Generic Bidirectional Typing for Dependent Type Theories

Thiago Felicissimo

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$$t \mathbin{@}_{A,x.B} u \qquad \langle t, u \rangle_{A,x.B} \qquad t ::_A l \qquad \ldots$$

What one gets when seeing type theory as an algebraic theory

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Syntax so common that many don't realize that an omission is being made

Typechecking without annotations

Omission has a cost Knowing annotations is needed for typing

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type} \qquad \Gamma \vdash t : \Pi x : A.B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B[u/x]}$$

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How to find A and B if they're not stored in syntax?

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Complements unannotated syntax, *locally* explains how to recover annotations

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2. For each theory, we define declarative and bidirectional type systems
3. We show, in a theory-independent fashion, their equivalence

BiTTs: A theory-independent bidirectional type-checker

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Implemented in the theory-independent bidirectional type-checker BiTTs

```
constructor List () (A : Ty) : Ty
constructor nil (A : Ty) () : Tm(List(A))
constructor cons (A : Ty) (a : Tm(A), l : Tm(List(A))) : Tm(List(A))

destructor ind_List     (A : Ty) [l : Tm(List(A))] (P {x : Tm(List(A))} : Ty, l_nil : Tm(P{nil}),
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let 0::1::2::3::nil : Tm(List(ℕ)) := cons(0, cons(S(0), cons(S(S(0)), cons(S(S(S(0))), nil))))

let sum_of_list : Tm(Π(List(ℕ), _. ℕ)) := λ(l. ind_List(l, _. ℕ, 0, x _ acc. @(@(+, x), acc)))

assert @(sum_of_list, 0::1::2::3::nil) = S(S(S(S(S(S(0))))))
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Compared with other theory-independent type-checkers (Dedukti, Andromeda) non-annotated syntax should allow for better performances

The theories

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Used to represent the judgment forms of the theory (as in GATs, SOGATs, …)

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Formally, of the form $c(\Theta)$ sort, with Θ metavariable context representing premises

Example in formal notation: $\mathrm{Ty}(\cdot)$ sort and $\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})$ sort

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Formally, constructor rules of the form $c(\Theta_1; \Theta_2) : U^{\mathsf{P}}$, with U^{P} pattern on Θ_1

Example in formal notation: $\Pi(\cdot;\ \mathsf{A} : \mathbf{Ty},\ \mathsf{B}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Ty}) : \mathbf{Ty}$ and $\lambda(\mathsf{A} : \mathbf{Ty},\ \mathsf{B}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Ty};\ \mathtt{t}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Tm}(\mathsf{B}\{x\})) : \mathbf{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$

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Rewrite rules Specify the definitional equality (aka conversion) $\equiv$ of the theory

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Full example Theory $\mathbb{T}_{\lambda\Pi}$

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Main typing rules instantiate the schematic rules of $\mathbb{T}$:

$$d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T} \; \frac{\Theta; \Gamma \vdash \mathbf{t}, t, \mathbf{u} : \Xi_1.(\mathsf{x} : T).\Xi_2}{\Theta; \Gamma \vdash d(t, \mathbf{u}) : U[\mathbf{t}, t, \mathbf{u}]} \, \textsc{Dest}$$

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(for $@(\mathsf{A} : \mathbf{Ty},\ \mathsf{B}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Ty};\ \mathsf{t} : \mathbf{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}));\ \mathsf{u} : \mathbf{Tm}(\mathsf{A})) : \mathbf{Tm}(\mathsf{B}\{\mathsf{u}\}) \in \mathbb{T}_{\lambda\Pi}$)

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Reading bottom-up, requires guessing A and B

Declarative type system

Each theory $\mathbb{T}$ defines a declarative type system, with main judgment $\Theta; \Gamma \vdash t : T$

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Reading bottom-up, requires guessing A and B

Properties of the declarative system Weakening, substitution property, sorts are well-typed, subject reduction, etc (see the paper)

Bidirectional typing system

Matching modulo rewriting

In bidirectional typing, we need matching modulo to recover missing arguments.

$$\frac{\Gamma \vdash t \Rightarrow U \qquad \dots}{\Gamma \vdash @(t, u) \Rightarrow}$$

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$$U \equiv \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))[A/\mathsf{A},\ x.B/\mathsf{B}]$$

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Solution We define an algorithmic[2] matching judgment $T^{\mathsf{P}} \prec U \rightsquigarrow \mathbf{v}$

We have $T^{\mathsf{P}}[\mathbf{v}] \equiv U$ iff $T^{\mathsf{P}} \prec U \rightsquigarrow \mathbf{v}'$ for some $\mathbf{v}' \equiv \mathbf{v}$

[2]Decidable when U is normalizing

Bidirectional syntax

Not all unannotated terms can be algorithmically typed

$$\dfrac{\dfrac{?}{\Gamma \vdash \lambda(x.t) \Rightarrow ?} \qquad \cdots}{\Gamma \vdash @(\lambda(x.t), u) \Rightarrow ?}$$

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Avoided by defining bidirectional system only for *inferrable* and *checkable* terms

$$\boxed{\mathrm{Tm}^{\mathrm{i}}} \ni \quad t^{\mathrm{i}}, u^{\mathrm{i}} ::= x \mid d(t^{\mathrm{i}}, \mathbf{t}^{\mathrm{c}})$$

$$\boxed{\mathrm{Tm}^{\mathrm{c}}} \ni \quad t^{\mathrm{c}}, u^{\mathrm{c}} ::= c(\mathbf{t}^{\mathrm{c}}) \mid \underline{t}^{\mathrm{i}}$$

$$\boxed{\mathrm{MSub}^{\mathrm{c}}} \ni \quad \mathbf{t}^{\mathrm{c}}, \mathbf{u}^{\mathrm{c}} ::= \epsilon \mid \mathbf{t}^{\mathrm{c}}, \vec{x}.t^{\mathrm{c}}$$

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Principal argument of a destructor can only be variable or another destructor

For most theories, $t^{\mathsf{c}}, u^{\mathsf{c}}, \ldots$ are the normal forms

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Each $\mathbb{T}$ defines a bidirectional system. Main judgments: $\Gamma \vdash t^{\mathsf{c}} \Leftarrow T$ and $\Gamma \vdash t^{\mathsf{i}} \Rightarrow T$

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The main typing rules instantiate the schematic rules of $\mathbb{T}$:[3]

$$d(\Xi_1; \mathsf{t} : T; \Xi_2) : U \in \mathbb{T} \quad \frac{\begin{array}{c}\Gamma \vdash t^{\mathsf{i}} \Rightarrow T' \qquad T \prec T' \leadsto \mathbf{v} \\ \Gamma \mid (\mathbf{v}, \ulcorner t^{\mathsf{i}} \urcorner) : (\Xi_1, \mathsf{x} : T) \vdash \mathbf{u}^{\mathsf{c}} \Leftarrow \Xi_2\end{array}}{\Gamma \vdash d(t^{\mathsf{i}}, \mathbf{u}^{\mathsf{c}}) \Rightarrow U[\mathbf{v}, \ulcorner t^{\mathsf{i}} \urcorner, \ulcorner \mathbf{u}^{\mathsf{c}} \urcorner]}\ \textsc{Dest}$$

[3] Given t^{i} or u^{c}, I write $\ulcorner t^{\mathsf{i}} \urcorner$ or $\ulcorner u^{\mathsf{c}} \urcorner$ for the underlying regular term.

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Reading bottom-up, no more need to guess A and B!

[3] Given t^{i} or u^{c}, I write $\ulcorner t^{\mathsf{i}} \urcorner$ or $\ulcorner u^{\mathsf{c}} \urcorner$ for the underlying regular term.

Correctness with respect to declarative typing

Suppose underlying theory $\mathbb{T}$ is valid

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Soundness If $\Gamma \vdash$ and $\Gamma \vdash t^{\mathsf{i}} \Rightarrow T$ then $\Gamma \vdash \ulcorner t^{\mathsf{i}} \urcorner : T$

If $\Gamma \vdash T$ sort and $\Gamma \vdash t^{\mathsf{c}} \Leftarrow T$ then $\Gamma \vdash \ulcorner t^{\mathsf{c}} \urcorner : T$

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Completeness If $\Gamma \vdash \ulcorner t^{\mathsf{i}} \urcorner : T$ then $\Gamma \vdash t^{\mathsf{i}} \Rightarrow T'$ with $T' \equiv T$
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Decidability If $\mathbb{T}$ normalizing, then inference is decidable for inferable terms, and checking is decidable for checkable terms

More examples

Dependent sums

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}) : \mathrm{Ty}}$$

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\mathsf{A}) \qquad \mathsf{u} : \mathrm{Tm}(\mathsf{B}\{\mathsf{t}\})}{\mathrm{pair}(\mathsf{t}, \mathsf{u}) : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}{\mathrm{proj}_1(\mathsf{t}) : \mathrm{Tm}(\mathsf{A})}$$

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}{\mathrm{proj}_2(\mathsf{t}) : \mathrm{Tm}(\mathsf{B}\{\mathrm{proj}_1(\mathsf{t})\})}$$

$$\mathrm{proj}_1(\mathrm{pair}(\mathsf{t}, \mathsf{u})) \longmapsto \mathsf{t} \qquad \mathrm{proj}_2(\mathrm{pair}(\mathsf{t}, \mathsf{u})) \longmapsto \mathsf{u}$$

Lists

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathtt{A} : \mathrm{Ty}}{\mathrm{List}(\mathtt{A}) : \mathrm{Ty}} \qquad \frac{\mathtt{A} : \mathrm{Ty}}{\mathrm{nil} : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))} \qquad \frac{\mathtt{A} : \mathrm{Ty} \qquad \mathtt{x} : \mathrm{Tm}(\mathtt{A}) \qquad \mathtt{l} : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))}{\mathrm{cons}(\mathtt{x}, \mathtt{l}) : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))}$$

$$\frac{\mathtt{A} : \mathrm{Ty} \qquad \mathtt{l} : \mathrm{Tm}(\mathrm{List}(\mathtt{A})) \qquad x : \mathrm{Tm}(\mathrm{List}(\mathtt{A})) \vdash \mathtt{P} : \mathrm{Ty} \qquad \mathtt{pnil} : \mathrm{Tm}(\mathtt{P}\{\mathrm{nil}\}) \qquad x : \mathrm{Tm}(\mathtt{A}), y : \mathrm{Tm}(\mathrm{List}(\mathtt{A})), z : \mathrm{Tm}(\mathtt{P}\{y\}) \vdash \mathtt{pcons} : \mathrm{Tm}(\mathtt{P}\{\mathrm{cons}(x, y)\})}{\mathrm{ListRec}(\mathtt{l}, x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) : \mathrm{Tm}(\mathtt{P}\{\mathtt{l}\})}$$

$$\mathrm{ListRec}(\mathrm{nil}, x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) \longmapsto \mathtt{pnil}$$

$$\mathrm{ListRec}(\mathrm{cons}(\mathtt{x}, \mathtt{l}), x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) \longmapsto$$

$$\mathtt{pcons}\{\mathtt{x}, \mathtt{l}, \mathrm{ListRec}(\mathtt{l}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\})\}$$

W types

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}) : \mathrm{Ty}} \qquad \frac{\begin{array}{c}\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \mathsf{f} : \mathrm{Tm}(\Pi(\mathsf{B}\{\mathsf{a}\}, \_.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})))\end{array}}{\mathrm{sup}(\mathsf{a}, \mathsf{f}) : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c}\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \qquad x : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \vdash \mathsf{P} : \mathrm{Ty} \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))), z : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathsf{P}\{@(y, x')\})) \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathrm{sup}(x, y)\})\end{array}}{\mathrm{WRec}(\mathsf{t}, x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{t}\})}$$

$$\mathrm{WRec}(\mathrm{sup}(\mathsf{a}, \mathsf{f}), x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}) \longmapsto \mathsf{p}\{\mathsf{a}, \mathsf{f}, \lambda(x.\mathrm{WRec}(@(\mathsf{f}, x), x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}))\}$$

Universes

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{}{\mathsf{U} : \mathsf{Ty}} \qquad \frac{\mathsf{a} : \mathsf{Tm}(\mathsf{U})}{\mathsf{El}(\mathsf{a}) : \mathsf{Ty}}$$

Tarski-style Adds codes for all types

$$\frac{}{\mathsf{u} : \mathsf{Tm}(\mathsf{U})} \qquad \mathsf{El}(\mathsf{u}) \longmapsto \mathsf{U}$$

$$\frac{\mathsf{a} : \mathsf{Tm}(\mathsf{U}) \qquad x : \mathsf{Tm}(\mathsf{El}(\mathsf{a})) \vdash \mathsf{b} : \mathsf{Tm}(\mathsf{U})}{\pi(\mathsf{a}, x.\mathsf{b}\{x\}) : \mathsf{Tm}(\mathsf{U})}$$

$$\mathsf{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\})) \longmapsto \Pi(\mathsf{El}(\mathsf{a}), x.\mathsf{El}(\mathsf{b}\{x\}))$$

(Weak) Coquand-style

Adds a code constructor c

$$\frac{\mathsf{A} : \mathsf{Ty}}{\mathsf{c}(\mathsf{A}) : \mathsf{Tm}(\mathsf{U})}$$

$$\mathsf{El}(\mathsf{c}(\mathsf{A})) \longmapsto \mathsf{A}$$

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$$@(\lambda(x.t) :: T, u)$$

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3. Type-directed equalities (η-rules, proof irrelevance), generically?
   Alternatively, treat conversion with a black-box approach

Thank you for your attention!