Generic Bidirectional Typing for Dependent Type Theories

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$$t \mathbin{@_{A,x.B}} u \qquad \langle t, u \rangle_{A,x.B} \qquad t ::_A l \qquad \ldots$$

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Syntax so common that many don't realize that an omission is being made

Typechecking without annotations

Omission has a cost Knowing annotations is needed for typing

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type} \qquad \Gamma \vdash t : \Pi x : A.B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B[u/x]}$$

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$$\frac{\Gamma \vdash t \Rightarrow C \qquad C \longrightarrow^* \Pi x : A.B \qquad \Gamma \vdash u \Leftarrow A}{\Gamma \vdash t\ u \Rightarrow B[u/x]}$$

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Complements unannotated syntax, *locally* explains how to recover annotations

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1. We give a general definition of type theories (or equivalently, a *logical framework*) supporting non-annotated syntaxes
2. For each theory, we define declarative and bidirectional type systems
3. We show, in a theory-independent fashion, their equivalence

BiTTs: A theory-independent bidirectional type-checker

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Implemented in the theory-independent bidirectional type-checker BiTTs

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constructor Eq () (A : Ty, x : Tm(A), y : Tm(A)) : Ty
constructor refl (A : Ty, x : Tm(A)) () (x / y : Tm(A)) : Tm(Eq(A, x, y))

destructor J (A : Ty, x : Tm(A), y : Tm(A)) [t : Tm(Eq(A, x, y))]
             (P{y : Tm(A), e : Tm(Eq(A, x, y))} : Ty, p : Tm(P{x, refl})) : Tm(P{y, t})

equation J(refl, y e. P{y, e}, p) --> p

let sym : Tm(Π(U, a. Π(El(a), x. Π(El(a), y. Π(Eq(El(a), x, y), _. Eq(El(a), y, x))))))
    :=  λ(a. λ(x. λ(y. λ(p. J(p, z q. Eq(El(a), z, x), refl)))))

let transp : Tm(Π(U, a. Π(U, b. Π(Eq(U, a, b), _. Π(El(a), _. El(b))))))
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Many theories supported: flavours of MLTT, OTT, HOL (see the implementation)

Compared with other theory-independent type-checkers (Dedukti, Andromeda) non-annotated syntax should allow for better performances

The theories

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A *theory* $\mathbb{T}$ is made of *schematic typing rules* and *rewrite rules*.

3 schematic typing rules: *sort rules*, *constructor rules* and *destructor rules*

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Used to represent the judgment forms of the theory (as in GATs, SOGATs, …)

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Formally, of the form $c(\Theta)$ sort, with Θ metavariable context representing premises.

Example in formal notation: $\mathrm{Ty}(\cdot)$ sort and $\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})$ sort

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Formally, constructor rules of the form $c(\Theta_1; \Theta_2) : U^{\mathsf{P}}$, with U^{P} pattern on Θ_1

Example in formal notation: $\Pi(\cdot;\ \mathsf{A} : \mathbf{Ty},\ \mathsf{B}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Ty}) : \mathbf{Ty}$ and $\lambda(\mathsf{A} : \mathbf{Ty},\ \mathsf{B}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Ty};\ \mathtt{t}\{x : \mathbf{Tm}(\mathsf{A})\} : \mathbf{Tm}(\mathsf{B}\{x\})) : \mathbf{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$.

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The theories

Rewrite rules Define the definitional equality (aka conversion) $\equiv$ of the theory.

$$@(\lambda(x.\mathsf{t}\{x\}), \mathsf{u}) \longmapsto \mathsf{t}\{\mathsf{u}\}$$

In general, of the form $d(c(\mathbf{t}_1^{\mathsf{P}}), \mathbf{t}_2^{\mathsf{P}}) \longmapsto r$ with $(\mathrm{metas}(\mathbf{t}_1^{\mathsf{P}}) \cap \mathrm{metas}(\mathbf{t}_2^{\mathsf{P}}) = \emptyset)$.

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Full example Theory $\mathbb{T}_{\lambda\Pi}$.

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Main typing rules instantiate the schematic rules of $\mathbb{T}$:

$$d(\Xi_1; \mathsf{x} : T; \Xi_2) : U \in \mathbb{T} \; \frac{\Theta; \Gamma \vdash \mathbf{t}, t, \mathbf{u} : \Xi_1.(\mathsf{x} : T).\Xi_2}{\Theta; \Gamma \vdash d(\mathbf{t}, t, \mathbf{u}) : U[\mathbf{t}, t, \mathbf{u}]} \textsc{Dest}$$

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Reading bottom-up, requires guessing A and B

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Each theory $\mathbb{T}$ defines a declarative type system, with main judgment $\Theta; \Gamma \vdash t : T$

Main typing rules instantiate the schematic rules of $\mathbb{T}$:

$$\frac{\begin{array}{c}\Theta; \Gamma \vdash \qquad \Theta; \Gamma \vdash A : \mathrm{Ty} \qquad \Theta; \Gamma, x : \mathrm{Tm}(A) \vdash B : \mathrm{Ty} \\ \Theta; \Gamma \vdash t : \mathrm{Tm}(\Pi(A, x.B)) \qquad \Theta; \Gamma \vdash u : \mathrm{Tm}(A)\end{array}}{\Theta; \Gamma \vdash @(t, u) : \mathrm{Tm}(B[u/x])}$$

(for $@(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}));\ \mathsf{u} : \mathrm{Tm}(\mathsf{A})) : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\}) \in \mathbb{T}_{\lambda\Pi}$)

Reading bottom-up, requires guessing A and B

Properties of the declarative system Weakening, substitution property, sorts are well-typed, subject reduction, etc (see the paper)

Bidirectional typing system

Matching modulo rewriting

In bidirectional typing, we need matching modulo to recover missing arguments.

$$\frac{\Gamma \vdash t \Rightarrow U \qquad \ldots}{\Gamma \vdash @(t, u) \Rightarrow}$$

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If $@(t, u)$ is well-typed (in the declarative system), for some A, B we have

$$U \equiv \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))[A/\mathsf{A},\ x.B/\mathsf{B}]$$

but how to recover A and B from U?

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Solution We define an algorithmic[2] matching judgment $T^{\mathsf{P}} < U \rightsquigarrow \mathbf{v}$

We have $T^{\mathsf{P}}[\mathbf{v}] \equiv U$ iff $T^{\mathsf{P}} < U \rightsquigarrow \mathbf{v}'$ for some $\mathbf{v}' \equiv \mathbf{v}$

[2]Decidable when U is normalizing

Bidirectional syntax

Not all unannotated terms can be algorithmically typed

$$\dfrac{\dfrac{?}{\Gamma \vdash \lambda(x.t) \Rightarrow ?} \qquad \cdots}{\Gamma \vdash @(\lambda(x.t), u) \Rightarrow ?}$$

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Bidirectional system defined over *inferrable* and *checkable* terms

$$\boxed{\mathsf{Tm}^{\mathsf{i}}} \ni \quad t^{\mathsf{i}}, u^{\mathsf{i}} ::= x \mid d(t^{\mathsf{i}}, \mathbf{t}^{\mathsf{c}}) \mid t^{\mathsf{c}} :: T^{\mathsf{c}}$$

$$\boxed{\mathsf{Tm}^{\mathsf{c}}} \ni \quad t^{\mathsf{c}}, u^{\mathsf{c}} ::= c(\mathbf{t}^{\mathsf{c}}) \mid \underline{t}^{\mathsf{i}}$$

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\end{array}$$

When destructor meets a constructor, we need an *ascription*, in the style of McBride:

$$@(\lambda(x.t^{\mathsf{c}}) :: T^{\mathsf{c}}, u^{\mathsf{c}})$$

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The main typing rules instantiate the schematic rules of $\mathbb{T}$:[3]

$$d(\Xi_1; \mathsf{t} : T; \Xi_2) : U \in \mathbb{T}\ \frac{\begin{array}{c}\Gamma \vdash t^{\mathsf{i}} \Rightarrow T' \qquad T \prec T' \rightsquigarrow \mathbf{v} \\ \Gamma \mid (\mathbf{v}, \ulcorner t^{\mathsf{i}} \urcorner) : (\Xi_1, \mathsf{x} : T) \vdash \mathbf{u}^{\mathsf{c}} \Leftarrow \Xi_2\end{array}}{\Gamma \vdash d(t^{\mathsf{i}}, \mathbf{u}^{\mathsf{c}}) \Rightarrow U[\mathbf{v}, \ulcorner t^{\mathsf{i}} \urcorner, \ulcorner \mathbf{u}^{\mathsf{c}} \urcorner]}\ \textsc{Dest}$$

[3]Given t^{i} or u^{c}, I write $\ulcorner t^{\mathsf{i}} \urcorner$ or $\ulcorner u^{\mathsf{c}} \urcorner$ for the underlying regular term.

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Reading bottom-up, no more need to guess A and B!

[3] Given t^{i} or u^{c}, I write $\ulcorner t^{\mathsf{i}} \urcorner$ or $\ulcorner u^{\mathsf{c}} \urcorner$ for the underlying regular term.

Correctness with respect to declarative typing

Suppose underlying theory $\mathbb{T}$ is valid.

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Soundness If $\Gamma \vdash$ and $\Gamma \vdash t^{\mathsf{i}} \Rightarrow T$ then $\Gamma \vdash \ulcorner t^{\mathsf{i}} \urcorner : T$.
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Decidability If $\mathbb{T}$ normalizing, then inference is decidable for inferable terms, and checking is decidable for checkable terms.

More examples

Dependent sums

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}) : \mathrm{Ty}}
\qquad
\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\mathsf{A}) \qquad \mathsf{u} : \mathrm{Tm}(\mathsf{B}\{\mathsf{t}\})}{\mathrm{pair}(\mathsf{t}, \mathsf{u}) : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}{\mathrm{proj}_1(\mathsf{t}) : \mathrm{Tm}(\mathsf{A})}
\qquad
\frac{\mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}{\mathrm{proj}_2(\mathsf{t}) : \mathrm{Tm}(\mathsf{B}\{\mathrm{proj}_1(\mathsf{t})\})}$$

$$\mathrm{proj}_1(\mathrm{pair}(\mathsf{t}, \mathsf{u})) \longmapsto \mathsf{t} \qquad \mathrm{proj}_2(\mathrm{pair}(\mathsf{t}, \mathsf{u})) \longmapsto \mathsf{u}$$

Lists

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathtt{A} : \mathrm{Ty}}{\mathrm{List}(\mathtt{A}) : \mathrm{Ty}} \qquad \frac{\mathtt{A} : \mathrm{Ty}}{\mathrm{nil} : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))} \qquad \frac{\mathtt{A} : \mathrm{Ty} \qquad \mathtt{x} : \mathrm{Tm}(\mathtt{A}) \qquad \mathtt{l} : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))}{\mathrm{cons}(\mathtt{x}, \mathtt{l}) : \mathrm{Tm}(\mathrm{List}(\mathtt{A}))}$$

$$\frac{\mathtt{A} : \mathrm{Ty} \qquad \mathtt{l} : \mathrm{Tm}(\mathrm{List}(\mathtt{A})) \qquad x : \mathrm{Tm}(\mathrm{List}(\mathtt{A})) \vdash \mathsf{P} : \mathrm{Ty} \qquad \mathtt{pnil} : \mathrm{Tm}(\mathsf{P}\{\mathrm{nil}\}) \qquad x : \mathrm{Tm}(\mathtt{A}), y : \mathrm{Tm}(\mathrm{List}(\mathtt{A})), z : \mathrm{Tm}(\mathsf{P}\{y\}) \vdash \mathsf{pcons} : \mathrm{Tm}(\mathsf{P}\{\mathrm{cons}(x, y)\})}{\mathrm{ListRec}(\mathtt{l}, x.\mathsf{P}\{x\}, \mathtt{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) : \mathrm{Tm}(\mathsf{P}\{\mathtt{l}\})}$$

$$\mathrm{ListRec}(\mathrm{nil}, x.\mathsf{P}\{x\}, \mathtt{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) \longmapsto \mathtt{pnil}$$

$$\mathrm{ListRec}(\mathrm{cons}(\mathtt{x}, \mathtt{l}), x.\mathsf{P}\{x\}, \mathtt{pnil}, xyz.\mathsf{pcons}\{x, y, z\}) \longmapsto$$

$$\mathsf{pcons}\{\mathtt{x}, \mathtt{l}, \mathrm{ListRec}(\mathtt{l}; x.\mathsf{P}\{x\}, \mathtt{pnil}, xyz.\mathsf{pcons}\{x, y, z\})\}$$

Equality

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathsf{A} : \mathrm{Ty} \qquad \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \mathsf{b} : \mathrm{Tm}(\mathsf{A})}{\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{b}) : \mathrm{Ty}} \qquad \frac{\mathsf{A} : \mathrm{Ty} \qquad \mathsf{a} : \mathrm{Tm}(\mathsf{A})}{\mathrm{refl} : \mathrm{Tm}(\mathrm{Eq}(\mathsf{A}, \mathsf{a}, \mathsf{a}))}$$

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Vectors

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\mathtt{A} : \mathrm{Ty} \qquad \mathtt{n} : \mathrm{Tm}(\mathrm{Nat})}{\mathrm{Vec}(\mathtt{A}, \mathtt{n}) : \mathrm{Ty}} \qquad \frac{\begin{array}{c}\mathtt{A} : \mathrm{Ty} \\ \mathtt{n} \mapsto 0 : \mathrm{Tm}(\mathrm{Nat})\end{array}}{\mathrm{nil} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{n}))} \qquad \frac{\begin{array}{c}\mathtt{A} : \mathrm{Ty} \qquad \mathtt{m} : \mathrm{Tm}(\mathrm{Nat}) \qquad \mathtt{x} : \mathrm{Tm}(\mathtt{A}) \\ \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{m})) \qquad \mathtt{n} \mapsto \mathrm{S}(\mathtt{m}) : \mathrm{Tm}(\mathrm{Nat})\end{array}}{\mathrm{cons}(\mathtt{m}, \mathtt{x}, \mathtt{l}) : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{n}))}$$

$$\frac{\begin{array}{c}\mathtt{A} : \mathrm{Ty} \qquad \mathtt{n} : \mathrm{Tm}(\mathrm{Nat}) \qquad \mathtt{l} : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, \mathtt{n})) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)) \vdash \mathtt{P} : \mathrm{Ty} \qquad \mathtt{pnil} : \mathrm{Tm}(\mathtt{P}\{0, \mathrm{nil}\}) \\ x : \mathrm{Tm}(\mathrm{Nat}), y : \mathrm{Tm}(\mathtt{A}), z : \mathrm{Tm}(\mathrm{Vec}(\mathtt{A}, x)), w : \mathrm{Tm}(\mathtt{P}\{x, z\}) \vdash \mathtt{pcons} : \mathrm{Tm}(\mathtt{P}\{\mathrm{S}(x), \mathrm{cons}(x, y, z)\})\end{array}}{\mathrm{VecRec}(\mathtt{l}, xy.\mathtt{P}\{x, y\}, \mathtt{pnil}, xyzw.\mathtt{pcons}\{x, y, z, w\}) : \mathrm{Tm}(\mathtt{P}\{\mathtt{n}, \mathtt{l}\})}$$

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Other examples

In the implementation, you can also find:

- Higher-order logic
- Tarksi-style universes, with cumulativity (lifts $\uparrow$)
- (Weak) Coquand-style universes, with cumulativity and universe polymorphism
- Flavous of Observational Type Theory

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2. Type-directed equalities (η-rules, proof irrelevance), generically?
   Alternatively, treat conversion with a black-box approach
3. More abstract declarative type system (fully-annotated syntax, typed equality, fully-quotiented terms)?
   Generic bidirectional elaboration for a class of SOGATs?

Thank you for your attention!