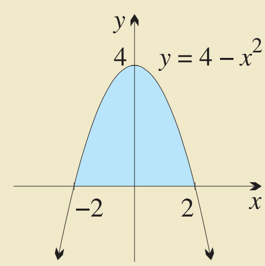
5 Integration

The calculation of areas has so far been restricted to regions bounded by straight lines or parts of circles. This chapter will extend the study of areas to regions bounded by more general curves.

For example, it will be possible to calculate the area of the shaded region in the diagram to the right, bounded by the parabola $y = 4 - x^2$ and the x-axis.



The method developed in this chapter is called *integration*. We will soon show that finding tangents and finding areas are inverse processes, so that integration is the inverse process of differentiation. This surprising result is called the *fundamental theorem of calculus* — the word 'fundamental' is well chosen because the theorem is the basis of the way in which calculus is used throughout mathematics and science.

Graphing software that can also estimate selected areas is useful in the chapter to illustrate how answers change as the curves and boundaries are varied.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

ISBN 978-1-108-76630-2

Cambridge University Press

5A Areas and the definite integral

All area formulae and calculations of area are based on two principles:

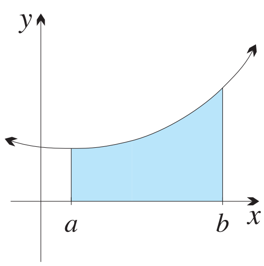
1. Area of a rectangle $=$ length $\times$ breadth.
2. When a region is dissected, the area is unchanged.

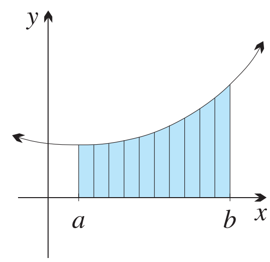
A region bounded by straight lines, such as a triangle or a trapezium, can be cut up and rearranged into rectangles with a few well-chosen cuts. Dissecting a curved region into rectangles, however, requires an infinite number of rectangles, and so must be a limiting process, just as differentiation is.

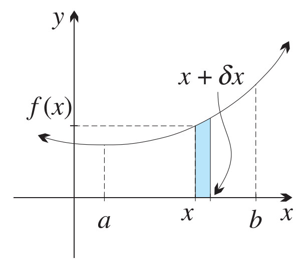
A new symbol — the definite integral

Some new notation is needed to reflect this process of infinite dissection as it applies to functions and their graphs.

The diagram on the left below shows the region contained between a given curve $y = f(x)$ and the x-axis, from $x = a$ to $x = b$, where $a \le b$. The curve must be continuous and, for the moment, never go below the x-axis.







In the middle diagram, the region has been dissected into a number of thin strips. Each strip is approximately a rectangle, but only roughly so, because the upper boundary is curved. The area of the region is the sum of the areas of all the strips.

The third diagram shows just one of the strips, above the value x on the x-axis. Its height at the left-hand end is $f(x)$, and provided the strip is very thin, the height is still about $f(x)$ at the right-hand end. Let the width of the strip be δx, where δx is, as usual in calculus, thought of as being very small. Then, roughly,

$$\begin{aligned} \text{area of strip} &\doteqdot \text{height} \times \text{width} \\ &\doteqdot f(x)\,\delta x. \end{aligned}$$

Adding up all the strips, using sigma notation for the sum,

$$\begin{aligned} \text{area of shaded region} &\doteqdot \sum_{x=a}^{b} (\text{area of each strip}) \\ &\doteqdot \sum_{x=a}^{b} f(x)\,\delta x. \end{aligned}$$

Now imagine that there are infinitely many of these strips, each infinitesimally thin, so that the inaccuracy disappears. This involves taking the limit, and we might expect to see something like this

$$\text{area of shaded region} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x,$$

but instead, we use the brilliant and flexible notation introduced by Leibnitz.

Mathematics Extension 1 Year 12
Cambridge Maths Stage 6
ISBN 978-1-108-76630-2

The width δx is replaced by the symbol dx, which suggests an infinitesimal width, and an old form $\int$ of the letter S is used to suggest an infinite sum under a smooth curve. The result is the strange-looking symbol $\int_a^b f(x)\,dx$. We now *define* this symbol to be the shaded area,

$$\int_a^b f(x)\,dx = \text{area of shaded region.}$$

The definite integral

This new object $\int_a^b f(x)\,dx$ is called a *definite integral*. The rest of the chapter is concerned with evaluating definite integrals and applying them.

1 THE DEFINITE INTEGRAL

Let $f(x)$ be a function that is continuous in a closed interval $[a, b]$, where $a \le b$.
For the moment, suppose that $f(x)$ is never negative in the interval.

- The *definite integral* $\int_a^b f(x)\,dx$ is defined to be the area of the region between the curve and the x-axis, from $x = a$ to $x = b$.
- The function $f(x)$ is called the *integrand*, and the values $x = a$ and $x = b$ are called the *lower* and *upper limits* (or *bounds*) of the integral.

The name 'integration' suggests putting many parts together to make a whole. The notation arises from building up the region from an infinitely large number of infinitesimally thin strips. Integration is 'making a whole' from these thin slices.

Evaluating definite integrals using area formulae

When the function is linear or circular, the definite integral can be calculated from the graph using well-known area formulae, although a quicker method will be developed later for linear functions.

Here are the relevant area formulae:

2 AREA FORMULAE FOR TRIANGLE, TRAPEZIUM AND CIRCLE

Triangle:	Area $= \frac{1}{2}bh$	$= \frac{1}{2} \times$ base $\times$ height
Trapezium:	Area $= \frac{1}{2}(a + b)h$	$=$ average of parallel sides $\times$ width
Circle:	Area $= \pi r^2$	$= \pi \times$ square of the radius

For a trapezium, h is the perpendicular distance between the parallel sides. Depending on the orientation, the word 'height' or 'width' may be more appropriate. Similarly, any side of a triangle may be taken as its 'base'.

ISBN 978-1-108-76630-2

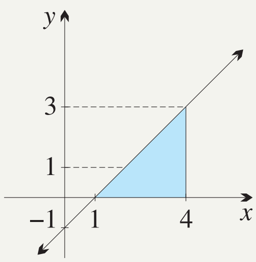
Example 1 5A

Evaluate using a graph and area formulae:

a $\int_1^4 (x - 1)\,dx$ **b** $\int_2^4 (x - 1)\,dx$ **c** $\int_{-2}^2 |x|\,dx$ **d** $\int_{-5}^5 \sqrt{25 - x^2}\,dx$

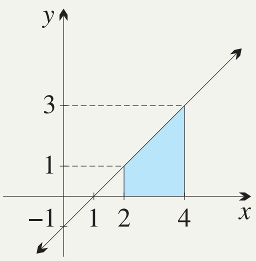
SOLUTION

a The area represented by the integral is the shaded triangle, with base $4 - 1 = 3$ and height 3.



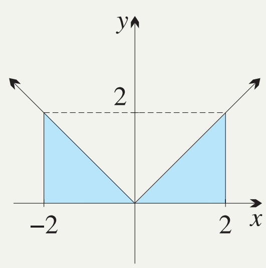
$$\text{Hence } \int_1^4 (x - 1)\,dx = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times 3 \times 3 = 4\tfrac{1}{2}.$$

b The area represented by the integral is the shaded trapezium, with width $4 - 2 = 2$ and parallel sides of length 1 and 3.



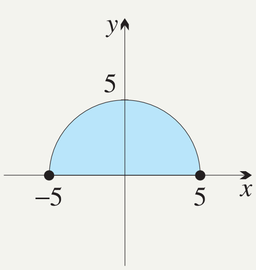
$$\text{Hence } \int_2^4 (x - 1)\,dx = \text{average of parallel sides} \times \text{width} = \frac{1 + 3}{2} \times 2 = 4.$$

c Each shaded triangle has base 2 and height 2.



$$\text{Hence } \int_{-2}^2 |x|\,dx = 2 \times \left(\tfrac{1}{2} \times \text{base} \times \text{height}\right) = 2 \times \left(\tfrac{1}{2} \times 2 \times 2\right) = 4.$$

d The shaded region is a semi-circle with radius 5.



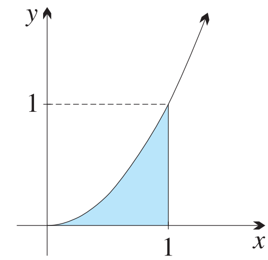
$$\text{Hence } \int_{-5}^5 \sqrt{25 - x^2}\,dx = \frac{1}{2} \times \pi r^2 = \frac{1}{2} \times 5^2 \times \pi = \frac{25\pi}{2}.$$

ISBN 978-1-108-76630-2

Cambridge University Press

Using upper and lower rectangles to trap an integral

First-principles integration calculations are more elaborate than those in first-principles differentiation. But they are used when proving the fundamental theorem of calculus, which will soon make calculations of integrals straightforward. One should therefore carry out a very few such calculations in order to understand what is happening. The technique, in geometric forms, was already highly developed by the Greeks.

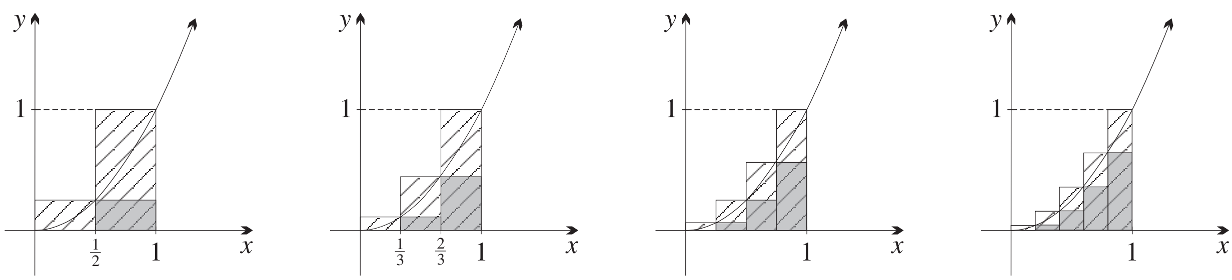


We shall find the integral $A = \int_0^1 x^2\, dx$ sketched above.

The first step in the process is to trap the integral using *upper rectangles* (or *outer rectangles*) that the shaded region lies inside, and *lower rectangles* (or *inner rectangles*) that lie inside the shaded region. To begin the process, notice that the shaded area is completely contained within the unit square, so

$$0 < A < 1.$$

In the four pictures below, the region has been sliced successively into two, three, four and five strips, then upper and lower rectangles have been constructed so that the region is trapped, or sandwiched, between the upper and lower rectangles. Calculating the areas of these upper and lower rectangles provides tighter and tighter bounds on the area A.



In the first picture,

$$\tfrac{1}{2} \times \left(\tfrac{1}{2}\right)^2 < A < \tfrac{1}{2} \times \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} \times 1^2$$
$$\tfrac{1}{8} < A < \tfrac{5}{8}.$$

In the second picture,

$$\tfrac{1}{3} \times \left(\tfrac{1}{3}\right)^2 + \tfrac{1}{3} \times \left(\tfrac{2}{3}\right)^2 < A < \tfrac{1}{3} \times \left(\tfrac{1}{3}\right)^2 + \tfrac{1}{3} \times \left(\tfrac{2}{3}\right)^2 + \tfrac{1}{3} \times 1^2$$
$$\tfrac{5}{27} < A < \tfrac{14}{27}.$$

In the third picture,

$$\tfrac{1}{4}\left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2\right) < A < \tfrac{1}{4}\left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 + \left(\tfrac{4}{4}\right)^2\right)$$
$$\tfrac{14}{64} < A < \tfrac{30}{64}.$$

In the fourth picture,

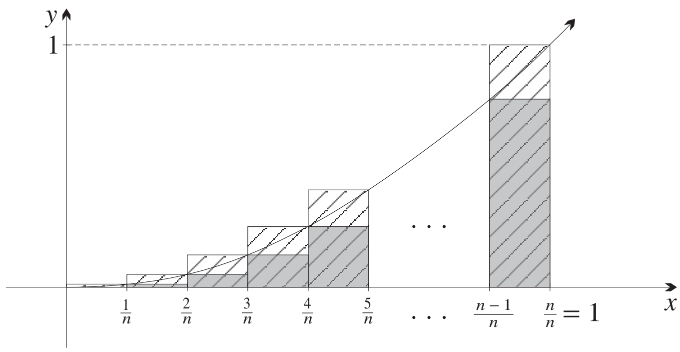
$$\tfrac{1}{5}\left(\left(\tfrac{1}{5}\right)^2 + \left(\tfrac{2}{5}\right)^2 + \left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2\right) < A < \tfrac{1}{5}\left(\left(\tfrac{1}{5}\right)^2 + \left(\tfrac{2}{5}\right)^2 + \left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2 + \left(\tfrac{5}{5}\right)^2\right)$$
$$\tfrac{30}{125} < A < \tfrac{55}{125}.$$

The limiting process

The bounds on the area are getting tighter, but the exact value of the area can only be obtained if this sandwiching process can be turned into a limiting process. The calculations below will need the formula for the sum of the first n squares proven in Question **2f** of Exercise 2A by induction:

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

ISBN 978-1-108-76630-2



A Divide the interval $0 \le x \le 1$ into n subintervals, each of width $\dfrac{1}{n}$.

On each subinterval form the upper rectangle and the lower rectangle.
Then the required region is entirely contained within the upper rectangles, and, in turn, the lower rectangles are entirely contained within the required region. Thus however many strips the region has been dissected into,

$$(\text{sum of lower rectangles}) \le A \le (\text{sum of upper rectangles}).$$

B The heights of the successive upper rectangles are $\dfrac{1^2}{n^2}, \dfrac{2^2}{n^2}, \dots, \dfrac{n^2}{n^2}$, and so, using the formula quoted above,

$$\begin{aligned}
\text{sum of upper rectangles} &= \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right) \\
&= \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2) \\
&= \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6}, \quad \text{(using the formula above)} \\
&= \frac{1}{3} \times \frac{n}{n} \times \frac{n+1}{n} \times \frac{2n+1}{2n} \\
&= \frac{1}{3} \times \left(1 + \frac{1}{n}\right) \times \left(1 + \frac{1}{2n}\right),
\end{aligned}$$

hence the sum of the upper rectangles has limit $\frac{1}{3}$ as $n \to \infty$.

C The heights of the successive lower rectangles are $0, \dfrac{1^2}{n^2}, \dfrac{2^2}{n^2}, \dots, \dfrac{(n-1)^2}{n^2}$, so substituting $n - 1$ for n into the quoted formula,

$$\begin{aligned}
\text{sum of lower rectangles} &= \frac{1}{n}\left(0 + \frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right) \\
&= \frac{1}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2) \\
&= \frac{1}{n^3} \times \frac{(n-1)n(2n-1)}{6} \\
&= \frac{1}{3} \times \frac{n}{n} \times \frac{n-1}{n} \times \frac{2n-1}{2n} \\
&= \frac{1}{3} \times \left(1 - \frac{1}{n}\right) \times \left(1 - \frac{1}{2n}\right),
\end{aligned}$$

hence the sum of the lower rectangles also has limit $\frac{1}{3}$ as $n \to \infty$.

ISBN 978-1-108-76630-2

D Finally, (sum of lower rectangles) $\le A \le$ (sum of upper rectangles), and both these sums have the same limit $\frac{1}{3}$, so it follows that $A = \frac{1}{3}$.

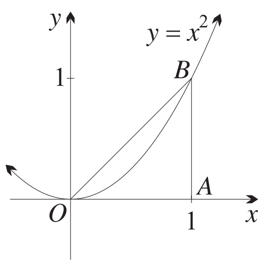
The conclusion of all this heavy machinery of limits is simply that $\int_0^1 x^2\,dx = \frac{1}{3}$.

You will be relieved to know that we will soon have much quicker methods.

Exercise 5A

FOUNDATION

1

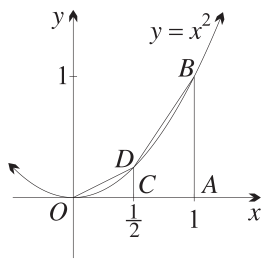


a Find the area of $\triangle OAB$ in the diagram above.

b Hence explain why

$$\int_0^1 x^2\,dx < \tfrac{1}{2}.$$

2



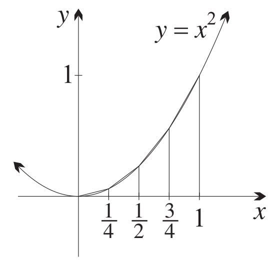
a Find the area of $\triangle OCD$ in the diagram above.

b Find the area of the trapezium $CABD$.

c Hence explain why

$$\int_0^1 x^2\,dx < \tfrac{3}{8}.$$

3

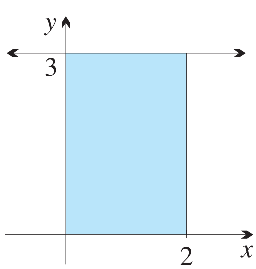


a Use the diagram above to show that $\int_0^1 x^2\,dx < \frac{11}{32}$.

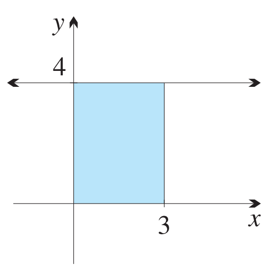
b Explain why $\frac{11}{32}$ is a better approximation to $\int_0^1 x^2\,dx$ than $\frac{3}{8}$ is.

4 Use area formulae to calculate these definite integrals (sketches are given below).

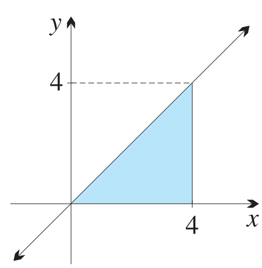
a $\displaystyle\int_0^2 3\,dx$



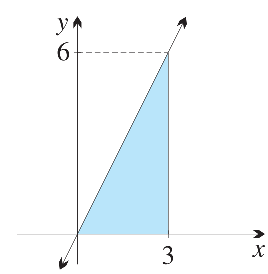
b $\displaystyle\int_0^3 4\,dx$



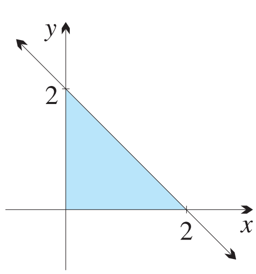
c $\displaystyle\int_0^4 x\,dx$



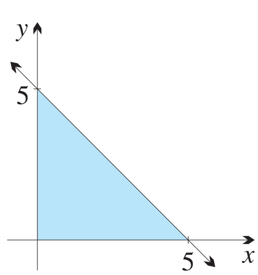
d $\displaystyle\int_0^3 2x\,dx$



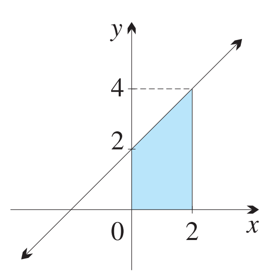
e $\displaystyle\int_0^2 (2 - x)\,dx$



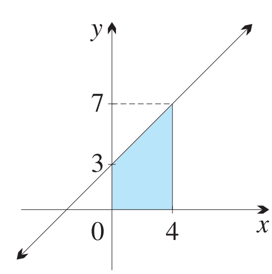
f $\displaystyle\int_0^5 (5 - x)\,dx$



g $\displaystyle\int_0^2 (x + 2)\,dx$



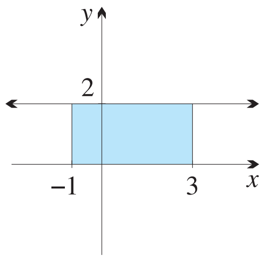
h $\displaystyle\int_0^4 (x + 3)\,dx$



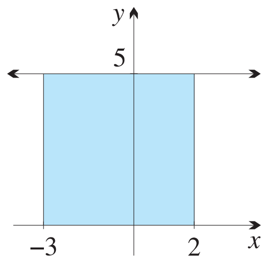
ISBN 978-1-108-76630-2

5 Use area formulae to calculate the sketched definite integrals.

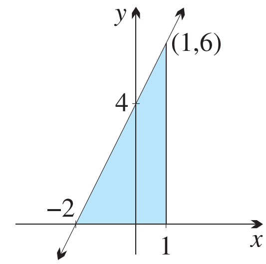
a $\int_{-1}^{3} 2\,dx$



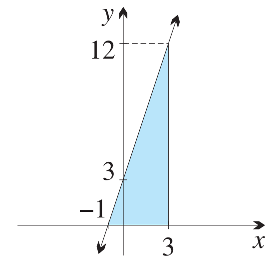
b $\int_{-3}^{2} 5\,dx$



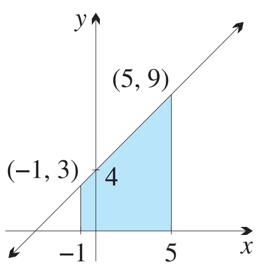
c $\int_{-2}^{1} (2x + 4)\,dx$



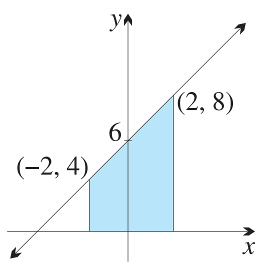
d $\int_{-1}^{3} (3x + 3)\,dx$



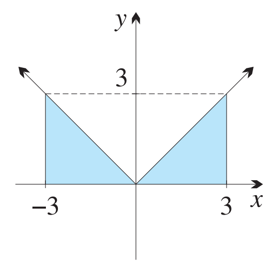
e $\int_{-1}^{5} (x + 4)\,dx$



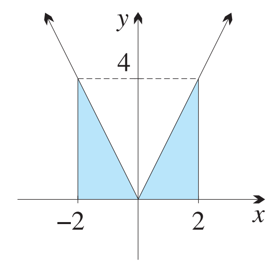
f $\int_{-2}^{2} (x + 6)\,dx$



g $\int_{-3}^{3} |x|\,dx$



h $\int_{-2}^{2} |2x|\,dx$



6 Sketch a graph of each definite integral, then use an area formula to calculate it.

a $\int_{0}^{3} 5\,dx$ **b** $\int_{-3}^{0} 5\,dx$ **c** $\int_{-1}^{4} 5\,dx$ **d** $\int_{-2}^{6} 5\,dx$

e $\int_{-5}^{0} (x + 5)\,dx$ **f** $\int_{0}^{2} (x + 5)\,dx$ **g** $\int_{2}^{4} (x + 5)\,dx$ **h** $\int_{-1}^{3} (x + 5)\,dx$

i $\int_{4}^{8} (x - 4)\,dx$ **j** $\int_{4}^{10} (x - 4)\,dx$ **k** $\int_{5}^{7} (x - 4)\,dx$ **l** $\int_{6}^{10} (x - 4)\,dx$

m $\int_{-2}^{2} |x|\,dx$ **n** $\int_{-4}^{4} |x|\,dx$ **o** $\int_{0}^{5} |x - 5|\,dx$ **p** $\int_{5}^{10} |x - 5|\,dx$

7 Sketch a graph of each definite integral, then use an area formula to calculate it.

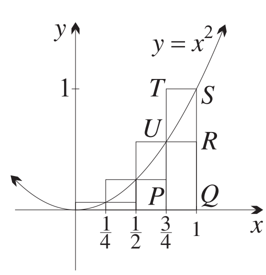
a $\int_{-4}^{4} \sqrt{16 - x^2}\,dx$ **b** $\int_{-5}^{0} \sqrt{25 - x^2}\,dx$

DEVELOPMENT

8 **a** In the diagram to the right, add the areas of the lower rectangles. (For example, $PQRU$ is a lower rectangle.)

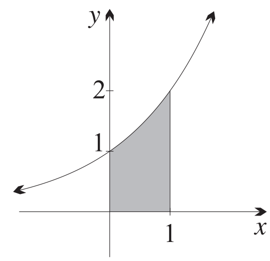
b Add the areas of the upper rectangles. (For example, $PQST$ is an upper rectangle.)

c Hence explain why $\frac{7}{32} < \int_{0}^{1} x^2\,dx < \frac{15}{32}$.



ISBN 978-1-108-76630-2

9 The area of the region in the diagram to the right is given by $\int_0^1 2^x\,dx$.



a Use one lower and one upper rectangle to show that $1 < \int_0^1 2^x\,dx < 2$.

b Use 2 lower and 2 upper rectangles of equal width to show that (with decimals rounded to one place)

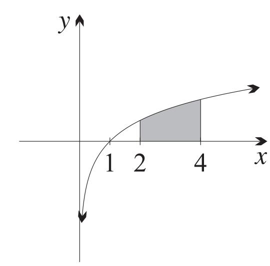
$1.2 < \int_0^1 2^x\,dx < 1.7$.

c Use 4 lower and 4 upper rectangles of equal width to show that

$1.3 < \int_0^1 2^x\,dx < 1.6$.

d What trend can be identified in the parts above?

10 The area of the region in the diagram to the right is given by $\int_2^4 \ln\,x\,dx$.



a Use 2 lower and 2 upper rectangles to show that (with decimals rounded to 2 places) $1.79 < \int_2^4 \ln\,x\,dx < 2.48$.

b Use 4 lower and 4 upper rectangles of equal width to show that

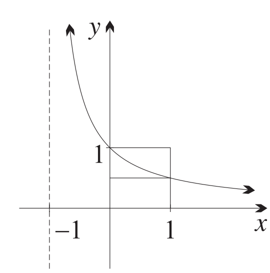
$1.98 < \int_2^4 \ln\,x\,dx < 2.33$.

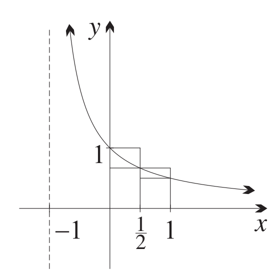
c Use 8 lower and 8 upper rectangles of equal width to show that

$2.07 < \int_2^4 \ln\,x\,dx < 2.24$.

d What trend can be identified in the parts above?

11 Let $A = \int_0^1 \frac{1}{x+1}\,dx$.





a Use the areas of the lower and upper rectangles in the top diagram to show that $\frac{1}{2} < A < 1$.

b Use the areas of the 2 lower and 2 upper rectangles in the bottom diagram to show that $\frac{7}{12} < A < \frac{5}{6}$. (That is, $0.58 < A < 0.83$, correct to 2 decimal places.)

c Use 3 lower and 3 upper rectangles of equal width to show that $\frac{37}{60} < A < \frac{47}{60}$. (That is, $0.62 < A < 0.78$, correct to 2 decimal places.)

d Finally, use 4 lower and 4 upper rectangles of equal width to show that $\frac{533}{840} < A < \frac{319}{420}$. (That is, correct to 2 decimal places, $0.63 < A < 0.76$.)

e As the number of rectangles increases, is the interval within which A lies getting bigger or smaller?

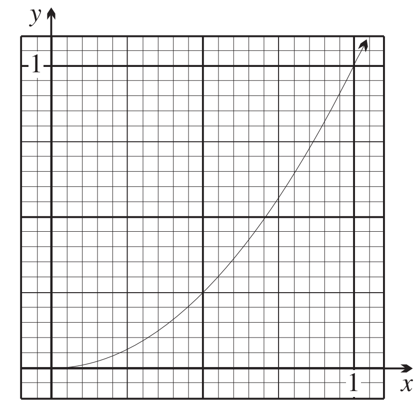
f The exact value of A is $\ln 2 = 0.693147\ldots$. Do the lower and upper limits of the intervals in parts **a** to **d** seem to be approaching the exact value?

ISBN 978-1-108-76630-2

12 [Technology] Some of the previous questions involve summing the areas of lower and upper rectangles to approximate a definite integral. Many software programs can do this automatically, using any prescribed number of rectangles. Steadily increasing the number of rectangles will show the sums of the lower and upper rectangles converging to the exact area, which can be checked either using area formulae or using the exact value of the definite integral as calculated later in the course.

Investigate some of the definite integrals from Questions 1–3 and 8–13 in this way.

13 The diagram to the right shows the graph of $y = x^2$ from $x = 0$ to $x = 1$, drawn on graph paper.
The scale is 20 little divisions to 1 unit. This means that 400 little squares make up 1 square unit.



a Count how many little squares there are under the graph from $x = 0$ to $x = 1$ (keeping reasonable track of fragments of squares), then divide by 400 to approximate $\int_0^1 x^2\,dx$.
Write your answer correct to 2 decimal places.

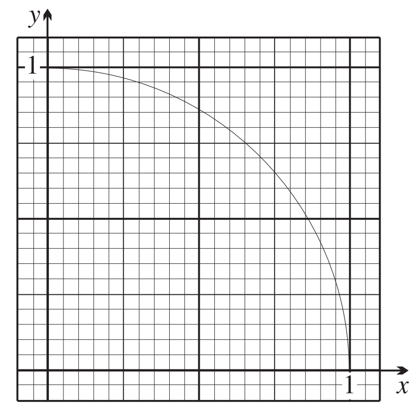
b By counting the appropriate squares, approximate:

i $\int_0^{\frac{1}{2}} x^2\,dx$ **ii** $\int_{\frac{1}{2}}^1 x^2\,dx$

Confirm that the sum of the answers to parts **i** and **ii** is the answer to part **a**.

14 The diagram to the right shows the quadrant

$$y = \sqrt{1 - x^2}, \text{ from } x = 0 \text{ to } x = 1.$$



As before, the scale is 20 little divisions to 1 unit.

a Count how many little squares there are under the graph from $x = 0$ to $x = 1$.

b Divide by 400 to approximate $\int_0^1 \sqrt{1 - x^2}\,dx$.
Write your answer correct to 2 decimal places.

c Hence, using the fact that a quadrant has area $\frac{1}{4}\pi r^2$, find an approximation for π. Give your answer correct to 2 decimal places.

ENRICHMENT

15 Using exactly the same setting out as in the example in the notes above this exercise, prove from first principles that $\int_0^1 x^3\,dx = \frac{1}{4}$. Use upper and lower rectangles, and take the limit as the number of rectangular strips approaches infinity.

Note: This calculation will need the formula $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$ for the sum of the first n cubes, which was proven by mathematical induction in worked Example 1 of Section 2A.

ISBN 978-1-108-76630-2

16 Prove the following two definite integrals from first principles. Use upper and lower rectangles, and take the limit as the number of rectangular strips approaches infinity.

a $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}$

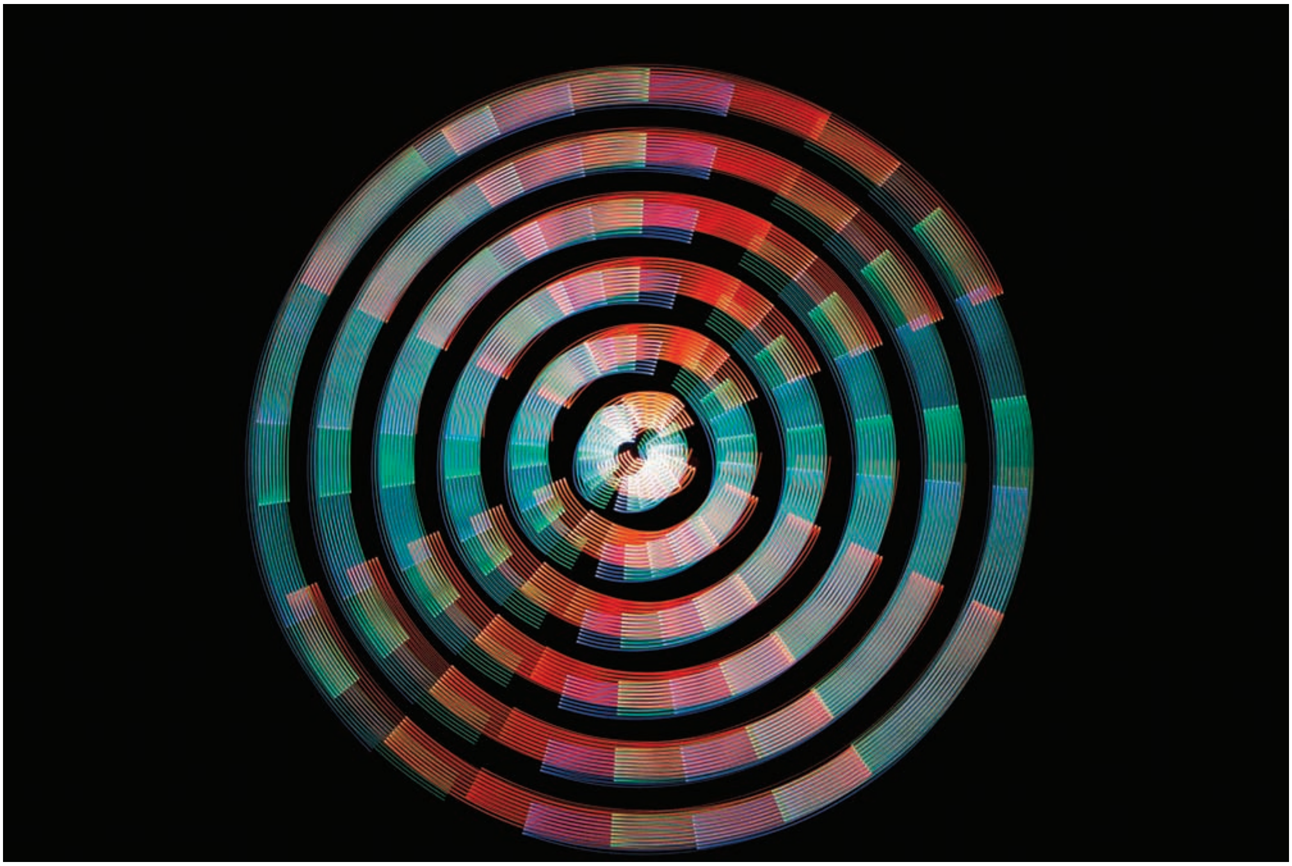
b $\displaystyle\int_0^a x^3\,dx = \frac{a^4}{4}$

17 Draw a large sketch of $y = x^2$ for $0 \le x \le 1$, and let U be the point $(1, 0)$. For some positive integer n, let $P_0 (= O), P_1, P_2, \ldots, P_n$ be the points on the curve with x-coordinates $x = 0$, $x = \frac{1}{n}$, $x = \frac{2}{n}, \ldots$, $x = \frac{n}{n} = 1$. Join the chords $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$, and join P_nU.

a Use the area formula for a trapezium to find the area of the polygon $P_0P_1P_2 \cdots P_nU$.

b Explain geometrically why this area is always greater than $\displaystyle\int_0^1 x^2\,dx$.

c Show that its limit as $n \to \infty$ is $\frac{1}{3}$. This is half of the proof that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$.



ISBN 978-1-108-76630-2

5B The fundamental theorem of calculus

There is a remarkably simple formula for evaluating definite integrals, based on taking the primitive of the function. The formula is called the *fundamental theorem of calculus* because the whole of calculus depends on it. We have delayed its challenging proof until Section 5D, when its usefulness will have been established.

Primitives

Let us first review from the last section of Chapter 4 what primitives are, and the first step in finding them.

3 PRIMITIVES

- A function $F(x)$ is called a *primitive* or *anti-derivative* of a function $f(x)$ if its derivative is $f(x)$:
  $F(x)$ is a primitive of $f(x)$ if $F'(x) = f(x)$.
- To find the general primitive of a power x^n, where $n \neq -1$:
  $$\text{If } \frac{dy}{dx} = x^n, \text{ then } y = \frac{x^{n+1}}{n+1} + C, \text{ for some constant } C.$$

'Increase the index by 1 and divide by the new index.'

Statement of the fundamental theorem

The fundamental theorem says that a definite integral can be evaluated by writing down any primitive $F(x)$ of $f(x)$, then substituting the upper and lower limits into it and subtracting.

4 THE FUNDAMENTAL THEOREM OF CALCULUS

Let $f(x)$ be a function that is continuous in a closed interval $[a, b]$. Then
$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) \text{ is any primitive of } f(x).$$

This result is extraordinary because it says that taking areas and taking tangents are inverse processes, which is not obvious.

Using the fundamental theorem to evaluate an integral

The conventional way to set out these calculations is to enclose the primitive in square brackets, writing the lower and upper limits as subscript and superscript respectively.

ISBN 978-1-108-76630-2

Example 2 5B

Evaluate these definite integrals. Then draw diagrams to show the regions that they represent, and check the answers using area formulae.

a $\displaystyle\int_0^2 2x\,dx$ **b** $\displaystyle\int_2^4 (2x - 3)\,dx$

SOLUTION

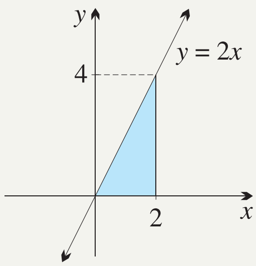
a $\displaystyle\int_0^2 2x\,dx = \Big[x^2\Big]_0^2$ (x^2 is a primitive of $2x$)

$= 2^2 - 0^2$ (substitute 2, then substitute 0 and subtract)

$= 4$

Using areas, area of triangle $= \dfrac{1}{2} \times 2 \times 4$

$= 4.$



b $\displaystyle\int_2^4 (2x - 3)\,dx = \Big[x^2 - 3x\Big]_2^4$ (take the primitive of each term)

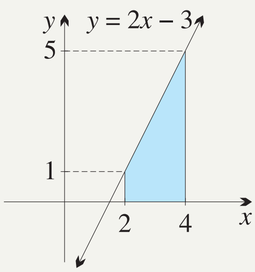
$= (16 - 12) - (4 - 6)$ (substitute 4, then substitute 2)

$= 4 - (-2)$

$= 6$

Using areas, area of trapezium $= \dfrac{1 + 5}{2} \times 2$

$= 6.$



Note: Whenever the primitive has two or more terms, brackets are needed when substituting the upper and lower limits of integration.

Example 3 5B

Evaluate these definite integrals.

a $\displaystyle\int_0^1 5x^2\,dx$ **b** $\displaystyle\int_0^4 (25 - x^2)\,dx$

SOLUTION

a $\displaystyle\int_0^1 5x^2\,dx = \left[\frac{5}{3}x^3\right]_0^1$ (increase the index from 2 to 3, then divide by 3)

$= \frac{5}{3} - 0$ (substitute 1, then substitute 0 and subtract)

$= \frac{5}{3}$

b $\displaystyle\int_0^4 (25 - x^2)\,dx = \left[25x - \tfrac{1}{3}x^3\right]_0^4$

$= (100 - \frac{1}{3} \times 64) - (0 - 0)$ (never omit a substitution of 0)

$= 78\frac{2}{3}$

ISBN 978-1-108-76630-2

Expanding brackets in the integrand

As with differentiation, it is often necessary to expand the brackets in the integrand before finding a primitive. With integration, there is no 'product rule' that could avoid the expansion.

Example 4 5B

Expand the brackets in each integral, then evaluate it.

a $\displaystyle\int_1^6 x(x+1)\,dx$

b $\displaystyle\int_0^3 (x-4)(x-6)\,dx$

SOLUTION

a $\displaystyle\int_1^6 x(x+1)\,dx$

$\displaystyle= \int_1^6 (x^2+x)\,dx$

$\displaystyle= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^6$

$\displaystyle= (72+18)-\left(\frac{1}{3}+\frac{1}{2}\right)$

$= 90 - \frac{5}{6}$ (care with the fractions)

$= 89\frac{1}{6}$

b $\displaystyle\int_0^3 (x-4)(x-6)\,dx$

$\displaystyle= \int_0^3 (x^2-10x+24)\,dx$

$\displaystyle= \left[\frac{x^3}{3}-5x^2+24x\right]_0^3$

$= (9-45+72)-(0-0+0)$

$= 36$

Writing the integrand as separate fractions

If the integrand is a fraction with two or more terms in the numerator, it should normally be written as separate fractions, as with differentiation. With integration, there is no 'quotient rule' that could avoid this.

Example 5 5B

Write each integrand as two separate fractions, then evaluate the integral.

a $\displaystyle\int_1^2 \frac{3x^4-2x^2}{x^2}\,dx$

b $\displaystyle\int_{-3}^{-2} \frac{x^3-2x^4}{x^3}\,dx$

SOLUTION

a $\displaystyle\int_1^2 \frac{3x^4-2x^2}{x^2}\,dx$

$\displaystyle= \int_1^2 (3x^2-2)\,dx$

$\displaystyle= \Big[x^3-2x\Big]_1^2$

$= (8-4)-(1-2)$

$= 4-(-1)$

$= 5$

b $\displaystyle\int_{-3}^{-2} \frac{x^3-2x^4}{x^3}\,dx$

$\displaystyle= \int_{-3}^{-2} (1-2x)\,dx$

$\displaystyle= \Big[x-x^2\Big]_{-3}^{-2}$

$= (-2-4)-(-3-9)$

$= -6-(-12)$

$= 6$

ISBN 978-1-108-76630-2

Negative indices

The fundamental theorem also works when the indices are negative. Care is needed when converting between negative powers of x and fractions.

Example 6

Use negative indices to evaluate these definite integrals.

a $\displaystyle\int_1^5 x^{-2}\,dx$

b $\displaystyle\int_1^2 \frac{1}{x^4}\,dx$

SOLUTION

a Increase the index from -2 to -1, and divide by -1.

$$\begin{aligned}\int_1^5 x^{-2}\,dx &= \left[\frac{x^{-1}}{-1}\right]_1^5\\ &= \left[-\frac{1}{x}\right]_1^5\\ &= -\frac{1}{5} - (-1)\\ &= -\frac{1}{5} + 1\\ &= \frac{4}{5}\end{aligned}$$

b Increase the index from -4 to -3, and divide by -3.

$$\begin{aligned}\int_1^2 \frac{1}{x^4}\,dx &= \int_1^2 x^{-4}\,dx\\ &= \left[\frac{x^{-3}}{-3}\right]_1^2\\ &= \left[-\frac{1}{3x^3}\right]_1^2\\ &= -\frac{1}{24} - \left(-\frac{1}{3}\right)\\ &= \frac{7}{24}\end{aligned}$$

Note: The negative index -1 cannot be handled by this rule, because it would generate division by 0, which is nonsense:

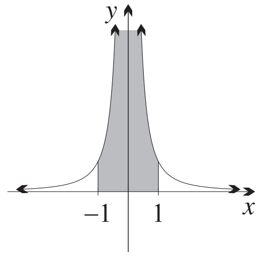
$$\int_1^2 \frac{1}{x}\,dx = \int_1^2 x^{-1}\,dx = \left[\frac{x^0}{0}\right]_2^1 = \text{nonsense.}$$

Chapter 6 on exponential and logarithmic functions will handle this integral.

Warning: Do not integrate across an asymptote

The following calculation seems just as valid as part (b) above:

$$\begin{aligned}\int_{-1}^1 x^{-4}\,dx &= \left[\frac{x^{-3}}{-3}\right]_{-1}^1\\ &= -\frac{1}{3} - \frac{1}{3}\\ &= -\frac{2}{3}.\end{aligned}$$

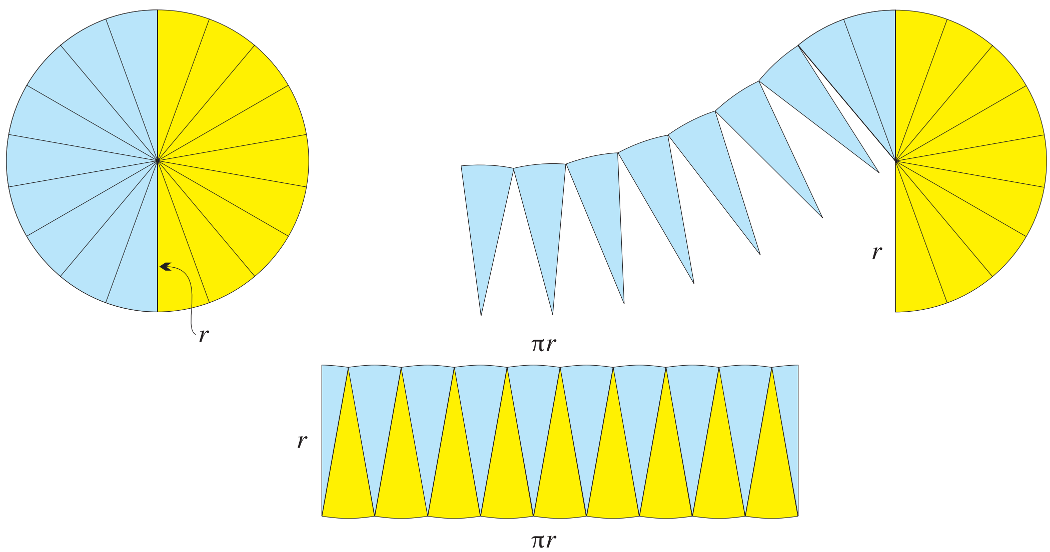


But in fact the calculation is nonsense — the function has an asymptote at $x = 0$, so it is not even defined there. (And the function $y = x^{-4}$ is always positive, so how could it give a negative answer for an integral?) You cannot integrate across an asymptote, and you always need to be on the lookout for such meaningless integrals.

ISBN 978-1-108-76630-2

The area of a circle

In earlier years, the formula $A = \pi r^2$ for the area of a circle was developed. Because the boundary is a curve, some limiting process had to be used in its proof. For comparison with the notation for the definite integral explained at the start of this section, here is the most common version of that argument — a little rough in its logic, but very quick. It involves dissecting the circle into infinitesimally thin sectors and then rearranging them into a rectangle.



The height of the rectangle in the lower diagram is r. Because the circumference $2\pi r$ is divided equally between the top and bottom sides, the length of the rectangle is πr. Hence the rectangle has area πr^2, which is therefore the area of the circle.

Exercise 5B

FOUNDATION

Technology: Many programs allow definite integrals to be calculated automatically. This allows not just quick checking of the answers, but experimentation with further definite integrals. It would be helpful to generate screen sketches of the graphs and the regions associated with the integrals.

1 Evaluate these definite integrals using the fundamental theorem.

a $\int_0^1 2x\,dx$ **b** $\int_1^4 2x\,dx$ **c** $\int_1^3 4x\,dx$

d $\int_2^5 8x\,dx$ **e** $\int_2^3 3x^2\,dx$ **f** $\int_0^3 5x^4\,dx$

g $\int_1^2 10x^4\,dx$ **h** $\int_0^1 12x^5\,dx$ **i** $\int_0^1 11x^{10}\,dx$

2 **a** Evaluate these definite integrals using the fundamental theorem.

i $\int_0^1 4\,dx$ **ii** $\int_2^7 5\,dx$ **iii** $\int_4^5 dx$

b Check your answers by sketching the graph of the region involved.

ISBN 978-1-108-76630-2

Cambridge University Press

3 Evaluate these definite integrals using the fundamental theorem.

a $\int_3^6 (2x + 1)\,dx$ **b** $\int_2^4 (2x - 3)\,dx$ **c** $\int_0^3 (4x + 5)\,dx$

d $\int_2^3 (3x^2 - 1)\,dx$ **e** $\int_1^4 (6x^2 + 2)\,dx$ **f** $\int_0^1 (3x^2 + 2x)\,dx$

g $\int_1^2 (4x^3 + 3x^2 + 1)\,dx$ **h** $\int_0^2 (2x + 3x^2 + 8x^3)\,dx$ **i** $\int_3^5 (3x^2 - 6x + 5)\,dx$

4 Evaluate these definite integrals using the fundamental theorem. You will need to take care when finding powers of negative numbers.

a $\int_{-1}^0 (1 - 2x)\,dx$ **b** $\int_{-1}^0 (2x + 3)\,dx$ **c** $\int_{-2}^1 3x^2\,dx$

d $\int_{-1}^2 (4x^3 + 5)\,dx$ **e** $\int_{-2}^2 (5x^4 + 6x^2)\,dx$ **f** $\int_{-2}^{-1} (4x^3 + 12x^2 - 3)\,dx$

5 Evaluate these definite integrals using the fundamental theorem. You will need to take care when adding and subtracting fractions.

a $\int_1^4 (x + 2)\,dx$ **b** $\int_0^2 (x^2 + x)\,dx$ **c** $\int_0^3 (x^3 + x^2)\,dx$

d $\int_{-1}^1 (x^3 - x + 1)\,dx$ **e** $\int_{-2}^3 (2x^2 - 3x + 1)\,dx$ **f** $\int_{-4}^{-2} (16 - x^3 - x)\,dx$

DEVELOPMENT

6 By expanding the brackets first, evaluate these definite integrals.

a $\int_2^3 x(2 + 3x)\,dx$ **b** $\int_0^2 (x + 1)(3x + 1)\,dx$ **c** $\int_{-1}^1 x^2(5x^2 + 1)\,dx$

d $\int_{-1}^2 (x - 3)^2\,dx$ **e** $\int_{-1}^0 x(x - 1)(x + 1)\,dx$ **f** $\int_{-1}^0 (1 - x^2)^2\,dx$

7 Write each integrand as separate fractions, then evaluate the integral.

a $\int_1^3 \frac{3x^3 + 4x^2}{x}\,dx$ **b** $\int_1^2 \frac{4x^4 - x}{x}\,dx$ **c** $\int_2^3 \frac{5x^2 + 9x^4}{x^2}\,dx$

d $\int_1^2 \frac{x^3 + 4x^2}{x}\,dx$ **e** $\int_1^3 \frac{x^3 - x^2 + x}{x}\,dx$ **f** $\int_{-2}^{-1} \frac{x^3 - 2x^5}{x^2}\,dx$

8 Evaluate these definite integrals using the fundamental theorem. You will need to take care when finding powers of fractions.

a $\int_0^{\frac{1}{2}} x^2\,dx$ **b** $\int_0^{\frac{2}{3}} (2x + 3x^2)\,dx$ **c** $\int_{\frac{1}{2}}^{\frac{3}{4}} (6 - 4x)\,dx$

ISBN 978-1-108-76630-2

9 **a** Evaluate these definite integrals.

i $\int_{5}^{10} x^{-2}\,dx$ **ii** $\int_{2}^{3} 2x^{-3}\,dx$ **iii** $\int_{\frac{1}{2}}^{1} 4x^{-5}\,dx$

b By writing them with negative indices, evaluate these definite integrals.

i $\int_{1}^{2} \frac{1}{x^2}\,dx$ **ii** $\int_{1}^{4} \frac{1}{x^3}\,dx$ **iii** $\int_{\frac{1}{2}}^{1} \frac{3}{x^4}dx$

10 **a** **i** Show that $\int_{2}^{k} 3\,dx = 3k - 6$.

ii Hence find the value of k, given that $\int_{2}^{k} 3\,dx = 18$.

b **i** Show that $\int_{0}^{k} x\,dx = \frac{1}{2}k^2$.

ii Hence find the positive value of k, given that $\int_{0}^{k} x\,dx = 18$.

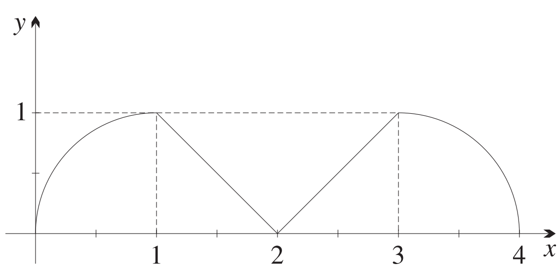
11 Find the value of k if $k > 0$ and.

a $\int_{k}^{3} 2\,dx = 4$ **b** $\int_{k}^{8} 3\,dx = 12$ **c** $\int_{2}^{3} (k - 3)\,dx = 5$

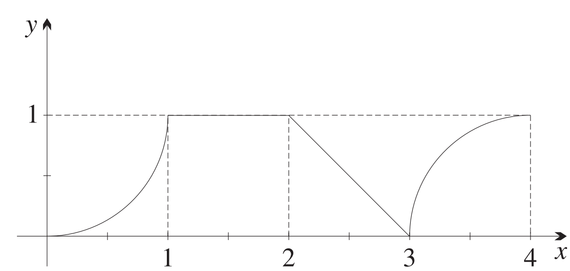
d $\int_{3}^{k} (x - 3)\,dx = 0$ **e** $\int_{1}^{k} (x + 1)\,dx = 6$ **f** $\int_{1}^{k} (k + 3x)\,dx = \frac{13}{2}$

12 Use area formulae to find $\int_{0}^{4} f(x)\,dx$ in each sketch of $f(x)$.

a



b



13 Evaluate each definite integral.

a $\int_{1}^{2} \frac{1 + x^2}{x^2}\,dx$ **b** $\int_{-2}^{-1} \frac{1 + 2x}{x^3}\,dx$ **c** $\int_{-3}^{-1} \frac{1 - x^3 - 4x^5}{2x^2}\,dx$

14 Evaluate these definite integrals.

a $\int_{1}^{3} \left(x + \frac{1}{x}\right)^2 dx$ **b** $\int_{1}^{2} \left(x^2 + \frac{1}{x^2}\right)^2 dx$ **c** $\int_{-2}^{-1} \left(\frac{1}{x^2} + \frac{1}{x}\right)^2 dx$

15 **a** Explain why the function $y = \frac{1}{x^2}$ is never negative.

b Sketch the integrand and explain why the argument below is invalid.

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

c Without evaluating any integrals, say which of these integrals are meaningless.

i $\int_{0}^{2} \frac{1}{(3 - x)^2}\,dx$ **ii** $\int_{2}^{4} \frac{1}{(3 - x)^2}\,dx$ **iii** $\int_{4}^{6} \frac{1}{(3 - x)^2}\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press

16 **a** Use the differential form of the fundamental theorem $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ to find:

i $\frac{d}{dx}\int_1^x t^2\,dt$

ii $\frac{d}{dx}\int_2^x (t^3 + 3t)\,dt$

iii $\frac{d}{dx}\int_a^x \frac{1}{t}\,dt$

iv $\frac{d}{dx}\int_a^x (t^3 - 3)^4\,dt$

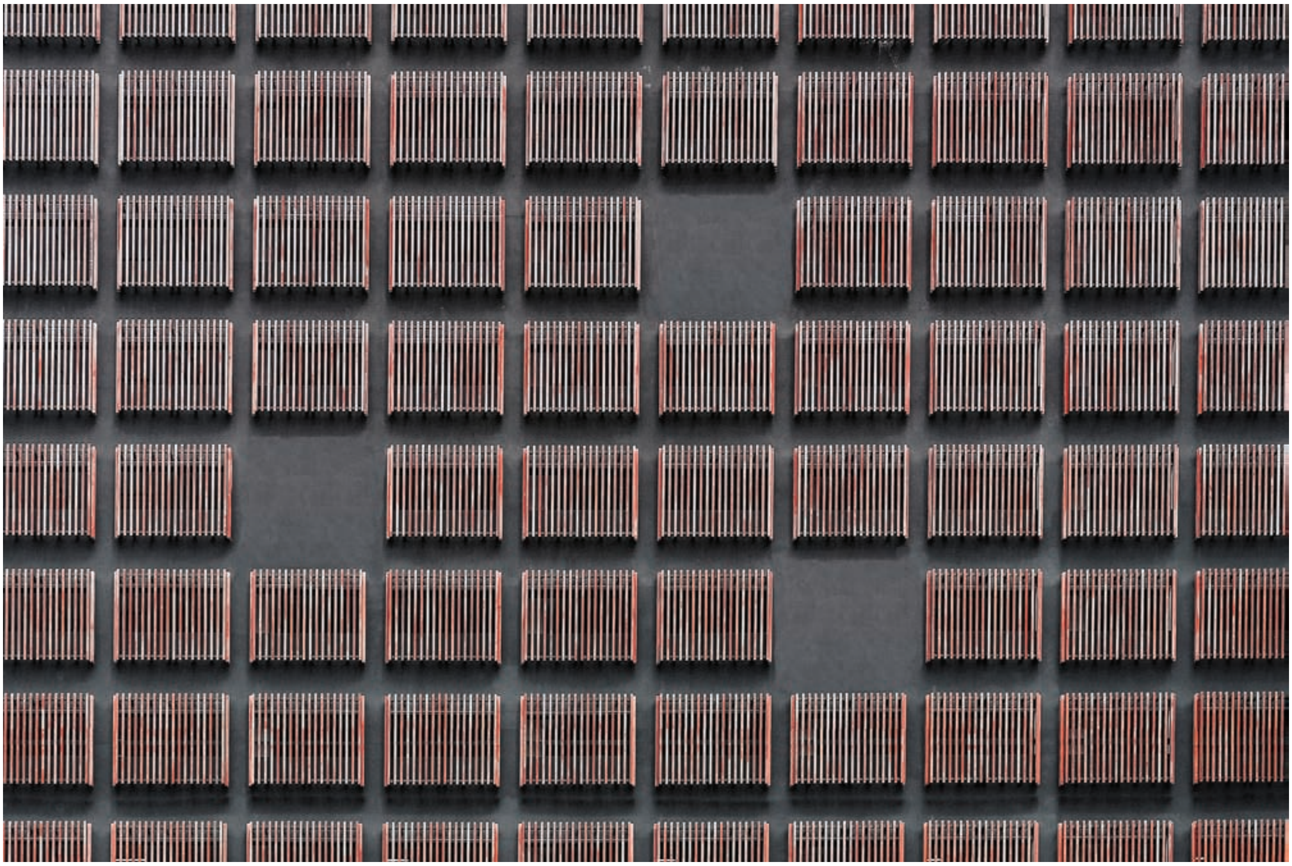
b Confirm your answers to parts (i) and (ii) above by evaluating the definite integral and then differentiating with respect to x.

ENRICHMENT

17 The derivative of the function $U(x)$ is $u(x)$.

a Find $V'(x)$, where $V(x) = (a - x)U(x) + \int_0^x U(t)\,dt$ and a is a constant.

b Hence prove that $\int_0^a U(x)\,dx = aU(0) + \int_0^a (a - x)\,u(x)\,dx$.



ISBN 978-1-108-76630-2

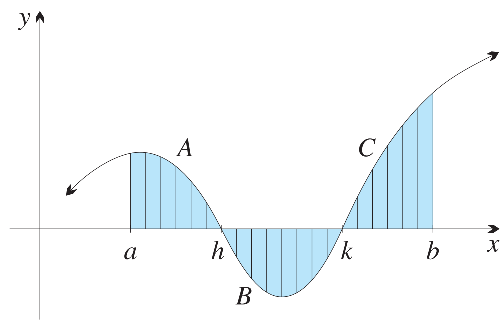
Cambridge University Press

5C The definite integral and its properties

This section will first extend the theory to functions with negative values. Then some simple properties of the definite integral will be established using arguments about the dissection of regions.

Integrating functions with negative values

When a function has negative values, its graph is below the x-axis, so the 'heights' of the little rectangles in the dissection are negative numbers. This means that any areas below the x-axis should contribute negative values to the value of the final integral.



For example, in the diagram above, where $a < h < k < b$, the region B is below the x-axis because the function $f(x)$ is negative in $[h, k]$. So the heights of all the infinitesimal strips making up B are negative, and B therefore contributes a negative number to the definite integral,

$$\int_a^b f(x)\,dx = +\text{area } A - \text{area } B + \text{area } C.$$

Thus we attach the sign + or − to each area, depending on whether the curve, and therefore the region, is above or below the x-axis. For this reason, the three terms in the sum above are often referred to as *signed areas under the curve*, because a sign has been attached to the area of each region.

5 THE DEFINITE INTEGRAL AS THE SUM OF SIGNED AREAS

Let $f(x)$ be a function continuous in the closed interval $[a, b]$, where $a \le b$, and suppose that we are taking the definite integral over $[a, b]$.

- For regions where the curve is above the x-axis, we attach + to the area.
  For regions where the curve is below the x-axis, we attach − to the area.
  These areas, with signs attached, are called *signed areas under the curve*.
- The *definite integral* $\int_a^b f(x)\,dx$ is the sum of these signed areas under the curve in the interval $[a, b]$.

The whole definite integral $\int_a^b f(x)\,dx$ is often also referred to as *the signed area under the curve*.

ISBN 978-1-108-76630-2

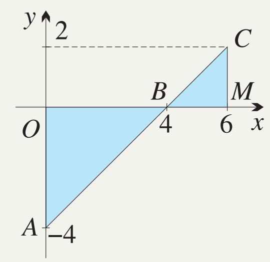
Example 7 5C

Evaluate these definite integrals.

a $\int_0^4 (x - 4)\,dx$ **b** $\int_4^6 (x - 4)\,dx$ **c** $\int_0^6 (x - 4)\,dx$

Sketch the graph of $y = x - 4$ and shade the regions associated with these integrals. Then explain how each result is related to the shaded regions.

SOLUTION



a $\int_0^4 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^4$
$= (8 - 16) - (0 - 0)$
$= -8$

b $\int_4^6 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_4^6$
$= (18 - 24) - (8 - 16)$
$= -6 - (-8)$
$= 2$

c $\int_0^6 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^6$
$= (18 - 24) - (0 - 0)$
$= -6$

Triangle OAB is below the x-axis and has area 8, and triangle BMC is above the x-axis, and has area 2, so the answers -8 in part **a** and $+2$ in part **b** are expected. This integral in part **c** represents the area of $\triangle BMC$ minus the area of $\triangle OAB$, so the value of the integral is $2 - 8 = -6$ is expected.

Odd and even functions

In the first example below, the function $y = x^3 - 4x$ is an odd function, with point symmetry in the origin. Thus the area of each shaded hump is the same. Hence the whole integral from $x = -2$ to $x = 2$ is zero, because the equal humps above and below the x-axis cancel out.

In the second diagram, the function $y = x^2 + 1$ is even, with line symmetry in the y-axis. Thus the areas to the left and right of the y-axis are equal, so there is a doubling instead of a cancelling.

6 INTEGRATING ODD AND EVEN FUNCTIONS

- If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.
- If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

ISBN 978-1-108-76630-2

Cambridge University Press

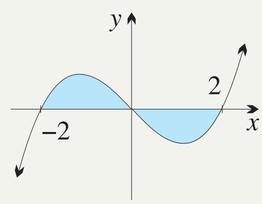
Example 8 5C

Sketch these integrals, then evaluate them using symmetry.

a $\displaystyle\int_{-2}^{2} (x^3 - 4x)\,dx$ **b** $\displaystyle\int_{-2}^{2} (x^2 + 1)\,dx$

SOLUTION

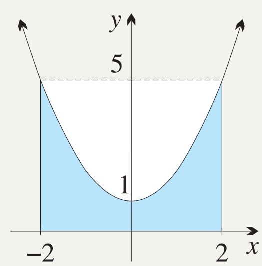
a $\displaystyle\int_{-2}^{2} (x^3 - 4x)\,dx = 0,$ because the integrand is odd.



Without this simplification, the calculation is:

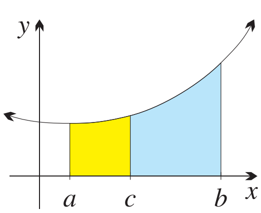
$$\begin{aligned}\int_{-2}^{2} (x^3 - 4x)\,dx &= \Big[\tfrac{1}{4}x^4 - 2x^2\Big]_{-2}^{2}\\ &= (4 - 8) - (4 - 8)\\ &= 0, \text{ as before.}\end{aligned}$$

b Because the integrand is even,



$$\begin{aligned}\int_{-2}^{2} (x^2 + 1)\,dx &= 2\int_{0}^{2} (x^2 + 1)\,dx\\ &= 2\Big[\tfrac{1}{3}x^3 + x\Big]_{0}^{2}\\ &= 2\big((2\tfrac{2}{3} + 2) - (0 + 0)\big)\\ &= 9\tfrac{1}{3}.\end{aligned}$$

Dissection of the interval



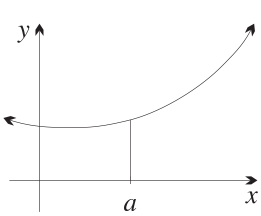
When a region is dissected, its area remains the same. In particular, we can always dissect the region by dissecting the interval $a \le x \le b$ over which we are integrating.

Thus if $f(x)$ is being integrated over the closed interval $[a, b]$ and the number c lies in this interval, then:

7 DISSECTION OF THE INTERVAL

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

Intervals of zero width



Suppose that a function is integrated over an interval $a \le x \le a$ of width zero, and that the function is defined at $x = a$. In this situation, the region also has width zero, so the integral is zero.

8 INTERVALS OF ZERO WIDTH

$$\int_{a}^{a} f(x)\,dx = 0$$

ISBN 978-1-108-76630-2

Running an integral backwards from right to left

A further small qualification must be made to the definition of the definite integral. Suppose that the limits of the integral are reversed, so that the integral 'runs backwards' from right to left over the interval. Then its value reverses in sign:

9 REVERSING THE INTERVAL

Let $f(x)$ be continuous in the closed interval $[a, b]$, where $a \le b$. Then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

This agrees perfectly with calculations using the fundamental theorem, because

$$F(a) - F(b) = -\big(F(b) - F(a)\big).$$

Example 9 5C

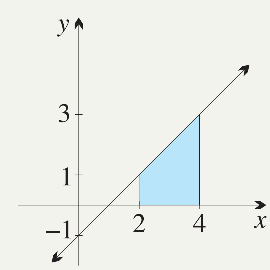
Evaluate and compare these two definite integrals using the fundamental theorem.

a $\displaystyle\int_2^4 (x - 1)\,dx$ **b** $\displaystyle\int_4^2 (x - 1)\,dx$

SOLUTION

a $$\int_2^4 (x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_2^4$$
$$= (8 - 4) - (2 - 2)$$
$$= 4,$$

which is positive, because the region is above the x-axis.



b $$\int_4^2 (x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_4^2$$
$$= (2 - 2) - (8 - 4)$$
$$= -4,$$

which is the opposite of part **a**, because the integral runs backwards from right to left, from $x = 4$ to $x = 2$.

Sums of functions

When two functions are added, the two regions are piled on top of each other, so that:

10 INTEGRAL OF A SUM

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

ISBN 978-1-108-76630-2

Example 10 5C

Evaluate these two expressions and show that they are equal.

a $\int_0^1 (x^2 + x + 1)\,dx$

b $\int_0^1 x^2\,dx + \int_0^1 x\,dx + \int_0^1 1\,dx$

SOLUTION

a $$\begin{aligned}\int_0^1 (x^2 + x + 1)\,dx &= \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_0^1 \\ &= \left(\tfrac{1}{3} + \tfrac{1}{2} + 1\right) - (0 + 0 + 0) \\ &= 1\tfrac{5}{6}.\end{aligned}$$

b $$\begin{aligned}\int_0^1 x^2\,dx + \int_0^1 x\,dx + \int_0^1 1\,dx &= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 + \Big[x\Big]_0^1 \\ &= \left(\tfrac{1}{3} - 0\right) + \left(\tfrac{1}{2} - 0\right) + (1 - 0) \\ &= 1\tfrac{5}{6}, \quad \text{the same as in part } \mathbf{a}.\end{aligned}$$

Multiples of functions

Similarly, when a function is multiplied by a constant, the region is stretched vertically by that constant, so that:

11 INTEGRAL OF A MULTIPLE

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

Example 11 5C

Evaluate these two expressions and show that they are equal.

a $\int_1^3 10x^3\,dx$

b $10\int_1^3 x^3\,dx$

SOLUTION

a $$\begin{aligned}\int_1^3 10x^3\,dx &= \left[\frac{10x^4}{4}\right]_1^3 \\ &= \frac{810}{4} - \frac{10}{4} \\ &= \frac{800}{4} \\ &= 200\end{aligned}$$

b $$\begin{aligned}10\int_1^3 x^3\,dx &= 10 \times \left[\frac{x^4}{4}\right]_1^3 \\ &= 10 \times \left(\frac{81}{4} - \frac{1}{4}\right) \\ &= 10 \times \frac{80}{4} \\ &= 200\end{aligned}$$

ISBN 978-1-108-76630-2

Inequalities with definite integrals

Suppose that a curve $y = f(x)$ is always underneath another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is less than the area under the curve $y = g(x)$.

In the language of definite integrals:

12 INEQUALITY

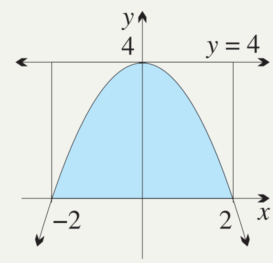
If $f(x) \le g(x)$ in the closed interval $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Example 12 **5C**

a Sketch the graph of $f(x) = 4 - x^2$, for $-2 \le x \le 2$.

b Explain why $0 \le \int_{-2}^{2} (4 - x^2)\,dx \le 16$.



SOLUTION

a The parabola and line are sketched opposite.

b Clearly $0 \le 4 - x^2 \le 4$ over the interval $-2 \le x \le 2$.
Hence the region associated with the integral is inside the square of side length 4 in the diagram opposite.

Exercise 5C

FOUNDATION

Technology: All the properties of the definite integral discussed in this section have been justified visually from sketches of the graphs. Computer sketches of the graphs in this exercises would be helpful in reinforcing these explanations. The simplification of integrals of even and odd functions is particularly important and is easily demonstrated visually by graphing programs.

1 Evaluate $\int_4^5 (2x - 3)\,dx$ and $\int_5^4 (2x - 3)\,dx$. What do you notice?

2 Show, by evaluating the definite integrals, that:

a $\int_0^1 6x^2\,dx = 6\int_0^1 x^2\,dx$

b $\int_{-1}^{2} (x^3 + x^2)\,dx = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} x^2\,dx$

c $\int_0^3 (x^2 - 4x + 3)\,dx = \int_0^2 (x^2 - 4x + 3)\,dx + \int_2^3 (x^2 - 4x + 3)\,dx$

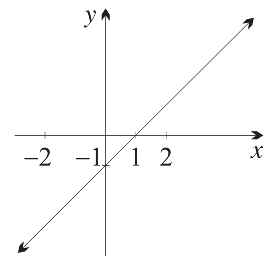
3 Without evaluating the definite integrals, explain why:

a $\int_2^2 (x^2 - 3x)\,dx = 0$

b $\int_{-2}^{2} x\,dx = 0$

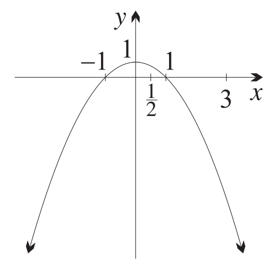
ISBN 978-1-108-76630-2

4 The diagram to the right shows the line $y = x - 1$. Without evaluating the definite integrals, explain why:



a $\int_0^1 (x - 1)\,dx$ is negative,
b $\int_1^2 (x - 1)\,dx$ is positive,
c $\int_0^2 (x - 1)\,dx$ is zero,
d $\int_{-2}^2 (x - 1)\,dx$ is negative.

5 The diagram to the right shows the parabola $y = 1 - x^2$. Without evaluating the definite integrals, explain why:



a $\int_{-1}^1 (1 - x^2)\,dx$ is positive,
b $\int_1^3 (1 - x^2)\,dx$ is negative,
c $\int_{-1}^0 (1 - x^2)\,dx = \int_0^1 (1 - x^2)\,dx$,
d $\int_0^{\frac{1}{2}} (1 - x^2)\,dx > \int_{\frac{1}{2}}^1 (1 - x^2)\,dx$.

6 a If $\int_1^3 f(x)\,dx = 7$, then what is the value of $\int_3^1 f(x)\,dx$?
b If $\int_2^1 g(x)\,dx = -5$, then what is the value of $\int_1^2 g(x)\,dx$?

7 Use a diagram to explain why $\int_0^1 2x\,dx > \int_0^1 x\,dx$.

8 Write $\int_{-2}^0 x^3\,dx + \int_0^1 x^3\,dx$ as a single integral, and then use a diagram to explain why this definite integral is negative.

DEVELOPMENT

9 In each part, evaluate the definite integrals. Then use the properties of the definite integral to explain the relationships amongst the integrals within that part.

a i $\int_0^2 (3x^2 - 1)\,dx$ ii $\int_2^0 (3x^2 - 1)\,dx$
b i $\int_0^1 20x^3\,dx$ ii $20\int_0^1 x^3\,dx$
c i $\int_1^4 (4x + 5)\,dx$ ii $\int_1^4 4x\,dx$ iii $\int_1^4 5\,dx$
d i $\int_0^2 12x^3\,dx$ ii $\int_0^1 12x^3\,dx$ iii $\int_1^2 12x^3\,dx$
e i $\int_3^3 (4 - 3x^2)\,dx$ ii $\int_{-2}^{-2} (4 - 3x^2)\,dx$

ISBN 978-1-108-76630-2

10 Use the properties of the definite integral to evaluate each integral without using a primitive function. Give reasons.

a $\displaystyle\int_3^3 \sqrt{9 - x^2}\,dx$ **b** $\displaystyle\int_4^4 (x^3 - 3x^2 + 5x - 7)\,dx$ **c** $\displaystyle\int_{-1}^1 x^3\,dx$

d $\displaystyle\int_{-5}^5 (x^3 - 25x)\,dx$ **e** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$ **f** $\displaystyle\int_{-2}^2 \frac{x}{1 + x^2}\,dx$

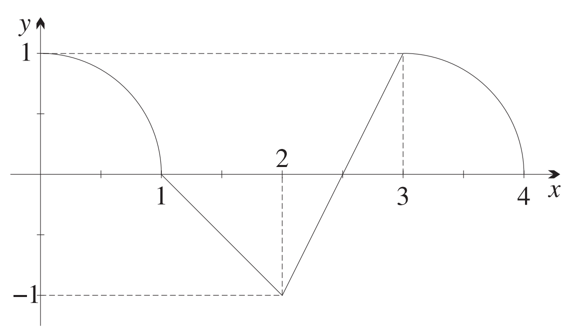
11 a On one set of axes sketch $y = x^2$ and $y = x^3$, clearly showing the points of intersection.

b Hence explain why $0 < \displaystyle\int_0^1 x^3\,dx < \int_0^1 x^2\,dx < 1$.

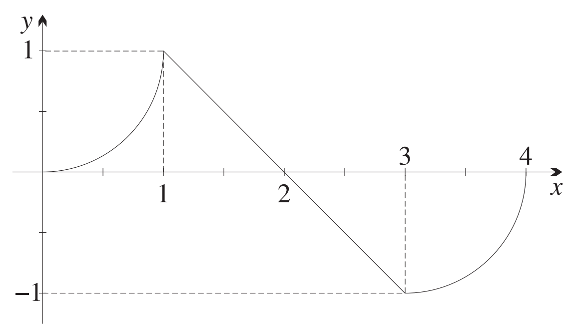
c Check the inequality in part **b** by evaluating each integral.

12 Use area formulae to find $\displaystyle\int_0^4 f(x)\,dx$, given the following sketches of $f(x)$.

a



b



13 a Calculate using a graph and an area formula.

i $\displaystyle\int_0^5 1\,dx$ **ii** $\displaystyle\int_0^5 x\,dx$

b Using these results, and the properties of integrals of sums and multiples, evaluate:

i $\displaystyle\int_0^5 2x\,dx$ **ii** $\displaystyle\int_0^5 (x + 1)\,dx$ **iii** $\displaystyle\int_0^5 (3x - 2)\,dx$

14 a Calculate these definite integrals using graphs and area formulae.

i $\displaystyle\int_{-\frac{1}{4}}^{\frac{1}{4}} (1 - 4x)\,dx$ **ii** $\displaystyle\int_{-5}^1 |x + 5|\,dx$ **iii** $\displaystyle\int_0^2 (|x| + 3)\,dx$

b Hence write down the value of:

i $\displaystyle\int_{\frac{1}{4}}^{-\frac{1}{4}} (1 - 4x)\,dx$ **ii** $\displaystyle\int_1^{-5} |x + 5|\,dx$ **iii** $\displaystyle\int_2^0 (|x| + 3)\,dx$

15 Using the properties of the definite integral, explain why:

a $\displaystyle\int_{-k}^k (ax^5 + cx^3 + e)\,dx = 0$

b $\displaystyle\int_{-k}^k (bx^4 + dx^2 + f)\,dx = 2\int_0^k (bx^4 + dx^2 + f)\,dx$

16 Sketch a graph of each integral and hence determine whether each statement is true or false.

a $\displaystyle\int_{-1}^1 2^x\,dx = 0$ **b** $\displaystyle\int_0^2 3^x > 0$ **c** $\displaystyle\int_{-2}^{-1} \frac{1}{x}\,dx > 0$ **d** $\displaystyle\int_2^1 \frac{1}{x}\,dx > 0$

ISBN 978-1-108-76630-2

17 a i Show that $\int_3^4 dx = \int_2^3 dx = \int_1^2 dx = 1$.

ii Show that $\frac{2}{7}\int_3^4 x\,dx = \frac{2}{5}\int_2^3 x\,dx = \frac{2}{3}\int_1^2 x\,dx = 1$.

iii Show that $\frac{3}{37}\int_3^4 x^2\,dx = \frac{3}{19}\int_2^3 x^2\,dx = \frac{3}{7}\int_1^2 x^2\,dx = 1$.

b Use the results above and the theorems on the definite integral to calculate:

i $\int_1^4 dx$ **ii** $\int_1^3 x\,dx$ **iii** $\int_2^1 x^2\,dx$

iv $\int_1^2 (x^2 + 1)\,dx$ **v** $\int_1^3 7x^2\,dx$ **vi** $\int_1^4 (3x^2 - 6x + 5)\,dx$

ENRICHMENT

18 State with reasons whether each statement is true or false.

a $\int_{-90}^{90} \sin^3 x^\circ\,dx = 0$

b $\int_{-30}^{30} \sin\ 4x^\circ \cos\ 2x^\circ\,dx = 0$

c $\int_{-1}^{1} 2^{-x^2}\,dx = 0$

d $\int_0^1 2^x\,dx < \int_0^1 3^x\,dx$

e $\int_{-1}^{0} 2^x\,dx < \int_{-1}^{0} 3^x\,dx$

f $\int_0^1 \frac{dt}{1 + t^n} \le \int_0^1 \frac{dt}{1 + t^{n+1}}$, where $n = 1, 2, 3, \ldots$.

19 Evaluate each definite integral, then establish the result that follows.
Provide a sketch of each situation.

a $\int_1^N \frac{1}{x^2}\,dx$ converges to 1 as $N \to \infty$.

b $\int_\varepsilon^1 \frac{1}{x^2}\,dx$ diverges to ∞ as $\varepsilon \to 0^+$.

c $\int_1^N \frac{1}{\sqrt{x}}\,dx$ diverges to ∞ as $N \to \infty$.

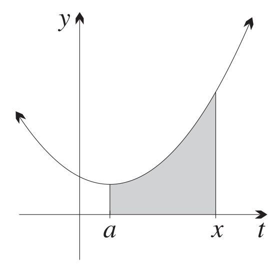
d $\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$ converges to 2 as $\varepsilon \to 0^+$.

ISBN 978-1-108-76630-2

5D Proving the fundamental theorem

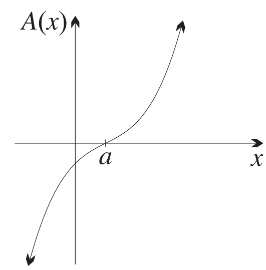
This section develops a proof of the fundamental theorem of calculus, as stated and used in Sections 5B and 5C. This would be the time to review the account of first-principles integration in Section 5A, because the approach used there of trap-ping an integral between upper and lower rectangles is used again in this proof.

The definite integral as a function of its upper limit



The value of a definite integral $\int_a^b f(x)\,dx$ changes when the value of b changes. This means that it is a function of its upper limit b. When we want to emphasise the functional relationship with the upper limit, we usually replace the letter b by the letter x, which is conventionally the variable of a function.

In turn, the original letter x needs to be replaced by some other letter, usually t. Then the definite integral is clearly represented as a function of its upper limit x, as in the first diagram above. This function is called the *signed area function* for $f(x)$ starting at $x = a$, and is defined by

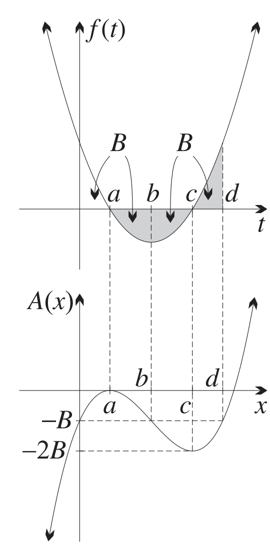


$$A(x) = \int_a^x f(t)\,dt$$

The function $A(x)$ is always zero at $x = a$. For the function sketched above, $A(x)$ increases at an increasing rate — see the second sketch. $A(x)$ is also defined to the left of a, where it is negative because the integrals run backwards.

The signed area function

The function in the sketch above was never negative. But the definite integral is the *signed area* under the curve, meaning that a negative sign is attached to areas of regions below the x-axis (provided that the integral is not running backwards).



The upper graph of $f(t)$ to the right is a parabola with axis of symmetry $x = b$. The four areas marked B are all equal. Here are some properties of the signed area function

$$A(x) = \int_a^x f(t)\,dt.$$

- $A(a) = 0$, as always.
- In the interval $[a, b]$, $f(t)$ is negative and decreasing, so $A(x)$ decreases at an increasing rate.
- In the interval $[b, c]$, $f(t)$ is negative and increasing, so $A(x)$ decreases at a decreasing rate.
- In the interval $[c, \infty)$, $f(t)$ is positive and increasing, so $A(x)$ increases at an increasing rate.
- $A(b) = -B$ and $A(c) = -2B$ and $A(d) = -B$.
- $A(x)$ is also defined in the interval $(-\infty, a]$ to the left of a, where it is negative because the integrals run backwards and the curve is above the x-axis.

The lower diagram above is a sketch of the *signed area function* $A(x)$ of $f(x)$.

ISBN 978-1-108-76630-2

13 THE SIGNED AREA FUNCTION

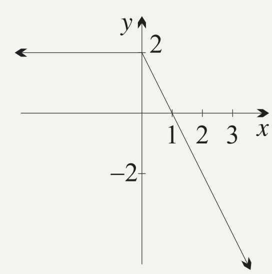
Suppose that $f(x)$ is a function defined in some interval I containing a. The *signed area function* for $f(x)$ starting at a is the function defined by the definite integral

$$A(x) = \int_a^x f(t)\,dt, \qquad \text{for all } x \text{ in the interval } I.$$

In Chapter 16 on continuous probability distributions, the cumulative distribution function of a continuous distribution will be defined in this way.

Example 13 5D

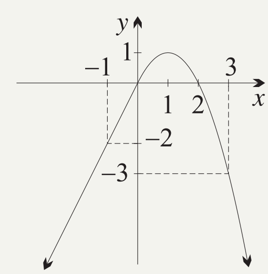
Let $A(x) = \int_0^x f(t)\,dt$ be the signed area function starting at $t = 0$ for the graph sketched to the right. Use area formulae to draw up a table of values for $y = A(x)$ in the interval $[-3, 3]$, then sketch $y = A(x)$.



SOLUTION

Use triangles for $x > 0$ and rectangles for $x < 0$.

x	-3	-2	-1	0	1	2	3
$A(x)$	-6	-4	-2	0	1	0	-3



For $x = -2$ and $x = -1$, $A(x)$ is negative because the integrals run backwards and the curve is above the x-axis. The area function $A(x)$ is increasing for $t < 1$ because $y > 0$, and is decreasing for $x > 1$ because $y < 0$.

The fundamental theorem — differential form

We can now state and prove the *differential form* of the fundamental theorem of calculus, from which we will derive the *integral form* used already in Sections 5B and 5C.

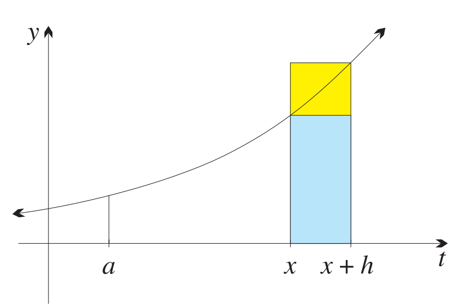
Theorem: If $f(x)$ is continuous, then the signed area function for $f(x)$ is a primitive of $f(x)$. That is,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof: Because the theorem is so fundamental, its proof must begin with the definition of the derivative as a limit,

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}.$$

Subtracting areas in the diagram to the right,



$$A(x+h) - A(x) = \int_x^{x+h} f(t)\,dt,$$

so $$A'(x) = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

ISBN 978-1-108-76630-2

Cambridge University Press

This limit is handled by means of a clever sandwiching technique.
Suppose that $f(t)$ is increasing in the interval $[x, x+h]$, as in the diagram above.
Then the lower rectangle on the interval $[x, x+h]$ has height $f(x)$,
and the upper rectangle on the interval $[x, x+h]$ has height $f(x+h)$,

so using areas, $h \times f(x) \le \int_x^{x+h} f(t)\,dt \le h \times f(x+h)$

$\boxed{\div h}$ $$f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x+h). \qquad (1)$$

Thus the middle expression is 'sandwiched' between $f(x)$ and $f(x+h)$.

Because $f(x)$ is continuous, $f(x+h) \to f(x)$ as $h \to 0$,

so by (1), $\lim\limits_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$, meaning that $A'(x) = f(x)$, as required.

If $f(x)$ is decreasing in the interval $[x, x+h]$, the same argument applies, but with the inequalities reversed.

Note: This theorem shows that the signed area function $A(x) = \int_a^x f(t)\,dt$ is a primitive of $f(x)$. It is therefore often written as $F(x)$ rather than $A(x)$.

Example 14 5D

Use the differential form of the fundamental theorem to simplify these expressions. Do not try to evaluate the integral and then differentiate it.

a $\frac{d}{dx}\int_0^x (t^2+1)^3\,dt$ **b** $\frac{d}{dx}\int_4^x (\log_e t)\,dt$ **c** $\frac{d}{dx}\int_{-3}^x e^{-t^2}\,dt$

SOLUTION

The differential form says that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Hence:

a $\frac{d}{dx}\int_0^x (t^2+1)^3\,dt = (x^2+1)^3$

b $\frac{d}{dx}\int_4^x (\log_e t)\,dt = \log_e x$

c $\frac{d}{dx}\int_{-3}^x e^{-t^2}\,dt = e^{-x^2}$

The fundamental theorem — integral form

The integral form of the fundamental theorem is the familiar form that we have been using in Sections 4B and 4C.

Theorem: Suppose that $f(x)$ is continuous in the closed interval $[a, b]$, and that $F(x)$ is a primitive of $f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ISBN 978-1-108-76630-2

Proof: We now know that $F(x)$ and $\int_a^x f(t)\,dt$ are both primitives of $f(x)$.

Because any two primitives differ only by a constant,

$$\int_a^x f(t)\,dt = F(x) + C, \quad \text{for some constant } C.$$

Substituting $x = a$, $\int_a^a f(t)\,dt = F(a) + C$,

but $\int_a^a f(t)\,dt = 0$, because the area in this definite integral has zero width,

so
$$\begin{aligned} 0 &= F(a) + C \\ C &= -F(a) \end{aligned}$$

Thus
$$\int_a^x f(t)\,dt = F(x) - F(a).$$

and changing letters from x to b and from t to x gives

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{as required.}$$

Example 15 5D

Use the integral form of the fundamental theorem to evaluate each integral. Then differentiate your result, thus confirming the consistency of the discussion above.

a $\dfrac{d}{dx}\displaystyle\int_1^x 6t^2\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_{-2}^x (t^3 - 9t^2 + 5)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_4^x \frac{1}{t^2}\,dt$

SOLUTION

a
$$\begin{aligned} \int_1^x 6t^2\,dt &= \Big[2t^3\Big]_1^x \\ &= 2x^3 - 2, \end{aligned}$$

so
$$\begin{aligned} \frac{d}{dx}\int_1^x 6t^2\,dt &= \frac{d}{dx}(2x^3 - 2) \\ &= 6x^2, \quad \text{consistent with the differential form.} \end{aligned}$$

b
$$\begin{aligned} \int_{-2}^x (t^3 - 9t^2 + 5)\,dt &= \Big[\tfrac{1}{4}t^4 - 3t^3 + 5t\Big]_{-2}^x \\ &= \left(\tfrac{1}{4}x^4 - 3x^3 + 5x\right) - (4 + 24 - 10) \\ &= \tfrac{1}{4}x^4 - 3x^3 + 5x - 18, \end{aligned}$$

so
$$\begin{aligned} \frac{d}{dx}\int_{-2}^x (t^3 - 9t^2 + 5)\,dt &= \frac{d}{dx}\left(\tfrac{1}{4}x^4 - 3x^3 + 5x - 18\right) \\ &= x^3 - 9x^2 + 5, \quad \text{consistent with the differential form.} \end{aligned}$$

c
$$\begin{aligned} \int_4^x \frac{1}{t^2}\,dt &= \int_4^x t^{-2}\,dt \\ &= \Big[-t^{-1}\Big]_4^x \\ &= -x^{-1} + \tfrac{1}{4}, \end{aligned}$$

so
$$\begin{aligned} \frac{d}{dx}\int_4^x \frac{1}{t^2}\,dt &= \frac{d}{dx}\left(-x^{-1} + \tfrac{1}{4}\right) \\ &= x^{-2}, \quad \text{consistent with the differential form.} \end{aligned}$$

ISBN 978-1-108-76630-2

Super challenge — continuous functions

In Section 9K of the Year 11 volume, we defined continuity at a point — you can draw the curve through the point without lifting the pencil off the paper.

Then throughout this chapter, we have been using the phrase, 'continuous in the closed interval $[a, b]$'. This idea is also straightforward, and informally means that you can place the pencil on the left-hand endpoint $\big(a, f(a)\big)$ and draw the curve to the right-hand endpoint $\big(b, f(b)\big)$ without lifting the pencil off the paper.

There is, however, a global notion of a *continuous function*:

A continuous function is a function continuous at every value in its domain.

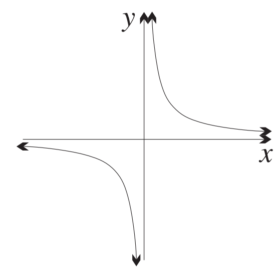
This may look obvious, like so many definitions in mathematics, but it is not.

It means, for example that the reciprocal function $y = \frac{1}{x}$ is a continuous function.

This is because $x = 0$ is not in its domain, so the function is continuous at every value of x in its domain.

Thus $y = \frac{1}{x}$ is a 'continuous function with a discontinuity'.

(Or perhaps the word 'discontinuity' is redefined). We recommend avoiding the concept completely unless some question specifically requires it.



Exercise 5D

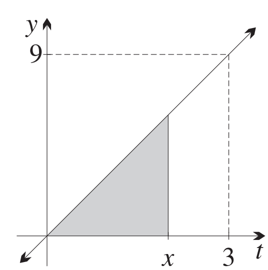
FOUNDATION

1 The graph to the right shows $y = 3t$, for $0 \le t \le 3$.

a Use the triangle area formula to find the signed area function

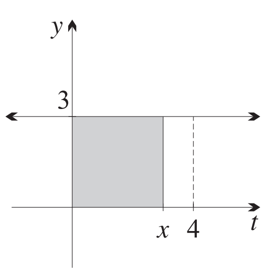
$$A(x) = \int_0^x 3t\,dt, \text{ for } 0 \le x \le 3.$$

b Differentiate $A(x)$ to show that $A'(x)$ is the original function, apart from a change of letter.

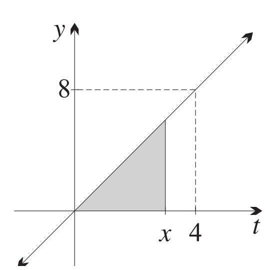


2 Write down the equation of each function, then use area formulae, not integration, to calculate the signed area function $A(x) = \int_0^x f(t)\,dt$, for $0 \le t \le 4$. Then differentiate $A(x)$ to confirm that $A'(x)$ is the original function, apart from a change of letter.

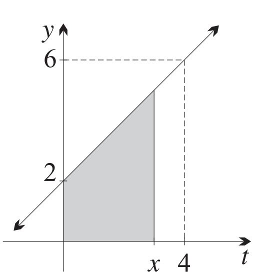
a



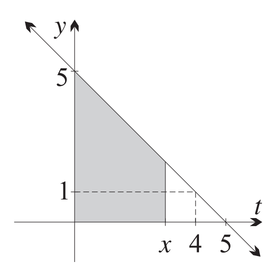
b



c



d

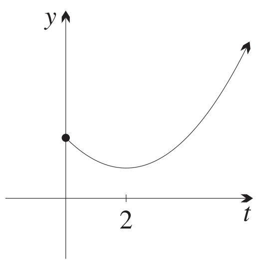


ISBN 978-1-108-76630-2

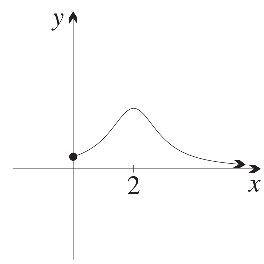
DEVELOPMENT

3 For each function sketched below, describe the behaviour of the signed area function $A(x) = \int_0^x f(t)\,dt$, for all $x \geq 0$, in the interval $[0, 2]$ and in the interval $[2, \infty)$. Then draw a freehand sketch of $y = A(x)$.

a



b



4 The differential form $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt$ of the fundamental theorem tells us that the derivative of the integral is the original function, with a change of letter. Use this to simplify these expressions. Do not attempt to find primitives.

a $\dfrac{d}{dx}\displaystyle\int_1^x \frac{1}{t}\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1 + t^3}\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_0^x e^{-\frac{1}{2}t^2}\,dt$

5 Use the differential form of the fundamental theorem to simplify these expressions. Then confirm the consistency of the discussion in this section by performing the integration and then differentiating.

a $\dfrac{d}{dx}\displaystyle\int_1^x (3t^2 - 12)\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + 4t)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_2^x \frac{1}{t^2}\,dt$

6 **a** Use the differential form of the fundamental theorem $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ to find:

i $\dfrac{d}{dx}\displaystyle\int_1^x t^2\,dt$ **ii** $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + 3t)\,dt$

iii $\dfrac{d}{dx}\displaystyle\int_a^x \frac{1}{t}\,dt$ **iv** $\dfrac{d}{dx}\displaystyle\int_a^x (t^3 - 3)^4\,dt$

b Confirm your answers to parts (i) and (ii) above by evaluating the definite integral and then differentiating with respect to x.

7 **a** Sketch $y = e^t$, then sketch the signed area function $A(x) = \int_0^x e^t\,dt$. How would you describe the behaviour of $y = A(x)$?

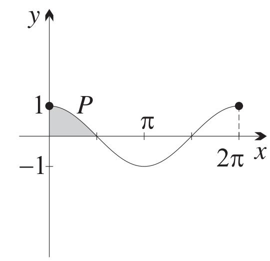
b Sketch $y = \log_e t$, then sketch the signed area function $A(x) = \int_1^x \log_e t\,dt$. How would you describe the behaviour of $y = A(x)$?

c Sketch $y = \dfrac{1}{t}$, then sketch the signed area function $A(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$. How would you describe the behaviour of $y = A(x)$?

ISBN 978-1-108-76630-2

ENRICHMENT

8 a Sketched to the right is $y = \cos x$, for $0 \le x \le 2\pi$. Copy and complete the table of values for the signed area function $A(x) = \int_0^x \cos t \, dt$, for $0 \le x \le 2\pi$, given that the region marked P has area exactly 1 (this is proven in Chapter 7). Then sketch $y = A(x)$.



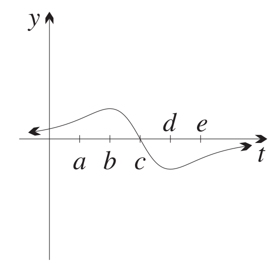
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$A(x)$					

What is your guess for the equation of $A(x)$, and what does this suggest the derivative of $\sin x$ is?

b Sketch $y = \sin x$, for $0 \le x \le 2\pi$, and repeat the procedures in part **a**.

9 The function $y = f(t)$ sketched to the right has point symmetry in $(c, 0)$.

Let $A(x) = \int_a^x f(t) \, dt$.



a Where is $A(x)$ increasing, and when it is decreasing?
b Where does $A(x)$ have a maximum turning point, and where does $A(x)$ have a minimum turning point?
c Where does $A(x)$ have inflections?
d Where are the zeroes of $A(x)$?
e Where is $A(x)$ positive, and where it is negative?
f Sketch $y = A(x)$.

10 This 'super challenge' question may illuminate the definition of a continuous function:

A continuous function is a function that is continuous at every number in its domain.

Classify these functions as continuous or not continuous according to the definition above.

a $y = x - 2$

b $y = \dfrac{1}{x - 2}$

c $y = \begin{cases} \dfrac{1}{x-2}, & \text{for } x \ne 2, \\ 0, & \text{for } x = 2. \end{cases}$

d $y = \sqrt{x}$

e $y = \dfrac{1}{\sqrt{x}}$

f $y = \begin{cases} \dfrac{1}{\sqrt{x}}, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$

ISBN 978-1-108-76630-2

5E The indefinite integral

Now that primitives have been established as the key to calculating definite integrals, this section turns again to the task of finding primitives. First, a new and convenient notation for the primitive is introduced.

The indefinite integral

Because of the close connection established by the fundamental theorem between primitives and definite integrals, the term *indefinite integral* is often used for the general primitive. The usual notation for the indefinite integral of a function $f(x)$ is an integral sign without any upper or lower limits. For example, *the primitive* or *the indefinite integral* of $x^2 + 1$ is

$$\int (x^2 + 1)\,dx = \frac{x^3}{3} + x + C, \quad \text{for some constant } C.$$

The word 'indefinite' suggests that the integral cannot be evaluated further because no limits for the integral have yet been specified.

The constant of integration

A definite integral ends up as a pure number. An indefinite integral, on the other hand, is a function of x — the pronumeral x is carried across to the answer.

It also contains an unknown constant C (or c, as it is often written) and the indefinite integral can also be regarded as a function of C (or of c). The constant is called a 'constant of integration' and is an important part of the answer — it must always be included.

The only exception to including the constant of integration is when calculating definite integrals, because in that situation any primitive can be used.

Note: Strictly speaking, the words 'for some constant C' or 'where C is a constant' should follow the first mention of the new pronumeral C, because no pronumeral should be used without having been formally introduced. There is a limit to one's patience, however (in this book there is often no room), and usually it is quite clear that C is the constant of integration. If another pronumeral such as D is used, it would be wise to introduce it formally.

Standard forms for integration

The two rules for finding primitives given in the last section of Chapter 4 can now be restated in this new notation.

14 STANDARD FORMS FOR INTEGRATION

Suppose that $n \neq -1$. Then

- $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for some constant } C.$$
- $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad \text{for some constant } C.$$

ISBN 978-1-108-76630-2

Cambridge University Press

The word 'integration' is commonly used to refer both to the finding of a primitive, and to the evaluating of a definite integral. Similarly, the unqualified term 'integral' is used to refer both to the indefinite integral and to the definite integral.

Example 16 5E

Use the standard form $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ to find:

a $\int 9\,dx$

b $\int 12x^3\,dx$

SOLUTION

a $\int 9\,dx = 9x + C$, for some constant C

Note: We know that $9x$ is the primitive of 9, because $\dfrac{d}{dx}(9x) = 9$.

But the formula still gives the correct answer, because $9 = 9x^0$, so increasing the index to 1 and dividing by this new index 1,

$$\begin{aligned}\int 9x^0\,dx &= \frac{9x^1}{1} + C, \quad \text{for some constant } C\\ &= 9x + C.\end{aligned}$$

b
$$\begin{aligned}\int 12x^3\,dx &= 12 \times \frac{x^4}{4} + C, \quad \text{for some constant } C\\ &= 3x^4 + C\end{aligned}$$

Example 17 5E

Use the standard form $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$ to find:

a $\int (3x+1)^5\,dx$

b $\int (5-2x)^2\,dx$

SOLUTION

a
$$\begin{aligned}\int (3x+1)^5\,dx &= \frac{(3x+1)^6}{3 \times 6} + C \qquad \text{(here } n = 5 \text{ and } a = 3 \text{ and } b = 1)\\ &= \tfrac{1}{18}(3x+1)^6 + C, \quad \text{for some content } C\end{aligned}$$

b
$$\begin{aligned}\int (5-2x)^2\,dx &= \frac{(5-2x)^3}{(-2) \times 3} + C \qquad \text{(here } n = 2 \text{ and } a = -2 \text{ and } b = 5)\\ &= -\tfrac{1}{6}(5-2x)^3 + C, \quad \text{for some content } C\end{aligned}$$

ISBN 978-1-108-76630-2

Negative indices

Both standard forms apply with negative indices as well as positive indices, as in the next worked example. The exception is the index -1, where the rule is nonsense because it results in division by zero. We shall deal with the integration of x^{-1} in Chapter 6.

Example 18 — 5E

Use negative indices to find these indefinite integrals.

a $\displaystyle\int \frac{12}{x^3}\,dx$

b $\displaystyle\int \frac{dx}{(3x+4)^2}$

SOLUTION

a
$$\int \frac{12}{x^3}\,dx = \int 12x^{-3}\,dx$$
$$= 12 \times \frac{x^{-2}}{-2} + C$$
$$= -\frac{6}{x^2} + C$$

b
$$\int \frac{dx}{(3x+4)^2} = \int (3x+4)^{-2}\,dx$$
$$= \frac{(3x+4)^{-1}}{3 \times (-1)} + C$$
$$= -\frac{1}{3(3x+4)} + C$$

Special expansions

In many integrals, brackets must be expanded before the indefinite integral can be found. The next worked example uses the special expansions. Part **b** also requires negative indices.

Example 19 — 5E

Find these indefinite integrals.

a $\displaystyle\int (x^3-1)^2\,dx$

b $\displaystyle\int \left(3-\frac{1}{x^2}\right)\left(3+\frac{1}{x^2}\right)dx$

SOLUTION

a
$$\int (x^3-1)^2\,dx = \int (x^6 - 2x^3 + 1)\,dx \qquad \text{(using } (A+B)^2 = A^2 + 2AB + B^2\text{)}$$
$$= \frac{x^7}{7} - \frac{x^4}{2} + x + C$$

b
$$\int \left(3-\frac{1}{x^2}\right)\left(3+\frac{1}{x^2}\right)dx = \int \left(9 - \frac{1}{x^4}\right)dx \qquad \text{(using } (A-B)(A+B) = A^2 - B^2\text{)}$$
$$= \int (9 - x^{-4})\,dx$$
$$= 9x - \frac{x^{-3}}{-3} + C$$
$$= 9x + \frac{1}{3x^3} + C$$

ISBN 978-1-108-76630-2

Fractional indices

The standard forms for finding primitives of powers also apply to fractional indices. These calculations require quick conversions between fractional indices and surds. The next worked example finds each indefinite integral and then uses it to evaluate a definite integral.

Example 20 5E

Use fractional and negative indices to evaluate:

a $\displaystyle\int_1^4 \sqrt{x}\,dx$

b $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

SOLUTION

a Increase the index to $\frac{3}{2}$, and divide by $\frac{3}{2}$.

$$\begin{aligned}\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{\frac{1}{2}}\,dx\\ &= \tfrac{2}{3} - \Big[x^{\frac{3}{2}}\Big]_1^4\\ &= \tfrac{2}{3} \times (8 - 1)\\ &= 4\tfrac{2}{3}\end{aligned}$$

b Increase the index to $\frac{1}{2}$, and divide by $\frac{1}{2}$.

$$\begin{aligned}\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{-\frac{1}{2}}\,dx\\ &= \tfrac{2}{1} - \Big[x^{\frac{1}{2}}\Big]_1^4\\ &= 2 \times (2 - 1)\\ &= 2\end{aligned}$$

Example 21 5E

a Use index notation to express $\dfrac{1}{\sqrt{9 - 2x}}$ as a power of $9 - 2x$.

b Hence find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{9 - 2x}}$.

SOLUTION

a $\dfrac{1}{\sqrt{9 - 2x}} = (9 - 2x)^{-\frac{1}{2}}$.

b Hence
$$\begin{aligned}\int \frac{1}{\sqrt{9 - 2x}} &= \int (9 - 2x)^{-\frac{1}{2}}\,dx\\ &= \frac{(9 - 2x)^{\frac{1}{2}}}{-2 \times \frac{1}{2}} + C, \quad \text{using} \int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)},\\ &= -\sqrt{9 - 2x} + C.\end{aligned}$$

ISBN 978-1-108-76630-2

Exercise 5E

FOUNDATION

Technology: Many programs that can perform algebraic manipulation are also able to deal with indefinite integrals. They can be used to check the questions in this exercise and to investigate the patterns arising in such calculations.

1 Find these indefinite integrals.

a $\int 4\,dx$ **b** $\int 1\,dx$ **c** $\int 0\,dx$ **d** $\int (-2)\,dx$

e $\int x\,dx$ **f** $\int x^2\,dx$ **g** $\int x^3\,dx$ **h** $\int x^7\,dx$

2 Find the indefinite integral of each function. Use the notation of the previous question.

a $2x$ **b** $4x$ **c** $3x^2$ **d** $4x^3$

e $10x^9$ **f** $2x^3$ **g** $4x^5$ **h** $3x^8$

3 Find these indefinite integrals.

a $\int (x + x^2)\,dx$ **b** $\int (x^4 - x^3)\,dx$ **c** $\int (x^7 + x^{10})\,dx$

d $\int (2x + 5x^4)\,dx$ **e** $\int (9x^8 - 11)\,dx$ **f** $\int (7x^{13} + 3x^8)\,dx$

g $\int (4 - 3x)\,dx$ **h** $\int (1 - x^2 + x^4)\,dx$ **i** $\int (3x^2 - 8x^3 + 7x^4)\,dx$

4 Find the indefinite integral of each function. (Leave negative indices in your answers.)

a x^{-2} **b** x^{-3} **c** x^{-8}

d $3x^{-4}$ **e** $9x^{-10}$ **f** $10x^{-6}$

5 Find these indefinite integrals. (Leave fractional indices in your answers.)

a $\int x^{\frac{1}{2}}\,dx$ **b** $\int x^{\frac{1}{3}}\,dx$ **c** $\int x^{\frac{1}{4}}\,dx$

d $\int x^{\frac{2}{3}}\,dx$ **e** $\int x^{-\frac{1}{2}}\,dx$ **f** $\int 4x^{\frac{1}{2}}\,dx$

6 By first expanding the brackets, find these indefinite integrals.

a $\int x(x + 2)\,dx$ **b** $\int x(4 - x^2)\,dx$ **c** $\int x^2(5 - 3x)\,dx$

d $\int x^3(x - 5)\,dx$ **e** $\int (x - 3)^2\,dx$ **f** $\int (2x + 1)^2\,dx$

g $\int (1 - x^2)^2\,dx$ **h** $\int (2 - 3x)(2 + 3x)\,dx$ **i** $\int (x^2 - 3)(1 - 2x)\,dx$

7 Write each integrand as separate fractions, then perform the integration.

a $\int \frac{x^2 + 2x}{x}\,dx$ **b** $\int \frac{x^7 + x^8}{x^6}\,dx$ **c** $\int \frac{2x^3 - x^4}{4x}\,dx$

ISBN 978-1-108-76630-2

DEVELOPMENT

8 Write these functions with negative indices and hence find their indefinite integrals.

a $\dfrac{1}{x^2}$ **b** $\dfrac{1}{x^3}$ **c** $\dfrac{1}{x^5}$ **d** $\dfrac{1}{x^{10}}$

e $\dfrac{3}{x^4}$ **f** $\dfrac{5}{x^6}$ **g** $\dfrac{7}{x^8}$ **h** $\dfrac{1}{3x^2}$

i $\dfrac{1}{7x^5}$ **j** $-\dfrac{1}{5x^3}$ **k** $\dfrac{1}{x^2}-\dfrac{1}{x^5}$ **l** $\dfrac{1}{x^3}+\dfrac{1}{x^4}$

9 Write these functions with fractional indices and hence find their indefinite integrals.

a $\sqrt{x}$ **b** $\sqrt[3]{x}$ **c** $\dfrac{1}{\sqrt{x}}$ **d** $\sqrt[3]{x^2}$

10 Use the indefinite integrals of the previous question to evaluate:

a $\displaystyle\int_0^9 \sqrt{x}\,dx$ **b** $\displaystyle\int_0^8 \sqrt[3]{x}\,dx$ **c** $\displaystyle\int_{25}^{49} \frac{1}{\sqrt{x}}\,dx$ **d** $\displaystyle\int_0^1 \sqrt[3]{x^2}\,dx$

11 By using the rule $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, find:

a $\displaystyle\int (x+1)^5\,dx$ **b** $\displaystyle\int (x+2)^3\,dx$ **c** $\displaystyle\int (4-x)^4\,dx$

d $\displaystyle\int (3-x)^2\,dx$ **e** $\displaystyle\int (3x+1)^4\,dx$ **f** $\displaystyle\int (4x-3)^7\,dx$

g $\displaystyle\int (5-2x)^6\,dx$ **h** $\displaystyle\int (1-5x)^7\,dx$ **i** $\displaystyle\int (2x+9)^{11}\,dx$

j $\displaystyle\int 3(2x-1)^{10}\,dx$ **k** $\displaystyle\int 4(5x-4)^6\,dx$ **l** $\displaystyle\int 7(3-2x)^3\,dx$

12 Find:

a $\displaystyle\int \left(\tfrac{1}{3}x-7\right)^4 dx$ **b** $\displaystyle\int \left(\tfrac{1}{4}x-7\right)^6 dx$ **c** $\displaystyle\int \left(1-\tfrac{1}{5}x\right)^3 dx$

13 Find:

a $\displaystyle\int \frac{1}{(x+1)^3}\,dx$ **b** $\displaystyle\int \frac{1}{(x-5)^4}\,dx$ **c** $\displaystyle\int \frac{1}{(3x-4)^2}\,dx$

d $\displaystyle\int \frac{1}{(2-x)^5}\,dx$ **e** $\displaystyle\int \frac{3}{(x-7)^6}\,dx$ **f** $\displaystyle\int \frac{8}{(4x+1)^5}\,dx$

g $\displaystyle\int \frac{2}{(3-5x)^4}\,dx$ **h** $\displaystyle\int \frac{4}{5(1-4x)^2}\,dx$ **i** $\displaystyle\int \frac{7}{8(3x+2)^5}\,dx$

14 By expanding the brackets, find:

a $\displaystyle\int \sqrt{x}(3\sqrt{x}-x)\,dx$ **b** $\displaystyle\int \left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)dx$ **c** $\displaystyle\int \left(2\sqrt{x}-1\right)^2 dx$

15 **a** Evaluate these definite integrals.

i $\displaystyle\int_0^1 x^{\frac{1}{2}}\,dx$ **ii** $\displaystyle\int_1^4 x^{-\frac{1}{2}}\,dx$ **iii** $\displaystyle\int_0^8 x^{\frac{1}{3}}\,dx$

ISBN 978-1-108-76630-2

b By writing them with fractional indices, evaluate these definite integrals.

i $\displaystyle\int_0^4 \sqrt{x}\,dx$ **ii** $\displaystyle\int_1^9 x\sqrt{x}\,dx$ **iii** $\displaystyle\int_1^9 \frac{dx}{\sqrt{x}}$

16 Expand the brackets and hence find:

a $\displaystyle\int_2^4 (2 - \sqrt{x})(2 + \sqrt{x})\,dx$ **b** $\displaystyle\int_0^1 \sqrt{x}\left(\sqrt{x} - 4\right)dx$ **c** $\displaystyle\int_4^9 \left(\sqrt{x} - 1\right)^2 dx$

17 Explain why the indefinite integral $\displaystyle\int \frac{1}{x}\,dx$ cannot be found in the usual way using the standard form $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.

18 Find each indefinite integral.

a $\displaystyle\int \sqrt{2x - 1}\,dx$ **b** $\displaystyle\int \sqrt{7 - 4x}\,dx$

c $\displaystyle\int \sqrt[3]{4x - 1}\,dx$ **d** $\displaystyle\int \frac{1}{\sqrt{3x + 5}}\,dx$

19 Evaluate these definite integrals.

a $\displaystyle\int_0^2 (x + 1)^4\,dx$ **b** $\displaystyle\int_2^3 (2x - 5)^3\,dx$ **c** $\displaystyle\int_{-2}^2 (1 - x)^5\,dx$

d $\displaystyle\int_0^5 \left(1 - \frac{x}{5}\right)^4 dx$ **e** $\displaystyle\int_0^1 \sqrt{9 - 8x}\,dx$ **f** $\displaystyle\int_2^7 \frac{1}{\sqrt{x + 2}}\,dx$

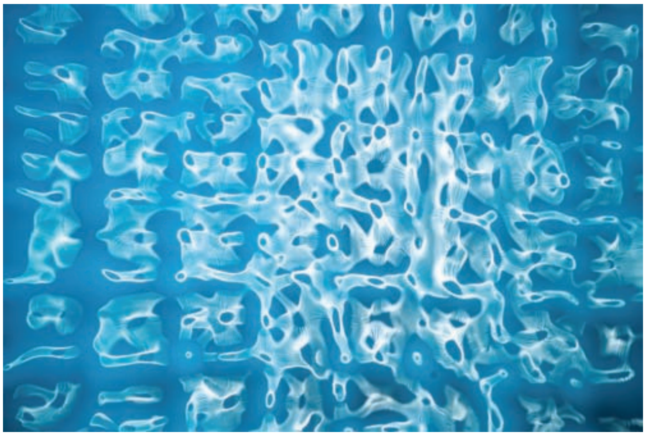
g $\displaystyle\int_{-2}^0 \sqrt[3]{x + 1}\,dx$ **h** $\displaystyle\int_1^5 \sqrt{3x + 1}\,dx$ **i** $\displaystyle\int_{-3}^0 \sqrt{1 - 5x}\,dx$

ENRICHMENT

20 **a** If u and v are differentiable functions of x, prove that $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

b Hence find:

i $\displaystyle\int x(x - 1)^4\,dx$ **ii** $\displaystyle\int x\sqrt{1 + x}\,dx$



ISBN 978-1-108-76630-2

5F Finding areas by integration

The aim of this section and the next is to use definite integrals to find the areas of regions bounded by curves, lines and the coordinate axes.

Sections 5F–5G ignore integrals that run backwards. Running an integral backwards reverses its sign, which would confuse the discussion of areas in these sections. When finding areas, we decide what integrals to create, and we naturally avoid integrals that run backwards.

Areas and definite integrals

Areas and definite integrals are closely related, but they are not the same thing.

- An area is always positive, whereas a definite integral may be positive or negative, depending on whether the curve is above or below the x-axis.

Problems on areas require care when finding the required integral or combination of integrals. Some particular techniques are listed below, but the general rule is to draw a diagram first to see which pieces need to be added or subtracted.

15 FINDING AN AREA

When using integrals to find the area of a region:

1 Draw a sketch of the curves, showing relevant intercepts and intersections.
2 Create and evaluate the necessary definite integral or integrals.
3 Write a conclusion, giving the required area in square units.

Regions above the x-axis

When a curve lies entirely above the x-axis, the relevant integral will be positive, and the area of the region will be equal to the integral, apart from needing units.

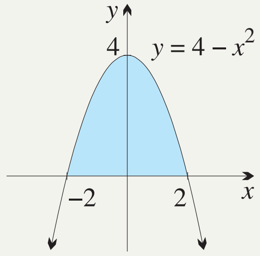
Example 22 **5F**

Find the area of the region bounded by the curve $y = 4 - x^2$ and the x-axis. (This was the example sketched in the introduction to this chapter.)

SOLUTION

The curve meets the x-axis at $(2, 0)$ and $(-2, 0)$.
The region lies entirely above the x-axis and the relevant integral is

$$\begin{aligned}\int_{-2}^{2}(4 - x^2)\,dx &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}\\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)\\ &= 5\tfrac{1}{3} - \left(-5\tfrac{1}{3}\right)\\ &= 10\tfrac{2}{3},\end{aligned}$$



which is positive because the curve lies entirely above the x-axis.
Hence the required area is $10\frac{2}{3}$ square units.

ISBN 978-1-108-76630-2

Regions below the x-axis

When a curve lies entirely below the x-axis, the relevant integral will be negative, and the area will be the opposite of this.

Example 23 5F

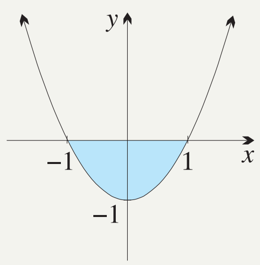
Find the area of the region bounded by the curve $y = x^2 - 1$ and the x-axis.

SOLUTION

The curve meets the x-axis at $(1, 0)$ and $(-1, 0)$.

The region lies entirely below the x-axis, and the relevant integral is

$$\begin{aligned}\int_{-1}^{1} (x^2 - 1)\, dx &= \left[\frac{x^3}{3} - x\right]_{-1}^{1}\\ &= \left(\tfrac{1}{3} - 1\right) - \left(-\tfrac{1}{3} + 1\right)\\ &= -\tfrac{2}{3} - \tfrac{2}{3}\\ &= -1\tfrac{1}{3},\end{aligned}$$



which is negative, because the curve lies entirely below the x-axis.

Hence the required area is $1\frac{1}{3}$ square units.

Curves that cross the x-axis

When a curve crosses the x-axis, the area of the region between the curve and the x-axis cannot usually be found by means of a single integral. This is because integrals representing regions below the x-axis have negative values.

Example 24 5F

a Sketch the cubic curve $y = x(x + 1)(x - 2)$, showing the x-intercepts.

b Shade the region enclosed between the x-axis and the curve, and find its area.

c Find $\displaystyle\int_{-1}^{2} x(x + 1)(x - 2)\, dx$ and explain why this integral does not represent the area of the region described in part **b**.

SOLUTION

a The curve has x-intercepts $x = -1$, $x = 0$ and $x = 2$, and is graphed below.

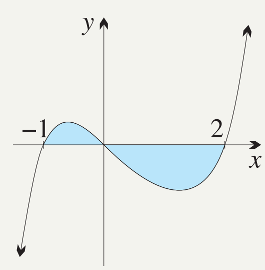
b Expanding the cubic,

$$\begin{aligned}y &= x(x + 1)(x - 2)\\ &= x(x^2 - x - 2)\\ &= x^3 - x^2 - 2x.\end{aligned}$$

For the region above the x-axis,

$$\begin{aligned}\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}\\ &= (0 - 0 - 0) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right)\\ &= \tfrac{5}{12},\end{aligned}$$

so area above $= \frac{5}{12}$ square units.



ISBN 978-1-108-76630-2

For the region below the x-axis,

$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2\\ &= \left(4 - 2\tfrac{2}{3} - 4\right) - (0 - 0 - 0)\\ &= -2\tfrac{2}{3},\end{aligned}$$

so area below $= 2\tfrac{2}{3}$ square units.

Adding these, total area $= \tfrac{5}{12} + 2\tfrac{2}{3}$

$= 3\tfrac{1}{12}$ square units.

c $$\begin{aligned}\int_{-1}^2 x(x+1)(x-2)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^2\\ &= \left(4 - 2\tfrac{2}{3} - 4\right) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right)\\ &= -2\tfrac{2}{3} + \tfrac{5}{12}\\ &= -2\tfrac{1}{4}.\end{aligned}$$

This integral represents the difference $2\tfrac{2}{3} - \tfrac{5}{12} = 2\tfrac{1}{4}$ of the two areas, and is negative because the area below is larger than the area above.

Areas associated with odd and even functions

As always in mathematics, calculations are often much easier if symmetries can be recognised.

Example 25 5F

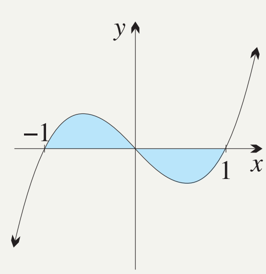
a Show that $y = x^3 - x$ is an odd function.

b Using part **a**, find the area between the curve $y = x^3 - x$ and the x-axis.

SOLUTION

a Let $f(x) = x^3 - x.$

Then $$\begin{aligned}f(-x) &= (-x)^3 - (-x)\\ &= -x^3 + x\\ &= -f(x), \text{ so } f(x) \text{ is odd.}\end{aligned}$$



b Factoring, $$\begin{aligned}y &= x(x^2 - 1)\\ &= x(x-1)(x+1),\end{aligned}$$

so the x-intercepts are $x = -1$, $x = 0$ and $x = 1$.

The two shaded regions have equal areas because the function is odd.

First, $$\begin{aligned}\int_0^1 (x^3 - x)\,dx &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1\\ &= \left(\tfrac{1}{4} - \tfrac{1}{2}\right) - (0 - 0)\\ &= -\tfrac{1}{4},\end{aligned}$$

so area below the x-axis $= \tfrac{1}{4}$ square units.

Doubling, total area $= \tfrac{1}{2}$ square units.

ISBN 978-1-108-76630-2

Area between a graph and the y-axis

Integration with respect to y rather than x can often give a result quickly, and avoid subtraction of areas.

Suppose then that we can find the inverse function, and write x is a function of y.

- A definite integral with respect to y represents the signed area of the region between the curve and the y-axis.
- This means that the definite integral is the sum of areas of regions to the right of the x-axis, minus the sum of areas of regions to the left of the x-axis.
- The limits of integration are values of y rather than values of x.

16 THE DEFINITE INTEGRAL AND INTEGRATION WITH RESPECT TO y

Let x be a continuous function of y in some closed interval $a \le y \le b$.

Then the definite integral $\int_a^b x\,dy$ is the sum of the areas of regions to the right of the y-axis, from $y = a$ to $y = b$, minus the sum of the areas of regions to the left of the y-axis.

When the function is given with y as a function of x, we first need to solve the equation for x. We are thus working with the inverse function.

Example 26 — 5F

a Sketch the lines $y = x + 1$ and $y = 5$, and shade the region between these lines to the right of the y-axis.

b Write the equation of the line so that x is a function of y.

c Use integration with respect to y to find the area of this region.

d Confirm the result by area formulae.

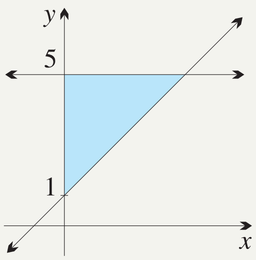
SOLUTION

a The lines are sketched below. They meet at $(4, 5)$.

b The given equation is $y = x + 1$.
Solving for x, $x = y - 1$.

c The required integral is

$$\begin{aligned}\int_1^5 (y-1)\,dy &= \left[\frac{y^2}{2} - y\right]_1^5\\ &= \left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\\ &= 7\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\\ &= 8,\end{aligned}$$



which is positive, because the region is to the right of the y-axis.
Hence the required area is $8\,\text{u}^2$.

d Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times 4 \times 4$
$= 8\,\text{u}^2$.

ISBN 978-1-108-76630-2

Cambridge University Press

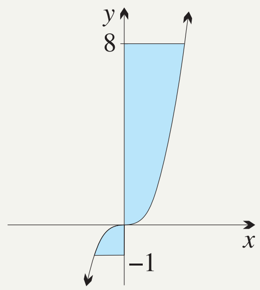
Example 27 5F

The curve in the diagram is the cubic $y = x^3$.

a Write the equation of the cubic so that x is a function of y.

b Use integration with respect to y to find the areas of the shaded regions to the right and left of the y-axis.

c Find the total area of the two shaded regions.



SOLUTION

a The given equation is $y = x^3$.
Solving for x, $x^3 = y$
$x = y^{\frac{1}{3}}$.

b For the region to the right of the y-axis,

$$\begin{aligned}\int_0^8 y^{\frac{1}{3}}\,dy &= \tfrac{3}{4}\Big[y^{\frac{4}{3}}\Big]_0^8 \\ &= \tfrac{3}{4}\times(16-0) \qquad (\text{because } 8^{\frac{4}{3}} = 2^4 = 16) \\ &= 12,\end{aligned}$$

so area $= 12$ square units.

For the region to the left of the y-axis,

$$\begin{aligned}\int_{-1}^0 y^{\frac{1}{3}}\,dy &= \tfrac{3}{4}\Big[y^{\frac{4}{3}}\Big]_{-1}^0 \\ &= \tfrac{3}{4}\times(0-1) \qquad (\text{because } (-1)^{\frac{4}{3}} = (-1)^4 = 1) \\ &= -\tfrac{3}{4},\end{aligned}$$

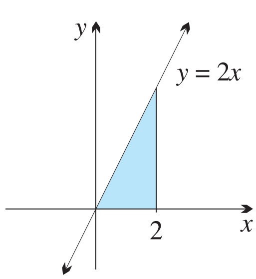
so area $= \frac{3}{4}$ square units.

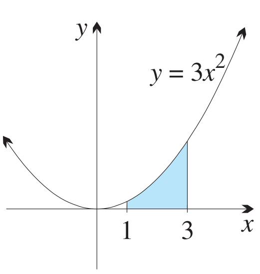
c Adding these, total area $= 12\frac{3}{4}$ square units.

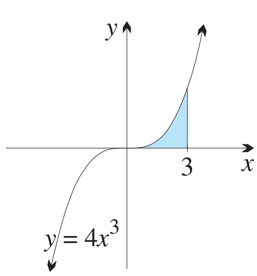
Exercise 5F FOUNDATION

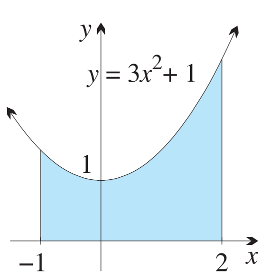
Technology: Graphing software would help in identifying the definite integrals that need to be evaluated to find the area of a given region.

1 Find the area of each shaded region below by evaluating the appropriate integral.

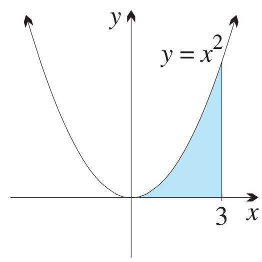
a


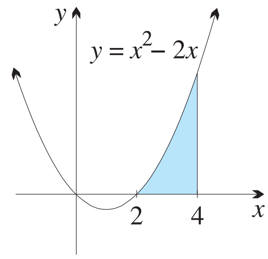
b


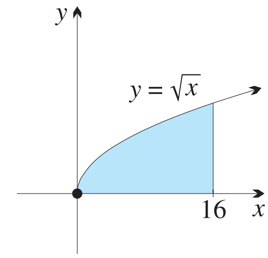
c


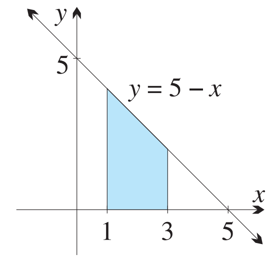
d


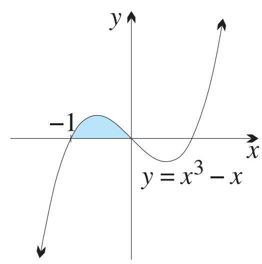
ISBN 978-1-108-76630-2

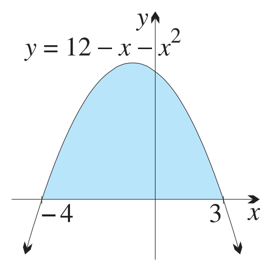
e


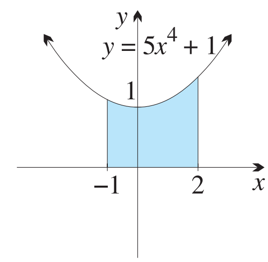
f


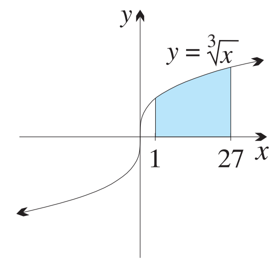
g


h


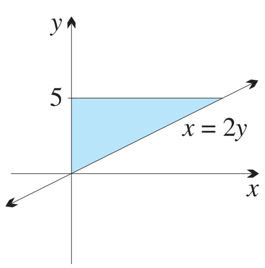
i


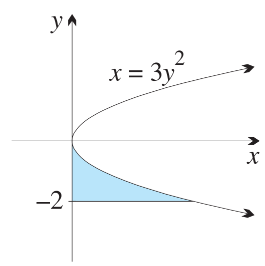
j


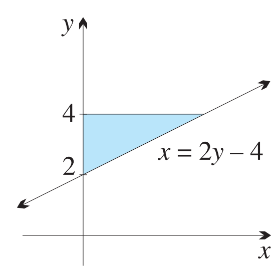
k


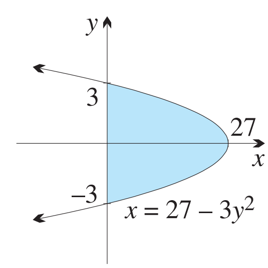
l


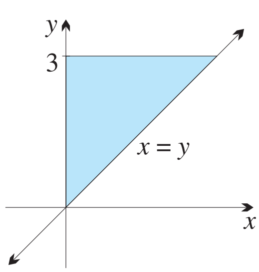
2 Find the area of each shaded region below by evaluating an integral of the form $\int_{y_1}^{y_2} g(y)\,dy$.

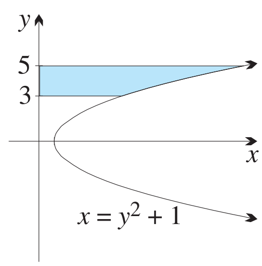
a


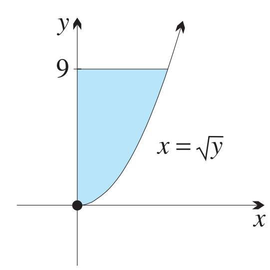
b


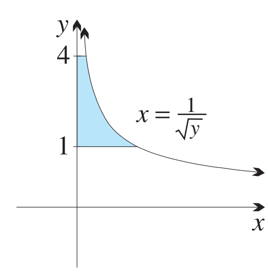
c


d


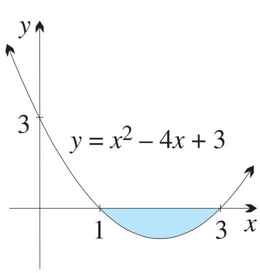
e


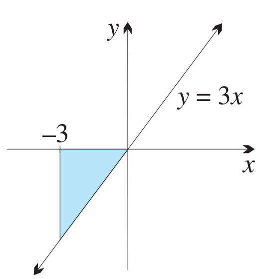
f


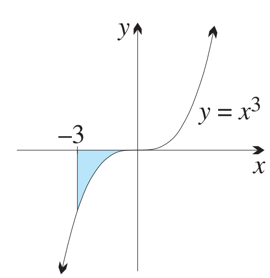
g


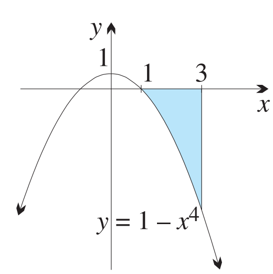
h


3 Find the area of each shaded region below by evaluating the appropriate integral.

a


b


c


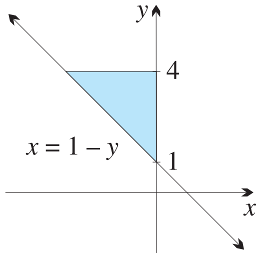
d


ISBN 978-1-108-76630-2

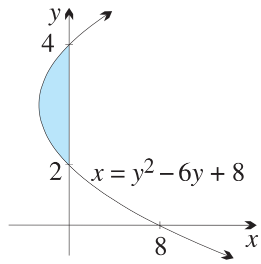
Cambridge University Press

4 Find the area of each shaded region below by evaluating the appropriate integral.

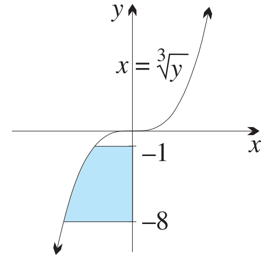
a



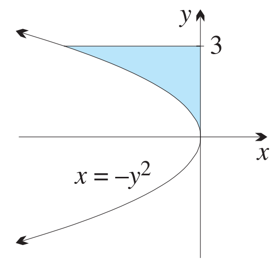
b



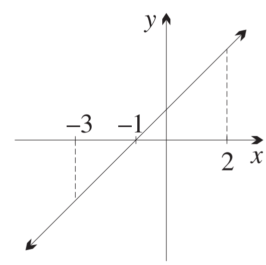
c



d



5 The line $y = x + 1$ is graphed on the right.



a Copy the diagram, and shade the region between the line $y = x + 1$ and the x-axis from $x = -3$ to $x = 2$.

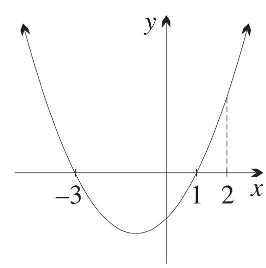
b By evaluating $\int_{-1}^{2} (x + 1)\, dx$, find the area of the shaded region above the x-axis.

c By evaluating $\int_{-3}^{-1} (x + 1)\, dx$, find the area of the shaded region below the x-axis.

d Hence find the area of the entire shaded region.

e Find $\int_{-3}^{2} (x + 1)\, dx$, and explain why this integral does not give the area of the shaded region.

6 The curve $y = (x - 1)(x + 3) = x^2 + 2x - 3$ is graphed.



a Copy the diagram, and shade the region between the curve $y = (x - 1)(x + 3)$ and the x-axis from $x = -3$ to $x = 2$.

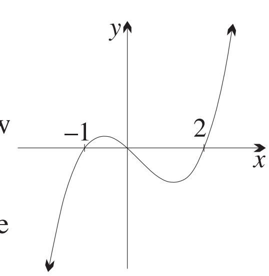
b By evaluating $\int_{-3}^{1} (x^2 + 2x - 3)\, dx$, find the area of the shaded region below the x-axis.

c By evaluating $\int_{1}^{2} (x^2 + 2x - 3)\, dx$, find the area of the shaded region above the x-axis.

d Hence find the area of the entire shaded region.

e Find $\int_{-3}^{2} (x^2 + 2x - 3)\, dx$, and explain why this integral does not give the area of the shaded region.

7 The curve $y = x(x + 1)(x - 2) = x^3 - x^2 - 2x$ is graphed.



a Copy the diagram, and shade the region bounded by the curve and the x-axis.

b By evaluating $\int_{0}^{2} (x^3 - x^2 - 2x)\, dx$, find the area of the shaded region below the x-axis.

c By evaluating $\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx$, find the area of the shaded region above the x-axis.

d Hence find the area of the entire region you have shaded.

e Find $\int_{-1}^{2} (x^3 - x^2 - 2x)\, dx$, and explain why this integral does not give the area of the shaded region.

ISBN 978-1-108-76630-2

DEVELOPMENT

8 In each part below, find the area of the region bounded by the graph of the given function and the x-axis between the specified values. Remember that areas above and below the x-axis must be calculated separately.

a $y = x^2$, between $x = -3$ and $x = 2$,
b $y = 2x^3$, between $x = -4$ and $x = 1$,
c $y = 3x(x - 2)$, between $x = 0$ and $x = 2$,
d $y = x - 3$, between $x = -1$ and $x = 4$,
e $y = (x - 1)(x + 3)(x - 2)$, between $x = -3$, and $x = 2$
f $y = -2x(x + 1)$, between $x = -2$ and $x = 2$.

9 In each part below, find the area of the region bounded by the graph of the given function and the y-axis between the specified values. Remember that areas to the right and to the left of the y-axis must be calculated separately.

a $x = y - 5$, between $y = 0$ and $y = 6$,
b $x = 3 - y$, between $y = 2$ and $y = 5$,
c $x = y^2$, between $y = -1$ and $y = 3$,
d $x = (y - 1)(y + 1)$, between $y = 3$ and $y = 0$.

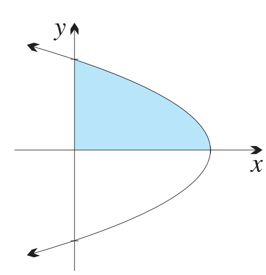
10 In each part below you should sketch the curve and look carefully for any symmetries that will simplify the calculation.

a Find the area of the region bounded by the given curve and the x-axis.
  i $y = x^7$, for $-2 \le x \le 2$,
  ii $y = x^3 - 16x = x(x - 4)(x + 4)$, for $-4 \le x \le 4$,
  iii $y = x^4 - 9x^2 = x^2(x - 3)(x + 3)$, for $-3 \le x \le 3$.

b Find the area of the region bounded by the given curve and the y-axis.
  i $x = 2y$, for $-5 \le y \le 5$,
  ii $x = y^2$, for $-3 \le y \le 3$,
  iii $x = 4 - y^2 = (2 - y)(2 + y)$, for $-2 \le y \le 2$.

11 Find the area of the region bounded by $y = |x + 2|$ and the x-axis, for $-2 \le x \le 2$.

12 The diagram shows the parabola $y^2 = 16(2 - x)$.



a Find the x-intercept and the y-intercepts.
b Find the exact area of the shaded region:
  i by integrating $y = 4\sqrt{2 - x}$ with respect to x,
  ii by integrating with respect to y. (You will need to make x the subject of the equation.)

13 The gradient of a curve is $y' = x^2 - 4x + 3$, and the curve passes through the origin.

a Find the equation of the curve.
b Show that the curve has turning points at $\left(1, 1\frac{1}{3}\right)$ and $(3, 0)$, and sketch its graph.
c Find the area of the region bounded by the curve and the x-axis between the two turning points.

ISBN 978-1-108-76630-2

14 Sketch $y = x^2$ and mark the points $A(a, a^2)$, $B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.

a Show that $\int_0^a x^2\,dx = \frac{2}{3}$ (area $\triangle OAP$).

b Show that $\int_{-a}^a x^2\,dx = \frac{1}{3}$ (area of rectangle $ABQP$).

15 Given positive real numbers a and n, let A, P and Q be the points (a, a^n), $(a, 0)$ and $(0, a^n)$ respectively. Find the ratios:

a $\int_0^a x^n\,dx$: (area of $\triangle AOP$)

b $\int_0^a x^n\,dx$: (area of rectangle $OPAQ$)

16 a Show that $x^4 - 2x^3 + x = x(x - 1)(x^2 - x - 1)$. Then sketch a graph of the function $y = x^4 - 2x^3 + x$ and shade the three regions bounded by the graph and the x-axis.

b If $a = \frac{1}{2}\left(1 + \sqrt{5}\right)$, evaluate a^2, a^4 and a^5.

c Show that the area of one shaded region equals the sum of the areas of the other two.

ENRICHMENT

17 Consider the function $G(x) = \int_0^x g(u)\,du$, where $g(u) = \begin{cases} 4 - \frac{4}{3}u, & \text{for } 0 \le u < 6, \\ u - 10, & \text{for } 6 \le u \le 12. \end{cases}$

a Sketch a graph of $g(u)$.

b Find the stationary points of the function $y = G(x)$ and determine their nature.

c Find the values of x for which $G(x) = 0$.

d Sketch the curve $y = G(x)$, indicating all important features.

e Find the area bounded by the curve $y = G(x)$ and the x-axis for $0 \le x \le 6$.

18 a Show that for $n < -1$, $\int_1^N x^n\,dx$ converges as $N \to \infty$, and find the limit.

b Show that for $n > -1$, $\int_\varepsilon^1 x^n\,dx$ converges as $\varepsilon \to 0^+$, and find the limit.

c Interpret these two results as areas.

ISBN 978-1-108-76630-2

5G Areas of compound regions

When a region is formed by two or more different curves, some dissection process is usually needed before integrals can be used to calculate its area. Thus a preliminary sketch of the region becomes all the more important.

Areas of regions under a combination of curves

Some regions are bounded by different curves in different parts of the x-axis.

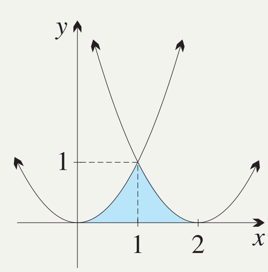
Example 28 5G

a Sketch the curves $y = x^2$ and $y = (x - 2)^2$ on one set of axes.
b Shade the region bounded by $y = x^2$, $y = (x - 2)^2$ and the x-axis.
c Find the area of this shaded region.

SOLUTION

a The two curves intersect at $(1, 1)$, because it is easily checked by substitution that this point lies on both curves.

b The whole region is above the x-axis, but it will be necessary to find separately the areas of the regions to the left and right of $x = 1$.



c First,
$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \tfrac{1}{3}.$$

Secondly,
$$\int_1^2 (x-2)^2\,dx = \left[\frac{(x-2)^3}{3}\right]_1^2 = 0 - \left(-\tfrac{1}{3}\right) = \tfrac{1}{3}.$$

Combining these,
$$\text{area} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}\text{ square units.}$$

Note: In this worked example, the second parabola is the first shifted right 2, and a parabola is symmetric about its axis of symmetry. This is why the two pieces have the same area.

Areas of regions between curves

Suppose that one curve $y = f(x)$ is always below another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area of the region between the curves from $x = a$ to $x = b$ can be found by subtraction.

17 AREA BETWEEN CURVES

If $f(x) \le g(x)$ in the interval $a \le x \le b$, then

$$\text{area between the curves} = \int_a^b \big(g(x) - f(x)\big)\,dx.$$

That is, take the integral of the top curve minus the bottom curve.

ISBN 978-1-108-76630-2

Cambridge University Press

The assumption that $f(x) \leq g(x)$ is important. If the curves cross each other, then separate integrals will need to be taken or else the areas of regions where different curves are on top will begin to cancel each other out.

Example 29 5G

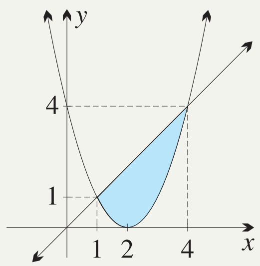
a Find the two points where the curve $y = (x-2)^2$ meets the line $y = x$.
b Draw a sketch and shade the area of the region between these two graphs.
c Find the shaded area.

SOLUTION

a Substituting $y = x$ into $y = (x-2)^2$ gives

$$\begin{aligned}(x-2)^2 &= x\\ x^2 - 4x + 4 &= x\\ x^2 - 5x + 4 &= 0\\ (x-1)(x-4) &= 0,\\ x &= 1 \text{ or } 4,\end{aligned}$$

so the two graphs intersect at $(1,1)$ and $(4,4)$.



b The sketch is drawn to the right.

c In the shaded region, the line is above the parabola.

$$\begin{aligned}\text{Hence area } &= \int_1^4 \left(x - (x-2)^2\right) dx\\ &= \int_1^4 \left(x - (x^2 - 4x + 4)\right) dx\\ &= \int_1^4 (-x^2 + 5x - 4)\, dx\\ &= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4\\ &= \left(-21\tfrac{1}{3} + 40 - 16\right) - \left(-\tfrac{1}{3} + 2\tfrac{1}{2} - 4\right)\\ &= 2\tfrac{2}{3} + 1\tfrac{5}{6}\\ &= 4\tfrac{1}{2} \text{ square units.}\end{aligned}$$

Note: The formula given in Box 17 for the area of the region between two curves holds even if the region crosses the x-axis.

To illustrate this point, the next example is the previous example shifted down 2 units so that the region between the line and the parabola crosses the x-axis. The area of course remains the same — and notice how the formula still gives the correct answer.

ISBN 978-1-108-76630-2

Cambridge University Press

Example 30 5G

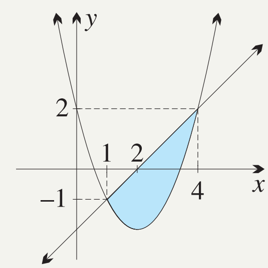
a Find the two points where the curves $y = x^2 - 4x + 2$ and $y = x - 2$ meet.
b Draw a sketch and find the area of the region between these two curves.

SOLUTION

a Substituting $y = x - 2$ into $y = x^2 - 4x + 2$ gives

$$\begin{aligned} x^2 - 4x + 2 &= x - 2 \\ x^2 - 5x + 4 &= 0 \\ (x-1)(x-4) &= 0 \\ x &= 1 \text{ or } 4. \end{aligned}$$

so the two graphs intersect at $(1, -1)$ and $(4, 2)$.



b Again, the line is above the parabola,

$$\begin{aligned} \text{so area} &= \int_1^4 \big((x-2) - (x^2 - 4x + 2)\big)\,dx \\ &= \int_1^4 (-x^2 + 5x - 4)\,dx \\ &= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4 \\ &= \left(-21\tfrac{1}{3} + 40 - 16\right) - \left(-\tfrac{1}{3} + 2\tfrac{1}{2} - 4\right) \\ &= 4\tfrac{1}{2} \text{ square units.} \end{aligned}$$

Areas of regions between curves that cross

Now suppose that one curve $y = f(x)$ is sometimes above and sometimes below another curve $y = g(x)$ in the relevant interval. In this case, the conditions of Box 17 no longer hold, and separate integrals will need to be calculated.

Example 31 5G

The diagram below shows the curves

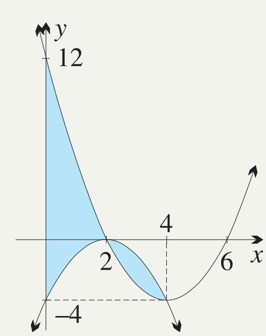
$$y = -x^2 + 4x - 4 \qquad \text{and} \qquad y = x^2 - 8x + 12$$

meeting at the points $(2, 0)$ and $(4, -4)$. Find the area of the shaded region.

SOLUTION

In the left-hand region, the second curve is above the first,

$$\begin{aligned} \text{So} \qquad \text{area} &= \int_0^2 \big((x^2 - 8x + 12) - (-x^2 + 4x - 4)\big)\,dx \\ &= \int_0^2 (2x^2 - 12x + 16)\,dx \\ &= \left[\frac{2x^3}{3} - 6x^2 + 16x\right]_0^2 \\ &= 5\tfrac{1}{3} - 24 + 32 \\ &= 13\tfrac{1}{3}\,\text{u}^2. \end{aligned}$$



ISBN 978-1-108-76630-2

In the right-hand region, the first curve is above the second,

$$\text{so} \qquad \text{area} = \int_2^4 \Big((-x^2 + 4x - 4) - (x^2 - 8x + 12) \Big)\, dx$$

$$= \int_2^4 (-2x^2 + 12x - 16)\, dx$$

$$= \left[-\frac{2x^3}{3} + 6x^2 - 16x \right]_2^4$$

$$= \left(-42\tfrac{2}{3} + 96 - 64\right) - \left(-5\tfrac{1}{3} + 24 - 32\right)$$

$$= -10\tfrac{2}{3} + 13\tfrac{1}{3}$$

$$= 2\tfrac{2}{3}\,\text{u}^2.$$

$$\text{Hence total area} = 13\tfrac{1}{3} + 2\tfrac{2}{3}$$

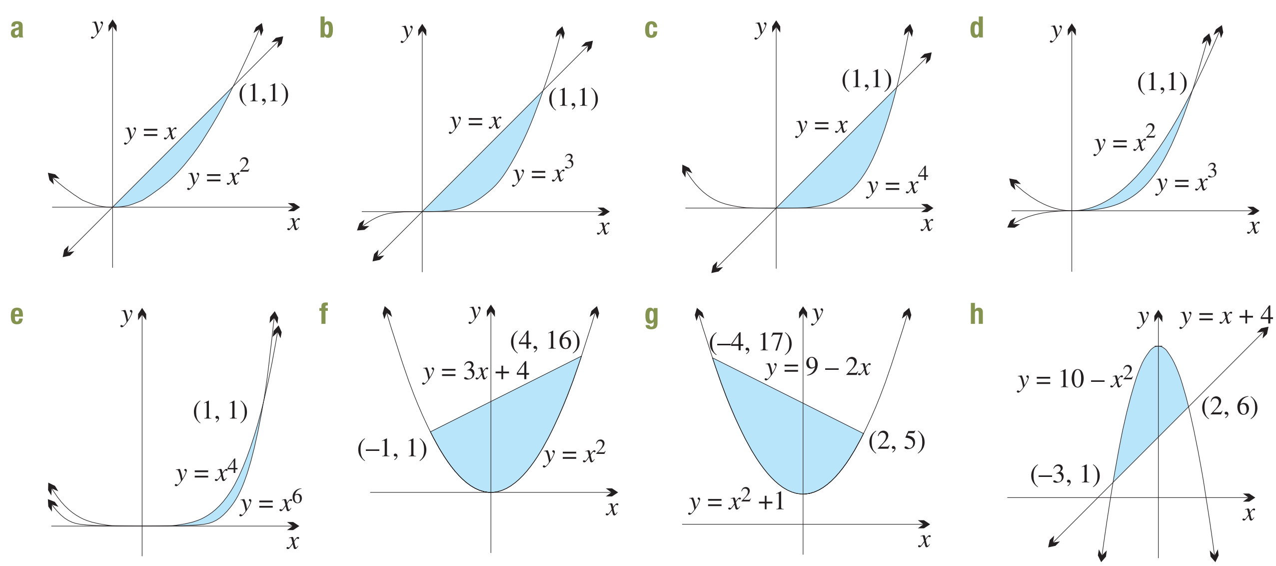
$$= 16\,\text{u}^2.$$

Exercise 5G

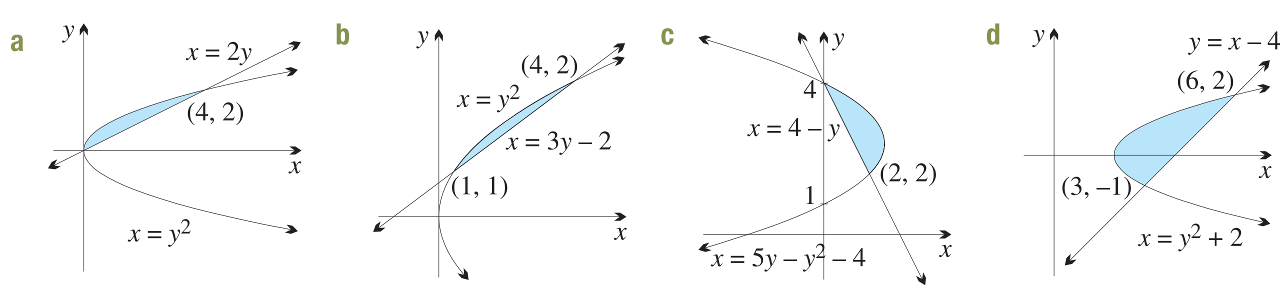
FOUNDATION

Technology: Graphing programs are particularly useful with compound regions because they allow the separate parts of the region to be identified clearly.

1 Find the area of the shaded region in each diagram below.



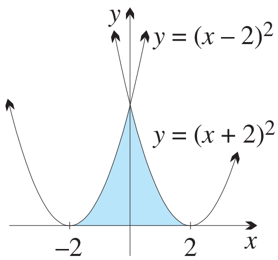
2 By considering regions between the curves and the y–axis, find the area of the shaded region in each diagram below.

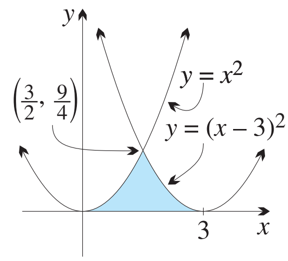


ISBN 978-1-108-76630-2

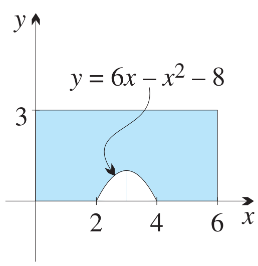
Cambridge University Press

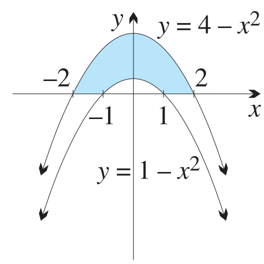
3 Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and add them.

a


b


4 Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and subtract one from the other.

a


b


DEVELOPMENT

5 a By solving the equations simultaneously, show that the parabola $y = x^2 + 4$ and the line $y = x + 6$ intersect at the points $(-1, 5)$ and $(2, 8)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{-1}^{2} \big((x + 6) - (x^2 + 4) \big)\, dx = \int_{-1}^{2} (x - x^2 + 2)\, dx,$$

and evaluate the integral.

6 a By solving the equations simultaneously, show that the parabola $y = 3x - x^2 = x(3 - x)$ and the line $y = x$ intersect at the points $(0, 0)$ and $(2, 2)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{0}^{2} (3x - x^2 - x)\, dx = \int_{0}^{2} (2x - x^2)\, dx,$$

and evaluate the integral.

7 a By solving the equations simultaneously, show that the parabola $y = (x - 3)^2$ and the line $y = 14 - 2x$ intersect at the points $(-1, 16)$ and $(5, 4)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{-1}^{5} \big((14 - 2x) - (x - 3)^2 \big)\, dx = \int_{-1}^{5} (4x + 5 - x^2)\, dx,$$

and evaluate the integral.

ISBN 978-1-108-76630-2

8 Solve simultaneously to find the points of intersection of each pair of graphs. Then sketch the graphs on the same diagram and shade the region enclosed between them. By evaluating the appropriate definite integral, find the area of the shaded region in each case.

a $y = x + 3$ and $y = x^2 + 1$

b $y = 9 - x^2$ and $y = 3 - x$

c $y = x^2 - x + 4$ and $y = -x^2 + 3x + 4$

9 **a** By solving the equations simultaneously, show that the parabola $y = x^2 + 2x - 8$ and the line $y = 2x + 1$ intersect at the points $(3, 7)$ and $(-3, -5)$.

b Sketch both graphs on the same diagram, and shade the region enclosed between them.

c Despite the fact that it crosses the x-axis, the region has area given by

$$\int_{-3}^{3} \Big((2x + 1) - (x^2 + 2x - 8) \Big)\,dx = \int_{-3}^{3} (9 - x^2)\,dx.$$

Evaluate the integral and hence find the area of the region enclosed between the curves.

10 **a** By solving the equations simultaneously, show that the parabola $y = x^2 - x - 2$ and the line $y = x - 2$ intersect at the points $(0, -2)$ and $(2, 0)$.

b Sketch both graphs on the same diagram, and shade the region enclosed between them.

c Despite the fact that it is below the x–axis, the region has area given by

$$\int_{0}^{2} \Big((x - 2) - (x^2 - x - 2) \Big)\,dx = \int_{0}^{2} (2x - x^2)\,dx.$$

Evaluate this integral and hence find the area of the region between the curves.

11 Solve simultaneously to find the points of intersection of each pair of graphs. Then sketch the graphs on the same diagram, and shade the region enclosed between them. By evaluating the appropriate definite integral, find the area of the shaded region in each case.

a $y = x^2 - 6x + 5$ and $y = x - 5$

b $y = -3x$ and $y = 4 - x^2$

c $y = x^2 - 1$ and $y = 7 - x^2$

12 **a** On the same number plane, sketch the graphs of the parabolas $y = x^2$ and $x = y^2$, clearly indicating their points of intersection. Shade the region enclosed between them.

b Explain why the area of this region is given by $\displaystyle\int_{0}^{1} \left(\sqrt{x} - x^2\right) dx$.

c Find the area of the region bounded by the two curves.

13 Tangents are drawn to the parabola $x^2 = 8y$ at the points $A(4, 2)$ and $B(-4, 2)$.

a Draw a diagram of the situation and note the symmetry about the y–axis.

b Find the equation of the tangent at the point A.

c Find the area of the region bounded by the curve and the tangents.

14 **a** Show that the tangent to the curve $y = x^3$ at the point where $x = 2$ has equation $y = 12x - 16$.

b Show by substitution that the tangent and the curve intersect again at the point $(-4, -64)$.

c Find the area of the region enclosed between the curve and the tangent.

15 Consider the curves $y = x^3 - 3$ and $y = -x^2 + 10x - 11$.

a Show by substitution that the curves intersect at three points whose x-values are -4, 1 and 2.

b Sketch the curves showing clearly their intersection points.

c Find the area of the region enclosed by the two curves.

ISBN 978-1-108-76630-2

16 **a** Find the points of intersection of the curves $y = x^2(1 - x)$ and $y = x(1 - x)^2$.
b Hence find the area bounded by the two curves.

17 **a** Given the two functions $f(x) = (x + 1)(x - 1)(x - 3)$ and $g(x) = (x + 1)(x - 1)$, for what values of x is $f(x) > g(x)$?
b Sketch a graph of the two functions on the same number plane, and find the area enclosed between them.

18 **a** Sketch a graph of the function $y = 12x - 32 - x^2$, clearly indicating the x-intercepts.
b Find the equation of the tangent to the curve at the point A where $x = 5$.
c If the tangent meets the x-axis at B, and C is the x-intercept of the parabola closer to the origin, find the area of the region bounded by AB, BC and the arc CA.

ENRICHMENT

19 Find the value of k for which the line $y = kx$ bisects the area enclosed by the curve $4y = 4x - x^2$ and the x-axis.

20 The *average value* of a continuous function $f(x)$ over an interval $a \le x \le b$ is defined to be $\dfrac{1}{b - a}\displaystyle\int_a^b f(x)\,dx$ if $a \ne b$, or $f(a)$ if $a = b$. If k is the average value of $f(x)$ on the interval $a \le x \le b$, show that the area of the region bounded by $f(x)$ above the line $y = k$ is equal to the area of the region bounded by $f(x)$ below the line $y = k$.

ISBN 978-1-108-76630-2

Cambridge University Press

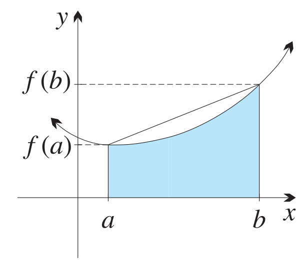
5H The trapezoidal rule

Methods of approximating definite integrals become necessary when exact calculations using primitives are not possible. This can happen for two reasons.

- The primitives of many important functions cannot be written down in a formula suitable for calculation — this is the case for the important normal distribution in Chapter 16.
- Some values of a function may be known only from experiments, and the function formula may be unknown.

The trapezoidal rule

Besides taking upper and lower rectangles, the most obvious way to approximate an integral is to replace the curve by a straight line, that is, by a chord joining $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. The resulting region is a trapezium, so this approximation method is called the *trapezoidal rule*.



Consider the trapezium in the diagram to the right.

Here width $= b - a$,

and average of parallel sides $= \dfrac{f(a) + f(b)}{2}$.

Hence area of trapezium $=$ width $\times$ average of parallel sides

$$= \frac{b-a}{2}\big(f(a) + f(b)\big).$$

The area of this trapezium is taken as an approximation of the integral.

18 THE TRAPEZOIDAL RULE USING ONE SUBINTERVAL

Let $f(x)$ be a function that is continuous in the closed interval $[a, b]$.

- Approximating the curve from $x = 1$ to $x = b$ by a chord allows the region under the curve to be approximated by a trapezium, giving

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}\big(f(a) + f(b)\big).$$

- If the function is linear, then the chord coincides with the curve and the formula is exact.
- Always start a trapezoidal-rule calculation by constructing a table of values.

Subdividing the interval

Given an integral over an interval $[a, b]$, we can split that interval $[a, b]$ up into a number of subintervals and apply the trapezoidal rule to each subinterval in turn. This will usually improve the accuracy of the approximation.

Here is the method applied to the reciprocal function $y = \dfrac{1}{x}$, whose primitive we will establish in Chapter 6.

ISBN 978-1-108-76630-2

Example 32 5H

Find approximations of $\int_1^5 \frac{1}{x}\,dx$ using the trapezoidal rule with:

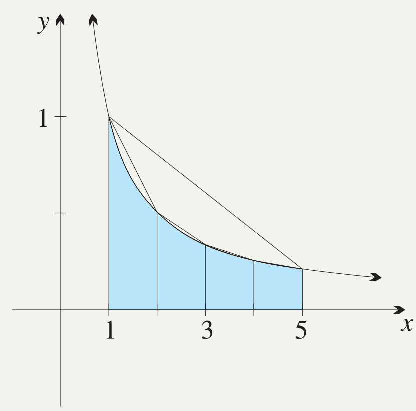
a one subinterval,

b four subintervals.

SOLUTION

Always begin with a table of values of the function.

x	1	2	3	4	5
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



a One application of the trapezoidal rule, using the whole interval as the subinterval, requires just two values of the function.

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \frac{5-1}{2} \times \big(f(1) + f(5)\big)\\ &\doteqdot 2 \times \big(1 + \tfrac{1}{5}\big)\\ &\doteqdot 2\tfrac{2}{5}\end{aligned}$$

b Four applications of the trapezoidal rule require five values of the function.

Dividing the interval $1 \le x \le 5$ into four equal subintervals,

$$\int_1^5 \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \int_3^4 \frac{1}{x}\,dx + \int_4^5 \frac{1}{x}\,dx$$

Each subinterval has width 1, so applying the trapezoidal rule to each integral,

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \tfrac{1}{2}\big(f(1) + f(2)\big) + \tfrac{1}{2}\big(f(2) + f(3)\big) + \tfrac{1}{2}\big(f(3) + f(4)\big) + \tfrac{1}{2}\big(f(4) + f(5)\big)\\ &\doteqdot \tfrac{1}{2}\big(\tfrac{1}{1} + \tfrac{1}{2}\big) + \tfrac{1}{2}\big(\tfrac{1}{2} + \tfrac{1}{3}\big) + \tfrac{1}{2}\big(\tfrac{1}{3} + \tfrac{1}{4}\big) + \tfrac{1}{2}\big(\tfrac{1}{4} + \tfrac{1}{5}\big)\\ &\doteqdot 1\tfrac{41}{60}.\end{aligned}$$

Note: Always take subintervals of equal width unless otherwise indicated.

Concavity and the trapezoidal rule

The curve in the example above is concave up, so every chord is above the curve, and every approximation found using the trapezoidal rule is therefore greater than the integral.

Similarly, if a curve is concave down, then every chord is below the curve, and every trapezoidal-rule approximation is less than the integral. The second derivative can be used to test concavity.

ISBN 978-1-108-76630-2

19 CONCAVITY AND THE TRAPEZOIDAL RULE

- If the curve is concave up, the trapezoidal rule overestimates the integral.
- If the curve is concave down, the trapezoidal rule underestimates the integral.
- If the curve is linear, the trapezoidal rule gives the exact value of the integral.

The second derivative $\dfrac{d^2y}{dx^2}$ can be used to test the concavity.

Example 33 5H

a Use the trapezoidal rule with one subinterval (that is, two function values) to approximate $\displaystyle\int_1^5 (200x - x^4)\,dx$.

b Use the second derivative to explain why the approximation underestimates the integral.

SOLUTION

a Construct a table of values for $y = 200x - x^4$.

x	1	5
y	199	375

$$\begin{aligned}\int_1^5 (200x - x^4)\,dx &\doteqdot \frac{5-1}{2} \times \big(f(1) + f(5)\big)\\ &\doteqdot 2 \times (199 + 375)\\ &\doteqdot 1148\end{aligned}$$

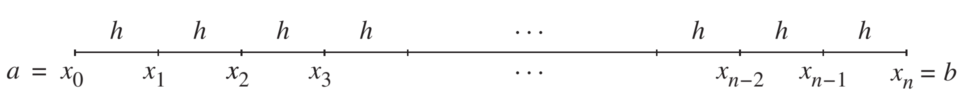
b The function is $y = 200x - x^4$.
Differentiating, $y' = 200 - 4x^3$
and $y'' = -12x^2$.
Because $y'' = -12x^2$ is negative throughout the interval $1 \le x \le 5$, the curve is concave down throughout this interval.
Hence the trapezoidal rule underestimates the integral.

A formula for multiple applications of the trapezoidal rule

When the trapezoidal rule is being applied two or three times, it is easier to perform the two or three calculations required. These separate calculations also reinforce the meaning of the approximation, and help to gain an intuitive understanding of the accuracy of the estimates.

But increasing accuracy with the trapezoidal rule requires larger numbers of applications of the rule, and this can quickly become tedious. Let us then develop a single formula that splits an integral into n subintervals of equal width and applies the trapezoidal rule to each — anyone writing a program or using a spreadsheet to estimate integrals would want to do this.

The first step is to divide the interval $[a, b]$ into n equal subintervals, each of width h, like this:



ISBN 978-1-108-76630-2

There are $n + 1$ points altogether, and they divide the interval into n equal subintervals. The endpoints are $a = x_0$ and $b = x_{n+1}$, and the $n - 1$ division points in between are $x_1, x_2, \ldots, x_{n-1}$.

There are n subintervals, so $nh = b - a$, and the width h of each subinterval is

$$h = \frac{b - a}{n}.$$

Thus starting with $a = x_0$, the successive values of the division points are

$$\begin{array}{lcl} x_0 = a & & x_{n-2} = a + (n-2)h \\ x_1 = a + h & \ldots & x_{n-1} = a + (n-1)h \\ x_2 = a + 2h & & x_n = a + nh = a + (b - a) = b \end{array}$$

That is, $x_r = a + rh$, for $r = 0, 1, 2, \ldots, n - 1, n$.
Now we can apply the trapezoidal rule to each subinterval in turn,

$$\begin{aligned} \int_a^b f(x)\,dx &= \int_a^{x_1} f(x)\,dx + \int_{x_2}^{x_1} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)\,dx + \int_{x_{n-1}}^{b} f(x)\,dx \\ &\doteqdot \frac{h}{2}\Big(f(a) + f(x_1)\Big) + \frac{h}{2}\Big(f(x_1) + f(x_2)\Big) + \cdots \\ &\quad + \frac{h}{2}\Big(f(x_{n-2}) + f(x_{n-1})\Big) + \frac{h}{2}\Big(f(x_{n-1}) + f(b)\Big) \\ &\doteqdot \frac{h}{2}\Big(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\Big). \end{aligned}$$

20 TRAPEZOIDAL-RULE FORMULA USING *n* SUBINTERVALS

Let $f(x)$ be a function that is continuous in the closed interval $[a, b]$. Then

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}\Big(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\Big)$$

where $h = \dfrac{b - a}{n}$ and $x_r = a + rh$, for $r = 1, 2, \ldots, n - 1$.

A common rearrangement of this formula, using three sets of nested brackets, is

$$\int_a^b f(x)\,dx \doteqdot \frac{b - a}{2n}\bigg(f(a) + f(b) + 2\Big(f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1})\Big)\bigg).$$

Using the formula for the trapezoidal rule

The formula may look complicated at first sight, but it is actually quite straightforward to use, provided that:

- We begin with a sensible value of the width h of each subinterval.
- We construct a clear table of values to work from.

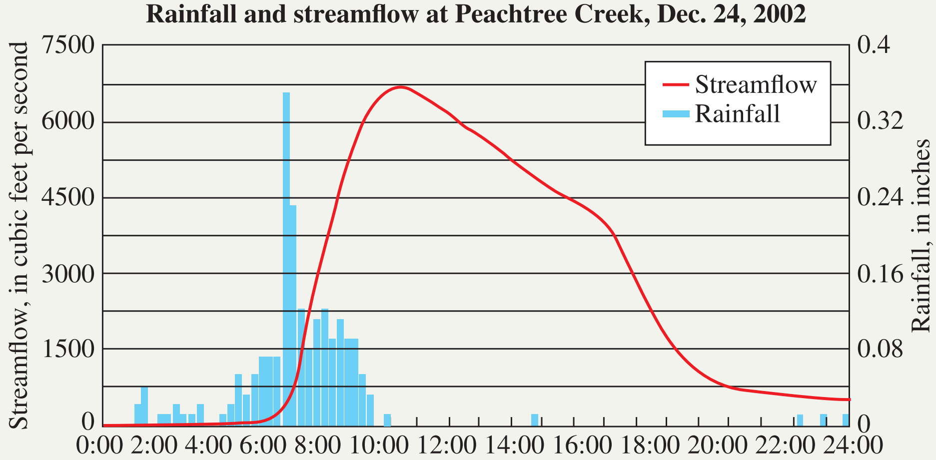
Here is an example where there is no equation of the function, but simply a set of experimental results gathered by recording equipment.

ISBN 978-1-108-76630-2

Example 34 5H

The flow at Peachtree Creek on 4th December 2002 after a storm is shown in the graph below. The flow rate in cubic feet per second is sketched as a function of the time t in hours.

Rainfall and streamflow at Peachtree Creek, Dec. 24, 2002



We can estimate the total amount of water that flowed down the creek after the storm that day by integrating from $t = 4$ to $t = 24$. Use the trapezoidal rule with two-hour subintervals to approximate the total amount of water.

SOLUTION

The graph is very inaccurate, like so much internet data, but here is a rough table of values of the flow rate in cubic feet per second as a function of time t in hours.

t	4	6	8	10	12	14	16	18	20	22	24
Flow rate	100	600	5500	6700	5800	4800	4100	1800	800	600	500

The units need attention. The time is in hours, so the flow rates must be converted to cubic feet per hour by multiplying by $60 \times 60 = 3600$.

To avoid zeroes, let R be the flow rate in millions of cubic feet per hour.

t	4	6	8	10	12	14	16	18	20	22	24
R	0.36	2.16	19.8	24.12	20.88	17.28	14.76	6.48	2.88	2.16	1.8

Here $h = 2$ and $n = 10$. Also $a = x_0 = 4, x_1 = 6, \ldots, x_{n-1} = 22, x_n = b = 24$.

$$\text{Hence } \int_4^{20} R\,dt \doteqdot \frac{2}{2}\Big(f(4) + 2f(6) + 2f(8) + \cdots + 2f(22) + f(24)\Big)$$
$$\doteqdot 0.36 + 4.32 + 39.6 + 48.24 + 34.56 + 29.52 + 12.96 + 5.76 + 4.32 + 1.8$$
$$\doteqdot 223.2.$$

Alternatively, using the second formula,

$$\int_4^{20} R\,dt \doteqdot \frac{24 - 4}{20}\Big(f(4) + f(24) + 2\big(f(6) + f(8) + \cdots + f(22)\big)\Big)$$
$$\doteqdot 0.36 + 1.8 + 2(2.16 + 19.8 + 24.12 + 17.28 + 14.76 + 62.48 + 1.44 + 1.08)$$
$$= 2.16 + 2 \times 110.52$$
$$= 223.2.$$

Thus about 223 million cubic feet of water flowed down the creek from 4:00 am to midnight.

ISBN 978-1-108-76630-2

Using a spreadsheet for calculations

The authors used a spreadsheet for all the calculations above — the trapezoidal-rule formula is well suited for machine computation. The next worked example shows how to use an Excel spreadsheet to carry out such a calculation, but any spreadsheet can be used. Note that:

- Excel commands and procedures have been changing over successive versions.
- Mac users will need some adjustments, particularly when implementing the ‘fill down’ and ‘fill right’ commands.

The calculation involves the integration of $\dfrac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. We will see in Chapter 16 that this function is the probability density function of the normal distribution, and is the most important function in statistics. There is no simple equation for its primitive, so approximations are always necessary.

Example 35 5H

Let $\phi(x) = \dfrac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. Approximate the integral $\displaystyle\int_0^1 \phi(x)\,dx$ using the trapezoidal rule with 10 subintervals.

(The symbol ϕ is the lower case Greek letter ‘phi’, corresponding to Latin f.)

SOLUTION

In these instructions, we enter a formula into a cell by typing the $=$ sign as the first character. The cell in column D and row is labelled D3, and in formulae, this refers to its contents. Normally leave the top row clear for later titles.

1 On a new sheet in Excel, enter 0 into Cell `A2` and press `Enter`.
— Select Cell `B2` and type `=0.1+A2` and press `Enter`.
— Cell `B2` should now show 0.1.

2 Select Cells `B2:K2` and press `Ctrl+R` to ‘fill right’.
— Cells `C2:K2` should now show 0.2, 0.3, . . . , 1.

3 Type `=EXP(-A2*A2/2)/SQRT(2*PI())` into Cell `A3`.
— Cell `A3` should now show $\phi(0) = 0.398942$.

4 Select Cells `A3:K3` and press `Ctrl+R` to ‘fill right’.
— Cells `B3:K3` should now show $\phi(0.1) = 0.396953, \dots, \phi(1) = 0.241971$.

We now have the table of values for the function $\phi(x)$, and we need to add

$$\phi(0) + 2\phi(0.1) + 2\phi(0.2) + \cdots + 2\phi(0.9) + \phi(1)$$

5 Select Cell `A4` and enter `=A3`. This should duplicate the value in `A3`.
— Select Cell `K4` and enter `=K3`. Again, this duplicates the value in `K3`.
— Select Cell `B4` and enter `=2*B3`. This should double the value in `B3`.
— Select Cells `B4:J4` and press `Ctrl+R` to ‘fill right’.
— Add the row by selecting Cell `L4` and typing `=SUM(A4:K4)`.

In this case, $h = 0.1$ so we multiply by $\frac{1}{2}h = 0.05$.

ISBN 978-1-108-76630-2

6 Select Cell L5 and type =L4*0.05 — this shows the final answer.

Hence $\int_0^1 \phi(x)\, dx \doteqdot 0.341$. We shall find in Chapter 16 that this is approximately the probability that a score in a normal distribution lies between the mean 0 and one standard deviations above the mean. The correct approximation to three decimal places is 0.398 — we will see in Chapter 16 that the curve is concave down in the interval $[0, 1]$, which explains why our estimate is a little smaller than it should be.

Readers may like to repeat the calculations above using 100 subintervals and see how close the approximation is then.

Exercise 5H

FOUNDATION

Technology: It is not difficult to write (or download) a program that will allow the calculations of the trapezoidal rule to be automated. It can then be applied to many examples from this exercise. The number of subintervals used can be steadily increased, and the approximations may then converge to the exact value of the integral. An accompanying screen sketch showing the curve and the chords would be helpful in giving a visual impression of the size and the sign of the error.

1 Approximate $\int_2^6 f(x)\, dx$ using the formula $\frac{1}{2}(a + b)h$ for the area of a trapezium.

a

x	2	6
$f(x)$	8	12

b

x	2	6
$f(x)$	6.2	4.8

c

x	2	6
$f(x)$	-4	-9

2 Three function values are given in the table to the right.

x	2	6	10
$f(x)$	12	20	30

a Approximate $\int_2^{10} f(x)\, dx$ by calculating the areas of two trapezia and then adding.

b Check your answer to part a by using the formula for the trapezoidal rule.

3 Use the trapezoidal rule with the three given function values to approximate $\int_{-5}^{5} f(x)\, dx$.

x	-5	0	5
$f(x)$	2.4	2.6	4.4

4 Show, by means of a diagram, that the trapezoidal rule will.

a overestimate $\int_a^b f(x)\, dx$, if $f''(x) > 0$ for $a \le x \le b$,

b underestimate $\int_a^b f(x)\, dx$, if $f''(x) < 0$ for $a \le x \le b$.

5 a Complete this table for the function $y = x(4 - x)$.

x	0	1	2	3	4
y					

b Hence use the trapezoidal rule with five function values to approximate $\int_0^4 x(4 - x)\, dx$.

ISBN 978-1-108-76630-2

c What is the exact value of $\int_0^4 x(4 - x)\, dx$, and why does it exceed the approximation? Sketch the curve and the four chords involved.

d Calculate the percentage error in the approximation (that is, divide the error by the exact answer and convert to a percentage).

6 **a** Complete this table for the function $y = \frac{6}{x}$.

x	1	2	3	4	5
y					

b Use the trapezoidal rule with the five function values above, that is with four subintervals, to approximate $\int_1^5 \frac{6}{x}\, dx$.

c Show that the second derivative of $y = \frac{6}{x}$ is $y'' = 12x^{-3}$, and use this result to explain why the approximation will exceed the exact value of the integral.

7 **a** Complete this table correct to four decimal places for the function $y = \sqrt{x}$.

x	9	10	11	12	12	13	14	15
y								

b Approximate $\int_9^{16} \sqrt{x}\, dx$, using the trapezoidal rule with the eight function values above. Give your answer correct to three significant figures.

c What is the exact value of $\int_9^{16} \sqrt{x}\, dx$? Show that the second derivative of $y = x^{\frac{1}{2}}$ is $y'' = -\frac{1}{4}x^{-\frac{3}{2}}$, and use this result to explain why the approximation is less than the value of the definite integral.

DEVELOPMENT

8 Use the trapezoidal rule with three function values to approximate each definite integral, writing your answer correct to two significant figures where necessary.

a $\int_0^1 2^{-x}\, dx$ **b** $\int_{-2}^0 2^{-x}\, dx$ **c** $\int_1^3 \sqrt[3]{9 - 2x}\, dx$ **d** $\int_{-13}^{-1} \sqrt{3 - x}\, dx$

9 Use the trapezoidal rule with five function values to approximate each definite integral, writing your answer correct to three significant figures where necessary.

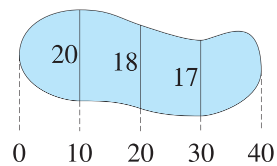
a $\int_2^6 \frac{1}{x}\, dx$ **b** $\int_0^2 \frac{1}{2 + \sqrt{x}}\, dx$ **c** $\int_4^8 \sqrt{x^2 - 3}\, dx$ **d** $\int_1^2 \log_{10} x\, dx$

10 An object is moving along the x-axis with values of the velocity v in m/s at various times t given in the table to the right. Given that the distance travelled may be found by calculating the area under the velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the particle in the first 5 seconds.

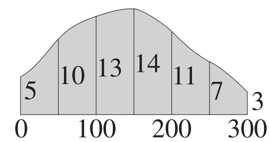
t	0	1	2	3	4	5
v	1.5	1.3	1.4	2.0	2.4	2.7

ISBN 978-1-108-76630-2

11 The diagram to the right shows the width of a lake at 10-metre intervals. Use the trapezoidal rule to estimate the surface area of the water.



12 The diagram to the right shows a vertical rock cutting of length 300 metres alongside a straight horizontal section of highway. The heights of the cutting are measured at 50-metre intervals. Use the trapezoidal rule to estimate the area of the vertical rock cutting.



13 **a** Use the trapezoidal rule with five function values to approximate $\int_0^1 \sqrt{1 - x^2}\, dx$, giving your answer correct to four decimal places.

a Use part **a** and the fact that $y = \sqrt{1 - x^2}$ is a semi-circle to approximate π. Give your answer correct to one decimal place, and explain why your approximation is less than π.

14 Use the trapezoidal rule with five function values, together with appropriate log laws, to show that $\int_1^5 \ln x\, dx \doteqdot \ln 54$.

ENRICHMENT

15 **a** Show that the function $y = \sqrt{x}$ is increasing for all $x > 0$.

a By dividing the area under the curve $y = \sqrt{x}$ into n equal subintervals, show that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \geq \int_0^n \sqrt{x}dx = \frac{2n\sqrt{n}}{3}.$$

b Use the trapezoidal rule to show that for all integers $n \geq 1$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leq \frac{\sqrt{n}(4n + 3)}{6}.$$

c Hence estimate $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{12\,000}$, correct to the nearest hundred.

An investigation using a spreadsheet for trapezoidal-rule calculations

XLS

16 Work through the spreadsheet example just above this exercise. Then use a spreadsheet estimate these integrals using the trapezoidal rule with 5, 10, 20 and perhaps more subintervals.

a $\int_1^{11} \frac{1}{x}\, dx$ **b** $\int_1^{11} \log_e x\, dx$ **c** $\int_0^{10} e^{-x^2}\, dx$

You will need to look at the results and perhaps vary the number of decimal places that you are using in the calculations and recording in your answers.

Possible spreadsheet projects

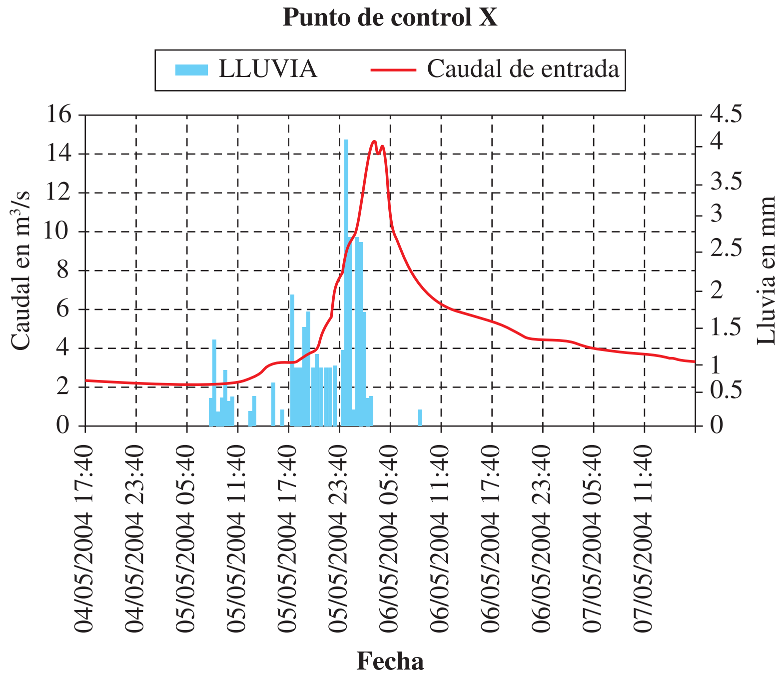
It is possible to program a spreadsheet so that the number of subintervals can be entered as a single variable. The construction of such a program and similar programs could be incorporated into a longer project examining the usefulness and accuracy of the trapezoidal rule, or examining some physical phenomena.

In the next diagram, it is clear that in parts of the graph where there is a lot of activity, the subintervals should be quite narrow, whereas in other calmer parts they can be far wider. Such variability could also be incorporated into the spreadsheet and its formulae.

ISBN 978-1-108-76630-2

An investigation (and possible project) integrating a graph from the web by the trapezoidal rule
There is a great deal of data available on the web for a sustained investigation of river flow. The following question suggests some interesting questions about one such situation, but there are many more situations and questions. Integrating graphs of all kinds from the web using the trapezoidal rule could be the basis of various different projects.

17



The hydrograph above shows the rate of flow through Control Point X on the Turia River in Spain over a three-day period in May 2004. The rate of flow ('Caudal') is given as a function of the date–times ('Fecha') — notice that the successive date–times on the horizontal axis are separated by exactly 6 hours. The rainfall ('Lluvia') is given by the vertical bars.

The units of time are hours, and the units of the flow rate are 'cubic metres per second'. The flow rate R should be converted to units of 'thousands of cubic meters per hour' so that time is in hours and there are fewer zeroes — multiply by $\dfrac{60 \times 60}{1000} = 3.6$.

a From the graph, copy and complete the table of values of the flow rate R at the first four date–times, 04/05/2004 17:40 to 05/05/2004 11:40.

t	17:40	23:40	05:40	11:40
R				

Then use the trapezoidal rule to estimate the total volume of water that flowed through the control point in those 18 hours.

b Draw up a similar table for the 18 hours of heavy flow from 05/05/2004 17:40 to 06/05/2004 11:40, but use 3 hours as the separation between successive times.

t	17:40	20:40	23:40	02:40	05:40	08:40	11:40
R							

Then use the trapezoidal rule to estimate the total volume of water that flowed through the Control Point in those 18 hours. Why are 3 hours suggested here in part **b** for the width of the subintervals, where 6 hours was used in part **a**?

c How many times more water flowed down the river in the second 18-hour period? Look at the rainfall record, and discuss how the river flow responded to the rainfall.

ISBN 978-1-108-76630-2

5I The reverse chain rule

When we use the chain rule to differentiate a composite function, the result is a product of two terms. For example, in the first worked example below,

$$\frac{d}{dx}(x^3 + 5)^4 = 4(x^3 + 5)^3 \times 3x^2.$$

This section deals with the problem of reversing a chain-rule differentiation.

Reversing a chain-rule differentiation

Finding primitives is the reverse process of differentiation. Thus once any differentiation has been performed, the process can then be reversed to give a primitive.

Example 36 5I

a Differentiate $(x^3 + 5)^4$ with full setting-out of the chain rule.
b Hence find a primitive of $12x^2(x^3 + 5)^3$.
c Hence find the primitive of $x^2(x^3 + 5)^3$.

SOLUTION

a Let $y = (x^3 + 5)^4$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 4(x^3 + 5)^3 \times 3x^2$

$= 12x^2(x^3 + 5)^3$.

Let $u = x^3 + 5$.
Then $y = u^4$.
Hence $\dfrac{du}{dx} = 3x^2$
and $\dfrac{dy}{du} = 4u^3$.

b By part **a**, $\dfrac{d}{dx}(x^3 + 5)^4 = 12x^2(x^3 + 5)^3$.

Reversing this, $\displaystyle\int 12x^2(x^3 + 5)^3\,dx = (x^3 + 5)^4$.

c Dividing by 12, $\displaystyle\int x^2(x^3 + 5)^3\,dx = \tfrac{1}{12}(x^3 + 5)^4 + C$, for some constant C.

Note: It is best not to add the arbitrary constant until the last line, because it would be pointless to divide C by 12 as well.

Example 37 5I

a Differentiate $\dfrac{1}{1 + x^2}$ with full setting-out of the chain rule.

b Hence find the primitive of $\dfrac{x}{(1 + x^2)^2}$.

ISBN 978-1-108-76630-2

SOLUTION

a Let $y = \dfrac{1}{1 + x^2}$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -(1 + x^2)^{-2} \times 2x$

$= \dfrac{-2x}{(1 + x^2)^2}$.

Let $u = 1 + x^2$.
Then $y = u^{-1}$.

Hence $\dfrac{du}{dx} = 2x$

and $\dfrac{dy}{du} = -u^{-2}$.

b By part **a**, $\quad \dfrac{d}{dx}\left(\dfrac{1}{1 + x^2}\right) = \dfrac{-2x}{(1 + x^2)^2}$.

Reversing this, $\quad \displaystyle\int \frac{-2x}{(1 + x^2)^2}\,dx = \frac{1}{1 + x^2}, + C$, for some constant C,

$\boxed{\div (-2)}$ $\quad \displaystyle\int \frac{x}{(1 + x^2)^2}\,dx = \frac{-1}{2(1 + x^2)} + C.$

21 REVERSING A CHAIN-RULE DIFFERENTIATION

Once a chain-rule differentiation, or any differentiation, has been performed, the result can be written down in reverse as an indefinite integral.

A formula for the reverse chain rule

There is a formula for the reverse chain rule. Start with the formula for differentiating a function using the chain rule — we gave the formula in two forms:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}f'(x)$$

and we can reverse both forms of the formula,

$$\int u^{n-1}\frac{du}{dx}\,dx = \frac{u^n}{n} \qquad \int \big(f(x)\big)^{n-1} f'(x)\,dx = \frac{\big(f(x)\big)^n}{n}.$$

Then replacing $n - 1$ by n and n by $n + 1$,

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n + 1} \qquad \text{OR} \qquad \int \big(f(x)\big)^n f'(x)\,dx = \frac{\big(f(x)\big)^{n+1}}{n + 1}.$$

Take your pick which formula you prefer to use. The difficult part is recognising what u or $f(x)$ should be in the function that you are integrating.

We shall do worked Example 36 **c** again using the formula.

ISBN 978-1-108-76630-2

Example 38 5I

Use the formula for the reverse chain rule to find $\int x^2(x^3 + 5)^3\,dx$.

SOLUTION

The key to finding this integral is to realise that x^2 is a multiple of the derivative of $x^3 + 5$. At this point, we can put $u = x^3 + 5$ or $f(x) = x^3 + 5$, and everything works smoothly after that. Here are two strong recommendations:

- Show working identifying u or $f(x)$ and its derivative on the right-hand side.
- Write down the standard form above substituting the particular value of n.

Using the first formula,

$$\begin{aligned}\int x^2(x^3+5)^3\,dx &= \tfrac{1}{3}\int 3x^2(x^3+5)^3\,dx \\ &= \tfrac{1}{3}\times\tfrac{1}{4}(x^3+5)^4 + C, \\ &\quad\text{for some constant } C, \\ &= \tfrac{1}{12}(x^3+5)^4 + C.\end{aligned}$$

Let $u = x^3 + 5$.

Then $\dfrac{du}{dx} = 3x^2$.

Here $\displaystyle\int u^3\frac{du}{dx}\,dx = \tfrac{1}{4}u^4$.

Using the second formula,

$$\begin{aligned}\int x^2(x^3+5)^3\,dx &= \tfrac{1}{3}\int 3x^2(x^3+5)^3\,dx \\ &= \tfrac{1}{3}\times\tfrac{1}{4}(x^3+5)^4 + C, \\ &\quad\text{for some constant } C, \\ &= \tfrac{1}{12}(x^3+5)^4 + C.\end{aligned}$$

Let $f(x) = x^3 + 5$.

Then $f'(x) = 3x^2$.

Here $\displaystyle\int \big(f(x)\big)^3 f'(x)\,dx = \tfrac{1}{4}\big(f(x)\big)^4$.

Notice that the two notations differ only in the working in the right-hand column.

Example 39 5I

Use the formula for the reverse chain rule to find:

a $\displaystyle\int x\sqrt{1-x^2}\,dx$ **b** $\displaystyle\int_0^2 x\sqrt{1-x^2}\,dx$

SOLUTION

a This integral is based on the recognition that $\dfrac{d}{dx}(1-x^2) = -2x$.

Using the first formula,

$$\begin{aligned}\int x\sqrt{1-x^2}\,dx &= -\tfrac{1}{2}\int (-2x)\times(1-x^2)^{\frac{1}{2}}\,dx \\ &= -\tfrac{1}{2}\times\tfrac{2}{3}(1-x^2)^{\frac{3}{2}}, \\ &\quad\text{for some constant } C, \\ &= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$.

Here $\displaystyle\int u^{\frac{1}{2}}\frac{du}{dx}\,dx = \tfrac{2}{3}u^{\frac{3}{2}}$.

ISBN 978-1-108-76630-2

Using the second formula,

$$\int x\sqrt{1 - x^2}\,dx = -\tfrac{1}{2}\int (-2x) \times (1 - x^2)^{\frac{1}{2}}\,dx$$
$$= -\tfrac{1}{2} \times \tfrac{2}{3}(1 - x^2)^{\frac{3}{2}}$$
for some constant C,
$$= -\tfrac{1}{3}(1 - x^2)^{\frac{3}{2}} + C,$$

Let $f(x) = 1 - x^2$.
Then $f'(x) = -2x$.
Here $\int \left(f(x)^{\frac{1}{2}}\right) f'(x)\,dx = \tfrac{2}{3}\left(f(x)\right)^{\frac{3}{2}}$.

b The definite integral is meaningless because $\sqrt{1 - x^2}$ is undefined for $x > 1$.

22 A FORMULA FOR THE REVERSE CHAIN RULE

- The reversing of the chain rule can be written as a formula in two ways:

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n + 1} + C \qquad \text{OR} \qquad \int \left(f(x)\right)^n f'(x)\,dx = \frac{\left(f(x)\right)^{n+1}}{n + 1} + C.$$

- The vital step in using this formula is to identity u or $f(x)$ and its derivative.

To use the formula, we have to write the integrand as a product. One factor is a power of a function $f(x)$ or u. The other factor is the derivative of that function.

Exercise 5I

FOUNDATION

1 **a** Find $\frac{d}{dx}(2x + 3)^4$.

b Hence find:

i $\int 8(2x + 3)^3\,dx$

ii $\int 16(2x + 3)^3\,dx$

2 **a** Find $\frac{d}{dx}(3x - 5)^3$.

b Hence find:

i $\int 9(3x - 5)^2\,dx$

ii $\int 27(3x - 5)^2\,dx$

3 **a** Find $\frac{d}{dx}(1 + 4x)^5$.

b Hence find:

i $\int 20(1 + 4x)^4\,dx$

ii $\int 10(1 + 4x)^4\,dx$

4 **a** Find $\frac{d}{dx}(1 - 2x)^4$.

b Hence find:

i $\int -8(1 - 2x)^3\,dx$

ii $\int -2(1 - 2x)^3\,dx$

ISBN 978-1-108-76630-2

Cambridge University Press

5 **a** Find $\frac{d}{dx}(4x + 3)^{-1}$.

b Hence find:

i $\int -4(4x + 3)^{-2}\,dx$

ii $\int (4x + 3)^{-2}\,dx$

6 **a** Find $\frac{d}{dx}(2x - 5)^{\frac{1}{2}}$.

b Hence find:

i $\int (2x - 5)^{-\frac{1}{2}}\,dx$

ii $\int \frac{1}{3}(2x - 5)^{-\frac{1}{2}}\,dx$

7 **a** Find $\frac{d}{dx}(x^2 + 3)^4$.

b Hence find:

i $\int 8x(x^2 + 3)^3\,dx$

ii $\int 40x(x^2 + 3)^3\,dx$

8 **a** Find $\frac{d}{dx}(x^3 - 1)^5$.

b Hence find:

i $\int 15x^2(x^3 - 1)^4\,dx$

ii $\int 3x^2(x^3 - 1)^4\,dx$

9 **a** Find $\frac{d}{dx}\sqrt{2x^2 + 3}$.

b Hence find:

i $\int \frac{2x}{\sqrt{2x^2 + 3}}\,dx$

ii $\int \frac{x}{\sqrt{2x^2 + 3}}\,dx$

10 **a** Find $\frac{d}{dx}(\sqrt{x} + 1)^3$.

b Hence find:

i $\int \frac{3\left(\sqrt{x} + 1\right)^2}{2\sqrt{x}}\,dx$

ii $\int \frac{\left(\sqrt{x} + 1\right)^2}{\sqrt{x}}\,dx$

11 **a** Find $\frac{d}{dx}(x^3 + 3x^2 + 5)^4$.

b Hence find:

i $\int 12(x^2 + 2x)(x^3 + 3x^2 + 5)^3\,dx$

ii $\int (x^2 + 2x)(x^3 + 3x^2 + 5)^3\,dx$

12 **a** Find $\frac{d}{dx}(5 - x^2 - x)^7$.

b Hence find:

i $\int (-14x - 7)(5 - x^2 - x)^6\,dx$

ii $\int (2x + 1)(5 - x^2 - x)^6\,dx$

ISBN 978-1-108-76630-2

DEVELOPMENT

13 Find these indefinite integrals using the reverse chain rule in either form

$$\int f'(x)\left(f(x)\right)^n dx = \frac{\left(f(x)\right)^{n+1}}{n+1} + C \qquad \text{OR} \qquad \int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$$

a $\int 5(5x+4)^3\,dx$ (Let $f(x) = 5x + 4$ or $u = 5x + 4$.)

b $\int -3(1-3x)^5\,dx$ (Let $f(x) = 1 - 3x$ or $u = 1 - 3x$.)

c $\int 2x(x^2-5)^7\,dx$ (Let $f(x) = x^2 - 5$ or $u = x^2 - 5$.)

d $\int 3x^2(x^3+7)^4\,dx$ (Let $f(x) = x^3 + 7$ or $u = x^3 + 7$.)

e $\int \frac{6x}{(3x^2+2)^2}\,dx$ (Let $f(x) = 3x^2 + 2$ or $u = 3x^2 + 2$.)

f $\int \frac{-6x^2}{\sqrt{9-2x^3}}\,dx$ (Let $f(x) = 9 - 2x^3$ or $u = 9 - 2x^3$.)

14 Find these indefinite integrals using the reverse chain rule.

a $\int 10x(5x^2+3)^2\,dx$

b $\int 2x(x^2+1)^3\,dx$

c $\int 12x^2(1+4x^3)^5\,dx$

d $\int x(1+3x^2)^4\,dx$

e $\int x^3(1-x^4)^7\,dx$

f $\int 3x^2\sqrt{x^3-1}\,dx$

g $\int x\sqrt{5x^2+1}\,dx$

h $\int \frac{2x}{\sqrt{x^2+3}}\,dx$

i $\int \frac{x+1}{\sqrt{4x^2+8x+1}}\,dx$

j $\int \frac{x}{(x^2+5)^3}\,dx$

15 Evaluate these definite integrals using the reverse chain rule.

a $\int_{-1}^{1} x^2(x^3+1)^4\,dx$

b $\int_{0}^{1} \frac{x}{(5x^2+1)^3}\,dx$

c $\int_{0}^{\frac{1}{2}} x\sqrt{1-4x^2}\,dx$

d $\int_{-3}^{-1} (x+5)(x^2+10x+3)^2\,dx$

ENRICHMENT

16 Use the reverse chain rule to find:

a $\int \frac{\left(1-\frac{1}{x}\right)^5}{x^2}\,dx$

b $\int_{1}^{4} \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

ISBN 978-1-108-76630-2

17 **a** What is the domain of the function $f(x) = x\sqrt{x^2 - 1}$?

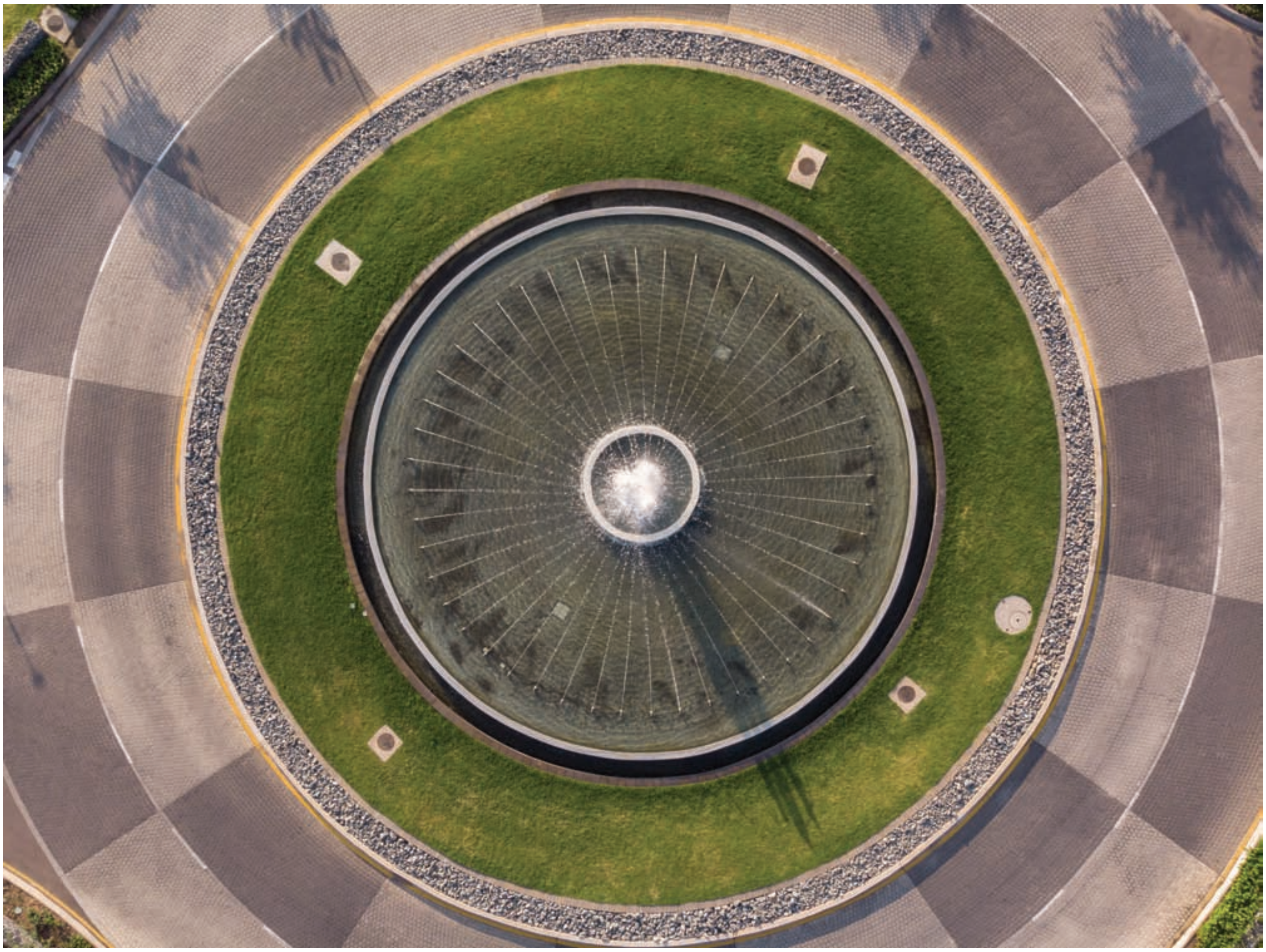
b Find $f'(x)$, and hence show that the function has no stationary points in its domain.

c Show that the function is odd, and hence sketch its graph.

d By evaluating the appropriate definite integral, find the area of the region bounded by the curve and the x-axis from $x = 1$ to $x = 3$.

18 **a** Sketch $y = x(7 - x^2)^3$, indicating all stationary points and intercepts with the axes.

b Find the area of the region enclosed between the curve and the x-axis.



ISBN 978-1-108-76630-2

Chapter 5 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 5 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Evaluate these definite integrals, using the fundamental theorem.

a $\displaystyle\int_0^1 3x^2\,dx$ **b** $\displaystyle\int_1^2 x\,dx$ **c** $\displaystyle\int_2^5 4x^3\,dx$

d $\displaystyle\int_{-1}^1 x^4\,dx$ **e** $\displaystyle\int_{-4}^{-2} 2x\,dx$ **f** $\displaystyle\int_{-3}^{-1} x^2\,dx$

g $\displaystyle\int_0^2 (x+3)\,dx$ **h** $\displaystyle\int_{-1}^4 (2x-5)\,dx$ **i** $\displaystyle\int_{-3}^1 (x^2-2x+1)\,dx$

2 By expanding the brackets where necessary, evaluate these definite integrals.

a $\displaystyle\int_1^3 x(x-1)\,dx$ **b** $\displaystyle\int_{-1}^0 (x+1)(x-3)\,dx$ **c** $\displaystyle\int_0^1 (2x-1)^2\,dx$

3 Write each integrand as separate fractions, then evaluate the integral.

a $\displaystyle\int_1^2 \frac{x^2-3x}{x}\,dx$ **b** $\displaystyle\int_2^3 \frac{3x^4-4x^2}{x^2}\,dx$ **c** $\displaystyle\int_{-2}^{-1} \frac{x^3-2x^4}{x^2}\,dx$

4 **a** **i** Show that $\displaystyle\int_4^k 5\,dx = 5k-20$.

ii Hence find the value of k if $\displaystyle\int_4^k 5\,dx = 10$.

b **i** Show that $\displaystyle\int_0^k (2x-1)\,dx = k^2-k$.

ii Hence find the positive value of k for which $\displaystyle\int_0^k (2x-1)\,dx = 6$.

5 Without finding a primitive, use the properties of the definite integral to evaluate these integrals, stating reasons.

a $\displaystyle\int_3^3 (x^3-5x+4)\,dx$ **b** $\displaystyle\int_{-2}^2 x^3\,dx$ **c** $\displaystyle\int_{-3}^3 (x^3-9x)\,dx$

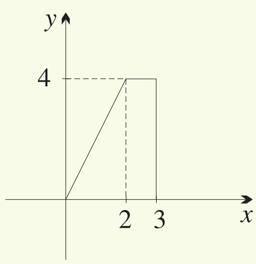
ISBN 978-1-108-76630-2

Cambridge University Press

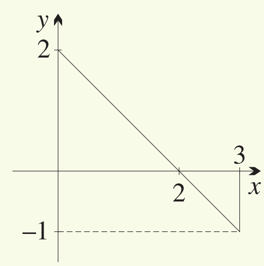
Review

6 Use area formulae to find $\int_0^3 f(x)\,dx$, given the following sketches of $f(x)$.

a



b



7 **a** Find each signed area function.

i $A(x) = \int_{-2}^{x} (4 - t)\,dt$ **ii** $A(x) = \int_{2}^{x} t^{-2}\,dt$

b Differentiate the results in part **a** to find:

i $\dfrac{d}{dx}\int_{-2}^{x} (4 - t)\,dt$ **ii** $\dfrac{d}{dx}\int_{2}^{x} t^{-2}\,dt$

c Without first performing the integration, use the fundamental theorem of calculus to find these functions.

i $\dfrac{d}{dx}\int_{7}^{x} (t^5 - 5t^3 + 1)\,dt$ **ii** $\dfrac{d}{dx}\int_{3}^{x} \dfrac{t^2 + 4}{t^2 - 1}\,dt$

8 Find these indefinite integrals.

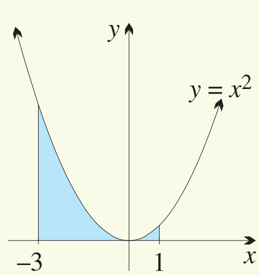
a $\int (x + 2)\,dx$ **b** $\int (x^3 + 3x^2 - 5x + 1)\,dx$ **c** $\int x(x - 1)\,dx$

d $\int (x - 3)(2 - x)\,dx$ **e** $\int x^{-2}\,dx$ **f** $\int \dfrac{1}{x^7}\,dx$

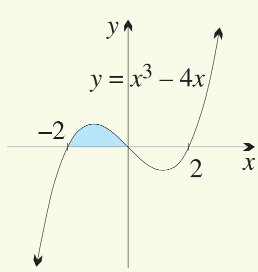
g $\int \sqrt{x}\,dx$ **h** $\int (x + 1)^4\,dx$ **i** $\int (2x - 3)^5\,dx$

9 Find the area of each shaded region below by evaluating a definite integral.

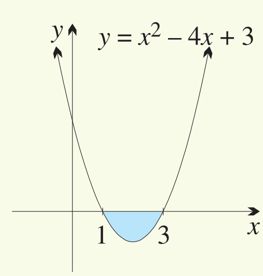
a



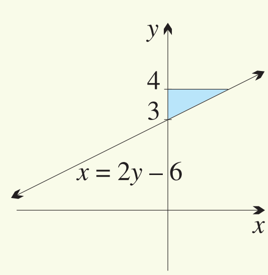
b



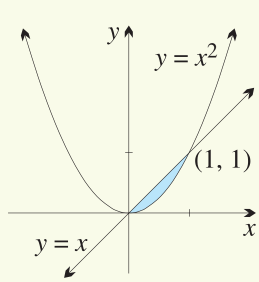
c



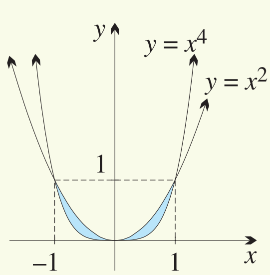
d



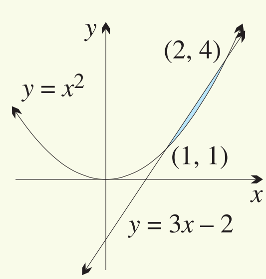
e



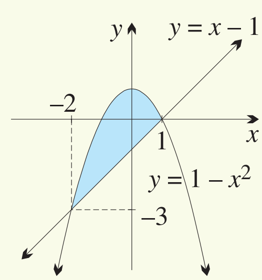
f



g



h



ISBN 978-1-108-76630-2

Cambridge University Press

10 a By solving the equations simultaneously, show that the curves $y = x^2 - 3x + 5$ and $y = x + 2$ intersect at the points $(1, 3)$ and $(3, 5)$.

b Sketch both curves on the same diagram and find the area of the region enclosed between them.

11 a Use the trapezoidal rule with three function values to approximate $\int_1^3 2^x\,dx$.

b Use the trapezoidal rule with five function values to approximate $\int_1^3 \log_{10} x\,dx$. Give your answer correct to two significant figures.

12 a Find $\dfrac{d}{dx}(3x + 4)^6$.

b Hence find:

i $\int 18(3x + 4)^5\,dx$

ii $\int 9(3x + 4)^5\,dx$

13 a Find $\dfrac{d}{dx}(x^2 - 1)^3$.

b Hence find:

i $\int 6x(x^2 - 1)^2\,dx$

ii $\int x(x^2 - 1)^2\,dx$

14 Find these indefinite integrals using the reverse chain rule.

a $\int 3x^2(x^3 + 1)^4\,dx$

b $\int \dfrac{2x}{(x^2 - 5)^3}\,dx$

15 Use the reverse chain rule to show that $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2 + 3}}\,dx = 2 - \sqrt{3}$.

ISBN 978-1-108-76630-2