + 2)^2}$$

OR

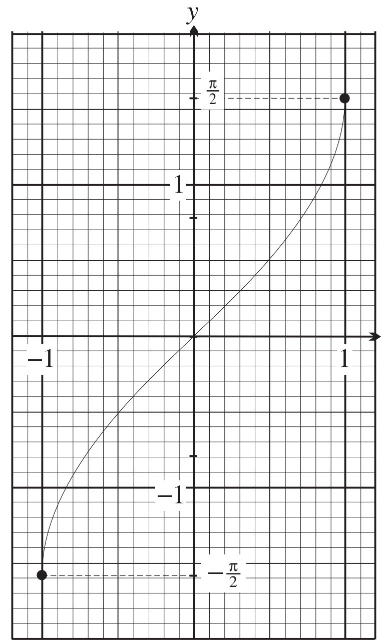
Here $f(x) = 3x^2 + 2$ and $f'(x) = 6x$,

$$\text{so } \frac{d}{dx}\big(\cos^{-1}(3x^2 + 2)\big) = \frac{f'(x)}{1 + \big(f'(x)\big)^2} = \frac{6x}{1 + (3x^2 + 2)^2}.$$

ISBN 978-1-108-76630-2

Exercise 12A

FOUNDATION



1 a Photocopy the graph of $y = \sin^{-1} x$ shown to the right. Then carefully draw a tangent at each x value in the table. Then, by measurement and calculation of rise/run, find the gradient of each tangent correct to two decimal places and fill in the second row of the table.

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\frac{dy}{dx}$									

b Check your gradients using $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$.

2 Differentiate with respect to x:

a $\cos^{-1} x$
b $\tan^{-1} x$
c $\sin^{-1} 2x$
d $\tan^{-1} 3x$
e $\cos^{-1} 5x$
f $\sin^{-1} (-x)$
g $\sin^{-1} x^2$
h $\tan^{-1} x^3$
i $\tan^{-1} (x + 2)$
j $\cos^{-1} (1 - x)$
k $x \sin^{-1} x$
l $(1 + x^2) \tan^{-1} x$
m $\sin^{-1} \frac{1}{5}x$
n $\tan^{-1} \frac{1}{4}x$
o $\cos^{-1} \sqrt{x}$
p $\tan^{-1} \sqrt{x}$
q $\tan^{-1} \frac{1}{x}$

3 Find the gradient of the tangent to each curve at the point indicated:

a $y = 2 \tan^{-1} x$, at $x = 0$
b $y = \sqrt{3} \sin^{-1} x$, at $x = \frac{1}{2}$
c $y = \tan^{-1} 2x$, at $x = -\frac{1}{2}$
d $y = \cos^{-1} \frac{x}{2}$, at $x = \sqrt{3}$

4 Find, in the form $y = mx + b$, the equation of the tangent and the normal to each curve at the point indicated:

a $y = 2 \cos^{-1} 3x$, at $x = 0$
b $y = \sin^{-1} \frac{x}{2}$, at $x = \sqrt{2}$

5 a Show that $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$.

b Hence explain why $\sin^{-1} x + \cos^{-1} x$ is a constant function, and use any convenient value of x in its domain to find the value of the constant.

6 Use the method of the previous question to show that each of these functions is a constant function, and find the value of the constant.

a $\cos^{-1} x + \cos^{-1} (-x)$
b $2 \sin^{-1} \sqrt{x} - \sin^{-1} (2x - 1)$

DEVELOPMENT

7 a If $f(x) = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2)$, show that $f''(x) = \frac{1}{1 + x^2}$.

b Is the graph of $y = f(x)$ concave up or concave down at $x = -1$?

ISBN 978-1-108-76630-2

8 Show that the gradient of the curve $y = \dfrac{\sin^{-1} x}{x}$ at the point where $x = \frac{1}{2}$ is $\frac{2}{3}(2\sqrt{3} - \pi)$.

9 Find the derivative of each function in simplest form:

a $x\cos^{-1} x - \sqrt{1 - x^2}$ **b** $\sin^{-1} e^{3x}$ **c** $\sin^{-1} \frac{1}{4}(2x - 3)$

d $\tan^{-1} \frac{1}{1 - x}$ **e** $\sin^{-1} e^{x}$ **f** $\log_e \sqrt{\sin^{-1} x}$

g $\sin^{-1} \sqrt{\log_e x}$ **h** $\sqrt{x}\sin^{-1} \sqrt{1 - x}$ **i** $\tan^{-1} \frac{x + 2}{1 - 2x}$

10 a i If $y = (\sin^{-1} x)^2$, show that $y'' = \dfrac{2 + \frac{2x\sin^{-1} x}{\sqrt{1 - x^2}}}{1 - x^2}$.

ii Hence show that $(1 - x^2)y'' - xy' - 2 = 0$.

b Show that $y = e^{\sin^{-1} x}$ satisfies the differential equation $(1 - x^2)y'' - xy' - y = 0$.

11 Consider the function $y = \sin^{-1} 2x$.

a Write down the range of the function.

b Make x the subject of the equation.

c Find $\dfrac{dx}{dy}$, and explain why it is never negative.

d Use the result that $\dfrac{dy}{dx}$ is the reciprocal of $\dfrac{dx}{dy}$ to find $\dfrac{dy}{dx}$.

12 Use the approach in the previous question to find $\dfrac{dy}{dx}$ given:

a $y = \sin^{-1} \frac{x}{2}$ **b** $y = \cos^{-1}(x - 1)$ **c** $y = \tan^{-1} \sqrt{x}$

13 Consider the function $f(x) = \cos^{-1} x^2$.

a What is the domain of $f(x)$?

b About which line is the graph of $y = f(x)$ symmetric?

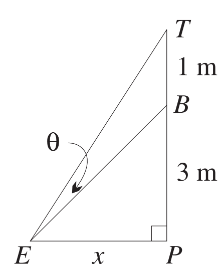
c Find $f'(x)$.

d Show that $y = f(x)$ has a maximum turning point at $x = 0$.

e Show that $f'(x)$ is undefined at the endpoints of the domain. What is the geometrical significance of this?

f Sketch the graph of $y = f(x)$.

14 A picture TB that is 1 metre tall is hung on a wall so that its bottom edge B is 3 metres above the eye E of a viewer. Let the distance EP be x metres, and let θ be the angle that the picture subtends at E.

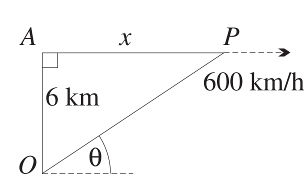


a Show that $\theta = \tan^{-1} \frac{4}{x} - \tan^{-1} \frac{3}{x}$.

b Show that θ is maximised when the viewer is $2\sqrt{3}$ metres from the wall.

c Show that the maximum angle subtended by the picture at E is $\tan^{-1} \frac{\sqrt{3}}{12}$.

15 A plane P at a constant altitude of 6 km and at a constant speed of 600 km/h is flying directly away from an observer at O on the ground. The point A on the path of the plane lies directly above O. Let the distance AP be x km, and let the angle of elevation of the plane from the observer be θ.



a Show that $\theta = \tan^{-1} \frac{6}{x}$.

b Show that $\dfrac{d\theta}{dt} = \dfrac{-3600}{x^2 + 36}$ radians per hour.

c Hence find, in radians per second, the rate at which θ is decreasing at the instant when the distance AP is 3 km.

ISBN 978-1-108-76630-2

16 a State the domain of $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$, and its symmetry.

b Show that $f'(x) = 0$ for all values of x in the domain.

c Show that $f(x) = \begin{cases} \frac{\pi}{2}, & \text{for } x < 0, \\ -\frac{\pi}{2}, & \text{for } x > 0 \end{cases}$ and hence sketch the graph of $f(x)$.

17 Consider the function $f(x) = \cos^{-1} \frac{1}{x}$.

a State the domain of $f(x)$. (Hint: Think about it rather than relying on algebra.)

b Recalling that $\sqrt{x^2} = |x|$, show that $f'(x) = \dfrac{1}{|x|\sqrt{x^2 - 1}}$.

c Comment on $f'(1)$ and $f'(-1)$.

d Use the expression for $f'(x)$ in part **b** to write down separate expressions for $f'(x)$ when $x > 1$ and when $x < -1$.

e Explain why $f(x)$ is increasing for $x > 1$ and for $x < -1$.

f Find:

i $\lim_{x \to \infty} f(x)$ **ii** $\lim_{x \to -\infty} f(x)$

g Sketch the graph of $y = f(x)$.

18 Use the formulae for the chain-rule standard forms in Box 3 to differentiate:

a $\tan^{-1} e^{3x}$ **b** $\sin^{-1} x^3$ **c** $\cos^{-1} (\log_e x)$

ENRICHMENT

19 a What is the domain of $g(x) = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$?

b Show that $g'(x) = \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{x}{|x|\sqrt{1 - x^2}}$.

c Hence determine the interval over which $g(x)$ is constant, and find this constant.

20 In question **9 i**, you proved that $\dfrac{d}{dx} \tan^{-1} \dfrac{x + 2}{1 - 2x}$ was $\dfrac{1}{1 + x^2}$, which is also the derivative of $\tan^{-1} x$. What is going on?

21 The function $f(x)$ is defined by the rule $f(x) = \sin^{-1} (\sin x)$.

a State the domain and range of $f(x)$, and whether it is even, odd or neither.

b Show that $f'(x) = \dfrac{\cos x}{|\cos x|}$.

c Is $f'(x)$ defined when $\cos x = 0$?

d What are the only two values that $f'(x)$ takes if $\cos x \neq 0$, and when does each of these values occur?

e Sketch the graph of $f(x)$ using the above information and a table of values if necessary.

22 In Question 14, construct the circle that passes through T and B and is tangent to the horizontal line ℓ through P. Then the point E on ℓ at which TB subtends the greatest angle is the point where the circle touches the line ℓ.

a Prove this using Euclidean geometry.

b Explain how to construct this circle, and E, using straight edge and compasses.

ISBN 978-1-108-76630-2

12B Inverse trigonometric functions — integrating

This section deals with the integrals associated with the inverse trigonometric functions, and with the standard applications of those integrals.

The basic standard forms

Differentiation of the inverse trigonometric functions yields purely algebraic functions,

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

These are remarkable results, and indicate once again that trigonometric functions are very closely related to algebraic functions associated with squares and square roots. The relationship was already clear when the trigonometric functions were defined using the circle, whose equation $x^2 + y^2 = r^2$ is Pythagoras' theorem, which is purely algebraic.

This section concerns integration, and we begin by reversing the three standard forms for differentiation.

4 STANDARD FORMS FOR INTEGRATION

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C \qquad \text{OR} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

Thus some purely algebraic functions require the inverse trigonometric functions for their integration. We have seen this sort of phenomenon before with the standard form $\int \frac{1}{x}\,dx = \log_e|x|$, where the logarithmic function was required for the integration of another purely algebraic function.

The functions $y = \dfrac{1}{\sqrt{1-x^2}}$ and $y = \dfrac{1}{1+x^2}$

The primitives of both these functions have now been obtained, and they should therefore be regarded as reasonably standard functions whose graphs should be known. The sketch of each function and some important definite integrals associated with them are developed in Questions 15 and 16 in Exercise 12B.

Example 6 12B

Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ using both standard forms given in Box 4.

SOLUTION

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx &= \Big[\sin^{-1}x\Big]_0^{\frac{1}{2}} \\ &= \sin^{-1}\tfrac{1}{2} - \sin^{-1}0 \\ &= \tfrac{\pi}{6} - 0 \\ &= \tfrac{\pi}{6}\end{aligned}$$

OR

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx &= \Big[-\cos^{-1}x\Big]_0^{\frac{1}{2}} \\ &= -\cos^{-1}\tfrac{1}{2} + \cos^{-1}0 \\ &= -\tfrac{\pi}{3} + \tfrac{\pi}{2} \\ &= \tfrac{\pi}{6}\end{aligned}$$

ISBN 978-1-108-76630-2

Example 7 12B

Evaluate these definite integrals exactly or correct to four significant figures.

a $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

b $\displaystyle\int_0^4 \frac{1}{1+x^2}\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1}x\Big]_0^1\\ &= \tan^{-1}1 - \tan^{-1}0\\ &= \tfrac{\pi}{4}\end{aligned}$$

b
$$\begin{aligned}\int_0^4 \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1}x\Big]_0^4\\ &= \tan^{-1}4 - \tan^{-1}0\\ &\doteqdot 1.326\end{aligned}$$

More general standard forms

When constants are involved, the calculation of the primitive becomes fiddly. These standard integrals are commonly used.

5 STANDARD FORMS WITH ONE CONSTANT

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \qquad \text{OR} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Proof

A
$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx\\ &= \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \times \frac{1}{a}\,dx\\ &= \sin^{-1}\frac{x}{a} + C\end{aligned}$$

Let $u = \dfrac{x}{a}$.

Then $\dfrac{du}{dx} = \dfrac{1}{a}$.

$$\int \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\,dx = \sin^{-1}u$$

B
$$\begin{aligned}\int \frac{1}{a^2+x^2}\,dx &= \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\,dx\\ &= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a}\,dx\\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C\end{aligned}$$

Let $u = \dfrac{x}{a}$.

Then $\dfrac{du}{dx} = \dfrac{1}{a}$.

$$\int \frac{1}{1+u^2}\frac{du}{dx}\,dx = \tan^{-1}u$$

These forms also hold when a is negative. Can you prove this?

ISBN 978-1-108-76630-2

Example 8 12B

Evaluate these four indefinite integrals. In parts **a** and **b**, the formulae can be applied immediately, but in parts **c** and **d**, the coefficients of x^2 need to be taken out first.

a $$\int \frac{2}{\sqrt{9 - x^2}}\,dx = 2\sin^{-1}\frac{x}{3} + C$$

b $$\int \frac{1}{8 + x^2}\,dx = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x}{2\sqrt{2}} + C$$

c $$\begin{aligned}&\int \frac{6}{49 + 25x^2}\,dx \\ &= \frac{6}{25}\int \frac{1}{\frac{49}{25} + x^2}\,dx \\ &= \frac{6}{25} \times \frac{5}{7}\tan^{-1}\frac{x}{7/5} + C \\ &= \frac{6}{35}\tan^{-1}\frac{5x}{7} + C\end{aligned}$$

d $$\begin{aligned}&\int \frac{1}{\sqrt{5 - 3x^2}}\,dx \\ &= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\frac{5}{3} - x^2}}\,dx \\ &= \tfrac{1}{3}\sqrt{3}\sin^{-1} x\sqrt{\tfrac{3}{5}} + C\end{aligned}$$

Because manipulating the constants in parts **c** and **d** is still difficult, some prefer to remember these fuller versions of the standard forms:

6 STANDARD FORMS WITH TWO CONSTANTS

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C \quad \text{OR} \quad -\frac{1}{b}\cos^{-1}\frac{bx}{a} + C$$

$$\int \frac{1}{a^2 + b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + C$$

These forms can be proven in the same manner as the forms with a single constant, or they can be developed from those forms in the same way as was done in parts **c** and **d** above. With these more general forms, parts **c** and **d** can be written down without any intermediate working.

Reverse chain rule

In the usual way, the standard forms can be extended to give forms appropriate for the reverse chain rule. These forms are the reversals of the standard forms at the end of Section 12A.

7 THE REVERSE CHAIN RULE

$$\int \frac{u'}{\sqrt{1 - u^2}}\,dx = \sin^{-1}u + C \quad \text{OR} \quad \int \frac{f'(x)}{\sqrt{1 - (f'(x))^2}}\,dx = \sin^{-1}f'(x) + C$$

$$\int \frac{u'}{\sqrt{1 - u^2}}\,dx = -\cos^{-1}u + C \qquad \int \frac{f'(x)}{\sqrt{1 - (f'(x))^2}}\,dx = -\cos^{-1}f'(x) + C$$

$$\int \frac{u'}{1 + u^2}\,dx = \tan^{-1}u + C \qquad \int \frac{f'(x)}{1 + (f'(x))^2}\,dx = \tan^{-1}f'(x) + C$$

ISBN 978-1-108-76630-2

Example 9 12B

Find primitives of:

a $\dfrac{e^x}{1 + e^{2x}}$

b $\dfrac{x}{\sqrt{1 - x^4}}$

SOLUTION

a Let $u = e^x$, then $u' = e^x$. OR Let $f(x) = e^x$, then $f'(x) = e^x$.

With both notations, $\displaystyle\int \frac{e^x}{1 + e^{2x}}\,dx = \int \frac{e^x}{1 + (e^x)^2}\,dx$

$$= \tan^{-1} e^x + C, \text{ for some constant } C$$

b Let $u = x^2$, then $u' = 2x$. OR Let $f(x) = x^2$, then $f'(x) = 2x$.

With both notations, $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx = \tfrac{1}{2}\int \frac{2x}{\sqrt{1 - (x^2)^2}}\,dx$

$$= \tfrac{1}{2}\sin^{-1} x^2 + C, \text{ for some constant } C.$$

Given a derivative, find an integral

As always, the result of a product-rule differentiation can be used to obtain an integral. In particular, this allows the primitives of the inverse trigonometric functions to be obtained.

Example 10 12B

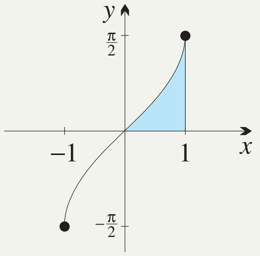
a Differentiate $x\sin^{-1} x$, and hence find a primitive of $\sin^{-1} x$.

b Find the shaded area under the curve $y = \sin^{-1} x$ from $x = 0$ to $x = 1$.

SOLUTION

a Let $y = x\sin^{-1} x$.

Using the product rule with $u = x$ and $v = \sin^{-1} x$,



$$\frac{dy}{dx} = \sin^{-1} x + \frac{x}{\sqrt{1 - x^2}}.$$

Hence $\displaystyle\int \sin^{-1} x\,dx + \int \frac{x}{\sqrt{1 - x^2}}\,dx = x\sin^{-1} x$

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}}\,dx.$$

Using the reverse chain rule,

$$-\int \frac{x}{\sqrt{1 - x^2}}\,dx = \tfrac{1}{2}\int (1 - x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$= \tfrac{1}{2} \times (1 - x^2)^{\frac{1}{2}} \times \frac{2}{1}$$

$$= \sqrt{1 - x^2},$$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$.

$$\int u^{-\frac{1}{2}}\frac{du}{dx}\,dx = u^{\frac{1}{2}} \times 2.$$

ISBN 978-1-108-76630-2

so $\displaystyle\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1 - x^2} + C,$

and $\displaystyle\int_0^1 \sin^{-1} x\,dx = \Big[x\sin^{-1} x + \sqrt{1 - x^2}\Big]_0^1$

$= (1 \times \frac{\pi}{2} + 0) - (0 + 1)$

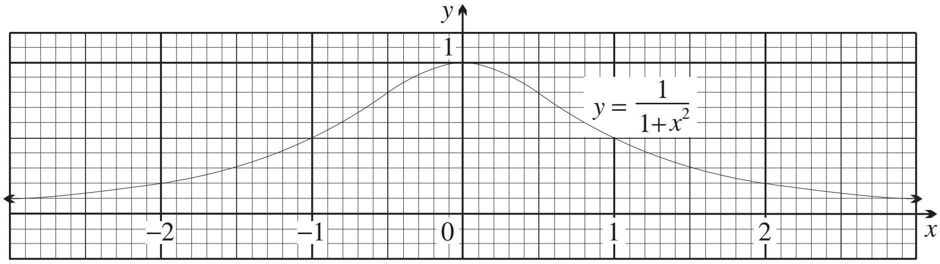
$= \frac{\pi}{2} - 1$ square units.

Note: We have already established in Section 7D that the area under $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ is exactly 1 square unit. This means that the area between $y = \sin^{-1} x$ and the y-axis is 1, and subtracting this area from the rectangle of area $\dfrac{\pi}{2}$ in the diagram above gives the same value $\frac{\pi}{2} - 1$ for the shaded area.

Exercise 12B

FOUNDATION

1 a



Find each definite integral correct to two decimal places from the graph by counting the number of little squares in the region under the curve:

i $\displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx$ **ii** $\displaystyle\int_0^2 \frac{1}{1 + x^2}\,dx$

iii $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1 + x^2}\,dx$ **iv** $\displaystyle\int_{-3}^{-1} \frac{1}{1 + x^2}\,dx$

b Check your answers to **a** by using the fact that $\tan^{-1} x$ is a primitive of $\dfrac{1}{1 + x^2}$.

2 Find:

a $\displaystyle\int \frac{-1}{\sqrt{1 - x^2}}\,dx$ **b** $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\,dx$ **c** $\displaystyle\int \frac{1}{9 + x^2}\,dx$

d $\displaystyle\int \frac{1}{\sqrt{\frac{4}{9} - x^2}}\,dx$ **e** $\displaystyle\int \frac{1}{2 + x^2}\,dx$ **f** $\displaystyle\int \frac{-1}{\sqrt{5 - x^2}}\,dx$

3 Find the exact value of:

a $\displaystyle\int_0^3 \frac{1}{\sqrt{9 - x^2}}\,dx$ **b** $\displaystyle\int_0^2 \frac{1}{4 + x^2}\,dx$ **c** $\displaystyle\int_0^1 \frac{1}{\sqrt{2 - x^2}}\,dx$

d $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\frac{1}{2}}{\frac{1}{4} + x^2}\,dx$ **e** $\displaystyle\int_{\frac{1}{6}\sqrt{3}}^{\frac{1}{6}} \frac{-1}{\sqrt{\frac{1}{9} - x^2}}\,dx$ **f** $\displaystyle\int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\sqrt{\frac{9}{4} - x^2}}\,dx$

4 Find the equation of the curve, given that:

a $y' = (1 - x^2)^{-\frac{1}{2}}$ and the curve passes through the point $(0, \pi)$.

b $y' = 4(16 + x^2)^{-1}$ and the curve passes through the point $(-4, 0)$.

ISBN 978-1-108-76630-2

Cambridge University Press

5 **a** If $y' = \dfrac{1}{\sqrt{36 - x^2}}$ and $y = \frac{\pi}{6}$ when $x = 3$, find the value of y when $x = 3\sqrt{3}$.

b Given that $y' = \dfrac{2}{4 + x^2}$ and that $y = \frac{\pi}{3}$ when $x = 2$, find y when $x = \frac{2}{\sqrt{3}}$.

DEVELOPMENT

6 Find:

a $\displaystyle\int \frac{1}{\sqrt{1 - 4x^2}}\,dx$ **b** $\displaystyle\int \frac{1}{1 + 16x^2}\,dx$ **c** $\displaystyle\int \frac{-1}{\sqrt{1 - 2x^2}}\,dx$

d $\displaystyle\int \frac{1}{\sqrt{4 - 9x^2}}\,dx$ **e** $\displaystyle\int \frac{1}{25 + 9x^2}\,dx$ **f** $\displaystyle\int \frac{-1}{\sqrt{3 - 4x^2}}\,dx$

7 Find the exact value of:

a $\displaystyle\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1 - 9x^2}}\,dx$ **b** $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1 + 4x^2}\,dx$ **c** $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1 - 3x^2}}\,dx$

d $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9 - 4x^2}}\,dx$ **e** $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9 - 4x^2}}\,dx$ **f** $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9 - 4x^2}}\,dx$

8 **a** Shade the region bounded by $y = \sin^{-1} x$, the x-axis and the vertical line $x = \frac{1}{2}$.

b Show that $\dfrac{d}{dx}\left(x \sin^{-1} x + \sqrt{1 - x^2}\right) = \sin^{-1} x$.

c Hence find the exact area of the region.

9 **a** Shade the region bounded by the curve $y = \sin^{-1} x$, the y-axis and the line $y = \frac{\pi}{6}$.

b Find the exact area of this region.

c Hence use an alternative approach to confirm the area in the previous question.

10 **a** Show that $\dfrac{d}{dx}\left(\cos^{-1}(2 - x)\right) = \dfrac{1}{\sqrt{4x - x^2 - 3}}$.

b Hence find $\displaystyle\int_1^2 \frac{1}{\sqrt{4x - x^2 - 3}}\,dx$.

11 **a** Differentiate $\tan^{-1} \frac{1}{2} x^3$.

b Hence find $\displaystyle\int \frac{x^2}{4 + x^6}\,dx$.

12 **a** Differentiate $x \tan^{-1} x$.

b Hence find $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

13 Without finding any primitives, use arguments from symmetry or geometry to evaluate:

a $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \sin^{-1} x\,dx$ **b** $\displaystyle\int_{-5}^{5} \tan^{-1} x\,dx$ **c** $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x\,dx$

d $\displaystyle\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{x}{\sqrt{1 - x^2}}\,dx$ **e** $\displaystyle\int_{-3}^{3} \frac{x}{1 + x^2}\,dx$ **f** $\displaystyle\int_{-6}^{6} \sqrt{36 - x^2}\,dx$

ISBN 978-1-108-76630-2

14 a Given that $f(x) = \dfrac{x}{1 + x^2} - \tan^{-1} x$:

i find $f(0)$,

ii show that $f'(x) = \dfrac{-2x^2}{(1 + x^2)^2}$.

b Hence:

i explain why $f(x) < 0$ for all $x > 0$,

ii find $\displaystyle\int_0^1 \frac{x^2}{(1 + x^2)^2}\,dx$.

15 Consider the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

a Sketch the graph of $y = \sqrt{4 - x^2}$.

b Hence sketch the graph of $y = f(x)$.

c Write down the domain and range of $f(x)$, and describe its symmetry.

d Find the area between the curve and the x-axis from $x = -1$ to $x = 1$.

e Find the total area between the curve and the x-axis. [**Note:** This is an example of an unbounded region having a finite area.]

16 Consider the function $f(x) = \dfrac{4}{x^2 + 4}$.

a What is the axis of symmetry of $y = f(x)$?

b What are the domain and range?

c Show that the graph of $f(x)$ has a maximum turning point at $(0, 1)$.

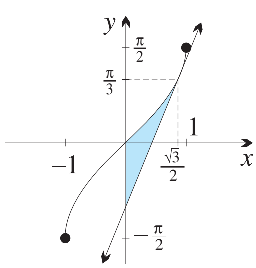
d Find $\lim\limits_{x \to \infty} f(x)$, and hence sketch $y = f(x)$. On the same axis, sketch $y = \frac{1}{4}(x^2 + 4)$.

e Calculate the area bounded by the curve and the x-axis from $x = -2\sqrt{3}$ to $x = \frac{2}{3}\sqrt{3}$.

f Find the exact area between the curve and the x-axis from $x = -a$ to $x = a$, where a is a positive constant.

g By letting a tend to infinity, find the total area between the curve and the x-axis. [**Note:** This is another example of an unbounded region having a finite area.]

17 The diagram to the right shows the region bounded by the curve $y = \sin^{-1} x$, the y-axis and the tangent to the curve at the point $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$. Show that the region has area $\frac{1}{4}$ unit^2.



18 a Use the trapezoidal rule with five points to approximate $I = \displaystyle\int_0^1 \frac{1}{1 + x^2}\,dx$, expressing your answer in simplest fraction form.

b Find the exact value of I, and hence show that $\pi \doteqdot \frac{5323}{1700}$.

19 Show that $\displaystyle\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1 + x^2}\,dx = \frac{\pi}{4}$.

20 Find, using the reverse chain rule:

a $\displaystyle\int \frac{1}{\sqrt{x}(1 + x)}\,dx$

b $\displaystyle\int_0^1 \frac{1}{e^{-x} + e^{x}}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press

ENRICHMENT

21 **a** Show that $\dfrac{d}{dx}\left(\tan^{-1}\left(\tfrac{3}{2}\tan x\right)\right) = \dfrac{6}{5\sin^2 x + 4}$.

b Hence find the area under the curve $y = \dfrac{1}{5\sin^2 x + 4}$ $x = 0$ to $x = 7$.

c Why is the calculation asked for in part **b** invalid and completely wrong?

22 [The power series for $\tan^{-1} x$]

Let x be a positive real number.

a Find the sum of the geometric series $1 - t^2 + t^4 - t^6 + \cdots + t^{4n}$, and hence show that for $0 < t < x$,

$$\frac{1}{1 + t^2} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n}.$$

b Find $1 - t^2 + t^4 - t^6 + \cdots + t^{4n} - t^{4n+2}$, and hence show that for $0 < t < x$,

$$1 - t^2 + t^4 - t^6 + \cdots + t^{4n} < \frac{1}{1 + t^2} + t^{4n+2}.$$

c By integrating the inequalities of parts **a** and **b** from $t = 0$ to $t = x$, show that

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n + 1} < \tan^{-1} x + \frac{x^{4n+3}}{4n + 3}.$$

d By taking limits as $n \to \infty$, show that for $0 \le x \le 1$,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

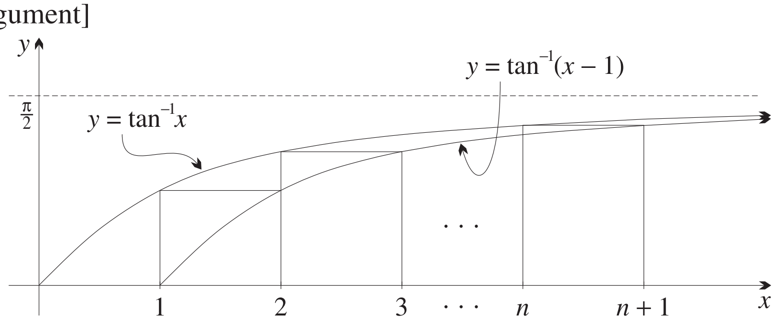
e Use the fact that $\tan^{-1} x$ is an odd function to prove this identity for $-1 \le x < 0$.

f [Gregory's series] Use a suitable substitution to prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

g By combining the terms in pairs, show that $\dfrac{\pi}{8} = \dfrac{1}{1 \times 3} + \dfrac{1}{5 \times 7} + \dfrac{1}{9 \times 11} + \cdots$, and use the calculator to find how close an approximation to π can be obtained by taking 10 terms.

23 [A sandwiching argument]



In the diagram, n rectangles are constructed between the two curves $y = \tan^{-1} x$ and $y = \tan^{-1}(x - 1)$ in the interval $1 \le x \le n + 1$.

a Write down an expression for S_n, the sum of the areas of the n rectangles.

b Differentiate $x \tan^{-1} x$ and hence find a primitive of $\tan^{-1} x$.

c Show that for all $n \ge 1$,

$$n \tan^{-1} n - \tfrac{1}{2}\ln(n^2 + 1) < S_n < (n + 1)\tan^{-1}(n + 1) - \tfrac{1}{2}\ln\left(\tfrac{n^2}{2} + n + 1\right) - \tfrac{\pi}{4}$$

d Deduce that $1562 < \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 + \cdots + \tan^{-1} 1000 < 1565$.

ISBN 978-1-108-76630-2

Cambridge University Press

12C Further trigonometric integrals

The principal purpose of this section is the integration of $\sin^2 x$ and $\cos^2 x$. Trigonometric integrals in general are quickly reviewed, particularly reverse-chain-rule integrations in preparation for the next two sections.

Six standard forms for integration

Reversing the derivatives of the six standard forms gives six standard integrals:

8 SIX STANDARD INTEGRALS

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad {}^*\int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$${}^*\int \sec x \tan x\,dx = \sec x + C \qquad {}^*\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$$

The full list is there only for completeness, and so that the patterns can be seen. The three integrals marked * need not be memorised — they are the reversals of derivatives in Question 19 of Exercise 7B.

The primitives of $\sin^2 x$ and $\cos^2 x$

These two integrals are very important. The keys to finding them are the two further forms of the $\cos 2\theta$ formulae that we established in Section 11A,

$$\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta \qquad \text{and} \qquad \cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta,$$

and it is better to remember these identities rather than the actual integrals.

Example 11 — 12C

Evaluate:

a $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

b $\displaystyle\int_0^{\frac{5\pi}{3}} \cos^2 \tfrac{1}{2}x\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx &= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx\\ &= \Big[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{2}}\\ &= \left(\tfrac{\pi}{4} - 0\right) - (0 - 0)\\ &= \tfrac{\pi}{4}\end{aligned}$$

b
$$\begin{aligned}\int_0^{\frac{5\pi}{3}} \cos^2 \tfrac{1}{2}x\,dx &= \int_0^{\frac{5\pi}{3}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos x\right) dx\\ &= \Big[\tfrac{1}{2}x - \tfrac{1}{2}\sin x\Big]_0^{\frac{5\pi}{3}}\\ &= \left(\tfrac{5\pi}{6} - \tfrac{1}{4}\right) - (0 - 0)\\ &= \tfrac{5\pi}{6} - \tfrac{1}{4}\end{aligned}$$

Here are the primitives of the squares of all six trigonometric functions. Two are given as standard forms. The other four are given only in terms of the identities that are needed to obtain them. The forms marked * are there for completeness.

ISBN 978-1-108-76630-2

9 INTEGRATING THE SQUARES OF THE TRIGONOMETRIC FUNCTIONS

$$\int \sin^2 x\,dx = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)dx \qquad \int \cos^2 x\,dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)dx$$

$$\int \sec^2 x\,dx = \tan x + C \qquad {}^*\int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx \qquad {}^*\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)\,dx$$

Trigonometric integrals using the reverse chain rule

It remains to demonstrate the use of the reverse chain rule in trigonometric integrals, in preparation for the next two sections on substitution.

Example 12 — 12C

Integrate $\cot x$ using the reverse chain rule.

SOLUTION

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$
$$= \log_e|\sin x| + C$$

Let $u = \sin x$.
Then $u' = \cos x$,
and $\displaystyle\int \frac{u'}{u}\,dx = \log_e|u|$.

Example 13 — 12C

Use the reverse chain rule to find primitives of:

a $y = \sin x \cos^4 x$

b $y = \cos x \sin^n x$

SOLUTION

a $$\int \sin x \cos^4 x\,dx = -\int (-\sin x)\cos^4 x\,dx$$
$$= -\tfrac{1}{5}\cos^5 x + C$$

Let $u = \cos x$.
Then $\dfrac{du}{dx} = -\sin x$,
and $\displaystyle\int u^4 \frac{du}{dx}\,dx = \tfrac{1}{5}u^5$.

b $$\int \cos x \sin^n x\,dx = \frac{\sin^{n+1} x}{n+1} + C$$

Let $u = \sin x$.
Then $\dfrac{du}{dx} = \cos x$,
and $\displaystyle\int u^n \frac{du}{dx}\,dx = \frac{u^n}{n+1}$.

ISBN 978-1-108-76630-2

Exercise 12C

FOUNDATION

1 Use double-angle results to show that:

a $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

b $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

c $\sin 3x \cos 3x = \frac{1}{2}\sin 6x$

d $2\sin^2 \frac{x}{2} = 1 - \cos x$

2 Without a calculator, find the value of:

a $\cos^2 15°$

b $\sin^2 \frac{5\pi}{12}$

c $\sin 105° \cos 105°$

d $\sin^2 \frac{7\pi}{8}$

3 Express $\sin^2\theta$ in terms of $\cos 2\theta$, and hence find:

a $\int \sin^2 x \, dx$

b $\int \sin^2 2x \, dx$

c $\int \sin^2 \frac{1}{4}x \, dx$

d $\int \sin^2 3x \, dx$

4 Express $\cos^2\theta$ in terms of $\cos 2\theta$, and hence find:

a $\int \cos^2 x \, dx$

b $\int \cos^2 6x \, dx$

c $\int \cos^2 \frac{1}{2}x \, dx$

d $\int \cos^2 10x \, dx$

DEVELOPMENT

5 Use the results $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ to evaluate:

a $\int_0^{\pi} \sin^2 x \, dx$

b $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

c $\int_0^{\frac{\pi}{6}} \sin^2 \frac{1}{2}x \, dx$

d $\int_0^{\frac{\pi}{16}} \cos^2 2x \, dx$

e $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2\left(x + \frac{\pi}{12}\right) dx$

f $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\left(x - \frac{\pi}{6}\right) dx$

6 Use the products-to-sums formulae reviewed in Section 11A to find:

a $\int \sin 3x \cos 2x \, dx$

b $\int \cos 3x \sin x \, dx$

c $\int_0^{\frac{\pi}{4}} 2\cos 2x \cos x \, dx$

d $\int_0^{\frac{\pi}{3}} \sin 5x \sin 2x \, dx$

7 a Sketch the graph of $y = \cos 2x$, for $0 \le x \le 2\pi$.

b Hence sketch, on the same diagram, $y = \frac{1}{2}(1 + \cos 2x)$ and $y = \frac{1}{2}(1 - \cos 2x)$.

c Deduce graphically that $\cos^2 x + \sin^2 x = 1$.

8 Use the reverse chain rule to find:

a $\int \sin^3 x \cos x \, dx$ (Let $u = \sin x$.)

b $\int \sin^6 x \cos x \, dx$

c $\int \cos^5 x \sin x \, dx$ (Let $u = \cos x$.)

d $\int \cos^8 x \sin x \, dx$

e $\int e^x \sin e^x \, dx$ (Let $u = e^x$.)

f $\int e^x \cos 5e^x \, dx$

g $\int \tan x \, dx$ (Let $u = \cos x$)

h $\int \cot 7x \, dx$

ISBN 978-1-108-76630-2

9 **a** Find the range of the function $y = \cos x \sin x$.

b Integrate $y = \cos x \sin x$ in three ways:

i using the $\sin 2x$ formula,

ii using the reverse chain rule in two different ways.

c Reconcile the three results.

10 **a** By writing $\sin^4 x$ as $(\sin^2 x)^2$, show that $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$.

b Find a similar result for $\cos^4 x$.

c Hence find:

i $\displaystyle\int_0^{\pi} \sin^4 x\, dx$ **ii** $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4 x\, dx$

d Use the Pythagorean identity to find $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\, dx$.

11 Evaluate using the standard forms $\int \sec^2 x\, dx = \tan x$ and $\int \operatorname{cosec}^2 x\, dx = -\cot x$:

a $\displaystyle\int \tan^2 2x\, dx$ **b** $\displaystyle\int \cot^2 \tfrac{1}{2}x\, dx$ **c** $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3\tan^2 3x\, dx$ **d** $\displaystyle\int_{\frac{\pi}{24}}^{\frac{\pi}{8}} \cot^2 4x\, dx$

12 Integrate these functions, using the reverse chain rule or otherwise.

a $\tan x \sec^2 x$ **b** $\dfrac{\sin^2 x}{1 + \cos x}$ **c** $\dfrac{1 + \cos^3 x}{\cos^2 x}$

ENRICHMENT

13 Define $F(x) = \displaystyle\int_0^x \sin^2 t\, dt$, where $0 \le x \le 2\pi$.

a Show that $F(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$.

b Explain why $F'(x) = \sin^2 x$. Hence state the values of x in the given domain for which $F(x)$ is:

i stationary, **ii** increasing, **iii** decreasing.

c Explain why $F(x)$ never differs from $\frac{1}{2}x$ by more than $\frac{1}{4}$.

d Find any points of inflection of $F(x)$ in the given domain.

e Sketch, on the same diagram, the graphs of $y = F(x)$ and $y = F'(x)$ over the given domain, and observe how they are related.

f **i** For what value of k is $\displaystyle\int_0^k \sin^2 x\, dx = \frac{3\pi}{2}$?

ii For what values of k is $\displaystyle\int_0^k \sin^2 x\, dx = \frac{n\pi}{2}$, where n is an integer?

14 Find the value of $\displaystyle\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\, dt\right)$, explaining your reasoning carefully.

ISBN 978-1-108-76630-2

Cambridge University Press

12D Integration by substitution

The reverse chain rule as we have been using it so far does not cover all the situations where the chain rule can be used in integration. This section and the next develop a more general method called *integration by substitution*.

The first stage, covered in this section, begins by translating the reverse chain rule into a slightly more flexible notation. It involves substitutions of the form

'Let $u =$ some function of x.'

The reverse chain rule — an example

Here is an example of the reverse chain rule as we have been using it. The working is set out in full on the right.

Example 14 — 12D

Find $\int x(1 - x^2)^4\,dx$.

SOLUTION

$$\begin{aligned}\int x(1 - x^2)^4\,dx &= -\tfrac{1}{2}\int (-2x)\,(1 - x^2)^4\,dx\\ &= -\tfrac{1}{2} \times \tfrac{1}{5}(1 - x^2)^5 + C\\ &= -\tfrac{1}{10}(1 - x^2)^5 + C\end{aligned}$$

Let $u = 1 - x^2$.
Then $\dfrac{du}{dx} = -2x$,
and $\displaystyle\int u^4\,\frac{du}{dx}\,dx = \tfrac{1}{5}u^5$.

Rewriting this example as integration by substitution

We shall now rewrite this using a new notation. The key to this new notation is that the derivative $\dfrac{du}{dx}$ is treated as a fraction — the du and the dx are split apart, so that the statement

$\dfrac{du}{dx} = -2x$ is written instead as $du = -2x\,dx$.

The new variable u no longer remains in the working column on the right, but is brought over into the main sequence of the solution on the left.

Example 15 — 12D

Find $\int x(1 - x^2)^4\,dx$, using the substitution $u = 1 - x^2$.

SOLUTION

$$\begin{aligned}\int x(1 - x^2)^4\,dx &= \int u^4\left(-\tfrac{1}{2}\right)du\\ &= -\tfrac{1}{2} \times \tfrac{1}{5}u^5 + C\\ &= -\tfrac{1}{10}(1 - x^2)^5 + C\end{aligned}$$

Let $u = 1 - x^2$.
Then $du = -2x\,dx$,
and $x\,dx = -\tfrac{1}{2}\,du$.

ISBN 978-1-108-76630-2

Example 16 12D

Find $\int \sin x \sqrt{1 - \cos x}\, dx$, using the substitution $u = 1 - \cos x$.

SOLUTION

$$\int \sin x \sqrt{1 - \cos x}\, dx = \int u^{\frac{1}{2}}\, du$$
$$= \tfrac{2}{3}u^{\frac{3}{2}} + C$$
$$= \tfrac{2}{3}(1 - \cos x)^{\frac{3}{2}} + C$$

Let $u = 1 - \cos x$.
Then $du = \sin x\, dx$.

An advance on the reverse chain rule

Some integrals that can be done in this way could only be done by the reverse chain rule in a rather clumsy manner.

Example 17 12D

Find $\int x\sqrt{1 - x}\, dx$, using the substitution $u = 1 - x$.

SOLUTION

$$\int x\sqrt{1 - x}\, dx = \int (1 - u)\sqrt{u}\,(-du)$$
$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$
$$= \tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}} + C$$
$$= \tfrac{2}{5}(1 - x)^{\frac{5}{2}} - \tfrac{2}{3}(1 - x)^{\frac{3}{2}} + C$$

Let $u = 1 - x$.
Then $du = -dx$,
and $x = 1 - u$.

Substituting the limits of integration in a definite integral

A great advantage of this new method is that the limits of integration can be changed from values of x to values of u. There is then no need ever to go back to x. The first worked example below repeats the previous integrand, but this time within a definite integral.

ISBN 978-1-108-76630-2

Example 18 12D

Find $\int_0^1 x\sqrt{1-x}\,dx$, using the substitution $u = 1 - x$.

SOLUTION

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= -\int_1^0 (1-u)\sqrt{u}\,du \\ &= -\int_1^0 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du \\ &= -\left[\tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right]_1^0 \\ &= -0 + \left(\tfrac{2}{3} - \tfrac{2}{5}\right) \\ &= \tfrac{4}{15}\end{aligned}$$

Let $u = 1 - x$.
Then $du = -dx$,
and $x = 1 - u$.
When $x = 0, u = 1$,
when $x = 1, u = 0$.

Example 19 12D

Find $\int_0^\pi \sin x \cos^6 x\,dx$, using the substitution $u = \cos x$.

SOLUTION

$$\begin{aligned}\int_0^\pi \sin x\cos^6 x\,dx &= -\int_1^{-1} u^6\,du \\ &= -\tfrac{1}{7}\Big[u^7\Big]_1^{-1} \\ &= -\tfrac{1}{7}\times(-1) + \tfrac{1}{7}\times 1 \\ &= \tfrac{2}{7}\end{aligned}$$

Let $u = \cos x$.
Then $du = -\sin x\,dx$.
When $x = 0, u = 1$,
when $x = \pi, u = -1$.

Exercise 12D FOUNDATION

1 Consider the integral $\int 2x(1+x^2)^3\,dx$, and the substitution $u = 1 + x^2$.

a Show that $du = 2x\,dx$.

b Show that the integral can be written as $\int u^3\,du$.

c Hence find the primitive of $2x(1+x^2)^3$.

d Check your answer by differentiating it.

2 Repeat the previous question for each indefinite integral and substitution.

a $\int 2(2x+3)^3\,dx$ (Let $u = 2x + 3$.)

b $\int 3x^2(1+x^3)^4\,dx$ (Let $u = 1 + x^3$.)

c $\int \frac{2x}{(1+x^2)^2}\,dx$ (Let $u = 1 + x^2$.)

d $\int \frac{3}{\sqrt{3x-5}}\,dx$ (Let $u = 3x - 5$.)

e $\int \sin^3 x\cos x\,dx$ (Let $u = \sin x$.)

f $\int \frac{4x^3}{1+x^4}\,dx$ (Let $u = 1 + x^4$.)

ISBN 978-1-108-76630-2

3 Consider the integral $\int \frac{x}{\sqrt{1-x^2}}\,dx$, and the substitution $u = 1 - x^2$.

a Show that $x\,dx = -\frac{1}{2}\,du$.

b Show that the integral can be written as $-\frac{1}{2}\int u^{-\frac{1}{2}}\,du$.

c Hence find the primitive of $\frac{x}{\sqrt{1-x^2}}$.

4 Repeat the previous question for each indefinite integral and substitution.

a $\int x^3(x^4+1)^5\,dx$ (Let $u = x^4 + 1$.)

b $\int x^2\sqrt{x^3-1}\,dx$ (Let $u = x^3 - 1$.)

c $\int x^2 e^{x^3}\,dx$ (Let $u = x^3$.)

d $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^3}\,dx$ (Let $u = 1 + \sqrt{x}$.)

e $\int \tan^2 2x \sec^2 2x\,dx$ (Let $u = \tan 2x$.)

f $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$ (Let $u = \frac{1}{x}$.)

5 Find the exact value of each definite integral, using the given substitution.

a $\int_0^1 x^2(2+x^3)^3\,dx$ (Let $u = 2 + x^3$.)

b $\int_0^1 \frac{2x^3}{\sqrt{1+x^4}}\,dx$ (Let $u = 1 + x^4$.)

c $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\,dx$ (Let $u = \cos x$.)

d $\int_{\frac{1}{2\sqrt{3}}}^1 x\sqrt{1-x^2}\,dx$ (Let $u = 1 - x^2$.)

e $\int_1^{e^2} \frac{\ln x}{x}\,dx$ (Let $u = \ln x$.)

f $\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}}\,dx$ (Let $u = \sqrt{x}$.)

g $\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x\,dx$ (Let $u = \sin 2x$.)

h $\int_0^1 \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}}\,dx$ (Let $u = \sin^{-1} x$.)

i $\int_0^2 \frac{x+1}{\sqrt[3]{x^2+2x}}\,dx$ (Let $u = x^2 + 2x$.)

j $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$ (Let $u = \tan x$.)

DEVELOPMENT

6 Use the substitution $u = x^3$ to find the exact area bounded by the curve $y = \frac{x^2}{1+x^6}$, the x-axis and the line $x = 1$.

7 Evaluate each definite integral, using the substitution $u = \sin x$.

a $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx$

b $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$

c $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

d $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x}\,dx$

8 Find each indefinite integral, using the given substitution.

a $\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$ (Let $u = e^{2x}$.)

b $\int \frac{1}{x \ln x}\,dx$ (Let $u = \ln x$.)

c $\int \frac{\tan x}{\ln \cos x}\,dx$ (Let $u = \ln \cos x$.)

d $\int \tan^3 x \sec^4 x\,dx$ (Let $u = \tan x$.)

ISBN 978-1-108-76630-2

9 **a** A curve has gradient function $\dfrac{e^{2x}}{1+e^{4x}}$ and passes through the point $\left(0, \frac{\pi}{8}\right)$. Use the substitution $u = e^{2x}$ to find its equation.

b If $y'' = \dfrac{x}{(4-x^2)^{\frac{3}{2}}}$, and when $x = 0$, $y' = 1$ and $y = \frac{1}{2}$, use the substitution $u = 4 - x^2$ to find y' and then find y as a function of x.

10 **a** Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

b Use the substitution $u = \sec x$, and the standard form $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$, to find:

i $\displaystyle\int_0^{\frac{\pi}{3}} 2^{\sec x} \sec x \tan x\,dx$ **ii** $\displaystyle\int_0^{\frac{\pi}{4}} \sec^5 x \tan x\,dx$

11 Evaluate each integral, using the given substitution.

a $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\sin^2 x}\,dx$ (Let $u = \sin^2 x$.) **b** $\displaystyle\int_1^{e} \frac{\ln x + 1}{(x\ln x + 1)^2}\,dx$ (Let $u = x\ln x$.)

12 Use the substitution $u = \sqrt{x-1}$ to find $\displaystyle\int \frac{1}{2x\sqrt{x-1}}\,dx$.

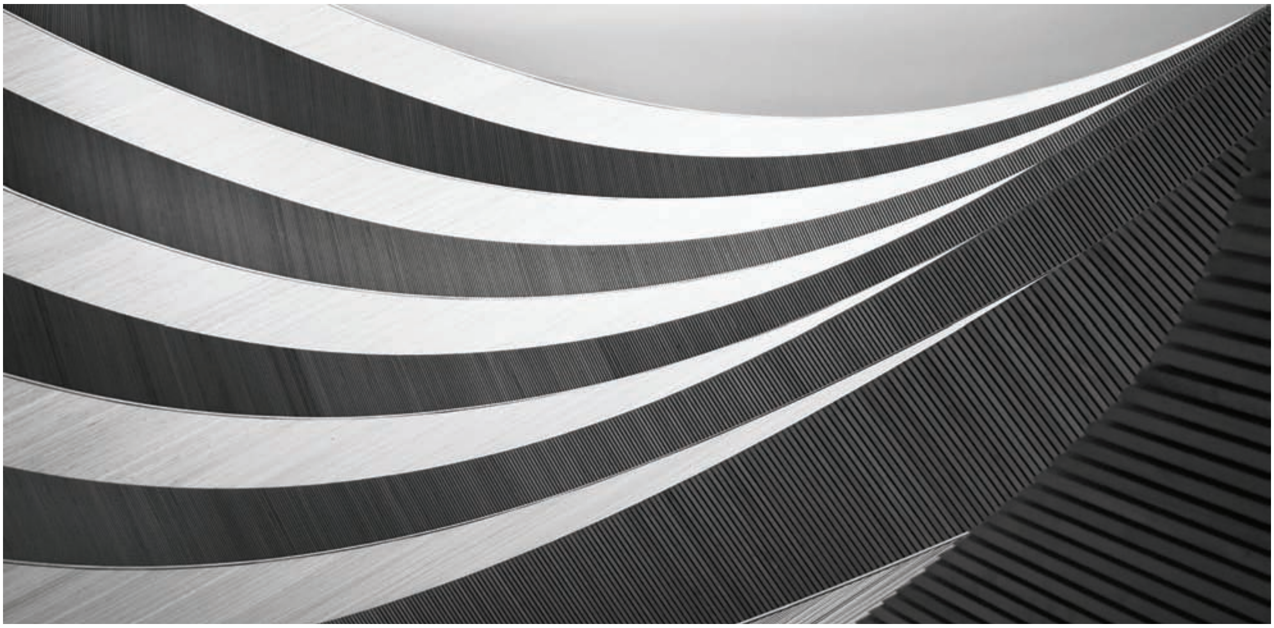
13 **a** Use the substitution $u = \sqrt{x}$ to find $\displaystyle\int \frac{1}{\sqrt{x(1-x)}}\,dx$.

b Evaluate the integral in part **a** again, using the substitution $u = x - \frac{1}{2}$.

c Hence show that $\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} - \frac{\pi}{2}$, for $0 < x < 1$.

ENRICHMENT

14 Use the substitution $u = x - \dfrac{1}{x}$ to show that $\displaystyle\int_1^{\frac{1}{2}(\sqrt{6}+\sqrt{2})} \frac{1+x^2}{1+x^4}\,dx = \frac{\pi}{4\sqrt{2}}$.



ISBN 978-1-108-76630-2

12E Further integration by substitution

The second stage of integration by substitution reverses the previous procedure and replaces x by a function of u. The substitutions are therefore of the form

'Let x = some function of u.'

Substituting x by a function of u

As a first example, here is a quite different substitution which solves the integral given in a worked example of the last section.

Example 20 12E

Find $\int_0^1 x\sqrt{1-x}\,dx$, using the substitution $x = 1 - u^2$.

SOLUTION

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= \int_1^0 (1-u^2)u(-2u)\,du\\ &= -2\int_1^0 (u^2-u^4)\,du\\ &= -2\Big[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\Big]_1^0\\ &= -0 + 2\left(\tfrac{1}{3}-\tfrac{1}{5}\right)\\ &= \tfrac{4}{15}\end{aligned}$$

Let $x = 1 - u^2$.
Then $dx = -2u\,du$,
and $\sqrt{1-x} = u$.
When $x = 0, u = 1$,
when $x = 1, u = 0$.

This question is a good example of how an integral may be evaluated in contrasting ways. The next integral uses a trigonometric substitution, but can also be done using areas of segments.

Example 21 12E

Find $\int_{3\sqrt{2}}^{6} \sqrt{36-x^2}\,dx$:

a using the substitution $x = 6\sin u$,

b using the formula for the area of a segment.

SOLUTION

a $$\begin{aligned}\int_{3\sqrt{2}}^{6}\sqrt{36-x^2}\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 6\cos u \times 6\cos u\,du\\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 36\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2u\right)du\\ &= \Big[\,18u + 9\sin 2u\,\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\\ &= (9\pi + 0) - \left(\tfrac{9\pi}{2}+9\right)\\ &= \tfrac{9}{2}(\pi - 2)\end{aligned}$$

Let $x = 6\sin u$.
Then $dx = 6\cos u\,du$,
and $\sqrt{36-x^2} = 6\cos u$.
When $x = 3\sqrt{2}$, $u = \frac{\pi}{4}$,
when $x = 6$, $u = \frac{\pi}{2}$.

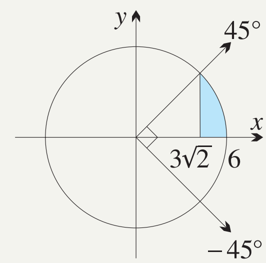
ISBN 978-1-108-76630-2

Cambridge University Press

b The integral is sketched opposite. The shaded area is half the segment subtending an angle of 90°.

Hence $\int_{3\sqrt{2}}^{6} \sqrt{36 - x^2}\, dx = \frac{1}{2} \times \frac{1}{2} \times 6^2 \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$= 9\left(\frac{\pi}{2} - 1\right).$



Note: Careful readers may notice a problem here, in that given the value $x = 3\sqrt{2}$, u is determined by $\sin u = \frac{1}{2}\sqrt{2}$, so there are infinitely many possible values of u. A similar problem occurred in the previous worked example, where $0 = 1 - u^2$ had two solutions. These problems arise because the functions involved in the substitutions were $x = 1 - u^2$ and $x = 6 \sin u$, whose inverses were not functions. A full account of all this would require substitutions by restrictions of the functions given above so that they had inverse functions. In practice, however, this is rarely necessary, and it is certainly not a concern of this course.

As a rule of thumb, work with positive square roots, and with trigonometric functions, work in the same quadrants as were involved in the definitions of the inverse trigonometric functions in Chapter 17 of the Year 11 book.

Here is a summary of Sections 12D and 12E.

10 INTEGRATION USING THE REVERSE CHAIN RULE AND SUBSTITUTION

The chain rule can be used in reverse in various ways to find integrals.

- The reverse chain rule straightforwardly reverses the steps of differentiation by the chain rule.
- A reverse-chain-rule formula can be developed for each standard form.
- Substitution can be done by, 'Let u be a function of x.'
- Substitution can be done by, 'Let x be a function of u.'
- When substituting into a definite integral, the limits of integration can be substituted as well, so that there is no need ever to return from u to x.

Exercise 12E

FOUNDATION

1 Consider the integral $I = \int x(x - 1)^5\, dx$, and let $x = u + 1$.

a Show that $dx = du$.

b Show that $I = \int u^5(u + 1)\, du$.

c Hence find I.

d Check your answer by differentiating it.

2 Using the same substitution as in the previous question, find:

a $\int \frac{x}{\sqrt{x - 1}}\, dx$

b $\int \frac{x}{(x - 1)^2}\, dx$

ISBN 978-1-108-76630-2

3 Consider the integral $J = \int x\sqrt{x + 1}\,dx$, and let $x = u^2 - 1$.

a Show that $dx = 2u\,du$.

b Show that $J = 2\int (u^4 - u^2)\,du$.

c Hence find J.

d Check your answer by differentiating it.

4 Using the same substitution as in the previous question, find:

a $\int x^2\sqrt{x + 1}\,dx$

b $\int \frac{2x + 3}{\sqrt{x + 1}}\,dx$

5 Find each indefinite integral using the given substitution.

a $\int \frac{x - 2}{x + 2}\,dx$ (Let $x = u - 2$.)

b $\int \frac{2x + 1}{\sqrt{2x - 1}}\,dx$ $\left(\text{Let } x = \frac{1}{2}(u + 1).\right)$

c $\int 3x\sqrt{4x - 5}\,dx$ $\left(\text{Let } x = \frac{1}{4}(u^2 + 5).\right)$

d $\int \frac{1}{1 + \sqrt{x}}\,dx$ $\left(\text{Let } x = (u - 1)^2.\right)$

6 Evaluate, using the given substitution:

a $\int_0^1 x(x + 1)^3\,dx$ (Let $x = u - 1$.)

b $\int_0^{\frac{1}{2}} \frac{1 + x}{1 - x}\,dx$ (Let $x = 1 - u$.)

c $\int_0^1 \frac{3x}{\sqrt{3x + 1}}\,dx$ $\left(\text{Let } x = \frac{1}{3}(u - 1).\right)$

d $\int_0^1 \frac{2 - x}{(2 + x)^3}\,dx$ (Let $x = u - 2$.)

e $\int_0^4 x\sqrt{4 - x}\,dx$ $\left(\text{Let } x = 4 - u^2.\right)$

f $\int_1^5 \frac{x}{(2x - 1)^{\frac{3}{2}}}\,dx$ $\left(\text{Let } x = \frac{1}{2}(u^2 + 1).\right)$

g $\int_0^4 \frac{1}{3 + \sqrt{x}}\,dx$ $\left(\text{Let } x = (u - 3)^2.\right)$

h $\int_0^7 \frac{x^2}{\sqrt[3]{x + 1}}\,dx$ $\left(\text{Let } x = u^3 - 1.\right)$

DEVELOPMENT

7 a Consider the integral $I = \int \frac{1}{\sqrt{5 - 4x - x^2}}\,dx$, and let $x = u - 2$.

Show that $I = \int \frac{1}{\sqrt{9 - u^2}}\,du$, and hence find I.

b Use a similar approach to find:

i $\int \frac{1}{x^2 + 2x + 4}\,dx$ (Let $x = u - 1$.)

ii $\int \frac{1}{\sqrt{4 - 2x - x^2}}\,dx$ (Let $x = u - 1$.)

iii $\int_1^2 \frac{1}{\sqrt{3 + 2x - x^2}}\,dx$ (Let $x = u + 1$.)

iv $\int_3^7 \frac{1}{x^2 - 6x + 25}\,dx$ (Let $x = u + 3$.)

8 a Consider the integral $J = \int \frac{1}{\sqrt{4 - x^2}}\,dx$, and let $x = 2\sin\theta$.

Show that $J = \int 1\,d\theta$, and hence show that $J = \sin^{-1}\frac{x}{2} + C$.

b Using a similar approach, find:

i $\int \frac{1}{9 + x^2}\,dx$ (Let $x = 3\tan\theta$.)

ii $\int \frac{-1}{\sqrt{3 - x^2}}\,dx$ (Let $x = \sqrt{3}\cos\theta$.)

ISBN 978-1-108-76630-2

Cambridge University Press

iii $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$ (Let $x = \frac{1}{2}\sin\theta$.)

iv $\displaystyle\int \frac{1}{1+16x^2}\,dx$ (Let $x = \frac{1}{4}\tan\theta$.)

v $\displaystyle\int_0^3 \frac{1}{\sqrt{36-x^2}}\,dx$ (Let $x = 6\sin\theta$.)

vi $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}\,dx$ (Let $x = \frac{2}{3}\tan\theta$.)

9 a Consider the integral $I = \displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$, and let $x = \sin\theta$.

Show that $I = \displaystyle\int \sec^2\theta\,d\theta$, and hence show that $I = \dfrac{x}{\sqrt{1-x^2}} + C$.

b Similarly, use the given substitution to find:

i $\displaystyle\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$ (Let $x = 2\tan\theta$.)

ii $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$ (Let $x = \sin\theta$.)

iii $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ (Let $x = 2\sin\theta$.)

iv $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ (Let $x = 5\cos\theta$.)

v $\displaystyle\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$ (Let $x = 3\tan\theta$.)

vi $\displaystyle\int_2^4 \frac{1}{x^2\sqrt{x^2-4}}\,dx$ (Let $x = 2\sec\theta$.)

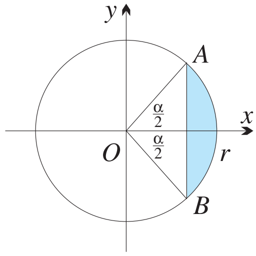
10 Find the equation of the curve $y = f(x)$ if $f'(x) = \dfrac{\sqrt{x^2-9}}{x}$ and $f(3) = 0$.
(Hint: Use the substitution $x = 3\sec\theta$.)

11 Find the exact area of the region bounded by $y = \dfrac{x^3}{\sqrt{3-x^2}}$, the x-axis and the line $x = 1$.
(Hint: Use the substitution $x = \sqrt{3}\sin\theta$, followed by the substitution $u = \cos\theta$.)

12 [These are confirmations rather than proofs, because the calculus of trigonometric functions was developed on the basis of the formulae in parts **a** and **b**.]

a Use integration to confirm that the area of a circle is πr^2.
(Hint: Find the area bounded by the semi-circle $y = \sqrt{r^2 - x^2}$ and the x-axis and double it. Use the substitution $x = r\sin\theta$.)

b The shaded area in the diagram to the right is the segment of a circle of radius r cut off by the chord AB subtending an angle α at the centre O.



i Show that the area is $I = 2\displaystyle\int_{r\cos\frac{1}{2}\alpha}^{r} \sqrt{r^2-x^2}\,dx$.

ii Let $x = r\cos\theta$, and show that $I = -2r^2\displaystyle\int_{\frac{1}{2}\alpha}^{0} \sin^2\theta\,d\theta$.

iii Hence confirm that $I = \frac{1}{2}r^2(\alpha - \sin\alpha)$.

c Use a similar approach to confirm that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. Then justify the formula by regarding the ellipse as the unit circle stretched horizontally by a factor of a and vertically by a factor of b.

ENRICHMENT

13 a Use the substitution $x = -u$ to show that $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx = \int_{-2}^{2} \frac{x^2e^x}{e^x+1}\,dx$.

b Hence find $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx$.

ISBN 978-1-108-76630-2

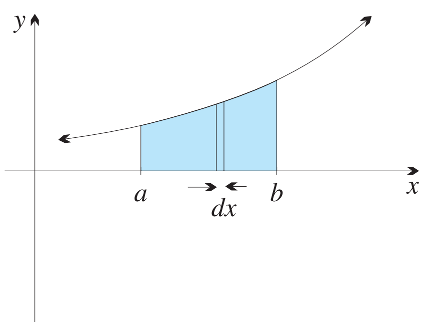
12F Volumes of rotation

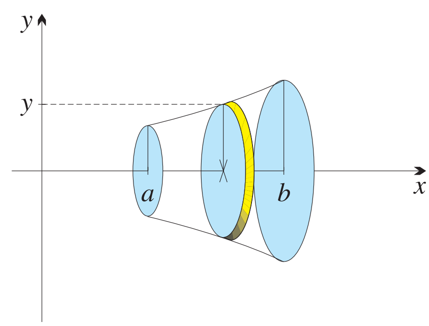
When a region in the coordinate plane is rotated about either the x-axis or the y-axis, a solid region in three dimensions is generated, called a *solid of revolution*. The process is similar to shaping wood on a lathe, or making pottery on a wheel, because such shapes have rotational symmetry and circular cross-sections.

The volumes of such solids can be found using a simple integration formula. The well-known formulae for the volumes of cones and spheres can finally be proven by this method.

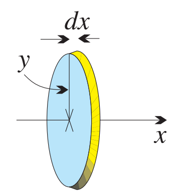
Rotating a region about the *x*-axis

The first diagram below shows the region under the curve $y = f(x)$ in the interval $a \le x \le b$, and the second shows the solid generated when this region is rotated about the x-axis.





Imagine the solid sliced like salami perpendicular to the x-axis into infinitely many circular slices, each of width dx. One of the slices is shown in the right-hand diagram, and again in more detail below. The vertical strip in the left-hand diagram is what generates this slice when it is rotated about the x-axis.



Now radius of circular slice $= y$, the height of the strip,

so area of circular slice $= \pi y^2$, because it is a circle.

The slice is a thin cylinder of infinitesimal thickness dx,

so volume of circular slice $= \pi y^2\, dx$ (area $\times$ thickness).

To get the total volume, we add all the slices from $x = a$ to $x = b$,

so volume of solid $= \displaystyle\int_a^b \pi y^2\, dx$.

11 VOLUMES OF REVOLUTION ABOUT THE *x*-AXIS

The volume of the solid generated when the region between a curve and the x-axis from $x = a$ to $x = b$ is rotated about the x-axis is

$$\text{volume} = \int_a^b \pi y^2\, dx \text{ cubic units.}$$

If the curve is below the x-axis, so that y is negative, then the volume calculated is still positive, because y^2 rather than y occurs in the formula.

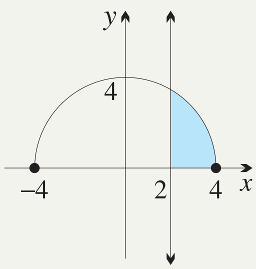
Unless other units are specified, 'cubic units' (u^3), should be used, by analogy with the areas of regions discussed in Section 3F.

ISBN 978-1-108-76630-2

Example 22 12F

The shaded region cut off the semi-circle $y = \sqrt{16 - x^2}$ by the line $x = 2$ is rotated about the x-axis. Find the volume generated.

SOLUTION



Squaring, $y^2 = 16 - x^2$,

$$\begin{aligned} \text{so} \quad \text{volume} &= \int_2^4 \pi y^2\,dx \\ &= \pi \int_2^4 (16 - x^2)\,dx \\ &= \pi \Big[16x - \tfrac{1}{3}x^3 \Big]_2^4 \\ &= \pi \left(64 - \tfrac{64}{3} - 32 + \tfrac{8}{3}\right) \\ &= \tfrac{40\pi}{3} \text{ cubic units.} \end{aligned}$$

Volumes of revolution about the y-axis

To calculate the volume when a region is rotated about the y-axis, we exchange x and y.

12 VOLUMES OF REVOLUTION ABOUT THE y-AXIS

The volume of the solid generated when the region between a curve and the y-axis from $y = a$ to $y = b$ is rotated about the y-axis is

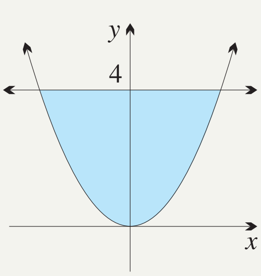
$$\text{volume} = \int_a^b \pi x^2\,dy \text{ cubic units.}$$

When y is given as a function of x, the equation will need to be written with x^2 as the subject.

Example 23 12F

Find the volume of the solid formed by rotating the region between $y = x^2$ and the line $y = 4$ about the y-axis.

SOLUTION



$$\begin{aligned} \text{Because } x^2 = y, \text{ volume} &= \int_0^4 \pi x^2\,dy \\ &= \pi \int_0^4 y\,dy \\ &= \pi \Big[\tfrac{1}{2}y^2 \Big]_0^4 \\ &= 8\pi \text{ cubic units.} \end{aligned}$$

Volume by subtraction

When rotating the region between two curves lying above the x-axis, the two integrals need to be subtracted, as if the outer volume has been formed first, and the inner volume then cut away from it. The two volumes

ISBN 978-1-108-76630-2

can always be calculated separately and subtracted, but if the two integrals have the same limits of integration, it may be more convenient to combine them.

13 ROTATING THE REGION BETWEEN CURVES

- The volume of the solid generated when the region between two curves from $x = a$ to $x = b$ is rotated about the x-axis is
$$\text{volume} = \int_a^b \pi({y_2}^2 - {y_1}^2)\,dx \quad (\text{where } y_2 > y_1 > 0).$$
- Similarly, the volume of the solid generated when the region between two curves from $y = a$ to $y = b$ is rotated about the y-axis is
$$\text{volume} = \int_a^b \pi({x_2}^2 - {x_1}^2)\,dy \quad (\text{where } x_2 > x_1 > 0).$$

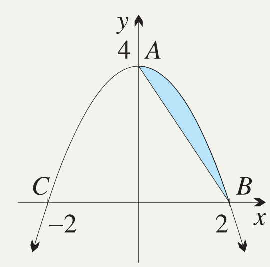
Example 24 12F

The curve $y = 4 - x^2$ meets the y-axis at $A(0, 4)$ and the x-axis at $B(2, 0)$ and $C(-2, 0)$. Find the volume generated when the region between the curve and the line AB is rotated:

a about the x-axis, **b** about the y-axis.

SOLUTION

a The parabola is $y_2 = 4 - x^2$, and the line AB is $y_1 = 4 - 2x$,



$$\begin{aligned}
\text{so volume} &= \int_0^2 \pi({y_2}^2 - {y_1}^2)\,dx \\
&= \pi\int_0^2 \big((16 - 8x^2 + x^4) - (16 - 16x + 4x^2)\big)\,dx \\
&= \pi\int_0^2 (x^4 - 12x^2 + 16x)\,dx \\
&= \pi\Big[\tfrac{1}{5}x^5 - 4x^3 + 8x^2\Big]_0^2 \\
&= \pi\big(\tfrac{32}{5} - 32 + 32\big) \\
&= \tfrac{32\pi}{5} \text{ cubic units.}
\end{aligned}$$

b The parabola is ${x_2}^2 = 4 - y$, and the line is $x_1 = 2 - \frac{1}{2}y$,

$$\begin{aligned}
\text{so volume} &= \int_0^4 \pi({x_2}^2 - {x_1}^2)\,dy \\
&= \pi\int_0^4 \big((4 - y) - (4 - 2y + \tfrac{1}{4}y^2)\big)\,dy \\
&= \pi\int_0^4 (-\tfrac{1}{4}y^2 + y)\,dy \\
&= \pi\Big[-\tfrac{1}{12}y^3 + \tfrac{1}{2}y^2\Big]_0^4 \\
&= \pi\big(-\tfrac{16}{3} + 8\big) \\
&= \tfrac{8\pi}{3} \text{ cubic units.}
\end{aligned}$$

ISBN 978-1-108-76630-2

Note: One would not normally expect the volumes of revolution about the two different axes to be equal. This is because an element of area will generate a larger element of volume if it is moved further away from the axis of rotation.

Cones and spheres

The formulae for the volumes of cones and spheres may have been learnt earlier, but they cannot be proven without arguments involving integration. The proofs of both results are developed in Question 12 of Exercise 12F, and these questions should be carefully worked through.

14 VOLUME OF CONES AND SPHERES

- For a cone, $V = \frac{1}{3}\pi r^2 h$.
- For a sphere, $V = \frac{4}{3}\pi r^3$.

Exercise 12F

FOUNDATION

1 **a** Sketch the region bounded by the line $y = 3x$ and the x-axis between $x = 0$ and $x = 3$.

b When this region is rotated about the x-axis, a right circular cone is formed. Find the radius and height of the cone and hence find its volume.

c Evaluate $\pi\int_0^3 y^2\,dx = \pi\int_0^3 9x^2\,dx$ in order to check your answer.

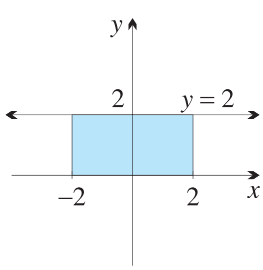
2 **a** Sketch the region bounded by the curve $y = \sqrt{9 - x^2}$ and the x-axis between $x = -3$ and $x = 3$.

b When this region is rotated about the x-axis, a sphere is formed. Find the radius of the sphere and hence find its volume.

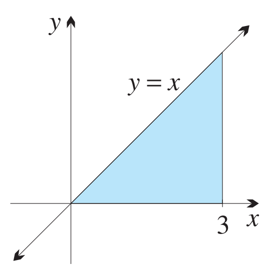
c Evaluate $\pi\int_{-3}^3 y^2\,dx = \pi\int_{-3}^3 (9 - x^2)\,dx$ in order to check your answer.

3 Calculate the volume generated when each shaded region is rotated about the x-axis.

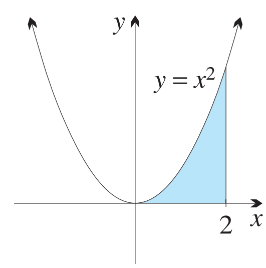
a



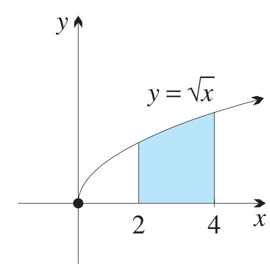
b



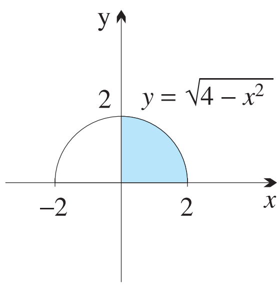
c



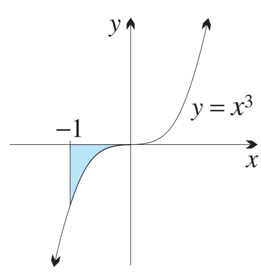
d



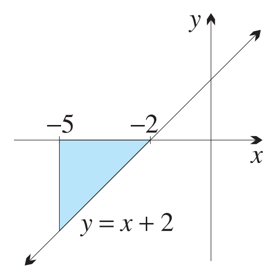
e



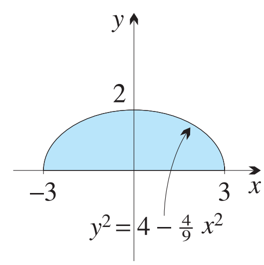
f



g



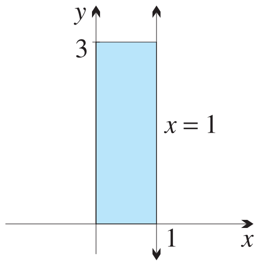
h



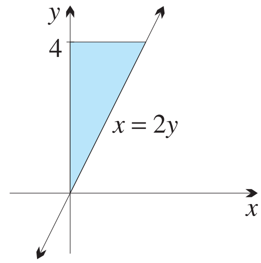
ISBN 978-1-108-76630-2

4 Calculate the volume generated when each shaded region is rotated about the y-axis.

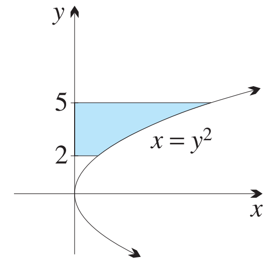
a



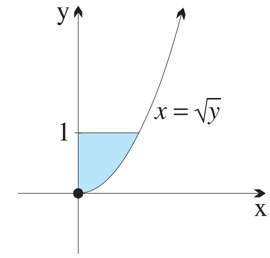
b



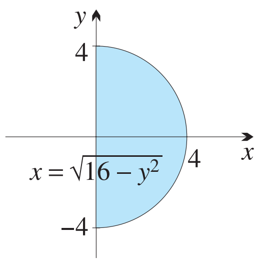
c



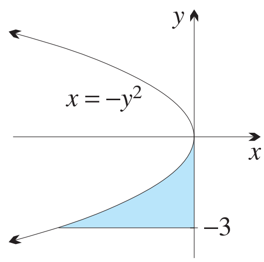
d



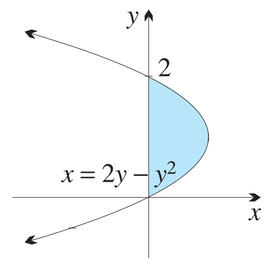
e



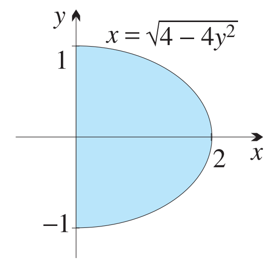
f



g



h



DEVELOPMENT

5 The region between the curve $y = e^x$ and the x-axis from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the solid generated.

6 Find the volume generated when the region between the curve $y = \dfrac{1}{\sqrt{x}}$ and the x-axis, from $x = 2$ to $x = 4$, is rotated about the x-axis.

7 The region between the curve $y = \dfrac{1}{x^2}$ (called a *truncus*) and the y-axis from $y = 1$ to $y = 6$ is rotated about the y-axis. Calculate the exact volume of the solid formed.

8 **a** Write $\tan^2 x$ in terms of $\sec^2 x$.

b The region bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$ is rotated about the x-axis. Find the volume of the solid generated.

9 **a** Write $\sin^2 x$ in terms of $\cos 2x$.

b The region bounded by the curve $y = \sin x$, the x-axis and the vertical line $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the volume of the solid generated.

10 Find the volume of the solid formed by rotating the region with the given boundaries about the x-axis. (A sketch of each region will be needed.)

a $y = x + 3$, $x = 3$, $x = 5$ and $y = 0$ **b** $y = 1 + \sqrt{x}$, $x = 1$, $x = 4$ and $y = 0$

c $y = 5x - x^2$ and $y = 0$ **d** $y = x^3 - x$ and $y = 0$

11 Find the volume of the solid formed by rotating the region with the given boundaries about the y-axis. (A sketch of each region will be needed.)

a $x = y - 2$, $y = 1$ and $x = 0$ **b** $x = y^2 + 1$, $y = 0$, $y = 1$ and $x = 0$

c $x = y(y - 3)$ and $x = 0$ **d** $y = 1 - x^2$ and $y = 0$

ISBN 978-1-108-76630-2

12 A vat is designed by rotating about the x-axis the region between the curve $y = 1 + \dfrac{1}{x}$ and the x-axis from $x = \frac{1}{2}$ to $x = 3$. Show that its volume is $\dfrac{\pi}{6}(25 + 12\ln 6)$ u^3.

13 A metal stud is created by rotating about the x-axis the region contained between the curve $y = e^x - e^{-x}$ and the x-axis from $x = 0$ to $x = \frac{1}{2}$. Show that the volume of the stud is $\frac{\pi}{2}(e - 2 - e^{-1})$ u^3.

14 A champagne flute is designed by rotating about the x-axis the region between the curve $y = 4 + 4\sin\frac{x}{4}$ and the x-axis from $x = 4\pi$ to $x = 6\pi$. Find, correct to 4 significant figures, the capacity of the flute, if 1 unit = 1 cm.

15 The region R is bounded by the parabola $y = x^2$ and the x-axis from $x = 0$ to $x = 4$.

a Find the volume of the cylinder formed when the region between the line $x = 4$ and the y-axis from $y = 0$ to $y = 16$ is rotated about the y-axis.

b Find the volume of the solid formed when the region between the parabola $y = x^2$ and the y-axis from $y = 0$ to $y = 16$ is rotated about the y-axis.

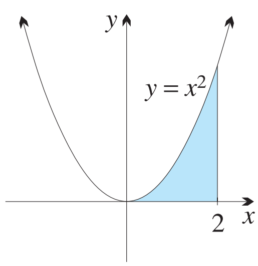
c Hence find the volume of the solid formed when R is rotated about the y-axis.

16 Find the volume of the solid generated by rotating each region about:

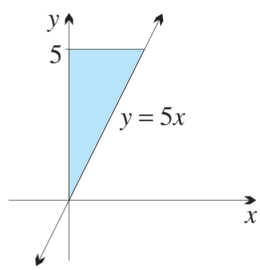
i the x-axis,

ii the y-axis. (Hint: In some cases a subtraction of volumes will be necessary.)

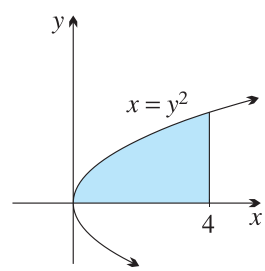
a



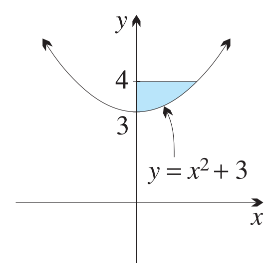
b



c



d



17 a Sketch the region bounded by the curves $y = x^2$ and $y = x^3$.

b Find the volume of the solid generated when this region is rotated about:

i the x-axis, **ii** the y-axis.

18 a On the same number plane sketch the graphs of $xy = 5$ and $x + y = 6$, clearly indicating their points of intersection.

b Find the volume of the solid generated when the region bounded by the two curves is rotated about the x-axis.

19 The region bounded by the hyperbola $y = 2 - \dfrac{2}{x}$, the vertical line $x = 1$ and the horizontal line $y = 1$ is rotated about the x-axis. Find the volume of the solid formed.

20 a Find the equation of the tangent to the curve $y = x^3 + 2$ at the point where $x = 1$.

b Draw a diagram showing the region bounded by the curve, the tangent and the y-axis.

c Calculate the volume of the solid formed when this region is rotated about:

i the x-axis, **ii** the y-axis.

21 A rubber washer is generated by rotating the region between the curve $y = \log_e x$, the x-axis and the line $x = 2$ about the y-axis. Find the exact volume of the washer.

ISBN 978-1-108-76630-2

Cambridge University Press

22 In this question some standard volume formulae will be proven.

a A right circular cone of height h and radius r is generated by rotating about the x-axis the region bounded by the line $y = \dfrac{rx}{h}$, the x-axis and the line $x = h$. Show that the volume of the cone is $\frac{1}{3}\pi r^2 h$.

b A cylinder of height h and radius r is generated by rotating about the x-axis the region bounded by the line $y = r$, the x-axis, the y-axis and the line $x = h$. Show that the volume of the cylinder is $\pi r^2 h$.

c i A sphere of radius r is generated by rotating about the x-axis the region between the semi-circle $y = \sqrt{r^2 - x^2}$ and the x-axis. Show that the volume of the sphere is $\frac{4}{3}\pi r^3$.

ii A spherical cap of height h is formed by rotating about the x-axis the region between the semi-circle $y = \sqrt{r^2 - x^2}$ and the x-axis from $x = r - h$ to $x = r$. Show that the volume of the cap is $\frac{1}{3}\pi h^2(3r - h)$.

ENRICHMENT

23 a Multiply $\sec\theta$ by $\dfrac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$, and hence find $\displaystyle\int \sec\theta\, d\theta$.

b The region R is bounded by $y = \dfrac{x}{\sqrt{x^2 + 16}}$, the x-axis and the line $x = 4$. Show that the volume generated by rotating R about the y-axis is $16\pi\left(\sqrt{2} - \ln(\sqrt{2} + 1)\right)$ units^3. (Hint: Use the substitution $y = \sin\theta$ and the result in part **a**.)

24 Consider the curves $f(x) = x^n$ and $f(x) = x^{n+1}$, where n is a positive integer.

a Find the points of intersection of the curves.

b Show that the volume V_n of the solid generated when the region bounded by the two curves is rotated around the x-axis is given by $\pi\left(\dfrac{1}{2n + 1} - \dfrac{1}{2n + 3}\right)$ cubic units.

c Describe the solid whose volume is given by $\lim\limits_{n\to\infty}(V_1 + V_2 + V_3 + \cdots + V_n)$.

d Find the volume of the solid in part **c**.

e Deduce that the series $\dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \dfrac{1}{7 \times 9} + \cdots$ has a limiting sum of $\dfrac{1}{6}$.

25 a Sketch the region bounded by the parabola $y = -x^2 + 6x - 8$ and the x-axis.

b By completing the square, or otherwise, show that the equation of the curve is $x = 3 + \sqrt{1 - y}$ for $3 \le x \le 4$, and $x = 3 - \sqrt{1 - y}$ for $2 \le x \le 3$.

c The region in part **a** is rotated about the y-axis. Show that the volume of the solid formed is given by $V = \displaystyle\int_0^1 12\pi\sqrt{1 - y}\, dy$, and hence calculate the exact volume.

ISBN 978-1-108-76630-2

Chapter 12 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 12 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Find the derivative of:

a $y = \sin^{-1} 3x$ **b** $y = \tan^{-1} \frac{x}{3}$ **c** $y = \cos^{-1}(1 - x)$
d $y = x^2 \tan^{-1} x$ **e** $y = \tan^{-1} \left(\frac{1}{2}x + 1\right)$ **f** $y = \sin^{-1} \frac{1}{x}$

2 Show that $y = \tan^{-1} x$ satisfies the differential equation $(1 + x^2)y'' + 2xy' = 0$.

3 **a** Show that the functions $y = \cos^{-1} x$ and $y = \sin^{-1} \sqrt{1 - x^2}$ have the same derivative for $0 \le x \le 1$.

b Explain the significance of the result in **a**.

4 Find:

a $\displaystyle\int \frac{1}{3 + x^2}\,dx$ **b** $\displaystyle\int \frac{1}{\sqrt{3 - x^2}}\,dx$

c $\displaystyle\int \frac{1}{9 + 4x^2}\,dx$ **d** $\displaystyle\int \frac{-1}{\sqrt{16 - 9x^2}}\,dx$

5 Evaluate:

a $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + 9x^2}\,dx$ **b** $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}}\,dx$

6 Find:

a $\displaystyle\int \cos^2 x\,dx$ **b** $\displaystyle\int \sin^2 x\,dx$

c $\displaystyle\int \cos^2 2x\,dx$ **d** $\displaystyle\int \sin^2 4x\,dx$

7 Find the exact value of:

a $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 3x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{6}} \cos^2 \tfrac{1}{2}x\,dx$

8 Show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 2x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 4x\,dx = \frac{\pi}{4}$.

ISBN 978-1-108-76630-2

Review

9 Find each indefinite integral using the given substitution.

a $\int 5(5x-1)^5\,dx$ [Let $u = 5x - 1$.]

b $\int 2x(x^2+2)^2\,dx$ [Let $u = x^2 + 2$.]

c $\int \frac{4x^3}{(x^4+1)^2}\,dx$ [Let $u = x^4 + 1$.]

d $\int \frac{1}{\sqrt{4x+3}}\,dx$ [Let $u = 4x + 3$.]

e $\int \sin^2 x \cos x\,dx$ [Let $u = \sin x$.]

f $\int \tan^3 x \sec^2 x\,dx$ [Let $u = \tan x$.]

10 Evaluate each definite integral using the given substitution.

a $\int_{-1}^{0} x^2(1+x^3)^4\,dx$ [Let $u = 1 + x^3$.]

b $\int_{0}^{\frac{\pi}{2}} \cos^3 x \sin x\,dx$ [Let $u = \cos x$.]

c $\int_{1}^{\sqrt{2}} x\sqrt{x^2-1}\,dx$ [Let $u = x^2 - 1$.]

d $\int_{1}^{e} \frac{(\ln x)^2}{x}\,dx$ [Let $u = \ln x$.]

e $\int_{\frac{1}{2}}^{1} \frac{e^{\frac{1}{x}}}{x^2}\,dx$ [Let $u = \frac{1}{x}$.]

f $\int_{0}^{\frac{\pi}{8}} \frac{\sec^2 2x}{1+\tan 2x}\,dx$ [Let $u = \tan 2x$.]

11 Find each indefinite integral using the given substitution.

a $\int \frac{x}{x-1}\,dx$ [Let $x = u + 1$.]

b $\int \frac{x-1}{\sqrt{x+2}}\,dx$ [Let $x = u - 2$.]

c $\int x\sqrt{2x+1}\,dx$ [Let $x = \frac{1}{2}u^2 - \frac{1}{2}$.]

d $\int \frac{1}{4+\sqrt{x}}\,dx$ [Let $x = (u-4)^2$.]

12 Evaluate, using the given substitution:

a $\int_{1}^{2} x(x-1)^4\,dx$ [Let $x = u + 1$.]

b $\int_{1}^{5} \frac{x}{x+3}\,dx$ [Let $x = u - 3$.]

c $\int_{0}^{15} \frac{x}{\sqrt{x+1}}\,dx$ [Let $x = u^2 - 1$.]

d $\int_{2}^{3} \frac{1}{2}x\sqrt{x-2}\,dx$ [Let $x = u^2 + 2$.]

13 a State the domain and range of the function $y = \sqrt{9-x}$.

b Sketch the graph of the function.

c Calculate the area of the region bounded by the curve and the coordinate axes.

d Calculate the volume of the solid formed when this region is rotated about:

i the x-axis, **ii** the y-axis.

14 A horn is created by rotating about the x-axis the region between the curve $y = \frac{1}{\sqrt{4-x}}$ and the x-axis from $x = 0$ to $x = 3\frac{3}{4}$. Find the volume of the horn.

15 A vase is designed by rotating the parabola $y^2 = 18(x-6)$ from $y = -6$ to $y = 6$ about the y-axis. Find the exact volume of the vase.

ISBN 978-1-108-76630-2

Review

16 a Write $\cos^2 2x$ in terms of $\cos 4x$.

b The region bounded by the curve $y = \cos 2x$ and the x-axis from $x = -\frac{\pi}{6}$ to $x = \frac{\pi}{6}$ is rotated about the x-axis. Show that the solid generated has volume $\frac{\pi}{24}(4\pi + 3\sqrt{3})\ u^3$.

17 A tank is created by rotating about the x-axis the region between the curve $y = 1 + e^{-x}$ and the x-axis from $x = 1$ to $x = 3$. Find its volume correct to two decimal places.

18 a Sketch $y = 1 - \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes.

b Find the volume generated when R is rotated about the x-axis.

19 a Find the stationary points of the curve $y = x + \dfrac{1}{x}$ and sketch its graph.

b Show that the line $y = \frac{5}{2}$ intersects the curve when $x = \frac{1}{2}$ or $x = 2$.

c Find the volume of the solid generated when the region between the curve and the line $y = \frac{5}{2}$ is rotated about the x-axis.

20 a On the same set of axes sketch the curves $y = \sqrt{9 - x^2}$ and $y = 18 - 2x^2$.

b Find the area bounded by the two curves.

c Find the volume of the solid formed when this region is rotated about the x-axis

21 The region under the graph $y = 2^{x+1}$ between $x = 1$ and $x = 3$ is rotated about the x-axis.

a Write down the definite integral representing the volume of the solid that is formed.

b Use the trapezoidal rule with five function values to approximate the volume of the solid, giving your answer as multiple of π.

c Find the exact value of the volume, then approximate is as an integer multiple of π. Why is the exact answer smaller than the trapezoidal-rule approximation?

ISBN 978-1-108-76630-2