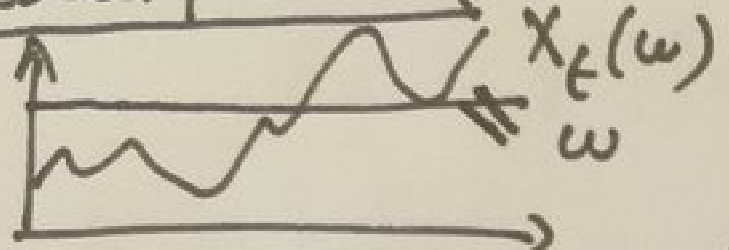
Processus d'Itô

$$\boxed{dX_t = a_t\,dt + b_t\,dB_t}$$

des fonctions aléatoires.
adaptées au processus (B_t)



ω

$\boxed{\text{Markovien Continu.}}$

Diffusion d'Itô

$$dX_t = \underline{a(X_t)}\,dt + b(X_t)\,dB_t$$

$$\left(C_0([0,T], \mathbb{R}) \equiv \Omega\right)$$

$$dX_t = B_t^2\,dt + dB_t$$

Définition d'un proc

(Ω, F, P)

ensemble

σ-algèbre

mesure de proba.

on définit une v.a. $X : (\Omega, F) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

fonction mesurable

$\forall O \in \mathcal{B}(\mathbb{R}),\ X^{-1}(O) \in F$

F_X la tribue engendrée par une v.a. X. : $F_X \subset F$

$X : (\Omega, F_X) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

(Ω, F)

$A \in F$

$(\Omega, F) \longrightarrow (\mathbb{R}, B(\mathbb{R}))$

$$\chi_A : \omega \longmapsto \begin{cases} 1 & \text{si } \omega \in A \\ 0 & \text{sinon} \end{cases}$$

est fonction mesurable

$\chi_A^{-1}(\{0\}) = \bar{A}$

$\chi_A^{-1}(\{1\}) = A$

$$\boxed{F_{\chi_A} = \{\emptyset, \Omega, A, \bar{A}\}}$$

$\chi_A : (\Omega, F_{\chi_A}) \longrightarrow (\mathbb{R}, B(\mathbb{R}))$

mesurable

$X = (X_0, X_1, X_2)$

$\boxed{\sigma(X_0, X_1, X_2)}$ plus petite tribue engendrée par X_0, X_1, X_2

Lemme de Doob-Dynkin

Soit Y une variable aléatoire

X ______

Y est F_X-mesurable $\underline{\text{ssi}}$ ($\Leftrightarrow$)

$$\exists g : (\mathbb{R}, B(\mathbb{R})) \xrightarrow[\text{mesurable}]{} (\mathbb{R}, B(\mathbb{R}))$$

tq $Y = g(X)$

Exemple :

$Y \in \sigma(X_0, X_1, X_2)$

$\exists g$ tq $Y = g(X_0, X_1, X_2)$

$g : (\mathbb{R}^n, B(\mathbb{R}^n)) \longrightarrow (\mathbb{R}, B(\mathbb{R}))$

$\omega \in \Omega$

$Y(\omega) = g(X_0(\omega), X_1(\omega), X_2(\omega))$

Exemple X est $\mathcal{N}(0,1)$

$Y = X^2$ Y est-elle F_X-mesurable

$g : x \mapsto x^2$

$$\left.\begin{array}{l} a_t \in \underbrace{\sigma\left((B_s)_{s\leq t}\right)} \\ b_t \in \underline{\qquad\qquad} \\ \text{et} \\ \mathbb{E}\left[\int_0^T a_t^2\,dt\right] < +\infty \\ \mathbb{E}\left[\int_0^T b_t^2\,dt\right] < +\infty \end{array}\right] \equiv$$

$(a_t, b_t)_{t\in[0,T]}$ sont des <u>processus adaptés au Brownien</u> et <u>d'énergie moyenne finie</u>

$$dX_t = B_t^2\,dt + dB_t$$

$$a_t = B_t^2 \in \sigma(B_t) \subset \sigma\left((B_s)_{s\leq t}\right)$$

$$b_t = 1$$

$$A^{(\omega)} = \int_0^T a_t^2(\omega)\,dt$$

$$\mathbb{E}_P\left[A^{(\omega)}\right]$$

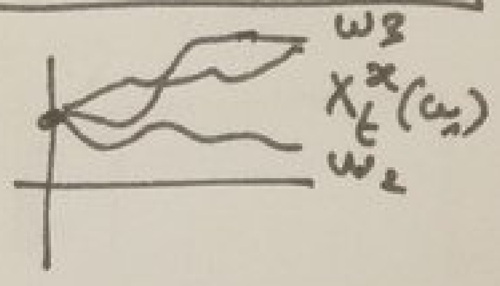
Dynamique observable / mesure dans le cadre stochastique

↑ marginale

$$\frac{dx}{dt} = a(x) + b(x)\,\eta\,\frac{1}{\sqrt{\delta t}}$$

① $\boxed{dX_t = a(X_t)\,dt + b(X_t)\,dB_t}$ une diffusion d'Itô

$X_0 = x$



une observable f est une fonction bornée

$\boxed{X_t^x}$ le processus solution de ①
et issue de la CI $X_0 = x$
$\forall \omega,\ X_0(\omega) = x$

Soit f_0 une observable $f(t,x) = f_0[X_t^x]$

$$f(t,x)(\omega) = f_0[X_t^x(\omega)]$$

on definit $\boxed{f(t,x) \equiv \mathbb{E}[f_0(X_t^x)]}$

déterministe

or $Y_t = f_0[X_t^x] \rightarrow d$

Formule d'Itô.

$$dY_t = f_0'(X_t^x)\,dX_t^x + \frac{1}{2} f_0''(X_t^x)\,dX_t^x \cdot dX_t^x$$

$$= \left[f_0'(X_t^x)\cdot a(X_t) + \frac{b^2(X_t)}{2} f''(X_t^x)\right] dt$$

$$+ \dots dB_t$$

$$Y_t = Y_0 + \int \cdot\, dt + \int \cdot\, dB_t$$

$$\mathbb{E}[Y_t] = \mathbb{E}[Y_0] + \mathbb{E}\left[\int \cdot\, dt\right] + 0$$

$$f(t+dt,x) = \mathbb{E}[f_0(X^x_{t+dt})]$$

$$= \underbrace{\mathbb{E}[f_0(X^x_t)]}_{f(t,x)} + \mathbb{E}\left[f_0'(X^x_t)\,a(X^x_t) + \frac{b^2(X^x_t)}{2} f_0''(X^x_t)\right] dt$$

$$\boxed{\partial_t f = \mathbb{E}[\cancel{\partial_t}\mathcal{L} f_0] \quad \boxed{\mathcal{L} \equiv \underset{a(x)}{a}\,\partial_x + \frac{\underset{b(x)}{b^2}}{2}\partial_x^2}}$$

$$\boxed{\mathcal{L} f_0 = a(x)\,\partial_x f_0(x) + \frac{b(x)^2}{2}\,\partial_x^2 f_0(x)}$$

~~alors~~

$$(\mathcal{L} f_0)(X^x_t(\omega)) =$$

$$\int a\,\partial_x f \cdot p = -\int f \cdot \partial_x (a p)$$

$$\boxed{f(t,x) \equiv \mathbb{E}[f_0(X^x_t)]} \quad f_0 \cdot C^2 \ldots$$

vérifie la "dynamique des observable"

$$\boxed{\partial_t f = \mathcal{L} f} \qquad \mathbb{E}[\mathcal{L} f] \to \mathcal{L}(\mathbb{E}[f]) \qquad \Longrightarrow$$

$f(t,\cdot)$ est C^2

$\mathbb{E}[f_0(X^x_t)]$

$$\boxed{f(t,x) = e^{\mathcal{L}t} f(0,x)}$$

"dynamique des mesures"

$$\boxed{\partial_t p = \mathcal{L}^* p}$$

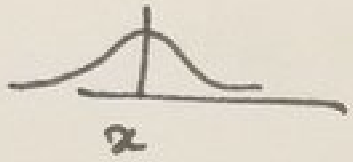
adjoint formel de $\mathcal{L}$.

$$\boxed{\mathcal{L}^* p = -\partial_x(a p) + \frac{1}{2}\partial_x^2(b p)}$$

Application : distribution limite

$$\partial_t p = \mathcal{L}^* p$$

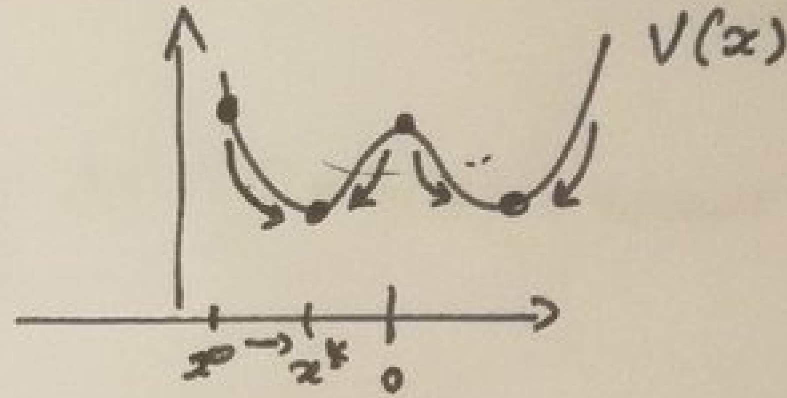
$$\boxed{\mathcal{L}^* p^\infty = 0}$$



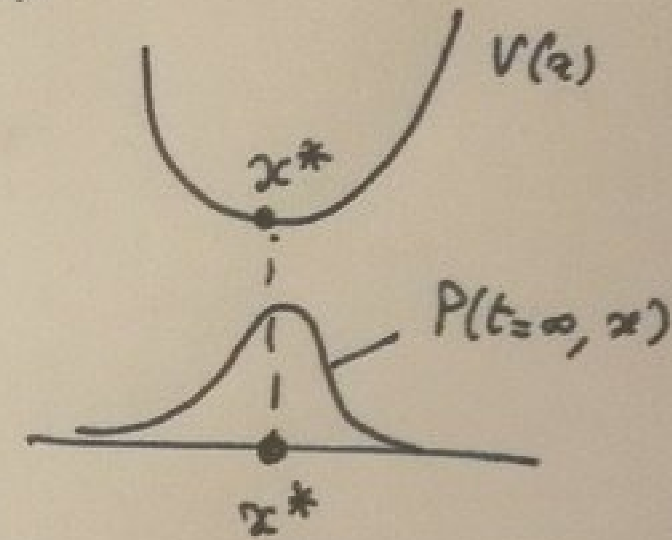
$p^\infty(x)$

$$\boxed{\frac{dx}{dt} = -\nabla V(x)}$$

dynamique à gradient
"flot-gradient"

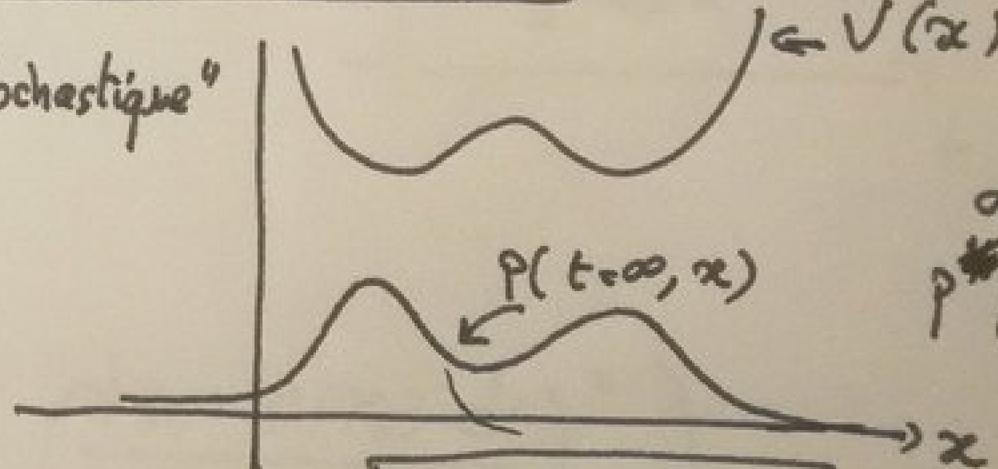


$$\frac{d}{dt} V(x) < 0$$
$$= -|\nabla V(x)|^2$$



$$\boxed{dX = -\nabla V(x)\,dt + \boxed{\sigma(X_t)}\,dB_t}$$

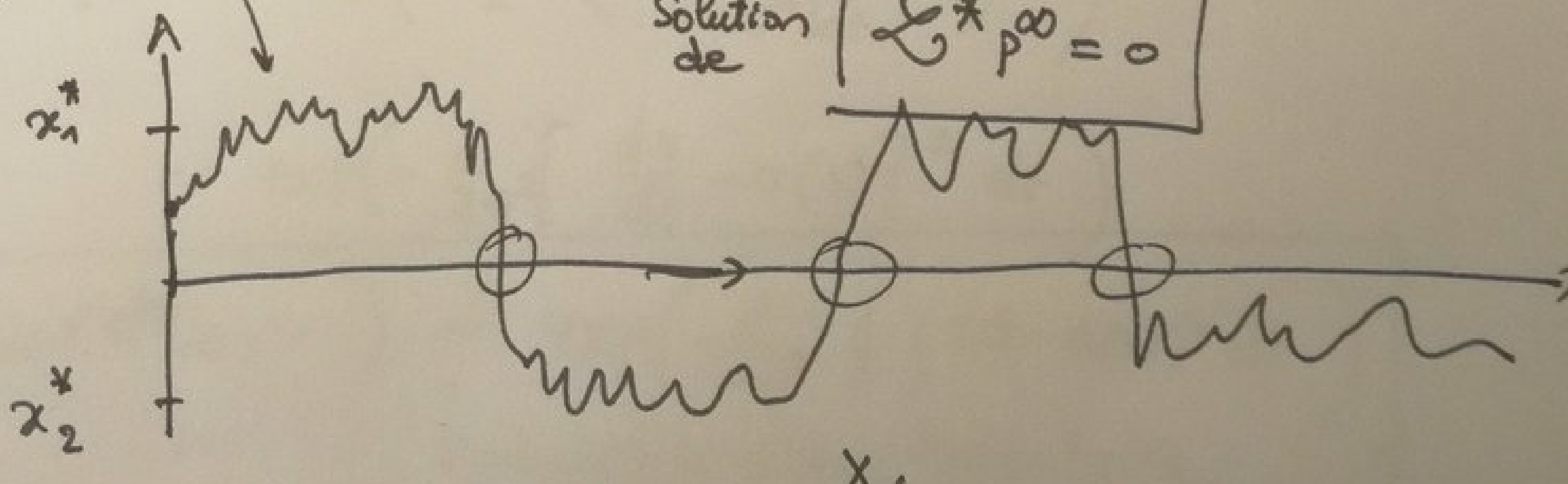
"Flot gradient stochastique"



$$p^\infty(x) = \lim_{t\to\infty} P(t,x)$$

Solution de $\boxed{\mathcal{L}^* p^\infty = 0}$

Exemple de trajectoire pour la dynamique Flot gradient stochastique



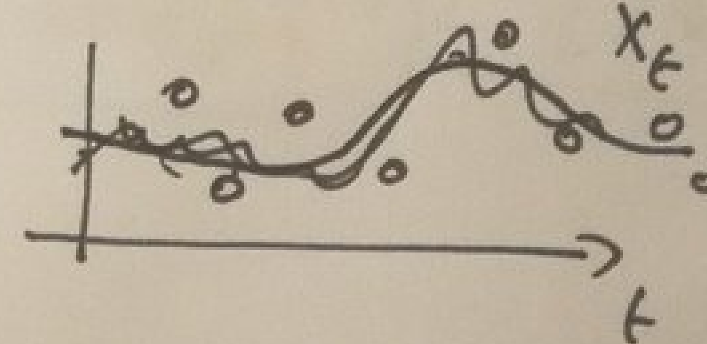
X_t

11-12-8

$$\left.\begin{array}{l} dx_t = a(x_t)\,dt + \underbrace{\sigma}_{\text{constante}}\, dB_t \end{array}\right\} \iff \frac{dx}{dt} = a(x) + \text{"erreur"}$$

$$dB_t = \xi\sqrt{dt}$$

$$\frac{dx_t - a(x_t)\,dt}{\sigma} = \underbrace{\xi}_{\mathcal{N}(0,1)}\sqrt{dt}$$



$$\frac{dx_t}{dt} = a.$$

$$dx_t = x_{t+dt} - x_t$$

$$P(x_{t+dt} / x_t) \sim \exp\left(-\frac{1}{2}\left(\frac{dx_t - a(x_t)\,dt}{\sigma\sqrt{dt}}\right)^2\right)$$

$$\sim \exp\left(-\frac{1}{2\sigma^2}\left[\frac{dx}{dt} - a(x)\right]^2 dt\right)$$

$$\boxed{P(x_0 \in dx_0, x_{dt} \in dx_{dt}, \cdots) \sim \exp\left(-\frac{1}{2\sigma^2} \int (\dot{x} - a(x))^2\, dt\right)}$$

Probabilité d'une trajectoire

$$J[x] = \frac{1}{2\sigma^2}\int_0^t (\dot{x} - a(x))^2\,dt + \boxed{\frac{1}{2\sigma_0^2}\int_0^t (y_{o_t} - H(x_t))^2\,dt}$$

fonctionnelle type 4D-Var en continue.

$$\boxed{dy_{o_t} = H(x_t)\,dt + dW_t}$$

erreur d'obs

Brownien.

$$J_{4Dvar} = \frac{1}{2}\|x - x^b\|^2_{B^{-1}} + \boxed{\sum \frac{1}{2}\|\dot{x} - a(x)\|^2_{Q^{-1}}} + \boxed{\sum \frac{1}{2}\|y_o - Hx_t\|^2_{R^{-1}}}$$

Pour le cas continu

ici Y_{o_t} est un processus d'observation

$$dY_{o_t} = H(x_t)\,dt + \sigma^o\,dW_t$$

(autre Brownien indépendant de β_t)

$$\boxed{B = (B_t)_{t \in [0,T]}} \qquad B_0 = 0 \qquad \Omega = [-1,1]^n$$

$$\boxed{B_{t+dt} = B_t + \xi \sqrt{dt}} \qquad \Omega = \mathbb{R}$$

$$\omega \to B(\omega) = (B_t(\omega))_{t \in [0,T]} \qquad (\Omega, \mathcal{F}, P)$$

?

$$\mathcal{F}_t = \sigma((B_s)_{s \le t})$$

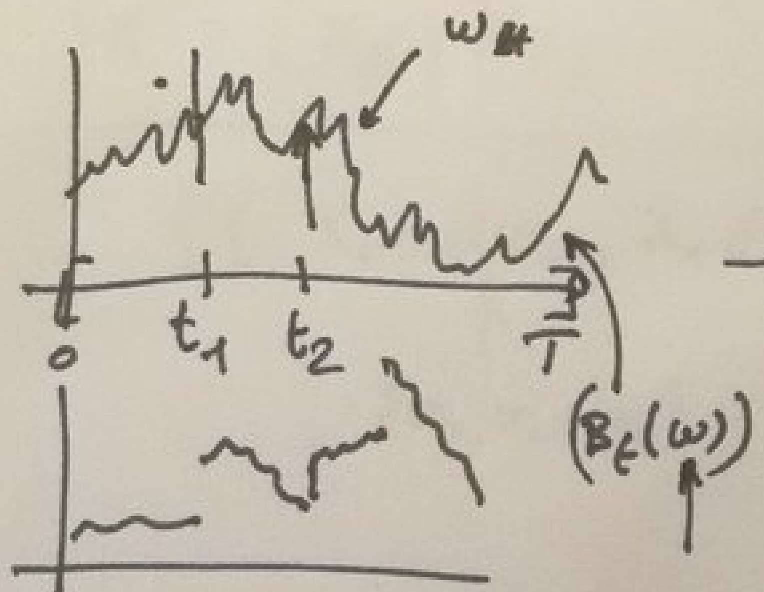
$$y_{q+1} = y_q + K \sin y_q$$

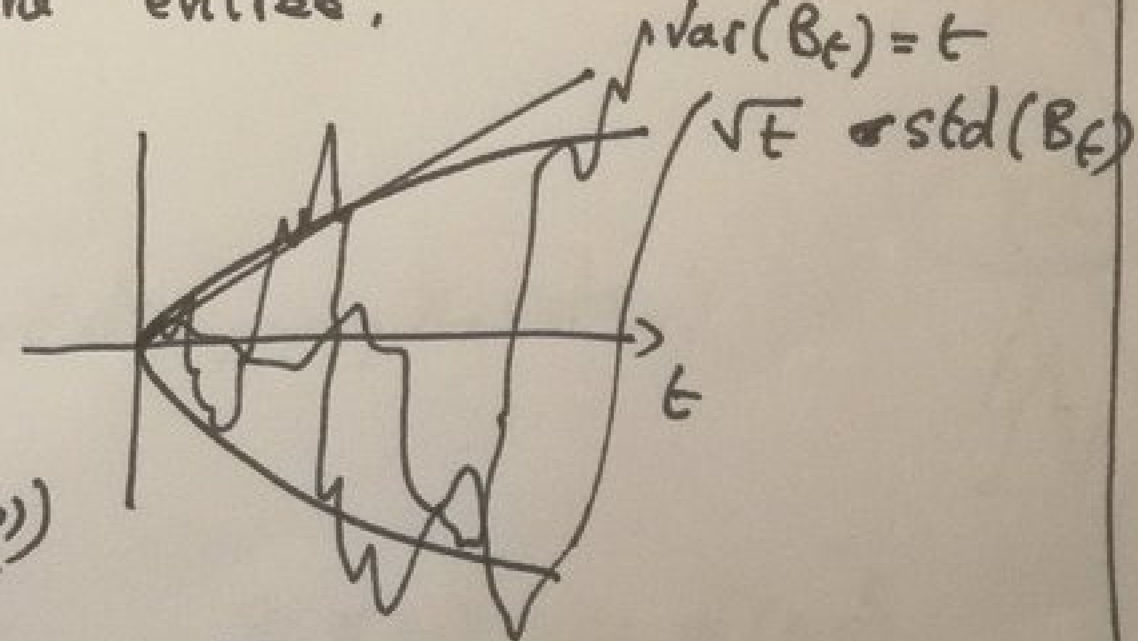
$$y = (y_0 \cdots y_\infty)$$

$$B^{(\omega)} = (B_t(\omega))_{[0,T]}$$

<u>cumsum</u> $\Leftrightarrow$ $\boxed{B_{q+1} = B_q + \xi_q \sqrt{dt}}$

ω ? $\equiv$ touche "entrée".





$\boxed{\omega = 1}$

$$\boxed{\Omega = \cancel{\mathbb{R}}\ \cancel{[0,T]}} \qquad \boxed{\mathcal{L}^2([0,T], \mathbb{R})}$$

1112-11

$$\boxed{dB_t = \sqrt{\delta t}\ \underbrace{\xi_t}_{\mathcal{N}(0,1)}} \qquad \boxed{B_{t+\delta t} = B_t + \xi_t \sqrt{\delta t}} \qquad \boxed{\mathbb{E}\left[|B_{t+\delta t} - B_t|^{\alpha}\right] \leq \delta t^{1+\beta}}$$

$$\boxed{\mathbb{E}\left[|B_{t+\delta t} - B_t|^{4}\right] \leq 3\, dt^{4}}$$

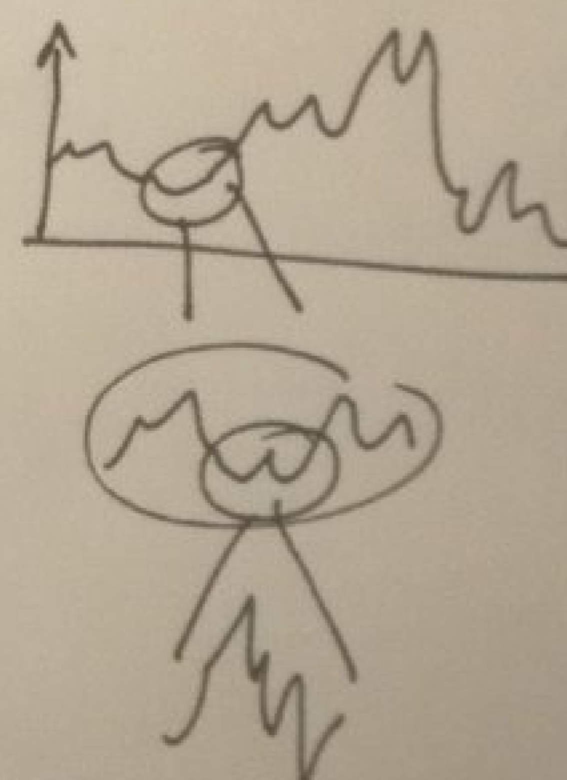
$$\boxed{p(x_{t+T}, t+T \,/\, x_t, t) \propto \exp\left(-\frac{1}{2}\frac{(x_{t+T}-x_t)^2}{T}\right)}$$

$$\boxed{x_{t+T} = x_t + \xi_t \sqrt{T}}$$

$P(\dots$

$$P(\underset{\uparrow}{B_0}, B_1, \dots, B_n) = \prod (\quad)$$

$B_{t_q} \quad t_q = q\,\delta t$



$$\boxed{X = (X_0, X_1, \dots)}$$

$$P(X_0 \in \underbrace{dx_0}_{[x_0, x_0+dx_0]}, X_1 \in dx_1, \dots, X_n \in dx_n)$$

Si X est une $\mathcal{N}(0,\sigma^2)$

$$\mathbb{E}[X^4] = 3\,\mathbb{E}[X^2]^2 = 3\sigma^4$$

Exemple d'application de la formule de Wick valable pour distribution Gaussienne centrée

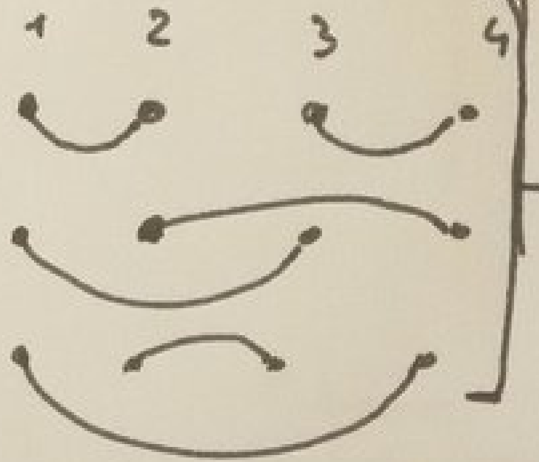
$$\underbrace{\mathbb{E}[X_1 X_2 X_3 X_4]}_{\text{moment d'ordre 4}} = \underbrace{\mathbb{E}[X_1 X_2]}_{\text{Covariance}} \underbrace{\mathbb{E}[X_3 X_4]}_{\text{Covariance}} + \mathbb{E}[X_1 X_3]\,\mathbb{E}[X_2 X_4] + \mathbb{E}[X_1 X_4]\,\mathbb{E}[X_2 X_3]$$

$X_i \sim$ ~~normale~~ gaussienne

Covariance

moment d'ordre 4

Formule de Wick



représentation graphique des termes contribuant au Calcul

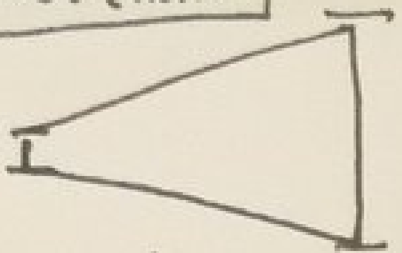
Calcul moment d'ordre 6

$$\mathbb{E}[X_1 X_2 X_3 X_4 X_5 X_6] = \mathbb{E}[X_1 X_2] \cdot \mathbb{E}[X_3 X_4] \cdot \mathbb{E}[X_5 X_6]$$

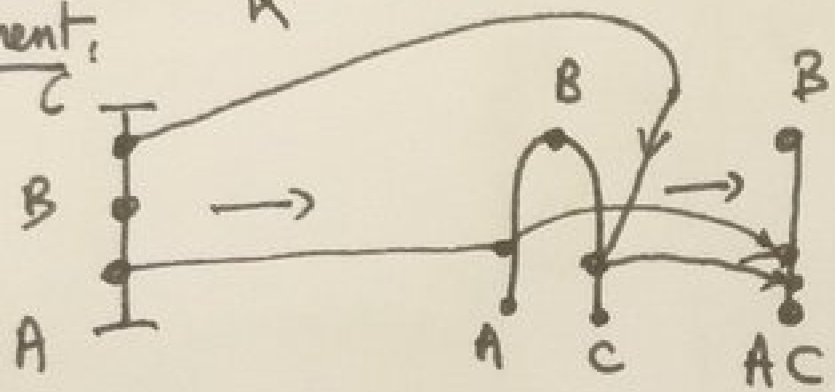
1112-13

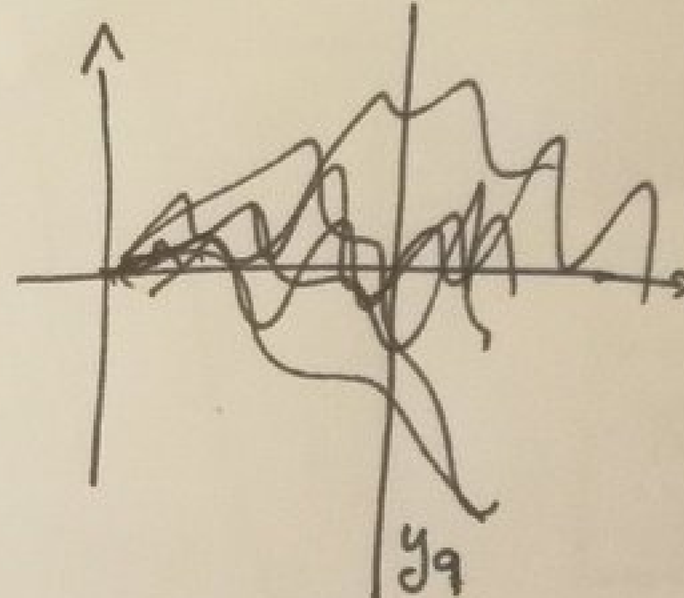
ingredient du chaos :

étirement :



repliement :





à q donné on a une Gaussienne (TCL)

itérés

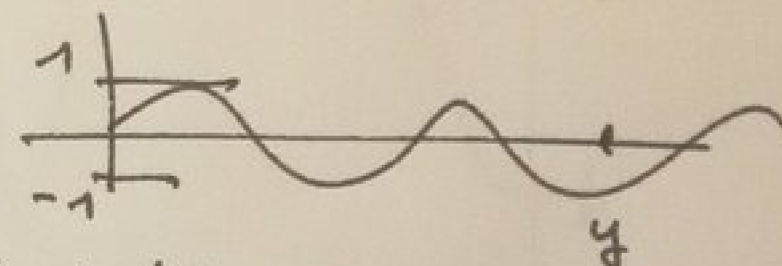
$y_q(\omega)$

avec $y_0(\omega) = y_0 + \xi \cdot \varepsilon$

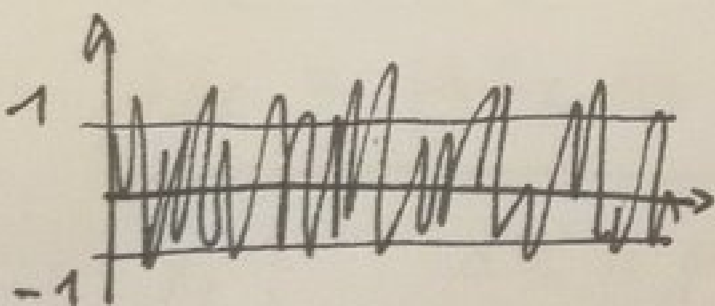
$\xi \sim \mathcal{N}(0,1)$

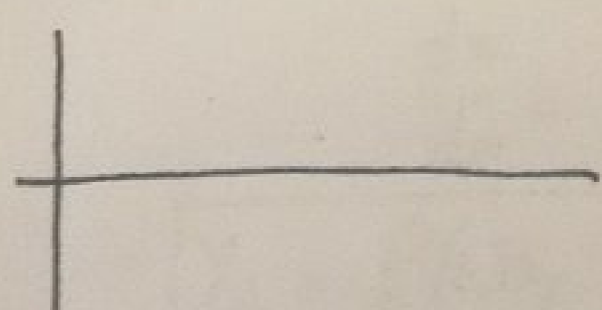
$\varepsilon = 10^{-6}$

$\sin(y) \in [-1, 1]$



$$\boxed{y_{q+1} = \boxed{K \sin y_q}}$$





$\mathcal{N}(0, t+dt)$ $\mathcal{N}(0,t)$

$$\boxed{B_{t+dt} = B_t + \underbrace{\xi \sqrt{dt}}}$$

$$y_{q+1} = y_q + \boxed{K \sin y_q}$$

$$y_q = \sum_{k=0}^{q} \underbrace{K \sin y_k}_{\eta_k} \sim \boxed{\mathcal{N}(0, q\sigma^2)}$$

$$\frac{1}{q} \sum_{k=0}^{q} \eta_k \underset{\uparrow}{\sim} \mathcal{N}(0, \sigma^2)$$

TCL (Théorème de la limite Centrale)

$t_q = q\,\delta t$

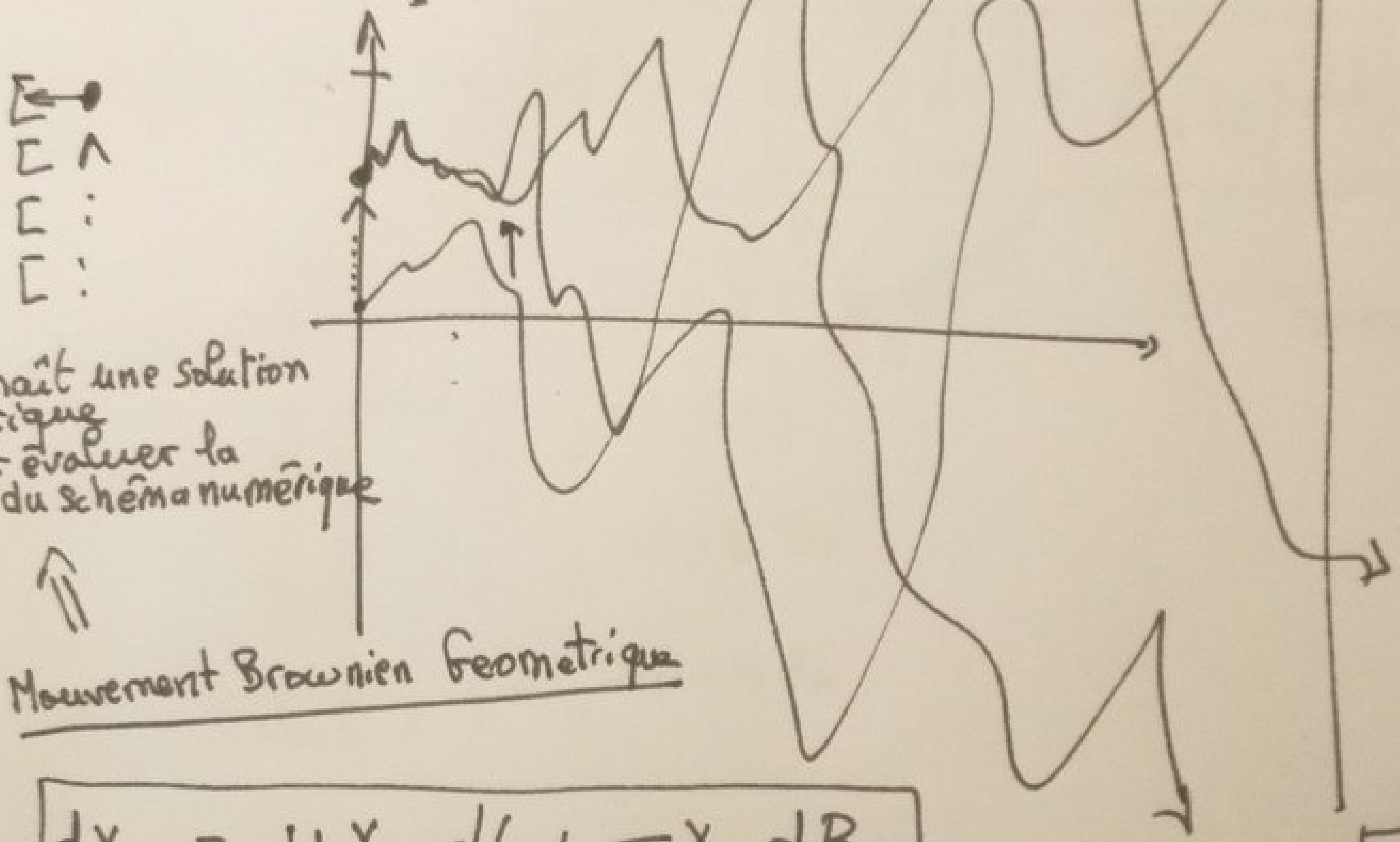
$\updownarrow$

$t_q = q\sigma^2$

Suite "renormalisée" $\simeq$ Brownien

$$\boxed{\tilde{y}_q = \frac{1}{\sigma}\sqrt{\delta t}\; y_q}$$

convergence $(y_q^0) \leftarrow$ pour $y_0 + \varepsilon \xrightarrow[\varepsilon \to 0]{} y_0$



$$\boxed{X_t \in \sigma\left((B_s)_{0 \le s \le t}\right)}$$

- on connaît une solution analytique
- on peut évaluer la qualité du schéma numérique

$\Uparrow$

Mouvement Brownien Géométrique

$$\boxed{dX_t = \mu X_t \, dt + \sigma X_t \, dB_t}$$

$$\boxed{Y_t = f(X_t)}$$

X_t ? $\quad dx = \mu x \, dt$

$$\frac{dx}{dt} = \mu x \rightarrow x(t) = e^{\mu t} x(0)$$

$$\boxed{Y_t = f(t, B_t) = e^{at + bB_t}} \quad \text{avec} \quad \boxed{f(t,x) = e^{at+bx}}$$

$\Rightarrow$ Utilisation de la formule d'Itô pour le cas f dépendant de "t".

11-12-15

$$Y_t = f(X_t) \longrightarrow dY_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,dX_t \cdot dX_t$$

$$Y_t = f(t, X_t) \longrightarrow \boxed{dY_t = \boxed{\partial_t f\, dt} + f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,dX_t \cdot dX_t}$$

$$\cancel{\frac{1}{2}\partial_t^2 f\, dt^2}$$

Formule d'Itô avec f dépendant de "t".

$$\partial_t f^{(t,B_t)} = a\, e^{at + bB_t}$$

$$\partial_x f = b\, e^{at + bB_t}$$

$$\partial_x^2 f = b^2\, e^{at + bB_t}$$

Evaluation d'un schéma numérique

Rappel cas déterministe
$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(t) \end{cases}$$

$$y_{q+1} = y_q + \delta t\, f(y_q)$$

$$y_q \simeq \boxed{x(t_q)}$$

$$\boxed{y_T \overset{?}{=} x(T)}$$

$$\boxed{|y_T - x(T)| \leq C\,\delta t^{\alpha}}$$

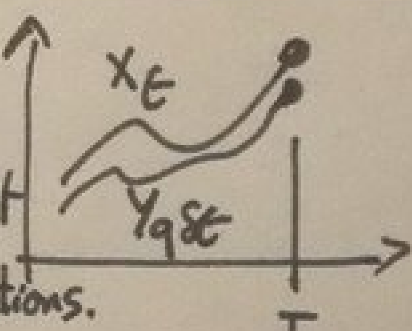
4

Plusieurs manières de comparer Y_T et X_T

cadre stochastique

$$Y_T(\omega) \simeq X_T(\omega) \quad \text{p.s.} \longrightarrow \boxed{\mathbb{E}\left[\,|Y_T(\omega) - X_T(\omega)|\,\right] \leq C\,\delta t^{\alpha_1}}$$

permet de contrôler le comportement du processus ie des ≠ réalisations.



peuvent être ≠.

ou $\text{Loi}(Y_T) = \text{loi}(X_T)$

ou $\mathbb{E}[Y_T] = \mathbb{E}[X_T]$

$$\left|\,\mathbb{E}[Y_T] - \mathbb{E}[X_T]\,\right| \leq C\,\delta t^{\alpha_2}$$

$$\left|\,\mathbb{E}[g(Y_T)] - \mathbb{E}[g(X_T)]\,\right| \leq C\,\delta t^{\beta}$$

Pour $\forall g$ bornée

Pour Euler-Maruyama
ordre convergence faible : 1
— — forte : $\frac{1}{2}$

Rq. :
logique car $\mathbb{P}(A) \equiv \mathbb{E}[\chi_A]$ donc g bornées suffisent !