A Tractable Paraconsistent Fuzzy Description Logic

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Abstract. In this paper, we introduce the tractable $pf\text{-}\mathcal{EL}^{++}$ logic, a paraconsistent version of the fuzzy logic $f\text{-}\mathcal{EL}^{++}$. Within $pf\text{-}\mathcal{EL}^{++}$, it is possible to tolerate contradictions under incomplete and vague knowledge. $pf\text{-}\mathcal{EL}^{++}$ extends the $f\text{-}\mathcal{EL}^{++}$ language with a paraconsistent negation in order to represent contradictions. This paraconsistent negation is defined under Belnap's bilattices. Besides its syntax and semantics, we also prove that $pf\text{-}\mathcal{EL}^{++}$ is a conservative extension of $f\text{-}\mathcal{EL}^{++}$, thus assuring that the polynomial-time reasoning algorithm used in $f\text{-}\mathcal{EL}^{++}$ can also be used in $pf\text{-}\mathcal{EL}^{++}$.

1 Introduction

A difficult task in a knowledge base that aims to formalise a real world application is to deal with incomplete, imprecise and contradictory information. To illustrate this problem, suppose that a doctor diagnoses the illness of a patient as a benign tumour based on first symptoms and some laboratory exams. However, in the presence of additional exams, he realizes that the tumour is indeed a cancer. Hence, it is unreasonable to expect that a knowledge base which allows realistic reasoning based on partial knowledge must always be kept logically consistent.

An important tool in symbolic knowledge representation is the logical formalism. In the last century, logics were designed to handle inconsistencies without deriving anything from a contradiction. These logics were called paraconsistent and the former ones were from Jaśkowski [13] and Da Costa [9]. Here, we are particularly interested in the paraconsistent logic introduced by Belnap [7]. Besides the true and false values, Belnap's logic introduces two additional truth values: the *underdefined* one, for the denotation of a proposition that is neither true or false, and the *overdefined* one, for the denotation of what is both true and false, that is, contradictory.

There are also some logical approaches that attempt to formalise reasoning under incomplete and imprecise knowledge. Some are symbolic ones, as the nonmonotonic logics, but here we will focus on a numeric approach: the fuzzy logic introduced by Zadeh [22]. This logic is based not only on fuzzy sets but also on the fuzzy logic theory, and it has been used in several real world applications [23].

Although expressive enough to deal with incomplete, imprecise and contradictory information, the satisfiability problem for paraconsistent and fuzzy logics is undecidable. Since real world applications demand efficient inference systems, a family of logics, the Description Logics (DLs) [5], have been proposed. DLs are decidable fragments of classical first-order logic, and they have been customarily used in the definition of ontologies and applications for the Semantic Web [2].

A lot of work has been carried out in extending DLs to increase its expressivity for knowledge representation. However, some of their extensions are shown to be intractable [6]. Fortunately, there are some examples of DLs with a good balance between expressivity and complexity, namely: $\mathcal{EL}^{++}$ [6], Horn-DLs [14] and DL-Lite [4].

It was shown that $\mathcal{EL}^{++}$ has a polynomial-time subsumption algorithm. Because of it, many knowledge base applications for classification tasks were designed based in $\mathcal{EL}^{++}$ as, for instance, systems that represent a medical nomenclature [19,8]. Note that, in a medical knowledge base, some definitions, diseases, symptoms, procedures may contain partial and inconsistent information which are not expressible in $\mathcal{EL}^{++}$. Hence, extensions of $\mathcal{EL}^{++}$ which handle vague, imprecise or contradictory information can be useful in practice.

In [17], a fuzzy logic f-$\mathcal{EL}^{++}$ was specially defined to deal with imprecise and vague knowledge without losing tractability. Unfortunately, this logic cannot express negative information. In fact, it was proved that the introduction of the classical negation in DLs leads to undecidability [6].

In this paper, we introduce the tractable pf-$\mathcal{EL}^{++}$ logic, a paraconsistent version of f-$\mathcal{EL}^{++}$ that is able to tolerate contradiction under incomplete and vague knowledge. It extends the f-$\mathcal{EL}^{++}$ language with a paraconsistent negation in order to represent contradictions.

The paper is structured as follows. We first review the syntax and semantics of the fuzzy logic f-$\mathcal{EL}^{++}$ in Section 2. In the sequel, we present Belnap's bilattices. In sections 4 and 5, we introduce the pf-$\mathcal{EL}^{++}$ syntax, semantics and a translation algorithm to f-$\mathcal{EL}^{++}$ proving that the former is a conservative extension of the latter. With this translation, we assure the tractability of pf-$\mathcal{EL}^{++}$. We discuss conclusions and future work in Section 6.

2 The f-$\mathcal{EL}^{++}$ Logic

The logic $\mathcal{EL}^{++}$ is a well-known description logic with a polynomial-time subsumption algorithm [6]. Because of such a great performance, some works have proposed tractable extensions for it [21,17]. In this section we focus on one of them: f-$\mathcal{EL}^{++}$ [17], a fuzzy extension of $\mathcal{EL}^{++}$ tailored to tackle with imprecise and vague knowledge. In the sequel, we present its syntax and semantics:

Definition 1 (Alphabet) *The f-$\mathcal{EL}^{++}$ alphabet is composed of the symbols $\sqcap, \exists, \top, \bot$ and three disjoint sets: the set of individuals N_I, the set of concept names N_C and the set of role names N_R.*

An individual represents an element of the universe of discourse (domain). Concept names represent unary predicates related to individuals, For instance, let a be an individual and A be a concept name; to represent that a is an instance of A, we write $A(a)$. Role names represent binary predicates between individuals. For example, let a and b be individuals and R be a role name; to describe that the individual a is related with the individual b by R, we write $R(a, b)$. Elementary descriptions as $A(a)$ and $R(a, b)$

are called, respectively, *atomic concepts* and *atomic roles*. General descriptions can be inductively built from them with concept constructors as follows:

Definition 2 (Concept) *Concepts in f-$\mathcal{EL}^{++}$ are defined by the syntax rules below, where C and D are concepts, A is an atomic concept, a is an individual and R is an atomic role: $C, D \longrightarrow A \mid \top \mid \bot \mid \{a\} \mid C \sqcap D \mid \exists R.C$.*

$\{a\}$ represents the nominal concept. Nominal is a singleton $\{a\}$ that can be understood as the definition of a constant. As an example of what can be expressed in f-$\mathcal{EL}^{++}$, suppose that Item and New are atomic concepts, then $\mathsf{Item} \sqcap \mathsf{New}$ is a concept describing those items that are new. In addition, suppose that $\mathsf{Expensive}$ is an atomic concept and Cost is an atomic role; we can create the concept $\mathsf{Item} \sqcap \exists\mathsf{Cost}.\mathsf{Expensive}$, denoting those items which have an expensive cost. We can also define the concept Item as a nominal, for example $\{car\}$.

Note that concepts like New and $\mathsf{Expensive}$ may not express a precise idea. We call *fuzzy concepts* all those which can communicate imprecise information. In order to represent imprecision, fuzzy concepts can be evaluated in many degrees of truth, usually denoted by a real interval from 0 to 1. For example, if an expensive item was evaluated with the value of 0.3, it could mean that this item is "little" expensive. The formal semantics of f-$\mathcal{EL}^{++}$ concepts contains is given as follows:

Definition 3 (Concept Semantics) *The semantics of f-$\mathcal{EL}^{++}$ concepts is given by a fuzzy interpretation I = (Δ^I, $.^I$), where the domain Δ^I is a nonempty set of elements and $.^I$ is a mapping function defined by: each individual $a \in N_I$ is mapped to $a^I \in \Delta^I$; each atomic concept name $A \in N_C$ is mapped to $A^I : \Delta^I \to [0,1]$; each atomic role name $R \in N_R$ is mapped to $R^I : \Delta^I \times \Delta^I \to [0,1]$.*

Concepts can be inductively interpreted as follows, where for all $x \in \Delta^I$,

Name	Syntax	Semantics
top	$\top$	$\top^I(x) = 1$
bottom	$\bot$	$\bot^I(x) = 0$
nominal	$\{a\}$	$\{a\}^I(x) = \begin{cases} 1 \text{ if } x = a^I \\ 0 \text{ otherwise} \end{cases}$
conjunction	$C \sqcap D$	$(C \sqcap D)^I(x) = min(C^I(x), D^I(x))$
existential restriction	$\exists R.C$	$(\exists R.C)^I(x) = \sup_{y \in \Delta^I} (min(R^I(x,y), C^I(y)))$

Now we define the fuzzy extension of TBox and ABox used in f-$\mathcal{EL}^{++}$:

Definition 4 (Fuzzy TBox) *A fuzzy TBox (Terminological Box) consists of a finite set of fuzzy general concept inclusions (f-GCIs) and role inclusion axioms (RIAs). f-GCIs are defined as $C \sqsubseteq_n D$, where $n \in [0,1]$. RIAs are defined as $R_1 \circ \ldots \circ R_k \sqsubseteq S$, where the $\circ$ denotes role composition.*

Given $C \sqsubseteq_n D$, this axiom expresses that the degree of subsethood of C to D is at-least equal to n. Note that the fuzzy subsumption ($\sqsubseteq_n$) was chosen not to be used in RIAs [20], i.e. it is the same thing that we say $R_1 \circ \ldots \circ R_k \sqsubseteq_1 S$. RIAs can be used to express various important role properties in ontology applications: *role hierarchies* $R \sqsubseteq S$; *transitive roles*, which can be expressed by $R \circ R \sqsubseteq R$; *reflexive roles*, which can be expressed by $R \sqsubseteq R$; and also *right-identity rules* $R \circ S \sqsubseteq R$.

Definition 5 (Fuzzy ABox) *A fuzzy ABox (Assertional Box) consists of a finite set of assertion axioms of the form $C(a) \geq n$ and $R(a, b) \geq n$, where $n \in [0, 1]$.*

These assertions denote that the membership degree of an individual a to concept C is greater or equal to n and the membership degree of the individuals (a, b) to role R is greater or equal to n, respectively. Note that we can represent the crisp case using the notation $C(a) \geq 1$ and $R(a, b) \geq 1$.

Definition 6 (Ontology) *An ontology or knowledge base in f-$\mathcal{EL}^{++}$ is a set composed by a fuzzy TBox and a fuzzy ABox.*

The semantics of inclusion and assertion axioms is given by the following table, where for all $x, y \in \Delta^I$:

Axiom Name	Syntax	Semantics
f-GCI	$C \sqsubseteq_n D$	$min(C^I(x), n) \leq D^I(x)$
RIA	$R_1 \circ \ldots \circ R_k \sqsubseteq S$	$[R_1^I \circ^t \ldots \circ^t R_k^I](x, y) \leq S^I(x, y)$
Concept assertion	$C(a) \geq n$	$C^I(a^I) \geq n$
Role assertion	$R(a, b) \geq n$	$R^I(a^I, b^I) \geq n$

Definition 7 (Satisfiability) *The notion of satisfiability of an axiom α by a fuzzy interpretation I, denoted $I \models \alpha$, is defined as follows: $I \models C \sqsubseteq_n D$ iff $\forall x \in \Delta^I$, $min(C^I(x), n)) \leq D^I(x)$. $I \models R_1 \circ \ldots \circ R_k \sqsubseteq S$ iff $\forall x, y \in \Delta^I$, $[R_1^I \circ^t \ldots \circ^t R_k^I](x, y) \leq S^I(x, y)$. $I \models C(a) \geq n$ iff $C^I(a^I) \geq n$. $I \models R(a, b) \geq n$ iff $R^I(a^I, b^I) \geq n$. For a set of axioms ε, we say that I satisfies ε iff I satisfies each element in ε. If $I \models \alpha$ we say that I is a model of α. I satisfies an ontology O, denoted by $I \models O$, iff I is a model of each axiom of the ontology O.*

Definition 8 (Logical Consequence) *An axiom α is a logical consequence of an ontology O, denoted by $O \models \alpha$ iff every model of O satisfies α.*

Unfortunately, f-$\mathcal{EL}^{++}$ cannot express negative negation on concepts, due to the loss of tractability [6]. Our contribution in this paper focuses on presenting an extension capable of representing negative knowledge and consequently dealing with incomplete and inconsistent information, but keeping the same decidability complexity. Before presenting pf-$\mathcal{EL}^{++}$, we need to show some algebraic structures called bilattices, which will help us in the representation of paraconsistent information and which will play an important role in the definition of syntax and semantics of pf-$\mathcal{EL}^{++}$.

3 Bilattices

In [7] Belnap introduced a logic intended to deal with inconsistent and incomplete information. This logic is capable of representing four truth values: t (true), f (false), $\ddot{\top}$

(overdefined) and $\ddot{\bot}$ (underdefined). The underdefined value represents the total lack of knowledge, while the overdefined one represents the excess of knowledge (conflicts between information).

Belnap's logic was generalized by Ginsberg [12], who introduced the notion of bilattices, which are algebraic structures containing an arbitrary number of truth values simultaneously arranged in two partial orders. Further, these structures were studied by Fitting [11], who attested that they are particularly adequate to represent knowledge in situations where we can find uncertainty, incompleteness, and inconsistency. Indeed, the use of bilattices for supporting paraconsistent reasoning has been tackled by works as [10,3,1]. In the sequel, we will show the definition of bilattices and introduce the particular billatice employed in the representation of fuzzy truth-values in our proposal:

Definition 9 (Complete Bilattice) *Given two complete lattices*[1] $\langle C, \leq_1\rangle$ *and* $\langle D, \leq_2\rangle$, *the structure* $\mathcal{B}(C,D)$=$\langle C\times D, \leq_k, \leq_t, \neg\rangle$ *is a complete bilattice, in which:* $\langle c_1, d_1\rangle \leq_k \langle c_2, d_2\rangle$ if $c_1 \leq_1 c_2$ and $d_1 \leq_2 d_2$, $\langle c_1, d_1\rangle \leq_t \langle c_2, d_2\rangle$ if $c_1 \leq_1 c_2$ and $d_2 \leq_2 d_1$. *Furthermore,* $\neg : C \times D \rightarrow D \times C$ *is a negation operation such that: (1)* $a \leq_k b \Rightarrow \neg a \leq_k \neg b$*, (2)* $a \leq_t b \Rightarrow \neg b \leq_t \neg a$*, (3)* $\neg\neg a = a$.

The intuition here is that C provides the evidence for (and D evidence against) believing in the truth of a statement. If $\langle c_1, d_1\rangle \leq_k \langle c_2, d_2\rangle$, then situation 2 has more information than 1, i.e. knowledge is increased, both in truth and in falsity. If $\langle c_1, d_1\rangle \leq_t \langle c_2, d_2\rangle$, then we have more reasons to believe in situation 2 than in 1, because the reasons to believe in the statement 2 is increased and/or the reasons against it are weaker, i.e. $\langle c_2, d_2\rangle$ is truer than $\langle c_1, d_1\rangle$. The negation is defined just as an operator that should reverse the degree of truth, but preserve that of knowledge.

In our paper, we will focus our attention on the bilattice $\mathcal{B}^2$ based on the interval $[0, 1]$:

Definition 10 (Bilattice $\mathcal{B}^2$) *The structure* $\mathcal{B}^2 = \langle [0,1] \times [0,1], \leq_t, \leq_k, \neg\rangle$ *is a complete bilattice where the* $\neg$ *satisifies* $\neg\langle x_1, x_2\rangle = \langle x_2, x_1\rangle$.

It is also important to observe that if we replace $[0, 1] \times [0, 1]$ by $\{0, 1\} \times \{0, 1\}$ in $\mathcal{B}^2$, we will obtain a bilattice which gives us an isomorphic copy of Belnap's four-valued logic [7]. Thus, we can see $\mathcal{B}^2$ as an extension of Belnap's proposal in which instead of a bilattice with four values, we are employing a bilattice with infinite values in $[0, 1] \times [0, 1]$. In Figure 1 we illustrate both bilattices: note that whereas Belnap's bilattice has four values, in $\mathcal{B}^2$ the inner area of the bilattice is filled with a pair $\langle x, y\rangle$, where $x, y \in [0, 1]$ is a truth-value.

In an element $x = \langle x_1, x_2\rangle$ in $[0, 1] \times [0, 1]$, x_1 and x_2 represent, respectively, the membership and non-membership degrees of x in $[0, 1]$. This means that x_2 can be any value in $[0, 1]$ and not necessarily $1 - x_1$ as one would expect in the classical case. It is a very important distinction because it will allow us to identify contradictory truth-values. In the sense of Fitting's work: a truth-value $x = \langle x_1, x_2\rangle$ is *contradictory* whenever $x_1 + x_2 > 1$.

[1] Let L be a nonempty set and $\leq$ a partial order on L. The pair $\langle L, \leq\rangle$ is a complete lattice if every subset of L has both a least upper bound and a greatest lower bound according to $\leq$.

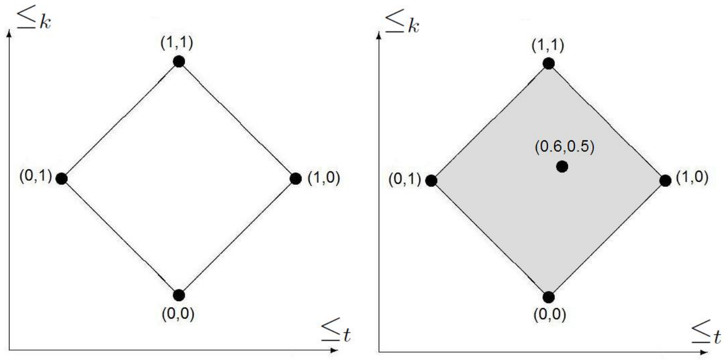

Fig. 1. Belnap's bilattice and $\mathcal{B}^2$

In the next section we will employ $\mathcal{B}^2$ to represent the truth-values of the logic pf-$\mathcal{EL}^{++}$. Unlike the classical negation in DLs, the negation introduced in pf-$\mathcal{EL}^{++}$ is related to the operator $\neg$ defined on $\mathcal{B}^2$. As we will detail in the forthcoming sections, an advantage of this approach is to guarantee a paraconsistent treatment for contradictory information without losing the tractability of its complexity algorithm.

4 The pf-$\mathcal{EL}^{++}$ Logic

In this section we propose a new Description Logic, pf-$\mathcal{EL}^{++}$, by extending f-$\mathcal{EL}^{++}$ [17] with the negation operator $\neg$. The resulting logic is general enough to couple with incomplete/vague/fuzzy and contradictory information. Such an enriched language also gives room to introduce two kinds of concept inclusion as we will detail further. Motivated by the paraconsistent extension to description logic $\mathcal{ALC}$ [18] proposed by [16,15], we will employ a bilattice of truth-values in order to represent explicitly the degree of inclusion and non-inclusion of an individual to a concept. We now proceed by describing the syntax of pf-$\mathcal{EL}^{++}$ logic:

The unique difference between the syntax of pf-$\mathcal{EL}^{++}$ and f-$\mathcal{EL}^{++}$ concepts is that in our proposal we introduce the negation in the alphabet and it is now possible to use negation to concepts and atomic roles. Individuals and atomic concepts/roles are interpreted as follows:

Definition 11 (Concept Semantics) *The semantics of pf-$\mathcal{EL}^{++}$ individuals and atomic concepts/roles is given by a paraconsistent fuzzy interpretation I = (Δ^I, $.^I$), where the domain Δ^I is a nonempty set of elements and $.^I$ is a mapping function defined by: each individual $a \in N_I$ is mapped to $a^I \in \Delta^I$; each atomic concept name $A \in N_C$ is mapped to $A^I : \Delta^I \to [0,1] \times [0,1]$; each atomic role name $R \in N_R$ is mapped to $R^I : \Delta^I \times \Delta^I \to [0,1] \times [0,1]$.*

Each atomic concept/role C is mapped to a pair $\langle P, N \rangle$, where $P, N \in [0,1]$. Intuitively, P denotes the degree in which an element belongs to C, while N denotes the degree in which it does not belong to C. For example, given an atomic concept name *Rich* and an individual *John*, let the fuzzy interpretation be $(Rich(John))^I = (0.8, 0.4)$, whose meaning is John is rich with the degree of 0.8 and John is not rich with the degree of 0.4. Note that $P + N$ is not necessarily equal to 1 as in the classical case. Now we define the functions $proj^+ \langle P, N\rangle = P$ and $proj^- \langle P, N\rangle = N$ to access the positive and negative values $\langle P, N\rangle$, respectively. Concepts can be interpreted inductively as follows, where for all $x \in \Delta^I$:

Syntax	**Semantics**
$\top$	$\top^I(x) = \langle 1, 0\rangle$
$\bot$	$\bot^I(x) = \langle 0, 1\rangle$
$\neg C$	$(\neg C)^I(x) = \langle N, P\rangle$, if $C^I(x) = \langle P, N\rangle$
$\{a\}$	$\{a\}^I(x) = \begin{cases} \langle 1, 0\rangle \text{ if } x = a^I \\ \langle 0, 1\rangle \text{ otherwise} \end{cases}$
$C \sqcap D$	$(C \sqcap D)^I(x) = \langle min(P_1, P_2), min(N_1, N_2)\rangle$, if $C^I(x) = \langle P_1, N_1\rangle$ and $D^I(x) = \langle P_2, N_2\rangle$
$\exists R.C$	$(\exists R.C)^I(x) = \langle \sup\limits_{y \in \Delta^I} (min(proj^+(R^I(x, y)), proj^+(C^I(y)))), \sup\limits_{y \in \Delta^I} (min(proj^+(R^I(x, y)), proj^-(C^I(y))))\rangle$

The controversial part refers to $\sqcap$ and $\exists$. As one can easily check, the conditions for concepts conjunction and role restriction are designed in a way that $\neg(C \sqcap D)^I(x) = (\neg C \sqcap \neg D)^I(x)$ and $\neg(\exists R.C)^I(x) = (\exists R.\neg C)^I(x)$. Thus, these interpretations have as effect the internalisation of the negation. In practice, our approach interprets the operator $\neg$ as if it were restricted to atomic concepts. In contradistinction, one would expect (see [16]) $\neg(C \sqcap D)^I(x) = (\neg C \sqcup \neg D)^I(x)$ and $\neg(\exists R.C)^I(x) = (\forall R.\neg C)^I(x)$. However, since our aim is to present a tractable paraconsistent fuzzy extension for $\mathcal{EL}^{++}$, the inclusions of disjunction ($\sqcup$) and universal restriction ($\forall$) in $\mathcal{EL}^{++}$ are not allowed. Otherwise, as proved in [6], the algorithm of decidibility will grow exponentially!

We now focus on the syntax and semantics of inclusion axioms and assertion axioms. In fact, owing to a richer semantics than in f-$\mathcal{EL}^{++}$, that allows us to introduce negation, the semantics of pf-$\mathcal{EL}^{++}$ permits us to introduce two kinds of inclusion axioms: $C \sqsubset_n D$ (fuzzy internal inclusion axiom) and $C \rightarrow_n D$ (fuzzy strong inclusion axiom), where $n \in [0, 1]$. $C \sqsubset_n D$ denotes that the degree of subsethood of the beliefs in C to the beliefs in D is at least equal to n. In other terms, $C \sqsubset_n D$ means that for each individual (whose membership degree to C and D are respectively $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$) of the language, $\alpha_2 \geq \alpha_1$ or $\alpha_2 \geq n$. In particular, for $n = 1$, $\sqsubset_n$ reduces to $\sqsubset$. Unlike the classical settlement, we do not take as granted that $C \sqsubset_n D$ implies $\neg D \sqsubset_n \neg C$ (contraposition). As for $\rightarrow_n$, the satisfaction of contraposition is included, i.e., $C \rightarrow_n D$ if and only if $C \sqsubset_n D$ and $\neg D \sqsubset_n \neg C$. Below, we define the notion of TBox and ABox in pf-$\mathcal{EL}^{++}$. For now on, consider $T_1, \ldots, T_k, T$ refer to atomic roles or the negation of them. The semantics of negation of roles is similar to negation of concepts.

Definition 12 (Paraconsistent Fuzzy TBox) *A paraconsistent fuzzy TBox (Terminological Box) in pf-$\mathcal{EL}^{++}$ is a finite set of internal fuzzy inclusion axioms ($C \sqsubset_n D$),* *strong fuzzy inclusion axioms ($C \rightarrow_n D$), internal role inclusion axioms ($T_1 \circ \ldots \circ T_k \sqsubset T$) and strong role inclusion axioms ($T_1 \circ \ldots \circ T_k \rightarrow T$).*

In a paraconsistent fuzzy TBox, note that the negation of roles is only introduced by applying role inclusion axioms, i.e. concept inclusion axioms still deals exclusively with atomic roles. With this negative role constructor, we can increase the expressivity of the logic, and it is easy to see that inconsistencies may arise on role axioms. Fuzzy assertional boxes can be defined as follows:

Definition 13 (Paraconsistent Fuzzy ABox) *A paraconsistent fuzzy ABox (Assertional Box) in $pf\text{-}\mathcal{EL}^{++}$ consists of a finite set of assertion axioms of the form $C(a) \geq n$ and $T(a,b) \geq n$, where $n \in [0,1]$.*

These assertions denote respectively that the degree of belief an individual a belongs to a concept C is greater or equal to n and the degree of belief the individuals (a,b) belong to a role T is greater or equal to n. Now we present the notion of ontology:

Definition 14 (Ontology) *An ontology or knowledge base in $pf\text{-}\mathcal{EL}^{++}$ is a set composed by a paraconsistent fuzzy TBox and a paraconsistent fuzzy ABox.*

The semantics of both paraconsistent fuzzy general concept inclusions, role inclusions, concept assertion and role assertion is given by the following table, where for all $x, y \in \Delta^I$:

Axiom Name	Syntax	Semantics
Internal f-GCI	$C_1 \sqsubset_n C_2$	$min(proj^+(C_1^I(x)), n) \leq proj^+(C_2^I(x))$
Strong f-GCI	$C_1 \rightarrow_n C_2$	$min(proj^+(C_1^I(x)), n) \leq proj^+(C_2^I(x))$, $min(proj^-(C_2^I(x)), n) \leq proj^-(C_1^I(x))$
Internal RIA	$T_1 \circ \ldots \circ T_k \sqsubset T$	$proj^+([T_1^I \circ^t \ldots \circ^t T_k^I](x,y)) \leq proj^+(T^I(x,y))$
Strong RIA	$T_1 \circ \ldots \circ T_k \rightarrow T$	$proj^+([T_1^I \circ^t \ldots \circ^t T_k^I](x,y)) \leq proj^+(T^I(x,y))$, $proj^-(T^I(x,y)) \leq proj^-([T_1^I \circ^t \ldots \circ^t T_k^I](x,y))$
Concept assertion	$C(a) \geq n$	$proj^+(C^I(a^I)) \geq n$
Role assertion	$T(a,b) \geq n$	$proj^+(T^I(a^I, b^I)) \geq n$

Finally, we show the notions of satisfiability and logical consequence in $pf\text{-}\mathcal{EL}^{++}$:

Definition 15 (Satisfiability) *The notion of satisfiability of an axiom α by a fuzzy interpretation I, denoted $I \models \alpha$, is defined as follows: $I \models C_1 \sqsubseteq_n C_2$ iff $\forall x \in \Delta^I$, $min(proj^+(C_1^I(x)), n) \leq proj^+(C_2^I(x))$. The notion is similarly applied to the other axioms shown in the table above. For a set of axioms ε, we say that I satisfies ε iff I satisfies each element in ε. If $I \models \alpha$ we say that I is a model of α. I satisfies an ontology O, denoted by $I \models O$, iff I is a model of each axiom of ontology O.*

Definition 16 (Logical Consequence) *An axiom α is a logical consequence of an ontology O, denoted by $O \models \alpha$, iff every model of O satisfies α.*

Paraconsistency comes to deal with the principle that $\alpha, \neg\alpha \not\models \bot$, where α is an axiom. Note that in $pf\text{-}\mathcal{EL}^{++}$, $\bot$ is not logical consequence of α and $\neg\alpha$. For example, consider the axioms $(C(a) \geq 0)$, $(\neg C(a) \geq 0)$ and $(\bot(a) \geq 1)$. We have that $(C(a) \geq 0), (\neg C(a) \geq 0) \not\models (\bot(a) \geq 1)$, because there is an interpretation I (say $C^I(a^I) = \langle 0, 0\rangle$) such that $(C(a) \geq 0)^I$ and $(\neg C(a) \geq 0)^I$ are true and $(\bot(a) \geq 1)^I$ is false.

5 Translating pf-$\mathcal{EL}^{++}$ into f-$\mathcal{EL}^{++}$

In this section, we show that pf-$\mathcal{EL}^{++}$ is a conservative extension of f-$\mathcal{EL}^{++}$. A logic L_2 is a conservative extension of a logic L_1 if there is a translation map π from L_1 to L_2 that preserves logical consequence, i.e. $O \models \alpha$ iff $\pi(O) \models \pi(\alpha)$. In [16], it is presented a linear algorithm to translate a four-valued paraconsistent extension of $\mathcal{ALC}$, called $\mathcal{ALC}_4$, into its standard two-valued $\mathcal{ALC}$. The result obtained is that paraconsistent reasoning could be simulated by using standard $\mathcal{ALC}$ reasoning algorithms. Inspired by this translation function, we map axioms of the language pf-$\mathcal{EL}^{++}$ to the language f-$\mathcal{EL}^{++}$, and prove that pf-$\mathcal{EL}^{++}$ is a conservative extension of f-$\mathcal{EL}^{++}$. Since this translation is linear on the size of the ontology and since it was proved that f-$\mathcal{EL}^{++}$ has a polynomial-time decision problem [17], we then proved that pf-$\mathcal{EL}^{++}$ decision problem is also polynomial and, hence, tractable.

The main idea behind the translation function is the decomposability of paraconsistent language, i.e. each concept C and role R in that language can be divided into two concepts C_1, C_2 and two roles R_1, R_2 of the standard language. First, we will show how to translate individuals, concepts and roles in pf-$\mathcal{EL}^{++}$ into f-$\mathcal{EL}^{++}$, then we proceed with the translation of axiom inclusions and assertions. Finally, we prove the correspondence between these two logics.

Definition 17 (Concept Translation) *Let π be the translation function which maps the elements of the pf-$\mathcal{EL}^{++}$ language to f-$\mathcal{EL}^{++}$ language union with $\{A^+, A^-, \{a\}^+, \{a\}^-, R^+, R^- \mid$ A is an atomic concept, $\{a\}$ is a nominal and R is an atomic role $\}$. For any given concept C, its translation $\pi(C)$ is the concept obtained from C by the following inductively defined transformation: $\pi(A)$=A^+; $\pi(\neg A)$=A^-; $\pi(\top)$=$\top$; $\pi(\neg\top)$=$\bot$; $\pi(\bot)$=$\bot$; $\pi(\neg\bot)$=$\top$; $\pi(\{a\})$=$\{a\}^+$; $\pi(\neg\{a\})$=$\{a\}^-$; $\pi((C_1 \sqcap C_2))$ = $(\pi(C_1) \sqcap \pi(C_2))$; $\pi(\neg(C_1 \sqcap C_2))$ = $(\pi(\neg C_1) \sqcap \pi(\neg C_2))$; $\pi(\exists R.C_1)$=$\exists R.\pi(C_1)$; $\pi(\neg(\exists R.C_1))$=$\exists R.\pi(\neg C_1)$; $\pi(\neg\neg C_1)$=$\pi(C_1)$.*

Based on this, axioms and assertions are translated as follows:

Definition 18 (Axiom Translation) *For any given axiom inclusions, concept assertion and role assertions, their translation are obtained by the following rules: $\pi(C_1 \sqsubset_n C_2)$ = $\pi(C_1) \sqsubseteq_n \pi(C_2)$; $\pi(C_1 \rightarrow_n C_2)$ = $\{ \pi(C_1) \sqsubseteq_n \pi(C_2), \pi(\neg C_2) \sqsubseteq_n \pi(\neg C_1)\}$; $\pi(T_1 \circ \ldots \circ T_k \sqsubset T)$ = $\pi([T_1 \circ \ldots \circ T_k]) \sqsubseteq \pi(T)$; $\pi(T_1 \circ \ldots \circ T_k \rightarrow T)$ = $\{ \pi([T_1 \circ \ldots \circ T_k]) \sqsubseteq \pi(T), \pi(\neg T) \sqsubseteq \pi(\neg[T_1 \circ \ldots \circ T_k])\}$; $\pi(C(a) \geq n)$ = $\pi(C)(a) \geq n$; $\pi(T(a,b) \geq n)$ = $\pi(T(a,b)) \geq n$.*

Definition 19 (Induced Interpretation) *Let I = ($\Delta^I, .^I$) be a paraconsistent fuzzy interpretation of pf-$\mathcal{EL}^{++}$ in an ontology O. $\pi(O)$ is the fuzzy induced ontology, where $\overline{I} = (\Delta^{\overline{I}}, .^{\overline{I}})$ is the fuzzy induced interpretation, defined as follows: I and $\overline{I}$ have the same domain, i.e. $\Delta^{\overline{I}} = \Delta^I$; the individual's interpretation in I and $\overline{I}$ is the same, i.e. $\pi(a)^{\overline{I}} = a^I$; for any atomic concept A, $A^I = \langle P, N\rangle$ iff $(A^+)^{\overline{I}} = P$ and $(A^-)^{\overline{I}} = N$; for any atomic role R, $R^I = \langle P, N\rangle$ iff $(R^+)^{\overline{I}} = P$ and $(R^-)^{\overline{I}} = N$; for nominals $\{a\}$, $\{a\}^I = \langle P, N\rangle$ iff $\{a^+\}^{\overline{I}} = P$ and $\{a^-\}^{\overline{I}} = N$; the semantics of concepts is obtained according to their translation.*

Now we prove that the semantics of pf-$\mathcal{EL}^{++}$ can be reduced to the semantics of f-$\mathcal{EL}^{++}$:

Lemma 1 *Let O be a pf-$\mathcal{EL}^{++}$ ontology, C be a concept and T be a role. For any paraconsistent fuzzy interpretation I, $C^I = \langle P, N\rangle$ if and only if $\pi(C)^{\overline{I}} = P$, $\pi(\neg C)^{\overline{I}} = N$ and $T^I = \langle P, N\rangle$ if and only if $\pi(T)^{\overline{I}} = P$, $\pi(\neg T)^{\overline{I}} = N$, where $\overline{I}$ is the fuzzy induced interpretation of I.*

Proof. We prove this result by resorting to induction on concept structure as displayed in the sequel:

- Case: C is an atomic concept A or C=$\{a\}$. The result follows from Definition 19.
- Case: $C = \neg C_1$, $\pi(C) = \pi(\neg C_1)$, $\pi(\neg C) = \pi(C_1)$,
  - Suppose $C^I = \langle P, N\rangle$. Then $C_1^I = \langle N, P\rangle$. By induction assumption, we know $\pi(C_1)^{\overline{I}} = N$ e $\pi(\neg C_1)^{\overline{I}} = P$. That is $\pi(C)^{\overline{I}} = P$ e $\pi(\neg C)^{\overline{I}} = N$.
  - Suppose $\pi(C)^{\overline{I}} = P$ and $\pi(\neg C)^{\overline{I}} = N$. Then $\pi(C_1)^{\overline{I}} = N$ and $\pi(\neg C_1)^{\overline{I}} = P$. By induction assumption, we know $C_1^I = \langle N, P\rangle$. Through the semantics of negation, we know $C^I = \langle P, N\rangle$.
- Case: $C = (C_1 \sqcap C_2)$, $\pi(C) = (\pi(C_1) \sqcap \pi(C_2))$, $\pi(\neg C) = (\pi(\neg C_1) \sqcap \pi(\neg C_2))$,
  - Suppose $C^I = \langle P, N\rangle$, $C_1^I = \langle P_1, N_1\rangle$, $C_2^I = \langle P_2, N_2\rangle$.
    Then $P = min(P_1, P_2)$ and $N = min(N_1, N_2)$. By induction assumption, we know $\pi(C_1)^{\overline{I}} = P_1$, $\pi(\neg C_1)^{\overline{I}} = N_1$, $\pi(C_2)^{\overline{I}} = P_2$, $\pi(\neg C_2)^{\overline{I}} = N_2$. Therefore, $\pi(C)^{\overline{I}} = min(\pi(C_1)^{\overline{I}}, \pi(C_2)^{\overline{I}}) = min(P_1, P_2) = P$, $\pi(\neg C)^{\overline{I}} = min(\pi(\neg C_1)^{\overline{I}}, \pi(\neg C_2)^{\overline{I}}) = min(N_1, N_2) = N$.
  - Suppose $\pi(C)^{\overline{I}} = P$, $\pi(\neg C)^{\overline{I}} = N$, $\pi(C_1)^{\overline{I}} = P_1$, $\pi(\neg C_1)^{\overline{I}} = N_1$, $\pi(C_2)^{\overline{I}} = P_2$, $\pi(\neg C_2)^{\overline{I}} = N_2$. By semantic definitions, $P = min(P_1, P_2)$ and $N = min(N_1, N_2)$. By induction assumption, we know $C_1^I = \langle P_1, N_1\rangle$, $C_2^I = \langle P_2, N_2\rangle$. Therefore, $C^I = \langle min(P_1, P_2), min(N_1, N_2)\rangle = \langle P, N\rangle$.
- Case: $C = (\exists R.C_1)$, $\pi(C) = (\exists R.\pi(C_1))$, $\pi(\neg C) = (\exists R.\pi(\neg C_1))$
  - Suppose $C^I = \langle P, N\rangle$, $C_1^I = \langle P_1, N_1\rangle$. By semantic definitions, we know
    $P = \sup_{y \in \Delta_I} (min(proj^+(R^I(x, y)), proj^+(C^I(y))))$,
    $N = \sup_{y \in \Delta_I} (min(proj^+(R^I(x, y)), proj^-(C^I(y))))$. By induction assumption, we know $\pi(C_1)^{\overline{I}} = P_1$, $\pi(\neg C_1)^{\overline{I}} = N_1$. Furthermore, $P_1 = proj^+(C_1^I)$, $N_1 = proj^-(C_1^I)$. Therefore,
    $\pi(C)^{\overline{I}} = \sup_{y \in \Delta_I} (min(R^I(x, y), \pi(C_1)^{\overline{I}})) = \sup_{y \in \Delta_I} (min(R^I(x, y), P_1)) = P$,
    $\pi(\neg C)^{\overline{I}} = \sup_{y \in \Delta_I} (min(R^I(x, y), \pi(\neg C_1)^{\overline{I}})) = \sup_{y \in \Delta_I} (min(R^I(x, y), N_1)) = N$.
  - Suppose $\pi(C)^{\overline{I}} = P$, $\pi(\neg C)^{\overline{I}} = N$, $\pi(C_1)^{\overline{I}} = P_1$, $\pi(\neg C_1)^{\overline{I}} = N_1$. By the definition of semantics,
    $P = \pi(C)^{\overline{I}} = (\exists R.\pi(C_1))^{\overline{I}} = \sup_{y \in \Delta_I} (min(R^I(x, y), P_1))$,
    $N = \pi(\neg C)^{\overline{I}} = (\exists R.\pi(\neg C_1))^{\overline{I}} = \sup_{y \in \Delta_I} (min(R^I(x, y), N_1))$

By induction assumption, we know $C_1^I = \langle P_1, N_1 \rangle$. Then, by definition of semantics of pf-$\mathcal{EL}^{++}$, we have
$C^I = \langle \sup_{y \in \Delta_I} (min(R^I(x,y), P_1)), \sup_{y \in \Delta_I} (min(R^I(x,y), N_1)) \rangle$.
– The proof for any role T follows similarly. $\Box$

The following theorem shows that paraconsistent reasoning can indeed be simulated on standard reasoner by means of the translation given. This implies that paraconsistent reasoning in our paradigm is not more expensive than classical reasoning. Note that the translation algorithm is linear on the size of the ontology.

Theorem 1 (Conservative Extension) *For any ontology O in pf-$\mathcal{EL}^{++}$, we have $O \models \alpha$ if and only if $\pi(O) \models \pi(\alpha)$, where α is an axiom of O.*

Proof. (Necessity) For any paraconsistent fuzzy interpretation I of O, let $\overline{I}$ be the fuzzy induced interpretation. According to the relationship between O and $\pi(O)$:

Let $C_1^I = \langle P_1, N_1 \rangle$ and $C_2^I = \langle P_2, N_2 \rangle$. By Lemma 1, $\pi(C_1)^{\overline{I}} = P_1$, $\pi(C_2)^{\overline{I}} = P_2$, $\pi(\neg C_1)^{\overline{I}} = N_1$, $\pi(\neg C_2)^{\overline{I}} = N_2$. We have

– If $I \models C_1 \to_n C_2$, then, $min(P_1, n) = min(proj^+(C_1^I), n) \leq proj^+(C_2^I) = P_2$, $min(N_2, n) = min(proj^-(C_2^I), n) \leq proj^-(C_1^I) = N_1$, that is, $\overline{I}$ satisfies $\{\pi(C_1) \sqsubseteq_n \pi(C_2), \pi(\neg C_2) \sqsubseteq_n \pi(\neg C_1)\}$.
– $I \models C_1 \sqsubset_n C_2$, then $min(P_1, n) = min(proj^+(C_1^I), n) \leq proj^+(C_2^I) = P_2$, that is, $\overline{I}$ satisfies $\pi(C_1) \sqsubseteq_n \pi(C_2)$.
– The proof for role inclusion axioms follows similarly.
– For any assertion of the form $\pi(C)(a) \geq n$, $C(a) \geq n$ belongs to O. Suppose $C^I = \langle P, N \rangle$, $a^I = \delta_0 \in \Delta^I$, then $(\pi(C))^{\overline{I}} = P$, $(\pi(\neg C))^{\overline{I}} = N$, $\overline{a}^{\overline{I}} = \delta_0$. I satisfies $C(a) \geq n$. Therefore, $proj^+(\langle P, N \rangle) = C^I(a^I) \geq n$, that is, $\overline{I}$ satisfies $\pi(C)(a) \geq n$. It is similar for $T(a,b) \geq n$ assertion.

(Sufficiency) For any fuzzy induced interpretation $\overline{I} = (\Delta^{\overline{I}}, \cdot^{\overline{I}})$ of $\pi(O)$, let I be the paraconsistent fuzzy interpretation of $\overline{I}$. Similarly, we can prove the proposition to be right. Observe that for all fuzzy interpretation for $\pi(O)$ and $\pi(\alpha)$, there is a paraconsistent fuzzy interpretation, i.e. the set of fuzzy induced interpretations corresponds exactly to the set of the possible fuzzy interpretations .$\Box$

6 Conclusions and Future Works

In this paper, we introduced a paraconsistent extension of the fuzzy description logic $\mathcal{EL}^{++}$, that deals with negation on concepts and roles. Besides the syntax and semantics we showed how to translate pf-$\mathcal{EL}^{++}$ into f-$\mathcal{EL}^{++}$, preserving logical consequence, and under linear time and space in the size of the ontology. Since it is presented in [20], an algorithm for deciding fuzzy concept subsumptions that operates in polynomial time, we conclude that paraconsistency can be simulated by f-$\mathcal{EL}^{++}$ without the loss of tractability.

Regarding future works, we plan to investigate and extend another approach to fuzzy $\mathcal{EL}$, presented by Vojtás [21], where conjunction is interpreted as a fuzzy aggregation function rather than fuzzy intersection. Another line of research is to extend tractable DLs to deal with probabilistic and possibilistic knowledge.

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