Filtering Process for ARMA-GARCH / APARCH estimation

- GARCH(1, 1) process:

$$data_t = mu + Z_t \longrightarrow \text{find } Z_t$$

$$h_t^2 = w + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1}^2 \longrightarrow \text{find } h_t$$

All I need to compute the LLH.

$$Z_t \overset{iid}{\sim} D_\vartheta(0,1) \longrightarrow \text{standard distribution}$$

STEPS:

(1) $Z_t = data_t - mu$

(2) $$e = \begin{bmatrix} w + \alpha_1 \times mean(Z_t^2) \\ w + \alpha_1 \cdot Z_1^2 \\ \vdots \\ w + \alpha_1 \cdot Z_{n-1}^2 \end{bmatrix}$$

(3) $h = \text{filter}(e, \text{beta}, \text{"r"}, \text{init} = mean(Z_t^2))$ — FILTER

$h[1] = e[1] + \beta_1 \times h[0]$

$h[2] = e[2] + \beta_1 \times h[1]$

$\vdots$

$h[n] = e[n] + \beta_1 \times h[n-1]$

$$\Longrightarrow h = \begin{bmatrix} w + \alpha_1 mean(Z_t^2) + \beta_1 \times mean(Z_t^2) \\ w + \alpha_1 Z_1^2 + \beta_1 h[1] \\ w + \alpha_1 Z_2^2 + \beta_1 h[2] \\ \vdots \\ w + \alpha_1 Z_{n-1}^2 + \beta_1 h[n-1] \end{bmatrix}$$

Note: here we call $h \equiv h^2$ and then find

$h = \text{sqrt}(h^2)$.

- GARCH(p, q), $p, q \geq 1$.

STEPS: ① Init $\boxed{Z_t}$ of size $p \geq 1$. ($Init_1, Init_2, \ldots, Init_p$)

② $Z_t = data_t - mu$

③

$$e = \begin{array}{c} 1 \\ 2 \\ \\ p \\ p+1 \\ \\ m \end{array} \begin{bmatrix} w + \alpha_1 . Init_p + \alpha_2 Init_{p-1} + \cdots + \alpha_p . Init_1 \\ w + \alpha_1 \times Z_1 + \alpha_2 Init_p + \cdots + \alpha_p . Init_2 \\ \vdots \\ w + \alpha_1 . Z_{p-1} + \alpha_2 Z_{p-2} + \cdots + \alpha_p . Init_p \\ w + \alpha_1 \times Z_p + \alpha_2 Z_{p-1} + \cdots + \alpha_p \times Z_1 \\ \vdots \\ w + \alpha_1 Z_{N-1} + \alpha_2 Z_{N-2} + \cdots + \alpha_p . Z_{m-p} \end{bmatrix} =$$

$$= W_{N \times 1} + \alpha_1 \times \begin{bmatrix} Init_p \\ Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_{p-1} \\ Z_p \\ \vdots \\ Z_{N-1} \end{bmatrix} + \alpha_2 \begin{bmatrix} Init_{p-1} \\ Init_p \\ Z_1 \\ Z_2 \\ \vdots \\ \vdots \\ Z_{N-2} \end{bmatrix} + \alpha_3 \begin{bmatrix} Init_{p-2} \\ Init_{p-1} \\ Init_{p-} \\ Z_1 \\ Z_2 \\ \vdots \\ Z_{N-3} \end{bmatrix} + \alpha_p \begin{bmatrix} Init_1 \\ Init_2 \\ \vdots \\ Init_p \\ Z_1 \\ Z_2 \\ \vdots \\ Z_{N-p} \end{bmatrix} =$$

Let the vector

$$V_1 := \begin{bmatrix} Init_1 \\ Init_2 \\ \vdots \\ Init_p \\ Z_1 \\ Z_2 \\ \vdots \\ Z_{N-1} \end{bmatrix}_{(N-1+p) \times 1}$$

$$= \alpha_1 \times V_1 \left[p : p + (N-1) \right] + \alpha_2 . V_1 \left[(p-1) : (p-1) + (N-1) \right] + \cdots + \alpha_p V_1 \left[1 : (1 + (N-1)) \right]$$

Filtering Process for ARMA estimation

ARMA(m, n) process, $m, n \geq 1$.

$$data_t = mu + X_t.$$

$$X_t = a_1 X_{t-1} + \dots + a_m X_{t-m} + b_1 Z_{t-1} + \dots + b_n Z_{t-n} + Z_t$$

$$Z \overset{iid}{\sim} D_\nu(0, \sigma)$$ → sclale parameter of the Conditional distribution.

STEPS:

① $X_t = data_t - mu$

② Init X_t of size m (start as 0) [Init 1, Init 2, ..., Initm]

③ Init $\cancel{Z_t}$ of size n (start as 0)

④ $Z_t = \underbrace{X_t - a_1 X_{t-1} - \dots - a_m X_{t-m}}_{\text{Find this part First. "Xparcial}} - b_1 Z_{t-1} - \dots - b_n Z_{t-n}.$

$$X_{parcial} \begin{bmatrix} X_1 - a_1 X_{initm} - a_2 X_{initm-1} - \dots - a_m X_{init1} \\ X_2 - a_1 X_1 - a_2 X_{initm} - \dots - a_m X_{init2} \\ \vdots \qquad\qquad - a_m X_{initm} \\ X_m - a_1 X_{m-1} - a_2 X_{m-2} - \dots - a_m X_1 \\ X_{m+1} - a_1 X_m - a_2 X_{m-1} - \dots - a_m X_2 \\ \vdots \qquad\qquad - a_m X_{N-m} \\ X_N - a_1 X_{N-1} - a_2 X_{N-2} - \dots \end{bmatrix}$$

$$X_{parcial} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} - a_1 \times \begin{bmatrix} x_{init\,m} \\ x_1 \\ \vdots \\ x_{m-1} \\ x_m \\ \vdots \\ x_{N-1} \end{bmatrix} - a_2 \begin{bmatrix} x_{init\,m-1} \\ x_{init\,m} \\ \vdots \\ \vdots \\ x_{N-2} \end{bmatrix} - a_m \begin{bmatrix} x_{init1} \\ x_{init2} \\ \vdots \\ x_{init\,m} \\ x_1 \\ x_2 \\ \vdots \\ x_{N-m} \end{bmatrix}.$$

Let $V := \begin{bmatrix} x_{init1} \\ x_{init2} \\ \vdots \\ x_{init\,m} \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}$. Then, $X_{parcial} =$

$$\underset{\sim}{X} - a_1 \times V\left[m : m + (N-1)\right] - a_2 V\left[(m-1) : (m-1+(N-1))\right] - \dots - a_m \times V\left[1 : (1+(N-1))\right]$$

⑤ Apply the R filter function:

$$Z = \text{filter}\left(X_{parcial},\ -b,\ \text{"recursive"},\ \text{init} = \overset{\text{init } Z}{\emptyset}\right)$$

Obs: The parameters are estimated better with $X_{init} = Z_{init} = 0$.