Generic Bidirectional Typing for Dependent Type Theories

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Type annotations in dependent type theory

Dependent type theory suffers from verbosity of type annotations

Application: $t @_{A,x.B} u$

Dependent pair: $\langle t, u \rangle_{A,x.B}$

Cons: $t ::_A l$

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Syntax so common that many don't realize that an omission is being made

Typechecking without annotations

Omission has a cost Knowing annotations is needed for typing

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/x]}{\Gamma \vdash \langle t, u \rangle : \Sigma x : A.B}$$

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Bidirectional typing Decompose $t : A$ in modes check $t \Leftarrow A$ and infer $t \Rightarrow A$

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Allow specify flow of type information in typing rules, explain how to use them

$$\frac{C \longrightarrow^* \Sigma x : A.B \qquad \Gamma \vdash t \Leftarrow A \qquad \Gamma \vdash u \Leftarrow B[t/x]}{\Gamma \vdash \langle t, u \rangle \Leftarrow C}$$

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Complements unannotated syntax very well, explains how to recover annotations

Contribution

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However, general guidelines have remained informal, no unified framework

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4. We derive decidability of typing for weak normalizing theories

The theories

The intrinsically-scoped syntax

$$
\begin{array}{rcrcll}
\boxed{\mathrm{Tm}\ \theta\ \gamma} & \ni & t, u, T, U & ::= & \mid x & \text{if } x \in \gamma \\
 & & & & \mid \mathsf{x}\{\vec{t}\ \in \mathrm{Sub}\ \theta\ \gamma\ \delta\} & \text{if } \mathsf{x}\{\delta\} \in \theta \\
 & & & & \mid c(\mathbf{t}\ \in \mathrm{MSub}\ \theta\ \gamma\ \xi) & \text{if } c(\xi) \in \Sigma \\
 & & & & \mid d(t\ \in \mathrm{Tm}\ \theta\ \gamma; \mathbf{t}\ \in \mathrm{MSub}\ \theta\ \gamma\ \xi) & \text{if } d(\xi) \in \Sigma \\
\boxed{\mathrm{Sub}\ \theta\ \gamma\ \delta} & \ni & \vec{t}, \vec{u}, \vec{s}, \vec{v} & ::= & \mid \epsilon & \text{if } \delta = \cdot \\
 & & & & \mid \vec{t}'\ \in \mathrm{Sub}\ \theta\ \gamma\ \delta', t\ \in \mathrm{Tm}\ \theta\ \gamma & \text{if } \delta = \delta', x \\
\boxed{\mathrm{MSub}\ \theta\ \gamma\ \xi} & \ni & \mathbf{t}, \mathbf{u}, \mathbf{s}, \mathbf{v} & ::= & \mid \epsilon & \text{if } \xi = \cdot \\
 & & & & \mid \mathbf{t}'\ \in \mathrm{MSub}\ \theta\ \gamma\ \xi', \vec{x}_\delta.t\ \in \mathrm{Tm}\ \theta\ \gamma.\delta & \text{if } \xi = \xi', \mathsf{x}\{\delta\}
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The intrinsically-scoped syntax

$\boxed{\mathrm{Tm}\ \theta\ \gamma} \ni \quad t, u, T, U ::= \mid x \qquad \text{if } x \in \gamma$

$\mid \mathsf{x}\{\vec{t} \in \mathrm{Sub}\ \theta\ \gamma\ \delta\} \qquad \text{if } \mathsf{x}\{\delta\} \in \theta$

$\mid c(\mathbf{t} \in \mathrm{MSub}\ \theta\ \gamma\ \xi) \qquad \text{if } c(\xi) \in \Sigma$

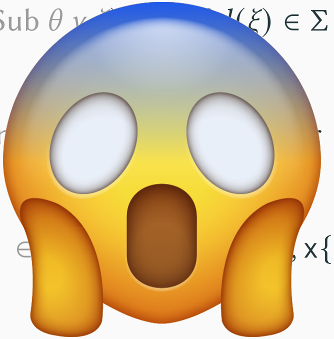
$\mid d(t \in \mathrm{Tm}\ \theta\ \gamma; \mathbf{t} \in \mathrm{MSub}\ \theta\ \gamma$ $(\xi) \in \Sigma$

$\boxed{\mathrm{Sub}\ \theta\ \gamma\ \delta} \ni \quad \vec{t}, \vec{u}, \vec{s}, \vec{v} ::= \mid \epsilon$

$\mid \vec{t}' \in \mathrm{Sub}\ \theta\ \gamma\ \delta', t \in \mathrm{Tr}$

$\boxed{\mathrm{MSub}\ \theta\ \gamma\ \xi} \ni \quad \mathbf{t}, \mathbf{u}, \mathbf{s}, \mathbf{v} ::= \mid \epsilon$

$\mid \mathbf{t}' \in \mathrm{MSub}\ \theta\ \gamma\ \xi', \vec{x}_\delta.t \in$ $, \mathsf{x}\{\delta\}$



The syntax

Terms and substitutions

Write x for variables, x for metavariables, c for constructors and d for destructors

$$\boxed{\mathrm{Tm}} \ni t, u, T, U ::= x \mid \mathsf{x}\{\vec{t}\} \mid c(\mathbf{t}) \mid d(t; \mathbf{t}) \quad (t \text{ is called the } \textit{principal argument})$$

$$\boxed{\mathrm{Sub}} \ni \vec{t}, \vec{u} ::= \epsilon \mid \vec{t}, t$$

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Example

$$\begin{aligned} \Sigma_{\lambda\Pi} = \quad & \Pi(\mathsf{A}, \mathsf{B}\{x\}),\ \lambda(\mathsf{t}\{x\}),\ \mathrm{Ty},\ \mathrm{Tm}(\mathsf{A}), && \text{(constructors)} \\ & @(\mathsf{u}) && \text{(destructors)} \end{aligned}$$

$$t, u, A, B ::= x \mid \mathsf{x}\{\vec{t}\} \mid \mathrm{Ty} \mid \mathrm{Tm}(A) \mid @(t; u) \mid \lambda(x.t) \mid \Pi(A, x.B)$$

The syntax

Contexts

$$\boxed{\mathrm{Ctx}} \ni \Gamma, \Delta ::= \cdot \mid \Gamma, x : T$$

$$\boxed{\mathrm{MCtx}} \ni \Theta, \Xi ::= \cdot \mid \Theta, \mathrm{x}\{\Gamma\} : T$$

Example: $\Gamma = A : \mathrm{Ty}, x : \mathrm{Tm}(A), y : \mathrm{Tm}(\Pi(A, z.A))$

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(Linear) Patterns

$$\boxed{\mathsf{Tm}^{\mathsf{P}}} \ni t^{\mathsf{P}}, u^{\mathsf{P}} ::= \mathsf{x}\{\vec{x}\} \mid c(\mathbf{t}^{\mathsf{P}})$$

$$\boxed{\mathsf{MSub}^{\mathsf{P}}} \ni \mathbf{t}^{\mathsf{P}}, \mathbf{u}^{\mathsf{P}} ::= \epsilon \mid \mathbf{t}^{\mathsf{P}}, \vec{x}.t^{\mathsf{P}} \quad (\mathsf{metas}(\mathbf{t}^{\mathsf{P}}) \cap \mathsf{metas}(t^{\mathsf{P}}) = \emptyset)$$

The theories

A *theory* $\mathbb{T}$ is made of *schematic typing rules* and *rewrite rules*.

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Sort rules Used to define judgment forms of the theory.

Example: In MLTT, 2 judgment forms: $\square$ type and $\square : A$ for a type A.

$$\frac{}{\vdash \mathrm{Ty}\ \mathsf{sort}} \qquad\qquad \frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{Tm}(\mathsf{A})\ \mathsf{sort}}$$

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Formally, of the form $c(\Theta)$ sort, where Θ represent premises.

Example in formal notation: $\mathrm{Ty}(\cdot)$ sort and $\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})$ sort

The theories

Constructor rules Two groups of premises: Θ_1 erased and Θ_2 kept in the syntax.

To recover Θ_1, rule will be used in mode check, so the type should be pattern containing metavariables of Θ_1.

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty}}{\vdash \Pi(\mathsf{A}, \mathsf{B}) : \mathbf{Ty}} \qquad \frac{\begin{array}{c}\vdash \mathsf{A} : \mathbf{Ty} \qquad x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathbf{Ty} \\ x : \mathbf{Tm}(\mathsf{A}) \vdash \mathsf{t} : \mathbf{Tm}(\mathsf{B}\{x\})\end{array}}{\vdash \lambda(\mathsf{t}) : \mathbf{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

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Formally, of the form $c(\Theta_1; \Theta_2) : U^{\mathsf{P}}$

Example in formal notation: $\Pi(\cdot;\ \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty}) : \mathrm{Ty}$ and $\lambda(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \mathsf{t}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Tm}(\mathsf{B}\{x\})) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$.

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Destructor rules Two groups of premises: Θ_1 erased and Θ_2 kept in the syntax.

But also a *principal argument* $\mathsf{x} : T^{\mathsf{P}}$, a pattern allowing to recover Θ_1.

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Formally, of the form $d(\Theta_1;\ \mathsf{x} : T^{\mathsf{P}};\ \Theta_2) : U$

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The theories

Rewrite rules Define the definitional equality (aka conversion) $\equiv$ of the theory.

$$@(\lambda(x.\mathtt{t}\{x\}); \mathtt{u}) \longmapsto \mathtt{t}\{\mathtt{u}\}$$

In general, of the form $d(t^{\mathsf{P}}; \mathbf{u}^{\mathsf{P}}) \longmapsto r$ with $(\mathrm{metas}(t^{\mathsf{P}}) \cap \mathrm{metas}(\mathbf{u}^{\mathsf{P}}) = \emptyset)$.

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Condition: no two left-hand sides unify.

Therefore, rewrite systems are orthogonal, hence confluent by construction!

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Full example Theory $\mathbb{T}_{\lambda\Pi}$.

$$\mathrm{Ty}(\cdot)\ \mathsf{sort} \qquad \mathrm{Tm}(\mathsf{A} : \mathrm{Ty})\ \mathsf{sort} \qquad \Pi(\cdot;\ \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty}) : \mathrm{Ty}$$

$$\lambda(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Tm}(\mathsf{B}\{x\})) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$$

$$@(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}));\ \ \mathsf{u} : \mathrm{Tm}(\mathsf{A})) : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\})$$

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Declarative typing

Declarative typing rules

Each theory $\mathbb{T}$ defines a declarative type system.

Judgments $\Theta; \Gamma \vdash T$ sort and $\Theta; \Gamma \vdash t : T$ and $\Theta; \Gamma \vdash \vec{t} : \Delta$ and $\Theta; \Gamma \vdash \mathbf{t} : \Xi$.

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$$c(\Xi) \text{ sort} \in \mathbb{T} \; \overset{\textsc{Sort}}{\dfrac{\Theta; \Gamma \vdash \mathbf{t} : \Xi}{\Theta; \Gamma \vdash c(\mathbf{t}) \text{ sort}}}$$

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$$\textsc{Cons}\quad \frac{c(\Xi_1;\Xi_2) : T \in \mathbb{T} \qquad \Theta;\Gamma \vdash \mathbf{t}, \mathbf{u} : \Xi_1.\Xi_2}{\Theta;\Gamma \vdash c(\mathbf{u}) : T[\mathbf{t}]}$$

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$$\textsc{Dest}\quad \frac{d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T} \qquad \Theta; \Gamma \vdash \mathbf{t}, t, \mathbf{u} : \Xi_1.(\mathsf{x} : T).\Xi_2}{\Theta; \Gamma \vdash d(t; \mathbf{u}) : U[\mathbf{t}, t, \mathbf{u}]}$$

Example

$$
\begin{array}{llllllll}
d(& \Xi_1 & ; & \mathsf{x} : T^{\mathsf{P}} & ; & \Xi_2) & & : U \\
@(& \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty} & ; & \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\})) & ; & \mathsf{u} : \mathrm{Tm}(\mathsf{A})) & & : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\})
\end{array}
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Properties of declarative system

In the following, we assume theory the $\mathbb{T}$ to be *valid* (definition not given here).

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Moreover, our valid theories all satisfy **subject reduction**!
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$$\Gamma \vdash t : T \text{ and } t \longrightarrow t' \text{ implies } \Gamma \vdash t' : T$$

Proof uses the following key lemma:

Key property of patterns If $\Theta \vdash t^{\mathsf{P}} : T$ and $\Delta \vdash t^{\mathsf{P}}[\mathbf{v}] : T[\mathbf{v}]$ then $\Delta \vdash \mathbf{v} : \Theta$

Bidirectional typing

Matching modulo rewriting

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We know

$$U \equiv \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))[A/\mathsf{A},\ x.B/\mathsf{B}]$$

but how to recover A and B from U?

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Given t^{P} and u, judgment $t^{\mathsf{P}} \prec u \rightsquigarrow \mathbf{v}$ tries to compute $\mathbf{v}$ with $t^{\mathsf{P}}[\mathbf{v}] \equiv u$.

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Proof: induction on matching rules

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Proof: Uses the fact that $\longrightarrow_{\mathrm{m/o}}$ is head-normalizing for orthogonal systems.

Inferable and checkable terms

Not all unannotated terms can be algorithmically typed

$$\frac{\dfrac{?}{\Gamma \vdash \lambda(x.t) \Rightarrow ?} \qquad \dots}{\Gamma \vdash @(\lambda(x.t); u) \Rightarrow ?}$$

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$$\boxed{\mathsf{Tm}^{\mathsf{i}}} \ni t^{\mathsf{i}}, u^{\mathsf{i}} ::= x \mid d(t^{\mathsf{i}}; \mathbf{t}^{\mathsf{c}})$$

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$$\boxed{\mathsf{MSub}^{\mathsf{c}}} \ni \mathbf{t}^{\mathsf{c}}, \mathbf{u}^{\mathsf{c}} ::= \epsilon \mid \mathbf{t}^{\mathsf{c}}, \vec{x}.t^{\mathsf{c}}$$

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Principal argument of a destructor can only be variable or another destructor.

For most theories: Tm^{c} = normal forms, and Tm^{i} = neutrals

Bidirectional typing rules

Judgments $\Gamma \vdash T^{\mathsf{c}} \Leftarrow \mathsf{sort}$ and $\Gamma \vdash t^{\mathsf{c}} \Leftarrow T$ and $\Gamma \vdash t^{\mathsf{i}} \Rightarrow T$ and $\Gamma \mid \mathbf{v} : \Xi \vdash \mathbf{t}^{\mathsf{c}} \Leftarrow \Theta$.

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$$\textsc{Cons}\quad \frac{c(\Xi_1;\Xi_2) : T \in \mathbb{T} \qquad T \prec U \rightsquigarrow \mathbf{v} \qquad \Gamma \mid \mathbf{v} : \Xi_1 \vdash \mathbf{u}^{\mathsf{c}} \Leftarrow \Xi_2}{\Gamma \vdash c(\mathbf{u}^{\mathsf{c}}) \Leftarrow U}$$

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$$\textsc{Dest}\quad d(\Xi_1; \mathtt{t} : T; \Xi_2) : U \in \mathbb{T}\ \frac{\Gamma \vdash t^{\mathsf{i}} \Rightarrow V \qquad T < V \rightsquigarrow \mathbf{v} \qquad \Gamma \mid (\mathbf{v}, t) : (\Xi_1, \mathtt{t} : T) \vdash \mathbf{u}^{\mathsf{c}} \Leftarrow \Xi_2}{\Gamma \vdash d(t^{\mathsf{i}}; \mathbf{u}^{\mathsf{c}}) \Rightarrow U[\mathbf{v}, t, \mathbf{u}]}$$

Example

$$\begin{array}{llclclcl} d(& \Xi_1 & ; & \mathsf{x} : T^{\mathsf{P}} & ; & \Xi_2) & & : U \\ @(& \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty} & ; & \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\})) & ; & \mathsf{u} : \mathrm{Tm}(\mathsf{A})) & & : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\}) \end{array}$$

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Suppose underlying theory $\mathbb{T}$ is valid.

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Decidability If $\mathbb{T}$ weak normalizing, then inference is decidable for inferable terms, and checking is decidable for checkable terms.

More examples

Dependent sums

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \Sigma(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}}$$

$$\frac{\begin{array}{cc} \vdash \mathsf{A} : \mathrm{Ty} & x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{t} : \mathrm{Tm}(\mathsf{A}) & \vdash \mathsf{u} : \mathrm{Tm}(\mathsf{B}\{\mathsf{t}\}) \end{array}}{\vdash \mathrm{pair}(\mathsf{t}, \mathsf{u}) : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{cc} \vdash \mathsf{A} : \mathrm{Ty} & x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \multicolumn{2}{c}{\vdash \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))} \end{array}}{\vdash \mathrm{proj}_1(\mathsf{t}; \cdot) : \mathrm{Tm}(\mathsf{A})}$$

$$\frac{\begin{array}{cc} \vdash \mathsf{A} : \mathrm{Ty} & x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \multicolumn{2}{c}{\vdash \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))} \end{array}}{\vdash \mathrm{proj}_2(\mathsf{t}; \cdot) : \mathrm{Tm}(\mathsf{B}\{\mathrm{proj}_1(\mathsf{t})\})}$$

$$\mathrm{proj}_1(\mathrm{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{t} \qquad \mathrm{proj}_2(\mathrm{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{u}$$

Lists

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{List}(\mathsf{A}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{nil} : \mathrm{Tm}(\mathrm{List}(\mathsf{A}))} \qquad \frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{x} : \mathrm{Tm}(\mathsf{A}) \\ \vdash \mathsf{l} : \mathrm{Tm}(\mathrm{List}(\mathsf{A}))\end{array}}{\vdash \mathrm{cons}(\mathsf{x}, \mathsf{l}) : \mathrm{Tm}(\mathrm{List}(\mathsf{A}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{l} : \mathrm{Tm}(\mathrm{List}(\mathsf{A})) \qquad x : \mathrm{Tm}(\mathrm{List}(\mathsf{A})) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{pnil} : \mathrm{Tm}(\mathsf{P}\{\mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{List}(\mathsf{A})), z : \mathrm{Tm}(\mathsf{P}\{y\}) \vdash \mathsf{pcons} : \mathrm{Tm}(\mathsf{P}\{\mathrm{cons}(x, y)\})\end{array}}{\vdash \mathrm{ListRec}(\mathsf{l}; \mathsf{P}, \mathsf{pnil}, \mathsf{pcons}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{l}\})}$$

$$\mathrm{ListRec}(\mathrm{nil}; x.\mathsf{P}\{x\}, \mathsf{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) \longmapsto \mathsf{pnil}$$

$$\mathrm{ListRec}(\mathrm{cons}(\mathsf{x}, \mathsf{l}); x.\mathsf{P}\{x\}, \mathsf{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) \longmapsto$$
$$\mathsf{pcons}\{\mathsf{x}, \mathsf{l}, \mathrm{ListRec}(\mathsf{l}; x.\mathsf{P}\{x\}, \mathsf{pnil}, xyz.\mathsf{pcons}\{x, y, z\})\}$$

W types

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \mathrm{W}(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}} \qquad \frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{f} : \mathrm{Tm}(\Pi(\mathsf{B}\{\mathsf{a}\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})))\end{array}}{\vdash \mathrm{sup}(\mathsf{a}, \mathsf{f}) : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \vdash \mathsf{t} : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \qquad x : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \vdash \mathsf{P} : \mathrm{Ty} \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))), z : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathsf{P}\{@(y, x')\})) \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathrm{sup}(x, y)\})\end{array}}{\vdash \mathrm{WRec}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{t}\})}$$

$$\mathrm{WRec}(\mathrm{sup}(\mathsf{a}, \mathsf{f}); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}) \longmapsto \mathsf{p}\{\mathsf{a}, \mathsf{f}, \lambda(x.\mathrm{WRec}(@(\mathsf{f}, x); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}))\}$$

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

$$\mathrm{J}(\mathrm{refl}, xy.\mathsf{P}\{x, y\}, \mathsf{p}) \longmapsto \mathsf{p}$$

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

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Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\vdash \mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{t} : \mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathsf{a}, \mathrm{refl}\})\end{array}}{\vdash \mathrm{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

$$\mathrm{J}(\mathrm{refl}, xy.\mathsf{P}\{x, y\}, \mathsf{p}) \longmapsto \mathsf{p}$$

We add conversion premises to constructor rules (not shown in this talk)

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathbf{Ty} \qquad \vdash \mathsf{a} : \mathbf{Tm}(\mathsf{A}) \qquad \vdash \mathsf{b} : \mathbf{Tm}(\mathsf{A})}{\vdash \mathbf{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathbf{Ty}} \qquad \frac{\begin{array}{cc}\vdash \mathsf{A} : \mathbf{Ty} & \vdash \mathsf{a} : \mathbf{Tm}(\mathsf{A}) \\ \vdash \mathsf{b} : \mathbf{Tm}(\mathsf{A}) & \mathsf{a} = \mathsf{b}\end{array}}{\vdash \mathbf{refl} : \mathbf{Tm}(\mathbf{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}))}$$

$$\frac{\begin{array}{cccc}\vdash \mathsf{A} : \mathbf{Ty} & \vdash \mathsf{a} : \mathbf{Tm}(\mathsf{A}) & \vdash \mathsf{b} : \mathbf{Tm}(\mathsf{A}) & \vdash \mathsf{t} : \mathbf{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) \\ \multicolumn{3}{c}{x : \mathbf{Tm}(\mathsf{A}), y : \mathbf{Tm}(\mathbf{Eq}(\mathsf{A}, \mathsf{a}, x)) \vdash \mathsf{P} : \mathbf{Ty}} & \vdash \mathsf{p} : \mathbf{Tm}(\mathsf{P}\{\mathsf{a}, \mathbf{refl}\})\end{array}}{\vdash \mathbf{J}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathbf{Tm}(\mathsf{P}\{\mathsf{b}, \mathsf{t}\})}$$

$$\mathbf{J}(\mathbf{refl}, xy.\mathsf{P}\{x, y\}, \mathsf{p}) \longmapsto \mathsf{p}$$

We add conversion premises to constructor rules (not shown in this talk)

Vectors

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{n} : \mathrm{Tm}(\mathrm{Nat})}{\vdash \mathrm{Vec}(\mathtt{A}, \mathtt{n}) : \mathrm{Ty}} \qquad \frac{\vdash \mathtt{A} : \mathrm{Ty}}{\vdash \mathrm{nil} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, 0))} \qquad \frac{\vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{x} : \mathrm{Tm}(\mathtt{A}) \qquad \vdash \mathtt{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}))}{\vdash \mathrm{cons}(\mathtt{n}, \mathtt{x}, \mathtt{l}) : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathrm{S}(\mathtt{n})))}$$

$$\frac{\begin{array}{c} \vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)) \vdash \mathtt{P} : \mathrm{Ty} \qquad \vdash \mathtt{pnil} : \mathrm{Tm}(\mathtt{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathtt{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)), w : \mathrm{Tm}(\mathtt{P}\{x, z\}) \vdash \mathtt{pcons} : \mathrm{Tm}(\mathtt{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\}) \end{array}}{\vdash \mathrm{VecRec}(\mathtt{l}; \mathtt{P}, \mathtt{pnil}, \mathtt{pcons}) : \mathrm{Tm}(\mathtt{P}\{\mathtt{n}, \mathtt{l}\})}$$

$$\mathrm{VecRec}(\mathrm{nil}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyzw.\mathtt{pcons}\{x, y, z, w\}) \longmapsto \mathtt{pnil}$$

$$\mathrm{VecRec}(\mathrm{cons}(\mathtt{n}, \mathtt{x}, \mathtt{l}); x.\mathtt{P}\{x\}, \mathtt{pnil}, xyzw.\mathtt{pcons}\{x, y, z, w\}) \longmapsto$$

$$\mathtt{pcons}\{\mathtt{n}, \mathtt{x}, \mathtt{l}, \mathrm{VecRec}(\mathtt{l}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyzw.\mathtt{pcons}\{x, y, z, w\})\}$$

Vectors

Extends $\mathbb{T}_{\lambda\Pi}$ with

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$$\frac{\begin{array}{c}\vdash \mathtt{A} : \mathrm{Ty} \qquad \vdash \mathtt{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)) \vdash \mathtt{P} : \mathrm{Ty} \qquad \vdash \mathtt{pnil} : \mathrm{Tm}(\mathtt{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathtt{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)), w : \mathrm{Tm}(\mathtt{P}\{x, z\}) \vdash \mathtt{pcons} : \mathrm{Tm}(\mathtt{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\})\end{array}}{\vdash \mathrm{VecRec}(\mathtt{l}; \mathtt{P}, \mathtt{pnil}, \mathtt{pcons}) : \mathrm{Tm}(\mathtt{P}\{\mathtt{n}, \mathtt{l}\})}$$

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Vectors

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$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{n}' : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathsf{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathsf{x} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A})) \qquad \mathsf{n}' = \mathrm{S}(\mathsf{n})}{\vdash \mathrm{cons}(\mathsf{n}, \mathsf{x}, \mathtt{l}) : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, \mathsf{n}'))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad \vdash \mathsf{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \vdash \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, \mathsf{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, x)) \vdash \mathsf{P} : \mathrm{Ty} \qquad \vdash \mathsf{pnil} : \mathrm{Tm}(\mathsf{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathsf{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathsf{A}, x)), w : \mathrm{Tm}(\mathsf{P}\{x, z\}) \vdash \mathsf{pcons} : \mathrm{Tm}(\mathsf{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\})\end{array}}{\vdash \mathrm{VecRec}(\mathtt{l}; \mathsf{P}, \mathsf{pnil}, \mathsf{pcons}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{n}, \mathtt{l}\})}$$

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Universes

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{}{\vdash \mathrm{U}(\cdot) : \mathrm{Ty}} \qquad \frac{\vdash a : \mathrm{Tm}(\mathrm{U})}{\vdash \mathrm{El}(a; \cdot) : \mathrm{Ty}}$$

Then, two options:

Universes

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Then, two options:

Tarski-style Adds codes for all types

$$\frac{}{\vdash \mathrm{u}(\cdot) : \mathrm{Tm}(\mathrm{U})} \qquad \mathrm{El}(\mathrm{u}; \varepsilon) \longmapsto \mathrm{U}$$

$$\frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U}) \qquad x : \mathrm{Tm}(\mathrm{El}(\mathsf{a})) \vdash \mathsf{b} : \mathrm{Tm}(\mathrm{U})}{\vdash \pi(\mathsf{a}, \mathsf{b}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\}); \varepsilon) \longmapsto \Pi(\mathrm{El}(\mathsf{a}; \varepsilon), x.\mathrm{El}(\mathsf{b}\{x\}; \varepsilon))$$

Universes

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Then, two options:

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$$\frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U}) \qquad x : \mathrm{Tm}(\mathrm{El}(\mathsf{a})) \vdash \mathsf{b} : \mathrm{Tm}(\mathrm{U})}{\vdash \pi(\mathsf{a}, \mathsf{b}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\}); \varepsilon) \longmapsto \Pi(\mathrm{El}(\mathsf{a}; \varepsilon), x.\mathrm{El}(\mathsf{b}\{x\}; \varepsilon))$$

(Weak) Coquand-style

Adds a code constructor c

$$\frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{c}(\mathsf{A}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\mathrm{c}(\mathsf{A}); \varepsilon) \longmapsto \mathsf{A}$$

Conclusion

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We have given a generic account of bidirectional type for a class of type theories

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Bidirectional system implemented in a prototype, available at

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1. Going beyond the constructor/destructor separation (do we want this?)

$$\mathrm{Tm}(\mathrm{U}) \longrightarrow \mathrm{Ty}$$

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Future work Extending class of type theories we can handle by

1. Going beyond the constructor/destructor separation (do we want this?)

$$\mathrm{Tm}(\mathrm{U}) \longrightarrow \mathrm{Ty}$$

2. Going beyond rewriting rules, needed for eta/proof-irrelevance

Thank you for your attention!