Plan du cours

I Filtrage stochastique

→ filtrage non linéaire ≡ Formule de Bayes
→ Filtre Kalman
→ filtre particulaire

II Prévision deterministe

→ évolution distribution de proba

dualité $\boxed{\text{"observable"} \leftrightarrow \text{proba}}$

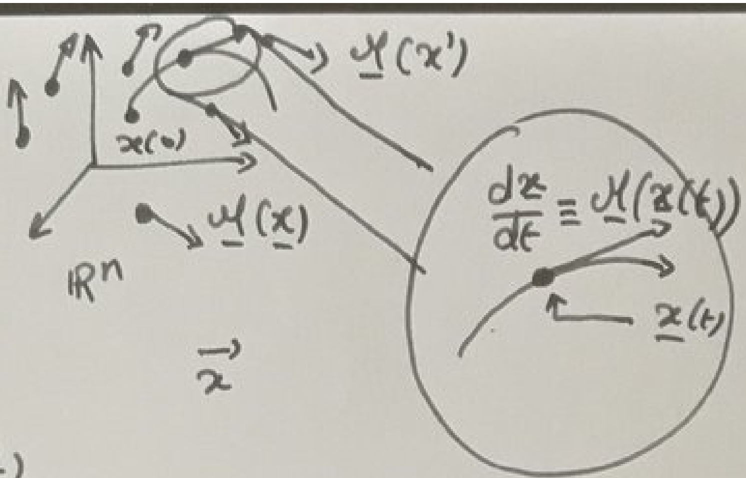
III Prévision Stochastique

→ integrale d'Itô
→ ——— de Stratonovitch
→ 4DVar dans une formulation c°.

Exercice : obtention des equations du filtre de Kalman !

I] Filtrage Sto

& $\underline{x}(t)$



$\vec{x}$

(1) $\boxed{\frac{d\underline{x}}{dt} = \underline{\mathcal{M}}(\underline{x})}$

$\underline{x}(0) \longrightarrow \underline{x}(t)$

$\underline{\underline{A}} = [a_{ij}]$

cas 1

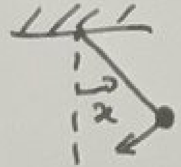
(2) $\boxed{\frac{d\underline{x}}{dt} = \underline{\underline{M}}\,\underline{x}} \longrightarrow \boxed{\underline{x}(t) = e^{\underline{\underline{M}}t}\,\underline{x}(0)}$ (3)

cas où $\underline{\mathcal{M}}(\underline{x}) = \underline{\underline{M}}\,\underline{x}$

cas 2 général

$\underline{\mathcal{M}}(\underline{x})$ non nécessairement linéaire

(4) $\boxed{\ddot{x} + \omega^2 \sin(x) = 0}$



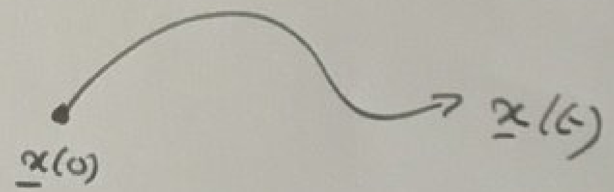
$x(t)$

$$\underline{x} \left| \begin{array}{l} x_1 \equiv x \\ x_2 \equiv \dot{x} \end{array} \right.$$

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = \frac{d\dot{x}}{dt} = \ddot{x} = -\omega^2 \sin(x) \end{cases}$$

$$\boxed{\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\omega^2 \sin(x_1) \end{cases}} \quad (5)$$

$$\boxed{\underline{\mathcal{M}}(\underline{x}) \left| \begin{array}{l} x_2 \\ -\omega^2 \sin(x_1) \end{array} \right.} \quad (6)$$



$\underline{x}(t) \longrightarrow \underline{x}(t+dt) \longrightarrow \underline{x}(t+2dt)$

$o(\delta t)$

$\underline{x}(t+dt) = \underline{x}(t) + dt\,\frac{d\underline{x}(t)}{dt} + O(\delta t^2)$

(7) $$\boxed{\underline{x}(t+\delta t) = \underline{x}(t) + \delta t \underbrace{\frac{d\underline{x}}{dt}(t)}_{\equiv\ \underline{\mathcal{M}}(\underline{x}(t))} + o(\delta t)}$$

$$y_q \approx \underline{x}(t_q) \qquad t_q = q\,\delta t$$

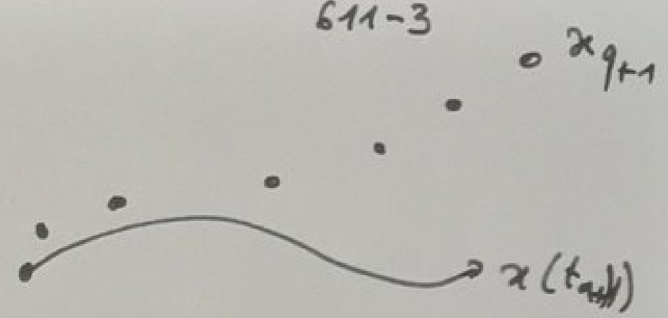
(8) $$\boxed{y_{q+1} = y_q + \delta t\, \underline{\mathcal{M}}(y_q)}$$ Euler

(9) $$\boxed{\underline{x}(t+\delta t) = \underline{x}(t) + \delta t \underbrace{\frac{d\underline{x}}{dt}}_{\underline{\mathcal{M}}(\underline{x}(t))} + \frac{\delta t^2}{2} \boxed{\frac{d^2\underline{x}}{dt^2}} + \dots}$$

Passage Scalaire

$$\frac{d^2\underline{x}}{dt^2} = \frac{d}{dt}\left[\frac{d\underline{x}}{dt}\right] = \frac{d}{dt}\left[\underline{\mathcal{M}}(\underline{x})\right] \approx \frac{dx}{dt}\cdot \mathcal{M}'(x)$$

(10) $$\boxed{\begin{cases} y_* = y_q + \frac{\delta t}{2}\mathcal{M}(y_q) \\ y_{q+1} = y_q + \delta t\, \mathcal{M}(y_*) \end{cases}}$$ RK2



x_q : état du système à l'instant t_q

y_q : observation du système à l'instant t_q.

$$t \longrightarrow t+T$$

$$\begin{array}{ccccc} x_q & \longrightarrow & x_{q+1} & \longrightarrow & x_{q+2} \\ & & \uparrow & & \\ & & y_{q+1} & & \end{array}$$

$$\boxed{\begin{array}{c} x_q \\ \uparrow \\ \text{obs.} \end{array}} \longrightarrow \begin{array}{c} x_{q+1} \\ \uparrow \\ + \text{obs} \end{array} \longrightarrow$$

cycle d'assimilation et de prévision.

$\cancel{p(x_q)}$

$p(x_q / y_{0:q-1})$

$$\boxed{p(x_q / y_{0:q}) \propto p(y_q / x_q) \cdot p(x_q / y_{0:q-1})}$$

Formule de Bayes

Cas Gaussien

forecast

$$p(x_q / y_{0:q-1}) \equiv \mathcal{N}(x_q^f, P_q^f)$$

$$y_q = H x_q + \varepsilon_q^o \quad \text{où} \quad \varepsilon_q^o \sim \mathcal{N}(0, \underline{\underline{R}}_q)$$

erreur d'observation

$$p(y_q / x_q) \equiv \mathcal{N}(H x_q, \underline{\underline{R}}_q)$$

$$\boxed{\cancel{\mathcal{N}(\underline{m}, \underline{\underline{P}}) = \frac{1}{Z} e^{\frac{1}{2}(v-m)^T \underline{\underline{P}} (v-m)}}} \quad \underline{m}^T \underline{\underline{P}}^{-1} \underline{m} \equiv \| m$$

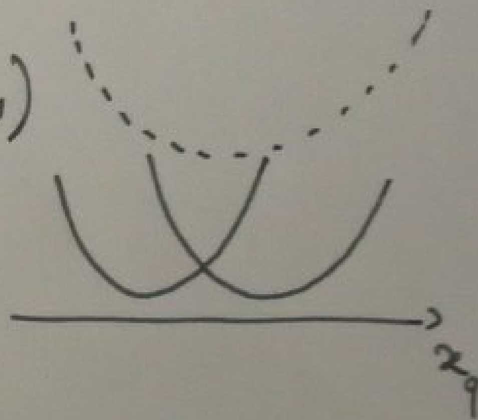
$\cancel{\mathcal{N}(\underline{m}, \underline{\underline{P}}) = \frac{1}{Z} \text{ex}}$

$$\underline{V} \sim \mathcal{N}(\underline{m}, \underline{\underline{P}})$$

$$p(\underline{v}) = \frac{1}{Z} \exp\left(\frac{1}{2} \| \underline{v} - \underline{m} \|^2_{\underline{\underline{P}}^{-1}} \right)$$

$$\text{où} \quad \| \underline{v} - \underline{m} \|^2_{\underline{\underline{P}}^{-1}} \equiv (\underline{v} - \underline{m})^T \underline{\underline{P}}^{-1} (\underline{v} - \underline{m})$$

$$p(x_q / y_{0:q}) \underset{\substack{\uparrow \\ \text{proportionnel}}}{\propto} p(y_q / x_q) \cdot p(x_q / y_{0:q-1})$$



$$p(x_q / y_{0:q}) \propto \exp\left(-\tfrac{1}{2} \| y_q - H x_q \|^2_{\underline{\underline{R}}_q^{-1}}\right) \cdot \exp\left(-\tfrac{1}{2} \| x_q - x_q^f \|^2_{\underline{\underline{P}}_q^{f-1}}\right)$$

$$\propto \exp\left(-\tfrac{1}{2} \| y_q - H x_q \|^2_{\underline{\underline{R}}_q^{-1}} - \tfrac{1}{2} \| x_q - x_q^f \|^2_{\underline{\underline{P}}_q^{f-1}}\right)$$

$$\propto \exp\left(-\tfrac{1}{2} \| x_q - x_q^a \|^2_{(\underline{\underline{P}}^a)^{-1}}\right)$$

$$J(x_q) = \tfrac{1}{2} \| x_q - x_q^f \|^2_{(\underline{\underline{P}}_q^f)^{-1}} + \tfrac{1}{2} \| y_q - H x_q \|^2_{\underline{\underline{R}}_q^{-1}}$$

$$= \tfrac{1}{2} \| x_q - x_q^a \|^2_{(\underline{\underline{P}}_q^a)^{-1}} + C$$

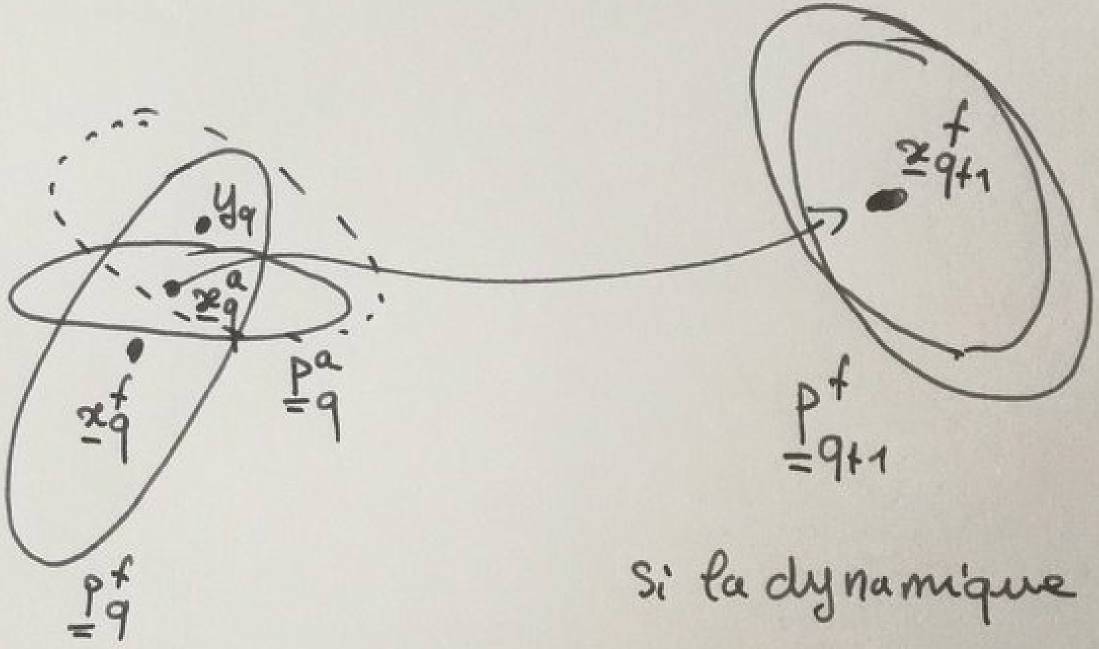
$$\boxed{\begin{aligned} &\nabla J_{x_q^a} = 0 \rightarrow x_q^a \\ &J'' = (P_q^a)^{-1} \end{aligned}}$$

(1) $\boxed{\underline{x}_q^a = \underline{x}_q^f + \underline{\underline{K}}_q (y_q - \underline{\underline{H}}\, \underline{x}_q^f)}$

où $\boxed{\underline{\underline{K}}_q = \underline{\underline{P}}_q^f \underline{\underline{H}}_q^T \left(\underline{\underline{H}}_q \underline{\underline{P}}_q^f \underline{\underline{H}}_q^T + \underline{\underline{R}}_q\right)^{-1}}$

(2) $\boxed{\underline{\underline{P}}_q^a = (\underline{\underline{I}} - \underline{\underline{K}}_q \underline{\underline{H}})\, \underline{\underline{P}}_q^f}$

Filtre Kalman pour l'analyse



Si la dynamique s'exprim

$$\underline{x}_{q+1} = \underline{\underline{M}}\, \underline{x}_q$$

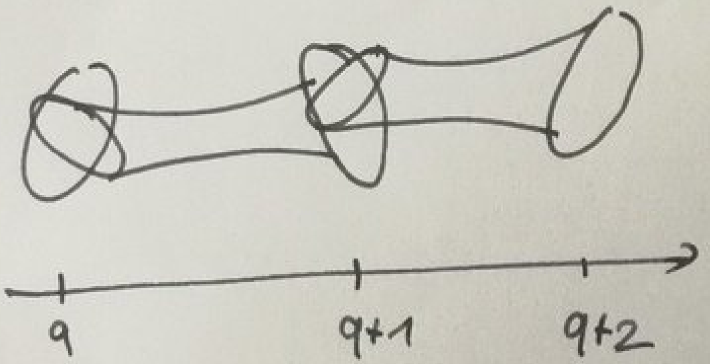
$x^f \leftarrow$ forecast

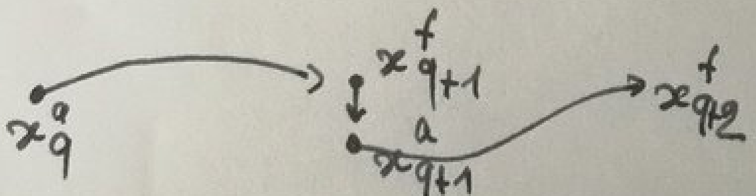
$x^a \leftarrow$ analysis.

$$(3) \quad \boxed{\underline{x}_{q+1}^f = \underline{\underline{M}}\, \underline{x}_q^a}$$

$$(4) \quad \boxed{\underline{\underline{P}}_{q+1}^f = \underline{\underline{M}}\, \underline{\underline{P}}_q^a\, \underline{\underline{M}}^T}$$

Kalman pour prévision





$$\left\{\begin{array}{l}\underline{x}_q^f \\ \underline{\underline{P}}_q^f\end{array}\right. \xrightarrow[\uparrow\, y_q]{\text{analyse}} \left\{\begin{array}{l}\underline{x}_q^a \\ \underline{\underline{P}}_q^a\end{array}\right. \xrightarrow{\text{prevision}} \left\{\begin{array}{l}\underline{x}_{q+1}^f \\ \underline{\underline{P}}_{q+1}^f\end{array}\right.$$

$$\cdots \longrightarrow \mathcal{N}(\underline{x}_q^f, \underline{\underline{P}}_q^f) \longrightarrow \mathcal{N}(\underline{x}_q^a, \underline{\underline{P}}_q^a) \longrightarrow \mathcal{N}(\underline{x}_{q+1}^f, \underline{\underline{P}}_{q+1}^f) \longrightarrow \cdots$$

$\ddot{x} + \omega^2 x = 0 \longrightarrow x(t) = a\cos(\omega t+\varphi)$ (a, φ) données à partir de $(x(0), \dot{x}(0))$

$$\underline{y} \equiv \left| \begin{array}{l} x \\ \dot{x} \end{array}\right. \qquad \dot{\underline{y}} = \left| \begin{array}{l} \dfrac{dy_1}{dt} = \dot{x} \equiv y_2 \\[2mm] \dfrac{dy_2}{dt} = -\omega^2 x \equiv -\omega^2 y_1 \end{array}\right.$$

$$\underline{y} = \left| \begin{array}{l} y_1 \\ y_2 \end{array}\right. \qquad \frac{d\underline{y}}{dt}$$

$$\boxed{\frac{d\underline{y}}{dt} = \underline{f}(\underline{y})}$$

$$\underline{f}(\underline{y}) \left| \begin{array}{l} f_1(\underline{y}) \equiv y_2 \\ f_2(\underline{y}) \equiv -\omega^2 y_1 \end{array}\right.$$

mais comme $\underline{f}$ est linéaire <u>ici</u>

on peut écrire

$$\boxed{\frac{d\underline{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \underline{y}}$$

$$\boxed{\frac{d\underline{y}}{dt} = \underline{\underline{A}}\, \underline{y}}$$

Solution ?

$$\boxed{\underline{y}(t) = \underline{y}(0)\, e^{\underline{\underline{A}} t}}$$

$\cancel{\underline{y}(t) = \left| \cos(\omega t + \right.}$

$$\underline{y}(t) = \left| \begin{array}{l} a\cos(\omega t + \varphi) \\ -\omega a \sin(\omega t + \varphi) \end{array}\right.$$

$$\boxed{\frac{dy}{dt} = \underline{\underline{A}}\, y}$$

$$\boxed{\underline{\underline{M}}_{t \leftarrow 0} \equiv e^{\underline{\underline{A}} t}}$$ propagateur

<u>Solution :</u>

proposition 1 : $y(t) = \cancel{y(0)}\, e^{\underline{\underline{A}}t}$

proposition 2 : $\boxed{y(t) = e^{\underline{\underline{A}}t}\, y(0)}$

$$\boxed{y(t) = \underline{\underline{M}}_{t \leftarrow 0}\, y(0)}$$

cas scalaire :

$$\frac{dy}{dt} = a y \rightarrow y(t) = e^{at} y(0) = y(0) e^{at}$$

avec le formalisme de l'assimilation

$$x_{q+1} = M_{q+1 \leftarrow q}\, x_q = M_{q+1 \leftarrow 0}\, x_0$$

$$e^{\underline{\underline{A}}t}$$

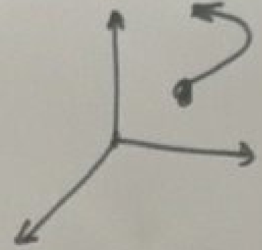
$\underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}$ cas $\underline{\underline{A}}$ est diagonalisable

$$e^{\underline{\underline{A}}t} = e^{\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1} t} = \sum \frac{t^k}{k!} (\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1})^k = P\left(\sum \frac{t^k}{k!} \underline{\underline{D}}^k\right) P^{-1} = P e^{\underline{\underline{D}}t} P^{-1}$$

$$(PDP^{-1})^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{P^{-1}P}_{I}DP^{-1} = PD^2P^{-1}$$

$\frac{dy}{dt} = \underline{\underline{A}}\, y$ où $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$

$\frac{dy}{dt} = f(y) \longrightarrow$



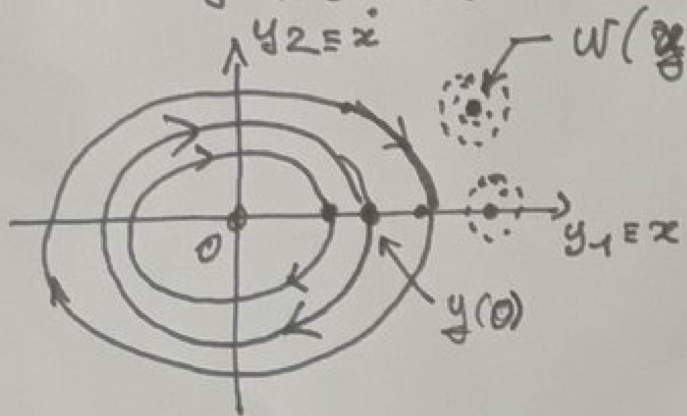
formellement $y(t) = e^{\underline{\underline{A}}t}\, y(0)$

ie $y(t) = \underline{\underline{M}}_{t\leftarrow 0}\, y(0)$

avec $\underline{\underline{M}}_{t\leftarrow 0} \equiv e^{\underline{\underline{A}}t}$

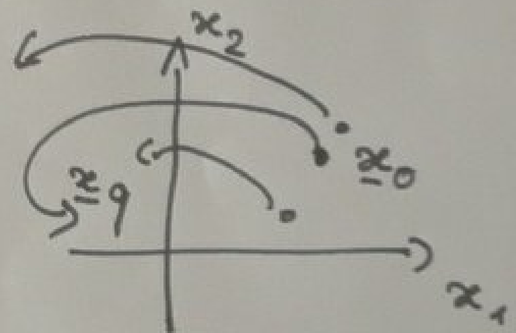
Pour $x(t) = a\cos(\omega t + \varphi)$

dans le plan $y = (y_1, y_2)$ on a

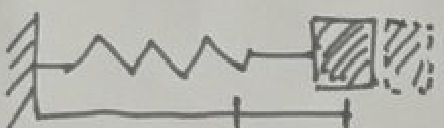


$\longleftrightarrow$

Assimilation:



$\dot{x}(0) = 0$



$\Leftrightarrow \quad \ddot{x} + \omega^2 x = 0$

$E = \omega^2 x^2 + \dot{x}^2$

$$\boxed{E(y) = \omega^2 y_1^2 + y_2^2 \equiv c^{te} \equiv \omega^2 x(0)^2 + \dot{x}(0)^2}$$

13-11-4

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \qquad \underline{x} = \begin{vmatrix} x \\ \dot{x} \end{vmatrix} \Leftrightarrow \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$$E = \omega^2 x_1^{\,2} + x_2^{\,2}$$
$$= x_1^{\,2} + x_2^{\,2} = C^{te}$$

cas où $\omega = 1$

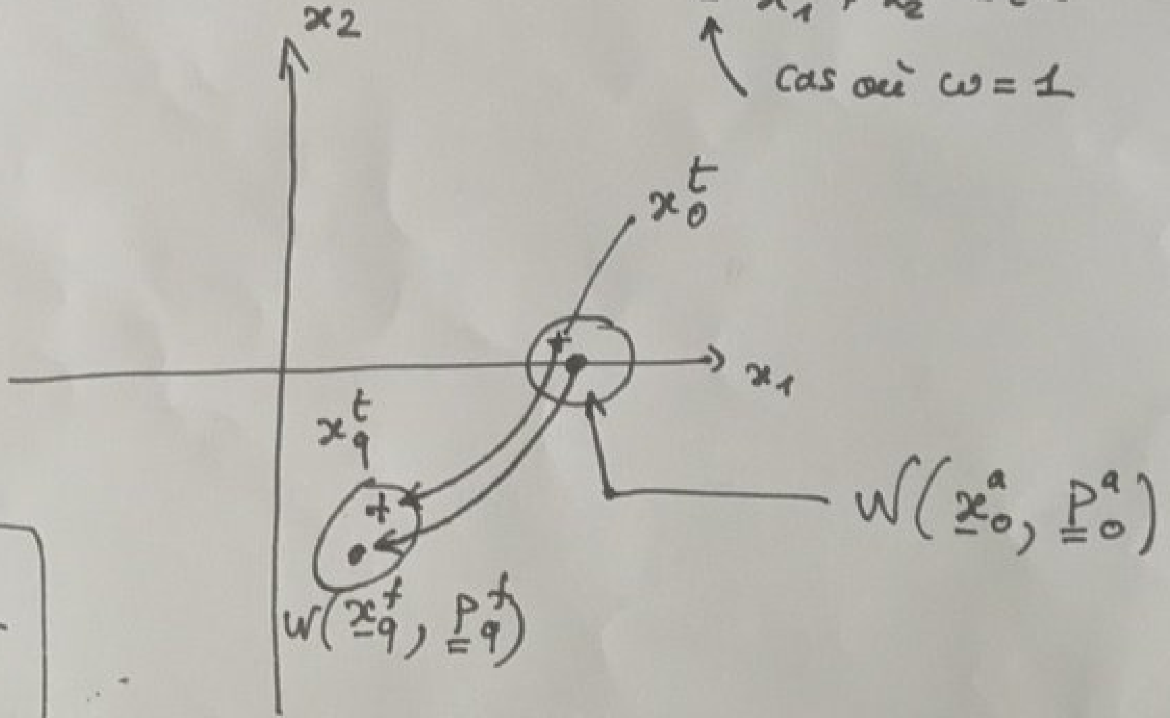
$$\underline{x}(t) = \underline{\underline{M}}_{t\leftarrow 0}\,\underline{x}(0)$$

$$\underline{\underline{M}}_{t\leftarrow 0} = e^{\underline{\underline{A}}t}$$

$$\boxed{\mathcal{N}(\underline{x}_0^a, \underline{\underline{P}}_0^a)}$$

$$\downarrow\ t_q \leftarrow 0$$

$$\boxed{\begin{aligned} \underline{x}_q^f &= \underline{\underline{M}}_{t_q\leftarrow 0}\, x_0^a \\ \underline{\underline{P}}_q^f &= \underline{\underline{M}}_{t_q\leftarrow 0}\, \underline{\underline{P}}_0^a \left(\underline{\underline{M}}_{t_q\leftarrow 0}\right)^T \end{aligned}}$$



analyse   prévision

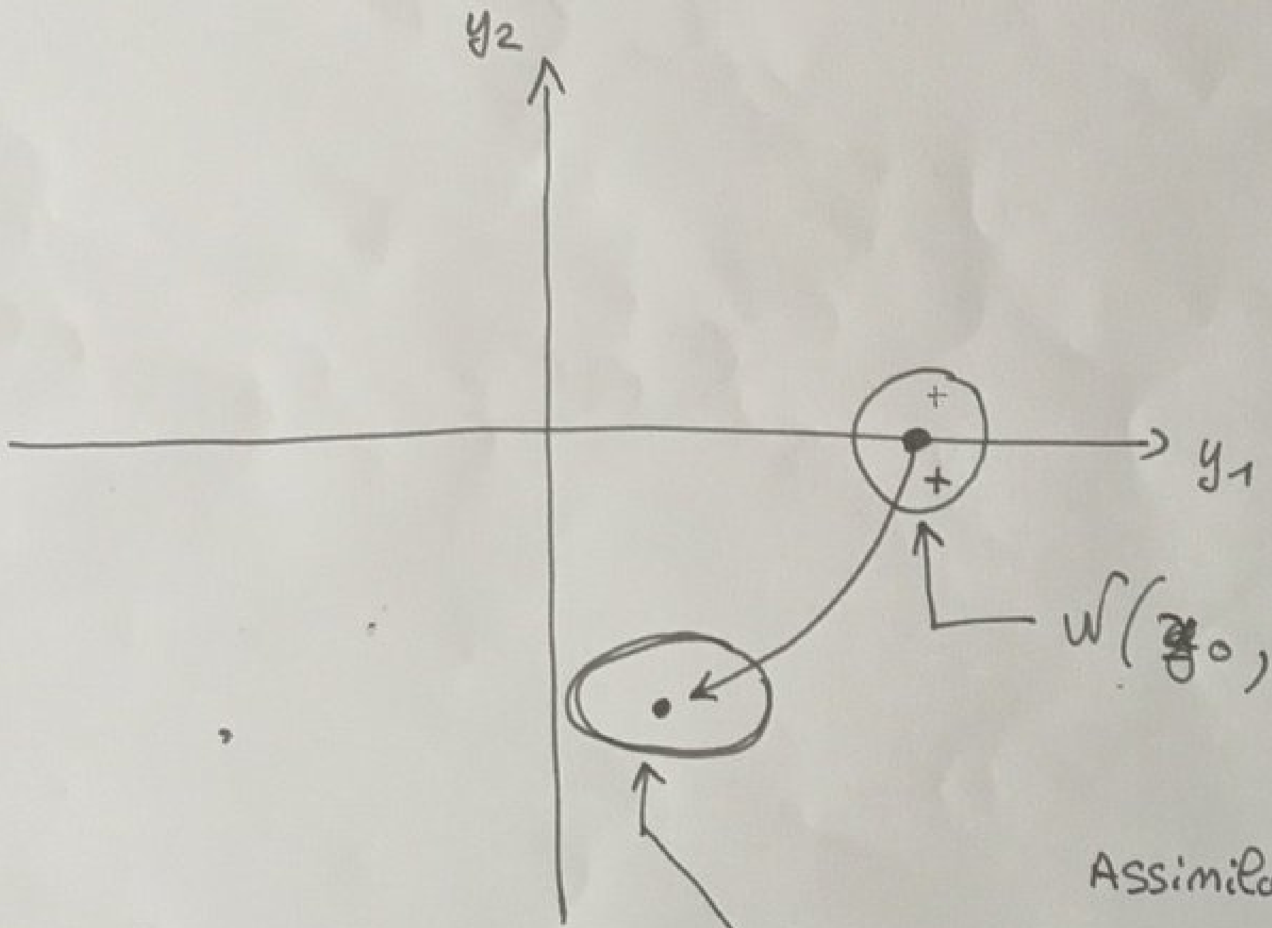
$0 \quad x_0 \longrightarrow (\underline{x}_q^f, \underline{\underline{P}}_q^f) \longrightarrow (\underline{x}_q^a, \underline{\underline{P}}_q^a) \longrightarrow (\underline{x}_{q+1}^f, \underline{\underline{P}}_{q+1}^f) \longrightarrow \ldots$

$\uparrow$ y

...

$t=0$

$(\underline{x}_0^a, \underline{\underline{P}}_0^a) \longrightarrow \quad \ldots$



$\mathcal{N}(\underline{x}_0, \underline{\underline{P}}_0)$

Assimilation

$\mathcal{N}(\underline{x}_0^a, \underline{\underline{P}}_0^a)$

$\downarrow$ prévision $0 \longrightarrow t_q$

$\boxed{\mathcal{N}(\underline{x}_q^f, \underline{\underline{P}}_q^f)}$

1311-6

$$\ddot{x} + \omega^2 x = 0$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \qquad \underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

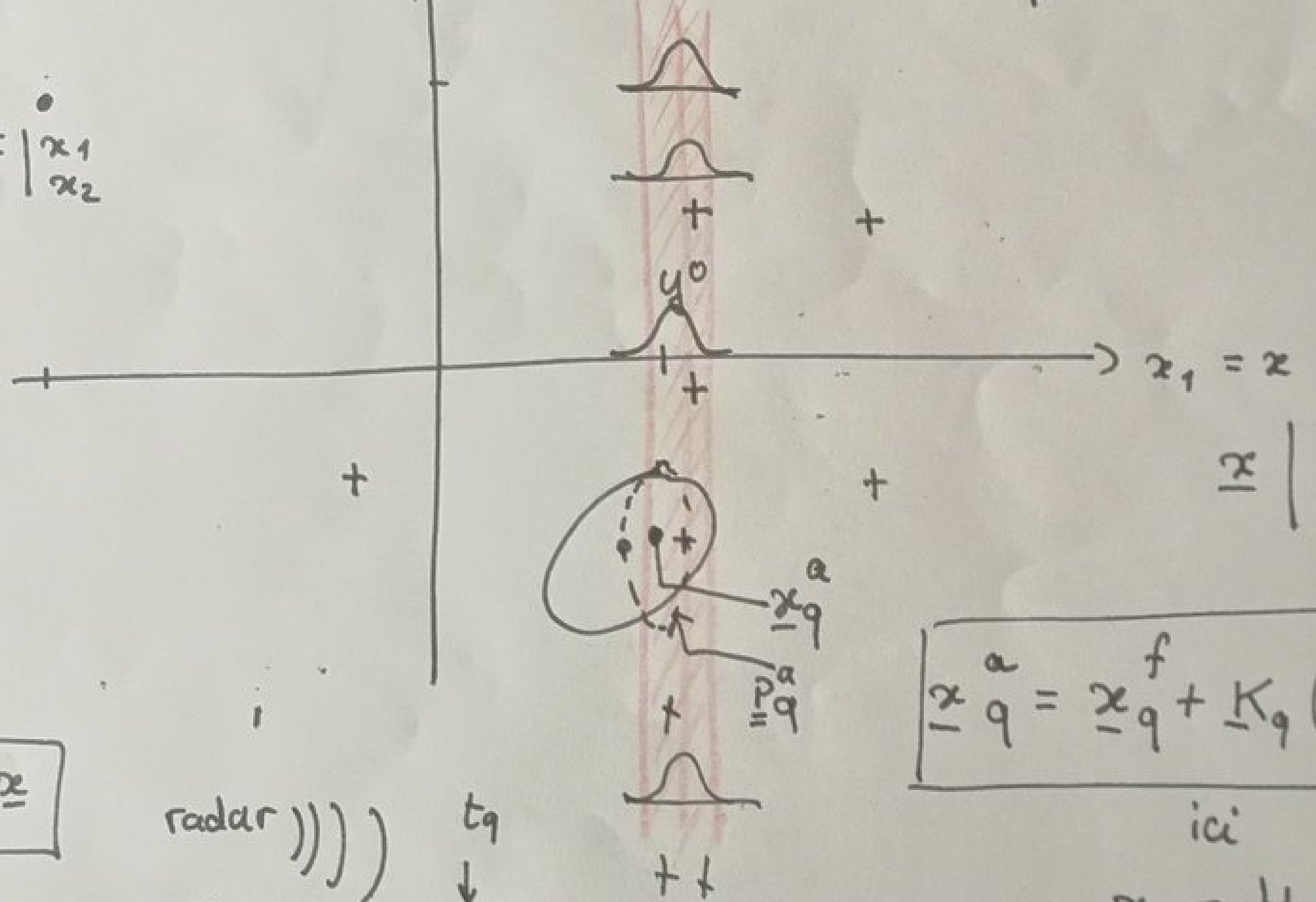
$$\underline{x}(t) = \underbrace{e^{\underline{\underline{A}}t}}_{\underline{\underline{M}}_{t \leftarrow 0}} \underline{x}(0)$$

$$\boxed{\underline{\underline{P}}^f(t) = e^{\underline{\underline{A}}t}\, \underline{\underline{P}}^f(0) \left(e^{\underline{\underline{A}}t}\right)^T}$$

$\underbrace{p(x_q^a / y_{0:q})} \propto \underbrace{p(y_q / x_q)} \cdot \underbrace{p(x_q / y_{0:q-1})}$ 1311-7

$\underline{x}_q^f = \underline{\underline{M}}_{t_q \leftarrow 0} \; x_0^a$

$\underline{x} \left| \begin{matrix} x_1 \\ x_2 \end{matrix} \right.$

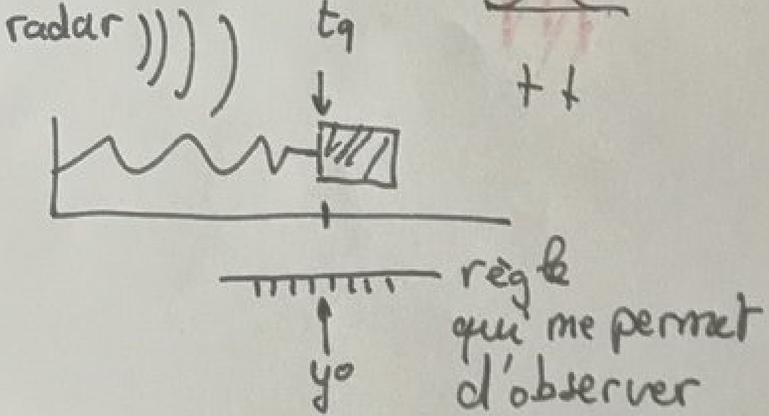


$\underline{x} \left| \begin{matrix} x_1 \\ x_2 \end{matrix} \right.$

$$\boxed{\underline{x}_q^a = \underline{x}_q^f + \underline{K}_q \left(\underline{y}_q^o - \underline{\underline{H}}_q \, \underline{x}_q^f \right)}$$

$\boxed{x_1 = \underline{\underline{H}}\, \underline{x}}$

y^o



ici

$x_1 = \underline{\underline{H}}\, \underline{x}$

avec $\underline{\underline{H}} = [1 \;\; 0]$

$x_1 = \underbrace{[1 \;\; 0]}_{\underline{\underline{H}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}}$

$$p(x_q / y_{0:q}) \propto p(y_q / x_q)\; p(x_q / y_{0:q-1})$$

$$\mathcal{N}(\underline{x}_q^f, \underline{\underline{P}}_q^f)$$

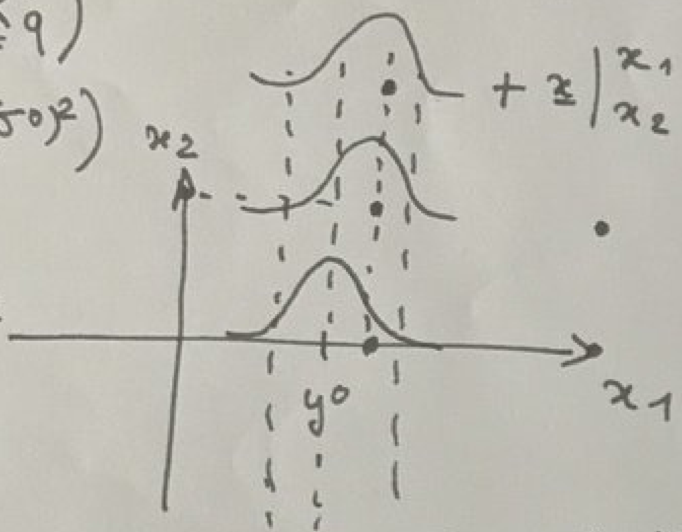
$$p(y_q / x_q) = \cancel{\quad}$$

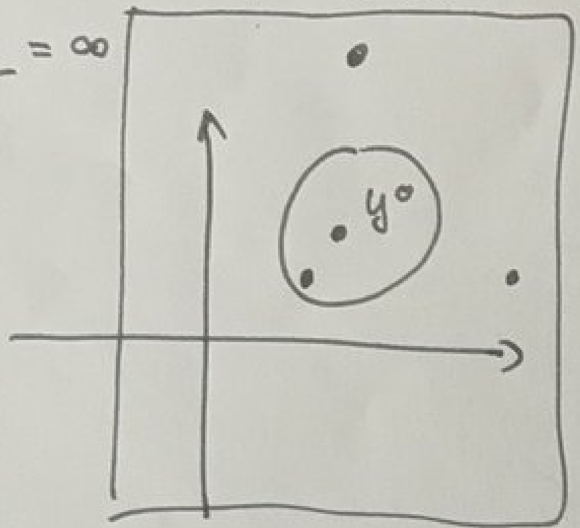
$$= \mathcal{N}(\underbrace{\underline{\underline{H}}\,\underline{x}_q^t}, \underline{\underline{R}}_q^o)$$

$$\mathcal{N}\left((x_q^t)_1, (\sigma^o)^2\right)$$

$$\iint \boxed{e^{-\frac{1}{2}\frac{1}{(\sigma^o)^2}\left(y_q^o - (x_q^t)_1\right)^2}}\, dx_1 dx_2 = \infty$$

$$= \int 1 \, dx_2 = \infty$$





$\underline{\underline{I}}$ cas où $\underline{\underline{H}} = \underline{\underline{I}}$

Si $\underline{y} \sim \mathcal{N}(\underline{\underline{H}}\,\underline{x}^t, (\sigma^o)^2\, \underline{\underline{I}}_2)$

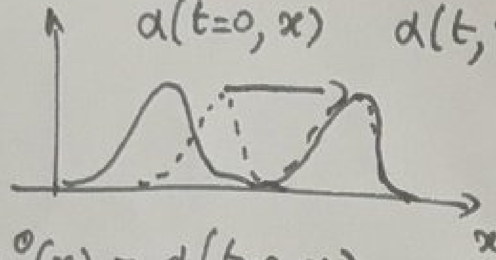
on observe à la fois la position <u>et</u> la vitesse

① $\boxed{\partial_t \alpha + u \, \partial_x \alpha = 0}$

$u = u^0$ est constant

$u^0 > 0$

transport à la vitesse u^0



$\boxed{\alpha(t,x) = \alpha^0(x - ut)}$

$\alpha^0(x) = \alpha(t=0, x)$

Condition initiale

soit 2 solutions de ①

$\alpha(t,x)$

$\beta(t,x)$

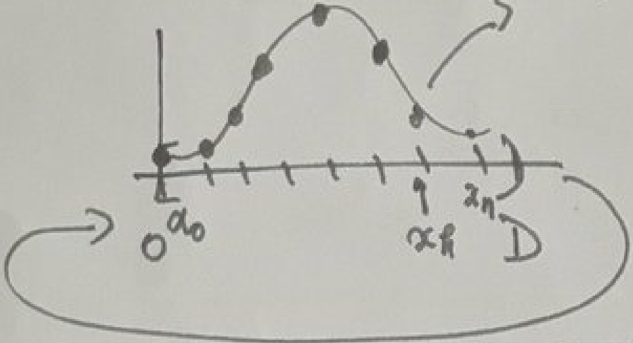
et $\nu \in \mathbb{R}$

$\gamma = \alpha + \nu \beta$

$\gamma(t,x)$ est aussi une solution de ①

$\Longrightarrow$ ① est linéaire.

$\alpha(t_q, x_k) \equiv \alpha_k^q$



discrétisation spatiale

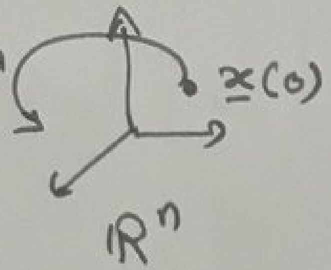
① $\Longleftrightarrow$ $\boxed{\dfrac{d\underline{x}}{dt} = \underline{\underline{M}} \, \underline{x}}$

$\underline{x} \left| \begin{array}{l} \alpha_0(t) \\ \vdots \\ \alpha_n(t) \end{array} \right.$

$\underline{x}(t) = \underline{\underline{M}}_{t \leftarrow 0} \, \underline{x}(0)$

$$\boxed{\frac{\alpha_k^{q+1} - \alpha_k^q}{\delta t} = -u \, \frac{\alpha_k^q - \alpha_{k-1}^q}{\delta x}}$$

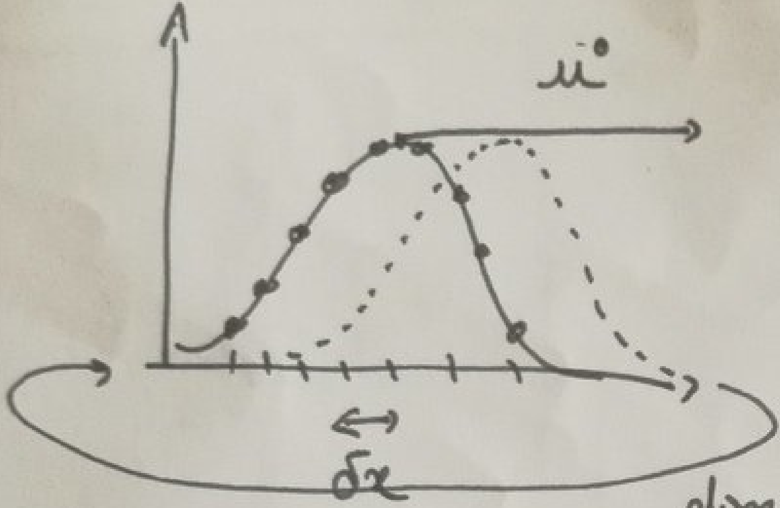
discrétisation temporelle



Euler + discrétisation spatiale

de type différence finie

1311-10

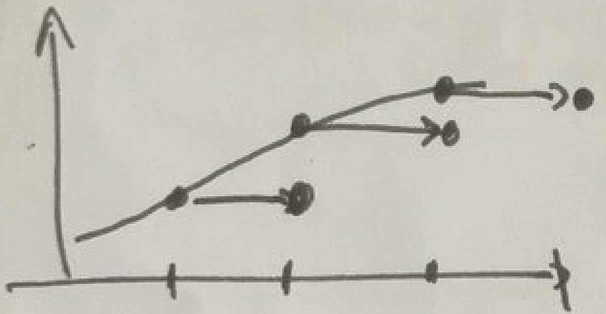


domaine périodique

$$\partial_t \alpha + u\,\partial_x \alpha = 0$$

$$\boxed{\delta t = \frac{\delta x}{u^\circ}}$$

avec ce δt



$\underline{\underline{M}} \equiv$ Matrice de permutation

$$\begin{array}{ccc}
\begin{bmatrix} \alpha_n \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} \equiv &
\begin{bmatrix} 0 & 0 & \cdots & & 1 \\ 1 & 0 & \cdots & & 0 \\ & \ddots & \ddots & (0) & \\ & (0) & & \ddots & \\ & & & 1 & 0 \end{bmatrix} &
\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix} \\
t_{q+1} \xleftarrow{+\delta t} & & t_q
\end{array}$$

construction d'une incertitude initiale

2011-1

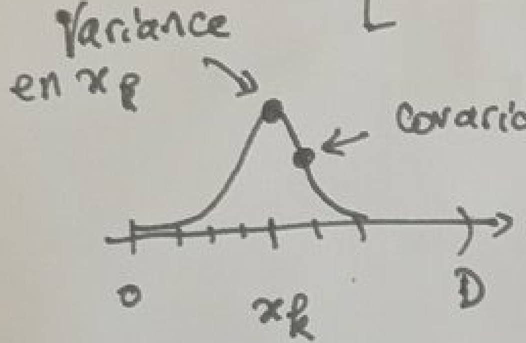
$(\underline{x}_0^a, \underline{\underline{P}}_0^a)$

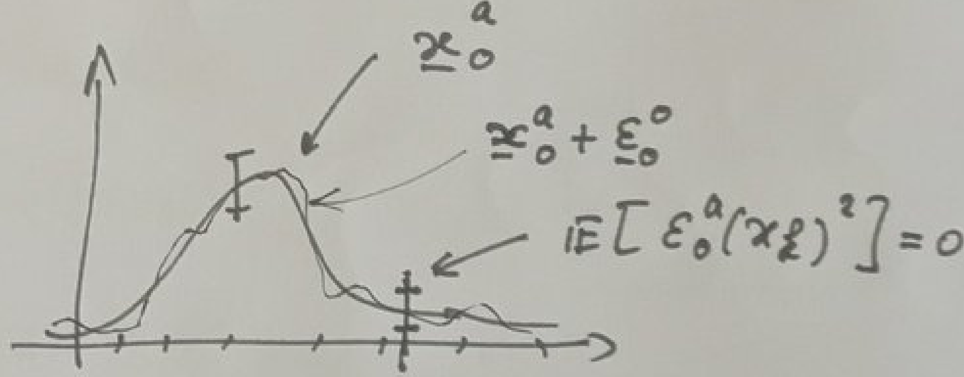
$$\begin{array}{cc} & k \text{ ième} \\ \underline{\underline{P}}_0^a = & \begin{bmatrix} & \Big| & \end{bmatrix} \end{array}$$

k ième $\equiv$ la k ième colonne est la fonction de covariance relative au point x_k

Sous $\mathbb{E}[\varepsilon_0^a] = 0$

$$P_0^a(x_k, x_{k+1}) = \mathbb{E}\left[\varepsilon_0^a(x_k) \cdot \varepsilon_0^a(x_{k+1})\right]$$



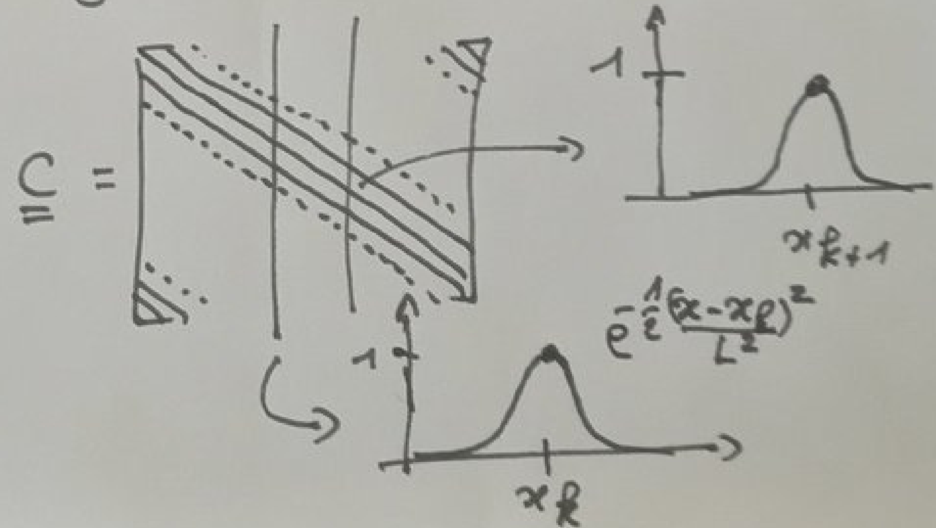


la gaussienne est une fonction de covariance

$$\boxed{\underline{\underline{P}}_0^a = \underline{\underline{\Sigma}}\ \underline{\underline{C}}\ \underline{\underline{\Sigma}}^T}$$

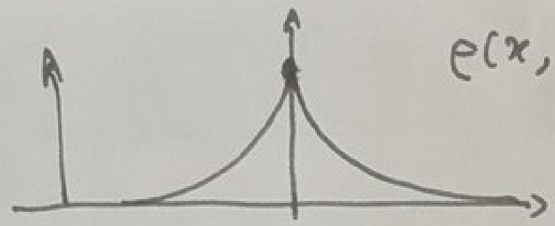
diagonale des écart-type

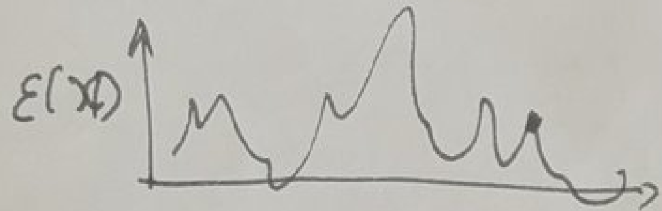
matrice de corrélation

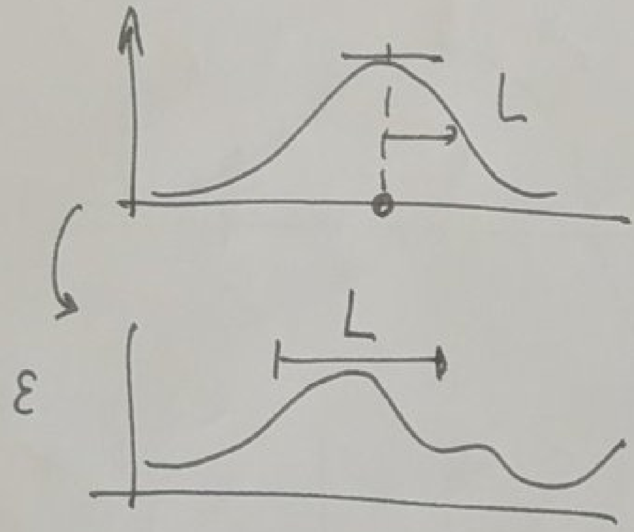


2011-1

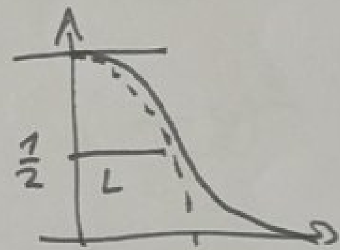
$$\rho(x,y) = e^{-\frac{1}{L}|x-y|}$$







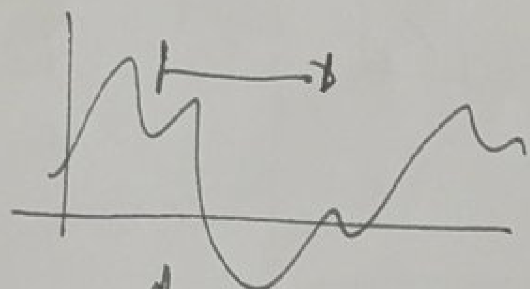
Pour corrélation lisse à l'origine

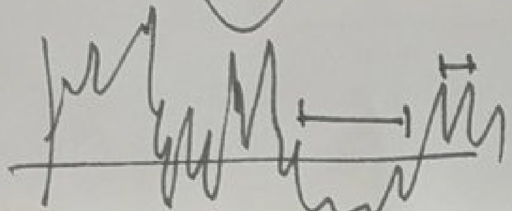


$$\rho(x,y) \underset{y\sim x}{=} e^{-\frac{1}{2}(x-y)^2\frac{1}{L^2}}$$

$$\simeq 1 - \frac{(x-y)^2}{2L^2} + o(|x-y|^2)$$

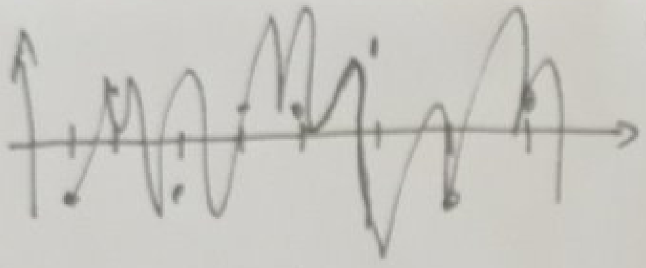
$$\rho(x, x+\delta x) \simeq 1 - \frac{1}{2}\,\frac{\delta x^2}{L^2}$$

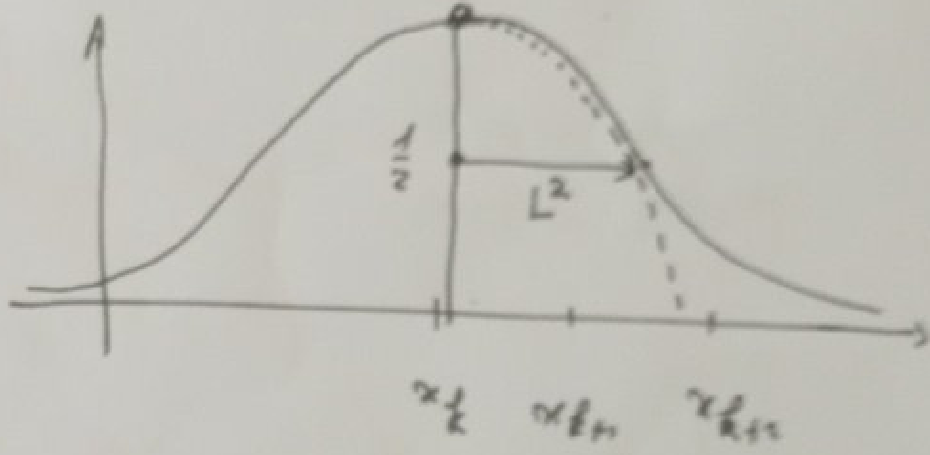




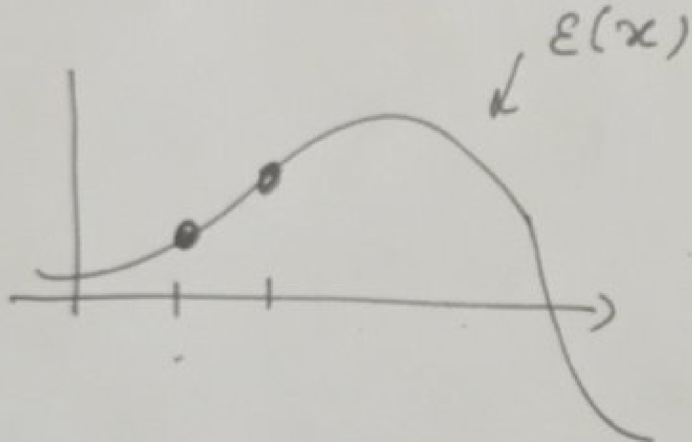
cas bruit blanc

2011-3

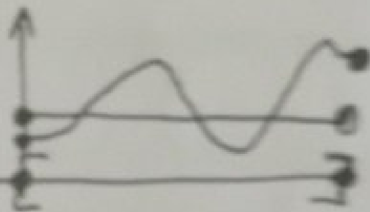


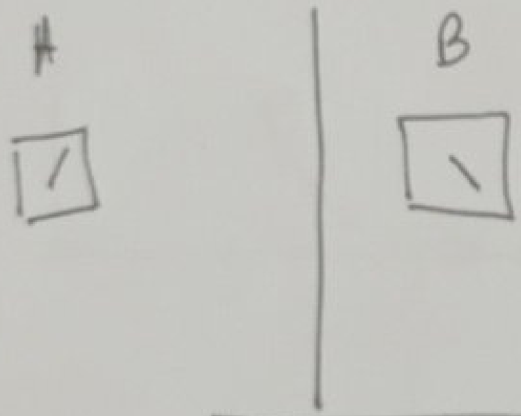


$$\underline{\varepsilon}^a \sim \mathcal{N}(\underline{0}, \underline{\underline{I}})$$

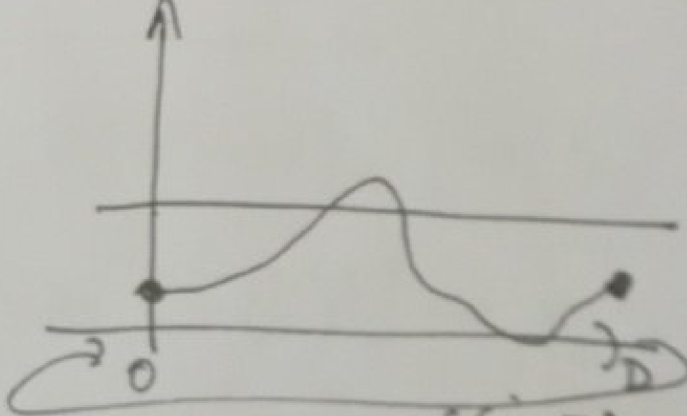


$$\rho(x, y) = \exp\left(-\frac{1}{2}\frac{(x-y)^2}{L^2}\right)$$
$$\| $$
$$\mathbb{E}[\varepsilon^a(x)\,\varepsilon^a(y)]$$





cas où $L = \infty$

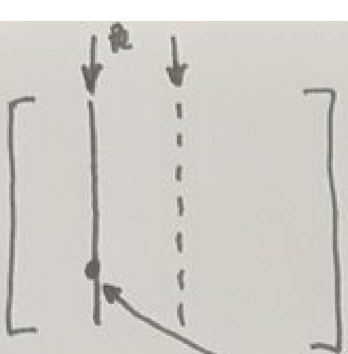


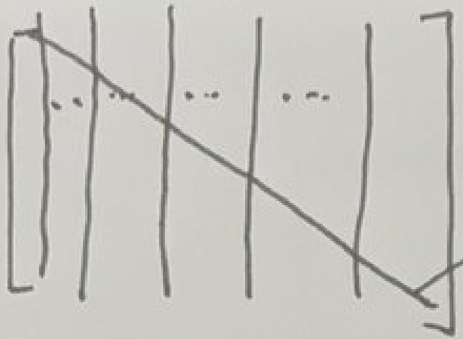
$$\underline{\underline{C}} \longrightarrow \boxed{\underline{\varepsilon} \sim \mathcal{N}(0, \underline{\underline{C}})}$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{C}}^{1/2}\, \underline{\xi} \rightarrow \mathcal{N}(0, \underline{\underline{I}})$$

$$\mathbb{E}[\varepsilon\varepsilon^T] = \underline{\underline{C}}^{1/2}\, \underline{\underline{I}}\, \underline{\underline{C}}^{T/2} = \underline{\underline{C}}$$

$\boxed{P^f} =$



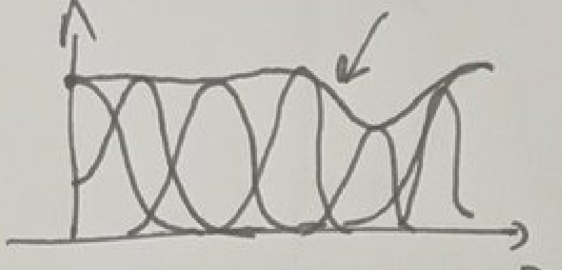


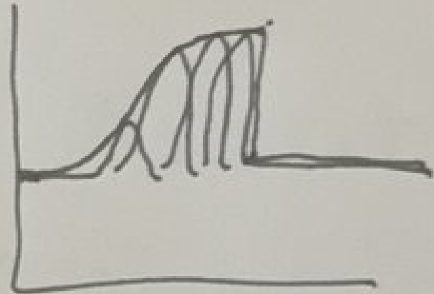
Carte de variance

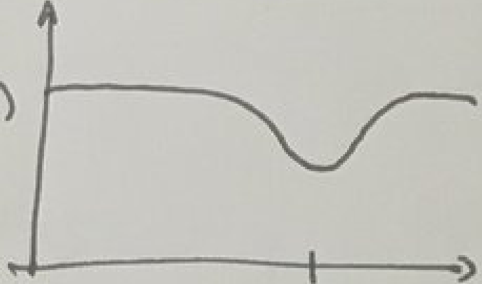
$V(x) = \mathbb{E}[\varepsilon^f(x)^2]$

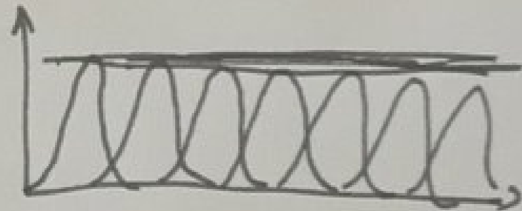
$P^f_{(x,y)} = \mathbb{E}[\varepsilon^f(x)\ \varepsilon^f(y)]$

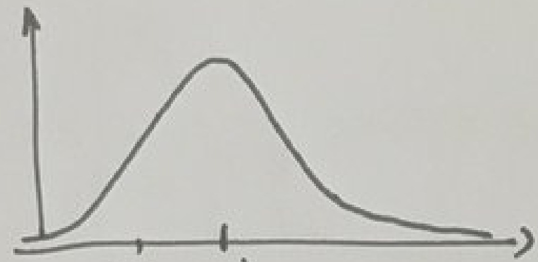
$\boxed{P^f(x,y)}$





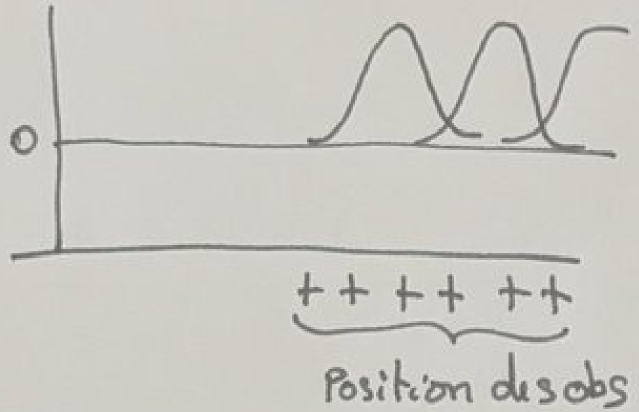


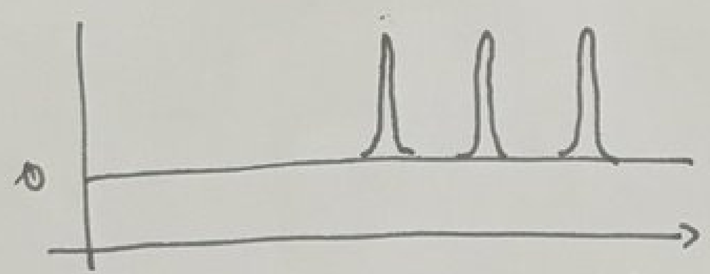




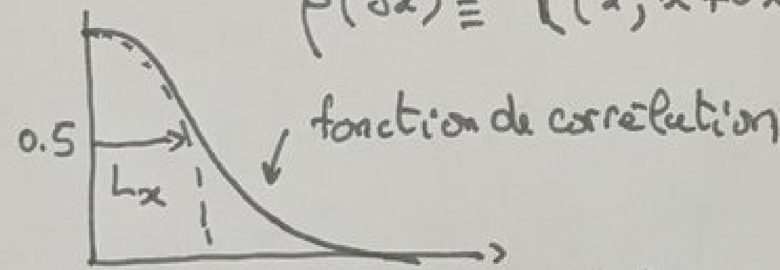
$$P^f_{(x,y)} = \mathbb{E}[\varepsilon^f(x)\ \varepsilon^f(y)] \longrightarrow C^f_{(x,y)} = \frac{P^f(x,y)}{\sqrt{V(x)}\ \sqrt{V(y)}}$$

$$\underline{\underline{P}}^f = \mathbb{E}[\underline{\varepsilon}^f \cdot (\underline{\varepsilon}^f)^T] \longrightarrow \boxed{\underline{\underline{C}}^f = \underline{\underline{\Sigma}}^{-1}\ \underline{\underline{P}}^f\ \underline{\underline{\Sigma}}^{-T}}$$



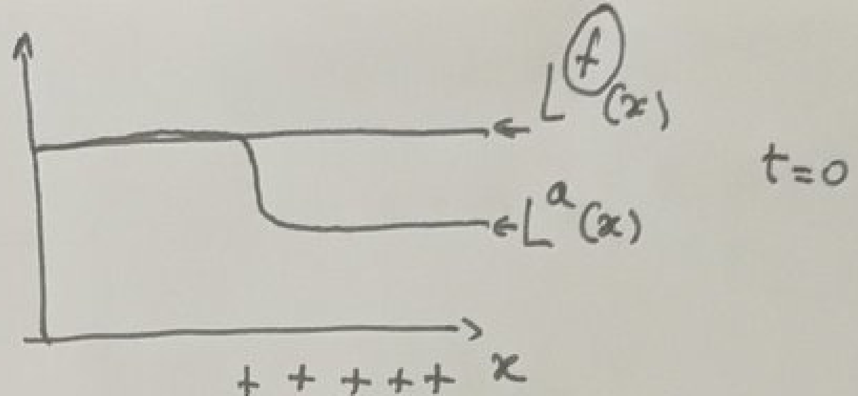


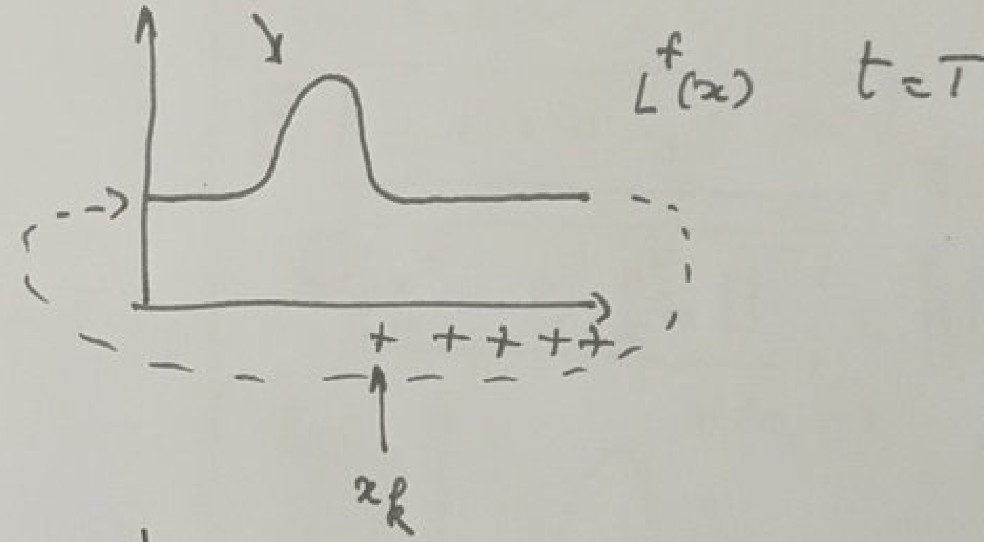
$$\rho(\delta x) \equiv \rho(x, x+\delta x)$$

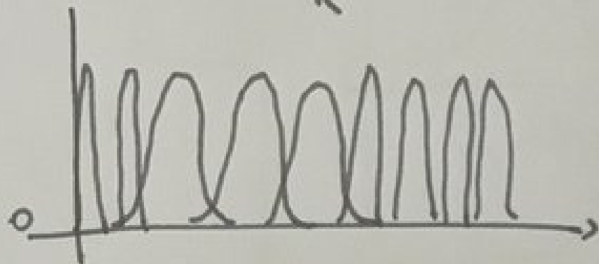


$$\rho(\delta x) = 1 - \frac{\delta x^2}{2 L_x^2} + o(\delta x^2)$$

longueur de portée







(Suite) 2011-6

① $\boxed{\partial_t \alpha + u \partial_x \alpha = 0}$

② $\boxed{\alpha = \mathbb{E}[\alpha] + \alpha'}$

$\alpha' = \alpha - \mathbb{E}[\alpha]$

dynamique de l'erreur de prévision.

③ $\boxed{\partial_t \varepsilon^f + u \partial_x \varepsilon^f = 0}$ $\varepsilon^f(t,x)$

on peut montrer que la <u>variance</u> vérifie une <u>dynamique</u> :

④ $\boxed{V(x) = \mathbb{E}[\varepsilon^f(x)^2]}$ $V(t,x)$

$$\partial_t V(x) = \partial_t \mathbb{E}[(\varepsilon^f)^2]$$
$$= \mathbb{E}[\partial_t(\varepsilon^{f2})]$$
$$= \mathbb{E}[2\varepsilon^f \partial_t \varepsilon^f] \quad \text{or } \partial_t \varepsilon^f = -u \partial_x \varepsilon^f$$
$$= 2\,\mathbb{E}[\varepsilon^f (-u) \partial_x \varepsilon^f]$$
$$= -2u\,\mathbb{E}[\varepsilon^f \partial_x \varepsilon^f] \quad \partial_x(\varepsilon^f)^2 = 2\varepsilon^f \partial_x \varepsilon^f$$
$$= -u\,\mathbb{E}[\partial_x (\varepsilon^f)^2]$$

(Suite)

$$\partial_t V(x) = -u\,\mathbb{E}[\partial_x (\varepsilon^f)^2]$$
$$= -u\,\partial_x [\underbrace{\mathbb{E}[\varepsilon^{f2}]}_{\text{c'est } V(t,x)}]$$
$$= -u\,\partial_x V$$

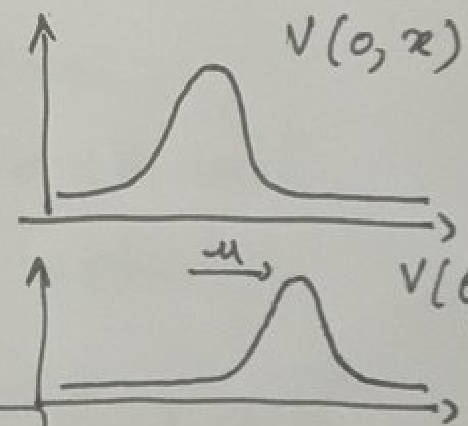
Conclusion

⑤ $\boxed{\partial_t V + u \partial_x V = 0}$

⑥ $\boxed{\underline{\underline{P}}^f = \underline{\underline{M}}\, \underline{\underline{P}}^f_{(0)}\, \underline{\underline{M}}^T} = \underline{\underline{M}}(\underline{\underline{M}}\,\underline{\underline{P}}^f_{(0)})^T$

$(\underline{\underline{P}}^f)^T = \underline{\underline{P}}^f$

<u>Ex:</u>



2011-7

$$P(x_q^f / y_{0:q-1}) \xrightarrow{} P(x_q^a / y_{0:q}) \xrightarrow{} P(x_{q+1}^f / y_{0:q})$$

a priori information du passé

Analyse y_q^o

a posteriori

Prévision

$\equiv$ Formule de Bayes

$$\boxed{P(x_q^a / y_{0:q}) \propto P(y_q^o / x_q) \, P(x_q / y_{0:q-1})}$$

Analyse

$$\begin{cases} x^a = x^f + K(y^o - Hx^f) \\ P^a = (I - KH) P^f \end{cases}$$

Prévision

$$\begin{cases} \underline{x}^f = \underline{\underline{M}}\, \underline{x}^a \\ \underline{\underline{P}}^f = \underline{\underline{M}}\, \underline{\underline{P}}^a\, \underline{\underline{M}}^T \end{cases}$$

Cas linéaire Gaussien $\Rightarrow$ Filtre de Kalman.

Filtre de Kalman d'ensemble

$\Longleftrightarrow$ x_q^k N échantillon

$\mathcal{N}(\underline{x}_q^f, \underline{\underline{P}}_q^f)$

N intégration

$\sim$ 20 à 30

100.

$n \sim 10^9$

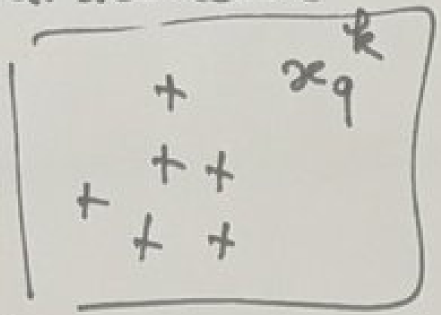
$$\underline{x}_q^f \simeq \frac{1}{N} \sum_{k=1}^{N} x_q^k = \boxed{\hat{x}_q^f}$$

$$\underline{\underline{P}}^f = \underline{\underline{M}}\, \underline{\underline{P}}^a\, \underline{\underline{M}}^T$$

2n intégration

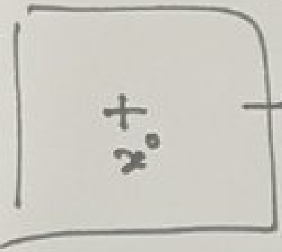
$$\underline{\underline{P}}_q^f \simeq \frac{1}{N} \sum_{k=1}^{N} (x_q^k - \hat{x}_q^f)(x_q^k - \hat{x}_q^f)^T$$

Filtre particulaire



$$P(x_q) \underset{N}{\simeq} \frac{1}{N} \sum_{k=1}^{N} \delta(x_q - x_q^k) = \bar{P}$$

distribution de Dirac



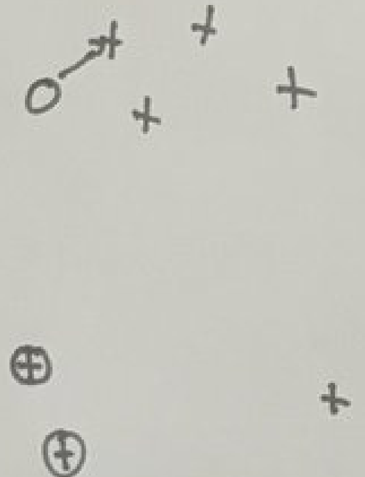
$\rightarrow \delta(x - x^\circ)$

au sens faible ie

$$\mathbb{E}_P[f(x_q)] \simeq \mathbb{E}_{\bar{P}}[f(x_q)] \quad \forall f \text{ bornée}$$

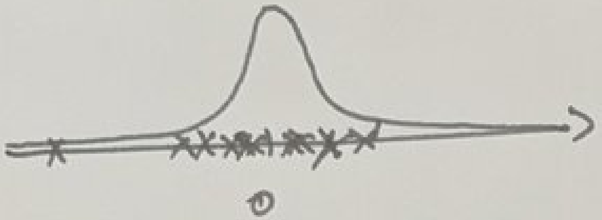
$P(y_q^\circ / x_q^k) \rightarrow$ poids $\equiv$ vraisemblance

Formule de Bayes : au sens faible

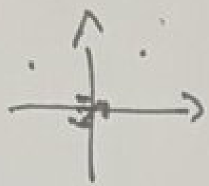


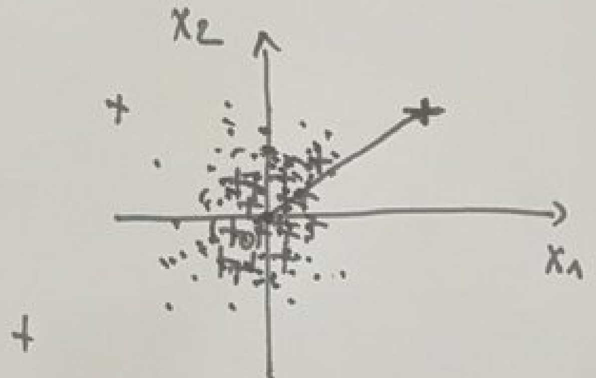
$$\boxed{\begin{array}{l} P(x_q / y_{0:q}) \simeq \sum w_k \, \delta(x_q - x_q^k) \\[1em] w_k = \dfrac{P(y_q^\circ / x_q^k)}{\sum_j P(y_q^\circ / x_q^j)} \end{array}}$$

2011-9



$X \sim \mathcal{N}(0, 1)$





$X \sim \mathcal{N}(0, \underline{\underline{I}}_2)$

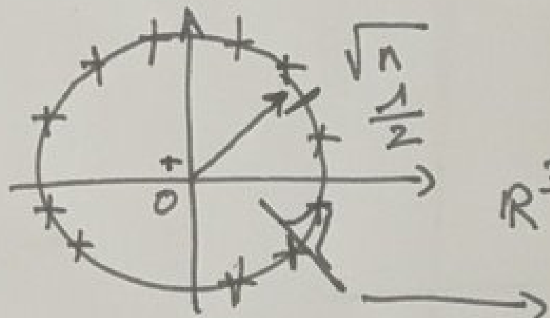
$X = (X_1, X_2)$

$X_i \sim \mathcal{N}(0,1)$

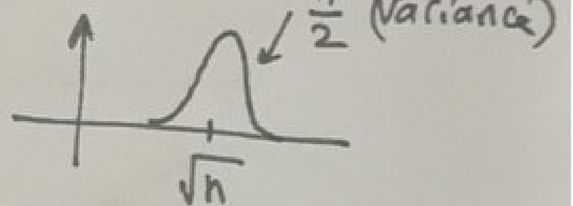
$\|X\|^2 = \sum X_i^2$

$X \sim \mathcal{N}(0, \underline{\underline{I}}_n)$

$n = 30$ ou 1000 ?



$\mathbb{R}^{30}$



$\underline{X} = (X_1 \cdots X_{30})$

$X_i \sim \mathcal{N}(0, 1)$

$\|\underline{X}(\omega)\|^2 = \boxed{\sum X_i^2} = \boxed{n}$

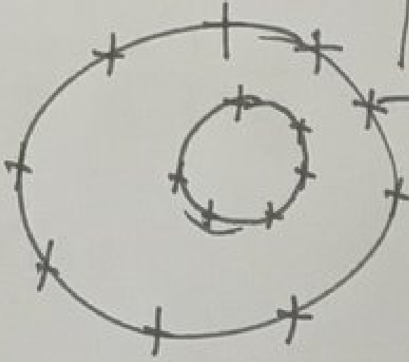
$\Omega \cdots$

$\mathbb{E}[\|X\|^2] = \sum_{i=1}^{n} \underbrace{\mathbb{E}[X_i^2]}_{=1} = n$

$\underline{X} = (0 \cdots 0)$

$\boxed{M_n = \frac{1}{n} \sum_{i=1}^{n} Z_i}$

Estimateur d'une moyenne



v.a.

Z_i va iid

Loi des grands nombres.

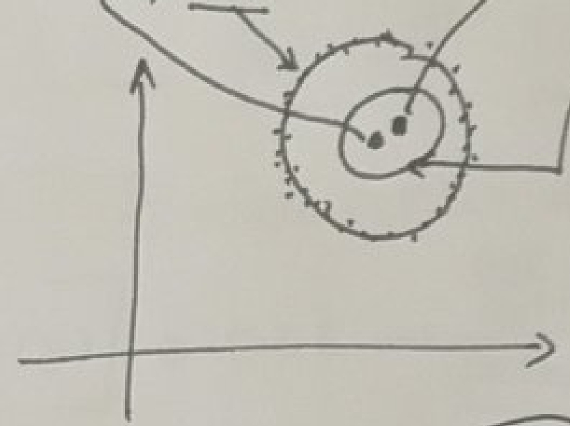
$\boxed{\lim_n M_n = \mathbb{E}[Z]} = \int P_Z(z)\, z\, dz$

$$M_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$$

$$M_n \sim \mathcal{N}\left(\mathbb{E}[Z] , \frac{V(Z)}{n} \right)$$

les fluctuations autour de la moyenne

$$\mathcal{N}(x_q^f, p_q^f) \to \mathcal{N}(x_q^a, p_q^a)$$

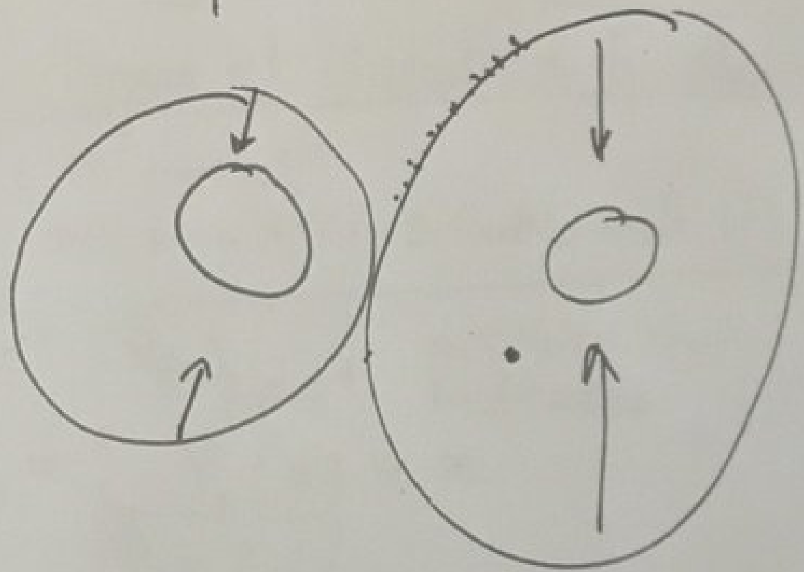


$$\|X\|^2 = \sum_{i=1}^{n} X_i^2 \equiv n$$

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 \sim \mathcal{N}\left(1, \frac{\mathbb{E}[X_i^2]}{n}\right)$$

$$\sum_{i=1}^{n} X_i^2 \sim \underbrace{\mathcal{N}\left(n , n^2 \frac{V(X^2)}{n}\right)}$$

$$N_n = n + \underset{\uparrow \atop \mathcal{N}(0,1)}{\xi} \sqrt{n} \sqrt{V(X^2)}$$



$$\boxed{\|X\| \sim \mathcal{N}\left(\sqrt{n}, \frac{1}{2}\right)}$$

Comparaison des filtres : étape d'analyse

Filtre "non linéaire"
Formule de Bayes

Linéaire Gaussien $\longrightarrow$

Filtre de Kalman

$$\boxed{P(x_q / y_q) \propto \underbrace{P(y_q / x_q)}_{\text{pondération via vraissemblance}} P(x_q / y_{q-1})}$$

~~discrétisation "particulaire"~~

$$\boxed{x_q^a = x_q^f + \underbrace{K(y_q - H x_q^f)}_{\text{correction de } x^f \text{ via les obs}}}$$

$$\boxed{P_q^a = (I - KH) P_q^f}$$

mise à jour mat. covariance

discrétisation

Filtre Particulaire

$$\boxed{\mathbb{P}^f(x_q) = \frac{1}{N} \sum_k \delta(x - x_q^k)}$$

où $(x_q^k)_{k \in [1, N]}$ un échantillon

Filtre de Kalman d'ensemble

$\boxed{(x_q^k) \text{ discrétisation de } \mathcal{N}(x_q^f, P_q^f)}$

avec

$$\boxed{\bar{x}_q = \frac{1}{N} \sum_k x_q^k \simeq x_q^f}$$

$$\boxed{\bar{P}_q^f = \frac{1}{N} \sum_k (x_q^k - \bar{x})(x_q^k - \bar{x})^T \quad \text{tq } \bar{P}_q^f \simeq P_q^f}$$

Analyse

$$\boxed{\omega_q^k = \frac{P(y_q / x_q^k)}{\sum_j P(y_q / x_q^j)}}$$

$$\boxed{\mathbb{P}^a(x_q) = \sum_k \omega_q^k \, \delta(x - x_q^k)}$$

ce sont les mêmes x_q^k que pour $\mathbb{P}_q^f$

$$\boxed{\tilde{x}_q^k = x_q^k + \bar{K}(y_q^k - H x_q^k)}$$

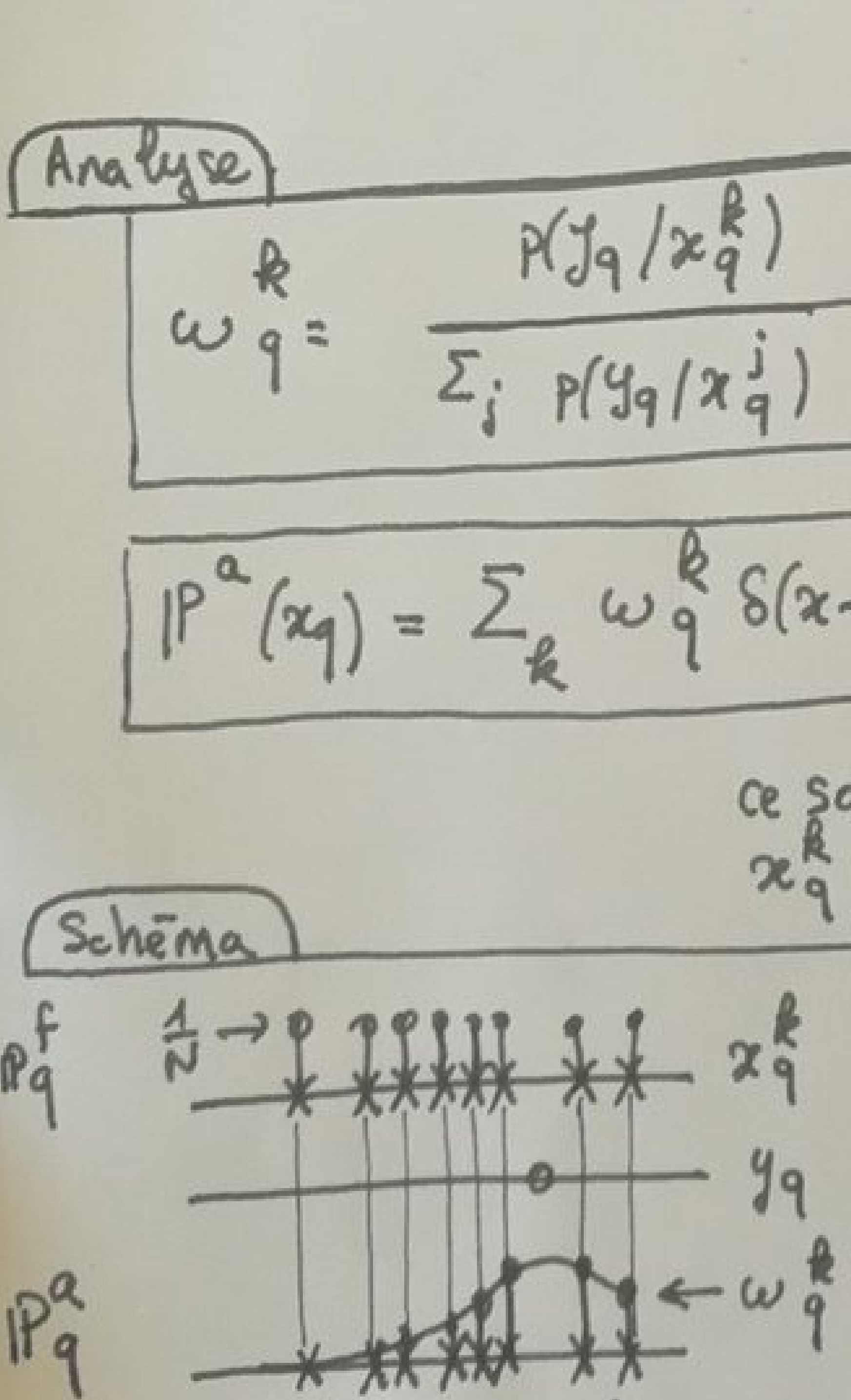
avec $y_q^k = y_q + R_q^{1/2} \xi^k$

ensemble d'analyse tel que

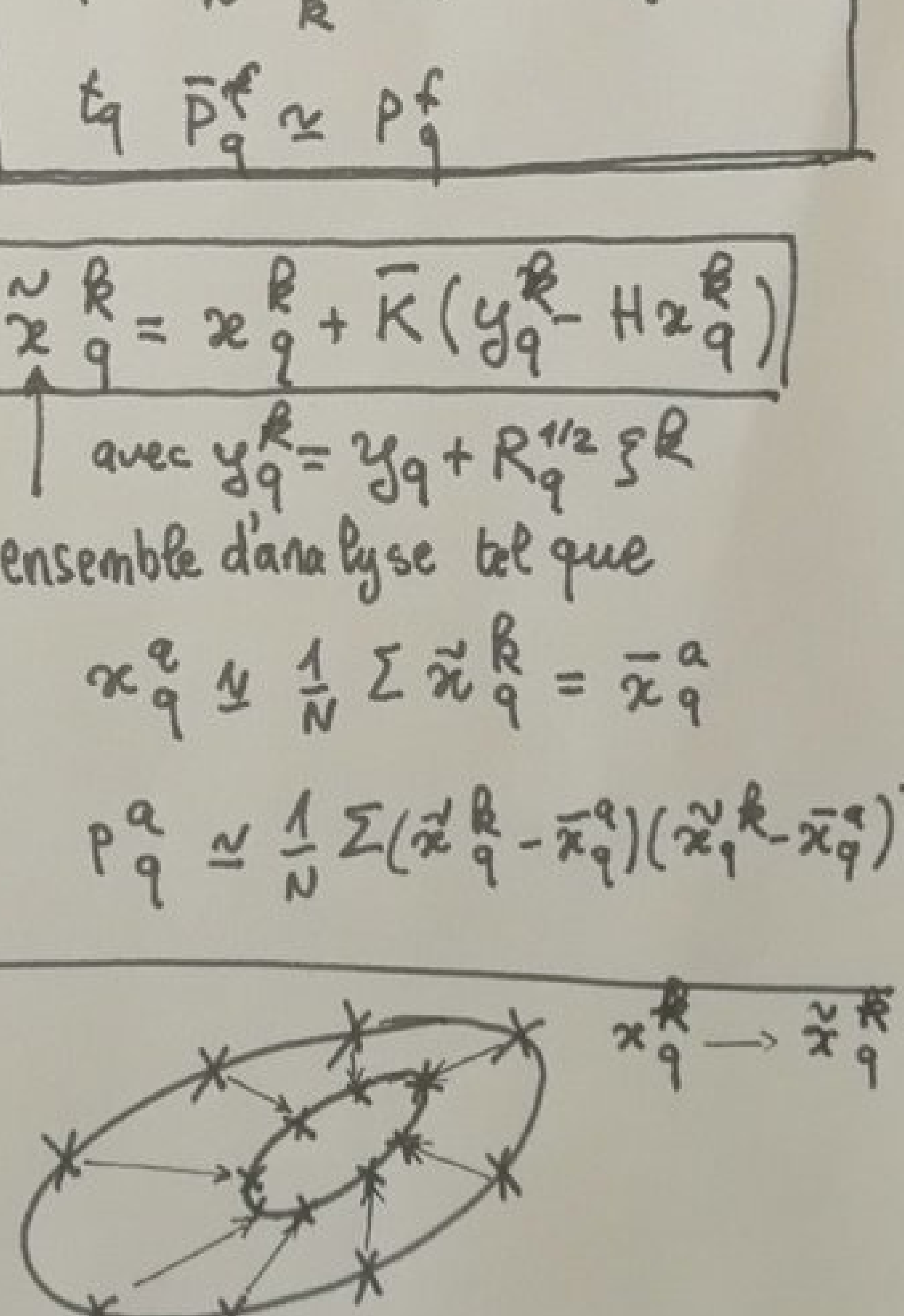
$$x_q^a \simeq \frac{1}{N} \sum \tilde{x}_q^k = \bar{\tilde{x}}_q^a$$

$$P_q^a \simeq \frac{1}{N} \sum (\tilde{x}_q^k - \bar{\tilde{x}}_q^a)(\tilde{x}_q^k - \bar{\tilde{x}}_q^a)^T$$

Schéma



reponderation du même ensemble !



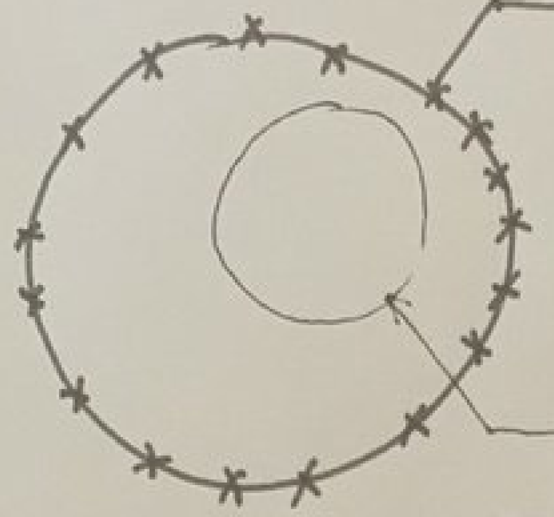
Transformation des membres de l'ensemble

Problématique de la grande dimension

on considère $x_q \sim \mathcal{N}(x_q^f, P_q^f)$ cas gaussien, $x \in \mathbb{R}^n$

avec $x_q^f = 0$ et $P_q^f = \underline{\underline{I}}$ $\qquad n \geqslant 30$

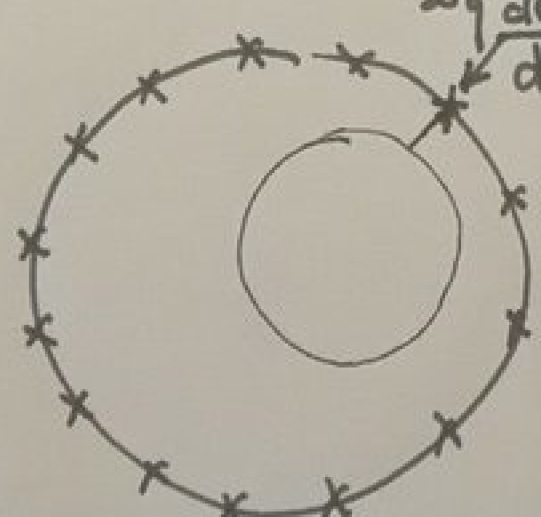
Avec une méthode de discrétisation tous (pour un ensemble fini de taille raisonnable) les échantillons se trouvent sur la sphère de rayon $\sqrt{n}$



Si tous les points sont observés, $\boxed{P_q^a = (I - KH)\, P_q^f \simeq \frac{1}{2}\, P_q^f}$

or dans cas gaussien $\qquad$ Formule de Bayes $\equiv$ Filtre de Kalman

Filtrage Particulaire



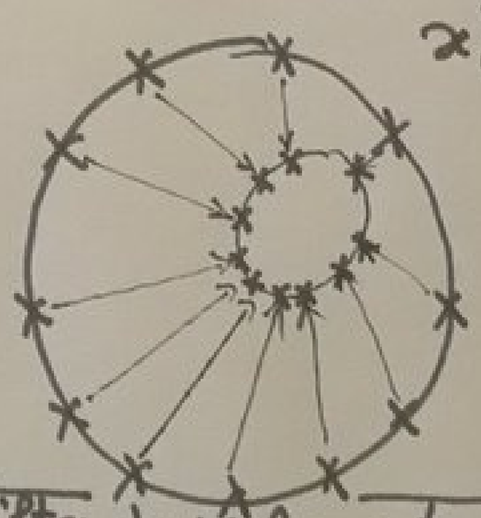
$\boxed{\mathbb{P}^a = \delta(x - x_q^*)}$ est singulière !

Aucun échantillon ne se trouve sur la distribution d'analyse

$\mathbb{P}^a \equiv$ repondération des échantillon x_q^k

Filtre de Kalman d'ensemble

$x_q^k \rightarrow \underbrace{\tilde{x}_q^k}_{\text{ensemble analysé}}$



Le filtre de Kalman transforme l'échantillon de prévision en un échantillon d'analyse

$\Rightarrow$ Le Filtre de Kalman / EnKF ne souffre pas de la malédiction de la dimension car résolution analytique du cas Linéaire - Gaussien