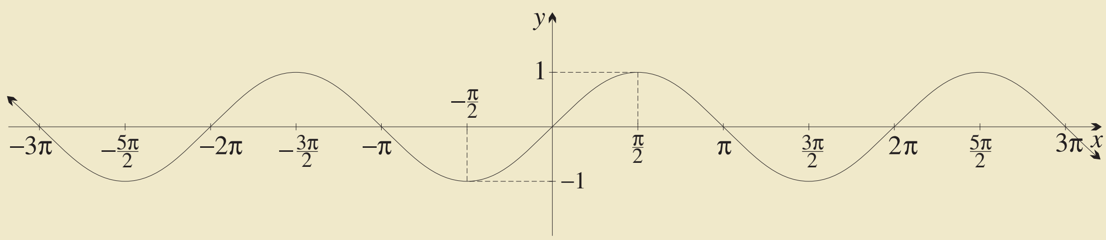
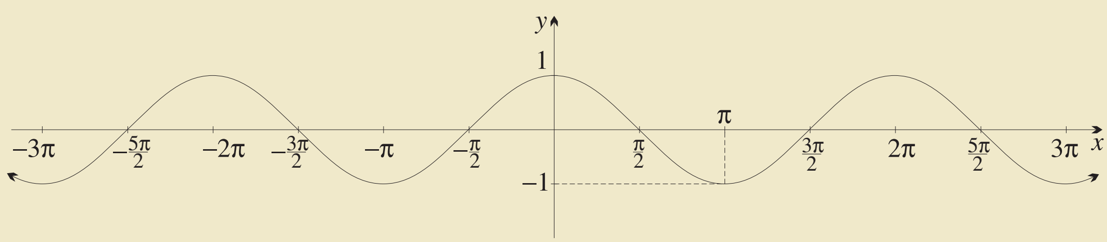
7 The trigonometric functions

This chapter extends calculus to the trigonometric functions. The sine and cosine functions are extremely important because their graphs are waves. They are therefore essential in the modelling of all the many wave-like phenomena such as sound waves, light and radio waves, vibrating strings, tides, and economic cycles. The alternating current that we use in our homes fluctuates in a sine wave. Most of the attention in this chapter is given to these two functions.

$y = \sin x$



$y = \cos x$



In the second half of Chapter 11 of the Year 11 book, we introduced radian measure, promising that it was the correct way to measure angles when doing calculus. We drew the six trigonometric graphs in radians and discussed their symmetries in some detail, and also developed area formula for calculating arc length and the areas of sectors and segments. Then in the last section of Chapter 3, we applied translations, reflections and dilations to the trigonometric graphs and developed the three ideas of amplitude, period and phase. All this previous work is required in the present chapter.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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7A The behaviour of sin x near the origin

This section proves an important limit that is the crucial step in finding the derivative of sin x in the next section. This limit establishes that the curve $y = \sin x$ has gradient 1 when it passes through the origin. Geometrically, this means that the line $y = x$ is the tangent to $y = \sin x$ at the origin.

Note: The limit established in this section provides the geometric basis for differentiating the trigonometric functions on Section 7B. The section could well be left to a second reading of the chapter at a later time.

A fundamental inequality

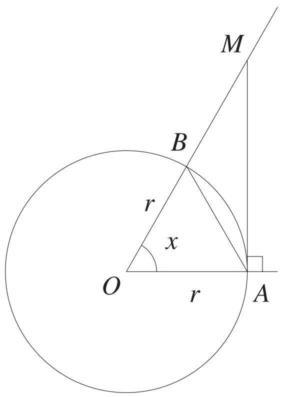
First, an appeal to geometry is needed to establish an inequality concerning x, $\sin x$ and $\tan x$.

1 AN INEQUALITY FOR sin x AND tan x NEAR THE ORIGIN

- For all acute angles x, $\sin x < x < \tan x$.
- For $-\frac{\pi}{2} < x < 0$, $\sin x > x > \tan x$.

Proof

A Let x be an acute angle.
Construct a circle of centre O and any radius r,
and a sector AOB subtending the angle x at the centre O.
Let the tangent at A meet the radius OB at M
(the radius OB will need to be produced) and join the chord AB.



In $\triangle OAM$, $\dfrac{AM}{r} = \tan x$,

so $AM = r\tan x$.

It is clear from the diagram that

$$\text{area } \triangle OAB < \text{area sector } OAB < \text{area } \triangle OAM,$$

and using area formulae for triangles and sectors,

$$\tfrac{1}{2}r^2 \sin x < \tfrac{1}{2}r^2 x < \tfrac{1}{2}r^2 \tan x$$

$$\boxed{\div \tfrac{1}{2}r^2} \qquad \sin x < x < \tan x.$$

B Because x, $\sin x$ and $\tan x$ are all odd functions,

$$\sin x > x > \tan x, \quad \text{for } -\tfrac{\pi}{2} < x < 0.$$

The main theorem

This inequality now allows two fundamental limits to be proven:

2 TWO FUNDAMENTAL LIMITS

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

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Proof

When x is acute, $\sin x < x < \tan x$.

Dividing through by $\sin x$, $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$.

As $x \to 0^+$, $\cos x \to 1$, so $\dfrac{x}{\sin x} \to 1$ as $x \to 0^+$.

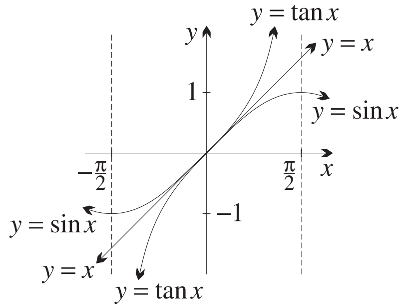
But $\dfrac{x}{\sin x}$ is even, so $\dfrac{x}{\sin x} \to 1$ as $x \to 0^-$.

Combining these two limits, $\dfrac{x}{\sin x} \to 1$ as $x \to 0$.

Finally,
$$\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$$
$$\to 1 \times 1, \text{ as } x \to 0.$$

The diagram to the right shows what has been proven about the graphs of $y = x$, $y = \sin x$ and $y = \tan x$ near the origin.

- The line $y = x$ is a common tangent at the origin to both $y = \sin x$ and $y = \tan x$.
- On both sides of the origin, $y = \sin x$ curls away from the tangent towards the x-axis.
- On both sides of the origin, $y = \tan x$ curls away from the tangent in the opposite direction.



3 THE BEHAVIOUR OF sin x AND tan x NEAR THE ORIGIN

- The line $y = x$ is a tangent to both $y = \sin x$ and $y = \tan x$ at the origin.
- When $x = 0$, the derivatives of both $\sin x$ and $\tan x$ are exactly 1.

Approximations to the trigonometric functions for small angles

For ‘small’ angles, positive or negative, the limits above yield good approximations for the three trigonometric functions (the angle must, of course, be expressed in radians).

4 SMALL-ANGLE APPROXIMATIONS

- Suppose that x is a ‘small’ angle (written in radians). Then
  $\sin x \doteqdot x$ and $\cos x \doteqdot 1$ and $\tan x \doteqdot x$.

In order to use these approximations, one needs to get some idea about how good the approximations are. Two questions in Exercise 7A below ask for tables of values for $\sin x$, $\cos x$ and $\tan x$ for progressively smaller angles.

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Example 1 7A

Use the small-angle approximations in Box 4 to give approximate values of:

a $\sin 1°$ **b** $\cos 1°$ **c** $\tan 1°$

SOLUTION

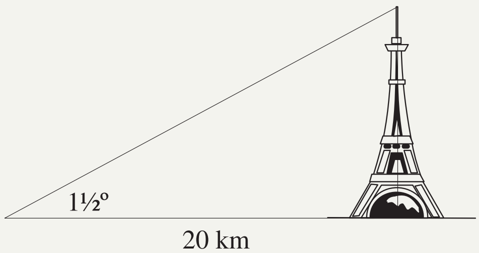
The 'small angle' of $1°$ is $\frac{\pi}{180}$ radians. Hence, using the approximations above:

a $\sin 1° \doteqdot \frac{\pi}{180}$ **b** $\cos 1° \doteqdot 1$ **c** $\tan 1° \doteqdot \frac{\pi}{180}$

Example 2 7A

Approximately how high is a tower that subtends an angle of $1\frac{1}{2}°$ when it is 20 km away?

SOLUTION



Convert 20 km to 20000 metres.

Then from the diagram, using simple trigonometry,

$$\frac{\text{height}}{20000} = \tan 1\tfrac{1}{2}°$$

$$\text{height} = 20000 \times \tan 1\tfrac{1}{2}°.$$

But the 'small' angle $1\frac{1}{2}°$ expressed in radians is $\frac{\pi}{120}$,

so $\tan 1\frac{1}{2}° \doteqdot \frac{\pi}{120}$.

Hence, approximately,
$$\text{height} \doteqdot 20000 \times \frac{\pi}{120}$$
$$\doteqdot \frac{500\pi}{3} \text{ metres}$$
$$\doteqdot 524 \text{ metres.}$$

Example 3 7A

The sun subtends an angle of $0°31'$ at the Earth, which is 150000000 km away. What is the sun's approximate diameter?

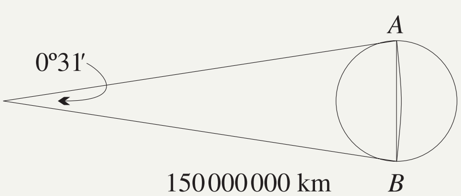
Note: This problem can be done similarly to the previous problem, but like many small-angle problems, it can also be done by approximating the diameter to an arc of the circle.

SOLUTION

First,
$$0°31' = \frac{31°}{60}$$
$$= \frac{31}{60} \times \frac{\pi}{180} \text{ radians.}$$

Because the diameter AB is approximately equal to the arc length AB,

$$\text{diameter} \doteqdot r\theta$$
$$\doteqdot 150000000 \times \frac{31}{60} \times \frac{\pi}{180}$$
$$\doteqdot 1353000 \text{ km.}$$



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Exercise 7A

FOUNDATION

1 **a** Copy and complete the following table of values, giving entries correct to six decimal places. (Your calculator must be in radian mode.)

angle size in radians	1	0.5	0.2	0.1	0.08	0.05	0.02	0.01	0.005	0.002
$\sin x$										
$\dfrac{\sin x}{x}$										
$\tan x$										
$\dfrac{\tan x}{x}$										
$\cos x$										

b What are the limits of $\dfrac{\sin x}{x}$ and $\dfrac{\tan x}{x}$ as $x \to 0$?

2 [Technology]

The previous question is perfect for a spreadsheet approach. The spreadsheet columns can be identical to the rows above. Various graphs can then be drawn using the data from the spreadsheet.

3 **a** Express $2°$ in radians.

b Explain why $\sin 2° \doteqdot \dfrac{\pi}{90}$.

c Taking π as 3.142, find $\sin 2°$, correct to four decimal places, *without* using a calculator.

4 **a** Copy and complete the following table of values, giving entries correct to four significant figures. For each column, hold x in the calculator's memory until the column is complete:

angle size in degrees	$60°$	$30°$	$10°$	$5°$	$2°$	$1°$	$20'$	$5'$	$1'$	$30''$	$10''$
angle size x in radians											
$\sin x$											
$\dfrac{\sin x}{x}$											
$\tan x$											
$\dfrac{\tan x}{x}$											
$\cos x$											

b Write x, $\sin x$ and $\tan x$ in ascending order, for acute angles x.

c Although $\sin x \to 0$ and $\tan x \to 0$ as $x \to 0$, what are the limits, as $x \to 0$, of:

i $\dfrac{\sin x}{x}$?

ii $\dfrac{\tan x}{x}$?

d Experiment with your calculator, or a spreadsheet, to find how small x must be in order for $\dfrac{\sin x}{x} > 0.999$ to be true.

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5 [Technology]

A properly prepared spreadsheet makes it easy to ask a sequence of questions like part **d** of the previous question. One can ask how small x must be for each of the following three functions to be closer to 1 than 0.1, 0.001, 0.0001, 0.00001, . . .

$$\frac{\sin x}{x} \quad \text{and} \quad \frac{\tan x}{x} \quad \text{and} \quad \cos x$$

DEVELOPMENT

6 Find the following limits:

a $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

b $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

c $\lim\limits_{x \to 0} \dfrac{\sin x}{2x}$

d $\lim\limits_{x \to 0} \dfrac{\sin 3x}{2x}$

e $\lim\limits_{x \to 0} \dfrac{5x}{\sin 3x}$

f $\lim\limits_{x \to 0} \dfrac{\sin 3x + \sin 5x}{x}$

7 A car travels 1 km up a road that is inclined at 5° to the horizontal. Through what vertical distance has the car climbed? (Use the fact that $\sin x \doteqdot x$ for small angles, and give your answer correct to the nearest metre.)

8 A tower is 30 metres high. What angle, correct to the nearest minute, does it subtend at a point 4 km away? (Use the fact that when x is small, $\tan x \doteqdot x$.)

9 [Technology]

Draw on one screen the graphs $y = \sin x$, $y = \tan x$ and $y = x$, noting how the two trigonometric graphs curl away from $y = x$ in opposite directions. Zoom in on the origin until the three graphs are indistinguishable.

10 [Technology]

Draw the graph of $y = \dfrac{\sin x}{x}$. It is undefined at the y-intercept, but the curve around this point is flat, and clearly has limit 1 as $x \to 0$. Other features of the graph can be explained, and the exercise can be repeated with the function $y = \dfrac{\tan x}{x}$.

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11 The moon subtends an angle of $31'$ at an observation point on Earth, 400 000 km away. Use the fact that the diameter of the moon is approximately equal to an arc of a circle whose centre is the point of observation to show that the diameter of the moon is approximately 3600 km. (Hint: Use a diagram like that in the last worked exercise in the notes above.)

12 A regular polygon of 300 sides is inscribed in a circle of radius 60 cm. Show that each side is approximately 1.26 cm.

13 [A better approximation for $\cos x$ when x is small]

The chord AB of a circle of radius r subtends an angle x at the centre O.

a Find AB^2 by the cosine rule, and find the length of the arc AB.

b By equating arc and chord, show that for small angles, $\cos x \doteqdot 1 - \dfrac{x^2}{2}$.

Explain whether the approximation is bigger or smaller than $\cos x$.

c Check the accuracy of the approximation for angles of $1°$, $10°$, $20°$ and $30°$.

14 [Technology]

Sketch on one screen the graphs of $y = \cos x$ and $y = 1 - \frac{1}{2}x^2$ as discussed in the previous question. Which one is larger, and why? A spreadsheet may help you to identify the size of the error for different values of x.

ENRICHMENT

15 a Write down the compound-angle formula for $\sin\,(A - B)$.

b Hence show that, for small x, $\sin\,(\theta - x) \doteqdot \sin\theta - x\cos\theta$.

c Use the result in **b** to show that $\sin\,29°57' \doteqdot \dfrac{3600 - \sqrt{3}\pi}{7200}$.

d To how many decimal places is the approximation in **c** accurate?

e Use similar methods to obtain approximations to $\sin 29°$, $\cos 31°$, $\tan 61°$, $\cot 59°$ and $\sin 46°$, checking the accuracy of your approximations using the calculator.

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7B Differentiating the trigonometric functions

Using the limit from Section 7A, we can now establish the derivatives of the three trigonometric functions $\sin x$, $\cos x$ and $\tan x$.

Standard forms

Here are the formulae, proven below, for these derivatives.

5 STANDARD DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

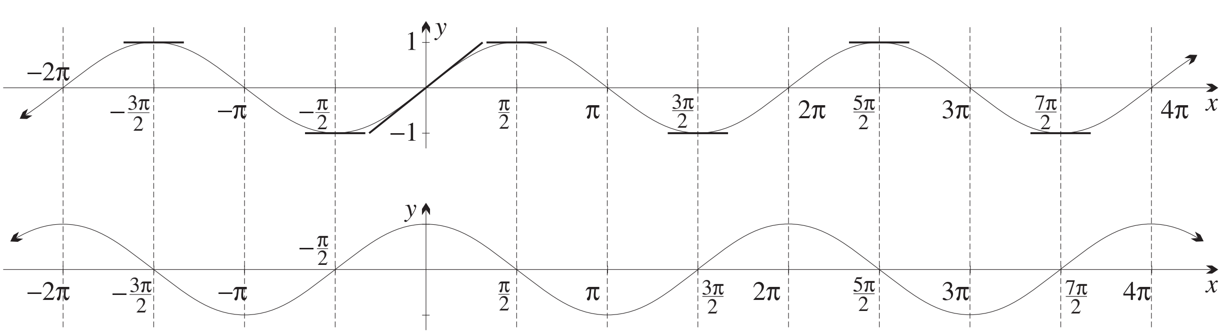
- $\dfrac{d}{dx}\sin x = \cos x$
- $\dfrac{d}{dx}\cos x = -\sin x$
- $\dfrac{d}{dx}\tan x = \sec^2 x$

The exercises ask for derivatives of the secant, cosecant and cotangent functions.

A graphical demonstration that the derivative of sin *x* is cos *x*

The upper graph in the sketch below is $y = \sin x$. The lower graph is a rough sketch of the derivative of $y = \sin x$. This second graph is straightforward to construct simply by paying attention to where the gradients of tangents to $y = \sin x$ are zero, maximum and minimum. The lower graph is periodic, with period 2π, and has a shape unmistakably like a cosine graph.

Moreover, it was proven in the previous section that the gradient of $y = \sin x$ at the origin is exactly 1. This means that the lower graph has a maximum of 1 when $x = 0$. By symmetry, all its maxima are 1 and all its minima are -1. Thus the lower graph not only has the distinctive shape of the cosine curve, but has the correct amplitude as well.



This doesn't prove conclusively that the derivative of $\sin x$ is $\cos x$, but it is very convincing.

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Proving that the derivative of sin *x* is cos *x*

Completing the proof requires the fundamental limit from Section 7A, together with a trigonometric identity from Exercise 11G of the Year 11 book. The first-principles differentiation formula is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and applying this formula to the function $f(x) = \sin x$ gives

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

We now appeal to the formula for the difference of sines, proven in Question **10** of Exercise 17G in the Year 11 book,

$$\sin P - \sin Q = 2\cos \tfrac{1}{2}(P+Q)\sin \tfrac{1}{2}(P-Q).$$

$$\begin{aligned}
\text{Then } \frac{d}{dx}(\sin x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{2\cos\frac{1}{2}(2x+h)\sin\frac{1}{2}h}{h} \\
&= \lim_{h \to 0} \left(\cos(x + \tfrac{1}{2}h) \times \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \right) \\
&= (\cos x) \times 1, \text{ because } \lim_{h \to 0} \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} = 1 \text{ as proven in Section 7A,} \\
&= \cos x.
\end{aligned}$$

The derivatives of cos *x* and tan *x*

These calculations are now straightforward:

$$\frac{d}{dx}\cos x = -\sin x \qquad \text{and} \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Proof:

A Let $y = \cos x$.

Then $y = \sin\left(\frac{\pi}{2} - x\right)$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \quad \text{(chain rule)} \\
&= -\cos\left(\tfrac{\pi}{2} - x\right) \\
&= -\sin x.
\end{aligned}$$

Let $u = \frac{\pi}{2} - x$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = -1$
and $\dfrac{dy}{du} = \cos u$.

B Let $y = \tan x$.

Then $y = \dfrac{\sin x}{\cos x}$.

$$\begin{aligned}
y' &= \frac{vu' - uv'}{v^2} \quad \text{(quotient rule)} \\
&= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}, \quad \text{because } \cos^2 x + \sin^2 x = 1, \\
&= \sec^2 x.
\end{aligned}$$

Let $u = \sin x$
and $v = \cos x$.
Then $u' = \cos x$
and $v' = -\sin x$.

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Differentiating using the three standard forms

These worked examples use the standard forms to differentiate functions involving $\sin x$, $\cos x$ and $\tan x$.

Example 4 7B

Differentiate these functions.

a $y = \sin x + \cos x$ **b** $y = x - \tan x$

Hence find the gradient of each curve when $x = \frac{\pi}{4}$.

SOLUTION

a The function is $y = \sin x + \cos x$.

Differentiating, $y' = \cos x - \sin x$.

When $x = \frac{\pi}{4}$, $y' = \cos\frac{\pi}{4} - \sin\frac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}$

$= 0$.

b The function is $y = x - \tan x$.

Differentiating, $y' = 1 - \sec^2 x$.

When $x = \frac{\pi}{4}$, $y' = 1 - \sec^2\frac{\pi}{4}$

$= 1 - \left(\sqrt{2}\right)^2$

$= -1$.

Example 5 7B

If $f(x) = \sin x$, find $f'(0)$. Hence find the equation of the tangent to $y = \sin x$ at the origin, then sketch the curve and the tangent.

SOLUTION

Here $f(x) = \sin x$,

and substituting $x = 0$, $f(0) = 0$,

so the curve passes through the origin.

Differentiating, $f'(x) = \cos x$,

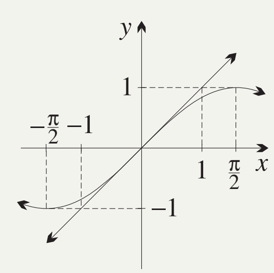
and substituting $x = 0$, $f'(0) = \cos 0$

$= 1$,

so the tangent to $y = \sin x$ at the origin has gradient 1.

Hence its equation is $y - 0 = 1(x - 0)$

$y = x$.



Note: This result was already clear from the limit $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ proven in the previous section. The simplicity of the result confirms that radian measure is the correct measure to use for angles when doing calculus.

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Using the chain rule to generate more standard forms

A simple pattern emerges when the chain rule is used to differentiate functions such as $\cos(3x + 4)$, where the angle $3x + 4$ is a linear function.

Example 6 7B

Use the chain rule to differentiate:

a $y = \cos(3x + 4)$ **b** $y = \tan(5x - 1)$ **c** $y = \sin(ax + b)$

SOLUTION

a Here $y = \cos(3x + 4)$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin(3x + 4) \times 3 = -3\sin(3x + 4).$$

Let $u = 3x + 4$.
Then $y = \cos u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = -\sin u$.

b Here $y = \tan(5x - 1)$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \sec^2(5x - 1) \times 5 = 5\sec^2(5x - 1).$$

Let $u = 5x - 1$.
Then $y = \tan u$.
Hence $\dfrac{du}{dx} = 5$
and $\dfrac{dy}{du} = \sec^2 u$.

c Here $y = \sin(ax + b)$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos(ax + b) \times a = a\cos(ax + b).$$

Let $u = ax + b$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \cos u$.

The last result in the previous worked example can be extended to the other trigonometric functions, giving the following standard forms:

6 STANDARD DERIVATIVES OF FUNCTIONS OF *ax* + *b*

- $\dfrac{d}{dx}\sin(ax + b) = a\cos(ax + b)$
- $\dfrac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$
- $\dfrac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$

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Example 7 7B

Use the extended standard forms given in Box 6 above to differentiate:

a $y = \cos 7x$ **b** $y = 4\sin\left(3x - \frac{\pi}{3}\right)$ **c** $y = \tan\frac{3}{2}x$

SOLUTION

a The function is $y = \cos 7x$, so $a = 7$ and $b = 0$,

and $$\frac{dy}{dx} = -7\sin 7x.$$

b The function is $y = 4\sin\left(3x - \frac{\pi}{3}\right)$, so $a = 3$ and $b = -\frac{\pi}{3}$,

and $$\frac{dy}{dx} = 12\cos\left(3x - \frac{\pi}{3}\right).$$

c The function is $y = \tan\frac{3}{2}x$, so $a = \frac{3}{2}$ and $b = 0$,

and $$\frac{dy}{dx} = \frac{3}{2}\sec^2\frac{3}{2}x$$

Using the chain rule with trigonometric functions

The chain rule can also be applied in the usual way to differentiate compound functions.

Example 8 7B

Use the chain rule to differentiate:

a $y = \tan^2 x$ **b** $y = \sin\left(x^2 - \frac{\pi}{4}\right)$

SOLUTION

a Here $y = \tan^2 x$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2\tan x \sec^2 x.$$

Let $u = \tan x$.
Then $y = u^2$.
Hence $\dfrac{du}{dx} = \sec^2 x$
and $\dfrac{dy}{du} = 2u$.

b Here $y = \sin\left(x^2 - \frac{\pi}{4}\right)$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2x\cos\left(x^2 - \frac{\pi}{4}\right).$$

Let $u = x^2 - \frac{\pi}{4}$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = \cos u$.

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Using the product rule with trigonometric functions

A function such as $y = e^x \cos x$ is the product of the two functions $u = e^x$ and $v = \cos x$. It can therefore be differentiated using the product rule.

Example 9 7B

Use the product rule to differentiate:

a $y = e^x \cos x$

b $y = 5 \cos 2x \cos \frac{1}{2}x$

SOLUTION

a Here $y = e^x \cos x$.

Applying the product rule,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^x \cos x - e^x \sin x$$
$$= e^x(\cos x - \sin x).$$

Let $u = e^x$
and $v = \cos x$.
Then $\frac{du}{dx} = e^x$
and $\frac{dv}{dx} = -\sin x$.

b Here $y = 5 \cos 2x \cos \frac{1}{2}x$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= -10 \sin 2x \cos \tfrac{1}{2}x - \tfrac{5}{2}\cos 2x \sin \tfrac{1}{2}x.$$

Let $u = 5 \cos 2x$
and $v = \cos \frac{1}{2}x$.
Then $u' = -10 \sin 2x$
and $v' = -\frac{1}{2} \sin \frac{1}{2}x$.

Using the quotient rule with trigonometric functions

A function such as $y = \dfrac{\sin x}{x}$ is the quotient of the two functions $u = \sin x$ and $v = x$. Thus it can be differentiated using the quotient rule.

Example 10 7B

Use the quotient rule to differentiate:

a $y = \dfrac{\sin x}{x}$

b $y = \dfrac{\cos 2x}{\cos 5x}$

SOLUTION

a Here $y = \dfrac{\sin x}{x}$.

Applying the quotient rule,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{x \cos x - \sin x}{x^2}.$$

Let $u = \sin x$
and $v = x$.
Then $\frac{du}{dx} = \cos x$
and $\frac{dv}{dx} = 1$.

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b Here $y = \dfrac{\cos 2x}{\cos 5x}$.

Let $u = \cos 2x$
and $v = \cos 5x$.
Then $u' = -2 \sin 2x$
and $v' = -5 \sin 5x$.

Applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2}$$

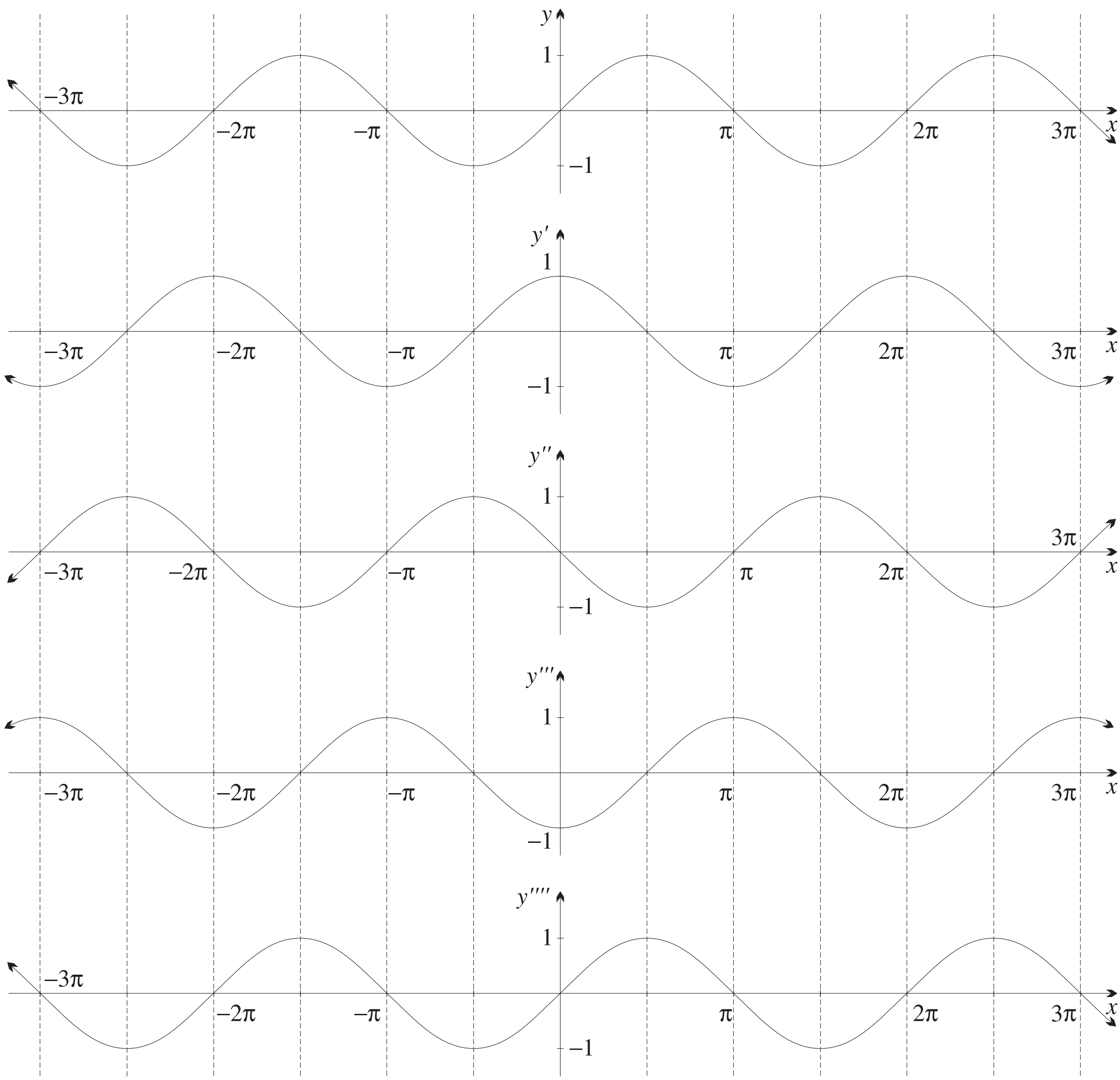
$$= \frac{-2 \sin 2x \cos 5x + 5 \cos 2x \sin 5x}{\cos^2 5x}.$$

Successive differentiation of sine and cosine

Differentiating $y = \sin x$ repeatedly,

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} = -\sin x, \qquad \frac{d^3y}{dx^3} = -\cos x, \qquad \frac{d^4y}{dx^4} = \sin x.$$

Thus differentiation is an *order 4 operation* on the sine function, meaning that when differentiation is applied four times, the original function returns. Sketched below are the graphs of $y = \sin x$ and its first four derivatives.



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Each application of differentiation shifts the wave left $\frac{\pi}{2}$, which is a quarter of the period 2π. Thus differentiation advances the phase by $\frac{\pi}{2}$, meaning that

$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \tfrac{\pi}{2}\right) \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right)$$

Double differentiation shifts the wave left π, which is a half the period 2π, and thus advances the phase by π. Double differentiation also exchanges $y = \sin x$ with its opposite function $y = -\sin x$, with each graph being the reflection of the other in the x-axis. It has similar effects on the cosine function. Thus both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'' = -y$.

Four differentiations shift the wave left 2π, which is one full period, where it coincides with itself again. Thus the differentiation transformation acting on the sine and cosine functions has *order 4*, and both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'''' = y$.

7 DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS AS PHASE SHIFT

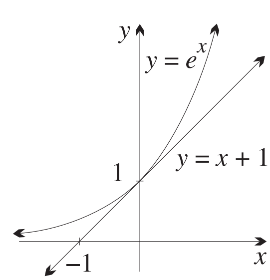
- Differentiation of $y = \sin x$ and $y = \cos x$ shifts each curve left $\frac{\pi}{2}$, advancing the phase $\frac{\pi}{2}$,
$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \tfrac{\pi}{2}\right) \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right)$$
- The second derivatives of $\sin x$ and $\cos x$ reflect each curve in the x-axis,
$$\frac{d^2}{dx^2}\sin x = -\sin x \quad \text{and} \quad \frac{d^2}{dx^2}\cos x = -\cos x$$
- Differentiation of $\sin x$ and $\cos x$ has order 4,
$$\frac{d^4}{dx^4}\sin x = \sin x \quad \text{and} \quad \frac{d^4}{dx^4}\cos x = \cos x$$

The properties of the exponential function $y = e^x$ are quite similar. The first derivative of $y = e^x$ is $y' = e^x$ and the second derivative of $y = e^{-x}$ is $y'' = e^{-x}$. This means there are now four functions whose fourth derivatives are equal to themselves:

$$y = \sin x, \qquad y = \cos x, \qquad y = e^x, \qquad y = e^{-x}.$$

This is one clue amongst many others in the course that the trigonometric functions and the exponential functions are closely related. See also Question **14(d)** in Exercise 7B.

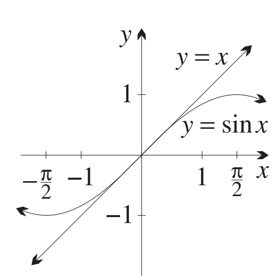
Some analogies between π and e



In the previous chapter, and in Chapter 11 of the Year 11 book, we discussed how choosing the special number e as the base of the exponential function makes the derivative of $y = e^x$ is exactly $y' = e^x$.

In particular, the tangent to $y = e^x$ at the y-intercept has gradient exactly 1.

The choice of radian measure, based on the special number π, was motivated in exactly the same way. As has just been explained, the derivative of $y = \sin x$ using radian measure is exactly $y' = \cos x$.



In particular, the tangent to $y = \sin x$ at the origin has gradient exactly 1.

Both numbers $\pi = 3.141\,592\ldots$ and $e = 2.718\,281\ldots$ are irrational. The number π is associated with the area of a circle and e is associated with areas under the rectangular hyperbola. These things are further hints of connections between trigonometric and exponential functions.

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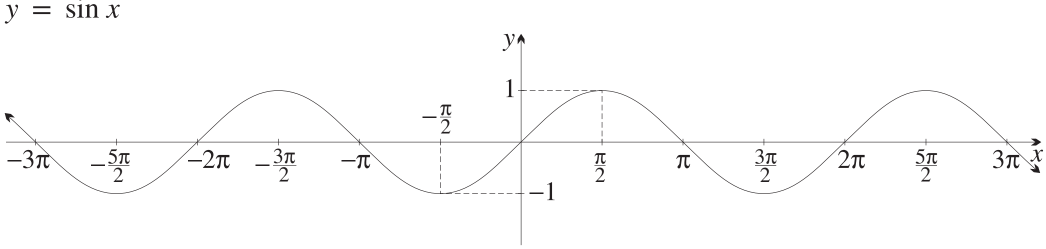
Sketches of the six trigonometric functions in radians

In Section 11J of the Year 11 book, we sketched all six trigonometric functions in radians. The six graphs are repeated on this page. Here are the key properties:

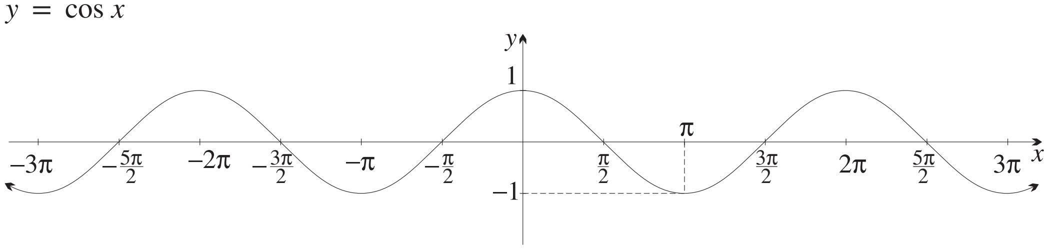
- $\sin x$ and $\cos x$ each have amplitude 1. The others do not have an amplitude.
- $\sin x$ and $\cos x$ (and their reciprocals $\sec x$ and $\operatorname{cosec} x$ each have period 2π, $\tan x$ (and its reciprocal $\cot x$) have period π.

Exercise 11J in the Year 11 book investigated in some detail the symmetry properties of these six trigonometric functions.

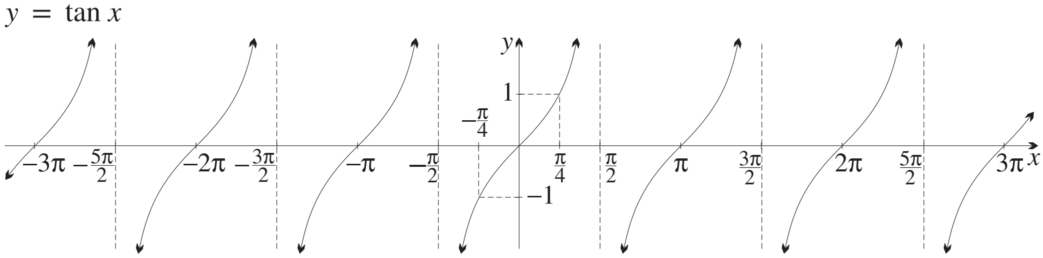
$y = \sin x$



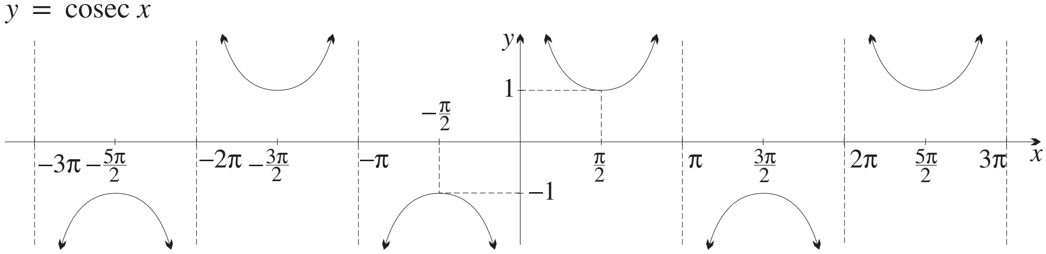
$y = \cos x$



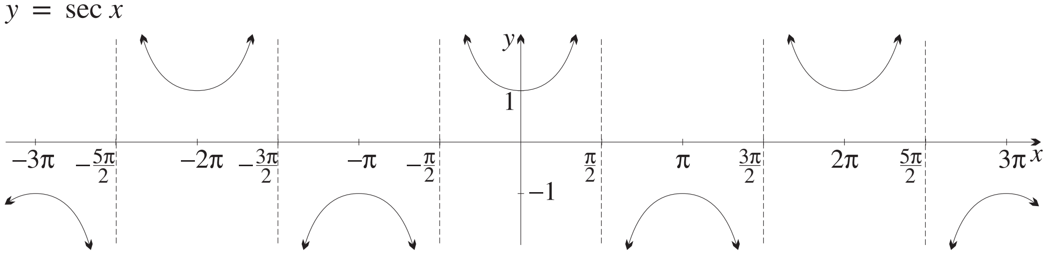
$y = \tan x$



$y = \operatorname{cosec} x$



$y = \sec x$



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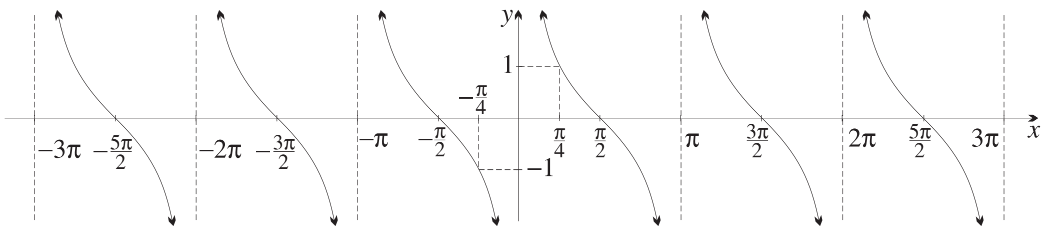
$y = \cot x$



Exercise 7B

FOUNDATION

1 Use the standard forms to differentiate with respect to x.

a $y = \sin x$ **b** $y = \cos x$ **c** $y = \tan x$
d $y = 2\sin x$ **e** $y = \sin 2x$ **f** $y = 3\cos x$
g $y = \cos 3x$ **h** $y = \tan 4x$ **i** $y = 4\tan x$
j $y = 2\sin 3x$ **k** $y = 2\tan 2x$ **l** $y = 4\cos 2x$
m $y = -\sin 2x$ **n** $y = -\cos 2x$ **o** $y = -\tan 2x$
p $y = \tan \frac{1}{2}x$ **q** $y = \cos \frac{1}{2}x$ **r** $y = \sin \frac{x}{2}$
s $y = 5\tan \frac{1}{5}x$ **t** $y = 6\cos \frac{x}{3}$ **u** $y = 12\sin \frac{x}{4}$

2 Differentiate with respect to x.

a $\sin 2\pi x$ **b** $\tan \frac{\pi}{2}x$ **c** $3\sin x + \cos 5x$
d $4\sin \pi x + 3\cos \pi x$ **e** $\sin(2x - 1)$ **f** $\tan(1 + 3x)$
g $2\cos(1 - x)$ **h** $\cos(5x + 4)$ **i** $7\sin(2 - 3x)$
j $10\tan(10 - x)$ **k** $6\sin\left(\frac{x+1}{2}\right)$ **l** $15\cos\left(\frac{2x+1}{5}\right)$

3 Find the first, second, third and fourth derivatives of:

a $y = \sin 2x$ **b** $y = \cos 10x$ **c** $y = \sin \frac{1}{2}x$ **d** $y = \cos \frac{1}{3}x$

4 If $f(x) = \cos 2x$, find $f'(x)$ and then find:

a $f'(0)$ **b** $f'\left(\frac{\pi}{12}\right)$ **c** $f'\left(\frac{\pi}{6}\right)$ **d** $f'\left(\frac{\pi}{4}\right)$

5 If $f(x) = \sin\left(\frac{1}{4}x + \frac{\pi}{2}\right)$, find $f'(x)$ and then find:

a $f'(0)$ **b** $f'(2\pi)$ **c** $f'(-\pi)$ **d** $f'(\pi)$

DEVELOPMENT

6 Find $\frac{dy}{dx}$ using the product rule.

a $y = x\sin x$ **b** $y = 2x\tan 2x$ **c** $y = x^2\cos 2x$ **d** $y = x^3\sin 3x$

7 Find $\frac{dy}{dx}$ using the quotient rule.

a $y = \dfrac{\sin x}{x}$ **b** $y = \dfrac{\cos x}{x}$ **c** $y = \dfrac{x^2}{\cos x}$ **d** $y = \dfrac{x}{1 + \sin x}$

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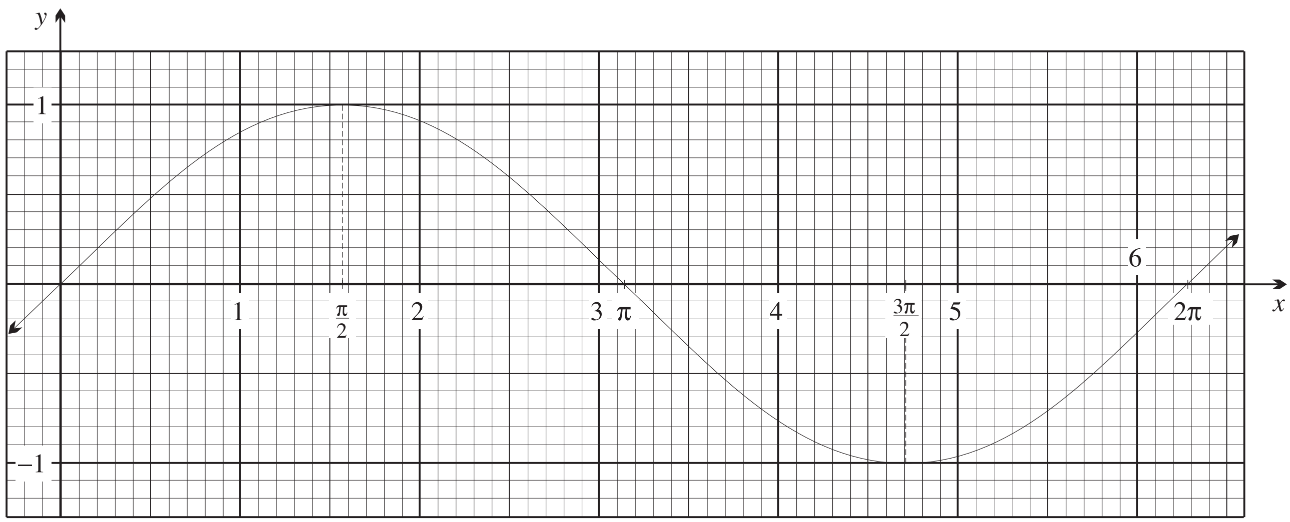
8 Find $\frac{dy}{dx}$ using the chain rule. (Hint: Remember that $\cos^2 x$ means $(\cos x)^2$.)

a $y = \sin(x^2)$ **b** $y = \sin(1 - x^2)$ **c** $y = \cos(x^3 + 1)$

d $y = \sin \frac{1}{x}$ **e** $y = \cos^2 x$ **f** $y = \sin^3 x$

g $y = \tan^2 x$ **h** $y = \tan\sqrt{x}$

9



a Photocopy the sketch above of $f(x) = \sin x$. Carefully draw tangents at the points where $x = 0, 0.5, 1, 1.5, \ldots, 3$, and also at $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

b Measure the gradient of each tangent correct to two decimal places, and copy and complete the following table.

x	0	0.5	1	1.5	$\frac{\pi}{2}$	2	2.5	3	π	3.5	4	4.5	$\frac{3\pi}{2}$	5	5.5	6	2π
$f'(x)$																	

c Use these values to plot the graph of $y = f'(x)$.

d What is the equation of this graph?

10 [Technology]

Most graphing programs can graph the derivative of a function. Start with $y = \sin x$, as in the previous question, then graph y', y'', y''' and y'''', and compare your results with the graphs printed in the theory introducing this exercise.

11 Differentiate:

a $f(x) = e^{\tan x}$ **b** $f(x) = e^{\sin 2x}$ **c** $f(x) = \sin\left(e^{2x}\right)$

d $f(x) = \log_e(\cos x)$ **e** $f(x) = \log_e(\sin x)$ **f** $f(x) = \log_e(\cos 4x)$

12 Differentiate these functions.

a $y = \sin x \cos x$ **b** $y = \sin^2 7x$ **c** $y = \cos^5 3x$

d $y = (1 - \cos 3x)^3$ **e** $y = \sin 2x \sin 4x$ **f** $y = \tan^3(5x - 4)$

13 Find $f'(x)$, given that:

a $f(x) = \dfrac{1}{1 + \sin x}$ **b** $f(x) = \dfrac{\sin x}{1 + \cos x}$

c $f(x) = \dfrac{1 - \sin x}{\cos x}$ **d** $f(x) = \dfrac{\cos x}{\cos x + \sin x}$

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14 a Sketch $y = \cos x$, for $-3\pi \le x \le 3\pi$.

b Find y', y'', y''' and y'''', and sketch them underneath the first graph.

c What geometrical interpretations can be given of the facts that:

i $y'' = -y$? **ii** $y'''' = y$?

15 [Technology]

The previous question is well suited to a graphing program, and the results should be compared with those of successive differentiation of $\sin x$.

16 a If $y = e^x \sin x$, find y' and y'', and show that $y'' - 2y' + 2y = 0$.

b If $y = e^{-x} \cos x$, find y' and y'', and show that $y'' + 2y' + 2y = 0$.

17 Consider the function $y = \frac{1}{3}\tan^3 x - \tan x + x$.

a Show that $\dfrac{dy}{dx} = \tan^2 x \sec^2 x - \sec^2 x + 1$.

b Hence use the identity $\sec^2 x = 1 + \tan^2 x$ to show that $\dfrac{dy}{dx} = \tan^4 x$.

18 a Copy and complete: $\log_b\left(\frac{P}{Q}\right) = \ldots.$

b If $f(x) = \log_e\left(\dfrac{1 + \sin x}{\cos x}\right)$, show that $f'(x) = \sec x$.

19 a By writing $\sec x$ as $(\cos x)^{-1}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

b Similarly, show that $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

c Similarly, show that $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

20 Show that $\dfrac{d}{dx}\left(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right) = \sin^4 x \cos^3 x$.

ENRICHMENT

21 a If $y = \sin x$, prove:

i $\dfrac{dy}{dx} = \sin\left(\frac{\pi}{2} + x\right)$ **ii** $\dfrac{d^2y}{dx^2} = \sin(\pi + x)$ **iii** $\dfrac{d^3y}{dx^3} = \sin\left(\frac{3\pi}{2} + x\right)$

b Deduce an expression for $\dfrac{d^ny}{dx^n}$.

22 a Show that $\frac{1}{2}(\sin(m + n)x + \sin(m - n)x) = \sin mx \cos nx$.

b Hence, without using the product rule, differentiate $\sin mx \cos nx$.

c Simplify $\frac{1}{2}(\cos(m + n)x + \cos(m - n)x)$, and hence differentiate $\cos mx \cos nx$.

23 Show that the function $y = e^{-x}(\cos 2x + \sin 2x)$ is a solution of the differential equation $y'' + 2y' + 5y = 0$.

24 a If $y = \ln(\tan 2x)$, show that $\dfrac{dy}{dx} = 2\sec 2x \operatorname{cosec} 2x$.

b If $y = \ln\left(\dfrac{\sqrt{2} - \cos x}{\sqrt{2} + \cos x}\right)$, show that $\dfrac{dy}{dx} = \dfrac{2\sqrt{2}\sin x}{1 + \sin^2 x}$.

25 At the start of this section, we differentiated $\sin x$ by first principles. Using that working as a guide, differentiate $\cos x$ by first principles. You will need this sums-to-products identity from Question **10** of Exercise 17G in the Year 11 book,

$$\cos P - \cos Q = -2\sin\tfrac{1}{2}(P + Q)\sin\tfrac{1}{2}(P - Q).$$

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7C Applications of differentiation

Differentiation of the trigonometric functions can be applied in the usual way to the analysis of a number of functions that are very significant in the practical application of calculus. It can also be used to solve *optimisation problems* meaning problems about maxima and minima).

Tangents and normals

As always, the derivative is used to find the gradients of the relevant tangents, then point–gradient form is used to find their equations.

Example 11 7C

Find the equation of the tangent to $y = 2\sin x$ at the point P where $x = \frac{\pi}{6}$.

SOLUTION

When $x = \frac{\pi}{6}$, $y = 2\sin\frac{\pi}{6}$

$= 1$ $\left(\text{because } \sin\frac{\pi}{6} = \frac{1}{2}\right)$,

so the point P has coordinates $\left(\frac{\pi}{6}, 1\right)$.

Differentiating, $\dfrac{dy}{dx} = 2\cos x$.

When $x = \frac{\pi}{6}$, $\dfrac{dy}{dx} = 2\cos\frac{\pi}{6}$

$= \sqrt{3}$ $\left(\text{because } \cos\frac{\pi}{6} = \frac{1}{2}\sqrt{3}\right)$,

so the tangent at $P\left(\frac{\pi}{6}, 1\right)$ has gradient $\sqrt{3}$.

Hence its equation is $y - y_1 = m(x - x_1)$ (point−gradient form)

$$y - 1 = \sqrt{3}\left(x - \tfrac{\pi}{6}\right)$$

$$y = x\sqrt{3} + 1 - \tfrac{\pi}{6}\sqrt{3}.$$

Example 12 7C

a Find the equations of the tangents and normals to the curve $y = \cos x$ at $A\left(-\frac{\pi}{2}, 0\right)$ and $B\left(\frac{\pi}{2}, 0\right)$.

b Show that the four lines form a square, sketch, and find the other two vertices.

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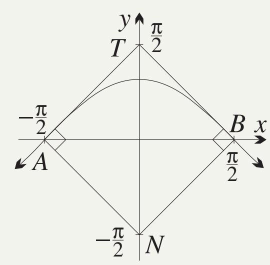
SOLUTION

a The function is $y = \cos x$,
and the derivative is $y' = -\sin x$.

Hence gradient of tangent at $A\left(-\frac{\pi}{2}, 0\right) = -\sin\left(-\frac{\pi}{2}\right)$
$= 1$,
and gradient of normal at $A\left(-\frac{\pi}{2}, 0\right) = -1$.

Similarly, gradient of tangent at $B\left(\frac{\pi}{2}, 0\right) = -\sin \frac{\pi}{2}$
$= -1$,
and gradient of normal at $B\left(\frac{\pi}{2}, 0\right) = 1$.

Hence the tangent at A is $y - 0 = 1 \times \left(x + \frac{\pi}{2}\right)$
$y = x + \frac{\pi}{2}$,
and the normal at A is $y - 0 = -1 \times \left(x + \frac{\pi}{2}\right)$
$y = -x - \frac{\pi}{2}$.
Similarly, the tangent at B is $y - 0 = -1 \times \left(x - \frac{\pi}{2}\right)$
$y = -x + \frac{\pi}{2}$,
and the normal at B is $y - 0 = 1 \times \left(x - \frac{\pi}{2}\right)$
$y = x - \frac{\pi}{2}$.



b Hence the two tangents meet on the y-axis at $T\left(0, \frac{\pi}{2}\right)$, and the two normals meet on the y-axis at $N\left(0, -\frac{\pi}{2}\right)$. Because adjacent sides are perpendicular, $ANBT$ is a rectangle, and because the diagonals are perpendicular, it is also a rhombus, so the quadrilateral $ANBT$ is a square.

Example 13 7C

a Find the equation of the tangent to $y = \tan 2x$ at the point on the curve where $x = \frac{\pi}{8}$.
b Find the x-intercept and y-intercept of this tangent.
c Sketch the situation.
d Find the area of the triangle formed by this tangent and the coordinate axes.

SOLUTION

a The function is $y = \tan 2x$,
and differentiating, $y' = 2\sec^2 2x$.
When $x = \frac{\pi}{8}$, $y = \tan \frac{\pi}{4}$
$= 1$
and $y' = 2\sec^2 \frac{\pi}{4}$
$= 2 \times \left(\sqrt{2}\right)^2$
$= 4$,
so the tangent is $y - 1 = 4\left(x - \frac{\pi}{8}\right)$
$y = 4x - \frac{\pi}{2} + 1$.

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b When $x = 0$, $y = 1 - \frac{\pi}{2}$

$= \dfrac{2 - \pi}{2}$,

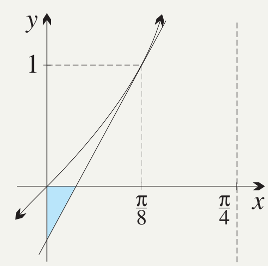
and when $y = 0$, $0 = 4x - \frac{\pi}{2} + 1$

$4x = \frac{\pi}{2} - 1$

$4x = \dfrac{\pi - 2}{2}$

$\boxed{\div 4}$ $x = \dfrac{\pi - 2}{8}$.

c The sketch is drawn opposite.



d Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times \dfrac{\pi - 2}{2} \times \dfrac{\pi - 2}{8}$

$= \dfrac{(\pi - 2)^2}{32}$ square units.

Curve sketching

Curve-sketching problems involving trigonometric functions can be long, with difficult details. Nevertheless, the usual steps of the 'curve-sketching menu' still apply and the working of each step is done exactly the same as usual.

Sketching these curves using either a computer package or a graphics calculator would greatly aid understanding of the relationships between the equations of the curves and their graphs.

Note: With trigonometric functions, it is often easier to determine the nature of stationary points from an examination of the second derivative than from a table of values of the first derivative.

Example 14 7C

Consider the curve $y = \sin x + \cos x$ in the interval $0 \le x \le 2\pi$.

a Find the values of the function at the endpoints of the domain.

b Find the x-intercepts of the graph.

c Find any stationary points and determine their nature.

d Find any points of inflection and sketch the curve.

SOLUTION

a When $x = 0$, $y = \sin 0 + \cos 0 = 1$,
and when $x = 2\pi$, $y = \sin 2\pi + \cos 2\pi = 1$.

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b To find the x-intercepts, put $y = 0$.

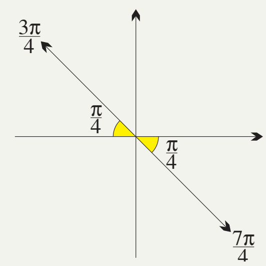
Then $\sin x + \cos x = 0$

$$\sin x = -\cos x$$

$$\tan x = -1 \quad \text{(dividing through by } \cos x\text{)}.$$

Hence x is in quadrant 2 or 4, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.



c Differentiating, $\quad y' = \cos x - \sin x$,

so y' has zeroes when $\sin x = \cos x$,

that is, $\quad \tan x = 1 \quad$ (dividing through by $\cos x$).

Hence x is in quadrant 1 or 3, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

When $x = \frac{\pi}{4}$, $\quad y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$$= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= \sqrt{2},$$

and when $x = \frac{5\pi}{4}$, $\quad y = -\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}$

$$= -\sqrt{2}.$$

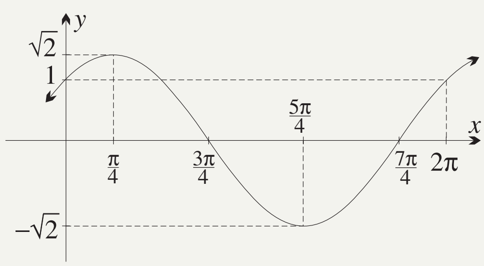
Differentiating again, $\quad y'' = -\sin x - \cos x$,

so when $x = \frac{\pi}{4}$, $\quad y'' = -\sqrt{2}$,

and when $x = \frac{5\pi}{4}$, $\quad y'' = \sqrt{2}$.

Hence $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is a maximum turning point,

and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a minimum turning point.



d The second derivative y'' has zeroes when $-\sin x - \cos x = 0$,

that is, at the zeroes of y, which are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

x	0	$\frac{3\pi}{4}$	π	$\frac{7\pi}{4}$	2π
y''	-1	0	1	0	-1
	$\frown$	$\cdot$	$\smile$	$\cdot$	$\frown$

Hence the x-intercepts $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$ are also inflections.

Note: The final graph is simply a wave with the same period 2π as $\sin x$ and $\cos x$, but with amplitude $\sqrt{2}$. It is actually $y = \sqrt{2}\cos x$ shifted right by $\frac{\pi}{4}$. Any function of the form $y = a\sin x + b\cos x$ has a similar graph.

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Example 15 7C

[A harder example]

Sketch the graph of $f(x) = x - \sin x$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$.

e Find any stationary points and examine their nature.

f Find any points of inflection.

Note: This function is essentially the function describing the area of a segment, if the radius in the formula $A = \frac{1}{2}r^2(x - \sin x)$ is held constant while the angle x at the centre varies.

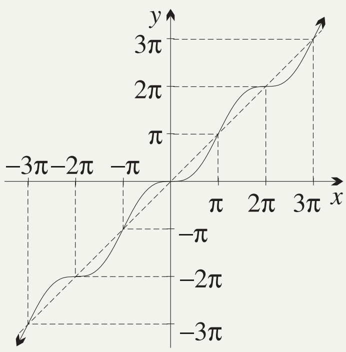
SOLUTION

a The domain of $f(x) = x - \sin x$ is the set of all real numbers.

b $f(x)$ is odd, because both $\sin x$ and x are odd.

c The function is zero at $x = 0$ and nowhere else,
because $\sin x < x$, for $x > 0$,
and $\sin x > x$, for $x < 0$.

d The value of $\sin x$ always remains between -1 and 1,
so for $f(x) = x - \sin x$, $f(x) \to \infty$ as $x \to \infty$,
and $f(x) \to -\infty$ as $x \to -\infty$.



e Differentiating, $f'(x) = 1 - \cos x$,
so $f'(x)$ has zeroes whenever $\cos x = 1$,
that is, for $x = \dots, -2\pi, 0, 2\pi, 4\pi, \dots$.
But $f'(x) = 1 - \cos x$ is never negative, because $\cos x$ is never greater than 1,
thus the curve $f(x)$ is always increasing except at its stationary points.
Hence each stationary point is a stationary inflection,
and these points are $\dots, (-2\pi, -2\pi), (0, 0), (2\pi, 2\pi), (4\pi, 4\pi), \dots$.

f Differentiating again, $f''(x) = \sin x$,
which is zero for $x = \dots, -\pi, 0, \pi, 2\pi, 3\pi, \dots$.
We know that $\sin x$ changes sign around each of these points,
so $\dots, (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi), \dots$ are also inflections.
Because $f'(\pi) = 1 - (-1) = 2$, the gradient at these other inflections is 2.

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Exercise 7C

FOUNDATION

Technology: The large number of sketches in this exercise should allow many of the graphs to be drawn first on a computer. Such sketching should be followed by an algebraic explanation of the features.

Many graphing packages allow tangents and normals to be drawn at specific points so that diagrams can be drawn of the earlier questions in the exercise.

1 Find the gradient of the tangent to each of the following curves at the point indicated.

a $y = \sin x$ at $x = 0$ **b** $y = \cos x$ at $x = \frac{\pi}{2}$ **c** $y = \sin x$ at $x = \frac{\pi}{3}$

d $y = \cos x$ at $x = \frac{\pi}{6}$ **e** $y = \sin x$ at $x = \frac{\pi}{4}$ **f** $y = \tan x$ at $x = 0$

g $y = \tan x$ at $x = \frac{\pi}{4}$ **h** $y = \cos 2x$ at $x = \frac{\pi}{4}$ **i** $y = -\cos \frac{1}{2}x$ at $x = \frac{2\pi}{3}$

j $y = \sin \frac{x}{2}$ at $x = \frac{2\pi}{3}$ **k** $y = \tan 2x$ at $x = \frac{\pi}{6}$ **l** $y = \sin 2x$ at $x = \frac{\pi}{12}$

2 a Show that the line $y = x$ is the tangent to the curve $y = \sin x$ at $(0, 0)$.

b Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.

c Show that the line $y = \frac{\pi}{2} - x$ is the tangent to the curve $y = \cos x$ at $\left(\frac{\pi}{2}, 0\right)$.

3 Find the equation of the tangent at the given point on each of the following curves.

a $y = \sin x$ at $(\pi, 0)$ **b** $y = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$

c $y = \cos x$ at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ **d** $y = \cos 2x$ at $\left(\frac{\pi}{4}, 0\right)$

e $y = \sin 2x$ at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **f** $y = x \sin x$ at $(\pi, 0)$

4 Find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on each of the following curves where the gradient of the tangent is zero.

a $y = 2 \sin x$ **b** $y = 2 \sin x - x$

c $y = 2 \cos x + x$ **d** $y = 2 \sin x + \sqrt{3}x$

5 The point $P\left(\frac{\pi}{6}, \frac{1}{2}\right)$ lies on the curve $y = 2 \sin x - \cos 2x$.

a Show that the tangent at P has equation $2\sqrt{3}x - y = \frac{1}{3}\pi\sqrt{3} - \frac{1}{2}$.

b Show that the normal at P has equation $x + 2\sqrt{3}y = \frac{\pi}{6} + \sqrt{3}$.

6 a Show that $y = \sin^2 x$ has derivative $y' = 2 \sin x \cos x$.

b Find the gradients of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

c Find the equations of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

d Suppose that the tangent meets the x-axis at P, the normal meets the y-axis at Q and O is the origin. Show that $\triangle OPQ$ has area $\frac{1}{32}(\pi^2 - 4)$ units^2.

7 a Differentiate $y = e^{\sin x}$.

b Hence find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on the curve $y = e^{\sin x}$ where the tangent is horizontal.

8 a Differentiate $y = e^{\cos x}$.

b Hence find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on the curve $y = e^{\cos x}$ where the tangent is horizontal.

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DEVELOPMENT

9 **a** Find the first and second derivatives of $y = \cos x + \sqrt{3}\sin x$.

b Find the stationary points in the domain $0 \le x \le 2\pi$, and use the second derivative to determine their nature.

c Find the points of inflection.

d Hence sketch the curve, for $0 \le x \le 2\pi$.

10 **a** Find the derivative of $y = x + \sin x$, and show that $y'' = -\sin x$.

b Find the stationary points in the domain $-2\pi < x < 2\pi$, and determine their nature.

c Find the points of inflection.

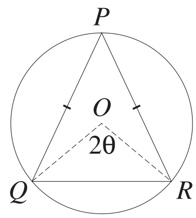
d Hence sketch the curve, for $-2\pi \le x \le 2\pi$.

11 Find any stationary points and inflections of the curve $y = 2\sin x + x$ in the interval $0 \le x \le 2\pi$, then sketch the curve.

12 An isosceles triangle has equal sides of length 10 cm. The angle θ between these equal sides is increasing at the rate of 3° per minute. Show that the area of the triangle is increasing at $\frac{5\sqrt{3}\pi}{12}$ cm^2 per minute at the instant when $\theta = 30°$.

13 A rotating light L is situated at sea 180 metres from the nearest point P on a straight shoreline. The light rotates through one revolution every 10 seconds. Show that the rate at which a ray of light moves along the shore at a point 300 metres from P is 136π m/s.

14 An isosceles triangle PQR is inscribed in a circle with centre O of radius 1 unit, as shown in the diagram to the right. Let $\angle QOR = 2\theta$, where θ is acute.



a Join PO and extend it to meet QR at M. Then prove that $QM = \sin\theta$ and $OM = \cos\theta$.

b Show that the area A of $\triangle PQR$ is $A = \sin\theta(\cos\theta + 1)$.

c Hence show that, as θ varies, $\triangle PQR$ has its maximum possible area when it is equilateral.

15 **a** Show that $\dfrac{d}{d\theta}\left(\dfrac{2 - \sin\theta}{\cos\theta}\right) = \dfrac{2\sin\theta - 1}{\cos^2\theta}$.

b Hence find the maximum and minimum values of the expression $\dfrac{2 - \sin\theta}{\cos\theta}$ in the interval $0 \le \theta \le \frac{\pi}{4}$, and state the values of θ for which they occur.

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16 **a** Find the first and second derivatives of $y = 2\sin x + \cos 2x$.

b Show that $y' = 0$ when $\cos x = 0$ or $\sin x = \frac{1}{2}$. (You will need to use the formula $\sin 2x = 2\sin x \cos x$.)

c Hence find the stationary points in the interval $-\pi \le x \le \pi$ and determine their nature.

d Sketch the curve for $-\pi \le x \le \pi$ using this information.

17 **a** Find the first and second derivatives of $y = e^{-x}\cos x$. (Note that this function models damped oscillations.)

b Find the stationary points for $-\pi \le x \le \pi$ and determine their nature.

c Find the points of inflection for $-\pi \le x \le \pi$.

d Hence sketch the curve for $-\pi \le x \le \pi$.

ENRICHMENT

18 A straight line passes through the point $(2, 1)$ and has positive x- and y-intercepts at P and Q respectively. Suppose $\angle OPQ = \alpha$, where O is the origin.

a Explain why the line has gradient $-\tan \alpha$.

b Find the x- and y-intercepts in terms of α.

c Show that the area of $\triangle OPQ$ is given by $A = \dfrac{(2\tan \alpha + 1)^2}{2\tan \alpha}$.

d Hence show that this area is maximised when $\tan \alpha = \frac{1}{2}$.

19 **a** Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.

b Using a diagram, explain why $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

c Let $f(x) = \dfrac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$. Find $f'(x)$ and show that $f'(x) < 0$ in the given domain.

d Sketch the graph of $f(x)$ over the given domain, and hence explain why $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$.

20 Find the stationary points and hence sketch, for $0 \le x \le 2\pi$:

a $y = \sin^2 x + \cos x$ **b** $y = \sin^3 x \cos x$ **c** $y = \tan^2 x - 2\tan x$

21 Let $f(x) = \dfrac{\sin x}{x}$. Remember that we proved in Section 7A that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

a Write down the domain of $f(x)$, show that $f(x)$ is even, find the zeroes of $f(x)$, and determine $\lim\limits_{x \to \infty} f(x)$.

b Differentiate $f(x)$, and hence show that $f(x)$ has stationary points when $\tan x = x$.

c Sketch $y = \tan x$ and $y = x$ on one set of axes, and hence use your calculator to estimate the turning points for $0 \le x \le 4\pi$. Give the x-coordinates in the form $\lambda\pi$, with λ to no more than two decimal places.

d Using this information, sketch $y = f(x)$.

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7D Integrating the trigonometric functions

As always, the standard forms for differentiation can be reversed to give standard forms for integration.

The standard forms for integrating the trigonometric functions

When the standard forms for differentiating $\sin x$, $\cos x$ and $\tan x$ are reversed, they give three new standard integrals.

First, $\frac{d}{dx}\sin x = \cos x$, and reversing this, $\int \cos x\,dx = \sin x$.

Secondly, $\frac{d}{dx}\cos x = -\sin x$, and reversing this, $\int (-\sin x)\,dx = \cos x$

$\boxed{\times\ (-1)}$ $\int \sin x\,dx = -\cos x$.

Thirdly, $\frac{d}{dx}\tan x = \sec^2 x$, and reversing this, $\int \sec^2 x\,dx = \tan x$.

This gives three new standard integrals. These three standard forms should be carefully memorised — pay attention to the signs in the first two standard forms.

8 STANDARD TRIGONOMETRIC INTEGRALS

- $\int \cos x\,dx = \sin x + C$, for some constant C
- $\int \sin x\,dx = -\cos x + C$, for some constant C
- $\int \sec^2 x\,dx = \tan x + C$, for some constant C

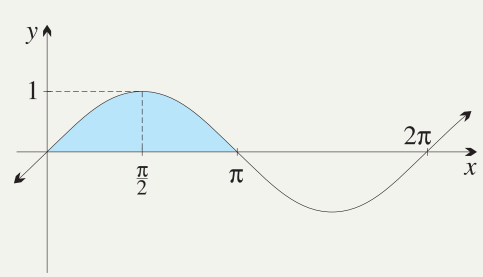
No calculation involving a primitive may cross an asymptote.

Example 16 7D

The curve $y = \sin x$ is sketched below. Show that the first arch of the curve, as shaded in the diagram, has area 2 square units.

SOLUTION

Because the region is entirely above the x-axis,



$$
\begin{aligned}
\text{area} &= \int_0^{\pi} \sin x\,dx \\
&= \Big[-\cos x\Big]_0^{\pi} \\
&= -\cos \pi + \cos 0 \\
&= -(-1) + 1
\end{aligned}
$$

(the graph of $y = \cos x$ shows that $\cos \pi = -1$)

$= 2$ square units.

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Note: This simple answer confirms again that radians are the right units to use for calculus with trigonometric functions. Similar simple results were obtained earlier when e was used as the base for powers. For example, Question 39c of the Chapter 6 Review gathered together three remarkably simple results:

$$\int_1^e \frac{1}{x}\,dx = \int_1^e \log_e x\,dx = \int_0^1 xe^x\,dx = 1$$

Example 17 — 7D

Evaluate these definite integrals.

a $\displaystyle\int_0^{\pi} \cos x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ **c** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\pi} \cos x\,dx &= \Big[\sin x\Big]_0^{\pi}\\ &= \sin\pi - \sin 0\\ &= 0\end{aligned}$$

(Use the graph of $y = \sin x$ to see that $\sin\pi = 0$ and $\sin 0 = 0$.)

b
$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec^2 x\,dx &= \Big[\tan x\Big]_0^{\frac{\pi}{3}}\\ &= \tan\tfrac{\pi}{3} - \tan 0\\ &= \sqrt{3}\end{aligned}$$

(Here $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan 0 = 0$.)

c This integral is meaningless because it crosses the asymptote at $x = \frac{\pi}{2}$.

Replacing x by $ax + b$

Reversing the standard forms for derivatives in Section 7B gives a further set of standard forms. Again, the constants of integration have been ignored until the boxed statement of the standard forms.

First, $$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b),$$

so $$\int a\cos(ax+b)\,dx = \sin(ax+b)$$

and dividing by a, $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b).$$

Secondly, $$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b),$$

so $$\int -a\sin(ax+b)\,dx = \cos(ax+b)$$

and dividing by $-a$, $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b).$$

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Thirdly, $\dfrac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$,

so $\displaystyle\int a\sec^2(ax+b)\,dx = \tan(ax+b)$.

and dividing by a, $\displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$.

The result is extended forms of the three standard integrals. These extended standard forms should also be carefully memorised.

9 STANDARD INTEGRALS FOR FUNCTIONS OF $ax + b$

- $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$, for some constant C
- $\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$, for some constant C
- $\displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$, for some constant C

Example 18 7D

Evaluate these definite integrals.

a $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x\,dx$ **b** $\displaystyle\int_\pi^{2\pi} \sin \tfrac{1}{4}x\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos 3x\,dx &= \tfrac{1}{3}\Big[\sin 3x\Big]_0^{\frac{\pi}{6}}\\ &= \tfrac{1}{3}\left(\sin\tfrac{\pi}{2} - 3\sin 0\right)\\ &= \tfrac{1}{3}\end{aligned}$$

b
$$\begin{aligned}\int_\pi^{2\pi} \sin\tfrac{1}{4}x\,dx &= -4\Big[\cos\tfrac{1}{4}x\Big]_\pi^{2\pi} \qquad \text{(because the reciprocal of } \tfrac{1}{4} \text{ is 4)}\\ &= -4\cos\tfrac{\pi}{2} + 4\cos\tfrac{\pi}{4}\\ &= 0 + 4\times\frac{\sqrt{2}}{2}\\ &= 2\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx &= \tfrac{1}{2}\Big[\tan(2x+\pi)\Big]_0^{\frac{\pi}{8}}\\ &= \tfrac{1}{2}\left(\tan\tfrac{5\pi}{4} - \tan\pi\right)\\ &= \tfrac{1}{2}(1-0) \qquad \left(\tfrac{5\pi}{4} \text{ is in quadrant 3 with related angle } \tfrac{\pi}{4}\right)\\ &= \tfrac{1}{2}\end{aligned}$$

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The primitives of tan x and cot x

The primitives of tan x and cot x can be found by using the ratio formulae $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$ and then applying the standard form from the previous chapter, in either one of its two versions,

$$\int \frac{u'}{u}\,dx = \log_e |u| + C \qquad \text{or} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C.$$

Example 19 — 7D

Find primitives of these functions.

a $\cot x$ **b** $\tan x$

SOLUTION

a
$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log_e |\sin x| + C.$$

Let $u = \sin x$.
Then $u' = \cos x$.
$$\int \frac{u'}{u}\,dx = \log_e |u|$$

OR

Let $f(x) = \sin x$.
Then $f'(x) = \cos x$.
$$\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$$

b
$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$$

Let $u = \cos x$.
Then $u' = -\sin x$.
$$\int \frac{u'}{u}\,dx = \log_e |u|$$

OR

Let $f(x) = \cos x$.
Then $f'(x) = -\sin x$.
$$\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$$

$= -\log_e |\cos x| + C.$ (This can also be written as $\log_e |\sec x|$.)

Note: Do not run across a zero of $\sin x$ when using part **a**, or a zero of $\cos x$ when using part **b**.

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Finding a function whose derivative is known

If the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be found.

Example 20 7D

The derivative of a certain function is $y' = \cos x$, and the graph of the function has y-intercept $(0, 3)$. Find the original function $f(x)$ and then find $f\left(\frac{\pi}{2}\right)$.

SOLUTION

Here $\quad y' = \cos x,$

and taking the primitive, $\quad y = \sin x + C, \quad$ for some constant C.

When $x = 0$, $y = 3$, so substituting $x = 0$,

$$3 = \sin 0 + C$$
$$C = 3.$$

Hence $\quad y = \sin x + 3.$

When $x = \frac{\pi}{2}$, $\quad y = \sin \frac{\pi}{2} + 3$

$\quad = 4, \quad$ because $\sin \frac{\pi}{2} = 1$.

Example 21 7D

Given that $f'(x) = \sin 2x$ and $f(\pi) = 1$:

a find the function $f(x)$,

b find $f\left(\frac{\pi}{4}\right)$.

SOLUTION

a Here $\quad f'(x) = \sin 2x,$

and taking the primitive, $\quad f(x) = -\frac{1}{2}\cos 2x + C, \quad$ for some constant C.

It is known that $f(\pi) = 1$, so substituting $x = \pi$,

$$1 = -\tfrac{1}{2}\cos 2\pi + C$$
$$1 = -\tfrac{1}{2} \times 1 + C$$
$$C = 1\tfrac{1}{2}.$$

Hence $\quad f(x) = -\frac{1}{2}\cos 2x + 1\frac{1}{2}.$

b Substituting $x = \frac{\pi}{4}$, $\quad f\left(\frac{\pi}{4}\right) = -\frac{1}{2} \times \cos \frac{\pi}{2} + 1\frac{1}{2}$

$\quad = 1\frac{1}{2}, \quad$ because $\cos \frac{\pi}{2} = 0$.

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Given a chain-rule derivative, find an integral

As always, the results of a chain-rule differentiation can be reversed to give a primitive.

Example 22 7D

a Use the chain rule to differentiate $\cos^5 x$.

b Hence find $\displaystyle\int_0^{\pi} \sin x \cos^4 x \, dx$.

SOLUTION

a Let $y = \cos^5 x$.

By the chain rule, $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -5 \sin x \cos^4 x.$$

Let $u = \cos x$.
Then $y = u^5$.
Hence $\dfrac{du}{dx} = -\sin x$
and $\dfrac{dy}{du} = 5u^4$.

b From part **a**, $\dfrac{d}{dx}(\cos^5 x) = -5 \sin x \cos^4 x$.

Reversing this, $\displaystyle\int (-5 \sin x \cos^4 x)\, dx = \cos^5 x$.

$\boxed{\div (-5)}$ $\displaystyle\int \sin x \cos^4 x \, dx = -\tfrac{1}{5} \cos^5 x$.

Hence
$$\begin{aligned}\int_0^{\pi} \sin x \cos^4 x \, dx &= -\tfrac{1}{5}\Big[\cos^5 x\Big]_0^{\pi} \\ &= -\tfrac{1}{5}(-1 - 1) \\ &= \tfrac{2}{5}.\end{aligned}$$

Using a formula for the reverse chain rule

The integral in the worked example above could have been done using the reverse chain rule for powers of u or $f(x)$.

Example 23 7D

Use the reverse chain rule to find $\displaystyle\int \sin x \cos^4 x \, dx$.

SOLUTION

$$\begin{aligned}&\int_0^{\pi} \sin x \cos^4 x \, dx \\ &= -\int (-\sin x) \cos^4 x \, dx \\ &= -\tfrac{1}{5} \cos^5 x + C.\end{aligned}$$

Let $u = \cos x$.
Then $u' = -\sin x$.
$$\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1}$$

OR

Let $f(x) = \cos x$.
Then $f'(x) = -\sin x$.
$$\int \big(f(x)\big)^n \frac{du}{dx}\, dx = \frac{\big(f(x)\big)^{n+1}}{n+1}$$

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Exercise 7D

FOUNDATION

1 Find the following indefinite integrals.

a $\int \sec^2 x\,dx$ **b** $\int \cos x\,dx$ **c** $\int \sin x\,dx$

d $\int -\sin x\,dx$ **e** $\int 2\cos x\,dx$ **f** $\int \cos 2x\,dx$

g $\int \frac{1}{2}\cos x\,dx$ **h** $\int \cos \frac{1}{2}x\,dx$ **i** $\int \sin 2x\,dx$

j $\int \sec^2 5x\,dx$ **k** $\int \cos 3x\,dx$ **l** $\int \sec^2 \frac{1}{3}x\,dx$

m $\int \sin \frac{x}{2}\,dx$ **n** $\int -\cos \frac{1}{5}x\,dx$ **o** $\int -4\sin 2x\,dx$

p $\int \frac{1}{4}\sin \frac{1}{4}x\,dx$ **q** $\int 12\sec^2 \frac{1}{3}x\,dx$ **r** $\int 2\cos \frac{x}{3}\,dx$

2 Find the value of:

a $\int_0^{\frac{\pi}{2}} \cos x\,dx$ **b** $\int_0^{\frac{\pi}{6}} \cos x\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$

d $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ **e** $\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx$ **f** $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$

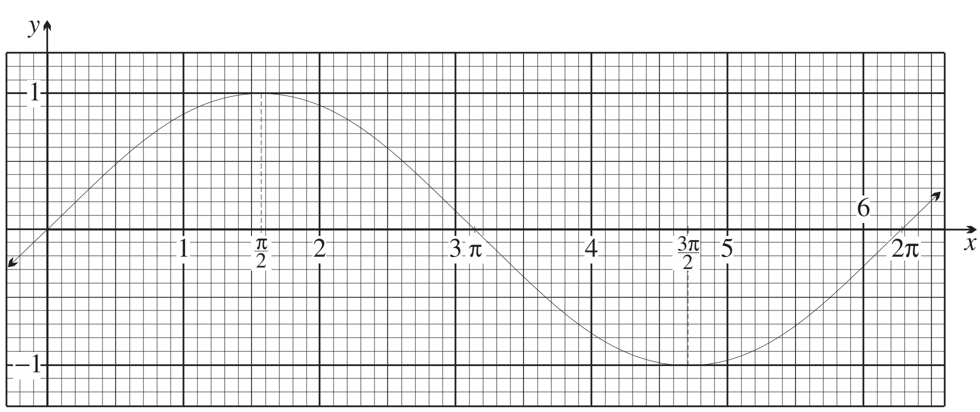
g $\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right)dx$ **h** $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{1}{2}x\right)dx$ **i** $\int_0^{\pi} (2\sin x - \sin 2x)\,dx$

3 **a** The gradient function of a certain curve is given by $\dfrac{dy}{dx} = \sin x$. If the curve passes through the origin, find its equation.

b Another curve passing through the origin has gradient function $y' = \cos x - 2\sin 2x$. Find its equation.

c If $\dfrac{dy}{dx} = \sin x + \cos x$, and $y = -2$ when $x = \pi$, find y as a function of x.

4



The graph of $y = \sin x$ is sketched above.

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a The first worked exercise in the notes for this section proved that $\int_0^{\pi} \sin x \, dx = 2$. Count squares on the graph of $y = \sin x$ above to confirm this result.

b On the same graph of $y = \sin x$, count squares and use symmetry to find:

i $\int_0^{\frac{\pi}{4}} \sin x \, dx$ **ii** $\int_0^{\frac{\pi}{2}} \sin x \, dx$ **iii** $\int_0^{\frac{3\pi}{4}} \sin x \, dx$

iv $\int_0^{\frac{5\pi}{4}} \sin x \, dx$ **v** $\int_0^{\frac{3\pi}{2}} \sin x \, dx$ **vi** $\int_0^{\frac{7\pi}{4}} \sin x \, dx$

c Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$, and confirm the results of part **b**.

5 [Technology]
Programs that sketch the graph and then approximate definite integrals would help reinforce the previous very important investigation. The investigation could then be continued past $x = \pi$, after which the definite integral decreases again.

Similar investigation with the graphs of $\cos x$ and $\sec^2 x$ would also be helpful, comparing the results of computer integration with the exact results obtained by integration using the standard primitives.

6 Find the following indefinite integrals.

a $\int \cos(x + 2) \, dx$ **b** $\int \cos(2x + 1) \, dx$ **c** $\int \sin(x + 2) \, dx$

d $\int \sin(2x + 1) \, dx$ **e** $\int \cos(3x - 2) \, dx$ **f** $\int \sin(7 - 5x) \, dx$

g $\int \sec^2(4 - x) \, dx$ **h** $\int \sec^2\left(\frac{1-x}{3}\right) dx$ **i** $\int \sin\left(\frac{1-x}{3}\right) dx$

7 a Find $\int \left(6 \cos 3x - 4 \sin \tfrac{1}{2}x\right) dx$.

b Find $\int \left(8 \sec^2 2x - 10 \cos \tfrac{1}{4}x + 12 \sin \tfrac{1}{3}x\right) dx$.

8 a If $f'(x) = \pi \cos \pi x$ and $f(0) = 0$, find $f(x)$ and $f\left(\tfrac{1}{3}\right)$.

b If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(x)$ and $f\left(\tfrac{1}{6}\right)$.

c If $f''(x) = 18 \cos 3x$ and $f'(0) = f\left(\tfrac{\pi}{2}\right) = 1$, find $f(x)$.

DEVELOPMENT

9 Find the following indefinite integrals, where a, b, u and v are constants.

a $\int a \sin(ax + b) \, dx$ **b** $\int \pi^2 \cos \pi x \, dx$

c $\int \frac{1}{u} \sec^2(v + ux) \, dx$ **d** $\int \frac{a}{\cos^2 ax} \, dx$

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10 **a** Copy and complete $1 + \tan^2 x = \ldots$, and hence find $\int \tan^2 x \, dx$.

b Simplify $1 - \sin^2 x$, and hence find the value of $\int_0^{\frac{\pi}{3}} \frac{2}{1 - \sin^2 x} \, dx$.

11 **a** Copy and complete $\int \frac{f'(x)}{f(x)} \, dx = \ldots.$

b Hence show that $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x} \, dx \doteqdot 0.4$.

12 **a** Use the fact that $\tan x = \frac{\sin x}{\cos x}$ to show that $\int_0^{\frac{\pi}{4}} \tan x \, dx = \frac{1}{2} \ln 2$.

b Use the fact that $\cot x = \frac{\cos x}{\sin x}$ to find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \, dx$.

13 **a** Find $\frac{d}{dx}(\sin^5 x)$, and hence find $\int \sin^4 x \cos x \, dx$.

b Find $\frac{d}{dx}(\tan^3 x)$, and hence find $\int \tan^2 x \sec^2 x \, dx$.

14 **a** Differentiate $e^{\sin x}$, and hence find the value of $\int_0^{\frac{\pi}{2}} \cos x e^{\sin x} \, dx$.

b Differentiate $e^{\tan x}$, and hence find the value of $\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x} \, dx$.

15 **a** Show that $\frac{d}{dx}(\sin x - x \cos x) = x \sin x$, and hence find $\int_0^{\frac{\pi}{2}} x \sin x \, dx$.

b Show that $\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right) = \sin^3 x$, and hence find $\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$.

16 Use the reverse chain rule $\int f'(x) \left(f(x)\right)^n dx = \frac{\left(f(x)\right)^{n+1}}{n + 1}$, to evaluate:

a $\int_0^{\pi} \sin x \cos^8 x \, dx$ **b** $\int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx$ **c** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin^7 x \, dx$

d $\int_0^{\frac{\pi}{6}} \cos x \sin^n x \, dx$ **e** $\int_0^{\frac{\pi}{3}} \sec^2 x \tan^7 x \, dx$ **f** $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^n x \, dx$

17 **a** Show, by finding the integral in two different ways, that for constants C and D,

$$\int \sin x \cos x \, dx = \tfrac{1}{2}\sin^2 x + C = -\tfrac{1}{4}\cos 2x + D.$$

b How may the two answers be reconciled?

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18 Find $\dfrac{d}{dx}\big(x\sin 2x\big)$, and hence find $\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$.

19 **a** Show that $\dfrac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$.

b Hence find $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$.

ENRICHMENT

20 **a** Show that $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.

b Hence find:

i $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin 3x\cos 2x\,dx$

ii $\displaystyle\int_0^{\pi} \sin 3x\cos 4x\,dx$

c Show that $\displaystyle\int_{-\pi}^{\pi} \sin mx\cos nx\,dx = 0$ for positive integers m and n:

i using the primitive,

ii using symmetry arguments.

21 **a** Find the values of A and B in the identity

$$A(2\sin x + \cos x) + B(2\cos x - \sin x) = 7\sin x + 11\cos x.$$

b Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\,dx = \frac{1}{2}(5\pi + 6\ln 2)$.

22 [The power series for $\sin x$ and $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .]$$

a We know that $\cos t \le 1$, for t positive. Integrate this inequality over the interval $0 \le t \le x$, where x is positive, and hence show that $\sin x \le x$.

b Change the variable to t, integrate the inequality $\sin t \le t$ over $0 \le t \le x$, and hence show that $\cos x \ge 1 - \dfrac{x^2}{2!}$.

c Do it twice more, and show that:

i $\sin x \ge x - \dfrac{x^3}{3!}$

ii $\cos x \le 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

d Now use induction (informally) to show that for all positive integers n,

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n + 1)!} \le \sin x + \frac{x^{4n+3}}{(4n + 3)!},$$

and use this inequality to conclude that $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$ converges, with limit $\sin x$.

e Proceeding similarly, prove that $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$ converges, with limit $\cos x$.

f Use evenness and oddness to extend the results of (d) and (e) to negative values of x.

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7E Applications of integration

The trigonometric integrals can now be used to find areas in the usual way.

Finding areas by integration

As always, *a sketch is essential*, because areas below the x-axis are represented as a negative number by the definite integral.

It is best to evaluate the separate integrals first and then make a conclusion about areas.

Example 24 — 7E

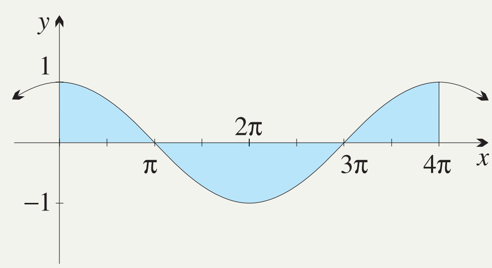
a Sketch $y = \cos \frac{1}{2}x$ in the interval $0 \le x \le 4\pi$, marking both x-intercepts.

b Hence find the area between the curve and the x-axis, for $0 \le x \le 4\pi$.

SOLUTION

a The curve $y = \cos \frac{1}{2}x$ has amplitude 1, and the period is $2\pi \div \frac{1}{2} = 4\pi$.
The two x-intercepts in the interval are $x = \pi$ and $x = 3\pi$.

b We must integrate separately over the three intervals $[0, \pi]$ and $[\pi, 3\pi]$ and $[3\pi, 4\pi]$.



First,
$$\int_0^{\pi} \cos \tfrac{1}{2}x\,dx = \Big[2\sin \tfrac{1}{2}x\Big]_0^{\pi}$$
$$= 2\sin \tfrac{\pi}{2} - 2\sin 0$$
$$= 2 - 0$$
$$= 2,$$
which is positive, because the curve is above the x-axis for $0 \le x \le \pi$.

Secondly,
$$\int_{\pi}^{3\pi} \cos \tfrac{1}{2}x\,dx = \Big[2\sin \tfrac{1}{2}x\Big]_{\pi}^{3\pi}$$
$$= 2\sin \tfrac{3\pi}{2} - 2\sin \tfrac{\pi}{2}$$
$$= -2 - 2$$
$$= -4,$$
which is negative, because the curve is below the x-axis for $\pi \le x \le 3\pi$.

Thirdly,
$$\int_{3\pi}^{4\pi} \cos \tfrac{1}{2}x\,dx = \Big[2\sin \tfrac{1}{2}x\Big]_{3\pi}^{4\pi}$$
$$= 2\sin 2\pi - 2\sin \tfrac{3\pi}{2}$$
$$= 0 - (-2)$$
$$= 2,$$
which is positive, because the curve is above the x-axis for $3\pi \le x \le 4\pi$.

Hence
$$\text{total area} = 2 + 4 + 2$$
$$= 8 \text{ square units.}$$

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Finding areas between curves

The next worked example uses the principle that if $y = f(x)$ is above $y = g(x)$ throughout some interval $a \le x \le b$, then the area between the curves is given by the formula

$$\text{area between the curves} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 25 7E

a Show that the curves $y = \sin x$ and $y = \sin 2x$ intersect when $x = \frac{\pi}{3}$.
b Sketch these curves in the interval $[0, \pi]$.
c Find the area contained between the curves in the interval $\left[0, \frac{\pi}{3}\right]$.

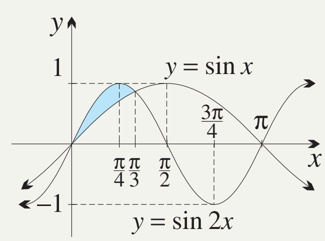
SOLUTION

a The curves intersect at $x = \frac{\pi}{3}$ because $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{1}{2}\sqrt{3}$.

b The curves are sketched to the right below.

c In the interval $0 \le x \le \frac{\pi}{3}$, the curve $y = \sin 2x$ is always above $y = \sin x$,

$$\begin{aligned}\text{so area between} &= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx\\ &= \Big[-\tfrac{1}{2}\cos 2x + \cos x\Big]_0^{\frac{\pi}{3}}\\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right).\end{aligned}$$



Because $\cos 0 = 1$ and $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{2\pi}{3} = -\frac{1}{2}$,

$$\begin{aligned}\text{area} &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right) - \left(-\tfrac{1}{2} + 1\right)\\ &= \tfrac{1}{4}\text{ square units.}\end{aligned}$$

Example 26 7E

a Show that in the interval $0 \le x \le 2\pi$, the curves $y = \sin x$ and $y = \cos x$ intersect when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$.
b Sketch the curves in this interval and find the area contained between them.

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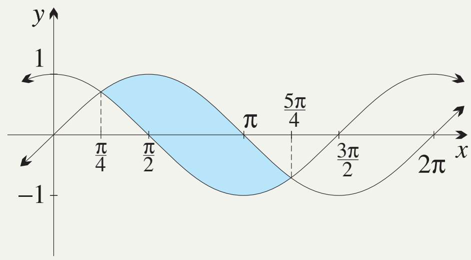
SOLUTION

a Put $\sin x = \cos x.$
Then $\tan x = 1$
$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$,
so the curves intersect at the points
$\left(\frac{\pi}{4}, \frac{1}{2}\sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{2}\sqrt{2}\right)$.



b Area between $= \displaystyle\int_{\frac{\pi}{4}}^{-\frac{5\pi}{4}} (\sin x - \cos x)\,dx$

$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= -\left(-\frac{1}{2}\sqrt{2}\right) - \left(-\frac{1}{2}\sqrt{2}\right) + \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}$

$= 2\sqrt{2}$ square units.

Exercise 7E

FOUNDATION

Technology: Some graphing programs can perform numerical integration on specified regions. Such programs would help to confirm the integrals in this exercise and to investigate quickly further integrals associated with these curves.

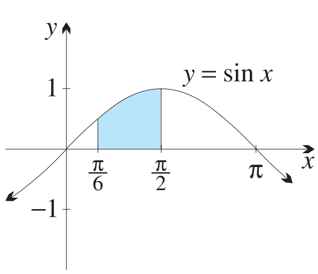
1 Find the exact area between the curve $y = \cos x$ and the x-axis:
a from $x = 0$ to $x = \frac{\pi}{2}$,
b from $x = 0$ to $x = \frac{\pi}{6}$.

2 Find the exact area between the curve $y = \sec^2 x$ and the x-axis:
a from $x = 0$ to $x = \frac{\pi}{4}$,
b from $x = 0$ to $x = \frac{\pi}{3}$.

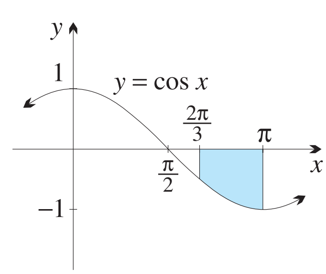
3 Find the exact area between the curve $y = \sin x$ and the x-axis:
a from $x = 0$ to $x = \frac{\pi}{4}$,
b from $x = 0$ to $x = \frac{\pi}{6}$.

4 Calculate the area of the shaded region in each diagram below (and then observe that the two regions have equal area).

a



b



5 Find the area enclosed between each curve and the x-axis over the specified domain.
a $y = \sin x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$
b $y = \sin 2x$, from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$
c $y = \cos x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$
d $y = \cos 3x$, from $x = \frac{\pi}{12}$ to $x = \frac{\pi}{6}$
e $y = \sec^2 x$, from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$
f $y = \sec^2 \frac{1}{2}x$, from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

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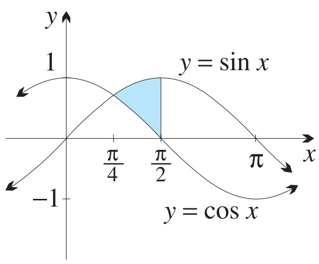
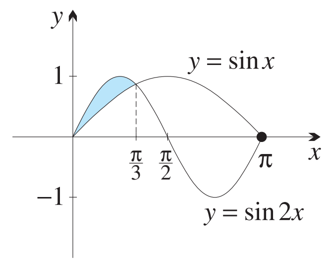
6 Calculate the area of the shaded region in each diagram below.

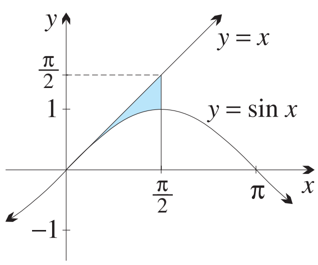
a



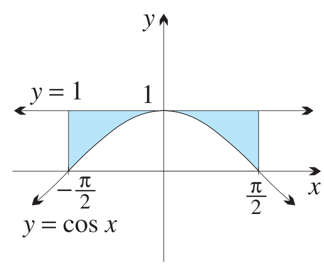
b



c

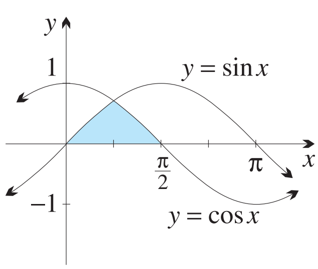


d

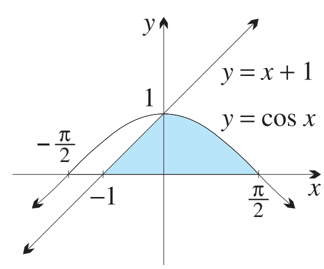


7 Calculate the area of the shaded region in each diagram below.

a



b



DEVELOPMENT

8 Find, using a diagram, the area bounded by one arch of each curve and the x-axis.

a $y = \sin x$

b $y = \cos 2x$

9 Sketch the area enclosed between each curve and the x-axis over the specified domain, and then find the exact value of the area. (Make use of symmetry wherever possible.)

a $y = \cos x$, from $x = 0$ to $x = \pi$

b $y = \sin x$, from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$

c $y = \cos 2x$, from $x = 0$ to $x = \pi$

d $y = \sin 2x$, from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$

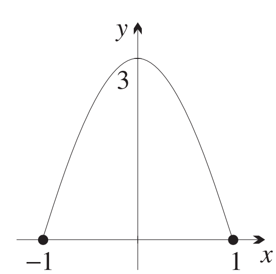
e $y = \sin x$, from $x = -\frac{5\pi}{6}$ to $x = \frac{7\pi}{6}$

f $y = \cos 3x$, from $x = \frac{\pi}{6}$ to $x = \frac{2\pi}{3}$

10 a Sketch the curve $y = 2\cos \pi x$, for $-1 \le x \le 1$, clearly marking the two x-intercepts.

b Find the exact area bounded by the curve $y = 2\cos \pi x$ and the x-axis, between the two x-intercepts.

11 An arch window 3 metres high and 2 metres wide is made in the shape of the curve $y = 3\cos\left(\frac{\pi}{2}x\right)$, as shown to the right. Find the area of the window in square metres, correct to one decimal place.



12 The graphs of $y = x - \sin x$ and $y = x$ are sketched together in a worked exercise in Section 7C. Find the total area enclosed between these graphs, from $x = 0$ to $x = 2\pi$.

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13 The region R is bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$. Show that R has area $\ln 2$ square units.

14 a Sketch the region bounded by the graphs of $y = \sin x$ and $y = \cos x$, and by the vertical lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

b Find the area of the region in part **a**.

15 a Show by substitution that $y = \sin x$ and $y = \cos 2x$ meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

b On the same number plane, sketch $y = \sin x$ and $y = \cos 2x$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{6}$.

c Hence find the area of the region bounded by the two curves.

16 a Show that $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sin x + \cos x$.

b Hence, or otherwise, find the exact area under one arch of the curve $y = \sin x + \cos x$.

17 a Show that for all positive integers n:

i $\displaystyle\int_0^{2\pi} \sin nx\, dx = 0$ **ii** $\displaystyle\int_0^{2\pi} \cos nx\, dx = 0$

b Sketch each of the following graphs, and then find the area between the curve and the x-axis, from $x = 0$ to $x = 2\pi$.

i $y = \sin x$ **ii** $y = \sin 2x$ **iii** $y = \sin 3x$ **iv** $y = \sin nx$ **v** $y = \cos nx$

18 a Show that $\displaystyle\int_0^{n} (1 + \sin 2\pi x)\,dx = n$, for all positive integers n.

b Sketch $y = 1 + \sin 2\pi x$, and interpret the result geometrically.

19 Sketch $y = |\sin x|$ for $0 \le x \le 6\pi$, and hence evaluate $\displaystyle\int_0^{6\pi} |\sin x|\, dx$.

20 a Using the fact that $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$, explain why $x^2\sin x < x^3 < x^2\tan x$ for $0 < x < \frac{\pi}{2}$.

b Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} x^2 \sin x\, dx < \frac{\pi^4}{4^5} < \int_0^{\frac{\pi}{4}} x^2 \tan x\, dx$.

21 a Given that $y = \dfrac{1}{1 + \sin x}$, show that $y' = -\dfrac{\cos x}{(1 + \sin x)^2}$.

b Hence explain why the function $y = \dfrac{1}{1 + \sin x}$ is decreasing for $0 < x < \frac{\pi}{2}$.

c Sketch the curve for $0 \le x \le \frac{\pi}{2}$, and hence show that $\displaystyle\frac{\pi}{4} < \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x}dx < \frac{\pi}{2}$.

22 Use symmetry arguments to help evaluate:

a $\displaystyle\int_{-4\pi}^{4\pi} \sin 3x\, dx$ **b** $\displaystyle\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x\, dx$

c $\displaystyle\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}} \cos x\, dx$ **d** $\displaystyle\int_{-\pi}^{\pi} \sec^2 \tfrac{1}{3} x\, dx$

e $\displaystyle\int_{-\pi}^{\pi} (3 + 2x + \sin x)\, dx$ **f** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 2x + \cos 3x + 3x^2)\, dx$

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ENRICHMENT

23 a Show that $\frac{d}{dx}\left(-\tfrac{1}{2}e^{-x}(\sin x + \cos x)\right) = e^{-x}\sin x$.

b Find $\int_0^N e^{-x}\sin x\,dx$, and show that $\int_0^\infty e^{-x}\sin x\,dx$ converges to $\frac{1}{2}$.

c Find $\int_0^\pi e^{-x}\sin x\,dx$, $\int_{2\pi}^{3\pi} e^{-x}\sin x\,dx, \ldots$, and show that the areas of the arches above the x-axis form a GP with limiting sum $\frac{e^\pi}{2(e^\pi - 1)}$.

d Show that the areas of the arches below the x-axis also form a GP, and hence show that the total area contained between the curve and the x-axis, to the right of the y-axis, is $\frac{e^\pi + 1}{2(e^\pi - 1)}$. Also confirm by subtraction the result of part **b**.

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Chapter 7 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 7 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Differentiate with respect to x.

a $y = 5\sin x$ **b** $y = \sin 5x$ **c** $y = 5\cos 5x$

d $y = \tan(5x - 4)$ **e** $y = x\sin 5x$ **f** $y = \dfrac{\cos 5x}{x}$

g $y = \sin^5 x$ **h** $y = \tan(x^5)$ **i** $y = e^{\cos 5x}$

j $y = \log_e(\sin 5x)$

2 Find the gradient of the tangent to $y = \cos 2x$ at the point on the curve where $x = \frac{\pi}{3}$.

3 **a** Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{3}$.

b Find the equation of the tangent to $y = x\cos x$ at the point where $x = \frac{\pi}{2}$.

4 Find the x-coordinates of the stationary points on each curve, for $0 \le x \le 2\pi$.

a $y = x + \cos x$ **b** $y = \sin x - \cos x$

5 Find:

a $\displaystyle\int 4\cos x\,dx$ **b** $\displaystyle\int \sin 4x\,dx$ **c** $\displaystyle\int \sec^2 \tfrac{1}{4}x\,dx$

6 Find the value of:

a $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x\,dx$ **c** $\displaystyle\int_0^{\frac{1}{3}} \pi\sin \pi x\,dx$

7 Find the value of $\displaystyle\int_0^{\frac{1}{4}} \sin 3x\,dx$, correct to three decimal places.

8 A curve has gradient function $y' = \cos\frac{1}{2}x$ and passes through the point $(\pi, 1)$. Find its equation.

9 **a** Sketch the curve $y = 2\sin 2x$, for $0 \le x \le \pi$, and then shade the area between the curve and the x-axis from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

b Calculate the shaded area in part **a**.

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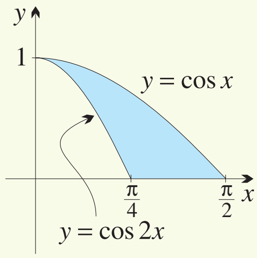
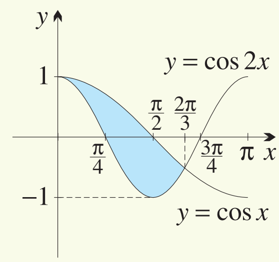
10 Find the area of the shaded region in each diagram below.

a

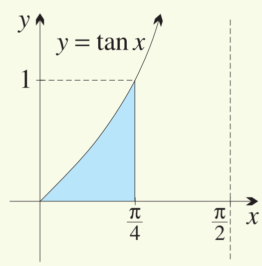


b



11 a Write $\tan x$ in terms of $\sin x$ and $\cos x$.

b Hence find the exact area of the shaded region in the diagram below.



The following questions are more difficult:

12 Consider the curve $y = 2\cos x + \sin 2x$ for $-\pi \le x \le \pi$.

a Find the x- and y-intercepts.

b Find the stationary points and determine their nature. (You will need to use the identity $\cos 2x = 1 - 2\sin^2 x$.)

c Hence sketch the curve over the given domain.

13 a Find the first and second derivatives of $y = e^x \sin x$. (Note that this function models oscillations such as feedback loops which grow exponentially.)

b Find the stationary points for $-\pi \le x \le \pi$ and determine their nature.

c Find the points of inflexion for $-\pi \le x \le \pi$.

d Hence sketch the curve for $-\pi \le x \le \pi$.

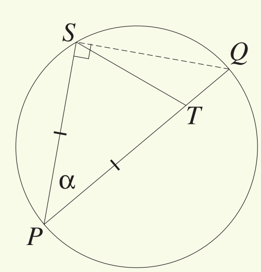
14 The angle θ between two radii OP and OQ of a circle of radius 6 cm is increasing at the rate of 0.1 radians per minute.

a Show that the area of sector OPQ is increasing at the rate of 1.8 cm^2/min.

b Find the rate at which the area of $\triangle OPQ$ is increasing at the instant when $\theta = \frac{\pi}{4}$.

c Find the value of θ for which the rate of increase of the area of the segment cut off by the chord PQ is at its maximum.

15 PQ is a diameter of the given circle and S is a point on the circumference. T is the point on PQ such that $PS = PT$. Let $\angle SPT = \alpha$.



a Show that the area A of $\triangle SPT$ is $A = \frac{1}{2}d^2\cos^2\alpha\sin\alpha$, where d is the diameter of the circle.

b Hence show that the maximum area of $\triangle SPT$ as S varies on the circle is $\frac{1}{9}d^2\sqrt{3}$ units^2.

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16 Find:

a $\int e^{2x} \cos e^{2x}\, dx$

b $\int \frac{\sin e^{-2x}}{e^{2x}}\, dx$

c $\int \frac{\sec^2 x}{3\tan x + 1}\, dx$

d $\int \frac{3\sin x}{4 + 5\cos x}\, dx$

e $\int \frac{1 - \cos^3 x}{1 - \sin^2 x}\, dx$

f $\int_0^{\pi} \sin x \cos^2 x\, dx$

17 a Show that $\tan^3 x = \tan x \sec^2 x - \tan x$.

b Hence find:

i $\int \tan^3 x\, dx$

ii $\int \tan^5 x\, dx$

18 a Sketch $y = 1 - \tan x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes.

b Find the area of R.

19 a Show that $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$.

b Hence find:

i $\int_0^{\frac{\pi}{2}} 2\cos 3x \cos 2x\, dx$

ii $\int_0^{\pi} \cos 3x \cos 4x\, dx$

c Find $\int \cos mx \cos nx\, dx$, where m and n are:

i distinct positive integers,

ii equal positive integers.

d Show that for positive integers m and n, $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$

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