11 Trigonometric equations

Trigonometric equations occur whenever trigonometric functions are being analysed, and careful study of them is essential. Various approaches to them have already been developed in Section 6H, Section 11H and part of Chapter 17 of the Year 11 book, and in parts of Chapter 7 of this book.

This chapter develops those methods further, with particular emphasis on the use of the compound-angle formulae.

Expressions of the form $a\cos\theta + b\sin\theta$ that are the sum of a sine wave and a cosine wave dominate Sections 11B and 11C. Such sums of waves turn out to be extremely important in physics and science generally. The methods of Section 11B lead eventually to the analysis of every periodic function into the sum of sine and cosine waves, and so are essential, for example, in the analysis of data sent by radio through the air, or by light waves through a fibre.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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11A Equations involving compound angles

This section reviews the various compound-angle formulae discussed so far and uses them to solve trigonometric equations. Two further forms of the $\cos 2\theta$ formula are needed, and will be needed again in the next chapter when integrating.

A review of the compound-angle formulae

For reference, here are the compound-angle formulae that were proven in Chapter 17 of the Year 11 book.

The basic compound-angle formulae:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

The double-angle formulae and the $\cos 2\theta$ formulae:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\end{aligned}$$

The t-formulae: Let $t = \tan\frac{1}{2}\theta$. Then

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

Products to sums:

$$\begin{aligned}2\sin A\cos B &= \sin(A+B) + \sin(A-B)\\ 2\cos A\sin B &= \sin(A+B) - \sin(A-B)\\ 2\cos A\cos B &= \cos(A+B) + \cos(A-B)\\ -2\sin A\sin B &= \cos(A+B) - \cos(A-B)\end{aligned}$$

The enrichment section of Exercise 17G in the Year 11 book also introduced the sums-to-products formulae that reverse the products-to-sums formulae given here.

Expressing $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$

Using the second and third $\cos 2\theta$ formulae above, we can write the squares $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$.

$$\begin{aligned}\text{From } \cos 2\theta &= 2\cos^2\theta - 1,\\ 2\cos^2\theta &= 1 + \cos 2\theta\\ \cos^2\theta &= \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

$$\begin{aligned}\text{From } \cos 2\theta &= 1 - 2\sin^2\theta,\\ 2\sin^2\theta &= 1 - \cos 2\theta\\ \sin^2\theta &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

1 EXPRESSING $\sin^2\theta$ AND $\cos^2\theta$ IN TERMS OF $\cos 2\theta$

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta \qquad \sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

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Adding these two identities gives

$$\cos^2\theta + \sin^2\theta = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) + \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) = 1,$$

as expected from the Pythagorean identities. This observation may help you to memorise the two new formulae.

Example 1 11A

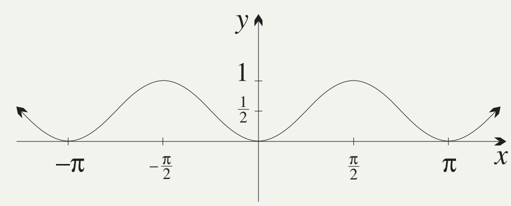
Sketch $y = \sin^2 x$, and state its amplitude, period and range.

SOLUTION

Using the identities above,

$$y = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x.$$

This is the graph of $y = \cos 2x$ turned upside down, then stretched vertically by the factor $\tfrac{1}{2}$, then shifted up $\tfrac{1}{2}$. Its period is π, and its amplitude is $\tfrac{1}{2}$. Because it oscillates around $\tfrac{1}{2}$ rather than 0, its range is $0 \le y \le 1$.



Equations with more than one trigonometric function, but the same angle

This is where trigonometric identities are essential.

2 EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION

- Trigonometric identities can usually be used to produce an equation in only one trigonometric function.
- Many trigonometric equations can be solved by more than one method.

Example 2 11A

Solve the equation $2\tan\theta = \sec\theta$, for $0° \le \theta \le 360°$:

a using the ratio identities,

b by squaring both sides.

SOLUTION

a
$$2\tan\theta = \sec\theta$$
$$\frac{2\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}$$
$$\sin\theta = \tfrac{1}{2}$$
$$\theta = 30° \text{ or } 150°$$

b Squaring,
$$4\tan^2\theta = \sec^2\theta$$
$$4\sec^2\theta - 4 = \sec^2\theta$$
$$\sec^2\theta = \tfrac{4}{3}$$
$$\cos\theta = \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3}$$
$$\theta = 30°,\ 150°,\ 210° \text{ or } 330°.$$

Checking each solution, $\theta = 30°$ or $150°$.

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The dangers of squaring an equation

In part **b** of the previous worked example, we had to check each solution because the equation had been squared.

Squaring can introduce extra solutions. For example:

- $\sin x = 1$, for $0 \le x \le 2\pi$, has one solution $x = \frac{\pi}{2}$.
- $\sin^2 x = 1$, for $0 \le x \le 2\pi$, has two solutions $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Equations involving different angles

When different angles are involved in the one trigonometric equation, use compound-angle identities to change all the trigonometric functions to functions of the one angle.

3 EQUATIONS INVOLVING DIFFERENT ANGLES

- Use compound-angle identities to change all the trigonometric functions to functions of the one angle.

Example 3 — 11A

Solve $\tan 4x = -\tan 2x$, for $0 \le x \le \frac{\pi}{2}$, using the $\tan 2\theta$ formula.

SOLUTION

$$\begin{aligned}
\tan 4x &= -\tan 2x \\
\frac{2\tan 2x}{1 - \tan^2 2x} &= -\tan 2x \\
2\tan 2x &= -\tan 2x + \tan^3 2x \\
\tan^3 2x - 3\tan 2x &= 0 \\
\tan 2x(\tan^2 2x - 3) &= 0 \\
\tan 2x &= 0, \text{ or } \tan 2x = \sqrt{3}, \text{ or } \tan 2x = -\sqrt{3}.
\end{aligned}$$

The restriction on $2x$ is $0 \le 2x \le \pi$, so the solutions are

$$\begin{aligned}
2x &= 0 \text{ or } \pi, \text{ or } 2x = \tfrac{\pi}{3}, \text{ or } 2x = \tfrac{2\pi}{3} \\
x &= 0 \text{ or } \tfrac{\pi}{6} \text{ or } \tfrac{\pi}{3} \text{ or } \tfrac{\pi}{2}.
\end{aligned}$$

Alternatively, substitute $t = \tan 2x$ after the second line.

Equations involving different angles and functions

The six trigonometric functions are very closely related. The best approach is usually:

4 APPROACHING TRIGONOMETRIC EQUATIONS

- First, try to get all the angles the same.
- Then try to get all the trigonometric functions the same.

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Example 4 11A

Solve $\cos 2x = 4\sin^2 x - 14\cos^2 x$, for $0 \le x \le 2\pi$:
a by changing all the angles to x, **b** by changing all the angles to $2x$.

SOLUTION

a
$$\begin{aligned} \cos 2x &= 4\sin^2 x - 14\cos^2 x \\ \cos^2 x - \sin^2 x &= 4\sin^2 x - 14\cos^2 x \\ 15\cos^2 x &= 5\sin^2 x \\ \tan x &= \sqrt{3} \text{ or } -\sqrt{3} \\ x &= \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3} \end{aligned}$$

b
$$\begin{aligned} \cos 2x &= 4\sin^2 x - 14\cos^2 x \\ \cos 2x &= 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) - 14\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) \\ 10\cos 2x &= -5 \\ \cos 2x &= -\tfrac{1}{2} \\ 2x &= \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \tfrac{8\pi}{3} \text{ or } \tfrac{10\pi}{3} \\ x &= \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3} \end{aligned}$$

Homogeneous equations

Expressions *homogeneous in* $\sin x$ *and* $\cos x$ were mentioned in Section 17F of the Year 11 book in the context of proving the t-formulae. The idea is also useful when solving trigonometric equations.

5 HOMOGENEOUS EQUATIONS

- An equation is called *homogeneous* in $\sin x$ and $\cos x$ if the sum of the indices of $\sin x$ and $\cos x$ in each term is the same.
- To solve an equation homogeneous in $\sin x$ and $\cos x$, divide through by a suitable power of $\cos x$ to produce an equation in $\tan x$ alone.

The expansions of $\sin 2x$ and $\cos 2x$ are homogeneous of degree 2 in $\sin x$ and $\cos x$.
Also, $1 = \sin^2 x + \cos^2 x$ can be regarded as being homogeneous of degree 2.

Example 5 11A

Solve $\sin 2x + \cos 2x = \sin^2 x + 1$, for $0 \le x \le 2\pi$.

SOLUTION

Expanding, $2\sin x\cos x + (\cos^2 x - \sin^2 x) = \sin^2 x + (\sin^2 x + \cos^2 x)$
$$3\sin^2 x - 2\sin x\cos x = 0$$
$\boxed{\div \cos^2 x}$
$$\begin{aligned} 3\tan^2 x - 2\tan x &= 0 \\ \tan x(3\tan x - 2) &= 0 \\ \tan x = 0 \text{ or } \tan x &= \tfrac{2}{3}. \end{aligned}$$
Hence $x = 0, \pi$ or 2π, or $x \doteqdot 0.588$ or 3.730.

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Exercise 11A

FOUNDATION

1 Consider the equation $\sin 2x - \cos x = 0$.

a By using a double-angle formula and then factoring, show that $\cos x = 0$ or $\sin x = \frac{1}{2}$.

b Hence solve the equation for $0 \le x \le 2\pi$.

2 Consider the equation $\cos 2x - \cos x = 0$.

a By using a double-angle formula and then factoring, show that $\cos x = 1$ or $-\frac{1}{2}$.

b Hence solve the equation for $0 \le x \le 2\pi$.

3 Consider the equation $\sin\left(x + \frac{\pi}{4}\right) = 2\cos\left(x - \frac{\pi}{4}\right)$.

a Use compound-angle formulae to show that $\tan x = -1$.

b Hence solve the equation for $0 \le x \le 2\pi$.

DEVELOPMENT

4 Use compound-angle formulae to solve, for $0 \le \theta \le 2\pi$:

a $\sin\left(\theta + \frac{\pi}{6}\right) = 2\sin\left(\theta - \frac{\pi}{6}\right)$

b $\cos\left(\theta - \frac{\pi}{6}\right) = 2\cos\left(\theta + \frac{\pi}{6}\right)$

c $\cos 4\theta\cos\theta + \sin 4\theta\sin\theta = \frac{1}{2}$

d $\cos 3\theta = \cos 2\theta\cos\theta$

(Hint: In part d, write $\cos 3\theta$ as $\cos(2\theta + \theta)$.)

5 Use double-angle formulae to solve, for $0 \le x \le 2\pi$:

a $\sin 2x = \sin x$

b $\sin 2x + \sqrt{3}\cos x = 0$

c $3\sin x + \cos 2x = 2$

d $\cos 2x + 3\cos x + 2 = 0$

e $\tan 2x + \tan x = 0$

f $\sin 2x = \tan x$

6 Solve, for $0° \le \theta \le 360°$, giving solutions correct to the nearest minute where necessary:

a $2\sin 2\theta + \cos\theta = 0$

b $2\cos^2\theta + \cos 2\theta = 0$

c $2\cos 2\theta + 4\cos\theta = 1$

d $8\sin^2\theta\cos^2\theta = 1$

e $3\cos 2\theta + \sin\theta = 1$

f $\cos 2\theta = 3\cos^2\theta - 2\sin^2\theta$

g $10\cos\theta + 13\cos\frac{1}{2}\theta = 5$

h $\tan\theta = 3\tan\frac{1}{2}\theta$

i $\cos^2 2\theta = \sin^2\theta$

(Hint: Use $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.)

j $\cos 2\theta + 3 = 3\sin 2\theta$

(Hint: Write 3 as $3\cos^2\theta + 3\sin^2\theta$.)

7 Consider the equation $\tan\left(\frac{\pi}{4} + \theta\right) = 3\tan\left(\frac{\pi}{4} - \theta\right)$.

a Show that $\tan^2\theta - 4\tan\theta + 1 = 0$.

b Hence use the quadratic formula to solve the equation for $0 \le \theta \le \pi$.

8 Given the equation $2\cos x - 1 = 2\cos 2x$:

a Show that $\cos x = \frac{1}{4}\left(1 + \sqrt{5}\right)$ or $\cos x = \frac{1}{4}\left(1 - \sqrt{5}\right)$.

b Hence solve the equation for $0 \le x \le 2\pi$.

9 a Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

b Hence solve the equation $\sin^2 3\theta - \sin^2\theta = \sin 2\theta$, for $0 \le \theta \le \pi$.

10 a Show that $\sin 3x = 3\sin x - 4\sin^3 x$.

b Hence solve the equation $\sin 3x + \sin 2x = \sin x$, for $0 \le x \le 2\pi$.

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11 a Given the equation $\sin\left(\theta + \frac{\pi}{6}\right) = \cos\left(\theta - \frac{\pi}{4}\right)$, show that $\tan\theta = \sqrt{6} - \sqrt{3} - \sqrt{2} + 2$.

b Hence solve the equation for $0 \le \theta \le 2\pi$.

12 Use compound-angle formulae, other trigonometric identities, and factoring to solve for $0° \le \alpha \le 360°$, giving solutions correct to the nearest minute where necessary.

a $\sec^2\alpha = 2\sec\alpha$

b $\sec^2\alpha - \tan\alpha - 3 = 0$

c $\operatorname{cosec}^3 2\alpha = 4\operatorname{cosec} 2\alpha$

d $\sqrt{3}\operatorname{cosec}^2\frac{1}{2}\alpha + \cot\frac{1}{2}\alpha = \sqrt{3}$

e $\sqrt{3}\operatorname{cosec}^2\alpha = 4\cot\alpha$

f $\cot\alpha + 3\tan\alpha = 5\operatorname{cosec}\alpha$

ENRICHMENT

13 a Use the product-to-sum identity $2\cos A\cos B = \cos(A + B) + \cos(A - B)$ to prove the sum-to-product identity $\cos P + \cos Q = 2\cos\left(\frac{P + Q}{2}\right)\cos\left(\frac{P - Q}{2}\right)$.

b Hence solve the equation $\cos 4x + \cos x = 0$, for $0 \le x \le \pi$.

14 Consider the equation $\sin\theta + \cos\theta = \sin 2\theta$, for $0° \le \theta \le 360°$.

a By squaring both sides, show that $\sin^2 2\theta - \sin 2\theta - 1 = 0$.

b Hence solve for θ over the given domain, giving solutions correct to the nearest minute. (Hint: Beware of the fact that squaring can create invalid solutions.)

15 a Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

b By substituting $x = 2\cos\theta$, show that the equation $x^3 - 3x - 1 = 0$ has roots $2\cos 20°$, $-2\sin 10°$ and $-2\cos 40°$.

c Use a similar approach to find, correct to three decimal places, the three real roots of the equation $x^3 - 12x = 8\sqrt{3}$.

16 a If $t = \tan x$, show that $\tan 4x = \dfrac{4t(1 - t^2)}{1 - 6t^2 + t^4}$.

b If $\tan 4x\tan x = 1$, show that $5t^4 - 10t^2 + 1 = 0$.

c Show that $\sin A\sin B = \frac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)$ and that $\cos A\cos B = \frac{1}{2}\big(\cos(A - B) + \cos(A + B)\big)$.

d Hence show that $\frac{\pi}{10}$ and $\frac{3\pi}{10}$ both satisfy $\tan 4x\tan x = 1$.

e Hence write down, in trigonometric form, the four real roots of the polynomial equation $5x^4 - 10x^2 + 1 = 0$.

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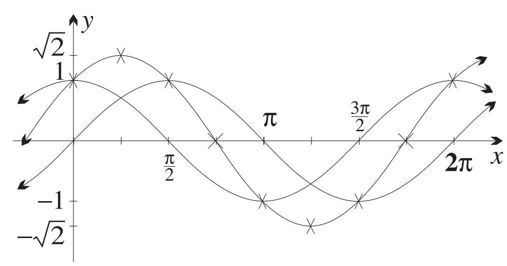
11B The sum of sine and cosine functions

We have seen that the sine and cosine curves are the same, except that the sine wave is the cosine wave shifted right by $\frac{\pi}{2}$. This section analyses what happens when the sine and cosine curves are added, and, more generally, when multiples of the two curves are added. The surprising result is that $y = a\sin x + b\cos x$ is still a sine or cosine wave, whatever the values of a and b are, but shifted sideways so that usually the zeroes no longer lie on multiples of $\frac{\pi}{2}$, and stretched vertically.

These forms for $a\sin x + b\cos x$ give a systematic method of solving any equation of the form $a\cos x + b\sin x = c$. In the next section, an alternative method of solution using the t-formulae is developed.

Sketching $y = \sin x + \cos x$ by graphical methods



The diagram to the right shows the two graphs of $y = \sin x$ and $y = \cos x$. From these two graphs, the sum function $y = \sin x + \cos x$ has been drawn on the same diagram — the crosses represent obvious points to mark on the graph of the sum.

- The new graph has the same period 2π as $y = \sin x$ and $y = \cos x$. It looks like a wave, and within $[0, 2\pi]$ there are zeroes at the two values $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ where $\sin x$ and $\cos x$ take opposite values.
- The new amplitude is bigger than 1. The value at $x = \frac{\pi}{4}$ is $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$, so if the maximum occurs there, as seems likely, the amplitude is $\sqrt{2}$.

This would indicate that the resulting sum function is $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, because it is the stretched sine curve shifted left $\frac{\pi}{4}$. We can check this by expansion,

$$\begin{aligned}\sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right) &= \sqrt{2}\left(\sin x\cos\tfrac{\pi}{4} + \cos x\sin\tfrac{\pi}{4}\right)\\ &= \sin x + \cos x, \text{ because } \cos\tfrac{\pi}{4} = \sin\tfrac{\pi}{4} = \frac{1}{\sqrt{2}}.\end{aligned}$$

That is exactly what we expected from the sketches of the graphs.

The general algebraic approach — the auxiliary angle

It is true in general that any function of the form $f(x) = a\sin x + b\cos x$ can be written as a single wave function. There are four possible forms in which the wave can be written, and the process is done by expanding the standard form and equating coefficients of $\sin x$ and $\cos x$.

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6 AUXILIARY-ANGLE METHOD

- Any function of the form $f(x) = a\sin x + b\cos x$, where a and b are constants (not both zero), can be written in any one of the four forms:

$$y = R\sin(x - \alpha) \qquad y = R\cos(x - \alpha)$$
$$y = R\sin(x + \alpha) \qquad y = R\cos(x + \alpha)$$

  where $R > 0$ and $0° \le \alpha < 360°$. The constant $R = \sqrt{a^2 + b^2}$ is the same for all forms, but the *auxiliary angle* α depends on which form is chosen.

- To begin the process, expand the standard form and equate coefficients of $\sin x$ and $\cos x$. Be careful to identify the quadrant in which α lies.

The next worked example continues with the example given at the start of the section, and shows the systematic algorithm used to obtain the required form.

EXAMPLE 6 11B

Express $y = \sin x + \cos x$ in the two forms:

a $R\sin(x + \alpha)$,

b $R\cos(x + \alpha)$,

where, in each case, $R > 0$ and $0 \le \alpha < 2\pi$. Then sketch the curve, showing all intercepts and turning points in the interval $0 \le x \le 2\pi$.

SOLUTION

a Expanding, $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$,

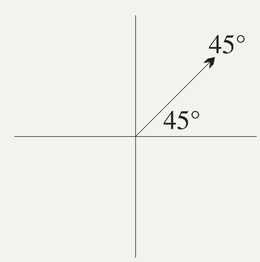
so for all x, $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$.

Equating coefficients of $\sin x$, $R\cos\alpha = 1$, (1)

equating coefficients of $\cos x$, $R\sin\alpha = 1$. (2)

Squaring and adding, $R^2 = 2$,

and because $R > 0$, $R = \sqrt{2}$.



From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)

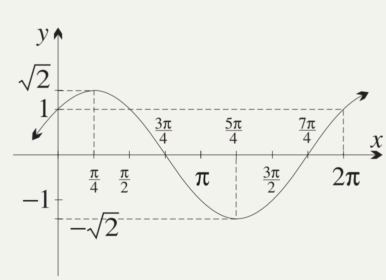
and from (2), $\sin\alpha = \dfrac{1}{\sqrt{2}}$, (2A)

so α is in the 1st quadrant, with related angle $\frac{\pi}{4}$.

Hence $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

The graph is $y = \sin x$ shifted left by $\frac{\pi}{4}$

and stretched vertically by a factor of $\sqrt{2}$.



Thus the x-intercepts are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$,

there is a maximum of $\sqrt{2}$ when $x = \frac{\pi}{4}$,

and a minimum of $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.

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b Expanding, $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$,
so for all x, $\sin x + \cos x = R\cos x\cos\alpha - R\sin x\sin\alpha$.
Equating coefficients of $\cos x$, $R\cos\alpha = 1$, (1)
equating coefficients of $\sin x$, $R\sin\alpha = -1$. (2)
Squaring and adding, $R^2 = 2$,
and because R > 0, $R = \sqrt{2}$.
From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)
and from (2), $\sin\alpha = -\dfrac{1}{\sqrt{2}}$, (2A)
so α is in the 4th quadrant, with related angle $\frac{\pi}{4}$.
Hence $\sin x + \cos x = \sqrt{2}\cos\left(x + \frac{7\pi}{4}\right)$.
The graph above could equally well be obtained from this.
It is $y = \cos x$ shifted left by $\frac{7\pi}{4}$ and stretched vertically by a factor of $\sqrt{2}$.

Approximating the auxiliary angle

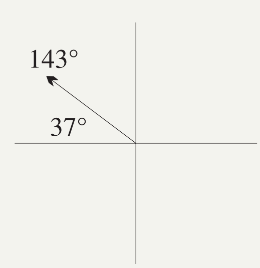
Unless special angles are involved, the auxiliary angle will need to be approximated on the calculator. Degrees or radian measure may be used, but the next worked example uses degrees to make the working a little more intuitive.

Example 7 11B

a Express $y = 3\sin x - 4\cos x$ in the form $y = R\cos(x-\alpha)$, where $R > 0$ and $0° \le \alpha < 360°$, giving α correct to the nearest degree.
b Sketch the curve, showing, correct to the nearest degree, all intercepts and turning points in the interval $-180° \le x \le 180°$.

SOLUTION

a Expanding, $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$,
so far all x, $3\sin x - 4\cos x = R\cos x\cos\alpha + R\sin x\sin\alpha$.
Equating coefficients of $\cos x$, $R\cos\alpha = -4$, (1)
equating coefficients of $\sin x$, $R\sin\alpha = 3$. (2)
Squaring and adding, $R^2 = 25$,
and because $R > 0$, $R = 5$.
From (1), $\cos\alpha = -\frac{4}{5}$, (1A)
and from (2), $\sin\alpha = \frac{3}{5}$, (2A)



so α is in the 2nd quadrant, with related angle about 37°.
Hence $3\sin x - 4\cos x = 5\cos(x-\alpha)$, where $\alpha \doteqdot 143°$.

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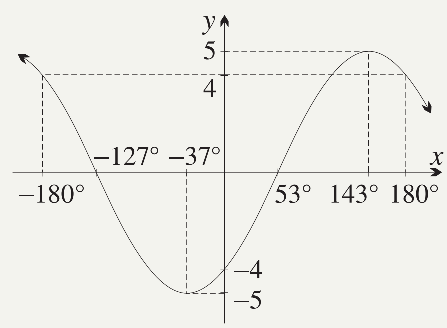
b The graph is $y = \cos\ x$ shifted right by $\alpha \doteqdot 143°$ and stretched vertically by a factor of 5. Thus the x-intercepts are $x \doteqdot 53°$ and $x \doteqdot -127°$, there is a maximum of 5 when $x \doteqdot 143°$, and a minimum of -5 when $x \doteqdot -37°$.



A note on the calculator and approximations for the auxiliary angle

In the previous worked examples, the exact value of α is $\alpha = 180° - \sin^{-1}\frac{3}{5}$, because α is in the second quadrant. It is this value that is obtained on the calculator, and if there are subsequent calculations to do, as in the equation solved below, this value should be stored in memory and used whenever the auxiliary angle is required. Re-entry of the approximation may lead to rounding errors.

Solving equations of the form $a \sin x + b \cos x = c$, and inequations

Once the LHS has been put into one of the four standard forms, the solutions can easily be obtained. It is always important to keep track of restrictions on the compound angle. The worked example below continues with the previous example.

Example 8 11B

a Using the previous worked example, solve the equation $3 \sin x - 4 \cos x = -2$, for $-180° \le x \le 180°$, correct to the nearest degree.

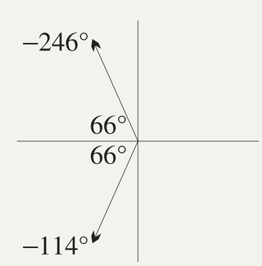
b Hence use the graph to solve $3 \sin x - 4 \cos x \le -2$, for $-180° \le x \le 180°$.

SOLUTION

a Using $3 \sin x - 4 \cos x = 5 \cos\ (x - \alpha)$, where $\alpha \doteqdot 143°$,

$$5 \cos\ (x - \alpha) = -2, \text{ where } -323° \le x - \alpha \le 37°$$

$$\cos\ (x - \alpha) = -\frac{2}{5}.$$



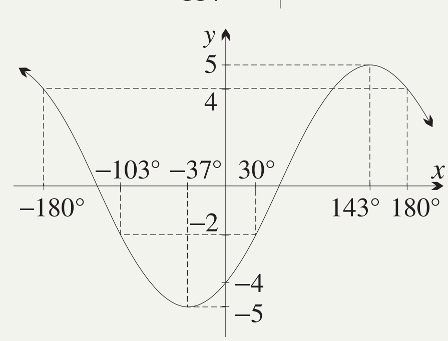
Hence $x - \alpha$ is in quadrant 2 or 3, with related angle about 66°,

so $x - \alpha \doteqdot -114°$ or $-246°$

$x \doteqdot 30°$ or $-103°$.

Be careful to use the calculator's memory here.
Never re-enter approximations of the angles.

b The graph to the right shows the previously drawn graph of $y = 3 \sin x - 4 \cos x$ with the horizontal line $y = -2$ added. This roughly verifies the two answers obtained in part **a**. It also shows that the solution of the inequality $3 \sin x - 4 \cos x \le -2$ is $-103° \le x \le 30°$.



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7 THE AUXILIARY-ANGLE METHOD FOR EQUATIONS OF THE FORM $a \sin x + b \cos x = c$

- Get the LHS into one of the four forms

  $R \sin(x + \alpha)$ or $R \sin(x - \alpha)$ or $R \cos(x + \alpha)$ or $R \cos(x - \alpha)$.

  Then solve the resulting equation.

Exercise 11B

FOUNDATION

1 Find R and α exactly, if $R > 0$ and $0 \le \alpha < 2\pi$, and:

a $R \sin \alpha = \sqrt{3}$ and $R \cos \alpha = 1$,

b $R \sin \alpha = 3$ and $R \cos \alpha = 3$.

2 Find R (exactly) and α (correct to the nearest minute), if $R > 0$ and $0° \le \alpha < 360°$, and:

a $R \sin \alpha = 5$ and $R \cos \alpha = 12$,

b $R \cos \alpha = 2$ and $R \sin \alpha = 4$.

3 **a** If $\cos x - \sin x = A \cos(x + \alpha)$, show that $A \cos \alpha = 1$ and $A \sin \alpha = 1$.

b Find the positive value of A by squaring and adding.

c Find α, if $0 \le \alpha < 2\pi$.

d State the maximum and minimum values of $\cos x - \sin x$, and the first positive values of x for which they occur.

e Solve the equation $\cos x - \sin x = -1$, for $0 \le x \le 2\pi$.

f Write down the amplitude and period of $\cos x - \sin x$. Hence sketch $y = \cos x - \sin x$, for $0 \le x \le 2\pi$. Indicate on your sketch the line $y = -1$ and the solutions of the equation in part **e**.

4 Sketch $y = \cos x$ and $y = \sin x$ on one set of axes. Then, by taking differences of heights, sketch $y = \cos x - \sin x$. Compare your sketch with that in the previous question.

5 **a** If $\sqrt{3} \cos x - \sin x = B \cos(x + \theta)$, show that $B \cos \theta = \sqrt{3}$ and $B \sin \theta = 1$.

b Find B, if $B > 0$, by squaring and adding.

c Find θ, if $0 \le \theta < 2\pi$.

d State the greatest and least possible values of $\sqrt{3} \cos x - \sin x$, and the values of x closest to $x = 0$ for which they occur.

e Solve the equation $\sqrt{3} \cos x - \sin x = 1$, for $0 \le x \le 2\pi$.

f Sketch $y = \sqrt{3} \cos x - \sin x$, for $0 \le x \le 2\pi$. On the same diagram, sketch the line $y = 1$. Indicate on your diagram the solutions of the equation in part **e**.

6 Let $4 \sin x - 3 \cos x = A \sin(x - \alpha)$, where $A > 0$ and $0° \le \alpha < 360°$.

a Show that $A \cos \alpha = 4$ and $A \sin \alpha = 3$.

b Show that $A = 5$ and $\alpha = \tan^{-1} \frac{3}{4}$.

c Hence solve the equation $4 \sin x - 3 \cos x = 5$, for $0° \le x \le 360°$. Give the solution(s) correct to the nearest minute.

7 Consider the equation $2 \cos x + \sin x = 1$.

a Let $2 \cos x + \sin x = B \cos(x - \theta)$, where $B > 0$ and $0° \le \theta < 360°$. Show that $B = \sqrt{5}$ and $\theta = \tan^{-1} \frac{1}{2}$.

b Hence find, correct to the nearest minute where necessary, the solutions of the equation, for $0° \le x \le 360°$.

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8 Let $\cos x - 3\sin x = D\cos(x + \phi)$, where $D > 0$ and $0° \le \phi < 360°$.

a Show that $D = \sqrt{10}$ and $\phi = \tan^{-1}3$.

b Hence solve $\cos x - 3\sin x = 3$, for $0° \le x \le 360°$. Give the solutions correct to the nearest minute where necessary.

9 Consider the equation $\sqrt{5}\sin x + 2\cos x = -2$.

a Transform the LHS into the form $C\sin(x + \alpha)$, where $C > 0$ and $0° \le \alpha < 360°$.

b Find, correct to the nearest minute where necessary, the solutions of the equation, for $0° \le x \le 360°$.

DEVELOPMENT

10 Solve each equation, for $0° \le x \le 360°$, by transforming the LHS into a single-term sine or cosine function. Give solutions correct to the nearest minute.

a $3\sin x + 5\cos x = 4$

b $6\sin x - 5\cos x = 7$

c $7\cos x - 2\sin x = 5$

d $9\cos x + 7\sin x = 3$

11 Find A and α exactly, if $A > 0$ and $0 \le \alpha < 2\pi$, and:

a $A\sin\alpha = 1$ and $A\cos\alpha = -\sqrt{3}$,

b $A\cos\alpha = -5$ and $A\sin\alpha = -5$.

12 Find A (exactly) and α (correct to the nearest minute), if $A > 0$ and $0° \le \alpha < 360°$, and:

a $A\cos\alpha = 5$ and $A\sin\alpha = -4$,

b $A\sin\alpha = -11$ and $A\cos\alpha = -2$.

13 a i Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and $0 < \theta < 2\pi$.

ii Hence solve $\sqrt{3}\cos x + \sin x = 1$, for $0 \le x < 2\pi$.

b i Express $\cos x - \sin x$ in the form $B\sin(x + \alpha)$, where $B > 0$ and $0 < \alpha < 2\pi$.

ii Hence solve $\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

c i Express $\sin x - \sqrt{3}\cos x$ in the form $C\sin(x + \beta)$, where $C > 0$ and $0 < \beta < 2\pi$.

ii Hence solve $\sin x - \sqrt{3}\cos x = -1$, for $0 \le x < 2\pi$.

d i Express $-\cos x - \sin x$ in the form $D\cos(x - \phi)$, where $D > 0$ and $0 < \phi < 2\pi$.

ii Hence solve $-\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

14 a i Express $2\cos x - \sin x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 360°$. (Write α correct to the nearest minute.)

ii Hence solve $2\cos x - \sin x = 1$, for $0° \le x < 360°$. Give the solutions correct to the nearest minute where necessary.

b i Express $-3\sin x - 4\cos x$ in the form $S\cos(x - \beta)$, where $S > 0$ and $0 < \beta < 2\pi$. (Write β correct to four decimal places.)

ii Hence solve $-3\sin x - 4\cos x = 2$, for $0 \le x < 2\pi$. Give the solutions correct to two decimal places.

15 Solve, for $0° \le x \le 360°$, giving solutions correct to the nearest minute:

a $2\sec x - 2\tan x = 5$

b $2\operatorname{cosec} x + 5\cot x = 3$

16 a Given the equation $\sin\theta + \cos\theta = \cos 2\theta$, show that $\tan\theta = -1$ or $\cos\theta - \sin\theta = 1$.

b Hence solve $\sin\theta + \cos\theta = \cos 2\theta$, for $0 \le \theta < 2\pi$.

17 Solve, for $0 \le x \le 2\pi$:

a $\sin x - \cos x = \sqrt{1.5}$

b $\sqrt{3}\sin 2x - \cos 2x = 2$

c $\sin 4x + \cos 4x = 1$

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18 a Show that $\left(\sqrt{3}+1\right)\cos 2x+\left(\sqrt{3}-1\right)\sin 2x = 2\sqrt{2}\cos\left(2x-\frac{\pi}{12}\right)$.

b Hence solve $\left(\sqrt{3}+1\right)\cos 2x+\left(\sqrt{3}-1\right)\sin 2x = 2$, for $-\pi \le x \le \pi$.

19 a i Show that $\sin x - \cos x = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$.

ii Hence sketch the graph of $y = \sin x - \cos x$, for $0 \le x \le 2\pi$.

iii Use your sketch to determine the values of x in the domain $0 \le x \le 2\pi$ for which $\sin x - \cos x > 1$.

b Use a similar approach to part **a** to solve, for $0 \le x \le 2\pi$:

i $\sin x + \sqrt{3}\cos x \le 1$

ii $\sin x - \sqrt{3}\cos x < -1$

iii $\left|\sqrt{3}\sin x + \cos x\right| < 1$

iv $\cos x - \sin x \ge \frac{1}{2}\sqrt{2}$

ENRICHMENT

20 a Prove that:

i $\sin\theta = \cos\left(\theta-\frac{\pi}{2}\right)$

ii $\cos\theta = \sin\left(\theta+\frac{\pi}{2}\right)$

b Using the identity $\sin x + \sqrt{3}\cos x = 2\sin\left(x+\frac{\pi}{3}\right)$, and the identities in part **a**, express $\sin x + \sqrt{3}\cos x$ in each of the other three standard forms.

c Repeat part **b**, this time starting with the identity $\cos x - \sin x = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$.

21 a Prove that $\sin(\theta+\pi) = -\sin\theta$.

b Given the identity $\sqrt{3}\sin x + \cos x = 2\sin\left(x+\frac{\pi}{6}\right)$, use reflection in the y-axis, the fact that $\sin x$ is odd and $\cos x$ is even, and part **a**, to prove that:

i $-\sqrt{3}\sin x + \cos x = 2\sin\left(x+\frac{5\pi}{6}\right)$

ii $-\sqrt{3}\sin x - \cos x = 2\sin\left(x+\frac{7\pi}{6}\right)$

iii $\sqrt{3}\sin x - \cos x = 2\sin\left(x-\frac{\pi}{6}\right)$

22 a Show that if $\cos(x-\alpha) = \cos\beta$, then $\tan x = \tan(\alpha+\beta)$ or $\tan x = \tan(\alpha-\beta)$.

b Show that $2\cos x + 11\sin x = 5\sqrt{5}\cos\left(x-\tan^{-1}\frac{11}{2}\right)$.

c Consider the equation $2\cos x + 11\sin x = 10$, for $0 \le x < 2\pi$.

i By writing the equation in the form $\cos(x-\alpha) = \cos\beta$ and using part **a**, show that $\tan x = \frac{4}{3}$ or $-\frac{24}{7}$.

ii Deduce that the equation has roots $\tan^{-1}\frac{4}{3}$ and $\pi - \tan^{-1}\frac{24}{7}$.

iii Prove that one of the roots is twice the other.

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11C Using the t-formula to solve equations

The t-formulae provide a quite different method of solving an equation of the form $a\sin x + b\cos x = c$. Simply substitute $t = \tan\frac{1}{2}x$, and then everything will quickly be reduced to a quadratic equation. The advantages of this method are that it is far more automatic, and that only a single approximation is involved. But there are two important disadvantages.

- The intuition about the LHS being a shifted wave function is lost.
- If $x = 180°$ happens to be a solution, it will not be found by this method, because $\tan\frac{1}{2}x$ is not defined at $x = 180°$.

The t-formulae are included in the formula review at the start of Section 11A.

Example 9 — 11C

Solve $3\sin x - 4\cos x = -2$, for $-180° \le x \le 180°$, correct to the nearest minute, using the substitution $t = \tan\frac{1}{2}x$.

SOLUTION

Using $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$, the equation becomes

$$\frac{6t}{1+t^2} - \frac{4-4t^2}{1+t^2} = -2, \text{ provided that } x \ne 180°$$

$$6t - 4 + 4t^2 = -2 - 2t^2$$

$$6t^2 + 6t - 2 = 0$$

$$3t^2 + 3t - 1 = 0, \text{ which has discriminant } \triangle = 21,$$

$$\tan\tfrac{1}{2}x = -\tfrac{1}{2} + \tfrac{1}{6}\sqrt{21} \text{ or } -\tfrac{1}{2} - \tfrac{1}{6}\sqrt{21}.$$

Because $-180° \le x \le 180°$, the restriction on $\frac{1}{2}x$ is $-90° \le \frac{1}{2}x \le 90°$,

so $\frac{1}{2}x = 14.775\,961\ldots°$ or $-51.645\,859\ldots°$ (only round at the last step)

$x \doteqdot 29°33'$ or $-103°18'$.

The problem when $x = 180°$ is a solution

The substitution $t = \tan\frac{1}{2}x$ fails when $x = 180°$, because $\tan 90°$ is undefined. We must always be aware of this possibility, and be prepared to add this answer to the final solution. The situation can easily be recognised in either of the following ways:

- The terms in t^2 cancel out, leaving a linear equation in t.
- The coefficient of $\cos x$ is the opposite of the constant term.

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Example 10 11C

Solve $7\sin x - 4\cos x = 4$, for $0° \le x \le 360°$, by using the substitution $t = \tan \frac{1}{2}x$.

SOLUTION

Substituting $t = \tan \frac{1}{2}x$ gives

$$\frac{14t}{1+t^2} - \frac{4-4t^2}{1+t^2} = 4, \text{ provided that } x \ne 180°,$$
$$14t - 4 + 4t^2 = 4 + 4t^2$$
$$14t = 8.$$

Warning: The terms in t^2 have cancelled out — check $t = 180°$!

Hence $\tan \frac{1}{2}x = \frac{4}{7}$

$x \doteqdot 59°29'$.

But $x = 180°$ is also a solution, because then LHS $= 7 \times 0 - 4 \times (-1) =$ RHS,

so $x = 180°$ or $x \doteqdot 59°29'$.

A summary of methods of solving $a\sin x + b\cos x = c$

Here then is a summary of the two approaches to the solution.

8 SOLVING EQUATIONS OF THE FORM $a\sin x + b\cos x = c$:

- **The auxiliary-angle method:** Get the LHS into one of the forms
  $R\sin(x+\alpha)$ or $R\sin(x-\alpha)$ or $R\cos(x+\alpha)$ or $R\cos(x-\alpha)$,
  then solve the resulting equation.

- **Using the t-formulae:** Substitute $t = \tan \frac{1}{2}x$ and then solve the resulting quadratic in t.
  Be aware that $x = 180°$ may also be a solution if:
  — the terms in t^2 cancel out, leaving a linear equation in t, or equivalently,
  — the coefficient of $\cos x$ is the opposite of the constant term.

Exercise 11C

FOUNDATION

1 Consider the equation $\cos x - \sin x = 1$, where $0 \le x \le 2\pi$.

a Using the substitutions $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$, where $t = \tan \frac{1}{2}x$, show that the equation can be written as $t^2 + t = 0$.

b Hence show that $\tan \frac{1}{2}x = 0$ or -1, where $0 \le \frac{1}{2}x \le \pi$.

c Hence solve the given equation for x.

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2 Consider the equation $\sqrt{3}\sin x + \cos x = 1$.

a Show that the equation can be written as $t^2 = \sqrt{3}\,t$, where $t = \tan\frac{1}{2}x$.

b Hence solve the equation, for $0 \le x \le 2\pi$.

3 a Show that the equation $4\cos x + \sin x = 1$ can be written as $(5t + 3)(t - 1) = 0$, where $t = \tan\frac{1}{2}x$.

b Hence solve the equation, for $0° \le x \le 360°$. Give the solutions correct to the nearest minute where necessary.

4 a Show that the equation $3\sin x - 2\cos x = 2$ can be written as $3t - 2 = 0$, where $t = \tan\frac{1}{2}x$.

b Hence solve the equation for $0° \le x \le 360°$, giving solutions correct to the nearest minute where necessary. (Remember to check $x = 180°$ as a possible solution, given that the resulting equation in t is linear.)

5 a Show that the equation $6\sin x - 4\cos x = 5$ can be written as $t^2 - 12t + 9 = 0$, where $t = \tan\frac{1}{2}x$.

b Show that $\tan\frac{1}{2}x = 6 + 3\sqrt{3}$ or $6 - 3\sqrt{3}$.

c Hence show that $77°35'$ and $169°48'$ are the solutions (correct to the nearest minute) of the given equation over the domain $0° \le x \le 360°$.

DEVELOPMENT

6 Solve each equation, for $0° \le x \le 360°$, using the t-formulae, where $= \tan\frac{1}{2}x$. Give solutions correct to the nearest minute where necessary.

a $5\sin x + 4\cos x = 5$

b $7\cos x - 6\sin x = 2$

c $3\sin x - 2\cos x = 1$

d $5\cos x + 6\sin x = -5$

7 Use t-formulae to prove that the equation $8\tan\theta - 4\sec\theta = 1$ has solutions $2\tan^{-1}\frac{1}{3}$ and $2\tan^{-1}5$ in the domain $0 \le \theta \le 2\pi$.

8 Solve the equation $2\sin 2x + \cos 2x = 2$, for $0° \le x \le 360°$, using the substitution $t = \tan x$. Give solutions correct to the nearest tenth of a degree where necessary.

9 Suppose that $a\cos x = 1 + \sin x$, where $0° < x < 90°$.

a Prove that $\dfrac{a - 1}{a + 1} = t$, where $t = \tan\frac{1}{2}x$.

b Hence find, correct to the nearest minute, the acute angle x that satisfies the equation $2\cos x - \sin x = 1$.

10 Let θ_1 and θ_2, where $\theta_1 > \theta_2$, be the solutions of the equation $6\cos\theta + 17\sin\theta = 18$ in the domain $0 \le \theta \le 2\pi$. Show that $\tan\left(\dfrac{\theta_1 - \theta_2}{2}\right) = \dfrac{1}{18}$.

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ENRICHMENT

11 Consider the equation $a\cos x + b\sin x = c$, where a, b and c are constants.

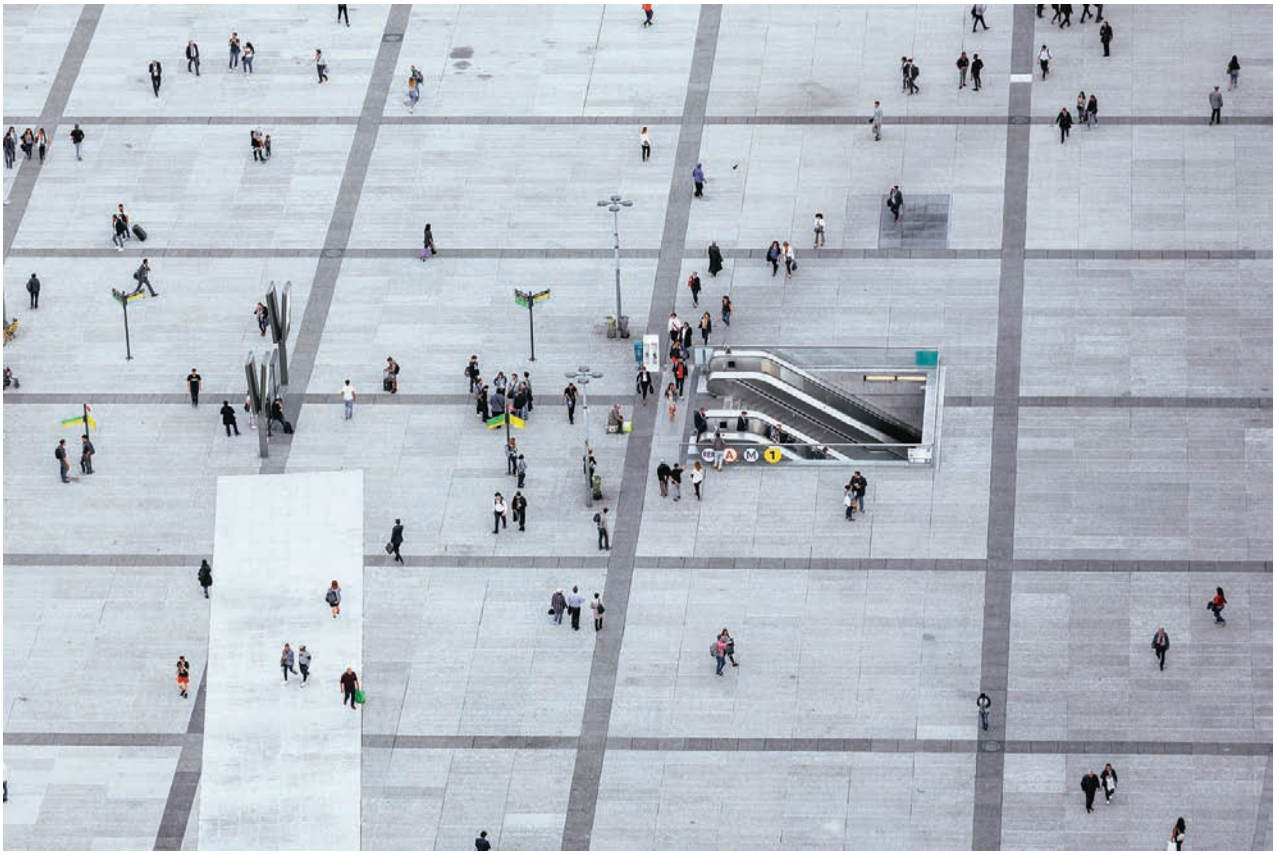
a Show that the equation can be written in the form $(a + c)t^2 - 2bt - (a - c) = 0$, where $t = \tan\frac{1}{2}x$.

b Show that the root(s) of the equation are real if $c^2 \le a^2 + b^2$.

c Suppose that $\tan\frac{1}{2}\alpha$ and $\tan\frac{1}{2}\beta$ are distinct real roots of the quadratic equation in part **a**. Prove that $\tan\frac{1}{2}(\alpha + \beta) = \dfrac{b}{a}$.

12 Given the equation $(2k - 1)\cos\theta + (k + 2)\sin\theta = 2k + 1$, where k is a constant, use t-formulae to prove that $\tan\theta = \dfrac{4}{3}$ or $\dfrac{2k}{k^2 - 1}$.

13 Let θ_1, θ_2, θ_3 and θ_4 be solutions of the equation $a\cos 4\theta + b\sin 4\theta = c$ such that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are distinct. Use the product of the roots of a quartic equation to prove that $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 = 1$.



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Chapter 11 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 11 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Solve, for $0 \le x \le 2\pi$:

a $\sin 2x + \sin x = 0$

b $\cos 2x + \cos x = 0$

c $\cos 2x + 5\sin x + 2 = 0$

d $2\sin\left(x - \frac{\pi}{6}\right) = \cos\left(x - \frac{\pi}{3}\right)$

2 **a** Express $\sin x - \cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Hence solve $\sin x - \cos x = \sqrt{2}$, for $0 \le x \le 2\pi$.

3 **a** Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x - \theta)$, where $A > 0$ and $0 < \theta < \frac{\pi}{2}$.

b Hence solve $\sqrt{3}\cos x + \sin x = -1$, for $0 \le x \le 2\pi$.

4 **a** Express $2\sin x + \sqrt{5}\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and α is acute.

b Hence solve $2\sin x + \sqrt{5}\cos x = 3$, for $0° \le x \le 360°$, writing the solution in degrees correct to one decimal place.

5 **a** Express $3\cos x - 2\sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and θ is acute.

b Hence solve $3\cos x - 2\sin x = 1$, for $0° \le x \le 360°$, writing the solutions correct to the nearest minute.

6 Use a suitable t-formula to solve $\sin x = \tan\frac{1}{2}x$, for $0 \le x \le 2\pi$.

7 Consider the equation $7\sin x + \cos x = 5$.

a Show that the equation can be written as $3t^2 - 7t + 2 = 0$, where $t = \tan\frac{x}{2}$.

b Hence show that the equation has solutions $x = 2\tan^{-1}\frac{1}{3}$ and $x = 2\tan^{-1}2$.

8 Use t-formulae to solve $4\sin x - 2\cos x = 3$, for $0 \le x \le 2\pi$. Write the solutions correct to 2 decimal places.

9 **a** Use compound- and double-angle formulae to prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

b Hence solve $\cos 3x + \sin 2x + \cos x = 0$, for $0 \le x \le 2\pi$.

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