1 Issue Introduction

This document aims to investigate a "bug" in Weaver Metafont. Basically we have the piece of code:

```
(4, 3) .. (2, 3) .. {-2, -1}(3, 3)
```

If we run Knuth's METAFONT, then this is evaluated as:

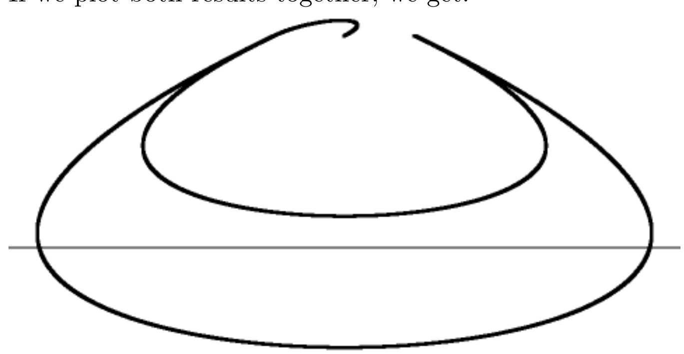
```
(4, 3)
  .. controls (11.22821, -0.42822)  and (-5.22821, -0.42822) ..
(2, 3)
  .. controls (2.60104, 3.28506)  and (3.59143, 3.29572) ..
{-2, -1}(3, 3)
```

But when running Weaver Metafont, the result is:

```
(4, 3)
  .. controls (16.487888, -2.922737)  and (-10.487888, -2.922737) ..
(2, 3)
  .. controls (2.601037, 3.285059)  and (3.591435 3.295718) ..
{-2, -1}(3, 3)
```

While both produce very similar results in the third and fouth pair of control points, the first and second pair are clearly very different.

If we plot both results together, we get:



Weaver Metafont draw a path below the baseline, but Knuth's METAFONT draw the path above the baseline.

Our objective here is try to discover what should be the correct value and if possible, why both versions do not agree with the result. For this, we will interpret the path code manually and step-to-step following the METAFONT Book instruction.

First the code must be placed in the following format (METAFONT Book, pág 131):

> $z_0\{w_0\}$ .. tension α_0 and β_1 .. $\{w_1\}z_1\{w_1\}$ .. tension α_1 and β_2 .. $\{w_2\}z_2$
> $\langle$etc.$\rangle$ $z_{n-1}\{w_{n-1}\}$ .. tension α_{n-1} and β_n .. $\{w_n\}z_n$.

Now, following from METAFONT Book pág 130:

> Special abbreviations are also allowed, so that the long forms of basic path joins can usually be avoided: '..' by itself stands for '.. tension 1 and 1 ..', while '.. tension α ..' stands for '.. tension α and α ..', and '.. controls u ..' stands for '.. controls u and u ..'.

This paragraph tell us that the original code:

```
(4, 3) .. (2, 3) .. {-2, -1}(3, 3)
```

Is interpreted as:

```
(4, 3) .. tension 1 and 1 .. (2, 3) .. tension 1 and 1 .. {-2, -1}(3, 3)
```

Therefore, we have:

```
z0 .. tension a0 and b1 .. z1 .. tension a1 and b2 .. {w2}z2
```

What we are missing is the direction specifiers `w0` and `w1`.

How should we fill the `w0` direction specifier is written in METAFONT Book pág 130:

> Now let's consider the rules by which empty direction specifiers can inherit specifications from their environment. An empty direction specifier at the beginning or end of a path, or just next to the '&' operator, is effectively replaced by '{curl 1}'. This rule should be interpreted properly with respect to cyclic paths, which have no beginning or end; for example, 'z_0 .. z_1 & z_1 .. z_2 .. cycle' is equivalent to 'z_0 .. z_1\{curl 1\}&\{curl 1\}z_1 .. z_2 .. cycle'.

Therefore, our path is:

```
(4, 3){curl 1} .. tension 1 and 1 .. (2, 3) .. tension 1 and 1 .. {-2, -1}(3, 3)
```

Only direction specifiers around point $(2, 3)$ is missing. And for this, METAFONT Book shows us how to compute it on page 130 to 131:

> After the previous three rules have been applied, we might still be left with cases in which there are points surrounded on both sides by empty direction specifiers. METAFONT must choose appropriate directions at such points, and it does so by applying the following algorithm due to John Hobby [*Discrete and Computational Geometry* **1** (1986), 123–140]: Given a sequence
>
> $z_0\{d_0\}$ .. tension α_0 and β_1 .. z_1 .. tension α_1 and β_2 .. z_2
> $\langle$etc.$\rangle$ z_{n-1} .. tension α_{n-1} and β_n .. $\{d_n\}z_n$
>
> for which interior directions need to be determined, we will regard the z's as if they were complex numbers. Let $l_k = |z_k - z_{k-1}|$ be the distance from z_{k-1} to z_k, and let $\psi_k = \arg\big((z_{k+1}-z_k)/(z_k-z_{k-1})\big)$ be the turning angle at z_k. We wish to find direction vectors $w_0, w_1, \ldots, w_n$ so that the given sequence can effectively be replaced by
>
> $z_0\{w_0\}$ .. tension α_0 and β_1 .. $\{w_1\}z_1\{w_1\}$ .. tension α_1 and β_2 .. $\{w_2\}z_2$
> $\langle$etc.$\rangle$ $z_{n-1}\{w_{n-1}\}$ .. tension α_{n-1} and β_n .. $\{w_n\}z_n$.

The previous "three rules" mentioned is the rule that we used to place a `curl 1` and two other rules not relevant for our situation.

The paragraph above introduce for us two values: l_1 and l_2:

$$l_1 = |(2,3) - (4,3)| = |(-2,0)| = \sqrt{(-2)^2} = 2$$

$$l_2 = |(3,3) - (2,3)| = |(1,0)| = 1$$

It also introduces the value:

$$\psi_1 = arg\left(\frac{(1,0)}{(-2,0)}\right) = arg\left(-\frac{1}{2} + 0i\right) = \pi$$

A função arg de um número negativo sem parte complexa é simplesmente igual a π.

More instructions from METAFONT Book, page 131:

> Since only the directions of the w's are significant, not the magnitudes, it suffices to determine the angles $\theta_k = \arg(w_k/(z_{k+1} - z_k))$. For convenience, we also let $\phi_k = \arg((z_k - z_{k-1})/w_k)$, so that
>
> $$\theta_k + \phi_k + \psi_k = 0. \qquad (*)$$

Therefore, we know that:

$$\theta_0 = arg\left(\frac{w_0}{z_1 - z_0}\right) = arg\left(\frac{w_0}{(2,3) - (4,3)}\right) = arg\left(-\frac{w_0}{2}\right)$$

$$\theta_1 = arg\left(\frac{w_1}{z_2 - z_1}\right) = arg\left(\frac{w_1}{(3,3) - (2,3)}\right) = arg(w_1)$$

$$\phi_1 = arg\left(\frac{z_1 - z_0}{w_1}\right) = arg\left(-\frac{2}{w_1}\right)$$

$$\phi_2 = arg\left(\frac{z_2 - z_1}{w_2}\right) = arg\left(\frac{1}{-2 - i}\right) = arg\left(-\frac{2}{5} + \frac{1}{5}i\right) = arctan(-\frac{1}{2}) + \pi$$

As we do not know the values for w_0 and w_1, but only for w_2 that was specified directly, we were able to compute only ϕ_2. But we know that the following equation must be true:

$$\theta_1 + \phi_1 + \psi_1 = 0$$

The next equation is given by METAFONT Book:

> Hobby's paper introduces the notion of "mock curvature" according to which the following equations should hold at interior points:
>
> $$\beta_k^2 l_k^{-1}(\alpha_{k-1}^{-1}(\theta_{k-1} + \phi_k) - 3\phi_k) = \alpha_k^2 l_{k+1}^{-1}(\beta_{k+1}^{-1}(\theta_k + \phi_{k+1}) - 3\theta_k). \qquad (**)$$

We have a single interior point. For us, the formula is:

$$\beta_1^2 l_1^{-1}(\alpha_0^{-1}(\theta_0 + \phi_1) - 3\phi_1) = \alpha_1^2 l_2^{-1}(\beta_2^{-1}(\theta_1 + \phi_2) - 3\theta_1)$$

As the tension for us is equal 1, then $\alpha_1 = \beta_1 = \alpha_0 = \beta_2 = 1$:

$$l_1^{-1}((\theta_0 + \phi_1) - 3\phi_1) = l_2^{-1}((\theta_1 + \phi_2) - 3\theta_1)$$

As $l_1 = 2$ and $l_2 = 1$:

$$\frac{1}{2}((\theta_0 + \phi_1) - 3\phi_1) = ((\theta_1 + \phi_2) - 3\theta_1)$$

Expanding, we have:

$$\frac{1}{2}\theta_0 + \frac{1}{2}\phi_1 - \frac{3}{2}\phi_1 = \theta_1 + \phi_2 - 3\theta_1$$

Which is equal:

$$\frac{1}{2}\theta_0 - \phi_1 = \phi_2 - 2\theta_1$$

Next rule from METAFONT Book:

> We also need to consider boundary conditions. If d_0 is an explicit direction vector w_0, we know θ_0; otherwise d_0 is 'curl γ_0' and we set up the equation
>
> $$\alpha_0^2(\beta_1^{-1}(\theta_0 + \phi_1) - 3\theta_0) = \gamma_0\beta_1^2(\alpha_0^{-1}(\theta_0 + \phi_1) - 3\phi_1). \qquad (***)$$

Indeed, d_0 is `curl 1`. Therefore, $\gamma_0 = 1$. As our tension is 1 for all values, $\alpha 0 = \beta_1 = 1$:

$$(\theta_0 + \phi_1) - 3\theta_0 = (\theta_0 + \phi_1) - 3\phi_1$$

Which yields::

$$\theta_0 = \phi_1$$

Recalling, all our formulas are:

$$\frac{1}{2}\theta_0 - \phi_1 = \phi_2 - 2\theta_1$$

$$\theta_1 + \phi_1 + \psi_1 = 0$$

$$\theta_0 = \phi_1$$

$$\psi_1 = \pi$$

$$\phi_2 = arctan(-\frac{1}{2}) + \pi$$

We can remove the unknowns ϕ_2 and ψ_1, as we know their values:

$$\frac{1}{2}\theta_0 - \phi_1 = arctan(-\frac{1}{2}) + \pi - 2\theta_1$$

$$\theta_1 + \phi_1 + \pi = 0$$

$$\theta_0 = \phi_1$$

As $\theta_0 = \phi_1$, we can replace ϕ_1 by θ_0:

$$\frac{1}{2}\theta_0 - \theta_0 = arctan(-\frac{1}{2}) + \pi - 2\theta_1$$
$$\theta_1 + \theta_0 + \pi = 0$$

The first formula now can be rewritten:

$$-\frac{1}{2}\theta_0 = arctan(-\frac{1}{2}) + \pi - 2\theta_1$$
$$\theta_1 + \theta_0 + \pi = 0$$

Finally, from the second formula, we can replace θ_1 by $-\theta_0 - \pi$:

$$-\frac{1}{2}\theta_0 = arctan(-\frac{1}{2}) + \pi - 2(-\theta_0 - \pi)$$
$$-\frac{1}{2}\theta_0 = arctan(-\frac{1}{2}) + \pi + 2\theta_0 + 2\pi$$
$$-\frac{5}{2}\theta_0 = arctan(-\frac{1}{2}) + 3\pi$$

Which reveals the value for θ_0:

$$\theta_0 = -\frac{2}{5}arctan\left(-\frac{1}{2}\right) - \frac{6}{5}\pi$$

We know that the negative of this value, minus pi is θ_1:

$$\theta_1 = (-\theta_0 - \pi)$$
$$\theta_1 = \frac{2}{5}arctan\left(-\frac{1}{2}\right) + \frac{6}{5}\pi - \pi$$
$$\theta_1 = \frac{2}{5}arctan\left(-\frac{1}{2}\right) + \frac{1}{5}\pi$$

We also know that $\theta_0 = \phi_1$:

$$\phi_1 = -\frac{2}{5}arctan\left(-\frac{1}{2}\right) - \frac{6}{5}\pi$$

Which was the last unknown value. We can write again all their values.

			Computed by Weaver Metafont
θ_0	$-\frac{2}{5}arctan\left(-\frac{1}{2}\right) - \frac{6}{5}\pi$	-3.5844521407	-3.584452
θ_1	$\frac{2}{5}arctan\left(-\frac{1}{2}\right) + \frac{1}{5}\pi$	0.4428594871	0.442859
ψ_1	π	3.1415926536	3.141593
ϕ_1	$-\frac{2}{5}arctan\left(-\frac{1}{2}\right) - \frac{6}{5}\pi$	-3.5844521407	-3.584452
ϕ_2	$arctan\left(-\frac{1}{2}\right) + \pi$	2.6779450446	2.677945

And the final rule to finally compute the control points:

Whew — these rules have determined the directions at all points. To complete the job of path specification, we need merely explain how to change a segment like '$z_0\{w_0\}$ .. tension α and β .. $\{w_1\}z_1$' into a segment of the form 'z_0 .. controls u and v .. z_1'; i.e., we finally want to know METAFONT's magic recipe for choosing the control points u and v. If $\theta = \arg\big(w_0/(z_1 - z_0)\big)$ and $\phi = \arg\big((z_1 - z_0)/w_1\big)$, the control points are

$$u = z_0 + e^{i\theta}(z_1 - z_0)f(\theta,\phi)/\alpha, \qquad v = z_1 - e^{-i\phi}(z_1 - z_0)f(\phi,\theta)/\beta,$$

where $f(\theta,\phi)$ is another formula due to John Hobby:

$$f(\theta,\phi) = \frac{2 + \sqrt{2}\,(\sin\theta - \frac{1}{16}\sin\phi)(\sin\phi - \frac{1}{16}\sin\theta)(\cos\theta - \cos\phi)}{3\,(1 + \frac{1}{2}(\sqrt{5} - 1)\cos\theta + \frac{1}{2}(3 - \sqrt{5}\,)\cos\phi)}.$$

In the first segment, we have: $\theta_0 = \phi_1$. Therefore: $(cos\theta_0 - cos\phi_1) = 0$. Computing $f(\theta_0, \phi_1)$ becomes simpler as we can ignore most of the numerator from $f(\theta_0, \phi_1)$. Therefore, the first control point is:

$$(4{+}3i){+}\frac{e^{i(-\frac{2}{5}arctan\left(-\frac{1}{2}\right)-\frac{6}{5}\pi)}(-2)2}{3(1+\frac{1}{2}(\sqrt{5}-1)cos(-\frac{2}{5}arctan\left(-\frac{1}{2}\right)-\frac{6}{5}\pi)+\frac{1}{2}(3-\sqrt{5})cos(-\frac{2}{5}arctan\left(-\frac{1}{2}\right)-\frac{6}{5}\pi))}$$

And the second is:

$$(2{+}3i){-}\frac{e^{-i(-\frac{2}{5}arctan\left(-\frac{1}{2}\right)-\frac{6}{5}\pi)}(-2)2}{3(1+\frac{1}{2}(\sqrt{5}-1)cos(-\frac{2}{5}arctan\left(-\frac{1}{2}\right)-\frac{6}{5}\pi)+\frac{1}{2}(3-\sqrt{5})cos(-\frac{2}{5}arctan\left(-\frac{1}{2}\right)-\frac{6}{5}\pi))}$$

We can write the following part of the computation as k_1:

$$k_1 = cos(-\frac{2}{5}arctan\left(-\frac{1}{2}\right) - \frac{6}{5}\pi)$$

Let's write as k_2 the other computation involving $arctan(x)$:

$$k_2 = (-\frac{2}{5}arctan\left(-\frac{1}{2}\right) - \frac{6}{5}\pi)$$

Now our control points can be simplified as:

$$u = (4+3i) + \frac{e^{ik_2}(-2)2}{3(1+\frac{1}{2}(\sqrt{5}-1)k_1 + \frac{1}{2}(3-\sqrt{5})k_1)}$$

$$v = (2+3i) - \frac{e^{-ik_2}(-2)2}{3(1+\frac{1}{2}(\sqrt{5}-1)k_1 + \frac{1}{2}(3-\sqrt{5})k_1)}$$

And:

$$u = (4+3i) + \frac{-4e^{ik_2}}{3 + (\frac{3\sqrt{5}}{2} - \frac{3}{2})k_1 + (\frac{9}{2} - \frac{3\sqrt{5}}{2})k_1}$$

$$v = (2+3i) - \frac{-4e^{-ik_2}}{3 + (\frac{3\sqrt{5}}{2} - \frac{3}{2})k_1 + (\frac{9}{2} - \frac{3\sqrt{5}}{2})k_1}$$

We can compute k_1 with arbitrary precision using `bc` program:

```
k1=c(-0.4*a(-0.5)-1.2*pi)
```

When aiming for 10 decimal places, we have:

$$k_1 = -0.9035299975$$

The value k_2 can be computed with:

```
pi=3.14159265358979323846264338327950288
k2=(-0.4*a(-0.5)-1.2*(pi))
```

When aiming with 10 decimal places, we have:

$$k_2 = -3.5844521407$$

Having k_1 and k_2, we can compute k_3 such that $u = k_3 e^{ik_2}$ with:

```
-4/(3+(3*sqrt(5)/2-1.5)*k1+(4.5-3*sqrt(5)/2)*k1)
```

Finally, we have:

$$u = (4+3i) - 13.8212221351e^{-3.5844521407i}$$

$$v = (2+3i) + 13.8212221351e^{3.5844521407i}$$

But this gives us values:

$$u = 16.4878888 - 2.9227371i$$

$$v = -10.4878888 + 8.9227371i$$

This agrees with Weaver Metafont values. And shows that Knuth's METAFONT produces an error of approximately 35% for both u and v.