Almgren-Chriss model and its extensions

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M203 Electronic markets project

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Project objectives overview

- Introduction of **Almgren-Chriss** model for trade execution [AC00].
- Balance of **market** risk and **liquidity** costs during significant trades.
- Extension of the model to a **continuous** setting.
- Taking inspirations from [PL24], [DBB22] and [Gué16].
- Examination of **benchmarks** like Target Close and TWAP.
- Comparison of discretized **Bellman** resolutions to analytic solutions.
- Implementation of **deep learning** for optimal liquidation strategies.
- Aims to lift key model **assumptions** via technological leverage.

Team split details

- Exercise 1 - **q1** and **q2**: Thibault
- Exercise 1 - **q3** and **q4**: François
- Exercise 2 - **q1** and **q2**: Thibault
- Exercise 2 - **q3**: François
- Exercise 3 - **q1**: François
- Exercise 4 - **q1** and **q3**: Thibault

Exercise 1: Target Close and TWAP orders - **q1**

- The stock inventory x_t follows the dynamics $dx_t = -n_t dt$.
- The spot price S_t is modeled by $dS_t = \sigma dW_t - g(n_t)dt$.
- Cash balance C_t updates as $dC_t = n_t(S_t - h(n_t))dt$.
- The system captures the convexity of the execution strategy.
- W_t is the Wiener process, modeling price as an ABM.

Exercise 1: Target Close and TWAP orders - **q1**

- Boundary conditions set inventory $x_0 = x$ and $x_T = 0$.
- Initially, the cash balance is zero $C_0 = 0$.
- The goal is to liquidate a given quantity x by the end of the process.
- No permanent market impact is assumed for this paper.

Exercise 1: Target Close and TWAP orders - q1

Implementation shortfall is the difference between the decision price and the execution price for a trade.

$$C_T - x_0 S_0 = \sigma \int_0^T x_t \, dW_t - \int_0^T n_t h(n_t) \, dt$$

We then would like to find the optimal strategy under the Mean-Variance framework, with λ being a *risk aversion* parameter.

$$\min_x \lambda V[C_T - x_0 S_0] - E[C_T - x_0 S_0]$$

Exercise 1: Target Close and TWAP orders - **q1**

The optimal liquidation strategy $x(t)$ is given by:

$$x(t) = \frac{x_0}{\sinh(kT)} \sinh(k(T-t))$$

We denote $k = \sigma\sqrt{\frac{\lambda}{\eta}}$.

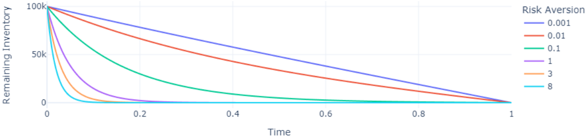

Figure: IS optimal liquidation for varying *risk aversion*

Exercise 1: Target Close and TWAP orders - **q1**

Target Close is when the trader aims at executing a trade at a price close to the closing price of the stock on a particular day.

$$C_T - x_0 S_T = \sigma \int_0^T x_t \, dW_t - \int_0^T n_t h(n_t) \, dt - x_0 \sigma W_T$$

We then would like to find the optimal strategy under the Mean-Variance framework, with λ being a *risk aversion* parameter.

$$\min_x \lambda V[C_T - x_0 S_0] - E[C_T - x_0 S_0]$$

Exercise 1: Target Close and TWAP orders - **q1**

The optimal liquidation strategy $x(t)$ is given by:

$$x(t) = x_0 \left(1 - \frac{\sinh(kt)}{\sinh(kT)}\right)$$

We denote $k = \sigma\sqrt{\frac{\lambda}{\eta}}$.

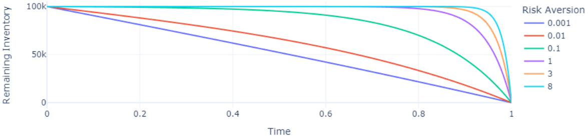

Figure: TC optimal liquidation for varying *risk aversion*

Exercise 1: Target Close and TWAP orders - **q1**

Time Weighted Average Price (**TWAP**) aims to minimize market impact of large trades by spreading the order out over a specific time period.

$$C_T - \frac{x_0}{T}\int_0^T S_t \, dt = \sigma \int_0^T x_t - x_0\left(1 - \frac{t}{T}\right) dW_t - \int_0^T n_t h(n_t)\, dt$$

We then would like to find the optimal strategy under the Mean-Variance framework, with λ being a *risk aversion* parameter.

$$\min_x \lambda V\left[C_T - \frac{x_0}{T}\int_0^T S_t \, dt\right] - E\left[C_T - \frac{x_0}{T}\int_0^T S_t \, dt\right]$$

Exercise 1: Target Close and TWAP orders - **q1**

The optimal liquidation strategy $x(t)$ is given by:

$$x(t) = x_0 \left(1 - \frac{t}{T}\right)$$

The plot would only be a straight decreasing curve with no further interest, regardless of the market parameters.

Exercise 1: Target Close and TWAP orders - **q2**

The efficient frontier represents a set of optimal portfolios that offer the highest expected return for a defined level of risk. For all three types of orders, we express the moments as functions of σ, x_0, t, λ and η.

IS efficient frontier

$$\mathbb{V}[C_T - x_0 S_0] = \left(\frac{x_0 \sigma}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) - 2kT}{4k}\right)$$

$$\mathbb{E}[C_T - x_0 S_0] = -\eta \left(\frac{x_0 k}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) + 2kT}{4k}\right)$$

Exercise 1: Target Close and TWAP orders - **q2**

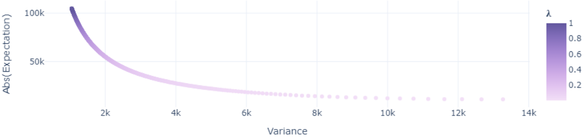

Figure: IS efficient frontier for varying *risk aversion*

Exercise 1: Target Close and TWAP orders - **q2**

TC efficient frontier

$$\mathbb{V}[C_T - x_0 S_T] = \left(\frac{\sigma x_0}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) - 2T}{4k}\right)$$

$$\mathbb{E}[C_T - x_0 S_0] = -\eta \left(\frac{x_0 k}{\sinh(kT)}\right)^2 \left(\frac{\sinh(2kT) + 2kT}{4k}\right)$$

Exercise 1: Target Close and TWAP orders - **q2**

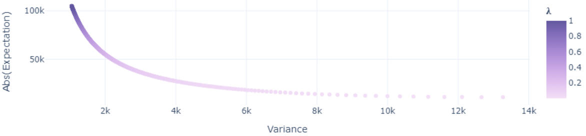

Figure: TC efficient frontier for varying *risk aversion*

TWAP efficient frontier

$$\mathbb{V}\left[C_T - \frac{x_0}{T}\int_0^T S_t\,dt\right] = 0$$

$$\mathbb{E}\left[C_T - \frac{x_0}{T}\int_0^T S_t\,dt\right] = -\frac{\eta x_0^2}{T}$$

The inventory is liquidated with constant speed and thus no variance. The expected loss is independent of the risk aversion coefficient λ and the efficient frontier would be represented as a single dot.

Exercise 1: Target Close and TWAP orders - **q3**

We consider a utility function of the form:

$$u : x \mapsto -exp(-\gamma x)$$

The Bellman value function of our problem is such that the very next iteration value comes from the action that maximizes an expected utility for a game, ie:

$$\nu(t+dt, x_{t+dt}, S_{t+dt}, C_{t+dt}) = \sup_{\psi} \mathbb{E}\left[u(C_T) | S_t = S, x_t = x, C_t = C\right]$$

Using Taylor-Itô expansion, and computing the expectation, we have:

$$\mathbb{E}[d\nu] = \psi\left[-\frac{\partial \nu}{\partial x} + \frac{\partial \nu}{\partial C}(S - h(\psi))\right]dt + \left[\frac{\partial \nu}{\partial t} + \frac{\partial^2 \nu}{\partial S^2}\frac{\sigma^2}{2}\right]dt$$

Exercise 1: Target Close and TWAP orders - **q3**

To reduce the number of variables, we propose the following form for the solutions:

$$\nu : t, x, S, C \mapsto -exp(-\gamma(Sx + C - \theta_{t,x}))$$

Simplifying further the expectation, we get the following optimization problem:

$$\frac{\partial \nu}{\partial t} + \frac{\gamma}{2}(\sigma x)^2 + \inf_{\psi} \psi \left[h(\psi) - \frac{\partial \nu}{\partial x} \right] = 0$$

IS Bellman equation

We know from Bellman theory that the current state value is the supremium of the expectation for the very next state. Thus, from the mean-variance framework introduced in the previous slides, we have to solve:

$$\nu_{t,x} = \inf_{\psi} \left[\nu(t,x) + \frac{\partial \nu}{\partial t} dt - \psi \frac{\partial \nu}{\partial x} dt + (\lambda \sigma^2 x^2 + \eta \psi^2) dt \right]$$

Which gives the following HJB optimization problem:

$$\frac{\partial \nu}{\partial t} + \lambda (\sigma x)^2 + \inf_{\psi} \psi \left[\eta \psi - \frac{\partial \nu}{\partial x} \right] = 0$$

Exercise 1: Target Close and TWAP orders - **q4**

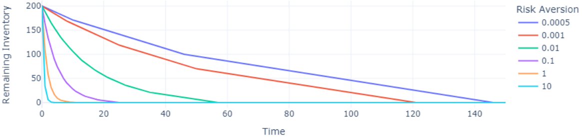

Figure: IS Bellman optimal liquidation for varying *risk aversion*

Exercise 1: Target Close and TWAP orders - **q4**

TC Bellman equation
In the same way, from the mean-variance optimization problem, we have to solve:

$$\nu_{t,x} = \inf_{\psi} \left[\nu(t,x) + \frac{\partial \nu}{\partial t} dt - \psi \frac{\partial \nu}{\partial x} dt + (\lambda \sigma^2 (x - x_0)^2 + \eta \psi^2) dt \right]$$

Which yields the following HJB equation:

$$\frac{\partial \nu}{\partial t} + \lambda \sigma^2 (x - x_0)^2 + \inf_{\psi} \psi \left[\eta \psi - \frac{\partial \nu}{\partial x} \right] = 0$$

Exercise 1: Target Close and TWAP orders - **q4**

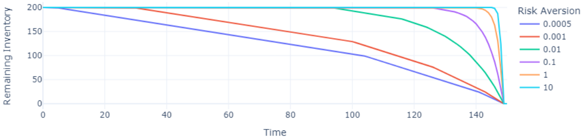

Figure: TC Bellman optimal liquidation for varying *risk aversion*

Exercise 2: Optimal POV - **q1**

In this exercise we restrict the set of strategies to *Percentage of Volume* orders, where the execution of the trade is contingent upon a specified percentage of the total trading volume of the security at the time.

We assume **constant** market volume and **constant** liquidation speed $\dot{x}_t = -n_t = -v$ until portfolio emptiness.

Exercise 2: Optimal POV - **q1**

The optimal pace of liquidation v can be found as a closed-form:

$$dC_t = n_t(S_t - h(n_t))\,dt = v(S_t - \eta v)\,dt$$

$$\int_0^T dC_t = v\left(\int_0^T S_t\,dt - \eta v T\right) = -\eta v^2 T + v\left(\int_0^T S_0 + \sigma W_t\,dt\right)$$

Exercise 2: Optimal POV - **q1**

We derive the expectation and variance to solve the optimization problem under the MV framework:

$$C_T - C_0 = -\eta v^2 T + v \left(TS_0 + \sigma \int_0^T (T - t)\, dW_t \right)$$

We thus compute $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ to derive the analytical solution:

$$\mathbb{V}(C_T - x_0 S_0) = \frac{x_0^3 \sigma^2}{3v}$$
$$\mathbb{E}(C_T - x_0 S_0) = -\eta v x_0$$

Plugging both moments into the MV problem under the constraints:

$$v = \sqrt{\frac{x_0^2 \sigma^2 \lambda}{3\eta}}$$

Exercise 2: Optimal POV - **q2**

Given our constant liquidation speed computed, we can now study the POV strategy under different sets of market parameters. We will thus be able to compare them to IS curves. Let us first write the analytical formula of liquidation for POV order given v:

$$dx_t = -vdt \quad \Rightarrow \quad x(t) = x_0 - vt$$

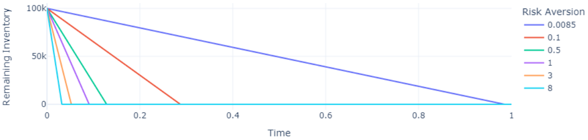

Figure: POV optimal liquidation for varying *risk aversion*

Exercise 2: Optimal POV - **q2**

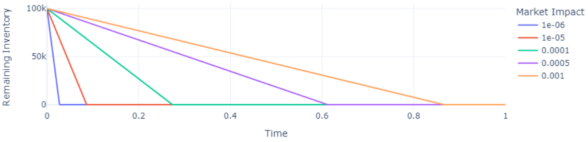

Figure: POV optimal liquidation for varying *market impact*

The bigger the temporary market impact η and the more dispersed will the execution scheme be.

Overall, POV and IS are highly similar in their intuition, only the IS incorporates a convexity in the pace of liquidation.

Exercise 2: Optimal POV - **q3**

Given the closed-form liquidation speed, we can simplify both moments:

$$\mathbb{V}(C_T - x_0 S_0) = \frac{\sigma^2 x_0^3}{3\nu}$$

$$\mathbb{E}(C_T - x_0 S_0) = -\sqrt{\frac{\eta\lambda\sigma^2}{3x_0}}$$

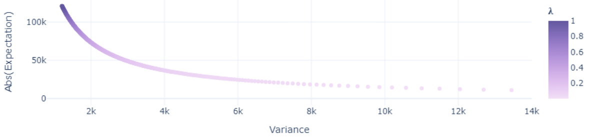

Figure: POV efficient frontier for varying *market impact*

Exercise 2: Optimal POV - **q3**

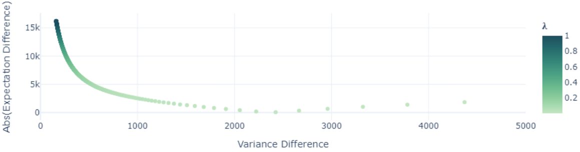

Figure: IS and POV efficient frontier differential for varying *market impact*

The POV frontier is slightly higher, especially at the curvature.
For each expected level of variance, the trader is willing to be exposed to more losses under the POV liquidation strategy

Exercise 3: Liquidating two assets - **q1**

Our system becomes, for $j = \{1, 2\}$:

$$\begin{cases} dx_t^{(j)} = \sigma_j \, dW_t^{(j)} \\ dS_t^{(j)} = \sigma_j \, dW_t^{(j)} \\ dC_t = \sum_{j=1}^{2} \left[n_t^{(j)} \left(S_t^{(j)} - h^{(j)}(n_t^{(j)}) \right) \right] dt \end{cases}$$

With boundary conditions:

$$\begin{cases} x_0 = (x_0^1, 0) \\ x_T = (0, 0) \\ C_0 = 0 \end{cases}$$

Exercise 3: Liquidating two assets - **q1**

Drawing upon our previous developments of the mean-variance framework in exercise 1, we want to solve the following Bellman value function:

$$\nu_{t,x} = \min_{x} \int_0^T \left(\lambda \left[\sigma_1^2 \left(x_t^{(1)} \right)^2 + \sigma_2^2 \left(x_t^{(2)} \right)^2 + 2\sigma_1 \sigma_2 x_t^{(1)} x_t^{(2)} \rho \right] + N_t \right) dt$$

Which yields:

$$\frac{\partial \nu_t}{\partial t} + \lambda \left[\sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j x_t^{(i)} x_t^{(j)} \rho_{i,j} \right] + \inf_{\psi} \left(N_t - \psi^{(1)} \frac{\partial \nu_x}{\partial x^{(1)}} - \psi^{(2)} \frac{\partial \nu_x}{\partial x^{(2)}} \right) = 0$$

Exercise 3: Liquidating two assets - **q1**

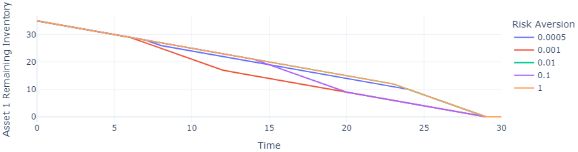

Figure: Two-dimensional IS liquidiation for varying *market impact*

Exercise 4: Deep learning approach to liquidation - **q1**

- Role of neural networks in liquidation strategy optimization as a function approximator.
- Mapping the state of a trading system to profitable actions through complex non-linear relationships.
- Inputs: time, current inventory level, current price of the asset.
- Output: the quantity to sell at each time step.

Exercise 4: Deep learning approach to liquidation - **q1**

- Neural network structure: a succession of connected layers to capture market complexities.
- Role of neurons: learning specific aspects of the trading strategy.
- Importance of each aspect measured by neuron weights, adjusted through backpropagation.
- Goal: optimize trade-off between execution pace and market impact, minimizing total cost.
- Learning phase: training the network to improve liquidation iteratively.

Exercise 4: Deep learning approach to liquidation - **q1**

- Steps to structure the deep learning model for liquidation strategy.
- Differentiation of nets in terms of liquidation and inventory costs as per previous sections.
- Standard architecture and initial market parameters across nets, benchmarking against spot dynamics simulated.
- Synthetic infinite loss to drive inventory liquidation before maturity.
- Plotting of the model for different values of λ to assess consistency.
- Implementation of a feedforward neural network with three hidden layers, decreasing neurons, and dropout layers to prevent overfitting.
- Use of ReLu activation function and the Adam optimizer.

Exercise 4: Deep learning approach to liquidation - **q1**

- Training of model with variable λ for plotting different trajectories.
  - Model trained over 250 episodes.
  - Objective to reduce portfolio to 1€ to avoid loss explosion.
  - Results scattered to accelerate computation.
- Application of new method to **implementation shortfall strategy**.
  - Model learns the convexity inherent to liquidation quickly.
  - Struggles with capturing risk aversion's influence, which can accelerate liquidation at high levels.
- Assessment of **participation of volume strategy**.
  - Inability to isolate optimal liquidation strategy for large/close orders.
  - With POV orders, the algorithm captures the overall shape of the liquidation curve and accounts for risk aversion.

Exercise 4: Deep learning approach to liquidation - **q1**

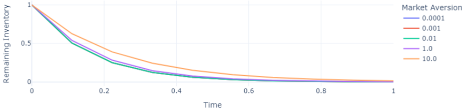

Figure: Neural net optimal IS liquidation for varying *risk aversion*

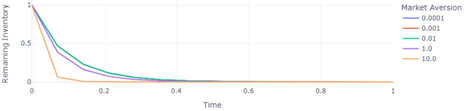

Figure: Neural net optimal POV liquidation for varying *risk aversion*

Exercise 4: Deep learning approach to liquidation - **q3**

- Exploration of stochastic dynamics for the volatility of the underlying.
- Utilization of the Cox-Ingersoll-Ross model for liquidation strategy under stochastic volatility.
- Adaptation of the model originally designed for interest rates to one asset liquidation strategy.
- The stochastic differential equation (SDE) includes:
  - Volatility of volatility (σ) parameter.
  - Reversion to mean (κ) parameter.
  - Mean-reversion and non-negativity as natural attributes for volatility.
- Transformation into a discrete version for the neural net framework.
- Strategy comparison with dynamic volatility updates across iterations.
- Continuity with previous sections via parameter consistency.

Exercise 4: Deep learning approach to liquidation - **q3**

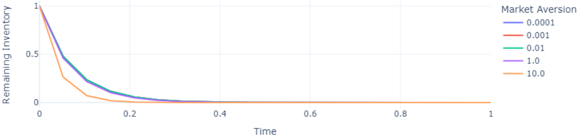

Figure: Neural net optimal IS liquidation with stochastic volatility

Conclusion

- In this project we implemented optimal liquidation strategies for different types of orders [AC00].
- Our modest contribution was through the proposition of an alternative method for finding the optimal liquidation through neural nets
- We then leveraged this technology to lift the assumption of constant volatility and make our model more realistic.

Setbacks and remarks

- Lifting the assumption of no permanent market impact would have yielded more realistic results, at the cost of increased mathematical complexity.
- Difficulties to implement 2D-Bellman in *Python* and to fully capture the logic behind the multi-dimensional backward algorithm.
- Struggled to develop a neural network capable of fully learning the TC order logic and strategy.

References

[AC00] Robert Almgren and Neil Chriss. *Optimal Execution of Portfolio Transactions*. Dec. 2000. URL: https://docplayer.net/20786814-Optimal-execution-of-portfolio-transactions.html.

[Gué16] Olivier Guéant. *The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, 2016.

[DBB22] Sebastien David, Arthur Bagourd, and Mounah Bizri. *A Tale of Two Models: Implementing the Almgren-Chriss framework through nonlinear and dynamic programming*. Aug. 2022.

[PL24] Jiang Pu and Gaelle Le Fol. *Electronic Markets*. Course taught at Université Paris-Dauphine, Master 203 program. 2024.