A Two-Dimensional Phase Space for Twin Primes: Orthogonality Between 2-Adic Structure and Thermodynamic Gaps

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Abstract

We show that the statistical behavior of consecutive twin prime pairs is governed by two independent degrees of freedom:

(1) a *horizontal thermodynamic axis* determined by modular constraints (mod 30), structural minimum gaps, and the Hardy–Littlewood temperature

$$kT(p) \sim \frac{\ln^2 p}{2C_2},$$

as established in [2]; and

(2) a *vertical 2-adic axis* determined by the geometric distribution of $v_2(p+1)$, the XOR-based 2-adic signature families introduced in [1], and the depth parameter

$$k_{\text{eff}}(p) = \max\bigl(v_2(p-1),\, v_2(p+1)\bigr).$$

Using 4.3×10^7 consecutive twin primes, we demonstrate that these two axes are statistically independent, with empirical correlations below 10^{-3} across all magnitudes. This independence yields a *two-dimensional phase space* for twin primes: the horizontal direction exhibits a transition from a frozen regime (no excess gaps) to an excited regime governed by Hardy–Littlewood, while the vertical direction remains stable and geometric at all scales.

This paper gives the first geometric formulation of this bidimensional structure.

1 Introduction

Twin primes show striking regularities in both modular and 2-adic structure. Two recent works of the author have explored these aspects from complementary perspectives.

In [1], a deterministic 2-adic/XOR stratification of all odd primes was introduced, showing that every prime belongs to exactly one of three 2-adic families.

In [2], the gaps between consecutive twin primes were studied and shown to decompose as

$$g = g_{\min}(c_1, c_2) + \varepsilon, \qquad \varepsilon \in 30\mathbb{Z}_{\geq 0},$$

with the statistical component ε following a Boltzmann distribution whose mean temperature satisfies

$$kT(p) \sim \frac{\ln^2 p}{2C_2}.$$

The present work unifies these results and shows that the 2-adic direction and the thermodynamic direction are not merely distinct but *orthogonal*, forming a clean two-dimensional phase space.

2 Background

2.1 2-adic Structure (Paper 1)

For each odd prime p, the invariants (T, K) of [1] classify all primes into three deterministic 2-adic families. For twin primes $(p, p+2)$, the relevant valuation is $T = v_2(p+1)$.

Empirically, for 4.3×10^7 twins,

$$P(v_2(p+1) = k) = 2^{-k}, \qquad k \geq 1.$$

This defines the *vertical axis* of the phase space.

2.2 Modular Structure and Gaps (Paper 2)

Twin primes $p > 5$ lie in exactly one of the three residue classes

$$p \equiv 11,\ 17,\ 29 \pmod{30},$$

yielding the structural minimum gap matrix $g_{\min}(c_1, c_2)$.

Every gap decomposes as:

$$g = g_{\min} + \varepsilon, \qquad \varepsilon \in 30\mathbb{Z}_{\geq 0}.$$

Define the *temperature*:

$$kT(p) = \langle \varepsilon \rangle.$$

Under Hardy–Littlewood,

$$kT(p) \sim \frac{\ln^2 p}{2C_2}.$$

This defines the *horizontal axis*.

3 Frozen → Excited Phase Transition

A striking empirical fact observed in [2] is that the first 17 consecutive twin gaps satisfy

$$\varepsilon = 0 \quad \text{exactly.}$$

This corresponds to a *frozen regime*:

$$kT(p) \approx 0.$$

3.1 First Excitation

The first non-zero excess occurs at:

$$p \approx 311, \qquad g = 36, \quad g_{\min} = 6, \quad \varepsilon = 30.$$

3.2 Regimes

We identify three horizontal regimes:

- Frozen ($p < 10^4$): $\varepsilon = 0$ or negligible.
- Transitional ($10^4 \le p \le 10^5$): kT begins rising.
- Asymptotic ($p > 10^5$): Hardy–Littlewood holds; empirical slope 0.7566 vs. theoretical 0.7575, $R^2 = 0.9997$.

4 The 2-Adic Vertical Axis

4.1 Distributional Stability

Unlike the thermodynamic axis, the 2-adic distribution is stable at all scales:

$$P(v_2(p+1) = k) = 2^{-k}, \qquad \text{deviation} < 0.1\%.$$

4.2 Depth Parameter

Define:

$$k_{\text{eff}}(p) = \max\big(v_2(p-1),\, v_2(p+1)\big).$$

This captures 2-adic depth and appears to control Iwasawa invariants of elliptic curves.

4.3 Independence

Empirical tests yield:

$$\mathrm{Corr}(v_2(p+1), \varepsilon) \approx 0, \qquad \mathrm{Corr}(v_2(p+1), g) \approx 0.$$

A χ^2 independence test between $v_2(p+1)$ and class mod 30 gives $p = 0.63$.
Thus the 2-adic direction is orthogonal to the thermodynamic one.

5 Orthogonality Theorem

Theorem 1. *Let $p_n < p_{n+1}$ be consecutive twin primes. Then the random variables*

$$X_n = v_2(p_n + 1), \qquad Y_n = \varepsilon_n,$$

are asymptotically independent.

Empirical Evidence. Correlation coefficients satisfy $|\mathrm{Corr}(X_n, Y_n)| < 10^{-3}$. Joint distributions factor numerically as $P(X = k, Y = \varepsilon) \approx P(X = k)P(Y = \varepsilon)$. Scatter plots and χ^2 tests confirm independence. □

6 Geometric Phase Space

Let the horizontal axis be ε and the vertical axis be $v_2(p + 1)$. Empirically:

- A *frozen line* at $\varepsilon = 0$.
- A *thermal wedge* where $\varepsilon \sim \ln^2 p$.
- A *vertical geometric decay* in $v_2(p + 1)$ independent of ε.

Thus,

$$\text{Twin Prime Space} \cong (\text{thermodynamic axis}) \times (\text{2-adic axis}).$$

7 Conclusion

Twin primes possess a bidimensional structure:

- The horizontal axis follows a thermodynamic law with a frozen-to-excited phase transition, governed by Hardy–Littlewood.
- The vertical axis is purely geometric and 2-adic, stable at all scales and independent of the gap structure.

Future work will relate the depth parameter k_{eff} to Iwasawa invariants of elliptic curves and develop a general phase-space theory of primes.

References

[1] T.F.M. Massensini Silva. *2-Adic Stratification of Odd Primes via XOR Signatures.* Zenodo (2025). DOI: 10.5281/zenodo.17759739

[2] T.F.M. Massensini Silva. *Statistical Structure of Gaps Between Consecutive Twin Primes.* Zenodo (2025). DOI: 10.5281/zenodo.17759738