Prévision Stochastique

Rappel

① $\boxed{\frac{d\underline{x}}{dt} = \underline{X}(\underline{x})}$ déterministe

φ_t => dynamique des observables

=> dynamique des mesures de probabilité.

② $\boxed{\partial_t p = \mathcal{L}_X^*(p)}$

③ $\boxed{\partial_t p = -\operatorname{div}(\underline{X}\, p)}$ equation de conservation

④ $\boxed{\partial_t p + X \cdot \nabla p = -p \cdot \operatorname{div}(\underline{X})}$

$\operatorname{div}(\underline{X}) = \sum_i \partial_{x^i} X^i$

or ③ ~~p(x,t)~~

$p(t,x)$ doit vérifier...

$$\int p(t,x)\,dx = 1$$

$$\int \operatorname{div}(\underline{X}\, p)\,dx = 0 \qquad \frac{d}{dt}\left(\int p\,dx\right) = 0$$

$P(t,x)$

$P(0,x) \longrightarrow P(t,x)$

$P(0,x) \longrightarrow P(t_1,x) \longrightarrow P(t,x)$

$$\cancel{P(t,x) = \int \boxed{P(t,x/t_1,x_1)}\, P(0,x_0)\, dx_1}$$

$P(t, x/0, x_0)$

$$\frac{dp}{dt} = \underline{\underline{A}}\, p \qquad \underline{\underline{A}}_{t\leftarrow 0} = e^{\underline{\underline{A}}\, t}$$

$$\underline{\underline{A}}_{t\leftarrow 0} = \underline{\underline{A}}_{t\leftarrow t_1} \cdot \underline{\underline{A}}_{t_1\leftarrow 0}$$

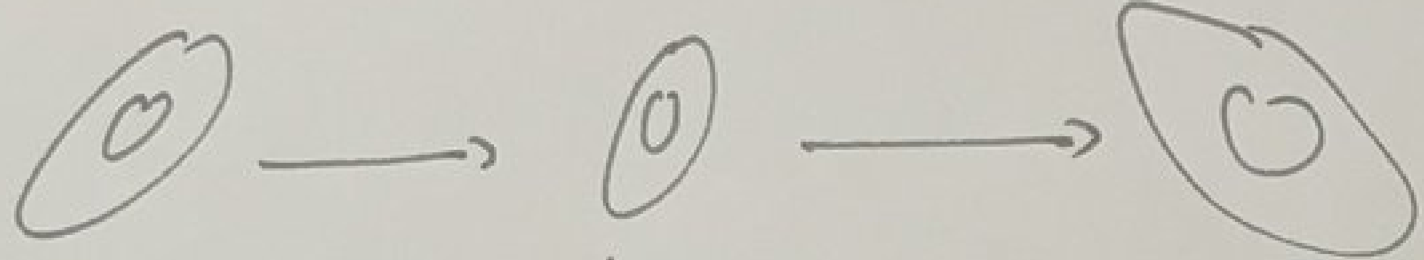
$$\boxed{P(A \cap B \cap C) = P(A/B\cap C) \cdot P(B/C)\, P(C)}$$

(t,x) (t_1,x_1) $(0,x_0)$

$$P(t,x/(0,x_0),(t_1,x_1))\, P(t_1,x_1/0,x_0)\, P(0,x_0)$$

$$\boxed{P(t,x/0,x_0) = \int P(t,x/t_1,x_1)\, P(t_1,x_1/0,x_0)\, dx_1}$$

Chapman - Kolmogorov



$t=0$ $\quad t_1$ $\quad t$

Processus Markovien.

≡ pendant déterministe dans le cadre stochastique

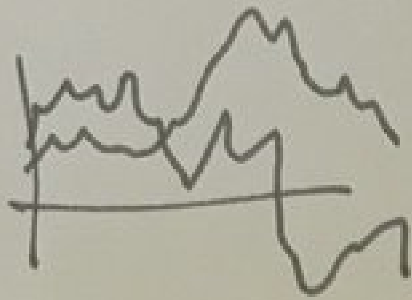
$$X^{(\omega)} = \left(X_0^{(\omega)}, X_1^{(\omega)}, \dots, \right) \in \mathbb{R}^{\mathbb{N}} \qquad (\Omega, F, P)$$

Processus iid

X_i sont des variables aléatoires

$$P(t,x/(t_0,x_0),(t_1,x_1),(t_2,x_2)\dots,(t_5,x_5)) = P(t,x/t_5,x_5)$$

$$X = (X_t)_{t\in[0,T]}$$



$X_t(\omega)$ continu en temps.

$X_t(\omega_2)$ également continu.

Processus d'Itô ≡ processus Markovien continu en temps.

$$\boxed{\frac{d\underline{x}}{dt} = \underline{m}(\underline{x}) \iff \underline{x}(t) = \underline{x}(0) + \boxed{\int_0^t \underline{m}(\underline{x}(s))\,ds}}$$

intégrale de Riemann.

$$\frac{dx}{dt} = m(x) + \eta_t$$

$$x(t) = x(0) + \int_0^t m(x_s)\,ds + \boxed{\int_0^t \eta_s\,ds}$$

X un processus. $(X_t)_{t\in[0,T]}$

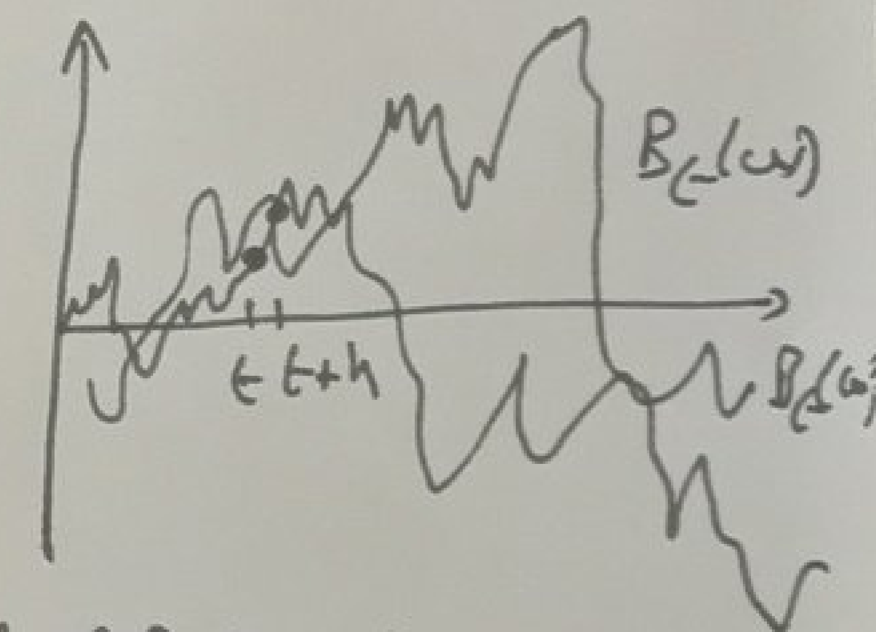
$$\boxed{X_T = X_0 + \int_0^t m(X_s)\,ds + \boxed{\int_0^t \sigma(X_s)\,dB_s}}$$

intégrale d'Itô.

$$\boxed{dB_t = \sqrt{\delta t}\cdot\xi \qquad \xi \sim \mathcal{N}(0,1)}$$

$$B_t = \overset{0}{\overset{\shortparallel}{B_0}} + \boxed{\int_0^t dB_s}$$

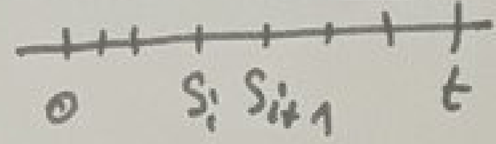
Mouvement Brownien.



$$Var(B_t) = t$$

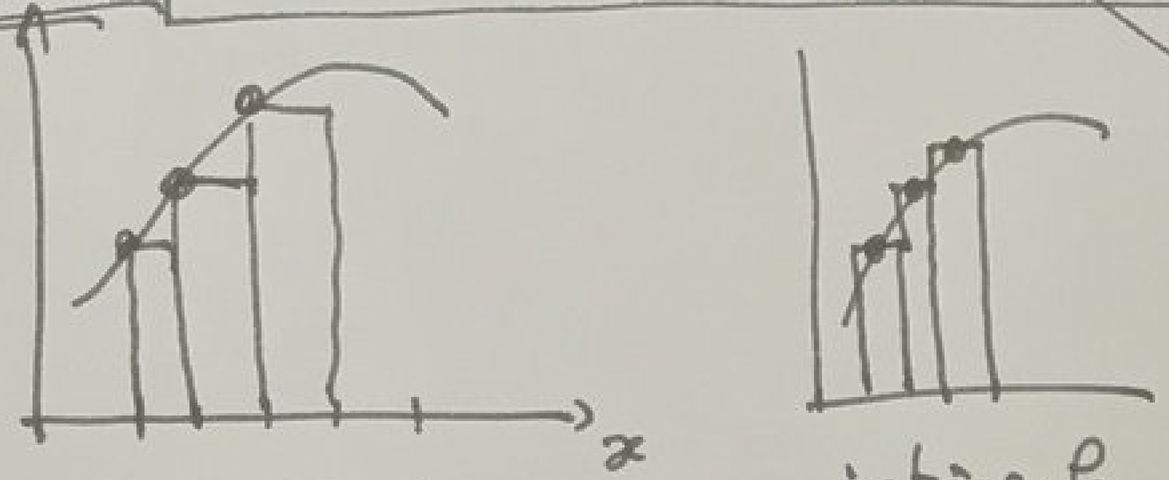
$$Var(B_{t+h} - B_t) = h$$

$$\boxed{Y_t = \int_0^t f_s \, dB_s = \lim_{\Delta s \to 0} \sum_i f_{s_i} \left(B_{s_{i+1}} - B_{s_i}\right)} \quad \boxed{\text{Intégrale d'Itô}}$$



$B_{s_i} = B$

Rappel Riemann



intégrale de Riemann $\Leftrightarrow$ intégrale de Riemann.

$\xi \sim \mathcal{N}(0,1)$

$Y = \xi^2$

$Y(\omega) = [\xi(\omega)]^2$

$f_{s_i} = F(B_0, B_{s_1}, \dots, B_{s_i})$

Pour ω donné $\boxed{f_s}(\omega)$ dans ce cadre

f_s est une fonction qui s'exprime à l'aide du processu $(B_u)_{0 \le u \le s}$

$B = (B_t)_{t \in [0,T]}$ on peut utiliser (B_t) pour construire des variable aléatoire

f_s est un processus compatible avec B $\Leftrightarrow$ $\boxed{f_s = F((B_u)_{u \le s})}$

Pour f un processus adapté au Brownien B

$$\boxed{Y_t = \int_0^t f_s \, dB_s = \lim_{\Delta s \to 0} \sum_i f_i \, \Delta B_i}$$

$\xi_i \sqrt{\Delta s}$

on vient de définir

$$\boxed{X_t = X_0 + \int_0^t m(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s}$$

Equation différentielle Stochastique !

$X = (X_t)_{t \in [0,T]}$ est un processus d'Itô

convention : notation différentielle

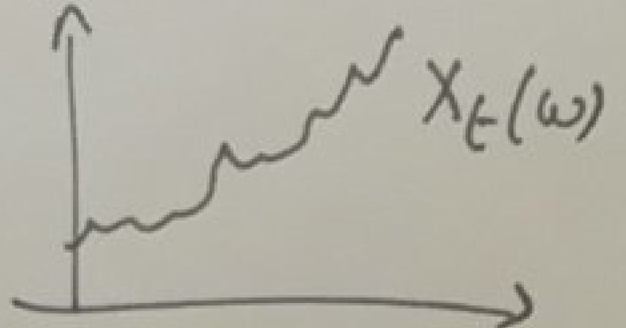
$$\boxed{dX_t = m(X_t)\,dt + \sigma(X_t)\,dB_t}$$

stochastique

(~ $\frac{dx}{dt} = m(x)$ déterministe)

$\Rightarrow$ X est un processus Markovien

et $X_t(\omega)$ est p.s. continue



$$dX_t = m(X_t)\,dt + \sigma(X_t)\,dB_t$$

$\uparrow$

$X_{t+dt} - X_t$

$$\boxed{X_{t+dt} = X_t + m(X_t)\,dt + \sigma(X_t)\,\underbrace{dB_t}_{\sqrt{dt}}}$$

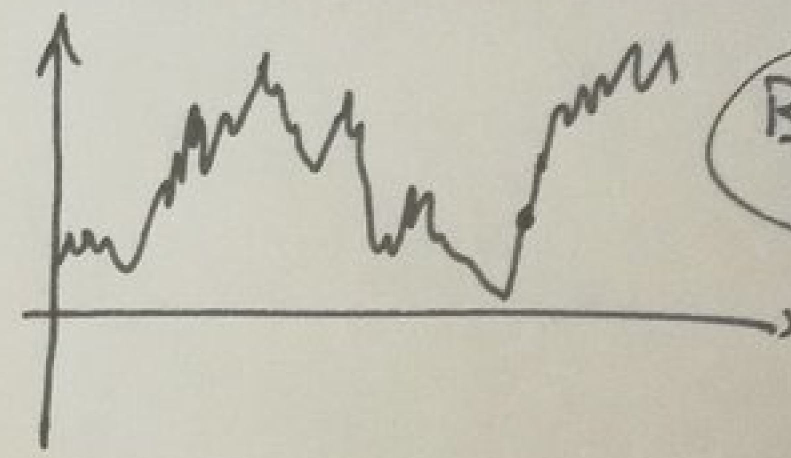
$\simeq$ Taylor

Rappel Taylor

$$\boxed{f(x+h) = f(x) + f'(x)\,h + f''(x)\,\frac{h^2}{2} + \dots \qquad \boxed{h \gg h^2 \gg h^3 \gg \dots}}$$

$dt \to 0$ quid de dt versus $\sqrt{dt}$

$$\boxed{X_{t+dt} = X_t + \sigma(X_t)\,\underbrace{dB_t}_{\sqrt{dt}} + m(X_t)\,\underbrace{dt}_{(\sqrt{dt})^2}}$$



$B_t(\omega)$

continu

$$\boxed{B_{t+dt} = B_t + \underbrace{\xi}_{\sim \mathcal{N}(0,1)}\sqrt{dt}}$$

$$\lim_{dt \to 0} \frac{|B_{t+dt} - B_t|}{dt} = \frac{|\xi|}{\sqrt{dt}} = +\infty$$

B_t continu mais pas differentiable (p.s.)

Problème : Calculer $\int_0^L x^2 dx = \left[\frac{x^3}{3} \right]_0^L = \frac{L^3}{3} - 0$

$$\uparrow$$

$$\lim_{\Delta x \to 0} \sum x_i^2 \Delta x_i$$

<u>étape 1</u>

$y = f(x)$

avec $f(x) = \frac{x^3}{3}$

<u>étape2</u>

$dy = x^2 dx$

$\int dy = \int x^2 dx \iff y_L - y_0 = \left[\frac{x^3}{3} \right]_0^L$

<u>Formule fondamentale du calcul intégrale</u>

$$\boxed{y = f(x)} \quad \frac{x^3}{3}$$

$$\downarrow \qquad\qquad \updownarrow$$

$$\boxed{dy = f'(x) dx} \quad x^2$$

Pour l'intégrale stochastique... on fait "pareil"

Soit (X_t) un processus d'Itô $\iff$ $\boxed{dX_t = m(X_t)dt + \sigma(X_t) dB_t}$

$Y_t = f(X_t)$ <u>hyp.</u> Y_t processus d'Itô.

$$\uparrow$$

une fonction "sympathique"

f au moins C^1

$$dY_t = \underset{a_t}{a\cancel{(Y_t)}}\, dt + \underset{b_t}{b\cancel{(Y_t)}}\, dB_t$$

on souhaite exprimer $\boxed{dY_t = a(Y_t)\,dt + b(Y_t)\,dB_t}$

$$dY_t \equiv Y_{t+dt} - Y_t = f(X_{t+dt}) - f(X_t)$$

$$= f(X_t + dX_t) - f(X_t)$$

$$= \cancel{f(X_t)} + f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,dX_t \cdot dX_t - \cancel{f(X_t)}$$

$$\boxed{dY_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,dX_t \cdot dX_t}$$

f est C^2 en x

$$(dX_t)^2 = \left(m(X_t)\,dt + \sigma(X_t)\,dB_t \right)^2$$

$$= \underbrace{\cancel{(m(X_t)\,dt)^2}}_{\cancel{dt^2}} + \cancel{2\,m(X_t)\sigma(X_t)\underbrace{dt \cdot dB_t}_{dt^{3/2}}} + \boxed{\underbrace{\sigma(X_t)^2 (dB_t)^2}_{dt}}$$

$dt \cdot dt = 0$ $\quad (o(dt))$

$dt \cdot dB_t = dB_t \cdot dt = 0$ $\quad (o(dt))$

$dB_t \cdot dB_t = dt$

$\mathbb{E}[\xi^2] = 1$

$$dB_t = \xi\sqrt{dt} \iff (dB_t)^2 = \xi^2\,dt$$

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$$\boxed{dY_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\, dX_t \cdot dX_t}$$

$dX_t \cdot dX_t = \sigma^2(X_t)\,dt$

$$= f'(X_t)\left[m(X_t)\,dt + \sigma(X_t)\,dB_t\right] + \frac{1}{2} f''(X_t)\,\sigma^2(X_t)\,dt$$

$$\boxed{dY_t = \left[m(X_t) f'(X_t) + \frac{1}{2} f''(X_t)\cdot \sigma^2(X_t)\right] dt + f'(X_t)\,\sigma(X_t)\,dB_t}$$

$\cancel{a(X_t)}$
a_t $\qquad$ b_t

$$\int_0^x e^{-\frac{t^2}{2}}\,dt \qquad \boxed{dY_t = a_t\,dt + b_t\,dB_t} \qquad \begin{matrix} a_t = 0 \\ b_t = B_t \end{matrix}$$

<u>Exemple</u> $\displaystyle\int_0^t B_s\,dB_s =$

étape 1 : identifier X et f.
étape 2 : $Y = f(X)$

$$Y_t = f(B_t) = \frac{B_t^2}{2} \qquad f(x) = \frac{x^2}{2}$$

ici $X_t = B_t$

$$dX_t \hookrightarrow dB_t = \overset{m}{0}\cdot dt + \overset{\sigma}{1}\,dB_t$$

$$dY_t = \left[\cancel{0 B_t} + \frac{1}{2}\right] dt + B_t\,dB_t$$

or $f'(x) = x$
$f''(x) = 1$

$$\boxed{d\left(\frac{B_t^2}{2}\right) = \left(\cancel{0 B_t} + \frac{1}{2}\right) dt + B_t\,dB_t}$$

$\displaystyle\int_0^t x\,dx =$

$f(x) = \frac{x^2}{2}$

$dy = f'(x)\,dx$
$= x\,dx$