Q1. Nurse Scheduling

Hospital administrators must schedule nurses so that the hospital's patients are provided with adequate care. At the same time, in the face of tighter competition in the health care industry, they must pay careful attention to keeping costs down. From historical records, administrators estimated the minimum number of nurses to have on hand for the various times of the day, as shown in the following table.

Shift	Time	Minimum number of nurses needed
1	Midnight-4am	5
2	4am-8am	12
3	8am-noon	14
4	noon-4pm	8
5	4pm-8pm	14
6	8pm-Midnight	10

Nurses work 8 hours a day in two consecutive shifts. As a result, in each shift, there are two types of nurses: those that started in the previous shift (and are now working their second shift), and those that just started in this shift (and will be working in the next shift as well). Note that if a nurse who starts at the 8pm-Midnight shift would finish work on the next day's Midnight-4am shift.

Formulate a linear optimization problem to minimize the total number of nurses subject to being able to fulfill all business constraints. In other words, complete Steps 1 and 2 (English description and concrete formulation) for the problem.

Step 1:
Decision Variables:
Number of nurses who start working at each shift
Objective:
Minimize number of nurses hired
Constraints:
- Minimum nurses required in each shift
- Nurses work for only two consecutive shifts

Step 2:
Decision Variables:
x_n: Number of nurses who start working at shift n (Integer)
Objective:

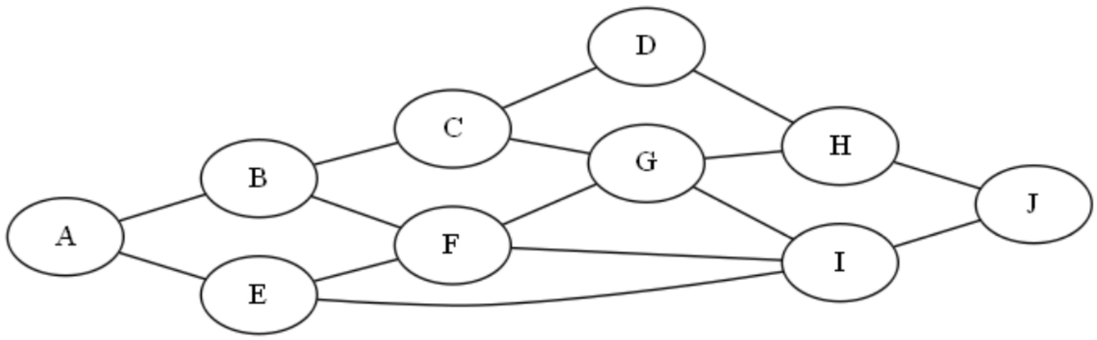
$$\text{Minimize: } x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Constraints:

(Nurses required in shift 1)	$x_6 + x_1 \geq 5$
(Nurses required in shift 2)	$x_1 + x_2 \geq 12$
(Nurses required in shift 3)	$x_2 + x_3 \geq 14$
(Nurses required in shift 4)	$x_3 + x_4 \geq 8$
(Nurses required in shift 5)	$x_4 + x_5 \geq 14$
(Nurses required in shift 6)	$x_5 + x_6 \geq 10$
(Non-Negativity)	$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Q2. Off-Campus Security

USC would like to protect every intersection around USC by stationing security staff, so that every intersection either has a staff stationed, or is connected directly to another intersection that has a staff stationed. For example, in the following sample map with 10 intersections, a staff stationed at intersection A is able to protect the intersections A, B and E; a staff stationed at intersection E is able to protect intersections A, E, F and I.



For the above map, formulate a linear optimization problem to minimize the total number of staff needed, subject to protecting all 10 intersections, as well as satisfying the following constraints:

- Staff cannot be stationed at both intersections A and B.
- At least three of E, F, G and H must have a staff directly stationed.
- No one can be stationed at intersection J.
- If someone is stationed at D, then at least one of H or J must be unstationed. (Even though the previous constraint makes this one automatically satisfied, still formulate this constraint as if the previous one didn't exist.)

a) What is the decision (in words)?
   - Whether staff will be stationed at each intersection

b) What is the objective (in words)?
   - Minimize total number of staff

c) Other than the constraints listed in bullets above, describe another constraint implied by the problem text (in words). Please list just one of many of these.
   - Every intersection must be protected

d) Write a concrete formulation using the correct mathematical notation, specifying the decision variables, the objective function and all constraints.

Decision Variables:

x_i: Whether staff is stationed at intersection i (Binary)

Objective:

$$\text{Minimize: } x_A + x_B + x_C + x_D + x_E + x_F + x_G + x_H + x_I + x_J$$

Constraints:

(Section A and B limitation)	$x_A + x_B \leq 1$
(Section E, F, G and H limitation)	$x_E + x_F + x_G + x_H \geq 3$
(Section J limitation)	$x_J = 0$
(Section D limitation)	$x_D + x_H + x_J \leq 2$
(Intersection A)	$x_A + x_B + x_E \geq 1$
(Intersection B)	$x_A + x_B + x_F \geq 1$
(Intersection C)	$x_B + x_C + x_G \geq 1$
(Intersection D)	$x_C + x_D + x_H \geq 1$
(Intersection E)	$x_E + x_F + x_I \geq 1$
(Intersection F)	$x_B + x_E + x_F + x_G + x_I \geq 1$
(Intersection G)	$x_C + x_F + x_G + x_H + x_I \geq 1$
(Intersection H)	$x_D + x_G + x_H + x_J \geq 1$
(Intersection I)	$x_E + x_F + x_G + x_I + x_J \geq 1$
(Intersection J)	$x_H + x_I + x_J \geq 1$

Q3. Course Selection

Aithne is currently enrolled in a Masters program at USC and is planning her courses for the next 2 semesters. There are five elective courses she would like to take, which we refer to as Courses A, B, C, D, and E. Based on her conversations with past students and prospective employers, she has estimated an "importance score" for each course, as well as the "workload" in terms of hours of work needed per week. Moreover, the schedules for the next two semesters have already been published, and this gives her information about scheduling conflicts as well as how much time she can afford to spend on these electives after accounting for her mandatory courses and other responsibilities. **Each course is a single semester long and can be taken only once, but the same course may be offered in both semesters, so she can choose when to take each course as well as whether to take it.** These information are summarized in the three tables below.

Course	**A**	**B**	**C**	**D**	**E**
Importance Score	5	3	2	4	5
Workload (hours/week)	15	10	10	5	10

Schedule	Semester 1	Semester 2
A	Tue/Thu 11-12:20	Mon/Wed 12:30-13:50
B	Tue 9-12:00	Tue 9:00-12:00
C	Mon/Wed 12:30-13:50	Not offered
D	Mon 12:00-15:00	Tue/Thu 11:00-12:20
E	Mon/Wed 14:00-15:50	Tue/Thu 11:00-12:20
Total time she can afford to spend	20 hours/week	15 hours/week

Furthermore, **Course A is a pre-requisite of Course D**, which means that if Aithne wishes to take Course D, she must take it in Semester 2 after taking Course A in Semester 1. Moreover, **Course A is a co-requisite of Course E**, which means that if she takes Course E in a semester, she must either be concurrently taking Course A or has already completed the course in a previous semester. (However, Course A can be taken by itself, without concurrently taking Course E.)

Aithne would like to plan a schedule that maximizes the total importance score of courses she takes. For example, if she takes only courses A and B in the two semesters, the total importance score would be 5 + 3 = 8. **Write a concrete formulation of a LP/MIP to help Aithne plan her course selection for the two semesters. The objective and all constraints must be linear.** For this question, you don't have to answer Step 1.

Decision Variables:

x_i: Whether course will be taken in semester 1 (Binary)

y_i: Whether course will be taken in semester 2 (Binary)

Objective:

$$\text{Maximize: } 5x_A + 3x_B + 2x_C + 4x_D + 5x_E + 5y_A + 3y_B + 2y_C + 4y_D + 5y_E$$

Constraints:

(Each course can be taken once)	$x_i + y_i \leq 1$
(Semester 1 workload)	$15x_A + 10x_B + 10x_C + 5x_D + 10x_E \leq 20$
(Semester 2 workload)	$15y_A + 10y_B + 10y_C + 5y_D + 10y_E \leq 15$
(Conflict schedule A-B in semester 1)	$x_A + x_B \leq 1$
(Conflict schedule C-D in semester 1)	$x_C + x_D \leq 1$
(Conflict schedule D-E in semester 1)	$x_D + x_E \leq 1$
(Conflict schedule B-D-E in semester 2)	$y_B + y_D + y_E \leq 1$
(Course C is not offered in semester 2)	$y_C = 0$
(Pre-requisite of course D in semester 1)	$x_D = 0$
(Pre-requisite of course D in semester 2)	$x_A \geq y_D$
(A and E is Co-requisite Semester 1)	$x_A \geq x_E$
(A and E is Co-requisite Semester 2)	$x_A + y_A \geq y_E$