10 Projectile motion

This short chapter deals with just one case of motion in two dimensions — the motion of a projectile, such as a thrown ball or a shell fired from a gun.

A *projectile* is something that is thrown or fired into the air, and subsequently moves under the influence of gravity alone. We regard it as a point.

- Missiles and aeroplanes are not projectiles, because they have motors on them that keep pushing them forwards.
- We shall ignore air resistance, so we will not be dealing with objects such as sheets of metal or pieces of paper where air resistance cannot reasonably be ignored.
- Our projectiles are always moving close to the Earth's surface, where the acceleration due to gravity is a constant $g \doteqdot 9.8\ \text{m/s}^2$.

Everyone can see that a projectile moves in a path that looks like a parabola. Our task is to set up the equations that describe this motion so that we can solve problems.

Vectors are useful in describing projectile motion, particularly when it comes to describing the velocity, which keeps changing direction. But apart from their components, they are not necessary in this topic. Their great advantage is that they provide a more concise description of the situation.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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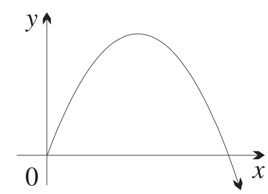
10A Projectile motion — the time equations



The diagram to the right shows the sort of path we would expect a projectile to move in. The two-dimensional space in which it moves has been made into a number plane by choosing an origin — in this case the point from which the projectile was fired — and measuring horizontal distance x and vertical distance y from this origin.

We could put time t on the graph, which would require a third dimension for the t-axis. But we will instead treat t as a parameter, because at each time t during the flight, the projectile is at some point $P(x, y)$ on its path.

These pronumerals x, y and t for horizontal distance, vertical distance and time, will often be used in this chapter without further introduction.

Using position vectors to specify a point

The choice of basis vectors should coincide with the chosen axes. Thus $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the positive directions of the x-axis and y-axis respectively.

Then a point such as $P(30, 15)$ can also be described by stating that its position vector is $\overrightarrow{OP} = 30\underset{\sim}{i} + 15\underset{\sim}{j}$.

Specifying the velocity

With two-dimensional motion, *velocity* is a vector. Its *speed* is its magnitude, and it is travelling in particular *direction*. When an object is moving through the air, we can describe its velocity at a particular time t in two different ways:

- We can give its speed and the angle at which it is moving. For example, at some instant of time a ball may be moving at 12 m/s with an angle of inclination of $60°$ or $-60°$. This *angle of inclination* is always measured from the horizontal, and is taken as negative if the object is travelling downwards.
- We can also specify the velocity at that instant by giving the rates $\dot{x}$ and $\dot{y}$ at which the horizontal displacement x and the vertical displacement y are changing. Using the basis vector $\underset{\sim}{i}$ and $\underset{\sim}{j}$, we can write the velocity as a vector
  $$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}.$$

The horizontal and vertical velocities are projections of $\underset{\sim}{v}$ onto the basis vectors,

$$\dot{x} = \underset{\sim}{v} \cdot \underset{\sim}{i} \text{ and } \dot{y} = \underset{\sim}{v} \cdot \underset{\sim}{j}.$$

1 TWO WAYS TO SPECIFY VELOCITY

The velocity of a projectile at some particular time t can be described in two ways:

- We can give the speed and angle of inclination. The *angle of inclination* is the acute angle between the path and the horizontal. It is positive if the object is travelling upwards, and negative if the object is travelling downwards.

OR

- We can give the horizontal and vertical components $\dot{x}$ and $\dot{y}$. This allows the velocity to be written as a vector
  $$\underset{\sim}{v} = x\underset{\sim}{i} + \dot{y}\underset{\sim}{j} \text{ where } \dot{x} = \underset{\sim}{v} \cdot \underset{\sim}{i} \text{ and } \dot{y} = \underset{\sim}{v} \cdot \underset{\sim}{j}.$$

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The resolution of velocity

The conversion from one way of describing velocity to the other requires a diagram to *resolve the velocity into its horizontal and vertical components*, or to convert back from components to speed and direction, as in the worked examples below. Vectors are the basis of the calculations, but usually the geometric diagram is sufficient for the working.

Example 1 — 10A

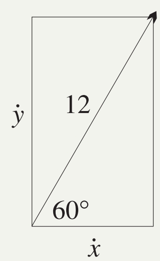
Use a diagram to resolve the velocity into its horizontal and vertical components, for a projectile moving with speed 12 m/s and angle of inclination:

a $60°$, **b** $-60°$.

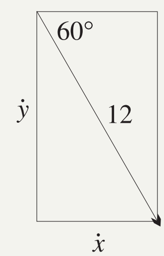
Give your answer as an expression for the vector velocity $\underset{\sim}{v}$ in vector components.

SOLUTION

a
$$\begin{aligned}\dot{x} &= 12\cos 60° \\ &= 6\text{ m/s}, \\ \dot{y} &= 12\sin 60° \\ &= 6\sqrt{3}\text{ m/s}, \\ \text{so } \underset{\sim}{v} &= 6\underset{\sim}{i} + 6\sqrt{3}\,\underset{\sim}{j}\text{ m/s}.\end{aligned}$$



b
$$\begin{aligned}\dot{x} &= 12\cos 60° \\ &= 6\text{ m/s}, \\ \dot{y} &= -12\sin 60° \\ &= -6\sqrt{3}\text{ m/s}, \\ \text{so } v &= 6\underset{\sim}{i} - 6\sqrt{3}\,\underset{\sim}{j}\text{ m/s}.\end{aligned}$$



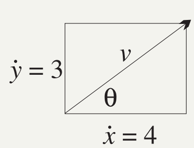
Example 2 — 10A

Find the speed v and angle of inclination θ (correct to the nearest degree) of a projectile whose velocity vector is:

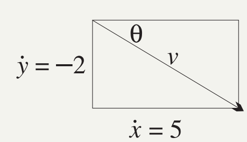
a $\underset{\sim}{v} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ m/s **b** $\underset{\sim}{v} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$ m/s

SOLUTION

a
$$\begin{aligned}v^2 &= 4^2 + 3^2 \\ v &= 5\text{ m/s}, \\ \tan\theta &= \tfrac{3}{4} \\ \theta &\doteqdot 37°.\end{aligned}$$



b
$$\begin{aligned}v^2 &= 5^2 + 2^2 \\ v &= \sqrt{29}\text{ m/s}, \\ \tan\theta &= -\tfrac{2}{5} \\ \theta &\doteqdot -22°.\end{aligned}$$



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2 RESOLUTION OF VELOCITY

To convert between velocity given in terms of speed v and angle of inclination θ, and velocity given in terms of horizontal and vertical components $\dot{x}$ and $\dot{y}$:

- Use a diagram to *resolve the velocity into its horizontal and vertical components*.
- Alternatively, use the conversion equations

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \end{cases} \qquad \text{and} \qquad \begin{cases} v^2 = \dot{x}^2 + \dot{y}^2 \\ \tan\theta = \dfrac{\dot{y}}{\dot{x}} \end{cases}$$

(where ambiguity between θ and $180° - \theta$ may need to be clarified).

The independence of the vertical and horizontal motion

We have already seen that gravity affects every object free to move by accelerating it downwards with the same constant acceleration g, where $g \doteqdot 9.8\ \text{m/s}^2$ (or $10\ \text{m/s}^2$ in round figures).

In this course, a projectile is unaffected by air resistance, and has no motor. No force acts on a projectile except for the downwards force of gravity.

- Because this acceleration is downwards, it affects the vertical component $\dot{y}$ of the velocity according to $\ddot{y} = -g$.
- It has no effect, however, on the horizontal component $\dot{x}$, and thus $\ddot{x} = 0$.

Every projectile motion is therefore governed by this same pair of equations,

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g.$$

In vector form, the acceleration vector $\underset{\sim}{a}$ is $\underset{\sim}{a} = -g\underset{\sim}{j}$, because the horizontal component of acceleration is zero.

3 THE FUNDAMENTAL EQUATIONS OF PROJECTILE MOTION

- Projectile motion is governed by the acceleration vector $\underset{\sim}{a} = -g\underset{\sim}{j}$.
- In practice, however, we work with the equations for the horizontal and vertical components of acceleration,

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g.$$

- Unless otherwise indicated, every question on projectile motion should begin with these two equations.
- The working will usually involve four integrations, two for $\ddot{x}$, and two for $\ddot{y}$.
- There will be four corresponding substitutions of the boundary conditions.

The integrations will yield six equations — the original equations for vertical and horizontal acceleration, two equations for vertical and horizontal velocity, and two equations for vertical and horizontal displacement.

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Example 3 — 10A

A ball is thrown with initial speed 40 m/s and angle of inclination 30° from the top of a stand 25 metres above the ground.

a Using the stand as the origin and $g = 10\ \text{m/s}^2$, find the six equations of motion.

b Find how high it rises, how long it takes to get there, what its speed is then, and how far it is horizontally from the stand.

c Find the flight time, the horizontal range, and the impact speed and angle.

SOLUTION

Initially, $x = y = 0$, and $\dot{x} = 40\cos 30° = 20\sqrt{3}$, $\quad \dot{y} = 40\sin 30° = 20$.

a To begin, $\ddot{x} = 0 \qquad (1)$

Integrating, $\dot{x} = C_1$.

When $t = 0$, $\dot{x} = 20\sqrt{3}$

$20\sqrt{3} = C_1$,

so $\dot{x} = 20\sqrt{3}. \qquad (2)$

Integrating, $x = 20t\sqrt{3} + C_2$.

When $t = 0$, $x = 0$

$0 = C_2$,

so $x = 20t\sqrt{3}. \qquad (3)$

To begin, $\ddot{y} = -10. \qquad (4)$

Integrating, $\dot{y} = -10t + C_3$.

When $t = 0$, $\dot{y} = 20$

$20 = C_3$,

so $\dot{y} = -10t + 20. \qquad (5)$

Integrating, $y = -5t^2 + 20t + C_4$.

When $t = 0$, $y = 0$

$0 = C_4$,

so $y = -5t^2 + 20t. \qquad (6)$

b At the top of its flight, the vertical component of velocity is zero,

so put $\dot{y} = 0$.

From (5), $-10t + 20 = 0$

$t = 2$ seconds (the time taken).

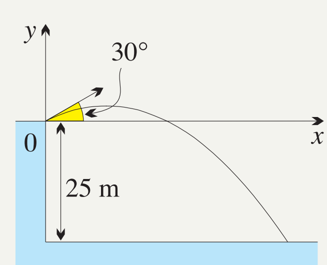
When $t = 2$, from (6), $y = -20 + 40$

$= 20$ metres (the maximum height above the stand).

When $t = 2$, from (3), $x = 40\sqrt{3}$ metres (the horizontal distance from the stand).

Because the vertical component of velocity is zero, the speed there is $\dot{x} = 20\sqrt{3}$ m/s.

c It hits the ground when it is 25 metres below the stand,



so put $y = -25$.

From (6), $-5t^2 + 20t = -25$

$t^2 - 4t - 5 = 0$

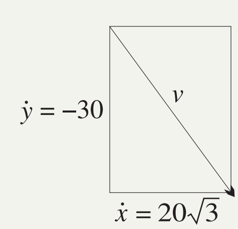
$(t - 5)(t + 1) = 0$,

so it hits the ground when $t = 5$ ($t = -1$ is inadmissible).

From (3), when $t = 5$, $x = 100\sqrt{3}$ metres,

so the horizontal range is $100\sqrt{3}$ metres.

Using a diagram to resolve the velocity at the impact,



$\dot{x} = 20\sqrt{3}$ and $\dot{y} = -50 + 20 = -30$,

so $v^2 = 1200 + 900$

$v = 10\sqrt{21}$ m/s (the impact speed),

and $\tan\theta = -\dfrac{30}{20\sqrt{3}}$

$\theta \doteqdot -40°54'$, and the impact angle is about $40°54'$.

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Using pronumerals for initial speed and angle of inclination

Many problems in projectile motion require the initial speed or angle of inclination to be found so that the projectile behaves in some particular fashion. Often the muzzle speed of a gun will be fixed, but the angle at which it is fired can easily be altered — in such situations there are usually two solutions, corresponding to a low-flying shot and a 'lobbed' shot that goes high into the air.

Example 4 10A

A gun at O fires shells with an initial speed of 200 m/s, but a variable angle of inclination α.
Take $g = 10$ m/s^2.

a Find the two possible angles at which the gun can be set so that it will hit a fortress F that is 2 km away on top of a mountain 1000 metres high.

b Show that the inclination of the lower angle to OF is the same as the inclination of the higher angle to the vertical.

c Find the corresponding flight times and the impact speeds and angles.

SOLUTION

Place the origin at the gun, so that initially, $x = y = 0$.
Resolving the initial velocity, $\dot{x} = 200\cos\alpha \quad \dot{y} = 200\sin\alpha$

To begin, $\ddot{x} = 0. \quad (1)$
Integrating, $\dot{x} = C_1$.
When $t = 0$, $\dot{x} = 200\cos\alpha$
$200\cos\alpha = C_1$,
so $\dot{x} = 200\cos\alpha. \quad (2)$
Integrating, $x = 200t\cos\alpha + C_2$.
When $t = 0$, $x = 0$
$0 = C_2$,
so $x = 200t\cos\alpha. \quad (3)$

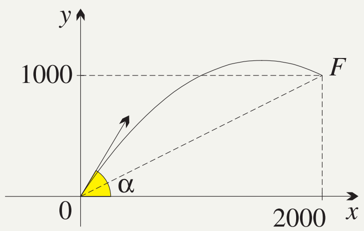
To begin, $\ddot{y} = -10. \quad (4)$
Integrating, $\dot{y} = -10t + C_3$.
When $t = 0$, $\dot{y} = 200\sin\alpha$
$200\sin\alpha = C_1$,
so $\dot{y} = -10t + 200\sin\alpha. \quad (5)$
Integrating, $y = -5t^2 + 200t\sin\alpha + C_4$.
When $t = 0$, $y = 0$
$0 = C_4$,
so $y = -5t^2 + 200t\sin\alpha. \quad (6)$

a Because the fortress is 2 km away, $x = 2000$,
so from (3), $200t\cos\alpha = 2000$

$$t = \frac{10}{\cos\alpha}.$$

Because the mountain is 1000 metres high, $y = 1000$,
so from (6), $-5t^2 + 200t\sin\alpha = 1000$.

Hence
$$-\frac{500}{\cos^2\alpha} + \frac{2000\sin\alpha}{\cos\alpha} - 1000 = 0$$
$$\sec^2\alpha - 4\tan\alpha + 2 = 0.$$



But $\sec^2\alpha = \tan^2\alpha + 1$,
so
$$\tan^2\alpha - 4\tan\alpha + 3 = 0$$
$$(\tan\alpha - 3)(\tan\alpha - 1) = 0$$
$$\tan\alpha = 1 \text{ or } 3$$
$$\alpha = 45° \text{ or } \tan^{-1}3 \ (\doteqdot 71°34').$$

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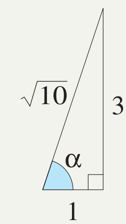
b $\angle OFX = \tan^{-1}\frac{1}{2} \doteqdot 26°34'$, so the 45° shot is inclined at 18°26′ to OF, and the 71°34′ shot is inclined at 18°26′ to the vertical.

c When $\alpha = 45°$, from (a), $t = \dfrac{10}{\cos\alpha} = 10\sqrt{2}$ seconds,

and when $t = 10\sqrt{2}$, from (5), $\dot{y} = -100\sqrt{2} + 200 \times \frac{1}{2}\sqrt{2} = 0$,

so from (2), the shell hits horizontally at $100\sqrt{2}$ m/s.



When $\alpha = \tan^{-1} 3$, $\cos\alpha = \dfrac{1}{\sqrt{10}}$, and $\sin\alpha = \dfrac{3}{\sqrt{10}}$,

so from (a), $t = \dfrac{100}{\cos\alpha} = 10\sqrt{10}$ seconds,

and when $t = 10\sqrt{10}$, from (5), $\dot{y} = -100\sqrt{10} + 60\sqrt{10} = -40\sqrt{10}$,

and from (2), $\dot{x} = 20\sqrt{10}$,

so $v^2 = 16\,000 + 4000 = 20\,000$

$v = 100\sqrt{2}$ m/s,

and using resolution of velocity, $\tan\theta = \dfrac{\dot{y}}{\dot{x}} = -2$

$\theta = -\tan^{-1} 2 \ (\doteqdot -63°26')$

so the shell hits at $100\sqrt{2}$ m/s, at about 63°26′ to the horizontal.

Exercise 10A

FOUNDATION

Note: In this exercise take $g = 10$ m/s^2 unless otherwise indicated.

1 A particle is projected from the origin with a speed of $30\sqrt{2}$ m/s at an angle of 45° to the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 30$ and $\dot{y} = -10t + 30$.
- **b** Hence find x and y in terms of t.
- **c** Put $y = 0$ to find when the particle returns to the x-axis.
- **d** Hence find the horizontal distance travelled by the particle.
- **e** Put $\dot{y} = 0$ to find when the particle reaches its greatest height above the x-axis.
- **f** Find the greatest height.

2 A particle is projected from horizontal ground with a speed of 40 m/s at an angle of 30° to the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 20\sqrt{3}$ and $\dot{y} = -10t + 20$.
- **b** Hence find x and y in terms of t.
- **c** Find:
  - **i** when the particle returns to the ground,
  - **ii** the horizontal distance travelled by the particle,
  - **iii** the greatest height reached above the ground.

3 A particle is projected from the origin with a speed of 20 m/s at an angle of 60° to the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 10$ and $\dot{y} = -10t + 10\sqrt{3}$.
- **b** Hence find x and y in terms of t.

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c Use Pythagoras' theorem to find, correct to one decimal place:
   i the distance of the particle from the origin after one second,
   ii the speed of the particle after one second.

4 A particle is projected from ground level with a speed of 60 m/s at an angle of $\tan^{-1}\frac{4}{3}$ to the horizontal.
   a Show that $\dot{x} = 36$ and $\dot{y} = -10t + 48$.
   b Find x and y as functions of t.
   c Find, correct to one decimal place, the distance of the particle from the point of projection after 3 seconds.
   d Show that the velocity of the particle after 3 seconds is $18\sqrt{5}$ m/s at an angle of $\tan^{-1}\frac{1}{2}$ above the horizontal. (Note that velocity is a vector quantity, so both the speed and the direction must be specified.)

5 A particle is projected from the origin. Its initial velocity vector is $8\underset{\sim}{i} + 6\underset{\sim}{j}$.
   a Express its velocity at time t in the form $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$.
   b Express its displacement at time t in the form $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$.
   c Find:
      i the initial speed of the particle,
      ii the position vector of the particle after 2 seconds,
      iii the position vector of the particle when it reaches its greatest height.

6 A particle is projected from the origin. Its horizontal and vertical components of displacement after t seconds are $x = 40t$ and $y = -5t^2 + 25t$ respectively.
   a Find the initial values of $\dot{x}$ and $\dot{y}$.
   b Hence show that the initial velocity is $5\sqrt{89}$ at an angle of approximately 32° to the horizontal.

DEVELOPMENT

7 A particle is projected from the origin with initial speed V at an angle of α to the horizontal. Two seconds later it passes through the point with position vector $8\underset{\sim}{i} - 12\underset{\sim}{j}$.
   a Show that $V\cos\alpha = 4$ and $V\sin\alpha = 4$, and hence write down the initial velocity vector of the particle.
   b Find the position vector of the particle after 0.5 seconds.

8 A particle is projected from a point O with initial speed V m/s at an angle of θ to the horizontal. After 2 seconds its horizontal and vertical displacements from O are both 30 m.
   a Show that $V\cos\theta = 15$ and $V\sin\theta = 25$.
   b Hence show that $V = 5\sqrt{34}$ m/s and $\theta = \arctan\frac{5}{3}$.

9 A stone is thrown from the top of an 11 m high vertical tower standing on level ground. The initial speed is 12 m/s and the initial direction is 30° *below* the horizontal.
   a Starting with the horizontal and vertical components of acceleration, show that $\dot{x} = 6\sqrt{3}$ and $\dot{y} = -10t - 6$.
   b If the origin is at the point of projection, find x and y in terms of t.
   c How long will it take for the stone to hit the ground?
   d How far from the base of the tower, in metres correct to one decimal place, will the stone hit the ground?
   e Show that the stone will hit the ground at an angle of $\tan^{-1}\frac{8\sqrt{3}}{9}$ below the horizontal.

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10 A ball is tossed with initial speed 6 m/s at an angle of 45° to the horizontal. It hits the ground at a point that is 2 m below the point of projection. Find, correct to two decimal places:

a the time of flight of the ball,

b the horizontal distance travelled by the ball.

11 A projectile was fired from level ground and just cleared a wall that is 10 m high and 20 m from the point of projection. If the angle of projection was 36°, find the initial speed of the projectile in m/s correct to the nearest integer.

12 A ball was projected from ground level and, when at its highest point, it just cleared a 3-metre wall. Given that the initial velocity of the ball was 20 m/s at an angle of α to the horizontal, prove that $\alpha = \sin^{-1}\frac{\sqrt{15}}{10}$.

13 Two particles P_1 and P_2 are projected from the origin with initial velocity vectors $20\underset{\sim}{i} + 30\underset{\sim}{j}$ and $60\underset{\sim}{i} + 50\underset{\sim}{j}$ respectively. If P_2 is projected 2 seconds after P_1, determine whether the particles collide and, if so, when they collide.

14 A particle is projected from the origin with initial speed $20\sqrt{2}$ m/s. On its flight it passes through the point $(20, 15)$ at time t. Let θ be the angle of projection.

a Show that $\sqrt{2}\,t\cos\theta = 1$ and $4\sqrt{2}\,t\sin\theta - t^2 = 3$.

b Hence show that $\tan^2\theta - 8\tan\theta + 7 = 0$.

c Hence find, correct to the nearest minute where necessary, the two possible values of θ.

15 A particle is projected with initial speed 50 m/s. On its flight it passes through a point P that is at a horizontal distance of 100 m and a vertical distance of 25 m from the point of projection.

a Show that the angle of projection α is $\tan^{-1}\frac{1}{2}$ or $\tan^{-1}\frac{9}{2}$.

b For each of the possible angles of projection, find:

i the time it takes for the particle to reach P,

ii the velocity of the particle as it passes through P, giving answers correct to one decimal place where necessary.

16 A particle is projected from level ground with initial speed V m/s at an angle of θ to the horizontal.

a Prove that:

i the greatest height is $\dfrac{V^2\sin^2\theta}{2g}$ metres,

ii the horizontal range is $\dfrac{V^2\sin 2\theta}{g}$ metres.

b If the horizontal range is five times the greatest height, prove that $\theta = \arctan\frac{4}{5}$.

ENRICHMENT

17 A particle is projected from the floor of a horizontal tunnel that is 2 m high. If the particle just touches the ceiling of the tunnel, prove that the horizontal range inside the tunnel is $\sqrt{\frac{16}{g}(V^2 - 4g)}$ metres.

18 A projectile is fired from level ground with initial speed V at an angle of θ to the horizontal. Suppose that the greatest height of the projectile is h.

a Prove that $V^2\sin^2\theta = 2gh$.

b Prove that the particle is at height $\dfrac{h}{2}$ when $t = \dfrac{\left(\sqrt{2} + 1\sqrt{h}\right)}{\sqrt{g}}$ or $t = \dfrac{\left(\sqrt{2} - 1\right)\sqrt{h}}{\sqrt{g}}$.

c If the ratio of the speed of the particle at height $\dfrac{h}{2}$ to the speed at height h is $\sqrt{5}:\sqrt{2}$, find θ.

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10B Projectile motion — the equation of path

The formulae for x and y in terms of t give a parametric equation of the physical path of the projectile through the x–y plane. Eliminating t will give the cartesian equation of the path, which is always an upside-down parabola. Many questions are solved more elegantly by consideration of the equation of path. Unless the question gives it, however, the equation of path must be derived each time.

The general case

The working below derives the equation of path in the general case of a projectile fired from the origin with initial speed V and angle of elevation α.

Resolving the initial velocity, $\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha$.

To begin, $\ddot{x} = 0$,	(1)	To begin, $\ddot{y} = -g$.	(4)
Integrating, $\dot{x} = C_1$.		Integrating, $\dot{y} = -gt + C_3$.	
When $t = 0$, $\dot{x} = V\cos\alpha$		When $t = 0$, $\dot{y} = V\sin\alpha$	
$V\cos\alpha = C_1$,		$V\sin\alpha = C_3$,	
so $\dot{x} = V\cos\alpha$.	(2)	so $\dot{y} = -gt + V\sin\alpha$.	(5)
Integrating, $x = Vt\cos\alpha + C_2$.		Integrating, $y = -\frac{1}{2}gt^2 + Vt\sin\alpha + C_4$.	
When $t = 0$, $x = 0$		When $t = 0$, $y = 0$	
$0 = C_2$,		$0 = C_4$,	
so $x = Vt\cos\alpha$.	(3)	so $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$.	(6)

From (3), $$t = \frac{x}{V\cos\alpha}.$$

Substituting into (6), $$y = -\frac{gx^2}{2V^2\cos^2\alpha} + \frac{Vx\sin\alpha}{V\cos\alpha}$$

$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha.$$

The Pythagorean identity $\dfrac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$ gives an alternative form

$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\alpha) + x\tan\alpha.$$

This working must always be shown unless the equation is given in the question.

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4 THE EQUATION OF PATH

- The path of a projectile fired from the origin with initial speed V and angle of elevation α is
$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \qquad \text{(not to be memorised)}.$$
- The Pythagorean identities give an alternative form
$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\alpha) + x\tan\alpha.$$
- This last equation is quadratic in x, $\tan\alpha$ and V, and is linear in g and y.
- Differentiation of the equation of path gives the gradient of the path for any value of x. This provides an alternative approach to finding the angle of inclination of a projectile in flight.

Example 5 10B

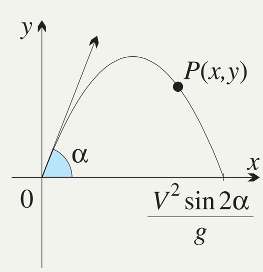
Use the equation of path above in these questions.

a Show that the range on level ground is $\dfrac{V^2}{g}\sin 2\alpha$, and hence find the maximum range.

b Arrange the equation of path as a quadratic in $\tan\alpha$, and hence show that with a given initial speed V and variable angle α of elevation, a projectile can be fired through the point $P(x, y)$ if and only if $2V^2gy \le V^4 - g^2x^2$.

c What does this mean geometrically?

SOLUTION



a Put $y = 0$, then $\dfrac{gx^2\sec^2\alpha}{2V^2} = x\tan\alpha$,

so $x = 0$ or $\dfrac{gx}{2V^2\cos^2\alpha} = \dfrac{\sin\alpha}{\cos\alpha}$

$$x = \frac{2V^2}{g}\cos\alpha\sin\alpha$$
$$x = \frac{V^2}{g}\sin 2\alpha.$$

Hence the projectile lands $\dfrac{V^2}{g}\sin 2\alpha$ away from the origin.

Because $\sin 2\alpha$ has a maximum value of 1 when $\alpha = 45°$,

the maximum range is $\dfrac{V^2}{g}$ when $\alpha = 45°$.

b Multiplying the equation of path through by $2V^2$,
$$2V^2y = -gx^2(1 + \tan^2\alpha) + 2V^2x\tan\alpha$$
$$2V^2y + gx^2 + gx^2\tan^2\alpha - 2V^2x\tan\alpha = 0$$
$$gx^2\tan^2\alpha - 2V^2x\tan\alpha + (2V^2y + gx^2) = 0.$$

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This equation is now a quadratic in $\tan\alpha$. Using the standard theory of quadratics, it will have a solution for $\tan\alpha$ when $\triangle \ge 0,$

$$(2V^2x)^2 - 4 \times gx^2 \times (2V^2y + gx^2) \ge 0$$

$$4V^4x^2 - 8x^2V^2gy - 4g^2x^4 \ge 0$$

$\boxed{\div\ 4x^2}$ $$2V^2gy \le V^4 - g^2x^2.$$

c Rearranging again, $y \le \dfrac{V^4 - g^2x^2}{2V^2g}$.

This means that the target $P(x,\ y)$ must lie on or under the parabola $y = \dfrac{V^4 - g^2x^2}{2V^2g}$,

whose vertex $\left(0, \dfrac{V^2}{2g}\right)$ lies on the y-axis, and whose zeroes are $x = \dfrac{V^2}{g}$ and $x = -\dfrac{V^2}{g}$.

Exercise 10B

FOUNDATION

Note: In this exercise take $g = 10\ \text{m/s}^2$ unless otherwise indicated.

1 Suppose that a particle is projected from the origin and its parabolic path has equation
$y = -\frac{5}{324}x^2 + \frac{4}{3}x$.

- **a** Find the height of the particle when it has travelled a horizontal distance of 12 m.
- **b** Find how far the particle has travelled horizontally when its height is 19 m.
- **c** Find $\dfrac{dy}{dx}$ and hence show that:
  - **i** the angle of projection is $\tan^{-1}\frac{4}{3}$,
  - **ii** when the horizontal distance travelled is 18 m, the direction of motion is $\tan^{-1}\frac{7}{9}$ above the horizontal,
  - **iii** when the horizontal distance travelled is 54 m, the direction of motion is $\tan^{-1}\frac{1}{3}$ below the horizontal.

2 A particle is projected from a point O. Its parabolic motion is governed by the parametric equations $x = 48t$ and $y = -5t^2 + 20t$, where x metres and y metres are the respective horizontal and vertical components of the displacement from O after t seconds.

- **a** Show that the parabolic path has Cartesian equation $y = -\frac{5}{2304}x^2 + \frac{5}{12}x$.
- **b** Use the Cartesian equation of the parabola to find:
  - **i** the horizontal range of the particle,
  - **ii** the greatest height of the particle,
  - **iii** the angle of projection (correct to one decimal place),
  - **iv** the direction in which the particle is moving when $x = 120$ (correct to the nearest degree).

3 A particle is projected from the origin O with initial velocity vector $3\underset{\sim}{i} + \underset{\sim}{j}$.

- **a** What is the initial speed of the particle?
- **b** Show that the parabolic path has parametric equations $x = 3t$ and $y = t - 5t^2$.
- **c** Hence show that the Cartesian equation of the path is $9y = 3x - 5x^2$.
- **d** Find, correct to one decimal place, the direction of motion when:
  - **i** $x = 0.15$
  - **ii** $t = 0.15$

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4 An object is tossed from the top of a 20 m tower with initial velocity vector $5\underset{\sim}{i}$.

a If the point of projection is the origin, find the Cartesian equation of the path of the object.

b Find how far the object lands from the base of the tower.

c Find, correct to the nearest degree, the direction in which the object is travelling when it hits the ground.

DEVELOPMENT

5 A ball is tossed from the origin O that is 6 m above ground level. The initial velocity is 24 m/s at an angle of 30° above the horizontal.

a Find $\dot{x}$, $\dot{y}$, x and y as functions of t.

b Show that the Cartesian equation of the path of the ball is $y = \frac{1}{\sqrt{3}}x - \frac{5}{432}x^2$.

c If D metres is the horizontal distance that the ball has travelled when it strikes the ground, show that $5D^2 - 144\sqrt{3}\,D - 2592 = 0$.

d Hence find D correct to one decimal place.

6 A stone is thrown from the top of a 60 m cliff at an angle of 27° below the horizontal. It lands in the ocean 35 m from the base of the cliff.

a Taking the origin at the point of projection, show that the path of the stone has Cartesian equation $y = -x\tan 27° - \dfrac{5x^2}{V^2\cos^2 27°}$, where V is the initial speed.

b Hence find:

i the initial speed of the stone correct to one decimal place,

ii the direction in which the stone is moving, correct to the nearest degree, when it lands in the ocean.

7 A particle is projected from the origin with initial speed V at an angle of θ to the horizontal. You may assume that the equation of its path is $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2V^2}$.

a Show that the horizontal range of the particle is $\dfrac{V^2\sin 2\theta}{g}$.

b If the initial speed is 30 m/s and the horizontal range is to be 75 m, find the two possible values of θ correct to one decimal place.

8 A particle is projected from the origin with initial speed V at an angle of θ to the horizontal. You may assume the equation of its parabolic path (as given, for example, in Question 7).

a Find $\dfrac{dy}{dx}$, and hence show that the vertex of the parabola is $\left(\dfrac{V^2\sin 2\theta}{2g}, \dfrac{V^2\sin^2\theta}{2g}\right)$.

b If the initial speed is 20 m/s and the greatest height is 15 m, find θ.

9 A gun can fire a shell with a constant initial speed V and a variable angle of elevation α. Assume that t seconds after being fired, the horizontal and vertical displacements x and y of the shell from the gun are given by

$$x = Vt\cos\alpha \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + Vt\sin\alpha.$$

a Show that the Cartesian equation of the shell's path may be written as

$$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0.$$

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b Suppose that $V = 200\text{ m/s}$, and that the shell hits a target positioned 3 km horizontally and 0.5 km vertically from the gun. Show that $\tan\alpha = \dfrac{4+\sqrt{3}}{3}$ or $\tan\alpha = \dfrac{4-\sqrt{3}}{3}$, and hence find the two possible values of α, correct to the nearest minute.

10 A particle is projected from a point O with speed 34 m/s at an angle of θ above the horizontal.

a Show that the parabolic path of the particle has equation $y = x\tan\theta - \dfrac{5x^2\sec^2\theta}{1156}$.

b During its flight the particle passes through a point that is 11 m above O and 30 m horizontally from O. Show that the two possible angles of projection are $\theta = \arctan\frac{8}{15}$ and $\theta = \arctan\frac{538}{75}$.

ENRICHMENT

11 A boy throws a ball with speed $V\text{ m/s}$ at an angle of $45°$ to the horizontal.

a Derive expressions for the horizontal and vertical components of the displacement of the ball from the point of projection.

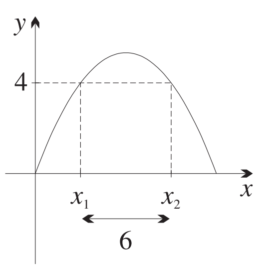
b Hence show that the Cartesian equation of the path of the ball is $y = x - \dfrac{gx^2}{V^2}$.

c The boy is now standing on a hill inclined at an angle θ to the horizontal. He throws the ball at the same angle of elevation of $45°$ and at the same speed of $V\text{ m/s}$. If he can throw the ball 60 metres measured down the hill, but only 30 metres measured up the hill, use the result in part (b) to show that

$$\tan\theta = 1 - \frac{30g\cos\theta}{V^2} = \frac{60g\cos\theta}{V^2} - 1,$$

and hence that $\theta = \tan^{-1}\frac{1}{3}$.

12 A particle is projected from the origin with speed $V\text{ m/s}$ at an angle α to the horizontal.



a Assuming that the coordinates of the particle at time t are $\left(Vt\cos\alpha,\, Vt\sin\alpha - \frac{1}{2}gt^2\right)$, prove that the horizontal range R of the particle is $\dfrac{V^2\sin 2\alpha}{g}$.

b Hence prove that the path of the particle has equation $y = x\left(1 - \dfrac{x}{R}\right)\tan\alpha$.

c Suppose that $\alpha = 45°$ and that the particle passes through two points 6 metres apart and 4 metres above the point of projection, as shown in the diagram. Let x_1 and x_2 be the x-coordinates of the two points.

i Show that x_1 and x_2 are the roots of the equation $x^2 - Rx + 4R = 0$.

ii Use the identity $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_2x_1$ to find R.

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Chapter 10 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 10 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Note: In this exercise take $g = 10\ \text{m/s}^2$ unless otherwise indicated.

1 A particle is projected from horizontal ground with a speed of $60\ \text{m/s}$ at an angle of $40°$ above the horizontal.

- **a** Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that $\dot{x} = 60\cos 40°$ and $\dot{y} = -10t + 60\sin 40°$.
- **b** Hence find x and y in terms of t.
- **c** Find, correct to one decimal place:
  - **i** when the particle returns to the ground,
  - **ii** the horizontal distance travelled by the particle,
  - **iii** the greatest height reached above the ground.

2 A rock is thrown from the top of a vertical cliff of height 40 metres. Its initial velocity is $30\ \text{m/s}$ at an angle of $60°$ above the horizontal. Find, correct to two decimal places where necessary:

- **a** the greatest height reached by the rock above the point of projection,
- **b** the distance that the rock lands from the base of the cliff,
- **c** the speed at which the rock hits the ground.

3 A particle is projected from the origin O with a speed of $25\ \text{m/s}$ at an angle of $\tan^{-1}\frac{4}{3}$ to the horizontal.

- **a** Show that $\dot{x} = 15$ and $\dot{y} = -10t + 20$.
- **b** Find x and y as functions of t.
- **c** Find the distance of the particle from the point of projection after one second.
- **d** Show that the velocity of the particle after one second is $5\sqrt{13}\ \text{m/s}$ at an angle of $\tan^{-1}\frac{2}{3}$ above the horizontal.
- **e** If R is the horizontal range of the particle and H is the greatest height, show that $R = 3H$.

4 An object is projected from level ground with initial speed $10\ \text{m/s}$ at an angle of $45°$ to the horizontal.

- **a** Find $\dot{x}$, x, $\dot{y}$ and y by integration from $\ddot{x} = 0$ and $\ddot{y} = -10$. Then, by eliminating t, show that the equation of the parabolic path is $y = x - \frac{1}{10}x^2$.
- **b** Use the equation of the path to find the horizontal range and the maximum height.
- **c** Suppose first that the stone hits a wall 8 metres away.
  - **i** Find how far up the wall the stone hits.
  - **ii** Differentiate the equation of the path, and hence find the direction of the object when it hits the wall.

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d Suppose now that the stone hits a ceiling 2.1 metres high.
  i Find the horizontal distance travelled before impact.
  ii Find the angle at which the stone hits the ceiling.

5 A particle is projected from the origin. Its initial velocity vector is $24\underset{\sim}{i} + 18\underset{\sim}{j}$.
  a Express its velocity at time t in component vector form.
  b Express its displacement at time t in component vector form.
  c **i** Find the initial speed of the particle,
  ii Find the position vector of the particle after 4 seconds,
  iii Find the position vector of the particle when it reaches its greatest height.

6 A ball is thrown with initial speed V at an angle of α to the horizontal. Two seconds later it passes through the point with position vector $24\sqrt{5}\underset{\sim}{i} + 28\underset{\sim}{j}$.
  a Find the initial velocity vector of the ball.
  b Find the position vector of the ball after 3 seconds.
  c Is the ball rising or falling after 3 seconds? Justify your answer.

7 A particle is projected from a point O on level ground with initial speed V m/s at an angle of θ to the horizontal. Its horizontal range is 108 m, and its flight time is 3 seconds.
  a Show that $V = 39$ m/s and $\theta = \arctan \frac{5}{12}$.
  b Find the greatest height reached by the particle.

8 Steve tosses an apple to Adam, who is sitting near him. Adam catches the apple at exactly the same height that Steve released it. Suppose that the initial speed of the apple is $V = 5$ m/s, and the initial angle α of elevation is given by $\tan \alpha = 2$.
  a Use a velocity resolution diagram to find the initial values of $\dot{x}$ and $\dot{y}$.
  b Find $\dot{x}$, x, $\dot{y}$ and y by integrating $\ddot{x} = 0$ and $\ddot{y} = -10$, taking the origin at Steve's hands.
  c Show by substitution into y that the apple is in the air for less than 1 second.
  d Find the greatest height above the point of release reached by the apple.
  e Show that the flight time is $\frac{2}{5}\sqrt{5}$ seconds, and hence find the horizontal distance travelled by the apple.
  f Find $\dot{x}$ and $\dot{y}$ at the time Adam catches the apple. Then use a diagram to resolve the velocity and show that the final speed equals the initial speed, and the final angle of inclination is the opposite of the initial angle of elevation.
  g The path of the apple is a parabolic arc. By eliminating t from the equations for x and y, find its equation in Cartesian form.

9 A golf ball was hit from level ground and just cleared a 12 m tall tree that was 30 m from the point of projection. If the angle of projection was 26°, find the initial speed of the ball in m/s correct to the nearest integer.

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10 A particle is projected from a point O with initial speed V at an angle of θ above the horizontal.

a Assuming that the displacement functions are $x = Vt\cos\theta$ and $y = -gt^2 + Vt\sin\theta$, prove that the Cartesian equation of the parabolic path is $y = x\tan\,\theta - \dfrac{gx^2\sec^2\theta}{2V^2}$.

b During its flight the particle passes through a point that is 6 m above O and 30 m horizontally from O. If $V = 25$ m/s, show that the two possible angles of projection are $\theta = \arctan\frac{1}{2}$ and $\theta = \arctan\frac{11}{3}$.

11 A ball was thrown uphill from the base of a hill inclined at 30° to the horizontal. The initial velocity was 15 m/s at 60° to the horizontal.

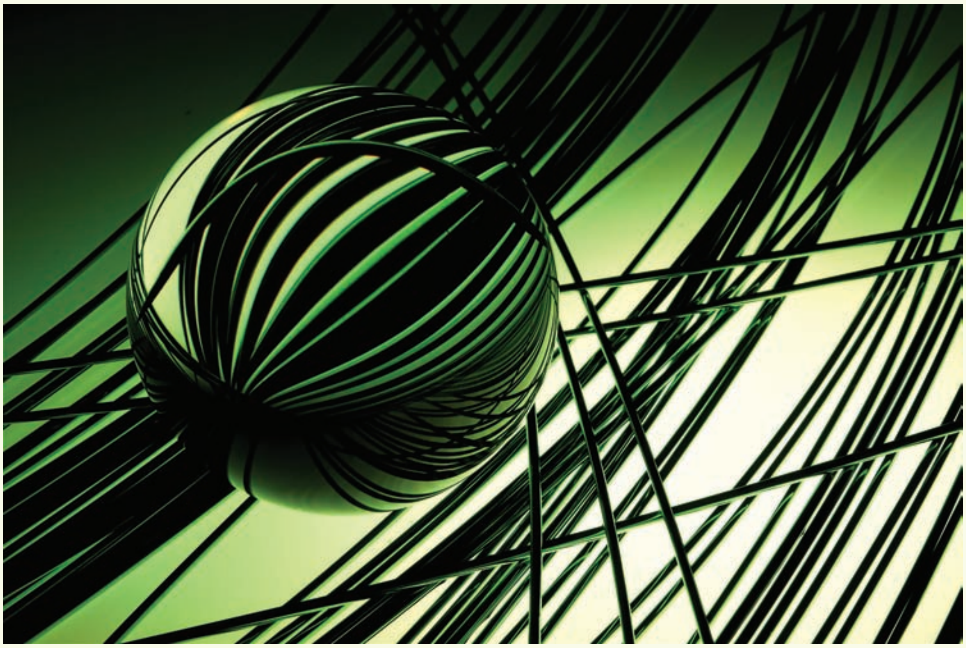
a Show that the parabolic path of the ball has parametric equations

$$x = \tfrac{15}{2}t \qquad \text{and} \qquad y = -5t^2 + \tfrac{15\sqrt{3}}{2}t.$$

b Hence find the Cartesian equation of the path.

c **i** Find how far up the hill the ball landed (measuring along the slope),

ii Find the time of flight.



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