Almgren-Chriss model and its extensions

M203 Electronic markets project

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1 Introduction

The paper aims to present enhancements to the Almgren-Chriss framework [AC00]. The original model by Almgren and Chriss was designed to balance market risk and the costs of liquidity during significant trade executions. This research expands that model to a continuous setting and explores variations.

It examines different benchmarks beyond initial pricing, such as using the *Target Close* benchmark and the *Time-Weighted Average Price*. Further analysis includes comparing the discretized Bellman Equation's results to existing analytic solutions. Additionally, the study explores trading velocity and its impact on liquidity and execution cost efficiency, as well as strategies for liquidating a second, more liquid asset in correlation with the first. Our contribution will mostly be through the implementation of deep learning methods to find an optimal liquidation for various orders and strategies, while observing what happens when we decide to lift some key assumptions.

2 Target Close and TWAP orders

2.1 Continuous Almgren-Chriss model presentation

The model first states the following dynamics on the stock inventory x_t, the spot S_t and the cash balance C_t.

$$\begin{cases} dx_t = -n_t dt \\ dS_t = \sigma dW_t - g(n_t)dt \\ dC_t = n_t(S_t - h(n_t))dt \end{cases} \tag{1}$$

In the above system, the remaining stock inventory is diffused such as to exhibit the convexity of the execution strategy. The spot price follows an *Arithmetic Brownian* motion through the Wiener process W_t with constant volatility. The potential permanent market impact of trading is denoted $g(n_t)$, but unless stated otherwise we consider it absent for the remainder of this paper. The cash balance grows by the given traded quantity valued at the current stock price corrected of a temporary market impact written $h(n_t)$.

$$\begin{cases} x_0 = x \\ x_T = 0 \\ C_0 = 0 \end{cases} \tag{2}$$

At inception we are attributed a quantity X we want to liquidate fully at the end of the process. We also assume that the cash balance at the start is zero.

2.2 Implementation Shortfall optimal strategy

Implementation shortfall is the difference between the decision price and the execution price for a trade. We remind there is no permanent market impact.

$$C_T - x_0 S_0 = \sigma \int_0^T x_t \, dW_t - \int_0^T g(n_t) x_t \, dt - \int_0^T n_t h(n_t) \, dt$$

$$C_T - x_0 S_0 = \sigma \int_0^T x_t dW_t - \int_0^T n_t h(n_t) dt \tag{3}$$

We then would like to find the optimal strategy under the *Mean-Variance* framework. This leads to solving the following optimization:

$$\min_x \lambda \mathbb{V}[C_T - x_0 S_0] - \mathbb{E}[C_T - x_0 S_0] \tag{4}$$

With λ being a risk aversion parameter. We below detail the expectation and variance for this system.

$$\mathbb{E}[C_T - x_0 S_0] = \sigma \mathbb{E}\left[\int_0^T x_t dW_t\right] - \mathbb{E}\left[\int_0^T n_t h(n_t) dt\right] = -\int_0^T n_t h(n_t) dt$$

$$\mathbb{V}[C_T - x_0 S_0] = \mathbb{V}\left[\sigma \int_0^T x_t dW_t - \int_0^T n_t h(n_t) dt\right] = \sigma^2 \int_0^T x_t^2 dt$$

We will denote $n_t = -\frac{dx_t}{dt} = -\dot{x}_t$ by considering the Euler-Lagrange notations:

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \tag{5}$$

This allows us to further simplify the optimization problem under (2):

$$\min_x \int_0^T \left[\lambda \sigma^2 x_t^2 - \dot{x}_t h(-\dot{x}_t)\right] dt$$

In the original model, h is assumed to be linear: the impact is directly proportional to the rate or volume of trading. We thus write $h(x) = \eta x$ and continue:

$$\min_x \int_0^T \left[\lambda\sigma^2 x_t^2 + \eta \dot{x_t}^2\right] dt \tag{6}$$

These simplifications make the model simple and tractable. Other models use more complex forms of the temporary market impact, as the accurate modelling of trade execution costs heavily rely on having chosen the right form.

At this stage, having combined Almgren-Chriss dynamics with mean-variance optimization led to a static optimization problem, with two terms appear: a *liquidation* penalty $\eta\dot{x}_t^2$ and an *inventory* penalty $\lambda\sigma^2 x_t^2$. They give us a clearer picture of how the optimal execution chronology might look like.

The liquidation penalty advises the trader not to execute too quickly, thus inferring on a lesser steep slope. Moreover, the inventory penalty term incorporates convexity in the case of a risk-averse trader. The optimal execution will then not be a straight decreasing line. Moving on with the problem, we define:

$$F(t, x_t, \dot{x_t}) = \lambda\sigma^2 x_t^2 + \eta \dot{x_t}^2 \tag{7}$$

To which we apply the predefined Euler-Lagrange equation:

$$\frac{\partial F}{\partial x_t} = 2\lambda\sigma^2 x_t; \quad \frac{\partial F}{\partial \dot{x_t}} = 2\eta\dot{x_t};$$

Using (5) we thus obtain with $y = t$ and $f = F$:

$$\frac{\partial F}{\partial x_t} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x_t}}\right) \implies 2\lambda\sigma^2 x_t = 2\eta\ddot{x}_t \iff \ddot{x}_t = \frac{\lambda\sigma^2}{\eta} x_t$$

The general solution to this differential equation is a linear combination of hyperbolic sine and cosine functions. We will use again (2) to conclude.

$$x(t) = A \cdot \cosh\left(\sigma t\sqrt{\frac{\lambda}{\eta}}\right) + B \cdot \sinh\left(\sigma t\sqrt{\frac{\lambda}{\eta}}\right)$$

We denote $k = \sigma\sqrt{\frac{\lambda}{\eta}}$.

$$\begin{cases} x_0 = A \\ \ \ 0 = A\cosh(kT) + B\sinh(kT) \end{cases} \implies \begin{cases} A = x_0 \\ B = -x_0\cosh(kT)\sinh^{-1}(kT) \end{cases}$$

$$
\begin{aligned}
x(t) &= x_0 \left(\cosh(kt) - \frac{\cosh(kT)\sinh(kt)}{\sinh(kT)} \right) \\
&= x_0 \left(\frac{\sinh(kT)\cosh(kt) - \cosh(kT)\sinh(kt)}{\sinh(kT)} \right) \\
&= x_0 \left(\frac{\sinh(k(T-t))}{\sinh(kT)} \right) = \frac{x_0}{\sinh(kT)} \sinh(k(T-t)) \qquad (8)
\end{aligned}
$$

We now display implementation shortfall optimal liquidation strategies for different parameters. We will initialize our market conditions at $x_0 = 100k£$, $\sigma = 20\%$, $T = 1$ day and $\eta = 1.1 \times 10^{-4}$.

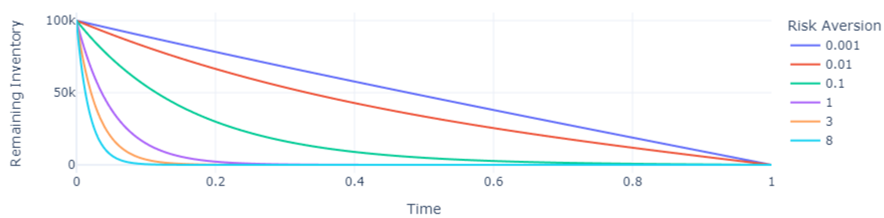

Figure 1: IS optimal liquidation for varying *risk aversion*

We observe a convex execution for well-chosen parameters. Traders with extreme risk aversion will want to liquidate as fast as possible by fear of any residual inventory at the end of the day. We now fix $\lambda = 0.1$ with varying η.

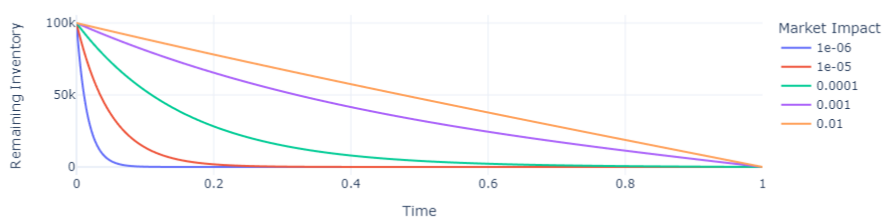

Figure 2: IS optimal liquidation for varying *temporary market impact*

For a fixed risk aversion, we notice that the bigger the temporary market impact, the smoother the execution process will be. This is quite expected: clustering orders within a restricted time frame would hurt, so the trader tries her best to proceed to an homogeneous execution.

2.3 Target Close optimal strategy

Target Close is when the trader aims at executing a trade at a price close to the closing price of the stock on a particular day. This approach is typically used to minimize the market impact and the transaction cost of large orders by leveraging the increased liquidity near the market's close.

$$
\begin{aligned}
C_T - x_0 S_0 &= \sigma \int_0^T x_t \, dW_t - \int_0^T g(n_t) x_t \, dt - \int_0^T n_t h(n_t) \, dt \\
C_T - x_0 S_T &= \sigma \int_0^T x_t dW_t - \int_0^T n_t h(n_t) dt + x_0 (S_0 - S_T) \\
C_T - x_0 S_T &= \sigma \int_0^T x_t dW_t - \int_0^T n_t h(n_t) dt - x_0 \sigma W_T
\end{aligned} \tag{9}
$$

We once again would like to find the optimal strategy under the *Mean-Variance* framework which gives the following, the benchmark in this case being S_T:

$$\min_x \lambda \mathbb{V}[C_T - x_0 S_T] - \mathbb{E}[C_T - x_0 S_T] \tag{10}$$

To solve the system we compute both moments $\mathbb{E}(\cdot)$ and $\mathbb{V}(\cdot)$:

$$\mathbb{E}[C_T - x_0 S_T] = \mathbb{E}\left[\int_0^T \sigma(x_t - x_0) \, dW_t - \int_0^T n_t h(n_t) \, dt\right] = -\int_0^T n_t h(n_t) \, dt$$

$$\mathbb{V}[C_T - x_0 S_T] = \mathbb{V}\left[\int_0^T \sigma(x_t - x_0) \, dW_t - \int_0^T n_t h(n_t) \, dt\right] = \sigma^2 \int_0^T (x_t - x_0)^2 \, dt$$

Continuing on the same path as for the previous problem and considering the same definitions for n_t, $\dot{x}_t$ and h, we get the following under (2):

$$\min_x \int_0^T \left[\lambda \sigma^2 (x_t - x_0)^2 - \dot{x}_t h(-\dot{x}_t)\right] dt$$

We again write $h(x) = \eta x$ and thus have:

$$\min_x \int_0^T \left[\lambda \sigma^2 (x_t - x_0)^2 + \eta \dot{x_t}^2\right] dt \tag{11}$$

To continue, we define a functional $F(\cdot)$ to which we apply Euler-Lagrange:

$$F(t, x_t, \dot{x}_t) = \lambda \sigma^2 (x_t - x_0)^2 + \eta \dot{x_t}^2 \tag{12}$$

$$\frac{\partial F}{\partial x_t} = 2\lambda \sigma^2 (x_t - x_0); \quad \frac{\partial F}{\partial \dot{x}_t} = 2\eta \dot{x}_t;$$

Using (5) we thus obtain with $y = t$ and $f = F$:

$$\frac{\partial F}{\partial x_t} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_t}\right) \implies 2\lambda\sigma^2(x_t - x_0) = 2\eta\ddot{x}_t \iff \ddot{x}_t = \frac{\lambda\sigma^2}{\eta}(x_t - x_0)$$

The general solution to this differential equation is also a linear combination of hyperbolic sine and cosine functions. We will use again (2) to conclude.

$$x(t) - x_0 = A \cdot \cosh\left(\sigma t\sqrt{\frac{\lambda}{\eta}}\right) + B \cdot \sinh\left(\sigma t\sqrt{\frac{\lambda}{\eta}}\right)$$

We denote $k = \sigma\sqrt{\frac{\lambda}{\eta}}$.

$$\begin{cases} A = 0 \\ B = -x_0 \sinh^{-1}(kT) \end{cases}$$

$$x(t) = x_0 - \frac{x_0}{\sinh(kT)}\sinh(k(t)) = x_0\left(1 - \frac{\sinh(kt)}{\sinh(kT)}\right) \tag{13}$$

We now display implementation shortfall optimal liquidation strategies for different parameters. We consider the same set of base market parameters than in section 2.2 for consistency. Let us first fix all parameters but the risk aversion.

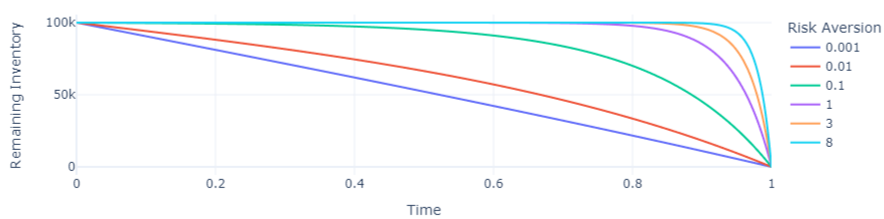

Figure 3: TC optimal liquidation for varying *risk aversion*

If a trader is heavily risk averse, she will tend to liquidate a bigger part of her inventory around the benchmark price, which in that case is S_T. This explains why the curve is skewed to the upper right part as opposed to the previous implementation where we considered S_0 as the reference. As in the previous section, let us now observe what happens when we fix all but the temporary market impact, fixing $\lambda = 0.1$ as before.

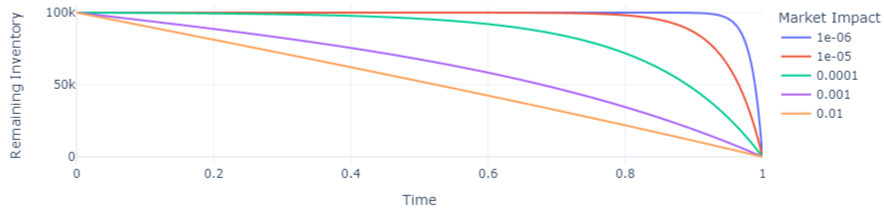

Figure 4: TC optimal liquidation for varying *temporary market impact*

Once again, the bigger the temporary market impact of an execution, the smoother the curve will be so as to spread out the impact on the timeline.

2.4 TWAP optimal strategy

Time Weighted Average Price aims to minimize the market impact of large trades by spreading the order out over a specific time period. Its main goal is to execute the trade as close to the average price of the security as possible, within the chosen time frame.

$$C_T - x_0 S_0 = \sigma \int_0^T x_t \, dW_t - \int_0^T g(n_t) x_t \, dt - \int_0^T n_t h(n_t) \, dt$$

$$C_T - \frac{x_0}{T} \int_0^T S_t \, dt = \sigma \int_0^T x_t \, dW_t - \int_0^T n_t h(n_t) \, dt + x_0 \left(S_0 - \frac{1}{T} \int_0^T S_t \, dt \right)$$

However we do know from (1) that:

$$S_t = S_0 + \sigma W_t \implies S_0 - \frac{1}{T} \int_0^T S_t \, dt = S_0 - \frac{1}{T} S_0 T - \frac{\sigma}{T} \int_0^T W_t \, dt = -\frac{\sigma}{T} \int_0^T W_t \, dt$$

Hence we further simplify our equation:

$$C_T - \frac{x_0}{T} \int_0^T S_t \, dt = \sigma \int_0^T x_t \, dW_t - \int_0^T n_t h(n_t) \, dt - \frac{x_0 \sigma}{T} \int_0^T W_t \, dt$$

$$C_T - \frac{x_0}{T} \int_0^T S_t \, dt = \sigma \int_0^T x_t \, dW_t - \int_0^T n_t h(n_t) \, dt - \frac{x_0 \sigma}{T} \int_0^T (T - t) \, dW_t$$

$$C_T - \frac{x_0}{T} \int_0^T S_t \, dt = \sigma \int_0^T x_t - x_0 \left(1 - \frac{t}{T} \right) dW_t - \int_0^T n_t h(n_t) \, dt \qquad (14)$$

The optimal strategy is found by solving the following optimization system:

$$\min_x \lambda \mathbb{V}\left[C_T - \frac{x_0}{T}\int_0^T S_t\, dt\right] - \mathbb{E}\left[C_T - \frac{x_0}{T}\int_0^T S_t\, dt\right] \tag{15}$$

Let us compute both moments $\mathbb{E}(\cdot)$ and $\mathbb{V}(\cdot)$:

$$\mathbb{E}\left[C_T - \frac{x_0}{T}\int_0^T S_t\, dt\right] = -\mathbb{E}\left[\int_0^T n_t h(n_t)\, dt\right] = -\int_0^T n_t h(n_t)\, dt$$

$$\begin{aligned}\mathbb{V}\left[C_T - \frac{x_0}{T}\int_0^T S_t\, dt\right] &= \mathbb{V}\left[\sigma\int_0^T x_t - x_0\left(1-\frac{t}{T}\right) dW_t\right] \\ &= \sigma^2\int_0^T \left(x_t - x_0\left(1-\frac{t}{T}\right)\right)^2 dt\end{aligned}$$

Again following the same path as for the previous two systems and considering the same definitions for n_t, $\dot{x_t}$ and h, we get the following under (2):

$$\min_x \int_0^T \left[\lambda\sigma^2\left(x_t - x_0\left(1-\frac{t}{T}\right)\right)^2 - \dot{x_t}h(-\dot{x_t})\right] dt$$

We again write $h(x) = \eta x$ and thus have:

$$\min_x \int_0^T \left[\lambda\sigma^2\left(x_t - x_0\left(1-\frac{t}{T}\right)\right)^2 + \eta\dot{x_t}^2\right] dt \tag{16}$$

To continue, we define a functional $F(\cdot)$ to which we apply Euler-Lagrange:

$$F(t, x_t, \dot{x_t}) = \lambda\sigma^2\left(x_t - x_0\left(1-\frac{t}{T}\right)\right)^2 + \eta\dot{x_t}^2 \tag{17}$$

$$\frac{\partial F}{\partial x_t} = 2\lambda\sigma^2\left(x_t - x_0\left(1-\frac{t}{T}\right)\right); \quad \frac{\partial F}{\partial \dot{x_t}} = 2\eta\dot{x_t};$$

Using (5) we thus obtain with $y = t$ and $f = F$:

$$\frac{\partial F}{\partial x_t} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x_t}}\right) \implies 2\lambda\sigma^2\left(x_t - x_0\left(1-\frac{t}{T}\right)\right) = 2\eta\ddot{x}_t$$

$$\ddot{x}_t = \frac{\lambda\sigma^2}{\eta}\left(x_t - x_0\left(1-\frac{t}{T}\right)\right)$$

The general solution to this differential equation is again a linear combination of hyperbolic sine and cosine functions, that we use with (2) to conclude.

$$x(t) - x_0\left(1 - \frac{t}{T}\right) = A \cdot \cosh\left(\sigma t\sqrt{\frac{\lambda}{\eta}}\right) + B \cdot \sinh\left(\sigma t\sqrt{\frac{\lambda}{\eta}}\right)$$

$$\begin{cases} A = 0 \\ B = 0 \end{cases} \implies x(t) = x_0\left(1 - \frac{t}{T}\right) \tag{18}$$

The liquidation strategy is smooth and does not account for any of the two risks defined above in section 2.2. The plot would only be a straight decreasing curve, regardless of the market parameters.

2.5 Implementation Shortfall efficient frontier

The efficient frontier represents a set of optimal portfolios that offer the highest expected return for a defined level of risk. Given the optimal deterministic strategy obtained in (8):

$$x(t) = \frac{x_0}{\sinh(kT)}\sinh(k(T-t))$$

We thus express our moments as functions of (8). Thus we will be able to plot each portfolio on a graph of axes $(\mathbb{E}[\cdot], \mathbb{V}[\cdot])$.

$$\mathbb{V}[C_T - x_0 S_0] = \sigma^2 \int_0^T x_t^2\, dt = \sigma^2 \int_0^T \left(\frac{x_0}{\sinh(kT)}\sinh(k(T-t))\right)^2 dt$$

$$\mathbb{V}[C_T - x_0 S_0] = \left(\frac{x_0\sigma}{\sinh(kT)}\right)^2 \int_0^T \sinh(k(T-t))^2\, dt$$

We focus on solving the remaining integral through the change of variable:

$$t^* = T - t \implies \int_0^T \sinh(k(T-t))^2\, dt = -\int_T^0 \sinh(kt^*)^2\, dt^*$$

Using some trigonometry rules to move forward we obtain:

$$\sinh(kt^*)^2 = \cosh(2kt^*) - \cosh(kt^*)^2 = \cosh(2kt^*) - \sinh(kt^*)^2 - 1$$

$$\sinh(kt^*)^2 = \frac{\cosh(2kt^*) - 1}{2}$$

So that we can further write the variance as the following:

$$-\int_T^0 \sinh(kt^*)^2\, dt^* = \frac{1}{2}\int_0^T \cosh(2kt^*)\, dt^* - \frac{T}{2} = \frac{1}{4}\left(\frac{\sinh(2kT)}{k} - 2T\right)$$

$$\mathbb{V}[C_T - x_0 S_0] = \left(\frac{x_0\sigma}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) - 2kT}{4k}\right)$$

We can then write $\mathbb{E}[C_T - x_0 S_0]$ knowing $n_t = -\frac{\partial x_t}{\partial t}$, which we first develop:

$$\frac{\partial}{\partial t}x_t = -\frac{x_0 k \cosh(k(T-t)}{\sinh(kT)}$$

$$\mathbb{E}[C_T - x_0 S_0] = -\int_0^T \eta n_t^2\, dt = -\eta\left(\frac{x_0 k}{\sinh(kT)}\right)^2 \int_0^T \cosh(kt^*)^2\, dt^*$$

Again using some hyperbolic trigonometry tricks we have:

$$\cosh(kt^*)^2 = \cosh(2kt^*) - \sinh(kt^*)^2 = \cosh(2kt^*) - \cosh(kt^*)^2 + 1$$

$$\cosh(kt^*)^2 = \frac{\cosh(2kt^*) + 1}{2}$$

So that simplifying forward becomes easy:

$$\int_0^T \cosh(kt^*)^2\, dt^* = = \frac{1}{2}\int_0^T \cosh(2kt^*)\, dt^* + \frac{T}{2} = \frac{1}{4}\left(\frac{\sinh(2kT)}{k} + 2T\right)$$

$$\mathbb{E}[C_T - x_0 S_0] = -\eta\left(\frac{x_0 k}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) + 2kT}{4k}\right)$$

We can then plot the frontier efficient for different values of λ in (4).

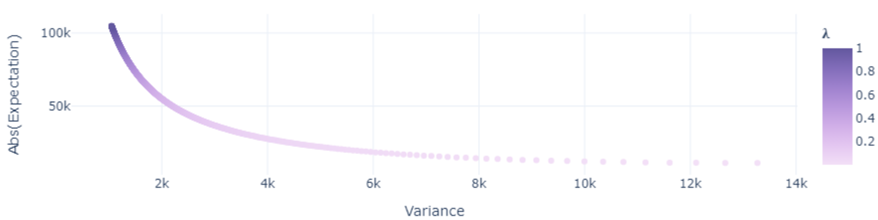

Figure 5: IS efficient frontier for varying *risk aversion*

We considered the same set of parameters as in section 2.2. The *y-axis* represents the absolute loss in cash. We observe that for a maximum λ, the trader is willing to bear a maximal loss exposure, which is in line with what is expected.

2.6 Target Close efficient frontier

We again derive hyperbolic functions to define the moments defined in (10). Given the optimal strategy computed in (13):

$$x(t) = x_0 \left(1 - \frac{\sinh(kt)}{\sinh(kT)}\right)$$

$$\mathbb{V}[C_T - x_0 S_T] = \sigma^2 \int_0^T (x_t - x_0)^2 \, dt = \left(\frac{\sigma x_0}{\sinh(kT)}\right)^2 \int_0^T \sinh(kt)^2 dt$$

$$\mathbb{V}[C_T - x_0 S_T] = \left(\frac{\sigma x_0}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) - 2T}{4k}\right)$$

We heavily relied on the mathematical developments from section 2.5 to integrate. We can then write $\mathbb{E}[C_T - x_0 S_0]$ knowing $n_t = -\frac{\partial x_t}{\partial t}$, which gives:

$$\frac{\partial}{\partial t} x_t = -\frac{x_0 k \cosh(kt)}{\sinh(kT)}$$

$$\mathbb{E}[C_T - x_0 S_0] = -\int_0^T \eta n_t^2 \, dt = -\eta \left(\frac{x_0 k}{\sinh(kT)}\right)^2 \int_0^T \cosh(kt)^2 \, dt$$

$$\mathbb{E}[C_T - x_0 S_0] = -\eta \left(\frac{x_0 k}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) + 2kT}{4k}\right)$$

For which we also plot the optimal portfolios following the system in (10).

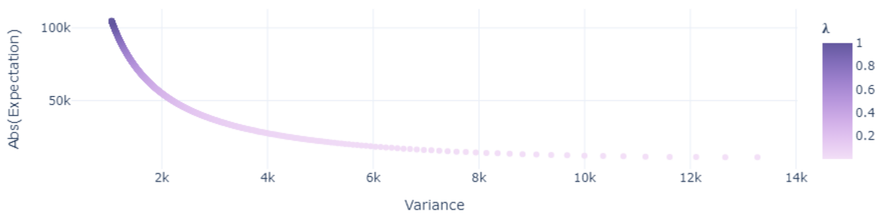

Figure 6: TC efficient frontier for varying *risk aversion*

Here we remark the graph is exactly the same as in 2.5. Mathematically it is because the formulas for $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ are equal.

2.7 TWAP efficient frontier

The efficient frontier for the TWAP is derived from its problem in (15). Given the closed-form optimal strategy in (18) we compute both moments:

$$x(t) = x_0 \left(1 - \frac{t}{T}\right)$$

$$\mathbb{V}\left[C_T - \frac{x_0}{T}\int_0^T S_t\, dt\right] = \sigma^2 \int_0^T \left(x_t - x_0\left(1 - \frac{t}{T}\right)\right)^2 dt = 0$$

We can then write the expectancy knowing $n_t = -\frac{\partial x_t}{\partial t} = -\frac{x_0}{T}$, which gives:

$$\mathbb{E}\left[C_T - \frac{x_0}{T}\int_0^T S_t\, dt\right] = -\int_0^T n_t h(n_t)\, dt = -\eta\left(\frac{x_0}{T}\right)^2 \int_0^T dt = -\frac{\eta x_0^2}{T}$$

The inventory is liquidated with constant speed and thus no variance. The expected loss is independent of the risk aversion coefficient λ and the efficient frontier is represented as a single dot.

2.8 Implementation Shortfall discrete Bellman resolution

Here we aim to identify the most effective liquidation tactics using a Bellman equation applied to a discretized grid. We'll demonstrate that by formulating the previously solved optimization problem as a dynamic problem, we can derive a Hamilton-Jacobi-Bellman equation. Furthermore, we will establish an equivalence between solving the problem within the Markovian/CARA framework and address it within the static Mean-Variance framework.

Let us first derive the HJB equation within the Markovian framework, which characterizes the optimal value function associated with the problem and provides optimal control strategies. We consider a utility function of the form:

$$u : x \to -\exp(-\gamma x) \tag{19}$$

The *Bellman* value function of our problem is such that:

$$v(t, x_{t+dt}, S_{t+dt}, C_{t+dt}) = \sup_n \mathbb{E}\left(u(C_T) \,|\, S_t = S, x_t = x, C_t = C\right) \tag{20}$$

We now want to derive an expression for $v(t+dt, x_{t+dt}, S_{t+dt}, C_{t+dt}) = v_{t+dt}$:

$$v_{t+dt} = v_t + \frac{\partial v_t}{\partial t}dt + \frac{\partial v_t}{\partial S}dS_t - \frac{\partial v_t}{\partial x}dx_t + \frac{\partial v_t}{\partial C}dC_t + \frac{1}{2}\frac{\partial^2 v_t}{\partial S^2}\left(dS_t\right)^2$$

$$dv = \frac{\partial v_t}{\partial t}dt + \frac{\partial v_t}{\partial S}\sigma dW_t - \frac{\partial v_t}{\partial x}n_t dt + \frac{\partial v_t}{\partial C}n_t(S_t - h(n_t))dt + \frac{\partial^2 v_t}{\partial S^2}\frac{\sigma^2}{2}dt$$

Next we compute the expectation of the above to average out stochastic terms:

$$\mathbb{E}[dv] = \mathbb{E}\left[\frac{\partial v_t}{\partial t}dt - \frac{\partial v_t}{\partial x}n_t dt + \frac{\partial v_t}{\partial C}n_t(S_t - h(n_t))dt + \frac{\partial^2 v_t}{\partial S^2}\frac{\sigma^2}{2}dt\right]$$

$$\mathbb{E}[dv] = n\left[-\frac{\partial v_t}{\partial x}dt + \frac{\partial v_t}{\partial C}(S - h(n))\right]dt + \left[\frac{\partial v_t}{\partial t} + \frac{\partial^2 v_t}{\partial S^2}\frac{\sigma^2}{2}\right]dt$$

We separated the static elements to the right for convenience. We go further by reducing the number of variables of this problem by guessing the form of their solutions. We propose the following ansatz:

$$v : t, x, S, C \to -\exp(-\gamma(Sx + C - \theta_{t,x})) \tag{21}$$

Such exponential utility is common as it ensures smoothness and can help ensure that the value function is well-defined. We can thus simplify further:

$$\frac{\partial v_t}{\partial t} + \frac{\partial^2 v_t}{\partial S^2}\frac{\sigma^2}{2} = \left[\frac{\partial \theta_t}{\partial t} + \frac{\gamma(\sigma x)^2}{2}\right]\gamma v$$

$$-\frac{\partial v_t}{\partial x}dt + \frac{\partial v_t}{\partial C}(S - h(n)) = \left[-\frac{\partial \theta_x}{\partial x} + h(n))\right]\gamma v$$

The problem is unchanged if we divide by γv, and can be easily inverted if we change signs. Thus we have the following optimization problem under boundary condition $\theta_{t,x} = 0$:

$$-\frac{\partial \theta_t}{\partial t} - \frac{\gamma(\sigma x)^2}{2} + \sup_n n\left[\frac{\partial \theta_x}{\partial x} - h(n))\right] = 0 \tag{22}$$

In this case, θ can be interpreted as the cost of keeping the stock. Let us now try this out from the *mean variance* optimization problem introduced above (4).

$$v_{t,x} = \min_{x} \int_{t}^{T} \left[\lambda\sigma^2 x_s^2 - \eta\dot{x_s}^2\right] ds$$

We know from Bellman theory that the current state value is the supremum of the expectation for the very next state. Thus we solve:

$$v_{t,x} = \inf_{\psi} \left[v(t+dt, x-\psi dt) + (\lambda\sigma^2 x^2 - \eta\psi^2)\, dt\right]$$

$$v_{t,x} = \inf_{\psi} \left[v(t,x) + \frac{\partial v}{\partial t}dt - \psi\frac{\partial v}{\partial x}dt + (\lambda\sigma^2 x^2 - \eta\psi^2)\, dt\right]$$

Which later gives the following HJB optimization problem:

$$\frac{\partial v_t}{\partial t} + \lambda\sigma^2 x^2 + \inf_{\psi} \psi \left[\eta\psi - \frac{\partial v}{\partial x}\right] = 0 \tag{23}$$

We will continue with the practical implementation later.

2.9 Target Close discrete Bellman resolution

Let us also proceed from the *mean variance* optimization problem introduced above (10) and its subsequent value function:

$$v_{t,x} = \min_{x} \int_{t}^{T} \left[\lambda\sigma^2 (x_s - x_0)^2 + \eta\dot{x_s}^2\right] ds$$

We continue to solve following Bellman theory by finding optimal action ψ:

$$v_{t,x} = \inf_{\psi} \left[v(t+dt, x-\psi dt) + (\lambda\sigma^2 (x-x_0)^2 + \eta\psi^2)\, dt\right]$$

$$v_{t,x} = \inf_{\psi} \left[v(t,x) + \frac{\partial v}{\partial t}dt - \psi\frac{\partial v}{\partial x}dt + (\lambda\sigma^2 (x-x_0)^2 + \eta\psi^2)\, dt\right]$$

Hence the resulting HJB equation:

$$\frac{\partial v_t}{\partial t} + \lambda\sigma^2 (x-x_0)^2 + \inf_{\psi} \psi \left[\eta\psi - \frac{\partial v}{\partial x}\right] = 0 \tag{24}$$

3 Optimal POV

In this exercise we restrict the set of strategies to *Percentage of Volume* orders, where the execution of the trade is contingent upon a specified percentage of the total trading volume of the security at the time. We assume constant market volume and constant liquidation speed $\dot{x_t} = -n_t = -v$ until portfolio emptiness.

3.1 IS optimal speed of liquidation in MV framework

We already worked on *Implementation Shortfall* orders within the *mean-variance* framework in section 2.2. The optimal pace of liquidation v can be found as a closed-form solution following the proof below, first using (1):

$$dC_t = n_t(S_t - h(n_t))\,dt = v(S_t - \eta v)\,dt$$

$$\int_0^T dC_t = v\left(\int_0^T S_t\,dt - \eta v T\right) = -\eta v^2 T + v\left(\int_0^T S_0 + \sigma W_t\,dt\right)$$

$$C_T - C_0 = -\eta v^2 T + v\left(TS_0 + \sigma\int_0^T (T-t)\,dW_t\right)$$

$$C_T - x_0 S_0 = -\eta v^2 T + S_0(vT - x_0) + \sigma v\int_0^T (T-t)\,dW_t \tag{25}$$

We will thus compute $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ to arrive at an optimization problem under the MV framework. Then we will derive the analytical solution and conclude.

$$\mathbb{V}(C_T - x_0 S_0) = (\sigma v)^2\,\mathbb{V}\left[\int_0^T (T-t)\,dW_t\right] = (\sigma v)^2\int_0^T (T-t)^2\,dt$$

$$\mathbb{V}(C_T - x_0 S_0) = (\sigma v)^2\int_0^T (T^2 - 2Tt + t^2)\,dt = (\sigma v)^2\,\frac{T^3}{3} = (\sigma v)^2\,\frac{T^3}{3}$$

$$\mathbb{V}(C_T - x_0 S_0) = \sigma^2\frac{x_0^3}{3v}\ \text{ because } \dot{x_t} = -v \implies x_0 = v\int_0^T dt = vT$$

The variance was easy to compute, most of the terms being constants. The first moment will also be straightforward as the Weiner process has mean 0.

$$\mathbb{E}(C_T - x_0 S_0) = -\eta v^2 T + S_0(vT - x_0) + \sigma v\,\mathbb{E}\left[\int_0^T (T-t)\,dW_t\right]$$

$$\mathbb{E}(C_T - x_0 S_0) = -\eta v^2 T + S_0(vT - x_0) = -\eta v x_0$$

We now plug both moments in the MV problem (4) under the constraints (2):

$$\min_v \left[\lambda\sigma^2 \frac{x_0^3}{3v} + \eta v x_0\right] = \min_v x_0 \left[\lambda\sigma^2 \frac{x_0^2}{3v} + \eta v\right] = \min_v x_0 L(v)$$

$$L(v) = \lambda\sigma^2 \frac{x_0^2}{3v} + \eta v \implies \frac{\partial L}{\partial v} = \eta - \lambda\sigma^2 \frac{x_0^2}{3v^2}$$

$$\frac{\partial L}{\partial v} = 0 \implies v = \sqrt{\frac{x_o^2\,\lambda\,\sigma^2}{3\eta}} \tag{26}$$

3.2 Visualizing optimal POV strategies

Given our constant liquidation speed computed in (26), we can now study the POV strategy under different sets of market parameters. We will thus be able to compare them to IS curves. Let us first write the analytical formula of liquidation for POV order given v:

$$dx_t = -vdt \implies x(t) = x_o - vt \tag{27}$$

We will try two sets of parameters and compare to the IS method already studied in section 2.2. We will observe high similarity in intuition.

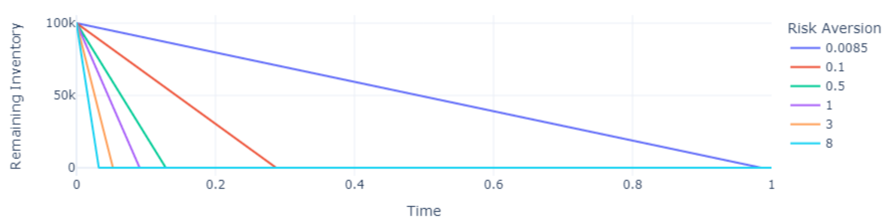

Figure 7: POV optimal liquidation for varying *risk aversion*

This figure shows as expected the constant pace of liquidation, through an execution scheme following a straight line. Very risk averse traders will also prefer to empty their inventory as fast as possible, with no particular degree of smoothness that was however observed in section 2.2 for the IS liquidation. Therefore, some might find themselves in a situation where they finished execution before the end of the day.

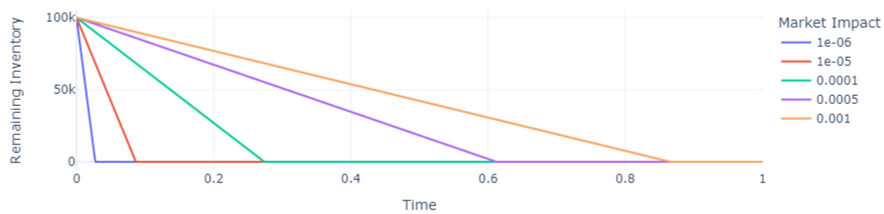

Figure 8: POV optimal liquidation for varying *temporary market impact*

As was observed in (2), the bigger the temporary market impact η and the more dispersed will the execution scheme be. This still makes sense as traders in both scenarios have high incentives to be prudent within an unstable market. Overall, POV and IS are highly similar in their intuition, only the IS incorporates a convexity in the pace of liquidation, generally spreading the execution process until the very end of the day.

3.3 POV efficient frontier

Given the closed-form liquidation in (26) we simplify both moments:

$$\mathbb{V}(C_T - x_0 S_0) = \sigma^2 \frac{x_0^3}{3v}$$

$$\mathbb{E}(C_T - x_0 S_0) = -\mu \sqrt{\frac{\lambda \sigma^2}{3\eta}} x_0^2$$

The plot is thus straightforward. By comparing to our IS resulting efficient frontier in (5), we notice that the POV frontier is slightly higher, especially at the curvature. This means that for each expected level of variance, the trader is willing to be exposed to more losses under the POV liquidation strategy.

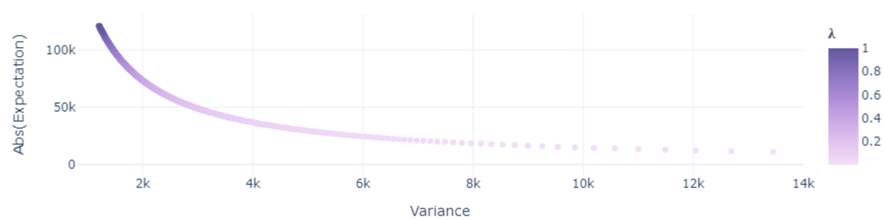

Figure 9: POV efficient frontier for varying *risk aversion*

We plotted below the difference between the two efficient frontier to provide a straightforward visual representation of this difference. We notice this difference is rather significant towards extreme cases, but remains existent for more regular couples.

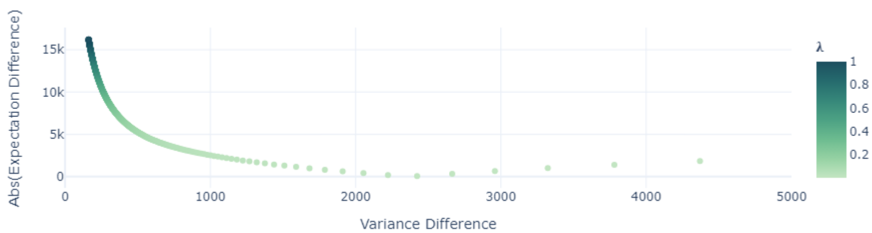

Figure 10: IS and POV efficient frontier differential for varying *risk aversion*

The reason why (10) shows POV is more efficient than IS is mainly explained by the inherent convex nature of IS. We observed earlier that for most risk averse traders, the liquidation under IS goes on until the very end of the period T, whereas the scheme might be done earlier when following a POV strategy. This gives the trader going through IS a longer exposure to market, where she has to bear inventory risk for the entire duration.

4 Liquidating two assets

In this exercise we consider two risky assets S_1 and S_2. We complete the initial set (1) with ρ the correlation of both Brownian motions. For $j = \{1, 2\}$:

$$\begin{cases} dx_t^{(j)} = \sigma_j dW_t^{(j)} \\ dS_t^{(j)} = \sigma_j dW_t^{(j)} \\ dC_t = \sum_{j=1}^{2} \left[n_t^{(j)} \left(S_t^{(j)} - h^{(j)} \left(n_t^{(j)} \right) \right) \right] dt \end{cases}$$

Boundary conditions are essentially the same as in (2), with $x_0 = (x_0^1, 0)$. As a first step we will draw upon our previous developments of the mean-variance framework. As in previous sections, we will compute $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ to thus solve the MV optimization problem under modified constraints (2). We then will do some stochastic control under HJM to retrieve the Bellman equation.

$$\int_0^T n_t^{(1)} h^{(1)} \left(n_t^{(1)} \right) + n_t^{(2)} h^{(2)} \left(n_t^{(2)} \right) dt = \int_0^T N_t dt \tag{28}$$

$$C_T - x_0^{(1)} S_0^{(1)} = \sigma_1 \int_0^T x_t^{(1)} dW_t^{(1)} + \sigma_2 \int_0^T x_t^{(2)} dW_t^{(2)} - \int_0^T N_t dt$$

We thus compute both moments, supposing an execution following IS:

$$\mathbb{E}\left(C_T - x_0^{(1)} S_0^{(1)}\right) = \sigma_1 \mathbb{E}\left[\int_0^T x_t^{(1)} dW_t^{(1)}\right] + \sigma_2 \mathbb{E}\left[\int_0^T x_t^{(2)} dW_t^{(2)}\right] - \int_0^T N_t dt$$

$$\mathbb{E}\left(C_T - x_0^{(1)} S_0^{(1)}\right) = -\int_0^T N_t dt$$

The first moment was almost as straightforward as the single asset models. The variance will however exhibit a covariance term:

$$\mathbb{V}\left(C_T - x_0^{(1)} S_0^{(1)}\right) = \mathbb{V}\left[\sigma_1 \int_0^T x_t^{(1)} dW_t^{(1)} + \sigma_2 \int_0^T x_t^{(2)} dW_t^{(2)}\right]$$

$$\mathbb{V}\left(C_T - x_0^{(1)} S_0^{(1)}\right) = \int_0^T \left[\sigma_1^2 \left(x_t^{(1)}\right)^2 + \sigma_2^2 \left(x_t^{(2)}\right)^2 + 2\,\sigma_1 \sigma_2 x_t^{(1)} x_t^{(2)} \rho\right] dt$$

Solving for the optimal strategy within MV, starting with (4). We assume:

$$\int_0^T N_t dt = \int_0^T \eta^{(1)} \left(\dot{x}_t^{(1)}\right)^2 + \eta^{(2)} \left(\dot{x}_t^{(2)}\right)^2 dt \tag{29}$$

We again assumed linearity of the temporary market impact. Going further:

$$\min_x \lambda \mathbb{V}\left[C_T - x_0^{(1)} S_0^{(1)}\right] - \mathbb{E}\left[C_T - x_0^{(1)} S_0^{(1)}\right]$$

$$\min_x \lambda \left[\int_0^T \left[\sigma_1^2 \left(x_t^{(1)}\right)^2 + \sigma_2^2 \left(x_t^{(2)}\right)^2 + 2\,\sigma_1 \sigma_2 x_t^{(1)} x_t^{(2)} \rho\right] dt\right] + \int_0^T N_t dt$$

$$\min_x \int_0^T \left(\lambda \left[\sigma_1^2 \left(x_t^{(1)}\right)^2 + \sigma_2^2 \left(x_t^{(2)}\right)^2 + 2\,\sigma_1 \sigma_2 x_t^{(1)} x_t^{(2)} \rho\right] + N_t\right) dt$$

We assume this system is equivalent to solving the below:

$$\min_x \int_0^T F\left(t, x_t^{(1)}, \dot{x}_t^{(1)}, x_t^{(2)}, \dot{x}_t^{(2)}\right)\, dt \tag{30}$$

As in section 2.2, we apply Euler-Lagrange notations for $i \neq j$:

$$\frac{\partial F}{\partial x_t^{(j)}} = 2\left[\lambda \sigma_j^2 x_t^{(j)} + \sigma_1 \sigma_2 x_t^{(i)} \rho\right]$$

$$\frac{\partial F}{\partial \dot{x_t}^{(j)}} = \frac{\partial N}{\partial \dot{x_t}^{(j)}} = 2\eta_j \dot{x_t}^{(j)} \implies \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x_t}^{(j)}}\right) = 2\eta_j \ddot{x_t}^{(j)}$$

Then assuming (5) holds:

$$\lambda\sigma_j^2 x_t^{(j)} + \sigma_1\sigma_2 x_t^{(i)}\rho = \eta_j \ddot{x_t}^{(j)} \implies \ddot{x_t}^{(j)} = x_t^{(j)}\frac{\lambda\sigma_j^2}{\eta_j} + x_t^{(i)}\frac{\sigma_1\sigma_2\rho}{\eta_j}$$

This system of equations can be written in matrix form if we let:

$$\ddot{x}_t = Ax_t \implies A = \begin{pmatrix} \frac{\lambda\sigma_1^2}{\eta_1} & \frac{\sigma_1\sigma_2\rho}{\eta_1} \\ \frac{\sigma_1\sigma_2\rho}{\eta_2} & \frac{\lambda\sigma_2^2}{\eta_2} \end{pmatrix}, \quad x_t = \begin{pmatrix} x_t^{(1)} \\ x_t^{(2)} \end{pmatrix} \tag{31}$$

We furthermore need to find the shape of the previous differential equation. We proceed with the transformation of this second-order differential equation into a first-order system. We first define $y_t = \dot{x_t} \implies \dot{y_t} = \ddot{x_t}$. Thus we write:

$$\begin{cases} y_t = \dot{x_t} \\ \dot{y_t} = A\ddot{x_t} \end{cases} \implies \dot{z_t} = B_t z_t\,, \quad B_t = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

The solution is given by $z_t = e^{B_t} z_0$. We would have to check if the matrix is diagonalizable. We will push this idea later.

4.1 Bellman equation on discretized grid

Based on the system in (30) we solve the Bellman value function, $x = \left(x^{(1)}, x^{(2)}\right)$:

$$v_{t,x} = \min_x \int_0^T \left(\lambda\left[\sigma_1^2\left(x_t^{(1)}\right)^2 + \sigma_2^2\left(x_t^{(2)}\right)^2 + 2\,\sigma_1\sigma_2 x_t^{(1)} x_t^{(2)}\rho\right] + N_t\right) dt$$

We continue to solve by finding optimal action $\psi = \left(\psi^{(1)}, \psi^{(2)}\right)$:

$$\int_0^T N_t dt = \int_0^T \eta^{(1)}\left(\psi_t^{(1)}\right)^2 + \eta^{(2)}\left(\psi_t^{(2)}\right)^2 dt \tag{32}$$

$$v\left(t+dt, x^{(1)} - \psi^{(1)}dt, x^{(2)} - \psi^{(1)}dt\right) = v_{t+dt,x-\psi dt}$$

$$v_{t,x} = \inf_\psi \left[v_{t+dt,x-\psi dt} + \left(\lambda\left[\sigma_1^2\left(x_t^{(1)}\right)^2 + \sigma_2^2\left(x_t^{(2)}\right)^2 + 2\,\sigma_1\sigma_2 x_t^{(1)} x_t^{(2)}\rho\right] + N_t\right)\right]$$

We simplify using Taylor expansion:

$$v_{t+dt,x-\psi dt} = v_{t,x} + \left(\frac{\partial v_t}{\partial t} - \psi^{(1)} \frac{\partial v_x}{\partial x^{(1)}} - \psi^{(2)} \frac{\partial v_x}{\partial x^{(2)}} \right) dt$$

$$\inf_{\psi} \frac{\partial v_t}{\partial t} - \psi^{(1)} \frac{\partial v_x}{\partial x^{(1)}} - \psi^{(2)} \frac{\partial v_x}{\partial x^{(2)}} + \lambda \left[\sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j x_t^{(i)} x_t^{(j)} \rho_{i,j} \right] + N_t = 0$$

$$\frac{\partial v_t}{\partial t} + \lambda \left[\sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j x_t^{(i)} x_t^{(j)} \rho_{i,j} \right] - \inf_{\psi} \left(\psi^{(1)} \frac{\partial v_x}{\partial x^{(1)}} + \psi^{(2)} \frac{\partial v_x}{\partial x^{(2)}} - N_t \right) = 0$$

We will thus have to implement the above. There is still heavy mathematical work to do.