Generic Bidirectional Typing for Dependent Type Theories

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Type annotations in dependent type theory

Dependent type theory suffers from verbosity of type annotations

Application: $t @_{A,x.B} u$

Dependent pair: $\langle t, u \rangle_{A,x.B}$

Cons: $t ::_A l$

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Most presentation restore usability by eliding type annotations from syntax

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Syntax so common that many don't realize that an omission is being made

Typechecking without annotations

Omission has a cost Knowing annotations is needed for typing

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/x]}{\Gamma \vdash \langle t, u \rangle : \Sigma x : A.B}$$

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Bidirectional typing Decompose $t : A$ in modes check $t \Leftarrow A$ and infer $t \Rightarrow A$

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Allow specify flow of type information in typing rules, explain how to use them

$$\frac{C \longrightarrow^* \Sigma x : A.B \qquad \Gamma \vdash t \Leftarrow A \qquad \Gamma \vdash u \Leftarrow B[t/x]}{\Gamma \vdash \langle t, u\rangle \Leftarrow C}$$

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Complements unannotated syntax very well, explains how to recover annotations

Contribution

Bidirectional type systems have been studied and proposed for many theories

However, general guidelines have remained informal, no unified framework

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4. We derive decidability of typing for weak normalizing theories

The theories

The intrinsically-scoped syntax

$$
\begin{array}{rrll}
\boxed{\mathrm{Tm}\ \theta\ \gamma} \ni & t, u, T, U ::= & \mid x & \text{if } x \in \gamma \\
& & \mid \mathsf{x}\{\vec{t} \in \mathrm{Sub}\ \theta\ \gamma\ \delta\} & \text{if } \mathsf{x}\{\delta\} \in \theta \\
& & \mid c(\mathbf{t} \in \mathrm{MSub}\ \theta\ \gamma\ \xi) & \text{if } c(\xi) \in \Sigma \\
& & \mid d(t \in \mathrm{Tm}\ \theta\ \gamma; \mathbf{t} \in \mathrm{MSub}\ \theta\ \gamma\ \xi) & \text{if } d(\xi) \in \Sigma \\
\boxed{\mathrm{Sub}\ \theta\ \gamma\ \delta} \ni & \vec{t}, \vec{u}, \vec{s}, \vec{v} ::= & \mid \epsilon & \text{if } \delta = \cdot \\
& & \mid \vec{t}' \in \mathrm{Sub}\ \theta\ \gamma\ \delta', t \in \mathrm{Tm}\ \theta\ \gamma & \text{if } \delta = \delta', x \\
\boxed{\mathrm{MSub}\ \theta\ \gamma\ \xi} \ni & \mathbf{t}, \mathbf{u}, \mathbf{s}, \mathbf{v} ::= & \mid \epsilon & \text{if } \xi = \cdot \\
& & \mid \mathbf{t}' \in \mathrm{MSub}\ \theta\ \gamma\ \xi', \vec{x}_\delta.t \in \mathrm{Tm}\ \theta\ \gamma.\delta & \text{if } \xi = \xi', \mathsf{x}\{\delta\}
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 & & & & \mid \vec{t}' \ \in \mathsf{Sub}\ \theta\ \gamma\ \delta', t \ \in \mathsf{Tr}\ldots & \ldots \\
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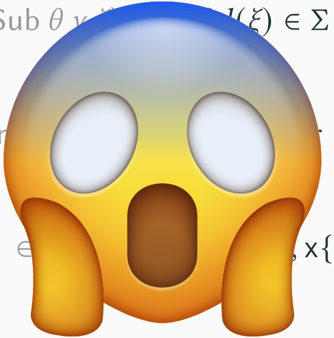
 & & & & \mid \mathbf{t}' \ \in \mathsf{MSub}\ \theta\ \gamma\ \xi', \vec{x}_\delta.t \ \in \ldots & \ldots, \mathsf{x}\{\delta\}
\end{array}
$$



The syntax

Terms and substitutions

Write x for variables, x for metavariables, c for constructors and d for destructors

$$\boxed{\mathrm{Tm}} \ni t, u, T, U ::= x \mid \mathsf{x}\{\vec{t}\} \mid c(\mathbf{t}) \mid d(t; \mathbf{t}) \quad (t \text{ is called the } \textit{principal argument})$$

$$\boxed{\mathrm{Sub}} \ni \vec{t}, \vec{u} ::= \epsilon \mid \vec{t}, t$$

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Example

$$\begin{aligned} \Sigma_{\lambda\Pi} = \quad & \Pi(\mathsf{A}, \mathsf{B}\{x\}),\ \lambda(\mathsf{t}\{x\}),\ \mathrm{Ty},\ \mathrm{Tm}(\mathsf{A}), && \text{(constructors)} \\ & @(\mathsf{u}) && \text{(destructors)} \end{aligned}$$

$$t, u, A, B ::= x \mid \mathsf{x}\{\vec{t}\} \mid \mathrm{Ty} \mid \mathrm{Tm}(A) \mid @(t; u) \mid \lambda(x.t) \mid \Pi(A, x.B)$$

The syntax

Contexts

$$\boxed{\mathrm{Ctx}} \ni \Gamma, \Delta ::= \cdot \mid \Gamma, x : T$$

$$\boxed{\mathrm{MCtx}} \ni \Theta, \Xi ::= \cdot \mid \Theta, \mathrm{x}\{\Gamma\} : T$$

Example: $\Gamma = A : \mathrm{Ty}, x : \mathrm{Tm}(A), y : \mathrm{Tm}(\Pi(A, z.A))$

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(Linear) Patterns

$$\boxed{\mathrm{Tm}^{\mathsf{P}}} \ni t^{\mathsf{P}}, u^{\mathsf{P}} ::= \mathsf{x}\{\vec{x}\} \mid c(\mathbf{t}^{\mathsf{P}})$$

$$\boxed{\mathrm{MSub}^{\mathsf{P}}} \ni \mathbf{t}^{\mathsf{P}}, \mathbf{u}^{\mathsf{P}} ::= \epsilon \mid \mathbf{t}^{\mathsf{P}}, \vec{x}.t^{\mathsf{P}} \quad (\mathrm{metas}(\mathbf{t}^{\mathsf{P}}) \cap \mathrm{metas}(t^{\mathsf{P}}) = \emptyset)$$

The theories

A *theory* $\mathbb{T}$ is made of *schematic typing rules* and *rewrite rules*.

3 schematic typing rules: *sort rules*, *constructor rules* and *destructor rules*

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Sort rules Used to define judgment forms of the theory.

Example: In MLTT, 2 judgment forms: $\square$ type and $\square : A$ for a type A.

$$\frac{}{\vdash \mathrm{Ty}\ \mathsf{sort}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{Tm}(\mathsf{A})\ \mathsf{sort}}$$

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Formally, of the form $c(\Theta)$ sort, where Θ represent premises.

Example in formal notation: $\mathrm{Ty}(\cdot)$ sort and $\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})$ sort

The theories

Constructor rules Two groups of premises: Θ_1 erased and Θ_2 kept in the syntax.

To recover Θ_1, rule will be used in mode check, so the type should be pattern containing metavariables of Θ_1.

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \Pi(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}} \qquad \frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{t} : \mathrm{Tm}(\mathsf{B}\{x\})\end{array}}{\vdash \lambda(\mathsf{t}) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

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Formally, of the form $c(\Theta_1; \Theta_2) : U^{\mathsf{P}}$

Example in formal notation: $\Pi(\cdot;\ \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty}) : \mathrm{Ty}$ and $\lambda(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \mathtt{t}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Tm}(\mathsf{B}\{x\})) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$.

The theories

Destructor rules Two groups of premises: Θ_1 erased and Θ_2 kept in the syntax.

But also a *principal argument* $\mathsf{x} : T^{\mathsf{P}}$, a pattern allowing to recover Θ_1.

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Formally, of the form $d(\Theta_1;\ \mathsf{x} : T^{\mathsf{P}};\ \Theta_2) : U$

Example in formal notation:
$@(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}));\ \ \mathsf{u} : \mathrm{Tm}(\mathsf{A})) : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\})$.

The theories

Rewrite rules Define the definitional equality (aka conversion) $\equiv$ of the theory.

$$@(\lambda(x.\mathtt{t}\{x\});\mathtt{u}) \longmapsto \mathtt{t}\{\mathtt{u}\}$$

In general, of the form $d(t^{\mathsf{P}};\mathbf{u}^{\mathsf{P}}) \longmapsto r$ with $(\mathrm{metas}(t^{\mathsf{P}}) \cap \mathrm{metas}(\mathbf{u}^{\mathsf{P}}) = \emptyset)$.

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Condition: no two left-hand sides unify.

Therefore, rewrite systems are orthogonal, hence confluent by construction!

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Full example Theory $\mathbb{T}_{\lambda\Pi}$.

$$\mathrm{Ty}(\cdot)\ \mathrm{sort} \qquad \mathrm{Tm}(\mathsf{A}:\mathrm{Ty})\ \mathrm{sort} \qquad \Pi(\cdot;\ \mathsf{A}:\mathrm{Ty},\ \mathsf{B}\{x:\mathrm{Tm}(\mathsf{A})\}:\mathrm{Ty}):\mathrm{Ty}$$

$$\lambda(\mathsf{A}:\mathrm{Ty},\ \mathsf{B}\{x:\mathrm{Tm}(\mathsf{A})\}:\mathrm{Ty};\ \ \mathtt{t}\{x:\mathrm{Tm}(\mathsf{A})\}:\mathrm{Tm}(\mathsf{B}\{x\})):\mathrm{Tm}(\Pi(\mathsf{A},x.\mathsf{B}\{x\}))$$

$$@(\mathsf{A}:\mathrm{Ty},\ \mathsf{B}\{x:\mathrm{Tm}(\mathsf{A})\}:\mathrm{Ty};\ \ \mathtt{t}:\mathrm{Tm}(\Pi(\mathsf{A},x.\mathsf{B}\{x\}));\ \ \mathtt{u}:\mathrm{Tm}(\mathsf{A})):\mathrm{Tm}(\mathsf{B}\{\mathtt{u}\})$$

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Declarative typing

Declarative typing rules

Each theory $\mathbb{T}$ defines a declarative type system.

Judgments $\Theta; \Gamma \vdash T$ sort and $\Theta; \Gamma \vdash t : T$ and $\Theta; \Gamma \vdash \vec{t} : \Delta$ and $\Theta; \Gamma \vdash \mathbf{t} : \Xi$.

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$$c(\Xi) \text{ sort} \in \mathbb{T} \; \frac{\Theta; \Gamma \vdash \mathbf{t} : \Xi}{\Theta; \Gamma \vdash c(\mathbf{t}) \text{ sort}} \; \textsc{Sort}$$

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$$\textsc{Sort}\quad \frac{c(\Xi)\ \mathrm{sort} \in \mathbb{T} \qquad \Theta; \Gamma \vdash \mathbf{t} : \Xi}{\Theta; \Gamma \vdash c(\mathbf{t})\ \mathrm{sort}} \qquad\qquad \textsc{Cons}\quad \frac{c(\Xi_1; \Xi_2) : T \in \mathbb{T} \qquad \Theta; \Gamma \vdash \mathbf{t}, \mathbf{u} : \Xi_1.\Xi_2}{\Theta; \Gamma \vdash c(\mathbf{u}) : T[\mathbf{t}]}$$

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$$\textsc{Dest}\;\frac{d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T} \qquad \Theta; \Gamma \vdash \mathbf{t}, t, \mathbf{u} : \Xi_1.(\mathsf{x} : T).\Xi_2}{\Theta; \Gamma \vdash d(t; \mathbf{u}) : U[\mathbf{t}, t, \mathbf{u}]}$$

Example

$$\begin{array}{llllll} d(& \Xi_1 & ; & \mathsf{x} : T^{\mathsf{P}} & ; & \Xi_2) & : U \\ @(& \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty} & ; & \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\})) & ; & \mathsf{u} : \mathrm{Tm}(\mathsf{A})) & : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\}) \end{array}$$

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$$\frac{\begin{array}{c} \Theta; \Gamma \vdash (A,\ x.B) : (\mathsf{A} : \mathrm{Ty},\ \mathsf{B}(x : \mathrm{Tm}(\mathsf{A})) : \mathrm{Ty}) \\ \Theta; \Gamma \vdash t : \mathrm{Tm}(\Pi(A, x.B)) \qquad \Theta; \Gamma \vdash u : \mathrm{Tm}(A) \end{array}}{\Theta; \Gamma \vdash @(t; u) : \mathrm{Tm}(B[u/x])}$$

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Properties of declarative system

In the following, we assume theory the $\mathbb{T}$ to be *valid* (definition not given here).

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Moreover, our valid theories all satisfy **subject reduction**!
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$$\Gamma \vdash t : T \text{ and } t \longrightarrow t' \text{ implies } \Gamma \vdash t' : T$$

Proof uses the following key lemma:

Key property of patterns If $\Theta \vdash t^{\mathsf{P}} : T$ and $\Delta \vdash t^{\mathsf{P}}[\mathbf{v}] : T[\mathbf{v}]$ then $\Delta \vdash \mathbf{v} : \Theta$

Bidirectional typing

Matching modulo rewriting

In bidirectional typing, we need matching modulo rewriting to recover missing arguments.

$$\frac{\Gamma \vdash t \Rightarrow U \qquad \ldots}{\Gamma \vdash @(t; u) \Rightarrow}$$

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$$\frac{\Gamma \vdash t \Rightarrow U \qquad \dots}{\Gamma \vdash @(t; u) \Rightarrow}$$

We know

$$U \equiv \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))[A/\mathsf{A},\ x.B/\mathsf{B}]$$

but how to recover A and B from U?

Matching modulo rewriting

Given t^{P} and u, judgment $t^{\mathsf{P}} \prec u \rightsquigarrow \mathbf{v}$ tries to compute $\mathbf{v}$ with $t^{\mathsf{P}}[\mathbf{v}] \equiv u$.

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Proof: induction on matching rules

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Proof: Uses the fact that $\longrightarrow_{\mathrm{m/o}}$ is head-normalizing for orthogonal systems.

Inferable and checkable terms

Not all unannotated terms can be algorithmically typed

$$\dfrac{\dfrac{?}{\Gamma \vdash \lambda(x.t) \Rightarrow ?} \qquad \dots}{\Gamma \vdash @(\lambda(x.t); u) \Rightarrow ?}$$

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Avoided by defining bidirectional typing only for *inferrable* and *checkable* terms.

$$\boxed{\mathrm{Tm}^{\mathrm{i}}} \ni t^{\mathrm{i}}, u^{\mathrm{i}} ::= x \mid d(t^{\mathrm{i}}; \mathbf{t}^{\mathrm{c}})$$

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Principal argument of a destructor can only be variable or another destructor.

For most theories: Tm^{c} = normal forms, and Tm^{i} = neutrals

Bidirectional typing rules

Judgments $\Gamma \vdash T^{\mathsf{c}} \Leftarrow \mathrm{sort}$ and $\Gamma \vdash t^{\mathsf{c}} \Leftarrow T$ and $\Gamma \vdash t^{\mathsf{i}} \Rightarrow T$ and $\Gamma \mid \mathbf{v} : \Xi \vdash \mathbf{t}^{\mathsf{c}} \Leftarrow \Theta$.

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$$d(\Xi_1; \mathsf{t} : T; \Xi_2) : U \in \mathbb{T}\ \frac{\Gamma \vdash t^{\mathsf{i}} \Rightarrow V \qquad T < V \rightsquigarrow \mathbf{v} \qquad \Gamma \mid (\mathbf{v}, t) : (\Xi_1, \mathsf{t} : T) \vdash \mathbf{u}^{\mathsf{c}} \Leftarrow \Xi_2}{\Gamma \vdash d(t^{\mathsf{i}}; \mathbf{u}^{\mathsf{c}}) \Rightarrow U[\mathbf{v}, t, \mathbf{u}]}\ \textsc{Dest}$$

Example

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$$\textsc{Dest}\ \frac{\Gamma \vdash t^{\mathsf{i}} \Rightarrow V \qquad T \prec V \rightsquigarrow \mathbf{v} \qquad \Gamma \mid (\mathbf{v}, t) : (\Xi_1, \mathsf{t} : T) \vdash \mathbf{u}^{\mathsf{c}} \Leftarrow \Xi_2}{\Gamma \vdash d(t^{\mathsf{i}}; \mathbf{u}^{\mathsf{c}}) \Rightarrow U[\mathbf{v}, t, \mathbf{u}]}$$

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Equivalence with declarative typing

Suppose underlying theory $\mathbb{T}$ is valid.

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Decidability If $\mathbb{T}$ weak normalizing, then inference is decidable for inferable terms, and checking is decidable for checkable terms.

More examples

Dependent sums

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty}}{\vdash \Sigma(\mathsf{A}, \mathsf{B}) : \mathbf{Ty}}
\qquad
\frac{\begin{array}{c}\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \\ \vdash \mathsf{t} : \mathbf{Tm}(\mathsf{A}) \qquad \vdash \mathsf{u} : \mathbf{Tm}(\mathsf{B}\{\mathsf{t}\})\end{array}}{\vdash \mathbf{pair}(\mathsf{t}, \mathsf{u}) : \mathbf{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \\ \vdash \mathsf{t} : \mathbf{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))\end{array}}{\vdash \mathrm{proj}_1(\mathsf{t}; \cdot) : \mathbf{Tm}(\mathsf{A})}
\qquad
\frac{\begin{array}{c}\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \\ \vdash \mathsf{t} : \mathbf{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))\end{array}}{\vdash \mathrm{proj}_2(\mathsf{t}; \cdot) : \mathbf{Tm}(\mathsf{B}\{\mathrm{proj}_1(\mathsf{t})\})}$$

$$\mathrm{proj}_1(\mathbf{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{t} \qquad \mathrm{proj}_2(\mathbf{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{u}$$

Lists

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathtt{A} : \mathrm{Ty}}{\vdash \mathrm{List}(\mathtt{A}) : \mathrm{Ty}} \qquad \frac{\vdash \mathtt{A} : \mathrm{Ty}}{\vdash \mathrm{nil} : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))} \qquad \frac{\vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{x} : \mathrm{Tm}(\mathtt{A}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))}{\vdash \mathrm{cons}(\mathtt{x}, \mathtt{l}) : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))}$$

$$\frac{\vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{List}(\mathtt{A})) \qquad x : \mathrm{Tm}(\mathrm{List}(\mathtt{A})) \vdash \mathtt{P} : \mathrm{Ty} \qquad \vdash \mathtt{pnil} : \mathrm{Tm}(\mathtt{P}\{\mathrm{nil}\}) \qquad x : \mathrm{Tm}(\mathtt{A}), y : \mathrm{Tm}(\mathrm{List}(\mathtt{A})), z : \mathrm{Tm}(\mathtt{P}\{y\}) \vdash \mathtt{pcons} : \mathrm{Tm}(\mathtt{P}\{\mathrm{cons}(x, y)\})}{\vdash \mathrm{ListRec}(\mathtt{l}; \mathtt{P}, \mathtt{pnil}, \mathtt{pcons}) : \mathrm{Tm}(\mathtt{P}\{\mathtt{l}\})}$$

$$\mathrm{ListRec}(\mathrm{nil}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) \longmapsto \mathtt{pnil}$$

$$\mathrm{ListRec}(\mathrm{cons}(\mathtt{x}, \mathtt{l}); x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) \longmapsto$$
$$\mathtt{pcons}\{\mathtt{x}, \mathtt{l}, \mathrm{ListRec}(\mathtt{l}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\})\}$$

W types

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \mathrm{W}(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}} \qquad \frac{\begin{array}{c} \vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{f} : \mathrm{Tm}(\Pi(\mathsf{B}\{\mathsf{a}\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))) \end{array}}{\vdash \mathrm{sup}(\mathsf{a}, \mathsf{f}) : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c} \vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \vdash \mathsf{t} : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \qquad x : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \vdash \mathsf{P} : \mathrm{Ty} \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))), z : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathsf{P}\{@(y, x')\})) \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathrm{sup}(x, y)\}) \end{array}}{\vdash \mathrm{WRec}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{t}\})}$$

$$\mathrm{WRec}(\mathrm{sup}(\mathsf{a}, \mathsf{f}); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}) \longmapsto \mathsf{p}\{\mathsf{a}, \mathsf{f}, \lambda(x.\mathrm{WRec}(@(\mathsf{f}, x); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}))\}$$

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

$$\mathrm{J}(\mathrm{refl}, xy.\mathsf{P}\{x, y\}, \mathsf{p}) \longmapsto \mathsf{p}$$

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

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Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

$$\mathrm{J}(\mathrm{refl}, xy.\mathsf{P}\{x, y\}, \mathsf{p}) \longmapsto \mathsf{p}$$

We add conversion premises to constructor rules (not shown in this talk)

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\begin{array}{cc}\vdash \mathsf{A} : \mathrm{Ty} & \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \\ \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) & \mathsf{a} = \mathsf{b}\end{array}}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}))}$$

$$\frac{\begin{array}{cccc}\vdash \mathsf{A} : \mathrm{Ty} & \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) & \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) & \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ \multicolumn{3}{c}{x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty}} & \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

$$\mathrm{J}(\mathrm{refl}, xy.\mathsf{P}\{x, y\}, \mathsf{p}) \longmapsto \mathsf{p}$$

We add conversion premises to constructor rules (not shown in this talk)

Vectors

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{n} : \mathrm{Tm}(\mathrm{Nat})}{\vdash \mathrm{Vec}(\mathsf{A}, \mathsf{n}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{nil} : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, 0))} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathsf{x} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{l} : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}))}{\vdash \mathrm{cons}(\mathsf{n}, \mathsf{x}, \mathsf{l}) : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, \mathrm{S}(\mathsf{n})))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathsf{l} : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, \mathsf{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{pnil} : \mathrm{Tm}(\mathsf{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathsf{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, x)), w : \mathrm{Tm}(\mathsf{P}\{x, z\}) \vdash \mathsf{pcons} : \mathrm{Tm}(\mathsf{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\})\end{array}}{\vdash \mathrm{VecRec}(\mathsf{l}; \mathsf{P}, \mathsf{pnil}, \mathsf{pcons}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{n}, \mathsf{l}\})}$$

$$\mathrm{VecRec}(\mathrm{nil}; x.\mathsf{P}\{x\}, \mathsf{pnil}, xyzw.\mathsf{pcons}\{x, y, z, w\}) \longmapsto \mathsf{pnil}$$

$$\mathrm{VecRec}(\mathrm{cons}(\mathsf{n}, \mathsf{x}, \mathsf{l}); x.\mathsf{P}\{x\}, \mathsf{pnil}, xyzw.\mathsf{pcons}\{x, y, z, w\}) \longmapsto$$

$$\mathsf{pcons}\{\mathsf{n}, \mathsf{x}, \mathsf{l}, \mathrm{VecRec}(\mathsf{l}; x.\mathsf{P}\{x\}, \mathsf{pnil}, xyzw.\mathsf{pcons}\{x, y, z, w\})\}$$

Vectors

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$$\frac{\begin{array}{c}\vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)) \vdash \mathtt{P} : \mathrm{Ty} \qquad \vdash \mathtt{pnil} : \mathrm{Tm}(\mathtt{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathtt{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)), w : \mathrm{Tm}(\mathtt{P}\{x, z\}) \vdash \mathtt{pcons} : \mathrm{Tm}(\mathtt{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\})\end{array}}{\vdash \mathrm{VecRec}(\mathtt{l}; \mathtt{P}, \mathtt{pnil}, \mathtt{pcons}) : \mathrm{Tm}(\mathtt{P}\{\mathtt{n}, \mathtt{l}\})}$$

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$$\frac{\begin{array}{c} \vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathsf{l} : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, \mathsf{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{pnil} : \mathrm{Tm}(\mathsf{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathsf{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, x)), w : \mathrm{Tm}(\mathsf{P}\{x, z\}) \vdash \mathsf{pcons} : \mathrm{Tm}(\mathsf{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\}) \end{array}}{\vdash \mathrm{VecRec}(\mathsf{l}; \mathsf{P}, \mathsf{pnil}, \mathsf{pcons}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{n}, \mathsf{l}\})}$$

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Universes

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{}{\vdash \mathrm{U}(\cdot) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U})}{\vdash \mathrm{El}(\mathsf{a}; \cdot) : \mathrm{Ty}}$$

Then, two options:

Universes

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Then, two options:

Tarski-style Adds codes for all types

$$\frac{}{\vdash \mathrm{u}(\cdot) : \mathrm{Tm}(\mathrm{U})} \qquad \mathrm{El}(\mathrm{u}; \varepsilon) \longmapsto \mathrm{U}$$

$$\frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U}) \qquad x : \mathrm{Tm}(\mathrm{El}(\mathsf{a})) \vdash \mathsf{b} : \mathrm{Tm}(\mathrm{U})}{\vdash \pi(\mathsf{a}, \mathsf{b}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\}); \varepsilon) \longmapsto \Pi(\mathrm{El}(\mathsf{a}; \varepsilon), x.\mathrm{El}(\mathsf{b}\{x\}; \varepsilon))$$

Universes

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Tarski-style Adds codes for all types

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$$\frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U}) \qquad x : \mathrm{Tm}(\mathrm{El}(\mathsf{a})) \vdash \mathsf{b} : \mathrm{Tm}(\mathrm{U})}{\vdash \pi(\mathsf{a}, \mathsf{b}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\}); \varepsilon) \longmapsto \Pi(\mathrm{El}(\mathsf{a}; \varepsilon), x.\mathrm{El}(\mathsf{b}\{x\}; \varepsilon))$$

(Weak) Coquand-style

Adds a code constructor c

$$\frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{c}(\mathsf{A}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\mathrm{c}(\mathsf{A}); \varepsilon) \longmapsto \mathsf{A}$$

Conclusion

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We have given a generic account of bidirectional type for a class of type theories

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1. Going beyond the constructor/destructor separation (do we want this?)

$$\mathrm{Tm}(\mathrm{U}) \longrightarrow \mathrm{Ty}$$

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We have given a generic account of bidirectional type for a class of type theories

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Future work Extending class of type theories we can handle by

1. Going beyond the constructor/destructor separation (do we want this?)

$$\mathrm{Tm}(\mathrm{U}) \longrightarrow \mathrm{Ty}$$

2. Going beyond rewriting rules, needed for eta/proof-irrelevance

Thank you for your attention!