14 Series and finance

Chapter 1 introduced sequences and series, principally arithmetic sequences (or APs) and geometric sequences (or GPs). The treatment there was mostly theoretical, because the intention was to give a wider mathematical context for linear and exponential functions, the derivative, and the definite integral.

The first two sections of this chapter review sequences and series, with particular attention to the use of logarithms, and apply them to many more practical problems. The remaining three sections deal entirely with the role of sequences and series in financial situations — simple and compound interest, depreciation and inflation, superannuation, and paying off a loan.

Readers may or may not need the review of the theory in the first two sections, but the applications are new and need attention. The large number of questions in the financial sections are a result of the variety of ways in which questions can be asked — there are too many for a first encounter, and many of them could be left for later revision.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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14A Applications of APs and GPs

This section and the next will review the main results about APs and GPs from Chapter 1 and apply them to a variety of problems, in preparation for the later sections on finance. Section 14B is particularly concerned with the use of logarithms in solving the exponential equations that arise when working with GPs.

Formulae for arithmetic sequences

Here are the essential definitions and formulae that are needed for problems involving APs.

1 ARITHMETIC SEQUENCES

- A sequence T_n is called an *arithmetic sequence* or *AP* if the difference between successive terms is constant. That is,

  $$T_n - T_{n-1} = d, \text{ for all } n \geq 2,$$

  where d is a constant, called the *common difference*.
- The nth term of an AP with first term a and common difference d is

  $$T_n = a + (n - 1)d.$$
- Three numbers a, x and b are in AP if

  $$b - x = x - a, \qquad \text{that is,} \qquad x = \frac{a + b}{2}.$$
- The sum S_n of the first n terms of an AP is

  $$S_n = \tfrac{1}{2}n(a + \ell) \qquad \text{(use when the last term } \ell = T_n \text{ is known),}$$

  or $$S_n = \tfrac{1}{2}n\big(2a + (n - 1)d\big) \qquad \text{(use when the difference } d \text{ is known).}$$

The word *series* is usually used when we are adding up the terms of a sequence. Typically, we will refer to

the sequence 1, 3, 5, . . . and the series $1 + 3 + 5 + \cdots$.

Example 1 [Salaries and APs] 14A

Georgia earned \$50 000 in her first year at Information Holdings, and her salary then increased every year by \$6000. She worked at the company for 12 years.

a What was her annual salary in her final year?

b What were her total earnings over the 12 years?

SOLUTION

Her annual salaries form a series, $50000 + 56000 + 62000 + \cdots$ with 12 terms.
This is an AP with $a = 50000$, $d = 6000$ and $n = 12$.

a Her final salary is the twelfth term T_{12} of the series.

$$\begin{aligned} \text{Final salary} &= a + 11d \qquad \text{(using the formula for } T_{12}\text{)} \\ &= 50\,000 + 66\,000 \\ &= \$116\,000. \end{aligned}$$

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b Her total earnings are the sum S_{12} of the first twelve terms of the series.

Using the first formula for S_n,

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2}n(a+\ell)\\ &= \tfrac{1}{2}\times 12\times(a+\ell)\\ &= 6\times(50000+116000)\\ &= \$996000.\end{aligned}$$

Using the second formula for S_n,

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2}n\big(2a+(n-1)d\big)\\ &\tfrac{1}{2}\times 12\times(2a+11d)\\ &= 6\times(100000+66000)\\ &= \$996000.\end{aligned}$$

Counting when the years are named

Problems in which events happen in particular named years are notoriously tricky. The following problem becomes clearer when the years are stated in terms of 'years after 2005'.

Example 2 — 14A

Gulgarindi Council is very happy. It had 2870 complaints in 2006, but only 2170 in 2016. The number of complaints decreased by the same amount each year.

a What was the total number of complaints during these years?

b By how much did the number of complaints decrease each year?

c If the trend continued, in what year would there be no complaints at all?

SOLUTION

The first year is 2006, the second year is 2007, and the 11th year is 2016.
In general, the nth year of the problem is the nth year after 2005.
The successive numbers of complaints form an AP with $a = 2870$, $\ell = 2170$ and $n = 11$.

a
$$\begin{aligned}\text{Total number of complaints} &= \tfrac{1}{2}n(a+\ell) \quad \text{(using the first formula for } S_n)\\ &= \tfrac{1}{2}\times 11\times(2870+2170)\\ &= 27720.\end{aligned}$$

b Put

$$\begin{aligned}T_{11} &= 2170\\ a+10d &= 2170 \qquad \text{(using the formula for } T_{11})\\ 2870+10d &= 2170\\ 10d &= -700\\ d &= -70.\end{aligned}$$

Hence the number of complaints decreased by 70 each year.

c
$$\begin{aligned}\text{The number of complaints is } T_n &= a+(n-1)d \qquad \text{(using the formula for } T_n)\\ &= 2870-70(n-1)\\ &= 2940-70n.\end{aligned}$$

Put

$$\begin{aligned}T_n &= 0 \qquad \text{(to find the year in which there are no complaints)}\\ 2940-70n &= 0\\ 70n &= 2940\\ n &= 42.\end{aligned}$$

Thus there would be no complaints at all in the year $2005 + 42 = 2047$.

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Formulae for geometric sequences

The formulae for GPs correspond roughly to the formulae for APs, except that the formula for the limiting sum of a GP has no analogy for arithmetic sequences.

2 GEOMETRIC SEQUENCES

- A sequence T_n is called a *geometric sequence* if the ratio of successive terms is constant. That is,
$$\frac{T_n}{T_{n-1}} = r, \text{ for all } n \geq 2,$$
where r is a constant, called the *common ratio*.
- The nth term of a GP with first term a and common ratio r is
$$T_n = ar^{n-1}.$$
- Neither the ratio nor any term of a GP can be zero.
- Three numbers a, x and b are in GP if
$$\frac{b}{x} = \frac{x}{a}, \qquad \text{that is,} \qquad x^2 = ab.$$
- The sum S_n of the first n terms of a GP is
$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \text{(easier when } r > 1\text{)},$$
$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \text{(easier when } r < 1\text{)}.$$
- The limiting sum S_∞ exists if and only if $-1 < r < 1$, that is, $|r| < 1$, and in this case,
$$S_\infty = \frac{a}{1 - r}, \qquad \text{that is,} \qquad \lim_{n \to \infty} S_n = \frac{a}{1 - r}.$$

Example 3 [An example with $r < 1$] 14A

Sales from the Gumnut Softdrinks Factory in the mountain town of Wadelbri are declining by 6% every year. In 2016, 50000 bottles were sold.

a How many bottles will be sold in 2025?

b How many bottles will be sold in total in the years 2016–2025?

SOLUTION

Here 2016 is the first year, 2017 is the second year, . . . , and 2025 is the 10th year.

The annual sales form a GP with $a = 50000$ and $r = 0.94$.

a The sales in 2025 are the 10th term T_{10}, because 2016–2025 consists of 10 years.

$$\begin{aligned} \text{Sales in 2025} &= ar^9 \\ &= 50\,000 \times 0.94^9 && \text{(using the formula for } T_{10}\text{)} \\ &\doteqdot 28\,650 && \text{(correct to the nearest bottle).} \end{aligned}$$

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b Total sales $= \dfrac{a(1-r^{10})}{1-r}$ (using the second formula for S_{10})

$= \dfrac{50\,000 \times (1-0.94^{10})}{0.06}$

$\doteqdot 384\,487$ (correct to the nearest bottle).

Limiting sums

If the ratio of a GP is between -1 and 1, that is, $0 < |r| < 1$, then the sum S_n of the first n terms of the GP converges to the limit $S_\infty = \dfrac{a}{1-r}$ as $n \to \infty$. In applications, this allows us to speak about the sum of the terms 'eventually', or 'as time goes on'.

Example 4 14A

Consider again the Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year and 50000 bottles were sold in 2016. Suppose now that the company continues in business indefinitely.

a What would the total sales from 2016 onwards be eventually?

b What proportion of those sales would occur by the end of 2025?

SOLUTION

The sales form a GP with $a = 50000$ and $r = 0.94$.

Because $-1 < r < 1$, the limiting sum exists.

a Eventual sales $= S_\infty$

$= \dfrac{a}{1-r}$

$= \dfrac{50\,000}{0.06}$

$\doteqdot 833\,333$ (correct to the nearest bottle).

b Using the results from part **a**, and from the previous worked example,

$\dfrac{\text{sales in 2016–2025}}{\text{eventual sales}} = \dfrac{50\,000 \times (1-0.94^{10})}{0.06} \times \dfrac{0.06}{50\,000}$ (using the exact values)

$= 1 - 0.94^{10}$

$\doteqdot 46.14\%$ (correct to the nearest 0.01%).

Exercise 14A FOUNDATION

Note: The theory for this exercise was discussed in Chapter 1 and is reviewed in this section. The exercise is a medley of problems on APs and GPs, with two introductory questions to revise the formulae for APs and GPs.

1 a Five hundred terms of the series $102 + 104 + 106 + \cdots$ are added. What is the total?

b In a particular arithmetic series, there are 48 terms between the first term 15 and the last term -10. What is the sum of all the terms in the series?

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c **i** Show that the series $100 + 97 + 94 + \cdots$ is an AP, and find the common difference.
ii Show that the nth term is $T_n = 103 - 3n$, and find the first negative term.
iii Find an expression for the sum S_n of the first n terms, and show that 68 is the minimum number of terms for which S_n is negative.

2 **a** The first few terms of a particular series are $2000 + 3000 + 4500 + \cdots$.
i Show that it is a geometric series, and find the common ratio.
ii What is the sum of the first five terms?
iii Explain why the series does not converge.
b Consider the series $18 + 6 + 2 + \cdots$.
i Show that it is a geometric series, and find the common ratio.
ii Explain why this geometric series has a limiting sum, and find its value.
iii Show that the limiting sum and the sum of the first ten terms are equal, correct to the first three decimal places.

3 A secretary starts on an annual salary of \$60000, with annual increments of \$4000.
a Use the AP formulae to find his annual salary, and his total earnings, at the end of 10 years.
b In which year will his salary be \$84000?

4 An accountant receives an annual salary of \$80000, with 5% increments each year.
a Show that her annual salary forms a GP and find the common ratio.
b Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.

5 **a** What can be said about the terms of an AP in which:
i the common difference is zero,
ii the common difference is negative?
b Why can't the common ratio of a GP be zero?
c What can be said about the terms of a GP with common ratio r in which:
i $r < 0$, **ii** $r = 1$, **iii** $r = -1$, **iv** $0 < |r| < 1$?

6 Lawrence and Julian start their first jobs on low wages. Lawrence starts at \$50000 per annum, with annual increases of \$5000. Julian starts at the lower wage of \$40000 per annum, with annual increases of 15%.
a Find Lawrence's annual wages in each of the first three years and explain why they form an arithmetic sequence.
b Find Julian's annual wages in each of the first three years and explain why they form a geometric sequence.
c Show that the first year in which Julian's annual wage is the greater of the two will be the sixth year, and find the difference, correct to the nearest dollar.

7 **a** An initial salary of \$50000 increases by \$3000 each year.
i Find a formula for T_n, the salary in the nth year.
ii In which year will the salary first be at least twice the original salary?
b An initial salary of \$50000 increases by 4% each year. What will the salary be in the tenth year, correct to the nearest dollar?

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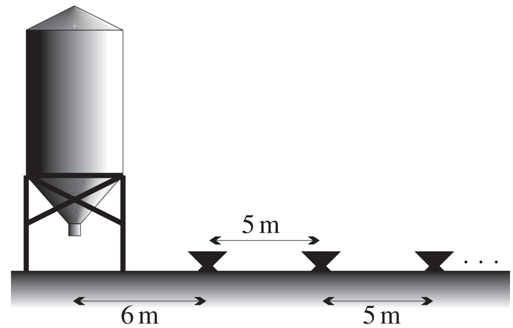
DEVELOPMENT

8 A farmhand is filling a row of feed troughs with grain. The distance between adjacent troughs is 5 metres, and the silo that stores the grain is 6 metres from the closest trough. He decides that he will fill the closest trough first and work his way to the far end. (He can only carry enough grain to fill one trough with each trip.)



a How far will the farmhand walk to fill the 1st trough and return to the silo? How far for the 2nd trough? How far for the 3rd trough?

b How far will the farmhand walk to fill the nth trough and return to the silo?

c If he walks a total of 62 metres to fill the furthest trough:

i how many feed troughs are there,

ii what is the total distance he will walk to fill all the troughs?

9 One Sunday, 120 days before Christmas, Aldsworth store publishes an advertisement saying '120 shopping days until Christmas'. Aldsworth subsequently publishes similar advertisements every Sunday until Christmas.

a How many times does Aldsworth advertise?

b Find the sum of the numbers of days published in all the advertisements.

c On which day of the week is Christmas?

10 The number of infections in an epidemic rose from 10000 on 1st July to 160000 on 1st of September.

a If the number of infections increased by a constant difference each month, what was the number of infections on 1st August?

b If the number of infections increased by a constant ratio each month, what was the number of infections on 1st August?

11 Theodor earns \$60000 in his first year of work, and his salary increases each year by a fixed amount \$$D$.

a Find D if his salary in his tenth year is \$117600.

b Find D if his total earnings in the first ten years are \$942000.

c If $D = 4400$, in which year will his salary first exceed \$120000?

d If $D = 4000$, show that his total earnings first exceed \$1200000 during his 14th year.

12 Margaret opens a hardware store. Sales in successive years form a GP, and sales in the fifth year are half the sales in the first year. Find the total sales of the company as time goes on, as a multiple of the first year's sales \$$F$, correct to two decimal places.

13 [Limiting sums of trigonometric series]

a Consider the series $1 - \tan^2 x + \tan^4 x - \cdots$, where $0 < |x| < \frac{\pi}{2}$.

i For what values of x does the series converge?

ii What is the limit when it does converge?

iii What happens when $x = 0$?

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b Consider the series $1 + \cos^2 x + \cos^4 x + \cdots$.
  i Show that the series is a GP, and find its common ratio.
  ii For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?
  iii Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \operatorname{cosec}^2 x$.
  iv We omitted a qualification. What happens when $\cos x = 0$?

c Consider the series $1 + \sin^2 x + \sin^4 x + \cdots$.
  i Show that the series is a GP, and find its common ratio.
  ii For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?
  iii Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \sec^2 x$.
  iv We omitted a qualification. What happens when $\sin x = 0$?

14 Two bulldozers are sitting in a construction site facing each other. Bulldozer A is at $x = 0$ and bulldozer B is 36 metres away at $x = 36$. There is a bee sitting on the scoop at the very front of bulldozer A.

At 7:00 am the workers start up both bulldozers and start them moving towards each other at the same speed V m/s. The bee is disturbed by the commotion and flies at twice the speed of the bulldozers to land on the scoop of bulldozer B.

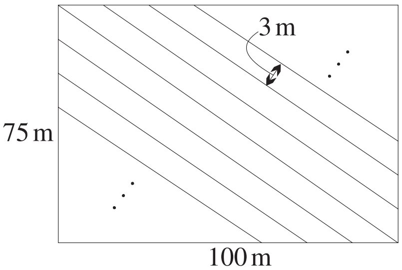
a Show that the bee reaches bulldozer B when it is at $x = 24$.

b Immediately the bee lands, it takes off again and flies back to bulldozer A. Where is bulldozer A when the two meet?

c Assume that the bulldozers keep moving towards each other and the bee keeps flying between the two, so that the bee will eventually have three feet on each bulldozer.
  i Where will this happen?
  ii How far will the bee have flown?

ENRICHMENT

15 The area available for planting in a particular paddock of a vineyard measures 100 metres by 75 metres. In order to make best use of the sun, the grape vines are planted in rows diagonally across the paddock, as shown in the diagram, with a 3-metre gap between adjacent rows.



a What is the length of the diagonal of the field?

b What is the length of each row on either side of the diagonal?

c Confirm that each row two away from the diagonal is 112.5 metres long.

d Show that the lengths of these rows form an arithmetic sequence.

e Hence find the total length of all the rows of vines in the paddock.

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14B The use of logarithms with GPs

None of the exercises in the previous section asked about the number of terms in a given GP. Such questions require either trial-and-error or logarithms.

Trial-and-error may be easier to understand, but it is a clumsy method when the numbers are larger. Logarithms provide a better approach, but require understanding of the relationship between logarithms and indices.

Solving exponential inequations using trial-and-error

The next worked example shows how to solve such an equation using trial-and-error. Notice that:

- The powers of 3 get bigger because 3 is greater than 1,

$$3^0 = 1,\ 3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ \ldots$$

- The powers of 0.95 get smaller because 0.95 is smaller than 1,

$$0.95^0 = 1,\ 0.95^1 = 0.95,\ 0.95^2 = 0.903,\ 0.95^3 = 0.857,\ 0.95^4 = 0.815,\ \ldots$$

Example 5 — 14B

Use trial-and-error on your calculator to find the smallest integer n such that:

a $3^n > 400\,000$

b $0.95^n < 0.01$

SOLUTION

a Using the function labelled $\boxed{x^y}$

$$\begin{aligned} 3^{11} &= 177\,147 \\ \text{and } 3^{12} &= 531\,441, \end{aligned}$$

so the smallest such integer is 12.

b Using the function labelled $\boxed{x^y}$

$$\begin{aligned} 0.95^{89} &= 0.010\,408\ldots \\ \text{and } 0.95^{90} &= 0.009\,888\ldots, \end{aligned}$$

so the smallest such integer is 90.

Note: In practice, quite a few more trial calculations are usually needed in order to trap the given number between two integer powers.

Solving exponential inequations using logarithms

To solve an exponential inequation using logarithms, the corresponding equation must first be solved. To do this:

- Convert the exponential equation to a logarithmic equation.
- Convert to logarithms base e or base 10 using the change-of-base formula

$$\log_b x = \frac{\log_e x}{\log_e b} \quad \text{'The log of the number over log of the base.'}$$

Logarithms base e or base 10 can be used.

An alternative approach is to take logarithms base e or base 10 of both sides of the exponential equation.

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Example 6 14B

Use logarithms to find the smallest integer n such that:

a $3^n > 400000$ **b** $0.95^n < 0.01$

SOLUTION

a Put $3^n = 400\,000$ (the corresponding equation)

Then $n = \log_3 400\,000.$

$$n = \frac{\log_e 400\,000}{\log_e 3} \quad \text{(change-of-base formula)}$$

$$= 11.741\ldots.$$

Thus the smallest such integer is 12, because $3^{11} < 400\,000$ and $3^{12} > 400\,000$.

b Put $0.95^n = 0.01$.

Then $n = \log_{0.95} 0.01$

$$= \frac{\log_e 0.01}{\log_e 0.95} \quad \text{(change-of-base formula)}$$

$$= 89.781\ldots.$$

Thus the smallest such integer is 90, because $0.95^{89} > 0.01$ and $0.95^{90} < 0.01$.

3 SOLVING EXPONENTIAL INEQUATIONS

To solve an exponential inequation such as $3^n > 400000$ or $0.95^n < 0.01$:

- The first approach is to use trial-and-error with the calculator.
- The second approach is to use logarithms base e or base 10.
  - Write down the corresponding equation $3^n = 400000$ or $0.95^n = 0.01$.
  - Solve for n, giving $n = \log_3 400000$ or $n = \log_{0.95} 0.01$.
  - Convert this to logarithms base e or base 10, and approximate.
  - Then write down the solution of the corresponding inequation.
- Be aware that powers of 3 get bigger as the index increases because 3 is greater than 1, and powers of 0.95 get smaller because 0.95 is smaller than 1.

A third approach is to take logarithms base e or base 10 of both sides.

Example 7 14B

The profits of the Extreme Sports Adventure Company have been increasing by 15% every year since its formation, when its profit was \$60000 in the first year.

a During which year did its profit first exceed \$1200000?

b During which year did its total profit since foundation first exceed \$4000000?

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SOLUTION

The successive profits form a GP with $a = 60000$ and $r = 1.15$.

a Put
$$\begin{aligned}
T_n &= 1\,200\,000 \quad \text{(the corresponding equation)}\\
ar^{n-1} &= 1\,200\,000\\
60\,000 \times 1.15^{n-1} &= 1\,200\,000\\
\boxed{\div 60\,000} \quad 1.15^{n-1} &= 20\\
n - 1 &= \log_{1.15} 20\\
n &= \frac{\log_e 20}{\log_e 1.15} + 1\\
n &\doteqdot 22.43,
\end{aligned}$$

so the profit first exceeds \$1200000 during the 23rd year.

b Here
$$\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \quad \text{(using this form because } r > 1)\\
&= \frac{60\,000 \times (1.15^n - 1)}{0.15}\\
&= 400\,000 \times (1.15^n - 1).
\end{aligned}$$

Put
$$\begin{aligned}
S_n &= 4\,000\,000 \quad \text{(the corresponding equation)}\\
400\,000 \times (1.15^n - 1) &= 4\,000\,000\\
\boxed{\div 400\,000} \quad 1.15^n - 1 &= 10\\
1.15^n &= 11\\
n &= \log_{1.15} 11\\
&= \frac{\log_e 11}{\log_e 1.15}\\
&\doteqdot 17.16,
\end{aligned}$$

so the total profit since foundation first exceeds \$4000000 during the 18th year.

Example 8 14B

Consider again the slowly failing Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year, with 50000 bottles sold in 2016.

During which year will sales first fall below 20000?

SOLUTION

The sales form a GP with $a = 50000$ and $r = 0.94$.

Put
$$\begin{aligned}
T_n &= 20\,000 \quad \text{(the corresponding equation)}\\
a\,r^{n-1} &= 20\,000\\
50\,000 \times 0.94^{n-1} &= 20\,000\\
\boxed{\div 50\,000} \quad 0.94^{n-1} &= 0.4\\
n - 1 &= \log_{0.94} 0.4\\
n &= \frac{\log_e 0.4}{\log_e 0.94} + 1\\
n &\doteqdot 15.81.
\end{aligned}$$

Hence sales will first fall below 20000 when $n = 16$, that is, in 2031.

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Exercise 14B

FOUNDATION

1 Use trial-and-error (and your calculator) to find the smallest integer n such that:

a $2^n > 30$
b $2^n > 15000$
c $3^n > 10$
d $3^n > 5000000$
e $\left(\frac{1}{2}\right)^n < 0.1$
f $\left(\frac{1}{2}\right)^n < 0.005$
g $\left(\frac{1}{2}\right)^n < 0.0001$
h $\left(\frac{1}{3}\right)^n < 0.00001$

2 Use logarithms to find the smallest integer n such that:

a $2^n > 7000$
b $3^n > 20000$
c $\left(\frac{1}{2}\right)^n < 0.004$
d $\left(\frac{1}{3}\right)^n < 0.0002$

3 a Show that 10, 11, 12.1, . . . is a geometric sequence.
b State the first term and the common ratio.
c Use the formula $T_n = ar^{n-1}$ to write down the fifteenth term.
d Find the number of terms less than 60 using trial-and-error on your calculator.
e Repeat part **d** using logarithms.

4 An accountant receives an annual salary of \$40000, with 5% increments each year.
a Show that her annual salary forms a GP, and find the common ratio.
b Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.
c In which year will her salary first exceed \$70000?

5 An initial salary of \$50000 increases by 4% each year. In which year will the salary first be at least twice the original salary?

DEVELOPMENT

6 A certain company manufactures three types of shade cloth. The product with code SC50 cuts out 50% of harmful UV rays, SC75 cuts out 75% and SC90 cuts out 90% of UV rays. In the following questions, you will need to consider the amount of UV light let through.
a What percentage of UV light does each cloth let through?
b Show that two layers of SC50 would be equivalent to one layer of SC75 shade cloth.
c Use logarithms to find the minimum number of layers of SC50 that would be required to cut out at least as much UV light as one layer of SC90.
d Similarly find how many layers of SC50 would be required to cut out 99% of UV rays.

7 Yesterday, a tennis ball used in a game of cricket in the playground was hit onto the science block roof. Luckily it rolled off the roof. After bouncing on the playground it reached a height of 3 metres. After the next bounce it reached 2 metres, then $1\frac{1}{3}$ metres, and so on.
a What was T_n, the height in metres reached after the nth bounce?
b What was the height of the roof that the ball fell from?
c The last time the ball bounced, its height was below 1 cm for the first time. After that it rolled away across the playground.
i Show that if $T_n < 0.01$, then $\left(\frac{3}{2}\right)^{n-1} > 300$.
ii How many times did the ball bounce?

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8 Olim, Pixi, Thi (pronounced 'tea'), Sid and Nee work in the sales division of a calculator company. Together they find that sales of scientific calculators are dropping by 150 per month, while sales of graphics calculators are increasing by 150 per month.

a Current sales of all calculators total 20 000 per month, and graphics calculators account for 10% of sales. How many graphics calculators are sold per month?

b How many more graphics calculators will be sold per month by the sales team six months from now?

c Assuming that current trends continue, how long will it be before all calculators sold by the company are graphics calculators?

9 Madeleine opens a business selling computer stationery. In its first year, the business has sales of \$200 000, and each year sales are 20% more than the previous year's sales.

a In which year do annual sales first exceed \$1 000 000?

b In which year do total sales since foundation first exceed \$2 000 000?

10 **a** Explain why 'increasing a quantity by 300%' means 'multiplying the quantity by 4'.

b A population is increasing by 25% every year. How many full years will it take the population to increase by over 300%?

11 Consider the geometric series $3, 2, \frac{4}{3}, \ldots$.

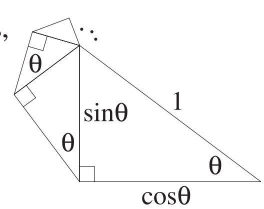
a Write down a formula for the sum S_n of the first n terms of the series.

b Explain why the geometric series has a limiting sum, and determine its value S.

c Find the smallest value of n for which $S - S_n < 0.01$.

ENRICHMENT

12 The diagram shows the first few triangles in a spiral of similar right-angled triangles, each successive one built with its hypotenuse on a side of the previous one.

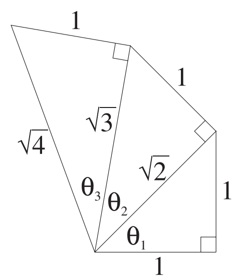


a What is the area of the largest triangle?

b Use the result for the ratio of areas of similar figures to show that the areas of successive triangles form a geometric sequence. What is the common ratio?

c Hence show that the limiting sum of the areas of the triangles is $\frac{1}{2}\tan\theta$.

13 The diagram shows the beginning of a spiral created when each successive right-angled triangle is constructed on the hypotenuse of the previous triangle. The altitude of each triangle is 1, and it is easy to show by Pythagoras' theorem that the sequence of hypotenuse lengths is $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$. Let the base angle of the nth triangle be θ_n. Clearly θ_n gets smaller, but does this mean that the spiral eventually stops turning? Answer the following questions to find out.



a Write down the value of $\tan\theta_n$.

b Show that $\displaystyle\sum_{n=1}^{k}\theta_n \ge \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$. (Hint: $\theta \ge \frac{1}{2}\tan\theta$, for $0 \le \theta \le \frac{\pi}{4}$.)

c By sketching $y = \dfrac{1}{x}$ and constructing the upper rectangle on each of the intervals

$1 \le x \le 2,\ 2 \le x \le 3,\ 3 \le x \le 4,\ \ldots$, show that $\displaystyle\sum_{n=1}^{k}\frac{1}{n} \ge \int_1^k \frac{1}{n}\,dn$.

d Does the total angle through which the spiral turns approach a limit?

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14C Simple and compound interest

This section reviews the formulae for simple and compound interest. Simple interest is both an arithmetic sequence and a linear function. Compound interest is both a geometric sequence and an exponential function.

Simple interest, arithmetic sequences and linear functions

The formula for simple interest I should be well known from earlier years,

$$I = PRn,$$

where P is the principal invested, n is the number of units of time (such as days, weeks, months or years), and R is the interest rate per unit time.

- Regard n as a real number. Then the interest I is a linear function of n.
- Substitute $n = 1, 2, 3, \ldots$. Then the interest forms an arithmetic sequence

  $$T_1 = PR, \quad T_2 = 2PR, \quad T_3 = 3PR, \quad \ldots$$

  with first term PR and difference PR. Substituting $n = 0$ gives $T_0 = 0$, the 0th term of the sequence, when the interest due is still zero.

Thus APs and linear functions are closely related, as discussed in Chapter 1.

The simple interest formula gives the interest alone. To find the total amount at the end of n units of time, add the original principal P to the interest.

4 SIMPLE INTEREST

Suppose that a principal $\$P$ earns simple interest at a rate R per unit time.

- The simple interest $\$I$ earned over n units of time is

  $$I = PRn.$$

- Thus the interest is a linear function of n.
- Substituting $n = 1, 2, 3 \ldots$ gives an AP with first term PR and difference PR.
- To find the total amount at the end of n units of time, add the principal P.

Note: The interest rate here is a number. If the rate is given as a percentage, such as 7% pa, then substitute $R = 0.07$. (The abbreviation ‘pa’ stands for *per annum*, which is Latin for ‘per year’ — always be careful of the units of time.)

Example 9 14C

A principal $\$P$ is invested at 6% pa simple interest.

a If the principal $\$P$ is \$3000, how much money will there be after seven years?

b Find the principal $\$P$, if the total at the end of five years is \$6500.

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SOLUTION

a Using the formula, interest $= PRn$

$= 3000 \times 0.06 \times 7$

$= \$1260.$

Hence final amount $= 3000 + 1260$ (principal + interest)

$= \$4260.$

b
$$\begin{aligned}
\text{Final amount after 5 years} &= 6500 \\
P + PRn &= 6500 \quad \text{(principal + interest)} \\
P + P \times 0.06 \times 5 &= 6500 \\
P + P \times 0.3 &= 6500 \\
P \times 1.3 &= 6500 \\
\boxed{\div 1.3} \qquad P &= \$5000.
\end{aligned}$$

Compound interest, geometric sequences and exponential functions

The formula for compound interest should also be well known from earlier years,

$$A_n = P(1 + R)^n,$$

where A_n is the final amount after n units of time (such as days, weeks, months or years), P is the principal, and R is the interest rate per unit time.

- Regard n as a real number. Then A_n is a multiple of an exponential function of n with base $1 + R$.
- Substitute the values $n = 1, 2, 3, \ldots$. Then the final amounts $A_1, A_2, A_3, \ldots$ after $1, 2, 3, \ldots$ units of time form a geometric sequence

  $$A_1 = P(1 + R)^1, \ A_2 = P(1 + R)^2, \ A_3 = P(1 + R)^3, \ldots$$

  with first term $P(1 + R)$ and common ratio $1 + R$. Substituting $n = 0$ gives $A_0 = P$, the 0th term of the sequence, when the amount due is still equal to the principal.

Thus GPs and exponential functions are closely related, as discussed in Chapter 1.

The compound interest formula gives the final amount. To find the interest, subtract the principal from the final amount.

5 COMPOUND INTEREST

Suppose that a principal $\$P$ earns compound interest at a rate R per unit time for n units of time, compounded every unit of time.

- The total amount A_n after n units of time is

  $$A_n = P(1 + R)^n.$$

- Thus the final amount is an exponential function with base $1 + R$.
- Substituting $n = 1, 2, 3, \ldots$ gives a GP with first term $P(1 + R)$ and ratio $1 + R$.
- To find the interest, subtract the principal from the final amount.

Note: The formula only works when compounding occurs after every unit of time. For example, if the interest rate is given as 24% per year with interest compounded monthly, then the units of time must be months, and the interest rate must be the rate per month, which is $R = 0.24 \div 12 = 0.02$.

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Proof Although the formula was developed in earlier years, it is important to understand how it arises and how the process of compounding generates a GP.

The initial principal is P, and the interest rate is R per unit time.
Hence the amount A_1 at the end of one unit of time is

$$A_1 = \text{principal} + \text{interest} = P + PR = P(1+R).$$

This means that adding the interest is effected by multiplying by $1 + R$.
Thus the amount A_2 is obtained by multiplying A_1 by $1 + R$:

$$A_2 = A_1(1+R) = P(1+R)^2.$$

Continuing the process for the amounts $A_3, A_4, \ldots,$

$$\begin{aligned} A_3 &= A_2(1+R) = P(1+R)^3, \\ A_4 &= A_3(1+R) = P(1+R)^4, \end{aligned}$$

so that when the money has been invested for n units of time,

$$A_n = A_{n-1}(1+R) = P(1+R)^n.$$

Example 10 14C

Amelda takes out a loan of \$5000 at a rate of 12% pa, compounded monthly. She makes no repayments.

a Find the total amount owing after five years, and the interest alone.

b Find when the amount owing doubles, correct to the nearest month.

SOLUTION

Because the interest is compounded every month, the units of time must be months. The interest rate is therefore 1% per month, so $R = 0.01$.

a
$$\begin{aligned} A_{60} &= P(1+R)^{60} \qquad \text{(converting 5 years to 60 months)} \\ &= 5000 \times 1.01^{60} \\ &\doteqdot \$9083. \end{aligned}$$

After five years, interest $= \$9083 - \$5000$ (subtracting the principal)
$= \$4083.$

b
$$\begin{aligned} \text{Put} \qquad A_n &= 2P. \\ \text{Then} \qquad P(1+R)^n &= 2P \\ P \times 1.01^n &= 2P \\ \boxed{\div P} \qquad 1.01^n &= 2 \qquad \text{(so the value } P = 5000 \text{ is irrelevant)} \\ n &= \log_{1.01} 2 \\ &= \frac{\log_e 2}{\log_e 1.01} \\ &\doteqdot 70 \text{ months.} \end{aligned}$$

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Depreciation

Depreciation is important when a business buys equipment because the equipment becomes worn or obsolete over time and loses its value. The company is required to record this loss of value as an expense in its accounts. Depreciation reduces the company's profit, which in turn reduces also the income tax payable. Depreciation is usually expressed as the loss per unit time of a percentage of the value of an item. The formula for depreciation is therefore the same as the formula for compound interest, except that the rate is negative.

6 DEPRECIATION

Suppose that goods originally costing $\$P$ depreciate at a rate R per unit time.

- The value A_n of the goods after n units of time is

$$A_n = P(1 - R)^n.$$

- Thus the final value is an exponential function of n with base $1 - R$.
- Substituting $n = 1, 2, 3, \ldots$ gives a GP with first term $P(1 - R)$ and ratio $1 - R$.
- To find the loss of value, subtract the final value from the original value.

Substituting $n = 0$ gives $A_0 = P$, the original value (0th term of the sequence).

Example 11 14C

An espresso machine bought for \$15 000 on 1st January 2016 depreciates at $12\frac{1}{2}\%$ pa.

a Find the depreciated value on 1st January 2025, and the loss of value over those nine years, correct to the nearest dollar.

b During which year will the value drop below 10% of the original cost?

SOLUTION

This is depreciation with $R = 0.125$, so $1 - R = 0.875$.

a Depreciated value $= A_9$ (from 01/01/2016 to 01/01/2025 is 9 years)

$$\begin{aligned} &= P(1 - R)^n, \\ &= 15000 \times 0.875^9 \\ &\doteqdot \$4510. \end{aligned}$$

Loss of value $\doteqdot 15000 - 4510$ (subtracting the depreciated value)

$$\doteqdot \$10490.$$

b Put

$$\begin{aligned} A_n &= 0.1P \\ P \times 0.875^n &= 0.1P \\ \boxed{\div P} \quad 0.875^n &= 0.1 \quad \text{(so the value } P = 15000 \text{ is irrelevant here)} \\ n &= \log_{0.875} 0.1 \\ &= \frac{\log_e 0.1}{\log_e 0.875} \\ &\doteqdot 17.24. \end{aligned}$$

Hence the depreciated value will drop below 10% during 2033.

(There are 17 years from 01/01/2016 to 01/01/2033, so the drop occurs during 2033.)

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Exercise 14C

FOUNDATION

1 Use the formula $I = PRn$ to find: **i** the simple interest, **ii** the total amount, when:
 a \$5000 is invested at 6% per annum for three years,
 b \$12 000 is invested at 6.15% per annum for seven years.

2 Use the formula $A = P(1 + R)^n$ to find, correct to the nearest cent: **i** the total value,
 ii the interest alone, of:
 a \$5000 invested at 6% per annum, compounded annually, for three years,
 b \$12 000 invested at 6.15% per annum, compounded annually, for seven years.

3 Use the formula $A = P(1 - R)^n$ to find, correct to the nearest cent: **i** the final value,
 ii the loss of value, of:
 a \$5000 depreciating at 6% per annum for three years,
 b \$12 000 depreciating at 6.15% per annum for seven years.

4 First convert the interest rate to the appropriate unit of time, then find the final value, correct to the nearest cent, when:
 a \$400 is invested at 12% per annum, compounded monthly, for two years,
 b \$10 000 is invested at 7.28% per annum, compounded weekly, for one year.

5 A man invested \$10 000 at 6.5% per annum simple interest.
 a Write down a formula for A_n, the total value of the investment at the end of the nth year.
 b Show that the investment exceeds \$20 000 at the end of 16 years, but not at the end of 15 years.

6 A company has just bought several cars for a total of \$229 000. The depreciation rate on these cars is 15% per annum.
 a What will be the net worth of the fleet of cars five years from now?
 b What will be the loss in value then?

7 Howard is arguing with Juno over who has the better investment. Each invested \$20 000 for one year. Howard has his invested at 6.75% per annum simple interest, while Juno has hers invested at 6.6% per annum compound interest.
 a If Juno's investment is compounded annually, who has the better investment, and what are the final values of the two investments?
 b Juno then points out that her interest is compounded monthly, not yearly. Now who has the better investment, and by how much?

DEVELOPMENT

8 **a** Find the total value of an investment of \$5000 earning 7% per annum simple interest for three years.
 b A woman invested an amount for nine years at a rate of 6% per annum. She earned a total of \$13 824 in simple interest. What was the initial amount she invested?
 c A man invested \$23 000 at 3.25% per annum simple interest. At the end of the investment period he withdrew all the funds from the bank, a total of \$31 222.50. How many years did the investment last?
 d The total value of an investment earning simple interest after six years is \$22 610. If the original investment was \$17 000, what was the interest rate?

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9 **a** The final value of an investment, after ten years earning 15% per annum, compounded yearly, was \$32 364. Find the amount invested, correct to the nearest dollar.

b The final value of an investment that earned 7% compound interest per annum for 18 years was \$40 559.20. What was the original amount, correct to the nearest dollar?

c A sum of money is invested at 4.5% interest per annum, compounded monthly. At the end of three years the value is \$22 884.96. Find the amount of the original investment, correct to the nearest dollar.

10 An insurance company recently valued my car at \$14 235. The car is three years old and the depreciation rate used by the insurance company was 10.7% per annum. What was the cost of the car, correct to the nearest dollar, when I bought it?

11 **a** What does \$6000 grow to at 8.25% per annum for three years, compounded monthly?

b How much interest is earned over the three years?

c What rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

12 An amount of \$10 000 is invested for five years at 4% pa interest, compounded monthly.

a Find the final value of the investment.

b What rate of simple interest, correct to two significant figures, would be needed to yield the same final balance?

c How many full months will it take for the money to exceed \$15 000?

13 The present value of a company asset is \$350 000. If it has been depreciating at $17\frac{1}{2}\%$ per annum for the last six years, what was the original value of the asset, correct to the nearest \$1000?

14 After six years compound interest, the final value of a \$30 000 investment was \$45 108.91. Find the rate of interest, correct to two significant figures, if it was compounded annually.

15 **a** Write down the formula for the total value A_n when a principal of \$6000 is invested at 12% pa compound interest for n years. Hence find the smallest number of years required for the investment to increase by a factor of 10.

b Xiao and Mai win a prize in the lottery and decide to put \$100 000 into a retirement fund offering 8.25% per annum interest, compounded monthly. How long will it be before their money has doubled? Give your answer correct to the nearest month.

c My brother bought a new car, and it is depreciating at 15% per annum. After how many years will its value first be less than 10% of its cost (correct to the nearest year)?

16 A bank customer earned \$7824.73 in interest on a \$40 000 investment at 6% per annum, compounded quarterly. Find the period of the investment, correct to the nearest quarter.

17 Thirwin, Neri, Sid and Nee each inherit \$10 000. Each invests the money for one year. Thirwin invests his money at 7.2% per annum simple interest. Neri invests hers at 7.2% per annum, compounded annually. Sid invests his at 7% per annum, compounded monthly. Nee invests in certain shares with a return of 8.1% per annum, but must pay stockbrokers' fees of \$50 to buy the shares initially and again to sell them at the end of the year. Who is furthest ahead at the end of the year?

18 **a** Find the interest on \$15 000 invested at 7% per annum simple interest for five years.

b Hence write down the total value of the investment.

c What rate of compound interest would yield the same amount if compounded annually? Give your answer correct to three significant figures.

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19 A student was asked to find the original value of an investment earning 9% per annum, compounded annually for three years, given its current value of \$54 391.22.

a She incorrectly thought that because she was working in reverse, she should use the depreciation formula. What value did she get?

b What is the correct answer, correct to the nearest dollar?

ENRICHMENT

20 **a** A bank lent \$1000 for one year at 12% pa compound interest. Find the amount owing at the end of the year, correct to the nearest cent, if the compounding occurred:

i annually, **ii** quarterly, **iii** monthly, **iv** daily.

b Question 17 of Exercise 6G outlined a proof that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. Use this limit to find the amount after one year if the compounding had been continuous.

c What would the difference have been between annual compounding and continuous compounding if the period of the loan had been 10 years?

21 **a** Find the total value A_n if P is invested at a simple interest rate R for n periods.

b Show, by means of the binomial expansion, that the total value of the investment when compound interest is applied may be written as $A_n = P + PRn + P\sum_{k=2}^{n} {}^nC_k\,R^k$.

c Explain what each of the three terms of the formula in part **b** represents.

A possible project

Interest rates change, sometimes only every couple of years, sometimes every month. The Reserve Bank of Australia sets a benchmark interest rate called the *cash rate*. The historical cash rates are available, and a spreadsheet can be set up that will calculate the value of an amount that has earned this benchmark rate of interest over a number of years. Alternatively, the historical term deposit interest rates from a major bank could be used.

Inflation keeps varying also. A good question to ask is whether the amount has kept up with inflation, which also changes from month to month and needs a spreadsheet to calculate. There is also the problem that an investor has to pay tax on the interest, and the rate of tax varies over time and varies with the investor's income.

Thus even though the dollar value of a monetary asset may have increased over the years, its purchasing power may have gone backwards and taxation will have been lost. All this can be set up in a spreadsheet over the last say 30–40 years using data gathered from the web. A significant comparison is the price of housing over the same period, but there are many other interesting comparisons to be made.

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14D Investing money by regular instalments

Investment schemes such as superannuation — often called annuities — require money to be invested at regular intervals, for example every month or every year. This complicates things because each individual instalment earns compound interest for a different length of time. Calculating the value of these investments at some future time requires adding the terms of a GP.

This section and the next are applications of GPs. Learning new formulae is not recommended, because they will all need to be derived within each question.

Developing the GP and summing it

The most straightforward way to solve these problems is to find what each instalment grows to as it accrues compound interest. These final amounts form a GP, which can then be summed.

7 FINDING THE FUTURE VALUE OF AN INVESTMENT SCHEME

- Find what each instalment will amount to as it earns compound interest.
- Add up all these amounts using the formula for the sum of a GP.

Example 12 **14D**

Rawen's parents invested \$1000 in his name on the day that he was born. They continued to invest \$1000 for him on each birthday until his 20th birthday. On his 21st birthday they gave him the value of the investment.

If all the money earned interest of 7% compounded annually, what was the final value of the scheme, correct to the nearest dollar?

SOLUTION

The 1st instalment is invested for 21 years, and so amounts to 1000×1.07^{21}.
The 2nd instalment is invested for 20 years, and so amounts to 1000×1.07^{20}.

. . . .

The 20th instalment is invested for 2 years, and so amounts to 1000×1.07^{2}.
The 21st and last instalment is invested for 1 year, and so amounts to 1000×1.07^{1}.
Thus the total amount A_{21} at the end of 21 years is the sum

$$\begin{aligned} A_{21} &= \text{instalments plus interest} \\ &= (1000 \times 1.07^1) + (1000 \times 1.07^2) + \cdots + (1000 \times 1.07^{21}). \end{aligned}$$

This is a GP with first term $a = 1000 \times 1.07$, ratio $r = 1.07$, and 21 terms.

$$\begin{aligned} \text{Hence } A_{21} &= \frac{a(r^{21} - 1)}{r - 1} \qquad \text{(using the formula for } S_n \text{ for a GP with } r > 1) \\ &= \frac{1000 \times 1.07 \times (1.07^{21} - 1)}{0.07} \\ &\doteqdot \$48\,006. \end{aligned}$$

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Example 13 14D

Robin and Robyn are investing \$10000 in a superannuation scheme on 1st July each year, beginning in the year 2010. The money earns compound interest at 8% pa, compounded annually.

a How much will the fund amount to by 30th June 2030?

b How much will the fund amount to by the end of n years?

c Show that 2031 is the year when the fund first exceeds \$500000 on 30th June.

d What annual instalment would have produced \$1000000 by 2030?

SOLUTION

a The 1st instalment is invested for 20 years, and so amounts to 10000×1.08^{20}.
The 2nd instalment is invested for 19 years, and so amounts to 10000×1.08^{19}.
The 19th instalment is invested for 2 years, and so amounts to 10000×1.08^{2}.
The 20th and last is invested for 1 year, and so amounts to 10000×1.08^{1}.
Thus the total amount A_{20} at the end of 20 years is the sum

$$\begin{aligned} A_{20} &= \text{instalments plus interest} \\ &= (10000 \times 1.08^1) + (10000 \times 1.08^2) + \cdots + (10000 \times 1.08^{20}). \end{aligned}$$

This is a GP with first term $a = 10000 \times 1.08$, ratio $r = 1.08$, and 20 terms.

$$\begin{aligned} \text{Hence } A_{20} &= \frac{a(r^{20} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{10000 \times 1.08 \times (1.08^{20} - 1)}{0.08} \\ &\doteqdot \$494229 \quad \text{(correct to the nearest dollar)}. \end{aligned}$$

b The 1st instalment is invested for n years, and so amounts to 10000×1.08^{n}.
The 2nd instalment is invested for $n - 1$ years, and so amounts to $10000 \times 1.08^{n-1}$.
The nth and last is invested for 1 year, and so amounts to 10000×1.08^{1}.
Thus the total amount A_n at the end of n years is the sum

$$\begin{aligned} A_{n} &= \text{instalments plus interest} \\ &= (10000 \times 1.08^1) + (10000 \times 1.08^2) + \cdots + (10000 \times 1.08^{n}). \end{aligned}$$

This is a GP with first term $a = 10000 \times 1.08$, ratio $r = 1.08$, and n terms.

$$\begin{aligned} \text{Hence } A_{n} &= \frac{a(r^{n} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{10000 \times 1.08 \times (1.08^{n} - 1)}{0.08} \\ &= 135000 \times (1.08^{n} - 1). \end{aligned}$$

c From part **a**, the total after 20 years is just under \$500000.
Substituting $n = 21$ into the formula in part **b**,

$$\begin{aligned} A_{21} &= 135000 \times (1.08^{21} - 1) \\ &\doteqdot \$544568. \end{aligned}$$

Hence 2031 is the year when the fund first exceeds \$500000 on 30th June.

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d Reworking part **b** with an instalment $\$M$ instead of \$10 000 gives the formula

$$A_n = 13.5 \times M \times (1.08^n - 1)\,.$$

Substituting $n = 20$ and $A_{20} = 1\,000\,000$ into this formula,

$$1\,000\,000 = 13.5 \times M \times (1.08^{20} - 1)$$

$$M = \frac{1\,000\,000}{13.5 \times (1.08^{20} - 1)} \qquad \text{(making } M \text{ the subject)}$$

$$\doteqdot \$20\,234 \qquad \text{(correct to the nearest dollar)}\,.$$

Example 14 [Using logarithms to find n] 14D

Continuing with the previous example, use logarithms to find the year in which the fund first exceeds \$700 000 on 30th June.

SOLUTION

Substituting $M = 10\,000$ and $A_n = 700\,000$ into the formula found in part **b**,

$$700\,000 = 135\,000 \times (1.08^n - 1)$$

$$\boxed{\div 135\,000} \quad 1.08^n - 1 = \tfrac{700}{135}$$

$$\boxed{+1} \quad 1.08^n = \tfrac{835}{135}$$

$$n = \log_{1.08} \tfrac{835}{135} \qquad \text{(converting to a logarithmic equation)}$$

$$= \frac{\log_{10} \frac{835}{135}}{\log_{10} 1.08} \qquad \text{(using the change-of-base formula)}$$

$$\doteqdot 23.68.$$

Hence the fund first exceeds \$700 000 on 30th June when $n = 24$, that is, in 2034.

Example 15 [Monthly and weekly compounding] 14D

a Charmaine has a superannuation scheme with monthly instalments of \$600 for 10 years and an interest rate of 7.8% pa, compounded monthly. What will the final value of her investment be?

b Charmaine was offered an alternative scheme with interest of 7.8% pa, compounded weekly, and weekly instalments. What weekly instalments would have yielded the same final value as the scheme in part **a**?

c Which scheme would have cost her more per year?

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SOLUTION

a The monthly interest rate is $0.078 \div 12 = 0.0065$.
There are 120 months in 10 years.
The 1st instalment is invested for 120 months and so amounts to 600×1.0065^{120}.
The 2nd instalment is invested for 119 months and so amounts to 600×1.0065^{119}.
The 120th and last is invested for 1 month and so amounts to 600×1.0065^{1}.
Thus the total amount A_{120} at the end of 120 months is the sum

$$\begin{aligned} A_{120} &= \text{instalments plus interest} \\ &= (600 \times 1.0065^1) + (600 \times 1.0065^2) + \cdots + (600 \times 1.0065^{120}). \end{aligned}$$

This is a GP with first term $a = 600 \times 1.0065$, ratio $r = 1.0065$, and 120 terms.

$$\begin{aligned} \text{Hence } A_{120} &= \frac{a(r^{120} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{600 \times 1.0065 \times (1.0065^{10} - 1)}{0.0065} \\ &\doteqdot \$109\,257 \qquad \text{(retained in the memory for part } \mathbf{b}). \end{aligned}$$

b The weekly interest rate is $0.078 \div 52 = 0.0015$.
Let $\$M$ be the weekly instalment. There are 520 weeks in 10 years.
The 1st instalment is invested for 520 weeks and so amounts to $M \times 1.0015^{520}$.
The 2nd instalment is invested for 519 weeks and so amounts to $M \times 1.0015^{519}$.
The 520th and last is invested for 1 week and so amounts to $M \times 1.0015^{1}$.
Thus the total amount A_{520} at the end of 520 weeks is the sum

$$\begin{aligned} A_{520} &= \text{instalments plus interest} \\ &= M \times 1.0015 + M \times 1.0015^2 + \cdots + M \times 1.0015^{520}. \end{aligned}$$

This is a GP with first term $a = M \times 1.0015$, ratio $r = 1.0015$ and 520 terms.

$$\begin{aligned} \text{Hence } A_{520} &= \frac{a(r^{520} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ A_{520} &= \frac{M \times 1.0015 \times (1.0015^{520} - 1)}{0.0015} \\ A_{520} &= \frac{M \times 10015 \times (1.0015^{520} - 1)}{15}. \end{aligned}$$

Writing this formula with M as the subject,

$$M = \frac{15 \times A_{520}}{10015 \times (1.0015^{520} - 1)}.$$

But the final value A_{520} is to be the same as the final value in part **a**,
so substituting the answer to part **a** for A_{520} gives

$$M \doteqdot \$138.65 \qquad \text{(retain in the memory for part } \mathbf{c}).$$

c The weekly scheme in part **b** therefore costs about \$7210.04 per year, compared with \$7200 per year for the monthly scheme in part **a**.

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An alternative approach using recursion

There is an alternative approach, using recursion, to developing the GPs involved in these calculations. Because the working is slightly longer, we have chosen not to display this method in the notes. It has, however, the great advantage that its steps follow the progress of a banking statement.

For those interested in the recursive method, it is developed in two structured questions, Questions 17 and 18, at the end of Exercise 14D. Most of the other questions can also be done by recursion if that is preferred (provided, of course, that some structuring within the question is ignored).

Exercise 14D

FOUNDATION

Note: Questions 1–5 of this exercise have been heavily structured to follow the approach given in the worked examples of this section. There are several other satisfactory approaches, including the recursive method outlined in Questions 17 and 18. If a different approach is chosen, the structuring in the first five questions below can be ignored.

1 Suppose that an instalment of \$500 is invested in a superannuation scheme on 1st January each year for four years, beginning in 2020. The money earns interest at 10% pa, compounded annually.

a **i** What is the value of the first instalment on 31st December 2023?

ii What is the value of the second instalment on 31st December 2023?

iii What is the value of the third instalment on this date?

iv What is the value of the fourth instalment?

v What is the total value of the superannuation on 31st December 2023?

b **i** Write down the four answers to parts **i** to **iv** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, and hence check your answer to part **v** of part **a**.

2 Suppose that an instalment of \$1200 is invested in a superannuation scheme on 1st April each year for five years, beginning in 2015. The money earns interest at 5% pa, compounded annually.

a In each part, round your answer correct to the nearest cent.

i What is the value of the first instalment on 31st March 2020?

ii What is the value of the second instalment on this date?

iii Do the same for the third, fourth and fifth instalments.

iv What is the total value of the superannuation on 31st March 2020?

b **i** Write down the answers to parts **i** to **iii** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part **iv** of part **a**.

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3 Joshua makes 15 contributions of \$1500 to his superannuation scheme on 1st April each year. The money earns compound interest at 7% per annum. He calculates what the scheme will be worth at a target date 15 years later.

a Let A_{15} be the total value of the fund at the target date.

i How much does the first instalment amount to at the target date?

ii How much does the second instalment amount to at the target date?

iii How much does the last contribution amount to invested for just one year?

iv Hence write down a series for A_{15}.

b Hence show that the final value of the fund is $A_{15} = \dfrac{1500 \times 1.07 \times (1.07^{15} - 1)}{0.07}$, and evaluate this correct to the nearest dollar.

4 Laura makes 24 contributions of \$250 to her superannuation scheme on the first day of each month. The money earns interest at 6% per annum, compounded monthly (that is, at 0.5% per month). She calculates the scheme's value at a target date 24 months later.

a Let A_{24} be the total value of the fund at the target date.

i How much does the first instalment amount to at the target date?

ii How much does the second instalment amount to at the target date?

iii What is the value of the last contribution, invested for just one month?

iv Hence write down a series for A_{24}.

b Hence show that the total value of the fund after contributions have been made for two years is $A_{24} = \dfrac{250 \times 1.005 \times (1.005^{24} - 1)}{0.005}$, and evaluate this correct to the nearest dollar.

5 A company makes contributions of \$3000 to the superannuation fund of one of its employees on 1st July each year. The money earns compound interest at 6.5% per annum. In this question, round all currency amounts correct to the nearest dollar.

a Let A_{25} be the value of the fund at the end of 25 years.

i How much does the first instalment amount to at the end of 25 years?

ii How much does the second instalment amount to at the end of 24 years?

iii How much does the last instalment amount to at the end of just one year?

iv Hence write down a series for A_{25}.

b Hence show that $A_{25} = \dfrac{3000 \times 1.065 \times (1.065^{25} - 1)}{0.065}$.

c What will be the value of the fund after 25 years, and what will be the total amount of the contributions?

DEVELOPMENT

6 Finster and Finster Superannuation offer a superannuation scheme with annual contributions of \$12 000 invested at an interest rate of 9% pa, compounded annually. Contributions are paid on 1st of January each year.

a Zoya decides to invest in the fund for the next 20 years. Show that the final value of her investment is given by $A_{20} = \dfrac{12000 \times 1.09 \times (1.09^{20} - 1)}{0.09}$.

b Evaluate A_{20}.

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c By how much does this exceed the total contributions Zoya made?

d The company agrees to let Zoya make a higher contribution to the scheme. Let this instalment be M. Show that in this case $A_{20} = \dfrac{M \times 1.09 \times (1.09^{20} - 1)}{0.09}$.

e What would Zoya's annual contribution have to be in order for her superannuation to have a total value of \$1 000 000 at the end of the 20 years?

7 The company that Itsushi works for makes contributions to his superannuation scheme on 1st January each year. Any amount invested in this scheme earns interest at the rate of 7.5% pa.

a Let M be the annual contribution. Show that the value of the investment at the end of the nth year is $A_n = \dfrac{M \times 1.075 \times (1.075^n - 1)}{0.075}$.

b Itsushi plans to have \$1 500 000 in superannuation when he retires in 25 years' time. Show that the company must contribute \$20 526.52 each year, correct to the nearest cent.

c The first year that Itsushi's superannuation is worth more than \$750 000, he decides to change jobs. Let this year be n.

i Show that n is the smallest integer solution of $(1.075)^n > \dfrac{750000 \times 0.075}{20526.52 \times 1.075} + 1$.

ii Evaluate the right-hand side and hence show that $(1.075)^n > 3.5492$.

iii Use logarithms or trial-and-error to find the value of n.

8 A person invests \$10 000 each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first payment is made on 1st January 2021 and the last payment is made on 1st January 2040.

a How much did the person invest over the life of the fund?

b Calculate, correct to the nearest dollar, the amount to which the 2021 payment has grown by the beginning of 2041.

c Find the total value of the fund when it is paid out on 1st January 2041.

d The person wants to reach a total value of \$1 000 000 in superannuation.

i Find a formula for A_n, the value of the investment after n years.

ii Show that the target is reached when $1.1^n > \dfrac{10}{1.1} + 1$.

iii At the end of which year will the superannuation be worth \$1 000 000?

e Suppose instead that the person wanted to achieve the same total investment of \$1 000 000 after only 20 years. What annual contribution would produce this amount? (Hint: Let M be the amount of each contribution.)

9 Each year on her birthday, Jane's parents put \$20 into an investment account earning $9\frac{1}{2}\%$ per annum compound interest. The first deposit took place on the day of her birth. On her 18th birthday, Jane's parents gave her the account and \$20 cash in hand.

a How much money had Jane's parents deposited in the account?

b How much money did she receive from her parents on her 18th birthday?

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10 A man about to turn 25 is getting married. He has decided to pay \$5000 each year on his birthday into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. If he dies before age 65, his wife will inherit the value of the insurance to that point. If he lives to age 65, the insurance company will pay out the value of the policy in full. Answer these questions correct to the nearest dollar.

a The man is in a dangerous job. What will be the payout if he dies just before he turns 30?

b The man's father died of a heart attack just before age 50. Suppose that the man also dies of a heart attack just before age 50. How much will his wife inherit?

c What will the insurance company pay the man if he survives to his 65th birthday?

11 In 2021, the school fees at a private girls' school are \$20 000 per year. Each year the fees rise by $4\frac{1}{2}\%$ due to inflation.

a Susan is sent to the school, starting in Year 7 in 2021. If she continues through to her HSC year, how much will her parents have paid the school over the six years?

b Susan's younger sister is starting in Year 1 in 2021. How much will they spend on her school fees over the next 12 years if she goes through to her HSC?

ENRICHMENT

12 A woman has just retired with a payment of \$500 000, having contributed for 25 years to a superannuation fund that pays compound interest at the rate of $12\frac{1}{2}\%$ per annum. What was the size of her annual premium, correct to the nearest dollar?

13 At age 20, a woman takes out a life insurance policy under which she agrees to pay premiums of \$500 per year until she turns 65, when she is to be paid a lump sum. The insurance company invests the money and gives a return of 9% per annum, compounded annually. If she dies before age 65, the company pays out the current value of the fund plus 25% of the difference between the current value and what the value would have been had she lived until 65.

a What is the value of the payout, correct to the nearest dollar, at age 65?

b Unfortunately she dies at age 53, just before her 35th premium is due.

i What is the current value of the life insurance?

ii How much does the life insurance company pay her family?

14 A person pays \$2000 into an investment fund every year, and it earns compound interest at a rate of 6% pa.

a How much is the fund worth at the end of 10 years?

b In which year will the fund reach \$70 000?

15 [Technology]
In the first column of a spreadsheet, enter the numbers from 1 to 30 on separate rows. In the first 30 rows of the second column, enter the formula

$$\frac{20256.52 \times 1.075 \times (1.075^n - 1)}{0.075}$$

for the value of a superannuation investment, where n is the value given in the first column.

a Which value of n is the first to give a superannuation amount greater than \$750 000?

b Compare this answer with your answer to question 7(c).

c Try to do question 8(d) in the same way.

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16 [Technology]
Try checking your answers to questions 3 to 11 using a spreadsheet and its built-in financial functions. In particular, the built-in Excel™ function `FV(rate, nper, pmt, pv, type)`, which calculates the future value of an investment, seems to produce an answer different from what might be expected. Investigate this and explain the difference.

Note: The last two questions illustrate an alternative approach to superannuation questions, using a recursive method to generate the appropriate GP. The advantage of the method is that its steps follow the progress of a banking statement.

17 [The recursive method]
At the start of each month, Cecilia deposits $\$M$ into a savings scheme paying 1% per month, compounded monthly. Let A_n be the amount in her account at the end of the nth month.

a Explain why $A_1 = 1.01M$.
b Explain why $A_2 = 1.01(M + A_1)$, and why $A_{n+1} = 1.01(M + A_n)$, for $n \ge 2$.
c Use the recursive formulae in part **b**, together with the value of A_1 in part **a**, to obtain expressions for $A_2, A_3, \ldots, A_n$.
d Using the formula for the sum of n terms of a GP, show that $A_n = 101M(1.01^n - 1)$.
e If each deposit is \$100, how much will be in the fund after three years?
f Hence find, correct to the nearest cent, how much each deposit M must be if Cecilia wants the fund to amount to \$30 000 at the end of five years.

18 [The recursive method]
A couple saves \$100 at the start of each week in an account paying 10.4% pa interest, compounded weekly. Let A_n be the amount in the account at the end of the nth week.

a Explain why $A_1 = 1.002 \times 100$, and why $A_{n+1} = 1.002 \times (100 + A_n)$, for $n \ge 2$.
b Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.
c Using GP formulae, show that $A_n = 50\,100 \times (1.002^n - 1)$.
d Hence find how many weeks it will be before the couple has \$100 000.

A possible project

As discussed at the end of Exercise 14C, interest rates vary over time, and inflation means that the purchasing power of a matured superannuation fund is less than its dollar-value may have suggested some years ago. Taking all this into account requires a spreadsheet. But with superannuation there are many other considerations, and building these things into a spreadsheet as well would involve an extended project.

- Most superannuation funds have an insurance component, insuring against early death or disability. The cost of this insurance is built into the policy, but the details are not straightforward.
- All superannuation funds charge fees, which are calculated in various ways, perhaps depending on the balance, perhaps depending on the future value, perhaps depending on the number of transactions. This could also be investigated and built into the spreadsheet.
- The contributions to the fund are almost certainly a proportion of the salary. Thus some estimates must be made of future salary.

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14E Paying off a loan

Long-term loans such as housing loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time. The calculations associated with paying off a loan are therefore similar to the investment calculations of the previous section.

Developing the GP and summing it

As with superannuation, the most straightforward method is to calculate the final value of each instalment as it earns compound interest, and then add these final values up as before, using the theory of GPs. But there is an extra complication — these instalments must be balanced against the initial loan, which is growing with compound interest. The loan is finally paid off when the amount owing is zero.

8 CALCULATIONS ASSOCIATED WITH PAYING OFF A LOAN

To find the amount A_n still owing after n units of time:

- Find what the principal, earning compound interest, would amount to if no instalments were paid.
- Find what each instalment will amount to as it earns compound interest, then add up all these amounts, using the formula for the sum of a GP.
- The amount A_n still owing at the end of n units of time is

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest}).$$

The loan is paid off when the amount A_n still owing is zero.

Note: When paying off a loan, the first payment is usually made one unit of time after the loan is taken out. But always read the question carefully!

Example 16 14E

Yianni and Eleni borrow \$20000 from the Town and Country Bank to go on a trip to Constantinople. Interest is charged at 12% per annum, compounded monthly. They start repaying the loan one month after taking it out, and their monthly instalments are \$300.

a How much will they still owe the bank at the end of six years?

b How much interest will they have paid in these six years?

SOLUTION

a The monthly interest rate is 1%, so $1 + R = 1.01$.

The initial loan of \$20000, after 72 months, amounts to 20000×1.01^{72}.

The 1st instalment is invested for 71 months, and so amounts to 300×1.01^{71}.

The 2nd instalment is invested for 70 months, and so amounts to 300×1.01^{70}.

The 71st instalment is invested for 1 month, and so amounts to 300×1.01^{1}.

The 72nd and last instalment is invested for no time at all, and so amounts to 300.

Hence the amount A_{72} still owing at the end of 72 months is

$$\begin{aligned} A_{72} &= (\text{principal plus interest}) - (\text{instalments plus interest}) \\ &= 20000 \times 1.01^{72} - (300 + 300 \times 1.01 + \cdots + 300 \times 1.01^{71}). \end{aligned}$$

The bit in brackets is a GP with first term $a = 300$, ratio $r = 1.01$, and 72 terms.

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$$\text{Hence } A_{72} = 20000 \times 1.01^{72} - \frac{a(r^{72}-1)}{r-1} \qquad \text{(finding the sum of the GP)}$$

$$= 20000 \times 1.01^{72} - \frac{300 \times (1.01^{72}-1)}{0.01}$$

$$= 20000 \times 1.01^{72} - 30000 \times (1.01^{72}-1)$$

$$\doteqdot \$9529 \qquad \text{(correct to the nearest dollar).}$$

b Total instalments over six years $= 300 \times 72$

$= \$21\,600.$

Reduction in loan over six years $= 20000 - 9529$

$= \$10\,471.$

Hence interest charged $= 21\,600 - 10\,471$

$= \$11\,129$ (more than half the original loan).

Example 17 [Finding what instalments should be paid] 14E

Ali takes out a loan of \$10000 to buy a car. He will repay the loan in five years, paying 60 equal monthly instalments, beginning one month after he takes out the loan. Interest is charged at 6% pa, compounded monthly.

Find how much the monthly instalment should be, correct to the nearest cent.

SOLUTION

The monthly interest rate is 0.5%, so $1 + R = 1.005$. Let each instalment be $\$M$.
First calculate the amount A_{60} still owing at the end of 60 months.
Then find M by setting A_{60} equal to zero.

The initial loan of \$10000, after 60 months, amounts to 10000×1.005^{60}.
The 1st instalment is invested for 59 months, and so amounts to $M \times 1.005^{59}$.
The 2nd instalment is invested for 58 months, and so amounts to $M \times 1.005^{58}$.
The 59th instalment is invested for 1 month, and so amounts to $M \times 1.005^{1}$.
The 60th and last instalment is invested for no time at all, and so amounts to M.
Hence the amount A_{60} still owing at the end of 60 months is

$$A_{60} = \text{(principal plus interest)} - \text{(instalments plus interest)}$$

$$= 10000 \times 1.005^{60} - (M + M \times 1.005 + \cdots + M \times 1.005^{59}).$$

The bit in brackets is a GP with first term $a = M$, ratio $r = 1.005$, and 60 terms.

$$\text{Hence } A_{60} = 10000 \times 1.005^{60} - \frac{a(r^{60}-1)}{r-1}$$

$$= 10000 \times 1.005^{60} - \frac{M(1.005^{60}-1)}{0.005}$$

$$= 10000 \times 1.005^{60} - 200M(1.005^{60}-1) \qquad \left(\text{because } 0.005 = \tfrac{1}{200}\right).$$

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But the loan is exactly paid off in these 5 years, so $A_{60} = 0$.

$$\begin{aligned}
\text{Hence} \quad 10000 \times 1.005^{60} - 200M(1.005^{60} - 1) &= 0 \\
200M(1.005^{60} - 1) &= 10000 \times 1.005^{60} \\
\boxed{\div 200} \quad M(1.005^{60} - 1) &= 50 \times 1.005^{60} \\
M &= \frac{50 \times 1.005^{60}}{1.005^{60} - 1} \\
&\doteqdot \$193.33.
\end{aligned}$$

Finding the length of the loan

A loan is fully repaid when the amount A_n still owing is zero. Thus finding the length of a loan means solving an equation for the index n, a process that requires logarithms.

Example 18 — 14E

Natasha and Richard take out a loan of \$200000 on 1st January 2002 to buy a house. They will repay the loan in monthly instalments of \$2200. Interest is charged at 12% pa, compounded monthly.

a Find a formula for the amount owing at the end of n months.

b How much is owing after five years?

c How long does it takes to repay:

i the full loan? **ii** half the loan?

d Why would instalments of \$1900 per month never repay the loan?

SOLUTION

a The monthly interest rate is 1%, so $1 + R = 1.01$.

The initial loan, after n months, amounts to 200000×1.01^n.

The 1st instalment is invested for $n - 1$ months and so amounts to $2200 \times 1.01^{n-1}$.

The 2nd instalment is invested for $n - 2$ months and so amounts to $2200 \times 1.01^{n-2}$.

The nth and last instalment is invested for no time at all and so amounts to 2200.

Hence the amount A_n still owing at the end of n months is

$$\begin{aligned}
A_n &= (\text{principal plus interest}) - (\text{instalments plus interest}) \\
&= 200000 \times 1.01^n - (2200 + 2200 \times 1.01 + \cdots + 2200 \times 1.01^{n-1}).
\end{aligned}$$

The bit in brackets is a GP with first term $a = 2200$, ratio $r = 1.01$, and n terms.

$$\begin{aligned}
\text{Hence } A_n &= 200000 \times 1.01^n - \frac{a(r^n - 1)}{r - 1} \\
&= 200000 \times 1.01^n - \frac{2200 \times (1.01^n - 1)}{0.01} \\
&= 200000 \times 1.01^n - 220000 \times (1.01^n - 1) \\
&= 220000 - 20000 \times 1.01^n.
\end{aligned}$$

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b To find the amount owing after 5 years, substitute $n = 60$,

$$\begin{aligned} A_{60} &= 220000 - 20000 \times 1.01^{60} \\ &\doteqdot \$183666 \qquad \text{(This is still almost as much as the original loan!)} \end{aligned}$$

c **i** To find when the loan is repaid, put $A_n = 0$,

$$\begin{aligned} 220000 = 20000 \times 1.01^n &= 0 \\ 20000 \times 1.01^n &= 220000 \\ \boxed{\div 20000} \qquad 1.01^n &= 11 \\ n &= \log_{1.01} 11 \qquad \text{(converting to a logarithmic equation)} \\ n &= \frac{\log_{10} 11}{\log_{10} 1.01} \qquad \text{(using the change-of-base formula)} \\ &\doteqdot 20 \text{ years and 1 month.} \end{aligned}$$

ii To find when the loan is half repaid, put $A_n = 100000$,

$$\begin{aligned} 220000 - 20000 \times 1.01^n &= 100000 \\ 20000 \times 1.01^n &= 220000 \\ \boxed{\div 20000} \qquad 1.01^n &= 6 \\ n &= \log_{1.01} 6 \qquad \text{(converting to a logarithmic equation)} \\ n &= \frac{\log_{10} 6}{\log_{10} 1.01} \qquad \text{(using the change-of-base formula)} \\ &\doteqdot 15 \text{ years.} \end{aligned}$$

Notice that this is about three-quarters, not half, the total time of the loan.

d With a loan of \$200000 at an interest rate of 1% per month,

$$\begin{aligned} \text{initial interest per month} &= 200000 \times 0.01 \\ &= \$2000. \end{aligned}$$

This means that at the start of the loan, \$2000 of the instalment is required just to pay the interest. Hence with repayments of only \$1900, the debt would increase rather than decrease.

The alternative approach using recursion

As with superannuation, the GP involved in a loan-repayment calculation can also be developed using a recursive method, whose steps follow the progress of a banking statement.

Again, this method is developed in two structured questions, Questions 18 and 19 at the end of Exercise 14E, and recursion can easily be applied to the other questions in the exercise provided that some internal structuring is ignored.

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Exercise 14E

FOUNDATION

Note: As in the previous exercise, Questions 1 and 2 have been heavily structured to follow the approach given in the worked examples of this section. There are several other satisfactory approaches, including the recursive method outlined in Questions 18 and 19. If a different approach is chosen, the structuring in the first three questions below can be ignored.

1 On 1st January 2020, Lizbet borrows \$501 from a bank for four years at an interest rate of 10% pa. She repays the loan with four equal instalments of \$158.05 at the end of each year.

a Use the compound interest formula to show that the initial loan amounts to \$733.51 at the end of four years.

b i What is the value of the first instalment on 31st December 2023, having been invested for three years?

ii What is the value of the second instalment on this date?

iii What is the value of the third instalment?

iv What is the value of the fourth (and last) instalment?

v Find the total value of all the instalments on 31st December 2023, and hence show that Lizbet has now repaid the loan.

c i Write down the four answers to parts i–iv above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part v of part b.

2 Suppose that on 1st April 2020 a loan of \$5600 is made, which is repaid with equal instalments of \$1293.46 made on 31st March each year for five years, beginning in 2021. The loan attracts interest at 5% pa, compounded annually.

a Use the compound interest formula to show that the initial loan amounts to \$7147.18 by 31st March 2025.

b In each part, round your answer correct to the nearest cent.

i What is the value of the first instalment on 31st March 2025?

ii What is the value of the second instalment on this date?

iii Do the same for the third, fourth and fifth instalments.

iv Find the total value of the instalments on 31st March 2025, and hence show that the loan has been repaid.

c i Write down your answers to parts i–iii above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part iv of part b.

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3 Lome took out a loan with Tornado Credit Union for \$15 000, to be repaid in 15 equal annual instalments of \$1646.92 on 1st April each year. Compound interest is charged at 7% per annum.

a Let A_{15} be the amount owed at the end of 15 years.

i Use the compound interest formula to show that $15000 \times (1.07)^{15}$ is owed on the initial loan after 15 years.

ii How much does the first instalment amount to at the end of the loan, having been invested for 14 years?

iii How much does the second instalment amount to at the end of 13 years?

iv What is the value of the second-last instalment?

v What is the worth of the last contribution, invested for no time at all?

vi Hence write down an expression involving a series for A_{15}.

b Show that the final amount owed is

$$A_{15} = 15000 \times (1.07)^{15} - \frac{1646.92 \times (1.07^{15} - 1)}{0.07}.$$

c Evaluate A_{15} and hence show that the loan has been repaid.

4 Matts signed a mortgage agreement for \$100 000 with a bank for 20 years at an interest rate of 6% per annum, compounded monthly (that is, at 0.5% per month).

a Let M be the size of each repayment to the bank, and let A_{240} be the amount owing on the loan after 20 years.

i What does the initial loan amount to after 20 years?

ii Write down the amount that the first repayment grows to by the end of the 240th month.

iii Do the same for the second repayment and for the last repayment.

iv Hence write down a series expression for A_{240}.

b Hence show that $A_{240} = 100000 \times 1.005^{240} - 200 \times M(1.005^{240} - 1)$.

c Explain why the bank puts $A_{240} = 0$.

d Hence find M, correct to the nearest cent.

e How much will Matts have paid the bank over the period of the loan?

5 I took out a personal loan of \$10 000 with a bank for five years at an interest rate of 18% per annum, compounded monthly (that is, at 1.5% per month).

a Let M be the size of each instalment to the bank, and let A_{60} be the amount owing on the loan after 60 months.

i What does the initial loan amount to after 60 months?

ii Write down the amount that the first instalment grows to by the end of the 60th month.

iii Do the same for the second instalment and for the last instalment.

iv Hence write down a series expression for A_{60}.

b Hence show that $0 = 10000 \times 1.015^{60} - \dfrac{M(1.015^{60} - 1)}{0.015}$.

c Hence find M, correct to the nearest dollar.

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DEVELOPMENT

6 A couple take out a \$165 000 mortgage on a house, and they agree to pay the bank \$1700 per month. The interest rate on the loan is 9% per annum, compounded monthly, and the contract requires that the loan be paid off within 15 years.

a Let A_{180} be the balance on the loan after 15 years. Find a series expression for A_{180}.

b Show that $A_{180} = 165000 \times 1.0075^{180} - \dfrac{1700\,(1.0075^{180} - 1)}{0.0075}$.

c Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

7 A couple take out a \$250 000 mortgage on a house, and they agree to pay the bank \$2000 per month. The interest rate on the loan is 7.2% per annum, compounded monthly, and the contract requires that the loan be paid off within 20 years.

a Let A_n be the balance on the loan after n months. Find a series expression for A_n.

b Hence show that $A_n = 250000 \times 1.006^{\,n} - \dfrac{2000\,(1.006^n - 1)}{0.006}$.

c Find the amount owing on the loan at the end of the tenth year, and state whether this is more or less than half the amount borrowed.

d Find A_{240}, and hence show that the loan is actually paid out in less than twenty years.

e If it is paid out after n months, show that $1.006^n = 4$, and hence that $n = \dfrac{\log 4}{\log 1.006}$.

f Find how many months early the loan is paid off.

8 A company borrows \$500 000 from the bank at an interest rate of 5.25% per annum, compounded monthly, to be repaid in monthly instalments. The company repays the loan at the rate of \$10 000 per month.

a Let A_n be the amount owing at the end of the nth month. Show that

$$A_n = 500000 \times 1.004375^n - \frac{10000\,(1.004375^n - 1)}{0.004375}.$$

b Given that the loan is paid off, use the result in part **a** to show that $1.004375^{\,n} = 1.28$.

c Use logarithms or trial-and-error to find how long it will take to pay off the loan. Give your answer in whole months.

9 As can be seen from these questions, the calculations involved with reducible loans are reasonably complex. For that reason, it is sometimes convenient to convert the reducible interest rate into a simple interest rate. Suppose that a mortgage is taken out on a \$180 000 house at 6.6% reducible interest per annum for a period of 25 years, with payments of amount M made monthly.

a Using the usual pronumerals, explain why $A_{300} = 0$.

b Show that $A_{300} = 180000 \times 1.0055^{300} - \dfrac{M\,(1.0055^{300} - 1)}{0.0055}$.

c Find the size of each repayment to the bank.

d Hence find the total paid to the bank, correct to the nearest dollar, over the life of the loan.

e What amount is therefore paid in interest? Use this amount and the simple interest formula to calculate the simple interest rate per annum over the life of the loan, correct to two significant figures.

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10 A personal loan of \$15 000 is borrowed from the Min Hua Finance Company at a rate of $13\frac{1}{2}\%$ per annum over five years, compounded monthly. Let M be the amount of each monthly instalment.

a Show that $15000(1.01125)^{60} - \dfrac{M(1.01125^{60} - 1)}{0.01125} = 0$.

b What is the monthly instalment necessary to pay back the loan? Give your answer correct to the nearest dollar.

11 [Problems with rounding]
Most questions so far have asked you to round monetary amounts correct to the nearest dollar. This is not always wise, as this question demonstrates. A personal loan for \$30 000 is approved with the following conditions. The reducible interest rate is 13.3% per annum, with payments to be made at six-monthly intervals over five years.

a Find the size of each instalment, correct to the nearest dollar.

b Using this amount, show that $A_{10} \neq 0$, that is, the loan is not paid off in five years.

c Explain why this has happened.

12 A couple have worked out that they can afford to pay \$19 200 each year in mortgage payments. The current home loan rate is 7.5% per annum, with equal payments made monthly over a period of 25 years.

a Let P be the principal borrowed and A_{300} the amount owing after 25 years. Show that

$$A_{300} = P \times 1.00625^{300} - \frac{1600(1.00625^{300} - 1)}{0.00625}.$$

b Hence determine the maximum amount that the couple can borrow and still pay off the loan. Round your answer down to the nearest dollar.

13 The current credit card rate of interest on Bankerscard is 23% per annum, compounded monthly.

a If a cardholder can afford to repay \$1500 per month on the card, what is the maximum value of purchases that can be made in one day if the debt is to be paid off in two months?

b How much would be saved in interest payments if the cardholder instead saved up the money for two months before making the purchase?

ENRICHMENT

14 Some banks offer a 'honeymoon' period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed \$170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay \$1650 per month to the bank. The standard rate of interest is $8\frac{1}{2}\%$ pa, but the bank also offers a special rate of 6% pa for one year to people buying their first home. (All interest rates are compounded monthly.)

a Calculate the amount the couple would owe at the end of the first year, using the special rate of interest.

b Use this value as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the loan contract?

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15 Over the course of years, a couple have saved \$300000 in a superannuation fund. Now that they have retired, they are going to draw on that fund in equal monthly pension payments for the next 20 years. The first payment is at the beginning of the first month. At the same time, any balance will be earning interest at $5\frac{1}{2}\%$ per annum, compounded monthly. Let B_n be the balance left immediately after the nth payment, and let M be the amount of the pension instalment. Also, let $P = 300000$ and R be the monthly interest rate.

a Show that $B_n = P \times (1 + R)^{n-1} - \dfrac{M\big((1 + R)^n - 1\big)}{R}$.

b Why is $B_{240} = 0$?

c What is the value of M?

16 A company buys machinery for \$500000 and pays it off by 20 equal six-monthly instalments, the first payment being made six months after the loan is taken out. If the interest rate is 12% pa, compounded monthly, how much will each instalment be?

XLS

17 [Technology]

In the first column of a spreadsheet, enter the numbers from 1 to 60 on separate rows. In the first 60 rows of the second column, enter the formula

$$500000 \times 1.004375^n - \frac{10000 \times (1.004375^n - 1)}{0.004375}$$

for the balance of a loan repayment, where n is the value given in the first column.

a Observe the pattern of figures in the second column. Notice that the balance decreases more slowly at first and more quickly towards the end of the loan.

b Which value of n is the first to give a balance less than or equal to zero?

c Compare this answer with your answer to question 8.

d Try to do question 7 **f** in the same way.

Note: The next two questions illustrate the alternative approach to loan repayment questions, using a recursive method to generate the appropriate GP.

18 [The recursive method]

A couple buying a house borrow $\$P = \$150000$ at an interest rate of 6% pa, compounded monthly. They borrow the money at the beginning of January, and at the end of every month, they pay an instalment of $\$M$. Let A_n be the amount owing at the end of n months.

a Explain why $A_1 = 1.005P - M$.

b Explain why $A_2 = 1.005A_1 - M$, and why $A_{n+1} = 1.005A_n - M$, for $n \geq 2$.

c Use the recursive formulae in part **b**, together with the value of A_1 in part **a**, to obtain expressions for $A_2, A_3, \ldots, A_n$.

d Using GP formulae, show that $A_n = 1.005^n P - 200M(1.005^n - 1)$.

e Hence find, correct to the nearest cent, what each instalment should be if the loan is to be paid off in 20 years?

f If each instalment is \$1000, how much is still owing after 20 years?

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19 [The recursive method]

Eric and Enid borrow $\$P$ to buy a house at an interest rate of 9.6% pa, compounded monthly. They borrow the money on 15th September, and on the 14th day of every subsequent month, they pay an instalment of $\$M$. Let A_n be the amount owing after n months have passed.

- **a** Explain why $A_1 = 1.008P - M$, and why $A_{n+1} = 1.008A_n - M$, for $n \geq 2$.
- **b** Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.
- **c** Using GP formulae, show that $A_n = 1.008^n P - 125M(1.008^n - 1)$.
- **d** If the maximum instalment they can afford is \$1200, what is the maximum they can borrow, if the loan is to be paid off in 25 years? (Answer correct to the nearest dollar.)
- **e** Put $A_n = 0$ in part **c**, and solve for n. Hence find how long will it take to pay off the loan of \$100 000 if each instalment is \$1000. (Round up to the next month.)

A possible project

The remarks at the end of Exercises 14C and 14D about varying interest rates and inflation hold also for housing loans. Many loans also contain insurance against death or disability or loss of employment, and again there are fees, which are not easily found.

All this historical data can be found on the web and built into a spreadsheet. The spreadsheet could also take into account the increasing value of housing over past years. An interesting comparison could be made between the relative wealth of a couple who rented for a long period and invested their savings elsewhere, and a couple on the same (increasing) salary who purchased a home with a large mortgage.

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Chapter 14 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 14 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Consider the series $31 + 44 + 57 + \cdots + 226$.

a Show that it is an AP and write down the first term and the common difference.

b How many terms are there in this series?

c Find the sum.

2 Consider the series $24 + 12 + 6 + \cdots$.

a Show that it is a geometric series and find the common ratio.

b Explain why this geometric series has a limiting sum.

c Find the limiting sum and the sum of the first 10 terms, and show that they are approximately equal, correct to the first three significant figures.

3 Use trial-and-error, and probably a calculator, to find the smallest integer such that:

a $2^n > 2000$ **b** $(1.08)^n > 2000$ **c** $(0.98)^n < 0.01$ **d** $\left(\frac{1}{2}\right)^n < 0.0001$

Then repeat parts **a**–**d** using logarithms.

4 On a certain day at the start of a drought, 900 litres of water flowed from the Neverfail Well. The next day, only 870 litres flowed from the well, and each day, the volume of water flowing from the well was $\frac{29}{30}$ of the previous day's volume. Find the total volume of water that would have flowed from the well if the drought had continued indefinitely.

5 The profits of a company are growing at 14% per year. If this trend continues, how many full years will it be before the profit has increased by over 2000%?

6 A chef receives an annual salary of \$35 000, with 4% increments each year.

a Show that her annual salaries form a GP and find the common ratio.

b Find her annual salary, and her total earnings, at the end of 10 years, each correct to the nearest dollar.

7 Darko's salary is \$47 000 at the beginning of 2004, and it will increase by \$4000 each year.

a Find a formula for T_n, his salary in the nth year.

b In which year will Darko's salary first be at least twice what it was in 2004?

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8 Miss Yamada begins her new job in 2005 on a salary of \$53 000, and it is increased by 3% each year. In which year will her salary be at least twice her original salary?

9 **a** Find the value of a \$12 000 investment that has earned 5.25% per annum, compounded monthly, for five years.

b How much interest was earned over the five years?

c What annual rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

10 A Wolfsrudel car depreciates at 12% per annum. Jake has just bought one that is four years old at its depreciated value of \$25 000.

a What will the car's depreciated value be in another four years?

b Find the average loss in value over the next four years.

c What was the new price of the car?

d Find the average loss in value over the four years from when it was new.

11 Katarina has entered a superannuation scheme into which she makes annual contributions of \$8000. The investment earns interest of 7.5% per annum, compounded annually, with contributions made on 1st October each year.

a Show that after 15 years of contributions, the value of Katarina's investment is given by

$$A_{15} = \frac{8000 \times 1.075 \times (1.075^{15} - 1)}{0.075}.$$

b Evaluate A_{15}.

c By how much does A_{15} exceed the total contributions Katarina made over these years?

d Show that after 17 years of contributions, the value A_{17} of the superannuation is more than double Katarina's contributions over the 17 years.

12 Ahmed wishes to retire with superannuation worth half a million dollars in 25 years' time. On 1st August each year he pays a contribution to a scheme that gives interest of 6.6% per annum, compounded annually.

a Let M be the annual contribution. Show that the value of the investment at the end of the nth year is

$$A_n = \frac{M \times 1.066 \times (1.066^n - 1)}{0.066}.$$

b Hence show that the amount of each contribution is \$7852.46.

13 Alonso takes out a mortgage on a flat for \$159 000, at an interest rate of 6.75% per annum, compounded monthly. He agrees to pay the bank \$1415 each month for 15 years.

a Let A_{180} be the balance of the loan after 15 years. Find a series expression for A_{180}.

b Show that $A_{180} = 159000 \times 1.005625^{180} - \dfrac{1415(1.005625^{180} - 1)}{0.005625}$.

c Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

d What monthly payment, correct to the nearest cent, is needed in order to pay off the loan in 15 years?

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Review

14 May-Eliane borrowed \$1.7 million from the bank to buy some machinery for her farm. She agreed to pay the bank \$18 000 per month. The interest rate is 4.5% per annum, compounded monthly, and the loan is to be repaid in 10 years.

a Let A_n be the balance of the loan after n months. Find a series expression for n.

b Hence show that $A_n = 1700000 \times 1.00375^n - \dfrac{18000(1.00375^n - 1)}{0.00375}$.

c Find the amount owing on the loan at the end of the fifth year, and state whether this is more or less than half the amount borrowed.

d Find A_{120}, and hence show that the loan is actually paid out in less than 10 years.

e If it is paid out after n months (that is, put $A_n = 0$), show that $1.00375^n = 1.5484$, and hence that

$$n = \frac{\log_{10} 1.5484}{\log_{10} 1.00375}.$$

f Find how many months early the loan is paid off.

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