3 Graphs and equations

This chapter completes non-calculus curve-sketching and the use of graphs to solve equations and inequations. Some material in the early sections and at the start of transformations is review, particularly of material covered in Chapter 5 of the Year 11 book, although the context is different. Readers who are confident of the earlier material may choose to be selective about the questions they attempt.

Sections 3A–3C discuss four approaches to curve-sketching — domain, odd and even symmetry, zeroes and sign, and asymptotes — and combine them into an informal menu for sketching any curve given its equation. This prepares the ground for Chapter 4, where the methods of calculus will be added to gain more information about the graph. Some necessary alternative notation for intervals and for composition of functions is introduced in Section 3A.

Section 3A also emphasises the role of the graph in solving inequations, and Sections 3D–3E develop graphical methods further to help solve various equations and inequations. Section 3F uses inequalities in two variables to specify regions in the coordinate plane.

Sections 3G–3I review transformations, and add stretching (or dilation) to the list of available transformations. The problem of combining two or more transformations is addressed, including a crucial and difficult question, 'Does it matter in which order the transformations are applied?'

Trigonometric graphs, in particular, benefit very greatly from a systematic approach using translations and dilations, because the amplitude, period, phase and mean value all depend on them. The final Section 3J deals with these graphs.

Whether explicitly suggested in an exercise or not, graphing software in any form is always very useful in confirming results and improving approximations. It is particularly suited to investigating what happens when changes are made to a function's equation.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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3A The sign of a function

The main purpose of this section is to review briefly the method of finding the sign of a function by using a table of signs with test values that dodge around zeroes and discontinuities.

This algorithm has the same two purposes as the whole chapter. First, when sketching a curve, we usually want to know very early where the curve is above the x-axis and where it is below. Secondly, solving an inequation is equivalent to finding the sign of a function, because we can always put all the terms on the left. For example,

$$x^3 + 1 \ge x^2 + x \qquad \text{can be written as} \qquad x^3 - x^2 - x + 1 \ge 0.$$

Before all this, however, two extra pieces of notation need to be introduced.

- Bracket interval notation is an alternative to inequality interval notation.
- Composition of functions has an alternative notation $f \circ g$.

Bracket interval notation

There is an alternative notation for intervals that will make the notation in this section and later a little more concise. The notation encloses the endpoints of the interval in brackets, using a square bracket if the endpoint is included and a round bracket if the endpoint is not included.

Here are the five examples from the Year 11 book written in both notations:

Diagram	Using inequalities	Using brackets
	$\frac{1}{3} \le x \le 3$	$\left[\frac{1}{3}, 3\right]$
Read this as, 'The closed interval from $\frac{1}{3}$ to 3'.		
	$-1 < x < 5$	$(-1, 5)$
Read this as, 'The open interval from -1 to 5'.		
	$-2 \le x < 3$	$[-2, 3)$
Read this as, 'The interval from -2 to 3, including -2 but excluding 3'.		
	$x \ge -5$	$[-5, \infty)$
Read this as, 'The closed ray from -5 to the right'.		
	$x < 2$	$(-\infty, 2)$
Read this as, 'The open ray from 2 to the left'.		

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The first interval $\left[\frac{1}{3}, 3\right]$ is *closed*, meaning that it contains all its endpoints.

The second interval $(-1, 5)$ is *open*, meaning that it does not contain any of its endpoints.

The third interval $[-2, 3)$ is neither open nor closed — it contains one of its endpoints, but does not contain the other endpoint.

The fourth interval $[-5, \infty)$ is *unbounded on the right*, meaning that it continues towards infinity. It only has one endpoint -5, which it contains, so it is closed.

The fifth interval $(-\infty, 2)$ is *unbounded on the left*, meaning that it continues towards negative infinity. It only has one endpoint 2, which it does not contain, so it is open.

'Infinity' and 'negative infinity', with their symbols ∞ and $-\infty$, are not numbers. They are ideas used in specific situations and phrases to make language and notation more concise. Here, they indicate that an interval is unbounded on the left or right, and the symbol '$(-\infty, 2)$' means 'all real numbers less than 2'.

Bracket interval notation has some details that need attention.

- The variable x or y or whatever is missing. This can be confusing when we are talking about domain and range, or solving an inequation for some variable. When, however, we are just thinking about 'all real numbers greater than 100', no variable is involved, so the notation $(100, \infty)$ is more satisfactory than $x > 100$.
- The notation can be dangerously ambiguous. For example, the open interval $(-1, 5)$ can easily be confused with the point $(-1, 5)$ in the coordinate plane.
- Infinity and negative infinity are not numbers, as remarked above.
- The set $\mathbb{R}$ of all real numbers can be written as $(-\infty, \infty)$.
- The notation $[4, 4]$ is the one-member set $\{4\}$, called a *degenerate interval* because it has length zero.
- Notations such as $(4, 4)$, $(4, 4]$, $[7, 3]$ and $[7, 3)$ all suggest the empty set, if they mean anything at all, and should be avoided in this course.

1 BRACKET INTERVAL NOTATION

- A square bracket means that the endpoint is included, and a round bracket means that the endpoint is not included.
- For $a < b$, we can form the four *bounded intervals* below. The first is closed, the last is open, and the other two are neither open nor closed.

$$[a, b] \quad \text{and} \quad [a, b) \quad \text{and} \quad (a, b] \quad \text{and} \quad (a, b).$$

- For any real number a, we can form the four *unbounded intervals* below. The first two are closed, and the last two are open.

$$[a, \infty) \quad \text{and} \quad (-\infty, a] \quad \text{and} \quad (a, \infty) \quad \text{and} \quad (-\infty, a).$$

- The notation $(-\infty, \infty)$ means the whole real number line $\mathbb{R}$.
- The notation $[a, a]$ is the one-member set $\{a\}$, called a *degenerate interval*.
- An interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

For those who enjoy precision, the interval $(-\infty, \infty)$ is both open and closed (it has no endpoints), and a degenerate interval $[a, a]$ is closed.

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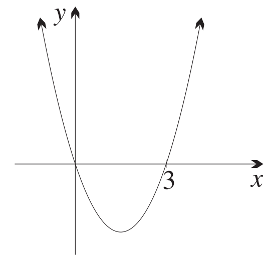
The union of intervals



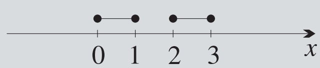
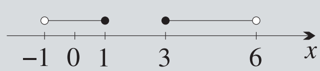
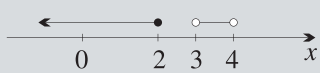
The graph to the right is the quadratic $y = x(x - 3)$. From the graph, we can see that the inequation

$$x(x - 3) \geq 0 \qquad \text{has solution} \qquad x \leq 0 \ \text{ or } \ x \geq 3.$$

This set is the *union* of the two intervals $(-\infty, 0]$ and $[3, \infty)$, so when using bracket interval notation, we write the set as

$$(-\infty, 0] \cup [3, \infty).$$

Here are some further examples using both types of interval notation. The close association between the word 'or' and the union of sets was discussed in Sections 12C and 12D of the Year 11 book in the context of probability.

Diagram	Using inequalities	Using brackets
	$0 \leq x \leq 1$ or $2 \leq x \leq 3$	$[0, 1] \cup [2, 3]$
	$-1 < x \leq 1$ or $3 \leq x < 6$	$(-1, 1] \cup [3, 6)$
	$x \leq 2$ or $3 < x < 4$	$(-\infty, 2] \cup (3, 4)$

Some alternative notation for composite functions:

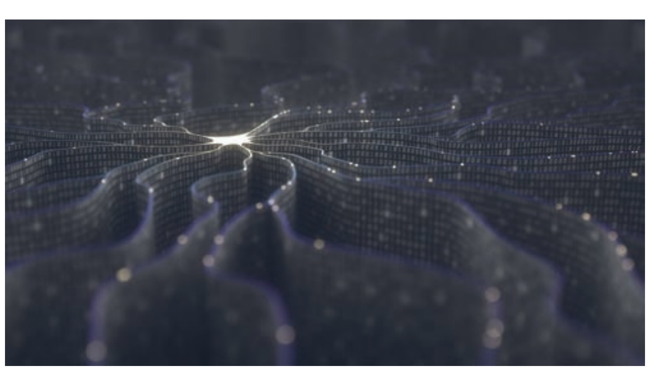
If $f(x)$ and $g(x)$ are two functions, the composite $g\big(f(x)\big)$ of two function $f(x)$ and $g(x)$ can also be written as $g \circ f(x)$. Thus

$$g \circ f(x) = g\big(f(x)\big) \text{ for all } x \text{ for which } f(x) \text{ and } g\big(f(x)\big) \text{ are defined.}$$

The advantage of this notation is that the composite function $g\big(f(x)\big)$ has a clear symbol $g \circ f$ that displays the composition of functions as a binary operator $\circ$ on the set of functions, with notation analogous to addition $a + b$, which is a binary operator on the set of numbers.

Be careful, however, when calculating $g \circ f(2)$, to apply the function f before the function g, because $g \circ f(x)$ means $g\big(f(x)\big)$. Section 4E of the Year 11 book developed composite functions in some detail, and Exercise 3A contains only a few mostly computational questions as practice of the new notation.

The composite $g \circ f(x)$ is often written with extra brackets as $(g \circ f)(x)$, and readers may prefer to add these extra brackets.



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Example 1 — 3A

If $f(x) = x + 3$ and $g(x) = x^2$, find:

a **i** $g \circ f(5)$ **ii** $f \circ g(5)$ **iii** $g \circ g(5)$ **iv** $f \circ f(5)$

b **i** $g \circ f(x)$ **ii** $f \circ g(x)$ **iii** $g \circ g(x)$ **iv** $f \circ f(x)$

SOLUTION

a **i** $g \circ f(5) = g(8)$
$= 64$

ii $f \circ g(5) = f(25)$
$= 28$

iii $g \circ g(5) = g(25)$
$= 625$

iv $f \circ f(5) = f(8)$
$= 11$

b **i** $g \circ f(x) = g(x + 3)$
$= (x + 3)^2$

ii $f \circ g(x) = f(x^2)$
$= x^2 + 3$

iii $g \circ g(x) = g(x^2)$
$= x^4$

iv $f \circ f(x) = f(x + 3)$
$= x + 6$

Finding the sign of a function using a table of signs

This algorithm was introduced in the Year 11 book — in the context of polynomials in Sections 3G and 10B, and for more general functions in Sections 5B — and need only be summarised here.

2 FINDING THE SIGN OF A FUNCTION USING A TABLE OF SIGNS

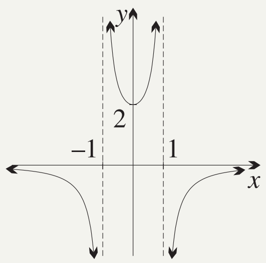
- The sign of a function tells us where its graph is above and below the x-axis.
- A function can only change sign (but might not) at a zero or a discontinuity.
- To examine the sign of a function, draw up a *table of signs*. This is a table of test values that dodge around any zeroes and discontinuities.
- When all the terms of an inequation are moved to the left, solving the inequation is equivalent to finding the sign of the LHS.

The worked examples below do not include polynomials, which have already been covered extensively.

Example 2 — 3A

Solve $f(x) < 0$ using the graph of $y = f(x)$ drawn to the right.

Write the answer using both interval notations.



SOLUTION

The solutions are where the curve is below the x-axis.

That is $x < -1$ or $x > 1$,

or alternatively, $(-\infty, -1) \cup (1, \infty)$.

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Example 3 3A

Solve the inequality $\dfrac{3}{2^x - 1} \le 1$ by moving everything to the left and constructing a table of signs. Give the answer in both notations.

SOLUTION

Moving 1 to the left, $\dfrac{3}{2^x - 1} - \dfrac{2^x - 1}{2^x - 1} \le 0$

$$\frac{4 - 2^x}{2^x - 1} \le 0$$

We now draw up a table of signs for $y = \dfrac{4 - 2^x}{2^x - 1}$.

This function has a zero at $x = 2$ and a discontinuity at $x = 0$.

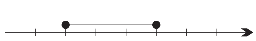
x	-1	0	1	2	3
y	-7	$*$	2	0	$-\frac{4}{7}$
sign	$-$	$*$	$+$	0	$-$

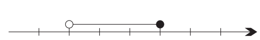
Hence the solution is $x < 0$ or $x \ge 2$, or alternatively $(-\infty, 0) \cup [2, \infty)$.

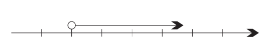
Exercise 3A FOUNDATION

1 For each number line, write the graphed interval using:

i inequality interval notation, **ii** bracket interval notation.

a 

b 

c 

2 For each interval given by inequality interval notation:

i draw the interval on a number line, **ii** write it using bracket interval notation.

a $-1 \le x < 2$ **b** $x \le 2$ **c** $x < 2$

3 For each interval given by bracket interval notation:

i draw the interval on a number line, **ii** write it using inequality interval notation.

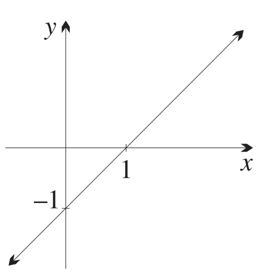
a $[-1, \infty)$ **b** $(-1, 2)$ **c** $(-\infty, \infty)$

4 If $f(x) = x + 1$ and $g(x) = 2^x$, find:

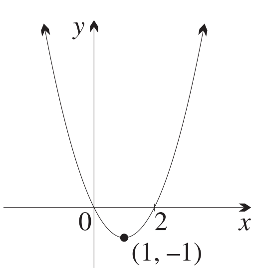
a **i** $g \circ f(3)$ **ii** $f \circ g(3)$ **iii** $g \circ g(3)$ **iv** $f \circ f(3)$

b **i** $g \circ f(x)$ **ii** $f \circ g(x)$ **iii** $g \circ g(x)$ **iv** $f \circ f(x)$

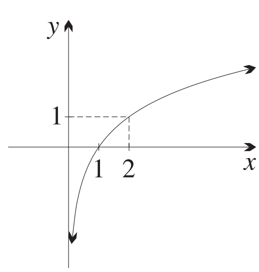
5 For each graph, use bracket interval notation to state where the function is negative.

a 

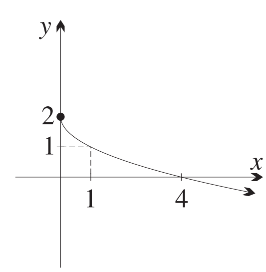
$f(x) = x - 1$

b 

$f(x) = x^2 - 2x$

c 

$f(x) = \log_2 x$

d 

$f(x) = 2 - \sqrt{x}$

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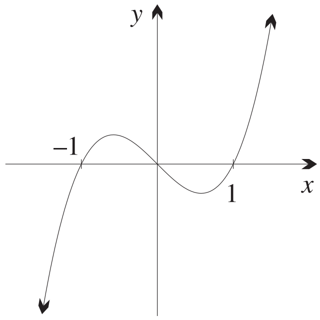
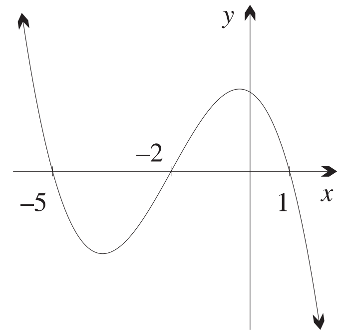
6 Use the given graph of the LHS to help solve each inequation.

a



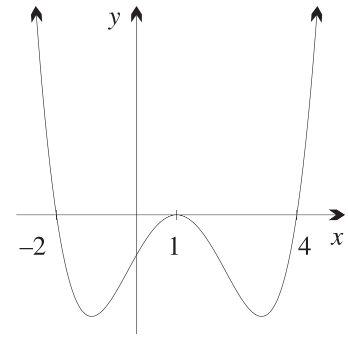
$x(x-1)(x+1) \geq 0$

b



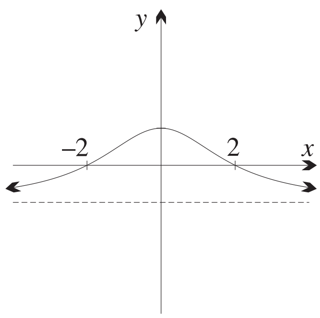
$(1-x)(x+2)(x+5) \leq 0$

c



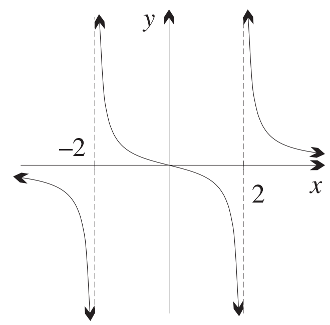
$(x-1)^2(x-4)(x+2) > 0$

d



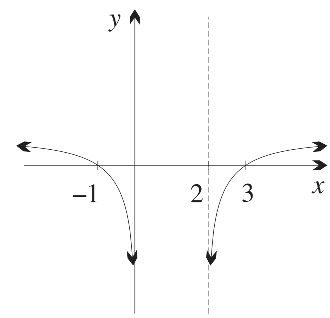
$\dfrac{4-x^2}{4+x^2} \geq 0$

e



$\dfrac{x}{x^2-4} < 0$

f



$\log_e\left(\dfrac{2x(x-2)}{x^2-2x+3}\right) \leq 0$

7 Find the natural domain of each function.

a $f(x) = \dfrac{1}{2x+3}$ **b** $g(x) = \sqrt{2-x}$ **c** $h(x) = \dfrac{e^x + e^{-x}}{2}$

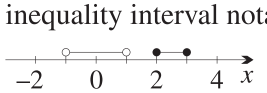
d $a(x) = \log_e(x+1)$ **e** $b(x) = \dfrac{1}{\sqrt{x+3}}$ **f** $c(x) = \log_e(x^2+2x+3)$

DEVELOPMENT

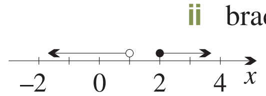
8 For each number line, write the graphed compound interval using:

i inequality interval notation, **ii** bracket interval notation.

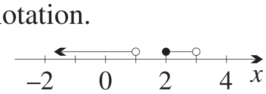
a



b



c



9 For each compound interval given by inequality interval notation:

i draw the number line graph, **ii** write it using bracket interval notation.

a $x = -1$ or $x \geq 2$ **b** $x \leq -1$ or $2 < x \leq 3$ **c** $-1 < x \leq 1$ or $x > 2$

10 For each compound interval given by bracket interval notation:

i draw the number line graph, **ii** write it using inequality interval notation.

a $[-1, 1] \cup [2, \infty)$ **b** $[-1, 1) \cup (2, 3]$ **c** $(-1, 1] \cup [3, 3]$

11 Re-write the solutions to Question **6** using bracket interval notation.

12 For each inequation, find the zeroes and discontinuities of the function on the left-hand side. Then use a table of signs to solve the inequation.

a $\dfrac{x-2}{x+1} \geq 0$ **b** $\dfrac{x-1}{x^2-2x-3} \geq 0$ **c** $\dfrac{x^2+2x+1}{x-2} < 0$

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13 Write the natural domain of each function using bracket interval notation.

a $f(x) = \dfrac{\sqrt{x}}{x^2 - 1}$

b $f(x) = \dfrac{1}{\sqrt{x^2 - 5x - 6}}$

c $f(x) = \dfrac{1}{\sqrt{3 + 2x - x^2}}$

d $f(x) = \dfrac{x - 1}{\sqrt{x^2 - 2x + 3}}$

14 Let $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$

a Carefully sketch $y = f(x)$.

b Hence solve $f(x) \geq 0$.

15 Let $h(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

a Find the domain of $h(x)$. Write your answer using inequality interval notation.

b Determine $h'(x)$ and hence show that $h'(x) < 0$ for all values of x in its domain.

16 **a** Consider the function $y = \dfrac{|x|}{x - 1}$.

i Determine the natural domain.

ii Find any intercepts with the axes.

iii Determine where the function is positive, negative or zero.

iv Use appropriate graphing software or applications to confirm your answers.

b Repeat the steps of part **a** for $y = \dfrac{|x|}{x^2 - 1}$.

17 **a** Let $f(x) = \sin\left(x + \dfrac{\pi}{3}\right)$, $g(x) = e^x$ and $h(x) = 1 - x^2$. Show that the composition of these three functions is associative. That is, show that

$$\big((f \circ g) \circ h\big)(x) = \big(f \circ (g \circ h)\big)(x)$$

b Prove that composition of functions is always associative. That is, prove that

$$\big((f \circ g) \circ h\big)(x) = \big(f \circ (g \circ h)\big)(x)$$

for all x where both sides are defined, regardless of the choice of functions.

18 An interval is *closed* if it contains all of its endpoints, and *open* if it does not contain any of its endpoints.

a Explain why the degenerate interval $[5, 5]$ is closed.

b Explain why the interval $(-\infty, \infty)$ is open.

c Explain why the interval $(-\infty, \infty)$ is closed.

ENRICHMENT

19 Let $f(x) = 1 + x + x^2 + x^3 + \cdots + x^{2n+1}$.

a Show that the only solution of $f(x) = 0$ is $x = -1$.

b Show that $(x + 1)^2$ is a factor of $f'(x) - (n + 1)x^{2n}$.

c Hence show that a tangent to $y = f(x)$ is never horizontal.

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3B Vertical and horizontal asymptotes

So far in this course we have discussed three steps in sketching an unknown function (leaving transformations aside for the moment). After factoring:

1 Identify the domain.
2 Test whether the function has even or odd symmetry or neither.
3 Identify the zeroes and discontinuities and draw up a table of signs.

This section introduces a fourth step:

4 Identify a curve's vertical and horizontal asymptotes.

This may also involve describing the curve's behaviour near them.

Vertical asymptotes were discussed at length in Section 5C of the Year 11 book, and are only summarised here. Finding horizontal asymptotes, however, needs some further techniques.

Vertical asymptotes

It is usually best to draw up a table of signs first so that the behaviour near any vertical asymptote can be quickly identified.

3 TESTING FOR VERTICAL ASYMPTOTES

Always factor the function first as far as possible.

- If the denominator has a zero at $x = a$, and the numerator is *not* zero at $x = a$, then the vertical line $x = a$ is an asymptote.
- The choice between $y \to \infty$ and $y \to -\infty$ can be made by looking at a table of signs.

Once vertical asymptotes have been identified, the behaviour of the curve near them can then be seen from the table of signs and described using the notation $x \to a^+$ and $x \to a^-$.

Be careful. The function $y = \dfrac{x^2 - 4}{x - 2}$ is $y = x + 2$, for $x \neq 2$. Its graph is the line $y = x - 2$ with the point $(2, 4)$ removed. There is a discontinuity at $x = 2$, but no asymptote there — the equation has a zero at $x = 2$ in the numerator as well as in the denominator.

Example 4 3B

Consider the function $f(x) = \dfrac{-2}{x + 2}$.

a Write down the domain, using both interval notations.
b Test whether the function is odd or even or neither.
c Find any zeroes and discontinuities and construct a table of signs.
d Find the vertical asymptote, and describe the behaviour near it.
e Find the horizontal asymptote, and describe the behaviour near it.
f Sketch the curve.

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SOLUTION

a The domain is $x \neq -2$. In bracket interval notation, this is $(-\infty, -2) \cup (-2, \infty)$.

b $f(-x) = \dfrac{-2}{-x + 2}$, which is neither $f(x)$ nor $-f(x)$, so $f(x)$ is neither even nor odd.

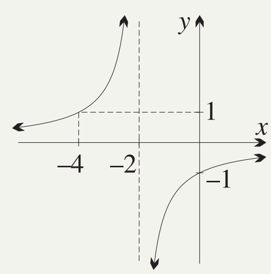
c There are no zeroes, and there is a discontinuity at $x = -2$.

x	-4	-2	0
$f(x)$	1	*	-1
sign	+	*	−

d At $x = -2$, the bottom is zero, but the top is non-zero,
so $x = 2$ is a vertical asymptote.
From the table of signs, $f(x) \to \infty$ as $x \to (-2)^-$,
and $f(x) \to -\infty$ as $x \to (-2)^+$.

e $f(x) \to 0$ as $x \to -\infty$,
and $f(x) \to 0$ as $x \to \infty$,
so the x-axis is a horizontal asymptote in both directions.

f



Horizontal asymptotes, and the behaviour as $x \to \infty$ and as $x \to -\infty$

It was very straightforward in the previous worked example to see that the x-axis is an asymptote to each curve. But it is not so straightforward to find the horizontal asymptotes, if indeed they exist, for curves such as

$$y = \frac{x - 1}{x - 4} \quad \text{or} \quad y = \frac{x - 1}{x^2 - 4}.$$

Such curves are called *rational functions* because they are the ratio of two polynomials. For rational functions, *dividing top and bottom by the highest power of x in the denominator* makes the situation clear.

4 BEHAVIOUR FOR LARGE X

- Divide top and bottom by the highest power of x in the denominator.
- Then use the fact that $\dfrac{1}{x} \to 0$ as $x \to \infty$ and as $x \to -\infty$.
- If $f(x)$ tends to a definite limit b as $x \to \infty$ or as $x \to -\infty$, then the horizontal line $y = b$ is a horizontal asymptote on the right or on the left.

Example 5 3B

a Examine the behaviour of the function $y = \dfrac{x - 1}{x - 4}$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

b Find any vertical asymptotes of the function $y = \dfrac{x - 1}{x - 4}$. Then use a table of signs to describe the behaviour of the curve near them.

c Sketch the curve.

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SOLUTION

a Dividing top and bottom by x gives

$$y = \frac{1-\frac{1}{x}}{1-\frac{4}{x}},$$

so $y \to \dfrac{1-0}{1-0} = 1$ as $x \to \infty$ and as $x \to -\infty$.

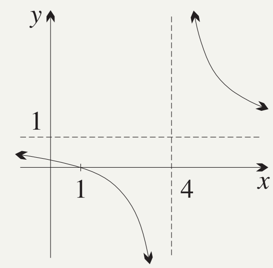
Hence $y = 1$ is a horizontal asymptote.

b When $x = 4$, the denominator vanishes, but the numerator does not, so $x = 4$ is an asymptote.

From the table of signs to the right, dodging around the zero at $x = 1$ and the discontinuity at $x = 4$:

As $x \to 4^-$, $y \to -\infty$, and as $x \to 4^+$, $y \to +\infty$,

c



x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	−	*	+

Example 6 — 3B

Examine the behaviour of these functions as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

a $y = \dfrac{x-1}{x^2-4}$ **b** $y = \dfrac{x^2-1}{x-4}$ **c** $y = \dfrac{3-5x-4x^2}{4-5x-3x^2}$

SOLUTION

a Dividing top and bottom by x^2, $y = \dfrac{\frac{1}{x}-\frac{1}{x^2}}{1-\frac{4}{x^2}}$.

Hence as $x \to \infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$, and the x-axis $y = 0$ is a horizontal asymptote.

b Dividing top and bottom by x, $y = \dfrac{x-\frac{1}{x}}{1-\frac{4}{x}}$.

Hence as $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$, and there are no horizontal asymptotes.

c Dividing top and bottom by x^2, $y = \dfrac{\frac{3}{x^2}-\frac{5}{x}-4}{\frac{4}{x^2}-\frac{5}{x}-3}$.

Hence as $x \to \infty$, $y \to \frac{4}{3}$, and as $x \to -\infty$, $y \to \frac{4}{3}$, and $y = \frac{4}{3}$ is a horizontal asymptote.

Horizontal asymptotes of functions with exponentials

In Chapter 13, when working with the logistic equation, we will need functions with exponentials in the top and bottom. The form suitable for the limit as $x \to \infty$ may be different from the form suitable for the limit as $x \to -\infty$.

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Example 7 3B

a Find any horizontal asymptotes of $y = \dfrac{2^x + 1}{2^x - 1}$.

SOLUTION

As $x \to -\infty$, $2^x + 1 \to 1$ and $2^x - 1 \to -1$, so $y \to -1$.

For the limit as $x \to \infty$, divide through by 2^x to give $y = \dfrac{1 + 2^{-x}}{1 - 2^{-x}}$.

Then as $x \to \infty$, $1 + 2^{-x} \to 1$ and $1 - 2^x \to 1$, so $y \to 1$.

Hence $y = -1$ is a horizontal asymptote on the left, and $y = 1$ on the right.

Exercise 3B FOUNDATION

1 Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator and taking the limit as $x \to \infty$ and as $x \to -\infty$:

a $f(x) = \dfrac{1}{x + 1}$ **b** $f(x) = \dfrac{x - 3}{x + 4}$ **c** $f(x) = \dfrac{2x + 1}{3 - x}$

d $f(x) = \dfrac{5 - x}{4 - 2x}$ **e** $\dfrac{1}{x^2 + 1}$ **f** $\dfrac{x}{x^2 + 4}$

2 Sketch the curve $y = \dfrac{x}{x - 2}$ after performing the following steps.

a Write down the natural domain.

b Find the intercepts and examine the sign.

c Show that $y = 1$ is the horizontal asymptote.

d Investigate the behaviour near the vertical asymptote.

3 Consider $y = \dfrac{x - 1}{x + 3}$.

a Where is the function undefined?

b Find the intercepts and examine the sign of the function.

c Identify and investigate the vertical and horizontal asymptotes.

d Hence sketch the curve.

e Is this function one-to-one or many-to-one?

4 Investigate the domain, intercepts, sign and asymptotes of the function $y = -\dfrac{1}{(x - 2)^2}$ and hence sketch its graph.

5 Let $y = \dfrac{2}{x^2 + 1}$.

a Determine the horizontal asymptote.

b Explain why there are no vertical asymptotes.

c Show that the tangent to the curve is horizontal at the y-intercept.

d Sketch $y = \dfrac{2}{x^2 + 1}$. Use a table of values if needed.

e What is the range of the function?

f Is this function one-to-one or many-to-one?

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6 Let $y = \dfrac{3}{(x + 1)(x - 3)}$.

- **a** State the natural domain.
- **b** Find the y-intercept.
- **c** Show that $y = 0$ is a horizontal asymptote.
- **d** Draw up a table of values.
- **e** Identify the vertical asymptotes, and use the table of values to write down its behaviour near them.
- **f** Sketch the graph of the function and state its range.

7 **a** Follow steps similar to those in Question 6 in order to sketch $y = \dfrac{4}{4 - x^2}$.

- **b** What is the range of $y = \dfrac{4}{4 - x^2}$?

DEVELOPMENT

8 Consider the function $y = \dfrac{3x}{x^2 + 1}$.

- **a** Show that it is an odd function.
- **b** Show that it has only one intercept with the axes at the origin.
- **c** Show that the x-axis is a horizontal asymptote.
- **d** Hence sketch the curve.

9 **a** Show that $y = \dfrac{4 - x^2}{4 + x^2}$ is even.

- **b** Find its three intercepts with the axes.
- **c** Determine the equation of the horizontal asymptote.
- **d** Sketch the curve.

10 In each case a rational function $f(x)$ has been given. Factor where needed, and hence find any vertical and horizontal asymptotes of the graph of $y = f(x)$.

a $\dfrac{x^2 + 5x + 6}{x^2 - 4x + 3}$ **b** $\dfrac{x^2 - 2x + 1}{x^2 + 5x + 4}$ **c** $\dfrac{x - 5}{x^2 + 3x - 10}$ **d** $\dfrac{1 - 4x^2}{1 - 9x^2}$

(Computer sketches of these curves may be useful to put these features in context.)

11 Consider the function $f(x) = \dfrac{x}{x^2 - 4}$.

- **a** Determine whether the function is even or odd.
- **b** State the domain of the function and the equations of any vertical asymptotes.
- **c** Use a table of test points of $f(x)$ to analyse the sign of the function.
- **d** Find the equation of the horizontal asymptote.
- **e** Find the derivative, and hence explain why the curve $y = f(x)$ is always decreasing.
- **f** Sketch the graph of $y = f(x)$, showing all important features.
- **g** Use the graph to state the range of the function.

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12 This question looks at graphs that have holes rather than vertical asymptotes.

a Show that $\dfrac{x^2 - 4}{x - 2} = x + 2$, provided $x \neq 2$. Hence sketch the graph of $y = \dfrac{x^2 - 4}{x - 2}$.

b Similarly sketch graphs of:

i $y = \dfrac{(x + 1)(3 - x)}{x + 1}$ **ii** $y = \dfrac{x^3 - 1}{x - 1}$ **iii** $y = \dfrac{(x + 2)(x - 2)}{(x - 2)(x + 1)}$

13 Consider the function $y = x + \dfrac{1}{x}$.

a Show that the function is odd. What symmetry does its graph have?

b State the domain of the function and the equation of the vertical asymptote.

c Use a table of values of y to analyse the sign of the function.

d Show that the derivative is $y' = \dfrac{x^2 - 1}{x^2}$.

e Find any points where the tangent is horizontal.

f Show that $\lim\limits_{|x| \to \infty} (y - x) = 0$. (This means that $y = x$ is an asymptote to the curve.)

g Sketch the graph of the function.

h Write down the range of the function.

14 In each question above, the horizontal asymptote was the same on the left as $x \to -\infty$, and on the right as $x \to \infty$. This is not always the case. Consider the function $y = \dfrac{1 - e^x}{1 + e^x}$.

a Find $\lim\limits_{x \to -\infty} y$.

b Multiply the numerator and denominator by e^{-x}, then find $\lim\limits_{x \to \infty} y$.

c Determine any intercepts with the axes.

d Using no other information, sketch this curve.

e Test algebraically whether the function is even, odd or neither.

f Similarly sketch $y = \dfrac{1 + e^x}{1 - e^x}$, taking care with the vertical asymptote.

ENRICHMENT

15 a Show that $\dfrac{x^2}{x - 1} = x + 1 + \dfrac{1}{x - 1}$, and deduce that $y = \dfrac{x^2}{x - 1}$ has an oblique asymptote $y = x + 1$. Then sketch the graph.

b Likewise sketch $y = \dfrac{x^2 - 4}{x + 1} = x - 1 - \dfrac{3}{x + 1}$, showing the oblique asymptote.

16 Investigate the asymptotic behaviour of the following functions, and graph them:

a $y = \dfrac{x^3 - 1}{x}$ **b** $y = \dfrac{1}{x} + \sqrt{x}$ **c** $y = |x| + \dfrac{1}{x}$

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3C A curve-sketching menu

We can now combine four approaches to curve-sketching into an informal four-step approach for sketching an unknown graph. This simple menu cannot possibly deal with every possible graph. Nevertheless, it will allow the main features of a surprising number of functions to be found. Two further steps involving calculus will be added in Chapter 4.

A 'sketch' of a graph is not an accurate plot. It is a neat diagram showing the main features of the curve.

5 SKETCHES

- A sketch should show any x- and y-intercepts if they are accessible, any vertical or horizontal asymptotes, and any other significant points on the curve.
- There should always be some indication of scale on both axes, and both axes should be labelled.

A curve-sketching menu

Here is our informal four-step approach to sketching an unknown function.

6 A CURVE SKETCHING MENU

0 **Preparation:** Combine any fractions using a common denominator, then factor top and bottom as far as possible.
1 **Domain:** What is the domain? (*Always* do this first.)
2 **Symmetry:** Is the function odd, or even, or neither?
3 **A** **Intercepts:** What are the y-intercept and the x-intercepts (zeroes)?
B **Sign:** Where is the function positive, and where is it negative?
4 **A** **Vertical asymptotes:** Examine the behaviour near any discontinuities, noting any vertical asymptotes.
B **Horizontal asymptotes:** Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

Finding the domain and finding the zeroes may both require factoring, which is the reason why the preparatory Step 0 is useful. Factoring, however, may not always be possible, even with the formula for the roots of a quadratic, and in such cases approximation methods may be useful.

Questions will normally give guidance as to what is required. Our menu is not an explicit part of the course, but rather a suggested way to organise the approaches presented in the course.

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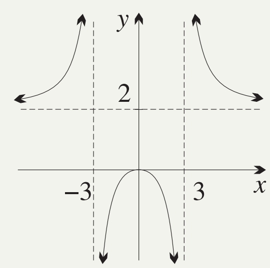
Putting it all together — first example

All that remains is to give two examples of the whole process.

Example 8 3C

Apply the steps in Box 6 to sketch $f(x) = \dfrac{2x^2}{x^2 - 9}$.

SOLUTION



0 Preparation: $f(x) = \dfrac{2x^2}{(x-3)(x+3)}$.

1 Domain: $x \neq 3$ and $x \neq -3$.

2 Symmetry: $f(-x) = \dfrac{2(-x)^2}{(-x)^2 - 9}$

$= \dfrac{2x^2}{x^2 - 9}$

$= f(x)$.

so $f(x)$ is even, with line symmetry in the$y$-axis.

3 Intercepts and sign: When $x = 0$, $y = 0$.

There is a zero at $x = 0$, and discontinuities at $x = 3$ and $x = -3$.

x	-4	-3	-1	0	1	3	4
$f(x)$	$\frac{32}{7}$	*	$-\frac{1}{4}$	0	$-\frac{1}{4}$	*	$\frac{32}{7}$
sign	$+$	*	$-$	0	$-$	*	$+$

4 Vertical asymptotes: At $x = 3$ and $x = -3$, the denominator vanishes, but the numerator does not, so $x = 3$ and $x = -3$ are vertical asymptotes. To make this more precise, it follows from the table of signs that

$$f(x) \to \infty \text{ as } x \to 3^+ \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to 3^-,$$

$$f(x) \to -\infty \text{ as } x \to (-3)^+ \quad \text{and} \quad f(x) \to \infty \text{ as } x \to (-3)^-$$

Horizontal asymptotes: Dividing through by x^2, $f(x) = \dfrac{2}{1 - \frac{9}{x^2}}$,

so $f(x) \to 2$ as $x \to \infty$ and as $x \to -\infty$, and $y = 2$ is a horizontal asymptote.

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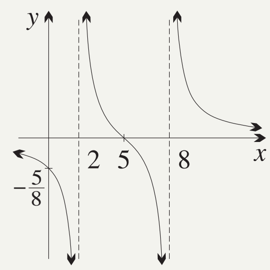
Putting it all together — second example

The second example requires a common denominator. The calculations involving intercepts and sign have been done with an alternative approach using signs rather than numbers.

Example 9 3C

Apply the steps in Box 6 to sketch $f(x) = \dfrac{1}{x-2} + \dfrac{1}{x-8}$.

SOLUTION



0 $f(x) = \dfrac{(x-8)+(x-2)}{(x-2)(x-8)}$

$= \dfrac{2x-10}{(x-2)(x-8)}$

$= \dfrac{2(x-5)}{(x-2)(x-8)}$.

1 The domain is $x \neq 2$ and $x \neq 8$.

2 $f(x)$ is neither even nor odd.

3 When $x = 0$, $y = -\dfrac{5}{8}$.

There is a zero at $x = 5$, and discontinuities at $x = 2$ and $x = 8$.

x	0	2	3	5	7	8	10
$x-2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-5$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-8$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	$*$	$+$	0	$-$	$*$	$+$

If only the signs are calculated, at least these three lines of working should be shown.

4 At $x = 2$ and $x = 8$ the denominator vanishes, but the numerator does not, so $x = 2$ and $x = 8$ are vertical asymptotes.

From the original form of the given equation, $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$, so $y = 0$ is a horizontal asymptote.

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Exercise 3C

FOUNDATION

1 Complete the following steps in order to sketch the graph of $y = \dfrac{9}{x^2 - 9}$.

a Factor the function as far as possible.
b State the domain using bracket interval notation.
c Show that the function is even. What symmetry does the graph have?
d Write down the coordinates of any intercepts with the axes.
e Investigate the sign of the function using a table of values. Where is $y \le 0$?
f Write down the equation of any vertical asymptote.
g What value does y approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
h Sketch the graph of the function showing these features.
i The graph appears to be horizontal at its y-intercept. Find y' and hence confirm that the graph is horizontal there.

2 Complete the following steps in order to sketch the graph of $y = \dfrac{x}{4 - x^2}$.

a Factor the function as far as possible.
b State the domain using bracket interval notation.
c Show that the function is odd. What symmetry does the graph have?
d Write down the coordinates of any intercepts with the axes.
e Investigate the sign of the function using a table of values. Where is $y \ge 0$?
f Write down the equation of any vertical asymptote.
g What value does y approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
h Sketch the graph of the function showing these features.
i Use the quotient rule to show that $y' = \dfrac{(x^2 + 4)}{(4 - x^2)^2}$, and hence explain why the graph always has a positive gradient.

3 Follow these steps to graph $y = f(x)$ where $f(x) = \dfrac{1}{x - 1} + \dfrac{1}{x - 4}$.

a Combine the two fractions using a common denominator, then factor the numerator and denominator as far as possible.
b State the domain.
c Determine whether $f(x)$ is even, odd or neither (the answers to part **a** may help).
d Write down the coordinates of any intercepts with the axes.
e Investigate the sign of the function using a table of values. Where is $y > 0$?
f Write down the equation of any vertical asymptote.
g What value does $f(x)$ approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
h Sketch the graph of the function showing these features.

4 Let $y = x^3 - 4x$.

a Factor this function.
b State the domain using bracket interval notation.

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c Write down the coordinates of any intercepts with the axes.
d Show that the function is odd. What symmetry does the graph have?
e Does this function have any asymptotes?
f Use this information and a table of values to sketch the curve.
g The graph seems to have a peak somewhere in the interval $-2 < x < 0$ and a trough in the interval $0 < x < 2$. Use calculus to find the x-coordinates of these points and add them to the diagram.

5 Let $y = \dfrac{3x - 3}{x^2 - 2x - 3}$.
a State the domain and any intercepts with the axes.
b Explain why the function is neither even nor odd.
(Hint: The answers to part **a** may help.)
c Write down the equations of the asymptotes.
d Sketch the graph of this curve.

6 Let $y = -x^3 - 6x^2 + 8x$.
a State the domain using inequality interval notation, and write down the coordinates of any intercepts with the axes.
b Use this information and a table of values to sketch the curve.
c The graph seems to be horizontal somewhere in the interval $0 < x < 2$, and again in the interval $2 < x < 4$. Use calculus to find the x-coordinates of these points and add them to the diagram.

DEVELOPMENT

7 Let $y = \dfrac{x^2 + 2x + 1}{x^2 + 2x - 3}$.
a State the domain and any intercepts with the axes.
b Explain why the function is neither even nor odd.
c Write down the equations of the asymptotes.
d Sketch the graph of this curve.
e What is the range of this function?

8 Let $f(x) = \dfrac{x^2 - 4}{x^2 - 4x}$. You may assume that $f(x)$ is neither even nor odd.
a State the domain of $f(x)$ and write down the intercepts of $y = f(x)$.
b Write down the equations of the asymptotes.
c Sketch the graph of $y = f(x)$.
d What is the range of this function?
e The graph crosses its horizontal asymptote in the interval $0 < x < 4$. Find the coordinates of this point and add it to the graph.

9 **a** Show that $y = \dfrac{1}{x + 1} - \dfrac{1}{x}$ can be written as $y = -\dfrac{1}{x(x + 1)}$. Then identify the domain and any zeroes, examine the asymptotes and sign, and hence sketch the graph.
b Likewise express $y = \dfrac{1}{x + 3} + \dfrac{1}{x - 3}$ with a common denominator and sketch it.

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10 **a** Examine the sign and asymptotes of $y = \dfrac{1}{x(x-2)}$ and hence sketch the curve.

b Likewise sketch $y = \dfrac{2}{x^2 - 4}$.

11 Use the curve-sketching menu as appropriate to obtain the graphs of:

a $y = \dfrac{1 + x^2}{1 - x^2}$ **b** $y = \dfrac{x + 1}{x(x - 3)}$ **c** $y = \dfrac{x - 1}{(x + 1)(x - 2)}$

d $y = \dfrac{x^2 - 2x}{x^2 - 2x + 2}$ **e** $y = \dfrac{x^2 - 4}{(x + 2)(x - 1)}$ **f** $y = \dfrac{x^2 - 2}{x}$

12 The curve $y = e^{-\frac{1}{2}x^2}$ is essential in statistics because it is related to the normal distribution, studied later in this course.

a Determine the domain and any intercepts of this function.

b Determine whether the function is even or odd, and investigate any asymptotes.

c By considering the maximum value of $-x^2$, find the highest point on this curve.

d Confirm your answer by showing that the tangent is horizontal there.

e Sketch the curve, and hence state its range.

f Which is higher, $y = e^{-\frac{1}{2}x^2}$ or $y = 2^{-\frac{1}{2}x^2}$?

ENRICHMENT

13 **a** **i** By considering the graphs of $y = e^x$ and $y = x$, or otherwise, determine which function is greater for $x \geq 0$.

ii Use part **i** to explain why $e^{-\frac{1}{2}x^2} < \dfrac{2}{x^2}$ for $x \neq 0$.

iii Hence, or otherwise, determine $\lim\limits_{x \to \infty} xe^{-\frac{1}{2}x^2}$.

b The graph of $y = e^{-\frac{1}{2}x^2}$, drawn in Question 12 appears to be steepest at $x = -1$ where it has positive gradient, and at $x = 1$ where it has negative gradient. In order to confirm this, the range of $y' = -xe^{-\frac{1}{2}x^2}$ is needed.

i Use the curve-sketching menu to sketch the graph of $f(x) = -xe^{-\frac{1}{2}x^2}$. You will need your answer to part **a** to determine the horizontal asymptote. Include on the sketch the x-coordinates of any points where the tangent is horizontal.

ii Hence prove the claim that $y = e^{-\frac{1}{2}x^2}$ is steepest at $x = 1$ or -1.

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3D Solving inequations

We can now turn our attention to equations, and more importantly to inequations (or ‘inequalities’ as they are often called). Again, the material in this section was covered in Section 5A of the Year 11 book, but it combines many approaches and needs review.

The basic approaches to solving inequations

Here is a summary of the basic methods that we have used in solving inequations.

7 SOLVING INEQUATIONS — A BASIC SUMMARY

- The graphical interpretation of an inequation $f(x) < g(x)$ is,
  ‘Where is the graph of $y = g(x)$ above the graph of $y = f(x)$?’
- Anything can be added to or subtracted from both sides of an inequation.
- When multiplying or dividing both sides by a negative number, the inequality symbol is reversed.
- As with equations, never multiply or divide by 0.
- A denominator can be removed by multiplying by its square, which is always positive or zero — the zeroes then become special cases.
- If the corresponding equation has been solved, and the discontinuities can be found, a table of signs will solve the inequation.

We will now review how these six dotpoints apply to various types of inequations.

Linear inequations: Linear inequations can be solved completely by the second and third dotpoints, and this needs no further review. Three considerations:

- If the graph is drawn, then the first dotpoint can be used.
- If the equation has been solved, then a table of signs can be used.
- If the linear equation has unknown constants, such as $ax > c$, then division by a could be disastrous, because a could be negative or even zero.

Quadratic and polynomial inequations: The standard method is to sketch the graph and read the solution off the graph. This was well covered in Year 11.

If the polynomial can be factored, as for example $(x - 2)(x - 5) < 0$, then an algebraic argument using the signs of the two factors can be used, but this easily leads to confusion. In this case, the two factors must have opposite signs, so $2 < x < 5$.

Inequations with a variable in the denominator: The standard approach is to multiply through by the denominator’s square, which is positive, so that we don’t generate cases depending on whether the inequality symbol is reversed.

- Keep expressions factored where possible — re-factoring may be difficult.
- If the denominator has zeroes, these must be treated as special cases.

The other approach, which is usually longer, is to sketch the graph and read the solution off the graph. The table of signs alone, however, may be sufficient.

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Example 10 3D

a Solve $\dfrac{3}{x-4} \geq -1$ by multiplying through by the square of the denominator.

b Use a sketch of $y = 1 + \dfrac{3}{x-4}$ to solve $\dfrac{3}{x-4} \geq -1$.

SOLUTION

a $\dfrac{3}{x-4} \geq -1$

We multiply both sides by $(x-4)^2$, which is positive, or zero if $x = 4$.

$$\boxed{\times (x-4)^2} \qquad \begin{aligned} 3(x-4) &\geq -(x-4)^2, \text{ where } x \neq 4, \\ (x-4)^2 + 3(x-4) &\geq 0 \\ (x-4)(x-4+3) &\geq 0 \\ (x-4)(x-1) &\geq 0. \end{aligned}$$

Hence after sketching the parabola $y = (x-4)(x-1)$,

$x \leq 1$ or $x \geq 4$.

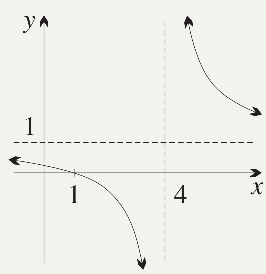
But $x = 4$ was excluded in the first step, so the solution is

$x \leq 1$ or $x > 4$.

b Moving everything to the LHS, $1 + \dfrac{3}{x-4} \geq 0$

$$\frac{x-1}{x-4} \geq 0.$$

From the graph, the solution is $x \leq 1$ or $x > 4$.



The LHS was graphed in worked Example 5, but the table of signs below is also sufficient working.

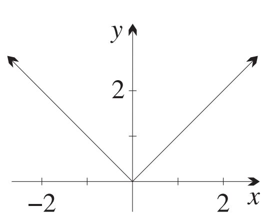
x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	−	*	+

From the graph or from the table, the solution is $x \leq 1$ or $x > 4$.

Solving absolute value equations and inequations

The six basic methods in Box 7 apply just as much to absolute value inequations, and many can be solved that way, particularly if a reasonable graph can be drawn quickly.

But absolute value equations and inequations are usually best solved using their own particular approaches. Absolute value was discussed in detail in Sections 4D, 5A and 5E of the Year 11 book, and these summaries do not do those sections justice.



$y = |x|$.

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8 ABSOLUTE VALUE

The meaning of absolute value:

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.
- $|x| = \begin{cases} x, & \text{for } x \ge 0, \\ -x, & \text{for } x < 0. \end{cases}$

Solving absolute value equations and inequations:

Let $a \ge 0$ be a real number.

- Rewrite an equation $|f(x)| = a$ as $f(x) = -a$ or $f(x) = a$.
- Rewrite an inequation $|f(x)| < a$ as $-a < f(x) < a$.
- Rewrite an inequation $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$.

Example 11 — 3D

a Solve $|10 - x^2| = 6$. **b** Solve $|10 - x^2| < 6$. **c** Solve $|10 - x^2| > 6$.

SOLUTION

a
$$10 - x^2 = 6 \quad\text{or}\quad 10 - x^2 = -6$$
$$x^2 = 4 \quad\text{or}\quad x^2 = 16$$
$$x = 2 \text{ or } -2 \text{ or } 4 \text{ or } -4.$$

b
$$-6 < 10 - x^2 < 6 \qquad \text{OR} \qquad \text{Use part } \textbf{a} \text{ and a table of signs.}$$
$\boxed{-10}$ $$-16 < -x^2 < -4$$
$\boxed{\times(-1)}$ $$4 < x^2 < 16$$
$$-4 < x < -2 \text{ or } 2 < x < 4.$$

c
$$10 - x^2 < -6 \text{ or } 10 - x^2 > 6 \qquad \text{OR} \qquad \text{Use part } \textbf{a} \text{ and a table of signs.}$$
$$x^2 > 16 \text{ or } x^2 < 4$$
$$x < -4 \text{ or } -2 < x < 2 \text{ or } x > 4.$$

Exercise 3D

FOUNDATION

1 Solve each inequation, and graph your solution on the number line.

a $x - 2 < 3$ **b** $3x \ge -6$ **c** $4x - 3 \le -7$

d $6x - 5 < 3x - 17$ **e** $\frac{1}{5}x - \frac{1}{2}x < 3$ **f** $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) \ge 2$

2 Solve each inequation, then write your answer using bracket interval notation.

a $3 - 2x > 7$ **b** $3 - 3x \le 19 + x$ **c** $12 - 7x > -2x - 18$

3 Write down and solve a suitable inequation to find the values of x for which the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.

4 Solve each double inequation, then write your answer in bracket interval notation.

a $-1 \le 2x \le 3$ **b** $-4 < -2x < 8$ **c** $-7 \le 5 - 3x < 4$

d $-5 < x - 3 \le 4$ **e** $-7 \le 5x + 3 < 3$ **f** $-4 < 1 - \frac{1}{3}x \le 3$

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5 **a** Sketch the lines $y = 1 - x$, $y = 2$ and $y = -1$ on a number plane and find the points of intersection.
b Solve the inequation $-1 < 1 - x \le 2$ and relate the answer to the graph.

6 In each case, solve for x by sketching an appropriate parabola.
a $x^2 + 2x - 8 < 0$ **b** $x^2 - 2x - 3 > 0$ **c** $-x^2 + 7x - 10 \ge 0$
d $x^2 + 4x + 3 \ge 0$ **e** $2x^2 - 11x + 5 > 0$ **f** $2x^2 + 13x + 20 \le 0$

7 Solve the following equations and inequations, and graph each solution on a number line.
a $|x - 4| = 1$ **b** $|2x - 3| = 7$ **c** $|x + 3| > 4$
d $|-x - 10| \le 6$ **e** $|3 - 2x| \le 1$ **f** $|3x + 4| > 2$

DEVELOPMENT

8 Multiply through by the square of the denominator and hence solve:
a $\dfrac{1}{x} \le 2$ **b** $\dfrac{3}{2 - x} > 1$ **c** $\dfrac{4}{3 - 2x} < 1$ **d** $\dfrac{5}{4x - 3} \ge -2$

9 Write down the equations of the two branches of the function, then sketch its graph:
a $y = |x - 2| + x + 1$ **b** $y = |2x + 4| - x + 5$ **c** $y = 3|x - 1| + x - 1$

10 Solve the following inequations involving logarithms and exponentials:
a $3^x \ge 27$ **b** $1 < 5^x \le 125$ **c** $\frac{1}{16} \le 2^x \le 16$
d $2^{-x} > 16$ **e** $\log_2 x < 3$ **f** $-2 \le \log_5 x \le 4$

11 In each case, use an appropriate inequation to solve the problem.
a Where is the parabola $y = x^2 - 2x$ below the line $y = x$?
b Where is the parabola $y = x^2 - 2x - 3$ above the line $y = 3 - 3x$?
c Where is the hyperbola $y = \dfrac{1}{x}$ below the line $y = -x$?
d Where is the hyperbola $y = \dfrac{2}{x - 1}$ above the line $y = \frac{1}{2}x - 2$?
e Where is the parabola $y = x^2 + 2x - 3$ below the parabola $y = 5 - 4x - x^2$?

12 **a** **i** Write down the equations of the two branches of the function $y = |2x| + x$.
ii Hence solve $|2x| + x > 1$.
b Likewise, solve these inequations.
i $3|x - 2| + x - 2 \le 2$. **ii** $|x + 1| - \frac{1}{2}x < 3$

13 Solve for x:
a $\dfrac{2x + 1}{x - 3} > 1$ **b** $\dfrac{x - 1}{x + 1} \le 2$ **c** $\dfrac{3x}{2x - 1} \ge 4$

14 **a** Where is $\cos x > 1 - \sin^2 x$ in the domain $0 \le x \le 2\pi$?
b Where is $\tan x \le \sec^2 x - 1$ in the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$?

15 Solve:
a $1 < |x + 2| \le 3$ **b** $1 \le |2x - 3| < 4$

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16 Determine whether these statements are true or false. If false, give a counter-example.
If true, provide examples with:

i $x > 0$ and $y > 0$, ii $x > 0$ and $y < 0$, iii $x < 0$ and $y > 0$, iv $x < 0$ and $y < 0$.

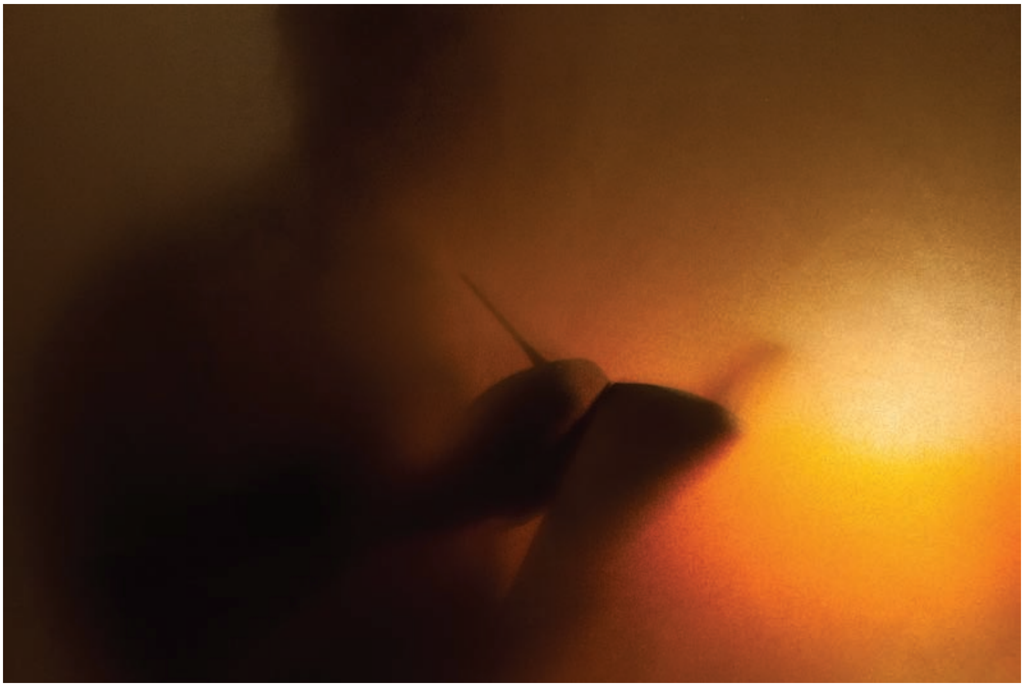
a $|x + y| = |x| + |y|$
b $|x + y| \le |x| + |y|$
c $|x - y| \le |x| - |y|$
d $|x - y| \le |x| + |y|$
e $|x - y| \ge \big||x| - |y|\big|$
f $2^{|x|} = 2^x$

ENRICHMENT

17 A student was asked to solve the inequation $\sqrt{5 - x} > x + 1$.

a The student decided to square both sides to get $5 - x > x^2 + 2x + 1$, and then solved this inequation. Explain why this gives the wrong answer.

b Find the correct solution.



18 a Draw the graph of $f(x) = |5 - 2x^2|$ by considering the equations of its branches.

b Hence or otherwise solve the inequation $|5 - 2x^2| \ge 3$.

c More generally, prove that if $|g(x)| \ge k$ then either $g(x) \le -k$ or $g(x) \ge k$.

19 Prove that $x^2 + xy + y^2 > 0$ for any non-zero values of x and y.

20 a Prove that $(x + y)^2 \ge 4xy$.

b Hence prove that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \ge \dfrac{4}{x^2 + y^2}$.

21 a Expand $(a - b)^2 + (b - c)^2 + (a - c)^2$, and hence prove that $a^2 + b^2 + c^2 \ge ab + bc + ac$.

b Expand $(a + b + c)\left((a - b)^2 + (b - c)^2 + (a - c)^2\right)$, and hence prove the identity $a^3 + b^3 + c^3 \ge 3abc$, for positive a, b and c.

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3E Using graphs to solve equations and inequations

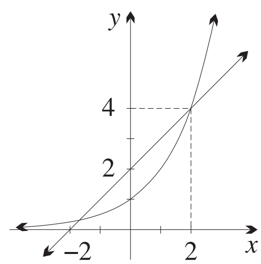
In this section, graphs are used to solve more general equations and inequations. The advantage of such an approach is that once the graphs are drawn, it is usually obvious from the picture how many solutions there are, and indeed if there are any solutions at all, as well as their approximate values. Exact solutions can sometimes then be calculated once the situation has been sorted out from the picture.

Constructing two functions from a given equation

Here is an equation that cannot be solved algebraically, so that a graphical approach is appropriate:

$$2^x = x + 2.$$

To the right, $y = \text{LHS}$ and $y = \text{RHS}$ are graphed together. (In other situations, some rearrangement of the equation first may be appropriate.)



The first thing to notice is that there are two solutions, because the graphs intersect twice.

The second thing is to examine what the values of the two solutions are. One solution is exactly $x = 2$, because

$$2^2 = 4 = 2 + 2, \qquad \text{so } (2, 4) \text{ lies on both graphs.}$$

The other solution is just to the right of $x = -2$. From the graph, we might guess $x \doteqdot -1.7$, and if necessary we can refine this solution in several ways:

- Plot the graphs carefully on graph paper (an old method that works).
- Use trial and error on a calculator (see Question 12 in Exercise 3E).
- Use graphing software that can generate approximations (if you have it).

Counting the number of solutions of an equation

Often, however, we only want to know how many solutions an equation has, and roughly where they are.

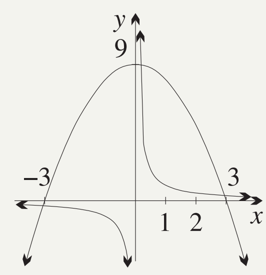
Example 12 3E

a Graph $y = 9 - x^2$ and $y = \dfrac{1}{x}$ on the one set of axes.

b Use your graph to investigate the equation $9 - x^2 = \dfrac{1}{x}$. How many solutions does the equation have, and approximately where are they?

SOLUTION

a The two functions are sketched to the right.



b There are three points of intersection of the two graphs.
Thus there are three solutions:
– one just to the left of $x = -3$,
– one just to the right of $x = 0$,
– and one just to the left of $x = 3$.

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9 GRAPHICAL SOLUTION OF AN EQUATION

- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
  — It may be appropriate to rearrange the original equation first.
- The solutions are the x-coordinates of any points of intersection.
- You may be interested only in the number of solutions.
- The graph will give a rough idea where any solution lies.

Solving an inequation using graphs

Now consider the inequation

$$2^x < x + 2.$$

From the sketch at the start of the section, the curve $y = 2^x$ is only below the curve $y = x + 2$ between the two points of intersection. Hence the solution of the inequation is approximately $-1.7 < x < 2$.

10 GRAPHICAL SOLUTION OF AN INEQUATION

- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
- Then examine which curve lies above the other at each value of x.

Absolute value equations and inequations — graphical solutions

The most straightforward approach to absolute value equations and inequations is to draw a sketch to sort out the situation. Then the exact values can usually be found algebraically.

The next worked example benefits greatly from the diagram that makes the situation so clear.

Example 13 3E

a Draw the graph of $y = |2x - 5|$.
b Write down the equations of the right-hand and left-hand branches.
c One the same diagram, draw the graph of $y = x + 2$.
d Hence find the points P and Q of intersection.
e Solve $|2x - 5| = x + 2$.
f Solve $|2x - 5| \ge x + 2$.

SOLUTION

a To find the x-intercept, put $x = 0$, then $2x - 5 = 0$

$$x = 2\tfrac{1}{2}$$

To find the y-intercept, put $x = 0$, then

$$\begin{aligned} y &= |0 - 5| \\ &= 5. \end{aligned}$$

We can now sketch the right-hand branch by symmetry (or use a table of values).

b For $x \ge 2\frac{1}{2}$, $y = 2x - 5$.
For $x < 2\frac{1}{2}$, $y = -2x + 5$.

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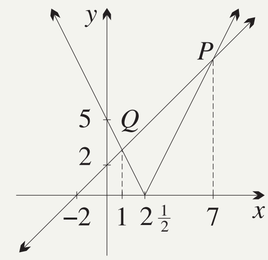


c The two graphs intersect at two points P and Q as shown.

d The points P and at Q can be now found algebraically.
P is the intersection of $y = x + 2$ with $y = 2x - 5$,

$$x + 2 = 2x - 5$$
$$x = 7, \text{ so } P = (7, 9),$$

and Q is the intersection of $y = x + 2$ with $y = -2x + 5$,

$$x + 2 = -2x + 5$$
$$x = 1, \text{ so } Q = (1, 3).$$

e Hence the solutions are $x = 7$ or $x = 1$.

f Look at where $y = |2x - 5|$ is on or above $y = x + 2$.
From the graph, $x \le 1$ or $x \ge 7$.

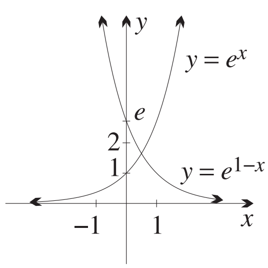
Exercise 3E

FOUNDATION

Note: Graphing software would be particularly useful in this exercise.

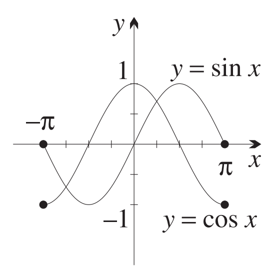
1 In each case, use the given graph to determine the number of solutions of the equation.

a



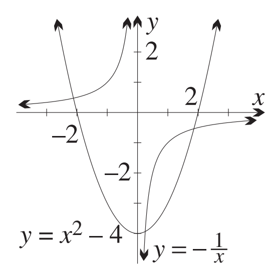
$e^x = e^{1-x}$

b



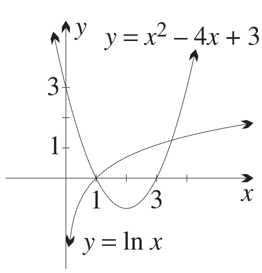
$\cos x = \sin x, \quad -\pi \le x \le \pi$

c



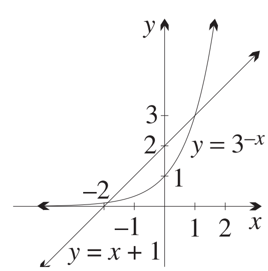
$x^2 - 4 = -\dfrac{1}{x}$

d



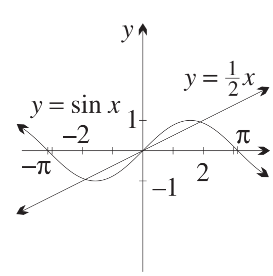
$\ln x = x^2 - 4x + 3$

e



$3^x = x + 2$

f



$\sin x = \dfrac{1}{2}x$

2 For each equation in Question 1, read the solutions from the graph, approximated correct to one decimal place where appropriate.

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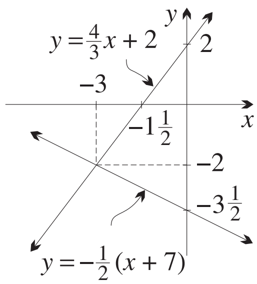
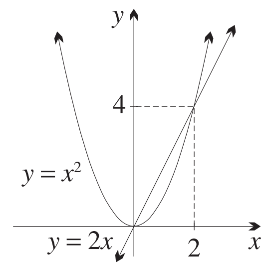
3 Use the given graphs to help solve each inequation.

a



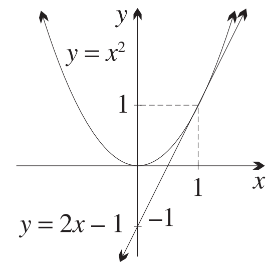
$\frac{4}{3}x + 2 \le -\frac{1}{2}(x + 7)$

b



$x^2 \le 2x$

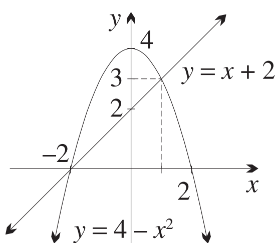
c



$x^2 \le 2x - 1$

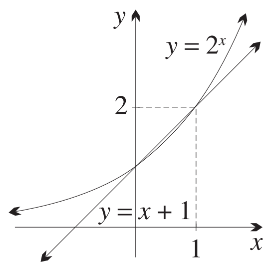
4 Solve these inequations using the given graphs.

a



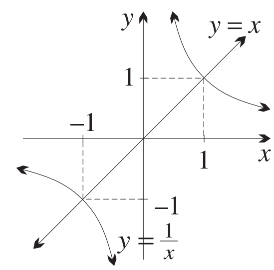
$4 - x^2 < x + 2$

b



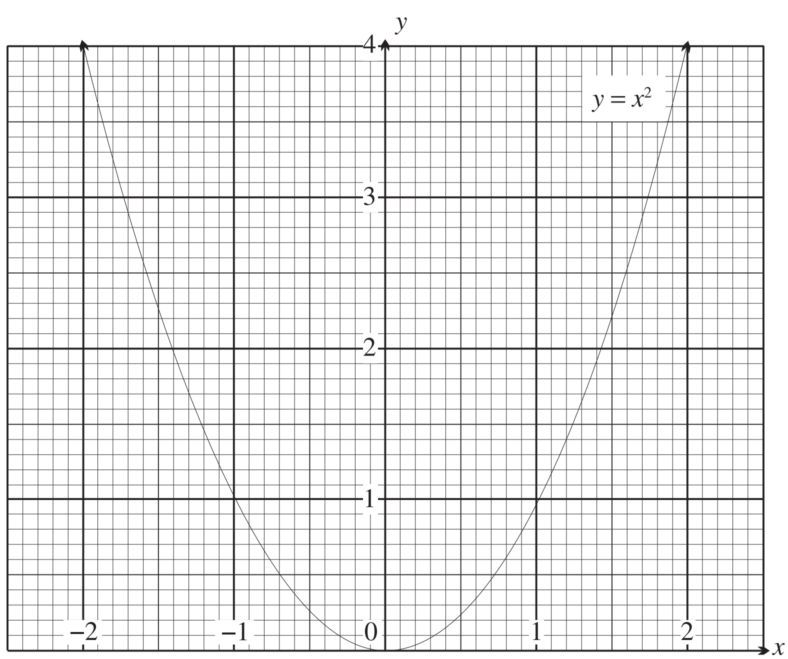
$2^x \le x + 1$

c



$\frac{1}{x} < x$

5



In preparation for the following questions, photocopy the sketch above, which shows the graph of the function

$$y = x^2, \text{ for } -2 \le x \le 2.$$

a Read $\sqrt{2}$ and $\sqrt{3}$ off the graph, correct to one decimal place, by locating 2 and 3 on the y-axis and reading the answer off the x-axis. Check your approximations using a calculator.

b Draw on the graph the line $y = x + 2$, and hence read off the graph the solutions of $x^2 = x + 2$. Then check your solution by solving $x^2 = x + 2$ algebraically.

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c From the graph, write down the solution of $x^2 > x + 2$.
d Draw a suitable line to solve $x^2 = 2 - x$ and $x^2 \le 2 - x$. Check your results by solving $x^2 = 2 - x$ algebraically.
e Draw $y = x + 1$, and hence solve $x^2 = x + 1$ approximately. Check your result algebraically.
f Find approximate solutions for these quadratic equations by rearranging each with x^2 as subject, and drawing a suitable line on the graph.
i $x^2 + x = 0$ **ii** $x^2 - x - \frac{1}{2} = 0$ **iii** $2x^2 - x - 1 = 0$

6 In each part:
i Carefully sketch each pair of equations. **ii** Read off the points of intersection.
iii Write down the equation satisfied by the x-coordinates of the points of intersection.
a $y = x - 2$ and $y = 3 - \frac{1}{4}x$ **b** $y = x$ and $y = 2x - x^2$
c $y = \dfrac{2}{x}$ and $y = x - 1$ **d** $y = x^3$ and $y = x$

7 Use your graphs from the previous question to solve the following inequations.
a $x - 2 \ge 3 - \frac{1}{4}x$ **b** $x < 2x - x^2$ **c** $\dfrac{2}{x} > x - 1$ **d** $x^3 > x$

8 **a** **i** Sketch on the same number plane the functions $y = |x + 1|$ and $y = \frac{1}{2}x - 1$.
ii Hence explain why all real numbers are solutions of the inequation $|x + 1| > \frac{1}{2}x - 1$.
b Draw a sketch of the curve $y = 2^x$ and the line $y = -1$. Hence explain why the inequation $2^x \le -1$ has no solutions.
c Draw $y = 2x - 1$ and $y = 2x + 3$ on the same number plane, and hence explain why the inequation $2x - 1 \le 2x + 3$ is true for all real values of x.

9 Sketch each pair of equations, and hence find the points of intersection.
a $y = |x + 1|$ and $y = 3$ **b** $y = |x - 2|$ and $y = x$
c $y = |2x|$ and $2x - 3y + 8 = 0$ **d** $y = |x| - 1$ and $y = 2x + 2$

10 Use your answers to the previous question to help solve:
a $|x + 1| \le 3$ **b** $|x - 2| > x$ **c** $|2x| \ge \dfrac{2x + 8}{3}$ **d** $|x| > 2x + 3$

DEVELOPMENT

11 Explain how the graphs of Question 1 parts **a**, **b** and **c** could be used to solve:
a $e^{2x} = e$ **b** $\tan x = 1$ **c** $x^3 - 4x + 1 = 0$.

12 [Solving an equation by trial and error]
a In Section 3E we sketched $y = 2^x$ and $y = x + 2$ on one set of axes, and saw that there is a solution a little to the right of $x = -2$. Fill in the table of values below, and hence find the negative solution of $2^x = x + 2$ correct to three decimal places.

x	-2	-1.7	-1.6	-1.68	-1.69	-1.691	-1.6905
2^x							
$x + 2$							

b For parts **c** and **e** of Question 1, use trial and error on the calculator to find the negative solution of the equation, correct to three decimal places.

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13 Sketch graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them.

a $1 - \frac{1}{2}x = x^2 - 2x$ **b** $|2x| = 2^x$ **c** $x^3 - x = \frac{1}{2}(x + 1)$

d $4x - x^2 = \dfrac{1}{x}$ **e** $2^x = 2x - x^2$ **f** $2^{-x} - 1 = \dfrac{1}{x}$

14 [Break-even point]

A certain business has fixed costs of \$900 plus costs of \$30 per item sold. The sale price of each item is \$50. If enough items are sold then the company is able to exactly pay its total costs. That point is called the *break-even point*. Companies may use several different methods to graph this information. Here are two such methods. In each case, let n be the number of items sold.

a **i** The gross profit per item is $\$50 - \$30 = \$20$. Sketch the graph of the gross profit for n items $y = \$20 \times n$.

ii On the same graph sketch the fixed costs $y = \$900$.

iii The point where these two lines cross is the break-even point. How many items need to be sold to break even?

b **i** On a new graph, draw $y = \$50 \times n$, the total sales for n items.

ii On the same graph, draw $y = \$900 + \$30 \times n$, the total cost for n items.

iii Does the break-even point for this graph agree with part **a**?

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15 **a** Sketch $y = x^2 - 2$, $y = x$ and $y = -x$ on the same number plane, and find all points of intersection of the three functions.

b Hence find the solutions of $x^2 - 2 = |x|$.

c Hence solve $x^2 - 2 > |x|$.

d Finally solve $x^2 - 2 \le -|x|$.

16 **a** Sketch $y = |2x + 1|$.

b Draw on the same number plane $y = x + c$ for $c = -1$, $c = 0$ and $c = 1$.

c For what values of c does $|2x + 1| = x + c$ have two solutions?

17 **a** Use a diagram and Pythagoras' theorem to show that for $b > 0$, the perpendicular distance from the line $x + y = 2b$ to the origin is $b\sqrt{2}$.

b Hence find the range of values of b for which the line intersects the circle $x^2 + y^2 = 9$ twice.

18 **a** Sketch $y = |7x - 4|$ and $y = 4x + 3$ on the same number plane to find the number of solutions of $|7x - 4| = 4x + 3$.

b Why is it inappropriate to use the graph to find the exact solutions?

c Find the solutions by separately considering the two branches of the absolute value.

19 Draw appropriate graphs, using a computer or graphics calculator, in order to find the solutions of these equations correct to one decimal place.

a $x^3 = 2(x - 2)^2$ **b** $x^3 = \sqrt{4 - x^2}$

c $2^x = -x(x + 2)$ **d** $2^{-x} = 2x - x^2$

ENRICHMENT

20 **a** Carefully sketch the graph of $y = |2x + 4| + |x - 1| - 5$ and write down the equation of each branch.

b On the same number plane draw the lines $y = -1$ and $y = 2$. Hence solve the inequation $-1 \le |2x + 4| + |x - 1| - 5 \le 2$.

21 **a** Show that $y = mx + b$ intersects $y = |x + 1|$ if $m > 1$ or $m < -1$.

b Given that $-1 \le m \le 1$, find the relationship between b and m so that the two graphs do not intersect.

c Generalise these results for $y = |px - q|$.

22 **a** Draw $y = a^x$ and $y = \log_a x$ for:

i $a = 3$ **ii** $a = 2$ **iii** $a = \sqrt{2}$

b Conclude how many solutions $a^x = \log_a x$ may have.

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3F Regions in the coordinate plane

The circle $x^2 + y^2 = 25$ divides the plane into two *regions* — inside the circle and outside the circle. The graph of the inequation $x^2 + y^2 > 25$ will be one of these regions. It remains to work out which of these regions should be shaded.

Graphing regions

Here is how to sketch the region described by an inequation.

11 GRAPHING THE REGION CORRESPONDING TO AN INEQUATION

1. **The boundary:** Replace the inequality symbol by the equal sign, and graph the curve. This will normally be the boundary of the region, and should be drawn broken if it is excluded, and unbroken if it is included.
2. **Shading:** Determine which parts are included and which are excluded, and shade the parts that are included. This can be done in two ways:
   - *Always possible:* Take one or more test points not on any boundary, and substitute into the LHS and RHS of the original inequation. The origin is the easiest test point, otherwise try to choose points on the axes.
   - *Quicker, but not always possible:* Alternatively, solve the inequation for y if possible, and shade the region above or below the curve. Or solve for x, and sketch the region to the right or left of the curve.
3. **Check boundaries and corners:**
   - Check that boundaries are correctly broken or unbroken.
   - Corner points must be marked clearly with a closed circle if they are included, or an open circle if excluded.

Example 14 3F

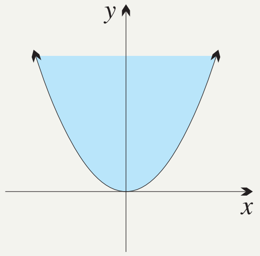
Sketch $y \geq x^2$.

SOLUTION

The boundary is $y = x^2$, and is included.
Because the inequation is $y \geq x^2$,
the region involved is the region *above* the curve.
Alternatively, take a test point $(0, 1)$, then

$$\text{LHS} = 1 \qquad \text{and} \qquad \text{RHS} = 0,$$

so LHS $\geq$ RHS, meaning that $(0, 1)$ is in the region.



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Example 15 3F

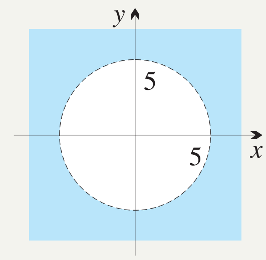
Sketch the region $x^2 + y^2 > 25$.

SOLUTION

The boundary is $x^2 + y^2 = 25$, and is excluded.
The inequation cannot be solved for x or y without complicated cases.
Instead, take a test point $(0, 0)$, then

$$\text{LHS} = 0 \quad \text{and} \quad \text{RHS} = 25,$$

so $LHS \not> RHS$, meaning that $(0, 0)$ is not in the region.



Points where the LHS or the RHS is undefined

There may be points in the plane where the LHS or the RHS of the inequation is undefined, as in the next worked example. If so, the set of all these points is a possible boundary of the region too, and will be excluded.

Example 16 3F

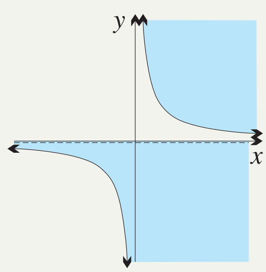
Sketch the region $x \geq \dfrac{1}{y}$.

SOLUTION

The boundary is $x = \dfrac{1}{y}$, and is included.

Also, the x-axis $y = 0$ is a boundary, because the RHS is undefined when $y = 0$. This boundary is excluded.

Because the inequation is $x \geq \dfrac{1}{y}$, the region to be shaded is the region to the right of the curve.



Alternatively, take test points, but this is more elaborate.

The point $(0, 1)$ is out because $0 \ngeq \dfrac{1}{1}$.

The point $(0, -1)$ is in because $0 \geq \dfrac{1}{-1}$.

The point $(2, 2)$ is in because $2 \geq \dfrac{1}{2}$.

The point $(-2, -2)$ is out because $-2 \ngeq \dfrac{1}{-2}$.

Troops on both sides of the border

Some dictators like to control their borders by having troops on both sides of the border. You cannot always assume that if one side of a boundary is in, then the other side is out, or vice versa.

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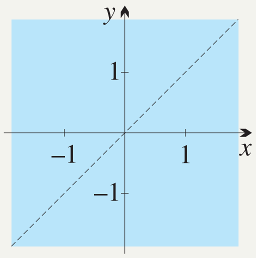
Example 17 3F

a Sketch $(y - x)^2 > 0$. **b** Describe $(y - x)^2 \le 0$.

SOLUTION



a The boundary is $(y - x)^2 = 0$, that is, $y = x$, and is excluded. But if $P(x, y)$ is any point off the line, then $x \ne y$, so $(x - y)^2 > 0$, and the points is included. Hence every point in the plane is in the region except for the points on the line $y = x$.

b Similarly, the region $(y - x)^2 \le 0$ is just the line $y = x$.

12 TWO QUALIFICATIONS OF THE METHOD FOR FINDING REGIONS

- Points where the LHS or the RHS is undefined form a possible boundary.
- Both sides of a boundary may be in, or both sides may be out.

Intersections and unions of regions

Some questions will ask explicitly for the intersection or union of two regions. Other questions will implicitly ask for intersections. For example,

$$|2x + 3y| < 6$$

means $-6 < 2x + 3y < 6$, so is the intersection of $2x + 3y > -6$ and $2x + 3y < 6$.

Or there may be a restriction on x or on y, as in

$$x^2 + y^2, \text{ where } x \le 3 \text{ and } y > -4,$$

which means the intersection of three different regions.

13 INTERSECTIONS AND UNIONS OF REGIONS

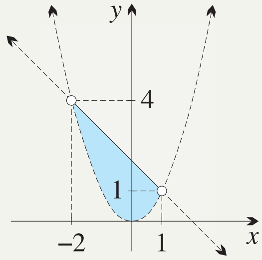
- Draw each region, then sketch the intersection or union.
- Pay particular attention to whether corner points are included or excluded.

Example 18 3F

Graph the intersection and union of the regions

$y > x^2$ and $x + y \le 2$.

SOLUTION



The boundary of the first region is $y = x^2$, and the region lies above the curve (with the boundary excluded).

The boundary of the second region is $x + y = 2$. Solving for y gives $y \le 2 - x$, so the region lies below the curve (with the boundary included).

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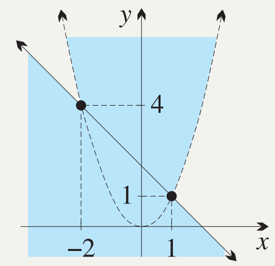
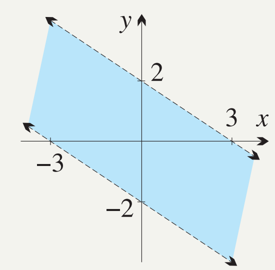
By inspection, or by simultaneous equations, the parabola and the line meet at $(1, 1)$ and $(-2, 4)$. These points are excluded from the intersection because they are not in the region $y > x^2$, but included in the union because they are in the region $x + y \le 2$.



Example 19 3F

Graph the region $|2x + 3y| < 6$.



SOLUTION

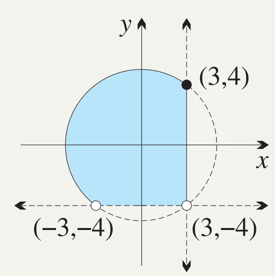
This is the region $-6 < 2x + 3y < 6$.
The boundaries are the parallel lines
$2x + 3y = 6$ and $2x + 3y = -6$, both of which are excluded.
The required region is the region between these two lines.

Example 20 3F

Graph the region

$$x^2 + y^2 \le 25, \quad \text{where } x \le 3 \text{ and } y > -4,$$

giving the coordinates of each corner point.



SOLUTION

The boundaries are $x^2 + y^2 = 25$ (included), and the vertical and horizontal lines $x = 3$ (included) and $y = -4$ excluded.
The points of intersection are $(3, 4)$ (included), $(3, -4)$ excluded and $(-3, -4)$ excluded.

Exercise 3F FOUNDATION

1 For each inequation:
   i sketch the boundary,
   ii shade the region above or below the boundary, as required.
   a $y < 1$ b $y > x - 1$ c $y \le 3 - x$ d $y < \frac{1}{2}x - 1$

2 For each inequation:
   i sketch the boundary,
   ii shade the region to the left or right of the boundary, as required.
   a $x > 1$ b $x \ge y + 2$ c $x < 2y - 1$ d $x > 3 - y$

3 For each inequation, sketch the boundary line, then use a suitable test point to decide which side of the line to shade.
   a $2x + 3y - 6 > 0$ b $x - y + 4 \ge 0$

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4 For each inequation, sketch the boundary circle, then use a suitable test point to decide which region to shade.

a $x^2 + y^2 < 4$ **b** $x^2 + y^2 \geq 1$

c $(x - 2)^2 + y^2 \leq 4$ **d** $(x + 1)^2 + (y - 2)^2 > 9$

5 Sketch the following regions (some of the quadratics need factoring):

a $y < x^2 - 2x - 3$ **b** $y \geq x^2 + 2x + 1$

c $y > 4 - x^2$ **d** $y < (5 - x)(1 + x)$

6 Draw the following regions of the number plane:

a $y > 2^x$ **b** $y \leq |x + 1|$ **c** $y > x^3$ **d** $y \leq \log_2 x$

7 **a** Find the intersection point of the lines $x = -1$ and $y = 2x - 1$.

b Hence sketch the intersection of the regions $x > -1$ and $y \leq 2x - 1$, paying careful attention to the boundaries and their point of intersection.

c Likewise sketch the union of the two regions.

8 **a** Sketch on separate number planes the two regions $y < x$ and $y \geq -x$. Hence sketch:

i the union of these two regions, **ii** the intersection of the regions.

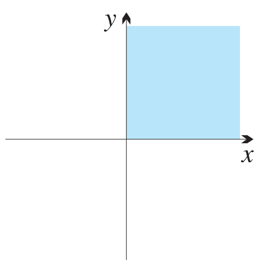
Pay careful attention to the boundaries and their points of intersection.

b Similarly, graph the union and intersection of $y > \frac{1}{2}x + 1$ and $y \leq -x - 2$.

DEVELOPMENT

9 Identify the inequations that correspond to the following regions:

a



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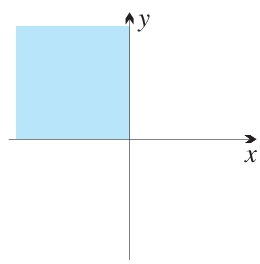
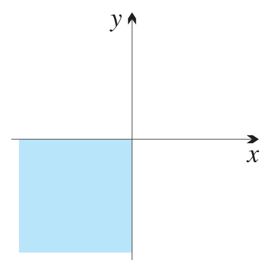
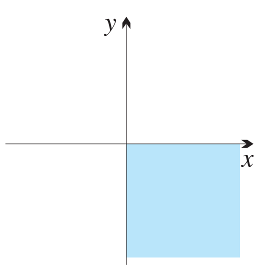


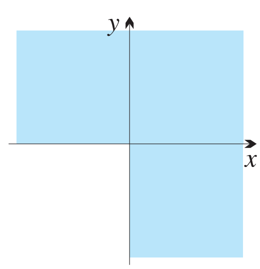
c



d



e



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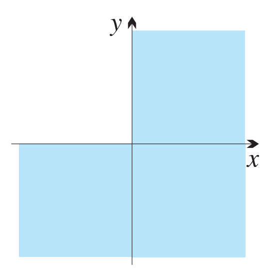
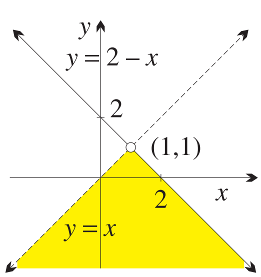


10 Write down intersections or unions that correspond to the following regions:

a



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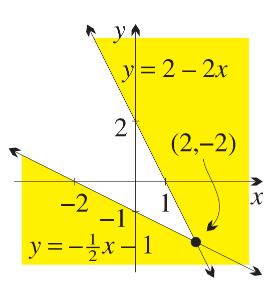
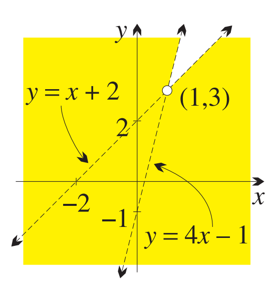


c



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11 **a** Show that the lines $y = x + 1$, $y = -\frac{1}{2}x - 2$, and $y = 4x - 2$ intersect at $(-2, -1)$, $(0, -2)$ and $(1, 2)$. Then sketch all three on the same number plane.

b Hence sketch the regions indicated by:

i $y < x + 1$ and $y \ge -\frac{1}{2}x - 2$

ii $y < x + 1$ and $y \ge -\frac{1}{2}x - 2$ and $y < 4x - 2$

iii $y > x + 1$ or $y < -\frac{1}{2}x - 2$ or $y < 4x - 2$

12 **a** Sketch the intersection of $x^2 + y^2 > 1$ and $x^2 + y^2 \le 9$.

b What is the union of these two regions?

13 **a** Sketch the union of $x^2 + y^2 \le 1$ and $y > 2 - x$.

b What is the intersection of these two regions?

14 **a** Find the intersection points of the line $y = 4 - x$ and the circle $x^2 + y^2 = 16$.

b Hence sketch

i the intersection of, and

ii the union, of $y \ge 4 - x$ and $x^2 + y^2 < 16$.

15 **a** The inequation $|x| < 2$ implies the intersection of two regions in the number plane. Write down the equations of these two regions. Hence sketch the region $|x| < 2$.

b The inequation $|x - y| \le 2$ implies the intersection of two regions. Write down the inequations of these two regions. Hence sketch $|x - y| \le 2$.

16 Sketch the region $x^2 + y^2 \ge 5$ for the domain $x > -1$ and range $y < 2$, and give the coordinates of each corner.

17 Sketch the region $y \le x^2 - 2x + 2$, where $y \ge 0$ and $0 \le x \le 2$.

18 **a** Draw the curve $y = \sqrt{x}$.

b Explain why the y-axis $x = 0$ is a boundary of the region $y < \sqrt{x}$.

c Hence sketch the region $y < \sqrt{x}$.

19 **a** Explain why $x = 0$ is a boundary of the region $y \ge \dfrac{1}{x}$.

b Hence sketch the region $y \ge \dfrac{1}{x}$.

20 Carefully sketch $y < \sqrt{4 - x^2}$, paying attention to implied boundaries.

21 Sketch the region defined by $x > |y + 1|$. (Hint: $x = |y + 1|$ is the inverse of what function?)

ENRICHMENT

22 **a** How many regions do the coordinate axes and the hyperbola $y = \dfrac{1}{x}$ divide the number plane into?

b Carefully sketch the following regions.(Hint: It may help to take test points in each of the regions found in the previous part.)

i $xy < 1$

ii $1 > \dfrac{1}{xy}$

23 Graph the regions:

a $|y| > |x|$

b $|xy| \ge 1$

c $\dfrac{1}{x} > \dfrac{1}{y}$

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3G Review of translations and reflections

This short section reviews translations and reflections in preparation for dilations.

Translations and reflections

Here are the rules from Chapter 4 of the Year 11 book.

14 A SUMMARY OF SHIFTING AND REFLECTING

Transformation	By replacement	By function rule
Shift horizontally h right	Replace x by $x - h$	$y = f(x) \to y = f(x - h)$
Shift vertically k up	Replace y by $y - k$	$y = f(x) \to y = f(x) + k$
Reflect in the y-axis	Replace x by $-x$	$y = f(x) \to y = f(-x)$
Reflect in the x-axis	Replace y by $-y$	$y = f(x) \to y = -f(x)$
Rotate 180° about O	Replace x by $-x$, y by $-y$	$y = f(x) \to y = -f(-x)$

The equation of a transformed relation can always be obtained *by replacement*, whether or not the relation is a function. The second method, *by function rule*, can only be used when the relation is a function.

Example 21 3G

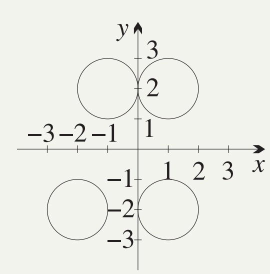
Write down the equation of the resulting graph when each transformation below is applied to the circle $(x - 1)^2 + (y + 2)^2 = 1$.

a shift left 3 units **b** reflect in the x-axis **c** rotate 180° about O

Then sketch all four circles on one set of axes.

SOLUTION

The original circle is $(x - 1)^2 + (y + 2)^2 = 1$.



a Shifting left 3 units gives $((x + 3) - 1)^2 + (y + 2)^2 = 1$

$(x + 2)^2 + (y + 2)^2 = 1$.

b Reflecting in the x-axis gives $(x - 1)^2 + (-y + 2)^2 = 1$

$(x - 1)^2 + (y - 2)^2 = 1$.

c Rotating 180° about O gives $(-x - 1)^2 + (-y + 2)^2 = 1$

$(x + 1)^2 + (y - 2)^2 = 1$.

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Example 22 3G

Write down the equation of the resulting graph when each transformation below is applied to the exponential curve $y = 2^x$.

a shift down 1 unit **b** reflect in the y-axis **c** rotate 180° about O

Then sketch all four curves on one set of axes.

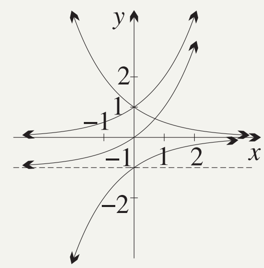
SOLUTION

The original curve is $y = 2^x$.

a Shifting down 1 unit gives $y = 2^x - 1$.

b Reflecting in the y-axis gives $y = 2^{-x}$.

c Rotating 180° about O gives $y = -2^{-x}$.



Exercise 3G FOUNDATION

1 Write down the new equation for each function or relation after the given shift has been applied. Draw a graph of the image after the shift.

a $y = x^2$: right 2 units
b $y = 2^x$: down 1 unit
c $y = x^2 - 1$: down 3 units
d $y = \frac{1}{x}$: right 3 units
e $x^2 + y^2 = 4$: up 1 unit
f $y = \log_2 x$: left 1 units
g $y = \sin x$: left $\frac{\pi}{2}$ units
h $y = \sqrt{x}$: up 2 units

2 Repeat the previous question for the given reflection line or rotation.

a x-axis **b** y-axis **c** x-axis **d** x-axis
e y-axis **f** rotate 180° **g** rotate 180° **h** y-axis

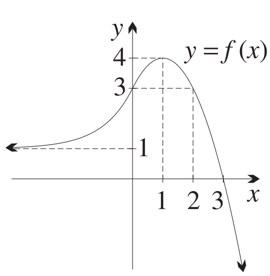
3 In which parts of Question 2 was the result the same as the original function? In each case, explain geometrically why that happened.

4 Use the shifting results and completion of the square where necessary to determine the centre and radius of each circle.

a $(x + 1)^2 + y^2 = 4$
b $(x - 1)^2 + (y - 2)^2 = 1$
c $x^2 - 4x + y^2 = 0$
d $x^2 + y^2 - 6y = 16$

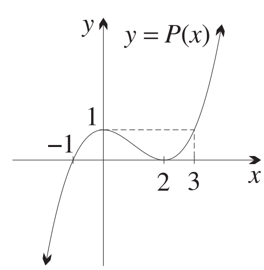
5 In each case an unknown function has been drawn. Draw the functions specified below each.

a



i $y = f(x - 2)$
ii $y = f(x) - 2$

b



i $y = P(x + 1)$
ii $y = P(x) + 1$

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6 The composition of functions can sometimes result in translations.

a Let $h(x) = x - 3$. Draw the following using the graph of $f(x)$ given in Question **5a**:

i $y = f \circ h(x)$ **ii** $y = h \circ f(x)$

b Let $k(x) = x + 2$. Draw the following using the graph of $P(x)$ given in Question **5b**:

i $y = P \circ k(x)$ **ii** $y = k \circ P(x)$

DEVELOPMENT

7 Write down the equation for each function after the given translation has been applied.

a $y = x^2$: left 1 unit, up 2 units

b $y = \frac{1}{x}$: right 2 units, up 3 units

c $y = \cos x$: right $\frac{\pi}{3}$ units, down 2 units

d $y = e^x$: left 2 units, down 1 unit

8 In each part explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each curve:

a From $y = -x$: **i** $y = -x + 2$ **ii** $y = -x - 2$ **iii** $y = x + 4$

b From $y = x^2$: **i** $y = (x + 1)^2$ **ii** $y = -(x + 1)^2$ **iii** $y = (x + 1)^2 - 1$

c From $y = \sqrt{x}$: **i** $y = \sqrt{x + 4}$ **ii** $y = -\sqrt{x}$ **iii** $y = -\sqrt{x + 4}$

d From $y = \dfrac{2}{x}$: **i** $y = \dfrac{2}{x} - 1$ **ii** $y = \dfrac{2}{x + 2} - 1$ **iii** $y = -\dfrac{2}{x}$

e From $y = \sin x$: **i** $y = \sin x - 1$ **ii** $y = \sin\left(x - \frac{\pi}{4}\right) - 1$ **iii** $y = -\sin x$

9 Answer the following questions about the cubic $y = x^3 - 3x$.

a Find the coordinates of the two points where the tangent is horizontal.

b The cubic is shifted 1 unit up.

i Write down the equation of this new cubic.

ii Show that the x-coordinates where the tangent is horizontal have not changed.

c The original cubic is shifted 1 unit left.

i Write down the equation of this third cubic, expanding any brackets.

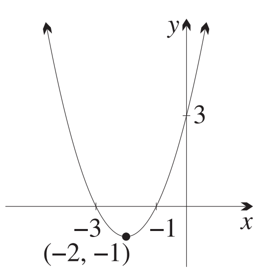
ii Show that the y-coordinates where the tangent is horizontal have not changed.

10 Complete the square then sketch each circle, stating the centre and radius. Find any intercepts with the axes by substituting $x = 0$ and $y = 0$.

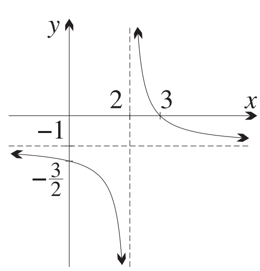
a $x^2 + 4x + y^2 - 8y = 0$ **b** $x^2 - 2x + y^2 + 4y = -1$

11 Describe each graph below as a standard curve transformed either by two shifts, or by a reflection followed by a shift. Hence write down its equation.

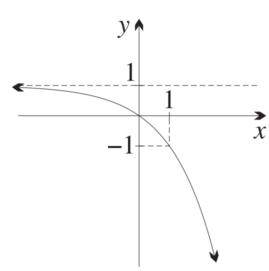
a



b



c



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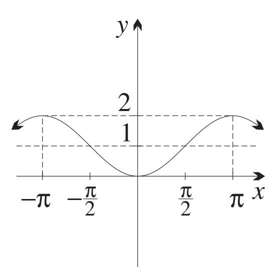

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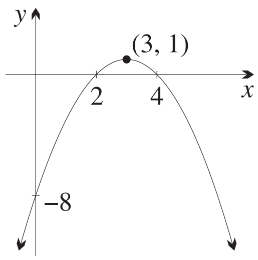
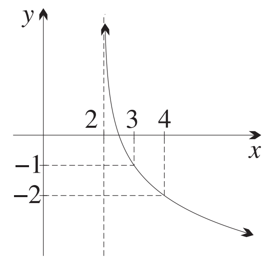
12 Describe each graph below as the given standard curve transformed by a reflection followed by two shifts, and hence write down its equation.

a



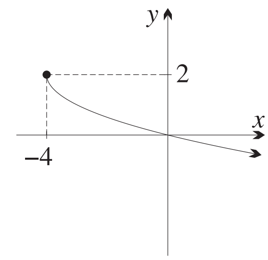
Start with $y = x^2$.

b



Start with $y = \log_2 x$.

c



Start with $y = \sqrt{x}$.

ENRICHMENT

13 **a** Let $\mathcal{I}$ be a reflection in the line $y = x$ and $\mathcal{H}$ be a reflection in the line $y = 0$.

- **i** Which functions are unchanged by applying $\mathcal{I}$, then $\mathcal{H}$, then $\mathcal{I}$, then $\mathcal{H}$? It may help to use a square piece of paper with something written on it.
- **ii** Do the transformations $\mathcal{I}$ and $\mathcal{H}$ commute? That is, if $\mathcal{I}$ is applied first and then $\mathcal{H}$, is the result the same as when $\mathcal{H}$ is applied first followed by $\mathcal{I}$?

b The graph of $y = f(x)$ is shifted left by a, reflected in the y-axis and finally shifted right by a.

- **i** What is the equation of the new graph?
- **ii** The new graph could instead be achieved by a single reflection in which line?
- **iii** Another function $y = g(x)$ undergoes the same transformation and it is noted that there is no change. What symmetry must $g(x)$ posses? Prove the result by substituting $x = a + t$.

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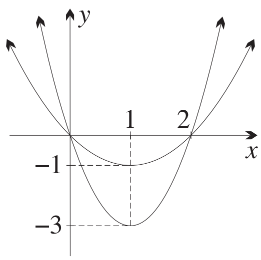
3H Dilations

A dilation is a stretch of a curve in one direction. For example, a dilation distorts a circle into an ellipse. Dilations are another kind of transformation of curves, and they belong naturally with the translations and reflections that were reviewed in the previous section. Most of the functions in the course can be reduced to very simple functions using a combination of translations, reflections and dilations.

Stretching a graph vertically

Compare the graphs of

$$y = x(x-2) \qquad \text{and} \qquad y = 3x(x-2).$$



Each value in the table below for $y = 3x(x-2)$ is three times the corresponding value in the table for $y = x(x-2)$. This means that the graph of $y = 3x(x-2)$ is obtained from the graph of $y = x(x-2)$ by *stretching away from the x-axis in the vertical* direction by a factor of 3:

x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8
$3x(x-2)$	24	9	0	-3	0	9	24

We can rewrite the equation $y = 3x(x-2)$ as $\frac{y}{3} = x(x-2)$. This makes it clear that the stretching has been obtained by replacing y by $\frac{y}{3}$.

The x-axis is the *axis of dilation*. Points on the x-axis do not move, and all other points on the graph triple their distance from the x-axis.

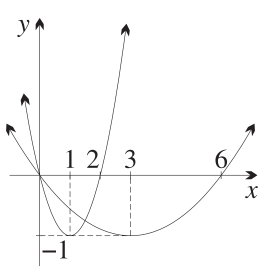
15 VERTICAL DILATIONS — STRETCHING A GRAPH VERTICALLY

- To stretch a graph in a vertical direction by a factor of a, replace y by $\dfrac{y}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = af(x)$.
- The *axis of dilation* for these transformations is the x-axis.

Stretching a graph horizontally

By analogy with the previous example, the graph of $y = x(x-2)$ can be *stretched horizontally away from the y-axis* by a factor of 3 by replacing x by $\dfrac{x}{3}$, giving the new function

$$y = \frac{x}{3}\left(\frac{x}{3} - 2\right) = \tfrac{1}{9}x(x-6).$$



Two table of values should make this clear. The first table is the original graph, the second is the new function.

x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8

x	-6	-3	0	3	6	9	12
$\frac{x}{3}\left(\frac{x}{3}-2\right)$	8	3	0	-1	0	3	8

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The y-coordinates in each table are the same, but we needed to treble the x-coordinates to produce those same y-coordinates. Thus the y-axis is the *axis of dilation*, and the *dilation factor* is 3, because the point $(0, 0)$ on the y-axis does not move, and all other points on the graph triple their distance from the y-axis.

16 HORIZONTAL DILATIONS — STRETCHING A GRAPH HORIZONTALLY

- To stretch the graph in a horizontal direction by a factor of a, replace x by $\dfrac{x}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = f\left(\dfrac{x}{a}\right)$.
- The *axis of dilation* for these transformations is the y-axis.

Example 23 3H

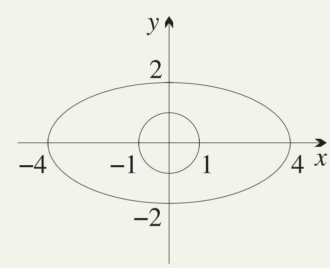
Obtain the graph of $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$.

SOLUTION

The equation can be rewritten as

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1,$$

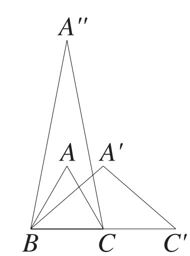
which is the unit circle stretched vertically by a factor of 2 and horizontally by a factor of 4.



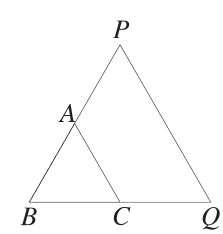
Note: Any curve of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is called an *ellipse*. It can be obtained from the unit circle $x^2 + y^2 = 1$ by stretching horizontally by a factor of a and vertically by a factor of b, so that its x-intercepts are a and $-a$ and its y-intercepts are b and $-b$.

Enlargements

The dilation of a figure is usually not similar to the original. For example, the equilateral triangle ABC in the figure to the right with its base on the x-axis is stretched to the squat isosceles triangle $A'BC'$ by a horizontal dilation with factor 2, and it is stretched to the skinny isosceles triangle $A''BC$ by a vertical dilation with factor 3.



But if two dilations with the same factor 2, one horizontal and the other vertical, are applied in order to the equilateral triangle ABC — the order does not matter — the result is the similar equilateral triangle PBQ. Such a combined transformation is called an *enlargement*, and the factor 2 is called the *enlargement factor* or *similarity factor*.



In the coordinate plane, the *centre* of an enlargement is normally taken as the origin.

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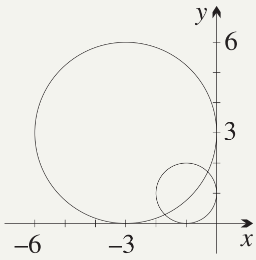
17 ENLARGMENTS

- An enlargement of a figure is similar to the original. In particular, matching angles are equal, and the ratios of matching lengths are equal.
- The composition of two dilations with the same factor, one horizontal and one vertical, is an *enlargement* with centre the origin.
- To apply an enlargement with factor a, replace x by $\dfrac{x}{a}$ and y by $\dfrac{y}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = af\left(\dfrac{x}{a}\right)$.

Example 24 3H

Apply an enlargement with centre the origin and factor 3 to the circle $(x + 1)^2 + (y - 1)^2 = 1$. Write down the new function, then sketch both curves.



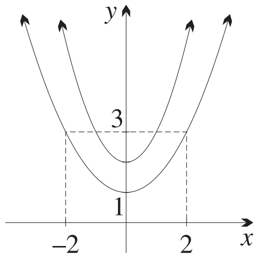
SOLUTION

The new function is $\left(\dfrac{x}{3} + 1\right)^2 + \left(\dfrac{y}{3} - 1\right)^2 = 1$

$\boxed{\times\ 3^2 = 9}$ $\qquad (x + 3)^2 + (y - 3)^2 = 9.$

The two circles are sketched to the right.

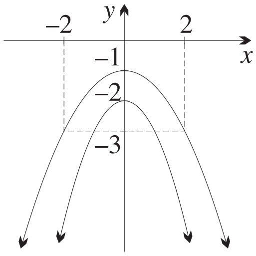
Stretching with a fractional or negative factor



In the upper diagram to the right, a vertical dilation with factor $\frac{1}{2}$ has been applied to the parabola $y = x^2 + 2$ to yield the parabola

$$\frac{y}{\frac{1}{2}} = x^2 + 2, \qquad \text{that is} \qquad y = \tfrac{1}{2}x^2 + 1.$$

The result is a compression, but we still call it a dilation.



In the lower diagram to the right, vertical dilations with factors -1 and $-\frac{1}{2}$ have been applied to the same parabola $y = x^2 + 2$. The results are the parabolas

$$y = -x^2 - 2 \qquad \text{and} \qquad y = -\tfrac{1}{2}x^2 - 1.$$

The first parabola is the reflection of the original in the x-axis. The second parabola is the reflection in the x-axis of the compressed image.

When the dilation factor is negative, the dilation can be thought of as a dilation with positive factor followed by a reflection.

In particular, a reflection is a dilation with factor -1.

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18 DILATIONS WITH A FRACTIONAL OR NEGATIVE FACTOR

- If the dilation factor is between 0 and 1, the graph is compressed.
- If the dilation factor is negative, the dilation is the composition of a dilation with positive factor and a reflection — the order does not matter.
- In particular:
  — A reflection is a dilation with factor -1.
  — A rotation of 180° about the origin is an enlargement with factor -1, and is often called a *reflection in the origin.*

Example 25 3H

Write down the new functions when each dilation is applied to the parabola $y = (x - 3)(x - 5)$. Then sketch the four curves on one set of axes.

a A horizontal dilation with factor -2.

b A horizontal dilation with factor $-\frac{1}{2}$.

c A vertical dilation with factor -1.

SOLUTION

a Replacing x by $\dfrac{x}{-2}$,

$y = (-\frac{x}{2} - 3)(-\frac{x}{2} - 5)$

$y = \frac{1}{4}(x + 6)(x + 10)$

b Replacing x by $\dfrac{x}{-1/2} = -2x$,

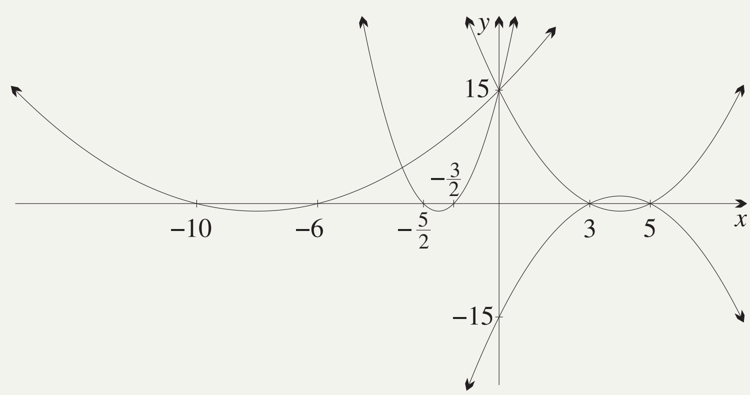
$y = (-2x - 3)(-2x - 5)$

$y = 4(x + 1\frac{1}{2})(x + 2\frac{1}{2})$

c Replacing y by $\dfrac{y}{-1} = -y$,

$-y = (x - 3)(x - 5)$

$y = -(x - 3)(x - 5)$



Note: The word 'dilation' is often used to mean 'enlargement', but in this course, it means a stretching in just one direction. Be careful when looking at other sources.

Exercise 3H FOUNDATION

1 Write down the new equation for each function or relation after the given dilation has been applied. Draw a graph of the image after the shift.

a $y = x^2$: horizontally by $\frac{1}{2}$

b $y = 2^x$: vertically by 2

c $y = x^2 - 1$: vertically by -1

d $y = \frac{1}{x}$: horizontally by 2

e $x^2 + y^2 = 4$: vertically by $\frac{1}{3}$

f $y = \log_2 x$: horizontally by -1

g $y = \sin x$: horizontally by $\frac{1}{2}$

h $y = \sqrt{x}$: vertically by -2

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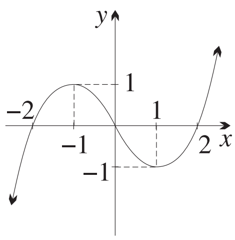
2 In each case an unknown function has been drawn. Use dilations to draw the new functions indicated beneath each.

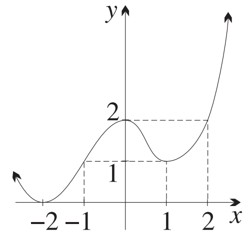
a $y = f(x)$



i $y = f(2x)$

ii $y = 2f(x)$

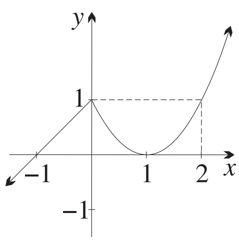
b $y = P(x)$



i $y = P\left(\frac{x}{2}\right)$

ii $y = \frac{1}{2}P(x)$

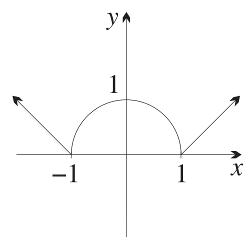
c $y = h(x)$



i $\frac{y}{2} = h(x)$

ii $y = h\left(\frac{x}{2}\right)$

d $y = g(x)$



i $2y = g(x)$

ii $y = g(2x)$

3 Sketch $x + y = 1$. Then explain how each graph below may be obtained by dilations of the first graph (there may be more than one answer), and sketch it.

a $\frac{x}{2} + y = 1$

b $\frac{x}{2} + \frac{y}{4} = 1$

c $2x + y = 1$

4 **a** The circle $(x - 3)^2 + y^2 = 4$ is enlarged by factor $\frac{1}{3}$ with centre the origin. Write down the new equation and draw both circles on the one set of axes.

b The hyperbola $y = \dfrac{1}{x}$ is enlarged by factor $\sqrt{3}$ with centre the origin. Write down the new equation and draw both hyperbolae on the one set of axes.

5 In each case graph the three given equations on one set of axes by using dilations:

a $y = x(4 + x)$, $y = 2x(4 + x)$, and $y = \frac{x}{2}(4 + \frac{x}{2})$.

b $x^2 + y^2 = 36$, $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 36$, and $(2x)^2 + (3y)^2 = 36$.

6 The composition of functions can sometimes result in dilations.

a Let $k(x) = 3x$. Draw the following using the graph of $h(x)$ given in Question 2c:

i $y = h \circ k(x)$ **ii** $y = k \circ h(x)$

b Let $\ell(x) = \frac{1}{3}x$. Draw the following using the graph of $g(x)$ given in Question 2d:

i $y = g \circ \ell(x)$ **ii** $y = \ell \circ g(x)$

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DEVELOPMENT

7 Sketch each group of three trigonometric functions on the one set of axes.

a $y = \sin x$, $y = 3\sin x$, $y = 3\sin 2x$ **b** $y = \cos x$, $y = \cos\frac{x}{2}$, $y = 2\cos\frac{x}{2}$

8 Answer the following questions about the cubic $y = x^3 - 3x$.

a Find the coordinates of the two points where the tangent is horizontal.

b The cubic is dilated vertically by factor 2.

i Write down the equation of this new cubic.

ii Show that the x-coordinates where the tangent is horizontal have not changed.

c The original cubic is dilated horizontally by factor 3.

i Write down the equation of this third cubic.

ii Show that the y-coordinates where the tangent is horizontal have not changed.

9 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = 3x^2 - 6x$ **b** $y = \dfrac{1}{x-4}$ and $y = \dfrac{1}{2x-4}$

c $y = \cos x$ and $y = \cos\frac{x}{4}$ **d** $y = \dfrac{1}{x+1}$ and $y = \dfrac{2}{x+1}$

10 Consider the hyperbola $y = \dfrac{1}{x}$.

a The hyperbola is stretched horizontally by factor 2. Write down its equation.

b The original hyperbola is stretched vertically by factor 2. Write down its equation.

c What do you notice about the answers to parts **a** and **b**?

d Can the hyperbolae in parts a or b be achieved by an enlargement?

e Investigate whether there are any other functions which exhibit similar behaviour.

11 Consider the parabola $y = x^2$.

a The parabola is dilated horizontally by factor $\frac{1}{2}$. Write down its equation.

b The original parabola is dilated vertically by factor 4. Write down its equation.

c What do you notice about the answers to parts **a** and **b**?

d Can the parabolas in parts a or b be achieved by an enlargement?

e Investigate whether there are any other functions which exhibit similar behaviour.

12 The mass M grams of a certain radioactive substance after t years is modeled by the formula $M = 3 \times 2^{-\frac{1}{53}t}$.

a Find the initial mass.

b Find the time taken for the mass to halve, called the *half-life*.

c Suppose now that the initial mass is doubled.

i Explain this in terms of a dilation and hence write down the new equation for M.

ii Show that the dilation does not change the value of the half-life.

13 Show that the equation $y = mx$ of a straight line through the origin is unchanged by any enlargement with centre the origin.

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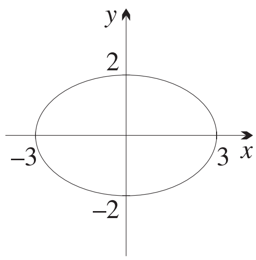
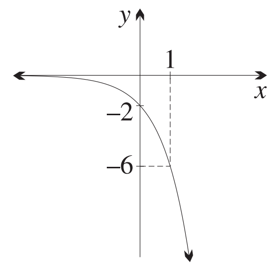
14 Describe each graph below as a standard curve transformed by dilations, and hence write down its equation.

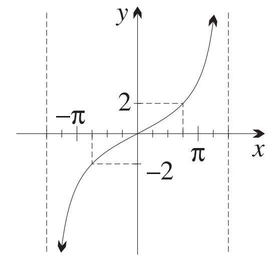
a



b



c



ENRICHMENT

15 **a** For each pair of curves, suggest two simple and distinct transformations by which the second equation may be obtained from the first:

i $y = 2^x,\ y = 2^{x+1}$ **ii** $y = \dfrac{1}{x},\ y = \dfrac{k^2}{x}$ **iii** $y = 3^x,\ y = 3^{-x}$

b Investigate other combinations of curves and transformations with similar ambiguity.

16 The parabola $y = x^2$ is stretched horizontally by factor a. Clearly a horizontal stretch by factor $\frac{1}{a}$ will restore the original parabola. What other stretch will produce a new parabola which appears identical to the original parabola $y = x^2$?

17 Determine how the curve $y = x^3 - x$ must be transformed in order to obtain the graph of $y = x^3 - 3x$. (Hint: Only stretchings are involved.)

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3I Combinations of transformations

We will now apply several transformations to a graph one after the other.

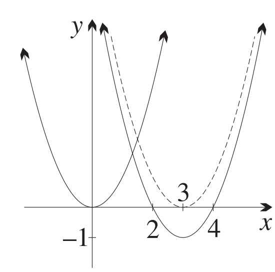
In this section we will mostly regard reflections in the axes as dilations with factor -1, and of course a rotation of 180° about the origin is the composition of two reflections. This reduces the types of transformations to just four — two translations and two dilations:

19 A SUMMARY OF TRANSFORMATIONS

Transformation	**By replacement**	**By function rule**
Shift horizontally right h	Replace x by $x - h$	$y = f(x) \rightarrow y = f(x - h)$
Shift vertically up k	Replace y by $y - k$	$y = f(x) \rightarrow y = f(x) + k$
Stretch horizontally factor a	Replace x by $\dfrac{x}{a}$	$y = f(x) \rightarrow y = f\left(\dfrac{x}{a}\right)$
Stretch vertically factor b	Replace y by $\dfrac{y}{b}$	$y = f(x) \rightarrow y = b\,f(x)$

We shall see that it sometimes matters in which order the two transformations are applied. Two transformations are said to *commute* if the order in which they are applied does not matter, whatever graph they are applied to.

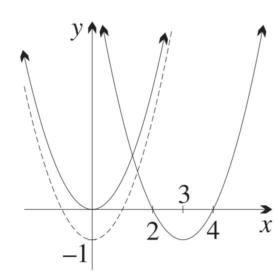
Two translations always commute



Suppose that the parabolic graph $y = x^2$ is shifted right 3 and then down 1.

$$\begin{aligned} \text{Shifting right 3,} \quad && y &= (x-3)^2, \\ \text{and shifting down 1,} \quad && y + 1 &= (x-3)^2 \\ && y &= (x-3)^2 - 1. \end{aligned}$$

The result is exactly the same if the graph $y = x^2$ is shifted down 1 and then right 3.



$$\begin{aligned} \text{Shifting down 1,} \quad && y + 1 &= x^2 \\ && y &= x^2 - 1, \\ \text{and shifting right 3,} \quad && y &= (x-3)^2 - 1. \end{aligned}$$

Thus the two translations commute.
In general, any two translations commute.

Two dilations always commute

Suppose that the circle graph $x^2 + y^2 = 1$ is stretched vertically with factor 2 and then horizontally with factor 3.

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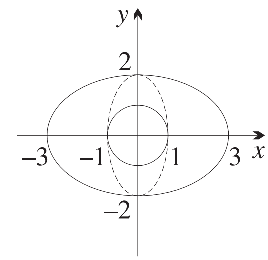


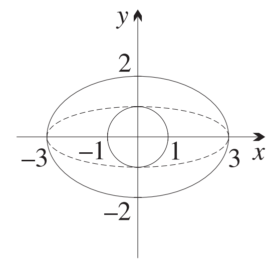
Stretching vertically factor 2, $x^2 + \left(\dfrac{y}{2}\right)^2 = 1$

$$x^2 + \frac{y^2}{4} = 1,$$

and stretching horizontally factor 3, $\left(\dfrac{x}{3}\right)^2 + \dfrac{y^2}{4} = 1$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The result is the same if the graph is stretched horizontally with factor 3 and then vertically with factor 2.



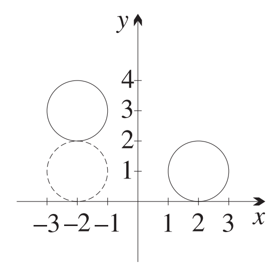
Stretching horizontally factor 3, $\dfrac{x^2}{9} + y^2 = 1,$

and stretching vertically factor 2, $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

Thus the two dilations commute, and in general, any two dilations commute.

A horizontal dilation and a vertical translation commute

Apply a reflection in the y-axis (horizontal dilation with factor -1), then shift up 2, successively to the circle $(x-2)^2 + (y-1)^2 = 1$.



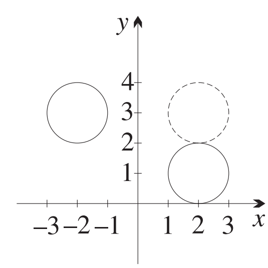
Reflecting in the y-axis, $(-x-2)^2 + (y-1)^2 = 1$

$$(x+2)^2 + (y-1)^2 = 1,$$

and shifting up 2, $(x+2)^2 + (y-2-1)^2 = 1$

$$(x+2)^2 + (y-3)^2 = 1.$$

The resulting circle $(x+2)^2 + (y-3)^2 = 1$ is the same if the transformations are done in the reverse order.



Shifting up 2, $(x-2)^2 + (y-3)^2 = 1,$

and reflecting in the y-axis, $(-x-2)^2 + (y-3)^2 = 1$

$$(x+2)^2 + (y-3)^2 = 1.$$

Thus the two transformations commute. In general, any horizontal dilation and any vertical translation commute. Similarly, any vertical dilation and any horizontal translation commute.

20 COMMUTING TRANSFORMATIONS

- Any two translations commute.
- Any two dilations commute (including reflections).
- A translation and a dilation commute if one is vertical and the other horizontal.

Transformations that do not commute

In the remaining case, the transformations do not commute. That is, a translation and a dilation do not commute when they are both horizontal or both vertical. The next worked examples give two examples of this.

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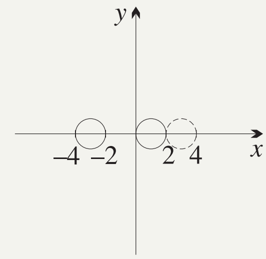
Example 26 3I

a The reflection in the y-axis (horizontal dilation with factor -1) and the translation left 2 are applied successively to the circle $(x+3)^2+y^2=1$. Find the equation of the resulting graph, and sketch it.

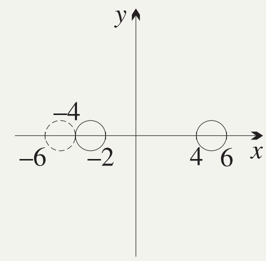
b Repeat when the transformations are done in the reverse order.

SOLUTION

a Applying the reflection,
$$(-x+3)^2+y^2=1$$
$$(x-3)^2+y^2=1,$$
and applying the translation,
$$(x+2-3)^2+y^2=1$$
$$(x-1)^2+y^2=1.$$



b Applying the translation,
$$(x+2+3)^2+y^2=1$$
$$(x+5)^2+y^2=1,$$
and applying the reflection,
$$(-x+5)^2+y^2=1$$
$$(x-5)^2+y^2=1.$$



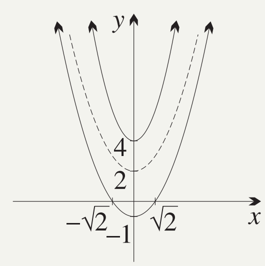
Example 27 3I

a The vertical dilation with factor $\frac{1}{2}$ and the translation down 3 units are applied successively to the parabola $y=x^2+4$. Find the equation of the resulting graph, and sketch it.

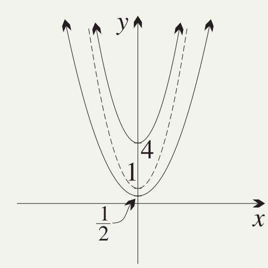
b Repeat when the transformations are done in the reverse order.

SOLUTION

a Applying the dilation,
$$y=\tfrac{1}{2}(x^2+4)$$
$$y=\tfrac{1}{2}x^2+2,$$
and applying the translation,
$$y=\tfrac{1}{2}x^2+2-3$$
$$y=\tfrac{1}{2}x^2-1.$$



b Applying the translation,
$$y=x^2+4-3$$
$$y=x^2+1,$$
and applying the dilation,
$$y=\tfrac{1}{2}(x^2+1)$$
$$y=\tfrac{1}{2}x^2+\tfrac{1}{2}.$$



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21 TRANSFORMATIONS THAT DO NOT COMMUTE

- A vertical translation and a vertical dilation do not commute.
- A horizontal translation and a horizontal dilation do not commute.

A universal formula involving all four transformations

When the graph is a function, there is a universal formula that allows the four transformations to be applied to any function $y = f(x)$. The formula is:

$$y = kf\big(a(x + b)\big) + c.$$

This formula is useful because it applies all four transformations to any function $y = f(x)$. It is useful for trigonometric functions, and for computer programs. The formula is tricky to use, however, and although readers must know the formula, most problems should be done using the methods already presented.

Here is how to analyse the successive transformations involved in this formula.

Start with	$y = f(x)$.
Stretching horizontally with factor $\frac{1}{a}$ gives	$y = f(ax)$.
Shifting left b gives	$y = f\big(a(x + b)\big)$
Stretching vertically with factor k gives	$y = kf\big(a(x + b)\big)$
Shifting up c gives	$y = kf\big(a(x + b)\big) + c$

Another way to analyse this formula is to rewrite it progressively so that the four successive transformations can be seen:

$$y = kf\big(a(x + b)\big) + c$$

$$\boxed{-c}\quad y - c = kf\big(a(x + b)\big)$$

$$\boxed{\div k}\quad \frac{y - c}{k} = f\big(a(x + b)\big)$$

$$\frac{y - c}{k} = f\left(\frac{x + b}{1/a}\right)$$

$$\frac{y - c}{k} = f\left(\frac{x - (-b)}{1/a}\right)$$

22 A UNIVERSAL FORMULA INVOLVING ALL FOUR TRANSFORMATIONS

The following sequence of transformations transforms the function $y = f(x)$ to

$$y = kf\big(a(x + b)\big) + c.$$

1. Stretch horizontally with factor $1/a$.
2. Shift left b.
3. Stretch vertically with factor k.
4. Shift up c.

Alternatively, the vertical dilation and translation (step 3 then step 4) could be done before the horizontal dilation and translation (steps 1 then step 2).

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Exercise 3I

FOUNDATION

1 Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.

a i The parabola is shifted right 1 unit. Sketch the situation and find its equation, expanding any brackets.

ii The parabola in part i is then shifted up 2 units. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is translated up 2 units. Sketch the result and find its equation.

ii The parabola in part i is then translated 1 unit right. Sketch the situation and find its equation, expanding any brackets.

c Parts a and b used the same translations, 1 unit right and 2 units up, but in a different order. Do these transformations commute?

2 As in Question 1, start with the parabola $y = x^2 - 2x$.

a i The parabola is dilated by factor 2 horizontally. Sketch the situation and find its equation.

ii The parabola in part i is then dilated by factor 3 vertically. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is stretched by 3 vertically. Sketch the result and find its equation.

ii The parabola in part i is then stretched by 2 horizontally. Sketch the situation and find its equation.

c Parts a and b used the same dilations, factor 2 horizontally and factor 3 vertically, but in a different order. Do these transformations commute?

3 Once again, start with the parabola $y = x^2 - 2x$.

a i The parabola is dilated by factor 2 horizontally. Sketch the situation and find its equation.

ii The parabola in part i is then translated 1 unit up. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is shifted 1 unit up. Sketch the result and find its equation.

ii The parabola in part i is then stretched by 2 horizontally. Sketch the situation and find its equation.

c Parts a and b used the same transformations, stretch by factor 2 horizontally and shift 1 unit up, but in a different order. Do these transformations commute?

4 Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.

a i The parabola is shifted right by 1 unit. Sketch the situation and find its equation, expanding any brackets.

ii The shifted parabola is then reflected in the y-axis. Sketch the new graph and find its equation.

b i The original parabola $y = x^2 - 2x$ is reflected in the y-axis. Sketch the result and find its equation.

ii The reflected parabola is then shifted 1 unit right. Sketch the situation and find its equation, expanding any brackets.

c Parts a and b used a shift 1 unit right and reflection in the y-axis, but in a different order. Do these transformations commute?

DEVELOPMENT

5 Which of the following pairs of transformations commute?

a reflection in the y-axis and horizontal translation

b vertical dilation and vertical translation

c vertical dilation and reflection in the x-axis

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d horizontal translation and vertical translation
e horizontal dilation and horizontal translation
f reflection in the x-axis and horizontal translation

6 Write down the new equation for each function or relation after the given transformations have been applied. Draw a graph of the image.
a $y = x^2$: right 2 units, then dilate by factor $\frac{1}{2}$ horizontally
b $y = 2^x$: down 1 unit then reflect in the y-axis
c $y = x^2 - 1$: down 3 units, then dilate by factor -1 vertically
d $y = \frac{1}{x}$: right 3 units then dilate by factor 2 vertically
e $x^2 + y^2 = 4$: up 2 units then dilate by factor $\frac{1}{2}$ vertically
f $y = \log_2 x$: left 1 units then dilate by factor 2 horizontally
g $y = \sin x$: left π units then reflect in the x-axis
h $y = \sqrt{x}$: up 2 units then dilate by factor -1 horizontally

7 Identify the various transformations to help graph these trigonometric functions. Make sure the transformations are applied in the correct order when they do not commute.
a $y = \sin 2x + 1$ b $y = 2 \sin x + 1$ c $y = 2 \sin(x + \frac{\pi}{4})$ d $y = \sin(2x + \frac{\pi}{4})$

8 Determine the equation of the curve after the given transformations have been applied in the order stated.
a $y = x^2$: left 1, down 4, dilate horizontally by 2,
b $y = x^2$: down 4, dilate horizontally by 2, left 1,
c $y = 2^x$: down 1, right 1, dilate vertically by -2,
d $y = \dfrac{1}{x}$: right 2, dilate by 2 vertically, up 1.

9 The parabola $y = (x - 1)^2$ is shifted 2 left and then reflected in the y-axis.
a Show that the new parabola has the same equation.
b Investigate why this has happened.

ENRICHMENT

10 Identify the transformations of these trigonometric functions and hence sketch them.
a $y = 3 \cos 2x + 1$ b $y = 2 \cos\left(x - \frac{\pi}{3}\right) + 2$
c $y = \cos\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 1$ d $y = \cos\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 1$

11 Sketch these transformed trigonometric functions.
a $y = 2 \sin\left(\frac{1}{2}x + \frac{\pi}{6}\right) + 1$ b $y = 3 \cos\left(2x + \frac{\pi}{4}\right) - 1$ c $y = 1 - \cos\left(2x - \frac{\pi}{3}\right)$

12 a Let H be a horizontal translation by a units and V be a vertical translation by b units. Show that H and V commute. That is, show that for all functions $y = f(x)$ the result of applying H followed by V is the same as applying V followed by H.
b Let E be a horizontal dilation by factor a and U be a vertical dilation by factor b. Show that E and U commute.
c Let F be a reflection in the y-axis and L be a reflection in the x-axis. Show that F and L commute.
d Now suppose that any two of the above transformations are applied in succession. Which pairs of transformations commute and why?

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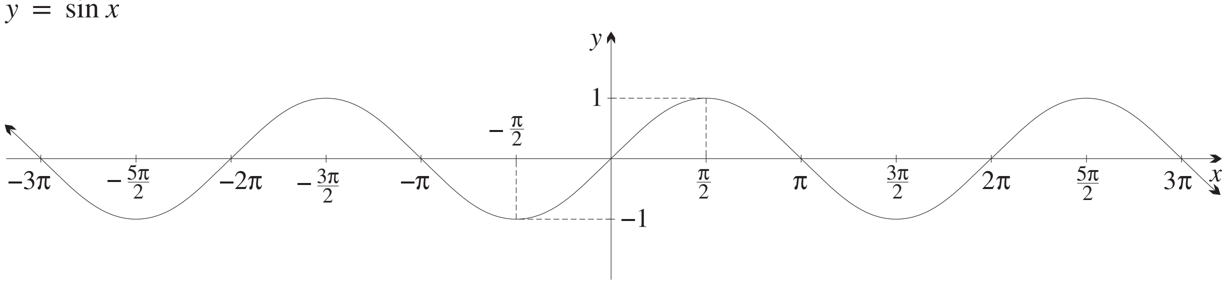
3J Trigonometric graphs

In Sections 11G–11J of the Year 11 book, we developed *radians* as a way of measuring angles. An angle measured in radians is a pure number, without units, and the important conversions between radians and the old familiar degrees are:

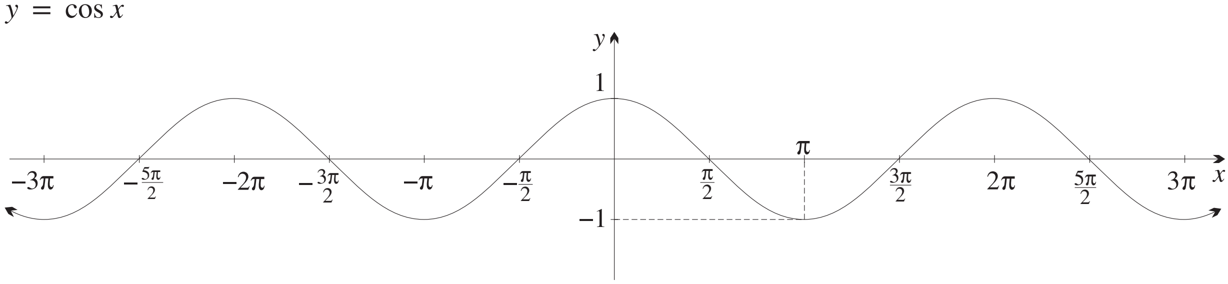
$$2\pi = 360°, \quad \pi = 180°, \quad \tfrac{\pi}{2} = 90°, \quad \tfrac{\pi}{3} = 60°, \quad \tfrac{\pi}{4} = 45°, \quad \tfrac{\pi}{6} = 30°.$$

In the final Section 11J we drew all six trigonometric graphs in radians. This present section deals only with $\sin x$, $\cos x$ and $\tan x$ — these three graphs are:

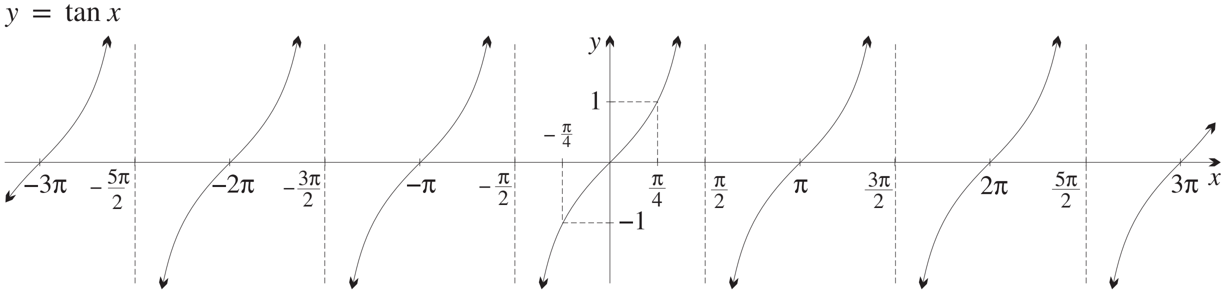
$y = \sin x$



$y = \cos x$



$y = \tan x$



We remarked in Section 11J of the Year 11 book that the trigonometric graphs were now drawn in the form appropriate for calculus. In particular, we shall prove in Chapter 7 that all three curves above have gradient 1 or -1 at all their x-intercepts.

Transformations are our concern here, however. The investigation Exercise 11J in the Year 11 book dealt thoroughly with the symmetries of these three graphs under translations, reflections in the axes, and rotations about the origin. We now have dilations, and this section shows how to generate any basic wave graph by combinations of translations and dilations.

Reflections and rotations do not need review, so all the dilations in this section have positive factors. We will consider separately, then in combination:

- Vertical dilations with positive factors — this leads to the *amplitude*.
- Horizontal dilations with positive factors — this leads to the *period*.
- Translations left and right — this leads to the *phase*.
- Translations up and down — this leads to the *mean value*.

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Vertical dilations and amplitude

The *amplitude* of a wave is the maximum height of the wave above its mean position. The graphs on the previous page both show that $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0 (the average of 1 and -1). Thus both have amplitude 1.

Now let us apply a vertical dilation with factor a to $y = \sin x$.

Replacing y by $\dfrac{y}{a}$ gives $\dfrac{y}{a} = \sin x$,

and multiplying by a, $\quad y = a \sin x$.

This function is also a wave, but its amplitude is now a, because it has maximum value $y = a$, minimum value $y = -a$, and mean value $y = 0$.

Exactly the same argument applies to $y = \cos x$.

23 VERTICAL DILATIONS AND AMPLITUDE

- The *amplitude* of a wave is the maximum height of the wave above its mean position.
- $y = \sin x$ and $y = \cos x$ both have amplitude 1.
- $y = a \sin x$ and $y = a \cos x$ both have amplitude a.
- $y = a \sin x$ and $y = a \cos x$ are the results of stretching $y = \sin x$ or $y = \cos x$ vertically with factor a.

We can stretch the function $y = \tan x$ vertically to $y = a \tan x$ in the usual way. But the function increases without bound near its asymptotes, so the idea of amplitude makes no sense. Instead, we can conveniently tie down the vertical scale of $y = a \tan x$ by using the fact that $\tan \frac{\pi}{4} = 1$, so when $x = \frac{\pi}{4}$, $y = a$.

Horizontal dilations and period

The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the previous page are waves, with a pattern that repeats every revolution. Thus they both have period 2π.

The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π.

24 THE PERIODS OF THE TRIGONOMETRIC FUNCTIONS

- The *period* of a function that repeats is the length of the smallest repeating unit.
- $y = \sin x$ and $y = \cos x$ have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

Now consider the function $y = \sin nx$.

We can write this as $\qquad y = \sin \dfrac{x}{1/n}$,

which shows that it is a horizontal dilation of $y = \sin x$ with factor $\dfrac{1}{n}$.

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Because $y = \sin x$ has period 2π, the dilation $y = \sin nx$ therefore has period $\dfrac{2\pi}{n}$.

The same arguments apply to $y = \cos nx$ and $y = \tan nx$.

25 HORIZONTAL DILATIONS AND PERIOD

- $y = \sin nx$ and $y = \cos nx$ have period $\dfrac{2\pi}{n}$.
- $y = \tan nx$ has period $\dfrac{\pi}{n}$.
- All three functions are the results of stretching $y = \sin x$, $y = \cos x$ or $y = \tan x$ horizontally with factor $\dfrac{1}{n}$.

The next worked example examines the amplitude and period together.

Example 28 — 3J

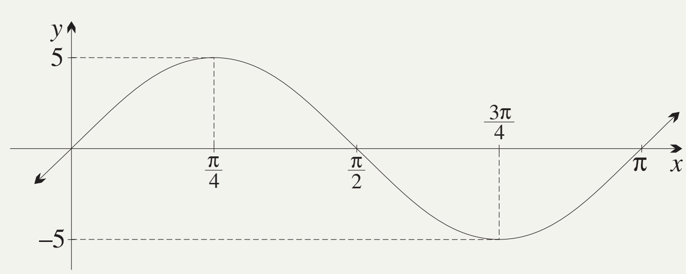
Find the period and amplitude of:

a $y = 5 \sin 2x$ **b** $y = 2 \tan \frac{1}{3}x$

Then sketch one period of the function, showing all intercepts, turning points and asymptotes.

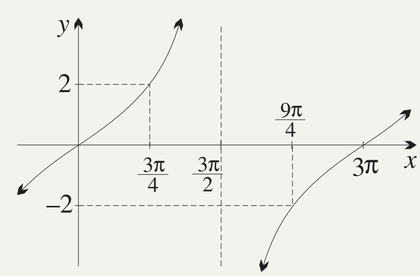
SOLUTION

a $y = 5 \sin 2x$ has an amplitude of 5, and a period of $\dfrac{2\pi}{2} = \pi$.



b $y = 2 \tan \frac{1}{3}x$ has period $\dfrac{\pi}{1/3} = 3\pi$.

It has no amplitude, but when $x = \frac{3\pi}{4}$, $y = 2 \tan \frac{\pi}{4} = 2$.



Horizontal translations and phase

The *initial phase angle*, or simply *phase*, of a trigonometric function is the angle when $x = 0$. Thus a function such as $y = \sin\left(x + \frac{\pi}{3}\right)$ has phase $\frac{\pi}{3}$, and $y = \sin x$ itself has phase 0.

Let us apply a translation left by α to the function $\quad y = \sin x$.
Replacing x by $x - (-\alpha) = x + \alpha$ gives $\quad y = \sin(x + \alpha)$,

which is a sine wave with phase α, because when $x = 0$, the angle is $0 + \alpha = \alpha$.

The same argument applies to $y = \cos x$ and $y = \tan x$.

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26 HORIZONTAL TRANSLATIONS AND PHASE

- The phase of a trigonometric function is the angle when $x = 0$.
- $y = \sin(x + \alpha)$, $y = \cos(x + \alpha)$ and $y = \tan(x + \alpha)$ all have phase α.
- All three functions are the result of shifting $y = \sin x$, $y = \cos x$ or $y = \tan x$ left by α.

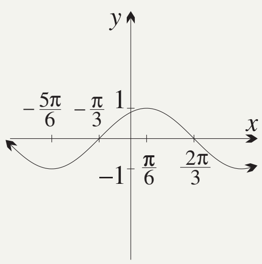
Example 29 3J

Use horizontal translations to sketch these functions, and state their phase.

a $y = \sin\left(x + \frac{\pi}{3}\right)$ **b** $y = \cos\left(x - \frac{5\pi}{2}\right)$ **c** $y = \tan\left(x + \frac{\pi}{4}\right)$

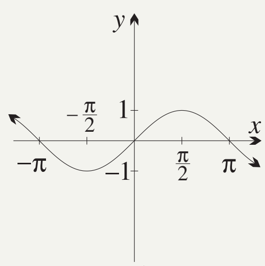
SOLUTION

a



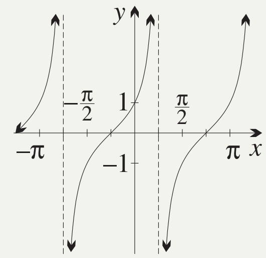
Shift left $\frac{\pi}{3}$, phase $\frac{\pi}{3}$

b



Shift right $\frac{5\pi}{2}$, phase $-\frac{5\pi}{2}$

c



Shift left $\frac{\pi}{4}$, phase $-\frac{\pi}{4}$

Note: The phase is not uniquely defined, because we can add and subtract multiples of the period. For example:

- In part **b**, it may be more convenient to write the phase as $-\frac{5\pi}{2} + 2\pi = -\frac{\pi}{2}$, or $-\frac{5\pi}{2} + 4\pi = \frac{3\pi}{2}$.
- In part **a**, we could also say that the phase is $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$.
- In part **c**, we could also say that the phase is $\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.

Combining period and phase

The two dilations and one translation that we have introduced into trigonometry so far all commute, except only that a horizontal dilation and a horizontal translation do not commute — this needs attention. Consider the function

$$y = \sin\left(2x + \tfrac{\pi}{3}\right) \qquad \text{or equivalently} \qquad y = \sin 2\left(x + \tfrac{\pi}{6}\right).$$

The period is $\dfrac{2\pi}{2} = \pi$. The phase is $0 + \frac{\pi}{3} = \frac{\pi}{3}$, or equivalently $2\left(0 + \frac{\pi}{6}\right) = \frac{\pi}{3}$.

The first form $y = \sin(2x + \frac{\pi}{3})$ of the equation regards the function as $y = \sin x$,

- shifted left $\frac{\pi}{3}$, giving $y = \sin\left(x + \frac{\pi}{3}\right)$,
- then stretched horizontally with factor $\frac{1}{2}$, giving $y = \sin\left(2x + \frac{\pi}{3}\right)$.

The second form $y = \sin 2(x + \frac{\pi}{6})$ of the equation regards it as $y = \sin x$,

- stretched horizontally with factor $\frac{1}{2}$, giving $y = \sin 2x$,
- then shifted left $\frac{\pi}{6}$, giving $y = \sin 2(x + \frac{\pi}{6})$.

The second way is what is suggested in the formula at the end of Section 3I, but either approach gets the result. In both cases, find the phase by putting $x = 0$.

Here is the general statement, but it is mostly better to work with transformations of each example individually than remember complicated formulae.

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27 COMBINING PERIOD AND PHASE

- The function $y = \sin(nx + \alpha)$ has period $\dfrac{2\pi}{n}$ and phase α.
- Written as $y = \sin(nx + \alpha)$ it suggests transforming $y = \sin x$ by
  — a shift left α, followed by a horizontal stretch with factor $\dfrac{1}{n}$.
- Written as $y = \sin n\left(x + \dfrac{\alpha}{n}\right)$ it suggests transforming $y = \sin x$ by
  — a horizontal stretch with factor $\dfrac{1}{n}$, followed by a shift left by $\dfrac{\alpha}{n}$.

Vertical translations and the mean value

If there is a vertical translation, do it last so that it does not get confused with the vertical stretch associated with the amplitude a vertical dilation and a vertical translation do not commute. A vertical translation shifts the mean value of the wave from 0 to some other value.

28 VERTICAL TRANSLATIONS AND THE MEAN VALUE

- The *mean value* of a wave is the mean of its maximum and minimum values.
- $y = \sin x + c$ and $y = \cos x + c$ both have mean value c.
- $y = \sin x + c$ is the result of shifting $y = \sin x$ up c.
- In a combination of transformations, do any vertical translation last.

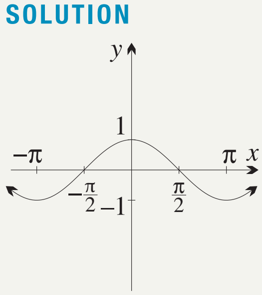
The function $y = \tan x + c$ is not a wave, and has no mean value.

The next worked example below puts all four transformations together.

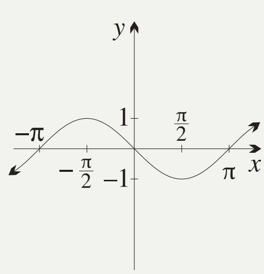
Example 30 3J

Use four successive transformations to sketch $y = 3\cos\left(2x + \frac{\pi}{2}\right) - 2$, and specify the amplitude, period, phase and mean value (and ignore x-intercepts this time).

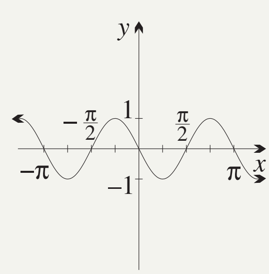
SOLUTION



1 Start with $y = \cos x$.



2 Shift $y = \cos x$ left $\frac{\pi}{2}$, giving $y = \cos\left(x + \frac{\pi}{2}\right)$.
Phase is now $0 + \frac{\pi}{2} = \frac{\pi}{2}$.



3 Stretch horizontally factor $\frac{1}{2}$, giving $y = \cos\left(2x + \frac{\pi}{2}\right)$.
The period is now $\frac{2\pi}{2} = \pi$.

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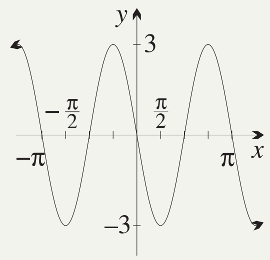
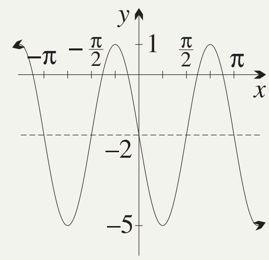


4 Then stretch vertically with factor 3, giving $y = 3\cos(2x + \frac{\pi}{2})$.
The amplitude is now 3.



5 Shift the whole thing down 2 units, giving $y = 3\cos\left(2x + \frac{\pi}{2}\right) - 2$.
The mean value is now -2.

Alternatively, rewrite the function as $y = 3\cos 2\left(x + \frac{\pi}{4}\right) - 2$. This suggests that the first two transformations are now:
- Stretch horizontally with factor $\frac{1}{2}$.
- Then shift left $\frac{\pi}{4}$.

Oddness and evenness of the trigonometric functions

Box 29 quickly reviews the oddness and evenness of the sine, cosine and tangent functions. These are crucially important properties of the three functions.

29 ODDNESS AND EVENNESS OF THE TRIGONOMETRIC FUNCTIONS

- The functions $y = \sin x$ and $y = \tan x$ are odd functions. Thus:
  $$\sin(-x) = -\sin x \qquad \text{and} \qquad \tan(-x) = -\tan x$$
  and they both have point symmetry in the origin.
- The function $y = \cos x$ is an even function. Thus:
  $$\cos(-x) = \cos x$$
  and it has line symmetry in the y-axis.

Graphical solutions of trigonometric equations

Many trigonometric equations cannot be solved by algebraic methods. Approximation methods using the graphs can usually be used instead and a graph-paper sketch will show:
- how many solutions there are, and
- the approximate values of the solutions.

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Example 31 3J

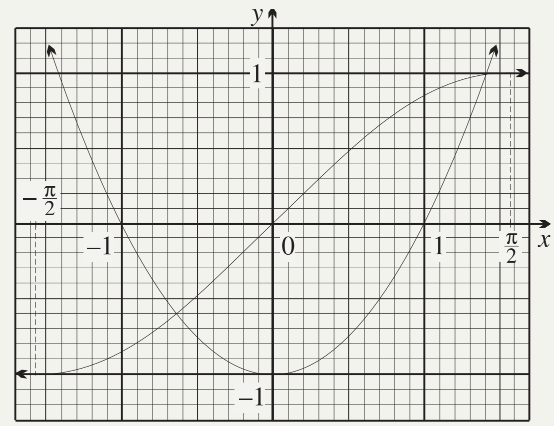
a Find, by drawing a graph, the number of solutions of $\sin x = x^2 - 1$.

b Then use the graph to find approximations correct to one decimal place.

SOLUTION

a Here are $y = \sin x$ and $y = x^2 - 1$.
Clearly the equation has two solutions.

b The positive solution is $x \doteqdot 1.4$,
and the negative solution is $x \doteqdot -0.6$.



Note: Technology is particularly useful here. It allows sketches to be drawn quickly, and many programs will give the approximate coordinates of the intersections.

Exercise 3J

FOUNDATION

Technology: Computer sketching can provide experience of a large number of graphs similar to the ones listed in this exercise. In particular, it is very useful in making clear the importance of period and amplitude and in reinforcing the formulae for them.

1 a Sketch the graph of each function for $0 \le x \le 2\pi$, stating the amplitude in each case.
  i $y = \frac{1}{2}\sin x$ **ii** $y = 2\sin x$ **iii** $y = 3\sin x$

b Describe the transformation from $y = \sin x$ to $y = k\sin x$. assume that k is positive.)

c How does the graph of $y = k\sin x$ change as k increases?

2 a Sketch the graph of each function for $0 \le x \le 2\pi$, and state the period in each case.
  i $y = \cos\frac{1}{2}x$ **ii** $y = \cos 2x$ **iii** $y = \cos 3x$

b Describe the transformation from $y = \cos x$ to $y = \cos nx$. (Assume that n is positive.)

c How does the graph of $y = \cos nx$ change as n increases?

3 a Sketch the graph of each function for $0 \le x \le 2\pi$, and state the period in each case.
  i $y = \tan x$ **ii** $y = \tan\frac{1}{2}x$ **iii** $y = \tan 2x$

b Describe the transformation from $y = \tan x$ to $y = \tan ax$. (Assume that a is positive.)

c How does the graph of $y = \tan ax$ change as a increases?

4 a Sketch the graph of each function for $0 \le x \le 2\pi$, and state the phase in each case.
  i $y = \sin\left(x + \frac{\pi}{2}\right)$ **ii** $y = \sin(x + \pi)$ **iii** $y = \sin(x + 2\pi)$

b Describe the transformation from $y = \sin x$ to $y = \sin(x + \alpha)$. (Assume that α is positive.)

c Describe the transformation when α is a multiple of 2π.

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5 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the mean value and the range in each case.

i $y = \cos x + 1$ **ii** $y = \cos x + 2$ **iii** $y = \cos x + \frac{1}{2}$

b Describe the transformation from $y = \cos x$ to $y = \cos x + c$. (Assume that c is positive.)

c How does the graph of $y = \cos x + c$ change as c increases?

DEVELOPMENT

6 State the amplitude and period of each function, then sketch its graph for $-\pi \le x \le \pi$.

a $y = 3\cos 2x$ **b** $y = 2\sin\frac{1}{2}x$

c $y = \tan\frac{3x}{2}$ **d** $y = 2\cos 3x$

7 Write down a sequence of transformations that will transform $y = \sin x$ to the given function, and hence sketch the given function for $0 \le x \le 2\pi$.

a $y = 3\sin 3x$ **b** $y = -2\sin\frac{x}{2}$ **c** $y = 3\sin\left(x - \frac{\pi}{2}\right) + 2$

8 Write down a sequence of transformations that will transform $y = \cos x$ to the given function, and hence sketch the given function for $-\pi \le x \le \pi$.

a $y = 5\cos\frac{1}{2}x$ **b** $y = -2\cos 2x - 2$ **c** $y = \cos\left(2\left(x - \frac{\pi}{2}\right)\right)$

9 Write down a sequence of transformations that will transform $y = \sin x$ to the given function.

a $y = \sin\left(3x + \frac{\pi}{2}\right)$ **b** $y = \frac{1}{4}\sin(4x - \pi) - 4$ **c** $y = -6\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$

10 **a** What is the period and phase of each function in Question 9?

b What are the period and phase of these functions?

i $y = 3\sin 2\left(x - \frac{\pi}{3}\right)$ **ii** $y = \frac{5}{2}\cos\frac{1}{3}(x + \pi)$ **iii** $y = 2\tan 3\left(x + \frac{\pi}{8}\right)$

11 Solve each equation, for $0 \le x \le 2\pi$. Then indicate the solutions on a diagram showing sketches of the functions on the LHS and RHS of the equation.

a $2\sin\left(x - \frac{\pi}{3}\right) = 1$ **b** $2\cos 2x = -1$

12 Solve each equation, for $0 \le x \le \pi$, giving solutions correct to 3 decimal places.

a $\cos(x + 0.2) = -0.3$ **b** $\tan 2x = 0.5$

13 **a** Find the vertex of the parabola $y = x^2 - 2x + 4$.

b Hence show graphically that $x^2 - 2x + 4 > 3\sin x$ for all real values of x.

14 **a** Sketch the graph of $y = 2\cos x$ for $-2\pi \le x \le 2\pi$.

b On the same diagram, carefully sketch the line $y = 1 - \frac{1}{2}x$, showing its x- and y-intercepts.

c How many solutions does the equation $2\cos x = 1 - \frac{1}{2}x$ have?

d Mark with the letter P the point on the diagram from which the negative solution of the equation in part **c** is obtained.

e Prove algebraically that if n is a solution of the equation in part **c**, then $-2 \le n \le 6$.

15 **a** What is the period of the function $y = \sin\frac{\pi}{2}x$?

b Sketch the curve $y = 1 + \sin\frac{\pi}{2}x$, for $0 \le x \le 4$.

c Through what fixed point does the line $y = mx$ always pass for varying values of m?

d By considering possible points of intersection of the graphs of $y = 1 + \sin\frac{\pi}{2}x$ and $y = mx$, find the values of m for which the equation $\sin\frac{\pi}{2}x = mx - 1$ has exactly one real solution in the domain $0 \le x \le 4$.

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16 The depth of water in Dolphin Bay varies according to the tides. The depth is modelled by the equation $x = 2\cos\left(\frac{\pi}{7}t\right) + 8$, where x metres is the depth and t hours is the time since the last high tide. Last Saturday, it was high tide at 7 am.

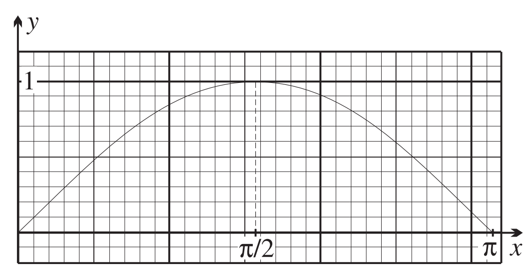
- **a** How deep is the bay at high tide?
- **b** How deep is the bay at low tide?
- **c** When did the first low tide after 7 am occur?
- **d** At what time last Saturday morning was the depth 9 metres?

17 **a** Sketch $y = 3\sin 2x$ and $y = 4\cos 2x$ on the same diagram, for $-\pi \le x \le \pi$.

- **b** Hence sketch the graph of $y = 3\sin 2x - 4\cos 2x$ on the same diagram, for $-\pi \le x \le \pi$.
- **c** Estimate the amplitude of the graph sketched in part **b**.

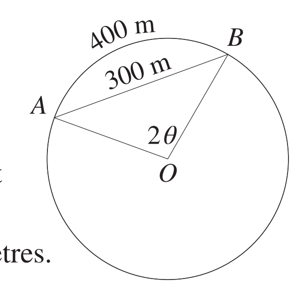
ENRICHMENT

18 **a** **i** Photocopy this graph of $y = \sin x$, for $0 \le x \le \pi$, and on it graph the line $y = \frac{3}{4}x$.



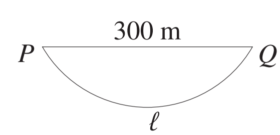
- **ii** Measure the gradient of $y = \sin x$ at the origin.
- **iii** For what values of k does $\sin x = kx$ have a solution, for $0 < x < \pi$?

b The diagram shows points A and B on a circle with centre O, where $\angle AOB = 2\theta$, chord AB has length 300 metres, and the minor arc AB has length 400 metres.



- **i** Show that $\sin\theta = \dfrac{3}{4}\theta$.
- **ii** Use the graph from part **a i** to determine θ, correct to one decimal place.
- **iii** Hence find $\angle\, AOB$ in radians, correct to one decimal place, and show that the radius of the circle is about 154 metres.

c P and Q are two points 300 metres apart. The circular arc PQ has length ℓ metres.



- **i** If C is the centre of the arc and $\angle PCQ = 2\alpha$, show that $\sin\alpha = \dfrac{300\alpha}{\ell}$.
- **ii** Use your answer to part **a iii** to find the possible range of values of ℓ.

19 Consider the equation $\dfrac{1}{1 + \cos x} = \dfrac{2x}{\pi}$.

- **a** Show that $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$ satisfy the equation.
- **b** On the same diagram, sketch $y = \dfrac{2x}{\pi}$ and $y = \dfrac{1}{1 + \cos x}$ for $0 \le x < \pi$.
- **c** Deduce that $4x\cos^2 \frac{1}{2}x > \pi$, for $\frac{\pi}{3} < x < \frac{\pi}{2}$.

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Chapter 3 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

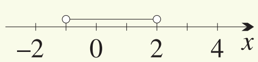
Chapter 3 Multiple-choice quiz

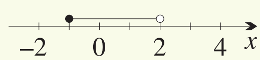
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

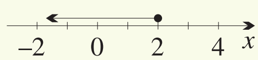
Chapter review exercise

1 For each number line, write the graphed interval using:

i inequality notation, **ii** bracket notation.

a 

b 

c 

2 If $f(x) = x^2 - 1$ and $g(x) = x + 1$, find:

a **i** $f \circ g(-2)$ **ii** $g \circ f(-2)$ **iii** $f \circ f(-2)$ **iv** $g \circ g(-2)$

b **i** $f \circ g(x)$ **iii** $g \circ f(x)$ **iv** $f \circ f(x)$ **v** $g \circ g(x)$

3 Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator, and taking the limit *as* $x \to \infty$ *and as* $x \to -\infty$.

a $f(x) = \dfrac{1}{x + 2}$ **b** $f(x) = \dfrac{x - 3}{2x + 5}$ **c** $f(x) = \dfrac{x}{x^2 + 1}$

4 Let $y = x^3 - 9x^2 + 18x$.

a State the domain using inequality interval notation.

b Write down the coordinates of any intercepts with the axes.

c Does this function have any asymptotes?

d Use this information and a table of values to sketch the curve.

e The graph seems to be horizontal somewhere in the interval $0 < x < 3$, and again in the interval $3 < x < 6$. Use calculus to find the x-coordinates of these points, and add them to the diagram.

5 Solve each double inequation, then write your answer in bracket interval notation.

a $-6 < -3x \le 12$ **b** $-2 < 2x + 1 < 1$

c $-7 \le 5 + 4x < 7$ **d** $-4 \le 1 - \frac{1}{2}x \le 3$

6 Carefully draw the graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them.

a $x - 2 = \log_2 x$

b $\cos x = 1 - x^2$

c $x(x - 2)(x + 2) = 2 - |x|$

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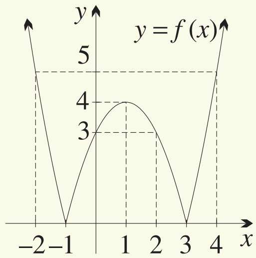
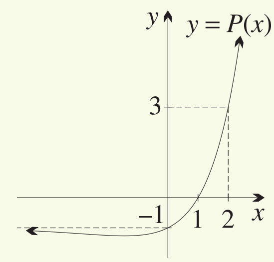
Review

7 In each case an unknown function has been drawn. Draw the functions specified below it.

a



b



i $y = f(x - 1)$ **ii** $y = f(x) + 1$

i $y = P(x + 1)$ **ii** $y = P(x) - 1$

8 In each case apply the indicated dilation to the corresponding function in Question **7** and draw the resulting graph.

a **i** $y = f\left(\frac{1}{2}x\right)$ **ii** $y = \frac{1}{2}f(x)$

b **i** $y = P(2x)$ **ii** $y = 2P(x)$

9 In each case, completely factor the given polynomial where necessary and hence sketch its graph. A table of values may also help. Then use the graph to solve $f(x) \leq 0$.

a $f(x) = (x + 1)(x - 3)$ **b** $f(x) = x(x - 2)(x + 1)$ **c** $f(x) = x^2 - 4x - 5$

d $f(x) = 3 - 2x - x^2$ **e** $f(x) = 2x - x^2 - x^3$ **f** $f(x) = x^3 + 4x^2 + 4x$

10 Let $y = \dfrac{4}{(x + 2)(2 - x)}$.

a State the natural domain. **b** Find the y-intercept.

c Show that $y = 0$ is a horizontal asymptote.

d Draw up a table of values.

e Identify the vertical asymptotes, and use the table of values to describe its behaviour near them.

f Sketch the graph of the function and state its range using bracket interval notation.

11 **a** Factor the right-hand side of $y = \dfrac{3x + 3}{x^2 + 2x - 3}$.

b State the domain and any intercepts with the axes.

c Explain why the function is neither even nor odd. (Hint: The answers to part **a** may help.)

d Find the equations of the asymptotes.

e Sketch the graph of this curve.

12 Solve these absolute value equations and inequations algebraically.

a $|2x| = 7$ **b** $|3x - 2| = 1$ **c** $|3x + 5| \leq 4$ **d** $|6x + 7| > 5$

13 Carefully sketch the functions on the LHS and RHS of each inequation on the same number plane. Then use the graph to solve the inequations.

a $x - 1 \geq 1 + \frac{1}{2}x$ **b** $\dfrac{1}{1 - x} > 1 - 2x$

c $|2x| \leq x + 3$ **d** $\left|\frac{1}{2}x + 1\right| > \frac{1}{4}(x + 5)$

14 **a** Find the points where $y = x^2 - 2x + 1$ intersects $y = 1 + 4x - x^2$.

b Hence sketch the region where both $y \geq x^2 + 2x + 1$ and $y \leq 1 + 4x - x^2$.

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Review

15 Write down the equation for each function after the given translations have been applied.

a $y = x^2$: right 2 units, up 1 unit

b $y = \frac{1}{x}$: left 2 units, down 3 units

c $y = \sin x$: left $\frac{\pi}{6}$ units, down 1 unit

d $y = e^x$: right 2 units, up 1 unit

16 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = \frac{1}{4}x^2 - x$

b $y = \dfrac{1}{x - 4}$ and $y = \dfrac{1}{2x - 8}$

c $y = \cos x$ and $y = \frac{1}{3}\cos x$

d $y = \dfrac{1}{x + 1}$ and $y = \dfrac{2}{x + 2}$

17 Which of the following pairs of transformations commute?

a reflection in the y-axis and reflection in the x-axis,

b vertical reflection and vertical translation,

c horizontal translation and horizontal dilation,

d vertical translation and horizontal dilation.

18 Identify the various transformations of the standard functions and hence graph each. Make sure the transformations are applied in the correct order when they do not commute.

a $y = 4 - 2^x$

b $y = \dfrac{1}{2}(x - 2)^2 - 1$

c $y = 2\sin\left(x + \frac{\pi}{6}\right) + 1$

19 Write down the amplitude and period, then sketch the graph for $-\pi \le x \le \pi$.

a $y = 4\sin 2x$

b $y = \frac{3}{2}\cos\frac{1}{2}x$

20 **a** Explain how the graph of $y = \tan x$ can be transformed into the graph of $y = 1 - \tan x$

b Hence sketch $y = 1 - \tan x$ for $-\pi \le x \le \pi$.

21 Write down a sequence of transformations that will transform $y = \cos x$ into:

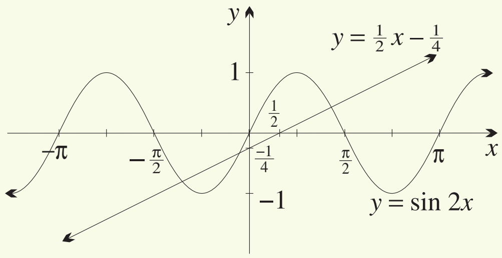
a $y = 3\cos(-x) - 2$

b $y = 4\cos\left(4\left(x + \frac{\pi}{2}\right)\right)$

c $y = \cos\left(2x - \frac{\pi}{3}\right)$

22 What is the phase of each function in Question **21**?

23 In the given diagram, the curve $y = \sin 2x$ is graphed for $-\pi \le x \le \pi$, and the line $y = \frac{1}{2}x - \frac{1}{4}$ is graphed.



a In how many points does the line $y = \frac{1}{2}x - \frac{1}{4}$ meet the curve $y = \sin 2x$?

b State the number of solutions of the equation $\sin 2x = \frac{1}{2}x - \frac{1}{4}$. How many of these solutions are positive?

c Briefly explain why the line $y = \frac{1}{2}x - \frac{1}{4}$ will not meet the curve $y = \sin 2x$ outside the domain $-\pi \le x \le \pi$.

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