Generic bidirectional typing for dependent type theories

Technical report

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Contents

1 The framework

1.1 Raw Syntax

Scopes and signatures We define the scopes, metavariable scopes and signatures by the following grammars.

$$\boxed{\mathsf{Scope}} \ni \quad \gamma, \delta ::= \cdot \mid \gamma, x$$

$$\boxed{\mathsf{MScope}} \ni \quad \theta, \xi ::= \cdot \mid \theta, \mathsf{x}\{\delta\}$$

$$\boxed{\mathsf{Sig}} \ni \quad \Sigma ::= \cdot \mid \Sigma, c(\theta) \mid \Sigma, d(\theta)$$

Given two scopes γ, γ' or two metavariable scopes θ, θ' we write $\gamma.\gamma'$ or $\theta.\theta'$ respectively for their concatenation.

Terms and spines Given a fixed signature Σ, we define the terms, substitutions and metavariables substitutions by the following grammar.

$$\begin{array}{rrll}
\boxed{\mathsf{Tm}\ \theta\ \gamma} \ni & t, u, v, T, U, V ::= & \mid x & \text{if } x \in \gamma \\
& & \mid \mathsf{x}\{\vec{t} \in \mathsf{Sub}\ \theta\ \gamma\ \delta\} & \text{if } \mathsf{x}\{\delta\} \in \theta \\
& & \mid c(\mathbf{t} \in \mathsf{MSub}\ \theta\ \gamma\ \xi) & \text{if } c(\xi) \in \Sigma \\
& & \mid d(t \in \mathsf{Tm}\ \theta\ \gamma; \mathbf{t} \in \mathsf{MSub}\ \theta\ \gamma\ \xi) & \text{if } d(\xi) \in \Sigma \\
\boxed{\mathsf{Sub}\ \theta\ \gamma\ \delta} \ni & \vec{t}, \vec{u}, \vec{s}, \vec{v} ::= & \mid \epsilon & \text{if } \delta = \cdot \\
& & \mid \vec{t} \in \mathsf{Sub}\ \theta\ \gamma\ \delta', t \in \mathsf{Tm}\ \theta\ \gamma & \text{if } \delta = \delta', x \\
\boxed{\mathsf{MSub}\ \theta\ \gamma\ \xi} \ni & \mathbf{t}, \mathbf{u}, \mathbf{s}, \mathbf{v} ::= & \mid \epsilon & \text{if } \xi = \cdot \\
& & \mid \mathbf{t} \in \mathsf{MSub}\ \theta\ \gamma\ \xi', \vec{x}_\delta.t \in \mathsf{Tm}\ \theta\ \gamma.\delta & \text{if } \xi = \xi', \mathsf{x}\{\delta\}
\end{array}$$

Given a metavariable substitution $\mathbf{t} \in \mathsf{MSub}\ \theta\ \gamma\ \xi$ and $\mathsf{x}\{\delta\} \in \xi$, we write $\mathbf{t}_\mathsf{x} \in \mathsf{Tm}\ \theta\ \gamma.\delta$ for the term in $\mathbf{t}$ at the position pointed by x. Similarly, given a substitution $\vec{t} \in \mathsf{Sub}\ \theta\ \gamma\ \delta$ and $x \in \delta$, we write $t_x \in \mathsf{Tm}\ \theta\ \gamma$ for the term in $\vec{t}$ at the position pointed by x.

Contexts Given a fixed signature Σ, we define the contexts and metavariable contexts by the following grammar, by induction-recursion with the function $|\_|$ computing their underlying scopes and metavariable scopes.

$$\begin{array}{rrl}
\boxed{\mathsf{Ctx}\ \theta\ \gamma} \ni & \Gamma, \Delta ::= & \mid \cdot \\
& & \mid \Gamma \in \mathsf{Ctx}\ \theta\ \gamma, x : T \in \mathsf{Tm}\ \theta\ \gamma.|\Gamma| \\
\boxed{\mathsf{MCtx}\ \theta} \ni & \Theta, \Xi ::= & \mid \cdot \\
& & \mid \Theta \in \mathsf{MCtx}\ \theta, \mathsf{x}\{\Gamma \in \mathsf{Ctx}\ \theta.|\Theta|\ \cdot\} : T \in \mathsf{Tm}\ \theta.|\Theta|\ |\Gamma|
\end{array}$$

$$
\begin{aligned}
&|\_| : \mathsf{Ctx}\ \theta\ \gamma \to \mathsf{Scope}\\
&|\cdot| := \cdot\\
&|\Gamma, x : T| := |\Gamma|, x
\end{aligned}
$$

$$
\begin{aligned}
&|\_| : \mathsf{MCtx}\ \theta \to \mathsf{MScope}\\
&|\cdot| := \cdot\\
&|\Theta, \mathsf{x}\{\Delta\} : T| := |\Theta|, \mathsf{x}\{|\Delta|\}
\end{aligned}
$$

We define the concatenation of two contexts as following, by recursion-recursion with Lemma 1.1.

$$
\begin{aligned}
&\_.\_ : (\Gamma \in \mathsf{Ctx}\ \theta\ \gamma) \to (\Delta \in \mathsf{Ctx}\ \theta\ \gamma.|\Gamma|) \to \mathsf{Ctx}\ \theta\ \gamma\\
&\Gamma.(\cdot) := \Gamma\\
&\Gamma.(\Delta, x : T) := \Gamma.\Delta, x : T
\end{aligned}
$$

Lemma 1.1 (Concatenation commutes with underlying scope). *We have* $|\Gamma.\Delta| = |\Gamma|.|\Delta|$.

Notation 1.2. If the underlying signature is not clear from the context, we write Tm_Σ, Sub_Σ, MSub_Σ, Ctx_Σ, MCtx_Σ in order to make it explicit.

Notation 1.3. We sometimes write $\mathsf{Ctx}\ \theta$ for $\mathsf{Ctx}\ \theta\ \cdot$, Ctx for $\mathsf{Ctx}\ \cdot\ \cdot$ and MCtx for $\mathsf{MCtx}\ \cdot$.

Notation 1.4. In the following we write $e \in \mathsf{Expr}\ \theta\ \gamma$ for $e \in \mathsf{Tm}\ \theta\ \gamma$ or $e \in \mathsf{Sub}\ \theta\ \gamma\ \delta$ or $e \in \mathsf{MSub}\ \theta\ \gamma\ \xi$ or $e \in \mathsf{Ctx}\ \theta\ \gamma$.

Observation 1.5. Because we work with a nameful syntax, we allow ourselves to implicitly weaken expressions: if $e \in \mathsf{Expr}\ \theta\ \gamma$ and θ is a subsequence of θ' and γ is a subsequence of γ' then we also have $e \in \mathsf{Expr}\ \theta'\ \gamma'$. Nevertheless, we expect that our proofs can be formally carried out using de Bruijn indices, by properly inserting weakenings whenever needed (and showing all the lemmata associated with it). In particular, this should be facilitated by the fact that we already work with an intrinsically-scoped syntax.

1.2 Substitution

For each scope γ we have an identity substitution $\mathsf{id}_\gamma \in \mathsf{Sub}\ \cdot\ \gamma\ \gamma$, and for each metavariable scope θ we have an identity metavariable substitution $\mathsf{id}_\theta \in \mathsf{MSub}\ \theta\ \cdot\ \theta$, which are defined by the following clauses.

$$
\begin{aligned}
&\mathsf{id}\_ : (\gamma \in \mathsf{Scope}) \to \mathsf{Sub}\ \cdot\ \gamma\ \gamma\\
&\mathsf{id}_{(\cdot)} := \varepsilon\\
&\mathsf{id}_{\gamma,x} := \mathsf{id}_\gamma, x
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{id}\_ : (\theta \in \mathsf{MScope}) \to \mathsf{MSub}\ \theta\ \cdot\ \theta\\
&\mathsf{id}_{(\cdot)} := \varepsilon\\
&\mathsf{id}_{\theta,\mathsf{x}\{\gamma\}} := \mathsf{id}_\theta, \vec{x}_\gamma.\mathsf{x}\{\mathsf{id}_\gamma\}
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{Tm}\ \theta\ \gamma_2 \to \mathsf{Sub}\ \theta\ \gamma_1\ \gamma_2 \to \mathsf{Tm}\ \theta\ \gamma_1 \\
&x[\vec{v}] := v_x \\
&\mathsf{x}\{\vec{t}\}[\vec{v}] := \mathsf{x}\{\vec{t}[\vec{v}]\} \\
&c(\mathbf{t})[\vec{v}] := c(\mathbf{t}[\vec{v}]) \\
&d(t; \mathbf{u})[\vec{v}] := d(t[\vec{v}]; \mathbf{u}[\vec{v}])
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{Sub}\ \theta\ \gamma_2\ \delta \to \mathsf{Sub}\ \theta\ \gamma_1\ \gamma_2 \to \mathsf{Sub}\ \theta\ \gamma_1\ \delta \\
&\varepsilon[\vec{v}] := \varepsilon \\
&(\vec{t}, u)[\vec{v}] := \vec{t}[\vec{v}], u[\vec{v}]
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{MSub}\ \theta\ \gamma_2\ \xi \to \mathsf{Sub}\ \theta\ \gamma_1\ \gamma_2 \to \mathsf{MSub}\ \theta\ \gamma_1\ \xi \\
&\varepsilon[\vec{v}] := \varepsilon \\
&(\mathbf{t}, \vec{x}_\delta.u)[\vec{v}] := \mathbf{t}[\vec{v}], \vec{x}.u[\vec{v}, \mathsf{id}_\delta]
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{Ctx}\ \theta\ \gamma_2 \to \mathsf{Sub}\ \theta\ \gamma_1\ \gamma_2 \to \mathsf{Ctx}\ \theta\ \gamma_1 \\
&(\cdot)[\vec{v}] := \cdot \\
&(\Gamma, x : T)[\vec{v}] := \Gamma[\vec{v}], x : T[\vec{v}, \mathsf{id}_\Gamma]
\end{aligned}
$$

Figure 1: Variable substitution

Notation 1.6. We sometimes abuse notation and write id_Γ for $\mathsf{id}_{|\Gamma|}$ and id_Θ for $\mathsf{id}_{|\Theta|}$.

Substitution application is defined by recursion-recursion by Figures 1 and 2, together with Lemma 1.7.

Lemma 1.7 (Underlying scope is invariant under substitution)**.** *We have* $|\Gamma[\vec{v}]| = |\Gamma|$ *and* $|\Gamma[\mathbf{v}]| = |\Gamma|$ *for all* $\Gamma \in \mathsf{Ctx}\ \theta_2\ \gamma_2$ *and* $\vec{v} \in \mathsf{Sub}\ \theta_2\ \gamma_1\ \gamma_2$ *and* $\mathbf{v} \in \mathsf{MSub}\ \theta_1\ \delta\ \theta_2$*.*

Proposition 1.8 (Unit laws for id)**.** *We have* $e[\mathsf{id}_\gamma] = e$ *and* $e[\mathsf{id}_\theta] = e$ *for all* $e \in \mathsf{Expr}\ \theta\ \gamma$*, and* $\mathsf{id}_\delta[\vec{t}] = \vec{t}$ *for all* $\vec{t} \in \mathsf{Sub}\ \theta\ \gamma\ \delta$*, and* $\mathsf{id}_\theta[\mathbf{v}] = \mathbf{v}$ *for all* $\mathbf{v} \in \mathsf{MSub}\ \xi\ \gamma\ \theta$*.*

Proof. We first show $e[\mathsf{id}_\gamma] = e$ by induction on e and $\mathsf{id}_\delta[\vec{t}] = \vec{t}$ by induction on δ. We then show $e[\mathsf{id}_\theta] = e$ by induction on e and $\mathsf{id}_\theta[\mathbf{v}] = \mathbf{v}$ by induction on θ. □

The following two properties are shown simultaneously.

Proposition 1.9 (Commutation lemmas)**.** *We have* $e[\mathbf{u}][\vec{v}, \mathsf{id}_\delta] = e[\mathbf{u}[\vec{v}]] \in \mathsf{Expr}\ \theta_1\ \gamma_1.\delta$ *for all* $e \in \mathsf{Expr}\ \theta_2\ \delta$ *and* $\mathbf{u} \in \mathsf{MSub}\ \theta_1\ \gamma_2\ \theta_2$ *and* $\vec{v} \in \mathsf{Sub}\ \theta_1\ \gamma_1\ \gamma_2$*. We have* $e[\vec{u}][\mathbf{v}] = e[\mathbf{v}][\mathsf{id}_\delta, \vec{u}[\mathbf{v}]] \in \mathsf{Expr}\ \theta_1\ \delta.\gamma_1$ *for all* $e \in \mathsf{Expr}\ \theta_2\ \gamma_2$ *and* $\vec{u} \in \mathsf{Sub}\ \theta_2\ \gamma_1\ \gamma_2$ *and* $\mathbf{v} \in \mathsf{MSub}\ \theta_1\ \delta\ \theta_2$*.*

$$
\begin{aligned}
&\_[\_] : \mathsf{Tm}\ \theta_2\ \gamma \to \mathsf{MSub}\ \theta_1\ \delta\ \theta_2 \to \mathsf{Tm}\ \theta_1\ \delta.\gamma\\
&x[\mathbf{v}] := x\\
&\mathsf{x}\{\vec{t}\}[\mathbf{v}] := \mathbf{v}_{\mathsf{x}}[\mathsf{id}_\delta, \vec{t}[\mathbf{v}]]\\
&c(\mathbf{t})[\mathbf{v}] := c(\mathbf{t}[\mathbf{v}])\\
&d(t;\mathbf{u})[\mathbf{v}] := c(t[\mathbf{v}];\mathbf{u}[\mathbf{v}])
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{Sub}\ \theta_2\ \gamma\ \gamma_0 \to \mathsf{MSub}\ \theta_1\ \delta\ \theta_2 \to \mathsf{Sub}\ \theta_1\ \delta.\gamma\ \gamma_0\\
&\cdot\ [\mathbf{v}] := \cdot\\
&(\vec{t}, u)[\mathbf{v}] := \vec{t}[\mathbf{v}], u[\mathbf{v}]
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{MSub}\ \theta_2\ \gamma\ \xi \to \mathsf{MSub}\ \theta_1\ \delta\ \theta_2 \to \mathsf{MSub}\ \theta_1\ \delta.\gamma\ \xi\\
&\cdot\ [\mathbf{v}] := \cdot\\
&(\mathbf{t}, \vec{x}.u)[\mathbf{v}] := \mathbf{t}[\mathbf{v}], \vec{x}.u[\mathbf{v}]
\end{aligned}
$$

$$
\begin{aligned}
&\_[\_] : \mathsf{Ctx}\ \theta_2\ \gamma \to \mathsf{MSub}\ \theta_1\ \delta\ \theta_2 \to \mathsf{Ctx}\ \theta_1\ \delta.\gamma\\
&(\cdot)[\mathbf{v}] := \cdot\\
&(\Gamma, x : T)[\mathbf{v}] := \Gamma[\mathbf{v}], x : T[\mathbf{v}]
\end{aligned}
$$

Figure 2: Metavariable substitution

Proposition 1.10 (Associativity of substitution)**.** *Let* $e \in \mathsf{Expr}\ \theta_3\ \gamma_3$*. For all* $\vec{v} \in \mathsf{Sub}\ \theta_3\ \gamma_2\ \gamma_3$ *and* $\vec{u} \in \mathsf{Sub}\ \theta_3\ \gamma_1\ \gamma_2$ *we have* $e[\vec{v}][\vec{u}] = e[\vec{v}[\vec{u}]]$*, and for all* $\mathbf{v} \in \mathsf{MSub}\ \theta_2\ \gamma_3\ \theta_3$ *and* $\mathbf{u} \in \mathsf{MSub}\ \theta_1\ \gamma_3\ \theta_2$ *we have* $e[\mathbf{v}][\mathbf{u}] = e[\mathbf{v}[\mathbf{u}]]$.

Proof. All results are shown by induction on e, on the following order: first $e[\vec{v}][\vec{u}] = e[\vec{v}[\vec{u}]]$, then $e[\mathbf{u}][\vec{v}, \mathrm{id}_\delta] = e[\mathbf{u}[\vec{v}]]$, then $e[\vec{u}][\mathbf{v}] = e[\mathbf{v}][\mathrm{id}_\delta, \vec{u}[\mathbf{v}]]$, then $e[\mathbf{v}][\mathbf{u}] = e[\mathbf{v}[\mathbf{u}]]$. □

In the following subsections, we allow ourselves to apply these basic properties about substitutions without announcement.

1.3 Patterns

Given a fixed signature Σ, we define the patterns by the following grammar.

$$
\begin{array}{lll}
\boxed{\mathsf{Tm}^{\mathsf{P}}\ \theta\ \gamma} \ni \quad t, u, v, s ::= & \mid \mathsf{x}\{\mathrm{id}_\gamma\} & \text{if } \theta = \mathsf{x}\{\gamma\} \\
& \mid c(\mathbf{t} \in \mathsf{MSub}^{\mathsf{P}}\ \theta\ \gamma\ \xi) & \text{if } c(\xi) \in \Sigma \\
\boxed{\mathsf{MSub}^{\mathsf{P}}\ \theta\ \gamma\ \xi} \ni \quad \mathbf{t}, \mathbf{u}, \mathbf{s}, \mathbf{v} ::= & \mid \epsilon & \text{if } \xi = \cdot \text{ and } \theta = \cdot \\
& \mid \mathbf{t} \in \mathsf{MSub}^{\mathsf{P}}\ \theta_1\ \gamma\ \xi', \vec{x}_\delta.t \in \mathsf{Tm}^{\mathsf{P}}\ \theta_2\ \gamma.\delta & \text{if } \xi = \xi', \mathsf{x}\{\delta\} \text{ and } \theta = \theta_1.\theta_2
\end{array}
$$

1.4 Theories

We define the theories by the following grammar, by induction-recursion with the the function $|\_|$ computing the underlying signature of a theory.

$$
\begin{array}{l}
\boxed{\mathsf{Thy}} \ni \mathbb{T} ::= \mid \cdot \\
\quad \mid \mathbb{T},\ c(\Xi \in \mathsf{MCtx}_{|\mathbb{T}|})\ \mathsf{sort} \\
\quad \mid \mathbb{T},\ c(\Xi_1 \in \mathsf{MCtx}_{|\mathbb{T}|};\ \Xi_2 \in \mathsf{MCtx}_{|\mathbb{T}|}\ |\Xi_1|) : T \in \mathsf{Tm}^{\mathsf{P}}_{|\mathbb{T}|}\ |\Xi_1|\ (\cdot) \\
\quad \mid \mathbb{T},\ d(\Xi_1 \in \mathsf{MCtx}_{|\mathbb{T}|};\ \mathsf{x} : T \in \mathsf{Tm}^{\mathsf{P}}_{|\mathbb{T}|}\ |\Xi_1|\ (\cdot);\ \Xi_2 \in \mathsf{MCtx}_{|\mathbb{T}|}\ (|\Xi_1|, \mathsf{x}) : U \in \mathsf{Tm}_{|\mathbb{T}|}\ (|\Xi_1|, \mathsf{x}, |\Xi_2|)\ (\cdot) \\
\quad \mid \mathbb{T},\ \theta_1; \theta_2 \Vdash d(t \in \mathsf{Tm}^{\mathsf{P}}_{|\mathbb{T}|}\ \theta_1\ (\cdot); \mathbf{v} \in \mathsf{MSub}^{\mathsf{P}}_{|\mathbb{T}|}\ \theta_2\ (\cdot)\ \xi) \longmapsto r \in \mathsf{Tm}_{|\mathbb{T}|}\ \theta_1.\theta_2\ (\cdot) \\
\qquad \text{if } d(\xi) \in |\mathbb{T}| \text{ and for no } \theta_1'; \theta_2' \Vdash d(t', \mathbf{v}') \longmapsto r' \in \mathbb{T} \text{ and } \mathbf{t}, \mathbf{t}' \text{ we have } d(t; \mathbf{v})[\mathbf{t}] = d(t'; \mathbf{v}')[\mathbf{t}']
\end{array}
$$

$$
\begin{array}{l}
|\_| : \mathsf{Thy} \to \mathsf{Sig} \\
|\cdot| := \cdot \\
|\mathbb{T}, c(\Xi)\ \mathsf{sort}| := |\mathbb{T}|, c(|\Xi|) \\
|\mathbb{T}, c(\Xi_1; \Xi_2) : T| := |\mathbb{T}|, c(|\Xi_2|) \\
|\mathbb{T}, d(\Xi_1; \mathsf{x} : T; \Xi_2) : U| := |\mathbb{T}|, d(|\Xi_2|) \\
|\mathbb{T}, \theta_1; \theta_2 \Vdash d(t; \mathbf{u}) \longmapsto r| := |\mathbb{T}|
\end{array}
$$

$$\boxed{t \longrightarrow u \quad (t \in \mathsf{Tm}\ \theta\ \gamma;\ u \in \mathsf{Tm}\ \theta\ \gamma)}$$

$$\frac{\vec{t} \longrightarrow \vec{t}'}{\mathsf{x}\{\vec{t}\} \longrightarrow \mathsf{x}\{\vec{t}'\}} \qquad \frac{\mathbf{v} \longrightarrow \mathbf{v}'}{c(\mathbf{v}) \longrightarrow c(\mathbf{v}')} \qquad \frac{\mathbf{v} \longrightarrow \mathbf{v}'}{d(t;\mathbf{v}) \longrightarrow d(t;\mathbf{v}')} \qquad \frac{t \longrightarrow t'}{d(t;\mathbf{v}) \longrightarrow d(t';\mathbf{v})}$$

$$\frac{\theta_1;\theta_2 \Vdash d(t;\mathbf{v}) \longmapsto r\ \in \mathbb{T} \qquad \mathbf{t} \in \mathsf{MSub}\ \theta\ \gamma\ \theta_1 \qquad \mathbf{u} \in \mathsf{MSub}\ \theta\ \gamma\ \theta_2}{d(t[\mathbf{t}];\mathbf{v}[\mathbf{u}]) \longrightarrow r[\mathbf{t},\mathbf{u}]}$$

$$\boxed{\vec{t} \longrightarrow \vec{u} \quad (\vec{t} \in \mathsf{Sub}\ \theta\ \gamma\ \delta;\ \vec{u} \in \mathsf{Sub}\ \theta\ \gamma\ \delta)}$$

$$\frac{\vec{t} \longrightarrow \vec{t}'}{\vec{t},u \longrightarrow \vec{t}',u} \qquad \frac{u \longrightarrow u'}{\vec{t},u \longrightarrow \vec{t},u'}$$

$$\boxed{\mathbf{v} \longrightarrow \mathbf{u} \quad (\mathbf{v} \in \mathsf{MSub}\ \theta\ \gamma\ \xi;\ \mathbf{u} \in \mathsf{MSub}\ \theta\ \gamma\ \xi)}$$

$$\frac{\mathbf{v} \longrightarrow \mathbf{v}'}{\mathbf{v},\vec{x}.u \longrightarrow \mathbf{v}',\vec{x}.u} \qquad \frac{u \longrightarrow u'}{\mathbf{v},\vec{x}.u \longrightarrow \mathbf{t},\vec{x}.u'}$$

$$\boxed{\Gamma \longrightarrow \Delta \quad (\Gamma \in \mathsf{Ctx}\ \theta\ \gamma;\ \Delta \in \mathsf{Ctx}\ \theta\ \gamma)}$$

$$\frac{\Gamma \longrightarrow \Gamma'}{\Gamma, x : T \longrightarrow \Gamma', x : T} \qquad \frac{T \longrightarrow T'}{\Gamma, x : T \longrightarrow \Gamma, x : T'}$$

Figure 3: Rewriting relation defined by theory $\mathbb{T}$

1.5 Rewriting

Given an underlying theory $\mathbb{T}$, we define its rewrite relation by Figure 3, by induction-recursion with Lemma 1.11.

Lemma 1.11 (Underlying scope is invariant under rewriting). *If* $\Gamma \longrightarrow \Gamma'$ *then* $|\Gamma| = |\Gamma'|$.

The relations $\longrightarrow^+$, $\longrightarrow^*$ and $\equiv$ are then defined as usual, respectively as the transitive, reflexive-transitive and reflexive-symmetric-transitive closures of $\longrightarrow$.

Notation 1.12. Whenever the underlying theory is not clear from the context, we write $\longrightarrow_{\mathbb{T}}$ and $\longrightarrow^+_{\mathbb{T}}$ and $\longrightarrow^*_{\mathbb{T}}$ and $\equiv_{\mathbb{T}}$.

Proposition 1.13 (Stability of rewriting under substitution). *Let* $e \in \mathsf{Expr}\ \theta\ \gamma$ *with* $e \longrightarrow^* e'$. *If* $\vec{v} \in \mathsf{Sub}\ \theta\ \delta\ \gamma$ *and* $\vec{v} \longrightarrow^* \vec{v}'$ *then* $e[\vec{v}] \longrightarrow^* e'[\vec{v}']$. *If* $\mathbf{v} \in \mathsf{MSub}\ \xi\ \gamma\ \theta$ *and* $\mathbf{v} \longrightarrow^* \mathbf{v}'$ *then* $e[\mathbf{v}] \longrightarrow^* e'[\mathbf{v}']$.

Proof. We first show $e[\vec{v}] \longrightarrow^* e[\vec{v}']$ for all e and $\vec{v} \longrightarrow^* \vec{v}'$, by induction on e. Using it as a lemma, we show that $e \longrightarrow^* e'$ and $\vec{v} \longrightarrow^* \vec{v}'$ imply $e[\vec{v}] \longrightarrow^* e'[\vec{v}']$, by outer induction on the number of rewrites in $e \longrightarrow^* e'$ and inner induction on the first rewrite $e \longrightarrow e''$ of $e \longrightarrow e'' \longrightarrow^* e'$. We then show $e[\mathbf{v}] \longrightarrow^* e[\mathbf{v}']$ for all e and $\mathbf{v} \longrightarrow^* \mathbf{v}'$, by induction on e. Finally, we show that $e \longrightarrow^* e'$ and $\mathbf{v} \longrightarrow^* \mathbf{v}'$ imply $e[\mathbf{v}] \longrightarrow^* e'[\mathbf{v}']$, by outer induction on the number of rewrites in $e \longrightarrow^* e'$ and inner induction on the first rewrite $e \longrightarrow e''$ of $e \longrightarrow e'' \longrightarrow^* e'$. □

Corollary 1.14 (Stability of conversion under substitution)**.** *Suppose $e \equiv e'$. We have $e[\vec{v}] \equiv e'[\vec{v}']$ for all $\vec{v} \equiv \vec{v}'$ and $e[\mathbf{v}] \equiv e'[\mathbf{v}']$ for all $\mathbf{v} \equiv \mathbf{v}'$.*

Proposition 1.15 (Confluence)**.** *If $e' \;{}^*\!\!\longleftarrow e \longrightarrow^* e''$ then $e' \longrightarrow^* e''' \;{}^*\!\!\longleftarrow e''$ for some e'''.*

Proof. Note that our rewrite rules are all *left-linear* and moreover in the definition of theory we demand that no two left-hand sides unify. Because the only possible overlaps are at the head, this means that there are no overlaps. It follows that our rewrite systems are *orthogonal* and therefore are confluent by [Mayr and Nipkow(1998), Theorem 6.11]. □

2 Declarative typing system

Given a fixed theory $\mathbb{T}$, the declarative type system is defined by the rules in Figure 4.

Notation 2.1. We write $\Theta; \Gamma \vdash \mathcal{J}$ for any of the following: $\Theta; \Gamma \vdash$ or $\Theta; \Gamma \vdash T$ sort or $\Theta; \Gamma \vdash t : T$ or $\Theta; \Gamma \vdash \vec{t} : \Delta$ or $\Theta; \Gamma \vdash \mathbf{t} : \Xi$.

Notation 2.2. We write $\mathbb{T} \triangleright \Theta; \Gamma \vdash \mathcal{J}$ when the underlying theory is not clear from the context.

Notation 2.3. We write $\Theta \vdash \mathcal{J}$ for $\Theta; \cdot \vdash \mathcal{J}$ and $\Gamma \vdash \mathcal{J}$ for $\cdot; \Gamma \vdash \mathcal{J}$.

2.1 Valid theories

The valid theories are defined by the following inference rules.

$$\frac{}{\cdot \text{ valid}} \qquad \frac{\mathbb{T}\text{ valid} \qquad \mathbb{T} \triangleright \Xi \vdash}{\mathbb{T}, c(\Xi)\text{ sort valid}}$$

$$\frac{\mathbb{T}\text{ valid} \qquad \mathbb{T} \triangleright \Xi_1.(\mathsf{x} : U).\Xi_2 \vdash T\text{ sort}}{\mathbb{T}, d(\Xi_1; \mathsf{x} : U; \Xi_2) : T\text{ valid}} \qquad \frac{\mathbb{T}\text{ valid} \qquad \mathbb{T} \triangleright \Xi_1 \vdash T\text{ sort} \qquad \mathbb{T} \triangleright \Xi_1.\Xi_2 \vdash}{\mathbb{T}, c(\Xi_1; \Xi_2) : T\text{ valid}}$$

$$\frac{\begin{array}{c}\mathbb{T}\text{ valid} \qquad d(\Xi_1; \mathsf{x} : U; \Xi_2) : V \in \mathbb{T} \qquad \text{for some } \Theta_1, \Theta_2 \text{ with } |\Theta_1| = \theta_1 \text{ and } |\Theta_2| = \theta_2 : \\ \mathbb{T} \triangleright \Xi_1.\Theta_1.\Theta_2 \vdash (\mathrm{id}_{\Xi_1}, t, \mathbf{u}) : \Xi_1.(\mathsf{x} : U).\Xi_2 \qquad \mathbb{T} \triangleright \Xi_1.\Theta_1.\Theta_2 \vdash r : V[\mathrm{id}_{\Xi_1}, t, \mathbf{u}]\end{array}}{\mathbb{T}, \theta_1; \theta_2 \Vdash d(t; \mathbf{u}) \longmapsto r\text{ valid}}$$

2.2 Basic metaproperties

Proposition 2.4 (Contexts are well-formed)**.** *The following rules are admissible.*

$$\frac{\Theta; \Gamma \vdash \mathcal{J}}{\Theta; \Gamma \vdash} \qquad \frac{\Theta; \Gamma \vdash \mathcal{J}}{\Theta \vdash}$$

$$\boxed{\Theta \vdash \quad (\Theta \in \mathsf{MCtx})} \qquad \boxed{\Theta; \Gamma \vdash \quad (\Theta \in \mathsf{MCtx};\ \Gamma \in \mathsf{Ctx}\ |\Theta|)}$$

$$\textsc{EmptyMCtx}\ \frac{}{\cdot \vdash} \qquad \textsc{ExtMCtx}\ \frac{\Theta; \Gamma \vdash T \text{ sort}}{\Theta, \mathsf{x}\{\Gamma\} : T \vdash} \qquad \textsc{EmptyCtx}\ \frac{\Theta \vdash}{\Theta; \cdot \vdash} \qquad \textsc{ExtCtx}\ \frac{\Theta; \Gamma \vdash T \text{ sort}}{\Theta; \Gamma, x : T \vdash}$$

$$\boxed{\Theta; \Gamma \vdash T \text{ sort} \quad (\Theta \in \mathsf{MCtx};\ \Gamma \in \mathsf{Ctx}\ |\Theta|;\ T \in \mathsf{Tm}\ |\Theta|\ |\Gamma|)}$$

$$\textsc{Sort}\ c(\Xi) \text{ sort} \in \mathbb{T}\ \frac{\Theta; \Gamma \vdash \mathbf{t} : \Xi}{\Theta; \Gamma \vdash c(\mathbf{t}) \text{ sort}}$$

$$\boxed{\Theta; \Gamma \vdash t : T \quad (\Theta \in \mathsf{MCtx};\ \Gamma \in \mathsf{Ctx}\ |\Theta|;\ T \in \mathsf{Tm}\ |\Theta|\ |\Gamma|;\ t \in \mathsf{Tm}\ |\Theta|\ |\Gamma|)}$$

$$\textsc{Var}\ x : T \in \Gamma\ \frac{\Theta; \Gamma \vdash}{\Theta; \Gamma \vdash x : T} \qquad \textsc{Cons}\ c(\Xi_1; \Xi_2) : T \in \mathbb{T}\ \frac{\Theta; \Gamma \vdash \mathbf{t}, \mathbf{u} : \Xi_1.\Xi_2}{\Theta; \Gamma \vdash c(\mathbf{u}) : T[\mathbf{t}]} \qquad \textsc{MVar}\ \mathsf{x}(\Delta) : T \in \Xi\ \frac{\Theta; \Gamma \vdash \vec{t} : \Delta}{\Theta; \Gamma \vdash \mathsf{x}\{\vec{t}\} : T[\vec{t}]}$$

$$\textsc{Dest}\ d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T}\ \frac{\Theta; \Gamma \vdash \mathbf{t}, t, \mathbf{u} : \Xi_1.(\mathsf{x} : T).\Xi_2}{\Theta; \Gamma \vdash d(t; \mathbf{u}) : U[\mathbf{t}, t, \mathbf{u}]} \qquad \textsc{Conv}\ T \equiv_{\mathbb{T}} U\ \frac{\Theta; \Gamma \vdash t : T \qquad \Theta; \Gamma \vdash U \text{ sort}}{\Theta; \Gamma \vdash t : U}$$

$$\boxed{\Theta; \Gamma \vdash \vec{t} : \Delta \quad (\Theta \in \mathsf{MCtx};\ \Gamma \in \mathsf{Ctx}\ |\Theta|;\ \Delta \in \mathsf{Ctx}\ |\Theta|;\ \vec{t} \in \mathsf{Sub}\ |\Theta|\ |\Gamma|\ |\Delta|)}$$

$$\textsc{EmptySub}\ \frac{\Theta; \Gamma \vdash}{\Theta; \Gamma \vdash \varepsilon : (\cdot)} \qquad \textsc{ExtSub}\ \frac{\Theta; \Gamma \vdash \vec{t} : \Delta \qquad \Theta; \Gamma \vdash t : T[\vec{t}]}{\Theta; \Gamma \vdash \vec{t}, t : (\Delta, x : T)}$$

$$\boxed{\Theta; \Gamma \vdash \mathbf{t} : \Xi \quad (\Theta \in \mathsf{MCtx};\ \Gamma \in \mathsf{Ctx}\ |\Theta|;\ \Xi \in \mathsf{MCtx};\ \mathbf{t} \in \mathsf{MSub}\ |\Theta|\ |\Gamma|\ |\Xi|)}$$

$$\textsc{EmptyMSub}\ \frac{\Theta; \Gamma \vdash}{\Theta; \Gamma \vdash \varepsilon : (\cdot)} \qquad \textsc{ExtMSub}\ \frac{\Theta; \Gamma \vdash \mathbf{t} : \Xi \qquad \Theta; \Gamma.\Delta[\mathbf{t}] \vdash t : T[\mathbf{t}]}{\Theta; \Gamma \vdash \mathbf{t}, \vec{x}_\Delta.t : (\Xi, \mathsf{x}\{\Delta\} : T)}$$

Figure 4: Declarative typing rules

Proof. By induction on $\Theta;\Gamma \vdash \mathcal{J}$. $\square$

Proposition 2.5 (Weakening). *Let us write $\Gamma \sqsubseteq \Delta$ if Γ is a subsequence of Δ, and $\Theta \sqsubseteq \Xi$ if Θ is a subsequence of Ξ. The following rules are admissible.*

$$\Gamma \sqsubseteq \Delta \frac{\Theta;\Gamma \vdash \mathcal{J} \qquad \Theta;\Delta \vdash}{\Theta;\Delta \vdash \mathcal{J}} \qquad\qquad \Theta \sqsubseteq \Xi \frac{\Theta;\Gamma \vdash \mathcal{J} \qquad \Xi \vdash}{\Xi;\Gamma \vdash \mathcal{J}}$$

Proof. In order for the induction to go through, we strengthen the first statement: instead we show that $\Theta;\Gamma.\Gamma' \vdash \mathcal{J}$ and $\Theta;\Delta \vdash$ and $\Gamma \sqsubseteq \Delta$ imply $\Theta;\Delta.\Gamma' \vdash \mathcal{J}$. The proof is then by induction on $\Theta;\Gamma.\Gamma' \vdash \mathcal{J}$ for the first statement, and on $\Theta;\Gamma \vdash \mathcal{J}$ for the second. $\square$

In order to state the substitution property, given $\Theta;\Delta \vdash \mathcal{J}$ we define the notations $(\Theta;\Delta \vdash \mathcal{J})[\vec{v}]$ and $(\Theta;\Delta \vdash \mathcal{J})[\mathbf{v}]$ by the following table.

$\Theta;\Delta \vdash \mathcal{J}$	$(\Theta;\Delta \vdash \mathcal{J})[\vec{v}]$ where $\Theta;\Gamma \vdash \vec{v} : \Delta$	$(\Theta;\Delta \vdash \mathcal{J})[\mathbf{v}]$ where $\Xi;\Gamma \vdash \mathbf{v} : \Theta$
$\Theta;\Delta \vdash$	$\Theta;\Gamma \vdash$	$\Xi;\Gamma.\Delta[\mathbf{v}] \vdash$
$\Theta;\Delta \vdash T \text{ sort}$	$\Theta;\Gamma \vdash T[\vec{v}] \text{ sort}$	$\Xi;\Gamma.\Delta[\mathbf{v}] \vdash T[\mathbf{v}] \text{ sort}$
$\Theta;\Delta \vdash t : T$	$\Theta;\Gamma \vdash t[\vec{v}] : T[\vec{v}]$	$\Xi;\Gamma.\Delta[\mathbf{v}] \vdash t[\mathbf{v}] : T[\mathbf{v}]$
$\Theta;\Delta \vdash \vec{t} : \Delta'$	$\Theta;\Gamma \vdash \vec{t}[\vec{v}] : \Delta'$	$\Xi;\Gamma.\Delta[\mathbf{v}] \vdash \mathrm{id}_\Gamma, \vec{t}[\mathbf{v}] : \Gamma.\Delta'[\mathbf{v}]$
$\Theta;\Delta \vdash \mathbf{t} : \Xi'$	$\Theta;\Gamma \vdash \mathbf{t}[\vec{v}] : \Xi'$	$\Xi;\Gamma.\Delta[\mathbf{v}] \vdash \mathbf{t}[\mathbf{v}] : \Xi'$

Proposition 2.6 (Substitution property). *The following rules are admissible.*

$$\frac{\Theta;\Gamma \vdash \vec{v} : \Delta \qquad \Theta;\Delta \vdash \mathcal{J}}{(\Theta;\Gamma \vdash \mathcal{J})[\vec{v}]} \qquad\qquad \frac{\Xi;\Gamma \vdash \mathbf{v} : \Theta \qquad \Theta;\Delta \vdash \mathcal{J}}{(\Theta;\Delta \vdash \mathcal{J})[\mathbf{v}]}$$

Proof. We start with the proof of the first property, however first we generalize so the induction can go through: we instead show that, for $\Theta;\Gamma \vdash \vec{v} : \Delta$, we have

- $\Theta;\Delta.\Gamma' \vdash$ implies $\Theta;\Gamma.\Gamma'[\vec{v}] \vdash$
- $\Theta;\Delta.\Gamma' \vdash T$ sort implies $\Theta;\Gamma.\Gamma'[\vec{v}] \vdash T[\vec{v}, \mathrm{id}_{\Gamma'}]$ sort
- $\Theta;\Delta.\Gamma' \vdash t : T$ implies $\Theta;\Gamma.\Gamma'[\vec{v}] \vdash t[\vec{v}, \mathrm{id}_{\Gamma'}] : T[\vec{v}, \mathrm{id}_{\Gamma'}]$
- $\Theta;\Delta.\Gamma' \vdash \vec{t} : \Delta'$ implies $\Theta;\Gamma.\Gamma'[\vec{v}] \vdash \vec{t}[\vec{v}, \mathrm{id}_{\Gamma'}] : \Delta'$
- $\Theta;\Delta.\Gamma' \vdash \mathbf{t} : \Xi'$ implies $\Theta;\Gamma.\Gamma'[\vec{v}] \vdash \mathbf{t}[\vec{v}, \mathrm{id}_{\Gamma'}] : \Xi'$

We show only the main cases.

- Case

$$x : T \in \Delta.\Gamma' \;\frac{\Theta;\Delta.\Gamma' \vdash}{\Theta;\Delta.\Gamma' \vdash x : T}\,\textsc{Var}$$

  We have either $x : T \in \Delta$ or $x : T \in \Gamma'$.

– Case $x : T \in \Gamma'$: Then we apply the i.h. to get $\Theta; \Gamma.\Gamma'[\vec{v}] \vdash$ and then conclude with the variable rule.

– Case $x : T \in \Delta$: Then from $\Theta; \Gamma \vdash \vec{v} : \Delta$ we can show $\Theta; \Gamma \vdash v_x : T[\vec{v}]$. By i.h. we have $\Theta; \Gamma.\Gamma'[\vec{v}] \vdash$, so we can apply Proposition 2.5 to get $\Theta; \Gamma.\Gamma'[\vec{v}] \vdash v_x : T[\vec{v}]$. Because we have $T[\vec{v}] = T[\vec{v}, \mathrm{id}_{\Gamma'}]$ we are done.

- Case

$$d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T} \; \frac{\Theta; \Delta.\Gamma' \vdash \mathbf{t}, t, \mathbf{u} : \Xi_1.(\mathsf{x} : T).\Xi_2}{\Theta; \Delta.\Gamma' \vdash d(t; \mathbf{u}) : U[\mathbf{t}, t, \mathbf{u}]} \textsc{Dest}$$

By i.h. we have $\Theta; \Gamma.\Gamma'[\vec{v}] \vdash (\mathbf{t}, t, \mathbf{u})[\vec{v}, \mathrm{id}_{\Gamma'}] : \Xi_1.(\mathsf{x} : T).\Xi_2$, therefore we can derive $\Theta; \Gamma.\Gamma'[\vec{v}] \vdash d(t[\vec{v}, \mathrm{id}_{\Gamma'}]; \mathbf{u}[\vec{v}, \mathrm{id}_{\Gamma'}]) : U[(\mathbf{t}, t, \mathbf{u})[\vec{v}, \mathrm{id}_{\Gamma'}]]$. Because we have $U[(\mathbf{t}, t, \mathbf{u})[\vec{v}, \mathrm{id}_{\Gamma'}]] = U[\mathbf{t}, t, \mathbf{u}][\vec{v}, \mathrm{id}_{\Gamma'}]$ we are done.

- Case

$$\frac{\Theta; \Delta.\Gamma' \vdash \mathbf{t} : \Xi \qquad \Theta; \Delta.\Gamma'.\Delta_{\mathsf{x}}[\mathbf{t}] \vdash t : T[\mathbf{t}]}{\Theta; \Delta.\Gamma' \vdash \mathbf{t}, \vec{x}_{\Delta}.t : (\Xi, \mathsf{x}\{\Delta_{\mathsf{x}}\} : T)} \textsc{ExtMSub}$$

By i.h. we have

$$\begin{aligned}
&\Theta; \Gamma.\Gamma'[\vec{v}] \vdash \mathbf{t}[\vec{v}, \mathrm{id}_{\Gamma'}] : \Xi \\
&\Theta; \Gamma.(\Gamma'.\Delta_{\mathsf{x}}[\mathbf{t}])[\vec{v}] \vdash t[\vec{v}, \mathrm{id}_{\Gamma'}, \mathrm{id}_{\Delta_{\mathsf{x}}}] : T[\mathbf{t}][\vec{v}, \mathrm{id}_{\Gamma'}, \mathrm{id}_{\Delta_{\mathsf{x}}}]
\end{aligned}$$

We moreover have

$$\begin{aligned}
&(\Gamma'.\Delta_{\mathsf{x}}[\mathbf{t}])[\vec{v}] = \Gamma'[\vec{v}].\Delta_{\mathsf{x}}[\mathbf{t}][\vec{v}, \mathrm{id}_{\Gamma'}] = \Gamma'[\vec{v}].\Delta_{\mathsf{x}}[\mathbf{t}[\vec{v}, \mathrm{id}_{\Gamma'}]] \\
&T[\mathbf{t}][\vec{v}, \mathrm{id}_{\Gamma'}, \mathrm{id}_{\Delta_{\mathsf{x}}}] = T[\mathbf{t}[\vec{v}, \mathrm{id}_{\Gamma'}]] \\
&(\mathbf{t}, \vec{x}_{\Delta_{\mathsf{x}}}.t)[\vec{v}, \mathrm{id}_{\Gamma'}] = \mathbf{t}[\vec{v}, \mathrm{id}_{\Gamma'}], \vec{x}_{\Delta_{\mathsf{x}}}.t[\vec{v}, \mathrm{id}_{\Gamma'}, \mathrm{id}_{\Delta_{\mathsf{x}}}]
\end{aligned}$$

We can therefore conclude $\Theta; \Gamma.\Gamma'[\vec{v}] \vdash (\mathbf{t}, \vec{x}_{\Delta_{\mathsf{x}}}.t)[\vec{v}, \mathrm{id}_{\Gamma'}] : (\Xi, \mathsf{x}\{\Delta_{\mathsf{x}}\} : T)$.

We now move to the proof of the second property, which is by induction on $\Theta; \Delta \vdash \mathcal{J}$. Once again, we only show the main cases.

- Case

$$\mathsf{x}(\Delta') : T \in \Theta \; \frac{\Theta; \Delta \vdash \vec{t} : \Delta'}{\Theta; \Delta \vdash \mathsf{x}\{\vec{t}\} : T[\vec{t}]} \textsc{MVar}$$

By i.h. we have $\Xi; \Gamma.\Delta[\mathbf{v}] \vdash \mathrm{id}_{\Gamma}, \vec{t}[\mathbf{v}] : \Gamma.\Delta'[\mathbf{v}]$. Moreover, from $\Xi; \Gamma \vdash \mathbf{v} : \Theta$ we can derive $\Xi; \Gamma.\Delta'[\mathbf{v}] \vdash \mathbf{v}_{\mathsf{x}} : T[\mathbf{v}]$, so by the substitution property for variable substitutions we get $\Xi; \Gamma.\Delta[\mathbf{v}] \vdash \mathbf{v}_{\mathsf{x}}[\mathrm{id}_{\Gamma}, \vec{t}[\mathbf{v}]] : T[\mathbf{v}][\mathrm{id}_{\Gamma}, \vec{t}[\mathbf{v}]]$. Because we have $T[\mathbf{v}][\mathrm{id}_{\Gamma}, \vec{t}[\mathbf{v}]] = T[\vec{t}][\mathbf{v}]$ we are done.

- Case

$$\frac{\Theta;\Delta\vdash}{\Theta;\Delta\vdash\varepsilon:(\cdot)}\;\textsc{EmptySub}$$

By i.h. we have $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash$. It is easy to see that we can show $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash \mathrm{id}_\Gamma:\Gamma$ and so we are done.

- Case

$$\frac{\Theta;\Delta\vdash\vec{t}:\Delta \qquad \Theta;\Delta\vdash t:T[\vec{t}]}{\Theta;\Delta\vdash\vec{t},t:(\Delta',x:T)}\;\textsc{ExtSub}$$

By i.h. we have $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash \mathrm{id}_\Gamma,\vec{t}[\mathbf{v}]:\Gamma.\Delta'[\mathbf{v}]$ and $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash t[\mathbf{v}]:T[\vec{t}][\mathbf{v}]$. We have $T[\vec{t}][\mathbf{v}]=T[\mathbf{v}][\mathrm{id}_\Gamma,\vec{t}[\mathbf{v}]]$ and therefore we can conclude $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash\mathrm{id}_\Gamma,\vec{t}[\mathbf{v}],t[\mathbf{v}]:(\Gamma.\Delta'[\mathbf{v}],x:T[\mathbf{v}])$.

- Case

$$\frac{\Theta;\Delta\vdash\mathbf{t}:\Xi \qquad \Theta;\Delta.\Delta'[\mathbf{t}]\vdash t:T[\mathbf{t}]}{\Theta;\Delta\vdash\mathbf{t},\vec{x}_\Delta.t:(\Xi,\mathsf{x}\{\Delta'\}:T)}\;\textsc{ExtMSub}$$

By i.h. we have $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash\mathbf{t}[\mathbf{v}]:\Xi$ and $\Xi;\Gamma.(\Delta.\Delta'[\mathbf{t}])[\mathbf{v}]\vdash t[\mathbf{v}]:T[\mathbf{t}][\mathbf{v}]$. We have

$$\Gamma.(\Delta.\Delta'[\mathbf{t}])[\mathbf{v}]=\Gamma.\Delta[\mathbf{v}].\Delta'[\mathbf{t}][\mathbf{v}]=\Gamma.\Delta[\mathbf{v}].\Delta'[\mathbf{t}[\mathbf{v}]]$$

and $T[\mathbf{t}][\mathbf{v}]=T[\mathbf{t}[\mathbf{v}]]$. We can therefore conclude $\Xi;\Gamma.\Delta[\mathbf{v}]\vdash\mathbf{t}[\mathbf{v}],\vec{x}.t[\mathbf{v}]:(\Xi,\mathsf{x}\{\Delta'\}:T)$. □

Proposition 2.7 (Sorts are well-typed). *The following rule is admissible when the underlying theory is valid.*

$$\frac{\Theta;\Gamma\vdash t:T}{\Theta;\Gamma\vdash T\ \mathsf{sort}}$$

Proof. By case analysis on $\Theta;\Gamma\vdash t:T$, and using Proposition 2.6. For the constructor and destructor cases we use the fact that the theory is valid to deduce $\Xi_1\vdash T\ \mathsf{sort}$ from $c(\Xi_1;\Xi_2):T\in\mathbb{T}$ and $\Xi_1.(\mathsf{x}:T).\Xi_2\vdash U\ \mathsf{sort}$ from $d(\Xi_1;\mathsf{x}:T;\Xi_2):U\in\mathbb{T}$. □

Proposition 2.8 (Conversion in context). *The following rule is admissible.*

$$\Gamma\equiv\Delta\frac{\Theta;\Gamma\vdash\mathcal{J} \qquad \Theta;\Delta\vdash}{\Theta;\Delta\vdash\mathcal{J}}$$

Proof. We first need to strengthen the statement: If $\Theta;\Gamma.\Gamma'\vdash\mathcal{J}$ and $\Theta;\Delta\vdash$ and $\Gamma\equiv\Delta$ then $\Theta;\Delta.\Gamma'\vdash\mathcal{J}$. Now the proof follows by induction on the derivation of $\Theta;\Gamma.\Gamma'\vdash\mathcal{J}$.

We show the only non-trivial case: rule VAR with a variable from Γ. We decompose $\Gamma = \Gamma_1.(x : T).\Gamma_2$ and $\Delta = \Delta_1.(x : U).\Delta_2$, where we have $\Gamma_1 \equiv \Delta_1$ and $\Gamma_2 \equiv \Delta_2$ and $T \equiv U$.

$$\textsc{Var}\ \frac{\Theta; \Gamma_1.(x : T).\Gamma_2.\Gamma' \vdash}{\Theta; \Gamma_1.(x : T).\Gamma_2.\Gamma' \vdash x : T}$$

By the i.h. applied to $\Theta; \Gamma_1.(x : T).\Gamma_2.\Gamma' \vdash$ we get $\Theta; \Delta_1.(x : U).\Delta_2.\Gamma' \vdash$, so we can show $\Theta; \Delta_1.(x : U).\Delta_2.\Gamma' \vdash x : U$. Because $T \equiv U$ we now want to apply the conversion rule, but for this we first have to show $\Theta; \Delta_1.(x : U).\Delta_2.\Gamma' \vdash T$ sort.

We can show that from the derivation of $\Theta; \Gamma_1.(x : T).\Gamma_2.\Gamma' \vdash$ we can extract a strictly smaller derivation of $\Theta; \Gamma_1 \vdash T$ sort. Moreover, from $\Theta; \Delta_1.(x : U).\Delta_2 \vdash$ we also get $\Theta; \Delta_1 \vdash$, so by applying the i.h. with $\Gamma_1 \equiv \Delta_1$ we get $\Theta; \Delta_1 \vdash T$ sort. By Proposition 2.5 we then get $\Theta; \Delta_1.(x : U).\Delta_2.\Gamma' \vdash T$ sort.

Now we can apply the conversion rule to $\Theta; \Delta_1.(x : U).\Delta_2.\Gamma' \vdash x : U$ to conclude. □

2.3 Subject reduction

Lemma 2.9 (Injectivity of patterns). *If $t \in \mathsf{Tm}^{\mathsf{P}}\ \theta\ \gamma$ and $t[\mathbf{v}] \equiv t[\mathbf{v}']$ for some $\mathbf{v} \in \mathsf{MSub}\ \theta'\ \delta\ \theta$ and $\mathbf{v}' \in \mathsf{MSub}\ \theta'\ \delta\ \theta$ then $\mathbf{v} \equiv \mathbf{v}'$.*

Proof. We need to show together a similar result for substitution patterns: if $\mathbf{t} \in \mathsf{MSub}^{\mathsf{P}}\ \theta\ \gamma\ \xi$ and $\mathbf{t}[\mathbf{v}] \equiv \mathbf{t}[\mathbf{v}']$ then $\mathbf{v} \equiv \mathbf{v}'$. The proof is done by induction on the pattern, using confluence for the case $c(\mathbf{t}[\mathbf{v}]) \equiv c(\mathbf{t}[\mathbf{v}'])$ to deduce $\mathbf{t}[\mathbf{v}] \equiv \mathbf{t}[\mathbf{v}']$. □

Proposition 2.10 (Typing a substitution through a pattern). *Let $\mathbf{v} \in \mathsf{MSub}\ \cdot\ |\Delta|\ |\Theta|$.*

- *If $t \in \mathsf{Tm}^{\mathsf{P}}\ |\Theta|\ \cdot$ and $\Theta \vdash t : T$ and $\Delta \vdash t[\mathbf{v}] : T[\mathbf{v}]$ then $\Delta \vdash \mathbf{v} : \Theta$*
- *If $T \in \mathsf{Tm}^{\mathsf{P}}\ |\Theta|\ \cdot$ and $\Theta \vdash T$ sort and $\Delta \vdash T[\mathbf{v}]$ sort then $\Delta \vdash \mathbf{v} : \Theta$*
- *If $\mathbf{t} \in \mathsf{MSub}^{\mathsf{P}}\ |\Theta|\ \cdot\ |\Xi|$ and $\Theta \vdash \mathbf{t} : \Xi$ and $\Delta \vdash \mathbf{t}[\mathbf{v}] : \Xi$ then $\Delta \vdash \mathbf{v} : \Theta$*

Proof. In order for the proof to go through, we instead show a stronger property. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathsf{MSub}\ \cdot\ |\Delta|\ |\Theta_1.\Theta_2.\Theta_2|$, and write $\mathbf{v}$ for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and Θ for $\Theta_1.\Theta_2.\Theta_3$. Suppose that $\Delta \vdash \mathbf{v}_1 : \Theta_1$ and that one of the following holds.

- $t \in \mathsf{Tm}^{\mathsf{P}}\ |\Theta_2|\ |\Gamma|$ and $\Theta; \Gamma \vdash t : T$ and $\Delta.\Gamma' \vdash t[\mathbf{v}] : T'$ with $\Gamma' \equiv \Gamma[\mathbf{v}]$ and $T' \equiv T[\mathbf{v}]$.
- $\mathbf{t}_2 \in \mathsf{MSub}^{\mathsf{P}}\ |\Theta_2|\ |\Gamma|\ |\Xi_2|$ and $\Theta; \Gamma \vdash \mathbf{t}_1, \mathbf{t}_2 : \Xi_1.\Xi_2$ and $\Delta.\Gamma' \vdash \mathbf{t}'_1, \mathbf{t}_2[\mathbf{v}] : \Xi_1.\Xi_2$ with $\Gamma' \equiv \Gamma[\mathbf{v}]$ and $\mathbf{t}'_1 \equiv \mathbf{t}_1[\mathbf{v}]$.
- $T \in \mathsf{Tm}^{\mathsf{P}}\ |\Theta_2|\ |\Gamma|$ and $\Theta; \Gamma \vdash T$ sort and $\Delta.\Gamma' \vdash T'[\mathbf{v}]$ sort with $\Gamma' \equiv \Gamma[\mathbf{v}]$.

Then we have $\Delta \vdash \mathbf{v}_1, \mathbf{v}_2 : \Theta_1.\Theta_2$. The proof is by induction on the pattern.

- $T = c(\mathbf{t} \in \mathsf{MSub}^{\mathsf{P}}\ |\Theta_2|\ |\Gamma|\ |\Xi|)$ for $c(\Xi)$ sort $\in \mathbb{T}$. By inversion on $\Theta; \Gamma \vdash T$ sort and $\Delta.\Gamma' \vdash T'[\mathbf{v}]$ sort we obtain the following.

  $$\frac{\Theta; \Gamma \vdash \mathbf{t} : \Xi}{\Theta; \Gamma \vdash c(\mathbf{t})\ \text{sort}} \qquad \frac{\Delta.\Gamma' \vdash \mathbf{t}[\mathbf{v}] : \Xi}{\Delta.\Gamma' \vdash c(\mathbf{t}[\mathbf{v}])\ \text{sort}}$$

  By i.h. we conclude.

- $t = c(\mathbf{t}_2 \in \mathsf{MSub}^{\mathsf{P}}\ |\Theta_2|\ |\Gamma|\ |\Xi_2|)$ for $c(\Xi_1;\Xi_2) : U \in \mathbb{T}$. By inversion of $\Theta;\Gamma \vdash t : T$ and $\Delta.\Gamma' \vdash t[\mathbf{v}] : T'$ we obtain the following.

$$T \equiv U[\mathbf{t}_1]\,\frac{\dfrac{\Theta;\Gamma \vdash \mathbf{t}_1, \mathbf{t}_2 : \Xi_1.\Xi_2}{\Theta;\Gamma \vdash c(\mathbf{t}_2) : U[\mathbf{t}_1]}}{\Theta;\Gamma \vdash c(\mathbf{t}_2) : T} \qquad T' \equiv U[\mathbf{t}'_1]\,\frac{\dfrac{\Delta.\Gamma' \vdash \mathbf{t}'_1, \mathbf{t}_2[\mathbf{v}] : \Xi_1.\Xi_2}{\Delta.\Gamma' \vdash c(\mathbf{t}_2[\mathbf{v}]) : U[\mathbf{t}'_1]}}{\Delta.\Gamma' \vdash c(\mathbf{t}_2[\mathbf{v}]) : T'}$$

  Because $U \in \mathsf{Tm}^{\mathsf{P}}\ |\Xi_1|\ \cdot$, then $U[\mathbf{t}'_1] \equiv T' \equiv T[\mathbf{v}] \equiv U[\mathbf{t}_1][\mathbf{v}] = U[\mathbf{t}_1[\mathbf{v}]]$ implies $\mathbf{t}'_1 \equiv \mathbf{t}_1[\mathbf{v}]$. We can therefore apply the i.h. to conclude.

- $t = \mathsf{x}\{\mathsf{id}_\Gamma\}$ for $\mathsf{x}\{\Gamma_\mathsf{x}\} : U \in \Theta$, in which case we must have $\Theta_2 = \mathsf{x}\{\Gamma_\mathsf{x}\} : U$. By inversion of $\Theta;\Gamma \vdash t : T$ we get $T \equiv U$ and $\Gamma \equiv \Gamma_\mathsf{x}$, and therefore $T' \equiv U[\mathbf{v}]$ and $\Gamma' \equiv \Gamma_\mathsf{x}[\mathbf{v}]$. Moreover, as the only metavariables of Θ appearing in Γ_x and U are those of Θ_1, we have $\Gamma_\mathsf{x}[\mathbf{v}_1] = \Gamma_\mathsf{x}[\mathbf{v}]$ and $U[\mathbf{v}_1] = U[\mathbf{v}]$, and therefore $T' \equiv U[\mathbf{v}_1]$ and $\Gamma' \equiv \Gamma_\mathsf{x}[\mathbf{v}_1]$. Then, because $\Theta \vdash$, we have $\Theta_1;\Gamma_\mathsf{x} \vdash U$ sort, so by applying Proposition 2.6 with $\Delta \vdash \mathbf{v}_1 : \Theta_1$ we get $\Delta.\Gamma_\mathsf{x}[\mathbf{v}_1] \vdash U[\mathbf{v}_1]$ sort. Now we can apply conversion and Proposition 2.8 to $\Delta.\Gamma' \vdash t[\mathbf{v}] : T'$ to get $\Delta.\Gamma_\mathsf{x}[\mathbf{v}_1] \vdash t[\mathbf{v}] : U[\mathbf{v}_1]$. Because $t[\mathbf{v}] = \mathbf{v}_\mathsf{x}$ then together with $\Delta \vdash \mathbf{v}_1 : \Theta_1$ we get $\Delta \vdash \mathbf{v}_1, \vec{x}_\Gamma.\mathbf{v}_\mathsf{x} : (\Theta_1, \mathsf{x}\{\Gamma_\mathsf{x}\} : U)$.

- $\mathbf{t}_2 = \varepsilon \in \mathsf{MSub}^{\mathsf{P}}\ (\cdot)\ |\Gamma|\ (\cdot)$. Then the result follows by hypothesis.

- $\mathbf{t}_2 = \mathbf{u} \in \mathsf{MSub}^{\mathsf{P}}\ |\Theta_{2l}|\ |\Gamma|\ |\Xi'_2|, \vec{x}.u \in \mathsf{Tm}^{\mathsf{P}}\ |\Theta_{2r}|\ |\Gamma|.|\Delta'|$ for $\Theta_2 = \Theta_{2l}.\Theta_{2r}$ and $\Xi_2 = \Xi'_2, \mathsf{x}\{\Delta'\} : U$. By inversion of $\Theta;\Gamma \vdash \mathbf{t}_1, \mathbf{t}_2 : \Xi_1.\Xi_2$ and $\Delta.\Gamma' \vdash \mathbf{t}'_1, \mathbf{t}_2[\mathbf{v}] : \Xi_1.\Xi_2$ we obtain the following.

$$\frac{\Theta;\Gamma \vdash \mathbf{t}_1, \mathbf{u} : \Xi_1.\Xi'_2 \qquad \Theta;\Gamma.\Delta'[\mathbf{t}_1, \mathbf{u}] \vdash u : U[\mathbf{t}_1, \mathbf{u}]}{\Theta;\Gamma \vdash \mathbf{t}_1, \mathbf{u}, \vec{x}.u : (\Xi_1.\Xi'_2, \mathsf{x}\{\Delta'\} : U)} \qquad \frac{\Delta.\Gamma' \vdash \mathbf{t}'_1, \mathbf{u}[\mathbf{v}] : \Xi_1.\Xi'_2 \qquad \Delta.\Gamma'.\Delta'[\mathbf{t}'_1, \mathbf{u}[\mathbf{v}]] \vdash u[\mathbf{v}] : U[\mathbf{t}'_1, \mathbf{u}[\mathbf{v}]]}{\Delta.\Gamma' \vdash \mathbf{t}'_1, \mathbf{u}[\mathbf{v}], \vec{x}.u[\mathbf{v}] : (\Xi_1.\Xi'_2, \mathsf{x}\{\Delta'\} : U)}$$

  Let $\mathbf{v}_2 = \mathbf{v}_{2l}, \mathbf{v}_{2l}$ be the splitting of $\mathbf{v}_2$ according to the decomposition $\Theta_2 = \Theta_{2l}.\Theta_{2r}$. By the i.h. applied to the first premises we get $\Delta \vdash \mathbf{v}_1, \mathbf{v}_{2l} : \Theta_1.\Theta_{2l}$.

  Then, note that we have

$$(\Gamma.\Delta'[\mathbf{t}_1, \mathbf{u}])[\mathbf{v}] = \Gamma[\mathbf{v}].\Delta[\mathbf{t}_1, \mathbf{u}][\mathbf{v}] = \Gamma[\mathbf{v}].\Delta[\mathbf{t}_1[\mathbf{v}], \mathbf{u}[\mathbf{v}]] \equiv \Gamma'.\Delta[\mathbf{t}'_1, \mathbf{u}[\mathbf{v}]]$$

  and

$$U[\mathbf{t}_1, \mathbf{u}][\mathbf{v}] \equiv U[\mathbf{t}_1[\mathbf{v}], \mathbf{u}[\mathbf{v}]] \equiv U[\mathbf{t}'_1, \mathbf{u}[\mathbf{v}]]$$

  so by the i.h. applied to the second premises we get $\Delta \vdash \mathbf{v}_1, \mathbf{v}_{2l}, \mathbf{v}_{2r} : \Theta_1.\Theta_{2l}.\Theta_{2r}$, concluding the proof. □

Theorem 2.11 (Subject reduction). *Suppose that the underlying theory is valid.*

- *If* $\Gamma \vdash T$ sort *and* $T \longrightarrow T'$ *then* $\Gamma \vdash T'$ sort
- *If* $\Gamma \vdash t : T$ *and* $t \longrightarrow t'$ *then* $\Gamma \vdash t' : T$
- *If* $\Gamma \vdash \mathbf{t} : \Xi$ *and* $\Xi \vdash$ *and* $\mathbf{t} \longrightarrow \mathbf{t}'$ *then* $\Gamma \vdash \mathbf{t}' : \Xi$

Proof. By induction on the rewrite judgment.

- Case

$$\frac{\theta_1;\theta_2 \Vdash d(t;\mathbf{u}) \longmapsto r \qquad \mathbf{v}_1 \in \mathsf{MSub}\ \cdot\ |\Gamma|\ \theta_1 \qquad \mathbf{v}_2 \in \mathsf{MSub}\ \cdot\ |\Gamma|\ \theta_2}{d(t[\mathbf{v}_1];\mathbf{u}[\mathbf{v}_2]) \longrightarrow r[\mathbf{v}_1,\mathbf{v}_2]}$$

Let $d(\Xi_1;\mathsf{x} : U;\Xi_2) : V \in \mathbb{T}$ be the rule for d in $\mathbb{T}$. Because $\mathbb{T}$ is valid, there are Θ_1 and Θ_2 with $|\Theta_1| = \theta_1$ and $|\Theta_2| = \theta_2$ such that

$$\Xi_1.\Theta_1.\Theta_2 \vdash (\mathsf{id}_{\Xi_1}, t, \mathbf{u}) : \Xi_1.(\mathsf{x} : U).\Xi_2$$
$$\Xi_1.\Theta_1.\Theta_2 \vdash r : V[\mathsf{id}_{\Xi_1}, t, \mathbf{u}]$$

By inversion on $\Gamma \vdash d(t[\mathbf{v}_1];\mathbf{u}[\mathbf{v}_2]) : T$ we get

$$T \equiv V[\mathbf{v}_0, t[\mathbf{v}_1], \mathbf{u}[\mathbf{v}_2]]\,\frac{\dfrac{\Gamma \vdash \mathbf{v}_0, t[\mathbf{v}_1], \mathbf{u}[\mathbf{v}_2] : \Xi_1.(\mathsf{x} : U).\Xi_2}{\Gamma \vdash d(t[\mathbf{v}_1];\mathbf{u}[\mathbf{v}_2]) : V[\mathbf{v}_0, t[\mathbf{v}_1], \mathbf{u}[\mathbf{v}_2]]}}{\Gamma \vdash d(t[\mathbf{v}_1];\mathbf{u}[\mathbf{v}_2]) : T}$$

Therefore, we have $\Gamma \vdash (\mathsf{id}_{\Xi_1}, t, \mathbf{u})[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] : \Xi_1.(\mathsf{x} : U).\Xi_2$, and because $\mathsf{id}_{\Xi_1}, t, \mathbf{u} \in \mathsf{MSub}^{\mathsf{P}}\ |\Xi_1.\Theta_1.\Theta_2| \cdot |\Xi_1.(\mathsf{x} : U).\Xi_2|$, then by Proposition 2.10 we get $\Gamma \vdash \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 : \Xi_1.\Theta_1.\Theta_2$. By applying Proposition 2.6 with $\Xi_1.\Theta_1.\Theta_2 \vdash r : V[\mathsf{id}_{\Xi_1}, t, \mathbf{u}]$ we get $\Gamma \vdash r[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] : V[\mathsf{id}_{\Xi_1}, t, \mathbf{u}][\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$. Finally, we have $\Gamma \vdash T$ sort by Proposition 2.7 applied to $\Gamma \vdash d(t[\mathbf{v}_1];\mathbf{u}[\mathbf{v}_2]) : T$, and because $r[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] = r[\mathbf{v}_1, \mathbf{v}_2]$ and

$$V[\mathsf{id}_{\Xi_1}, t, \mathbf{u}][\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] = V[\mathbf{v}_0, t[\mathbf{v}_1], \mathbf{u}[\mathbf{v}_2]] \equiv T$$

then by the conversion rule we conclude $\Gamma \vdash r[\mathbf{v}_1, \mathbf{v}_2] : T$.

- Case

$$\frac{\mathbf{v} \longrightarrow \mathbf{v}'}{c(\mathbf{v}) \longrightarrow c(\mathbf{v}')}$$

We have two possibilities: either $c(\Xi_1;\Xi_2) : U \in \mathbb{T}$ or $c(\Xi)$ sort $\in \mathbb{T}$.

  - Case $c(\Xi_1;\Xi_2) : U \in \mathbb{T}$: By inversion of typing we have

$$T \equiv U[\mathbf{u}]\,\frac{\dfrac{\Gamma \vdash \mathbf{u}, \mathbf{v} : \Xi_1.\Xi_2}{\Gamma \vdash c(\mathbf{v}) : U[\mathbf{u}]}}{\Gamma \vdash c(\mathbf{v}) : T}$$

    Because the theory is valid we have $\Xi_1.\Xi_2 \vdash$, so by i.h. we have $\Gamma \vdash \mathbf{u}, \mathbf{v}' : \Xi_1.\Xi_2$, and thus $\Gamma \vdash c(\mathbf{v}') : U[\mathbf{u}]$. By Proposition 2.7 applied to $\Gamma \vdash c(\mathbf{v}) : T$ we get $\Gamma \vdash T$ sort, therefore we can apply the conversion rule to conclude $\Gamma \vdash c(\mathbf{v}') : T$.

  - Case $c(\Xi)$ sort $\in \mathbb{T}$: By inversion of typing we have

$$\frac{\Gamma \vdash \mathbf{v} : \Xi}{\Gamma \vdash c(\mathbf{v})\ \mathsf{sort}}$$

    Because the theory is valid we have $\Xi \vdash$, so by the i.h. we get $\Gamma \vdash \mathbf{v}' : \Xi$, allowing us to conclude $\Gamma \vdash c(\mathbf{v}')$ sort.

- Case

$$\frac{\mathbf{v} \longrightarrow \mathbf{v}'}{d(t;\mathbf{v}) \longrightarrow d(t;\mathbf{v}')}$$

By inversion of typing we have

$$U[\mathbf{u},t,\mathbf{v}] \equiv T \frac{\dfrac{d(\Xi_1;\mathsf{x}:V;\Xi_2):U \in \mathbb{T} \qquad \Gamma \vdash \mathbf{u},t,\mathbf{v} : \Xi_1.(\mathsf{x}:V).\Xi_2}{\Gamma \vdash d(t;\mathbf{v}) : U[\mathbf{u},t,\mathbf{v}]}}{\Gamma \vdash d(t;\mathbf{v}) : T}$$

Because the theory is valid, we have $\Xi_1.(\mathsf{x} : V).\Xi_2 \vdash$, so by i.h. we get $\Gamma \vdash \mathbf{u},t,\mathbf{v}' : \Xi_1.(\mathsf{x} : V).\Xi_2$, from which we can show $\Gamma \vdash d(t;\mathbf{v}') : U[\mathbf{u},t,\mathbf{v}']$. We have $T \equiv U[\mathbf{u},t,\mathbf{v}] \equiv U[\mathbf{u},t,\mathbf{v}']$, and by Proposition 2.7 applied to $\Gamma \vdash d(t;\mathbf{v}) : T$ we get $\Gamma \vdash T$ sort, therefore we can apply the conversion rule to conclude $\Gamma \vdash d(t;\mathbf{v}') : T$.

- Case

$$\frac{t \longrightarrow t'}{d(t;\mathbf{v}) \longrightarrow d(t';\mathbf{v})}$$

By inversion of typing we have

$$U[\mathbf{u},t,\mathbf{v}] \equiv T \frac{\dfrac{d(\Xi_1;\mathsf{x}:V;\Xi_2):U \in \mathbb{T} \qquad \Gamma \vdash \mathbf{u},t,\mathbf{v} : \Xi_1.(\mathsf{x}:V).\Xi_2}{\Gamma \vdash d(t;\mathbf{v}) : U[\mathbf{u},t,\mathbf{v}]}}{\Gamma \vdash d(t;\mathbf{v}) : T}$$

Because the theory is valid, we have $\Xi_1.(\mathsf{x} : V).\Xi_2 \vdash$, so by i.h. we get $\Gamma \vdash \mathbf{u},t',\mathbf{v} : \Xi_1.(\mathsf{x} : V).\Xi_2$, from which we can show $\Gamma \vdash d(t';\mathbf{v}) : U[\mathbf{u},t',\mathbf{v}]$. We have $T \equiv U[\mathbf{u},t,\mathbf{v}] \equiv U[\mathbf{u},t',\mathbf{v}]$, and by Proposition 2.7 applied to $\Gamma \vdash d(t;\mathbf{v}) : T$ we get $\Gamma \vdash T$ sort, therefore we can apply the conversion rule to conclude $\Gamma \vdash d(t';\mathbf{v}) : T$.

- Case

$$\frac{\mathbf{v} \longrightarrow \mathbf{v}'}{\mathbf{v},\vec{x}.u \longrightarrow \mathbf{v}',\vec{x}.u}$$

By inversion of typing we have $\Xi = \Xi', \mathsf{x}\{\Delta\} : T$ with

$$\frac{\Gamma \vdash \mathbf{v} : \Xi' \qquad \Gamma.\Delta[\mathbf{v}] \vdash u : T[\mathbf{v}]}{\Gamma \vdash \mathbf{v},\vec{x}.u : (\Xi', \mathsf{x}\{\Delta\} : T)}$$

From $\Xi \vdash$ we get $\Xi' \vdash$ and $\Xi';\Delta \vdash T$ sort, so by i.h. we get $\Gamma \vdash \mathbf{v}' : \Xi'$. By Proposition 2.6 we then get $\Gamma.\Delta[\mathbf{v}'] \vdash T[\mathbf{v}']$ sort, and because $\Delta[\mathbf{v}] \equiv \Delta[\mathbf{v}']$ and $T[\mathbf{v}] \equiv T[\mathbf{v}']$ we can apply Proposition 2.8 and conversion to $\Gamma.\Delta[\mathbf{v}] \vdash u : T[\mathbf{v}]$ to derive $\Gamma.\Delta[\mathbf{v}'] \vdash u : T[\mathbf{v}']$. We can now conclude $\Gamma \vdash \mathbf{v}',\vec{x}.u : (\Xi', \mathsf{x}\{\Delta\} : T)$.

- Case

$$\frac{u \longrightarrow u'}{\mathbf{v}, \vec{x}.u \longrightarrow \mathbf{t}, \vec{x}.u'}$$

By inversion of typing we have $\Xi = \Xi', \mathsf{x}\{\Delta\} : T$ with

$$\frac{\Gamma \vdash \mathbf{v} : \Xi' \qquad \Gamma.\Delta[\mathbf{v}] \vdash u : T[\mathbf{v}]}{\Gamma \vdash \mathbf{v}, \vec{x}.u : (\Xi', \mathsf{x}\{\Delta\} : T)}$$

By i.h. we have $\Gamma.\Delta[\mathbf{v}] \vdash u' : T[\mathbf{v}]$ and thus we can conclude $\Gamma \vdash \mathbf{v}, \vec{x}.u' : (\Xi', \mathsf{x}\{\Delta\} : T)$. □

Remark 2.12. We show subject reduction only for terms without metavariables, but by strengthening Proposition 2.10 this could be extended to terms with metavariables. This generalization would however be of no use for the proofs to come.

3 Bidirectional type system

3.1 Matching modulo rewriting

Figure 5 defines the matching judgment.

$$\boxed{t \prec u \rightsquigarrow \mathbf{v} \quad (t \in \mathsf{Tm}^{\mathsf{P}}\ \theta\ \gamma;\ u \in \mathsf{Tm}\ \cdot\ \delta.\gamma;\ \mathbf{v} \in \mathsf{MSub}\ \cdot\ \delta\ \theta)}$$

$$\texttt{Rigid}_{\prec}\ \frac{u \longrightarrow^{\mathsf{h}}_{\mathsf{m/o}} c(\mathbf{u}) \qquad \mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v}}{c(\mathbf{t}) \prec u \rightsquigarrow \mathbf{v}} \qquad \texttt{Flex}_{\prec}\ \frac{}{\mathsf{x}\{\mathsf{id}_\gamma\} \prec u \rightsquigarrow \vec{x}_\gamma.u}$$

$$\boxed{\mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v} \quad (\mathbf{t} \in \mathsf{MSub}^{\mathsf{P}}\ \theta\ \gamma\ \xi;\ \mathbf{u} \in \mathsf{MSub}\ \cdot\ \delta.\gamma\ \xi;\ \mathbf{v} \in \mathsf{Sub}\ \cdot\ \delta\ \theta)}$$

$$\texttt{EmptySp}_{\prec}\ \frac{}{\varepsilon \prec \varepsilon \rightsquigarrow \varepsilon} \qquad \texttt{ExtSp}_{\prec}\ \frac{\mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v}_1 \qquad t \prec u \rightsquigarrow \mathbf{v}_2}{\mathbf{t}, \vec{x}.t \prec \mathbf{u}, \vec{x}.u \rightsquigarrow \mathbf{v}_1, \mathbf{v}_2}$$

Figure 5: Matching judgment

Recall from rewriting theory that a *(functional) strategy* $\mathfrak{S}$ is defined by a subrelation $\longrightarrow_{\mathfrak{S}} \subseteq \longrightarrow^{+}$ which has the same normal forms as $\longrightarrow$ and is functional in the sense that $t \longrightarrow_{\mathfrak{S}} u_1$ and $t \longrightarrow_{\mathfrak{S}} u_2$ imply $u_1 = u_2$.

Let $\longrightarrow_{\mathsf{m/o}}$ be the *maximal outermost* strategy, which contracts all outermost redexes in one step. We write $t \longrightarrow^{\mathsf{h}}_{\mathsf{m/o}} c(\mathbf{u})$ if $c(\mathbf{u})$ is the first term headed by a constructor to which t reduces by $\longrightarrow^{*}_{\mathsf{m/o}}$. Formally, this means that $t \longrightarrow^{*}_{\mathsf{m/o}} c(\mathbf{u})$ and for all $\mathbf{u}'$ with $t \longrightarrow^{*}_{\mathsf{m/o}} c(\mathbf{u}')$ we have $c(\mathbf{u}) \longrightarrow^{*}_{\mathsf{m/o}} c(\mathbf{u}')$.

Lemma 3.1. *If* $u \equiv c(\mathbf{t})$ *then* $u \longrightarrow^{\mathsf{h}}_{\mathsf{m/o}} c(\mathbf{u})$ *with* $\mathbf{t} \equiv \mathbf{u}$.

Proof. The strategy m/o is *outermost-fair*, so because our rewrite systems are orthogonal and *fully extended*, by [van Oostrom(1999), Theorem 2] we get that m/o is head-normalizing. By confluence, u reduces to a term of the form $c(\mathbf{v})$, and therefore $u \longrightarrow^{\mathsf{h}}_{\mathsf{m/o}} c(\mathbf{u})$ for some $\mathbf{u}$. We deduce $\mathbf{t} \equiv \mathbf{u}$ from $c(\mathbf{t}) \equiv c(\mathbf{u})$ using confluence. □

Proposition 3.2 (Soundness of matching)**.**

- *If* $t \prec u \rightsquigarrow \mathbf{v}$ *then* $u \longrightarrow^* t[\mathbf{v}]$.
- *If* $\mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v}$ *then* $\mathbf{u} \longrightarrow^* \mathbf{t}[\mathbf{v}]$.

Proof. By induction on the matching judgment. □

Proposition 3.3 (Completeness of matching)**.**

- *If* $t[\mathbf{v}] \equiv u$ *for some* $\mathbf{v} \in \mathsf{MSub}\ \cdot\ \delta\ \theta$ *then* $t \prec u \rightsquigarrow \mathbf{v}'$ *for some* $\mathbf{v}'$ *with* $\mathbf{v} \equiv \mathbf{v}'$.
- *If* $\mathbf{t}[\mathbf{v}] \equiv \mathbf{u}$ *for some* $\mathbf{v} \in \mathsf{MSub}\ \cdot\ \delta\ \theta$ *then* $\mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v}'$ *for some* $\mathbf{v}'$ *with* $\mathbf{v} \equiv \mathbf{v}'$.

Proof. By induction on the pattern, using Lemma 3.1. □

Recall that an expression is weak normalizing if it reduces to an expression in normal form.

Proposition 3.4 (Decidability of matching)**.**

- *If* u *is weak normalizing, then for all* t *the statement* $\exists \mathbf{v}.\ t \prec u \rightsquigarrow \mathbf{v}$ *is decidable.*
- *If* $\mathbf{u}$ *is weak normalizing, then for all* $\mathbf{t}$ *the statement* $\exists \mathbf{v}.\ \mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v}$ *is decidable.*

Proof. By induction on the pattern, and using the fact that m/o is normalizing [van Raamsdonk(1997), Theorem 10], so if u is weak normalizing then reducing it by m/o eventually ends. □

3.2 Bidirectional typing rules

Given an underlying signature Σ, we define the inferable and checkable terms and the checkable metavariable substitutions by the following grammars.

$$
\begin{array}{rrll}
\boxed{\mathsf{Tm}^{\mathsf{c}}\ \gamma} \ni & t,u,v ::= & \mid c(\mathbf{t} \in \mathsf{MSub}^{\mathsf{c}}\ \gamma\ \xi) & \text{if } c(\xi) \in \Sigma \\
 & & \mid \underline{t} \in \mathsf{Tm}^{\mathsf{i}}\ \gamma & \\
\boxed{\mathsf{Tm}^{\mathsf{i}}\ \gamma} \ni & t,u,v ::= & \mid x & \text{if } x \in \gamma \\
 & & \mid d(t \in \mathsf{Tm}^{\mathsf{i}}\ \gamma; \mathbf{t} \in \mathsf{MSub}^{\mathsf{c}}\ \gamma\ \xi) & \text{if } d(\xi) \in \Sigma \\
\boxed{\mathsf{MSub}^{\mathsf{c}}\ \gamma\ \xi} \ni & \mathbf{t},\mathbf{u},\mathbf{v} ::= & \mid \epsilon & \text{if } \xi = \cdot \\
 & & \mid \mathbf{t}' \in \mathsf{MSub}^{\mathsf{c}}\ \gamma\ \xi', \vec{x}_\delta.t \in \mathsf{Tm}^{\mathsf{c}}\ \gamma.\delta & \text{if } \xi = \xi', \mathsf{x}\{\delta\}
\end{array}
$$

Given $t \in \mathsf{Tm}^{\mathsf{c}}\ \gamma$ or $t \in \mathsf{Tm}^{\mathsf{i}}\ \gamma$ we write $\ulcorner t \urcorner \in \mathsf{Tm}\ \cdot\ \gamma$ for its underlying term, and for $\mathbf{t} \in \mathsf{MSub}^{\mathsf{c}}\ \gamma\ \xi$ we also write $\ulcorner \mathbf{t} \urcorner \in \mathsf{MSub}\ \cdot\ \gamma\ \xi$ for its underlying metavariable substitution.

Given a theory $\mathbb{T}$, we define its bidirectional type system by the rules in Figure 6.

$$\boxed{\Gamma \vdash t \Leftarrow T \quad (\Gamma \in \mathsf{Ctx};\ T \in \mathsf{Tm}\ \cdot\ |\Gamma|;\ t \in \mathsf{Tm}^{\mathsf{c}}\ |\Gamma|)}$$

$$c(\Xi_1;\Xi_2) : T \in \mathbb{T}\ \frac{T \prec U \rightsquigarrow \mathbf{v} \qquad \Gamma \mid \mathbf{v} : \Xi_1 \vdash \mathbf{u} \Leftarrow \Xi_2}{\Gamma \vdash c(\mathbf{u}) \Leftarrow U}\ \textsc{Cons} \qquad T \equiv_{\mathbb{T}} U\ \frac{\Gamma \vdash t \Rightarrow T}{\Gamma \vdash \underline{t} \Leftarrow U}\ \textsc{Switch}$$

$$\boxed{\Gamma \vdash t \Rightarrow T \quad (\Gamma \in \mathsf{Ctx};\ T \in \mathsf{Tm}\ \cdot\ |\Gamma|;\ t \in \mathsf{Tm}^{\mathsf{i}}\ |\Gamma|)}$$

$$d(\Xi_1;\mathsf{x} : T;\Xi_2) : U \in \mathbb{T}\ \frac{\Gamma \vdash t \Rightarrow V \qquad T \prec V \rightsquigarrow \mathbf{v} \qquad \Gamma \mid (\mathbf{v}, \ulcorner t \urcorner) : (\Xi_1, \mathsf{x} : T) \vdash \mathbf{u} \Leftarrow \Xi_2}{\Gamma \vdash d(t;\mathbf{u}) \Rightarrow U[\mathbf{v}, \ulcorner t \urcorner, \ulcorner \mathbf{u} \urcorner]}\ \textsc{Dest} \qquad x : T \in \Gamma\ \frac{}{\Gamma \vdash x \Rightarrow T}\ \textsc{Var}$$

$$\boxed{\Gamma \vdash T \Leftarrow \mathsf{sort} \quad (\Gamma \in \mathsf{Ctx};\ T \in \mathsf{Tm}^{\mathsf{c}}\ |\Gamma|)}$$

$$\mathbb{T} \ni c(\Xi)\ \mathsf{sort}\ \frac{\Gamma \mid \varepsilon : (\cdot) \vdash \mathbf{t} \Leftarrow \Xi}{\Gamma \vdash c(\mathbf{t}) \Leftarrow \mathsf{sort}}\ \textsc{Sort}$$

$$\boxed{\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi \quad (\Gamma \in \mathsf{Ctx};\ \Theta \in \mathsf{MCtx};\ \mathbf{v} \in \mathsf{MSub}\ \cdot\ |\Gamma|\ |\Theta|;\ \Xi \in \mathsf{MCtx}\ |\Theta|;\ \mathbf{t} \in \mathsf{MSub}^{\mathsf{c}}\ |\Gamma|\ |\Xi|)}$$

$$\frac{}{\Gamma \mid \mathbf{v} : \Theta \vdash \varepsilon \Leftarrow (\cdot)}\ \textsc{EmptyMSub} \qquad \frac{\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi \qquad \Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{t} \urcorner] \vdash t \Leftarrow T[\mathbf{v}, \ulcorner \mathbf{t} \urcorner]}{\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t}, \vec{x}_{\Delta}.t \Leftarrow (\Xi, \mathsf{x}\{\Delta\} : T)}\ \textsc{ExtMSub}$$

Figure 6: Bidirectional typing rules

3.3 Equivalence with declarative typing

Theorem 3.5 (Soundness). *Suppose that the underlying theory $\mathbb{T}$ is valid.*

- *If* $\Gamma \vdash$ *and* $\Gamma \vdash t \Rightarrow T$ *then* $\Gamma \vdash \ulcorner t \urcorner : T$
- *If* $\Gamma \vdash T$ sort *and* $\Gamma \vdash t \Leftarrow T$ *then* $\Gamma \vdash \ulcorner t \urcorner : T$
- *If* $\Gamma \vdash$ *and* $\Gamma \vdash T \Leftarrow$ sort *then* $\Gamma \vdash \ulcorner T \urcorner$ sort
- *If* $\Gamma \vdash \mathbf{v} : \Xi_1$ *and* $\Xi_1.\Xi_2 \vdash$ *and* $\Gamma \mid \mathbf{v} : \Xi_1 \vdash \mathbf{t} \Leftarrow \Xi_2$ *then* $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Xi_1.\Xi_2$.

Proof. By induction on the derivation.

- Case

  $$c(\Xi_1;\Xi_2) : T \in \mathbb{T}\ \dfrac{T \prec U \rightsquigarrow \mathbf{v} \qquad \Gamma \mid \mathbf{v} : \Xi_1 \vdash \mathbf{u} \Leftarrow \Xi_2}{\Gamma \vdash c(\mathbf{u}) \Leftarrow U}\ \textsc{Cons}$$

  By Proposition 3.2 we have $U \longrightarrow^* T[\mathbf{v}]$, so because we have $\Gamma \vdash U$ sort then by Theorem 2.11 we get $\Gamma \vdash T[\mathbf{v}]$ sort. We have $T \in \mathsf{Tm}^{\mathsf{P}}\ |\Xi_1|\ \cdot$, and validity of the theory also gives $\Xi_1 \vdash T$ sort, therefore by Proposition 2.10 we get $\Gamma \vdash \mathbf{v} : \Xi_1$. By validity of the theory we also have $\Xi_1.\Xi_2 \vdash$, therefore by applying the i.h. to the second premise we get $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{u} \urcorner : \Xi_1.\Xi_2$. We can therefore derive $\Gamma \vdash c(\ulcorner \mathbf{u} \urcorner) : T[\mathbf{v}]$, and because $T[\mathbf{v}] \equiv U$ and $\Gamma \vdash U$ sort we can apply conversion to conclude $\Gamma \vdash c(\ulcorner \mathbf{u} \urcorner) : U$.

- Case

  $$T \equiv_{\mathbb{T}} U\ \dfrac{\Gamma \vdash t \Rightarrow T}{\Gamma \vdash \underline{t} \Leftarrow U}\ \textsc{Switch}$$

  By i.h. we have $\Gamma \vdash \ulcorner t \urcorner : T$, and because we have $\Gamma \vdash U$ sort and $T \equiv U$ we can apply the conversion rule to conclude $\Gamma \vdash \ulcorner \underline{t} \urcorner : U$.

- Case

  $$d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T}\ \dfrac{\Gamma \vdash t \Rightarrow V \qquad T \prec V \rightsquigarrow \mathbf{v} \qquad \Gamma \mid (\mathbf{v}, \ulcorner t \urcorner) : (\Xi_1, \mathsf{x} : T) \vdash \mathbf{u} \Leftarrow \Xi_2}{\Gamma \vdash d(t; \mathbf{u}) \Rightarrow U[\mathbf{v}, \ulcorner t \urcorner, \ulcorner \mathbf{u} \urcorner]}\ \textsc{Dest}$$

  By i.h. we have $\Gamma \vdash \ulcorner t \urcorner : V$. By Proposition 3.2 we have $V \longrightarrow^* T[\mathbf{v}]$, so by Proposition 2.7 and Theorem 2.11 we get $\Gamma \vdash T[\mathbf{v}]$ sort. By validity of the theory, we have $\Xi_1.(\mathsf{x} : T).\Xi_2 \vdash U$ sort and therefore $\Xi_1 \vdash T$ sort, so because $T \in \mathsf{Tm}^{\mathsf{P}}\ |\Xi_1|\ \cdot$ we can apply Proposition 2.10 to derive $\Gamma \vdash \mathbf{v} : \Xi_1$. From $\Gamma \vdash \ulcorner t \urcorner : V$ and $V \equiv T[\mathbf{v}]$ and $\Gamma \vdash T[\mathbf{v}]$ sort we can derive $\Gamma \vdash \ulcorner t \urcorner : T[\mathbf{v}]$ and therefore $\Gamma \vdash \mathbf{v}, \ulcorner t \urcorner : \Xi_1, \mathsf{x} : T$. We can now apply the i.h. to the third premise to derive $\Gamma \vdash \mathbf{v}, \ulcorner t \urcorner, \ulcorner \mathbf{u} \urcorner : \Xi_1.(\mathsf{x} : T).\Xi_2$, from which we get $\Gamma \vdash d(\ulcorner t \urcorner; \ulcorner \mathbf{u} \urcorner) : U[\mathbf{v}, \ulcorner t \urcorner, \ulcorner \mathbf{u} \urcorner]$.

- Case

  $$x : T \in \Gamma\ \dfrac{}{\Gamma \vdash x \Rightarrow T}\ \textsc{Var}$$

  Trivial.

- Case

$$\frac{\mathbb{T} \ni c(\Xi)\ \mathsf{sort} \quad \Gamma \mid \varepsilon : (\cdot) \vdash \mathbf{t} \Leftarrow \Xi}{\Gamma \vdash c(\mathbf{t}) \Leftarrow \mathsf{sort}}\ \textsc{Jdg}$$

By validity of the theory we have $\Xi \vdash$ and therefore we can apply the i.h. to show $\Gamma \vdash \ulcorner \mathbf{t} \urcorner : \Xi$, from which we conclude $\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner)\ \mathsf{sort}$.

- Case

$$\frac{}{\Gamma \mid \mathbf{v} : \Theta \vdash \varepsilon \Leftarrow (\cdot)}\ \textsc{EmptyMSub}$$

Trivial.

- Case

$$\frac{\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi \qquad \Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{t} \urcorner] \vdash t \Leftarrow T[\mathbf{v}, \ulcorner \mathbf{t} \urcorner]}{\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t}, \vec{x}_\Delta.t \Leftarrow (\Xi, \mathsf{x}\{\Delta\} : T)}\ \textsc{ExtMSub}$$

By hypothesis we have $\Theta.\Xi, \mathsf{x}\{\Delta\} : T \vdash$, from which we get $\Theta.\Xi \vdash$ and $\Theta.\Xi; \Delta \vdash T$ sort. By the i.h. applied to the first premise we get $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Theta.\Xi$, so by Proposition 2.6 applied with $\Theta.\Xi; \Delta \vdash T$ sort we get $\Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{t} \urcorner] \vdash T[\mathbf{v}, \ulcorner \mathbf{t} \urcorner]$ sort. Now we can apply the i.h. to the second premise and get $\Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{t} \urcorner] \vdash \ulcorner t \urcorner : T[\mathbf{v}, \ulcorner \mathbf{t} \urcorner]$, from which we can conclude $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner, \vec{x}.t : \Theta.\Xi, \mathsf{x}\{\Delta\} : T$. □

Theorem 3.6 (Completeness)**.** *Suppose that the underlying theory $\mathbb{T}$ is valid.*

- *If t is inferable and $\Gamma \vdash \ulcorner t \urcorner : T$ then $\Gamma \vdash t \Rightarrow U$ with $T \equiv U$*
- *If t is checkable and $\Gamma \vdash \ulcorner t \urcorner : T$ then we have $\Gamma \vdash t \Leftarrow T$*
- *If T is checkable and $\Gamma \vdash \ulcorner T \urcorner$ sort then $\Gamma \vdash T \Leftarrow$ sort*
- *If $\mathbf{t}$ is checkable and $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Theta.\Xi$ then we have $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi$*

Proof. We instead need to show a stronger statement for the induction to go through. Suppose that the underlying theory $\mathbb{T}$ is valid.

- If t is inferable and $\Gamma \vdash \ulcorner t \urcorner : T$ then for all $\Gamma' \equiv \Gamma$ we have $\Gamma' \vdash t \Rightarrow U$ for some $U \equiv T$
- If t is checkable and $\Gamma \vdash \ulcorner t \urcorner : T$ then for all $\Gamma' \equiv \Gamma$ and $T' \equiv T$ we have $\Gamma' \vdash t \Leftarrow T'$
- If T is checkable and $\Gamma \vdash \ulcorner T \urcorner$ sort then for all $\Gamma' \equiv \Gamma$ we have $\Gamma' \vdash T \Leftarrow$ sort
- If $\mathbf{t}$ is checkable and $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Theta.\Xi$ then for all $\Gamma' \equiv \Gamma$ and $\mathbf{v}' \equiv \mathbf{v}$ we have $\Gamma' \mid \mathbf{v}' : \Theta \vdash \mathbf{t} \Leftarrow \Xi$

The proof is by induction on the inferable/checkable term or checkable substitution.

- Case $t = x$: By inversion of typing we have $x : T \in \Gamma$, so for all $\Gamma' \equiv \Gamma$ we have $\Gamma' \vdash x \Rightarrow T'$ for some $T' \equiv T$.

- Case $t = d(u; \mathbf{t})$: By inversion of typing on $\Gamma \vdash \ulcorner t \urcorner : T$ we have

$$T \equiv V[\mathbf{v}, \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner] \frac{\dfrac{d(\Xi_1; \mathsf{x} : U; \Xi_2) : V \in \mathbb{T} \qquad \Gamma \vdash \mathbf{v}, \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner : \Xi_1.(\mathsf{x} : U).\Xi_2}{\Gamma \vdash d(\ulcorner u \urcorner; \ulcorner \mathbf{t} \urcorner) : V[\mathbf{v}, \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner]}}{\Gamma \vdash d(\ulcorner u \urcorner; \ulcorner \mathbf{t} \urcorner) : T}$$

  Let $\Gamma' \equiv \Gamma$. From $\Gamma \vdash \mathbf{v}, \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner : \Xi_1.(\mathsf{x} : U).\Xi_2$ we then get $\Gamma \vdash \ulcorner u \urcorner : U[\mathbf{v}]$, so because u is inferrable, by the i.h. we obtain some $U' \equiv U[\mathbf{v}]$ such that $\Gamma' \vdash u \Rightarrow U'$. By Proposition 3.3 we then get $U \prec U' \rightsquigarrow \mathbf{v}'$ with $\mathbf{v} \equiv \mathbf{v}'$. Then, because $\mathbf{t}$ is checkable, by the i.h. we derive $\Gamma' \mid (\mathbf{v}', \ulcorner u \urcorner) : (\Xi_1, \mathsf{x} : U) \vdash \mathbf{t} \Leftarrow \Xi_2$. We therefore conclude $\Gamma' \vdash d(u; \mathbf{t}) \Rightarrow V[\mathbf{v}', \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner]$, where $V[\mathbf{v}', \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner] \equiv T$.

- Case $t = \underline{u}$: Let $\Gamma' \equiv \Gamma$ and $U \equiv T$. By i.h. we have $\Gamma' \vdash u \Rightarrow T'$ for some $T' \equiv T$. Therefore, by switch we get $\Gamma' \vdash \underline{u} \Leftarrow U$.

- Case $t = c(\mathbf{t})$: We have two cases to consider: either $\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner) : T$ or $\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner)$ sort.

  - Suppose we have $\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner) : T$. By inversion of typing we then have

$$T \equiv U[\mathbf{v}] \frac{\dfrac{c(\Xi_1; \Xi_2) : U \in \mathbb{T} \qquad \Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Xi_1.\Xi_2}{\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner) : U[\mathbf{v}]}}{\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner) : T}$$

    Let $\Gamma' \equiv \Gamma$ and $T' \equiv T$. By Proposition 3.3, we get $U \prec T' \rightsquigarrow \mathbf{v}'$ for some $\mathbf{v}' \equiv \mathbf{v}$. Then, because $\mathbf{t}$ is checkable, by the i.h. we derive $\Gamma' \mid \mathbf{v}' : \Xi_1 \vdash \mathbf{t} \Leftarrow \Xi_2$. Therefore, we conclude $\Gamma' \vdash c(\mathbf{t}) \Leftarrow T'$.

  - Suppose we have $\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner)$ sort. By inversion of typing we then have

$$\frac{c(\Xi) \text{ sort} \in \mathbb{T} \qquad \Gamma \vdash \ulcorner \mathbf{t} \urcorner : \Xi}{\Gamma \vdash c(\ulcorner \mathbf{t} \urcorner) \text{ sort}}$$

    Let $\Gamma' \equiv \Gamma$. By the i.h. we get $\Gamma' \mid \varepsilon : (\cdot) \vdash \mathbf{t} \Leftarrow \Xi$, and thus $\Gamma' \vdash c(\mathbf{t}) \Leftarrow$ sort.

- Case $\mathbf{t} = \varepsilon$: Trivial.

- Case $\mathbf{t} = \mathbf{u}, \vec{x}.t$: By inversion of typing on $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Theta.\Xi$ we get the following, where $\Xi = \Xi', \mathsf{x}\{\Delta\} : T$.

$$\frac{\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{u} \urcorner : \Theta.\Xi' \qquad \Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{u} \urcorner] \vdash \ulcorner t \urcorner : T[\mathbf{v}, \ulcorner \mathbf{u} \urcorner]}{\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{u} \urcorner, \vec{x}.\ulcorner t \urcorner : \Theta.\Xi', \mathsf{x}\{\Delta\} : T}$$

  Let $\Gamma' \equiv \Gamma$ and $\mathbf{v}' \equiv \mathbf{v}$. Because $\mathbf{u}$ is checkable, by i.h. we have $\Gamma' \mid \mathbf{v}' : \Theta \vdash \mathbf{u} \Leftarrow \Xi'$. Because t is checkable, by the i.h. we get $\Gamma'.\Delta[\mathbf{v}', \ulcorner \mathbf{u} \urcorner] \vdash t \Leftarrow T[\mathbf{v}', \ulcorner \mathbf{u} \urcorner]$. Therefore, we conclude $\Gamma' \mid \mathbf{v}' : \Theta \vdash \mathbf{u}, \vec{x}.t \Leftarrow (\Xi', \mathsf{x}\{\Delta\} : T)$.

$\square$

3.4 Consequences of the equivalence

Decidability of typing

Lemma 3.7 (Functionality of matching)**.**

- *For each t and u, there is at most one $\mathbf{v}$ such that $t \prec u \rightsquigarrow \mathbf{v}$*
- *For each $\mathbf{t}$ and $\mathbf{u}$, there is at most one $\mathbf{v}$ such that $\mathbf{t} \prec \mathbf{u} \rightsquigarrow \mathbf{v}$*

Proof. By induction on the pattern t or $\mathbf{t}$. □

Lemma 3.8 (Functionality of inference)**.** *For each Γ and t, there is at most one T such that $\Gamma \vdash t \Rightarrow T$.*

Proof. Given $\Gamma \vdash t \Rightarrow T_1$ and $\Gamma \vdash t \Rightarrow T_2$ we show $T_1 = T_2$ by induction on the first derivation, using Lemma 3.7. □

Lemma 3.9 (Decidability of conversion)**.** *If e and e' are weak normalizing, then $e \equiv e'$ is decidable.*

Proof. We have $e \equiv e'$ iff they reduce to the same normal form. Because our rewrite systems are orthogonal and *fully-extended*, it follows by [van Raamsdonk(1997), Theorem 10] that the maximal outermost strategy is normalizing, and therefore can be used to decide if e and e' have the same normal forms. □

Remark 3.10. In the proof of Lemma 3.9, it would also have been possible to check that $e \equiv e'$ by enumerating all the rewrite sequences issuing from e and e', which would eventually lead to their normal forms. It is however clear that the underlying algorithm of our proof is much more efficient, and would be the preferred choice for an implementation.

We say that a theory $\mathbb{T}$ is weak normalizing if for all expressions e with $\Gamma \vdash e$ sort or $\Gamma \vdash e : T$ or $\Gamma \vdash e : \Xi$ we have that e is weak normalizing.

Theorem 3.11 (Decidability of typing)**.** *Suppose that the underlying theory $\mathbb{T}$ is valid and weak normalizing.*

- *If t is inferable and $\Gamma \vdash$ then the statement $\exists T.\ (\Gamma \vdash \ulcorner t \urcorner : T)$ is decidable.*
- *If t is checkable and $\Gamma \vdash T$ sort then the statement $\Gamma \vdash \ulcorner t \urcorner : T$ is decidable.*
- *If T is checkable and $\Gamma \vdash$ then the statement $\Gamma \vdash \ulcorner T \urcorner$ sort is decidable.*
- *If $\mathbf{t}$ is checkable and $\Theta.\Xi \vdash$ and $\Gamma \vdash \mathbf{v} : \Theta$ then the statement $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{t} \urcorner : \Theta.\Xi$ is decidable.*

Proof. By Theorems 3.5 and 3.6 it suffices the show the following.

- If t is inferable and $\Gamma \vdash$ then the statement $\exists T.\ (\Gamma \vdash t \Rightarrow T)$ is decidable.
- If t is checkable and $\Gamma \vdash T$ sort then the statement $\Gamma \vdash t \Leftarrow T$ is decidable.
- If T is checkable and $\Gamma \vdash$ then the statement $\Gamma \vdash T \Leftarrow$ sort is decidable.
- If $\mathbf{t}$ is checkable and $\Theta.\Xi \vdash$ and $\Gamma \vdash \mathbf{v} : \Theta$ then the statement $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi$ is decidable.

We now proceed with the proof, which is by induction on the inferable/checkable term or checkable substitution.

- Case $t = c(\mathbf{u})$: We have two possibilities, either $c(\Xi_1;\Xi_2) : U \in \mathbb{T}$ or $c(\Xi)$ sort $\in \mathbb{T}$.
  - Suppose $c(\Xi_1;\Xi_2) : U \in \mathbb{T}$, in which case we are to decide $\Gamma \vdash c(\mathbf{u}) \Leftarrow T$. By weak normalization of the theory and $\Gamma \vdash T$ sort it follows that T has a normal form, and therefore by Proposition 3.4 we get that $\exists \mathbf{v}.\ U \prec T \rightsquigarrow \mathbf{v}$ is decidable. We now do a case analysis on this last statement.
    - If $U \prec T \rightsquigarrow \mathbf{v}$ does not hold for any $\mathbf{v}$, it follows that $\Gamma \vdash c(\mathbf{u}) \Leftarrow T$ is not derivable.
    - If $U \prec T \rightsquigarrow \mathbf{v}$ holds, we apply Proposition 3.2 to derive $T \longrightarrow^* U[\mathbf{v}]$. Then, by Theorem 2.11 applied to $\Gamma \vdash T$ sort we get $\Gamma \vdash U[\mathbf{v}]$ sort, and by validity of the theory we have $\Xi_1 \vdash U$ sort, so by Proposition 2.10 we derive $\Gamma \vdash \mathbf{v} : \Xi_1$. By validity of the theory once again we have $\Xi_1.\Xi_2 \vdash$, so by i.h. we get that $\Gamma \mid \mathbf{v} : \Xi_1 \vdash \mathbf{u} \Leftarrow \Xi_2$ is decidable. If $\Gamma \mid \mathbf{v} : \Xi_1 \vdash \mathbf{u} \Leftarrow \Xi_2$ holds then it follows that $\Gamma \vdash c(\mathbf{u}) \Leftarrow T$ holds. Otherwise $\Gamma \vdash c(\mathbf{u}) \Leftarrow T$ does not hold, as by Lemma 3.7 there can be no other $\mathbf{v}'$ with $U \prec T \rightsquigarrow \mathbf{v}'$ for which $\Gamma \mid \mathbf{v}' : \Xi_1 \vdash \mathbf{u} \Leftarrow \Xi_2$ holds.
  - Suppose $c(\Xi)$ sort $\in \mathbb{T}$, in which case we are to decide $\Gamma \vdash c(\mathbf{u}) \Leftarrow$ sort. By validity of the theory we have $\Xi \vdash$, so by i.h. we get that $\Gamma \mid \varepsilon : (\cdot) \vdash \mathbf{u} \Leftarrow \Xi$ is decidable. Because this holds iff $\Gamma \vdash c(\mathbf{u}) \Leftarrow$ sort is derivable, it follows that the latter is decidable.
- Case $t = x$: Trivial.
- Case $t = d(u;\mathbf{t})$: Let $d(\Xi_1; \mathsf{x} : U; \Xi_2) : V \in \mathbb{T}$. By i.h. it follows that $\exists U'.\ \Gamma \vdash u \Rightarrow U'$ is decidable. We now do a case analysis on this last statement.
  - If $\exists U'.\ \Gamma \vdash u \Rightarrow U'$ is not derivable, it follows that $\exists T.\ \Gamma \vdash d(u;\mathbf{t}) \Rightarrow T$ is not derivable.
  - If $\Gamma \vdash u \Rightarrow U'$ is derivable, then by Theorem 3.5 it follows that $\Gamma \vdash \ulcorner u \urcorner : U'$ holds. By Proposition 2.7 and weak normalization of the theory it follows that U' is weak normalizing, so by Proposition 3.4 it follows that $\exists \mathbf{v}.\ U \prec U' \rightsquigarrow \mathbf{v}$ is decidable. We now do a case analysis on this last statement.
    - If $\exists \mathbf{v}.\ U \prec U' \rightsquigarrow \mathbf{v}$ does not hold, it follows that $\exists T.\ \Gamma \vdash d(u;\mathbf{t}) \Rightarrow T$ does not hold neither. Indeed, by Lemma 3.8 there can be no other U'' with $\Gamma \vdash u \Rightarrow U''$ for which $\exists \mathbf{v}.\ U \prec U'' \rightsquigarrow \mathbf{v}$ holds.
    - If $\exists \mathbf{v}.\ U \prec U' \rightsquigarrow \mathbf{v}$ is derivable, then by Proposition 3.2 we get $U' \longrightarrow^* U[\mathbf{v}]$, so by Theorem 2.11 applied to $\Gamma \vdash U'$ sort we get $\Gamma \vdash U[\mathbf{v}]$ sort. By validity of the theory, we have $\Xi_1.(\mathsf{x} : U).\Xi_2 \vdash V$ sort and thus $\Xi_1 \vdash U$ sort, so because $U \in \mathsf{Tm}^{\mathsf{P}}\ |\Xi_1|\ \cdot$ we can apply Proposition 2.10 to derive $\Gamma \vdash \mathbf{v} : \Xi_1$. From $\Gamma \vdash \ulcorner u \urcorner : U'$ and $U' \equiv U[\mathbf{v}]$ and $\Gamma \vdash U[\mathbf{v}]$ sort we can derive $\Gamma \vdash \ulcorner u \urcorner : U[\mathbf{v}]$, and therefore we have $\Gamma \vdash \mathbf{v}, \ulcorner u \urcorner : \Xi_1, \mathsf{x} : U$. Now we can apply the i.h. to show that $\Gamma \mid (\mathbf{v}, \ulcorner u \urcorner) : \Xi_1, \mathsf{x} : U \vdash \mathbf{u} \Leftarrow \Xi_2$ is decidable. If this last statement holds, then we conclude that $\Gamma \vdash d(u;\mathbf{t}) \Rightarrow U[\mathbf{v}, \ulcorner u \urcorner, \ulcorner \mathbf{t} \urcorner]$ holds. Otherwise $\exists T.\ \Gamma \vdash d(u;\mathbf{t}) \Rightarrow T$ does not hold, as by Lemmas 3.8 and 3.7 it follows that there can be no other U'' and $\mathbf{v}'$ with $\Gamma \vdash u \Rightarrow U''$ and $U \prec U'' \rightsquigarrow \mathbf{v}'$ for which $\Gamma \mid (\mathbf{v}', \ulcorner u \urcorner) : \Xi_1, \mathsf{x} : U \vdash \mathbf{u} \Leftarrow \Xi_2$ holds.
- Case $t = \underline{u}$: By i.h. we have that $\exists U.\ \Gamma \vdash u \Rightarrow U$ is decidable. If this statement does not hold, it follows that $\Gamma \vdash \underline{u} \Leftarrow T$ does not hold, otherwise by Theorem 3.5 we get $\Gamma \vdash \ulcorner u \urcorner : U$,

which by Proposition 2.7 implies $\Gamma \vdash U$ sort. We also have $\Gamma \vdash T$ sort so it follows that both U and T are weak normalizing, therefore by Lemma 3.9 we can decide whether $T \equiv U$ holds. If this is the case, then it follows that $\Gamma \vdash \underline{u} \Leftarrow T$ holds. Otherwise $\Gamma \vdash \underline{u} \Leftarrow T$ cannot hold, as by Lemma 3.8 there can be no other U' with $\Gamma \vdash u \Rightarrow U'$ for which $T \equiv U'$ holds.

- Case $\mathbf{t} = \varepsilon$: Trivial.
- Case $\mathbf{t} = \mathbf{u}, \vec{x}.t$: We must have $\Xi = \Xi', \mathsf{x}\{\Delta\} : T$, otherwise it is clear that $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi$ is not derivable. From $\Theta.\Xi', \mathsf{x}\{\Delta\} : T \vdash$ we then get $\Theta.\Xi' \vdash$ and $\Theta.\Xi'; \Delta \vdash T$ sort. By i.h. we then get that $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{u} \Leftarrow \Xi'$ is decidable. If this does not hold, then it is clear that $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{t} \Leftarrow \Xi$ is not derivable, so in the following we assume that we have $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{u} \Leftarrow \Xi'$. Then, by Theorem 3.5 we get $\Gamma \vdash \mathbf{v}, \ulcorner \mathbf{u} \urcorner : \Theta.\Xi'$, so by applying Proposition 2.6 with $\Theta.\Xi'; \Delta \vdash T$ sort we get $\Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{u} \urcorner] \vdash T[\mathbf{v}, \ulcorner \mathbf{u} \urcorner]$ sort. By i.h. we therefore get that $\Gamma.\Delta[\mathbf{v}, \ulcorner \mathbf{u} \urcorner] \vdash t \Leftarrow T[\mathbf{v}, \ulcorner \mathbf{u} \urcorner]$ is decidable, hence by testing this statement we can decide $\Gamma \mid \mathbf{v} : \Theta \vdash \mathbf{u}, \vec{x}.t \Leftarrow \Xi', \mathsf{x}\{\Delta\} : T$. □

Unicity of sorts

Theorem 3.12 (Unicity of sorts)**.** *Suppose that the underlying theory $\mathbb{T}$ is valid. If t is inferable and $\Gamma \vdash \ulcorner t \urcorner : T$ and $\Gamma \vdash \ulcorner t \urcorner : U$ then $T \equiv U$.*

Proof. By Theorem 3.6 we get $\Gamma \vdash t \Rightarrow T'$ with $T \equiv T'$ from $\Gamma \vdash \ulcorner t \urcorner : T$ and $\Gamma \vdash t \Rightarrow U'$ with $U \equiv U'$ from $\Gamma \vdash \ulcorner t \urcorner : U$. By Lemma 3.8 we then get $T' = U'$ and therefore $T \equiv U$. □

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