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Stagnation flow towards a shrinking sheet

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Abstract

The stagnation flow towards a shrinking sheet is studied. A similarity transform reduces the Navier–Stokes equations to a set of non-linear ordinary differential equations which are then integrated numerically. Both two-dimensional and axisymmetric stagnation flows are considered. It is found that solutions do not exist for larger shrinking rates and may be non-unique in the two-dimensional case. The non-alignment of the stagnation flow and the shrinking sheet complicates the flow structure. Convective heat transfer decreases with the shrinking rate due to an increase in boundary layer thickness.

Keywords: Stagnation flow; Shrinking sheet; Similarity solution

1. Introduction

Stagnation flow, describing the fluid motion near the stagnation region, exists on all solid bodies moving in a fluid. The stagnation region encounters the highest pressure, the highest heat transfer, and the highest rates of mass deposition. Hiemenz [1] was first to study two-dimensional stagnation flow using a similarity transform to reduce the Navier–Stokes equations to non-linear ordinary differential equations. The axisymmetric case was solved by Homann [2]. On the other hand, the flow due to a stretching sheet, which is important in extrusion problems, also admits similarity solutions. Crane [3] found a closed form solution to two-dimensional stretching, and Wang [4] obtained similarity solutions to the axisymmetric case. Recently, the combination of both stagnation flow and stretching surface was considered by Mahapatra and Gupta [5,6] and extended to oblique stagnation flow by Lok et al. [7].

The present paper studies the stagnation flow towards a *shrinking* sheet. The situation occurs, for example, on a rising, shrinking balloon. Notice that solutions do not exist for a shrinking sheet in an otherwise still fluid, since vorticity could not be confined in a boundary layer. However, with an added stagnation flow to contain the vorticity, similarity solutions may exist. These solutions are also exact solutions of the Navier–Stokes equations. Fig. 1 shows the cases considered in this paper: two-dimensional stagnation flow on a two-dimensional shrinking sheet and axisymmetric stagnation flow on an axisymmetric shrinking sheet. The symmetry line of the stagnation flow and that of the sheet need not be aligned.

2. The two-dimensional case

Let (u, υ, w) be velocity components in the Cartesian coordinates (x, y, z), respectively. Consider a two-dimensional potential stagnation flow at infinity given by $u = ax$, $w = -az$ where a is the strength of the stagnation flow. On the stretching surface the velocities are $u = b(x + c)$, $w = 0$ where b is the stretching rate (shrinking if $b < 0$) and $-c$ is the location of the stretching origin. Notice that the stretching axis and the stagnation flow are, in general, not aligned. Previous authors [5–7] considered the aligned cases only, where $c = 0$.

The following similarity transforms are used:

$$u = axf'(\eta) + bch(\eta), \quad \upsilon = 0, \quad w = -\sqrt{\nu a}\, f(\eta), \tag{1}$$

$$\eta = \sqrt{a/\nu}\, z, \tag{2}$$

where ν is the kinematic viscosity. Notice that our normalization differs from [5–7] since we need to consider negative b values. Continuity is automatically satisfied. Upon substitution of Eqs. (1), (2) into the Navier–Stokes equations, a set of similarity

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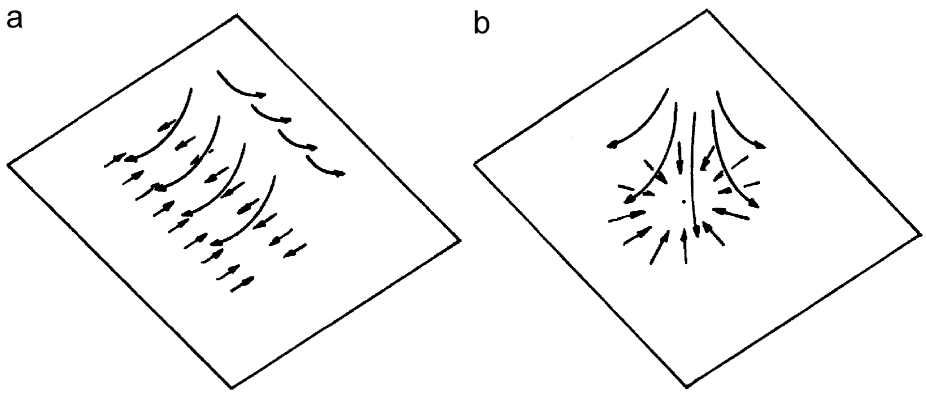

Fig. 1. (a) Two-dimensional stagnation flow on a two-dimensional shrinking sheet and (b) axisymmetric stagnation flow on an axisymmetric shrinking sheet.

Table 1
The initial values for the stretching sheet

α	0	0.1	0.2	0.5	1	2	5
$f''(0)$	1.232588	1.14656	1.05113	0.71330	0	−1.88731	−10.26475
			1.05129*	0.71334*		−1.88733*	−10.2648*
				0.71329•		−1.8874•	−10.265•
$h'(0)$	−0.811301	−0.86345	−0.91330	−1.05239	−1.25331	−1.58957	−2.33810

The superscript * is from [7] and • from [5], adjusted using our normalization.

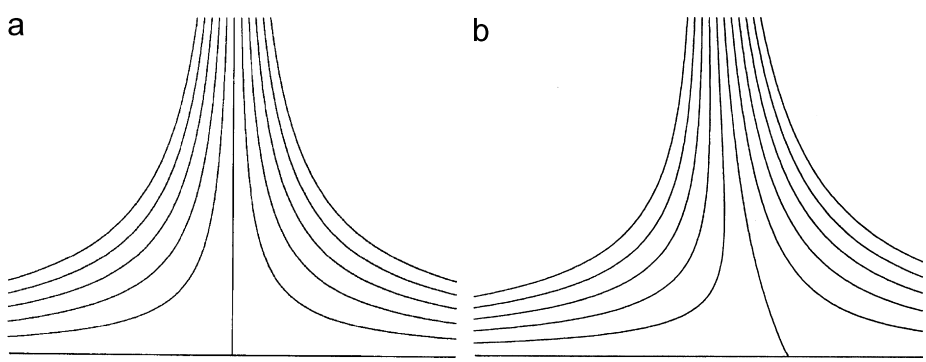

Fig. 2. Streamlines for the two-dimensional stretching case, $\alpha = 0.5$, $\Delta\psi/\sqrt{a\nu} = 0.1$: (a) aligned, $c = 0$ and (b) non- aligned, $c = -1$.

non-linear ordinary differential equations results

$$(f')^2 - ff'' = 1 + f''', \tag{3}$$

$$hf' - fh' = h''. \tag{4}$$

The boundary conditions are

$$f(0) = 0, \quad f'(0) = b/a \equiv \alpha, \tag{5}$$

$$f'(\infty) = 1, \tag{6}$$

$$h(0) = 1, \quad h(\infty) = 0. \tag{7}$$

The pressure p can be recovered by

$$p = p_0 - \rho a^2 x^2/2 - \rho w^2/2 + \rho \nu w_z. \tag{8}$$

Here p_0 is the stagnation pressure, and ρ is the fluid density.

By guessing $f''(0)$, Eqs. (3), (5) can be integrated as an initial value problem using a shooting method to satisfy Eq. (6). After f is obtained, Eqs. (4), (7) are then integrated similarly. For the stretching sheet, α is positive. Table 1 shows a comparison of initial values with those of the previous literature.

We see that the initial values for f compare well with existing published values. These values also agree with those of Wang [8] who studied a related impinging flow problem. The initial values for the non-alignment function h are new. Solutions exist for all $\alpha \geqslant 0$ and are unique. Notice that $\alpha = 0$ is Hiemenz flow, and $\alpha = 1$ is a degenerate inviscid flow where the stretching matches the conditions at infinity [9]. The non-alignment effect is not trivial for any α. Fig. 2a shows the symmetric stagnation flow towards a stretching sheet ($\alpha = 0.5$) when the origins of the stagnation flow and the stretching sheet are aligned. Fig. 2b shows that any non-alignment (here $c = -1$) destroys the symmetry.

The value of α is negative for the shrinking sheet. The numerical values are found in Table 2.

Table 2
Initial values for the shrinking sheet

α	-0.25	-0.5	-0.75	-1	-1.15	-1.2465
$f''(0)$	1.40224	1.49567	1.48930	1.32882	1.08223	0.55430
				0	0.116702	
$h'(0)$	-0.66857	-0.50145	-0.29376	0	0.297995	0.99904
				∞	0.276345	

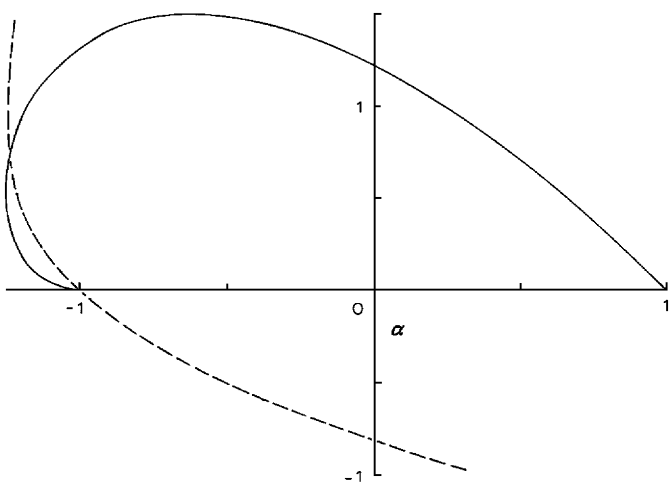

Fig. 3. Initial values versus α, two-dimensional case. — $f''(0)$, ---- $h'(0)$.

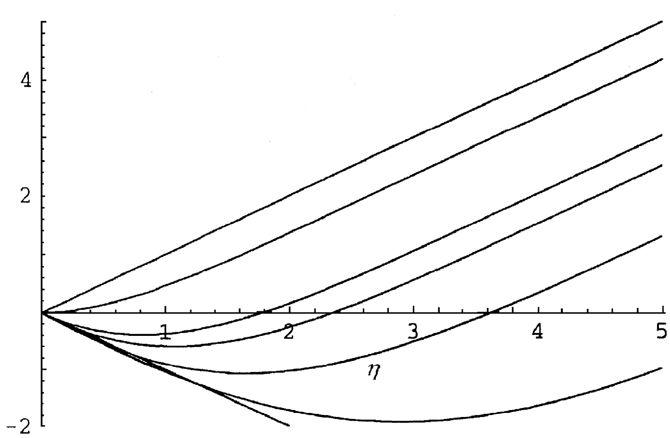

Fig. 4. The function $f(\eta)$ for the two-dimensional case. α values from top: 1, 0, -1, -1.15, -1.2465, -1.15, -1.

We find that the solution is unique for $\alpha > -1$, two solutions for $-1.2465 < \alpha \leqslant -1$ and no (similarity) solution for $\alpha < -1.2465$. Fig. 3 shows the trajectories of the initial values. The universal curves for the function f are shown in Fig. 4. All curves start from the origin and approach unit slope for large η. When $\alpha = 1$, the flow is inviscid, and no boundary layers are needed. The case $\alpha = 0$ is the Hiemenz flow towards a solid plate. For all shrinking cases ($\alpha < 0$) the function f is negative initially, showing regions of reverse cellular flow. Notice, that there are two solutions for $-1.2465 < \alpha \leqslant -1$. The curves do cross over somewhat for α close to -1. The boundary layer thickness increases as the recirculation zone increases (the states along the trajectory in Fig. 3) and finally the boundary layer thickness is infinite when at $\alpha = -1$, $f''(0) = 0$. The behavior for the universal function h is shown in Fig. 5. The effect of non-alignment is small for the stretching sheet, but large for the shrinking sheet. For two-dimensional flow, the normalized streamlines can be defined by

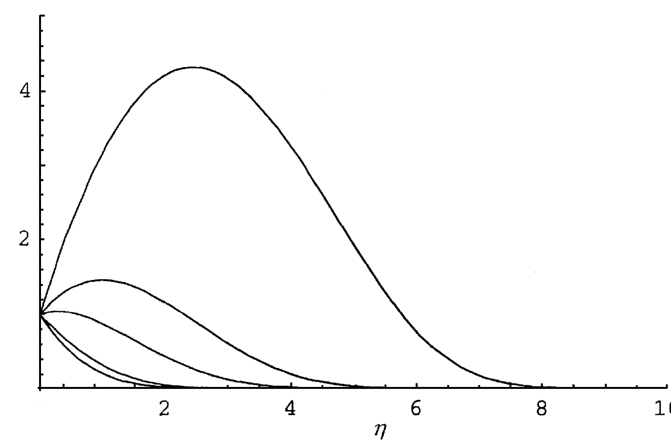

Fig. 5. The function $h(\eta)$ for the two-dimensional case. α values from top: -1.15, -1.2465, -1.15, 0, 1.

$$\frac{\psi}{\sqrt{a\nu}} = xf(\eta) + \alpha c \int_0^\eta h(\eta)\,\mathrm{d}\eta. \tag{9}$$

Fig. 6 shows the effect of non-alignment distance c.

3. Axisymmetric stagnation flow towards an axisymmetric shrinking surface

Owing to possible non-alignment, it is more appropriate to use Cartesian axes instead of cylindrical axes. Let

$$u = axf'(\eta) + bch(\eta), \quad v = ayf'(\eta), \quad w = -2\sqrt{\nu a}\,f(\eta). \tag{10}$$

The distance between the axis of the stagnation flow and the center of the shrinking surface is c. The Navier–Stokes equations are reduced to

$$(f')^2 - 2ff'' = 1 + f''', \tag{11}$$

$$hf' - 2fh' = h''. \tag{12}$$

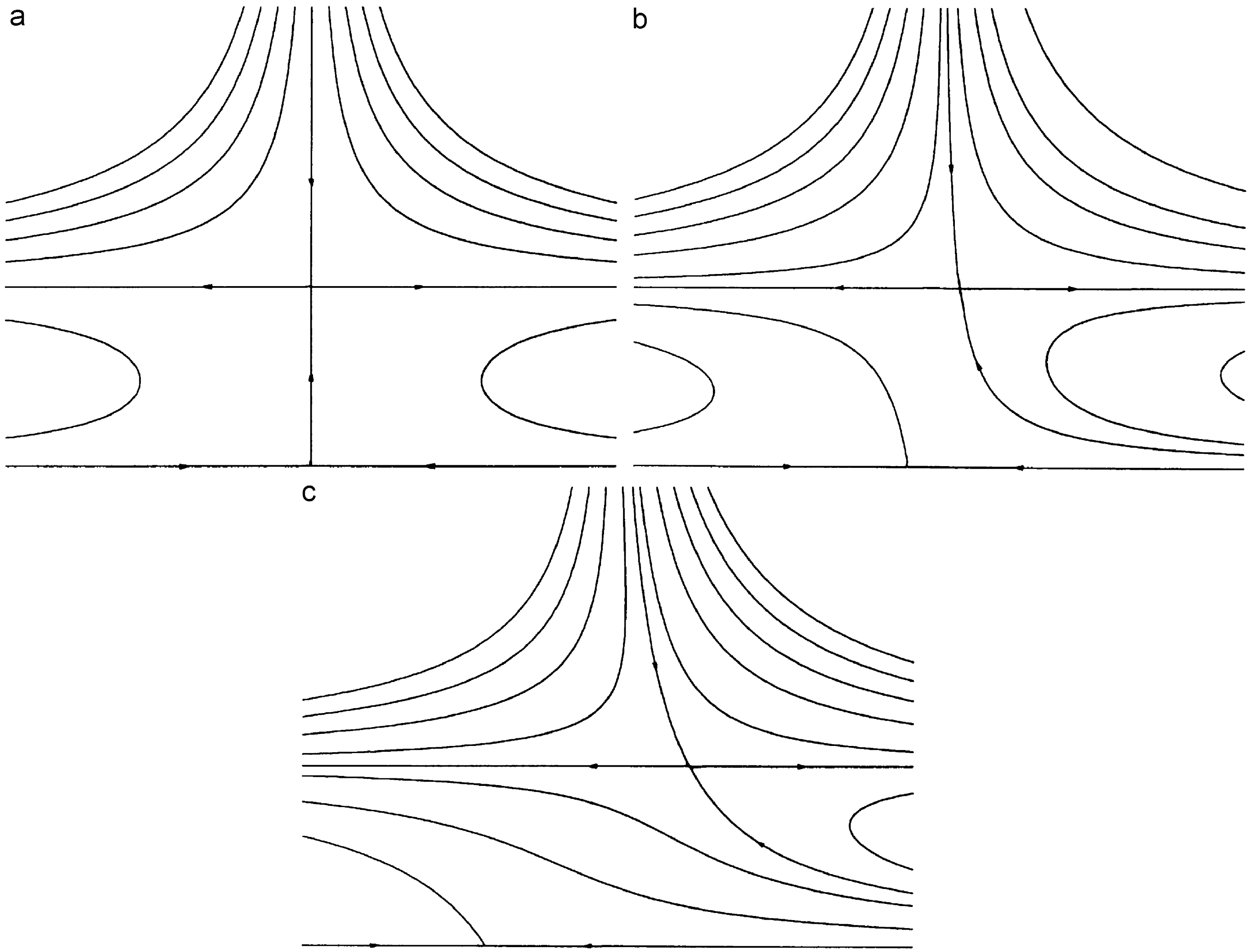

Fig. 6. Streamlines showing reverse flow for the shrinking sheet, $\alpha=-0.5$, $\Delta\psi/\sqrt{a\nu}=0.1$: (a) aligned, $c=0$; (b) non- aligned, $c=0.2$; and (c) non- aligned, $c=0.8$.

Table 3
Initial values for the axisymmetric case

α	0	0.1	0.2	0.5	1	2	5
$f''(0)$	1.311938	1.22911	1.13374	0.78032 0.77262*	0	−2.13107 −2.1182*	−11.8022 −11.744*
$h'(0)$	−0.93873	−1.0040	−1.0659	−1.2355	−1.4793	−1.8800	−2.7617

Asterisks are from Ref. [6], adjusted using our normalization.

The boundary conditions are similar to the two-dimensional case

$$f(0)=0, \quad f'(0)=b/a\equiv\alpha, \tag{13}$$

$$f'(\infty)=1, \tag{14}$$

$$h(0)=1, \quad h(\infty)=0. \tag{15}$$

The only previous work is due to Mahapatra and Gupta [6] for stretching ($\alpha>0$) and aligned ($c=0$). Our results for the stretching case are given in Table 3.

Notice that $\alpha=0$ is the axisymmetric stagnation flow towards a solid plate [2] and $\alpha=1$ corresponds to inviscid flow with no boundary layer.

The results for the shrinking case are given in Table 4. For the axisymmetric case, solutions seem to be unique for $\alpha\geqslant-1$. No solutions are found for $\alpha<-1$. Fig. 7 shows the trajectory of the initial values. The functions f and h are depicted in Figs. 8 and 9 for various α values. We comment that if the stagnation flow and the stretching (shrinking) surface are not aligned, the axial symmetry is completely destroyed. In contrast

Table 4
Initial values for the axisymmetric shrinking sheet

α	-0.25	-0.5	-0.75	-0.95	-0.9945	-1
$f''(0)$	1.45664	1.49001	1.35284	0.94690	0.5	0
$h'(0)$	-0.75639	-0.53237	-0.22079	0.26845	0.83183	∞

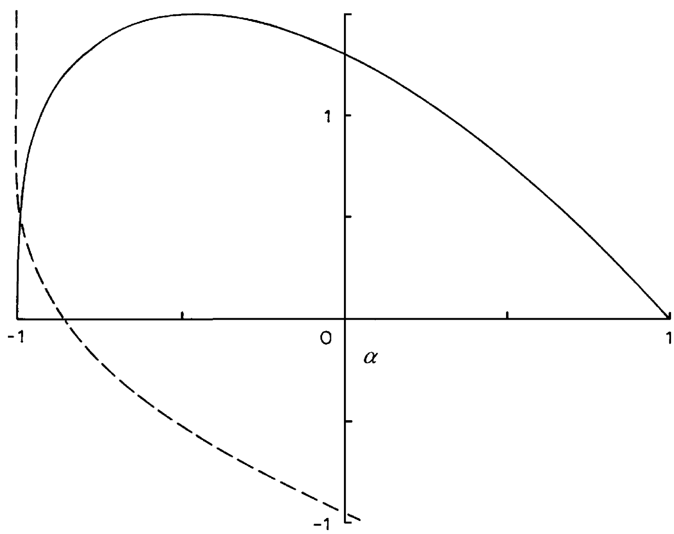

Fig. 7. Initial values versus α, axisymmetric case. — $f''(0)$, ---- $h'(0)$.

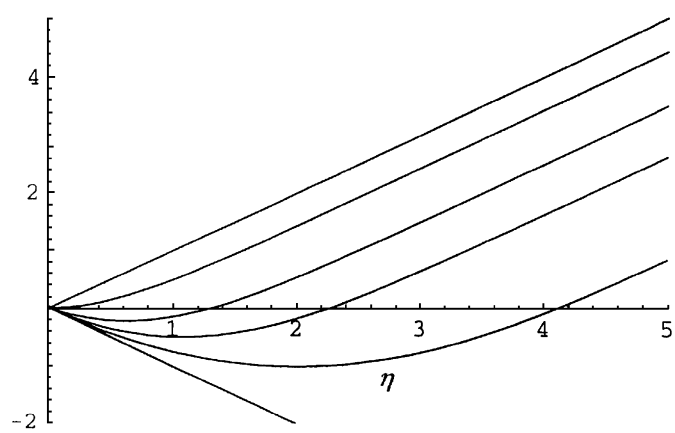

Fig. 8. The function $f(\eta)$ for the axisymmetric case. α values from top: 1, 0, -0.75, -0.95, -0.99945.

to the two-dimensional case, a stream function for the non-aligned axisymmetric case does not exist. The flow field is completely three dimensional. Although the path lines can be computed from the velocity distributions, the curves would be quite complicated and are not presented here.

4. The heat transfer problem

Consider first the two-dimensional case. The equation for convective heat transfer, neglecting viscous dissipation, is

$$uT_x + wT_z = \kappa(T_{xx} + T_{zz}). \tag{16}$$

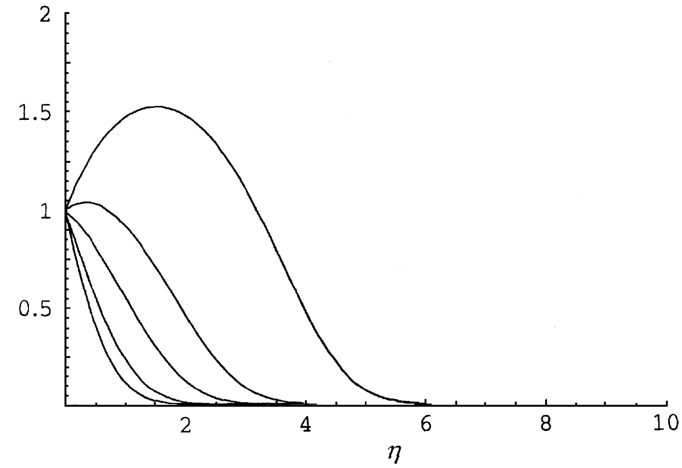

Fig. 9. The function $h(\eta)$ for the axisymmetric case. α values from top: -0.99945, -0.95, -0.75, 0, 1.

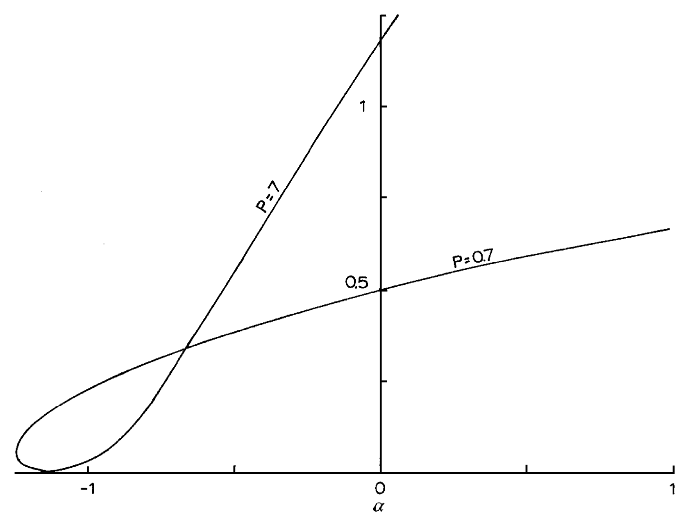

Fig. 10. Initial value $-\theta'(0)$ versus α, two-dimensional case.

Here T is the temperature and κ is the thermal diffusivity. Let the temperature at infinity and that on the sheet be T_∞, T_0, respectively. Define

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}. \tag{17}$$

Eq. (16) becomes

$$\theta''(\eta) + Pf\theta' = 0, \tag{18}$$

where $P = \nu/\kappa$ is the Prandtl number. The boundary conditions are

$$\theta(0) = 1, \quad \theta(\infty) = 0. \tag{19}$$

Using the solution for f, Eqs. (18), (19) are integrated numerically. Notice that heat transfer is not affected by the non-alignment function h. Fig. 10 shows the initial value $-\theta'(0)$ which is proportional to the heat loss from the surface. Since the boundary layer becomes thicker as shrinking is increased,

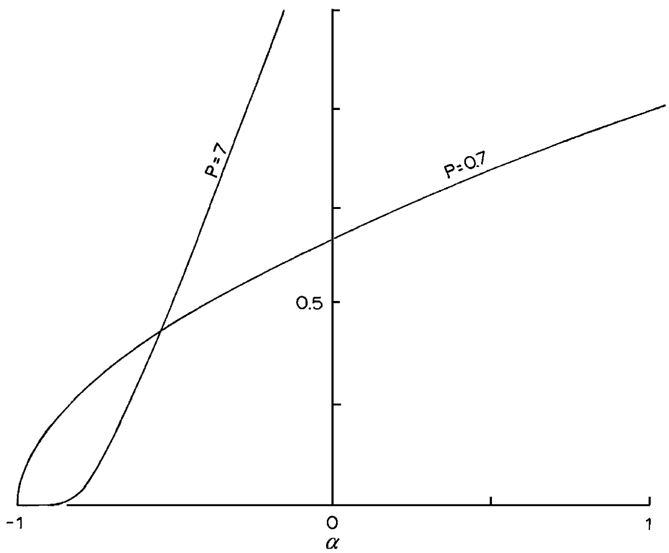

Fig. 11. Initial value $-\theta'(0)$ versus α, axisymmetric case.

the convective heat transfer decreases accordingly. The shrink ratio α causes a much larger change in heat transfer for water ($P = 7$) than for air ($P = 0.7$). Notice the non-uniqueness for $-1.2465 < \alpha \leqslant -1$.

The equation for the axisymmetric case is similar, with Eq. (18) replaced by

$$\theta''(\eta) + 2Pf\theta' = 0. \quad (20)$$

The results are shown in Fig. 11. The solutions are unique for $\alpha \geqslant -1$. The qualitative behaviors for other Prandtl numbers are similar to the curves for Prandtl numbers 0.7 and 7.

5. Discussion

This paper considers the stagnation flow towards a shrinking sheet. In comparison to the stretching sheet studied by previous authors, the shrinking sheet has many unique characteristics. A region of reverse flow always occurs near the surface, which hinders heat, mass and momentum transfer to the sheet. For larger shrinking rates, non-uniqueness and non-existence of the (similarity) solutions may exist.

The effect of non-alignment is also studied for the first time. We find that non-alignment of the stagnation flow and the shrinking (or stretching) of the sheet destroys the symmetry and complicates the flow field.

There are few exact solutions of the Navier–Stokes equations [10]. Our results represent a rare class of exact similarity solutions with reverse flow.

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