8 Vectors

A *vector* in a Euclidean plane consists of a length and a direction.

The most obvious reason to introduce vectors is their usefulness in science. Many quantities in science are *scalars*, meaning that they have magnitude, but no direction is associated with them. Many other quantities, however, are vectors, meaning that they have a magnitude and a direction.

The contrast between scalars and vectors occurs throughout science. For example:

- Distance is a scalar that can be positive or zero, but displacement is a vector, because we move a certain distance in a certain direction.
- Speed is a scalar, but velocity is a vector because it is speed in a certain direction.
- Time is a scalar that can be positive or negative depending on our choice of 'time zero'.
- Temperature is a scalar that cannot be less than absolute zero.
- The rotation of the Earth is a vector because its axis is tilted in a certain direction, but the Earth's mass is a scalar.
- Force is a vector because you always push something in a certain direction, but pressure is a scalar — it acts in all directions.

A vector combines the magnitude and direction of a quantity into a single mathematical object.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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8A Directed intervals and vectors

We can always specify a *vector* by giving its length and direction. For example,

'Walk 20 km east'.

The starting point is not part of the vector, so that the effect of this vector will be quite different depending on whether we start at Bathurst or at Bondi Beach.

We have already been working with vectors specified this way in trigonometry, such as when a ship sails at a certain speed in a certain direction. This section and the next develop two further ways of dealing with vectors:

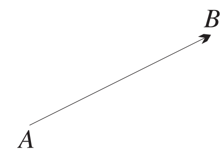
- Represent them by directed intervals that are free to wander around the plane.
- Represent them by *pairs of components*, which may be written in a column.

All three representations of a vector are equivalent, but the two ways introduced in this chapter allow a more general theory of vectors to be developed.

Vector notation

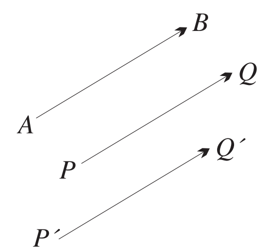
A vector can be handwritten as $\underset{\sim}{a}$ using a tilde underneath, or typeset in bold as $\mathbf{a}$. Boldface notation is clear in a printed text such as this book, but is unfortunately not appropriate for handwriting, so we have mostly avoided it. Remember that the one symbol $\underset{\sim}{a}$ or $\mathbf{a}$ holds both the length and the direction.

Directed intervals (or directed line segments)



Let A and B be distinct points in the Euclidean plane. The *directed interval* $\overrightarrow{AB}$, also called a *directed line segment*, is the interval AB together with the direction from A to B. The points A and B are called the *tail* and *head* of the directed interval, and the distance from A to B is called its *length*, written in this chapter as $|AB|$ to avoid ambiguity.

Representing a vector by directed intervals



Let $\underset{\sim}{a}$ be a vector, meaning that $\underset{\sim}{a}$ is a length and a direction. We can *represent* $\underset{\sim}{a}$ by any directed line interval $\overrightarrow{AB}$ that has the same length as $\underset{\sim}{a}$ and the same direction as $\underset{\sim}{a}$.

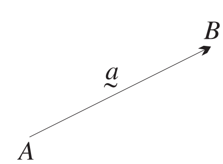
Now allow the directed interval $\overrightarrow{AB}$ to move around the plane using only translations. The resulting images $\overrightarrow{PQ}$, $\overrightarrow{P'Q'}$, . . . of $\overrightarrow{AB}$ are again directed intervals, and each has the same direction and length as $\overrightarrow{AB}$, so they all represent the same vector $\underset{\sim}{a}$. Thus $\overrightarrow{AB}$, $\overrightarrow{PQ}$, $\overrightarrow{P'Q'}$, . . . are distinct directed intervals, but they all represent the same vector.

This leaves the vector $\underset{\sim}{a}$ *free* to wander about all over the plane, the only restriction being that its length and direction are preserved as it wanders.

An alternative vector notation

We also write a vector as any directed interval that represents it, giving an alternative notation

$\underset{\sim}{a} = \overrightarrow{AB}$ (or using boldface notation, $\mathbf{a} = \overrightarrow{AB}$).



The notation $\overrightarrow{AB}$ for a vector is so common that from now on, the symbol $\overrightarrow{AB}$ will mean 'the vector $\overrightarrow{AB}$' unless it is explicitly referred to as 'the directed interval $\overrightarrow{AB}$'. In fact, the distinction between a vector and a directed interval is not strictly observed, either in language or in notation.

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The length of a vector

Denote by $|\underset{\sim}{a}|$ the length of a vector $\underset{\sim}{a}$. Thus $|\underset{\sim}{a}|$ is always positive or zero. The symbol $|\ldots|$ is generally used in mathematics for the magnitude of a quantity, for example, absolute value is written as $|-7| = 7$.

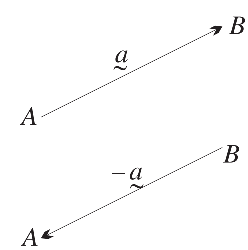
The word 'vector'

The Latin word *veho* means 'carry' or 'convey' from one place to another. In English, a 'vehicle' conveys passengers from where they are to their destination, and in medicine, a mosquito that carries a disease from one sick animal or person to another is called a 'vector'. Think of a geometric vector as moving things in the plane a certain distance in a certain direction.

1 VECTORS AND DIRECTED INTERVALS

- A *directed interval* $\overrightarrow{AB}$, or *directed line segment*, with *tail* A and *head* B, is the interval AB together with the *direction* from A to B. Its length is $|AB|$.
- A *vector* $\underset{\sim}{a}$ or **a** combines a length and a direction. It is *represented* by any directed interval $\overrightarrow{AB}$ with that length and direction, and we then write $\underset{\sim}{a} = \overrightarrow{AB}$.
- The length of a vector $\underset{\sim}{a}$ is written as $|\underset{\sim}{a}|$.

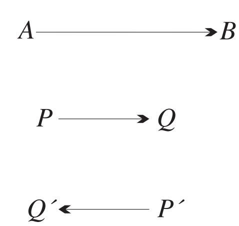
Opposite vectors and parallel vectors



The *opposite* of a directed interval $\overrightarrow{AB}$ is the directed interval $\overrightarrow{BA}$ with the same length and the opposite direction.

Correspondingly, the *opposite vector* $-\underset{\sim}{a}$ of a vector $\underset{\sim}{a} = \overrightarrow{AB}$ has the same length but opposite direction. It is thus represented by the opposite directed interval,

$$-\underset{\sim}{a} = \overrightarrow{BA}.$$



Two directed intervals $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are called *parallel* if the lines AB and PQ are parallel, whether in the same or the opposite direction. Similarly, two vectors $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{b} = \overrightarrow{PQ}$ are *parallel* if the lines AB and PQ are parallel, that is, if they are represented by parallel directed intervals.

Be careful: Parallel directed intervals or vectors may have the same direction or the opposite direction, because oppositely directed intervals or vectors are parallel.

The zero vector

When A and B are the same point, we have a *one-point directed interval* $\overrightarrow{AA}$, which is just the point A. It has no direction, and it has zero length.

The *zero vector* $\underset{\sim}{0}$ has length zero, and it is thus the only vector that does not have a direction. It can be represented by any one-point directed interval,

$$\underset{\sim}{0} = \overrightarrow{PP}, \quad \text{for all points } P.$$

The zero vector $\underset{\sim}{0}$ is the only vector that is its own opposite,

$$-\underset{\sim}{a} = \underset{\sim}{a} \quad \text{if and only if} \quad \underset{\sim}{a} = \underset{\sim}{0}.$$

Despite having no direction, the zero vector is defined to be *parallel to itself and to every other vector*.

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2 OPPOSITE VECTORS, PARALLEL VECTORS, AND THE ZERO VECTOR

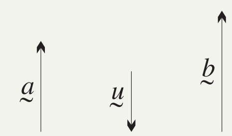
- The opposite of a vector $\underset{\sim}{a} = \overrightarrow{AB}$ is the vector $-\underset{\sim}{a} = \overrightarrow{BA}$ with the same length and the opposite direction.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are called *parallel* if the lines AB and PQ are parallel. Their directions may be the same or opposite.
- The zero vector $\underset{\sim}{0}$ has no length and no direction, and is the only vector that is its own opposite. It is parallel to every vector.

Example 1 8A

For non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{u}$ and $\underset{\sim}{b}$, we know from Euclidean geometry that:

$$\text{If } \underset{\sim}{a} \parallel \underset{\sim}{u} \text{ and } \underset{\sim}{u} \parallel \underset{\sim}{b}, \text{ then } \underset{\sim}{a} \parallel \underset{\sim}{b}.$$

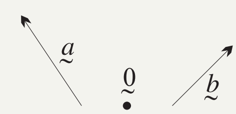
Is this statement true for all vectors?



SOLUTION

This statement is not true when the zero vector is allowed. Every vector is parallel to the zero vector, so if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are two vectors that are not parallel, then

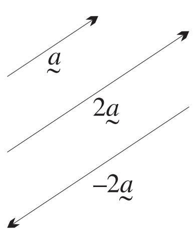
$$\underset{\sim}{a} \parallel \underset{\sim}{0} \text{ and } \underset{\sim}{0} \parallel \underset{\sim}{b}, \text{ but } \underset{\sim}{a} \nparallel \underset{\sim}{b}.$$



Multiplying a vector by a scalar

We can multiply a vector $\underset{\sim}{a}$ by a scalar λ.

- If either $\lambda = 0$ or $\underset{\sim}{a} = \underset{\sim}{0}$, then $\lambda\underset{\sim}{a} = \underset{\sim}{0}$ is the zero vector.
- Otherwise, $\lambda\underset{\sim}{a}$ has length $|\lambda| \times |\underset{\sim}{a}|$, and:
  — If $\lambda > 0$, then $\lambda\underset{\sim}{a}$ has the same direction as $\underset{\sim}{a}$.
  — If $\lambda < 0$, then $\lambda\underset{\sim}{a}$ has the opposite direction to $\underset{\sim}{a}$.



It is now easily checked that in all three cases,

$$|\lambda\underset{\sim}{a}| = |\lambda|\,|\underset{\sim}{a}| \qquad \text{and} \qquad \lambda\underset{\sim}{a} \parallel \underset{\sim}{a}.$$

It follows immediately that multiplication by 0, 1 and -1 behave as expected,

$$0 \times \underset{\sim}{a} = \underset{\sim}{0} \qquad \text{and} \qquad 1 \times \underset{\sim}{a} = \underset{\sim}{a} \qquad \text{and} \qquad (-1) \times \underset{\sim}{a} = -\underset{\sim}{a}.$$

and that multiplication by a scalar is associative, in the sense that if μ is a scalar,

$$\lambda(\mu\underset{\sim}{a}) = (\lambda\mu)\underset{\sim}{a}.$$

3 MULTIPLYING A VECTOR BY A SCALAR

A vector can be multiplied by a scalar, and for all vectors $\underset{\sim}{a}$ and scalars λ and μ:

Multiplying by zero:	$0\underset{\sim}{a} = \underset{\sim}{0}$
Multiplying by 1:	$1\underset{\sim}{a} = \underset{\sim}{a}$
Multiplying by -1:	$(-1)\underset{\sim}{a} = -\underset{\sim}{a}$
Associative law:	$\lambda(\mu\underset{\sim}{a}) = (\lambda\mu)\underset{\sim}{a}$
Lengths:	$|\lambda\underset{\sim}{a}| = |\lambda|\,|\underset{\sim}{a}|$
Parallels:	$\lambda\underset{\sim}{a} \parallel \underset{\sim}{a}$

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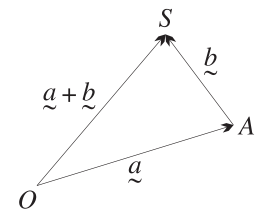
Vector addition



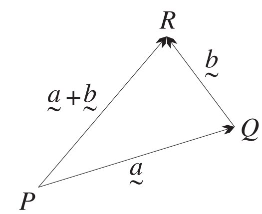
To add two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, represent them as

$$\underset{\sim}{a} = \overrightarrow{OA} \quad \text{and} \quad \underset{\sim}{b} = \overrightarrow{AS}$$

so that the head of the first is the tail of the second. The sum of the two vectors is then

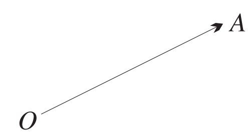
$$\overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}.$$

That is, the sum of a movement from O to A, followed by a movement from A to S, is a movement from O to S. Notice that the sum $\overrightarrow{OS}$ does not involve the point A. When I take a taxi from Orange to Sydney, all I care about is the result — I don't care if the driver goes through Armidale.



This definition is independent of the choice of the tail O used in the representation of the first vector. If we take another starting point P and construct $\underset{\sim}{a} = \overrightarrow{PQ}$ and $\underset{\sim}{b} = \overrightarrow{QR}$, then we can translate O to P, and the same translation will translate everything so that the directed interval $\overrightarrow{OB}$ is translated to the directed interval $\overrightarrow{PR}$.

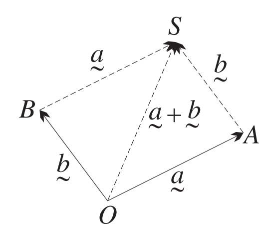
Zeroes and opposites



It is easily verified that addition of the zero vector changes nothing, and that adding a vector and its opposite gives the zero vector.

$$\overrightarrow{OA} + \overrightarrow{AA} = \overrightarrow{OA} \quad \text{and} \quad \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO}.$$

Addition by completing the parallelogram



For non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ that are not parallel, construct the sum $\underset{\sim}{a} + \underset{\sim}{b} = \overrightarrow{OS}$ as before. Now represent $\underset{\sim}{b}$ again as $\underset{\sim}{b} = \overrightarrow{OB}$ with tail O.

Then $OASB$ is a parallelogram because the opposite sides AS and OB are equal and parallel, and we have a second representation of $\underset{\sim}{a}$ as $\underset{\sim}{a} = \overrightarrow{BS}$.

This gives us a second method of adding vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

- Represent $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with a common tail O.
- Complete the parallelogram $AOBS$.
- Then $\underset{\sim}{a} + \underset{\sim}{b} = \overrightarrow{OS}$.

When $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel (including when one is zero), the resulting parallelogram is *degenerate* — meaning here that it is a one-dimensional (or zero-dimensional) object. In this case, we are just adding or subtracting lengths.

Addition is commutative

This follows immediately from the diagram above, where the parallelogram provides two representations of $\underset{\sim}{a}$ and two representations of $\underset{\sim}{b}$,

$$\begin{aligned} \underset{\sim}{a} + \underset{\sim}{b} &= \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}, \\ \text{and } \underset{\sim}{b} + \underset{\sim}{a} &= \overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BS} = \overrightarrow{OS}, \\ \text{so } \underset{\sim}{a} + \underset{\sim}{b} &= \underset{\sim}{b} + \underset{\sim}{a}, \quad \text{as required.} \end{aligned}$$

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Addition is associative

Given any three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ we can represent them by directed intervals head-to-tail as

$$\underset{\sim}{a} = \overrightarrow{AB}, \quad \underset{\sim}{b} = \overrightarrow{BC}, \quad \underset{\sim}{c} = \overrightarrow{CD}.$$

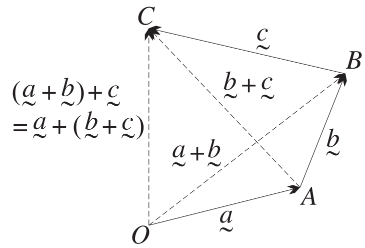
Then $(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c}$

$$\begin{aligned} &= (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} \\ &= \overrightarrow{AC} + \overrightarrow{CD} \\ &= \overrightarrow{AD}, \end{aligned}$$

and $\underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$

$$\begin{aligned} &= \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD}) \\ &= \overrightarrow{AB} + \overrightarrow{BD} \\ &= \overrightarrow{AD}, \end{aligned}$$

which proves the associative law $(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$.



Example 2 8A

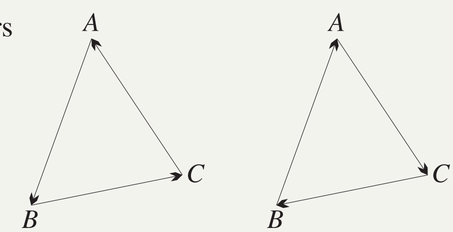
How can the sides of a triangle ABC be used to form three vectors whose sum is zero?

SOLUTION

Form the vectors so that they are head-to-tail.

First way: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \underset{\sim}{0}$

Second way: $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BA} = \overrightarrow{AA} = \underset{\sim}{0}$



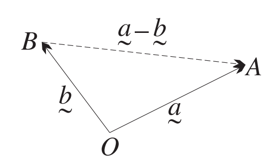
Subtraction of vectors

The *difference* $\underset{\sim}{a} - \underset{\sim}{b}$ of two vectors is defined in the usual way as the sum of $\underset{\sim}{a}$ and the opposite of $\underset{\sim}{b}$,

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b}).$$

The best way to represent subtraction is to use a triangle. Represent the vectors as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = OB$ with the same tail O. Then

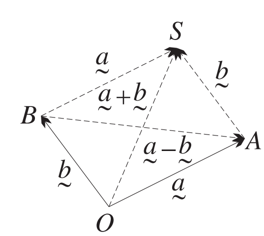
$$\begin{aligned} \underset{\sim}{a} - \underset{\sim}{b} &= \overrightarrow{OA} - \overrightarrow{OB} \\ &= \overrightarrow{OA} + \overrightarrow{BO} \\ &= \overrightarrow{BO} + \overrightarrow{OA} \\ &= \overrightarrow{BA}. \end{aligned}$$



Thus $\underset{\sim}{a} - \underset{\sim}{b} = \overrightarrow{OA} - \overrightarrow{OB}$ is the vector $\overrightarrow{BA}$ with tail B and head A.

Now complete the parallelogram $AOBS$.

- The two directions of the diagonal OS represents the sums
  $\overrightarrow{OS} = \overrightarrow{OA} + \overrightarrow{OB}$ and $\overrightarrow{SO} = \overrightarrow{AO} + \overrightarrow{BO}$.
- The two directions of the diagonal AB represents the differences
  $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

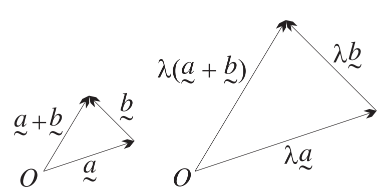


Two distributive laws

First, multiplying a vector by a scalar is distributive over vector addition,

$$\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}, \text{ for all scalars } \lambda \text{ and vectors } \underset{\sim}{a} \text{ and } \underset{\sim}{b}.$$

This is easily proven by similarity, as shown in the diagram to the right.



Secondly, multiplying a vector by a scalar is distributive over scalar addition, as is easily seen if a diagram is drawn.

$$(\lambda + \mu)\underset{\sim}{a} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{a}, \quad \text{for all scalars } \lambda \text{ and } \mu \text{ and vectors } \underset{\sim}{a}.$$

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4 VECTOR ADDITION AND SUBTRACTION

- To construct the sum $\underset{\sim}{a} + \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, place them head to tail as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = AB$. Then $\underset{\sim}{a} = \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}$.
- Alternatively, represent them as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with a common tail O, and complete the parallelogram $OASB$. Then $\underset{\sim}{a} + \underset{\sim}{b} = \overrightarrow{OS}$.
- To construct the difference $\underset{\sim}{a} - \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, represent them as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with a common tail O. Then the difference is the vector $\underset{\sim}{a} - \underset{\sim}{b} = \overrightarrow{BA}$ with tail B and head A.
- For all vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ and all scalars λ and μ:

Adding the zero vector:	$\underset{\sim}{a} + \underset{\sim}{0} = \underset{\sim}{a}$
Adding the opposite:	$\underset{\sim}{a} + (-\underset{\sim}{a}) = \underset{\sim}{0}$
Commutative law:	$\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$
Associative law:	$(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$
Two distributive laws:	$\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$
	$(\lambda + \mu)\underset{\sim}{a} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{a}$

The next worked example shows how vectors can be used to prove geometric theorems.

Example 3 8A

Let M be the midpoint of the side BC in $\triangle ABC$. Let $BA = \underset{\sim}{b}$ and $CA = \underset{\sim}{c}$.

a Explain why $\overrightarrow{BC} = \underset{\sim}{b} - \underset{\sim}{c}$, and hence express $\overrightarrow{BM}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

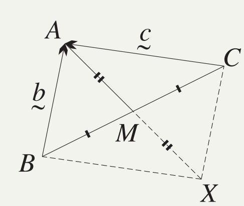
b Express $\overrightarrow{MA}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Why is the formula symmetric in $\underset{\sim}{b}$ and $\underset{\sim}{c}$?

c Extend AM to X so that $AM = MX$, and express $\overrightarrow{XA}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

d Express $\overrightarrow{XC}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

e What geometric theorem have you proven in part **d**?

SOLUTION



a Placing vectors head to tail,

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BA} - \overrightarrow{CA} = \underset{\sim}{b} - \underset{\sim}{c}.$$

Hence $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c})$.

b $\overrightarrow{MA} = \overrightarrow{MB} + \overrightarrow{BA} = -\overrightarrow{BM} + \overrightarrow{BA} = -\frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c}) + \underset{\sim}{b} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})$.
If we exchange B and C in the previous arguments, the result is the same.

c $\overrightarrow{XA} = 2 \times \overrightarrow{MA} = \underset{\sim}{b} + \underset{\sim}{c}$.

d $\overrightarrow{XC} = \overrightarrow{XA} + \overrightarrow{AC} = \overrightarrow{XA} - \overrightarrow{CA} = (\underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{c} = \underset{\sim}{b}$.

e Because $\overrightarrow{BA} = \underset{\sim}{b} = \overrightarrow{XC}$, the intervals BA and XC are equal and parallel, so $ABXC$ is a parallelogram.
We have proven that if the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

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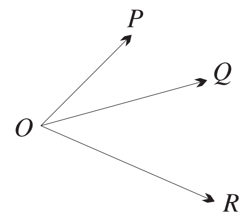
Choosing a reference point or origin — position vectors



Points are not vectors. It is often useful, however, in a diagram or a physical situation, to refer everything back to some conveniently chosen reference point or *origin* O. Then there is a correspondence between each point P in the plane and its *position vector* $\overrightarrow{OP}$ drawn with tail O and head P.

For example, surveyors in a town may place a marker on top of a nearby hill and refer every point in the town back to that reference point.

In this situation, every point P has a unique position vector $\overrightarrow{OP}$, and every vector $\underset{\sim}{a}$ is the position vector of a unique point P — represent the vector with tail O, and then the head is P.

5 AN ORIGIN AND POSITION VECTORS

- Choose a convenient *origin* O as a reference point.
- Each point P in the plane then corresponds to the *position vector* $\overrightarrow{OP}$.
- Conversely, any vector $\underset{\sim}{a}$ can be drawn as a position vector $\overrightarrow{OP}$, and thus corresponds to the point P at the head.
- Subtraction of position vectors is particularly straightforward, because they have a common tail O.

The next worked example illustrates the final dotpoint in Box 5.

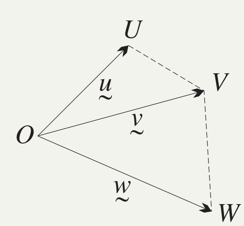
Example 4 8A

Three points U, V and W have position vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ with respect to a chosen origin O.

a Explain in two ways why $\overrightarrow{UV} = \underset{\sim}{v} - \underset{\sim}{u}$.

b Hence write down a condition on $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ for U, V and W to be collinear.

SOLUTION



a First, $\overrightarrow{UV} = \overrightarrow{UO} + \overrightarrow{OV} = -\overrightarrow{OU} + \overrightarrow{OV} = -\underset{\sim}{u} + \underset{\sim}{v}$.
Secondly, because $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are drawn as position vectors with a common tail O, $\underset{\sim}{v} - \underset{\sim}{u}$ can be represented as $\overrightarrow{UV}$ with tail U and head V.

b The points are collinear when W lies on UV,
that is, when $\overrightarrow{VW}$ is a multiple of $\overrightarrow{UV}$,
that is, when $\quad \underset{\sim}{w} - \underset{\sim}{v} = \lambda(\underset{\sim}{v} - \underset{\sim}{u})$, for some scalar λ,
better expressed as $\quad \underset{\sim}{w} = \underset{\sim}{v} + \lambda(\underset{\sim}{v} - \underset{\sim}{u})$, for some scalar λ.
(There are many other possible forms.)

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Exercise 8A

FOUNDATION

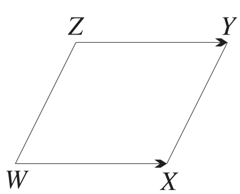
1 In each part, draw a diagram showing the displacement vector $\overrightarrow{AC}$. Then calculate the magnitude and direction of $\overrightarrow{AC}$. (Give magnitude correct to the nearest km where necessary, and direction as a true bearing correct to the nearest degree where necessary).

- **a** Sarah drives 100 km east from A to B, and then 40 km west from B to C.
- **b** William walks 6 km south from A to B, and then 4 km east from B to C.
- **c** A boat sails 25 km east from A to B, and then 15 km northeast from B to C.

2 Vikram cycled 28 km north from P to Q, then 19 km east from Q to R, and finally 12 km south from R to S.

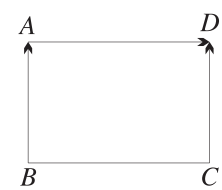
- **a** Draw a diagram representing his journey, and show the displacement vector $\overrightarrow{PS}$.
- **b** Determine the magnitude and direction of $\overrightarrow{PS}$ (in km correct to one decimal place, and as a true bearing correct to the nearest degree).

3



In the diagram above, $\overrightarrow{WX} = \overrightarrow{ZY}$.
Explain why $WXYZ$ is a parallelogram.

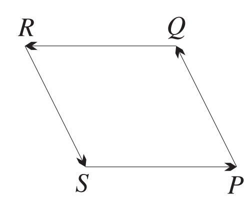
4



In the diagram above, $\overrightarrow{BA} = \overrightarrow{CD}$ and $\overrightarrow{BA} \perp \overrightarrow{AD}$.
What type of special quadrilateral is $ABCD$?
Give reasons for your answer.

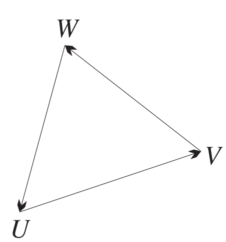
5 In the diagram to the right, $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$ and $\overrightarrow{SP}$ all have the same magnitude.

- **a** What type of special quadrilateral is $PQRS$? Justify your answer.
- **b** What can be said about the directions of $\overrightarrow{PQ}$ and $\overrightarrow{RS}$? Justify your answer.

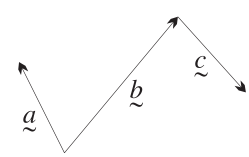


6 Suppose that u and v are non-zero vectors. By drawing a parallelogram, show that $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$.

7 In the diagram to the right, what is $\overrightarrow{UV} + \overrightarrow{VW} + \overrightarrow{WU}$?



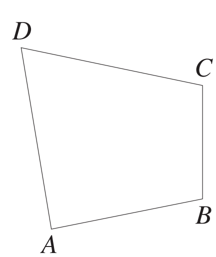
8



Copy the diagram above, and on it draw vectors representing $\underset{\sim}{b} - \underset{\sim}{a}$ and $\underset{\sim}{b} + \underset{\sim}{c}$.

9 From the diagram to the right, write down a single vector equal to:

- **a** $\overrightarrow{AC} + \overrightarrow{CD}$
- **b** $\overrightarrow{BC} + \overrightarrow{CA}$
- **c** $\overrightarrow{AD} - \overrightarrow{AB}$
- **d** $\overrightarrow{AC} - \overrightarrow{BC}$



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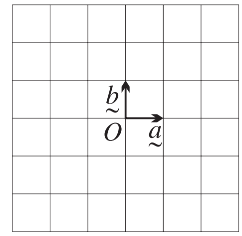
10 Copy the diagram to the right, and on it show *position vectors* representing:

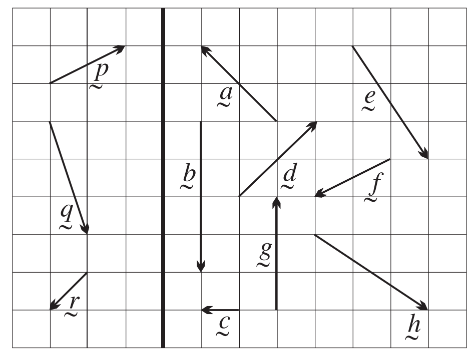


a $2\underset{\sim}{a} + 3\underset{\sim}{b}$

b $3\underset{\sim}{a} - \underset{\sim}{b}$

c $-2\underset{\sim}{a} - 2\underset{\sim}{b}$

11 Find the vector in the diagram to the right that represents:



a $-\underset{\sim}{p}$

b $-2\underset{\sim}{r}$

c $\underset{\sim}{p} + \underset{\sim}{q}$

d $\underset{\sim}{q} + \underset{\sim}{r}$

e $-\underset{\sim}{p} - \underset{\sim}{r}$

f $-\underset{\sim}{q} + \underset{\sim}{r}$

g $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r}$

h $\underset{\sim}{p} - \underset{\sim}{q} + \underset{\sim}{r}$

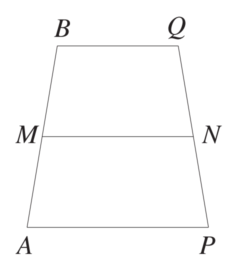
DEVELOPMENT

12 In $\triangle ABC$, the point M is the midpoint of BC. If $\overrightarrow{AC} = \underset{\sim}{u}$ and $\overrightarrow{BC} = \underset{\sim}{v}$, write in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$:

a $\overrightarrow{AB}$

b $\overrightarrow{AM}$

13 In the diagram, M and N are the midpoints of AB and PQ. Let $\underset{\sim}{u} = \overrightarrow{AM}$, $\underset{\sim}{v} = \overrightarrow{PN}$, $\underset{\sim}{a} = \overrightarrow{AP}$, $\underset{\sim}{b} = \overrightarrow{BQ}$ and $\underset{\sim}{p} = \overrightarrow{MN}$.



a Explain why $\overrightarrow{MB} = \underset{\sim}{u}$ and $\overrightarrow{NQ} = \underset{\sim}{v}$.

b Find $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

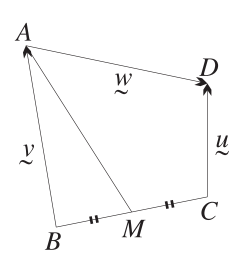
c Hence show that $\underset{\sim}{a} + \underset{\sim}{b} = 2\underset{\sim}{p}$.

14 In $\triangle ABC$, the point P lies on BC so that $BP:PC = 1:2$. Let $\overrightarrow{AB} = \underset{\sim}{p}$, $\overrightarrow{AC} = \underset{\sim}{q}$ and $\overrightarrow{AP} = \underset{\sim}{r}$.

a Draw a diagram, then write $\overrightarrow{BC}$, and then $\overrightarrow{BP}$, in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

b Prove that $\underset{\sim}{r} = \frac{2}{3}\underset{\sim}{p} + \frac{1}{3}\underset{\sim}{q}$.

15 In the diagram, $ABCD$ is a quadrilateral and M is the midpoint of BC. Let $\overrightarrow{CD} = \underset{\sim}{u}$, $\overrightarrow{BA} = \underset{\sim}{v}$ and $\overrightarrow{AD} = \underset{\sim}{w}$.



a Find $\overrightarrow{MB}$ in terms of $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

b Hence find $\overrightarrow{MA}$ in terms of $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

16 Suppose that $WXYZ$ is a quadrilateral, and that P, Q and R are the midpoints of WX, YZ and PQ respectively. Let $\overrightarrow{RW} = \underset{\sim}{w}$, $\overrightarrow{RX} = \underset{\sim}{x}$, $\overrightarrow{RY} = \underset{\sim}{y}$ and $\overrightarrow{RZ} = \underset{\sim}{z}$.

a Write $\overrightarrow{WX}$, and then $\overrightarrow{WP}$, in terms of $\underset{\sim}{w}$ and $\underset{\sim}{x}$.

b Find $\overrightarrow{RP}$ in terms of $\underset{\sim}{w}$ and $\underset{\sim}{x}$.

c Find $\overrightarrow{RQ}$ in terms of $\underset{\sim}{y}$ and $\underset{\sim}{z}$.

d Deduce that $\underset{\sim}{w} + \underset{\sim}{x} + \underset{\sim}{y} + \underset{\sim}{z} = \underset{\sim}{0}$.

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17 Suppose that P is a point on the line AB such that $\overrightarrow{AP} = k\overrightarrow{AB}$, where k is a constant.

a Show on a diagram the position of P relative to A and B if:

i $k > 1$ **ii** $0 < k < 1$ **iii** $k < 0$

b Find the value of k for which:

i $\overrightarrow{AP} = \frac{3}{2}\overrightarrow{PB}$ **ii** $\overrightarrow{AP} = -\frac{3}{2}\overrightarrow{PB}$ **iii** $\overrightarrow{AP} = -\frac{2}{3}\overrightarrow{PB}$

c Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{p}$ be the respective position vectors of the points A, B and P relative to an origin O. Prove that $\underset{\sim}{p} = (1 - k)\underset{\sim}{a} + k\underset{\sim}{b}$.

18 Just above Box 4 of the text, we claimed that the distributive law $\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$ is easily proven using the diagram. Explain this proof.

19 **a** Why is there only one zero vector?

b If you were the dictator of the world and you could put the origin anywhere on the surface of the Earth, where would you put it and why? What 'origins' have been chosen in the past, and which are still used?

ENRICHMENT

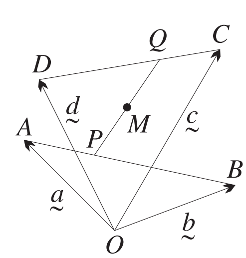
20 In $\triangle ABC$, X is the midpoint of BC, Y is the midpoint of AC and P is the point on AX such that $AP:PX = 2:1$. Let $\overrightarrow{AB} = \underset{\sim}{u}$ and $\overrightarrow{AC} = \underset{\sim}{v}$.

a Show that $\overrightarrow{PY} = \frac{1}{6}(\underset{\sim}{v} - 2\underset{\sim}{u})$.

b Hence show that the points B, P and Y are collinear.

c What geometric theorem have you proven?

21 In the diagram, the points A, B, C and D have respective position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ relative to an origin O. The point P divides AB in the ratio $1:3$, Q divides DC in the ratio $3:1$ and M is the midpoint of PQ.



a Show that P has position vector $\frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$.

b Express $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$.

c Hence show that M has position vector $\frac{1}{8}(3\underset{\sim}{a} + \underset{\sim}{b} + 3\underset{\sim}{c} + \underset{\sim}{d})$.

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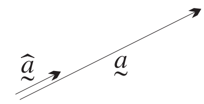
8B Components and column vectors

This section introduces *components*, which are another way to specify a vector. The components can be stacked in a *column vector*, which again allows a vector to be dealt with as a single object.

First, however, we need to define *unit vectors* and *perpendicular vectors*.

Unit vectors



A *unit vector* is a vector $\underset{\sim}{a}$ of length 1 , that is, $|\underset{\sim}{a}| = 1$. Every non-zero vector $\underset{\sim}{a}$ has a *corresponding unit vector* $\hat{\underset{\sim}{a}}$ with the same direction, namely

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \text{ the vector } \underset{\sim}{a} \text{ divided by its length } |\underset{\sim}{a}|.$$

There are two unit vectors parallel to $\underset{\sim}{a}$, namely $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ and $-\hat{\underset{\sim}{a}} = -\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

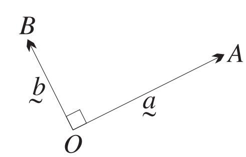
The first has the same direction as $\underset{\sim}{a}$, and the second has the opposite direction.

6 UNIT VECTORS

- A vector of length 1 is called a *unit vector*.
- If $\underset{\sim}{a} \neq \underset{\sim}{0}$, the vector $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ is a unit vector with the same direction as $\underset{\sim}{a}$,

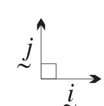
  and the vector $-\hat{\underset{\sim}{a}} = -\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ is a unit vector with the opposite direction.

Perpendicular vectors



Two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are called *perpendicular* if their directions are perpendicular. That is, when they are represented as $\underset{\sim}{a} = OA$ and $\underset{\sim}{b} = OB$ with common tail O, then the angle $\angle AOB$ is a right angle.

Choosing a basis



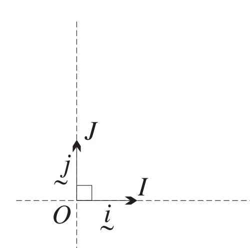
Choose a fixed *basis* consisting of two vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ such that:

- $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are both unit vectors,
- $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are perpendicular.

The correct name is *orthonormal basis*, but we will not be dealing with any other types of basis, so we shall omit the qualification.

Choosing an origin and forming axes

In Section 8A we described choosing a reference point or origin O in the plane. Then each point P in the plane has position vector $\overrightarrow{OP}$, and each vector $\underset{\sim}{u}$, when drawn with tail O as $\underset{\sim}{u} = \overrightarrow{OA}$, is the position vector of the head A.



In particular, we can represent $\underset{\sim}{i} = \overrightarrow{OI}$ and $\underset{\sim}{j} = \overrightarrow{OJ}$ as position vectors, thus forming a pair of *axes* at right angles. Each axis then becomes a number line by assigning 0 to the origin O and 1 to the point I and J.

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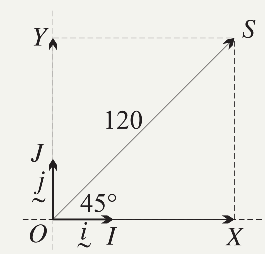
7 CHOOSING A BASIS AND AN ORIGIN, AND FORMING AXES

- Choose a *basis* consisting of perpendicular unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.
- Choose a point O as the *origin*.
- The basis vectors can now be drawn as position vectors $\underset{\sim}{i} = \overrightarrow{OI}$ and $\underset{\sim}{j} = \overrightarrow{OJ}$.
- The basis and the origin thus form a pair of *axes* OI and OJ.

The next worked example prepares the way for the central idea of components.

Example 5 8B

A ship S leaves a port O and sails north-east at 20 km/h. Choose a convenient origin and axes. Then describe its position after 6 hours by giving its position vector as a sum of multiples of the basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.



SOLUTION

The obvious choice of origin is the port O, and one obvious choice of basis vectors is to take $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the directions east and north. Then after 6 hours, the ship is 120 km from O.
Complete the rectangle $OXSY$, where X lies on OI and Y lies on OJ.

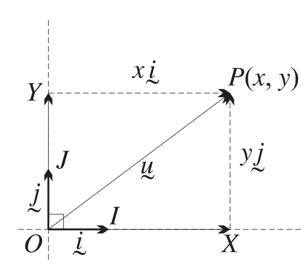
Then by simple trigonometry, $\overrightarrow{OX} = 60\sqrt{2}\underset{\sim}{i}$ and $\overrightarrow{OY} = 60\sqrt{2}\,\underset{\sim}{j}$.

The position vector $\overrightarrow{OS}$ is the vector sum of $\overrightarrow{OX}$ and $\overrightarrow{OY}$,

so $\quad \overrightarrow{OS} = 60\sqrt{2}\underset{\sim}{i} + 60\sqrt{2}\,\underset{\sim}{j}$.

Note: There is no problem with exchanging the chosen basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ so that $\underset{\sim}{i}$ points north and $\underset{\sim}{j}$ points east — in fact this would probably conform better to the conventions of navigation. In mathematics, however, we tend to take a basis in which $\underset{\sim}{j}$ is the vector $\underset{\sim}{i}$ rotated 90° anti-clockwise.

The components of a vector



We can quickly generalise the construction in the previous worked example. Suppose that we have an origin and basis, then take any vector $\underset{\sim}{u}$, and represent it as a position vector $\underset{\sim}{u} = \overrightarrow{OP}$. Complete the rectangle $PXOY$, where X lies on OI and Y lies on OJ.

Let $\overrightarrow{OX} = x\underset{\sim}{i}$ and $\overrightarrow{OY} = y\underset{\sim}{j}$. Then x and y are called the *scalar components* of the vector $\underset{\sim}{u}$, and we can write

$$P = (x, y) \qquad \text{and} \qquad \underset{\sim}{u} = \overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}.$$

This last form $\underset{\sim}{u} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$ is called the *component form* of the vector $\underset{\sim}{u}$, and the terms $x\,\underset{\sim}{i}$ and $y\underset{\sim}{j}$ are called the *vector components* of $\underset{\sim}{u}$. The rectangle $PXOY$ represents taking the sum of the two vector components $x\,\underset{\sim}{i}$ and $y\underset{\sim}{j}$.

The unqualified term *component* may mean either scalar or vector component. It is usually clear from the context which is intended.

These components are independent of the choice of the origin O, because a translation from O to a different choice O' of origin moves one figure onto the other. A different choice of basis vectors, however, would change the components.

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8 THE COMPONENTS OF A VECTOR

Suppose that a basis and an origin have been chosen, and let $\underset{\sim}{u}$ be any vector, represented as a position vector $\underset{\sim}{u} = \overrightarrow{OP}$.

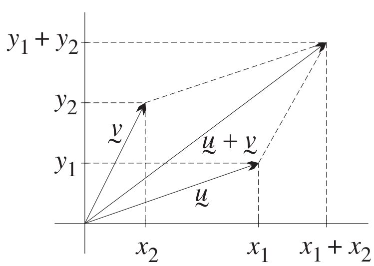
- Complete the rectangle $OXPY$, and let $\overrightarrow{OX} = x\underset{\sim}{i}$ and $\overrightarrow{OY} = y\underset{\sim}{j}$.
  Then x and y are called the *scalar components* of the vector $\underset{\sim}{u}$, and
  $$P = (x, y) \qquad \text{and} \qquad \underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j},$$
- The last form is called the *component form* of the vector $\underset{\sim}{u}$, and the vectors $\overrightarrow{OX} = x\underset{\sim}{i}$ and $\overrightarrow{OY} = y\underset{\sim}{j}$ are called the *vector components* of $\underset{\sim}{u}$.
- The rectangle $OXPY$ represents $\underset{\sim}{u}$ as the sum $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j}$.
- The components are independent of the choice of origin O, but depend very much on the choice of basis $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

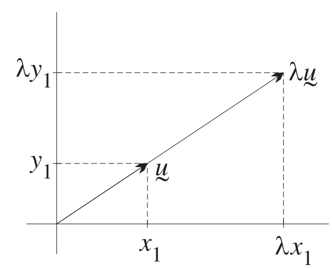
The point $P(x, y)$ naturally corresponds to the position vector $x\underset{\sim}{i} + y\underset{\sim}{j}$.

Two identities for components

Let $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ be vectors, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are a basis. Let λ be a scalar. Two identities are clear from the diagrams below them:

$$\underset{\sim}{u} + \underset{\sim}{v} = (x_1 + x_2)\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j} \qquad \text{and} \qquad \lambda\underset{\sim}{u} = (\lambda x_1)\underset{\sim}{i} + (\lambda y_1)\underset{\sim}{j}.$$





Alternatively, the identities can be proven using the distributive laws from Box 4 and the associative law from Box 3.

Column vector notation

Let $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ be a vector. We can now write $\underset{\sim}{u}$ as a *column vector*, placing its scalar components in a vertical stack,

$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad \text{or alternatively as} \qquad \underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

In column vector notation, the zero vector $\underset{\sim}{0}$ and the basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are

$$\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now let $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ be another vector and λ be a scalar.

The two identities for components above now become

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

and
$$\lambda\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda y_1 \end{bmatrix}.$$

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We can therefore write the vector $\underset{\sim}{u}$ in terms of the basis vectors and components as

$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(which in turn equals } x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\text{).}$$

9 COLUMN VECTORS

- Each vector $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ can be written as a *column vector*
$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{or} \quad \underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$
- The basis vectors are $\underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Addition and multiplication by a scalar are done component-wise:
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \quad \text{and} \quad \lambda\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda y_1 \end{bmatrix}.$$
- The vector $\underset{\sim}{u}$ can be written as a sum of the basis vectors as
$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

There is a natural correspondence between a point $P(x, y)$ and its position vector $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$, that is, between $P(x, y)$ and $\begin{bmatrix} x \\ y \end{bmatrix}$.

Column vectors can be written with square brackets or with round brackets, that is, as $\begin{bmatrix} x \\ y \end{bmatrix}$ or as $\begin{pmatrix} x \\ y \end{pmatrix}$, but we have mostly avoided the round brackets notation because of possible confusion with ${}^{x}\mathrm{C}_{y}$ in combinatorics, and because square brackets fit more neatly on a printed line.

Example 6 — 8B

a **i** Given $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$, find $2\underset{\sim}{u}$, $3\underset{\sim}{v}$, and $2\underset{\sim}{u} + 3\underset{\sim}{v}$.

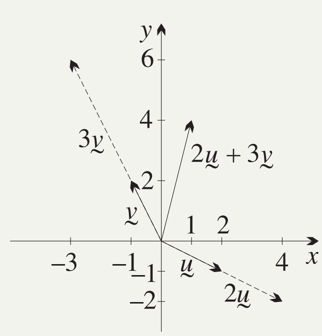
ii Draw all five vectors as position vectors. What figure do the origin and the heads of $2\underset{\sim}{u}$, $3\underset{\sim}{v}$, and $2\underset{\sim}{u} + 3\underset{\sim}{v}$ form?

b **i** Given $\underset{\sim}{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $\underset{\sim}{v} - \underset{\sim}{u}$.

ii Draw $\underset{\sim}{u} = \overrightarrow{OU}$ and $\underset{\sim}{v} = \overrightarrow{OV}$ as position vectors, and mark $\underset{\sim}{v} - \underset{\sim}{u}$.

SOLUTION

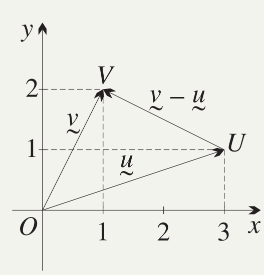
a



i $2\underset{\sim}{u} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$, $3\underset{\sim}{v} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$,
$2\underset{\sim}{u} + 3\underset{\sim}{v} = \underset{\sim}{i} + 4\underset{\sim}{j}$.

ii They form a parallelogram.

b



i $\underset{\sim}{v} - \underset{\sim}{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

ii $\underset{\sim}{v} - \underset{\sim}{u}$ has tail U and head V.

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Example 7 8B

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -5\underset{\sim}{i} + 4\underset{\sim}{j}$. Find λ and μ so that

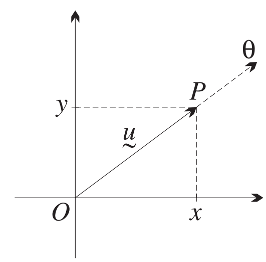
$$\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} = -4\underset{\sim}{i} - \underset{\sim}{j}.$$

SOLUTION

We need simultaneous equations for the solution.

$$\begin{array}{lrcll}
\text{First,} & \lambda\underset{\sim}{a} + \mu\underset{\sim}{b} & = & (2\underset{\sim}{i} - 3\underset{\sim}{j}) + \mu(-5\underset{\sim}{i} + 4\underset{\sim}{j}) & \\
 & & = & (2\lambda - 5\mu)\underset{\sim}{i} + (-3\lambda + 4\mu)\underset{\sim}{j}). & \\
\text{Hence} & 2\lambda - 5\mu & = & -4 & (1) \\
\text{and} & -3\lambda + 4\mu & = & -1. & (2) \\
\text{Taking } (1) \times 4, & 8\lambda - 20\mu & = & -16 & (1A) \\
\text{and } (2) \times 5, & -15\lambda + 20\mu & = & -5. & (2A) \\
\text{Adding } (1A) + (2A), & -7\lambda & = & -21 & \\
 & \lambda & = & 3, & \\
\text{and substituting into } (2), & -9 + 4\mu & = & -1 & \\
 & \mu & = & 2. &
\end{array}$$

Length and angle



When there are an origin and axes, we can import all the trigonometry from Chapter 6 of the Year 11 book. Let $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a non-zero vector, represented in the diagram to the right as a position vector $\overrightarrow{OP}$, and let θ be the angle of the ray OP.

- The length and angle are given by Pythagoras' theorem and trigonometry,

  $$|\underset{\sim}{u}|^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

  There will be two answers for θ, oriented in opposite directions, so the quadrant will need to be identified to distinguish between them.
- Conversely, the components of the vector can be recovered from θ and $|\underset{\sim}{u}|$,

  $$x = |\underset{\sim}{u}|\cos\theta \qquad \text{and} \qquad y = |\underset{\sim}{u}|\sin\theta.$$

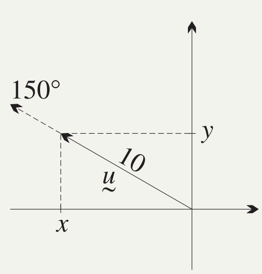
Example 8 8B

a Find the vector $\underset{\sim}{u}$ of length 10 whose position vector has angle 150°.

b Find the length and angle (nearest degree) of $\underset{\sim}{v} = -\underset{\sim}{i} - 2\underset{\sim}{j}$.

SOLUTION

a



$$\begin{aligned}
\underset{\sim}{u} &= (10\cos 150°)\underset{\sim}{i} + (10\sin 150°)\underset{\sim}{j} \\
&= 10 \times \left(-\frac{\sqrt{3}}{2}\right)\underset{\sim}{i} + 10 \times \tfrac{1}{2}\underset{\sim}{j} \\
&= -5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}
\end{aligned}$$

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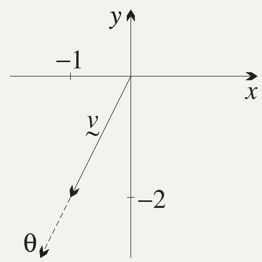
b



$$|\underset{\sim}{u}|^2 = (-1)^2 + (-2)^2 = 5$$
$$|\underset{\sim}{u}| = \sqrt{5}$$
$$\tan\theta = \frac{-2}{-1} = 2$$
$$\theta \doteqdot 63^\circ + 180^\circ \quad \text{(third quadrant)}$$
$$\doteqdot 243^\circ$$

10 LENGTH AND ANGLE OF POSITION VECTORS

Let $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a non-zero position vector relative to a chosen origin and basis, and let $\underset{\sim}{u}$ have trigonometric angle θ. Then

- $x = |\underset{\sim}{u}|\cos\theta$ and $y = |\underset{\sim}{u}|\sin\theta$,
- $|\underset{\sim}{u}|^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$ (the two answers for θ need to be distinguished).

Exercise 8B

FOUNDATION

1 If $\underset{\sim}{a} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$, find:

a $|\underset{\sim}{a}|$ **b** $2\underset{\sim}{a}$ **c** $|2\underset{\sim}{a}|$ **d** $-5\underset{\sim}{a}$ **e** $|-5\underset{\sim}{a}|$

2 Suppose that $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$. Find:

a $\underset{\sim}{a} + \underset{\sim}{b}$ **b** $|\underset{\sim}{a} + \underset{\sim}{b}|$ **c** $\underset{\sim}{a} - \underset{\sim}{b}$ **d** $|\underset{\sim}{a} - \underset{\sim}{b}|$ **e** $-3\underset{\sim}{a} - 2\underset{\sim}{b}$ **f** $|-3\underset{\sim}{a} - 2\underset{\sim}{b}|$

3 Given that $\underset{\sim}{a} = \begin{bmatrix} -17 \\ 3 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} -7 \\ -13 \end{bmatrix}$, find:

a $\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}$ **b** $|\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}|$ **c** $-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c}$ **d** $|-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c}|$

4 **a** For $\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$, write $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{u} + \underset{\sim}{v}$ as column vectors, and sketch them as position vectors.

b For $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, write $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ in component form and sketch them as position vectors. Also sketch $\underset{\sim}{a} - \underset{\sim}{b}$ as the vector subtraction of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

5 If $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{v} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{w} = \underset{\sim}{a} + \underset{\sim}{b}$, find:

a $\hat{\underset{\sim}{u}}$ **b** $\hat{\underset{\sim}{v}}$ **c** $\hat{\underset{\sim}{w}}$

6 Given $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -12 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -15 \\ -8 \end{bmatrix}$, show that:

a $|2\underset{\sim}{a}| = 2|\underset{\sim}{a}|$ **b** $|-\underset{\sim}{b}| = |\underset{\sim}{b}|$ **c** $|\underset{\sim}{a} + \underset{\sim}{b}| < |\underset{\sim}{a}| + |\underset{\sim}{b}|$ **d** $|\underset{\sim}{a} + \underset{\sim}{b}| > \big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big|$

7 **a** Test whether the points P, Q and R are collinear if

$$\overrightarrow{OP} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \quad \text{and} \quad \overrightarrow{OQ} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{OR} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}.$$

b Test whether the points A, B and C are collinear if

$$\overrightarrow{OA} = 3\underset{\sim}{i} + 8\underset{\sim}{j} \quad \text{and} \quad \overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j} \quad \text{and} \quad \overrightarrow{OC} = -4\underset{\sim}{i} - \underset{\sim}{j}.$$

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8 Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{m}$ be the respective position vectors representing the points $A(4,-7)$, $B(6,3)$ and M, where M is the midpoint of AB. Write in component form:

a $\underset{\sim}{a}$ **b** $\underset{\sim}{b}$ **c** $\underset{\sim}{m}$

9 Suppose that P is the point $(4,-1)$, Q is the point $(-3,5)$ and O is the origin. Write as column vectors:

a $\overrightarrow{OP}$ **b** $2\,\overrightarrow{OP} - \overrightarrow{OQ}$ **c** $\overrightarrow{PQ}$ **d** $\overrightarrow{QP}$

10 If A and B are the points $(1,-1)$ and $(7,3)$ respectively, find:

a $\overrightarrow{AB}$ in component form,
b $|AB|$,
c the unit vector in the direction of $\overrightarrow{AB}$.

DEVELOPMENT

11 Find the magnitude and direction (as an angle of inclination) of each vector.

a $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$ **b** $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ **c** $\underset{\sim}{c} = -3\sqrt{3}\underset{\sim}{i} + 3\underset{\sim}{j}$ **d** $\underset{\sim}{d} = \begin{bmatrix} -\sqrt{6} \\ -\sqrt{6} \end{bmatrix}$

12 Write, in component form, a vector whose magnitude and direction are:

a 4 and $-\frac{\pi}{4}$, **b** $2\sqrt{6}$ and $\frac{2\pi}{3}$, **c** 2 and $-\frac{5\pi}{6}$, **d** $2\sqrt{2}$ and $\frac{5\pi}{12}$.

13 Given that $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$, find the values of λ_1 and λ_2 such that $\underset{\sim}{c} = \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b}$.

14 The triangle ABC has vertices $A(2\sqrt{3},3)$, $B(3\sqrt{3},4)$ and $C(2\sqrt{3},5)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{CB}$ as column vectors.
b Hence find $|\overrightarrow{AB}|$ and $|\overrightarrow{CB}|$.
c What type of special triangle is ABC? Why?

15 The quadrilateral $ABCD$ has vertices $A(-7,-5)$, $B(8,-2)$, $C(10,9)$ and $D(-5,6)$. Show that $\overrightarrow{AD} = \overrightarrow{BC}$, and hence explain why $ABCD$ is a parallelogram.

16 The quadrilateral $PQRS$ has vertices $P(-3,-4)$, $Q(2,-2)$, $R(4,3)$ and $S(-1,1)$.

a Show that $\overrightarrow{PQ} = \overrightarrow{SR}$.
b Show that $|\overrightarrow{PQ}| = |\overrightarrow{QR}|$.
c What type of special quadrilateral is $PQRS$? Give reasons for your answer.

17 The quadrilateral $OABC$ has vertices $O(0,0)$, $A(5,-3)$, $B(7,4)$ and $C(2,7)$.

a Show that $\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{OB}$.
b What type of special quadrilateral is $OABC$? Give a reason for your answer.

18 The parallelogram $WXYZ$ has vertices $W(-6,4)$, $X(6,2)$, $Y(4,9)$ and $Z(a,b)$. Use vectors to find the values of a and b.

ENRICHMENT

19 The points A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. Find the three possible position vectors representing the point D if the points A, B, C and D are the vertices of a parallelogram.

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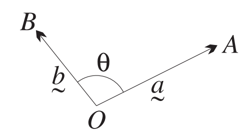
8C The dot product (or scalar product)

This section introduces an important product associated with vectors. The *dot product* or *scalar product* $\underset{\sim}{a} \cdot \underset{\sim}{b}$ of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is a scalar — the dot here is a raised dot. The dot product is closely associated with the cosine rule.

In Section 8E, we will introduce *projections*, and see that the dot product is also closely related to the projection of one vector onto another.

The angle between two non-zero vectors



To define the *angle between two non-zero vectors* $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$, represent the vectors as $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with the same tail. The non-reflex angle $\angle AOB$ is taken as the angle between the vectors.

This definition is independent of the choice of O, because if P is any other point, then the translation from O to P maps the whole figure across.

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be non-zero vectors with angle θ between them. Here are some immediate consequences of the definition.

- If λ is a positive scalar, then the angle between $\lambda\underset{\sim}{a}$ and $\underset{\sim}{a}$ is $0°$.
- If λ is a negative scalar, then the angle between $\lambda\underset{\sim}{a}$ and $\underset{\sim}{a}$ is $180°$.
- $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction if and only if the angle between them is $0°$.
- $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have opposite directions if and only if the angle between them is $180°$.
- The angle between $-\underset{\sim}{a}$ and $-\underset{\sim}{b}$ is θ.
- The angle between $\underset{\sim}{a}$ and $-\underset{\sim}{b}$, and between $-\underset{\sim}{a}$ and $\underset{\sim}{b}$, is $180° - \theta$.
- The angle between the unit vectors $\hat{\underset{\sim}{a}}$ and $\hat{\underset{\sim}{b}}$ is θ.

There is no meaning to the angle between a vector and the zero vector.

11 THE ANGLE BETWEEN TWO NON-ZERO VECTORS

Represent two non-zero vectors $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ with a common tail. Then the non-reflex angle $\angle AOB$ is called the *angle between the two vectors*.

The dot product or scalar product — geometric formula

There are two equivalent formulae for the dot product of two vectors. We will begin with the geometric formula using the cosine function, and then develop an alternative formula using the components of the vectors.

The *dot product* or *scalar product* of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined geometrically using the same diagram that was used above for the angle between two vectors.

- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are both non-zero vectors, then $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is the product of their lengths times the cosine of the angle between them,

  $$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \times \cos\theta, \quad \text{where } \theta \text{ is the angle between the vectors.}$$

- If either a or b is the zero vector, then the dot product is zero,

  $$\underset{\sim}{a} \cdot \underset{\sim}{0} = 0 \quad \text{and} \quad \underset{\sim}{0} \cdot \underset{\sim}{b} = 0.$$

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Hence for non-zero vectors, $\utilde{a} \cdot \utilde{b}$ is positive if and only if θ is acute or $0°$, and $\utilde{a} \cdot \utilde{b}$ is negative if and only if θ is obtuse or $180°$.

Notice also that $\cos 90° = 0$, so the dot product of two perpendicular vectors is zero. Most importantly, the dot product of two basis vectors is zero.

12 THE DOT PRODUCT OR SCALAR PRODUCT OF TWO VECTORS — GEOMETRIC FORMULA

- Define the dot or scalar product $\utilde{a} \cdot \utilde{b}$ of two vectors $\utilde{a}$ and $\utilde{b}$ as follows.
  — If both vectors are non-zero and the angle between them is θ, define
  $$\utilde{a} \cdot \utilde{b} = |\utilde{a}| \times |\utilde{b}| \times \cos\theta\,.$$
  — If $\utilde{a}$ or $\utilde{b}$ is the zero vector, define $\utilde{a} \cdot \utilde{b} = 0$.
- For all vectors $\utilde{a}$ and $\utilde{b}$ and all scalars λ,
  Perpendicular: $\utilde{a} \cdot \utilde{b} = 0$ if and only if $\utilde{a} = \utilde{0}$ or $\utilde{b} = \utilde{0}$ or $\utilde{a} \perp \utilde{b}$.
  Length: $\utilde{a} \cdot \utilde{a} = |\utilde{a}|^2$, which means that $|\utilde{a}| = \sqrt{\utilde{a} \cdot \utilde{a}}$.
  Commutative law: $\utilde{a} \cdot \utilde{a} = \utilde{b} \cdot \utilde{a}$
  Associative law: $\lambda(\utilde{a} \cdot \utilde{b}) = (\lambda\utilde{a}) \cdot \utilde{b}$
  Distributive law: $\utilde{a} \cdot (\utilde{b} + \utilde{c}) = \utilde{a} \cdot \utilde{b} + \utilde{a} \cdot \utilde{c}$

The second dotpoint contains five important laws for the dot product. The first four are all immediate consequences of the definition. The last is a distributive law, which will be easier to prove after we have developed the component formula.

The product $\utilde{a} \cdot \utilde{b}$ has two names — *dot product* and *scalar product*.
- It is called the *dot product* because the notation is a raised dot.
- It is called *scalar product* because the answer is a scalar.

(Naming $\utilde{u} \cdot \utilde{v}$ the 'dot product' after its notation links it with another vector product called the 'cross product' $\utilde{u} \times \utilde{v}$, which you will meet in later years.)

Be careful: If you decide to use the term 'scalar product', do not confuse $\lambda\utilde{v}$, which is 'multiplication by a scalar', with the 'scalar product' $\utilde{u} \cdot \utilde{v}$.

Example 9 8C

An isosceles right-angled triangle $\triangle ABC$ has sides of length $|AB| = 3\sqrt{2}$ and $|BC| = |CA| = 3$.
Let $\utilde{a} = \overrightarrow{AB}$, $\utilde{b} = \overrightarrow{BC}$ and $\utilde{c} = \overrightarrow{CA}$.

a Find:
i $\utilde{a} \cdot \utilde{a}$ **ii** $\utilde{a} \cdot \utilde{b}$ **iii** $3\utilde{a} \cdot (-2\utilde{c})$ **iv** $\utilde{c} \cdot (\utilde{b} - \utilde{a})$

b Without expanding the brackets, find $(\utilde{a} + \utilde{b} + \utilde{c}) \cdot (\utilde{a} + \utilde{b} + \utilde{c})$.

c Without calculating LHS and RHS, explain why $(\utilde{a} + \utilde{b}) \cdot \utilde{c} = \utilde{a} \cdot \utilde{c}$.

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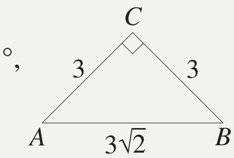
SOLUTION



We use the formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}| \cos\theta$. Notice that the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 135°, not 45°, and $\cos 135° = -\dfrac{1}{\sqrt{2}}$. Similarly, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{c}$ is 135°.

a **i** $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$
$\qquad = 18$

ii $\underset{\sim}{a} \cdot \underset{\sim}{b} = 3\sqrt{2} \times 3 \times \cos 135°$
$\qquad = -9$

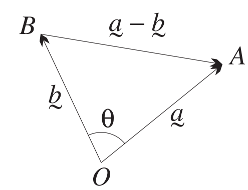
iii $3\underset{\sim}{a} \cdot (-2\underset{\sim}{c}) = -6 \times 3\sqrt{2} \times 3 \times \cos 135°$
$\qquad = 54$

iv $\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{c} \cdot \underset{\sim}{a}$
$\qquad = 0 - (-9)$
$\qquad = 9$

b Because $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, it follows that $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}).(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{0} \cdot \underset{\sim}{0} = 0$.

c $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c}$
$\qquad = \underset{\sim}{a} \cdot \underset{\sim}{c}$, because $\underset{\sim}{b} \perp \underset{\sim}{c}$.

The dot product, the cosine rule and Pythagoras' theorem



Let OAB be a triangle, and let $\theta = \angle AOB$.
Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$. Then the third side represents the vector $\underset{\sim}{a} - \underset{\sim}{b} = \overrightarrow{BA}$.
The cosine rule tells us that

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA|\,|OB| \cos\theta$$
$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$$

Replacing $|\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$ by $\underset{\sim}{a} \cdot \underset{\sim}{b}$,

$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}.$$

This can be solved for $\underset{\sim}{a} \cdot \underset{\sim}{b}$ to obtain another formula for the dot product,

$$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a} - \underset{\sim}{b}|^2.$$

We remarked in Section 6J of the Year 11 book that the cosine rule can be regarded as 'Pythagoras' theorem with an error term'. This last formula now characterises the dot product as half that error term.

13 THE DOT PRODUCT, THE COSINE RULE AND PYTHAGORAS' THEOREM

The cosine rule can be written in vector form as

$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}.$$

This can be solved for $\underset{\sim}{a} \cdot \underset{\sim}{b}$ to obtain another formula for the dot product,

$$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a} - \underset{\sim}{b}|^2.$$

Thus the dot product is half the error term in Pythagoras' theorem.

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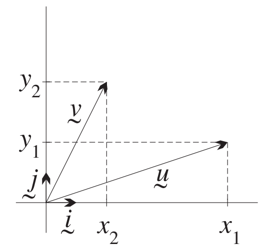
The dot product — component formula



Now suppose that we have a basis $\underset{\sim}{i}$ and $\underset{\sim}{j}$ and an origin O, and that

$$\underset{\sim}{u} = x_1 \underset{\sim}{i} + y_1 \underset{\sim}{j} \qquad \text{and} \qquad \underset{\sim}{v} = x_2 \underset{\sim}{i} + y_2 \underset{\sim}{j}.$$

Using the formula in Box 13 above,

$$2\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u} - \underset{\sim}{v}|^2,$$

and applying Pythagoras' theorem in the form of the distance formula,

$$2\underset{\sim}{u} \,.\, \underset{\sim}{v} = \left(x_1{}^2 + y_1{}^2\right) + \left(x_2{}^2 + y_2{}^2\right) - \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right)$$
$$= 2x_1x_2 + 2y_1y_2 \qquad \text{(after expanding)},$$

so $\quad \underset{\sim}{u} \,.\, \underset{\sim}{v} = x_1x_2 + y_1y_2. \qquad (*)$

In words, take the product of the scalar components in the $\underset{\sim}{i}$ direction, and add the product of the scalar components in the $\underset{\sim}{j}$ direction.

In column vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1x_2 + y_1y_2.$

Conversely, we could have taken the component formula as the definition of the dot product, and worked backwards through the working above to prove the original geometric formula.

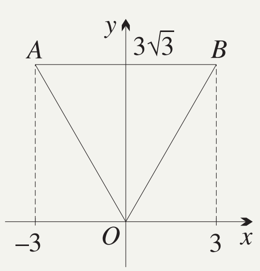
- If you are working with a basis $\underset{\sim}{i}$ and $\underset{\sim}{j}$, use the component formula.
- If you are working in a plane with no obvious basis, use the geometric form (or perhaps the error-in-Pythagoras'-theorem form).

14 THE DOT PRODUCT OR SCALAR PRODUCT — COMPONENT FORM

Let $\underset{\sim}{u} = x_1 \underset{\sim}{i} + y_1 \underset{\sim}{j}$ and $\underset{\sim}{v} = x_2 \underset{\sim}{i} + y_2 \underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are a basis.

- The dot product can be written using components as
  $$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2\,.$$
- Alternatively, using column vector notation,
  $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1x_2 + y_1y_2\,.$$
- This component form of the dot product is equivalent to the geometric form, and either can be taken as the definition of the dot product.

Example 10 8C



The diagram to the right shows a triangle OAB. Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ be the position vectors of A and B.

a Find $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\overrightarrow{AB}$ as column vectors.

b Using the component formula for the dot product, find:

i $\underset{\sim}{a} \cdot \underset{\sim}{a}$ **ii** $\underset{\sim}{a} \cdot \underset{\sim}{b}$

iii $\underset{\sim}{a} \cdot \overrightarrow{AB}$ **iv** $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \overrightarrow{AB}$

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SOLUTION

a $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 3 \\ 3\sqrt{3} \end{bmatrix}, \overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$

b **i**
$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{a} &= \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \\ &= 9 + 27 \\ &= 36. \end{aligned}$$

ii
$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3\sqrt{3} \end{bmatrix} \\ &= -9 + 27 \\ &= 18. \end{aligned}$$

iii
$$\begin{aligned} \underset{\sim}{a} \cdot \overrightarrow{AB} &= \begin{bmatrix} -3 \\ 3\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix} \\ &= -18 + 0 \\ &= -18. \end{aligned}$$

iv
$$\begin{aligned} (\underset{\sim}{a} + \underset{\sim}{b}) \cdot \overrightarrow{AB} &= \begin{bmatrix} 0 \\ 6\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix} \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Note: We suggest that the reader now prove that $\triangle ABC$ is equilateral, and then recalculate each part using the geometric formula for the dot product.

Proving the distributive law for dot product

We can now use the components and column vector notation to prove the distributive law stated in Box 12.

Let $\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ and $\underset{\sim}{w} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$.

$$\begin{aligned} \text{Then } \underset{\sim}{u} \cdot (\underset{\sim}{v} + \underset{\sim}{w}) &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 + x_3 \\ y_2 + y_3 \end{bmatrix} \\ &= x_1(x_2 + x_3) + y_1(y_2 + y_3) \\ &= (x_1x_2 + x_1x_3) + (y_1y_2 + y_1y_3) \\ &= (x_1x_2 + y_1y_2) + (x_1x_3 + y_1y_3) \\ &= \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}\,. \end{aligned}$$

Writing the components of a vector using the dot product

First, the basis vectors each have length 1 and are perpendicular, so

$$\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = 1 \qquad \text{and} \qquad \underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{i} = 0.$$

Now we can write the scalar components of any vector u using the dot product.

Using the distributive law and the dot products of the basis vectors,

$$\begin{aligned} \underset{\sim}{u} \cdot i &= (x\underset{\sim}{i} + yj) \cdot \underset{\sim}{i} \\ &= x(\underset{\sim}{i} \cdot \underset{\sim}{i}) + y(\underset{\sim}{j} \cdot \underset{\sim}{i}) \\ &= x\,. \end{aligned} \qquad \text{and} \qquad \begin{aligned} \underset{\sim}{u} \cdot j &= (x\underset{\sim}{i} + y\underset{\sim}{j}) \cdot \underset{\sim}{j} \\ &= x(\underset{\sim}{i} \cdot \underset{\sim}{j}) + y(\underset{\sim}{j} \cdot \underset{\sim}{j}) \\ &= y. \end{aligned}$$

Thus the scalar components of $\underset{\sim}{u}$ are $x = \underset{\sim}{u} \cdot \underset{\sim}{i}$ and $y = \underset{\sim}{u} \cdot \underset{\sim}{j}$, and

$$\underset{\sim}{u} = (\underset{\sim}{u} \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{u} \cdot \underset{\sim}{j})\underset{\sim}{j}, \quad \text{that is,} \quad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u} \cdot \underset{\sim}{i} \\ \underset{\sim}{u} \cdot \underset{\sim}{j} \end{bmatrix}.$$

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15 WRITING THE COMPONENTS OF A VECTOR USING THE DOT PRODUCT

- Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be a chosen basis. Then
  $$\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = 1 \quad \text{and} \quad \underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{i} = 0\,.$$
- Let $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j}$ be any vector.
  Then the scalar components of $\underset{\sim}{u}$ are $x = \underset{\sim}{u} \cdot \underset{\sim}{i}$ and $y = \underset{\sim}{u} \cdot \underset{\sim}{j}$, and
  $$\underset{\sim}{u} = (\underset{\sim}{u} \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{u} \cdot \underset{\sim}{j})\underset{\sim}{j}, \quad \text{that is,} \quad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u} \cdot \underset{\sim}{i} \\ \underset{\sim}{u} \cdot \underset{\sim}{j} \end{bmatrix}.$$

Exercise 8C

FOUNDATION

1 Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$ given:

a $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ **b** $\underset{\sim}{a} = \begin{bmatrix} -8 \\ -5 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 6 \\ -14 \end{bmatrix}$

c $\underset{\sim}{a} = \begin{bmatrix} 6u \\ -2v \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 3v \\ 9u \end{bmatrix}$ **d** $\underset{\sim}{a} = \begin{bmatrix} x-1 \\ x-2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} x-1 \\ x+2 \end{bmatrix}$

2 If θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$ given:

a $|\underset{\sim}{a}| = 6, |\underset{\sim}{b}| = 5$ and $\theta = 60°$, **b** $|\underset{\sim}{a}| = 4, |\underset{\sim}{b}| = 3$ and $\theta = 45°$.

3 Find the angle θ between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ (nearest degree if necessary) if

a $|\underset{\sim}{u}| = 4, |\underset{\sim}{v}| = 5$ and $\underset{\sim}{u} \cdot \underset{\sim}{v} = -10$, **b** $|\underset{\sim}{u}| = 3, |\underset{\sim}{v}| = 5$ and $\underset{\sim}{u} \cdot \underset{\sim}{v} = 12$.

4 Write down the value of:

a $4\underset{\sim}{i} \cdot 2\underset{\sim}{j}$ **b** $-5\underset{\sim}{i} \cdot 3\underset{\sim}{j}$ **c** $4\underset{\sim}{i} \cdot 2\underset{\sim}{i}$ **d** $-5\underset{\sim}{j} \cdot 3\underset{\sim}{j}$

5 Calculate the value of:

a $(4\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot (4\underset{\sim}{i} + 2\underset{\sim}{j})$ **b** $(-5\underset{\sim}{i} + 3\underset{\sim}{j}) \cdot (-5\underset{\sim}{i} + 3\underset{\sim}{j})$ **c** $(4\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot (-5\underset{\sim}{i} + 3\underset{\sim}{j})$

6 Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$ and hence determine in each part whether the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

a $\underset{\sim}{u} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}, \underset{\sim}{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ **b** $\underset{\sim}{u} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}, \underset{\sim}{v} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$ **c** $\underset{\sim}{u} = \begin{bmatrix} -1 \\ a^{-2} \end{bmatrix}, \underset{\sim}{v} = \begin{bmatrix} a^{-1} \\ a \end{bmatrix}$

DEVELOPMENT

7 Suppose that A, B and C are the points $(2, 5)$, $(5, 14)$ and $(-2, 13)$ respectively. It is known that the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $45°$.

a Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form.

b Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$ using the result $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2$.

c Confirm your answer to part **b** using the result $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$.

8 Suppose that P, Q and R are the points $(\sqrt{3}, 8)$, $(3\sqrt{3}, 14)$ and $(5\sqrt{3}, 12)$ respectively. It is known that the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $30°$.

a Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in component form.

b Find $\overrightarrow{PQ} \cdot \overrightarrow{PR}$ using the result $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2$.

c Confirm your answer to part **b** using the result $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$.

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9 If θ is the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, find the exact value of $\cos\theta$ given:

a $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}, \underset{\sim}{b} = 5\underset{\sim}{j}$ **b** $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **c** $\underset{\sim}{a} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$

10 Find the values of λ for which the vectors $\underset{\sim}{u} = \lambda^2\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} - (2 + 2\lambda)\underset{\sim}{j}$ are perpendicular.

11 a Given that $|\underset{\sim}{a}| = 6, |\underset{\sim}{b}| = 2$ and $\theta = \frac{\pi}{3}$, find:

i $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **ii** $2\underset{\sim}{a} \cdot (-5)\underset{\sim}{b}$ **iii** $4\underset{\sim}{a} \cdot 0\underset{\sim}{b}$

iv $\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ **v** $\underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ **vi** $(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

b Repeat part **a** if instead $\theta = \frac{2\pi}{3}$. **c** Repeat part **a** if instead $\theta = \frac{\pi}{2}$.

12 The quadrilateral $ABCD$ has vertices $A(-3, -6)$, $B(1, -4)$, $C(-2, 2)$ and $D(-6, 0)$.

a Show that $\overrightarrow{AB} = \overrightarrow{DC}$.

b Show that $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$.

c What type of special quadrilateral is $ABCD$? Give reasons for your answer.

13 The quadrilateral $PQRS$ has vertices $P(-8, 3)$, $Q(3, 7)$, $R(7, 18)$ and $S(-4, 14)$.

a Show that the diagonals PR and QS bisect each other by showing that $\frac{1}{2}\overrightarrow{PR} = \overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QS}$.

b Show that that the diagonals are perpendicular by showing that $\overrightarrow{PR} \cdot \overrightarrow{QS} = 0$.

c What type of special quadrilateral is $PQRS$? Give reasons for your answer.

14 Suppose that A, P and Q are the points $(-3, 3)$, $(2, 9)$ and $(10, 0)$ respectively.

a Write $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ as column vectors.

b Hence find $\angle PAQ$ correct to the nearest degree.

15 The quadrilateral $PQRS$ has vertices $P(1, 2)$, $Q(8, 3)$, $R(6, 13)$ and $S(4, 9)$. Use the scalar (that is, dot) product to find, correct to the nearest minute, the acute angle between the diagonals of the quadrilateral.

16 The point $P(r\cos\theta, r\sin\theta)$ varies on the circle $x^2 + y^2 = r^2$.
Let A and B be the points $(-r, 0)$ and $(r, 0)$ respectively.
Use the dot product to show that $\angle APB = 90°$, provided that $P \neq A$ or B.

17 Triangle ABC has vertices $A(2, 1)$, $B(10, 4)$ and $C(5, 13)$.

a Show that $\cos\angle ABC = \dfrac{13}{\sqrt{7738}}$.

b Find the exact value of $\sin\angle ABC$.

c Hence find the area of triangle ABC.

18 A triangle APB has area 10 u^2. Suppose that $\overrightarrow{PA} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\overrightarrow{PB} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $|\overrightarrow{PB}| = 4\sqrt{5}$.

a Show that $3a + b = 20$ or $3a + b = -20$.

b Hence find all the possibilities for $\overrightarrow{PB}$.

19 We claimed in Box 14 and in the text above it that the geometric formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$ for the dot product could be derived from the component formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2$, where $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are non-zero vectors with angle θ between them.

To prove this claim, let $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ be two vectors, drawn as position vectors $\underset{\sim}{u} = \overrightarrow{OA}$ and $\underset{\sim}{v} = \overrightarrow{OB}$, and let $\angle AOB = \theta$.

a Write down the cosine rule with AB^2 as the subject.

b Substitute expressions for the *squared lengths only* using the distance formula.

c Expand the brackets, and hence prove that $x_1x_2 + y_1y_2 = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$.

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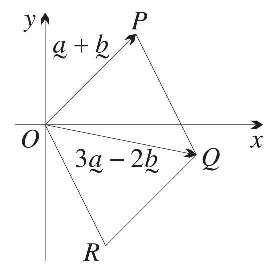
ENRICHMENT

20 In the diagram to the right, the points P and Q have respective position vectors $\underset{\sim}{a} + \underset{\sim}{b}$ and $3\underset{\sim}{a} - 2\underset{\sim}{b}$, and $OPQR$ is a parallelogram.



a Express $\overrightarrow{PR}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Now suppose that $OPQR$ is a square. Use dot products to prove that $|\underset{\sim}{a}|^2 = 2|\underset{\sim}{b}|^2$.

21 The points A, B, C and D are the vertices of a quadrilateral $ABCD$, and have respective position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ relative to an origin O.

a State, in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$, a condition for the diagonals AC and BD of the quadrilateral to be:

i perpendicular,

ii the same length.

b Suppose that $\underset{\sim}{a} = \begin{bmatrix}1\\5\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}5\\8\end{bmatrix}$, $\underset{\sim}{c} = \begin{bmatrix}8\\4\end{bmatrix}$ and $\underset{\sim}{d} = \begin{bmatrix}m\\n\end{bmatrix}$. If the diagonals AC and BD are perpendicular and have the same length, find the possible values of m and n.

22 a Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be three vectors with sum $\underset{\sim}{0}$. Expand $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$ using the distributive law to prove that

$$|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 = -2(\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c} + c \cdot \underset{\sim}{a}).$$

Explain why the sum of three squared lengths appears to be a negative number.

b In $\triangle ABC$, let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\underset{\sim}{c} = \overrightarrow{CA}$. Use the form of the cosine rule given in Box 13 to prove the identity in part **a**.

c Calculate the LHS and RHS of the identity separately for:

i an equilateral triangle of side length 1,

ii a right-angled isosceles triangle whose equal sides have length 1,

iii a right-angled triangle with hypotenuse of length 2 and one side of length 1.

23 a In a quadrilateral $ABCD$, let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$, $\underset{\sim}{c} = \overrightarrow{CD}$ and $\underset{\sim}{d} = \overrightarrow{DA}$, so that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} = \underset{\sim}{0}$. Expand $(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}) \cdot (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d})$ using the distributive law to prove that

$$\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{d} + \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{d} + \underset{\sim}{c} \cdot \underset{\sim}{d} = -\tfrac{1}{2}(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2).$$

b Evaluate all six terms on the LHS to confirm this result for:

i a rectangle with sides k and ℓ,

ii a parallelogram with sides k and ℓ and angle θ at A.

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8D Geometric problems

In Years 7–10, you will have developed and proven a systematic sequence of theorems in Euclidean geometry. The development of those theorems was based on Pythagoras' theorem, parallel lines, congruence, and similarity, and moved systematically through triangles, quadrilaterals and circles.

It is possible to prove those same geometric theorems using vectors, again in a systematic sequence of theorems. This short course in vectors cannot do this, however, because some of those theorems have already been used when we were developing vectors, and we would therefore be in danger of circular arguments.

Instead, Exercise 8D is an unsystematic demonstration of the use of vectors to prove geometric theorems. It should be clear in each question which definition of the geometric object is being used, and which properties and tests are being assumed in the question.

The striking thing about this exercise is that the vector proofs are usually dramatically different from the traditional proofs of Euclidean geometry. Geometry is extremely subtle, and many contrasting approaches are possible, even to its fundamental objects.

Proving theorems using vectors

As in all mathematics, there are no rules that are guaranteed to lead to a proof, but the following principles are important.

- Always begin by drawing a diagram and labelling all the points.
- Then introduce vectors — it may be useful to choose one of the vertices, or a point outside the figure, as a reference point or origin.
- Use the sum and product of vectors to complete a triangle.
- Parallel vectors are multiples of each other.
- The dot product is zero when two non-zero vectors are at right angles.
- Otherwise the dot product may allow access to $\cos\theta$.

Try to make your diagram, and your choice of letters, as symmetric as possible.

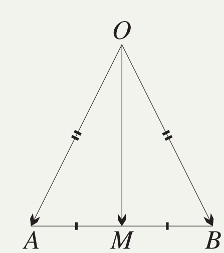
The three examples below, however, show that vector proofs of geometric theorems may be more elaborate than the Euclidean proof. It would be useful to compare the straightforward Euclidean proofs of these results, and perhaps of some others in Exercise 8D.

Example 11 8D

Prove that the line joining the apex of an isosceles triangle to the midpoint of the base is perpendicular to the base.

SOLUTION

Let $\triangle OAB$ be isosceles with $OA = OB$, and let M be the midpoint of AB.
Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, so that $|\underset{\sim}{a}| = |\underset{\sim}{b}|$.



$$\begin{aligned}\text{Then}\quad \overrightarrow{OM} &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}\\ &= \underset{\sim}{a} + \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})\\ &= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}),\\ \text{so}\quad \overrightarrow{OM}\cdot\overrightarrow{AB} &= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})\cdot(\underset{\sim}{b} - \underset{\sim}{a})\\ &= \tfrac{1}{2}(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2)\\ &= 0, \quad \text{which proves that } OM \perp AB.\end{aligned}$$

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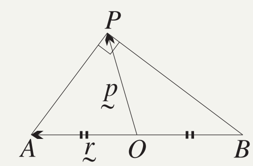
Example 12 8D

An interval AB with centre O subtends a right angle at a point P. Prove that the circle with diameter AB passes through P.

SOLUTION

Let $\underset{\sim}{r} = \overrightarrow{OA}$, where O is the midpoint of AB, and let $\underset{\sim}{p} = \overrightarrow{OP}$.



Then $\overrightarrow{OB} = -\underset{\sim}{r}$, so $\qquad \overrightarrow{BP} = \underset{\sim}{p} + \underset{\sim}{r}$,

and $\qquad \overrightarrow{AP} = \underset{\sim}{p} - \underset{\sim}{r}$.

Because $AP \perp BP$, $\qquad \overrightarrow{BP} \cdot \overrightarrow{AP} = 0$

$$
\begin{aligned}
(\underset{\sim}{p} + \underset{\sim}{r}) \cdot (\underset{\sim}{p} - \underset{\sim}{r}) &= 0 \\
\underset{\sim}{p} \cdot \underset{\sim}{p} - \underset{\sim}{p} \cdot \underset{\sim}{r} + \underset{\sim}{r} \cdot \underset{\sim}{p} - \underset{\sim}{r} \cdot \underset{\sim}{r} &= 0 \\
\underset{\sim}{p} \cdot \underset{\sim}{p} &= \underset{\sim}{r} \cdot \underset{\sim}{r} \\
|\underset{\sim}{p}| &= |\underset{\sim}{r}|
\end{aligned}
$$

Hence $OP = OA = OB$, and the circle with diameter AB passes through P.

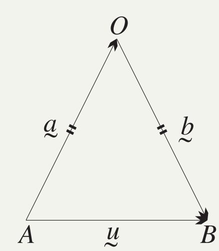
Note: This result is the converse of the well-known 'angle in a semi-circle' theorem proven in Question 7 of Exercise 8D.

Example 13 8D

Prove that the base angles of an isosceles triangle are equal.

SOLUTION

Let $\triangle OAB$ be isosceles with $OA = OB$.



Let $\underset{\sim}{a} = \overrightarrow{AO}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ and $\overrightarrow{AB} = \underset{\sim}{u}$.

Then $\underset{\sim}{u} = \underset{\sim}{a} + \underset{\sim}{b}$, so using the geometric dot product formula,

$$
\begin{aligned}
|\underset{\sim}{a}||\underset{\sim}{u}| \cos A &= \underset{\sim}{a} \cdot \underset{\sim}{u} = \underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} \\
|\underset{\sim}{b}||\underset{\sim}{u}| \cos B &= \underset{\sim}{b} \cdot \underset{\sim}{u} = \underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}.
\end{aligned}
$$

Subtracting, and using the fact that $|\underset{\sim}{b}| = |\underset{\sim}{a}|$, and so also that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$,

$$|\underset{\sim}{u}||\underset{\sim}{a}|(\cos A - \cos B) = 0.$$

Hence $\cos A = \cos B$, so $A = B$ because cosine is one-to-one in the interval $[0, \pi]$.

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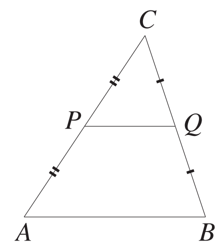
Exercise 8D

FOUNDATION

1 In $\triangle ABC$ to the right, P is the midpoint of AC and Q is the midpoint of BC.

Let $\overrightarrow{AC} = \underset{\sim}{a}$ and $\overrightarrow{CB} = \underset{\sim}{b}$.



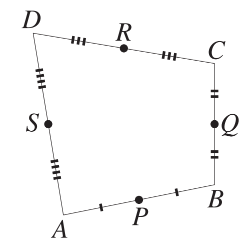
a Write $\overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Write $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

c Hence explain why $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$ and half its length.

2 In the diagram to the right, $ABCD$ is a quadrilateral. The points P, Q, R and S are the midpoints of AB, BC, CD and DA respectively.

Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$, $\overrightarrow{AD} = \underset{\sim}{d}$ and $\overrightarrow{DC} = \underset{\sim}{c}$.



a Explain why $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{d} + \underset{\sim}{c}$.

b Express $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

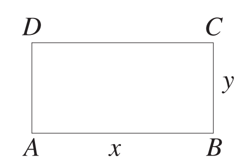
c Express $\overrightarrow{SR}$ in terms of $\underset{\sim}{d}$ and $\underset{\sim}{c}$.

d Hence show that $\overrightarrow{PQ} = \overrightarrow{SR}$.

e Deduce that the quadrilateral $PQRS$ is a parallelogram.

3 The rectangle $ABCD$ to the right has side lengths $|AB| = x$ and $|BC| = y$.

Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$ and $\overrightarrow{CD} = \underset{\sim}{c}$.



a Express $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and $\overrightarrow{BD}$ in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

b What is the value of $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

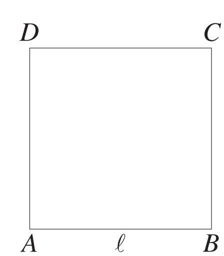
c Write $\underset{\sim}{a} \cdot \underset{\sim}{a}$ in terms of x.

d Show that $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ and $(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} + \underset{\sim}{c})$ are both equal to $x^2 + y^2$.

e What have we proven in part **d** about the diagonals of a rectangle?

4 The square $ABCD$ to the right has side length ℓ.

Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$.



a Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b What is the value of $\underset{\sim}{a} \cdot \underset{\sim}{b}$?

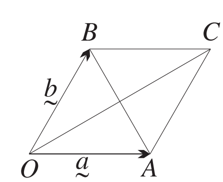
c Write $\underset{\sim}{a} \cdot \underset{\sim}{a}$ in terms of ℓ.

d Find $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$.

e What have we proven in part **d** about the diagonals of a square?

5 In the diagram to the right, $OACB$ is a rhombus.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



a Why is $|\underset{\sim}{a}| = |\underset{\sim}{b}|$?

b By squaring the result in part **a**, show that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$.

c Express $\overrightarrow{OC}$ and $\overrightarrow{BA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

d Hence show that $\overrightarrow{OC} \cdot \overrightarrow{BA} = 0$.

e What have we just proven about the diagonals of a rhombus?

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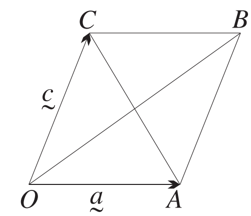
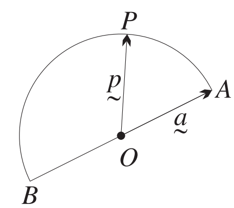
DEVELOPMENT

6 In the diagram to the right, $OABC$ is a parallelogram whose diagonals OB and AC are equal. The points A and C have respective position vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ relative to O.



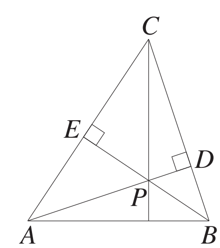
- **a** Explain why $\overrightarrow{CB} = \underset{\sim}{a}$.
- **b** Write $\overrightarrow{OB}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{a}$.
- **c** Write $\overrightarrow{AC}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{a}$.
- **d** Explain why $|\underset{\sim}{c} + \underset{\sim}{a}| = |\underset{\sim}{c} - \underset{\sim}{a}|$.
- **e** Use the result in part **d**, and the fact that $|\underset{\sim}{v}|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$, to show that $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$.
- **f** What can we now say about a parallelogram whose diagonals are equal?

7 In the diagram to the right, O is the centre of a semi-circle APB whose diameter is AB. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OP} = \underset{\sim}{p}$.



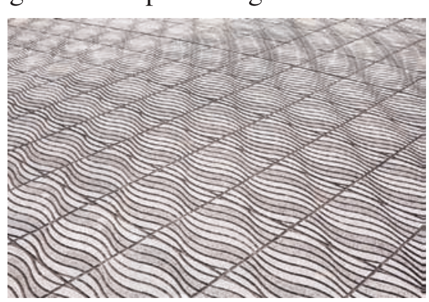
- **a** Write $\overrightarrow{OB}$ in terms of $\underset{\sim}{a}$.
- **b** Express $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{p}$.
- **c** Hence prove that $\angle APB = 90°$.
- **d** What circle geometry theorem have we just proven?

8 In the diagram to the right, AD and BE are altitudes of $\triangle ABC$ intersecting at P. Let $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{p}$ be the respective position vectors of A, B, C and P relative to an origin O.



- **a** Explain why $(\underset{\sim}{p} - \underset{\sim}{a}) \cdot (\underset{\sim}{c} - \underset{\sim}{b}) = 0$.
- **b** Explain why $(\underset{\sim}{p} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$.
- **c** Hence show that $(\underset{\sim}{p} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$.
- **d** Deduce that the three altitudes of a triangle are concurrent.

9 Prove, using vectors, that the diagonals of a parallelogram bisect each other.



10 Use vectors to prove that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides.

11 Suppose that $OABC$ is a parallelogram. Let M be the midpoint of OA and let P be the point of intersection of MC and OB. Prove, using vectors, that $OP = \frac{1}{3}OB$.

ENRICHMENT

12 Consider a triangle OAB, with P, Q and R the midpoints of OB, OA and AB respectively. Suppose that the medians AP and BQ intersect at C, and let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

- **a** Explain why $\overrightarrow{AC} = \lambda_1\left(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}\right)$ and $\overrightarrow{BC} = \lambda_2\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}\right)$, where λ_1 and λ_2 both lie between 0 and 1.
- **b** Hence prove that the three medians of a triangle are concurrent, and that C divides each median in the ratio $2:1$.

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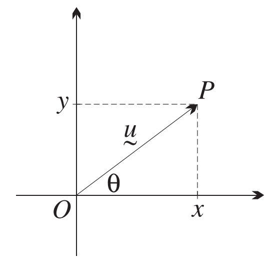
8E Projections

In Section 8C on the dot product, we expressed a position vector $\underset{\sim}{u} = \overrightarrow{OP}$ as a sum of its vector components

$$\underset{\sim}{u} = (\underset{\sim}{u} \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{u} \cdot \underset{\sim}{j})\underset{\sim}{j}$$



where $x = \underset{\sim}{u} \cdot \underset{\sim}{i}$
and $y = \underset{\sim}{u} \cdot \underset{\sim}{j}$

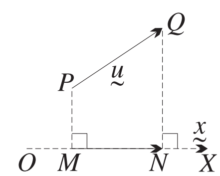
The necessary constructions were given in Section 8B and involved dropping perpendiculars to the horizontal and vertical axes. The constructions are examples of *projections* — in this case, projections onto the unit vector $\underset{\sim}{i}$ and the unit vector $\underset{\sim}{j}$.

This short section generalises the projection construction so that we can project any vector onto any other vector. Projection is a geometric idea, and its formulae can be expressed using the dot product.

Projections

Let $\underset{\sim}{x} = \overrightarrow{OX}$ be a non-zero reference vector.
We can *project* any vector $\underset{\sim}{u} = \overrightarrow{PQ}$ onto this vector $\underset{\sim}{x}$.

- Drop perpendiculars PM and QN to the line OX.
- The *projection of* $\underset{\sim}{u}$ *onto* $\underset{\sim}{x}$ is the vector $\overrightarrow{MN}$.
- We write the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ as $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN}$.



If $\underset{\sim}{u}$ is the zero vector, then N coincides with M, and the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ is $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN} = \underset{\sim}{0}$, the zero vector. Similarly, if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{OX}$ are perpendicular, then again M and N coincide and $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MM} = \underset{\sim}{0}$. In all other situations, the projection is non-zero.

16 PROJECTIONS

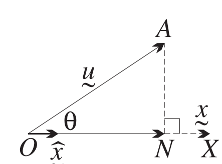
Let $\underset{\sim}{x} = \overrightarrow{OX}$ be a non-zero vector, and let $\underset{\sim}{u} = \overrightarrow{PQ}$ be any vector.

- To *project* $\underset{\sim}{u} = \overrightarrow{PQ}$ onto $\underset{\sim}{x}$, drop perpendiculars PM and QN to the line OX.
  The *projection* of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ is the vector $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN}$.

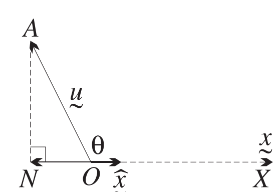
A better diagram for a geometric formula

It is easier to introduce angles if we represent $\underset{\sim}{u}$ as $\underset{\sim}{u} = \overrightarrow{OA}$ with tail at O. Let θ be the angle between $\underset{\sim}{u}$ and $\underset{\sim}{x}$, then $ON = |\underset{\sim}{u}|\cos\theta$ using trigonometry. The unit vector in the direction OX is $\underset{\sim}{\hat{x}}$, so we can write

$$\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{MN} = \overrightarrow{ON} = \left(|\underset{\sim}{u}|\cos\theta\right)\underset{\sim}{\hat{x}}.$$



When θ is obtuse, then $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ points in the opposite direction from $\underset{\sim}{x}$. The formula takes this into account because then $\cos\theta$ is negative. See the diagram to the right.



The cases where $\theta = 0°$ or $\theta = 90°$ or $\theta = 180°$ or $\underset{\sim}{u} = \underset{\sim}{0}$ are left to the reader. In all cases, the projection vector $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ has length $|\underset{\sim}{u}|\ |\cos\theta|$.

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17 PROJECTIONS — GEOMETRIC FORMULAE

Let $\underset{\sim}{u}$ and $\underset{\sim}{x} = \overrightarrow{OX}$ be non-zero vectors, and let θ be the angle between them.
Represent $\underset{\sim}{u} = OA$ with tail at O, and drop the perpendicular AN to OX.

- The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ is the vector $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \overrightarrow{ON}$.
- $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \left(|\underset{\sim}{u}|\cos\theta\right)\hat{\underset{\sim}{x}}$ and $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ has length $|\underset{\sim}{u}|\ |\cos\theta|$.

Projections and the dot product

Now we can combine projections with the dot product, using the definition of the dot product as $\underset{\sim}{u}\cdot\underset{\sim}{v} = |\underset{\sim}{u}|\ |\underset{\sim}{v}|\cos\theta$.

First,
$$\begin{aligned}\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} &= (|\underset{\sim}{u}|\cos\theta)\,\hat{\underset{\sim}{x}}\\ &= (|u||\hat{\underset{\sim}{x}}|\cos\theta)\hat{\underset{\sim}{x}}, \text{ because } |\hat{\underset{\sim}{x}}| = 1,\\ &= (\underset{\sim}{u}\cdot\hat{\underset{\sim}{x}})\,\hat{\underset{\sim}{x}}. \qquad (1)\end{aligned}$$

Secondly,
$$\hat{\underset{\sim}{x}} = \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|},$$

so from (1),
$$\begin{aligned}\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} &= \frac{(\underset{\sim}{u}\cdot\underset{\sim}{x})\,\underset{\sim}{x}}{|\underset{\sim}{x}|\times|\underset{\sim}{x}|}\\ &= \frac{\underset{\sim}{u}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}}\times\underset{\sim}{x}. \qquad (2)\end{aligned}$$

Projections and the vector components

In particular, replacing the vector $\underset{\sim}{x}$ in equation (1) above by either of the two basis vectors $\underset{\sim}{i}$ or $\underset{\sim}{j}$ gives the previous formulae for the vector components of $\underset{\sim}{u}$, that is,

$$\text{proj}_{\underset{\sim}{i}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{i})\,\underset{\sim}{i} \text{ and } \text{proj}_{\underset{\sim}{j}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{j})\underset{\sim}{j}.$$

Hence the vector components of any vector $\underset{\sim}{u}$ are the projections onto $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

The ‘completing the rectangle’ construction that we used below Box 7 in Section 8B can now be interpreted as the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i}$ and the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{j}$.

18 PROJECTIONS AND THE DOT PRODUCT — VECTOR COMPONENTS

Let $\underset{\sim}{x} = \overrightarrow{OX}$ be a non-zero vector, and let $\underset{\sim}{u}$ be a vector.

- The projection $\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u}$ of $\underset{\sim}{u}$ onto $\underset{\sim}{x}$ can be expressed using the dot product,
  $$\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\hat{\underset{\sim}{x}})\,\hat{\underset{\sim}{x}} \quad\text{and}\quad \text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}}\times\underset{\sim}{x}$$
- Projection satisfies the two laws,
  — $\text{proj}_{\underset{\sim}{x}}(\lambda\underset{\sim}{u}) = \lambda(\text{proj}_{\underset{\sim}{x}}\underset{\sim}{u})$, for any scalar λ.
  — $\text{proj}_{\underset{\sim}{x}}(\underset{\sim}{u}+\underset{\sim}{v}) = \text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} + \text{proj}_{\underset{\sim}{x}}\underset{\sim}{v}$, for any other vector $\underset{\sim}{v}$.
- The vector components of $\underset{\sim}{u}$ are the projections of $\underset{\sim}{u}$ onto $\underset{\sim}{i}$ and $\underset{\sim}{j}$,
  $$\text{proj}_{\underset{\sim}{i}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{i})\,\underset{\sim}{i} \quad\text{and}\quad \text{proj}_{\underset{\sim}{j}}\underset{\sim}{u} = (\underset{\sim}{u}\cdot\underset{\sim}{j})\underset{\sim}{j}.$$

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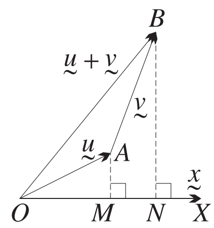
The two identities of the second dotpoint follow easily from the formulae in the first dotpoint of Box 18.

In particular, the second of these two identities is demonstrated geometrically by the diagram to the right,

$$\text{proj}_{\underset{\sim}{x}}(\underset{\sim}{u}+\underset{\sim}{v}) = \text{proj}_{\underset{\sim}{x}}\underset{\sim}{u} + \text{proj}_{\underset{\sim}{x}}\underset{\sim}{v}.$$



Example 14 8E

a Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, where $|OA| = 6$, $|OB| = 10$ and $\angle AOB = 60°$. Find $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$ as a multiple of $\underset{\sim}{b}$.

b Find the projection of $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ onto $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

SOLUTION

a
$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= (|\underset{\sim}{a}|\cos 60°)\,\hat{\underset{\sim}{b}}\\ &= 6 \times \tfrac{1}{2} \times \tfrac{1}{10}\underset{\sim}{b}\\ &= \tfrac{3}{10}\underset{\sim}{b}\end{aligned}$$

b
$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}\\ &= \frac{-3+12}{1+9} \times (-\underset{\sim}{i} + 3\underset{\sim}{j})\\ &= \tfrac{9}{10}(-\underset{\sim}{i} + 3\underset{\sim}{j})\\ &= -\tfrac{9}{10}\underset{\sim}{i} + \tfrac{27}{10}\underset{\sim}{j}\end{aligned}$$

Exercise 8E

FOUNDATION

1 Write down the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$ **b** $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{j}$ **c** $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$

2 Write down the length of the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$ **b** $\underset{\sim}{a} = -2\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{j}$ **c** $\underset{\sim}{a} = -6\sqrt{2}\underset{\sim}{i} + 8\sqrt{2}\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i}$

3 Write down the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \begin{bmatrix}2\\1\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}4\\0\end{bmatrix}$ **b** $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{j}$ **c** $\underset{\sim}{a} = \begin{bmatrix}5\\-3\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}-6\\0\end{bmatrix}$

4 Use trigonometry to find the length of the projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ if:

a $|\overrightarrow{OA}| = 6$ and $\angle AOB = 30°$, **b** $|\overrightarrow{OA}| = 6\sqrt{6}$ and $\angle AOB = 45°$.

DEVELOPMENT

5 Show that the projection of $\underset{\sim}{a} = \begin{bmatrix}10\\-2\end{bmatrix}$ onto $\underset{\sim}{b} = \begin{bmatrix}1\\-7\end{bmatrix}$ has length $\dfrac{12\sqrt{2}}{5}$.

6 Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}2\\2\end{bmatrix}$ **b** $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$ **c** $\underset{\sim}{a} = \begin{bmatrix}-5\\5\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}-6\\8\end{bmatrix}$

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7 Find the component of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}, \underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j}$

b $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}, \underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

8 Find the magnitude of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = -2\underset{\sim}{i}, \underset{\sim}{b} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$

b $\underset{\sim}{a} = 6\underset{\sim}{i} - 4\underset{\sim}{j}, \underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

9 Find the projection of $\overrightarrow{AB}$ onto $-6\underset{\sim}{i} + 4\underset{\sim}{j}$ given $A(-3, -7)$ and $B(1, 5)$.

10 Find the length of the projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$ given $A(1, 3)$, $B(6, 18)$, $C(9, 4)$ and $D(19, 24)$.

11 Find the possible values of λ if the projection of $\lambda\underset{\sim}{i} + 4\underset{\sim}{j}$ onto $12\underset{\sim}{i} - 5\underset{\sim}{j}$ has length $\frac{140}{13}$.

12 Use the identities in the first dotpoint of Box 18 to prove the two identities in the second dotpoint.

ENRICHMENT

13 Let P be the point $(25, -5)$ and ℓ be the line $x + 3y + 10 = 0$.

a What is the gradient of ℓ?

b Hence write down a vector $\underset{\sim}{v}$ that is parallel to ℓ.

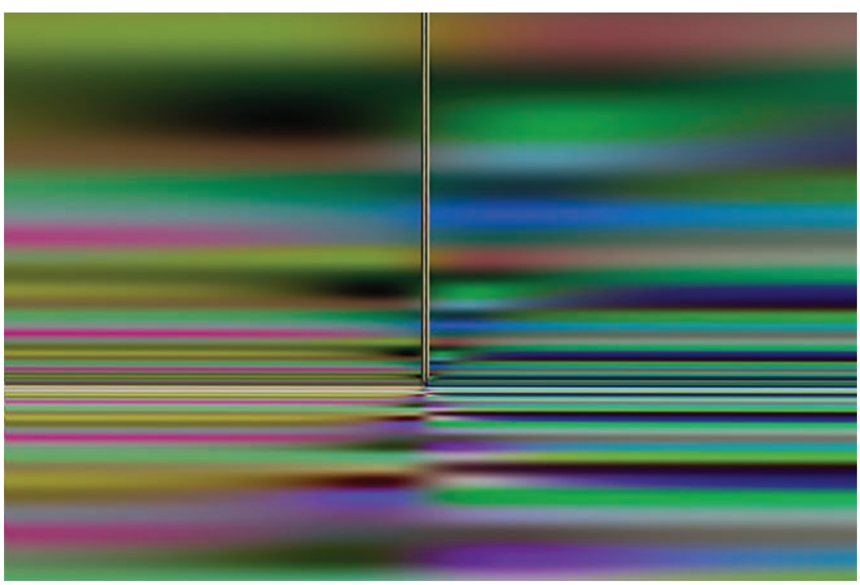
c Show that ℓ passes through the point $A(2, -4)$.

d Write down $\overrightarrow{AP}$ in component form.

e Find the length of the projection of $\overrightarrow{AP}$ onto $\underset{\sim}{v}$.

f Hence find the perpendicular distance from P to ℓ.

14 Use the approach in the previous question to prove that the perpendicular distance from the origin to the line $ax + by + c = 0$ is given by $\dfrac{|c|}{\sqrt{a^2 + b^2}}$.



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8F Applications to physical situations

Physics is full of problems in two- and three-dimensional space where vectors make the situation clearer. More advanced topics, such as electro-magnetic forces and waves, cannot be properly understood without vectors. At this stage, however, simple trigonometry will often solve problems quickly, and vectors may seem an unnecessary complication. The worked examples below use various methods, and each is done two ways. Readers should compare and contrast the methods used.

Without a great deal more physics, the only reasonable applications involve displacements, velocities, accelerations and forces. Some terminology is required, and some introduction to the relationship between force and acceleration.

Displacement and velocity

The first two worked examples involve only displacement and velocity.

Example 15 — 8F

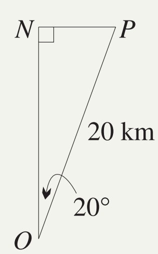
I walk 20 km in a direction N20°E. Find how far north I have gone:

a using a map of my journey,

b using projection vectors.

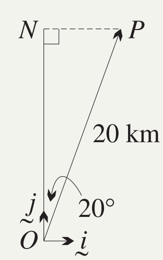
SOLUTION

Let the walk begin at O and end at P.
Let N be the point north of O and west of P.



a By trigonometry, $ON = 20\cos 20°$
$\doteqdot 18.8.$

b Using projections, let $\underset{\sim}{j}$ be a unit vector pointing north.



Then
$$\begin{aligned} |\overrightarrow{ON}| &= \text{proj}_{\underset{\sim}{j}}\overrightarrow{OP} \\ &= (20\cos 20°)\underset{\sim}{j} \\ &\doteqdot 18.8\underset{\sim}{j}\,. \end{aligned}$$

Hence I have gone 18.8 km north.

Example 16 — 8F

A ship leaves port and sails north-east in a straight line at an angle to the straight north-south coastline. Its speed along the coast is 20 km/h, and its speed in the water is 25 km/h (there is no current). Find its direction of motion and its speed away from the coast:

a using a velocity resolution diagram,

b using projection vectors.

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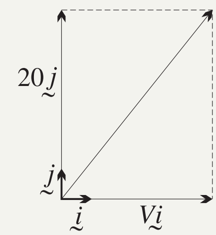
SOLUTION

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors east and north, and let V be the speed away from the coast.



Then the ship's velocity vector is $\underset{\sim}{v} = V\underset{\sim}{i} + 20\underset{\sim}{j}$.

We know that $V^2 + 20^2 = 25^2$,

so $V = 15$,

and the ship's velocity vector is $15\underset{\sim}{i} + 20\underset{\sim}{j}$.

Let θ be the acute angle between the ship's direction and shoreline.

a Using trigonometry, $\cos\theta = \dfrac{20}{25}$

$\theta \doteqdot 37°$.

b Using projections, velocity along the coast $= \text{proj}_{\underset{\sim}{j}}\underset{\sim}{v}$

$$20\underset{\sim}{j} = (|\underset{\sim}{v}|\cos\theta)\underset{\sim}{j}$$
$$20\underset{\sim}{j} = (25\cos\theta)\underset{\sim}{j}$$
$$\cos\theta = \tfrac{20}{25}$$
$$\theta \doteqdot 37°.$$

Hence the ship is travelling about N37°E, leaving the coast at 15 km/h.

The resultant of two vectors

The sum $\underset{\sim}{u} + \underset{\sim}{v}$ of two vectors is often called the *resultant* of the two vectors. This terminology is used particularly for the sum of two forces, when the sum or resultant of two forces can be regarded as a single force acting on the object, as in the next worked example.

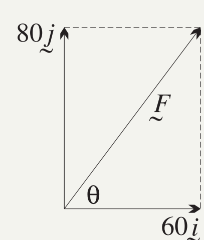
Example 17 8F

Peter and Paul are pulling a large box using ropes. They can never cooperate, and they end up pulling the box in different directions, Peter pulling east with a force of 60 newtons, and Paul pulling north with force of 80 newtons. Find the resultant force:

a using a forces diagram, **b** using projection vectors.

SOLUTION

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors east and north.



Then the resultant force is $\underset{\sim}{F} = 60\underset{\sim}{i} + 80\underset{\sim}{j}$,

so $|\underset{\sim}{F}| = 100$.

Let θ be the acute angle between $\underset{\sim}{i}$ and $\underset{\sim}{F}$.

a Using trigonometry, $\tan\theta = \tfrac{80}{60}$

$\theta \doteqdot 53°$.

b Using projections, $\text{proj}_{\underset{\sim}{i}}\underset{\sim}{F} = 60\underset{\sim}{i}$

$$(100\cos\theta)\underset{\sim}{i} = 60$$
$$\cos\theta = \tfrac{60}{100}$$
$$\theta \doteqdot 53°$$

Hence the resultant force is about 100 newtons in a direction N37°E.

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Forces and their units

Newton's second law of motion states that

$$F = ma, \qquad \text{meaning that} \qquad \text{force} = \text{mass} \times \text{acceleration}.$$

We will have a great deal more to say about acceleration in the next two chapters, but this law is needed now so that the units of force can be introduced — these units are called 'newtons', with symbol N, in honour of Sir Isaac Newton.

In words, Newton's second law says that if a body of mass m kg is accelerating at $a\ \text{m}/\text{s}^2$, then the sum of all the forces acting on the body has magnitude $F = ma$ newtons, and acts in the same direction as the acceleration. Thus 1 newton is the force required to accelerate a body at a rate of $1\ \text{m}/\text{s}^2$.

There is a second unit of force called 'kilograms weight'. A mass of m kilograms that is free to move is pulled downwards by gravity with a force that accelerates it at about $10\ \text{m}/\text{s}^2$. The physical value has symbol g, and a better approximation is $g = 9.8\ \text{m}/\text{s}^2$ ($g = 9.832\ \text{m}/\text{s}^2$ at the poles and $g = 9.780\ \text{m}/\text{s}^2$ at the equator).

This means that the downwards gravitational force on a mass m kg is mg newtons.

In particular, because $g \doteqdot 10\ \text{m}/\text{s}^2$, a force of 1 newton is about the downwards force that you feel when you hold a 100 g apple in your open hand.

19 NEWTON'S SECOND LAW AND THE UNITS OF FORCE

- One newton, written in symbols as 1 N, is the force required to accelerate a body at a rate of $1\ \text{m}/\text{s}^2$.
- Newton's second law of motion says that

  $$F = ma.$$

  'If a body of mass m kg is accelerating at $a\ \text{m}/\text{s}^2$, then the sum of all the forces acting on the body has magnitude $F = ma$ newtons, and acts in the same direction as the acceleration.'
- One kilogram weight is the downward force due to gravity on a mass of 1 kg at the Earth's surface.
- Acceleration due to gravity at the Earth's surface has the symbol g, whose approximate value is $9.8\ \text{m}/\text{s}^2$ (or $10\ \text{m}/\text{s}^2$ in round figures).
- One newton is therefore about $\frac{1}{10}$ kg weight — about the downward force due to gravity of a 100-gram apple on your open hand.

Example 18 8F

A parcel P of mass 8 kg is resting on a plane inclined at 30° to the horizontal.

a What component of the gravitation force is acting on the parcel in the direction down the plane? Answer the question:

i using a diagram of forces, **ii** using projection vectors.

b What is the force in newtons, taking $g \doteqdot 9.8\ \text{m}/\text{s}^2$?

c What frictional force is acting on the parcel in the direction up the plane?

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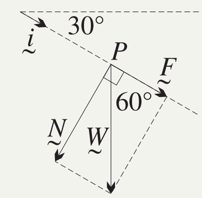
SOLUTION



Let $\underset{\sim}{W}$ be the weight of the parcel, represented as a vector down from P.

a **i** Regard the weight as the resultant of a force $\underset{\sim}{F}$ down the plane, and a force $\underset{\sim}{N}$ normal to the plane.

By simple trigonometry,
$$\begin{aligned}|\underset{\sim}{F}| &= |\underset{\sim}{W}| \cos 60° \\ &= 8 \times \tfrac{1}{2} \\ &= 4.\end{aligned}$$

ii Let $\underset{\sim}{W}$ be the weight of the parcel, represented as a vector down from P.

Let $\underset{\sim}{i}$ be a unit vector down the plane.

Then the force down the plane is the projection of $\underset{\sim}{W}$ onto i,

$$\begin{aligned}\underset{\sim}{F} &= \text{proj}_{\underset{\sim}{i}} \underset{\sim}{W} \\ &= (\underset{\sim}{W} \cdot \underset{\sim}{i})\underset{\sim}{i} \quad \text{OR} \quad (|\underset{\sim}{W}| \cos 60°)\underset{\sim}{i} \\ &= 4\underset{\sim}{i},\end{aligned}$$

Hence the force down the plane is 4 kg weight.

b In newtons, this is about $4 \times 9.8 = 39.2$ N.

c The parcel is not accelerating (it is not even moving), so the frictional force is equal and opposite to the component of the weight acting down the plane. Hence the frictional force is 4 kg weight up the plane.

Exercise 8F

FOUNDATION

In this exercise take $g = 9.8\ \text{m/s}^2$.

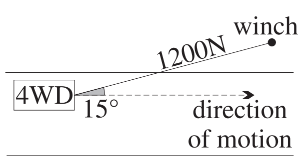
1 A ball is thrown at an angle of 30° to the horizontal with an initial speed of 20 m/s. Find the initial horizontal and vertical components of the velocity of the ball.

2 A particle has initial position vector $(4\underset{\sim}{i} + 5\underset{\sim}{j})$ metres. It moves with a constant velocity of $(3\underset{\sim}{i} - 2\underset{\sim}{j})$ m/s. Find its position vector after 7 seconds.

3 Find the magnitude of the resultant of the forces $(2\underset{\sim}{i} - 3\underset{\sim}{j})$ N, $(4\underset{\sim}{i} + \underset{\sim}{j})$ N and $(-3\underset{\sim}{i} + 3\underset{\sim}{j})$ N.

4 Two forces of magnitude 30 N and 16 N act away from a point P and are perpendicular. Find the magnitude and direction of the resultant force (measured from the 30 N force correct to the nearest degree).

5 In the diagram, a 4-wheel drive vehicle is bogged on a muddy road. A winch is pulling the vehicle with a force of 2000 N. The chain connecting the winch to the vehicle makes an angle of 15° with the direction of motion. Calculate, correct to the nearest newton, the magnitude of the component of the force:



a in the direction of motion,

b perpendicular to the direction of motion.

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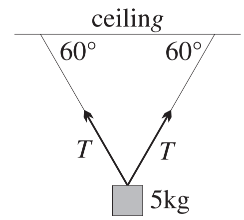
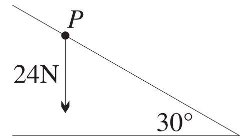
6 In the diagram, an object of mass 5 kg is suspended from a horizontal ceiling by two strings of equal length. Each string makes an angle of 60° with the ceiling. Calculate, correct to 3 significant figures, the equal tensions in the two strings.



7 The diagram shows an object of weight 24 N at rest at P on an inclined plane. Find the component of the weight:

a down the plane,

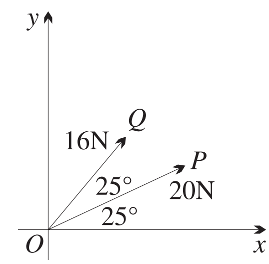
b perpendicular to the plane.



8 In the diagram, the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent forces of magnitude 20 N and 16 N respectively.

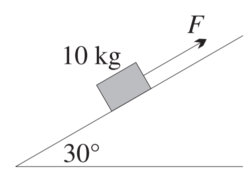
a Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in component form.

b Calculate, correct to 2 significant figures, the magnitude and direction of the resultant of the two forces.



DEVELOPMENT

9 In the diagram, an object of mass 10 kg is kept at rest on a smooth plane inclined at 30° to the horizontal by a force of F newtons acting parallel to the plane. Find the value of F.



10 A river is flowing at a speed of 1.5 m/s. Sam wants to row from point A on one bank to point B on the other bank directly opposite A. He intends to maintain a constant speed of 2.5 m/s. In what direction, correct to the nearest degree, should Sam row? Give your answer as an angle of inclination to the line AB.

11 Two dogs Brutus and Nitro are simultaneously tugging on a bone. Brutus is pulling with a force of 12 N in a direction 45° west of north, while Nitro is pulling with a force of 16 N in a direction 30° south of east. Calculate, correct to two significant figures, the magnitude and direction of the resultant force.

12 Three forces act on an object of mass 5 kg. These forces are represented by the vectors $9\underset{\sim}{i} - 2\underset{\sim}{j}$, $-3\underset{\sim}{i} + 10\underset{\sim}{j}$ and $18\underset{\sim}{i} - \underset{\sim}{j}$. Calculate the magnitude and direction of the acceleration of the object.

13 The position of a plane flying horizontally in a straight line at a constant speed is plotted on a radar screen. One unit on the screen represents 1 km in the air. At 12 noon the position vector of the plane is $40\underset{\sim}{i} + 16\underset{\sim}{j}$. Five minutes later its position vector is $33\underset{\sim}{i} + 40\underset{\sim}{j}$. Find:

a the position vector of the plane at 12:15 pm,

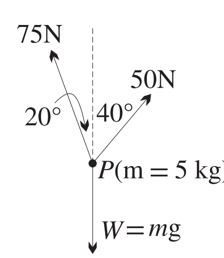
b the velocity of the plane as a vector in km/h.

14 The diagram shows an object of mass 5 kg being raised by forces of magnitude 75 N and 50 N.

a Find the weight of the object.

b Find, correct to the nearest newton, the magnitude of the resultant of the three forces acting on the object.

c Find, correct to the nearest degree, the angle this resultant makes with the upward vertical direction.



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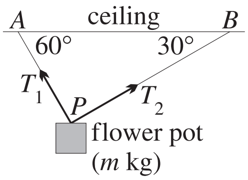
15 David can row at 5 m/s in still water. He starts rowing from a point on the south bank of a river that is flowing due east at 3 m/s and steers the boat at 90° to the bank. He is also being blown by a wind from the north-east at 4 m/s.

a Express the velocity of the boat as a component vector.

b Hence find the speed of the boat, correct to 2 significant figures, and the bearing on which the boat is travelling correct to the nearest tenth of a degree.

16 In the diagram, a flowerpot of mass m kg is hung from a ceiling by two chains. Let the tensions in the chains AP and BP be T_1 and T_2 newtons respectively. The third force acting at P is the weight of the flowerpot.

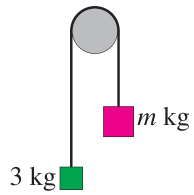


a By finding the horizontal component of the resultant of the three forces acting at P, show that $T_1 = \sqrt{3}\,T_2$.

b By finding the vertical component of the resultant of the three forces acting at P, show that $\sqrt{3}\,T_1 + T_2 = 19.6m$ newtons.

c Find the mass of the flowerpot, given that $T_2 = 98$ N.

17 The diagram shows objects of mass 3 kg and m kg attached to the ends of a light inextensible string that passes over a smooth pulley. The 3 kg object is accelerating at 4.9 m/s^2 upwards. Let the tension in the string be T newtons.

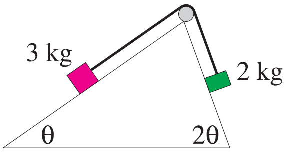


a Find the value of T.

b Find the value of m.

18 Two forces, of magnitude p newtons and q newtons, have a resultant of $2\sqrt{7}$ N when they act at 90° to each other. When they act at 30° to each other, however, the magnitude of the resultant is $2\sqrt{13}$ N. Find the values of p and q.

19 The diagram shows objects of mass 3 kg and 2 kg on connected smooth planes inclined at angles of θ and 2θ to the horizontal. The objects are attached to the ends of a light inextensible string that passes over a smooth pulley. Let T newtons be the tension in the string, and suppose that the 3 kg object is accelerating at a m/s^2 up its plane.



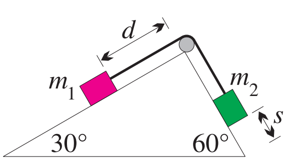
a Find, in terms of a, T, g and θ, an equation for the motion of the 3 kg object up its plane.

b Write down a similar equation for the motion of the 2 kg object down the other plane.

c Show that the system is in equilibrium when $\cos\theta = \frac{3}{4}$.

ENRICHMENT

20 In the diagram, objects of mass m_1 and m_2 are held at rest on adjoining smooth inclined planes. They are connected by a light inextensible string that passes over a smooth pulley.



a Show that when the objects are released, the object of mass m_1 will accelerate *towards the pulley* if $m_1 < \sqrt{3}\,m_2$.

b Assuming that the condition in part **a** is satisfied, show that the object of mass m_1 will hit the pulley with speed $\sqrt{\dfrac{dg(\sqrt{3}\,m_2 - m_1)}{m_1 + m_2}}$.

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Chapter 8 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 8 Multiple-choice quiz

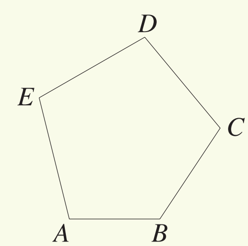
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 A ship sailed 133 km from port A to port B on a bearing of 068°T, then it sailed 98 km from port B to port C on a bearing of 116°T. Draw a diagram showing the displacement vector $\overrightarrow{AC}$, then calculate the magnitude and direction of $\overrightarrow{AC}$ (correct to the nearest tenth of a km, and as a true bearing correct to the nearest degree).

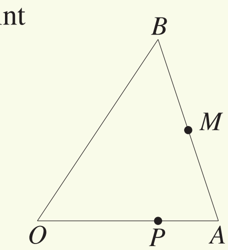
2 In the diagram to the right, write down a single vector equal to:

a $\overrightarrow{AE} + \overrightarrow{ED}$
b $\overrightarrow{AD} + \overrightarrow{DC}$
c $\overrightarrow{DA} + \overrightarrow{AC}$
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}$
e $\overrightarrow{AD} - \overrightarrow{AC}$
f $\overrightarrow{EB} - \overrightarrow{ED}$



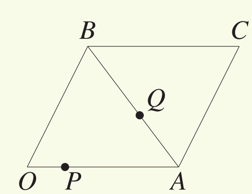
3 In the triangle OAB shown in the diagram, M is the midpoint of AB and P is the point on OA such that $OP:PA = 2:1$. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Express, in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

a $\overrightarrow{AB}$
b $\overrightarrow{OM}$
c $\overrightarrow{PM}$



4 In the diagram, $OACB$ is a parallelogram. The point P divides OA in the ratio $1:3$ and Q divides AB in the ratio $3:4$. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

a Express $\overrightarrow{PA}$ in terms of $\underset{\sim}{a}$.
b Express $\overrightarrow{AQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
c Show that $\overrightarrow{PQ} = \frac{9}{28}\underset{\sim}{a} + \frac{3}{7}\underset{\sim}{b}$.
d Show that $\overrightarrow{QC} = \frac{3}{7}\underset{\sim}{a} + \frac{4}{7}\underset{\sim}{b}$.
e Hence show that the points P, Q and C are collinear.



5 If A and B are the points $(-4, 2)$ and $(2, 10)$ respectively, find:

a $\overrightarrow{AB}$ in component form,
b $|AB|$,
c a unit vector in the direction of $\overrightarrow{AB}$.

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6 Given that $\underset{\sim}{v} = \begin{bmatrix} 2a \\ a \end{bmatrix}$, where $a > 0$, find:

a $|\underset{\sim}{v}|$

b $\hat{\underset{\sim}{v}}$

c $\underset{\sim}{v} + \underset{\sim}{v}$

d $\underset{\sim}{v} \cdot \underset{\sim}{v}$

7 Determine in each part whether the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

a $\underset{\sim}{a} = \begin{bmatrix} x - 1 \\ 1 - x \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} x + 1 \\ 1 + x \end{bmatrix}$

b $\underset{\sim}{a} = \begin{bmatrix} 5x \\ 5x - 1 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 1 - 2x \\ 2x \end{bmatrix}$

8 Find, correct to the nearest minute, the angle between the vectors $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$.

9 A quadrilateral $PQRS$ has vertices $P(-4, -5)$, $Q(10, 5)$, $R(5, 12)$ and $S(-9, 2)$.

a Show that $\overrightarrow{PQ} = \overrightarrow{SR}$.

b Show that $\overrightarrow{PQ} \cdot \overrightarrow{PS} = 0$.

c What type of special quadrilateral is $PQRS$?

10 Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ given:

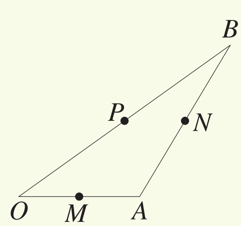
a $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$

b $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

11 Find $|\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a}|$ given $\underset{\sim}{a} = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$.

12 Suppose that A, B and C are the points $(-3, 1)$, $(4, 8)$ and $(2, -5)$ respectively. Use vector methods to find $\angle ABC$ correct to the nearest degree.

13 In $\triangle OAB$ to the right, M, N and P are the midpoints of OA, AB and OB respectively. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



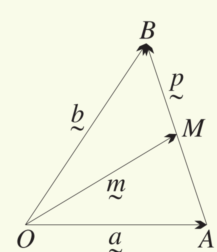
a Express $\overrightarrow{MA}$ in terms of $\underset{\sim}{a}$.

b Express $\overrightarrow{AN}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

c Hence show that $\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{b}$.

d Explain why $MNBP$ is a parallelogram.

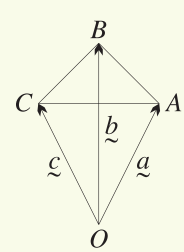
14 In $\triangle OAB$ to the right, M is the midpoint of AB. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{AB} = \underset{\sim}{p}$.



a Write $\underset{\sim}{p}$ and $\underset{\sim}{m}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Hence prove that $|\underset{\sim}{p}|^2 + 4|\underset{\sim}{m}|^2 = 2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{b}|^2$.

15 The diagram shows a kite $OABC$. Note that $OA = OC$ and $AB = CB$. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



a Explain why $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{c} \cdot \underset{\sim}{c}$.

b Explain why $(\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = (\underset{\sim}{b} - \underset{\sim}{c}) \cdot (\underset{\sim}{b} - \underset{\sim}{c})$.

c Hence prove that the diagonals of the kite are perpendicular.

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Review

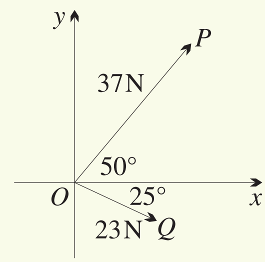
16 Draw a parallelogram $ABCD$ with $|AB| = |CD| = x$ and $|BC| = |DA| = y$. Let $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{b} = \overrightarrow{BC}$.

a Write $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Expand and simplify $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ and $(\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$.

c Hence prove that the diagonals of a parallelogram are equal if and only if the parallelogram is a rectangle.

17 In the diagram, the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent forces of magnitude 37 N and 23 N respectively.



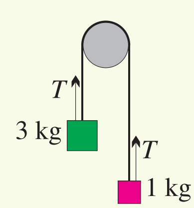
a Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in component form.

b Calculate, correct to 3 significant figures, the magnitude and direction of the resultant of the two forces.

18 A small fishing boat can travel at 8 km/h in still water. It is being steered due east, but there is a current running south at 2 km/h and a breeze blowing the boat south-west at 4 km/h . Find the resultant velocity of the boat, giving the speed correct to 2 decimal places and the direction correct to the nearest degree.

19 A light fitting of mass 2 kg is hung from a timber beam by two identical chains each inclined at 45° to the beam. Taking $g = 9.8\ \mathrm{m/s^2}$, find, correct to the nearest newton, the tension in each of the chains.

20 The diagram shows two objects of mass 3 kg and 1 kg attached to the ends of a light inextensible string that passes over a smooth pulley. The tension in the string is T Newtons. Take $g = 9.8\ \mathrm{m/s^2}$.



a Find the downwards acceleration of the 3 kg object.

b Find the value of T.

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