9 Motion and rates

Anyone watching objects in motion can see that they often make patterns with a striking simplicity and predictability. These patterns are related to the simplest objects in geometry and arithmetic. A thrown ball traces out a parabolic path. A cork bobbing in flowing water traces out a sine wave. A rolling billiard ball moves in a straight line, rebounding symmetrically off the table edge. The stars and planets move in more complicated, but highly predictable, paths across the sky. The relationship between physics and mathematics, logically and historically, begins with these and many similar observations.

The first three sections of this chapter, however, only begin to introduce the relationship between calculus and motion. Because this is a mathematics course, not a physics course, our attention will not be on the nature of space and time, but on the striking alternative interpretations that the physical world brings to the first and second derivatives. The first derivative of displacement is velocity, which we can see. The second derivative is acceleration, which we can feel.

Motion is just one example of a rate. We have met rates briefly several times in Year 11 — Section 8G (Applications of these functions — exponential rates), Section 9J (Rates of change — using the derivative), Sections 11F and 16B–16C (Exponential growth and decay) and Section 16A (Related rates). The last three sections of this chapter unify and extend examples of rates in general, using now a much larger array of functions. Related rates are reviewed here, but not exponential growth because that will be revisited in Chapter 13: Differential equations. Rates also provide the context to clarify the ideas of increasing and concave up in an interval, rather than at a point as in Chapter 4.

The examples of motion and rates in this chapter also provide models of the linear, quadratic, exponential and trigonometric functions, because all these functions can be brought into play at once.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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9A Average velocity and speed

This first section sets up the mathematical description of motion in one dimension, using a function to describe the relationship between time and the position of an object in motion. Average velocity is the gradient of the chord on this displacement–time graph. This will lead, in the next section, to the description of instantaneous velocity as the gradient of a tangent.

Motion in one dimension

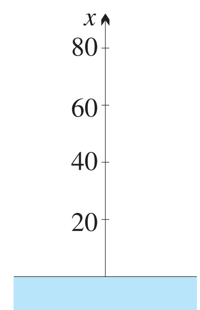
When a particle is moving in one dimension (meaning along a line) its position is varying over time. That position can be specified at any time t by a single number x, called the *displacement*, and the whole motion can be described by giving x as a function of the *time* t.

Suppose, for example, that a ball is hit vertically upwards from ground level and lands 8 seconds later in the same place. Its motion can be described approximately by the following quadratic equation and table of values,

$$x = 5t(8 - t)$$

t	0	2	4	6	8
x	0	60	80	60	0

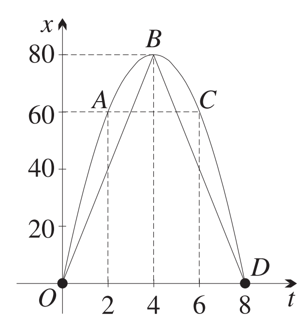
Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line.



This vertical line has been made into a number line, with the ground as the origin, upwards as the positive direction, and metres as the units of displacement.

Time has also become a number line. The origin of time is when the ball is thrown, and the units of time are seconds.

The graph to the right is the resulting graph of the equation of motion $x = 5t(8 - t)$. The horizontal axis is time and the vertical axis is displacement — the graph must not be mistaken as a picture of the ball's path.



The graph is a section of a parabola with vertex at $(4, 80)$, which means that the ball achieves a maximum height of 80 metres after 4 seconds. When $t = 8$, the height is zero, and the ball strikes the ground again. The equation of motion therefore has quite restricted domain and range,

$$0 \le t \le 8 \qquad \text{and} \qquad 0 \le x \le 80.$$

Most equations of motion have this sort of restriction on the domain of t. In particular, *it is a convention of this course that negative values of time are excluded unless the question specifically allows it.*

1 MOTION IN ONE DIMENSION

- Motion in one dimension is specified by giving the displacement x on the number line as a function of time t after time zero.
- Negative values of time are excluded unless otherwise stated.

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Example 1 9A

Consider the example above, where $x = 5t(8 - t)$.

a Find the height of the ball after 1 second.

b At what other time is the ball at this same height above the ground?

SOLUTION

a When $t = 1$, $x = 5 \times 1 \times 7$
$= 35.$

Hence the ball is 35 metres above the ground after 1 second.

b To find when the height is 35 metres, solve the equation $x = 35$.
Substituting into $x = 5t(8 - t)$ gives

$$\begin{aligned} 5t(8 - t) &= 35 \\ \boxed{\div 5} \quad t(8 - t) &= 7 \\ 8t - t^2 - 7 &= 0 \\ \boxed{\times (-1)} \quad t^2 - 8t + 7 &= 0 \\ (t - 1)(t - 7) &= 0 \\ t &= 1 \text{ or } 7. \end{aligned}$$

Hence the ball is 35 metres high after 1 second and again after 7 seconds.

Average velocity

During its ascent, the ball in the worked example above moved 80 metres upwards. This is a change in displacement of +80 metres in 4 seconds, giving an average velocity of 20 metres per second.

Average velocity thus equals the gradient of the chord OB on the displacement–time graph (be careful, because there are different scales on the two axes). Hence the formula for average velocity is the familiar gradient formula.

2 AVERAGE VELOCITY

Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$. Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

That is, on the displacement–time graph,

$$\text{average velocity} = \text{gradient of the chord.}$$

During its descent, the ball moved 80 metres downwards in 4 seconds, which is a change in displacement of $0 - 80 = -80$ metres. The average velocity is therefore -20 metres per second, which is equal to the gradient of the chord BD.

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Example 2 9A

Consider again the example $x = 5t(8 - t)$. Find the average velocities of the ball:

a during the first second,

b during the fifth second.

SOLUTION

The first second stretches from $t = 0$ to $t = 1$ and the fifth second stretches from $t = 4$ to $t = 5$. The displacements at these times are given in the table to the right.

t	0	1	4	5
x	0	35	80	75

a Average velocity during 1st second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{35 - 0}{1 - 0}$$
$$= 35 \text{ m/s.}$$

b Average velocity during 5th second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$
$$= \frac{75 - 80}{5 - 4}$$
$$= -5 \text{ m/s.}$$

Distance travelled

The change in displacement can be positive, negative or zero. *Distance*, however, is always positive or zero. In the previous example, the change in displacement during the 4 seconds from $t = 4$ to $t = 8$ is -80 metres, but the distance travelled is 80 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the total distance travelled by the ball is $80 + 80 = 160$ metres, even though the ball's change in displacement over the first 8 seconds is zero because the ball is back at its original position on the ground.

3 DISTANCE TRAVELLED

- The *distance travelled* takes into account any journey and return.
- Distance travelled can never be negative.

Average speed

The *average speed* is the distance travelled divided by the time taken. Average speed, unlike average velocity, can never be negative.

4 AVERAGE SPEED

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Average speed can never be negative.

During the 8 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 160 metres, so

$$\text{average velocity} = \frac{0 - 0}{8 - 0} = 0 \text{ m/s,} \qquad \text{average speed} = \frac{160}{8} = 20 \text{ m/s.}$$

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Example 3 9A

Find the average velocity and the average speed of the ball:

a during the eighth second,

b during the last six seconds.

SOLUTION

The eighth second stretches from $t = 7$ to $t = 8$ and the last six seconds stretch from $t = 2$ to $t = 8$. The displacements at these times are given in the table to the right.

t	0	2	7	8
x	0	60	35	0

a During the eighth second, the ball moves 35 metres down from $x = 35$ to $x = 0$.

$$\begin{aligned}\text{Hence}\quad \text{average velocity} &= \frac{0-35}{8-7}\\ &= -35\ \text{m/s}.\\ \text{Also}\quad \text{distance travelled} &= 35\ \text{metres,}\\ \text{so}\quad \text{average speed} &= 35\ \text{m/s}.\end{aligned}$$

b During the last six seconds, the ball rises 20 metres from $x = 60$ to $x = 80$, and then falls 80 metres from $x = 80$ to $x = 0$.

$$\begin{aligned}\text{Hence}\quad \text{average velocity} &= \frac{0-60}{8-2}\\ &= -10\ \text{m/s}.\\ \text{Also}\quad \text{distance travelled} &= 20 + 80\\ &= 100\ \text{metres,}\\ \text{so}\quad \text{average speed} &= \frac{100}{6}\\ &= 16\tfrac{2}{3}\ \text{m/s}.\end{aligned}$$

Exercise 9A

FOUNDATION

1 A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

t	0	1	2	3
x				

a Copy and complete the table of values to the right.

b Hence find the average velocity:

i during the first second,

ii during the first two seconds,

iii during the first three seconds,

iv during the third second.

c Use the table of values above to sketch the displacement–time graph. Then add the chords corresponding to the average velocities calculated in part **b**.

2 A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

t	0	1	2	3	4
x					

a Copy and complete the table of values to the right.

b Hence sketch the displacement–time graph.

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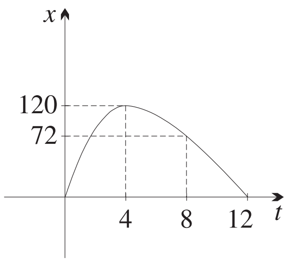
c Find the total distance travelled during the first 4 seconds. Then find the average speed during this time.

d Find the average velocity during the time:

i from $t = 0$ to $t = 2$, **ii** from $t = 2$ to $t = 4$, **iii** from $t = 0$ to $t = 4$.

e Add to your graph the chords corresponding to the average velocities in part **d**.

3 A piece of cardboard is shot 120 metres vertically into the air by an explosion and floats back to the ground, landing at the same place. The graph to the right gives its height x metres above the ground t seconds after the explosion.



a Copy and complete the following table of values.

t	0	4	8	12
x				

b What is the total distance travelled by the cardboard?

c Find the average speed of the cardboard during its travels, using the formula

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

d Find the average velocity during:

i the ascent, **ii** the descent, **iii** the full 12 seconds.

DEVELOPMENT

4 Michael the mailman rides his bicycle 1 km up a hill at a constant speed of 10 km/hr. He then turns around and rides back down the hill at a constant speed of 30 km/hr.

a How many minutes does he take to travel:

i the first kilometre, when he is riding up the hill,

ii the second kilometre, when he is riding back down again?

b Use these values to draw a displacement–time graph, with the time axis in minutes.

c What is his average speed over the total 2 km journey?

d What is the average of his speeds up and down the hill?

5 Sadie the snail is crawling up a 6-metre-high wall. She takes an hour to crawl up 3 metres, then falls asleep for an hour and slides down 2 metres, repeating the cycle until she reaches the top of the wall. Let x be Sadie's height in metres after t hours.

a Copy and complete the table of values of Sadie's height up the wall.

t	0	1	2	3	4	5	6	7
x								

b Hence sketch the displacement–time graph.

c How long does Sadie take to reach the top?

d What total distance does she travel, and what is her average speed?

e What is her average velocity over this whole time?

f Which places on the wall does she visit exactly three times?

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6 A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance x is in centimetres and time t is in seconds.

a Copy and complete the table of values to the right, and sketch the curve.

t					
x	0	2	4	6	8

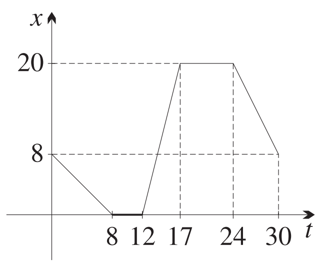
b Hence find the average velocity as the particle moves:

i from $x = 0$ to $x = 2$, **ii** from $x = 2$ to $x = 4$,

iii from $x = 4$ to $x = 6$, **iv** from $x = 0$ to $x = 6$.

c What does the equality of the answers to parts **ii** and **iv** of part **b** tell you about the corresponding chords in part **c**?

7 Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x in metres from the front gate after t seconds.



a What is her average velocity:

i during the first 8 seconds,

ii while she is driving forwards,

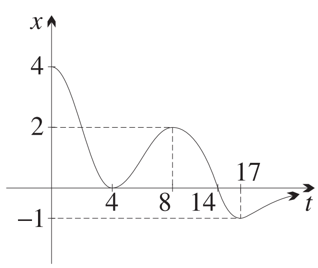
iii while she is reversing the second time?

b Find the total distance she travelled, and her average speed, over the 30 seconds.

c Find her change in displacement, and her average velocity, over the 30 seconds.

d What would her average speed have been if she had not paused at the gate and at the garage?

8 A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The graph shows the height x in metres of the weight above the water as a function of time t seconds after she first drops it.



a How many times is the weight:

i at $x = 3$, **ii** at $x = 1$, **iii** at $x = -\frac{1}{2}$?

b At what times is the weight:

i at the water surface, **ii** above the water surface?

c How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?

d What is the weight's greatest depth under the water and when does it occur?

e What happens to the weight eventually?

f What is its average velocity:

i during the first 4 seconds, **ii** from $t = 4$ to $t = 8$, **iii** from $t = 8$ to $t = 17$?

g What distance does it travel:

i over the first 4 seconds, **ii** over the first 8 seconds,

iii over the first 17 seconds, **iv** eventually?

h What is its average speed over the first:

i 4, **ii** 8, **iii** 17 seconds?

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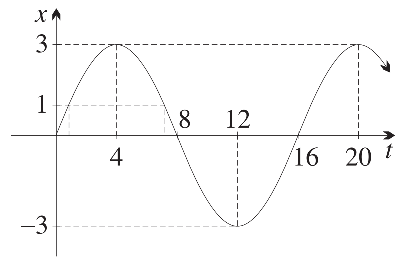
9 A particle is moving according to $x = 3\sin\frac{\pi}{8}t$, in units of centimetres and seconds. Its displacement–time graph is sketched to the right.



a Use $T = \dfrac{2\pi}{n}$ to confirm that the period is 16 seconds.

b Find the maximum and minimum values of the displacement.

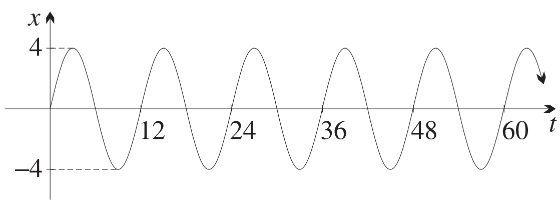
c Find the first two times when the displacement is maximum.

d Find the first two times when the particle returns to its initial position.

e When, during the first 20 seconds, is the particle on the negative side of the origin?

f Find the total distance travelled during the first 16 seconds, and the average speed.

10 A particle is moving according to the equation $x = 4\sin\frac{\pi}{6}t$, in units of metres and seconds. The graph of its displacement for the first minute is sketched to the right.



a Find the amplitude and period.

b How many times does the particle return to the origin by the end of the first minute?

c Find at what times it visits $x = 4$ during the first minute.

d Find how far it travels during the first 12 seconds, and its average speed in that time.

e Find the values of x when $t = 0$, $t = 1$ and $t = 3$. Hence show that the average speed during the first second is twice the average speed during the next 2 seconds.

11 A particle moves according to $x = 10\cos\frac{\pi}{12}t$, in units of metres and seconds.

a Find the amplitude, and use the formula $T = 2\pi/n$ to show that the period of the motion is 24 seconds.

b Sketch the displacement–time graph over the first 36 seconds.

c When, during the first 36 seconds, is the particle at the origin?

d Where is the particle initially? What is the maximum distance the particle reaches from its initial position, and when, during the 36 seconds, is it there?

e How far does the particle move during the first 36 seconds, and what is its average speed over this time?

f Use the fact that $\cos\frac{\pi}{3} = \frac{1}{2}$ to copy and complete the table of values to the right:

t	4	8	12	16	20	24
x						

g From the table, find the average velocity during the first 4 seconds, the second 4 seconds and the third 4 seconds.

h Use the graph and the table of values to find when, in the first 24 seconds, the particle is more than 15 metres from its initial position.

12 A balloon rises so that its height h in metres after t minutes is $h = 8000\left(1 - e^{-0.06t}\right)$.

a What height does it start from, and what happens to the height as $t \to \infty$?

b Copy and complete the table to the right, correct to the nearest metre.

t	0	10	20	30
x				

c Sketch the displacement–time graph of the motion.

d Find the balloon's average velocity during the first 10 minutes, the second 10 minutes and the third 10 minutes, correct to the nearest metre per minute.

e Use your calculator to show that the balloon has reached 99% of its final height after 77 minutes, but not after 76 minutes.

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13 Two engines, Thomas and Henry, move on close parallel tracks. They start at the origin, and are together again at time $t = e - 1$. Thomas' displacement–time equation, in units of metres and minutes, is $x = 300 \log(t + 1)$, and Henry's is $x = kt$, for some constant k.

a Sketch the two graphs.

b Show that $k = \dfrac{300}{e - 1}$.

c Use calculus to find the maximum distance between Henry and Thomas during the first $e - 1$ minutes, and the time when it occurs (in exact form, and then correct to the nearest metre or the nearest second).

ENRICHMENT

14 [Arithmetic mean, geometric mean and harmonic mean]

Let a and b be two positive numbers. The *arithmetic mean* and *(positive) geometric mean* of a and b are

$$\text{arithmetic mean} = \frac{a + b}{2} \qquad \text{and} \qquad \text{geometric mean} = \sqrt{ab},$$

and their *harmonic mean* is the number h such that

$$\frac{1}{h} \text{ is the arithmetic mean of } \frac{1}{a} \text{ and } \frac{1}{b}.$$

Suppose that town B lies on the road between town A and town C, and that a cyclist rides from A to B at a constant speed U, and then rides from B to C at a constant speed V.

a Prove that if town B lies midway between towns A and C, then the cyclist's average speed W over the total distance AC is the harmonic mean of U and V.

b Now suppose that the distances AB and BC are not equal.

i Show that if W is the arithmetic mean of U and V, then

$$AB : BC = U : V.$$

ii Show that if W is the geometric mean of U and V, then

$$AB : BC = \sqrt{U} : \sqrt{V}.$$

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9B Velocity and acceleration as derivatives

If I drive the 160 km from Sydney to Newcastle in 2 hours, my average velocity is 80 km per hour. But my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. Just as an average velocity corresponds to the gradient of a chord on the displacement–time graph, so an instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous velocity and speed

From now on, the words *velocity* and *speed* alone will mean instantaneous velocity and instantaneous speed.

5 INSTANTANEOUS VELOCITY AND INSTANTANEOUS SPEED

- The *instantaneous velocity* v of a particle in motion is the gradient of the tangent on the displacement–time graph,

$$v = \frac{dx}{dt} \qquad \text{which can also be written as} \qquad v = \dot{x}.$$

- The dot over any symbol means differentiation with respect to time.
- The *instantaneous speed* is the absolute value $|v|$ of the velocity.

The notation $\dot{x}$, originally introduced by Newton, is yet another way of writing the derivative. The dot over the x, or over any symbol, stands for differentiation with respect to time t. Thus the symbols v, $\dfrac{dx}{dt}$ and $\dot{x}$ are all symbols for the velocity.

Example 4 9B

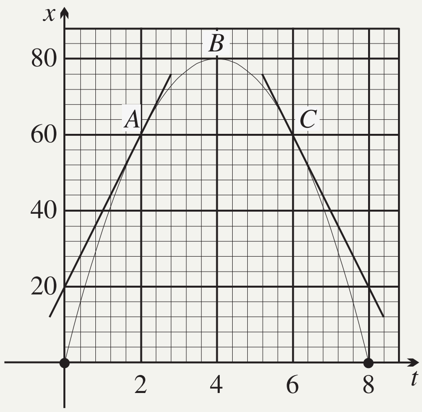
Here again is the displacement–time graph of the ball moving with equation $x = 5t(8 - t)$.

a Differentiate to find the equation for the velocity v, draw up a table of values at 2-second intervals and sketch the velocity–time graph.

b Measure the gradients of the tangents that have been drawn at A, B and C on the displacement–time graph and compare your answers with the table of values in part **a**.

c With what velocity was the ball originally hit?

d What is its impact speed when it hits the ground?



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SOLUTION

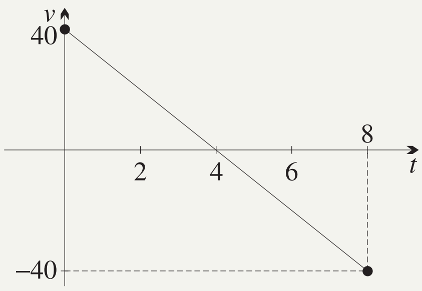
a The equation of motion is $x = 5t(8 - t)$

$x = 40t - 5t^2,$

and differentiating, $v = 40 - 10t.$

The graph of velocity is a straight line, with v-intercept 40 and gradient -10.

t	0	2	4	6	8
v	40	20	0	-20	-40



b These values agree with the measurements of the gradients of the tangents at A where $t = 2$, at B where $t = 4$, and at C where $t = 6$.
(Be careful to take account of the different scales on the two axes.)

c When $t = 0$, $v = 40$, so the ball was originally hit upwards at 40 m/s.

d When $t = 8$, $v = -40$, so the ball hits the ground again at 40 m/s.

Vector and scalar quantities

Displacement and velocity are *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards and a negative displacement would mean it was below ground level. Distance and speed, however, are *scalar quantities* — distance is the magnitude of the displacement, and speed is the magnitude of the velocity — and neither can be negative.

Finding when a particle is stationary

A particle is said to be *stationary* when its velocity v is zero, that is, when $\dfrac{dx}{dt} = 0$. This is the origin of the word 'stationary point', used in Chapter 4 and in Year 11 to describe a point on a graph where the derivative is zero. For example, the ball in the first example was stationary for an instant at the top of its flight when $t = 4$, because the velocity was zero at the instant when its motion changed from upwards to downwards.

6 FINDING WHEN A PARTICLE IS STATIONARY

- A particle is *stationary* when its velocity is zero.
- To find when a particle is stationary, put $v = 0$ and solve for t.

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Example 5 9B

A particle moves so that its distance in metres from the origin at time t seconds is given by $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

a Find the velocity and speed when $t = 4$.

b Find the times when the particle is stationary.

c Find its distance from the origin at these times.

SOLUTION

The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$,

and differentiating, $v = t^2 - 12t + 27$

$= (t - 3)(t - 9)$.

a When $t = 4$, $v = -5$,

so the velocity is -5 m/s and the speed is 5 m/s.

b When $v = 0$, $t = 3$ or 9,

so the particle is stationary after 3 seconds and again after 9 seconds.

c When $t = 3$, $x = 9 - 54 + 81 - 18$

$= 18$,

and when $t = 9$, $x = 243 - 486 + 243 - 18$

$= -18$.

Thus the particle is 18 metres from the origin on both occasions.

Acceleration as the second derivative

A particle is said to be *accelerating* if its velocity is changing. The *acceleration* of an object is defined to be the rate at which the velocity is changing. Thus acceleration a is the derivative $\frac{dv}{dt} = \dot{v}$ of the velocity with respect to time.

Velocity is already the derivative of displacement, so acceleration is the second derivative $\frac{d^2x}{dt^2} = \ddot{x}$ of displacement. The double-dot means the second derivative.

7 ACCELERATION AS A DERIVATIVE

- Acceleration is the first derivative of velocity with respect to time, and the second derivative of displacement with respect to time:

$$a = \frac{dv}{dt} = \dot{v} \qquad \text{and} \qquad a = \frac{d^2x}{dt^2} = \ddot{x}.$$

Note: Be very careful with the symbol a, because in this context a is the acceleration function, whereas elsewhere the letter a is usually used for a constant. Because of this issue, we tend to use $\ddot{x}$ for acceleration more often than a.

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Example 6 9B

Consider again the ball moving with displacement function $x = 5t(8 - t)$.

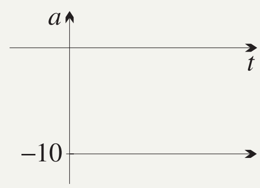
a Find the velocity function v and the acceleration function a, and sketch the acceleration function.

b Find and describe the displacement, velocity and acceleration when $t = 2$.

c State when the ball is speeding up and when it is slowing down, explaining why this can happen when the acceleration is constant.

SOLUTION

a The displacement is $x = 40t - 5t^2$.
Differentiating, $v = 40 - 10t$.
Differentiating again, $a = -10$, which is a constant.
Hence the acceleration is always $10\ \text{m/s}^2$ downwards.



b Substituting $t = 2$ into the functions x, v and a,

$$x = 60 \qquad \text{and} \qquad v = 20 \qquad \text{and} \qquad a = -10.$$

Thus when $t = 2$, the displacement is 60 metres above the ground, the velocity is 20 m/s upwards, and the acceleration is $10\ \text{m/s}^2$ downwards.

c During the first 4 seconds, the ball has positive velocity, meaning that it is rising, and the ball is slowing down by 10 m/s every second.

During the last 4 seconds, however, the ball has negative velocity, meaning that it is falling, and the ball is speeding up by 10 m/s every second.

Units of acceleration

In the previous example, the particle's velocity was decreasing by 10 m/s every second. The particle is said to be 'accelerating at −10 metres per second, per second', written in symbols as $-10\ \text{m/s}^2$ or as $-10\ \text{ms}^{-2}$.

The units of acceleration correspond with the indices of the second derivative $\dfrac{d^2x}{dt^2}$.

Acceleration would normally be regarded as a vector quantity, that is, with a direction built into it. This is why the particle's acceleration is written with a minus sign as $-10\ \text{m/s}^2$. Alternatively, one can omit the minus sign and specify the direction instead, writing '$10\ \text{m/s}^2$ in the downwards direction'.

Example 7 9B

In worked Example 5, we examined the function $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

a Find the acceleration function, and find when the acceleration is zero.

b Where is the particle at this time and what is its velocity?

SOLUTION

a The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.
Differentiating, $v = t^2 - 12t + 27$,
and differentiating again,
$$\begin{aligned} \ddot{x} &= 2t - 12 \\ &= 2(t - 6). \end{aligned}$$
Thus the acceleration is zero when $t = 6$.

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b When $t = 6$, $v = 36 - 72 + 27$
$= -9$,
and $x = 72 - 216 + 162 - 18$
$= 0$.
Thus when $t = 6$, the particle is at the origin, moving with velocity -9 m/s.

Trigonometric equations of motion

When a particle's motion is described by a sine or cosine function, it moves backwards and forwards and is therefore stationary over and over again. It is best to work with the graphs drawn.

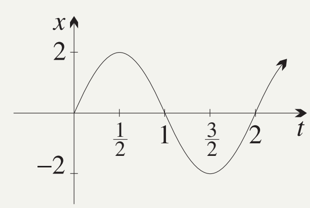
Example 8 9B

A particle's displacement function is $x = 2 \sin \pi t$.

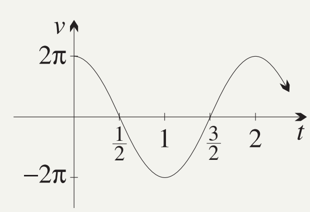
a Find its velocity and acceleration functions, and graph all three functions in the time interval $0 \le t \le 2$.
b Find the times within the time interval $0 \le t \le 2$ when the particle is at the origin, and find its speed and acceleration at those times.
c Find the times within the time interval $0 \le t \le 2$ when the particle is stationary, and find its displacement and acceleration at those times.
d Briefly describe the motion.

SOLUTION

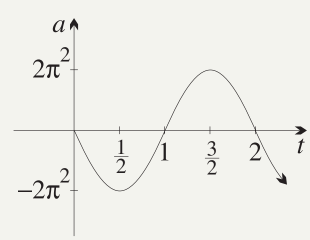
a The displacement function is $x = 2 \sin \pi t$,
which has amplitude 2 and period $\dfrac{2\pi}{\pi} = 2$.
Differentiating, $v = 2\pi \cos \pi t$,
which has amplitude 2π and period 2.
Differentiating again, $a = -2\pi^2 \sin \pi t$,
which has amplitude $2\pi^2$ and period 2.



b The particle is at the origin when $x = 0$,
and reading from the displacement graph,
this occurs when $t = 0, 1$ or 2.
Reading now from the velocity and acceleration graphs,
when $t = 0$ or 2, $v = 2\pi$ and $a = 0$,
and when $t = 1$, $v = -2\pi$ and $a = 0$,
so in all cases the speed is 2π and the acceleration is zero.



c The particle is stationary when $v = 0$,
and reading from the velocity graph above,
this occurs when $t = \frac{1}{2}$ or $1\frac{1}{2}$.
Reading from the displacement and acceleration graphs,
when $t = \frac{1}{2}$, $x = 2$ and $a = -2\pi^2$,
and when $t = 1\frac{1}{2}$, $x = -2$ and $a = 2\pi^2$.



d The particle oscillates forever between $x = -2$ and $x = 2$, with period 2, beginning at the origin and moving first to $x = 2$.

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Motion with exponential functions — limiting values of displacement and velocity

Sometimes a question will ask what happens to the particle 'eventually', or 'as time goes on'. This simply means taking the limit of the displacement and the velocity as $t \to \infty$. Particles whose motion is described by an exponential function are the most usual examples of this. Remember that $e^{-x} \to 0$ as $x \to \infty$.

Example 9 — 9B

A particle is moving so that its height x metres above the ground at time t seconds after time zero is $x = 2 - e^{-3t}$.

a Find the velocity and acceleration functions, and sketch the three graphs of displacement, velocity and acceleration.

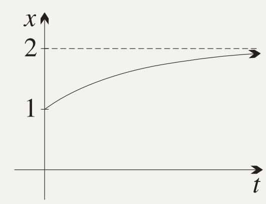
b Find the initial values of displacement, velocity and acceleration.

c What happens to the displacement, velocity and acceleration eventually?

d Briefly describe the motion.

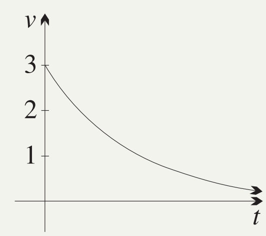
SOLUTION

a The displacement function is $\quad x = 2 - e^{-3t}$.
Differentiating, $\quad v = 3e^{-3t}$,
and differentiating again, $\quad a = -9e^{-3t}$.



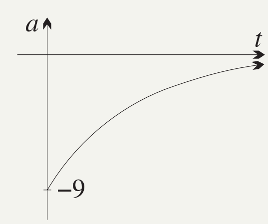
b Substituting $t = 0$ and using $e^0 = 1$,

$$x = 1 \quad \text{and} \quad v = 3 \quad \text{and} \quad a = -9.$$



c As t increases, that is, as $t \to \infty$, $e^{-3t} \to 0$.
Hence *eventually* (meaning as $t \to \infty$),

$$x \to 2 \quad \text{and} \quad v \to 0 \quad \text{and} \quad a \to 0.$$



d The particle starts 1 metre above the ground, with initial velocity of 3 m/s upwards.
It is constantly slowing down, and it moves towards a limiting position at height 2 metres.

Extension — Newton's second law of motion

Newton's second law of motion — a law of physics, not of mathematics — says that when a force is applied to a body free to move, the body accelerates with an acceleration proportional to the resultant force and inversely proportional to the mass of the body. Written symbolically,

$$F = ma,$$

where m is the mass, F is the resultant force, and a is the acceleration. (The units of force are chosen to make the constant of proportionality 1 — in units of kilograms, metres and seconds, the units of force are, appropriately, called *newtons*.)

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This means that acceleration is felt in our bodies as a force, as we all know when a car that we are in accelerates away from the lights, or comes to a stop quickly. In this way, the second derivative becomes directly observable to our senses as a force, just as the first derivative, velocity, is observable to our sight.

There is no need to formalise this, but it is helpful to have an intuitive idea that force and acceleration are closely related.

Exercise 9B

FOUNDATION

Note: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The mathematics should be well known, but the physical interpretations can be confusing.

1 A particle is moving with displacement function $x = 20 - t^2$, in units of metres and seconds.
  a Differentiate to find the velocity v as a function of time t.
  b Differentiate again to find the acceleration a.
  c Find the displacement, velocity and acceleration when $t = 3$.
  d What are the distance from the origin and the speed when $t = 3$?

2 A particle's displacement function is $x = t^2 - 10t$, in units of centimetres and seconds.
  a Differentiate to find v as a function of t.
  b What are the displacement, the distance from the origin, the velocity and the speed after 3 seconds?
  c When is the particle stationary and where is it then?

3 A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2$.
  a Differentiate to find v as a function of t, and differentiate again to find a.
  b Where is the particle initially and what are its speed and acceleration then?
  c At time $t = 3$, is the particle to the left or to the right of the origin?
  d At time $t = 3$, is the particle travelling to the left or to the right?
  e At time $t = 3$, is the particle accelerating to the left or to the right?
  f Show that the particle is stationary when $t = 4$ and find where it is at this time.
  g Show that the particle is at the origin when $t = 6$ and find its velocity and speed at this time.

4 A cricket ball is thrown vertically upwards. Its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.
  a Find v and a as functions of t, and show that the ball is always accelerating downwards. Then sketch graphs of x, v and a against t.
  b Find the speed at which the ball was thrown.
  c Find when it returns to the ground (that is, when $x = 0$) and show that its speed then is equal to the initial speed.
  d Find its maximum height above the ground and the time taken to reach this height.
  e Find the acceleration at the top of the flight, and explain why the acceleration can be non-zero when the ball is stationary.

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5 If $x = e^{-4t}$, find the velocity function $\dot{x}$ and the acceleration function $\ddot{x}$.

a Explain why neither x, nor $\dot{x}$, nor $\ddot{x}$ can ever change sign, and state their signs.

b Using the displacement function, find where the particle is:

i initially (substitute $t = 0$), **ii** eventually (take the limit as $t \to \infty$).

c What are the particle's velocity and acceleration:

i initially, **ii** eventually?

6 Find the velocity function v and the acceleration function a for a particle P moving according to $x = 2 \sin \pi t$.

a Show that P is at the origin when $t = 1$ and find its velocity and acceleration then.

b When $t = \frac{1}{3}$, in what direction is the particle:

i moving, **ii** accelerating?

DEVELOPMENT

7 A particle moves according to $x = t^2 - 8t + 7$, in units of metres and seconds.

a Find the velocity $\dot{x}$ and the acceleration $\ddot{x}$ as functions of time t.

b Sketch the graphs of the displacement x, velocity $\dot{x}$ and acceleration $\ddot{x}$.

c When is the particle:

i at the origin, **ii** stationary?

d What is the maximum distance from the origin, and when does it occur:

i during the first 2 seconds,

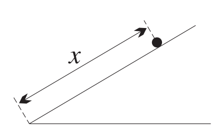
ii during the first 6 seconds,

iii during the first 10 seconds?

e What is the particle's average velocity during the first 7 seconds? When and where is its instantaneous velocity equal to this average?

f How far does it travel during the first 7 seconds, and what is its average speed?

8 A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.



a Find the functions for velocity v and acceleration a.

b Sketch the graphs of displacement x and velocity v.

c In which direction is the ice moving, and in which direction is it accelerating:

i when $t = 2$, **ii** when $t = 4$?

d When is the ice stationary, for how long is it stationary, where is it then, and is it accelerating then?

e Show that the average velocity over the first 2 seconds is 4 m/s. Then find the time and place at which the instantaneous velocity equals this average velocity.

f Show that the average speed during the first 3 seconds, the next 3 seconds and the first 6 seconds are all the same.

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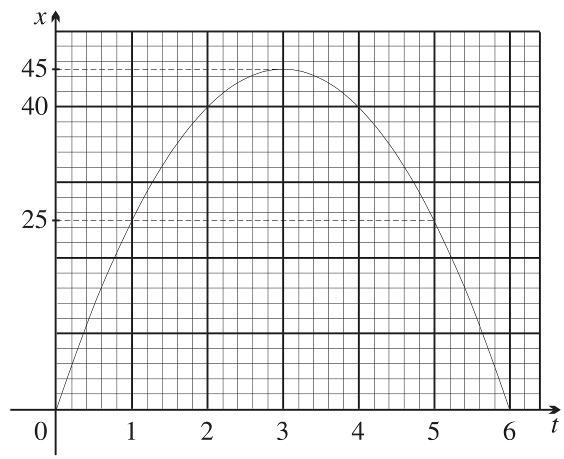
9 A stone was thrown vertically upwards. The graph to the right shows its height x metres at time t seconds after it was thrown.



a What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?

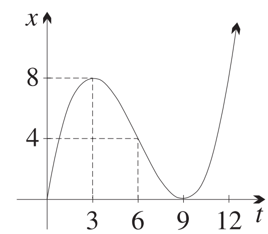
b Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ and 6.

c For what length of time was the stone stationary at the top of its flight?

d The graph is concave down everywhere. How is this relevant to the motion?

e Draw a graph of the instantaneous velocity of the stone from $t = 0$ to $t = 6$. What does this velocity–time graph tell you about what happened to the velocity during these 6 seconds?

10 A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.



a In the first 10 seconds, what is its maximum distance from the origin and when does it occur?

b By examining the gradient, find when the particle is:

i stationary, **ii** moving to the right,

iii moving to the left.

c When does it return to the origin, what is its velocity then, and in which direction is it accelerating?

d When is its acceleration zero, where is it then, and in what direction is it moving?

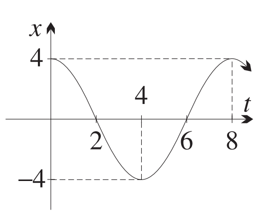
e By examining the concavity, find the time interval during which the particle's acceleration is negative.

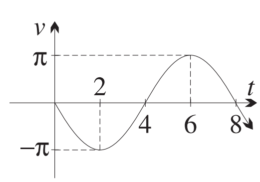
f At about what times are:

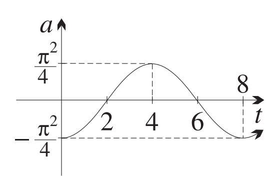
i the displacement, **ii** the velocity, about the same as those at $t = 2$?

g Sketch (roughly) the graphs of velocity v and acceleration a.

11 A particle is moving according to $x = 4\cos\frac{\pi}{4}t$, where the units are metres and seconds. The displacement, velocity and acceleration graphs are drawn below, for $0 \le t \le 8$.







a Differentiate to find the functions for velocity v and the acceleration a.

b What are the particle's maximum displacement, velocity and acceleration, and when, during the first 8 seconds, do they occur?

c How far does it travel during the first 20 seconds, and what is its average speed?

d Show by substitution that $x = 2$ when $t = 1\frac{1}{3}$ and when $t = 6\frac{2}{3}$. Hence use the graph to find when $x < 2$ during the first 8 seconds.

e When, during the first 8 seconds, is:

i $v = 0$, **ii** $v > 0$?

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12 A particle is oscillating on a spring so that its height is $x = 6\sin 2t$ cm at time t seconds.

a Find v and $\ddot{x}$ as functions of t, and sketch graphs of x, v and $\ddot{x}$, for $0 \le t \le 2\pi$.

b Show that $\ddot{x} = -kx$, for some constant k, and find k.

c When, during the first π seconds, is the particle:

i at the origin,

ii stationary,

iii moving with zero acceleration?

d When, during the first π seconds, is the particle:

i below the origin,

ii moving downwards,

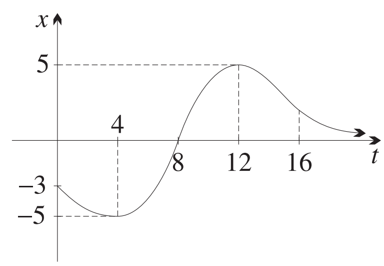
iii accelerating downwards?

e Find the first time the particle has:

i displacement $x = 3$,

ii speed $|v| = 6$.

13 A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



a At what times is this particle:

i below the origin,

ii moving downwards,

iii accelerating downwards?

b At about what time is its speed greatest?

c At about what times are:

i the distance from the origin,

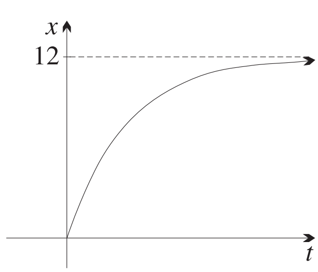
ii the velocity, about the same as those at $t = 3$?

d How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?

e How far will the particle eventually travel?

f Draw an approximate sketch of the graph of v as a function of time.

14 A large stone is falling through a layer of mud. Its depth x metres below ground level at time t minutes is given by $x = 12 - 12e^{-0.5t}$, and its displacement–time graph is drawn.



a Find the velocity and acceleration functions, and sketch them.

b In which direction is the stone always:

i travelling,

ii accelerating?

c What happens to the position, velocity and acceleration of the particle as $t \to \infty$?

d Find when the stone is halfway between the origin and its final position. Show that its speed is then half its initial speed, and its acceleration is half its initial acceleration.

e How long, correct to the nearest minute, will it take for the stone to reach within 1 mm of its final position?

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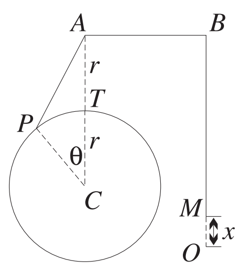
ENRICHMENT

15 The diagram to the right shows a point P that is rotating anticlockwise in a circle of radius r and centre C at a steady rate. A string passes over fixed pulleys at A and B, where A is distant r above the top T of the circle, and connects P to a mass M on the end of the string. At time zero, P is at T, and the mass M is at the point O. Let x be the height of the mass above the point O at time t seconds later, and θ be the angle $\angle TCP$ through which P has moved.



a Show that $x = -r + r\sqrt{5 - 4\cos\theta}$, and find the range of x.

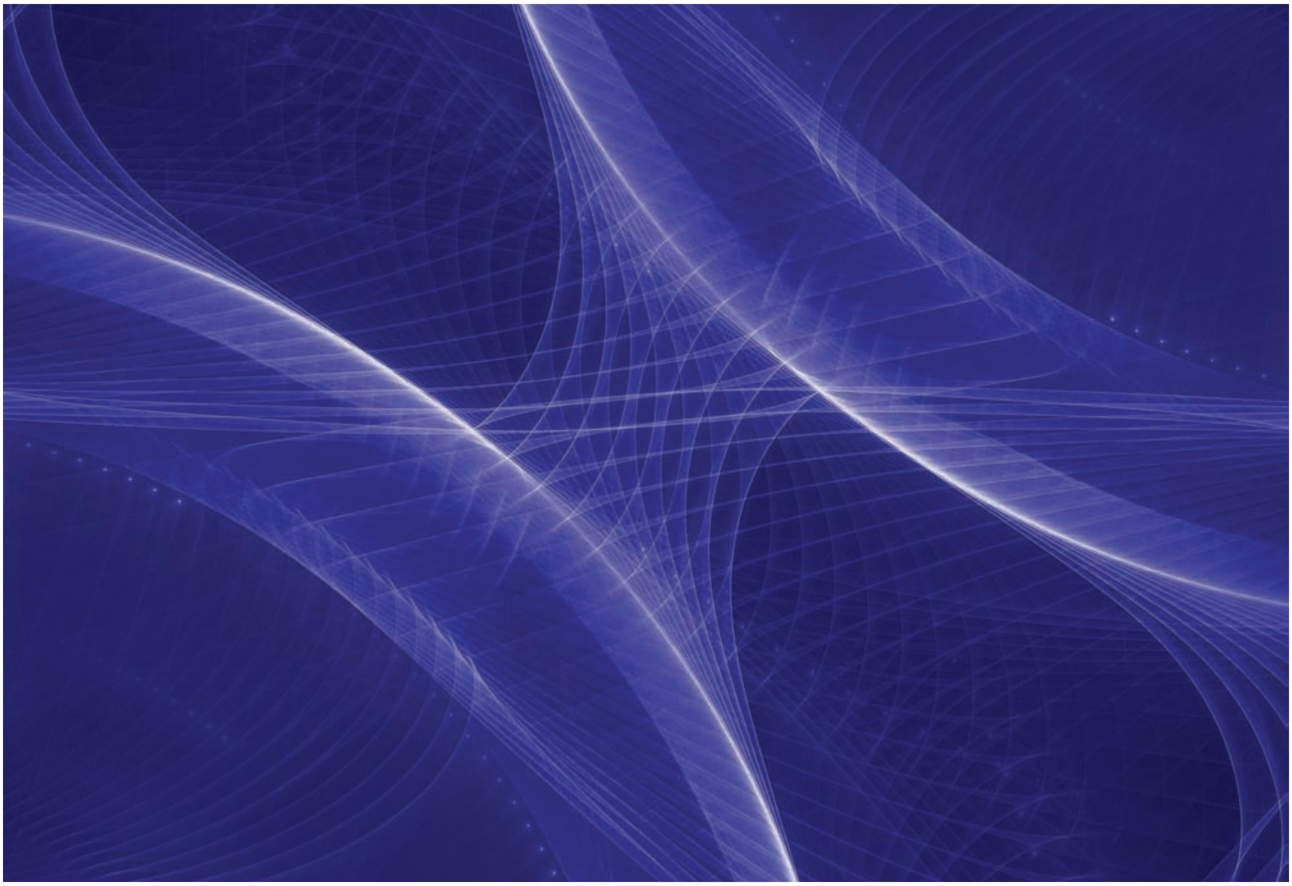
b Find $\dfrac{dx}{d\theta}$, and find for what values of θ the mass M is travelling:

i upwards, **ii** downwards.

c Show that $\dfrac{d^2x}{d\theta^2} = -\dfrac{2r(2\cos^2\theta - 5\cos\theta + 2)}{(5 - 4\cos\theta)^{\frac{3}{2}}}$. Find for what values of θ the speed of M is maximum, and find $\dfrac{dx}{d\theta}$ at these values of θ.

d Explain geometrically why these values of θ give the maximum speed, and why they give the values of $\dfrac{dx}{d\theta}$ they do.

16 [This question will require resolution of forces.]
At what angle α should the surface in Question **8** be inclined to the horizontal to produce these equations?



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9C Integrating with respect to time

The inverse process of differentiation is integration. Thus if the acceleration function is known, integration will generate the velocity function. In the same way, if the velocity function is known, integration will generate the displacement function.

Using initial conditions

Taking the primitive of a function always involves a constant of integration. Determining such a constant requires an *initial condition* (or *boundary condition*) to be known. For example, the problem may tell us the velocity when $t = 0$, or give us the displacement when $t = 3$.

In this chapter, the constants of integration cannot ever be omitted.

8 INTEGRATING WITH RESPECT TO TIME

- Given the acceleration function a, integrate to find the velocity function v.
- Given the velocity function v, integrate to find the displacement function x.
- Never omit the constants of integration.
- Use an initial or boundary condition to evaluate each constant of integration.

Example 10 9C

A particle's acceleration function is $\ddot{x} = 24t$. Initially it is at the origin, moving with velocity -12 cm/s.

a Integrate, substituting the initial condition, to find the velocity function.
b Integrate again to find the displacement function.
c Find when the particle is stationary and find the displacement then.
d Find when the particle returns to the origin and the acceleration then.

SOLUTION

a The given acceleration function is $\ddot{x} = 24t$. (1)
Integrating, $v = 12t^2 + C$, for some constant C.
When $t = 0$, $v = -12$, so $-12 = 0 + C$,
so $C = -12$, and $v = 12t^2 - 12$. (2)

b Integrating again, $x = 4t^3 - 12t + D$, for some constant D.
When $t = 0$, $x = 0$, so $0 = 0 - 0 + D$,
so $D = 0$, and $x = 4t^3 - 12t$. (3)

c Put $v = 0$. Then from (2),
$$12t^2 - 12 = 0$$
$$t^2 = 1$$
$$t = 1 \text{ (because } t \geq 0\text{)}.$$
Hence the particle is stationary after 1 second.
When $t = 1$, $x = -8$, so at this time its displacement is $x = -8$ cm.

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d Put $x = 0$. Then using (3), $4t^3 - 12t = 0$

$$4t(t^2 - 3) = 0,$$

so $t = 0$ or $t = \sqrt{3}$ (because $t \geq 0$).

Hence the particle returns to the origin after $\sqrt{3}$ seconds,

and at this time, $\ddot{x} = 24\sqrt{3}\ \text{cm/s}^2$.

The acceleration due to gravity

Since the time of Galileo, it has been known that near the surface of the Earth, a body that is free to fall accelerates downwards at a constant rate, whatever its mass and whatever its velocity, provided that air resistance is ignored. This acceleration is called the *acceleration due to gravity* and is conventionally given the symbol g. The value of this acceleration is about $9.8\ \text{m/s}^2$, or in rounder figures, $10\ \text{m/s}^2$.

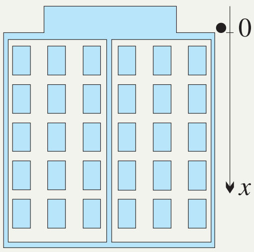
The acceleration is downwards. Thus if upwards is taken as positive, the acceleration is $-g$, but if downwards is taken as positive, the acceleration is g.

9 THE ACCELERATION DUE TO GRAVITY

- A body that is falling accelerates downwards at a constant rate $g \doteqdot 9.8\ \text{m/s}^2$, provided that air resistance is ignored.
- If upwards is taken as positive, start with the function $a = -g$ and integrate.
- If downwards is taken as positive, start with the function $a = g$ and integrate.

Example 11 9C

A stone is dropped from the top of a high building. How far has it travelled, and how fast is it going, after 5 seconds? Take $g = 9.8\ \text{m/s}^2$.



SOLUTION

Let x metres be the distance travelled t seconds after the stone is dropped.
This simple sentence puts the origin of space at the top of the building,
it puts the origin of time at the instant when the stone is dropped,
and it makes downwards the positive direction.

Then $a = 9.8$ (given). (1)

Integrating, $v = 9.8t + C$, for some constant C.
Because the stone was dropped, its initial speed was zero,
and substituting, $0 = 0 + C$,
so $C = 0$, and $v = 9.8t$. (2)

Integrating again, $x = 4.9t^2 + D$, for some constant D.
Because the initial displacement of the stone was zero,
$0 = 0 + D$,
so $D = 0$, and $x = 4.9t^2$. (3)

When $t = 5$, $v = 49$ (substituting into (2) above)
and $x = 122.5$ (substituting into (3) above).
Hence the stone has fallen 122.5 metres and is moving downwards at 49 m/s.

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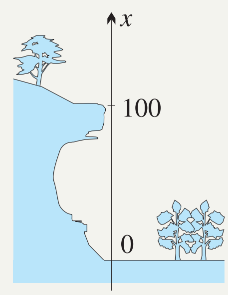
Making a convenient choice of the origin and the positive direction

Physical problems do not come with origins and directions attached. Thus it is up to us to choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible.

The previous worked example made reasonable choices, but the next worked example makes quite different choices. In all such problems, the physical interpretation of negatives and displacements is the mathematician's responsibility, and the final answer should be given in ordinary language.

Example 12 9C

A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of 40 m/s and it falls back past the lookout onto the valley floor below.



How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10$ m/s^2.)

SOLUTION

Let x be the distance above the valley floor t seconds after the stone is thrown. Again, this simple sentence puts the origin of space at the valley floor, and puts the origin of time at the instant when the stone is thrown. It also makes upwards positive, so that $a = -10$ because the acceleration is downwards.

As discussed, $\ddot{x} = -10.$ (1)

Integrating, $v = -10t + C$, for some constant C.
Because $v = 40$ when $t = 0$, $40 = 0 + C$,
so $C = 40$, and $v = -10t + 40.$ (2)

Integrating again, $x = -5t^2 + 40t + D$, for some constant D.
Because $x = 100$ when $t = 0$, $100 = 0 + 0 + D$,
so $D = 100$, and $x = -5t^2 + 40t + 100.$ (3)

The stone hits the ground when $x = 0$, so using (3) above,

$$
\begin{aligned}
-5t^2 + 40t + 100 &= 0 \\
t^2 - 8t - 20 &= 0 \\
(t - 10)(t + 2) &= 0 \\
t &= 10 \text{ or } -2.
\end{aligned}
$$

The ball was not in flight at $t = -2$, so the ball hits the ground after 10 seconds.
Substituting $t = 10$ into equation (2), $v = -100 + 40 = -60$,
so the ball hits the ground at 60 m/s.

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Formulae from physics cannot be used

This course requires that even problems where the acceleration is constant, such as the two above, must be solved by integrating the acceleration function. Many readers will know of three very useful equations for motion with constant acceleration a,

$$v = u + at \qquad \text{and} \qquad s = ut + \tfrac{1}{2}at^2 \qquad \text{and} \qquad v^2 = u^2 + 2as.$$

These equations automate the integration process, and so cannot be used in this course. Question **18** in Exercise 9C develops a proper proof of these results.

Integrating trigonometric functions

The next worked example applies the same methods of integration to motion involving trigonometric functions.

Example 13 — 9C

The velocity of a particle initially at the origin is $v = \sin\frac{1}{4}t$, in units of metres and seconds.

a Find the displacement function.

b Find the acceleration function.

c Find the values of displacement, velocity and acceleration when $t = 4\pi$.

d Briefly describe the motion, and sketch the displacement–time graph.

SOLUTION

a The velocity is $\quad v = \sin\frac{1}{4}t. \qquad (1)$

Integrating (1), $\quad x = -4\cos\frac{1}{4}t + C,$ for some constant C.

Substituting $x = 0$ when $t = 0$,

$$0 = -4 \times 1 + C,$$
$$C = 4.$$

Thus $\quad x = 4 - 4\cos\frac{1}{4}t. \qquad (2)$

b Differentiating (1), $\quad \ddot{x} = \frac{1}{4}\cos\frac{1}{4}t. \qquad (3)$

c When $t = 4\pi$, $\quad x = 4 - 4 \times \cos\pi$, using (2),

$= 8$ metres.

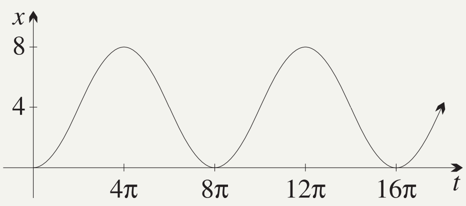
Also $\quad v = \sin\pi$, using (1),

$= 0\ \text{m/s}$,

and $\quad \ddot{x} = \frac{1}{4}\cos\pi$, using (3),

$= -\frac{1}{4}\ \text{m/s}^2$.

d The particle oscillates between $x = 0$ and $x = 8$ with period 8π seconds.



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Integrating exponential functions

The next worked example involves exponential functions. The velocity function approaches a limit 'as time goes on'.

Example 14 9C

The acceleration of a particle is given by $a = e^{-2t}$ (in units of metres and seconds), and the particle is initially stationary at the origin.

a Find the velocity and displacement functions.

b Find the displacement when $t = 10$.

c Sketch the velocity–time graph and describe briefly what happens to the velocity of the particle as time goes on.

SOLUTION

a The acceleration is $a = e^{-2t}$. (1)

Integrating, $v = -\frac{1}{2}e^{-2t} + C$, for some constant C.

It is given that when $t = 0$, $v = 0$,

so $0 = -\frac{1}{2} + C$

$C = \frac{1}{2}$,

and $v = -\frac{1}{2}e^{-2t} + \frac{1}{2}$. (2)

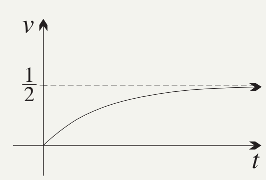
Integrating again, $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t + D$, for some constant D.

It is given that when $t = 0$, $x = 0$,

so $0 = \frac{1}{4} + D$

$D = -\frac{1}{4}$,

and $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$. (3)



b When $t = 10$, $x = \frac{1}{4}e^{-20} + 5 - \frac{1}{4}$

$= 4\frac{3}{4} + \frac{1}{4}e^{-20}$ metres.

c Using equation (2), the velocity is initially zero,
and increases so that the limiting velocity as time goes on is $\frac{1}{2}$ m/s.

Note: A car moving off from the kerb on level ground obeys this sort of equation if the accelerator is pressed down not too far and kept in that position.

Exercise 9C

FOUNDATION

1 A particle is moving with velocity function $v = 3t^2 - 6t$, in units of metres and seconds. At time $t = 0$, its displacement is $x = 4$.

a Integrate, substituting the initial condition, to find the displacement function.

b Show that the particle is at the origin when $t = 2$ and find its velocity then.

c Differentiate the given velocity function to find the acceleration function.

d Show that the acceleration is zero when $t = 1$, and find the displacement then.

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2 A stone is dropped from a lookout 80 metres high. Take $g = 10\ \text{m/s}^2$ and downwards as positive, so that the acceleration function is $a = 10$.

a Using the lookout as the origin, find the velocity and displacement as functions of t. (Hint: When $t = 0$, $v = 0$ and $x = 0$.)

b Find the time that the stone takes to fall, and its impact speed.

c Where is it, and what is its speed, halfway through its flight time?

d How long does it take to go halfway down, and what is its speed then?

3 A stone is thrown downwards from the top of a 120-metre building, with an initial speed of $25\ \text{m/s}$. Take $g = 10\ \text{m/s}^2$ and take upwards as positive, so that $a = -10$.

a Using the ground as the origin, find the acceleration, velocity and height x of the stone t seconds after it is thrown. (Hint: When $t = 0$, $v = -25$ and $x = 120$.)

b Find the time it takes to reach the ground, and the impact speed.

c What is the average speed of the stone during its descent?

4 **a** Find the velocity function $\dot{x}$ and the displacement function x of a particle whose initial velocity and displacement are zero if:

i $\ddot{x} = 6t$ **ii** $\ddot{x} = e^{-3t}$ **iii** $\ddot{x} = \cos \pi t$ **iv** $\ddot{x} = 12(t + 1)^{-2}$

b Find the acceleration function a and the displacement function x of a particle whose initial displacement is -2 if:

i $v = -4$ **ii** $v = e^{\frac{1}{2}t}$ **iii** $v = 8 \sin 2t$ **iv** $v = \sqrt{t}$

5 A particle is moving with acceleration $\ddot{x} = 12t$. Initially, it has velocity $-24\ \text{m/s}$ and is 20 metres on the positive side of the origin.

a Find the velocity function $\dot{x}$ and the displacement function x.

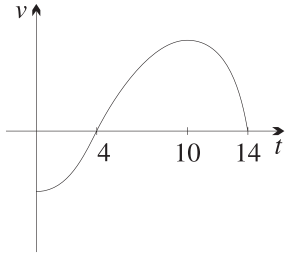
b When does the particle return to its initial position, and what is its speed then?

c What is the minimum displacement, and when does it occur?

d Find x when $t = 0, 1, 2, 3$ and 4, and sketch the displacement-time graph.

DEVELOPMENT

6 The graph to the right shows a particle's velocity–time graph.



a When is the particle moving forwards?

b When is the acceleration positive?

c When is it furthest from its starting point?

d When is it furthest in the negative direction?

e About when does it return to its starting point?

f Sketch the graphs of acceleration and displacement, assuming that the particle is initially at the origin.

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7 A car moves along a straight road from its front gate, where it is initially stationary. During the first 10 seconds, it has a constant acceleration of $2\ \text{m/s}^2$, it has zero acceleration during the next 30 seconds, and it decelerates at $1\ \text{m/s}^2$ for the final 20 seconds until it stops.

a What is the car's speed after 20 seconds?

b Show that the car travels:

i 100 metres during the first 10 seconds,

ii 600 metres during the next 30 seconds,

iii 200 metres during the last 20 seconds.

c Sketch the graphs of acceleration, velocity and distance from the gate.

8 A particle is moving with velocity $\dot{x} = 16 - 4t\ \text{cm/s}$ on a horizontal number line.

a Find a and x. (The function x will have a constant of integration.)

b When does it return to its original position, and what is its speed then?

c When is the particle stationary? Find the maximum distances right and left of the initial position during the first 10 seconds, and the corresponding times and accelerations.

d How far does it travel in the first 10 seconds, and what is its average speed?

9 A mouse emerges from his hole and moves out and back along a line. His velocity at time t seconds is

$$v = 4t(t-3)(t-6) = 4t^3 - 36t^2 + 72t\ \text{cm/s}.$$

a When does he return to his original position, and how fast is he then going?

b How far does he travel during this time, and what is his average speed?

c What is his maximum speed, and when does it occur?

d If a video of these 6 seconds were played backwards, could this be detected?

10 A body is moving with its acceleration proportional to the time elapsed, that is $\ddot{x} = kt$, for some constant of proportionality k. When $t = 1$, $v = -6$, and when $t = 2$, $v = 3$.

a Find the functions $\ddot{x}$ and v. (Hint: Integrate, using the usual constant C of integration. Then find C and k by substituting the two given values of t.)

b When does the body return to its original position?

11 A particle moves from $x = -1$ with velocity $v = \dfrac{1}{t+1}$. How long does it take to get to the origin, and what are its speed and acceleration then? Describe its subsequent motion.

12 A body moving vertically through air experiences an acceleration $a = -40e^{-2t}\ \text{m/s}^2$ (we are taking upwards as positive). Initially, it is thrown upwards with speed $15\ \text{m/s}$.

a Taking the origin at the point where it is thrown, find the velocity function $\dot{x}$ and the displacement function x, and find when the body is stationary.

b Find its maximum height and its acceleration then.

c Describe the velocity of the body as $t \to \infty$.

13 A moving particle is subject to an acceleration of $\ddot{x} = -2\cos t\ \text{m/s}^2$. Initially it is at $x = 2$, moving with velocity $1\ \text{m/s}$, and it travels for 2π seconds.

a Find the functions v and x.

b When is the acceleration positive?

c When and where is the particle stationary, and when is it moving backwards?

d What are the maximum and minimum velocities, and when and where do they occur?

e Find the change in displacement and the average velocity.

f Sketch the displacement–time graph, and hence find the distance travelled and the average speed.

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14 Once again, the trains Thomas and Henry are on parallel tracks, level with each other at time zero. Thomas is moving with velocity $v_T = \dfrac{20}{t + 1}$ and Henry with velocity $v_H = 5$.

- **a** Who is moving faster initially, and by how much?
- **b** Find the displacements x_T and x_H of the two trains, if they start at the origin.
- **c** Use your calculator to find during which second the trains are level, and find the speed at which the trains are drawing apart at the end of this second.
- **d** When is Henry furthest behind Thomas, and by how much (nearest metre)?

15 A ball is dropped from a lookout 180 metres high. At the same time, a stone is fired vertically upwards from the valley floor with speed V m/s. Take $g = 10$ m/s^2.

- **a** Find for what values of V a collision in the air will occur. Find, in terms of V, the time and the height when collision occurs, and prove that the collision speed is V m/s.
- **b** Find the value of V for which they collide halfway up the cliff, and the time taken.

16 A falling body experiences both the gravitational acceleration g and air resistance that is proportional to its velocity. Thus a typical equation of motion is $\ddot{x} = -10 - 2v$ m/s^2. Suppose that the body is dropped from the origin.

- **a** By writing $\ddot{x} = \dfrac{dv}{dt}$ and taking reciprocals, find t as a function of v, and hence find v as a function of t. Then find x as a function of t.
- **b** Describe the motion of the particle.

ENRICHMENT

17 [A proof of three constant-acceleration formulae from physics — not to be used elsewhere.]

- **a** A particle moves with constant acceleration a. Its initial velocity is u, and at time t it is moving with velocity v and its distance from its initial position is s. Show that:
  - **i** $v = u + at$
  - **ii** $s = ut + \frac{1}{2}at^2$
  - **iii** $v^2 = u^2 + 2as$
- **b** Solve questions 2 and 3 using formulae **ii** and **i**, and again using **iii** and **i**.

18 Particles P_1 and P_2 move with velocities $v_1 = 6 + 2t$ and $v_2 = 4 - 2t$, in units of metres and seconds. Initially, P_1 is at $x = 2$ and P_2 is at $x = 1$.

- **a** Find x_1, x_2 and the difference $D = x_1 - x_2$.
- **b** Prove that the particles never meet, and find the minimum distance between them.
- **c** Prove that the midpoint M between the two particles is moving with constant velocity, and find its distance from each particle after 3 seconds.

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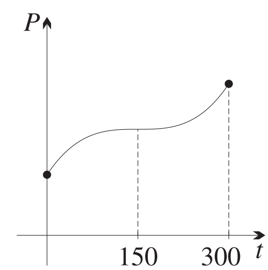
9D Rates and differentiation

When a quantity varies over time, we saw in Section 9J of the Year 11 book that we can differentiate to find the rate at which it is increasing or decreasing at each time t, and differentiate again to find the rate at which that rate is increasing or decreasing. This section reviews those methods in the larger context.

In Chapter 4, we carefully defined a function to be *increasing at* $x = a$ if $f'(a)$ is positive, and *decreasing at* $x = a$ if $f'(a)$ is negative. These were precise *pointwise* definitions. But at the same time we have continued, right from the beginning of Year 11, to use only vague language about a function being *increasing or decreasing over an interval*. These ideas will now be made precise as well.

Increasing and decreasing in an interval

Suppose that the price P of a share in newly-launched company t days after launching is given by the curve $P = P(t)$ drawn to the right.



The share price P is said to be *increasing in the interval* $0 \le t \le 300$ because every chord slopes upwards, that is

$$P(a) < P(b), \quad \text{for all } a < b \text{ in the interval } [0, 300].$$

Notice that except for the isolated point $t = 150$ where the tangent is horizontal, the function P is increasing at every point in the interval. Such a function is clearly increasing in the interval, because if every other tangent slopes upwards, then every chord without exception slopes upwards.

10 INCREASING AND DECREASING IN AN INTERVAL

Suppose that a function $f(x)$ is defined in an interval I. The interval may be bounded or unbounded, and may be open or closed or neither.

- The function $f(x)$ is called *increasing in the interval I* if every chord within the interval slopes upwards, that is,

$$f(a) < f(b), \quad \text{for all } a < b \text{ in the interval.}$$

- The function $f(x)$ is called *decreasing in the interval I* if every chord within the interval slopes downwards, that is,

$$f(a) > f(b), \quad \text{for all } a < b \text{ in the interval.}$$

If $f(x)$ is differentiable in the interval, and is increasing at every point in the interval (except perhaps for some isolated points where the tangent is horizontal), then clearly it is increasing in the interval.

Concave up and down in an interval

We have also been sloppy when talking about concavity over an interval rather than at a point. The graph above is said to be *concave down in the interval* $[0, 150]$ because every chord lies *under the curve*. The significance of this is that the share price, which is increasing in this interval, is *increasing at a decreasing rate*.

The graph is *concave up in the interval* $[150, 300]$ because every chord in this interval lies *above the curve*. The significance of this is that the share price, which is increasing in this interval as well, is *increasing at an increasing rate*.

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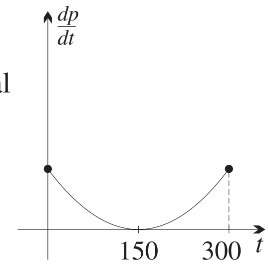
Sketched to the right is the gradient function $\frac{dP}{dt}$ of the share price P. This gradient function is decreasing in the interval $[0, 150]$, corresponding to the fact that the original curve is concave down in the interval $[0, 150]$. The gradient function $\frac{dP}{dt}$ is increasing in the remaining part $[150, 300]$ of the interval, corresponding to the fact that the original function P is concave up in the interval $[150, 300]$.



11 CONCAVE UP AND CONCAVE DOWN IN AN INTERVAL

Suppose that a function $f(x)$ is defined and continuous in an interval I, which may be bounded or unbounded, and may be open or closed or neither.

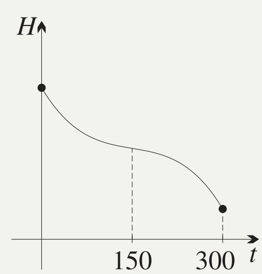
- The function $f(x)$ is called *concave up in the interval I* if every chord within the interval lies above the curve.
- The function $f(x)$ is called *concave down in the interval I* if every chord within the interval lies below the curve.

Concavity is usually a matter of common sense. These may or may not help.

- If $f(x)$ is differentiable in the interval, and $f'(x)$ is increasing in the interval, then $f(x)$ is concave up in the interval.
- If $f(x)$ is doubly differentiable in the interval and concave up at every point in the interval (except perhaps from some isolated points where $f''(x)$ is zero), then it is concave up in the interval.

Example 15 9D

Drought hit Kookaburra Valley last year, and the height of the Everyflow River dropped alarmingly, as shown in the graph to the right of the river height H at Emu Bridge t days after 1st January. Use the features of the graph to describe the behaviour of the river height. What happened in the period just before the 150th day?



SOLUTION

The river height was decreasing for the whole 300 days.
The graph is concave up in the interval $[0, 150]$,
and the height was decreasing at a decreasing rate.
The graph is concave down in the interval $[150, 300]$,
and the height was decreasing at an increasing rate.
It probably rained a little in the period just before $t = 150$.

Note: When we say that the height is 'decreasing at a decreasing rate', we are saying that $\frac{dH}{dt}$ is negative and that $\left|\frac{dH}{dt}\right|$ is decreasing. Similarly, when we say that the height is 'decreasing at an increasing rate', we are saying that $\frac{dH}{dt}$ is negative and that $\left|\frac{dH}{dt}\right|$ is increasing. People naturally use the right language here, but thinking about the situation may lead to confusion.

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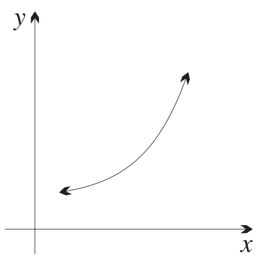
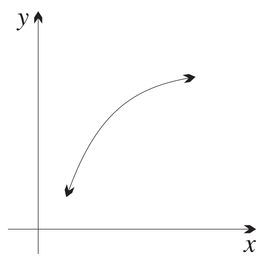
12 INCREASING OR DECREASING AT AN INCREASING OR DECREASING RATE

Suppose that a function $f(x)$ is defined and continuous in an interval I.

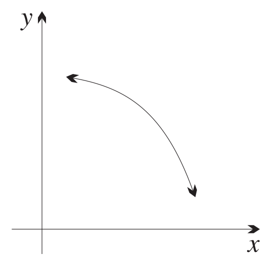
Increasing at an increasing rate: $f(x)$ is increasing and concave up in I.
Increasing at a decreasing rate: $f(x)$ is increasing and concave down in I.
Decreasing at an increasing rate: $f(x)$ is decreasing and concave down in I.
Decreasing at a decreasing rate: $f(x)$ is decreasing and concave up in I.



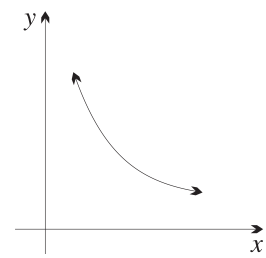
Increasing at an increasing rate



Increasing at a decreasing rate



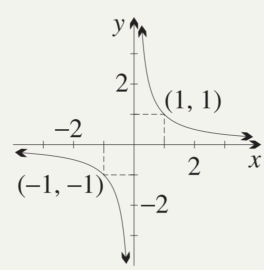
Decreasing at an increasing rate



Decreasing at a decreasing rate

Example 16 9D

Describe the branches of the function $y = \dfrac{1}{x}$ in the terms of this section.



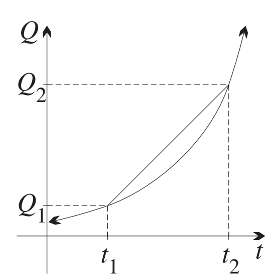
SOLUTION

In the interval $(-\infty, 0)$, the curve is decreasing at an increasing rate, and concave down. In the interval $(0, \infty)$, the curve is decreasing at a decreasing rate, and concave up.

Average rates and instantaneous rates

Worked Example 17 below is an example of a rates question that uses the new language of this section. It also uses differentiation to find a rate, and the second derivative to classify turning points and find inflections. First, however, here is a quick summary of average and instantaneous rates from Section 9J of the Year 11 book.

Suppose that a quantity Q is given as a function of time t, as in the diagram to the right. There are two types of rates.



- An *average rate of change* corresponds to a chord. From the usual gradient formula,

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- An *instantaneous rate of change* corresponds to a tangent. The instantaneous rate of change at time t_1 is the value of the derivative $\dfrac{dQ}{dt}$ at time $t = t_1$,

$$\text{instantaneous rate} = \frac{dQ}{dt}, \text{ evaluated at } t = t_1.$$

A rate of change always means the instantaneous rate of change unless otherwise stated, and is the gradient of the corresponding tangent.

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Example 17 9D

For the first 8 months after its first listing, the share price P in cents of the new company Avocado Marketing followed the cubic function $P = (t - 4)^3 - 12x + 100$, where the time t is in months after listing.

a What were the initial share price and its final share price?

b What were the rate of change of the price, and the rate of change of the rate of change?

c When was the share price at a local maximum or minimum, and what were those values?

d Find any points of inflection, and sketch the curve.

e Describe the behaviour of the price in different intervals of time using the terms in Box 12.

f What was the average rate of increase of the share price over the whole 8 months?

SOLUTION

a When $t = 0$,
$$\begin{aligned} P &= (-4)^3 + 0 + 100 \\ &= 36 \text{ cents.} \end{aligned}$$
When $t = 8$,
$$\begin{aligned} P &= 4^3 - 96 + 100 \\ &= 68 \text{ cents.} \end{aligned}$$

b Differentiating,
$$\frac{dP}{dt} = 3(t - 4)^2 - 12$$
$$\frac{d^2P}{dt^2} = 6(t - 4).$$

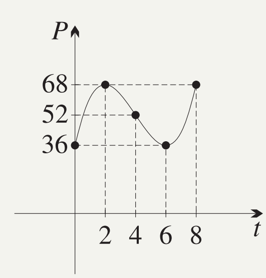
c Put $\dfrac{dP}{dt} = 0$ to find the stationary points,
$$\begin{aligned} 3(t - 4)^2 &= 12 \\ t - 4 &= 2 \text{ or } -2 \\ t &= 2 \text{ or } 6. \end{aligned}$$
When $t = 2$, $\dfrac{d^2P}{dt^2} = -12 < 0$ and $P = -8 - 24 + 100 = 68$,

and when $t = 6$, $\dfrac{d^2P}{dt^2} = 12 > 0$ and $P = 8 - 72 + 100 = 36$,

so the share price had a local maximum of 68 cents after 2 months, and a local minimum of 36 cents after 6 months.

d There is a zero of $\dfrac{d^2P}{dt^2}$ when $t = 4$, and the value there is $P = 0 - 48 + 100 = 52$.

x	2	4	6
$\dfrac{d^2P}{dt^2}$	-12	0	12
	$\frown$	$\cdot$	$\smile$



The table shows that $\dfrac{d^2P}{dt^2}$ changes sign around the point, so $(4, 52)$ is an inflection.

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e In the interval $[0, 2]$, the price is increasing at a decreasing rate.
In the interval $[2, 4]$, the price is decreasing at an increasing rate.
In the interval $[4, 6]$, the price is decreasing at a decreasing rate.
In the interval $[6, 8]$, the price is increasing at an increasing rate.

f The price at time $t = 0$ was 36 cents, and at time $t = 8$ it was 68 cents.

$$\begin{aligned}\text{Hence average rate of increase } &= \frac{68 - 36}{8}\\ &= 4 \text{ cents per month.}\end{aligned}$$

Exercise 9D

FOUNDATION

1 Grain is pouring into a storage silo. After t minutes there are V tonnes of grain in the silo, where $V = 20t$.
- **a** How much grain is in the silo after 4 minutes?
- **b** Show that the silo was empty to begin with.
- **c** If the silo takes 18 minutes to fill, what is its capacity?
- **d** At what rate is the silo being filled?

2 The amount F litres of fuel in a tank t minutes after it starts to empty is given by $F = 200(20 - t)^2$. Initially the tank is full.
- **a** Find the initial amount of fuel in the tank.
- **b** Find the quantity of fuel in the tank after 15 minutes.
- **c** Find the time taken for the tank to empty, and hence write down the domain of F.
- **d** Show that $\dfrac{dF}{dt} = -400(20 - t)$, and hence find the rate at which the tank is emptying after 5 minutes.
- **e** The value of $\dfrac{dF}{dt}$ is negative for all values of t in the domain. Explain why this is expected in this situation.

3 Grape juice is being pumped into a vat at the rate of $\dfrac{dV}{dt} = 300$ litres per minute, where V litres is the volume of grape juice in the tank after t minutes. The tank already has 1500 litres in it when the pump starts. Rachael correctly guesses that $V = kt + C$, but she does not know the values of k and C.
- **a** Use the initial value to determine C.
- **b** Substitute the formula for V into the equation for $\dfrac{dV}{dt}$ to find the value of k.
- **c** The tank can hold 6000 litres. How long does the pump need to run to fill the tank?

4 Using either the graph of $y = 2^x$ or a suitable reflection in one of the axes, draw a graph to represent a function that is:
- **a** increasing at a decreasing rate,
- **b** decreasing at an increasing rate,
- **c** decreasing at a decreasing rate,
- **d** increasing at an increasing rate.

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5 Consider the function $y = \sin x$ with domain $0 \leq x \leq 2\pi$. Sketch the function and then answer the following questions.

a For what values of x in the domain is the function:

i increasing at a decreasing rate,

ii decreasing at an increasing rate,

iii decreasing at a decreasing rate,

iv increasing at an increasing rate.

b Use the graph or your answers to part a to state where the function is:

i concave up,

ii concave down.

6 An object is projected vertically, and its height h metres at time t seconds is given by

$h = 180\left(1 - e^{-\frac{1}{3}t}\right) - 30t$.

a Find the rate at which the height is changing.

b What is the initial speed?

c The object reaches its maximum height at time T. Find T and find and the maximum height, correct to the nearest centimetre.

d Find the height, correct to the nearest centimetre, and the speed at time $2T$.

e What is the eventual speed of the object?

DEVELOPMENT

7 When a certain jet engine starts operating, the rate of fuel burn, R kg per minute, t minutes after startup is given by $R = 10 + \dfrac{10}{1 + 2t}$.

a What is the rate of fuel burn after:

i 2 minutes,

ii 7 minutes?

b What limiting value does R approach as t increases?

c Show that $\dfrac{dR}{dt} < 0$ and that $\dfrac{d^2R}{dt^2} > 0$ for $t \geq 0$.

d Describe the graph of R against t as either increasing or decreasing at either an increasing or decreasing rate.

e Draw a sketch of R as a function of t to confirm your answer to the previous part.

8 For a certain brand of medicine, the amount M present in the blood after t hours is given by $M = 9te^{-t}\mu$g, for $0 \leq t \leq 9$. (The symbol μg means 'micrograms'.)

a What are the values of M at $t = 0$ and $t = 9$? Evaluate the latter correct to one decimal place.

b Determine $\dfrac{dM}{dt}$ and hence find the turning point.

c Determine $\dfrac{d^2M}{dt^2}$ and hence find the inflection point.

d Sketch a graph of M against t, showing these features.

e When is the amount of medicine in the blood a maximum?

f When is the amount of medicine increasing most rapidly?

g When is the amount of medicine decreasing most rapidly?

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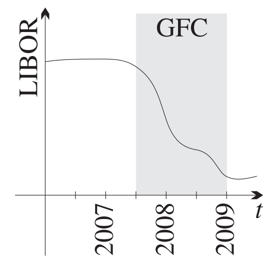
9 To the right is a simplified graph showing the effects of the Global Financial Crisis (GFC), from July in 2007 to the end of 2008, on the London Interbank Offered Rate (LIBOR). The LIBOR is sometimes used as a measure of the strength of the world economy.



a According to this graph, when was the crisis at its worst?

b What feature of the graph indicates the end of the crisis in January 2009.

c Why might an economist at the time have been optimistic in July of 2008?

d Sketch a possible graph of $\frac{dL}{dt}$, the derivative of the LIBOR, as a function of time t.

10 The number U of unemployed people at time t was studied over a period of time. At the start of this period, the number of unemployed was 600 000.

a Throughout the study, $\frac{dU}{dt} > 0$. What can be deduced about U over this period?

b The study also found that $\frac{d^2U}{dt^2} < 0$. What does this indicate about the changing unemployment level?

c Sketch a graph of U against t, showing this information.

11 A scientist studying an insect colony estimates the number $N(t)$ of insects after t months to be $N(t) = \frac{A}{2 + e^{-t}}$.

a When the scientist begins measuring, the number of insects in the colony is estimated to be 3×10^5. Find A.

b What is the population of the colony one month later?

c How many insects would you expect to find in the nest after a long time?

d Find an expression for the rate at which the population increases with time.

e Hence show that $\frac{dN}{dt} = \frac{N(1 - 2N)}{9 \times 10^5}$.

12 The inflation rate I as a percentage can be modeled using the Consumer Price Index C according to the equation

$$I = \frac{100}{C} \times \frac{dC}{dt}\ \%.$$

The treasury department in the nation of Mercatura has predicted that

$$C(t) = -\tfrac{1}{5}t^3 + 3t^2 + 200$$

for the next eight years, where t is the number of years from now.

a Find an expression for $I(t)$.

b Hence evaluate $I(4)$ correct to two decimal places.

c According to this model, there are two years in which the inflation rate is 0. What are those years, and why must the later value be rejected?

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ENRICHMENT

13 The standard normal distribution, which will be studied later in this course, has probability density function $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

a Show that $\phi(x)$ is an even function.

b Explain why $\phi(x) > 0$ for all values of x.

c Evaluate $\phi(0)$ and determine $\lim_{x \to \infty} \phi(x)$.

d Show that $\phi'(x) = -x\phi(x)$, and hence find where the function is decreasing.

e Show that $\phi''(x) = (x^2 - 1)\phi(x)$, and hence locate the two inflection points.

f Sketch the graph of $y = \phi(x)$ showing all these details.

g Using the graph or the signs of $\phi'(x)$ and $\phi''(x)$, determine where in the domain $\phi(x)$ is decreasing at an increasing rate, and where it is decreasing at a decreasing rate.

h How is the latter evident in the graph?

14 In an ideal situation, a sound wave decays away with distance according to the function $y = 2e^{-ax}\cos x$ for $x \geq 0$, where a is a positive constant.

a Find the y-intercept and x-intercepts.

b Use the product rule to show that $y' = -2e^{-ax}(a\cos x + \sin x)$.

c Use the product rule again to show that $y'' = 2e^{-ax}\big((a^2 - 1)\cos x + 2a\sin x\big)$.

d Show that when $a = \tan\frac{\pi}{12}$, the stationary points are at $x = \frac{11\pi}{12}, \frac{23\pi}{12}, \frac{35\pi}{12}, \ldots$

e It is known that $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. Show that $y'' = 0$ at $x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots$ You may assume that these are the inflection points of the curve.

f Sketch the curve showing this information, approximating the y-coordinates of the stationary and inflection points correct to one decimal place.

g Show that in any interval of 2π, the ratio of where y increases at an increasing rate to where y increases at a decreasing rate is 5 : 7.

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9E Review of related rates

In Section 16A of the Year 11 book, we applied the chain rule to rates so that we could deal easily with related rates. This section reviews the ideas and methods.

The example below is a bubble on water, growing over time. The radius and the volume are both increasing. Because volume is a function of radius, we can differentiate with respect to time using the chain rule to express the rate of change of the volume in terms of the rate of change of the radius.

Be very careful with units in all these calculations — this caution holds throughout the chapter, and anywhere where calculus is being applied.

Example 18 — 9E

A hemispherical bubble on the water surface is growing in size.

a Write down the formula for the volume V mm^3 in terms of the radius r mm, then differentiate with respect to time using the chain rule.

b Find the rate at which the volume is increasing when the radius is 6 mm, if the radius is increasing at 3 mm/s.

c Find the rate at which the radius is increasing when the volume is 18 mm^3, if the volume is increasing at 90 mm^3/s.

SOLUTION

a The volume is $V = \frac{1}{2} \times \frac{4\pi}{3} r^3$

$= \frac{2\pi}{3} r^3.$

Differentiating, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}.$

b Substituting $r = 6$ and $\frac{dr}{dt} = 3$,

$\frac{dV}{dt} = 2\pi \times 36 \times 3$

$= 216\pi$ mm^3/s.

c When $V = 18$, $18 = \frac{2\pi}{3} r^3$

$r^3 = 27\pi^{-1}$

$r = 3\pi^{-\frac{1}{3}}.$

Substituting $r = 3\pi^{-\frac{1}{3}}$ and $\frac{dV}{dt} = 90$,

$90 = 2\pi \times 9\pi^{-\frac{2}{3}} \times \frac{dr}{dt}$

$\frac{dr}{dt} = 5\pi^{-\frac{1}{3}}$ mm/s

$\doteqdot 3.4$ mm/s.

13 RELATED RATES

- Express one quantity as a function of the other quantity.
- Then differentiate with respect to time t using the chain rule.

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Exercise 9E

FOUNDATION

Note: This exercise reviews material covered in Exercise 16A of the Year 11 volume.

1 The sides of a square of side length x metres are increasing at a rate of 0.1 m/s.

a Show that the rate of increase of the area is given by $\dfrac{dA}{dt} = 0.2\,x$ m^2/s.

b At what rate is the area of the square increasing when its sides are 5 metres long?

c What is the side length when the area is increasing at 1.4 m^2/s?

d What is the area when the area is increasing at 0.6 m^2/s?

2 The diagonal of a square is decreasing at a rate of $\frac{1}{2}$ m/s.

a Find the area A of a square with a diagonal of length ℓ.

b Hence show that the rate of change of area is $\dfrac{dA}{dt} = -\frac{1}{2}\ell$ m^2/s.

c Find the rate at which the area is decreasing when:

i the diagonal is 10 metres,

ii the area is 18 m^2.

d What is the length of the diagonal when the area is decreasing at 17 m^2/s?

3 The radius r of a sphere is increasing at a rate of 0.3 m/s. In both parts, approximate π using a calculator and give your answer correct to three significant figures.

a Show that the sphere's rate of change of volume is $\dfrac{dV}{dt} = 1.2\pi r^2$, and find the rate of increase of its volume when the radius is 2 metres.

b Show that the sphere's rate of change of surface area is $\dfrac{dS}{dt} = 2.4\pi r$, and find the rate of increase of its surface area when the radius is 4 metres.

4 Jules is blowing up a spherical balloon at a constant rate of 200 cm^3/s.

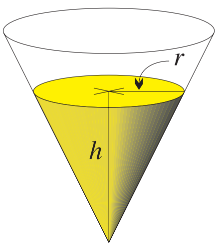
a Show that $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

b Hence find the rate at which the radius is growing when the radius is 15 cm.

c Find the radius and volume when the radius is growing at 0.5 cm/s.

DEVELOPMENT

5 An artist pours plaster of Paris into a conical mould at a rate of 5 cm^3/s. At time t, let the top of the plaster have radius r, and let the depth be h, where $h = 2r$.



a Find the volume of plaster, and show that it is the same as that of a hemisphere with the same radius.

b Find the rate at which the radius is increasing when the depth of the plaster is 10 cm.

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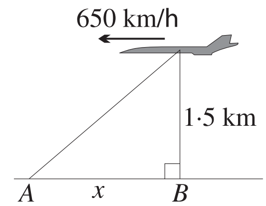
6 An observer at A in the diagram is watching a plane fly overhead, and he tilts his head so that he is always looking directly at the plane. The aircraft is flying at 650 km/h at an altitude of 1.5 km. Let θ be the angle of elevation of the plane from the observer, and suppose that the distance from A to B, directly below the aircraft, is x km.

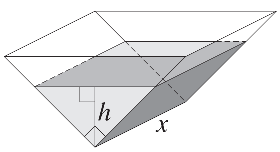


a By writing $x = \dfrac{3}{2\tan\theta}$, show that $\dfrac{dx}{d\theta} = -\dfrac{3}{2\sin^2\theta}$.

b Hence find the rate at which the observer's head is tilting when the angle of inclination to the plane is $\frac{\pi}{3}$. Convert your answer from radians per hour to degrees per second, correct to the nearest degree.

7 A boat is observed from the top of a 100-metre-high cliff. The boat is travelling towards the cliff at a speed of 50 m/min. How fast is the angle of depression changing when the angle of depression is 15°? Convert your answer from radians per minute to degrees per minute, correct to the nearest degree.

8 The water trough in the diagram is in the shape of an isosceles right triangular prism, x metres long. It is found that during summer in a drought, the amount of water lost to evaporation each day in cm^3 is equal to 10% of the surface area in cm^2.



a Find a formula for the volume V cm^3 of water in the trough when the depth is h cm.

b Show that the rate at which the depth of the water is changing is constant.

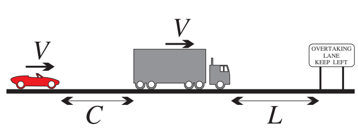
9 The volume of a sphere is increasing at a rate numerically equal to its surface area at that instant. Show that $\dfrac{dr}{dt} = 1$.

10 A point moves anticlockwise around the circle $x^2 + y^2 = 1$ at a uniform speed of 2 m/s.

a Find an expression for the rate of change of its x-coordinate in terms of x, when the point is above the x-axis. (The units on the axes are metres.)

b Use your answer to part **a** to find the rate of change of the x-coordinate as it crosses the y-axis at $P(0, 1)$. Why should this answer have been obvious without this formula?

11 A car is travelling C metres behind a truck, both travelling at a constant speed of V m/s. The road widens L metres ahead of the truck and there is an overtaking lane. The car accelerates at a uniform rate so that it is exactly alongside the truck at the beginning of the overtaking lane.



a What is the acceleration of the car?

b Show that the speed of the car as it passes the truck is $V\left(1 + \dfrac{2C}{L}\right)$.

c The objective of the driver of the car is to spend as little time alongside the truck as possible. What strategies could the driver employ?

d The speed limit is 100 km/h, and the truck is travelling at 90 km/h and is 50 metres ahead of the car. How far before the overtaking lane should the car begin to accelerate if applying the objective in part **c**?

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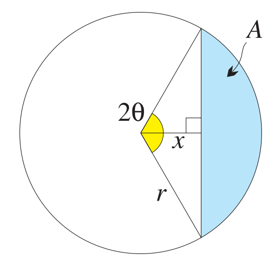
ENRICHMENT

12 The diagram shows a chord distant x from the centre of a circle. The radius of the circle is r, and the chord subtends an angle 2θ at the centre.



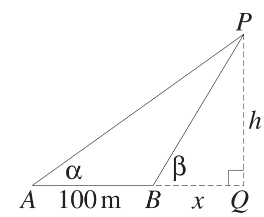
a Show that the area of the segment cut off by this chord is $A = r^2(\theta - \sin\theta\cos\theta)$.

b Explain why $\dfrac{dA}{dt} = \dfrac{dA}{d\theta} \times \dfrac{d\theta}{dx} \times \dfrac{dx}{dt}$.

c Show that $\dfrac{d\theta}{dx} = -\dfrac{1}{\sqrt{r^2 - x^2}}$.

d Given that $r = 2$, find the rate of increase in the area if $\dfrac{dx}{dt} = -\sqrt{3}$ when $x = 1$.

13 The diagram shows two radars at A and B 100 metres apart. An aircraft at P is approaching and the radars are tracking it, hence the angles α and β are changing with time.



a Show that $x\tan\beta = (x + 100)\tan\alpha$.

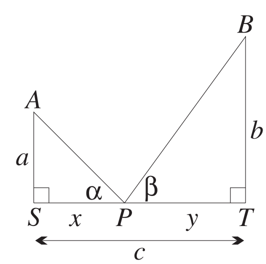
b Keeping in mind that x, α and β are all functions of time, use the chain and product rules to show that

$$\frac{dx}{dt} = \frac{\dot{\alpha}(x + 100)\sec^2\alpha - \dot{\beta}x\sec^2\beta}{\tan\beta - \tan\alpha}.$$

c Use part **a** to find the value of x and the height of the plane when $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$.

d At the angles given in part **c**, it is found that $\dfrac{d\alpha}{dt} = \frac{5}{36}\left(\sqrt{3} - 1\right)$ radians per second and $\dfrac{d\beta}{dt} = \frac{5}{18}\left(\sqrt{3} - 1\right)$ radians per second. Find the speed of the plane.

14 [A proof that for reflected light, the angle of incidence is equal to the angle of reflection.] Suppose that a light source is at A above a reflective surface ST, and the reflected light is observed at B. Further suppose that at the point of reflection P, the angle of incidence is $(90° - \alpha)$ and the angle of reflection is $(90° - \beta)$. Let $AS = a$, $BT = b$ and $ST = c$. Also let $SP = x$ and $PT = y$. We will assume that light travels between two points along a path that takes the shortest time and therefore AP and PB are straight lines. We will also assume that the speed of light is constant.



a Show that the distance APB is $s = \sqrt{a^2 + x^2} + \sqrt{b^2 + y^2}$.

b Let v be the speed of light, then $vt = s$. Also note that $y = c - x$. Use these results and the chain rule where necessary to show that

$$v\frac{dt}{dx} = \frac{\frac{x}{a}}{\sqrt{1 + \left(\frac{x}{a}\right)^2}} - \frac{\frac{y}{b}}{\sqrt{1 + \left(\frac{y}{b}\right)^2}}.$$

c Hence show that t has a stationary point when $\dfrac{x}{a} = \dfrac{y}{b}$.

d By considering the values of $\dfrac{dt}{dx}$ when P is at the points S and T, show that this stationary point is a minimum.

e Complete the proof by showing that $\alpha = \beta$.

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9F Rates and integration

In many situations, what is given is the rate $\frac{dQ}{dt}$ at which a quantity Q is changing. The original function Q can then be found by integration — and as with motion, never omit the constant of integration. To evaluate the constant of integration, use an *initial* or *boundary condition*, giving the value of Q at some particular time t.

14 FINDING THE QUANTITY FROM THE RATE

Suppose that the rate of change of a quantity Q is known as a function of time t.

- Integrate to find Q as a function of time.
- Never omit the constant of integration.
- Use an initial or boundary condition to evaluate the constant of integration.

Example 19 9F

A tank contains 40 000 litres of water. When the draining valve is opened, the volume V in litres of water in the tank decreases at a variable rate given by $\frac{dV}{dt} = -1500 + 30t$, where t is the time in seconds after opening the valve. Once the water stops flowing, the valve shuts off.

a When does the water stop flowing?

b Give a common-sense reason why the rate $\frac{dV}{dt}$ is negative up to this time.

c Integrate to find the volume of water in the tank at time t, and sketch the graph of volume V as a function of time t.

d How much water has flowed out of the tank and how much remains?

SOLUTION

a Put $\quad \frac{dV}{dt} = 0.$

Then $\quad -1500 + 30t = 0$

$\quad t = 50,$

so it takes 50 seconds for the flow to stop.

b During this 50 seconds, the water is flowing out of the tank.

Hence the volume V is decreasing, so the derivative $\frac{dV}{dt}$ is negative.

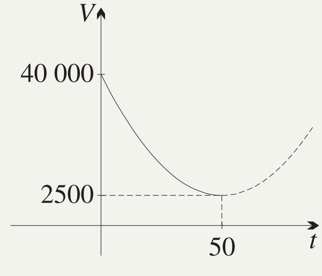
c Integrating, $\quad V = -1500t + 15t^2 + C,$ for some constant C.

It is given that when $t = 0$, $V = 40\,000$,

and substituting, $\quad 40\,000 = 0 + 0 + C$

$\quad C = 40\,000.$

Hence $\quad V = 40\,000 - 1500t + 15t^2.$



d When $t = 50$, $\quad V = 40\,000 - (1500 \times 50) + (15 \times 2500)$

$\quad = 2500.$

Hence the tank still holds 2500 litres when the valve closes,

so $40\,000 - 2500 = 37\,500$ litres has flowed out during the 50 seconds.

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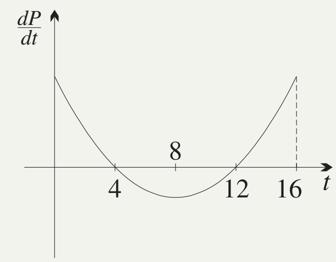
Questions with a diagram or a graph instead of an equation

In some problems about rates, a graph of some function is known, but its equation is unknown. Such problems require careful attention to zeroes and turning points and points of inflection. An approximate sketch of another graph often needs to be drawn.

Example 20 9F

Frog numbers were increasing in the Ranavilla district, but during a long drought, the rate of increase fell and actually became negative for a few years. The rate $\frac{dP}{dt}$ of population growth of the frogs has been graphed to the right as a function of the time t years after careful observations began.



a When was the frog population neither increasing nor decreasing?
b When was the frog population decreasing and when was it increasing?
c When was the frog population decreasing most rapidly?
d When, during the first 12 years, was the frog population at a maximum?
e When, during the years $4 \le t \le 16$, was the frog population at a minimum?
f Draw a possible graph of the frog population P against time t.

SOLUTION

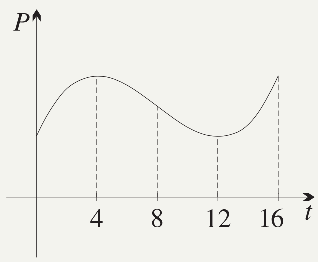
a The graph shows that $\frac{dP}{dt}$ is zero when $t = 4$ and again when $t = 12$.
These are the times when the frog population was neither increasing nor decreasing.

b The graph shows that $\frac{dP}{dt}$ is negative when $4 < t < 12$,
so the population was decreasing during the years $4 < t < 12$.
The graph shows that $\frac{dP}{dt}$ is positive when $0 < t < 4$ and when $12 < t < 16$,
so the population was increasing during the years $0 < t < 4$ and during $12 < t < 16$.

c The frog population was decreasing most rapidly when $t = 8$.

d The population was at a maximum when $x = 4$,
because from parts **a** and **b**, the population was rising before this and falling afterwards.

e Similarly, the population was minimum when $x = 12$.



f All that matters is to draw the possible graph of P so that its gradients are consistent with the graph of $\frac{dP}{dT}$. These things were discussed above in parts **a**–**d**. Also, the frog population must never fall below zero.

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Rates involving the exponential function

Many natural events involve a quantity that dies away gradually, with an equation that involves the exponential function. The next worked example uses the standard form $\int e^{ax+b} = \frac{1}{a}e^{ax+b} + C$ to evaluate the primitive of $3\,e^{-0.02t}$. The full working is

$$\begin{aligned}\int 3\,e^{-0.02t}dt &= 3 \times \frac{1}{-0.02} \times e^{-0.02t} + C\\ &= -3 \times \frac{100}{2} \times e^{-0.02t} + C\\ &= -150\,e^{-0.02t} + C.\end{aligned}$$

Example 21 9F

During a drought, the flow rate $\frac{dV}{dt}$ of water from Welcome Well gradually diminishes according to the formula $\frac{dV}{dt} = 3\,e^{-0.02t}$, where V is the volume in megalitres of water that has flowed out during the first t days after time zero.

a Show that $\frac{dV}{dt}$ is always positive, and explain the physical significance of this.

b Find the volume V as a function of time t.

c How much water will flow from the well during the first 100 days?

d Describe the behaviour of V as $t \to \infty$, and find what percentage of the total flow comes in the first 100 days. Then sketch the function.

SOLUTION

a Because $e^x > 0$ for all x, $\frac{dV}{dt} = 3\,e^{-0.02t}$ is always positive.
The volume V is always increasing, because V is the volume that has flowed out of the well, and the water doesn't flow backwards into the well.

b The given rate is $\frac{dV}{dt} = 3\,e^{-0.02t}$.

Integrating, $V = -150\,e^{-0.02t} + C$ (using the calculation above).

When $t = 0$, no water has flowed out, so $V = 0$,

and substituting, $0 = -150 \times e^0 + C$

$C = 150.$

Hence $V = -150\,e^{-0.02t} + 150$

$= 150\left(1 - e^{-0.02t}\right).$

c When $t = 100$, $V = 150(1 - e^{-2})$

$\doteqdot 129.7$ megalitres.

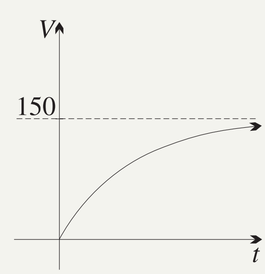
d As $t \to \infty$, $e^{-0.02t} \to 0$, so $V \to 150$.

Hence $\dfrac{\text{flow in first 100 days}}{\text{total flow}} = \dfrac{150(1 - e^{-2})}{150}$

$= 1 - e^{-2}$

$= 0.864\,66\ldots$

$\doteqdot 86.5\%.$



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Exercise 9F

FOUNDATION

1 Twenty-five wallabies are released on Wombat Island and the population is observed over the next six years. It is found that the rate of increase in the wallaby population is given by $\frac{dP}{dt} = 12t - 3t^2$, where time t is measured in years.

a Show that $P = 25 + 6t^2 - t^3$.
b After how many years does the population reach a maximum? (Hint: Let $\frac{dP}{dt} = 0$.)
c What is the maximum population?
d When does the population increase most rapidly? (Hint: Let $\frac{d^2P}{dt^2} = 0$.)

2 Water is flowing out of a tank at the rate of $\frac{dV}{dt} = 10t - 250$, where V is the volume in litres remaining in the tank at time t minutes after time zero.

a When does the water stop flowing?
b Given that the tank still has 20 litres left in it when the water flow stops, show that the volume V at any time is given by $V = 5t^2 - 250t + 3145$.
c How much water was initially in the tank?

3 The rate at which a perfume ball loses its scent over time is $\frac{dP}{dt} = -\frac{2}{t + 1}$, where t is measured in days.

a Find P as a function of t if the initial perfume content is 6.8.
b How long will it be before the perfume in the ball has run out and it needs to be replaced? (Answer correct to the nearest day.)

4 A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by $\frac{dV}{dt} = -2 + \frac{1}{10}t\ \mathrm{m^3/s}$.

a What is the initial flow rate, when the tap is fully on?
b How long does it take to turn the tap off?
c Given that when the tap has been turned off there are still 500 $\mathrm{m^3}$ of water left in the tank, find V as a function of t.
d Hence find how much water is released during the time it takes to turn the tap off.
e Suppose that it is necessary to let out a total of 300 $\mathrm{m^3}$ from the tank. How long should the tap be left fully on before gradually turning it off?

5 The velocity of a particle is given by $\frac{dx}{dt} = e^{-0.4t}$.

a Does the particle ever stop moving?
b If the particle starts at the origin, show that its displacement x as a function of t is given by $x = \frac{5}{2}\left(1 - e^{-0.4t}\right)$.
c When does the particle reach $x = 1$? (Answer correct to two decimal places.)
d Where does the particle eventually move to? (That is, find its limiting position.)

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DEVELOPMENT

6 A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time t, is given by $\dfrac{dx}{dt} = 250\left(e^{-0.2t} - 1\right)$, where x is the height above the ground.

a What is its initial speed?

b What is its eventual speed?

c Find x as a function of t, if it is initially 200 metres above the ground.

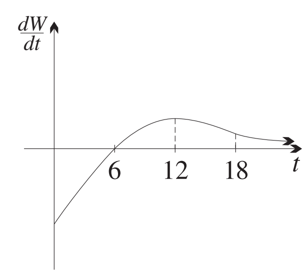
7 Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\dfrac{dI}{dt} = -5 + 4\cos\frac{\pi}{12}t$, where I is the tonnage of ice on the mountain at time t in hours since 2:00 am on 20th October.

a It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find I as a function of t.

b Explain, from the given rate, why the ice is always melting.

c The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.

8 The graph to the right shows the rate $\dfrac{dW}{dt}$ at which the average weight W of bullocks at St Vidgeon station was changing t months after a drought was officially proclaimed.



a When was the average weight decreasing and when was it increasing?

b When was the average weight at a minimum?

c When was the average weight increasing most rapidly?

d What appears to have happened to the average weight as time went on?

e Sketch a possible graph of the average weight W.

9 As a particle moves around a circle, its angular velocity is given by $\dfrac{d\theta}{dt} = \dfrac{1}{1 + t^2}$.

a Given that the particle starts at $\theta = \frac{\pi}{4}$, find θ as a function of t.

b Hence find t as a function of θ.

c Using the result of part **a**, show that $\frac{\pi}{4} \le \theta < \frac{3\pi}{4}$, and hence explain why the particle never moves through an angle of more than $\frac{\pi}{2}$.

10 The flow of water into a small dam over the course of a year varies with time and is approximated by $\dfrac{dW}{dt} = 1.2 - \cos^2\frac{\pi}{12}t$, where W is the volume of water in the dam, measured in thousands of cubic metres, and t is the time measured in months from the beginning of January.

a What is the maximum flow rate into the dam, and when does this happen?

b Given that the dam is initially empty, find W.

c The capacity of the dam is 25 200 m^3. Show that it will be full in three years.

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11 A certain brand of medicine tablet is in the shape of a sphere with diameter 5 mm. The rate at which the pill dissolves is $\frac{dr}{dt} = -k$, where r is the radius of the sphere at time t hours, and k is a positive constant.

a Show that $r = \frac{5}{2} - kt$.

b The pill dissolves completely in 12 hours. Find k.

12 Sand is poured onto the top of a pile in the shape of a cone at a rate of 0.5 m^3/s. The apex angle of the cone remains constant at 90°. Let the base have radius r and let the height of the cone be h.

a Find the volume of the cone, and show that it is one quarter of the volume of a sphere with the same radius.

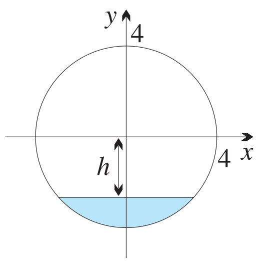
b Find the rate of change of the radius of the cone as a function of r.

c By taking reciprocals and integrating, find t as a function of r, given that the initial radius of the pile was 10 metres.

d Hence find how long it takes, correct to the nearest second, for the pile to grow another 2 metres in height.

ENRICHMENT

13 **a** The diagram shows the spherical cap formed when the region between the lower half of the circle $x^2 + y^2 = 16$ and the horizontal line $y = -h$ is rotated about the y-axis. Find the volume V so formed.



b The cap represents a shallow puddle of water left after some rain. When the sun comes out, the water evaporates at a rate proportional to its surface area (which is the circular area at the top of the cap).

i Find this surface area A.

ii We are told that $\frac{dV}{dt} = -kA$. Show that the rate at which the depth of the water changes is $-k$.

iii The puddle is initially 2 cm deep and the evaporation constant is known to be $k = 0.025\ \text{cm}/\text{min}$. Find how long it takes for the puddle to evaporate.

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Chapter 9 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 9 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity from $t = 2$ to $t = 4$. The units in each part are centimetres and seconds.

t	2	4
x		

a $x = 20 + t^2$ **b** $x = (t + 2)^2$ **c** $x = t^2 - 6t$ **d** $x = 3^t$

2 For each displacement function below, find the velocity function and the acceleration function. Then find the displacement, the velocity and the acceleration of the particle when $t = 5$. All units are metres and seconds.

a $x = 40t - t^2$ **b** $x = t^3 - 25t$ **c** $x = 4(t - 3)^2$
d $x = 50 - t^4$ **e** $x = 4\sin \pi t$ **f** $x = 7e^{3t-15}$

3 A ball rolls up an inclined plane and back down again. Its distance x metres up the plane after t seconds is given by $x = 16t - t^2$.

a Find the velocity function v and the acceleration function a.
b What are the ball's position, velocity, speed and acceleration after 10 seconds?
c When does the ball return to its starting point, and what is its velocity then?
d When is the ball farthest up the plane, and where is it then?
e Sketch the displacement–time graph, the velocity–time graph and the acceleration–time graph.

4 Differentiate each velocity function below to find the acceleration function a. Then integrate v to find the displacement function x, given that the particle is initially at $x = 4$.

a $v = 7$ **b** $v = 4 - 9t^2$ **c** $v = (t - 1)^2$
d $v = 0$ **e** $v = 12\cos 2t$ **f** $v = 12e^{-3t}$

5 For each acceleration function below, find the velocity function v and the displacement function x, given that the particle is initially stationary at $x = 2$.

a $a = 6t + 2$ **b** $a = -8$ **c** $a = 36t^2 - 4$
d $a = 0$ **e** $a = 5\cos t$ **f** $a = 7e^t$

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Review

6 A particle is moving with acceleration function $\ddot{x} = 6t$, in units of centimetres and seconds. Initially it is at the origin and has velocity $12\ \text{cm/s}$ in the negative direction.

a Find the velocity function $\dot{x}$ and the displacement function x.

b Show that the particle is stationary when $t = 2$.

c Hence find its maximum distance on the negative side of the origin.

d When does it return to the origin, and what are its velocity and acceleration then?

e What happens to the particle's position and velocity as time goes on?

7 A stone is thrown vertically upwards with velocity 40 m/s from a fixed point B situated 45 metres above the ground. Take $g = 10\ \text{m/s}^2$.

a Taking upwards as positive, explain why the acceleration function is $a = -10$.

b Using the ground as the origin, find the velocity function v and the displacement function x.

c Hence find how long the stone takes to reach its maximum height, and find that maximum height.

d Show that the time of flight of the stone until it strikes the ground is 9 seconds.

e With what speed does the stone strike the ground?

f Find the height of the stone after 1 second and after 2 seconds.

g Hence find the average velocity of the stone during the 2nd second.

8 The acceleration of a body moving along a line is given by $\ddot{x} = \sin t$, where x is the distance from the origin O at time t seconds.

a Sketch the acceleration–time graph.

b From your graph, state the first two times after $t = 0$ when the acceleration is zero.

c Integrate to find the velocity function, given that the initial velocity is $-1\ \text{m/s}$.

d What is the first time when the body stops moving?

e The body is initially at $x = 5$.

i Find the displacement function x.

ii Find where the body is when $t = \frac{\pi}{2}$.

9 The velocity of a particle is given by $v = 20\,e^{-t}$, in units of metres and seconds.

a What is the velocity when $t = 0$?

b Why is the particle always moving in a positive direction?

c Find the acceleration function a.

d What is the acceleration at time $t = 0$?

e The particle is initially at the origin. Find the displacement function x.

f What happens to the acceleration, the velocity and the displacement as t increases?

10 The stud farm at Benromach sold a prize bull to a grazier at Dalmore, 300 kilometres west. The truck delivering the bull left Benromach at 9:00 am, driving over the dirt roads at a constant speed of $50\ \text{km/hr}$. At 10:00 am, the driver realised that he had left the sale documents behind, so he drove back to Benromach at the same speed. He then drove the bull and the documents straight to Dalmore at $60\ \text{km/hr}$.

a Draw the displacement–time graph of his displacement x kilometres west of Benromach at time t hours after 9:00 am.

b What total distance did he travel?

c What was his average road speed for the whole journey?

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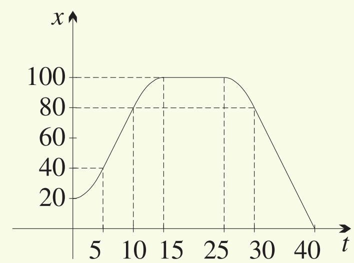
11 Crispin was trying out his bicycle in Abigail Street. The graph below shows his displacement in metres north of the oak tree after t seconds.



a Where did he start from, and what was his initial speed?

b What was his velocity:

i from $t = 5$ to $t = 10$,

ii from $t = 15$ to $t = 25$,

iii from $t = 30$ to $t = 40$?

c In what direction was he accelerating:

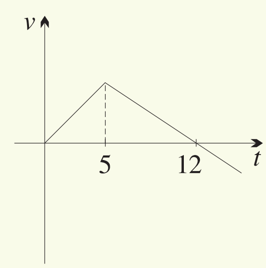
i from $t = 0$ to $t = 5$,

ii from $t = 10$ to $t = 15$,

iii from $t = 25$ to $t = 30$?

d Draw a possible sketch of the velocity–time graph.

12 A small rocket was launched vertically from the ground. The graph to the right shows its velocity–time graph. After a few seconds the motor cut out. A few seconds later the rocket reached its maximum height and then began to fall back towards the ground.



a When did the motor cut out?

b When was the rocket stationary, when was it moving upwards and when was it moving downwards?

c When was the rocket accelerating upwards, and when was it accelerating downwards?

d When was the rocket at its maximum height?

e Sketch the acceleration–time graph.

f Sketch the displacement–time graph.

13 **a** Draw rough sketches of these functions for $0 \le x \le \frac{\pi}{2}$.

i $y = \sin x$ **ii** $y = \cos x$ **iii** $y = \tan x$ **iv** $y = \cot x$

b For the graphs you drew, and over the given domain, which functions are:

i increasing at a decreasing rate **ii** decreasing at an increasing rate

iii decreasing at a decreasing rate **iv** increasing at an increasing rate

14 The kakapo is a critically endangered species of parrot in New Zealand. In 2017 the New Zealand Department of Conservation released a number of kakapo on Little Barrier Island. Biologists hoped that the population would grow slowly at first and later more quickly. It was also expected that after several more years the population would then grow at a slower and slower rate. Answer the following questions, assuming the biologists' predictions were correct.

a Explain why a graph of the kakapo population K over time t has an inflection point.

b Sketch a possible graph of K as a function of t, showing the inflection point.

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Review

15 The volume V litres of water in a tank at time t minutes is given by $V = 3(50 - 2t)^2$, for $0 \le t \le 25$.

a What is the initial volume of liquid in the tank?

b Determine $\dfrac{dV}{dt}$.

c Use $\dfrac{dV}{dt}$ to explain why the tank must be emptying.

d Does the outflow increase or decrease with time?

16 The velocity of a particle is given by $\dfrac{dx}{dt} = 3 - 2e^{-\frac{1}{5}t}$, with x measures in metres and t in seconds.

a Draw a graph of the velocity versus time.

b Is the particle accelerating or decelerating? Confirm your answer by finding $\dfrac{d^2x}{dt^2}$.

c What is the eventual velocity of the particle?

d The particle starts at the origin. Find x as a function of t.

17 James had a full drink bottle containing 500 ml of Gatorade™. He drank from it so that the volume V ml of Gatorade™ in the bottle changed at a rate given by $\dfrac{dV}{dt} = \left(\frac{2}{5}t - 20\right)$ ml/s.

a Find a formula for V.

b Show that it took James 50 seconds to drink the contents of the bottle.

c How long, correct to the nearest second, did it take James to drink half the contents of the bottle?

18 A 5-metre ladder is leaning against a wall, and the base is sliding away from the wall at 5 cm/s. Find the rate at which:

a the height,

b the angle of inclination, is changing when the foot is 1.4 m from the wall.

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