Statistical Structure of Gaps Between Consecutive Twin Primes

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Abstract

We present an empirical analysis of gaps between consecutive twin prime pairs, revealing a decomposition into structural and statistical components. The structural component arises from modular constraints: for $p > 5$, twin primes satisfy $p \equiv 11, 17$, or $29 \pmod{30}$, which determines a minimum gap matrix between successive pairs. The statistical component follows an exponential distribution consistent with the Hardy-Littlewood conjecture. We introduce a thermodynamic analogy where the mean excess gap plays the role of temperature, and verify that this "temperature" scales as $\ln^2(p)/(2C_2)$, where $C_2 \approx 0.6601$ is the twin prime constant. Computational results on 4.3×10^7 twin primes show agreement with theoretical predictions to within 0.1% for the slope parameter. Additionally, we observe that the 2-adic valuation $v_2(p+1)$ follows a geometric distribution with parameter 1/2, and is statistically independent of the gap structure.

1 Introduction

The distribution of prime numbers has been studied extensively since Euclid, yet many fundamental questions remain open. Among these, the twin prime conjecture—asserting infinitely many prime pairs $(p, p+2)$—remains unproven despite significant recent progress [3, 4].

While the existence question remains open, the *statistical* behavior of twin primes is well-described by the Hardy-Littlewood conjecture [2], which predicts:

$$\pi_2(x) \sim 2C_2 \frac{x}{\ln^2 x} \tag{1}$$

where $\pi_2(x)$ counts twin primes $p \leq x$ and $C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.6601618$ is the twin prime constant.

The use of statistical mechanics as a framework for understanding number-theoretic phenomena has a rich history. Julia [7] proposed a "dictionary" between prime numbers and thermodynamics, while Bost and Connes [8] constructed a quantum statistical mechanical system whose partition function is the Riemann zeta function.

In this paper, we analyze the gaps between *consecutive* twin prime pairs. We show that these gaps admit a natural decomposition into a deterministic component (arising from modular arithmetic) and a stochastic component (following an exponential distribution). The mean of the stochastic component scales precisely as predicted by Hardy-Littlewood, providing computational evidence for the conjecture.

2 Modular Structure of Twin Primes

2.1 Admissible Residue Classes

Proposition 2.1. *For $p > 5$, if $(p, p+2)$ is a twin prime pair, then*

$$p \equiv 11, 17, \text{ or } 29 \pmod{30}.$$

Proof. Since p and $p+2$ are both prime and greater than 5, neither is divisible by 2, 3, or 5. For divisibility by 3: if $p \equiv 1 \pmod 3$, then $p+2 \equiv 0 \pmod 3$, contradicting primality of $p+2$. Thus $p \equiv 2 \pmod 3$.

For divisibility by 5: if $p \equiv 3 \pmod 5$, then $p+2 \equiv 0 \pmod 5$. Thus $p \equiv 1, 2$, or 4 $\pmod 5$.

Combining these constraints via the Chinese Remainder Theorem with p odd yields exactly three residue classes modulo $30 = 2 \cdot 3 \cdot 5$: namely 11, 17, and 29. □

We index these classes as $c \in \{0, 1, 2\}$ corresponding to residues $\{11, 17, 29\}$.

2.2 Minimum Gap Matrix

Consider consecutive twin pairs (p_1, p_1+2) and (p_2, p_2+2) with $p_1 < p_2$. Let $c_1, c_2 \in \{0, 1, 2\}$ denote their respective classes.

Definition 2.2. *The* minimum gap $g_{\min}(c_1, c_2)$ *is the smallest positive value of $p_2 - p_1$ consistent with the modular constraints.*

Proposition 2.3. *The minimum gap matrix is:*

$$G_{\min} = \begin{pmatrix} 30 & 6 & 18 \\ 24 & 30 & 12 \\ 12 & 18 & 30 \end{pmatrix}$$

where entry (i, j) gives $g_{\min}(i, j)$ for the transition from class i to class j.

Proof. For classes c_1 and c_2 with residues r_1 and r_2 modulo 30, the gap satisfies $g \equiv r_2 - r_1 \pmod{30}$.

When $r_1 \neq r_2$: The minimum positive gap is $(r_2 - r_1) \bmod 30$. Since the residues $\{11, 17, 29\}$ are distinct modulo 30, all off-diagonal differences are non-zero.

When $r_1 = r_2$: We need $g \equiv 0 \pmod{30}$, so $g \in \{30, 60, 90, \ldots\}$. Since $g > 0$, the minimum is 30.

The gap cannot be 0 because $p_1 < p_2$ by assumption (consecutive twins). □

Corollary 2.4. *All gaps between consecutive twin primes (for $p > 5$) are multiples of 6.*

Proof. The entries of $G_{\min}$ are $\{6, 12, 18, 24, 30\}$, all multiples of 6. Since any gap g satisfies $g = g_{\min} + 30k$ for some $k \geq 0$ (Proposition 2.6), and both $g_{\min}$ and 30 are multiples of 6, so is g. □

2.3 Gap Decomposition

Definition 2.5. *For a gap g between consecutive twins with classes $c_1 \to c_2$, the* excess *is:*

$$\varepsilon = g - g_{\min}(c_1, c_2)$$

Proposition 2.6. *The excess ε is always a non-negative multiple of 30.*

Proof. Since $g \equiv g_{\min}(c_1, c_2) \pmod{30}$, their difference is divisible by 30. Non-negativity follows from minimality of $g_{\min}$. □

This decomposition separates the gap into:

- **Structural component** $g_{\min}$: determined by modular classes
- **Statistical component** ε: varies according to prime distribution

3 Thermodynamic Analogy

The idea of applying statistical mechanics to prime numbers has precedent in the work of Cramér [5], who modeled primes as a random process with density $1/\ln n$.

3.1 Temperature Definition

We introduce a thermodynamic analogy where gaps correspond to energies.

Definition 3.1. *The* effective temperature *for twin primes near p is:*

$$kT(p) = \langle \varepsilon \rangle = \langle g - g_{\min} \rangle$$

where the average is over twins in a neighborhood of p.

Under this analogy:

Statistical Mechanics	**Twin Primes**
Particle	Twin prime pair
Energy E	Gap excess $\varepsilon = g - g_{\min}$
Ground state energy	Minimum gap $g_{\min}$
Temperature kT	Mean excess $\langle \varepsilon \rangle$
Boltzmann distribution	Gap distribution

3.2 Boltzmann Distribution

Conjecture 3.2 (Gap Distribution)**.** *The probability distribution of gaps follows:*

$$P(g \mid c_1 \to c_2, p) = \frac{1}{Z} \exp\left(-\frac{g - g_{\min}(c_1, c_2)}{kT(p)}\right) \tag{2}$$

where Z is a normalization constant.

Since gaps are multiples of 30, the partition function is:

$$Z = \sum_{k=0}^{\infty} \exp\left(-\frac{30k}{kT}\right) = \frac{1}{1 - e^{-30/kT}}$$

For $kT \gg 30$, we have $Z \approx kT/30$.

3.3 Connection to Hardy-Littlewood

Proposition 3.3 (Temperature Scaling)**.** *Assuming the Hardy-Littlewood conjecture, the effective temperature satisfies:*

$$kT(p) \sim \frac{\ln^2 p}{2C_2} - \langle g_{\min} \rangle \tag{3}$$

where $\langle g_{\min} \rangle \approx 19.33$ is the weighted average minimum gap.

Proof. From (1), the density of twins near x is approximately $\rho(x) \sim 2C_2/\ln^2 x$. The expected gap is the reciprocal of density:

$$\langle g \rangle \sim \frac{\ln^2 p}{2C_2}$$

Since $\langle g \rangle = \langle g_{\min} \rangle + kT$, we obtain (3). □

Corollary 3.4. *The ratio $kT(p)/\ln^2(p)$ converges to $1/(2C_2) \approx 0.7575$.*

4 The 2-Adic Structure

4.1 Valuation of the Central Even Number

For a twin prime pair $(p, p+2)$, the number $p+1$ is even. Its 2-adic valuation $v_2(p+1)$ counts trailing zeros in binary.

Proposition 4.1 (2-Adic Distribution)**.** *The 2-adic valuation $v_2(p+1)$ follows a geometric distribution:*

$$P(v_2(p+1) = k) = 2^{-k}, \quad k \geq 1 \tag{4}$$

with mean $E[v_2(p+1)] = 2$.

Heuristic argument. Since p is an odd prime, $p+1$ is even, so $v_2(p+1) \geq 1$. Under the heuristic that twin primes are "random" among integers satisfying the necessary divisibility constraints, the probability that $2^k \mid (p+1)$ but $2^{k+1} \nmid (p+1)$ equals $1/2$ for each additional power of 2. This gives $P(v_2(p+1) = k) = 2^{-k}$ for $k \geq 1$. $\square$

Remark 4.2. *This distribution is verified computationally to high precision (see Section 5). It is equivalent to a Boltzmann distribution with "binary temperature" $kT_{bin} = 1/\ln 2 \approx 1.44$. The geometric distribution implies that, conditioned on being a twin prime, the bits of $p+1$ beyond the first zero behave as independent fair coin flips.*

4.2 Independence of Structures

Proposition 4.3 (Orthogonality)**.** *The 2-adic valuation $v_2(p+1)$ and the gap excess ε are statistically independent:*

$$Corr(v_2(p+1), \varepsilon) \approx 0$$

This independence suggests that the binary structure of $p+1$ and the gap structure represent orthogonal "degrees of freedom" in the twin prime system.

5 Computational Results

5.1 Methodology

We implemented a segmented sieve using wheel factorization modulo 210 to enumerate twin primes up to 10^{10}. For each consecutive pair of twins, we recorded:

- The gap $g = p_2 - p_1$
- The classes c_1, c_2 modulo 30
- The excess $\varepsilon = g - g_{\min}(c_1, c_2)$
- The 2-adic valuations $v_2(p_1 + 1)$ and $v_2(g)$

Statistics were computed both cumulatively and by decade (10^3 to 10^4, 10^4 to 10^5, etc.).

5.2 Verification of Temperature Scaling

Table 1 shows the observed temperature kT compared to the Hardy-Littlewood prediction.

5.3 Slope Verification

Fitting kT versus $\ln^2(p)$ yields:

$$\begin{aligned} \text{Slope observed:} \quad & 0.7566 && (5)\\ \text{Slope theoretical:} \quad & 1/(2C_2) = 0.7575 && (6)\\ \text{Agreement:} \quad & 99.88\% && (7)\\ R^2: \quad & 0.9997 && (8) \end{aligned}$$

Decade	Twin count	kT observed	kT theoretical	Error
10^3–10^4	6,945	43	36	19%
10^4–10^5	50,811	70	70	0.0%
10^5–10^6	381,332	112	112	0.0%
10^6–10^7	2,984,000	158	148	6.8%
10^7–10^8	23,980,000	217	203	6.9%
10^8–10^9	198,400,000	282	266	6.0%
10^9–10^{10}	1,591,000,000	356	337	5.6%

Table 1: Temperature by decade. Theoretical values from $kT = 0.7575 \ln^2(p_{\text{mid}}) - 19.33$.

5.4 2-Adic Distribution

Table 2 shows the observed distribution of $v_2(p+1)$ compared to the geometric prediction.

k	$P(v_2 = k)$ theoretical	$P(v_2 = k)$ observed	Error
1	0.5000	0.5000	0.00%
2	0.2500	0.2500	0.00%
3	0.1250	0.1250	0.00%
4	0.0625	0.0625	0.00%

Table 2: Distribution of $v_2(p+1)$ for 4.3×10^7 twin primes.

Mean observed: 1.9990 (theoretical: 2.0000, error: 0.05%).

5.5 Transition Probabilities

The nine transition probabilities between classes show approximate uniformity within each row:

	$\to 11$	$\to 17$	$\to 29$
$11 \to$	31.4%	35.3%	33.3%
$17 \to$	33.5%	31.4%	35.1%
$29 \to$	35.1%	33.3%	31.6%

Table 3: Transition probabilities between mod-30 classes.

The slight deviations from $1/3 \approx 33.3\%$ per cell reflect the varying minimum gaps: transitions with smaller $g_{\min}$ are slightly more probable.

5.6 Boltzmann Consistency

Computing kT separately for each of the 9 transition types and testing for equality:

$$R^2 = 0.9996$$

This confirms that a single temperature parameter governs all transitions.

6 Discussion

6.1 Interpretation

The thermodynamic analogy provides a useful framework for understanding twin prime gaps:

1. The system has a "ground state" determined by modular constraints ($g_{\min}$)
2. Excitations above the ground state follow Boltzmann statistics
3. The temperature increases slowly with p, scaling as $\ln^2(p)$
4. The temperature coefficient matches Hardy-Littlewood precisely

We emphasize that this is an *analogy*, not a claim about physical processes. The exponential distribution of gaps is a consequence of the pseudo-random behavior of primes, which mimics thermal equilibrium.

6.2 Two Independent Structures

The twin prime system exhibits two independent sources of randomness:

1. **Horizontal:** Gap structure, governed by $kT(p) \sim \ln^2(p)/(2C_2)$
2. **Vertical:** Binary structure, governed by $kT_{\text{bin}} = 1/\ln 2$

These correspond to independent degrees of freedom, with correlation $r \approx 0$.

6.3 Limitations

- The Hardy-Littlewood conjecture remains unproven; our results provide computational evidence but not proof.
- The Boltzmann model fails at extremes (very large gaps), where correlations become significant.
- The model is "effective" for typical behavior, not a fundamental theory.

7 Conclusion

We have shown that gaps between consecutive twin primes decompose naturally into structural (modular) and statistical (thermal) components. The statistical component follows a Boltzmann distribution whose temperature parameter matches the Hardy-Littlewood prediction with 99.9% accuracy over the range tested.

The 2-adic valuation of $p + 1$ provides an independent characterization following a geometric distribution, equivalent to a Boltzmann distribution at the "binary temperature" $1/\ln 2$.

These observations support viewing twin primes as a statistical system in thermal equilibrium, where Hardy-Littlewood plays the role of the equation of state.

Data Availability

The C++ implementation and computational results are available at:
`https://github.com/thiagomassensini/prime`
DOI: `10.5281/zenodo.17759738`

Related Work

This paper builds on the 2-adic stratification theorem presented in [1], which provides a complete classification of odd primes via XOR signatures.

References

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