6 The exponential and logarithmic functions

Chapter 11 of the Year 11 book began to extend calculus beyond algebraic functions to exponential functions and trigonometric functions. This chapter completes what is needed of the calculus of exponential functions, and introduces the calculus of the logarithmic functions. Chapter 7 will then bring the trigonometric functions into calculus as well.

The special number $e \doteqdot 2.7183$ was introduced as the most satisfactory base to use for the powers and logarithms discussed in Chapter 8 last year, and we established the two standard derivatives,

$$\frac{d}{dx}\,e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}\,e^{ax+b} = ae^{ax+b}.$$

We sketched the graphs of $y = e^x$ and its inverse function $y = \log_e x$, transformed them in various ways, and developed exponential growth and decay further in Chapter 16.

All this is assumed knowledge in the present chapter and is quickly reviewed in Section 6A and 6F, apart from exponential growth and decay, which will be reviewed in Chapter 9 on motion and rates, then generalised in Chapter 13 on differential equations.

Sections 6A–6E deal mostly with exponential functions base e, Sections 6F–6J deal mostly with logarithmic functions base e, and the final Section 6K uses the change-of-base formula to extend the topic to exponential and logarithmic functions with bases other than e.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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6A Review of exponential functions base e

Section 6A and Section 6F will review the ideas in Sections 11A–11F in the Year 11 book. Two small topics, however, are new in these two review sections.

- Dilations of exponential (Section 6A) and logarithmic (Section 6F) functions.
- Exponential and logarithmic equations reducible to quadratics (Section 6F).

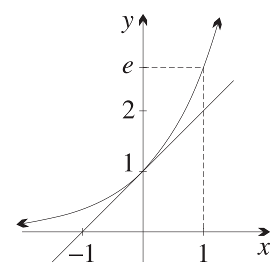
We will not list again the index laws and the logarithmic laws that were covered in Chapter 8 of that book and revisited in Chapter 11, but some early exercises will review them.

The exponential function $y = e^x$ is the subject of Sections 6A–6E. Section 6F then brings the logarithmic function $y = \log_e x$ into the discussion.

The number e and the function $y = e^x$

The fundamental result established in Chapter 11 of the Year 11 book is that the function $y = e^x$ is its own derivative,

$$\frac{d}{dx}e^x = e^x, \qquad \text{that is,} \qquad \text{gradient equals height.}$$



The number $e \doteqdot 2.7183$ is defined to be the base so that the exponential graph $y = e^x$ has gradient exactly 1 at the y-intercept. It is an irrational number, and it plays a role in exponential functions similar to the role that π plays in trigonometric functions.

To the right is a sketch of $y = e^x$. Its most significant properties are listed in Box 1.

1 THE FUNCTION $y = e^x$

- There is only one exponential function $y = e^x$ that is its own derivative, and the number $e \doteqdot 2.7183$ is defined to be the base of this function. Thus

$$\frac{d}{dx}e^x = e^x, \qquad \text{that is,} \qquad \text{at each point, gradient equals height.}$$

- The gradient at the y-intercept is 1.
- The domain is all real numbers, and the range is $y > 0$.
- The line $y = 0$ is a horizontal asymptote.
- The function is one-to-one, that is, the inverse relation is a function.
- Differentiating again, $\dfrac{d^2}{dx^2}e^x = e^x$,

  so the function is always concave up, increasing at an increasing rate.

Sections 6A–6E occasionally require the inverse function $\log_e x$ of e^x, and we need the two inverse function identities:

$$\log_e e^x = x \quad \text{for all real } x \qquad \text{and} \qquad e^{\log_e x} = x \quad \text{for } x > 0.$$

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Using the calculator

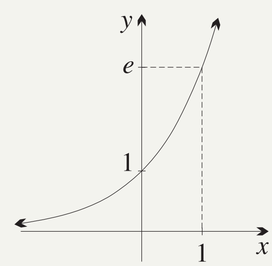
On the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. The function e^x is usually on the same button as $\log_e x$, and is accessed using $\boxed{\text{shift}}$ followed by $\boxed{\ln}$, or by some similar sequence.

Transformations of $y = e^x$

We applied translations and reflections to the curves, as in the next worked example. The first part shows the graph of $y = e^{-x}$, which is just as important in science as $y = e^x$ because $y = e^x$ governs exponential growth, and $y = e^{-x}$ governs exponential decay.

Example 1 6A

Sketch each function using a transformation of the graph of $y = e^x$ sketched to the right. Describe the transformation, show and state the y-intercept and the horizontal asymptote, and write down the range.

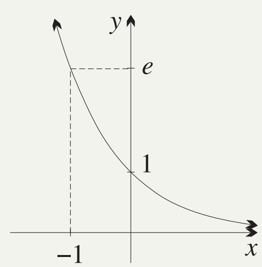


a $y = e^{-x}$ **b** $y = e^x + 3$ **c** $y = e^{x-2}$

Which transformations can also be done using a dilation?

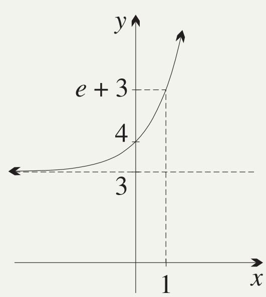
SOLUTION

a



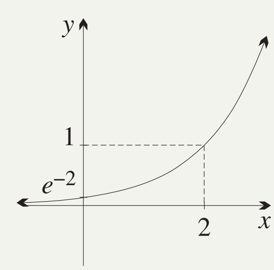
To graph $y = e^{-x}$,
reflect $y = e^x$ in y-axis.
y-intercept: $(0, 1)$
asymptote: $y = 0$
range: $y > 0$

b



To graph $y = e^x + 3$,
shift $y = e^x$ up 3.
y-intercept: $(0, 4)$
asymptote: $y = 3$
range: $y > 3$

c



To graph $y = e^{x-2}$,
shift $y = e^x$ right 2.
y-intercept: $(0, e^{-2})$
asymptote: $y = 0$
range: $y > 0$

- The equation $y = e^{-x}$ in part **a** is a reflection in the y-axis, and any reflection in the y-axis can be regarded as a horizontal dilation with factor -1.
- The equation $y = e^{x-2}$ in part **c** can be written as $y = e^{-2} \times e^x$, so it is also a vertical dilation of $y = e^x$ with factor $e^{-2} \doteqdot 0.135$.

Dilations of $y = e^x$

Dilations were only introduced in Section 3H of this book. In the context of exponential and logarithmic functions, dilations need further attention because some of them have an interesting property — they can be done with a shift in the other direction, as we have already seen in part **c** above.

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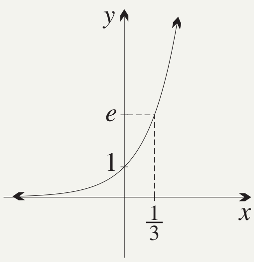
Example 2 6A

Use dilations of $y = e^x$ to generate a sketch of each function. Identify which dilation is also a shift in the other direction.

a $y = e^{3x}$ **b** $y = 3e^x$

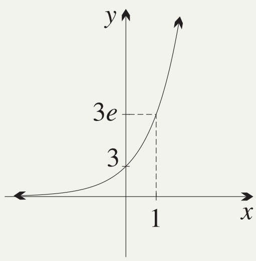
SOLUTION

a



Dilate $y = e^x$ vertically with factor $\frac{1}{3}$.

b



Dilate $y = e^x$ vertically with factor 3.

- $y = 3e^x$ can be written as $y = e^{\log_e 3} \times e^x = e^{x + \log_e 3}$, so it can also be regarded as a shift left by $\log_e 3$.

Tangents and normals to the exponential function

We applied the derivative to sketches of exponential functions. Here is a shortened form of the worked example given in Section 11D of the Year 11 book.

Example 3 6A

Let A be the point on the curve $y = 2e^x$ where $x = 1$.

a Find the equations of the tangent and normal at the point A.

b Show that the tangent at A passes through the origin, and find the point B where the normal meets the x-axis.

c Sketch the situation and find the area of $\triangle AOB$.

SOLUTION

a Substituting into $y = 2e^x$ shows that $A = (1, 2e)$.

Differentiating $y = 2e^x$ gives $y' = 2e^x$,

so at $A\,(1, 2e)$, where $x = 1$, $y' = 2e$ (which we know because gradient = height).

Hence, using point–gradient form, the tangent at A is

$$y - y_1 = m(x - x_1)$$
$$y - 2e = 2e(x - 1)$$
$$y = 2ex.$$

The normal at A has gradient $-\dfrac{1}{2e}$ (it is perpendicular to the tangent),

so its equation is $$y - 2e = -\frac{1}{2e}(x - 1)$$

$\boxed{\times\, 2e}$ $$2ey - 4e^2 = -x + 1$$
$$x + 2ey = 4e^2 + 1.$$

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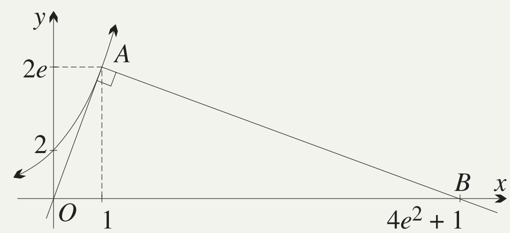
b The tangent passes through the origin O because its y-intercept is zero.
To find the x-intercept B of the normal,
put $y = 0$,
thus $x = 4e^2 + 1$,
so B has coordinates $(4e^2 + 1, 0)$.



c Hence area $\triangle AOB = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times (4e^2 + 1) \times 2e$
$= e(4e^2 + 1)$ square units.

Exercise 6A

FOUNDATION

Note: You will need the $\boxed{e^x}$ function on your calculator. This will require $\boxed{\text{shift}}$ followed by $\boxed{\ln}$, or some similar sequence of keys.

1 Simplify these expressions using the index laws.
a $2^3 \times 2^7$ **b** $e^4 \times e^3$ **c** $2^6 \div 2^2$ **d** $e^8 \div e^5$ **e** $(2^3)^4$ **f** $(e^5)^6$

2 Simplify these expressions using the index laws.
a $e^{2x} \times e^{5x}$ **b** $e^{10x} \div e^{8x}$ **c** $(e^{2x})^5$ **d** $e^{2x} \times e^{-7x}$ **e** $e^x \div e^{-4x}$ **f** $(e^{-3x})^4$

3 Write each expression as a power of e, then use your calculator to approximate it correct to four significant figures.
a e^2 **b** e^{-3} **c** e **d** $\dfrac{1}{e}$ **e** $\sqrt{e}$ **f** $\dfrac{1}{\sqrt{e}}$

4 a Write down the first and second derivatives of $y = e^x$.
b Hence copy and complete the sentence, 'The curve $y = e^x$ is always concave . . . , and is always . . . at . . . rate.'

5 a Find the gradient of the tangent to $y = e^x$ at $P(1, e)$, then find the equation of the tangent at P and show that it has x-intercept 0.
b Similarly find the equation of the tangent at $Q(0, 1)$, and show that its x-intercept is -1.
c Find the equation of the tangent at $R(-1, \frac{1}{e})$, and show that its x-intercept is -2.

6 a What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?
b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.
c Hence find the equations of the tangent and normal at P (in general form).

7 Sketch each curve using a single transformation of $y = e^x$, and describe the transformation.
a $y = e^x + 1$ **b** $y = e^x - 2$ **c** $y = \frac{1}{3}e^x$ **d** $y = e^{\frac{1}{2}x}$

8 Sketch each curve using a single transformation of $y = e^{-x}$, and describe the transformation.
a $y = e^{-x} - 1$ **b** $y = -e^{-x}$ **c** $y = e^{-2x}$

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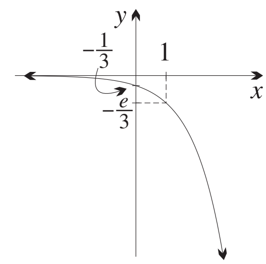
DEVELOPMENT

9 The graph to the right is a dilation of $y = e^x$. Describe the dilation, and write down the equation of the curve.



10 Expand and simplify:

a $(e^x + 1)(e^x - 1)$ **b** $(e^{4x} + 3)(e^{2x} + 3)$

c $(e^{-3x} - 2)e^{3x}$ **d** $(e^{-2x} + e^{2x})^2$

11 Write as a sum of powers of e:

a $\dfrac{e^{4x} + e^{3x}}{e^{2x}}$ **b** $\dfrac{e^{2x} - e^{3x}}{e^{4x}}$ **c** $\dfrac{e^{10x} + 5e^{20x}}{e^{-10x}}$ **d** $\dfrac{6e^{-x} + 9e^{-2x}}{3e^{3x}}$

12 **a** What is the gradient of the tangent to $y = e^x$ at its y-intercept?

b What transformation maps $y = e^x$ to $y = e^{-x}$?

c Use this transformation to find the gradient of $y = e^{-x}$ at its y-intercept.

d Sketch $y = e^x$ and $y = e^{-x}$ on one set of axes.

e How can the transformation be interpreted as a dilation?

13 Write down the first four derivatives of each function. For which curves is it true that at each point on the curve, the gradient equals the height?

a $y = e^x + 5$ **b** $y = e^x + x^3$ **c** $y = 4e^x$ **d** $y = 5e^x + 5x^2$

14 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = e^x$ at the points where:

a $x = 0$ **b** $x = 1$ **c** $x = -2$ **d** $x = 5$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

15 **a** What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?

b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.

c Hence find the equation of the tangent at the point P found in part **a**.

ENRICHMENT

16 **a** Use, and describe, a dilation to sketch $y = e^{2x}$.

b Use, and describe, a subsequent translation to sketch $y = e^{2(x-1)}$.

c Use, and describe, a subsequent dilation to sketch $y = \frac{1}{2}e^{2(x-1)}$.

d Use, and describe, a subsequent translation to sketch $y = \frac{1}{2}e^{2(x-1)} - 2$.

17 **a** Interpret the transformation from $y = e^x$ to $y = e^{x+2}$ as a translation. Then interpret it as a dilation.

b Interpret the transformation from $y = e^x$ to $y = 2e^x$ as a dilation. Then interpret it as a translation by first writing the coefficient 2 as $e^{\log_e 2}$.

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6B Differentiation of exponential functions

We can now develop the calculus of functions involving e^x, picking up the story at differentiation, where two standard forms were established in Chapter 11 (Year 11),

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}e^{ax+b} = ae^{ax+b}.$$

Using the two standard forms

The second standard form above requires the chain rule with $u = ax + b$. It is proven again in Question **6a** of Exercise 6B.

Example 4 — 6B

Differentiate:

a $y = e^x + e^{-x}$ **b** $y = 5e^{4x-3}$ **c** $y = e^{2-\frac{1}{2}x}$ **d** $y = \sqrt{e^x} + \dfrac{1}{\sqrt{e^x}}$

SOLUTION

a Given $y = e^x + e^{-x}$.
For e^{-x}, $a = -1$ and $b = 0$,
so $y' = e^x - e^{-x}$.

b Given $y = 5e^{4x-3}$.
Here $a = 4$ and $b = -3$,
so $y' = 20e^{4x-3}$.

c Given $y = e^{2-\frac{1}{2}x}$.
Here $a = -\frac{1}{2}$ and $b = 2$,
so $y' = -\frac{1}{2}e^{2-\frac{1}{2}x}$.

d Here $y = \sqrt{e^x} + \dfrac{1}{\sqrt{e^x}}$
$y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$,
so $y' = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x}$.

Differentiating using the chain rule

The chain rule can be applied in the usual way. As always, the full setting out should continue to be used until readers are very confident with missing some of the steps.

Example 5 — 6B

Use the chain rule to differentiate:

a $y = e^{1-x^2}$ **b** $y = (e^{2x} - 3)^4$

SOLUTION

a Here $y = e^{1-x^2}$.
Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -2xe^{1-x^2}.$$

Let $u = 1 - x^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = -2x$
and $\dfrac{dy}{du} = e^u$.

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b Here $y = (e^{2x} - 3)^4$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4(e^{2x} - 3)^3 \times 2e^{2x}$$
$$= 8e^{2x}(e^{2x} - 3)^3.$$

Let $u = e^{2x} - 3$.

Then $y = u^4$.

Hence $\dfrac{du}{dx} = 2e^{2x}$

and $\dfrac{dy}{du} = 4u^3$.

A formula for the chain rule

Some people prefer to learn a formula for chain rule differentiation that can be used for part **a** above. The formula can be written in two ways, using u and using $f(x)$,

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \qquad \text{OR} \qquad \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x).$$

If you use the formula, choose which form you prefer. In the next worked example, part **a** of the previous example is done again using both forms of the formula. Make sure that you are using the right formula, and that you show at least u or $f(x)$ on the right.

Notice that part **b** requires the formula for differentiating powers of functions of x, as reviewed in Section 5I,

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \text{OR} \qquad \frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}f'(x).$$

Example 6 — 6B

Use the chain rule, with a shorter setting out, to differentiate:

a $y = e^{1-x^2}$

b $y = (e^{2x} - 3)^4$

SOLUTION

a $y = e^{1-x^2}$

$y' = -2xe^{1-x^2}$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$

$\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$.

OR

Let $f(x) = 1 - x^2$.

Then $f'(x) = -2x$

$\dfrac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$.

b $y = (e^{2x} - 3)^4$

$y' = 8e^{2x} \times (e^{2x} - 3)^3$

Let $u = e^{2x} - 3$.

Then $\dfrac{du}{dx} = 2e^{2x}$

$\dfrac{d}{dx}u^4 = 4u^3\dfrac{du}{dx}$.

OR

Let $f(x) = e^{2x} - 3$.

Then $f'(x) = 2e^{2x}$

$\dfrac{d}{dx}\big(f(x)\big)^4 = 4\big(f(x)\big)^3 f'(x)$.

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2 THREE STANDARD DERIVATIVES FOR EXPONENTIAL FUNCTIONS

- $\dfrac{d}{dx}e^x = e^x$
- $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$
- $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$ OR $\dfrac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$

Using the product rule

A function such as $y = x^3e^x$ is the product of the two functions $u = x^3$ and $v = e^x$. Thus it can be differentiated by the product rule.

Often the result can be factored, allowing any stationary points to be found.

Example 7 6B

Find the derivatives of these functions. Then factor the derivative and write down all the stationary points.

a $y = x^3e^x$ **b** $y = xe^{5x-2}$

SOLUTION

a Here $y = x^3e^x$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= e^x \times 3x^2 + x^3 \times e^x,$$

and taking out the common factor x^2e^x,

$$y' = x^2e^x(3 + x).$$

Hence y' has zeroes at $x = 0$ and $x = -3$,

and the stationary points are $(0, 0)$ and $(-3, -27e^{-3})$.

Let $u = x^3$
and $v = e^x$.
Then $u' = 3x^2$
and $v' = e^x$.

b Here $y = xe^{5x-2}$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= e^{5x-2} \times 1 + x \times 5e^{5x-2}$$
$$= e^{5x-2}(1 + 5x).$$

Hence y' has a stationary point at $\left(-\tfrac{1}{5}, -\tfrac{1}{5}e^{-3}\right)$.

Let $u = x$
and $v = e^{5x-2}$.
Then $u' = 1$
and $v' = 5e^{5x-2}$.

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Using the quotient rule

A function such as $y = \dfrac{e^{5x}}{x}$ is the quotient of the two functions $u = e^{5x}$ and $v = x$. Thus it can be differentiated by the quotient rule.

Example 8 6B

Differentiate these functions, then find the x-values of all stationary points.

a $\dfrac{e^{5x}}{x}$ **b** $\dfrac{e^x}{1 - x^2}$

SOLUTION

a Let $y = \dfrac{e^{5x}}{x}$. Then applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{5xe^{5x} - e^{5x}}{x^2}$$
$$= \frac{e^{5x}(5x - 1)}{x^2}.$$

Let $u = e^{5x}$
and $v = x$.
Then $u' = 5e^{5x}$
and $v' = 1$.

Hence there is a stationary point where $x = \frac{1}{5}$.

b Let $y = \dfrac{e^x}{1 - x^2}$. Then applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{(1 - x^2)e^x + 2xe^x}{(1 - x^2)^2}$$
$$= \frac{e^x(1 + 2x - x^2)}{(1 - x^2)^2}.$$

Let $u = e^x$
and $v = 1 - x^2$.
Then $u' = e^x$
and $v' = -2x$.

Hence there is a stationary point where $x^2 - 2x - 1 = 0$,
and calculating $\Delta = 8$ first, $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$.

Exercise 6B

FOUNDATION

Technology: Programs that perform algebraic differentiation can be used to confirm the answers to many of these questions.

1 Use the standard form $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{7x}$ **b** $y = 4e^{3x}$ **c** $y = 6e^{\frac{1}{3}x}$ **d** $y = -\frac{1}{2}e^{-2x}$
e $y = e^{3x+4}$ **f** $y = e^{4x-3}$ **g** $y = e^{-3x+4}$ **h** $y = e^{-2x-7}$

2 Differentiate:

a $y = e^x + e^{-x}$ **b** $y = e^{2x} - e^{-3x}$ **c** $y = \dfrac{e^x - e^{-x}}{2}$

d $y = \dfrac{e^x + e^{-x}}{3}$ **e** $y = \dfrac{e^{2x}}{2} + \dfrac{e^{3x}}{3}$ **f** $y = \dfrac{e^{4x}}{4} + \dfrac{e^{5x}}{5}$

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3 Use the index laws to write each expression as a single power of e, then differentiate it.

a $y = e^x \times e^{2x}$ b $y = e^{3x} \times e^{-x}$ c $y = (e^x)^2$ d $y = (e^{2x})^3$

e $y = \dfrac{e^{4x}}{e^x}$ f $y = \dfrac{e^x}{e^{2x}}$ g $y = \dfrac{1}{e^{3x}}$ h $y = \dfrac{1}{e^{5x}}$

4 a i For the function $f(x) = e^{-x}$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

ii What is the pattern in these derivatives?

b i For the function $f(x) = e^{2x}$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

ii What is the pattern in these derivatives?

5 Expand the brackets and then differentiate:

a $e^x(e^x + 1)$ b $e^{-x}(2e^{-x} - 1)$ c $(e^x + 1)^2$ d $(e^x + 3)^2$

e $(e^x - 1)^2$ f $(e^x - 2)^2$ g $(e^x + e^{-x})(e^x - e^{-x})$ h $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

DEVELOPMENT

6 Use the chain rule with full setting-out to differentiate:

a $y = e^{ax+b}$ b $y = e^{x^2}$ c $y = e^{-\frac{1}{2}x^2}$ d $y = e^{x^2+1}$

e $y = e^{1-x^2}$ f $y = e^{x^2+2x}$ g $y = e^{6+x-x^2}$ h $y = \frac{1}{2}e^{3x^2-2x+1}$

7 Use the product rule to differentiate:

a $y = xe^x$ b $y = xe^{-x}$ c $y = (x - 1)e^x$ d $y = (x + 1)e^{3x-4}$

e $y = x^2e^{-x}$ f $y = (2x - 1)e^{2x}$ g $y = (x^2 - 5)e^x$ h $y = x^3e^{2x}$

8 Use the quotient rule to differentiate:

a $y = \dfrac{e^x}{x}$ b $y = \dfrac{x}{e^x}$ c $y = \dfrac{e^x}{x^2}$ d $y = \dfrac{x^2}{e^x}$

e $y = \dfrac{e^x}{x + 1}$ f $y = \dfrac{x + 1}{e^x}$ g $y = \dfrac{x - 3}{e^{2x}}$ h $y = \dfrac{1 - x^2}{e^x}$

9 Expand and simplify each expression, then differentiate it.

a $(e^x + 1)(e^x + 2)$ b $(e^{2x} + 3)(e^{2x} - 2)$ c $(e^{-x} + 2)(e^{-x} + 4)$

d $(e^{-3x} - 1)(e^{-3x} - 5)$ e $(e^{2x} + 1)(e^x + 1)$ f $(e^{3x} - 1)(e^{-x} + 4)$

10 Use the chain rule to differentiate:

a $y = (1 - e^x)^5$ b $y = (e^{4x} - 9)^4$ c $y = \dfrac{1}{e^x - 1}$ d $y = \dfrac{1}{(e^{3x} + 4)^2}$

11 a Show by substitution that the function $y = 3e^{2x}$ satisfies the equation $\dfrac{dy}{dx} = 2y$.

b Show by substitution that the function $y = 5e^{-4x}$ satisfies the equation $\dfrac{dy}{dx} = -4y$.

12 Find the first and second derivatives of each function below. Then evaluate both derivatives at the value given.

a $f(x) = e^{2x+1}$ at $x = 0$ b $f(x) = e^{-3x}$ at $x = 1$

c $f(x) = xe^{-x}$ at $x = 2$ d $f(x) = e^{-x^2}$ at $x = 0$

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13 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{ax}$ **b** $y = e^{-kx}$ **c** $y = Ae^{kx}$ **d** $y = Be^{-\ell x}$

e $y = e^{px+q}$ **f** $y = Ce^{px+q}$ **g** $y = \dfrac{e^{px} + e^{-qx}}{r}$ **h** $\dfrac{e^{ax}}{a} + \dfrac{e^{-px}}{p}$

14 Use the product, quotient and chain rules as appropriate to differentiate:

a $y = (e^x + 1)^3$ **b** $y = (e^x + e^{-x})^4$ **c** $y = (1 + x^2)\,e^{1+x}$

d $y = (x^2 - x)\,e^{2x-1}$ **e** $y = \dfrac{e^x}{e^x + 1}$ **f** $y = \dfrac{e^x + 1}{e^x - 1}$

15 Write each expression as the sum of simple powers of e, then differentiate it.

a $y = \dfrac{e^x + 1}{e^x}$ **b** $y = \dfrac{e^{2x} + e^x}{e^x}$ **c** $y = \dfrac{2 - e^x}{e^{2x}}$

d $y = \dfrac{3 + e^x}{e^{4x}}$ **e** $y = \dfrac{e^x + e^{2x} - 3e^{4x}}{e^x}$ **f** $y = \dfrac{e^{2x} + 2e^x + 1}{e^{2x}}$

16 **a** Show that $y = 2\,e^{3x}$ is a solution of each equation by substituting separately into the LHS and RHS:

i $y' = 3y$ **ii** $y'' - 9y = 0$

b Show that $y = \frac{1}{2}e^{-3x} + 4$ is a solution of $\frac{dy}{dx} = -3(y - 4)$ by substituting y into each side of the equation.

c Show that $y = e^{-3x} + x - 1$ is a solution of the differential equation $y'' + 2y' - 3y = 5 - 3x$.

d Show by substitution that each function is a solution of the equation $y'' + 2y' + y = 0$.

i $y = e^{-x}$ **ii** $y = x\,e^{-x}$

ENRICHMENT

17 Differentiate these functions.

a $y = \sqrt{e^x}$ **b** $y = \sqrt[3]{e^x}$ **c** $y = \dfrac{1}{\sqrt{e^x}}$ **d** $y = \dfrac{1}{\sqrt[3]{e^x}}$

e $e^{\sqrt{x}}$ **f** $e^{-\sqrt{x}}$ **g** $e^{\frac{1}{x}}$ **h** $e^{-\frac{1}{x}}$

i $e^{x-\frac{1}{x}}$ **j** e^{e^x}

18 Define the two functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

a Show that $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$.

b Find the second derivative of each function, and show that they both satisfy $y'' = y$.

c Show that $\cosh^2 x - \sinh^2 x = 1$.

19 **a** Show that $y = Ae^{kx}$ is a solution of:

i $y' = ky$ **ii** $y'' - k^2 y = 0$

b Show that $y = Ae^{kx} + C$ is a solution of $\frac{dy}{dx} = k(y - C)$.

c Show that $y = (Ax + B)\,e^{3x}$ is a solution of $y'' - 6y' + 9y = 0$.

20 Find the values of λ that make $y = e^{\lambda x}$ a solution of:

a $y'' + 3y' - 10y = 0$ **b** $y'' + y' - y = 0$

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6C Applications of differentiation

Differentiation can now be applied in the usual ways to examine functions involving e^x. Sketching of such curves is an important application. Some of these sketches require some subtle limits that would normally be given if a question needed them.

The graphs of e^x and e^{-x}

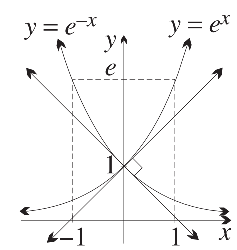
The graphs of $y = e^x$ and $y = e^{-x}$ are the fundamental graphs of this chapter. Because x is replaced by $-x$ in the second equation, the two graphs are reflections of each other in the y-axis.

For $y = e^x$:

x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$. The gradient of $y = e^x$ at $(0, 1)$ is 1, so by reflection, the gradient of $y = e^{-x}$ at $(0, 1)$ is -1. This means that the curves are perpendicular at their point of intersection.

As remarked earlier, the function $y = e^{-x}$ is just as important as $y = e^x$ in applications. It describes a great many physical situations where a quantity 'dies away exponentially', like the dying away of the sound of a plucked string.

An example of curve sketching

The following curve-sketching example illustrates the use of the six steps of our informal curve-sketching menu in the context of exponential functions. One special limit is given in part **d** so that the sketch may be completed.

Example 9 — 6C

Sketch the graph of $y = xe^{-x}$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function, and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. (You may assume that as $x \to \infty$, $xe^{-x} \to 0$.)

e Find any stationary points and examine their nature.

f Find any points of inflection, and examine the concavity.

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SOLUTION

a The domain of $y = xe^{-x}$ is the whole real number line.

b $f(-x) = -xe^{x}$, which is neither $f(x)$ nor $-f(x)$, so the function is neither even nor odd.

c The only zero is $x = 0$. From the table of signs, y is positive for $x > 0$ and negative for $x < 0$.

x	-1	0	1
y	$-e$	0	e^{-1}
sign	$-$	0	$+$

d As given in the question, $y \to 0$ as $x \to \infty$, so the x-axis is a horizontal asymptote on the right. Also, $y \to -\infty$ as $x \to -\infty$.

e Differentiating using the product rule,

$$\begin{aligned} f'(x) &= vu' + uv' \\ &= e^{-x} - xe^{-x} \\ &= e^{-x}(1 - x). \end{aligned}$$

Let $u = x$
and $v = e^{-x}$.
Then $u' = 1$
and $v' = -e^{-x}$.

Hence $f'(x) = 0$ when $x = 1$ (notice that e^{-x} can never be zero), so $\left(1, \frac{1}{e}\right)$ is the only stationary point.

Differentiating again by the product rule,

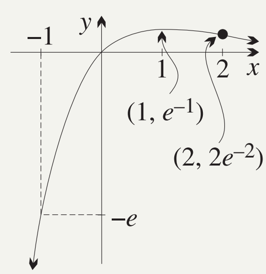
$$\begin{aligned} f''(x) &= vu' + uv' \\ &= -e^{-x} - (1 - x)e^{-x} \\ &= e^{-x}(x - 2), \end{aligned}$$

Let $u = 1 - x$
and $v = e^{-x}$.
Then $u' = -1$
and $v' = -e^{-x}$.

so $f''(1) = -e^{-1} < 0$, and $(1, e^{-1})$ is thus a maximum turning point.

f $f''(x) = e^{-x}(x - 2)$ has a zero at $x = 2$, and taking test values around $x = 2$,

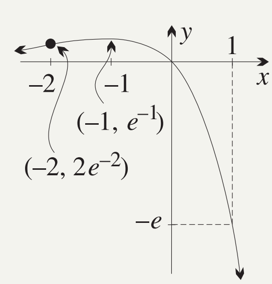
x	0	2	3
$f''(x)$	-2	0	e^{-3}
	$\frown$	$\cdot$	$\smile$



Thus there is an inflection at $(2, 2e^{-2}) \doteqdot (2, 0.27)$.
The curve is concave down for $x < 2$ and concave up for $x > 2$.

Example 10 — 6C

[Transforming graphs]
Use a suitable transformation of the graph sketched in the previous worked example to sketch $y = -xe^{x}$.



SOLUTION

$y = xe^{-x}$ becomes $y = -xe^{x}$ when x is replaced by $-x$. Graphically, this transformation is a reflection in the y-axis, hence the new graph is as sketched to the right.

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A difficulty with the limits of xe^x and xe^{-x}

Sketching the graph of $y = xe^{-x}$ above required knowing the behaviour of xe^{-x} as $x \to \infty$. This limit is puzzling, because when x is a large number, e^{-x} is a small positive number, and the product of a large number and a small number could be large, small, or anything in between.

In fact, e^{-x} gets small as $x \to \infty$ much more quickly than x gets large, and the product xe^{-x} gets small. The technical term for this is that e^{-x} *dominates* x. A table of values should make it reasonably clear that $\lim_{x \to \infty} xe^{-x} = 0$.

x	0	1	2	3	4	5	6	7	...
xe^{-x}	0	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	...
approx	0	0.37	0.27	0.15	0.073	0.034	0.015	0.006	...

Similarly, when x is a large negative number, e^x is a very small number, so it is unclear whether xe^x is large or small. Again, e^x dominates x, meaning that $xe^x \to 0$ as $x \to -\infty$. A similar table should make this reasonably obvious.

x	0	-1	-2	-3	-4	-5	-6	-7	...
xe^x	0	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	...
approx	0	-0.37	-0.27	-0.15	-0.073	-0.034	-0.015	-0.006	...

These limits would normally be given in any question where they are needed, but we will nevertheless box the general results for completeness.

3 DOMINANCE

- The function e^x *dominates* the function x, that is

$$\lim_{x \to \infty} xe^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} xe^x = 0.$$

- More generally, the function e^x *dominates* the function x^k, for all $k > 0$,

$$\lim_{x \to \infty} x^k e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} x^k e^x = 0.$$

More colourfully, in a battle between x and e^x, e^x always wins. We will provide a proof in Question **23** of Exercise 6J.

Exercise 6C

FOUNDATION

Technology: Graphing programs can be used in this exercise to sketch the curves and then investigate the effects on the curve of making small changes in the equations. It is advisable, however, to puzzle out most of the graphs first using the standard methods of the curve-sketching menu.

The results about dominance would normally be given in questions where they are needed, but this exercise is an exception because so many questions require them.

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1 **a** Find the y-coordinate of the point A on the curve $y = e^{2x-1}$ where $x = \frac{1}{2}$.

b Find the derivative of $y = e^{2x-1}$, and show that the gradient of the tangent at A is 2.

c Hence find the equation of the tangent at A, and prove that it passes through O.

2 **a** Write down the coordinates of the point R on the curve $y = e^{3x+1}$ where $x = -\frac{1}{3}$.

b Find $\dfrac{dy}{dx}$ and hence show that the gradient of the tangent at R is 3.

c What is the gradient of the normal at R?

d Hence find the equation of the normal at R in general form.

3 **a** Find the equation of the normal to $y = e^{-x}$ at the point $P(-1, e)$.

b Find the x- and y-intercepts of the normal.

c Find the area of the triangle whose vertices lie at the intercepts and the origin.

4 **a** Find the equation of the tangent to $y = e^{x}$ at its y-intercept $B(0, 1)$.

b Find the equation of the tangent to $y = e^{-x}$ at its y-intercept $B(0, 1)$.

c Find the points F and G where the tangents in parts **a** and **b** meet the x-axis.

d Sketch $y = e^{x}$ and $y = e^{-x}$ on the same set of axes, showing the two tangents.

e What sort of triangle is $\triangle BFG$, and what is its area?

5 **a** Show that the equation of the tangent to $y = 1 - e^{-x}$ at the origin is $y = x$.

b Deduce the equation of the normal at the origin without further use of calculus.

c What is the equation of the asymptote of this curve?

d Sketch the curve, showing the points T and N where the tangent and normal respectively cut the asymptote.

e Find the area of $\triangle OTN$.

6 **a** Find the first and second derivatives for the curve $y = x - e^{x}$.

b Deduce that the curve is concave down for all values of x.

c Find any stationary points, then determine their nature using the second derivative.

d Sketch the curve and write down its range.

e Finally, sketch $y = e^{x} - x$ by recognising the simple transformation.

DEVELOPMENT

7 **a** Show that the tangent to $y = e^{x}$ at $T(t, e^{t})$ has gradient e^{t}.

b Find the equation of the tangent at $x = t$, and show that its x-intercept is $t - 1$.

c Compare this result with Question **4** parts **a** and **c** above, and explain geometrically what is happening.

8 Consider the curve $y = x\,e^{x}$.

a Where is the function zero, positive and negative? Is it even, odd or neither?

b Show that $y' = (1 + x)\,e^{x}$ and $y'' = (2 + x)\,e^{x}$.

c Show that there is one stationary point, and determine its nature.

d Find the coordinates of the lone point of inflection.

e What happens to y, y' and y'' as $x \to \infty$?

f Given that $y \to 0$ as $x \to -\infty$, sketch the curve, then write down its range.

g Hence also sketch $y = -x\,e^{-x}$ by recognising the simple transformation.

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9 Consider the function $y = (1 - x)\, e^{x}$.

a Find the zero and draw up a table of signs.

b Find y' and y''.

c Show that the curve has a maximum turning point at its y-intercept, and a point of inflection at $(-1, 2\,e^{-1})$.

d What happens to y, y' and y'' as $x \to \infty$?

e Given that $y \to 0$ as $x \to -\infty$, sketch the graph and write down its range.

10 a Given that $y = x^2\, e^{-x}$, show that $y' = x(2 - x)\, e^{-x}$ and $y'' = (2 - 4x + x^2)\, e^{-x}$.

b Show that the function has a minimum turning point at the origin and a maximum turning point at $(2, 4\,e^{-2})$.

c i Show that $y'' = 0$ at $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$.

ii Use a table of values for y'' to show that there are inflection points at these values.

d Given that $y \to 0$ as $x \to \infty$, sketch the graph and write down its range.

11 a Find the intercepts of $y = (1 + x)^2\, e^{-x}$.

b Show that the curve has two turning points, and classify them.

c Examine the behaviour of y as $x \to \infty$ and hence deduce that it also has two inflections.

d Sketch the curve and write down its range.

12 Show that $y = (x^2 + 3x + 2)\, e^{x}$ has an inflection point at one of its x-intercepts. Sketch the curve and label all important features (ignore the y-coordinates of the stationary points).

13 a What is the natural domain of $y = \dfrac{e^{x}}{x}$?

b Show that the curve has a local minimum at $(1, e)$ but no inflection points.

c Sketch the curve and state its range.

14 a Given that $y = e^{-\frac{1}{2}x^2}$, find y' and y''.

b Show that this curve has a maximum turning point at its y-intercept, and has two points of inflection.

c Examine the behaviour of y as $x \to -\infty$ and $x \to \infty$.

d Sketch the graph and write down its range.

15 a Classify the stationary points of $y = x\, e^{-x^2}$.

b Locate the three inflection points, sketch the curve and write down its range.

16 [Technology] This question extends the remarks made in the text about *dominance*. Using a calculator or a spreadsheet, draw up a table of values to examine:

a the behaviour of $y = \dfrac{e^{x}}{x}$: **i** as $x \to -\infty$, **ii** as $x \to \infty$.

b the behaviour of $y = \dfrac{e^{-x}}{x}$: **i** as $x \to -\infty$, **ii** as $x \to \infty$.

c the behaviour of $x^k e^k$ as $x \to \infty$ and as $x \to -\infty$, for various real values of k.

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ENRICHMENT

17 Find the x-coordinates of the stationary points of $y = x\,e^{-|x|}$.

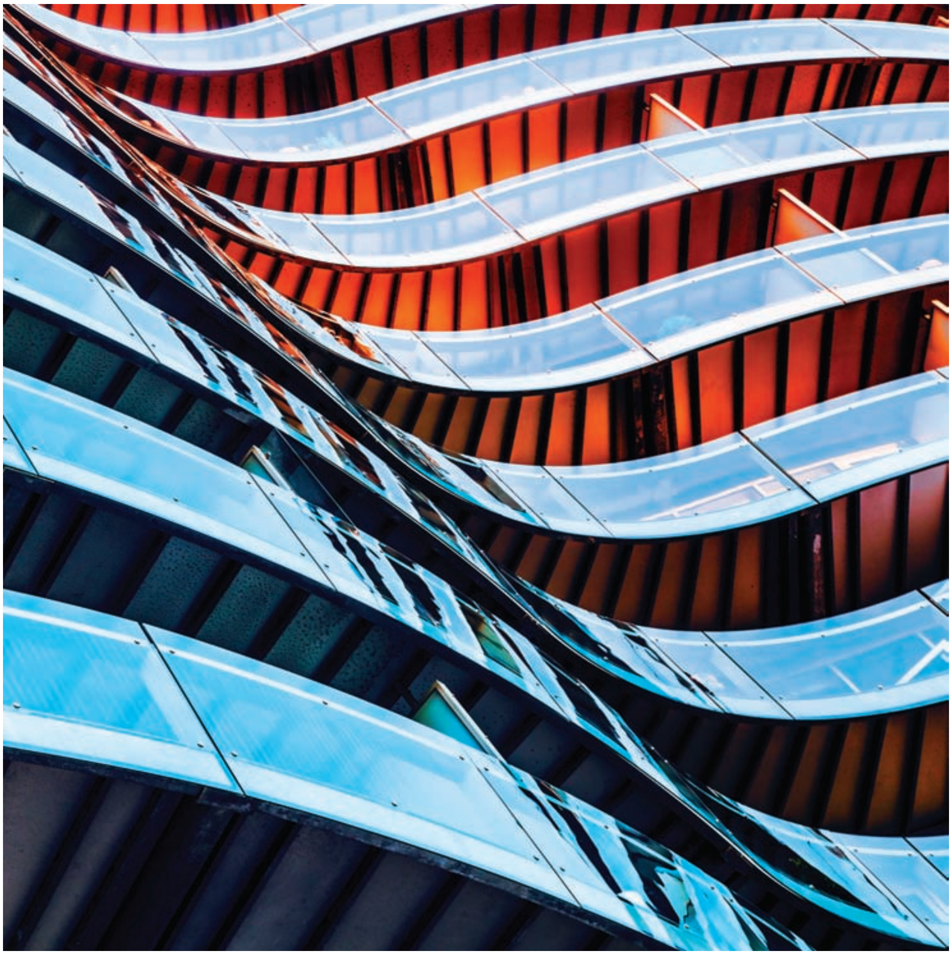
18 **a** What is the natural domain of $y = e^{\frac{1}{x}}$?

b Carefully determine the behaviour of y and y' as $x \to -\infty$, $x \to 0$ and $x \to \infty$.

c Deduce that there must be an inflection point and find it.

d Sketch the curve and give its range.

e Follow steps **a–d** to sketch the graph of $y = x\,e^{\frac{1}{x}}$.



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6D Integration of exponential functions

Finding primitives is the reverse of differentiation. Thus the new standard forms for differentiation can now be reversed to provide standard forms for integration.

Standard forms for integration

Reversing the standard forms for differentiating exponential functions gives the standard forms for integrating them.

Reversing $\dfrac{d}{dx}e^x = e^x$ gives $\displaystyle\int e^x\,dx = e^x + C$, for some constant C.

Reversing $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$ gives $\displaystyle\int ae^{ax+b}\,dx = e^{ax+b}$,

and dividing through by a, $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$, for some constant C.

4 STANDARD FORMS FOR INTEGRATION

- $\displaystyle\int e^x\,dx = e^x + C$, for some constant C
- $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$, for some constant C

There is also an associated formula for the reverse chain rule, but it is not included in the course. For reference, this formula has the usual two forms:

$$\int e^u\,\frac{du}{dx}\,dx = e^u + C \qquad \text{OR} \qquad \int e^{f(x)}f'(x)\,dx = e^{f(x)} + C.$$

Example 11 **6D**

Find these indefinite integrals.

a $\displaystyle\int e^{3x+2}\,dx$ **b** $\displaystyle\int (1 - x + e^x)\,dx$

SOLUTION

a $\displaystyle\int e^{3x+2}\,dx = \tfrac{1}{3}e^{3x+2} + C$ ($a = 3$ and $b = 2$)

b $\displaystyle\int (1 - x + e^x)\,dx = x - \tfrac{1}{2}x^2 + e^x + C$ (integrating each term separately)

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Definite integrals

Definite integrals are evaluated in the usual way by finding the primitive and substituting.

Example 12 6D

Evaluate these definite integrals.

a $\displaystyle\int_0^2 e^x\,dx$

b $\displaystyle\int_2^3 e^{5-2x}\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^2 e^x\,dx &= \Big[e^x\Big]_0^2\\ &= e^2 - e^0\\ &= e^2 - 1\end{aligned}$$

b
$$\begin{aligned}\int_2^3 e^{5-2x}\,dx &= -\tfrac{1}{2}\Big[e^{5-2x}\Big]_2^3 \qquad (a = -2 \text{ and } b = 5)\\ &= -\tfrac{1}{2}(e^{-1} - e)\\ &= -\frac{1}{2}\left(\frac{1}{e} - e\right)\\ &= \frac{e^2 - 1}{2e}\end{aligned}$$

Given the derivative, find the function

As before, if the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be determined.

Example 13 6D

It is known that $f'(x) = e^x$ and that $f(1) = 0$.

a Find the original function $f(x)$.

b Hence find $f(0)$.

SOLUTION

a It is given that $f'(x) = e^x$.

Taking the primitive, $f(x) = e^x + C$, for some constant C.

Substituting $f(1) = 0$, $0 = e^1 + C$

$C = -e$.

Hence $f(x) = e^x - e$.

b Substituting $x = 0$, $f(0) = e^0 - e$

$= 1 - e$.

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Example 14 6D

a If $y' = 1 + 2e^{-x}$ and $y(0) = 1$, find y.
b Hence find $y(1)$.

SOLUTION

a It is given that $y' = 1 + 2e^{-x}$.
Taking the primitive, $y = x - 2e^{-x} + C$, for some constant C.
Substituting $y(0) = 1$,
$$\begin{aligned} 1 &= 0 - 2\,e^{0} + C \\ 1 &= 0 - 2 + C \\ C &= 3. \end{aligned}$$
Hence $y = x - 2e^{-x} + 3$.

b Substituting $x = 1$,
$$\begin{aligned} y(1) &= 1 - 2e^{-1} + 3 \\ &= 4 - 2e^{-1}. \end{aligned}$$

Given a derivative, find an integral

The result of any differentiation can be reversed. This often allows a new primitive to be found.

Example 15 6D

a Use the chain rule to differentiate e^{x^2}.
b Hence find $\displaystyle\int_{-1}^{1} 2xe^{x^2}\,dx$.
c Why is this result obvious without any calculation?

SOLUTION

a Let $y = e^{x^2}$.
Applying the chain rule,
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2x\,e^{x^2}. \end{aligned}$$

Let $u = x^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = e^u$.

b Reversing part **a**, $\displaystyle\int 2xe^{x^2}\,dx = e^{x^2}$.

Hence
$$\begin{aligned} \int_{-1}^{1} 2xe^{x^2}\,dx &= \Big[e^{x^2}\Big]_{-1}^{1} \\ &= e^1 - e^1 \\ &= 0. \end{aligned}$$

c The function $2x\,e^{x^2}\,dx$ is odd, and the interval $[-1, 1]$ in the definite integral is symmetric in the y-axis, so the integral is zero.

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Using a formula for the reverse chain rule

There are some situations where the reverse chain rule formula from Section 5I can be used.

Example 16 6D

Use the reverse chain rule formula to find $\displaystyle\int \frac{e^{2x}}{(1 - e^{2x})^3}\,dx$.

SOLUTION

$$\int \frac{e^{2x}}{(1-e^{2x})^3}\,dx$$
$$= -\tfrac{1}{2}\int \frac{-2e^{2x}}{(1-e^{2x})^3}\,dx$$
$$= -\tfrac{1}{2} \times \left(-\tfrac{1}{2}\right) \times (1 - e^{2x})^{-2}$$
$$= \frac{1}{4\,(1-e^{2x})^2} + C, \quad \text{for some constant } C.$$

Let $u = 1 - e^{2x}$.
Then $u' = -2e^{2x}$.

$$\int u^{-3}\frac{du}{dx}\,dx = \frac{u^{-2}}{-2}$$

OR Let $f(x) = 1 - e^{2x}$.
Then $f'(x) = -2e^{2x}$.

$$\int \big(f(x)\big)^{-3} f'(x)\,dx = \frac{\big(f(x)\big)^{-2}}{-2}$$

Exercise 6D

FOUNDATION

Technology: Some algebraic programs can display the primitive and evaluate the exact value of an integral. These can be used to check the questions in this exercise and also to investigate the effect of making small changes to the function or to the limits of integration.

1 Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find each indefinite integral.

a $\int e^{2x}\,dx$ b $\int e^{3x}\,dx$ c $\int e^{\frac{1}{3}x}\,dx$

d $\int e^{\frac{1}{2}x}\,dx$ e $\int 10e^{2x}\,dx$ f $\int 12e^{3x}\,dx$

g $\int e^{4x+5}\,dx$ h $\int e^{4x-2}\,dx$ i $\int 6\,e^{3x+2}\,dx$

j $\int 4\,e^{4x+3}\,dx$ k $\int e^{7-2x}\,dx$ l $\int \frac{1}{2}e^{1-3x}\,dx$

2 Evaluate these definite integrals.

a $\int_0^1 e^x\,dx$ b $\int_1^2 e^x\,dx$ c $\int_{-1}^3 e^{-x}\,dx$ d $\int_{-2}^0 e^{-x}\,dx$

e $\int_0^2 e^{2x}\,dx$ f $\int_{-1}^2 20e^{-5x}\,dx$ g $\int_{-3}^1 8e^{-4x}\,dx$ h $\int_{-1}^3 9e^{6x}\,dx$

i $\int_{-1}^1 e^{2x+1}\,dx$ j $\int_{-2}^0 e^{4x-3}\,dx$ k $\int_{-2}^{-1} e^{3x+2}\,dx$ l $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x}\,dx$

m $\int_{-\frac{1}{3}}^{\frac{1}{3}} e^{2+3x}\,dx$ n $\int_1^2 6e^{3x+1}\,dx$ o $\int_2^3 12e^{4x-5}\,dx$ p $\int_1^2 12e^{8-3x}\,dx$

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3 Express each function using negative indices, and hence find its primitive.

a $\dfrac{1}{e^x}$ **b** $\dfrac{1}{e^{2x}}$ **c** $\dfrac{1}{e^{3x}}$ **d** $-\dfrac{3}{e^{3x}}$ **e** $\dfrac{6}{e^{2x}}$ **f** $\dfrac{8}{e^{-2x}}$

4 **a** A function $f(x)$ has derivative $f'(x) = e^{2x}$. Find the equation of $f(x)$, which will involve an arbitrary constant.

b It is also known that $f(0) = -2$. Find the arbitrary constant and hence write down the equation of $f(x)$.

c Find $f(1)$ and $f(2)$.

DEVELOPMENT

5 Find $f(x)$ and then find $f(1)$, given that:

a $f'(x) = 1 + 2e^x$ and $f(0) = 1$

b $f'(x) = 1 - 3e^x$ and $f(0) = -1$

c $f'(x) = 2 + e^{-x}$ and $f(0) = 0$

d $f'(x) = 4 - e^{-x}$ and $f(0) = 2$

e $f'(x) = e^{2x-1}$ and $f\left(\frac{1}{2}\right) = 3$

f $f'(x) = e^{1-3x}$ and $f\left(\frac{1}{3}\right) = \frac{2}{3}$

g $f'(x) = e^{\frac{1}{2}x+1}$ and $f(-2) = -4$

h $f'(x) = e^{\frac{1}{3}x+2}$ and $f(-6) = 2$

6 Expand the brackets and then find primitives of:

a $e^x(e^x + 1)$ **b** $e^x(e^x - 1)$ **c** $e^{-x}(2e^{-x} - 1)$

d $(e^x + 1)^2$ **e** $(e^x - 1)^2$ **f** $(e^x - 2)^2$

g $(e^x + e^{-x})(e^x - e^{-x})$ **h** $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

7 Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find these indefinite integrals.

a $\displaystyle\int e^{7x+q}\,dx$ **b** $\displaystyle\int e^{3x-k}\,dx$ **c** $\displaystyle\int e^{sx+1}\,dx$ **d** $\displaystyle\int e^{kx-1}\,dx$

e $\displaystyle\int pe^{px+q}\,dx$ **f** $\displaystyle\int me^{mx+k}\,dx$ **g** $\displaystyle\int Ae^{sx-t}\,dx$ **h** $\displaystyle\int Be^{kx-+}\,dx$

8 Express each function below as a power of e or a multiple of a power of e, and hence find its primitive.

a $\dfrac{1}{e^{x-1}}$ **b** $\dfrac{1}{e^{3x-1}}$ **c** $\dfrac{1}{e^{2x+5}}$

d $\dfrac{4}{e^{2x-1}}$ **e** $\dfrac{10}{e^{2-5x}}$ **f** $\dfrac{12}{e^{3x-5}}$

9 By writing each integrand as the sum of multiples of powers of e, find:

a $\displaystyle\int \frac{e^x + 1}{e^x}\,dx$ **b** $\displaystyle\int \frac{e^{2x} + 1}{e^x}\,dx$ **c** $\displaystyle\int \frac{e^x - 1}{e^{2x}}\,dx$

d $\displaystyle\int \frac{e^x - 3}{e^{3x}}\,dx$ **e** $\displaystyle\int \frac{2e^{2x} - 3e^x}{e^{4x}}\,dx$ **f** $\displaystyle\int \frac{2e^x - e^{2x}}{e^{3x}}\,dx$

10 **a** Find y as a function of x if $y' = e^{x-1}$, and $y = 1$ when $x = 1$. What is the y-intercept of this curve?

b The gradient of a curve is given by $y' = e^{2-x}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve? What is its horizontal asymptote?

c It is known that $f'(x) = e^x + \dfrac{1}{e}$ and that $f(-1) = -1$. Find $f(0)$.

d Given that $f''(x) = e^x - e^{-x}$ and that $y = f(x)$ is horizontal as it passes through the origin, find $f(x)$.

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11 By first writing each integrand as a sum of powers of e, find:

a $\int_0^1 e^x(2e^x - 1)\,dx$ **b** $\int_{-1}^1 (e^x + 2)^2\,dx$

c $\int_0^1 (e^x - 1)(e^{-x} + 1)\,dx$ **d** $\int_{-1}^1 (e^{2x} + e^{-x})(e^{2x} - e^{-x})\,dx$

e $\int_0^1 \frac{e^{3x} + e^x}{e^{2x}}\,dx$ **f** $\int_{-1}^1 \frac{e^x - 1}{e^{2x}}\,dx$

12 **a** **i** Differentiate e^{x^2+3}. **ii** Hence find $\int 2xe^{x^2+3}\,dx$.

b **i** Differentiate e^{x^2-2x+3}. **ii** Hence find $\int (x - 1)e^{x^2-2x+3}\,dx$.

c **i** Differentiate e^{3x^2+4x+1}. **ii** Hence find $\int (3x + 2)e^{3x^2+4x+1}\,dx$.

d **i** Differentiate $y = e^{x^3}$. **ii** Hence find $\int_{-1}^0 x^2e^{x^3}\,dx$.

13 Write each integrand as a power of e, and hence find the indefinite integral.

a $\int \frac{1}{(e^x)^2}\,dx$ **b** $\int \frac{1}{(e^x)^3}\,dx$ **c** $\int \sqrt{e^x}\,dx$

d $\int \sqrt[3]{e^x}\,dx$ **e** $\int \frac{1}{\sqrt{e^x}}\,dx$ **f** $\int \frac{1}{\sqrt[3]{e^x}}\,dx$

14 **a** Differentiate $y = x\,e^x$, and hence find $\int_0^2 x\,e^x\,dx$.

b Differentiate $y = x\,e^{-x}$, and hence find $\int_{-2}^0 x\,e^{-x}\,dx$.

15 By first simplifying each integrand, find:

a $\int \frac{e^x - e^{-x}}{\sqrt{e^x}}\,dx$ **b** $\int \frac{e^x + e^{-x}}{\sqrt[3]{e^x}}\,dx$

16 **a** Show that $f(x) = xe^{-x^2}$ is an odd function.

b Hence evaluate $\int_{-\sqrt{2}}^{\sqrt{2}} xe^{-x^2}\,dx$ without finding a primitive.

ENRICHMENT

17 Under Box 4, we gave the reverse chain rule for integrals of exponential functions,

$$\int e^u \frac{du}{dx}\,dx = e^u + C \quad \text{OR} \quad \int e^{f(x)} f'(x)\,dx = e^{f(x)} + C.$$

Use one or other of the formulae to find primitives of these functions

a xe^{x^2} **b** $4xe^{x^2-7}$ **c** $(3x + 2)e^{3x^2+4x+1}$

d $(x^2 - 2x)e^{x^3-3x^2}$ **e** $x^{-2}\,e^{x^{-1}}$ **f** $-\sqrt{x}\,e^{-x\sqrt{x}}$

18 Show that $\int \frac{e^x + 1}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}}\,dx = 2\,e^{\frac{1}{2}x} + C$.

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A power series for e^x:

19 The intention of this question is to outline a reasonably rigorous proof of the famous result that e^x can be written as the limit of the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

a For any positive number R, we know that $1 < e^t < e^R$ for $0 < t < R$, because e^x is an increasing function. Integrate this inequality over the interval $t = 0$ to $t = x$, where $0 < x < R$, and hence show that $x < e^x - 1 < e^R x$.

b Change the variable to t, giving $t < e^t - 1 < e^R t$. Then integrate this new inequality from $t = 0$ to $t = x$, and hence show that $\dfrac{x^2}{2!} < e^x - 1 - x < \dfrac{e^R x^2}{2!}$.

c Do this process twice more, and prove that:

i $\dfrac{x^3}{3!} < e^x - 1 - x - \dfrac{x^2}{2!} < \dfrac{e^R x^3}{3!}$

ii $\dfrac{x^4}{4!} < e^x - 1 - x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} < \dfrac{e^R x^4}{4!}$

d Now use induction to prove that $\dfrac{x^{n+1}}{(n+1)!} < e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^n}{n!} < \dfrac{e^R x^{n+1}}{(n+1)!}$.

e Show that as $n \to \infty$, the left and right expressions converge to zero. Hence prove that the infinite power series converges to e^x for $x > 0$. (Hint: Let k be the smallest integer greater than x and show that for $n > k$, each term in the sequence is less than the corresponding term of a geometric sequence with ratio $\frac{x}{k}$.)

f Prove that the power series also converges to e^x for $x < 0$.

20 a Use the power series in the previous question to show that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots.$$

b Find α, the value of the right-hand side, correct to four decimal places when $x = 0.5$.

c Let $u = e^{0.5}$. Show that $u^2 - 2\alpha u + 1 = 0$.

d Solve this quadratic equation and hence estimate both $e^{0.5}$ and $e^{-0.5}$ correct to two decimal places. Compare your answers with the values obtained directly from the calculator.

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6E Applications of integration

The normal methods of finding areas by integration can now be applied to functions involving e^x.

Finding the area between a curve and the x-axis

A sketch is essential here, because the definite integral attaches a negative sign to the area of any region below the x-axis (provided that the integral does not run backwards).

Example 17 — 6E

a Use shifting to sketch $y = e^x - e$, showing the intercepts and asymptote.

b Find the area of the region between this curve, the x-axis and the y-axis.

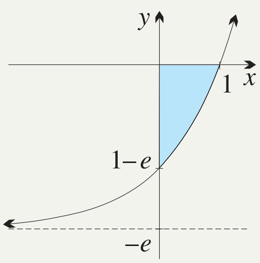
SOLUTION

a Move the graph of $y = e^x$ down e units.

When $x = 0$, $\quad y = e^0 - e$
$\quad\quad = 1 - e.$

When $y = 0$, $\quad e^x = e$
$\quad\quad x = 1.$

The horizontal asymptote moves down to $y = -e$.



b $$\int_0^1 (e^x - e)\,dx = \Big[e^x - ex\Big]_0^1 \quad \text{(the number } e \text{ is a constant)}$$
$$= (e^1 - e) - (e^0 - 0)$$
$$= -1.$$

This integral is negative because the region is below the x-axis.
Hence the required area is 1 square unit.

Finding areas between curves

If a curve $y = f(x)$ is always above $y = g(x)$ in an interval $a \le x \le b$, then the area of the region between the curves is

$$\text{area between the curves} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

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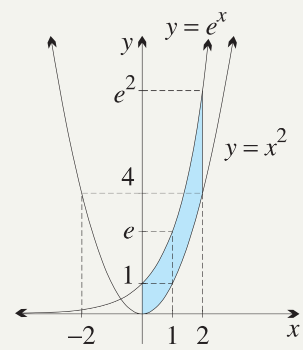
Example 18 6E

a Sketch the curves $y = e^x$ and $y = x^2$ in the interval $-2 \le x \le 2$.

b Find the area of the region between the curves, from $x = 0$ to $x = 2$.

SOLUTION



a The graphs are drawn to the right.

Note that for $x > 0$, $y = e^x$ is always above $y = x^2$.

b Using the standard formula above,

$$\begin{aligned} \text{area} &= \left[e^x - \tfrac{1}{3}x^3\right]_0^2 \\ &= \left(e^2 - \tfrac{8}{3}\right) - (e^0 - 0) \\ &= e^2 - 3\tfrac{2}{3} \text{ square units.} \end{aligned}$$

Exercise 6E

FOUNDATION

Technology: Graphing programs that can calculate the areas of specified regions may make the problems in this exercise clearer, particularly when no diagram has been given.

1 a Use the standard form $\int e^x dx = e^x + C$ to evaluate each definite integral. Then approximate it correct to two decimal places.

i $\int_0^1 e^x\,dx$ **ii** $\int_{-1}^0 e^x\,dx$ **iii** $\int_{-2}^0 e^x\,dx$ **iv** $\int_{-3}^0 e^x\,dx$

b The graph below shows $y = e^x$ from $x = -5$ to $x = 1$, with a scale of 10 divisions to 1 unit, so that 100 little squares equal 1 square unit. By counting squares under the curve from $x = 0$ to $x = 1$, find an approximation to $\int_0^1 e^x\,dx$, and compare it with the approximation obtained in part **a**.

c Count squares to the left of the y-axis to obtain approximations to:

i $\int_{-1}^0 e^x\,dx$, **ii** $\int_{-2}^0 e^x\,dx$, **iii** $\int_{-3}^0 e^x\,dx$,

and compare the results with the approximations obtained in part **a**.

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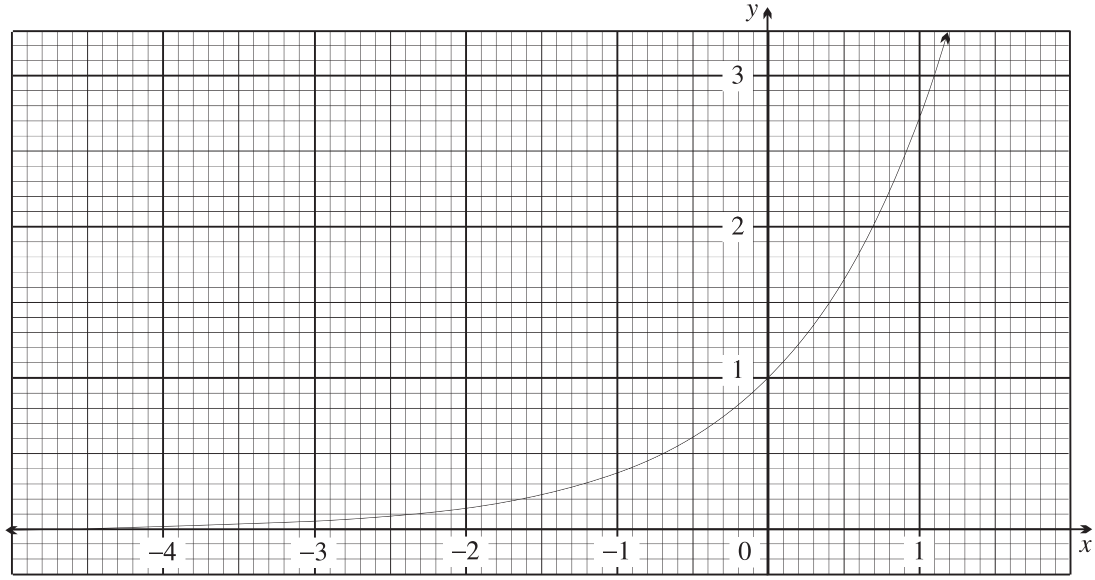
d Continue counting squares to the left of $x = -3$, and estimate the total area under the curve to the left of the y-axis.



2 Answer these questions first in exact form, then correct to four significant figures. In each case use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

a Find the area between the curve $y = e^{2x}$ and the x-axis from $x = 0$ to $x = 3$.

b Find the area between the curve $y = e^{-x}$ and the x-axis from $x = 0$ to $x = 1$.

c Find the area between the curve $y = e^{\frac{1}{3}x}$ and the x-axis from $x = -3$ to $x = 0$.

3 In each case find the area between the x-axis and the given curve between the given x-values. Use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

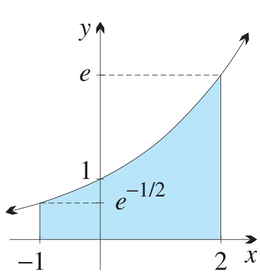
a $y = e^{x+3}$, for $-2 \le x \le 0$

b $y = e^{2x-1}$, for $0 \le x \le 1$

c $y = e^{-2x-1}$, for $-2 \le x \le -1$

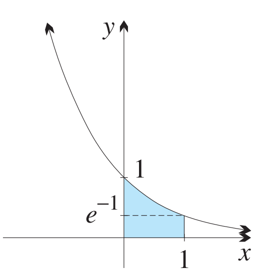
d $y = e^{\frac{1}{3}x+2}$, for $0 \le x \le 3$

4 **a**



Find the area of the region bounded by the curve $y = e^{\frac{1}{2}x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

b



Find the area of the region bounded by the curve $y = e^{-x}$, the x-axis, the y-axis and the line $x = 1$.

5 **a** Find the area between the curve $y = e^{x} + e^{-x}$ and the x-axis, from $x = -2$ to $x = 2$.

b Find the area between the curve $y = x^2 + e^{x}$ and the x-axis, from $x = -3$ to $x = 3$.

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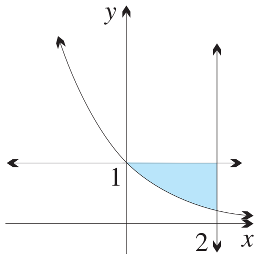
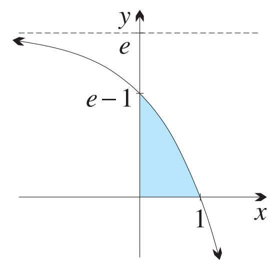
6 **a**



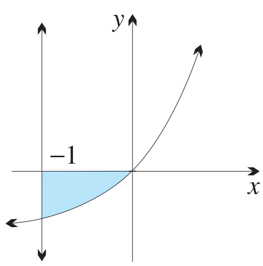
Find the area of the region bounded by the curve $y = e^{-x}$ and the lines $x = 2$ and $y = 1$.

b



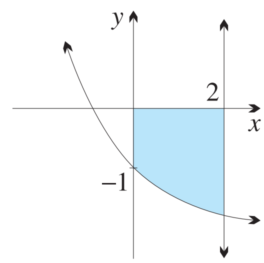
Find the area of the region in the first quadrant bounded by the coordinate axes and the curve $y = e - e^{x}$.

c



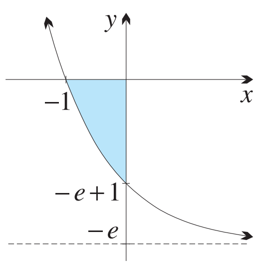
Find the area between the x-axis, the curve $y = e^{x} - 1$ and the line $x = -1$.

d



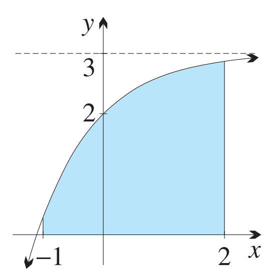
What is the area bounded by $x = 2$, $y = e^{-x} - 2$, the x-axis and the y-axis?

e



Find the area of the region bounded by the curve $y = e^{-x} - e$ and the coordinate axes.

f



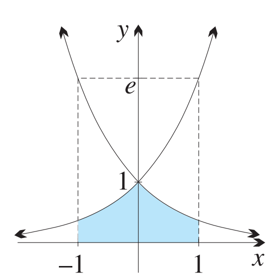
Find the area of the region bounded by the curve $y = 3 - e^{-x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

DEVELOPMENT

7 **a** Sketch the curves $y = e^{x}$ and $y = x + 1$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

b Sketch the curves $y = e^{x}$ and $y = 1 - x$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

8 The diagram to the right shows the region above the x-axis, below both $y = e^{x}$ and $y = e^{-x}$, between $x = -1$ and $x = 1$.



a Explain why the area of this region may be written as

$$\text{area} = 2\int_0^1 e^{-x}\,dx.$$

b Hence find the area of this region.

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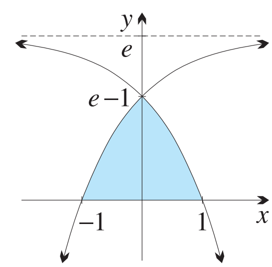
9 The diagram to the right shows the region above the x-axis, below both $y = e - e^{-x}$ and $y = e - e^{x}$.

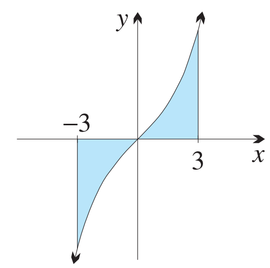


a Explain why the area of this region may be written as

$$\text{area} = 2\int_0^1 (e - e^x)\, dx.$$

b Hence find the area of this region.

10 The diagram to the right shows the region between the curve $y = e^x - e^{-x}$, the x-axis and the lines $x = -3$ and $x = 3$.



a Show that $y = e^x - e^{-x}$ is an odd function.

b Hence write down the value of $\int_{-3}^{3} (e^x - e^{-x})\, dx$ without finding a primitive.

c Explain why the area of this region may be written as

$$\text{area} = 2\int_0^3 (e^x - e^{-x})\, dx.$$

d Hence find the area of this region.

11 **a** Show that the curves $y = x^2$ and $y = e^{x+1}$ intersect at $x = -1$.

b Hence sketch the region in the second quadrant between these two curves and the y-axis.

c Find its area.

12 **a** Sketch the region between the graphs of $y = e^x$ and $y = x$, between the y-axis and $x = 2$, then find its area.

b Find the intercepts of the curve $y = 8 - 2^x$ and hence find the area of the region bounded by this curve and the coordinate axes.

13 In this question, give all approximations correct to four decimal places.

a Find the area between the curve $y = e^x$ and the x-axis, for $0 \le x \le 1$, by evaluating an appropriate integral. Then approximate the result.

b Estimate the area using the trapezoidal rule with two subintervals (that is, with three function values).

c Is the trapezoidal-rule approximation greater than or less than the exact value? Give a geometric explanation.

14 **a** Differentiate e^{-x^2} and hence write down a primitive of xe^{-x^2}.

b Hence find the area between the curve $y = xe^{-x^2}$ and the x-axis from $x = 0$ to $x = 2$, and from $x = -2$ to $x = 2$.

15 **a** **i** Evaluate the integral $\int_N^0 e^x dx$.

ii What is its limit as $N \to -\infty$?

b **i** Evaluate the integral $\int_0^N e^{-x} dx$.

ii What is its limit as $N \to \infty$?

c Similarly, evaluate $\int_0^N 2xe^{-x^2} dx$, and find its limit as $N \to \infty$.

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ENRICHMENT

16 **a** Find $\displaystyle\int_\delta^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

b What happens to the integral as $\delta \to 0^+$?

17 **a** Differentiate $x\,e^{-x}$, and hence find $\displaystyle\int_0^N x\,e^{-x}\,dx$.

b Find the limit of this integral as $N \to \infty$ (use the dominance results).

c Differentiate $x^2\,e^{-x}$, and hence find $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$.

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6F Review of logarithmic functions

Section 6A reviewed exponential functions from Sections 11A–11F of the Year 11 book, and this section will complete the review of those sections with a summary of logarithms base e. The two small topics that are new are:

- Dilations of logarithmic functions.
- Exponential and logarithmic equations reducible to quadratics.

The function $y = \log_e x$

As discussed in the Year 11 book, an exponential function to any base is one-to-one, so its inverse relation is also a function, and is called a *logarithmic function* to the same base. Remember that

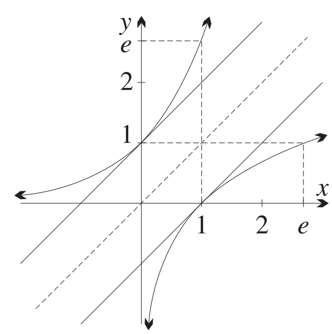
$$3 = \log_2 8 \text{ means } 8 = 2^3 \qquad \text{and} \qquad y = \log_e x \text{ means } x = e^y.$$

'The log is the index, when the number is written as a power of the base.'

Algebraically, the fact that $y = \log_e x$ is the inverse function of $y = e^x$ means that the composite of the two functions, in either order, is the identity function,

$$\log_e e^x = x, \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x, \text{ for all } x > 0.$$

Geometrically, when the functions are sketched on one graph, they are reflections of each other in the diagonal line $y = x$.



- Both graphs have gradient 1 at their intercepts, $y = e^x$ at its y-intercept, and $y = \log_e x$ at its x-intercept.
- Their domains and ranges are reversed, which is more easily seen with bracket interval notation:
  For $y = e^x$, domain $= (-\infty, \infty)$, range $= (0, \infty)$.
  For $y = \log_e x$, domain $= (0, \infty)$, range $= (-\infty, \infty)$.
- $y = e^x$ has a horizontal asymptote $y = 0$.
  $y = \log_e x$ has a vertical asymptote $x = 0$.
- Both are increasing throughout their domain, $y = e^x$ at an increasing rate, $y = \log_e x$ at a decreasing rate.

5 THE FUNCTION $y = \log_e x$ OR $\ln x$

- The function $y = \log_e x$ is the inverse function of $y = e^x$,
  $$y = \log_e x \qquad \text{means that} \qquad x = e^y.$$
- The composition of the functions $y = e^x$ and $y = \log_e x$, in any order, is the identity function,
  $$\log_e e^x = x, \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x, \text{ for all } x > 0.$$
- The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in $y = x$.
- This reflection exchanges the domain and range, exchanges the asymptotes, and exchanges the intercepts with the axes.
- The tangents to both curves at their intercepts have gradient 1.
- $y = e^x$ is always concave up, and $y = \log_e x$ is always concave down.
- Both graphs are one-to-one, and both graphs are increasing, $y = e^x$ at an increasing rate, $y = \log_e x$ at a decreasing rate.

The derivative of $y = \log_e x$ will be obtained in Section 6G.

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Notation and the calculator

Write the function as $y = \log_e x$ or as $y = \ln x$ ('logs naperian' or 'logs natural'). We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise to readers that the base is e, but $\ln x$ is also standard notation.

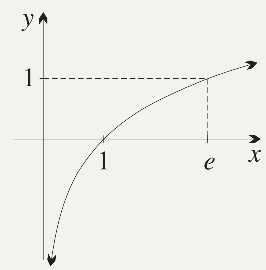
In mathematics, but not elsewhere, interpret $\log x$ as $\log_e x$. Be particularly careful on the calculator, where $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$.

Example 19 6F

Sketch each function using a transformation of the graph of $y = \log_e x$ sketched to the right. Describe the transformation, write down the domain, and show and state the x-intercept and the vertical asymptote.

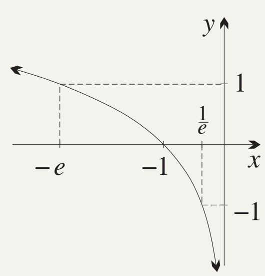
a $y = \log_e(-x)$ **b** $y = \log_e x - 2$ **c** $y = \log_e(x + 3)$

Which transformations can also be done using a dilation?



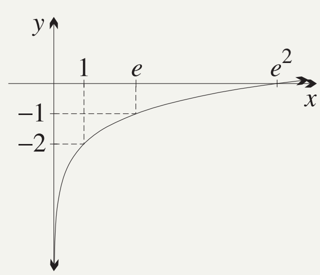
SOLUTION

a



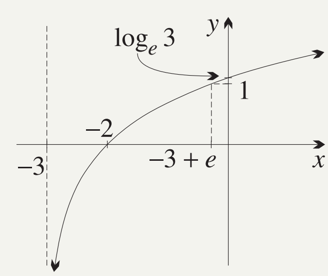
To graph $y = \log_e(-x)$,
reflect $y = \log_e x$ in y-axis.
domain: $x < 0$
x-intercept: $(-1, 0)$
asymptote: $x = 0$

b



To graph $y = \log_e x - 2$,
shift $y = \log_e x$ down 2.
domain: $x > 0$
x-intercept: $(e^2, 0)$
asymptote: $x = 0$

c



To graph $y = \log_e(x + 3)$,
shift $y = \log_e x$ left 3.
domain: $x > -3$
x-intercept: $(-2, 0)$
asymptote: $x = -3$

- The equation $y = \log_e(-x)$ in part **a** is a reflection in the y-axis, and any reflection in the y-axis can be regarded as a horizontal dilation with factor -1.
- In part **b**, the equation $y = \log_e x - 2 = \log_e x - \log_e e^2 = \log_e(e^{-2}x)$ can be regarded as a horizontal dilation with factor $e^2 \doteqdot 7.39$.

Dilations of $y = \log_e x$

Dilations of logarithmic functions share an interesting property with dilations of exponential functions — some of them can be done with a shift in the other direction, as we saw in part **b** above.

Example 20 6F

Use dilations of $y = \log_e x$ to generate a sketch of each function. Identify which dilation is also a shift in the other direction.

a $y = \log_e 3x$ **b** $y = 3\log_e x$

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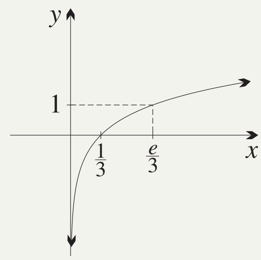
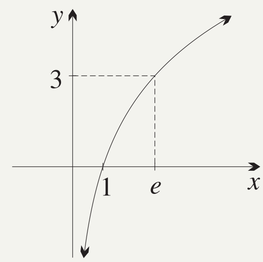
SOLUTION

a $y = \log_e 3x$



Dilate $y = \log_e x$ horizontally factor $\frac{1}{3}$.

b $y = 3\log_e x$



Dilate $y = \log_e x$ vertically factor 3.

- $y = \log_e 3x$ can be written as $y = \log_e x + \log_e 3$, so it is shift up $\log_e 3$.

Using the inverse identities

We conclude with a review of some of the manipulations needed when using logarithms base e. First, some simple examples of using the two inverse identities

$$\log_e e^x = x \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x \text{ for all } x > 0.$$

Example 21 — 6F

Simplify:

a $\log_e e^6$ **b** $\log_e e$ **c** $\log_e \frac{1}{e}$ **d** $\log_e \dfrac{1}{\sqrt{e}}$ **e** $e^{\log_e 10}$ **f** $e^{\log_e 0.1}$

SOLUTION

a $\log_e e^6 = 6$

b $\log_e e = \log_e e^1$
$= 1$

c $\log_e \frac{1}{e} = \log_e e^{-1}$
$= -1$

d $\log_e \dfrac{1}{\sqrt{e}} = \log_e e^{-\frac{1}{2}}$
$= -\frac{1}{2}$

e $e^{\log_e 10} = 10$

f $e^{\log_e 0.1} = 0.1$

Conversion between exponential statements and logarithm statements

We recommended that the following sentence be committed to memory:

$$\log_2 8 = 3 \qquad \text{because} \qquad 8 = 2^3.$$

- The base of the power is the base of the log.
- The log is the index, when the number is written as a power of the base.

This pattern applies in exactly the same way when the base is e.

$$\log_e x = y \qquad \text{means} \qquad x = e^y.$$

Example 22 — 6F

Convert each statement to the other form.

a $x = e^3$ **b** $\log_e x = -1$ **c** $x = \log_e 10$ **d** $e^x = \frac{1}{2}$

SOLUTION

a $\log_e x = 3$ **b** $x = \frac{1}{e}$ **c** $e^x = 10$ **d** $x = \log_e \frac{1}{2}$

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The change-of-base formula

We developed the general change of base formula. What is needed here is conversion to base e from a base b, which must be a positive number not equal to 1,

$$\log_b x = \frac{\log_e x}{\log_e b}, \quad \text{for all } x > 0.$$

Example 23 6F

a Locate $\log_2 100$ and $\log_3 100$ between two whole numbers.
b Use logarithms base e to solve $2^x = 100$ and $3^x = 100$ correct to three decimal places.

SOLUTION

a $2^6 < 100 < 2^7$, so $\log_2 100$ lies between 6 and 7.
$3^4 < 100 < 3^5$, so $\log_3 100$ lies between 4 and 5.

b
$$\begin{aligned} 2^x &= 100 \\ x &= \log_2 100 \\ &= \frac{\log_e 100}{\log_e 2} \\ &\doteqdot 6.644 \end{aligned} \qquad \begin{aligned} 3^x &= 100 \\ x &= \log_3 100 \\ &= \frac{\log_e 100}{\log_e 3} \\ &\doteqdot 4.192 \end{aligned}$$

Alternatively, take logarithms base e of both sides.

6 THE CHANGE-OF-BASE FORMULA

Suppose that the new base b is a positive number not equal to 1. Then

$$\log_b x = \frac{\log_e x}{\log_e b}.$$

'The log of the number over the log of the base.'

Exponential and logarithmic equations reducible to quadratics

Exponential and logarithmic equations can sometimes be reduced to quadratics with a substitution (although the working is sometimes easier without the substitution). This approach can be used whether or not the base is e.

Example 24 6F

a Use the substitution $u = 2^x$ to solve the equation $4^x - 7 \times 2^x + 12 = 0$.
b Use the substitution $u = e^x$ to solve the equation $3e^{2x} - 11e^x - 4 = 0$.
c Solve $\log_e x - \dfrac{9}{\log_e x} = 0$ with and without the substitution $u = \log_e x$.

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SOLUTION

a Writing $4^x = (2^x)^2$, the equation becomes

$$(2^x)^2 - 7 \times 2^x + 12 = 0.$$

Substituting $u = 2^x$,

$$\begin{aligned} u^2 - 7u + 12 &= 0 \\ (u-4)(u-3) &= 0 \\ u &= 4 \text{ or } 3, \end{aligned}$$

and returning to x,

$$\begin{aligned} 2^x &= 4 \text{ or } 2^x = 3 \\ x &= 2 \text{ or } \log_2 3. \end{aligned}$$

b Writing $e^{2x} = (e^x)^2$, the equation becomes

$$3(e^x)^2 - 11e^x - 4 = 0.$$

Substituting $u = e^x$,

$$\begin{aligned} 3u^2 - 11u - 4 &= 0 \quad (\alpha + \beta = -11, \alpha\beta = 3 \times (-4) = -12) \\ 3u^2 - 12u + u - 4 &= 0 \quad (\alpha \text{ and } \beta \text{ are } -12 \text{ and } 1) \\ 3u(u-4) + (u-4) &= 0 \\ (3u+1)(u-4) &= 0 \\ u &= -\tfrac{1}{3} \text{ or } 4, \end{aligned}$$

and returning to x,

$$e^x = -\tfrac{1}{3} \text{ or } e^x = 4.$$

Because e^x is never negative,

$$\begin{aligned} e^x &= 4 \\ x &= \log_e 4. \end{aligned}$$

c The equation is

$$\log_e x - \frac{9}{\log_e x} = 0.$$

Substituting $u = \log_e x$,

$$u - \frac{9}{u} = 0$$

$\boxed{\times\ u}$

$$\begin{aligned} (u^2 - 9) &= 0 \\ (u-3)(u+3) &= 0 \\ u &= 3 \text{ or } -3, \end{aligned}$$

and returning to x,

$$\log_e x = 3 \text{ or } -3.$$

Hence

$$x = e^3 \text{ or } e^{-3}.$$

Alternatively,

$$\log_e x - \frac{9}{\log_e x} = 0$$

$\boxed{\times\ \log_e x}$

$$\begin{aligned} (\log_e x)^2 - 9 &= 0 \\ (\log_e x)^2 &= 9 \\ \log_e x &= 3 \text{ or } -3 \\ x &= e^3 \text{ or } e^{-3}. \end{aligned}$$

Exercise 6F

FOUNDATION

Remember that on the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise the base.

1 Use the calculator's $\boxed{\ln}$ button to approximate, correct to four significant figures:

a $\log_e 10$ **b** $\log_e 0.1$ **c** $\ln 123\,456$ **d** $\ln 0.000\,006$ **e** $\log_e 50$ **f** $\log_e 0.02$

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2 Use the laws for logarithms to express as a single logarithm:

a $\ln 5 + \ln 4$ **b** $\ln 30 - \log_e 6$ **c** $\ln 12 - \ln 15 + \ln 10^2$

3 Use the identities $\log_e e^x = x$ for all real x, and $e^{\log_e x} = x$ for $x > 0$, to simplify:

a $\log_e e^3$ **b** $\log_e e^{-1}$ **c** $\log_e \frac{1}{e^2}$ **d** $\log_e \sqrt{e}$

e $e^{\ln 5}$ **f** $e^{\ln 0.05}$ **g** $e^{\ln 1}$ **h** $e^{\ln e}$

4 a Use your calculator to confirm that $\log_e 1 = 0$.

b Write 1 as a power of e, then use the identities in Question **3** to explain why $\log_e 1 = 0$.

c Use your calculator to confirm that $\log_e e = 1$. (You will need to find $e = e^1$ first.)

d Write e as a power of e, then use the identities in Question **3** to explain why $\log_e e = 1$.

5 Convert each exponential statement to logarithmic form, and each logarithmic statement to exponential form.

a $x = e^6$ **b** $\log_e x = -2$ **c** $x = \ln 24$ **d** $e^x = \frac{1}{3}$

6 Use the change-of base formula to express each logarithm in terms of logarithms base e. Then approximate it correct to four significant figures.

a $\log_2 7$ **b** $\log_{10} 25$ **c** $\log_3 0.04$

7 a What transformation maps $y = e^x$ to $y = \log_e x$, and how can this transformation be used to find the gradient of $y = \log_e x$ at its x-intercept?

b What transformation maps $y = \log_e x$ to $y = \log_e(-x)$, and how can this transformation also be interpreted as a dilation?

c Sketch $y = \log_e x$ and $y = \log_e(-x)$ on one set of axes.

8 Sketch each curve using a single transformation of $y = \log_e x$, and describe the transformation.

a $y = \log_e x + 1$ **b** $y = \log_e x - 2$ **c** $y = \log_e\left(\frac{1}{2}x\right)$ **d** $y = \frac{1}{3}\log_e x$

9 Sketch each curve using a single transformation of $y = \log_e(-x)$, and describe the transformation.

a $y = \log_e(-x) - 1$ **b** $y = -\log_e(-x)$ **c** $y = 3\log_e(-x)$

DEVELOPMENT

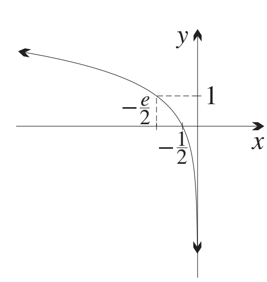
10 Simplify these expressions involving logarithms to the base e:

a $e \log_e e$ **b** $\frac{1}{e}\ln\frac{1}{e}$ **c** $3\log_e e^2$

d $\ln\sqrt{e}$ **e** $e\log_e e^3 - e\log_e e$ **f** $\log_e e + \log_e \frac{1}{e}$

g $\log_e e^e$ **h** $\log_e(\log_e e^e)$ **i** $\log_e\left(\log_e(\log_e e^e)\right)$

11 The graph drawn to the right is a dilation of $y = \log_e(-x)$. Describe the dilation, and write down the equation of the curve.



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12 **a** Use the substitution $u = 2^x$ to solve $4^x - 9 \times 2^x + 14 = 0$.

b Use the substitution $u = 3^x$ to solve $3^{2x} - 8 \times 3^x - 9 = 0$.

c Use similar substitutions, or none, to solve:

i $25^x - 26 \times 5^x + 25 = 0$

ii $9^x - 5 \times 3^x + 4 = 0$

iii $3^{2x} - 3^x - 20 = 0$

iv $7^{2x} + 7^x + 1 = 0$

v $3^{5x} = 9^{x+3}$

vi $4^x - 3 \times 2^{x+1} + 2^3 = 0$

13 Use the substitution $u = e^x$ or $u = e^{2x}$ to reduce these equations to quadratics and solve them. Write your answers as logarithms base e, unless they can be further simplified.

a $e^{2x} - 2e^x + 1 = 0$

b $e^{2x} + e^x - 6 = 0$

c $e^{4x} - 10e^{2x} + 9 = 0$

d $e^{4x} - e^{2x} = 0$

14 Use a substitution such as $u = 4^x$ to solve each equation. Give each solution as a rational number, or approximate correct to three decimal places.

a $2^{4x} - 7 \times 2^{2x} + 12 = 0$

b $100^x - 10^x - 1 = 0$

c $\left(\frac{1}{5}\right)^{2x} - 7 \times \left(\frac{1}{5}\right)^{x} + 10 = 0$

15 Use a substitution, or none, to solve:

a $(\log_e x)^2 - 5 \log_e x + 4 = 0$

b $(\log_e x)^2 = 3 \log_e x$

16 **a** Solve $\ln(x^2 + 5x) = 2\ln(x + 1)$

b Solve $\log_e(7x - 12) = 2\log_e x$.

ENRICHMENT

17 **a** Use, and describe, a dilation of $y = \log_e x$ to sketch $y = \log_e 2x$.

b Use, and describe, a subsequent translation to sketch $y = \log_e 2(x - 1)$.

c Use, and describe, a subsequent dilation to sketch $y = \frac{1}{2}\log_e 2(x - 1)$.

d Use, and describe, a subsequent translation to sketch $y = \frac{1}{2}\log_e 2(x - 1) - 2$.

18 In Box 6, when stating the change-of-base formula $\log_b x = \dfrac{\log_e x}{\log_e b}$, we required that the base b must be positive and not equal to 1. Why is this restriction on b necessary?

19 **a** Interpret the transformation from $y = \log_e x$ to $y = \log_e(5x)$ as a dilation. Then interpret it as a translation.

b Interpret the transformation from $y = \log_e x$ to $y = \log_e x + 2$ as a translation. Then interpret it as a dilation by writing 2 as $\log_e e^2$.

20 It can be shown (with some considerable difficulty) that the continued fraction to the right approaches the value e–1. With the help of a calculator, use this continued fraction to find a rational approximation for e that is accurate to four significant figures.

$$1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}}}}$$

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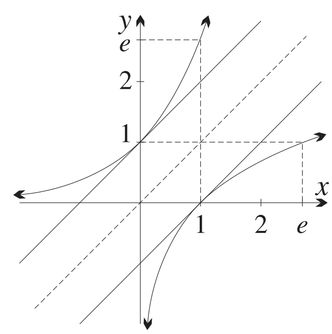
6G Differentiation of logarithmic functions

Calculus with the exponential function $y = e^x$ requires also the calculus of its inverse function $y = \log_e x$.



The diagram to the right shows once again the graphs of both curves drawn on the same set of axes — they are reflections of each other in the diagonal line $y = x$. Using this reflection, the important features of $y = \log_e x$ are:

- The domain is $x > 0$ and the range is all real x.
- The x-intercept is 1, and the gradient there is 1.
- The y-axis is a vertical asymptote.
- As $x \to \infty$, $y \to \infty$ (look at its reflection $y = e^x$ to see this).
- Throughout its whole domain, $\log_e x$ is increasing at a decreasing rate.

Differentiating the logarithmic function

The logarithmic function $= \log_e x$ can be differentiated easily using the known derivative of its inverse function e^x.

Let $y = \log_e x$.

Then $x = e^y$, by the definition of logarithms.

Differentiating, $\dfrac{dx}{dy} = e^y$, because the exponential function is its own derivative,

$= x$, because $e^y = x$,

and taking reciprocals, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Hence the derivative of the logarithmic function is the reciprocal function.

7 THE DERIVATIVE OF THE LOGARITHMIC FUNCTION IS THE RECIPROCAL FUNCTION

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

The next worked example uses the derivative to confirm that $y = \log_e x$ has two properties that were already clear from the reflection in the diagram above.

Example 25 **6G**

a Find the gradient of the tangent to $y = \log_e x$ at its x-intercept.

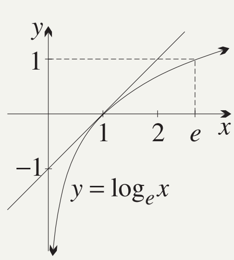
b Prove that $y = \log_e x$ is always increasing, and always concave down.

SOLUTION

a The function is $y = \log_e x$.

Differentiating, $y' = \dfrac{1}{x}$.

The graph crosses the x-axis at $(1, 0)$, and substituting $x = 1$ into y', gradient at x-intercept $= 1$.



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b The domain is $x > 0$, and $y' = \dfrac{1}{x}$ is positive for all $x > 0$.

Differentiating again, $y'' = -\dfrac{1}{x^2}$, which is negative for all $x > 0$.

Hence $y = e^x$ is always increasing, and always concave down.

Example 26 6G

Differentiate these functions using the standard form above.

a $y = x + \log_e x$

b $y = 5x^2 - 7\log_e x$

SOLUTION

a
$$y = x + \log_e x$$
$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

b
$$y = 5x^2 - 7\log_e x$$
$$\frac{dy}{dx} = 10x - \frac{7}{x}$$

Further standard forms

The next worked example uses the chain rule to develop two further standard forms for differentiation.

Example 27 6G

Differentiate each function using the chain rule. (Part **b** is a standard form.)

a $\log_e(3x + 4)$

b $\log_e(ax + b)$

c $\log_e(x^2 + 1)$

SOLUTION

a Let $y = \log_e(3x + 4)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{3x + 4} \times 3$$
$$= \frac{3}{3x + 4}.$$

Let $u = 3x + 4$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

b Let $y = \log_e(ax + b)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{ax + b} \times a$$
$$= \frac{a}{ax + b}.$$

Let $u = ax + b$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

c Let $y = \log_e(x^2 + 1)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (chain rule)

$$= \frac{1}{x^2 + 1} \times 2x$$
$$= \frac{2x}{x^2 + 1}.$$

Let $u = x^2 + 1$.
Then $y = \log_e u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

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Standard forms for differentiation

It is convenient to write down two further standard forms for differentiation based on the chain rule, giving three forms altogether.

8 THREE STANDARD FORMS FOR DIFFERENTIATING LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

$$\frac{d}{dx}\log_e(ax+b) = \frac{a}{ax+b}$$

$$\frac{d}{dx}\log_e u = \frac{u'}{u} \qquad \text{OR} \qquad \frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$$

The second of these standard forms was proven in part **b** of the previous worked example. Part **a** was an example of it.

The third standard form is a more general chain-rule extension — part **c** of the previous worked example was a good example of it. This standard form will be needed later for integration. For now, either learn it — in one of its two forms — or apply the chain rule each time.

Example 28 6G

Using the standard forms developed above, differentiate:

a $y = \log_e(4x - 9)$ **b** $y = \log_e(1 - \frac{1}{2}x)$ **c** $y = \log_e(4 + x^2)$

SOLUTION

a For $y = \log_e(4x - 9)$, use the second standard form with $ax + b = 4x - 9$.

Thus $y' = \dfrac{4}{4x - 9}$.

b For $y = \log_e(1 - \frac{1}{2}x)$, use the second standard form with $ax + b = -\frac{1}{2}x + 1$.

Thus $y' = \dfrac{-\frac{1}{2}}{-\frac{1}{2}x + 1}$

$= \dfrac{1}{x - 2}$, after multiplying top and bottom by -2.

c For $y = \log_e(4 + x^2)$,

$y' = \dfrac{2x}{4 + x^2}$.

Let $u = 4 + x^2$.
Then $u' = 2x$.
$\dfrac{d}{dx}\log_e u = \dfrac{u'}{u}$

OR

Let $f(x) = 4 + x^2$.
Then $f'(x) = 2x$.
$\dfrac{d}{dx}\log_e f(x) = \dfrac{f'(x)}{f(x)}$

Alternatively, use the chain rule, as in the previous worked example.

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Using the product and quotient rules

These two rules are used in the usual way.

Example 29 6G

Differentiate:

a $x^3 \ln x$ by the product rule,

b $\dfrac{\ln(1+x)}{x}$ by the quotient rule.

SOLUTION

a Let $y = x^3 \ln x$.

Then $y' = vu' + uv'$

$= 3x^2 \ln x + x^3 \times \dfrac{1}{x}$

$= x^2(1 + 3\ln x)$.

Let $u = x^3$

and $v = \ln x$.

Then $u' = 3x^2$

and $v' = \dfrac{1}{x}$.

b Let $y = \dfrac{\ln(1+x)}{x}$.

Then $y' = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{\dfrac{x}{1+x} - \ln(1+x)}{x^2}$

$= \dfrac{x - (1+x)\ln(1+x)}{x^2(1+x)}$.

Let $u = \ln(1+x)$

and $v = x$.

Then $u' = \dfrac{1}{1+x}$

and $v' = 1$.

Using the log laws to make differentiation easier

The next worked example shows the use of the log laws to avoid a combination of the chain and quotient rules.

Example 30 6G

Use the log laws to simplify each expression, then differentiate it.

a $\log_e 7x^2$

b $\log_e(3x - 7)^5$

c $\log_e \dfrac{1+x}{1-x}$.

SOLUTION

a Let $y = \log_e 7x^2$.

Then $y = \log_e 7 + \log_e x^2$

$= \log_e 7 + 2\log_e x$,

so $\dfrac{dy}{dx} = \dfrac{2}{x}$.

b Let $y = \log_e(3x - 7)^5$.

Then $y = 5\log_e(3x - 7)$,

so $\dfrac{dy}{dx} = \dfrac{15}{3x - 7}$.

c Let $y = \log_e \dfrac{1+x}{1-x}$.

Then $y = \log_e(1+x) - \log_e(1-x)$,

so $\dfrac{dy}{dx} = \dfrac{1}{1+x} + \dfrac{1}{1-x}$.

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Exercise 6G

FOUNDATION

Note: Remember that on the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise the base.

1 Use the standard form $\dfrac{d}{dx}\log_e(ax+b) = \dfrac{a}{ax+b}$ to differentiate:

- **a** $y = \log_e(x+2)$
- **b** $y = \log_e(x-3)$
- **c** $y = \log_e(3x+4)$
- **d** $y = \log_e(2x-1)$
- **e** $y = \log_e(-4x+1)$
- **f** $y = \log_e(-3x+4)$
- **g** $y = \ln(-2x-7)$
- **h** $y = 3\ln(2x+4)$
- **i** $y = 5\ln(3x-2)$

2 Differentiate these functions.

- **a** $y = \log_e 2x$
- **b** $y = \log_e 5x$
- **c** $y = \log_e 3x$
- **d** $y = \log_e 7x$
- **e** $y = 4\ln 7x$
- **f** $y = 3\ln 5x$
- **g** $y = 4\ln 6x$
- **h** $y = 3\ln 9x$

3 Find $\dfrac{dy}{dx}$ for each function. Then evaluate $\dfrac{dy}{dx}$ at $x = 3$.

- **a** $y = \log_e(x+1)$
- **b** $y = \log_e(2x-1)$
- **c** $y = \log_e(2x-5)$
- **d** $y = \log_e(4x+3)$
- **e** $y = 5\ln(x+1)$
- **f** $y = 6\ln(2x+9)$

4 Differentiate these functions.

- **a** $2 + \log_e x$
- **b** $5 - \log_e(x+1)$
- **c** $x + 4\log_e x$
- **d** $2x^4 + 1 + 3\log_e x$
- **e** $\ln(2x-1) + 3x^2$
- **f** $x^3 - 3x + 4 + \ln(5x-7)$

DEVELOPMENT

5 Use the log laws to simplify each function, then differentiate it.

- **a** $y = \ln x^3$
- **b** $y = \ln x^2$
- **c** $y = \ln x^{-3}$
- **d** $y = \ln x^{-2}$
- **e** $y = \ln\sqrt{x}$
- **f** $y = \ln\sqrt{x+1}$

6 Differentiate these functions.

- **a** $y = \log_e \frac{1}{2}x$
- **b** $y = \log_e \frac{1}{3}x$
- **c** $y = 3\log_e \frac{1}{5}x$
- **d** $y = -6\log_e \frac{1}{2}x$
- **e** $y = x + \log_e \frac{1}{7}x$
- **f** $y = 4x^3 - \log_e \frac{1}{5}x$

7 Use the full setting-out of the chain rule to differentiate:

- **a** $\ln(x^2+1)$
- **b** $\ln(2-x^2)$
- **c** $\ln(1+e^x)$

8 Use the standard form $\dfrac{d}{dx}\log_e u = \dfrac{u'}{u}$ OR $\dfrac{d}{dx}\log_e f(x) = \dfrac{f'(x)}{f(x)}$ to differentiate:

- **a** $\log_e(x^2+3x+2)$
- **b** $\log_e(1+2x^3)$
- **c** $\ln(e^x-2)$
- **d** $x + 3 - \ln(x^2+x)$
- **e** $x^2 + \ln(x^3-x)$
- **f** $4x^3 - 5x^2 + \ln(2x^2-3x+1)$

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9 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = \ln x$ at the points where:

a $x = 1$ **b** $x = 3$ **c** $x = \frac{1}{2}$ **d** $x = 4$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

10 Differentiate these functions using the product rule.

a $x \log_e x$ **b** $x \log_e(2x + 1)$ **c** $(2x + 1) \log_e x$ **d** $x^4 \log_e x$

e $(x + 3) \log_e(x + 3)$ **f** $(x - 1) \log_e(2x + 7)$ **g** $e^x \log_e x$ **h** $e^{-x} \log_e x$

11 Differentiate these functions using the quotient rule.

a $y = \dfrac{\log_e x}{x}$ **b** $y = \dfrac{\log_e x}{x^2}$ **c** $y = \dfrac{x}{\log_e x}$

d $y = \dfrac{x^2}{\log_e x}$ **e** $y = \dfrac{\log_e x}{e^x}$ **f** $y = \dfrac{e^x}{\log_e x}$

12 Use the log laws to simplify each function, then differentiate it.

a $y = \log_e 5x^3$ **b** $y = \log_e \sqrt[3]{x}$ **c** $y = \log_e \dfrac{3}{x}$

d $y = \ln \sqrt{2 - x}$ **e** $y = \log_e \dfrac{3}{x}$ **f** $y = \ln \dfrac{1 + x}{1 - x}$

g $y = \log_e 2^x$ **h** $y = \log_e e^x$ **i** $y = \log_e x^x$

13 Find the first and second derivatives of each function, then evaluate both derivatives at the value given.

a $f(x) = \log_e (x - 1)$ at $x = 3$ **b** $f(x) = \log_e (2x + 1)$ at $x = 0$

c $f(x) = \log_e x^2$ at $x = 2$ **d** $f(x) = x \log_e x$ at $x = e$

14 Differentiate each function using the chain, product or quotient rules. Then find any values of x for which the derivative is zero.

a $y = x \log_e x - x$ **b** $y = x^2 \log_e x$ **c** $y = \dfrac{\log_e x}{x}$

d $y = (\log_e x)^4$ **e** $y = (2 \log_e x - 3)^4$ **f** $y = \dfrac{1}{\log_e x}$

g $y = \log_e(\log_e x)$ **h** $y = x \ln x$ **i** $y = \dfrac{1}{x} + \ln x$

15 a Show that $y = \dfrac{x}{\log_e x}$ is a solution of the equation $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2$.

b Show that $y = \log_e(\log_e x)$ is a solution of the equation $x\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0$.

16 This result will be used in Section 6I.

a Copy and complete the statement $\log_e|x| = \begin{cases} \dots, & \text{for } x > 0, \\ \dots, & \text{for } x < 0. \end{cases}$

b Use part **a** to sketch the curve $y = \log_e|x|$.

c By differentiating separately the two branches in part **a**, show that

$$\frac{d}{dx}\log_e|x| = \frac{1}{x}, \text{ for all } x \neq 0.$$

d Why was $x = 0$ excluded in this discussion?

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ENRICHMENT

An alternative definition of e^x and e:

17 a If $y = \log_e x$, use differentiation by first principles to show that $y' = \lim\limits_{h \to 0} \log_e \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$.

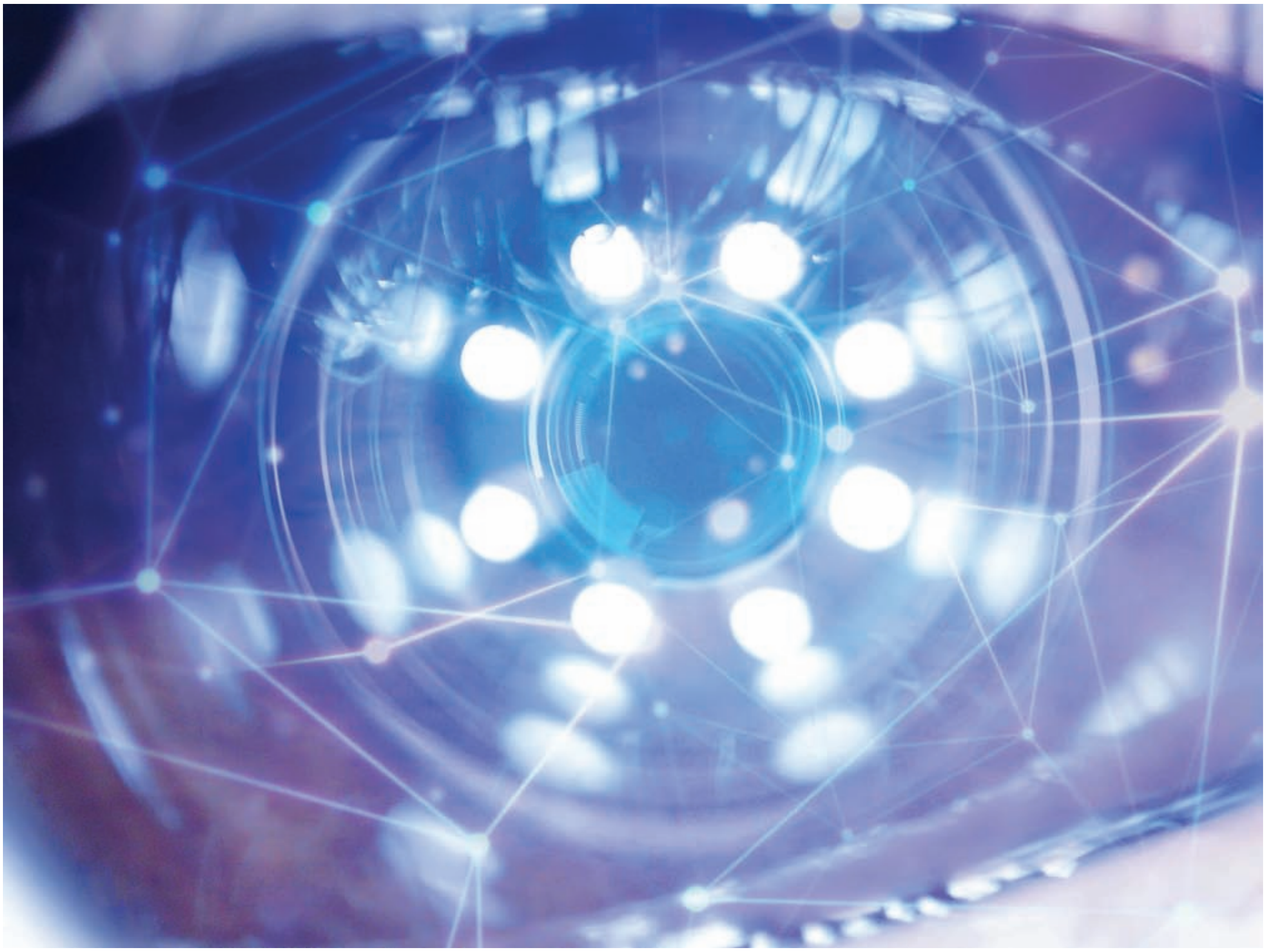
b Use the fact that $y' = \dfrac{1}{x}$ to show that $\lim\limits_{h \to 0} \log_e \left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \frac{1}{x}$.

c Substitute $n = \frac{1}{h}$ and $u = \frac{1}{x}$ to prove these two important limits:

i $\lim\limits_{n \to \infty} \left(1 + \dfrac{u}{n}\right)^n = e^u$ **ii** $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n = e$

d Investigate how quickly, or slowly, $\left(1 + \frac{1}{n}\right)^n$ converges to e by using your calculator with these values of n.

i 1 **ii** 10 **iii** 100 **iv** 1000 **v** 10 000



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6H Applications of differentiation of $\log_e x$

Differentiation can now be applied in the usual way to study the graphs of functions involving $\log_e x$.

The geometry of tangents and normals

The derivative can be used as usual to investigate the geometry of tangents and normals to a curve.

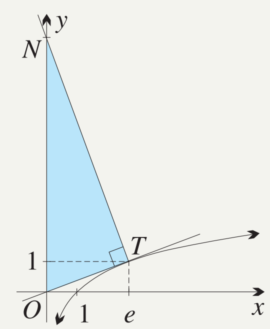
Example 31 6H

a Show that the tangent to $y = \log_e x$ at $T(e, 1)$ has equation $x = ey$.

b Find the equation of the normal to $y = \log_e x$ at $T(e, 1)$.

c Sketch the curve, the tangent and the normal, and find the area of the triangle formed by the y-axis and the tangent and normal at T.

SOLUTION



a Differentiating, $\dfrac{dy}{dx} = \dfrac{1}{x}$,

so the tangent at $T(e, 1)$ has gradient $\dfrac{1}{e}$,

and the tangent is
$$y - 1 = \frac{1}{e}(x - e)$$
$$ey - e = x - e$$
$$x = ey$$
$$y = \frac{x}{e}.$$

Notice that this tangent has gradient $\frac{1}{e}$ and passes through the origin.

b The tangent at $T(e, 1)$ has gradient $\frac{1}{e}$, so the normal there has gradient $-e$.

Hence the normal has equation $y - 1 = -e(x - e)$
$$y = -ex + (e^2 + 1).$$

c Substituting $x = 0$, the normal has y-intercept $N(0, e^2 + 1)$.
Hence the base ON of $\triangle ONT$ is $(e^2 + 1)$ and its altitude is e.
Thus the triangle $\triangle ONT$ has area $\frac{1}{2}e(e^2 + 1)$ square units.

An example of curve sketching

Here are the six steps of our informal curve-sketching menu applied to the function $y = x\log_e x$.

Example 32 6H

Sketch the graph of $y = x\log_e x$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. (You may assume that $x\log_e x \to 0$ as $x \to 0^+$.)

e Find any stationary points and examine their nature.

f Find any points of inflection, and examine the concavity.

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SOLUTION

a The domain is $x > 0$, because $\log_e x$ is undefined for $x \le 0$.

b The function is undefined when x is negative, so it is neither even nor odd.

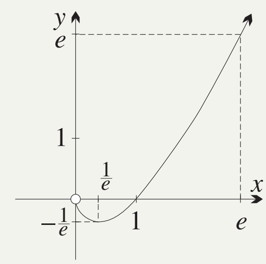
c The only zero is at $x = 1$, and the curve is continuous for $x > 0$.

We take test values at $x = e$ and at $= \dfrac{1}{e}$.

When $x = e$, $y = e\log_e e$
$\quad = e \times 1$
$\quad = e$.

When $x = e^{-1}$, $y = e^{-1}\log_e e^{-1}$
$\quad = e^{-1} \times (-1)$
$\quad = -e^{-1}$.

x	0	e^{-1}	1	e
y	*	$-e^{-1}$	0	e
sign	*	$-$	0	$+$



Hence y is negative for $0 < x < 1$ and positive for $x > 1$.

d As given in the hint, $y \to 0$ as $x \to 0^+$.
Also, $y \to \infty$ as $x \to \infty$.

e Differentiating by the product rule,

$$f'(x) = vu' + uv'$$
$$= \log_e x + x \times \frac{1}{x}$$
$$= \log_e x + 1,$$

and $f''(x) = \dfrac{1}{x}$.

Let $u = x$
and $v = \log_e x$.
Then $u' = 1$
and $v' = \dfrac{1}{x}$.

Putting $f'(x) = 0$ gives $\log_e x = -1$
$\quad x = e^{-1}$.

Substituting, $\quad f'(e^{-1}) = e > 0$
and $\quad f(e^{-1}) = -e^{-1}$, as above,
so $(e^{-1}, -e^{-1})$ is a minimum turning point.
Also $f'(x) \to -\infty$ as $x \to 0^+$, so the curve becomes vertical near the origin.

f Because $f''(x)$ is always positive, there are no inflections, and the curve is always concave up.

A difficulty with the limits of $x\log_e x$ and $\dfrac{\log_e x}{x}$

The curve-sketching example above involved knowing the behaviour of $x\log_e x$ as $x \to 0^+$. When x is a small positive number, $\log_e x$ is a large negative number, so it is not immediately clear whether the product $x\log_e x$ becomes large or small as $x \to 0^+$.

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In fact, $x\log_e x \to 0$ as $x \to 0^+$, and x is said to *dominate* $\log_e x$, in the same way that e^x dominated x in Section 6C. Here is a table of values that should make it reasonably clear that $\lim_{x\to 0^+} x\ \log_e x = 0$:

x	$\frac{1}{e}$	$\frac{1}{e^2}$	$\frac{1}{e^3}$	$\frac{1}{e^4}$	$\frac{1}{e^5}$	$\frac{1}{e^6}$	$\frac{1}{e^7}$	...
$x\log_e x$	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	...
approx.	−0.37	−0.27	−0.15	−0.073	−0.034	−0.015	−0.006	...

Such limits would normally be given in any question where they are needed.

A similar problem arises with the behaviour of $\dfrac{\log_e x}{x}$ as $x \to \infty$, because both top and bottom get large when x is large. Again, x dominates $\log_e x$, meaning that $\lim_{x\to\infty}\dfrac{\log_e x}{x} = 0$, as the following table should make reasonably obvious:

x	e	e^2	e^3	e^4	e^5	e^6	e^7	...
$\frac{\log_e x}{x}$	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	...
approx.	0.37	0.27	0.15	0.073	0.034	0.015	0.006	...

Again, this limit would normally be given if it is needed, but we box the results for completeness. We will provide a proof in Question **23** of Exercise 6J.

9 DOMINANCE

- The function x *dominates* the function $\log_e x$, that is

$$\lim_{x\to 0^+} x\ \log_e x = 0 \qquad \text{and} \qquad \lim_{x\to\infty}\frac{\log_e x}{x} = 0.$$

- More generally, the function x^k *dominates* the function $\log_e x$, for all $k > 0$.

Exercise 6H

FOUNDATION

1 a Find the tangent to $y = \log_e x$ at $P(e, 1)$, and prove that it passes through O.

b Find the tangent to $y = \log_e x$ at $Q(1, 0)$, and prove that it passes through $A(0, -1)$.

c Find the tangent to $y = \log_e x$ at $R(\frac{1}{e}, -1)$, and prove that it passes through $B(0, -2)$.

d Find the normal to $y = \log_e x$ at $A(1, 0)$, and its y-intercept.

2 a In Question **1a** you showed that the tangent at $P(e, 1)$ on the curve $y = \log_e x$ passes through the origin. Sketch the graph, showing the tangent, and explain graphically why no other tangent passes through the origin.

b Again arguing geometrically from the graph, classify the points in the plane according to whether 0, 1 or 2 tangents pass through them.

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3 Find, giving answers in the form $y = mx + b$, the equations of the tangent and normal to:

a $y = 4\log_e x$ at the point $Q(1, 0)$,
b $y = \log_e x + 3$ at the point $R(1, 3)$,
c $y = 2\log_e x - 2$ at the point $S(1, -2)$,
d $y = 1 - 3\log_e x$ at the point $T(1, 1)$.

4 **a** Show that the point $P(1, 0)$ lies on the curve $y = \log_e(3x - 2)$.
b Find the equations of the tangent and normal at P, and their y-intercepts.
c Find the area of the triangle formed by the tangent, the normal and the y-axis.

5 **a** Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient $\frac{1}{2}$. Then find the equation of the tangent and normal there, in the form $y = mx + b$.
b Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient 2. Then find the equation of the tangent and normal there, in the form $y = mx + b$.

6 **a** Write down the natural domain of $y = x - \log_e x$. What does this answer tell you about whether the function is even, odd or neither?
b Find its first two derivatives.
c Show that the curve is concave up for all values of x in its domain.
d Find the minimum turning point.
e Sketch the curve and write down its range.
f Finally sketch the curve $y = \log_e x - x$ by recognising the simple transformation.

7 **a** Write down the domain of $y = \dfrac{1}{x} + \ln x$.
b Show that the first and second derivatives may be expressed as single fractions as $y' = \dfrac{x - 1}{x^2}$ and $y'' = \dfrac{2 - x}{x^3}$.
c Show that the curve has a minimum at $(1, 1)$ and an inflection at $(2, \frac{1}{2} + \ln 2)$.
d Sketch the graph and write down its range.

8 Consider the curve $y = x\log_e x - x$.
a Write down the domain and x-intercept.
b Draw up a table of signs for the function.
c Show that $y' = \log_e x$ and find y''.
d Hence show that there is one stationary point and determine its nature.
e What does y'' tell you about the curve?
f Given that $y \to 0^-$ as $x \to 0^+$, and that the tangent approaches vertical as $x \to 0^+$, sketch the curve and write down its range.

9 **a** Write down the domain of $y = \log_e(1 + x^2)$.
b Is the curve, even, odd or neither?
c Find where the function is zero, and explain what its sign is otherwise.
d Show that $y' = \dfrac{2x}{1 + x^2}$ and $y'' = \dfrac{2(1 - x^2)}{(1 + x^2)^2}$.
e Hence show that $y = \log_e(1 + x^2)$ has one stationary point, and determine its nature.
f Find the coordinates of the two points of inflection.
g Hence sketch the curve, and then write down its range.

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DEVELOPMENT

10 **a** Find the domain of $y = (\ln x)^2$.

b Find where the function is zero, and explain what its sign is otherwise.

c Find y' and show that $y'' = \dfrac{2(1 - \ln x)}{x^2}$.

d Hence show that the curve has an inflection at $x = e$.

e Classify the stationary point at $x = 1$, sketch the curve, and write down the range.

11 **a** Find and classify the lone stationary point of $y = x^2 \log x$ in its natural domain.

b Show that there is an inflection at $x = e^{-\frac{3}{2}}$.

c Examine the behaviour of y and y' as $x \to 0^+$.

d Hence sketch the graph of this function, then write down its range.

12 **a** Write down the domain of $y = \dfrac{\log_e x}{x}$, then find any horizontal or vertical asymptotes.

b Find y' and y''.

c Find any stationary points and determine their nature.

d Find the exact coordinates of the lone point of inflection.

e Sketch the curve, and write down its range.

13 Carefully classify the significant points of $y = \dfrac{x}{\log x}$, and show that there is an inflection at $(e^2, \frac{1}{2}e^2)$.
Examine the behaviour of y and y' as $x \to 0^+$ and as $x \to \infty$, then sketch the curve and write down its range. Use your techniques of sketching the reciprocal to compare this graph with the graph in the previous question.

14 **a** Write down the domain of $y = \log_e\left(\dfrac{x^2}{x + 1}\right)$.

b Show that $y' = \dfrac{x + 2}{x(x + 1)}$, for x in the domain of the function.

c Show that substituting $x = -2$ into the equation in part **b** gives zero. Explain why there is nevertheless no stationary point at $x = -2$.

d How many inflection points does this curve have?

e Sketch the graph.

15 **a** What is the natural domain of $y = \ln(\ln x)$?

b What is the x-intercept?

c Find y' and y'', and explain why there are no stationary points.

d Explain why there is no inflection at $x = e^{-1}$.

e Sketch the curve.

16 [Technology]
This question confirms the remarks about *dominance* in the text of this section. Use technology to complete the two tables of values below. Do they confirm the values of $\lim\limits_{x \to \infty} \dfrac{\log_e x}{x}$ and $\lim\limits_{x \to 0^+} x \log_e x$ given in the text?

x	2	5	10	20	40	4000
$\dfrac{\log_e x}{x}$						

x	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{4000}$
$x \log_e x$						

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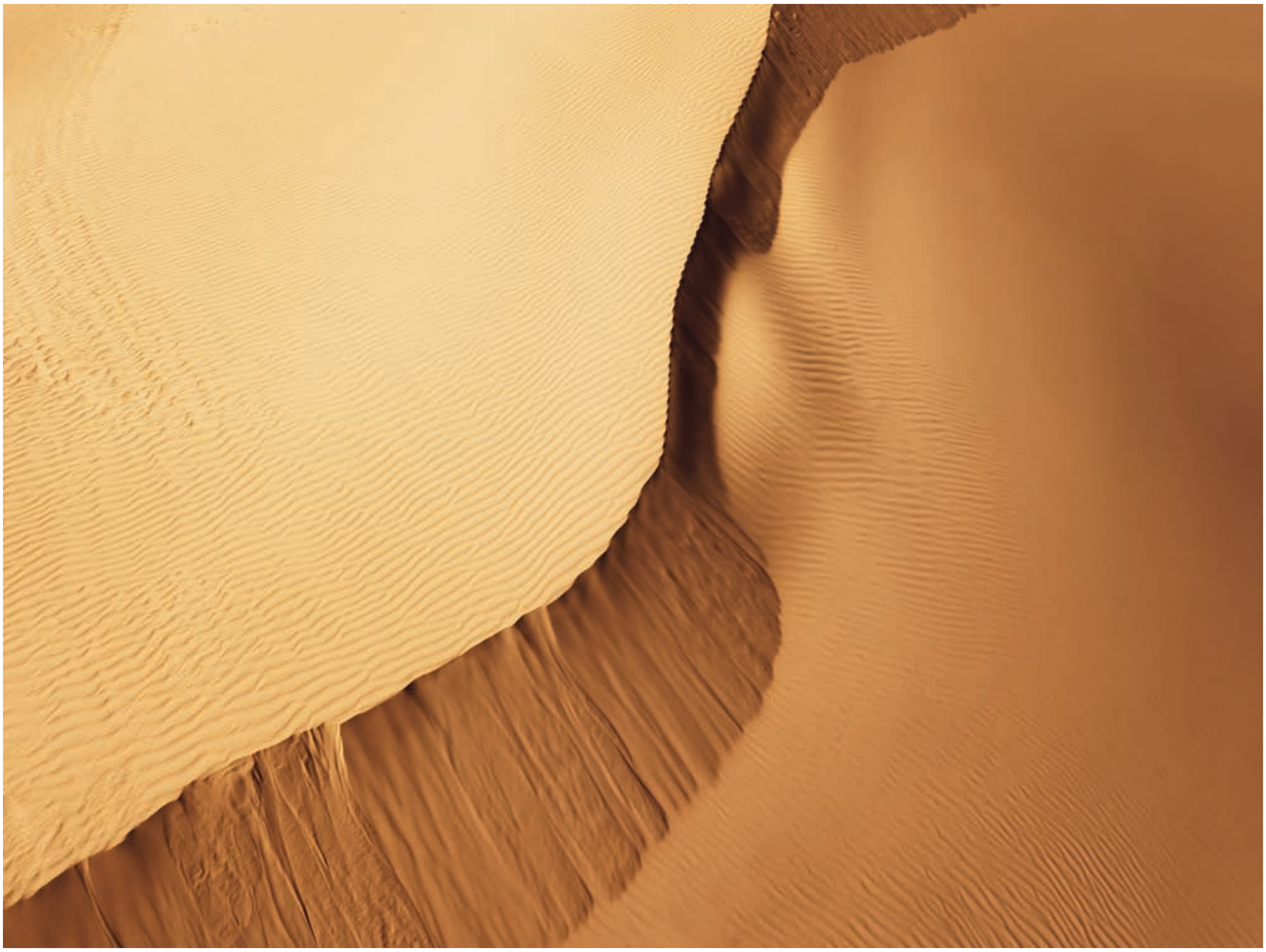
ENRICHMENT

17 Show that $y = x^{\frac{1}{\ln x}}$ is a constant function and find the value of this constant. What is the natural domain of this function? Sketch its graph.

18 a Differentiate $y = x^x$ by taking logs of both sides. Then examine the behaviour of $y = x^x$ near $x = 0$, and show that the curve becomes vertical as $x \to 0^+$.
b Locate and classify any stationary points, and where the curve has gradient 1.
c Sketch the function, and state its domain and range.

19 a Find the limits of $y = x^{\frac{1}{x}}$ as $x \to 0^+$ and as $x \to \infty$.
b Show that there is a maximum turning point when $x = e$.
c Show that $y = x^x$ and $y = x^{\frac{1}{x}}$ have a common tangent at $x = 1$.
d Sketch the graph of the function.



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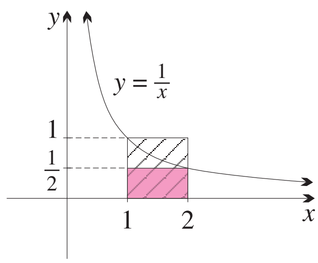
6I Integration of the reciprocal function

The reciprocal function $y = \frac{1}{x}$ is an important function — we have seen that it is required whenever two quantities are inversely proportional to each other. So far, however, it has not been possible to integrate the reciprocal function, because the usual rule for integrating powers of x gives nonsense:

$$\text{When } n = -1, \int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{gives} \int x^{-1}\,dx = \frac{x^0}{0},$$

which is nonsense because of the division by zero.



Yet the graph of $y = \frac{1}{x}$ to the right shows that there should be no problem with definite integrals involving $\frac{1}{x}$, provided that the integral does not cross the discontinuity at $x = 0$. For example, the diagram shows the integral $\int_1^2 \frac{1}{x}\,dx$, which the little rectangles show has a value between $\frac{1}{2}$ and 1.

Integration of the reciprocal function

Reversing the standard form for differentiating $\log x$ will now give the necessary standard forms for integrating $\frac{1}{x}$.

$$\text{Reversing} \quad \frac{d}{dx}\log_e x = \frac{1}{x} \text{ gives } \int \frac{1}{x}\,dx = \log_e x + C.$$

This is a new standard form for integrating the reciprocal function.

The only qualification is that $x > 0$, otherwise $\log_e x$ is undefined, so we have

$$\int \frac{1}{x}\,dx = \log_e x + C, \text{ for some constant } C, \text{ provided that } x > 0.$$

Example 33 6I

a Find the definite integral $\int_1^2 \frac{1}{x}\,dx$ sketched above.

b Approximate the integral correct to three decimal places and verify that

$$\tfrac{1}{2} < \int_1^2 \frac{1}{x}\,dx < 1.$$

SOLUTION

a $$\int_1^2 \frac{1}{x}\,dx = \Big[\log_e x\Big]_1^2$$
$$= \log_e 2 - \log_e 1$$
$$= \log_e 2, \quad \text{because } \log_e 1 = 0.$$

b Hence $\int_1^2 \frac{1}{x}\,dx \doteqdot 0.693$,

which is indeed between $\frac{1}{2}$ and 1, as the diagram above indicated.

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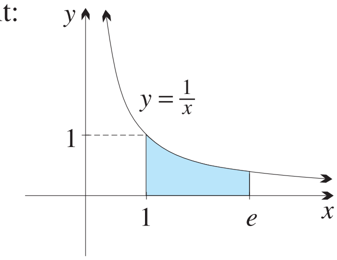
A characterisation of e

Integrating the reciprocal function from 1 to e gives an amazingly simple result:

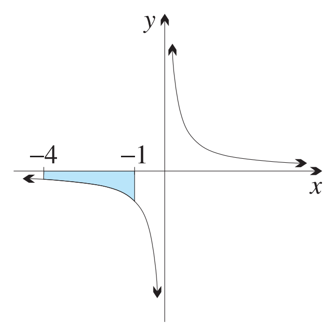


$$\begin{aligned}\int_1^e \frac{1}{x}\,dx &= \Big[\log_e x\Big]_1^e \\ &= \log_e e - \log_e 1 \\ &= 1 - 0 \\ &= 1.\end{aligned}$$

The integral is sketched to the right. The example is very important because it characterises e as the real number satisfying $\int_1^e \frac{1}{x}\,dx = 1$. In other expositions of the theory, this integral is taken as the definition of e.

The primitive of $y = \frac{1}{x}$ on both sides of the origin

So far our primitive is restricted by the condition $x > 0$, meaning that we can only deal with definite integrals on the right-hand side of the origin. The full graph of the reciprocal function $y = 1/x$, however, is a hyperbola, with two disconnected branches separated by the discontinuity at $x = 0$.



Clearly there is no reason why we should not integrate over a closed interval such as $-4 \le x \le -1$ on the left-hand side of the origin. We can take any definite integrals of $\frac{1}{x}$, provided only that we do not work across the asymptote at $x = 0$. If x is negative, then $\log(-x)$ is well defined, and using our previous standard forms,

$$\frac{d}{dx}\log(-x) = -\left(\frac{1}{-x}\right) = \frac{1}{x},$$

and reversing this, $\log(-x)$ is a primitive of $\frac{1}{x}$ when x is negative,

$$\int \frac{1}{x}\,dx = \log_e(-x) + C, \quad \text{provided that } x < 0.$$

The absolute value function is designed for just these situations. We can combine the two results into one standard form for the whole reciprocal function,

$$\int \frac{1}{x}\,dx = \log_e |x| + C, \quad \text{provided that } x \ne 0.$$

Question 16 of Exercise 6G gives more detail about this standard form.

Each branch may have its own constant of integration

Careful readers will realise that because $y = \frac{1}{x}$ has two disconnected branches, there can be different constants of integration in the two branches. So the general primitive of $\frac{1}{x}$ is

$$\int \frac{1}{x}\,dx = \begin{cases}\log_e x + A, & \text{for } x > 0,\\ \log(-x) + B, & \text{for } x < 0,\end{cases} \quad \text{where } A \text{ and } B \text{ are constants.}$$

If a boundary condition is given for one branch, this has no implication at all for the constant of integration in the other branch.

In any physical interpretation, however, the function would normally have meaning in only one of the two branches, so the complication discussed here is rarely needed, and the over-simplified forms in Box 10 below are standard and generally used — the qualification is understood and taken account of when necessary.

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Three standard forms

As always, reversing the other standard forms for differentiation gives two more standard forms.

10 STANDARD FORMS FOR INTEGRATING RECIPROCAL FUNCTIONS

- $$\int \frac{1}{x}\,dx = \log_e |x| + C$$
- $$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e |ax + b| + C$$
- $$\int \frac{u'}{u}\,dx = \log_e |u| + C \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$$

No calculation involving these primitives may cross an asymptote.

The final warning always applies to the primitive of any function, but it is mentioned here because it is such an obvious issue.

Example 34 6I

Evaluate these definite integrals using the first two standard forms above.

a $\displaystyle\int_e^{e^2} \frac{5}{x}\,dx$ **b** $\displaystyle\int_1^4 \frac{1}{1 - 2x}\,dx$ **c** $\displaystyle\int_1^5 \frac{1}{x - 2}\,dx$

SOLUTION

a
$$\begin{aligned}\int_e^{e^2} \frac{5}{x}\,dx &= 5\Big[\log_e |x|\Big]_e^{e^2}\\ &= 5(\log_e e^2 - \log_e e)\\ &= 5(2 - 1)\\ &= 5\end{aligned}$$

b
$$\begin{aligned}\int_1^4 \frac{1}{1 - 2x}\,dx &= -\tfrac{1}{2}\Big[\log_e |1 - 2x|\Big]_1^4 \qquad \text{(here } a = -2 \text{ and } b = 1)\\ &= -\tfrac{1}{2}(\log_e |-7| - \log_e |-1|)\\ &= -\tfrac{1}{2}(\log_e 7 - 0)\\ &= -\tfrac{1}{2}\log_e 7\end{aligned}$$

c This definite integral is meaningless because it crosses the asymptote at $x = 2$.

Using the third standard form

The vital point in using the third standard form,

$$\int \frac{u'}{u}\,dx = \log_e |u| \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|,$$

is that the top must be the derivative of the bottom. Choose whichever form of the reverse chain rule you are most comfortable with.

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Example 35 6I

Evaluate these definite integrals using the third standard form above.

a $\displaystyle\int_0^1 \frac{2x}{x^2+2}\,dx$ **b** $\displaystyle\int_4^5 \frac{x}{9-x^2}\,dx$ **c** $\displaystyle\int_0^2 \frac{3x}{1-x^3}\,dx$

SOLUTION

a Let $u = x^2 + 2$ OR $f(x) = x^2 + 2$.
Then $u' = 2x$ $f'(x) = 2x$.

Hence in the fraction $\dfrac{2x}{x^2+1}$, the top is the derivative of the bottom.

Thus, using $\displaystyle\int \frac{u'}{u}\,dx = \log_e |f(x)|$ OR $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$,

$$\int_0^1 \frac{2x}{x^2+2}\,dx = \Big[\log_e(x^2+2)\Big]_0^1$$
$$= \log_e 3 - \log_e 2.$$

Note: The use of absolute value signs here is unnecessary (but is not wrong) because $x^2 + 2$ is never negative.

b Let $u = 9 - x^2$ OR $f(x) = 9 - x^2$.
Then $u' = -2x$ $f'(x) = -2x$.
The first step is to make the top the derivative of the bottom,

$$\int_4^5 \frac{x}{9-x^2}\,dx = -\tfrac{1}{2}\int_4^5 \frac{-2x}{9-x^2}\,dx,\quad \text{of the form } \int \frac{u'}{u}\,dx \text{ OR } \int \frac{f'(x)}{f(x)}\,dx,$$
$$= -\tfrac{1}{2}\Big[\log_e |9-x^2|\Big]_4^5$$
$$= -\tfrac{1}{2}(\log_e |-16| - \log_e |-7|)$$
$$= -\tfrac{1}{2}(4\log_e 2 - \log_e 7)$$
$$= -2\log_e 2 + \tfrac{1}{2}\log_e 7.$$

c This definite integral is meaningless because it crosses the asymptote at $x = 1$.

Given the derivative, find the function

Finding the function from the derivative involves a constant that can be found if the value of y is known for some value of x.

Example 36 6I

a Find $f(x)$, if $f'(x) = \dfrac{2}{3-x}$ and the graph passes through the origin.

b Hence find $f(2)$.

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SOLUTION

a Here $\qquad f'(x) = \dfrac{2}{3 - x}$.

Taking the primitive, $\qquad f(x) = -2\ln|3 - x| + C$, for some constant C.

Because $f(0) = 0$, $\qquad 0 = -2\ln 3 + C$

$\qquad C = 2\ln 3$.

Hence $\qquad f(x) = 2\ln 3 - 2\ln|3 - x|$.

b Substituting $x = 2$ gives $f(2) = 2\ln 3 - 2\ln 1$

$\qquad = 2\ln 3$.

Note: As remarked above, this working is over-simplified, because each branch may have its own constant of integration. But there is no problem in this question because the asymptote is at $x = 3$, so that the given point $(0, 0)$ on the curve, and the value $x = 2$ in part **b**, are both on the same side of the asymptote.

A primitive of $\log_e x$

The next worked example is more difficult, but it is important because it produces a primitive of $\log_e x$, which the theory has not yielded so far. There is no need to memorise the result.

Example 37 — 6I

a Differentiate $x\log_e x$ by the product rule.

b Show by differentiation that $x\log_e x - x$ is a primitive of $\log_e x$.

c Use this result to evaluate $\displaystyle\int_1^e \log_e x\,dx$.

SOLUTION

a Differentiating by the product rule,

$$\frac{d}{dx}(x\log_e x) = vu' + uv'$$
$$= \log_e x + x \times \frac{1}{x},$$
$$= 1 + \log_e x.$$

Let $u = x$
and $v = \log_e x$.
Then $u' = 1$
and $v' = \dfrac{1}{x}$.

b Let $\qquad y = x\log_e x - x$.

Then $\qquad y' = (1 + \log_e x) - 1$, $\qquad$ using the result of part **a**,

$\qquad = \log_e x$.

Reversing this result gives the primitive of $\log_e x$,

$$\int \log_e x\,dx = x\log_e x - x + C.$$

c Part **b** can now be used to find the definite integral,

$$\int_1^e \log_e x\,dx = \Big[x\log_e x - x\Big]_1^e$$
$$= (e\log_e e - e) - (1\log_e 1 - 1)$$
$$= (e\log_e e - e) - (0 - 1)$$
$$= (e - e) + 1$$
$$= 1.$$

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Exercise 6I

FOUNDATION

1 First rewrite each integral using the result $\int \frac{k}{x}\,dx = k\int \frac{1}{x}\,dx$, where k is a constant. Then use the standard form $\int \frac{1}{x}\,dx = \log_e |x| + C$ to integrate it.

a $\int \frac{2}{x}\,dx$ b $\int \frac{1}{3x}\,dx$ c $\int \frac{4}{5x}\,dx$ d $\int \frac{3}{2x}\,dx$

2 Use the standard form $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e |ax+b| + C$ to find these indefinite integrals.

a $\int \frac{1}{4x+1}\,dx$ b $\int \frac{1}{5x-3}\,dx$ c $\int \frac{6}{3x+2}\,dx$

d $\int \frac{15}{5x+1}\,dx$ e $\int \frac{4}{4x+3}\,dx$ f $\int \frac{dx}{3-x}$

g $\int \frac{dx}{7-2x}$ h $\int \frac{4\,dx}{5x-1}$ i $\int \frac{12\,dx}{1-3x}$

3 Evaluate these definite integrals. Simplify your answers where possible.

a $\int_1^5 \frac{1}{x}\,dx$ b $\int_1^3 \frac{1}{x}\,dx$ c $\int_{-8}^{-2} \frac{1}{x}\,dx$

d $\int_{-3}^{9} \frac{1}{x}\,dx$ e $\int_1^4 \frac{dx}{2x}$ f $\int_{-15}^{-5} \frac{dx}{5x}$

4 Evaluate these definite integrals, then use the function labelled $\boxed{\ln}$ on your calculator to approximate each integral correct to four significant figures.

a $\int_0^1 \frac{dx}{x+1}$ b $\int_{-7}^{-5} \frac{dx}{x+2}$ c $\int_{-5}^{-2} \frac{dx}{2x+3}$

d $\int_1^2 \frac{3}{5-2x}\,dx$ e $\int_{-1}^{1} \frac{3}{7-3x}\,dx$ f $\int_0^{11} \frac{5}{2x-11}\,dx$

5 Evaluate these definite integrals. Simplify your answers where possible.

a $\int_1^e \frac{dx}{x}$ b $\int_1^{e^2} \frac{dx}{x}$ c $\int_e^{e^4} \frac{dx}{x}$ d $\int_{\sqrt{e}}^{e} \frac{dx}{x}$

6 Find primitives of these function by first writing them as separate fractions.

a $\frac{x+1}{x}$ b $\frac{x+3}{5x}$ c $\frac{1-8x}{9x}$

d $\frac{3x^2-2x}{x^2}$ e $\frac{2x^2+x-4}{x}$ f $\frac{x^4-x+2}{x^2}$

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DEVELOPMENT

7 In each case show that the numerator is the derivative of the denominator. Then use the form $\int \frac{u'}{u}\,dx = \log_e |u| + C$ or $\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$ to integrate the expression.

a $\frac{2x}{x^2 - 9}$ **b** $\frac{6x + 1}{3x^2 + x}$ **c** $\frac{2x + 1}{x^2 + x - 3}$

d $\frac{5 - 6x}{2 + 5x - 3x^2}$ **e** $\frac{x + 3}{x^2 + 6x - 1}$ **f** $\frac{3 - x}{12x - 3 - 2x^2}$

g $\frac{e^x}{1 + e^x}$ **h** $\frac{e^{-x}}{1 + e^{-x}}$ **i** $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

Why is it unnecessary (but not wrong) to use absolute value signs in the answers to parts **g**–**i**?

8 Use the standard form $\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e |ax + b| + C$ to find these indefinite integrals.

a $\int \frac{1}{3x - k}\,dx$ **b** $\int \frac{1}{mx - 2}\,dx$ **c** $\int \frac{p}{px + q}\,dx$ **d** $\int \frac{A}{sx - t}\,dx$

9 Find $f(x)$, and then find $f(2)$, given that:

a $f'(x) = 1 + \frac{2}{x}$ and $f(1) = 1$ **b** $f'(x) = 2x + \frac{1}{3x}$ and $f(1) = 2$

c $f'(x) = 3 + \frac{5}{2x - 1}$ and $f(1) = 0$ **d** $f'(x) = 6x^2 + \frac{15}{3x + 2}$ and $f(1) = 5 \ln 5$

10 a Given that the derivative of $f(x)$ is $\frac{x^2 + x + 1}{x}$ and $f(1) = 1\frac{1}{2}$, find $f(x)$.

b Given that the derivative of $g(x)$ is $\frac{2x^3 - 3x - 4}{x^2}$ and $g(2) = -3 \ln 2$, find $g(x)$.

11 a Find y as a function of x if $y' = \frac{1}{4x}$ and $y = 1$ when $x = e^2$. Where does this curve meet the x-axis on the right-hand side of the origin?

b The gradient of a curve is given by $y' = \frac{2}{x + 1}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve?

c Find $y(x)$, given that $y' = \frac{2x + 5}{x^2 + 5x + 4}$ and $y = 1$ when $x = 1$. Hence evaluate $y(0)$.

d Write down the equation of the family of curves with the property $y' = \frac{2 + x}{x}$. Hence find the curve that passes through $(1, 1)$ and evaluate y at $x = 2$ for this curve.

e Given that $f''(x) = \frac{1}{x^2}$, $f'(1) = 0$ and $f(1) = 3$, find $f(x)$ and hence evaluate $f(e)$.

12 Use one of the forms $\int \frac{u'}{u}\,dx = \log_e |u| + C$ OR $\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$ to find:

a $\int \frac{3x^2}{x^3 - 5}\,dx$ **b** $\int \frac{4x^3 + 1}{x^4 + x - 5}\,dx$ **c** $\int \frac{x^3 - 3x}{x^4 - 6x^2}\,dx$

d $\int \frac{10x^3 - 7x}{5x^4 - 7x^2 + 8}\,dx$ **e** $\int_2^3 \frac{3x^2 - 1}{x^3 - x}\,dx$ **f** $\int_e^{2e} \frac{2x + 2}{x^2 + 2x}\,dx$

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13 a i Differentiate $y = x\log_e x - x$.

ii Hence find $\int \log_e x\,dx$ and $\int_{\sqrt{e}}^{e} \log_e x\,dx$.

b i Find the derivative of $y = 2x^2\log_e x - x^2$.

ii Hence find $\int x\log_e x\,dx$ and $\int_e^2 x\log_e x\,dx$.

c Differentiate $(\log_e x)^2$, and hence find $\int_{\sqrt{e}}^{e} \dfrac{\log_e x}{x}\,dx$.

d Differentiate $\ln(\ln x)$ and hence determine the family of primitives of $\dfrac{1}{x\ln x}$.

14 Find the value of a if:

a $\int_1^a \frac{1}{x}\,dx = 5$ **b** $\int_a^e \frac{1}{x}\,dx = 5$ **c** $\int_a^{-1} \frac{1}{x}\,dx = -2$ **d** $\int_{-e}^{a} \frac{1}{x}\,dx = -2$

15 a Find $\int \dfrac{e^x}{e^x + 1}\,dx$.

b Find $\int_1^e \left(x + \dfrac{1}{x^2}\right)^2 dx$.

c Show that $x\,e^x = e^{x + \log_e x}$, and hence differentiate $x\,e^x$ without using the product rule.

16 Stella found the primitive of the function $\dfrac{1}{5x}$ by taking out a factor of $\frac{1}{5}$,

$$\int \frac{1}{5x}\,dx = \frac{1}{5}\int \frac{1}{x}\,dx = \tfrac{1}{5}\log_e|x| + C_1, \text{ for some constant } C_1.$$

Magar used the second standard form in Box 10 with $a = 5$ and $b = 0$,

$$\int \frac{1}{5x}\,dx = \tfrac{1}{5}\log_e|5x| + C_2, \text{ for some constant } C_2.$$

Explain what is going on. Will this affect their result when finding a definite integral?

ENRICHMENT

17 A certain curve has gradient $y' = \dfrac{1}{x}$ and its two branches pass through the two points $(-1, 2)$ and $(1, 1)$. Find the equation of the curve.

A power series for $\log_e(1 + x)$:

18 a Use the formula for the partial sum of a GP to prove that for $t \neq -1$,

$$1 - t + t^2 - t^3 + \cdots + t^{2n} = \frac{1}{1+t} + \frac{t^{2n+1}}{1+t}.$$

b Integrate both sides of this result from $t = 0$ to $t = x$ to show that for $x > -1$,

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t}\,dt.$$

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c Explain why $\dfrac{t^{2n+1}}{1+t} \le t^{2n+1}$, for $0 \le t \le 1$. Hence prove that for $0 \le x \le 1$, the integral

$\displaystyle\int_0^x \frac{t^{2n+1}}{1+t}\,dt$ converges to 0 as $n \to \infty$. Hence show that

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \text{ for } 0 \le x \le 1.$$

d **i** Use this series to approximate $\log_e \frac{3}{2}$ correct to two decimal places.

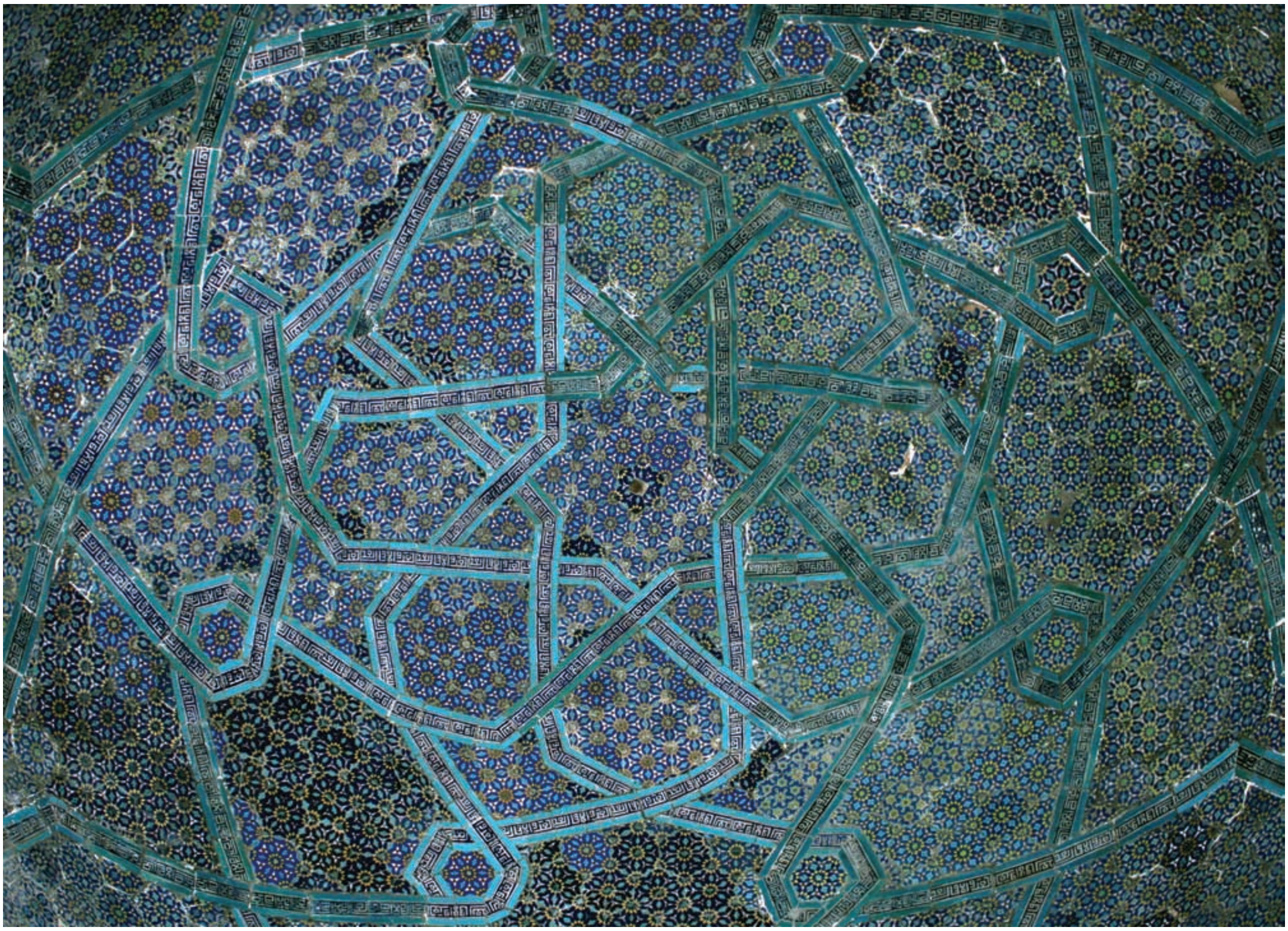
ii Write down the series converging to $\log_e 2$ — called the *alternating harmonic series*.

e With a little more effort, it can be shown that the series in part **c** converges to the given limit for $-1 < x \le 1$ (the proof is a reasonable challenge). Use this to write down the series converging to $\log_e(1-x)$ for $-1 \le x < 1$, and hence approximate $\log_e \frac{1}{2}$ correct to two decimal places.

f Use both series to show that for $-1 < x < 1$,

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Use this result and an appropriate value of x to find $\log_e 3$ correct to five significant figures.



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6J Applications of integration of $1/x$

The usual applications of integration can now be applied to the reciprocal function, whose primitive was previously unavailable.

Finding areas by integration

The next worked example involves finding the area between two given curves.

Example 38 — 6J

a Find where the hyperbola $xy = 2$ meets the line $x + y = 3$.

b Sketch the situation.

c Find the area of the region between the two curves, in exact form, and then correct to three decimal places.

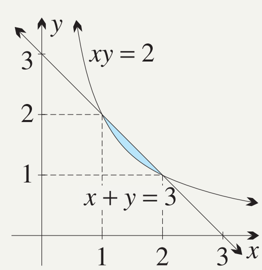
SOLUTION

a Substituting the line $y = 3 - x$ into the hyperbola,

$$\begin{aligned} x(3 - x) &= 2 \\ x^2 - 3x + 2 &= 0 \\ (x - 1)(x - 2) &= 0 \\ x &= 1 \text{ or } 2, \end{aligned}$$

so the curves meet at $(1, 2)$ and $(2, 1)$.

b The hyperbola $xy = 2$ has both axes as asymptotes.
The line $x + y = 3$ has x-intercept $(3, 0)$ and y-intercept $(0, 3)$.



c
$$\begin{aligned} \text{Area} &= \int_1^2 (\text{top curve} - \text{bottom curve})\,dx \\ &= \int_1^2 \left((3 - x) - \frac{2}{x}\right) dx \\ &= \Big[3x - \tfrac{1}{2}x^2 - 2\log_e|x|\Big]_1^2 \\ &= \left(6 - 2 - 2\log_e 2\right) - \left(3 - \tfrac{1}{2} - 2\log_e 1\right) \\ &= \left(4 - 2\log_e 2\right) - \left(2\tfrac{1}{2} - 0\right) \\ &= \left(1\tfrac{1}{2} - 2\log_e 2\right) \text{ square units} \\ &\doteqdot 0.114 \text{ square units.} \end{aligned}$$

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Exercise 6J

FOUNDATION

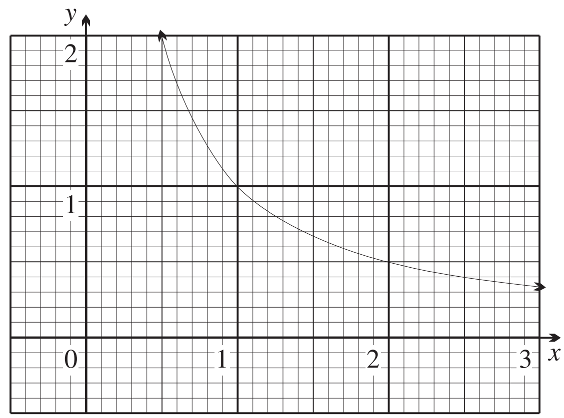
1 **a** Show that $\int_1^e \frac{1}{x}\,dx = 1$.

b This question uses the result in part **a** to estimate e from a graph of $y = \frac{1}{x}$.

The diagram to the right shows the graph of $y = \frac{1}{x}$ from $x = 0$ to $x = 3$.

The graph been drawn on graph paper with a scale of 10 little divisions to 1 unit, so that 100 of the little squares make 1 square unit.

Count the number of squares in the column from $x = 1.0$ to 1.1, then the squares in the column from $x = 1.1$ to 1.2, and so on.



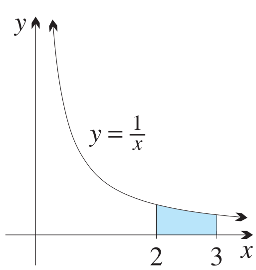
Continue until the number of squares equals 100 — the x-value at this point will be an estimate of e.

2 Find the area between the curve $y = \frac{1}{x}$ and the x-axis within the given interval. Answer first in exact form, and then correct to four significant figures.

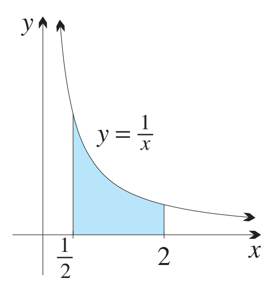
i $1 \le x \le 5$ **ii** $e \le x \le e^2$ **iii** $2 \le x \le 8$

3 Find the area of each shaded region.

a



b



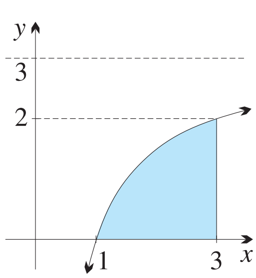
4 **a** Find the area between $y = \frac{1}{3x + 2}$ and the x-axis for $0 \le x \le 1$.

b Find the area between $y = \frac{3}{x - 1}$ and the x-axis for $2 \le x \le e^3 + 1$.

c Find the area between $y = \frac{1}{x} + x$ and the x-axis, from $x = \frac{1}{2}$ to $x = 2$.

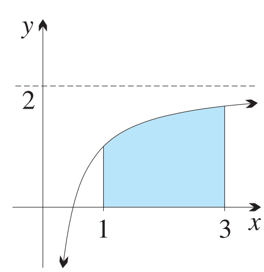
d Find the area between $y = \frac{1}{x} + x^2$ and the x-axis, from $x = 1$ to $x = 3$.

5 **a**



Find the area of the region bounded by $y = 3 - \frac{3}{x}$, the x-axis and $x = 3$.

b



Find the area of the region bounded by $y = 2 - \frac{1}{x}$, the x-axis, $x = 1$ and $x = 3$.

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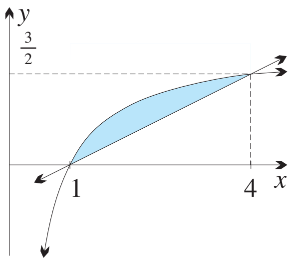
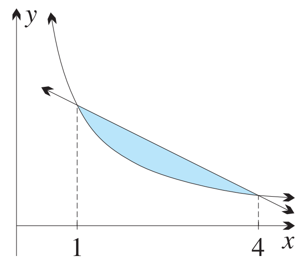
6 a



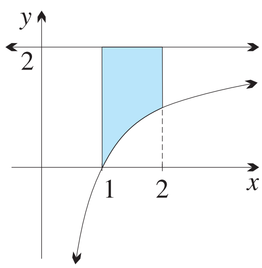
Find the area of the region bounded by $y = 2 - \dfrac{2}{x}$ and the line $y = \frac{1}{2}(x - 1)$.

b



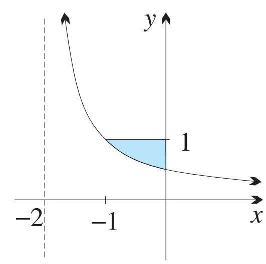
Find the area of the region between $y = \dfrac{2}{x}$ and the line $x + 2y - 5 = 0$.

7 a



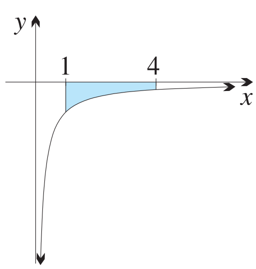
Find the area of the region in the first quadrant bounded by $y = 2 - \dfrac{2}{x}$ and $y = 2$, and lying between $x = 1$ and $x = 2$.

b



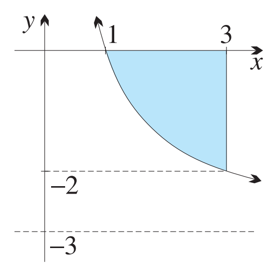
Find the area of the region bounded by the curve $y = \dfrac{1}{x + 2}$, the y-axis and the horizontal line $y = 1$.

8 a



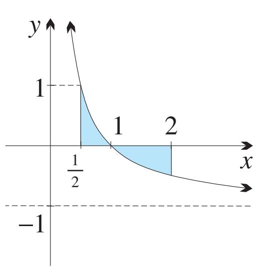
Find the area of the region bounded by $y = -\dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 4$.

b



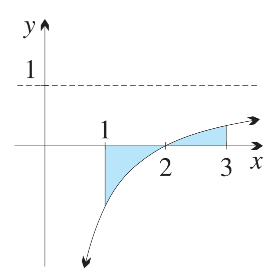
Find the area of the region bounded by $y = \dfrac{3}{x} - 3$, the x-axis and $x = 3$.

9 a



Find the area of the region bounded by $y = \dfrac{1}{x} - 1$, the x-axis, $x = \frac{1}{2}$ and $x = 2$.

b



Find the area of the region bounded by $y = 1 - \dfrac{2}{x}$, the x-axis, $x = 1$ and $x = 3$.

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DEVELOPMENT

10 a Sketch the region bounded by $y = 1$, $x = 8$ and the curve $y = \frac{4}{x}$.

b Find the area of this region with the aid of an appropriate integral.

11 a Find the two intersection points of the curve $y = \frac{1}{x}$ with the line $y = 4 - 3x$.

b Find the area between these two curves.

12 a Differentiate $x^2 + 1$, and hence find the area under the graph $y = \frac{x}{x^2 + 1}$, between $x = 0$ and $x = 2$.

b Differentiate $x^2 + 2x + 3$, and hence find the area under the graph $y = \frac{x + 1}{x^2 + 2x + 3}$, between $x = 0$ and $x = 1$.

13 a Sketch the region bounded by the x-axis, $y = x$, $y = \frac{1}{x}$ and $x = e$.

b Hence find the area of this region by using two appropriate integrals.

14 a Use the trapezoidal rule with four subintervals to approximate the area between the curve $y = \ln x$ and the x-axis, between $x = 1$ and $x = 5$. Answer correct to four decimal places.

b Differentiate $y = x \log_e x$. Hence find the exact value of the area, and approximate it correct to four decimal places.

c Is the estimate greater than or less than the exact value? Explain how the graph could have predicted this.

15 a Show that $4x = 2(2x + 1) - 2$.

b Hence evaluate the area under $y = \frac{4x}{2x + 1}$ between $x = 0$ and $x = 1$.

16 a Show that the curves $y = \frac{6}{x}$ and $y = x^2 - 6x + 11$ intersect when $x = 1, 2$ and 3.

b Graph these two curves and shade the two areas enclosed by them.

c Find the total area enclosed by the two curves.

17 a Use upper and lower rectangles to prove that $\frac{1}{2} < \int_{2^n}^{2^{n+1}} \frac{1}{x}\,dx < 1$, for $n \geq 0$.

b Hence prove that $\int_1^{2^n} \frac{1}{x}\,dx \to \infty$ as $n \to \infty$.

ENRICHMENT

18 Consider the two curves $y = 6e^{-x}$ and $y = e^x - 1$.

a Let $u = e^x$. Show that the x-coordinate of the point of intersection of these two curves satisfies $u^2 - u - 6 = 0$.

b Hence find the coordinates of the point of intersection.

c Sketch the curves on the same number plane, and shade the region bounded by them and the y-axis.

d Find the area of the shaded region.

19 The hyperbola $y = \frac{1}{x} + 1$ meets the x-axis at $(-1, 0)$. Find the area contained between the x-axis and the curve from:

a $x = -e$ to $x = -1$, **b** $x = -1$ to $x = -e^{-1}$, **c** $x = -e$ to $x = -e^{-1}$.

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20 Find a primitive of $\dfrac{1}{x + \sqrt{x}}$.

21 **a** Show that:

$$\frac{d}{dx}\log_e\left(x + \sqrt{x^2 + a^2}\,\right) = \frac{1}{\sqrt{x^2 + a^2}} \text{ and } \frac{d}{dx}\log_e\left|x + \sqrt{x^2 - a^2}\,\right| = \frac{1}{\sqrt{x^2 - a^2}}.$$

b Use these results to find:

i $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2 + 1}}\,dx$

ii $\displaystyle\int_4^8 \frac{1}{\sqrt{x^2 - 16}}\,dx$

22 Consider the area under $y = \dfrac{1}{x}$ between $x = n$ and $x = n + 1$.

a Show that $\dfrac{1}{n + 1} < \displaystyle\int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$.

b Hence show that $\dfrac{n}{n + 1} < \log_e\left(1 + \frac{1}{n}\right)^n < 1$.

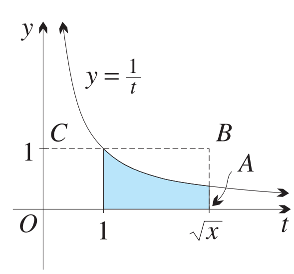
c Take the limit as $n \to \infty$ to show that $\displaystyle\lim_{n \to \infty}\left(1 + \tfrac{1}{n}\right)^n = e$.

d Repeat the above steps, replacing $n + 1$ with $n + t$ and show that $\displaystyle\lim_{n \to \infty}\left(1 + \tfrac{t}{n}\right)^n = e^t$.

Proving the dominance limits:

This question provides proofs of the basic dominance limits used in many of the curve-sketching questions in Exercises 6C and 6H.

23 **a** The shaded region in the diagram to the right shows the definite integral $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\,dt$.



i Explain why $0 < \displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\,dt <$ area $ABCO$.

ii Evaluate the integral, and prove that

$$0 < \frac{\log_e x}{x} < \frac{2}{\sqrt{x}}.$$

iii Hence prove that $\displaystyle\lim_{x \to \infty} \frac{\log_e x}{x} = 0$.

b Substitute $x = \dfrac{1}{u}$ into the limit in part **a** to prove that $\displaystyle\lim_{x \to 0^+} x\log_e x = 0$.

c Substitute $x = e^u$ into the limit in part **b** to prove that $\displaystyle\lim_{x \to -\infty} xe^x = 0$.

d Substitute $x = e^u$ into the limit in part **a** to prove that $\displaystyle\lim_{x \to \infty} xe^{-x} = 0$.

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6K Calculus with other bases

In applications of exponential functions where calculus is required, the base e can generally be used. For example, the treatment of exponential growth in Chapter 11 of the Year 11 book was done entirely using base e.

The change-of-base formula, however, allows calculus to be applied to exponential and logarithmic functions of any base without conversion to base e. In this section, we develop three further standard forms that allows calculus to be applied straightforwardly to functions such as $y = 2^x$ and $y = 10^x$.

Throughout this section, the other base a must be positive and not equal to 1.

Logarithmic functions to other bases

Any logarithmic function can be expressed easily in terms of $\log_e x$ by using the change-of-base formula. For example,

$$\log_2 x = \frac{\log_e x}{\log_e 2}.$$

Thus every other logarithmic function is just a constant multiple of $\log_e x$. This allows any other logarithmic function to be differentiated easily.

Example 39 — 6K

a Express the function $y = \log_5 x$ in terms of the function $\log_e x$.

b Hence use the calculator function labelled $\boxed{\ln}$ to approximate, correct to four decimal places:

i $\log_5 30$ **ii** $\log_5 2$ **iii** $\log_5 0.07$

c Check the results of part **b** using the function labelled $\boxed{x^y}$.

SOLUTION

a $\log_5 x = \dfrac{\log_e x}{\log_e 5}$

b **i** $\log_5 30 = \dfrac{\log_e 30}{\log_e 5} \doteqdot 2.1133$

ii $\log_5 2 = \dfrac{\log_e 2}{\log_e 5} \doteqdot 0.4307$

iii $\log_5 0.07 = \dfrac{\log_e 0.07}{\log_e 5} \doteqdot -1.6523$

c Checking these results using the function labelled $\boxed{x^y}$:

i $5^{2.1133} \doteqdot 30$ **ii** $5^{0.4307} \doteqdot 2$ **iii** $5^{-1.6523} \doteqdot 0.07$

11 LOGARITHMIC FUNCTIONS WITH OTHER BASES

Every logarithmic function can be written as a multiple of a logarithmic function base e:

$$\log_a x = \frac{\log_e x}{\log_e a}, \qquad \text{that is} \qquad \log_a x = \frac{1}{\log_e a} \times \log_e x.$$

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Differentiating logarithmic functions with other bases

Once the function is expressed as a multiple of a logarithmic function base e, it can be differentiated using the previous standard forms.

Example 40 — 6K

Use the change-of-base formula to differentiate:

a $y = \log_2 x$

b $y = \log_a x$

SOLUTION

a Here $y = \log_2 x$.

Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e 2}.$$

Because $\log_e 2$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e 2} = \frac{1}{x \log_e 2}.$$

b Here $y = \log_a x$.

Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e a}.$$

Because $\log_e a$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e a} = \frac{1}{x \log_e a}.$$

Part **b** above gives the formula in the general case:

12 DIFFERENTIATING LOGARITHMIC FUNCTIONS WITH OTHER BASES

- Either use the change-of-base formula to convert to logarithms base e.
- Or use the standard form $\frac{d}{dx} \log_a x = \frac{1}{x \log_e a}$.

Example 41 — 6K

a Differentiate $\log_{10} x$.

b Differentiate $\log_{1.05} x$.

SOLUTION

a $\frac{d}{dx} \log_{10} x = \frac{1}{x \log_e 10}$

b $\frac{d}{dx} \log_{1.05} x = \frac{1}{x \log_e 1.05}$

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A characterisation of the logarithmic function

We have already discussed in Section 6A that the tangent to $y = \log_e x$ at the x-intercept has gradient exactly 1.

The worked example below shows that this property distinguishes the logarithmic function base e from all other logarithmic functions.

Example 42 — 6K

a Show that the tangent to $y = \log_a x$ at the x-intercept has gradient $\dfrac{1}{\log_e a}$.

b Show that the function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

SOLUTION

a Here $\quad y = \log_a x$.

When $y = 0$, $\quad \log_a x = 0$

$\quad x = 1$,

so the x-intercept is $(1, 0)$.

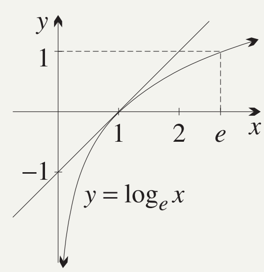
Differentiating, $\quad y' = \dfrac{1}{x \log_e a}$,

so when $x = 1$, $\quad y' = \dfrac{1}{\log_e a}$, as required.

b The gradient at the x-intercept is 1 if and only if

$$\begin{aligned} \log_e a &= 1 \\ a &= e^1 \\ &= e, \end{aligned}$$

that is, if and only if the original base a is equal to e.



13 THE GRADIENT AT THE x-INTERCEPT

The function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

Exponential functions with other bases

Before calculus can be applied to an exponential function $y = a^x$ with base a different from e, it must be written as an exponential function with base e. The important identity used to do this is

$$e^{\log_e a} = a,$$

which simply expresses the fact that the functions e^x and $\log_e x$ are inverse functions. Now a^x can be written as

$$\begin{aligned} a^x &= \left(e^{\log_e a}\right)^x, && \text{replacing } a \text{ by } e^{\log_e a}, \\ &= e^{x \log_e a}, && \text{using the index law } \left(e^k\right)^x = e^{kx}. \end{aligned}$$

Thus a^x has been expressed in the form e^{kx}, where $k = \log_e a$ is a constant.

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14 EXPONENTIAL FUNCTIONS WITH OTHER BASES

- Every positive real number can be written as a power of e:
$$a = e^{\log_e a}$$
- Every exponential function can be written as an exponential function base e:
$$a^x = e^{x \log_e a}$$

Example 43 6K

Express these numbers and functions as powers of e.

a 2 **b** 2^x **c** 5^{-x}

SOLUTION

a $2 = e^{\log_e 2}$

b $2^x = \left(e^{\log_e 2}\right)^x = e^{x \log_e 2}$

c $5^{-x} = \left(e^{\log_e 5}\right)^{-x} = e^{-x \log_e 5}$

Differentiating and integrating exponential functions with other bases

Write the function as a power of e. It can then be differentiated and integrated.

First, $$a^x = e^{\log_e a^x} = e^{x \log_e a}.$$

Differentiating,
$$\begin{aligned}\frac{d}{dx} a^x &= \frac{d}{dx} e^{x \log_e a}\\ &= e^{x \log_e a} \times \log_e a, \quad \text{because } \frac{d}{dx} e^{kx} = k e^{kx},\\ &= a^x \log_e a, \quad \text{because } e^{x \log_e a} = a^x.\end{aligned}$$

Integrating,
$$\begin{aligned}\int a^x\,dx &= \int e^{x \log_e a}\,dx\\ &= \frac{e^{x \log_e a}}{\log_e a}, \quad \text{because } \int e^{kx} = \frac{1}{k} e^{kx},\\ &= \frac{a^x}{\log_e a}, \quad \text{because } e^{x \log_e a} = a^x.\end{aligned}$$

This process can be carried through every time, or the results can be remembered as standard forms.

15 DIFFERENTIATION AND INTEGRATION WITH OTHER BASES

There are two approaches.

- Write all powers with base e before differentiating or integrating.
- Alternatively, use the standard forms:

$$\frac{d}{dx} a^x = a^x \log_e a \qquad \text{and} \qquad \int a^x\,dx = \frac{a^x}{\log_e a} + C$$

Note: The formulae for differentiating and integrating a^x both involve the constant $\log_e a$. This constant $\log_e a$ is 1 when $a = e$, so the formulae are simplest when the base is e. Again, this indicates that e is the appropriate base to use for calculus with exponential functions.

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Example 44 6K

Differentiate $y = 2^x$. Hence find the gradient of $y = 2^x$ at the y-intercept, correct to three significant figures.

SOLUTION

Here $\qquad y = 2^x$.

Using the standard form, $\quad y' = 2^x \log_e 2$.

Hence when $x = 0$, $\quad y' = 2^0 \times \log_e 2$

$$= \log_e 2$$

$$\doteqdot 0.693.$$

Note: This result may be compared with the results of physically measuring this gradient in Question 1 of Exercise 11A in the Year 11 book.

Example 45 6K

a Show that the line $y = x + 1$ meets the curve $y = 2^x$ at $A(0, 1)$ and $B(1, 2)$.

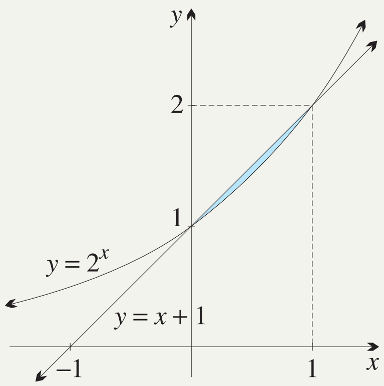
b Sketch the two curves and shade the region contained between them.

c Find the area of this shaded region, correct to four significant figures.

SOLUTION

a Simple substitution of $x = 0$ and $x = 1$ into both functions verifies the result.

b The graph is drawn to the right.



c
$$\text{Area} = \int_0^1 (\text{upper curve} - \text{lower curve})\,dx$$
$$= \int_0^1 (x + 1 - 2^x)\,dx$$
$$= \left[\tfrac{1}{2}x^2 + x - \frac{2^x}{\log_e 2}\right]_0^1$$
$$= \left(\tfrac{1}{2} + 1 - \frac{2}{\log_e 2}\right) - \left(0 + 0 - \frac{1}{\log_e 2}\right)$$
$$= 1\tfrac{1}{2} - \frac{1}{\log_e 2} \text{ square units}$$
$$\doteqdot 0.05730 \text{ square units.}$$

Exercise 6K FOUNDATION

1 Use the change-of-base formula $\log_a x = \dfrac{\log_e x}{\log_e a}$ and the function labelled $\boxed{\ln}$ on your calculator to evaluate each expression correct to three significant figures. Then check your answers using the function labelled $\boxed{x^y}$.

a $\log_2 3$ **b** $\log_2 10$ **c** $\log_5 26$ **d** $\log_3 0.0047$

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2 Use the change-of-base formula to express these with base e, then differentiate them.

a $y = \log_2 x$ **b** $y = \log_{10} x$ **c** $y = 3\log_5 x$

3 Use the standard form $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}$ to differentiate:

a $y = \log_3 x$ **b** $y = \log_7 x$ **c** $y = 5\log_6 x$

4 Express these functions as powers of e, then differentiate them.

a $y = 3^x$ **b** $y = 4^x$ **c** $y = 2^x$

5 Use the standard form $\frac{d}{dx}a^x = a^x \log_e a$ to differentiate:

a $y = 10^x$ **b** $y = 8^x$ **c** $y = 3 \times 5^x$

6 Convert each integrand to a power of e and then integrate.

a $\int 2^x\,dx$ **b** $\int 6^x\,dx$ **c** $\int 7^x\,dx$ **d** $\int 3^x\,dx$

7 Use the result $\int a^x\,dx = \frac{a^x}{\log_e a} + C$ to find each primitive, then evaluate the definite integral correct to four significant figures.

a $\int_0^1 2^x\,dx$ **b** $\int_0^1 3^x\,dx$ **c** $\int_{-1}^1 5^x\,dx$ **d** $\int_0^2 4^x\,dx$

8 **a** Complete the table of values to the right, giving your answers correct to two decimal places where necessary.

b Use this table of values to sketch the three curves $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ on the same set of axes.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$					
$\log_e x$					
$\log_4 x$					

DEVELOPMENT

9 **a** Differentiate $y = \log_2 x$. Hence find the gradient of the tangent to the curve at $x = 1$.

b Hence find the equation of the tangent there.

c Do likewise for:

i $y = \log_3 x$, **ii** $y = \log_5 x$.

10 Give the exact value of each integral, then evaluate it correct to four decimal places.

a $\int_1^3 2^x\,dx$ **b** $\int_{-1}^1 (3^x + 1)\,dx$ **c** $\int_0^2 (10^x - 10x)\,dx$

11 Use the change-of-base formula to express $y = \log_{10} x$ with base e, and hence find y'.

a Find the gradient of the tangent to this curve at the point $(10, 1)$.

b Thus find the equation of this tangent in general form.

c At what value of x will the tangent have gradient 1?

12 **a** Find the equations of the tangents to each of $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ at the points where $x = 3$.

b Show that the three tangents all meet at the same point on the x-axis.

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13 **a** Show that the curves $y = 2^x$ and $y = 1 + 2x - x^2$ intersect at $A(0, 1)$ and $B(1, 2)$.
b Sketch the curves and find the area between them.

14 Find the intercepts of the curve $y = 8 - 2^x$, and hence find the area of the region bounded by this curve and the coordinate axes.

15 **a** Sketch the curve $y = 3 - 3^x$, showing the intercepts and asymptote.
b Find the area contained between the curve and the axes.

16 **a** Show that the curves $y = x + 1$ and $y = 4^x$ intersect at the y-intercept and at $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
b Write the area of the region enclosed between these two curves as an integral.
c Evaluate the integral found in part **b**.

17 **a** Show that the tangent to $y = \log_3 x$ at $x = e$ passes through the origin.
b Show that the tangent to $y = \log_5 x$ at $x = e$ passes through the origin.
c Show that the same is true for $y = \log_a x$, for any base a.

18 **a** Differentiate $x \log_e x - x$, and hence find $\int \log_e x \, dx$.
b Use the change-of-base formula and the integral in part **a** to evaluate $\int_1^{10} \log_{10} x \, dx$.

ENRICHMENT

19 As always, the three standard forms in this section have linear extensions. The pronumeral m is used here instead of the usual a because a is being used for the base.

a Use the standard form $\frac{d}{dx} \log_a (mx + b) = \frac{m}{(mx + b)\log_e a}$ to differentiate:

i $y = \log_3 x$ **ii** $y = \log_7 (2x + 3)$ **iii** $y = 5 \log_6 (4 - 9x)$

b Use the standard form $\frac{d}{dx} a^{mx+b} = ma^{mx+b} \log_e a$ to differentiate:

i $y = 10^x$ **ii** $y = 8^{4x-3}$ **iii** $y = 3 \times 5^{2-7x}$

c Use the standard form $\int a^{mx+b} dx = \frac{a^{mx+b}}{m \log_e a} + C$ to find:

i $\int 3^{5x} dx$ **ii** $\int 6^{2x+7} dx$ **iii** $\int 5 \times 7^{4-9x} dx$

20 If the positive base a of $y = a^x$ and $y = \log_a x$ is small enough, then the two curves will intersect. What base must be chosen so that the two are tangent at the point of contact? Proceed as follows:
a Rewrite both equations with base e, and let $k = \log_e a$.
b Explain why the gradient of the tangent at the point of contact must be 1.
c Use the last part to obtain two equations for the gradient.
d Solve these simultaneously to find k, and hence write down the base a.

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Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 **a** Sketch the graphs of $y = e^x$ and $y = e^{-x}$ on the same number plane. Add the line that reflects each graph onto the other graph. Then draw the tangents at the y-intercepts, and mark the angle between them.

b Sketch the graphs of $y = e^x$ and $y = \log_e x$ on the same number plane. Add the line that reflects each graph onto the other graph. Then draw the tangents at the intercepts with the axes.

2 Use your calculator, and in some cases the change-of base formula, to approximate each expression correct to four significant figures.

a e^4 **b** e **c** $e^{-\frac{3}{2}}$ **d** $\log_e 2$

e $\log_{10} \frac{1}{2}$ **f** $\log_2 0.03$ **g** $\log_{1.05} 586$ **h** $\log_8 3\frac{3}{7}$

3 Use logarithms to solve these equations correct to four significant figures. You will need to apply the change-of-base formula before using your calculator.

a $3^x = 14$ **b** $2^x = 51$ **c** $4^x = 1345$ **d** $5^x = 132$

4 Simplify:

a $e^{2x} \times e^{3x}$ **b** $e^{7x} \div e^x$ **c** $\dfrac{e^{2x}}{e^{6x}}$ **d** $(e^{3x})^3$

5 Solve each equation using a suitable substitution to reduce it to a quadratic.

a $9^x - 7 \times 3^x - 18 = 0$ **b** $e^{2x} - 11e^x + 28 = 0$

6 Sketch the graph of each function on a separate number plane, and state its range.

a $y = e^x$ **b** $y = e^{-x}$ **c** $y = e^x + 1$ **d** $y = e^{-x} - 1$

7 **a** **i** Explain how $y = e^{x-3}$ can be obtained by translating $y = e^x$, and sketch it.

ii Explain how $y = e^{x-3}$ can be obtained by dilating $y = e^x$.

b **i** Explain how $y = \log_e 3x$ can be obtained by dilating $y = \log_e x$, and sketch it.

ii Explain how $y = \log_e 3x$ can be obtained by translating $y = \log_e x$.

8 Differentiate:

a $y = e^x$ **b** $y = e^{3x}$ **c** $y = e^{2x+3}$ **d** $y = e^{-x}$

e $y = e^{-3x}$ **f** $y = 3e^{2x+5}$ **g** $y = 4e^{\frac{1}{2}x}$ **h** $y = \frac{2}{3}e^{6x-5}$

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9 Write each function as a single power of e, and then differentiate it.

a $y = e^{3x} \times e^{2x}$ **b** $y = \dfrac{e^{7x}}{e^{3x}}$ **c** $y = \dfrac{e^{x}}{e^{4x}}$ **d** $y = (e^{-2x})^3$

10 Differentiate each function using the chain, product and quotient rules as appropriate.

a $y = e^{x^3}$ **b** $y = e^{x^2 - 3x}$ **c** $y = xe^{2x}$ **d** $y = (e^{2x} + 1)^3$

e $y = \dfrac{e^{3x}}{x}$ **f** $y = x^2 e^{x^2}$ **g** $y = (e^x - e^{-x})^5$ **h** $y = \dfrac{e^{2x}}{2x + 1}$

11 Find the first and second derivatives of:

a $y = e^{2x+1}$ **b** $y = e^{x^2+1}$

12 Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x-intercept and y-intercept of this tangent.

13 Consider the curve $y = e^{-3x}$.

a Find the gradient of the normal to the curve at the point where $x = 0$.

b Find y'' and hence determine the concavity of the curve at the point where $x = 0$.

14 Consider the curve $y = e^x - x$.

a Find y' and y''.

b Show that there is a stationary point at $(0, 1)$, and determine its nature.

c Explain why the curve is concave up for all values of x.

d Sketch the curve and write down its range.

15 Find the stationary point on the curve $y = xe^{-2x}$ and determine its nature.

16 Find:

a $\int e^{5x}\,dx$ **b** $\int 10e^{2-5x}\,dx$ **c** $\int e^{\frac{1}{5}x}\,dx$ **d** $\int 3e^{5x-4}\,dx$

17 Find the exact value of:

a $\int_0^2 e^x\,dx$ **b** $\int_0^1 e^{2x}\,dx$ **c** $\int_{-1}^0 e^{-x}\,dx$

d $\int_{-\frac{2}{3}}^0 e^{3x+2}\,dx$ **e** $\int_0^{\frac{1}{2}} e^{3-2x}\,dx$ **f** $\int_0^2 2e^{\frac{1}{2}x}\,dx$

18 Find the primitive of:

a $\dfrac{1}{e^{5x}}$ **b** $e^{3x} \times e^x$ **c** $\dfrac{6}{e^{3x}}$ **d** $(e^{3x})^2$

e $\dfrac{e^{3x}}{e^{5x}}$ **f** $\dfrac{e^{3x} + 1}{e^{2x}}$ **g** $e^{2x}(e^x + e^{-x})$ **h** $(1 + e^{-x})^2$

19 Find the exact value of:

a $\int_0^1 (1 + e^{-x})\,dx$ **b** $\int_0^2 (e^{2x} + x)\,dx$ **c** $\int_0^1 \dfrac{2}{e^x}\,dx$

d $\int_0^{\frac{1}{3}} e^{3x}(1 - e^{-3x})\,dx$ **e** $\int_0^1 \dfrac{e^{2x} + 1}{e^x}\,dx$ **f** $\int_0^1 (e^x + 1)^2\,dx$

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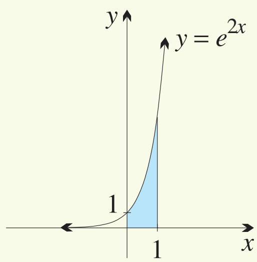
20 If $f'(x) = e^x - e^{-x} - 1$ and $f(0) = 3$, find $f(x)$ and then find $f(1)$.

21 **a** Differentiate e^{x^3}.

b Hence find $\int_0^1 x^2 e^{x^3}\,dx$.

22 Find the area of each region correct to three significant figures.

a



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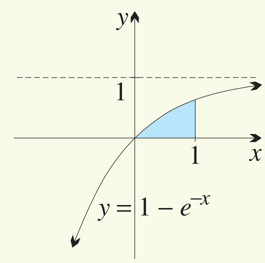
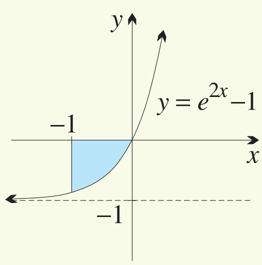


23 Find the exact area of the shaded region.

a



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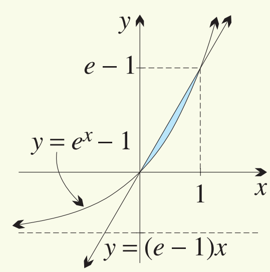

24 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_2 x$ **b** $y = -\log_2 x$ **c** $y = \log_2 (x - 1)$ **d** $y = \log_2 (x + 3)$

25 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_e x$ **b** $y = \log_e (-x)$ **c** $y = \log_e (x - 2)$ **d** $y = \log_e x + 1$

26 Use the log laws to simplify:

a $e \log_e e$ **b** $\log_e e^3$ **c** $\ln \frac{1}{e}$ **d** $2e \ln \sqrt{e}$

27 Differentiate these functions.

a $\log_e x$ **b** $\log_e 2x$ **c** $\log_e (x + 4)$

d $\log_e (2x - 5)$ **e** $2 \log_e (5x - 1)$ **f** $x + \log_e x$

g $\ln (x^2 - 5x + 2)$ **h** $\ln (1 + 3x^5)$ **i** $4x^2 - 8x^3 + \ln (x^2 - 2)$

28 Use the log laws to simplify each function and then find its derivative.

a $\log_e x^3$ **b** $\log_e \sqrt{x}$ **c** $\ln x(x + 2)$ **d** $\ln \frac{x}{x - 1}$

29 Differentiate these functions using the product or quotient rule.

a $x \log_e x$ **b** $e^x \log_e x$ **c** $\frac{x}{\ln x}$ **d** $\frac{\ln x}{x^2}$

30 Find the equation of the tangent to the curve $y = 3 \log_e x + 4$ at the point $(1, 4)$.

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31 Consider the function $y = x - \log_e x$.

a Show that $y' = \dfrac{x-1}{x}$.

b Hence show that the graph of $y = x - \log_e x$ has a minimum turning point at $(1, 1)$.

32 Find these indefinite integrals.

a $\displaystyle\int \frac{1}{x}\,dx$ **b** $\displaystyle\int \frac{3}{x}\,dx$ **c** $\displaystyle\int \frac{1}{5x}\,dx$ **d** $\displaystyle\int \frac{1}{x+7}\,dx$

e $\displaystyle\int \frac{1}{2x-1}\,dx$ **f** $\displaystyle\int \frac{1}{2-3x}\,dx$ **g** $\displaystyle\int \frac{2}{2x+9}\,dx$ **h** $\displaystyle\int \frac{8}{1-4x}\,dx$

33 Evaluate these definite integrals.

a $\displaystyle\int_0^1 \frac{1}{x+2}\,dx$ **b** $\displaystyle\int_1^4 \frac{1}{4x-3}\,dx$ **c** $\displaystyle\int_1^e \frac{1}{x}\,dx$ **d** $\displaystyle\int_{e^2}^{e^3} \frac{1}{x}\,dx$

34 Use the standard form $\displaystyle\int \frac{u'}{u}\,dx = \log_e|u| + C$ OR $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$ to find:

a $\displaystyle\int \frac{2x}{x^2+4}\,dx$ **b** $\displaystyle\int \frac{3x^2-5}{x^3-5x+7}\,dx$ **c** $\displaystyle\int \frac{x}{x^2-3}\,dx$ **d** $\displaystyle\int \frac{x^3-1}{x^4-4x}\,dx$

35 Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 4$.

36 **a** By solving the equations simultaneously, show that the curve $y = \dfrac{5}{x}$ and the line $y = 6 - x$ intersect at the points $(1, 5)$ and $(5, 1)$.

b By sketching both graphs on the same number plane, find the area of the region enclosed between them.

37 Find the derivatives of:

a e^x **b** 2^x **c** 3^x **d** 5^x

38 Find these indefinite integrals.

a $\displaystyle\int e^x\,dx$ **b** $\displaystyle\int 2^x\,dx$ **c** $\displaystyle\int 3^x\,dx$ **d** $\displaystyle\int 5^x\,dx$

39 **a** Differentiate $x\log_e x$, and hence find $\displaystyle\int \log_e x\,dx$.

b Differentiate xe^x, and hence find $\displaystyle\int xe^x\,dx$.

c Hence prove that $\displaystyle\int_1^e \frac{1}{x}\,dx = \int_1^e \log_e x\,dx = \int_0^1 xe^x\,dx = 1$.

40 **a** Find the gradient of $y = 2^x$ at $A(3, 8)$.

b Find the gradient of $y = \log_2 x$ at $B(8, 3)$.

c Explain geometrically why the two gradients are reciprocals of each other.

41 **a** Find $\displaystyle\int_0^3 2^x\,dx$ and $\displaystyle\int_{-3}^0 2^x\,dx$.

b Explain geometrically why the first is 8 times the second.

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