Monoidal weak ω-categories as models of a type theory

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Abstract

Weak ω-categories are notoriously difficult to define because of the very intricate nature of their axioms. Various approaches have been explored, based on different shapes given to the cells. Interestingly, homotopy type theory encompasses a definition of weak ω-groupoid in a globular setting, since every type carries such a structure. Starting from this remark, Brunerie could extract this definition of globular weak ω-groupoids, formulated as a type theory. By refining its rules, Finster and Mimram have then defined a type theory called CaTT, whose models are weak ω-categories. Here, we generalize this approach to *monoidal* weak ω-categories. Based on the principle that they should be equivalent to weak ω-categories with only one 0-cell, we are able to derive a type theory MCaTT whose models are monoidal categories. This requires changing the rules of the theory in order to encode the information carried by the unique 0-cell. The correctness of the resulting type theory is shown by defining a pair of translations between our type theory MCaTT and the type theory CaTT. Our main contribution is to show that these translations relate the models of our type theory to the models of the type theory CaTT consisting of ω-categories with only one 0-cell, by analyzing in details how the notion of models interact with the structural rules of both type theories.

Contents

Weak ω-categories are algebraic structures occurring naturally in modern algebraic topology and type theory. They consist of collections of cells in every dimension, which can be composed in various ways. The main difficulty in properly establishing a definition for those is due to the fact that the usual coherence axioms imposed on composition, such as associativity, are here relaxed and only supposed up to *invertible* higher cells, that we call *witnesses* for these axioms. Moreover, these witnesses themselves admit compositions, which satisfy axioms up to new witnesses, and so on, making the compositions and their coherence axioms intricate. There have been different approaches to propose a definition of weak ω-categories, which are summed up in a couple of surveys [17, 11]. These approaches are based on various shapes, such as simplicial sets or opetopic sets. In this article, we are interested in approaches based on globular sets. Examples of such approaches can be found in the work of Batanin [5] and Leinster [18], relying on the structure of globular operad. Independently Maltsioniotis proposed an alternative approach [21], inspired by a definition of weak ω-groupoid (i.e., weak ω-categories whose all cells are invertible) proposed by Grothendieck [16], and which is based on presheaves preserving some structure on a well-chosen category. The two approaches have been proved equivalent by Ara [2].

Type theoretical approach. An important observation which came along with the development of homotopy type theory [22] is the fact that the types in Martin-Löf type theory, and in homotopy type theory carry a structure of weak ω-groupoid, where the higher cells are given by identity types [19, 23, 1]. This allowed Brunerie to extract a minimal set of rules from homotopy type theory for generating the weak ω-categories [10], that he could prove to be equivalent to the definition of Grothendieck. Recently, Finster and Mimram proposed a generalization of Brunerie's type theory to weak ω-categories [14], parallel to the generalization Maltsiniotis proposed from Grothendieck definition. The type theory they introduced is called CaTT, and has been proved to be equivalent to the definition of Maltsiniotis [8].

Monoidal categories. In this article, we are interested in monoidal weak ω-categories. These are categories equipped with a tensor product allowing new ways to compose cells of every dimension. In particular, one cannot compose the cells of dimension 0 in weak ω-categories, but one can compose them in monoidal weak ω-categories. Moreover these new compositions are required to satisfy axioms like associativity, but since it happens in a weak setup, these axioms are again relaxed versions with witnesses in higher dimensions. This can be seen as a categorification of the notion of monoid. Our goal here is to provide a variant of the CaTT type theory in order to describe monidal weak ω-categories.

As noticed by Baez and Dolan [4, 3], monoidal weak ω-categories should be equivalent to weak ω-categories with only one 0-cell. In this correspondence, there is a shift in dimension: the 0-cells of the monoidal ω-category are the 1-cells of the ω-category with one 0-cell, and so on. The monoidal tensor product becomes the composition of arrows of the ω-category, and all its coherences are exactly those satisfied by these arrows in the ω-category.

Taking this correspondence as a starting point, we work out an explicit type theory MCaTT whose models are monoidal categories, by describing CaTT with the extra restriction that there should be only one 0-cells. To achieve this, we rely on the type theory CaTT and use its constituent to express the rules of the theory MCaTT We then define a pair of translations back and forth between CaTT and MCaTT, and use the interaction of these translations with the structure of the type theory to show that our proposed definition satisfies the correspondence we started with.

Plan of the paper. In section 1, we introduce the type theory CaTT along with the general tools we use to study type theories and their models. Then in section 2 we define the type theory MCaTT along with a pair of translations between the theories CaTT and MCaTT. We show that these translation exhibit a coreflective adjunction between the syntactic categories associated to the theory that lifts as an equivalence between a localization of the models of CaTT and the models of MCaTT.

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1 Type theory for weak ω-categories

Before working on monoidal weak ω-categories, we recall in this section the type theory CaTT whose models are weak ω-categories. We refer the reader to [14, 8] for a more detailed presentation of it. We begin by defining what we mean here by a type theory, and construct, in this setting, a type theory describing globular sets. Then, by adding extra rules, we show how to extend it in order to obtain a type theory describing weak ω-categories.

1.1 Type theoretical notations and conventions

We first recall some basic definitions in type theory in order to establish the notations, terminology and conventions that are used throughout this article. Note that our method is to formulate a theory by defining a type theory, as opposed to a developing structures internally to Martin-Löf type theory or homotopy type theory.

Expressions. A type theory manipulates various kind of objects, that we present here along with the convention we use for naming them.

- *variables*, which are elements of a given infinite countable set of variables, and that we denote $x, y, z, \ldots$,
- *terms*, which are built out of variables and constructors that will be introduced later on, they are denoted $t, u, \ldots$,
- *types*, which are built out of terms and constructors that will be introduced later on, they are denoted $A, B, \ldots$,
- *contexts*, which are supported by lists of associations of the form $x : A$, they are denoted $\Gamma, \Delta, \ldots$,
- *substitutions*, which are supported by lists of mappings of the form $x \mapsto t$ where x is a variable and t a term, they are denoted $\gamma, \delta, \ldots$.

All of these notions come with an associated set of variables, that we denote Var and that is the set of variables needed to build it out. Moreover, we also denote $\mathrm{Var}(t : A)$ for the union of $\mathrm{Var}(t)$ and $\mathrm{Var}(A)$.

Judgments. There are four kinds of judgments that are commons to all of our type theories, and that we refer to as *structural judgments*, they express the well-definedness of the previously introduced objects:

$$\begin{array}{lcl}
\Gamma \vdash & : & \Gamma \text{ is a valid context} \\
\Gamma \vdash A & : & A \text{ is a valid type in } \Gamma \\
\Gamma \vdash t : A & : & t \text{ is a valid term of type } A \text{ in } \Gamma \\
\Delta \vdash \gamma : \Gamma & : & \gamma \text{ is a valid substitution from } \Delta \text{ to } \Gamma
\end{array}$$

In order to distinguish, we sometimes refer to the raw syntax, or to context (resp. type, term, substitution) expressions for a syntactic entity which is not assumed to satisfy any of the above judgment.

Structural rules. These judgments are always subject to the same structural rules. The only difference between the various type theories that we introduce is that they have different type constructors term constructors, with different introduction rules. The structural rules that are common to all type theories are the following

$$\frac{}{\varnothing \vdash}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE})$$

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : \Gamma, x : A}(\text{SE})$$

Where in the rules (CE) and (SE) we assume that $x \notin \mathrm{Var}(\Gamma)$. Note that we do not suppose any term or type constructors at first, because we want to study different theories, and the constructors vary from one to the other. In particular, our type theories do not support Σ-types, Π-types, nor any construction of inductive types such as W-types or identity types.

Action of substitutions. We write $A[\gamma]$ for the application of a substitution γ on the type A, and $t[\gamma]$, which the application of γ on the term t. These notions have to be defined specifically for each type and term constructors that we introduce, and we thus present them along with the theories each individual type theories, we give here only the action of substitutions on variables, that is the same in all the type theories we consider, and defined by

$$x[\gamma] = \begin{cases} t & \text{If } x \mapsto t \text{ is the last mapping matching } x \text{ in } \gamma \\ x & \text{If there is no mapping matching } x \text{ in } \gamma \end{cases}$$

We admit that in all the cases, these definitions always make the following rules admissible

$$\frac{\Gamma \vdash \gamma : \Delta \qquad \Delta \vdash A}{\Gamma \vdash A[\gamma]}([]\text{-}\textsc{Ty}) \qquad\qquad \frac{\Gamma \vdash \gamma : \Delta \qquad \Delta \vdash t : A}{\Gamma \vdash t[\gamma] : A[\gamma]}([]\text{-}\textsc{Tm})$$

These actions also define a composition of substitutions given inductively by

$$\langle\rangle \circ \delta = \langle\rangle \qquad\qquad \langle \gamma, x \mapsto t\rangle \circ \delta = \langle \gamma \circ \delta, x \mapsto t[\delta]\rangle$$

It is always be possible to check by induction that the action of substitutions is compatible with the composition, and thus making the composition associative

$$t[\gamma][\delta] = t[\gamma \circ \delta]$$

$$\gamma \circ (\delta \circ \delta) = (\gamma \circ \delta) \circ \delta$$

Since there is also always an identity substitution, this shows that for a type theory $\mathfrak{T}$, one can construct a *syntactic category* $\mathcal{S}_{\mathfrak{T}}$ associated to $\mathfrak{T}$, whose objects are the contexts of the type theory, and whose morphisms are the substitutions.

Properties. The type theories that we consider all satisfy a few properties that are given by the structure of the type theory. These properties are standard, and proved by induction on the derivation trees. These proofs are not very interesting, and proving them for all the theory that we study requires either to introduce a very general framework to show them all at once, or to prove each of them for every theory, which is very repetitive. For the sake of simplicity, we admit that these properties hold in all our theories and refer the reader to a formalization in Agda [6] where we define some of the theories we study here and formally prove these properties.

Proposition 1. *In our type theories, all the entities that appear in a derivable judgment are also derivable, and these theories support the weakening. Moreover, valid expression only use variables declared in the context. More precisely, the following hold*

- *For every derivable judgment $\Delta \vdash A$, the judgment $\Delta \vdash$ is also derivable.*
- *For every derivable judgment $\Delta \vdash t : A$, the judgments $\Delta \vdash$ and $\Delta \vdash A$ are also derivable.*
- *For every derivable judgments $\Delta \vdash \gamma : \Gamma$, the judgments $\Delta \vdash$ and $\Gamma \vdash$ are also derivable.*
- *For every derivable judgment $(\Delta, x : A) \vdash$, if the judgment $\Delta \vdash B$ is derivable then so is $(\Delta, x : A) \vdash B$.*
- *For every derivable judgment $(\Delta, x : A) \vdash$, if the judgment $\Delta \vdash t : B$ is derivable then so is $(\Delta, x : A) \vdash t : B$.*
- *For every derivable judgment $(\Delta, x : A)$, if the judgment $\Delta \vdash \gamma : \Gamma$ is derivable then so is $(\Delta, x : A) \vdash \gamma : \Gamma$.*
- *For every derivable judgment $\Delta \vdash A$, we have $\mathrm{Var}(A) \subset \mathrm{Var}(\Delta)$.*
- *For every derivable judgment $\Delta \vdash t : A$, we have $\mathrm{Var}(t) \subset \mathrm{Var}(\Delta)$.*
- *For every derivable judgment $\Delta \vdash \gamma : \Gamma$, we have $\mathrm{Var}(\gamma) \subset \mathrm{Var}(\Delta)$, and moreover writing $\gamma = \langle x_i \mapsto t_i \rangle_{0 \leq i \leq n}$ and $\Gamma = (y_i : A_i)_{0 \leq i \leq m}$ we necessarily have $n = m$ and for all i, $x_i = y_i$.*

Category with families. We use the formalism of *categories with families*, introduced by Dybjer [12] as our categorical axiomatization of the models of such a theory. We write **Fam** for the category of families, where an object is a family $(A_i)_{i \in I}$ of sets A_i indexed in a set I and a morphism $f : (A_i)_{i \in I} \to (B_j)_{j \in J}$ is a pair consisting of a function $f : I \to J$ and a family of functions $(f_i : A_i \to B_{f(i)})_{i \in I}$.

Suppose given a category C equipped with a functor $T : C^{\mathrm{op}} \to \mathbf{Fam}$. Given an object Γ of C, its image will be denoted

$$T\Gamma = \left(\mathrm{Tm}_A^\Gamma\right)_{A \in \mathrm{Ty}^\Gamma}$$

i.e., we write Ty^Γ for the index set and Tm^Γ_A for the elements of the family. By analogy with a type theory, for a morphism $\gamma : \Delta \to \Gamma$ an element $A \in \mathrm{Ty}^\Gamma$ and an element $t \in \mathrm{Tm}^\Gamma_A$, we write $A[\gamma] = T\gamma(A)$ the image of A in Ty^Δ, and $t[\gamma] = T_A\gamma(t)$ the image of t in $\mathrm{Tm}^\Delta_{A[\gamma]}$. With those notations, the functoriality of T can be written as

$$A[\sigma \circ \delta] = A[\sigma][\delta] \qquad\qquad t[\sigma \circ \delta] = t[\sigma][\delta]$$

$$A[\mathrm{id}] = A \qquad\qquad t[\mathrm{id}] = t$$

for composable morphisms of C.

A *category with families* (or *CwF*) consists of a category C equipped with a functor as above $T : C^{\mathrm{op}} \to \mathbf{Fam}$, such that C has a terminal object, denoted $\varnothing$, and that there is a *context comprehension* operation: given a context Γ and type $A \in \mathrm{Ty}^\Gamma$, there is a context (Γ, A), together with a projection morphism $\pi : (\Gamma, A) \to \Gamma$ and a term $p \in \mathrm{Tm}^{(\Gamma,A)}_{A[\pi]}$, such that for every morphism $\sigma : \Delta \to \Gamma$ in C together with a term $t \in \mathrm{Tm}^\Delta_{A[\sigma]}$, there exists a unique morphism $\langle \sigma, t\rangle : \Delta \to (\Gamma, A)$ such that $p[\langle \sigma, t\rangle] = t$:

$$\begin{tikzcd} & (\Gamma, A) \arrow[d, "\pi"] \\ \Delta \arrow[ur, dotted, "\langle\sigma,t\rangle"] \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

We call an arrow of the form $\pi : (\Gamma, A) \to \Gamma$ a *display map*. The following result is well-known and allow for giving structure to the syntactic category of a type theory.

Proposition 2. *The syntactic category of a type theory is endowed with a structure of category with families, where for every context Γ, Ty^Γ is the set of derivable types in Γ and Tm^Γ_A is the set of derivable terms of type A in Γ*

Pullbacks along display maps. The structure of category with families encompasses a compatibility condition between context comprehension and the action of morphisms on the type, expressed by the following lemma. In particular, it states that all pullbacks along display maps exist and that they can be explicitly computed from the given structure.

Lemma 3. *In a category with families C, for every morphism $f : \Delta \to \Gamma$ in C and $A \in \mathrm{Ty}^\Gamma$,* *the square*

$$\begin{tikzcd} (\Delta, A[f]) \arrow[r, "\langle f\circ\pi', p'\rangle"] \arrow[d, "\pi'"'] & (\Gamma, A) \arrow[d, "\pi"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd}$$

is a pullback, where $\pi' : (\Delta, A[f]) \to \Delta$ *and* $p' \in \mathrm{Tm}^{(\Delta,A[f])}_{A[f][\pi']}$ *are obtained by context comprehension.*

Proof.

$$\begin{tikzcd} \Theta \arrow[drr, bend left, "\sigma"] \arrow[ddr, bend right, "\delta"'] & & \\ & (\Delta, A[f]) \arrow[r, "\langle f\circ\pi', p'\rangle"] \arrow[d, "\pi'"] & (\Gamma, A) \arrow[d, "\pi"] \\ & \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

Consider the term $p \in \mathrm{Tm}^{(\Gamma,A)}_{A[\pi]}$, then $p[\sigma] \in \mathrm{Tm}^{\Theta}_{A[\pi][\sigma]} = \mathrm{Tm}^{\Theta}_{A[f][\delta]}$. By context extension, we get a map $\langle \delta, p[\sigma]\rangle : \Theta \to (\Delta, A[f])$ such that $\pi' \circ \langle \delta, p[\sigma]\rangle = \delta$ and $p'[\langle \delta, p[\sigma]\rangle] = p[\sigma]$. Since moreover $p' = p[\langle f \circ \pi', p'\rangle]$, the term equality gives in fact $p[\sigma] = p[\langle f \circ \pi', p'\rangle \circ \langle \delta, p[\sigma]\rangle]$, which is a necessary condition for the upper triangle to commute, thus proving uniqueness of the map. We just have to show that this map makes the upper triangle commute. Notice that $\pi \circ \langle f \circ \pi', p'\rangle \circ \langle \delta, p[\sigma]\rangle = \pi \circ \sigma$, and $p[\sigma] = p[\langle f \circ \pi', p'\rangle \circ \langle \delta, p[\sigma]\rangle]$, by universal property of the extension for morphisms, this implies the commutativity of upper triangle. $\square$

Lemma 4. *Let C and D be two categories with families, together with a morphism $(F, \phi) : C \to D$, then for any object Γ in C together with an element $A \in \mathrm{Ty}^{\Gamma}$ and for any morphism $\gamma : \Delta \to \Gamma$ in C, the following equation is satisfied*

$$F(\Delta, A[\gamma]) = (F\Delta, (\phi_\Gamma A)[F\gamma])$$

Proof. By definition of a morphism of category with families, we have

$$F(\Delta, A[\gamma]) = (F(\Delta), (\phi_\Delta(A[\gamma])))$$

Moreover by naturality of ϕ, the following square commutes

$$\begin{array}{ccc} Ty^{\Gamma} & \xrightarrow{\phi_\Gamma} & Ty^{F(\Gamma)} \\ {\scriptstyle \_[f]}\downarrow & & \downarrow{\scriptstyle \_[F\gamma]} \\ Ty^{\Delta} & \xrightarrow{\phi_\Delta} & Ty^{F(\Delta)} \end{array}$$

thus $\phi_\Delta(A[\gamma]) = (\phi_\Gamma A)[F\gamma]$. □

Note that Lemma 3 allows to understand this result as the fact that F preserves the pullbacks along the display maps. In fact one can understand the formalism of category with families as a way to define a category with a choice of pullbacks along a certain class of maps, while also ensuring that this choice of pullback is split: the composition of two pullbacks is not only isomorphic to the pullback of the composition, but is equal on the nose. Syntactically, this translates into the equality $(\Delta, A[\gamma \circ \delta]) = (\Delta, A[\gamma][\delta])$.

Models of a type theory. A *morphism* between two categories with families $(\mathbf{C}, T)$ and $(\mathbf{C}', T')$, is a functor $F : \mathbf{C} \to \mathbf{C}'$ together with a natural transformation $\phi : T \to T' \circ F$, such that F preserves the terminal object and the context comprehension operation. A 2-*morphism* θ between two morphisms $(F, \phi) : T \to T'$ and $(F', \phi') : T \to T'$ is a natural transformation $\theta : F_1 \to F_2$ such that $T\theta \circ \phi = \phi'$.

The category **Set** comes equipped with a structure of (large) category with families: technically, one should in fact consider a category with proper classes of types and terms associated to be the structure associated to **Set**, but we ignore this size issue in this article. The category of *models* of the type theory $\mathfrak{T}$ is defined to be the category of morphisms of category with families from $\mathcal{S}_{\mathfrak{T}}$ to **Set**, and is denoted $\mathbf{Mod}\,(\mathcal{S}_{\mathfrak{T}})$.

The pullbacks along the display give a nice characterization of the models of a category with families

Lemma 5. *The category of models of a category with families C is isomorphic to category of functors $C \to$ **Set** that preserve the terminal object and the morphisms along the display maps.*

Proof. Lemma 4, the underlying functor of a morphism of category with families preserves the pullbacks along the display maps, and by definition, such functor has to preserve the initial object as well. So it suffices to prove that a functor $F : C \to \mathbf{Set}$ preserving the initial object and the pullbacks along display maps gives rise to a unique model. Consider such a functor F, together with an object Γ in C and a type $A \in \mathrm{Ty}^\Gamma$. Suppose defined ϕ such that (F, ϕ) is a model of C, then necessarily $F(\Gamma, A) = (F\Gamma, \phi_\Gamma A) = \phi_\Gamma A$ by definition of the context comprehension in $\mathbf{Set}$. Thus necessarily $\phi_\Gamma(A) = F(\Gamma, A)$. Consider a term $t \in \mathrm{Tm}^\Gamma_A$, then there is a morphism $\langle \mathrm{id}_\Gamma, t\rangle : \Gamma \to (\Gamma, A)$, and by definition of the category with families structure of $\mathbf{Set}$, we then have $F(\langle \mathrm{id}_\Gamma, t\rangle) = \langle \mathrm{id}_{F\Gamma}, \phi_{\Gamma,A}(t)\rangle = t$, which proves that necessarily $\phi_{\Gamma,A}(t) = F(\langle \mathrm{id}_\Gamma, t\rangle)$. Conversely, these assignments define a natural transformation ϕ, which make (F, ϕ) into a model of F. $\square$

This condition relies on the specific structure of category with families of $\mathbf{Set}$: It may not be true in general that the morphisms of categories with families between two arbitrary categories with families C and D are isomorphic to the functors preserving the display maps and the pullbacks along them. It also justifies retrospectively not to be too precise about the size issues with $\mathbf{Set}$, as one may as well ignore the structure of category with families on $\mathbf{Set}$ altogether, and define a model as a functor $C \to \mathbf{Set}$ that preserves the terminal object and the pullback along the display maps.

1.2 Globular sets

We study globular weak ω-categories, that is weak ω-categories whose underlying structure is a globular set, and hence we first define a type theory whose models are globular sets, following [14]. The *category of globes* $\mathbf{G}$ is the category generated by

$$[0] \underset{t}{\overset{s}{\rightrightarrows}} [1] \underset{t}{\overset{s}{\rightrightarrows}} [2] \underset{t}{\overset{s}{\rightrightarrows}} \cdots$$

with the relations $ts = ss$ and $st = tt$. The category of *globular sets* $\mathbf{GSet}$ is the category of presehaves over $\mathbf{G}$. It comes equipped with the *Yoneda embedding* $\mathrm{Y} : \mathbf{G} \to \mathbf{GSet}$. A globular set which is in the image of this functor is called *representable* or a *disk* and we denote it $D^n = \mathrm{Y}([n])$. Every globular set is a colimit of representables in a canonical way [20].

Type theory for globular sets. In order to manipulate globular sets, we introduce two type constructors $\star$, which is the type of the 0-cells, and $\to$, which constructs the types of higher cells

$$\frac{}{\varnothing}(\mathrm{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\mathrm{CE})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\longrightarrow} u}(\to\text{-INTRO})$$

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\mathrm{VAR})$$

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\mathrm{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t\rangle : (\Gamma, x : A)}(\mathrm{SE})$$

Figure 1: The type theory $\mathfrak{G}$

between two given cells. More precisely, we define a type expression to be either the expression $\star$ or the expression $t \underset{A}{\longrightarrow} u$, where A is a type expression, and t and u are term expressions. We also define the action of substitutions for these types to be

$$\star[\gamma] = \star \qquad\qquad (t \underset{A}{\longrightarrow} u)[\gamma] = t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]$$

These constructors are subject to the following introduction rules

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\longrightarrow} u}(\to\text{-INTRO})$$

In the other type theories, we study structures that are supported by a globular sets, thus we keep the same type constructors, the extra structure being added via new term constructors. For now there are no term constructors, so the only terms in this theory are variables, we call this theory $\mathfrak{G}$. We summarize all the rules of the theory $\mathfrak{G}$ in Fig. 1. It can be verified by induction on the rules of the theory that the rules ([]-TY) and ([]-TM) are admissible.

Syntactic category and models of $\mathfrak{G}$. Here, we present basic results about the theory $\mathfrak{G}$ without proofs, but by illustrating them with examples; a detailed presentation can be found in [14, 8].

Proposition 6. *The syntactic category of the theory $\mathfrak{G}$ is equivalent to the opposite of the category of finite globular sets.*

Example 7. To understand this correspondance, one can simply read a globular set on a context, and conversely. For instance, the context on the left below corresponds to the globular set depicted on the right:

$$x : \star, y : \star, z : \star$$
$$f_1 : x \to y, f_2 : x \to y, g : x \to z, h : y \to y$$
$$\alpha : f_1 \to f_2$$

$$\begin{tikzcd} x \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & y \arrow[loop right, "h"] \\ & z \arrow[ul, "g"] & \end{tikzcd}$$

Proposition 8. *The category of models of $\mathfrak{G}$ is equivalent to the category of globular sets.*

Note that it respects the Gabriel-Ülmer duality [15] : The theory is the opposite of the finitely generated objects, and it inludes in its models via a Yoneda embedding.

Disk contexts. Note that the category of finite globular sets contains in particular the representable objects D^n, thus the equivalence provides corresponding objects in the syntactic category, that we again denote D^n and that we call *disk contexts.* In low dimensions, these contexts are given (up to renaming of their variables), by

$$\begin{aligned} D^0 &\;:\; (x : \star) \\ D^1 &\;:\; (x : \star, y : \star, f : x \underset{\star}{\to} y) \\ D^2 &\;:\; (x : \star, y : \star, f : x \underset{\star}{\to} y, g : x \underset{\star}{\to} y, \alpha : f \xrightarrow[x \underset{\star}{\to} y]{} g) \end{aligned}$$

1.3 Pasting schemes and ps-contexts

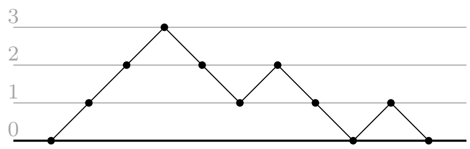
In order to indtroduce new terms in our type theory, we need special kind of contexts, that we describe using the correspondence with finite globular sets.

Globular products. In the category of globular sets, we call a *globular product* a limit in the syntactic category $\mathcal{S}_{\mathfrak{G}}$ of the form

$$\begin{tikzcd}[column sep=small]
D^{i_0} \arrow[dr] & & D^{i_1} \arrow[dl] \arrow[dr] & & D^{i_{k-1}} \arrow[dl] \arrow[dr] & & D^{i_k} \arrow[dl] \\
& D^{j_1} & & \cdots & & D^{j_{k-1}} &
\end{tikzcd}$$

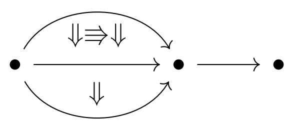
Where all the arrows pointing to the right are iterated sources and all the arrows pointing to the left are iterated targets. The *pasting schemes* are the globular sets that are obtained as a globular product, note that they necessarily are finite.

Combinatorial description. We represent the combinatorial data provided by a pasting scheme with a diagram of the following form, where we determine the corresponding globular product by reading the heights of the peaks.



$$\begin{tikzcd}[column sep=small]
D^3 \arrow[dr] & & D^2 \arrow[dl] \arrow[dr] & & D^1 \arrow[dl] \\
& D^1 & & D^0 &
\end{tikzcd}$$

These two representation are both description of the pasting scheme which, as a globular set, is the following



More formally, this combinatorial data can be encoded as non-decreasing parking functions, that is non-decreasing functions $f : \{0, \cdots, n\} \to \{0, \cdots, n\}$ such that $f(0) = 0$ and $f(n) = n$, where we plot the function $f(n) - n$. There are various other combinatorial descriptions of pasting schemes, like Batanin trees [5], or Dyck words.

Ps-contexts. We define *ps-contexts* to be the contexts of the type theory $\mathfrak{G}$ which correspond to pasting schemes. Those can be characterized by a simple algorithm, which can be itself described using rules of a type-theoretical flavor: a ps-context Γ is precisely a context such that the judgment $\Gamma \vdash_{\mathsf{ps}}$ described below is derivable. We thus introduce two new forms of judgments

$$\begin{array}{lcl} \Gamma \vdash_{\mathsf{ps}} & : & \text{the context } \Gamma \text{ is a ps-context} \\ \Gamma \vdash_{\mathsf{ps}} x : A & : & \Gamma \text{ is a partial ps-context with dangling variable } x \end{array}$$

The second judgment should be understood as an auxiliary function, where the dangling variable indicates the unique variable on which one is allowed to glue a cell at the next step. These two judgments are subject to the following rules

$$\frac{}{(x:\star) \vdash_{\mathsf{ps}} x : \star}\,(\text{PSS}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}{\Gamma \vdash_{\mathsf{ps}} y : A}\,(\text{PSD})$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\to} y \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}\,(\text{PSE}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}\,(\text{PS})$$

Source and target. Given an integer $i \in \mathbb{N}$, a ps-context Γ comes equipped with an *i-source* $\partial_i^- \Gamma$, defined inductively on the structure by $\partial_i^-(x : \star) = (x : \star)$ and

$$\partial_i^-(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^- \Gamma & \text{if } \dim A \geq i \\ \partial_i^- \Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

Similarly, a ps-context Γ also has an *i-target* $\partial_i^+ \Gamma$, defined inductively on its structure by $\partial_i^+(x : \star) = (x : \star)$ and

$$\partial_i^+(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^+ \Gamma & \text{if } \dim A > i \\ \mathrm{drop}(\partial_i^+ \Gamma), y : A & \text{if } \dim A = i \\ \partial_i^+ \Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

where $\mathrm{drop}(\Gamma)$ is the context Γ with its last variable removed, i.e.,

$$\mathrm{drop}(\Gamma, x : A) = \Gamma$$

Moreover, we write

$$\partial^- \Gamma = \partial^-_{\dim \Gamma - 1} \Gamma \qquad\qquad \partial^+ \Gamma = \partial^+_{\dim \Gamma - 1} \Gamma$$

Note that with these conventions, $\partial^-(x : \star)$ and $\partial^+(x : \star)$ are not defined.

1.4 The type theory CaTT

In order to axiomatize the weak ω-category structure, we add new terms, that we call operations and coherences. This is done in our theory by adding two term constructors op and coh, together with introduction rules. The work we have done so far with ps-contexts shows relevance here, since those index the introduction of coherences, dually to how pasting schemes index the operations in other definitions of weak ω-categories [5, 18, 21]. A term expression in this theory is thus defined to be either a variable or of the form $\mathsf{op}_{\Delta,A}[\delta]$ or $\mathsf{coh}_{\Delta,A}[\delta]$, where in both cases Δ is a context, A is a type and δ is a substitution. The application of substitutions is defined by the formulas

$$\mathsf{op}_{\Delta,A}[\delta][\gamma] = \mathsf{op}_{\Delta,A}[\delta \circ \gamma] \qquad\qquad \mathsf{coh}_{\Delta,A}[\delta][\gamma] = \mathsf{coh}_{\Delta,A}[\delta \circ \gamma]$$

Side conditions. The introduction rules for the term constructors op and coh have to verify some side conditions regarding the variables that are used in the type that we derive. Intuitively the introduction rule for the constructor op creates witnesses for *operations*, for instance the composition, or whiskering, which a priori have no reason to be invertible, whereas the introduction rule for the constructor coh creates witnesses for *coherences*, as for instance the associators, which are always weakly invertible. In order to simplify the notations, we encompass the requirements for these rules along with their side conditions in new judgments

$$\begin{array}{lcl} \Gamma \vdash_{\mathsf{op}} A & : & \text{The type } A \text{ defines an admissible operation in } \Gamma \\ \Gamma \vdash_{\mathsf{eq}} A & : & \text{The type } A \text{ defines an admissible coherence in } \Gamma \end{array}$$

These two judgments are subject to the following derivation rules, which express all the requirements for the introduction rules of the constructors op and coh

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A}{\Gamma \vdash_{\mathsf{op}} t \underset{A}{\longrightarrow} u} \quad \begin{cases} \mathrm{Var}(t : A) = \mathrm{Var}(\partial^-(\Gamma)) \\ \mathrm{Var}(u : A) = \mathrm{Var}(\partial^+(\Gamma)) \end{cases}$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash_{\mathsf{eq}} A} \quad \mathrm{Var}(A) = \mathrm{Var}(\Gamma)$$

Note that whenever $\Gamma \vdash_{\mathsf{op}} A$ is derivable, then a top-dimensional variable of Γ cannot appear in A, whereas whenever $\Gamma \vdash_{\mathsf{eq}} A$ is derivable, a top-dimensional variable of Γ has to appear in A, hence these two rules are mutually exclusive.

Interpretation. Intuitively, the side condition $\mathrm{Var}(t : A) = \mathrm{Var}(\Gamma)$ can be understood as the data of a term t that is a complete composition of the ps-context Γ. Hence the data of a type $\Gamma \vdash_{\mathsf{op}} t \to u$ can be thought of a the data of a ps-context together with a way of fully composing its source and a way of fully composing its target, defining the borders of a valid operation. Moreover, a ps-context has to contain at least one variable, and thus derivation $\Gamma \vdash_{\mathsf{eq}} A$ is only possible when $A = t \underset{B}{\longrightarrow} u$. Moreover, one can show that the condition $\mathrm{Var}(A) = \mathrm{Var}(\Gamma)$ is then equivalent to the pair of conditions $\mathrm{Var}(t : B) = \mathrm{Var}(\Gamma)$ and $\mathrm{Var}(u : B) = \mathrm{Var}(\Gamma)$. This proof is surprisingly involved, so we simply assume this property here and refer the reader to [8] for a better account of it. Under this correspondence, one can think of a type $\Gamma \vdash_{\mathsf{eq}} A$ as a ps-context equipped with two ways of composing it fully that are parallel, these are the border of a coherence.

Coherences. We can now give the introduction rules for the new term constructor coh.

$$\frac{\Gamma \vdash_{\mathsf{op}} A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma,A}[\gamma] : A[\gamma]}(\mathsf{op}\text{-}\textsc{intro}) \qquad\qquad \frac{\Gamma \vdash_{\mathsf{eq}} A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma,A}[\gamma] : A[\gamma]}(\mathsf{coh}\text{-}\textsc{intro})$$

The resulting type theory obtained by adding these rules is called CaTT. We present all the rules of the type theory CaTT together, in Fig. 2, and we admit that it makes the rules ([]-Ty)

$$\frac{}{\varnothing}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u}(\to\text{-INTRO})$$

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR}) \qquad \frac{\Gamma \vdash_{\mathsf{op}} A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma,A}[\gamma] : A[\gamma]}(\mathsf{op}\text{-INTRO})$$

$$\frac{\Gamma \vdash_{\mathsf{eq}} A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma,A}[\gamma] : A[\gamma]}(\mathsf{coh}\text{-INTRO})$$

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

Figure 2: The type theory CaTT

and ([]-TY) admissible.

Examples. We give a few examples of derivations in this system illustrating how it describes weak ω-categories. This shows the introduction of a new operation and a new coherence, and emphasizes the role of the substitutions taken as their arguments.

- Composition: In CaTT one can use an operation to derive a witness for composition of 1-cells. Start by considering the context

  $$\Gamma_{\mathsf{comp}} = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z)$$

  One can check that $\Gamma_{\mathsf{comp}} \vdash_{\mathsf{ps}}$, and compute its source $\partial^-(\Gamma_{\mathsf{comp}}) = (x : \star)$ and target $\partial^+(\Gamma_{\mathsf{comp}}) = (z : \star)$. Thus, the judgment $\Gamma_{\mathsf{comp}} \vdash_{\mathsf{op}} x \to z$ is derivable. Now considering any context Γ in CaTT, with two terms v, w, such that $\Gamma \vdash v : u \to u'$ and $\Gamma \vdash w : u' \to u''$ (i.e., two composable 1-cells u and v), there is a substitution $\gamma = \langle x \mapsto u, y \mapsto u', f \mapsto v, z \mapsto u'', g \mapsto w \rangle$ defined by this data, such that

  $$\Gamma \vdash \gamma : \Gamma_{\mathsf{comp}}$$

So the introduction rule for coherences applies and builds a witness of the composition of v and w

$$\Gamma \vdash \mathsf{op}_{\Gamma_{\mathsf{comp}}, x \to z}[\gamma] : u \to u''$$

We denote this term with the simpler and more usual notation

$$\Gamma \vdash \mathtt{comp}\ v\ w : u \to u''$$

- Associativity: Similarly, one can pose the context

$$\Gamma_{\mathsf{assoc}} = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z, w : \star, h : z \to w)$$

and one can check $\Gamma_{\mathsf{assoc}} \vdash_{\mathsf{ps}}$, and

$$\Gamma_{\mathsf{assoc}} \vdash_{\mathsf{eq}} \mathtt{comp}\ (\mathtt{comp}\ f\ g)\ h \to \mathtt{comp}\ f\ (\mathtt{comp}\ g\ h)$$

Whenever a context Γ defines three terms u, v, w that are composable 1-cells, the rule (coh-INTRO) provides a witness for the associativity of their compositions, denoted

$$\Gamma \vdash \mathtt{assoc}\ u\ v\ w : \mathtt{comp}\ (\mathtt{comp}\ u\ v)\ w \to \mathtt{comp}\ u\ (\mathtt{comp}\ v\ w)$$

Models. In this article we take the category of models of CaTT to be the definition of weak ω-categories. We refer the reader to [8] (in preparation at the time of writing) for further discussions about this definition of weak ω-categories together with its relation with other known definitions of the same notion, however it is not the goal of this article to expand on these aspects and we simply give here some intuition for the reason this definition corresponds to the intuition. Considering a model $F : \mathcal{S}_{\mathsf{CaTT}} \to \mathbf{Set}$, we define its set of objects to be $F(D^0)$, and given two objects $x, y \in F(D^0)$, the set of $1-cells$ between x and y in F to be

$$\mathrm{Hom}_F(x, y) = \left\{ f \in F(D^1), F(s)(f) = x, F(t)(f) = y \right\}$$

where $D^1 \vdash s : D^0$ is the source substitution and $D^1 \vdash t : D^0$ is the target substitution. Considering two 1-cells $f \in \mathrm{Hom}_F(x, y)$ and $g \in \mathrm{Hom}_F(y, z)$, this defines a unique element of $F(\Gamma_{\mathtt{comp}})$ the substitution $\Gamma_{\mathtt{comp}} \to D^1$ given by the composition term induces an element of $F(D^1)$, which is the composition of f and g. Unraveling further this definition shows that all the derivable terms yield corresponding properties on the models of the theory, which consolidate the intuition of the models of CaTT being the weak ω-categories.

Syntactic category. Note that the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$ can be concieved as the opposite category of the full subcategory of $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})$ whose objects are freely finitely generated. This is again an instance of the Gabriel-Ülmer duality [15], although in this case it is more complicated to define what it means to be freely finitely generated. For this reason, we do not expand on this aspect and only mention it to gain better intuition of the situation.

2 Type theory for monoidal weak ω-category

We now focus on monoidal categories and define a theory that we call MCaTT. The idea here is to adapt the type theory MCaTT, in order to enforce the constraint that our categories should always have exactly one 0-cell. In order to align ourselves with the language of monoidal categories, we consider this new unique object as of "dimension -1" in the monoidal setting, in such a way that the cells of dimension 0 within a monoidal categories really are the objects of the monoidal category.

The subcategory of context with one objects. The first candidate for describing the weak ω-categories with only one objects is the full subcategory of $\mathcal{S}_{\mathsf{CaTT}}$ whose objects are the context that define only one object, we denote it $\mathcal{S}_{\mathsf{CaTT},\bullet}$. It is not clear however that there is a structure of categories with families on this category, nor how to find a type theory the present it. We now introduce a the type theory MCaTT whose syntactic category we prove to be equivalent to $\mathcal{S}_{\mathsf{CaTT},\bullet}$, thus answering those questions.

2.1 Globular sets with a unit type

Unit type. We start by defining the type theory $\mathfrak{G}_{\mathbf{1}}$, which is obtained by formally replacing the type $\star$ in $\mathfrak{G}$ by a unit type that we denote $\mathbf{1}$, together with a constant that we denote $()$, which represent a term of this unit type. These new constructors are subject to the following introduction rules

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbf{1}}(\mathbf{1}\text{-INTRO}) \qquad\qquad \frac{\Gamma \vdash}{\Gamma \vdash () : \mathbf{1}}(()\text{-INTRO})$$

Having added these new constructors, we have to specify how substitutions compute on them, and we define the action to be trivial for any substitution

$$\mathbf{1}[\gamma] = \mathbf{1} \qquad\qquad ()[\gamma] = ()$$

Along with these constructors, we still add the constructor $\to$ as previously, introduced with the rule

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u}(\to\text{-INTRO})$$

Additionally, these constructors are subject to an additional computation rules that postulates a new definitional equality. We use the symbol $\equiv$ to denote such definitional equalities, and the required rule is the following

$$\frac{\Gamma \vdash x : \mathbf{1}}{\Gamma \vdash x \equiv () : \mathbf{1}}(\eta_{\mathbf{1}})$$

Along with a rule enforcing which enforce the typing to respect the definitional equality

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \equiv B}{\Gamma \vdash t : B}$$

We refer the reader to Fig. 3 for a summary of all the rules of the theory

Theories with definitional equalities The addition of definitional equality in general makes the semantics more complicated to study, as we have to slightly modify the definition of the

$$\frac{}{\varnothing}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbf{1}}(\mathbf{1}\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\rightarrow} u}(\rightarrow\text{-INTRO})$$

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR}) \qquad \frac{\Gamma \vdash}{\Gamma \vdash () : \mathbf{1}}(()\text{-INTRO})$$

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \equiv B}{\Gamma \vdash t : B} \qquad \frac{\Gamma \vdash t : \mathbf{1} \qquad \Gamma \vdash u : \mathbf{1}}{\Gamma \vdash t \equiv u : \mathbf{1}}(\eta_{\mathbf{1}})$$

Figure 3: The type theory $\mathfrak{G}_{\mathbf{1}}$

syntactic category. We set the objects to be equivalence classes of contexts up to definitional equality, and the morphisms to be equivalence classes of substitutions up to definitional equality. The structure of categories with families is also slightly modify, as the set of types and terms associated to a contexts have also to be taken up to definitional equality. Additionally, the second rule may make the type checking undecidable, as it relies on a definitional equality judgment. However, in our case, there is a single rule that introduces definitional equalities, and it just equates all the terms of type $\mathbf{1}$. Hence, we can define a normalization procedure that rewrites any term $\Gamma \vdash t : \mathbf{1}$ into the term $()$, and thus definitional equality is decidable and so is type checking. From now on, we only work with expressions (contexts, types, terms, substitutions) that are in normal form. For a type or a substitution to be in normal form means that all the terms that appear in it are in normal form and for a context to be in normal form means that all the types that appear in it are in normal form. In practice, this means that the only term of type $\mathbf{1}$ that we can ever refer to is $()$, and there is no constraints on terms of other terms. Working only with normal forms means that we do not need to check that the operations we define respect the definitional equality, as by definition the image of an expression that is not in normal form is the image of its normal form.

Dimension and type of terms. By convention, we set that the term $\Gamma \vdash () : \mathbf{1}$ is of dimension -1 and that a term $\Gamma \vdash t : u \underset{A}{\to} v$ is of dimension 1 more than the common dimension of u and v. Note that for every context Γ, there is only ever one type of dimension 0 that is in normal form, the type $\Gamma \vdash () \underset{\mathbf{1}}{\to} ()$, we denote this type $\star$ by analogy with the theory $\mathfrak{G}$. When we define operations acting on $\mathfrak{G}_{\mathbf{1}}$ we always ensure that they respect the definitional equality by defining them only in normal form. That means that in practice, we may treat $\star$ as a separate case if needed, keeping in mind that it is simply a short form for $() \underset{\mathbf{1}}{\to} ()$.

Semantics. We now show that the semantics of the theory $\mathfrak{G}_{\mathbf{1}}$ is the same as the semantics of the theory $\mathfrak{G}$.

Lemma 9. *The context $(x : \mathbf{1})$ is terminal in the category $\mathcal{S}_{\mathfrak{G}_{\mathbf{1}}}$*

Proof. For any context Γ we always have the substitution $\Gamma \vdash \langle x \mapsto ()\rangle : (x : \mathbf{1})$. Moreover, consider a substitution $\Gamma \vdash \gamma : (x : \mathbf{1})$. Then by construction of substitutions, γ is necessarily of the form $\gamma = \langle x \mapsto t\rangle$ with a derivation for the judgment $\Gamma \vdash t : \mathbf{1}$. The conversion rule $(\eta_{\mathbf{1}})$ then applies and gives a definitional equality $\Gamma \vdash t \equiv () : \mathbf{1}$. This produces a definitional equality between the substitutions $\Gamma \vdash \gamma \equiv \langle x \mapsto ()\rangle : (x : \mathbf{1})$. This proves that there is a unique morphism $\Gamma \to (x : \star)$ in the syntactic category $\mathcal{S}_{\mathfrak{G}}$. □

Lemma 10. *The syntactic categories $\mathcal{S}_{\mathfrak{G}}$ and $\mathcal{S}_{\mathfrak{G}_{\mathbf{1}}}$ are equivalent.*

Proof. We show that the natural inclusion functor $\mathcal{S}_{\mathfrak{G}} \to \mathcal{S}_{\mathfrak{G}_{\mathbf{1}}}$ is an equivalence, by showing that it is fully faithful and essentially surjective. First we can show that the functor is faithful by induction on the length of the target context: If the target context is empty, then it is terminal, so the statement is vacuous, consider two distinct substitutions $\Delta \vdash \langle \gamma, x \mapsto t\rangle : \Gamma$ and $\Delta \vdash \langle \gamma', x \mapsto t'\rangle : \Gamma$. Then either the two substitutions γ and γ' are distinct in $\mathfrak{G}$, in which case the induction hypothesis shows that they are distinct in $\mathfrak{G}_{\mathbf{1}}$, or necessarily the terms t and t' are distinct in $\mathfrak{G}$. Since these terms are in $\mathfrak{G}$, they are not of type $\mathbf{1}$, and hence there cannot be a definitional equality between them in $\mathfrak{G}_{\mathbf{1}}$, this proves that $\langle \gamma, x \mapsto t\rangle$ and $\langle \gamma', x \mapsto t'\rangle$ define distinct arrows in $\mathcal{S}_{\mathfrak{G}_{\mathbf{1}}}$, hence the functor is faithful. We can show that it is full by considering two contexts Δ and Γ of $\mathfrak{G}$ together with a substitution $\Delta \vdash \gamma : \Gamma$ in $\mathfrak{G}_{\mathbf{1}}$. Then the substitution is built out of terms of the context Δ, and since none of the variables in Γ has type $\mathbf{1}$, none of those terms have type $\mathbf{1}$, so they are all variables, of Δ, and hence the substitution γ is in fact definable in $\mathfrak{G}$. Finally, we prove that this functor is essentially surjective. Indeed, first note that for all context Γ, we have the pullback in $\mathcal{S}_{\mathsf{MCaTT}}$:

$$\begin{tikzcd}
(\Gamma, x : \mathbf{1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (x : \mathbf{1}) \arrow[d] \\
\Gamma \arrow[r] & \varnothing
\end{tikzcd}$$

Since both $(x : \mathbf{1})$ and $\varnothing$ are terminal objects in $\mathcal{S}_{\mathsf{MCaTT}}$, the display map $(\Gamma, x : \mathbf{1}) \to \Gamma$ is an isomorphism. Using this fact we can recursively eliminate all the variables of type $\mathbf{1}$ in a context and show that every context is isomorphic to one without any variable of type $\mathbf{1}$, that is, it is isomorphic to a context of $\mathfrak{G}$. □

Lemma 11. *The equivalence of categories between $\mathcal{S}_{\mathfrak{G}}$ and $\mathcal{S}_{\mathfrak{G}_{\mathbf{1}}}$ induces an equivalence of categories with families.*

Proof. This is straightforward, as the syntax is interpreted in the same way in both the theories. So the empty context $\varnothing$ is the same, as well as the context comprehension operation, and the extension of substitutions. □

Lemma 11 shows that from a logical point of view, the theory $\mathfrak{G}$ and the theory $\mathfrak{G}_{\mathbf{1}}$ are indistinguishable. They are two syntactic presentation of the same category with families (i.e., the same theory), and as such have the same models. This is expected since we have designed the type $\mathbf{1}$ as carrying no information. Under this correspondence, the notion of dimension of a term within the two theories coincide, and in fact we have set up the dimension of a term of type $\mathbf{1}$ to be -1 precisely for this reason.

The desuspension operation. In order to express the theory MCaTT we need an operation that we call the *desuspension*, that we first describe as associating, to every context Γ of $\mathfrak{G}_{\mathbf{1}}$, the context $\downarrow\!\Gamma$ in $\mathfrak{G}$. It is defined by induction on the raw syntax of the theory $\mathfrak{G}_{\mathbf{1}}$, together with the corresponding operation on types of $\mathfrak{G}_{\mathbf{1}}$

$$\downarrow\varnothing = \varnothing \qquad \downarrow(\Gamma, x : A) = (\downarrow\Gamma, x : \downarrow A)$$

$$\downarrow\star = \mathbf{1} \qquad x \underset{A}{\longrightarrow} y = x \underset{\downarrow A}{\longrightarrow} y$$

Proposition 12. *The desuspension respects the judgments in $\mathfrak{G}_{\mathbf{1}}$, more exactly:*

- *For any context $\Gamma \vdash$ the judgment $\downarrow\Gamma \vdash$ is derivable*
- *For any type $\Gamma \vdash A$ the judgment $\downarrow\Gamma \vdash \downarrow A$ is derivable*
- *For any term $\Gamma \vdash x : A$ the judgment $\Gamma \vdash x : \downarrow A$ is derivable.*

Proof. We prove this result by mutual induction on the derivation tree of valid judgments.

Induction case for contexts:

- For a derivable context obtained by the rule (EC), it is necessarily the context $\varnothing \vdash$, the rule (EC) gives a derivation of $\downarrow\varnothing$
- For a derivable context obtained by the rule (CE), it is of the form $(\Gamma, x : A)$, by induction case on types we have $\downarrow\Gamma \vdash \downarrow A$, and applying the rule (CE) yields a derivation for $\downarrow\Gamma, x : \downarrow A \vdash$.

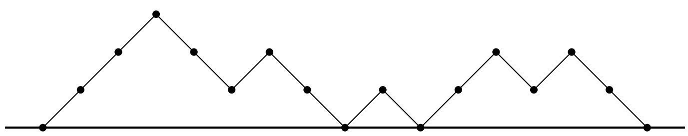
Induction for types:

- For a derivable type obtained by the rule ($\star$-INTRO), it is of the form $\Gamma \vdash \star$, we necessarily have $\Gamma \vdash$, and by the induction case for contexts, it implies $\downarrow\Gamma \vdash$, hence we can apply the rule (**1**-INTRO) to get a derivation for $\downarrow\Gamma \vdash \mathbf{1}$.
- For a derivable type obtained by the rule ($\rightarrow -intro$), it is of the form $\Gamma \vdash t \underset{A}{\rightarrow} u$, by the induction cases for types and variables, we have derivations for the judgments $\downarrow\Gamma \vdash$, $\downarrow\Gamma \vdash t : \downarrow A$ and $\downarrow\Gamma \vdash u : \downarrow A$. This lets us apply the rule ($\rightarrow$-INTRO) in order to get a derivation for the judgment $\downarrow\Gamma \vdash t \underset{\downarrow A}{\longrightarrow} u$

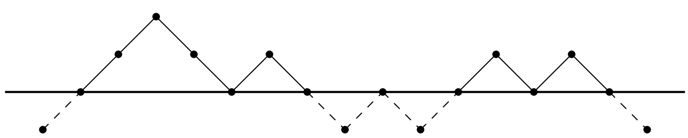
Induction for variables: For a term obtained by the rule (VAR), it is a variable $\Gamma \vdash x : A$, and necessarily we have $\Gamma \vdash$, which by induction for contexts implies $\downarrow\Gamma \vdash$, and $(x : A) \in \Gamma$, which implies $(x : \downarrow A) \in \downarrow\Gamma$. Thus applying the rule (VAR) lets us construct a derivation for $\downarrow\Gamma \vdash x : \downarrow A$. $\square$

From now on, when we perform proofs by induction on the derivation tree, we rely on the form of the judgment, for instance, we may write: "For the context $\varnothing \vdash$" to mean that we discriminate on the rule (EC) and that in this case the context is necessarily $\varnothing$. This is justified since the rule (EC) is the only one that allows to derive $\varnothing \vdash$. For the other rules, we justify this quicker notation by noticing that each particular form of a syntactic expression corresponds to exactly one introduction rule.

Interpretation of the desuspension. Intuitively, the desuspension operation on ps-contexts corresponds to a shift in dimensions, by decreasing all the dimensions by 1, and forgetting the information that lies in the dimension 0. Visually considering a ps-context with the following combinatorial structure



performing the desuspension yields the following structure, that describes a context that is no a ps-context anymore. In order to visualize, we have kept the dimension 0 as dashed in the diagram, even though in the syntax of the theory all the points that went below the horizon (became of dimension -1) are collapsed into a single point.



Examples We provide a few examples of the suspension, which intuitively merely consists in rewriting the type $\star$ into the type $\mathbf{1}$. For each of the context in $\mathfrak{S}_\mathbf{1}$ we also give a simpler context to which it is isomorphic.

Γ	$\downarrow\Gamma$
$(x : \star)$	$(x : \mathbf{1}) \simeq \varnothing$
$(x : \star, y : \star, f : x \underset{\star}{\to} y)$	$(x : \mathbf{1}, y : \mathbf{1}, f : x \underset{\mathbf{1}}{\to} y) \simeq (f : \star)$
$(x : \star, y : \star, f : x \underset{\star}{\to} y, g : y \underset{\star}{\to} x)$	$(x : \mathbf{1}, y : \mathbf{1}, f : x \underset{\mathbf{1}}{\to} y, g : z \underset{\mathbf{1}}{\to} y) \simeq (f : \star, g : \star)$

2.2 The theory MCaTT

The type theory MCaTT that we introduce relies on the theory CaTT on a fundamental level, in the sense that its inference rules use the derivability of judgments in CaTT, in order express these rules, we need to introduce a general version of the desuspension operation, defined on the raw syntax of the theory CaTT. In order to write this definition, we need to introduce the raw syntax of the theory MCaTT. It is obtained, similarly to the theory CaTT, by adding two term constructors to the theory $\mathfrak{G}_{\mathbf{1}}$, that we denote by analogy mop and mcoh. Thus, a terms expression is either a variables, or of the form $\mathsf{mop}_{\Delta,A}[\delta]$ or $\mathsf{mcoh}_{\Delta,A}[\delta]$ with Δ a ps-context and A a type and δ a substitution. Substitutions act on these term constructors via the following formulas

$$\mathsf{mop}_{\Delta,A}[\gamma][\delta] = \mathsf{mop}_{\Delta,A}[\gamma \circ \delta] \qquad\qquad \mathsf{mcoh}_{\Delta,A}[\gamma][\delta] = \mathsf{mcoh}_{\Delta,A}[\gamma \circ \delta]$$

The general desuspension operation. We generalize the desuspension operation to well-defined contexts, types, terms and substitutions of the theory CaTT, with the following definition

For contexts:

$$\downarrow\varnothing = \varnothing \qquad\qquad \downarrow(\Gamma, x : A) = \downarrow\Gamma, x : \downarrow A$$

For types:

$$\downarrow\star = \mathbf{1} \qquad\qquad \downarrow(t \underset{A}{\longrightarrow} u) = \downarrow t \underset{\downarrow A}{\longrightarrow} \downarrow u$$

For terms:

$$\downarrow x = x \qquad\qquad \downarrow \mathsf{op}_{\Gamma,A}[\gamma] = \mathsf{mop}_{\Gamma,A}[\downarrow\gamma]$$
$$\downarrow \mathsf{coh}_{\Gamma,A}[\gamma] = \mathsf{mcoh}_{\Gamma,A}[\downarrow\gamma]$$

For substitutions:

$$\downarrow\langle\rangle = \langle\rangle \qquad\qquad \downarrow\langle\gamma, x \mapsto t\rangle = \langle\downarrow\gamma, x \mapsto \downarrow t\rangle$$

The theory MCaTT. We consider the type theory MCaTT obtained, for every ps-context Γ and every type $\Gamma \vdash_{\mathsf{op}} A$ in CaTT, by assigning to the term $\mathsf{mop}_{\Gamma,A}$ the type $\downarrow A$. More generally, we assign a type to this term in every context by adjoining a substitution. The general rule for operations in MCaTT can thus be written as

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{op}} A \qquad \Delta \vdash \gamma : \downarrow\Gamma}{\Delta \vdash \mathsf{mop}_{\Gamma,A} : (\downarrow A)[\gamma]}(\mathsf{mop}\text{-}\textsc{intro})$$

Note that this rule explicitly use the translation, and that the judgments $\Gamma \vdash_{\mathsf{ps}}$ and $\Gamma \vdash_{\mathsf{op}} A$ are the judgments defined for the theory CaTT. In particular, although A is used as an index for the terms of the theory MCaTT, it may not be a valid type in this theory, but it has to be a valid type in the theory CaTT, for instance in the third of our following example, the type A uses the expression `comp` that we have defined as a constructor of CaTT and not of MCaTT. Similarly, we give the following rule for coherences in the theory MCaTT

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{eq}} A \qquad \Delta \vdash \gamma : \downarrow\Gamma}{\Delta \vdash \mathsf{mcoh}_{\Gamma,A} : (\downarrow A)[\gamma]}(\mathsf{mcoh}\text{-}\textsc{intro})$$

$$\frac{}{\varnothing}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbf{1}}(\mathbf{1}\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\rightarrow} u}(\rightarrow\text{-INTRO})$$

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{op}} A \qquad \Delta \vdash \gamma : \downarrow\Gamma}{\Delta \vdash \mathsf{mop}_{\Gamma,A}[\gamma] : \downarrow A[\gamma]}(\mathsf{mop}\text{-INTRO})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash () : \mathbf{1}}(()\text{-INTRO}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{eq}} A \qquad \Delta \vdash \gamma : \downarrow\Gamma}{\Delta \vdash \mathsf{mcoh}_{\Gamma,A}[\gamma] : \downarrow A[\gamma]}(\mathsf{mcoh}\text{-INTRO})$$

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \equiv B}{\Gamma \vdash t : B} \qquad \frac{\Gamma \vdash t : \mathbf{1} \qquad \Gamma \vdash u : \mathbf{1}}{\Gamma \vdash t \equiv u : \mathbf{1}}(\eta_{\mathbf{1}})$$

Figure 4: The type theory MCaTT

All the rules of the theory MCaTT are summarized in Fig. 4 in order to give a more self contained overview

Examples. Even though we do not provide an implementation for this theory, we can check by hand a few examples of derivations that are possible and that they correspond to the intuitive idea that we have about monoidal weak ω-categories.

- Monoidal product: We consider the ps-context for the composition of 1-cells,

$$\Gamma = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z)$$

  and denote

$$\texttt{prod} := \mathsf{mop}_{\Gamma, x \to z}$$

  We have $\downarrow\Gamma = (f : \star, g : \star)$, thus for any context Δ in the theory, a substitution $\Delta \vdash \gamma : \downarrow\Gamma$ is just a pair of terms t, u of type $\star$. Suppose that $\Delta \vdash t : \star$ and $\Delta \vdash u : \star$, then the rule (mop-INTRO) gives a derivation of the product of t and u:

$$\Delta \vdash \texttt{prod}\ t\ u : \star$$

- Associativity of monoidal product: similarly, we consider the ps-context

$$\Gamma = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z, w : \star, h : z \to w)$$

and denote

$$\mathsf{assoc} := \mathsf{mcoh}_{\Gamma, \texttt{comp}\ (\texttt{comp}\ f\ g)\ h \to \texttt{comp}\ f\ (\texttt{comp}\ g\ h)}$$

A substitution $\Delta \vdash \gamma : \downarrow\Gamma$ is now a triple of terms t, u, v of type $\star$ in Δ, given such a triple, the rule (mcoh-INTRO) gives a derivation of

$$\Gamma \vdash \texttt{assoc}\ t\ u\ v : \texttt{prod}\ t\ (\texttt{prod}\ u\ v) \underset{\star}{\to} \texttt{prod}\ (\texttt{prod}\ t\ u)\ v$$

- Neutral element: Consider the ps-context $\Gamma = (x : \star)$, and pose

$$\mathsf{e} := \mathsf{mcoh}_{\Gamma, x \to x}$$

A substitution $\Gamma \vdash \gamma : \downarrow\Gamma$ is necessarily the empty substitution, which lets us define by the rule (mcoh-INTRO) the term

$$\Delta \vdash \mathsf{e}\langle\rangle : \star$$

2.3 Properties of the desuspension

In order to define MCaTT, we have introduced the desuspension operation, and have used it freely on the raw syntax of the theory. We now study this operation, and show in particular that it respects the derivability of judgments. In order to show this, we need to prove the following result, of the interaction of the desuspension with the application of substitutions.

Lemma 13. *Given a substitution $\Delta \vdash \gamma : \Gamma$, for any type $\Gamma \vdash A$, we have $\downarrow(A[\gamma]) = \downarrow A[\downarrow\gamma]$, for any term $\Gamma \vdash t : A$, we have $\downarrow(t[\gamma]) = \downarrow t[\downarrow\gamma]$ and for any substitution $\Gamma \vdash \theta : \Theta$, we have*

$\downarrow(\theta \circ \gamma) = \downarrow\theta \circ \downarrow\gamma$.

Proof. We prove this by mutual induction

Induction for types:

- In the case of the type $\star$, we have $\downarrow(\star[\gamma]) = \mathbf{1}$ and $\downarrow\star[\downarrow\gamma] = \mathbf{1}$.
- In the case of a type of the form $t \underset{A}{\to} u$ with A distinct from $\star$, we have the following equalities

$$\downarrow((t \underset{A}{\to} u)[\gamma]) = \downarrow(t[\gamma]) \xrightarrow[\downarrow(A[\gamma])]{} \downarrow(u[\gamma])$$

$$\downarrow(t \underset{A}{\to} u)[\downarrow\gamma] = \downarrow t[\downarrow\gamma] \xrightarrow[\downarrow A[\downarrow\gamma]]{} \downarrow u[\downarrow\gamma]$$

  The induction case for types shows that $\downarrow(A[\gamma]) = \downarrow A[\downarrow\gamma]$ and the induction case for terms shows the equalities $\downarrow(t[\gamma]) = \downarrow t[\downarrow\gamma]$ and $\downarrow(u[\gamma]) = \downarrow u[\downarrow\gamma]$. These prove the equality between the two above expressions.

Induction for terms:

- In the case of a variable $\Gamma \vdash x : A$, denote $t = x[\gamma]$. Then the association $x \mapsto \downarrow t$ appears in $\downarrow\gamma$. Hence $x[\downarrow\gamma] = \downarrow(x[\gamma])$.
- In the case of a term of the form $\mathsf{op}_{\Gamma,A}[\delta]$, the following equalities hold

$$\downarrow(\mathsf{op}_{\Gamma,A}[\delta][\gamma]) = \mathsf{mop}_{\Gamma,A}[\downarrow(\delta \circ \gamma)]$$

$$\downarrow(\mathsf{op}_{\Gamma,A}[\delta])[\downarrow\gamma] = \mathsf{mop}_{\Gamma,A}[\downarrow\delta \circ \downarrow\gamma]$$

  The induction case for substitution then provides the equality between these two expressions.
- Similarly in the case of a term of the form $\mathsf{coh}_{\Gamma,A}[\delta]$, we have the equalities

$$\downarrow(\mathsf{coh}_{\Gamma,A}[\delta][\gamma]) = \mathsf{mcoh}_{\Gamma,A}[\downarrow(\delta \circ \gamma)]$$

$$\downarrow(\mathsf{coh}_{\Gamma,A}[\delta])[\downarrow\gamma] = \mathsf{mcoh}_{\Gamma,A}[\downarrow\delta \circ \downarrow\gamma]$$

The induction case for substitution then provides the equality between these two expressions.

Induction for substitutions:

- In the case of the empty substitution $\langle\rangle$, we have $\langle\rangle \circ \gamma = \langle\rangle$, and hence $\downarrow(\langle\rangle \circ \gamma) = \langle\rangle$. But we also have $\downarrow\langle\rangle \circ \gamma = \langle\rangle$.
- In the case of a substitution of the form $\langle\theta, x \mapsto t\rangle$, we have the following equalities

$$\downarrow\langle\theta, x \mapsto t\rangle \circ \gamma = \langle\downarrow(\theta \circ \gamma), x \mapsto \downarrow(t[\gamma])\rangle$$
$$\downarrow\langle\theta, x \mapsto t\rangle \circ \downarrow\gamma = \langle\downarrow\theta \circ \downarrow\gamma, x \mapsto \downarrow t[\downarrow\gamma]\rangle$$

Then the induction case for substitutions proves that $\downarrow(\theta \circ \gamma) = \downarrow\theta \circ \downarrow\gamma$, and the induction case for terms applies and shows that $\downarrow(t[\gamma]) = \downarrow t[\downarrow\gamma]$. This proves the equality between the two above expressions. $\square$

Correctness of the desuspension. We have defined the desuspension as a syntactic operation between the raw syntaxes of the theories CaTT and MCaTT. This operation actually preserves most of the structure of the theory, and in particular it preserves the derivability of the judgments, and can thus be lifted as an operation between the theories themselves.

Proposition 14. *The following statements hold*

- *For every context* $\Gamma \vdash$ *in* CaTT*, the judgment* $\downarrow\Gamma \vdash$ *is derivable in* MCaTT*.*
- *For every type* $\Gamma \vdash A$ *in* CaTT*, the judgment* $\downarrow\Gamma \vdash \downarrow A$ *is derivable in* MCaTT*.*
- *For every term,* $\Gamma \vdash t : A$*, the judgment* $\downarrow\Gamma \vdash \downarrow t : \downarrow A$ *is derivable in* MCaTT
- *For every substitution* $\Delta \vdash \gamma : \Gamma$ *in* CaTT*, the judgment* $\downarrow\Delta \vdash \downarrow\gamma : \downarrow\Gamma$ *is derivable in* MCaTT*.*

Proof. We prove this result by mutual induction on the derivation of the judgements.

Induction for contexts:

- For the empty context $\varnothing$, we have $\downarrow\varnothing = \varnothing$ and the rule (EC) gives a derivation of $\downarrow\varnothing \vdash$

- For a context of the form $(\Gamma, x : A) \vdash$, we necessarily have $\Gamma \vdash A$, so the induction case for types shows that $\downarrow\Gamma \vdash \downarrow A$, and hence the rule (CE) applies to give a derivation for $(\downarrow\Gamma, x : \downarrow A) \vdash$. Since we also have $\downarrow(\Gamma, x : A) = (\downarrow\Gamma, x : \downarrow A)$, this lets us conclude that $\downarrow(\Gamma, x : A) \vdash$.

Induction for types:

- For the type $\Gamma \vdash \star$, we necessarily have $\Gamma \vdash$, hence the induction case for contexts shows that $\downarrow\Gamma \vdash$, and applying the rule (**1**-INTRO) gives a derivation for $\downarrow\Gamma \vdash \downarrow\star$.

- For a type of the form $\Gamma \vdash t \underset{A}{\to} u$ we necessarily have a derivation of $\Gamma \vdash A$, which gives by the induction case for the types a derivation of $\downarrow\Gamma \vdash \downarrow A$. Moreover, we necessarily have two derivations $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$, the induction case for terms gives derivations of $\downarrow\Gamma \vdash \downarrow t : \downarrow A$ and $\downarrow\Gamma \vdash \downarrow u : \downarrow A$. These derivations assemble with the rule ($\to$-INTRO) to provide a derivation of $\downarrow\Gamma \vdash \downarrow t \underset{\downarrow A}{\longrightarrow} \downarrow u$.

Induction for terms:

- For a variable $\Gamma \vdash x : A$, we necessarily have $\Gamma \vdash$ which by the induction case for contexts provides a derivation of $\downarrow\Gamma \vdash$. Moreover, we have the condition $(x : A) \in \Gamma$, hence $(x : \downarrow A) \in \downarrow\Gamma$. This lets us apply the rule (VAR) in order to prove $\downarrow\Gamma \vdash x : \downarrow A$.

- For a term of the form $\Delta \vdash \mathsf{op}_{\Gamma,A}[\gamma] : A[\gamma]$, necessarily we have and derivation of the judgments $\Gamma \vdash_{\mathsf{op}} A$ and $\Delta \vdash \gamma : \Gamma$. By the induction case for substitutions, the latter provides a derivation for $\downarrow\Delta \vdash \downarrow\gamma : \downarrow\Gamma$. This lets us apply the rule (mop-INTRO) to construct the term $\downarrow\Delta \vdash \mathsf{mop}_{\Gamma,A}[\downarrow\gamma]$.

- Similarly, for a term of the form $\Delta \vdash \mathsf{coh}_{\Gamma,A}[\gamma] : A[\gamma]$, we have a derivation of the judgments $\Gamma \vdash_{\mathsf{eq}} A$ and $\Delta \vdash \gamma : \Gamma$. By the induction case for substitutions, the latter provides a derivation for $\downarrow\Delta \vdash \downarrow\gamma : \downarrow\Gamma$. This lets us apply the rule (mcoh-INTRO) to construct the term $\downarrow\Delta \vdash \mathsf{mcoh}_{\Gamma,A}[\downarrow\gamma]$.

Induction for substitutions:

- For the empty substitution $\Delta \vdash \langle\rangle : \varnothing$, we necessarily have a derivation of $\Delta \vdash$, and hence by induction case for the context, this produces a derivation of $\downarrow\!\Delta \vdash$. Applying the rule (ES), we get a derivation of $\downarrow\!\Delta \vdash \langle\rangle : \varnothing$.
- For a substitution of the form $\Delta \vdash \langle \gamma, x \mapsto t\rangle : (\Gamma, x : A)$, we necessarily have a derivation of $\Delta \vdash \gamma : \Gamma$, which by the induction case for the substitutions gives a derivation of $\downarrow\!\Delta \vdash \downarrow\!\gamma : \downarrow\!\Gamma$. Moreover, we have a derivation of $(\Gamma, x : A) \vdash$ which provides, by the induction rule for contexts a derivation of $\downarrow\!(\Gamma, x : A) \vdash$, which reduces to $(\downarrow\!\Gamma, x : \downarrow\!A)$. Finally, the substitution also gives derivation of $\Delta \vdash t : A[\gamma]$ and the induction case for terms applies to give a derivation of $\downarrow\!\Delta \vdash \downarrow\!t : \downarrow\!(A[\gamma])$. Lemma 13 applies and this judgment rewrites as $\downarrow\!\Delta \vdash \downarrow\!t : \downarrow\!A[\downarrow\!\gamma]$. We can then apply the rule (SE) in order to get a derivation of $\downarrow\!\Delta \vdash \downarrow\!\langle \gamma, x \mapsto t\rangle : (\Gamma, x : A)$ as follows

$$\frac{\downarrow\!\Delta \vdash \downarrow\!\gamma : \downarrow\!\Gamma \qquad \downarrow\!\Gamma, x : \downarrow\!A \vdash \qquad \downarrow\!\Delta \vdash \downarrow\!t : \downarrow\!A[\downarrow\!\gamma]}{\downarrow\!\Delta \vdash \langle \downarrow\!\gamma, x \mapsto \downarrow\!t\rangle : (\downarrow\!\Gamma, x : \downarrow\!A)}\text{(SE)}$$

$\square$

The desuspension as a functor. We can reformulate all the results proven in Lemma 13 and Proposition 14 in a more categorical fashion, taking advantage of the structure of category with families that the syntactic category of a type theory naturally carries. This gives the following more compact statement, that is useful for relating the semantics of the two theories.

Corollary 15. *The desuspension operation defines a morphism of categories with families*

$$\downarrow : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{S}_{\mathsf{MCaTT}}$$

Proof. We first show that this operation defines a functor, by sending each object Γ, which is a context in CaTT to the context $\downarrow\!\Gamma$, and each map $\Delta \vdash \gamma : \Gamma$ in $\mathcal{S}_{\mathsf{CaTT}}$, which is a substitution onto the substitution γ. Proposition 14 ensures that the judgments $\downarrow\!\Gamma \vdash$ and $\downarrow\!\Delta \vdash \downarrow\!\gamma : \downarrow\!\Gamma$ are derivable, hence defining an association from $\mathcal{S}_{\mathsf{CaTT}}$ to $\mathcal{S}_{\mathsf{MCaTT}}$. Moreover, Lemma 13 shows that $\downarrow\!\gamma \circ \delta = \downarrow\!\gamma \circ \downarrow\!\delta$, so to prove that it is a functor, it suffices to show that it sends the identity onto the identity, which we check by induction:

- For the empty context $\varnothing$, the identity is given by the empty substitution $\mathrm{id}_\varnothing = \langle\rangle$, and we have $\downarrow\langle\rangle = \langle\rangle$.
- For a context of the form $(\Gamma, x : A)$, the identity is given by $\langle \mathrm{id}_\Gamma, x \mapsto x\rangle$, whose image is $\langle \downarrow\mathrm{id}_\Gamma, x \mapsto x\rangle$. By induction, we have $\downarrow\mathrm{id}_\Gamma = \mathrm{id}_{\downarrow\Gamma}$, which gives the equality $\downarrow\mathrm{id}_{(\Gamma,x:A)} = \mathrm{id}_{\downarrow(\Gamma,x:A)}$.

This shows that $\downarrow$ sends the identity onto the identity, and thus defines a functor $\mathcal{S}_{\mathsf{CaTT}} \to \mathcal{S}_{\mathsf{MCaTT}}$. We now show that this functor is a morphism of categories with families, be defining the image of a type $\Gamma \vdash A$ to be $\downarrow A$ and the image of a term $\Gamma \vdash t : A$ to be $\downarrow t$. Proposition 14 shows that these elements live in the adequate set, and Lemma 13 moreover ensures that this association is functorial, so $\downarrow$ defines a morphism in the slice category **Cat/Fam**. Since moreover the operation $\downarrow$ is defined to preserve the terminal context and the context comprehension (by definition, we have $\downarrow\varnothing = \varnothing$ and $\downarrow(\Gamma, x : A) = (\downarrow\Gamma, x : \downarrow A)$ and $\downarrow\langle\gamma, x \mapsto t\rangle = \langle\downarrow\gamma, x \mapsto \downarrow t\rangle$), it is in fact a morphism of categories with families. $\square$

2.4 Reduced suspension

We now define another translation related to the desuspension that goes in the other direction: From an expression in the theory MCaTT, it associates an expression in the theory CaTT. We call this translation the *reduced suspension* and the intuition is that it exhibits MCaTT as the theory of the judgments in CaTT concerned with context that only have one object.

The reduced suspension operation. In order to define the reduced suspension, we assume the existence of a variable name that is completely fresh, and call it $\bullet$. This could be achieved by extending the set of variables that we use with the variable $\bullet$. Contrary to the desuspension, the reduced suspension is not defined on the raw syntax, but it is defined only for the derivable judgments of theory MCaTT, this is due to the fact that we need to respect definitional equality, hence we define this operation only on normal forms, but the raw syntax does not have normal form. We first define it as an operation that produces an expression in the raw syntax of the theory CaTT before verifying that this expression corresponds to a derivable judgment. This operation is defined inductively as follows

For the context $\varnothing \vdash$

$\Uparrow\varnothing = (\bullet : \star)$

For the context $\Gamma, x : \mathbf{1} \vdash$

$\Uparrow(\Gamma, x : \mathbf{1}) = \Uparrow\Gamma$

For the context $\Gamma, x : A \vdash$

$\Uparrow(\Gamma, x : A) = (\Uparrow\Gamma, x : \Uparrow A)$

For the type $\Gamma \vdash \mathbf{1}$

$\Uparrow\mathbf{1} = \star$

For the type $\Gamma \vdash \star$

$$\Uparrow\star = \bullet \underset{\star}{\to} \bullet$$

For the type $\Gamma \vdash t \underset{A}{\to} u$

$$\Uparrow\left(t \underset{A}{\to} u\right) = \Uparrow t \underset{\Uparrow A}{\longrightarrow} \Uparrow u$$

For the term $\Gamma \vdash () : \mathbf{1}$

$\Uparrow() = \bullet$

For a variable $\Gamma \vdash x : A$ $(A \neq \mathbf{1})$

$\Uparrow x = x$

For a term $\Gamma \vdash \mathsf{mop}_{\Theta,A}[\gamma] : A[\gamma]$

$\Uparrow\mathsf{mop}_{\Theta,A}[\gamma] = \mathsf{op}_{\Theta,A}[\bullet_\Theta \circ \Uparrow\gamma]$

For a term $\Gamma \vdash \mathsf{mcoh}_{\Theta,A}[\gamma] : A[\gamma]$

$\Uparrow\mathsf{mcoh}_{\Theta,A}[\gamma] = \mathsf{coh}_{\Theta,A}[\bullet_\Theta \circ \Uparrow\gamma]$

For the substitution $\Gamma \vdash \langle\rangle : \varnothing$

$\Uparrow\langle\rangle = \langle \bullet \mapsto \bullet \rangle$

For the substitution $\Gamma \vdash \langle \gamma, x \mapsto t \rangle : (\Delta, x : \mathbf{1})$

$\Uparrow\langle \gamma, x \mapsto t \rangle = \Uparrow\gamma$

For the substitution $\Gamma \vdash \langle \gamma, x \mapsto t \rangle : (\Delta, x : A)$

$\Uparrow\langle \gamma, x \mapsto t \rangle = \langle \Uparrow\gamma, x \mapsto \Uparrow t \rangle$

Where the substitution $\bullet_\Theta$ is defined by induction on the context $\Theta \vdash$ of CaTT by

$\bullet_\varnothing = \langle\rangle$ $\qquad$ $\bullet_{(\Theta, x:A)} = \langle \bullet_\Theta, x \mapsto \Uparrow\Downarrow x \rangle$

Note that in the case of a variable $\Gamma \vdash x : A$ we have to assume that $A \neq \mathbf{1}$ to ensure that the term is in normal form so that definitional equality is respected. Before showing that this operation respects the derivability of judgments, we first show some useful syntactic properties.

Lemma 16. *For all substitution* $\Delta \vdash \gamma : \Gamma$*, we have* $\bullet[\Uparrow\gamma] = \bullet$.

Proof. By definition of $\Uparrow\gamma$, the mapping $\bullet \mapsto \bullet$ appears in $\Uparrow\gamma$, and it is the only mapping with the variable $\bullet$ on the left. $\square$

Lemma 17. *Given a substitution $\Delta \vdash \gamma : \Gamma$ in the theory* MCaTT*, For any type $\Gamma \vdash A$ we have the equality $\Uparrow(A[\gamma]) = \Uparrow A[\Uparrow\gamma]$, for any term $\Gamma \vdash t : A$, we have the equality $\Uparrow(t[\gamma]) = \Uparrow t[\Uparrow\gamma]$ and for any substitution $\Gamma \vdash \theta : \Theta$, we have the equality $\Uparrow\delta \circ \gamma = \Uparrow\delta \circ \Uparrow\gamma$.*

Proof. We suppose given a substitution γ and prove these three results by mutual induction

Induction for types:

- For the type $\mathbf{1}$, we have $\mathbf{1}[\gamma] = \mathbf{1}$, hence $\Uparrow(\mathbf{1}[\gamma]) = \Uparrow\mathbf{1} = \star$. But we also have $\Uparrow\mathbf{1}[\Uparrow\gamma] = \star[\Uparrow\gamma] = \star$.
- For the type $t \underset{A}{\longrightarrow} u$, we have the two following equalities

$$\Uparrow((t \underset{A}{\longrightarrow} u)[\gamma]) = \Uparrow(t[\gamma]) \xrightarrow[\Uparrow(A[\gamma])]{} \Uparrow(u[\gamma])$$

$$(\Uparrow(t \underset{A}{\longrightarrow} u))[\Uparrow\gamma] = (\Uparrow t)[\Uparrow\gamma] \xrightarrow[(\Uparrow A)[\Uparrow\gamma]]{} (\Uparrow u)[\Uparrow\gamma]$$

  The induction case for type shows that $\Uparrow(A[\gamma]) = (\Uparrow A)[\Uparrow\gamma]$ and the induction case for terms proves the equalities $\Uparrow(t[\gamma]) = (\Uparrow t)[\Uparrow\gamma]$ and $\Uparrow(u[\gamma]) = (\Uparrow u)[\Uparrow\gamma]$. These three equalities show the equality between the two previous expressions.

Induction for terms:

- For the term $()$, we have $()[\gamma] = ()$ and $\Uparrow(()[\gamma]) = \bullet$. But we also have $\Uparrow()[\Uparrow\gamma] = \bullet[\Uparrow\gamma]$, and by Lemma 16, this shows $\Uparrow()[\Uparrow\gamma] = \bullet$.
- For a variable x of positive dimension, we have $x[\gamma] = t$, and hence $\Uparrow(x[\gamma]) = \Uparrow t$. Moreover the mapping $x \mapsto \Uparrow t$ appears in $\Uparrow\gamma$, and hence $x[\Uparrow\gamma] = \Uparrow t$.
- For a term of the form $\mathsf{mop}_{\Gamma,A}[\delta]$ we have the two following equalities

$$\Uparrow(\mathsf{mop}_{\Gamma,A}[\delta][\gamma]) = \mathsf{op}_{\Gamma,A}[\bullet_\Gamma \circ \Uparrow(\delta \circ \gamma)]$$

$$(\Uparrow\mathsf{mop}_{\Gamma,A}[\delta])[\Uparrow\gamma] = \mathsf{op}_{\Gamma,A}[(\bullet_\Gamma \circ \Uparrow\delta) \circ \Uparrow\gamma]$$

and the induction case for substitutions together with the associativity of composition for substitution show that these two terms are equal.

- Similarly for a term of the form $\mathsf{mcoh}_{\Gamma,A}[\delta]$ we have the two following equalities

$$\uparrow(\mathsf{mcoh}_{\Gamma,A}[\delta][\gamma]) = \mathsf{coh}_{\Gamma,A}[\bullet_\Gamma \circ \uparrow(\delta \circ \gamma)]$$
$$(\uparrow\mathsf{mcoh}_{\Gamma,A}[\delta])[\uparrow\gamma] = \mathsf{coh}_{\Gamma,A}[(\bullet_\Gamma \circ \uparrow\delta) \circ \uparrow\gamma]$$

  and by induction and associativity these two expressions are equal.

Induction for substitutions:

- In the case of the empty substitutions, we have $\langle\rangle \circ \gamma = \langle\rangle$, and hence $\uparrow(\langle\rangle \circ \gamma) = \langle \bullet \mapsto \bullet\rangle$. Moreover, $\uparrow\langle\rangle \circ \uparrow\gamma = \langle \bullet \mapsto \bullet[\uparrow\gamma]\rangle$. By Lemma 16 this shows that $\uparrow\langle\rangle \circ \uparrow\gamma = \langle \bullet \mapsto \bullet\rangle$.
- In the case of a substitution of the form $\Gamma \vdash \langle \theta, x \mapsto t\rangle : (\Theta, x : \mathbf{1})$, we have the following equalities

$$\uparrow(\langle \theta, x \mapsto t\rangle \circ \gamma) = \langle \uparrow(\theta \circ \gamma)\rangle$$
$$\uparrow\langle \theta, x \mapsto t\rangle \circ \uparrow\gamma = \langle \uparrow\theta \circ \uparrow\gamma\rangle$$

  and the induction case for substitutions provides the result.
- In the case of a substitution of the form $\Gamma \vdash \langle \theta, x \mapsto t\rangle : (\Theta, x : A)$ with $A \neq \mathbf{1}$, we have the following equalities

$$\uparrow(\langle \theta, x \mapsto t\rangle \circ \gamma) = \langle \uparrow(\theta \circ \gamma), x \mapsto \uparrow(t[\gamma])\rangle$$
$$\uparrow\langle \theta, x \mapsto t\rangle \circ \uparrow\gamma = \langle \uparrow\theta \circ \uparrow\gamma, x \mapsto (\uparrow t)[\uparrow\gamma]\rangle$$

  and the induction cases for substitutions and terms provide the result.

$\square$

Reduced suspension on the theory $\mathfrak{G}$. We first note that the theory $\mathfrak{G}$ can be included in the theory MCaTT, by sending any expression in the theory $\mathfrak{G}$ to the same expression, seen as an expression of MCaTT. Indeed, an expression in the theory $\mathfrak{G}$ is built with the type constructors $\star$ and $\to$ along with variables, which are also valid types in the theory MCaTT obeying the same rules (where the type $\star$ is a short form for $() \to ()$). We now focus on the restriction of the reduced suspension to the theory $\mathfrak{G}$. Since the image of a variable by the reduced suspension is again a variable, this shows that the image of an expression in $\mathfrak{G}$ by the reduced suspension is again an expression in $\mathfrak{G}$. So the reduced suspension as an operation from the raw syntax of $\mathfrak{G}$ to itself. Understanding this operation first is key in studying the general reduced suspension between MCaTT and CaTT, since general terms are introduced by substitutions to a ps-contexts, that are special cases of contexts in $\mathfrak{G}$.

Lemma 18. *In the theory $\mathfrak{G}$, the following properties are satisfied*

- *For any well-defined context $\Gamma \vdash$, the judgment $\uparrow\Gamma$ is derivable.*
- *For any type $\Gamma \vdash A$, the judgment $\uparrow\Gamma \vdash \uparrow A$ is derivable.*
- *For any term $\Gamma \vdash t : A$, the judgment $\uparrow\Gamma \vdash \uparrow t : \uparrow A$ is derivable.*
- *For any substitution $\Delta \vdash \gamma : \Gamma$, the judgment $\uparrow\Delta \vdash \uparrow\gamma : \uparrow\Gamma$ is derivable.*

Proof. We prove this result by mutual induction on contexts, types, terms and substitutions

Induction for contexts:

- In the case of the empty context $\varnothing$, we have $\uparrow\varnothing = (\bullet : \star)$, and the following derivation shows that $\uparrow\varnothing \vdash$

$$\dfrac{\dfrac{\dfrac{}{\varnothing \vdash}\,(\text{EC})}{\varnothing \vdash \star}\,(\star\text{-INTRO})}{\bullet : \star \vdash}\,(\text{CE})$$

- In the case of a context of the form $(\Gamma, x : A) \vdash$ (by assumption, $A \neq \mathbf{1}$ since we are in the theory $\mathfrak{G}$), we can extract a derivation of $\Gamma \vdash A$ and by the induction case for types, it gives a derivation for $\uparrow\Gamma \vdash \uparrow A$. by applying the rule (CE), we get a derivation for $(\uparrow\Gamma, x : \uparrow A) \vdash$

Induction for types:

- In the case of the type $\Gamma \vdash \star$, we can extract a derivation of $\Gamma \vdash$, which by the induction rule for contexts shows $\uparrow\Gamma \vdash$. Since moreover $(\bullet : \star) \in \uparrow\Gamma$, we can construct a derivation of $\uparrow\Gamma \vdash \uparrow\star$ as follows

$$\dfrac{\dfrac{\uparrow\Gamma \vdash}{\uparrow\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \dfrac{\uparrow\Gamma \vdash \qquad (\bullet : \star) \in \uparrow\Gamma}{\uparrow\Gamma \vdash \bullet : \star \qquad \uparrow\Gamma \vdash \bullet : \star}(\text{VAR})}{\uparrow\Gamma \vdash \bullet \underset{\star}{\to} \bullet}(\to\text{-INTRO})$$

- In the case of a type of the form $\Gamma \vdash t \underset{A}{\to} u$, we have a derivation for $\Gamma \vdash A$, which by the induction case for types gives a derivation for $\uparrow\Gamma \vdash \uparrow A$ and two derivations for $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$, which by the induction case for terms gives $\uparrow\Gamma \vdash \uparrow t : \uparrow A$ and $\uparrow\Gamma \vdash \uparrow u : \uparrow A$. These three derivations assemble with the rule ($\to$-INTRO) in order to produce a derivation for the judgment $\uparrow\Gamma \vdash \uparrow t \underset{\uparrow A}{\longrightarrow} \uparrow u$.

Induction for terms: A term is necessarily a variable $\Gamma \vdash x : A$. Since we have $\Gamma \vdash$, by the induction case for contexts we also get $\uparrow\Gamma \vdash$. Moreover, the condition $(x : A) \in \Gamma$ is also necessarily satisfied, and $A \neq \mathbf{1}$ since we are in the theory $\mathfrak{G}$. Then we necessarily have $(x : \uparrow A) \in \uparrow\Gamma$, and the equality $\uparrow x = x$ holds. Hence, applying the rule (VAR) yields a derivation for $\uparrow\Gamma \vdash x : \uparrow A$

$$\dfrac{\uparrow\Gamma \vdash \qquad (x : \uparrow A) \in \uparrow\Gamma}{\uparrow\Gamma \vdash x : \uparrow A}(\text{VAR})$$

Induction for substitutions:

- In the case of the empty substitution $\Delta \vdash \langle\rangle : \varnothing$, we necessarily have a derivation of $\Delta \vdash$, which by the induction case for contexts gives a derivation of $\uparrow\Delta \vdash$. Moreover, we also have by definition that $(\bullet : \star) \in \uparrow\Delta$. This lets us construct a derivation for $\uparrow\Delta \vdash \uparrow\langle\rangle : \uparrow\varnothing$ as follows

$$\dfrac{\dfrac{\uparrow\Delta \vdash}{\uparrow\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \bullet : \star \vdash \qquad \dfrac{\uparrow\Delta \vdash \qquad (\bullet : \star) \in \uparrow\Delta}{\uparrow\Delta \vdash \bullet : \star}(\text{VAR})}{\uparrow\Delta \vdash \langle \bullet \mapsto \bullet \rangle : (\bullet : \star)}(\text{SE})$$

- In the case of a substitution of the form $\Delta \vdash \langle \gamma, x \mapsto t\rangle : (\Gamma, x : A)$, since the judgment is derivable in $\mathfrak{G}$, $A \neq \mathbf{1}$. Then we necessarily have $\Delta \vdash \gamma : \Gamma$, which gives by the induction case for substitutions, $\upharpoonleft\Delta \vdash \upharpoonleft\gamma : \upharpoonleft\Gamma$. Moreover, we have a derivation of $(\Gamma, x : A) \vdash$, which gives by the induction case for contexts a derivation of $\upharpoonleft\Gamma, x : \upharpoonleft A \vdash$, and a derivation of $\Delta \vdash t : A[\gamma]$, which gives by the induction case for terms a derivation of $\upharpoonleft\Delta \vdash \upharpoonleft t : \upharpoonleft A[\gamma]$. Lemma 17 allows to rewrite this into a derivation for $\upharpoonleft\Delta \vdash \upharpoonleft t : \upharpoonleft A[\upharpoonleft\gamma]$. Thus these derivations can be assembled using the rule (SE) into a derivation of $\upharpoonleft\Delta \vdash \upharpoonleft\langle\gamma, x \mapsto t\rangle : \upharpoonleft(\Gamma, x : A)$ as follows

$$\dfrac{\upharpoonleft\Delta \vdash \upharpoonleft\gamma : \upharpoonleft\Gamma \qquad \upharpoonleft\Gamma, x : \upharpoonleft A \vdash \qquad \upharpoonleft\Delta \vdash \upharpoonleft t : \upharpoonleft A[\upharpoonleft\gamma]}{\upharpoonleft\Delta \vdash \langle\upharpoonleft\gamma, x \mapsto \upharpoonleft t\rangle : (\upharpoonleft\Gamma, x : \upharpoonleft A)}\,(\text{SE})$$

□

Properties of the substitution $\bullet_\Delta$. In order to study the reduced suspension operation, we also need to give the properties of the substitution $\bullet_\Delta$ for a ps-context Δ, on which the reduced suspension relies.

Lemma 19. *We have the following result for the action of the substitution $\bullet_\Delta$ on terms, types and substitutions.*

- *For any type $\Delta \vdash A$ in the theory CaTT, we have $A[\bullet_\Delta] = \upharpoonleft\downharpoonleft A$*
- *For any term $\Delta \vdash t : A$ in the theory CaTT, we have $t[\bullet_\Delta] = \upharpoonleft\downharpoonleft t$.*
- *For any substitution $\Delta \vdash \gamma : \Gamma$ in the theory CaTT, we have $\gamma \circ \bullet_\Delta = \bullet_\Gamma \circ \upharpoonleft\downharpoonleft\gamma$.*

Proof. We prove these equalities by mutual induction

Induction on types:

- For the type $\Delta \vdash \star$, we have $\upharpoonleft\downharpoonleft\star = \star$ and by definition of the application of substitutions, $\star[\bullet_\Delta] = \star$.

- For a type of the form $t \underset{A}{\to} u$

$$(t \underset{A}{\to} u)[\bullet_\Delta] = t[\bullet_\Delta] \xrightarrow[A[\bullet_\Delta]]{} u[\bullet_\Delta]$$

$$\upharpoonleft\!\downharpoonright(t \underset{A}{\to} u) = \upharpoonleft\!\downharpoonright t \xrightarrow[\upharpoonleft\downharpoonright A]{} \upharpoonleft\!\downharpoonright u$$

By the induction case on types $A[\bullet_\Delta] = \upharpoonleft\!\downharpoonright A$, and by the induction case on terms $t[\bullet_\Delta] = \upharpoonleft\!\downharpoonright t$ and $u[\bullet_\Delta] = \upharpoonleft\!\downharpoonright u$. This proves the equality $(t \underset{A}{\to} u)[\bullet_\Delta] = \upharpoonleft\!\downharpoonright(t \underset{A}{\to} u)$.

Induction on terms:

- In the case of a variable $\Delta \vdash x : A$, by definition the mapping $x \mapsto \upharpoonleft\!\downharpoonright x$ appears in $\bullet_\Delta$, and this is the only mapping for x, hence we have $x[\bullet_\Delta] = \upharpoonleft\!\downharpoonright x$.
- In the case of a term of the form $\Delta \vdash \mathsf{op}_{\Gamma,A}[\gamma] : A[\gamma]$, we have the following equations

$$\mathsf{op}_{\Gamma,A}[\gamma][\bullet_\Delta] = \mathsf{op}_{\Gamma,A}[\gamma \circ \bullet_\Delta]$$

$$\upharpoonleft\!\downharpoonright\mathsf{op}_{\Gamma,A}[\gamma] = \mathsf{op}_{\Gamma,A}[\bullet_\Gamma \circ \upharpoonleft\!\downharpoonright\gamma]$$

and the induction case for substitutions then gives the equality.
- Similarly, in the case of a term of the form $\Delta \vdash \mathsf{coh}_{\Gamma,A}[\gamma]$ we have

$$\mathsf{coh}_{\Gamma,A}[\gamma][\bullet_\Delta] = \mathsf{coh}_{\Gamma,A}[\gamma \circ \bullet_\Delta]$$

$$\upharpoonleft\!\downharpoonright\mathsf{coh}_{\Gamma,A}[\gamma] = \mathsf{coh}_{\Gamma,A}[\bullet_\Gamma \circ \upharpoonleft\!\downharpoonright\gamma]$$

and we conclude by the induction case for substitutions.

Induction case for substitutions:

- In the case of the empty substitution $\Delta \vdash \langle\rangle : \varnothing$, since $\bullet_\varnothing = \langle\rangle$, we have the two following equalities

$$\langle\rangle \circ \bullet_\Delta = \langle\rangle$$

$$\bullet_\varnothing \circ \upharpoonleft\!\downharpoonright\langle\rangle = \langle\rangle$$

- In the case of a substitution of the form $\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)$, we have the following equalities

$$\langle \gamma, x \mapsto t \rangle \circ \bullet_\Delta = \langle \gamma \circ \bullet_\Delta, x \mapsto t[\bullet_\Delta] \rangle$$
$$\langle \bullet_\Gamma, x \mapsto \uparrow\!\downarrow x \rangle \circ \langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle = \langle \bullet_\Gamma \circ \langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle, x \mapsto (\uparrow\!\downarrow x)[\langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle] \rangle$$

  Since the substitution $\bullet_\Gamma$ has source $\uparrow\!\downarrow \Gamma$ that do not use the variable x, we have

$$\bullet_\Gamma \circ \langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle = \bullet_\Gamma \circ \uparrow\!\downarrow t$$

  Moreover, if x is of type $\star$ we have $\uparrow\!\downarrow x[\langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle] = \bullet = \uparrow\!\downarrow t$, and if x is not of type $\star$, the expression $x[\langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle]$ simplifies to $\uparrow\!\downarrow t$. Hence we have the equality

$$\langle \bullet_\Gamma, x \mapsto x \rangle \circ \langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle = \langle \bullet_\Gamma \circ \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle$$

  the induction case for substitutions then shows that $\gamma \circ \bullet_\Delta = \bullet_\Gamma \circ \uparrow\!\downarrow \gamma$, and the induction case for terms shows that $t[\bullet_\Delta] = \uparrow\!\downarrow t$, which proves the equality

$$\langle \gamma, x \mapsto t \rangle \circ \bullet_\Delta = \langle \bullet_\Gamma, x \mapsto x \rangle \circ \langle \uparrow\!\downarrow \gamma, x \mapsto \uparrow\!\downarrow t \rangle$$

□

Lemma 20. *For any context $\Gamma \vdash$ in CaTT, the expression $\bullet_\Gamma$ is a well defined substitution $\uparrow\!\downarrow \Gamma \vdash \bullet_\Gamma : \Gamma$.*

Proof. This result is proved by induction on the context Γ.

- For the empty context $\varnothing$, we have already proven that $\uparrow\!\downarrow \varnothing \vdash$. Hence the rule (ES) proves that $\uparrow\!\downarrow \Gamma \vdash \langle\rangle : \varnothing$.

- For a context of the form $(\Gamma, x : A) \vdash$, we have by induction $\uparrow\!\downarrow \Gamma \vdash \bullet_\Gamma : \Gamma$. Hence by weakening (c.f. Proposition 1), this shows that we have a derivation of $\uparrow\!\downarrow (\Gamma, x : A) \vdash \bullet_\Gamma : \Gamma$. Moreover, by Lemma 18, we have that $\uparrow\!\downarrow (\Gamma, x : A) \vdash x : \uparrow\!\downarrow A$, and Lemma 19 then shows that we have $\uparrow\!\downarrow (\Gamma, x : A) \vdash x : A[\bullet_\Gamma]$. This lets us apply the rule (SE) to construct a derivation of $\uparrow\downarrow(\Gamma, x : A) \vdash \bullet_{(\Gamma,x:A)} : (\Gamma, x : A)$ as follows

$$\dfrac{\uparrow\downarrow(\Gamma, x : A) \vdash \bullet_\Gamma : \Gamma \qquad (\Gamma, x : A) \vdash \qquad \uparrow\downarrow(\Gamma, x : A) \vdash x : A[\bullet_\Gamma]}{\uparrow\downarrow(\Gamma, x : A) \vdash \langle \bullet_{(\Gamma,x:A)}, x \mapsto x \rangle : (\Gamma, x : A)}\,(\mathrm{SE})$$

$\square$

Correctness of the reduced suspension. We are now equipped to study the reduced suspension operation in its full generality, as an operation from the theory MCaTT to the theory CaTT. In particular we prove the following correctness result showing that this operation is a well defined translation between these two theories.

Proposition 21. *The reduced suspension operation preserves derivability. More precisely, we have*

- *For any context $\Delta \vdash$ derivable in the theory MCaTT, $\uparrow\Delta \vdash$ is derivable in the theory CaTT.*
- *For any type $\Delta \vdash A$ derivable in the theory MCaTT, $\uparrow\Delta \vdash \uparrow A$ is derivable in the theory CaTT.*
- *For any term $\Delta \vdash t : A$ derivable in the theory MCaTT, $\uparrow\Delta \vdash \uparrow t : \uparrow A$ is derivable in the theory CaTT.*
- *For any substitution $\Delta \vdash \gamma : \Gamma$ derivable in the theory MCaTT, $\uparrow\Delta \vdash \uparrow\gamma : \uparrow\Gamma$ is derivable in the theory CaTT.*

Proof. The proof of this result is essentially the same than the proof of Lemma 18 by mutual induction, and adding additional cases for the term constructors we added. For the sake of completeness and to avoid the reader to constantly refer to the previous proof, we still give the entire proof here, by mutual induction on the derivation trees, always keeping all the cases in normal form.

Induction for contexts:

- In the case of the empty context $\varnothing$, we have $\uparrow\varnothing = (\bullet : \star)$, and the following derivation shows that $\upharpoonleft\varnothing \vdash$

$$\dfrac{\dfrac{\dfrac{}{\varnothing \vdash}\,(\text{EC})}{\varnothing \vdash \star}\,(\star\text{-INTRO})}{\bullet_0 : \star \vdash}\,(\text{CE})$$

- In the case of a context of the form $(\Gamma, x : \mathbf{1}) \vdash$, we have $\upharpoonleft(\Gamma, x : \mathbf{1}) = \upharpoonleft\Gamma$ and the induction case for context shows that $\upharpoonleft\Gamma \vdash$.
- In the case of a context of the form $(\Gamma, x : A) \vdash$ with $A \neq \mathbf{1}$, we can extract a derivation of $\Gamma \vdash A$ and by the induction case for types, it gives a derivation for $\upharpoonleft\Gamma \vdash \upharpoonleft A$. by applying the rule (CE), we get a derivation for $(\upharpoonleft\Gamma, x : \upharpoonleft A) \vdash$

Induction for types:

- In the case of the type $\Gamma \vdash \mathbf{1}$, we necessarily have a derivation of $\Gamma \vdash$. The induction case for contexts then gives a derivation of $\upharpoonleft\Gamma \vdash$. The rule ($\star$-INTRO) then applies to provide a derivation of $\upharpoonleft\Gamma \vdash \star$.
- In the case of a type of the form $\Gamma \vdash t \underset{A}{\to} u$, we have a derivation for $\Gamma \vdash A$, which by the induction case for types gives a derivation for $\upharpoonleft\Gamma \vdash \upharpoonleft A$ and two derivations for $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$, which by the induction case for terms gives $\upharpoonleft\Gamma \vdash \upharpoonleft t : \upharpoonleft A$ and $\upharpoonleft\Gamma \vdash \upharpoonleft u \ : \upharpoonleft A$. These three derivations assemble with the rule ($\to$-INTRO) in order to produce a derivation for the judgment $\upharpoonleft\Gamma \vdash \upharpoonleft t \underset{\upharpoonleft A}{\longrightarrow} \upharpoonleft u$.

Induction for terms:

- For the term $\Gamma \vdash () : \mathbf{1}$, we have $\upharpoonleft() = \bullet$. By the induction case for contexts, we have $\upharpoonleft\Gamma \vdash$. Since moreover $(\bullet : \star) \in \upharpoonleft\Gamma$, the rule (VAR) gives a derivation

$$\dfrac{\upharpoonleft\Gamma \vdash \qquad (\bullet : \star) \in \upharpoonleft\Gamma}{\upharpoonleft\Gamma \vdash \bullet : \star}\,(\text{VAR})$$

- For a variable $\Gamma \vdash x : A$ with $A \neq \mathbf{1}$ for the term to be in normal form, we necessarily have $\Gamma \vdash$ which gives by the induction case for contexts $\upharpoonleft\Gamma \vdash$. The condition $(x : A) \in \Gamma$ is also necessarily satisfied, which implies that $(x : \Uparrow A) \in \Uparrow\Gamma$. Hence, applying the rule (VAR) yields a derivation for $\Uparrow\Gamma \vdash x : \Uparrow A$.

- For a term of the form $\mathsf{mop}_{\Gamma,A}[\gamma]$ (resp. $\mathsf{mcoh}_{\Gamma,A}[\gamma]$), the pair (Γ, A) defines a valid operation cut (resp. coherence cut), and we necessarily have a derivation of $\Delta \vdash \gamma : \Downarrow\Gamma$. By the induction case for substitutions, this gives a derivation for $\Uparrow\Delta \vdash \Uparrow\gamma : \Uparrow\Downarrow\Gamma$. Moreover, Lemma 20 ensures that we have $\Uparrow\Downarrow\Gamma \vdash \bullet_\Gamma : \Gamma$, hence we have a substitution $\Uparrow\Delta \vdash \bullet_\Gamma \circ \Uparrow\gamma : \Gamma$. By applying the rule ($\mathsf{op}$-INTRO) (resp. the rule (coh-INTRO)), this provides a derivation for the judgment $\Uparrow\Delta \vdash \mathsf{op}_{\Gamma,A}[\bullet_\Gamma \circ \Uparrow\gamma] : A[\bullet_\Gamma \circ \Uparrow\gamma]$ (resp. for the judgment $\Uparrow\Delta \vdash \mathsf{coh}_{\Gamma,A}[\bullet_\Gamma \circ \Uparrow\gamma] : A[\bullet_\Gamma \circ \Uparrow\gamma]$). We then have the following equalities, proving that the type is the reduced suspension of $(\Downarrow A)[\gamma]$

$$\begin{aligned} A[\bullet_\Gamma \circ \Uparrow\gamma] &= \Uparrow\Downarrow A[\Uparrow\gamma] && \text{By Lemma 19} \\ &= \Uparrow((\Downarrow A)[\gamma]) && \text{By Lemma 17} \end{aligned}$$

Induction for substitutions:

- In the case of the empty substitution $\Delta \vdash \langle\rangle : \varnothing$, we necessarily have a derivation of $\Delta \vdash$, which by the induction case for contexts gives a derivation of $\Uparrow\Delta \vdash$. Moreover, we also have by definition that $(\bullet : \star) \in \Uparrow\Delta$. This lets us construct a derivation for $\Uparrow\Delta \vdash \Uparrow\langle\rangle : \Uparrow\varnothing$ as follows

$$\dfrac{\dfrac{\Uparrow\Delta \vdash}{\Uparrow\Delta \vdash \langle\rangle : \varnothing}\,(\text{ES}) \qquad \bullet : \star \vdash \qquad \dfrac{\Uparrow\Delta \vdash \qquad (\bullet : \star) \in \Uparrow\Delta}{\Uparrow\Delta \vdash \bullet : \star}\,(\text{VAR})}{\Uparrow\Delta \vdash \langle \bullet \mapsto \bullet \rangle : (\bullet : \star)}\,(\text{SE})$$

- In the case of a substitution of the form $\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : \mathbf{1})$, we have the equalities

$$\Uparrow(\Gamma, x : \mathbf{1}) = \Uparrow\Gamma$$
$$\Uparrow\langle \gamma, x \mapsto t \rangle = \Uparrow\gamma$$

Moreover, we necessarily have $\Delta \vdash \gamma : \Gamma$, which gives by the induction case for substitutions, $\uparrow\Delta \vdash \uparrow\gamma : \uparrow\Gamma$.

- In the case of a substitution of the form $\Delta \vdash \langle\gamma, x \mapsto t\rangle : (\Gamma, x : A)$, we necessarily have $\Delta \vdash \gamma : \Gamma$, which gives by the induction case for substitutions, $\uparrow\Delta \vdash \uparrow\gamma : \uparrow\Gamma$. Moreover, we have a derivation of $(\Gamma, x : A) \vdash$, which gives by the induction case for contexts a derivation of $\uparrow\Gamma, x : \uparrow A \vdash$, and a derivation of $\Delta \vdash t : A[\gamma]$, which gives by the induction case for terms a derivation of $\uparrow\Delta \vdash \uparrow t : \uparrow A[\gamma]$. Lemma 17 allows to rewrite this into a derivation for $\uparrow\Delta \vdash \uparrow t : \uparrow A[\uparrow\gamma]$. Thus these derivations can be assembled using the rule (SE) into a derivation of $\uparrow\Delta \vdash \uparrow\langle\gamma, x \mapsto t\rangle : \uparrow(\Gamma, x : A)$ as follows

$$\frac{\uparrow\Delta \vdash \uparrow\gamma : \uparrow\Gamma \qquad \uparrow\Gamma, x : \uparrow A \vdash \qquad \uparrow\Delta \vdash \uparrow t : \uparrow A[\uparrow\gamma]}{\uparrow\Delta \vdash \langle\uparrow\gamma, x \mapsto \uparrow t\rangle : (\uparrow\Gamma, x : \uparrow A)}\,(\text{SE})$$

$\square$

Examples. We now give a few examples of contexts in MCaTT along with their translations in CaTT to build intuition on this translation:

$$\Gamma = (x : \star) \qquad \uparrow\Gamma = (\bullet : \star, x : \bullet \underset{\star}{\to} \bullet)$$

$$\Gamma = (x : \star, f : x \underset{\star}{\to} x, a : \mathbf{1}) \qquad \uparrow\Gamma = (\bullet : \star, x : \bullet \underset{\star}{\to} \bullet, f : x \underset{\bullet \to \bullet}{\longrightarrow} x)$$

$$\Gamma = (x : \star, y : \star; f : x \underset{\star}{\to} y) \qquad \uparrow\Gamma = (\bullet : \star, x : \bullet \underset{\star}{\to} \bullet, y : \bullet \underset{\star}{\to} \bullet, f : x \underset{\bullet \to \bullet}{\longrightarrow} y)$$

These examples help understand why it is important to require that the translation is defined on normal form. Indeed, in the second example, we have a derivation $\Gamma \vdash a : \mathbf{1}$ but the term is not in normal form. Putting it in normal form before applying the reduced suspension yields to the term $\Gamma \vdash \bullet : \mathbf{1}$ which is indeed derivable. However, had we simply defined the reduced suspension of any variable to be itself, without consideration of whether it is in normal form, we would have gotten the judgment $\uparrow\Gamma \vdash a : \star$ which is not derivable.

Reduced suspension as a functor. Similarly to what we have presented for the desuspension, we can reformulate our study of the category. However the result that we prove here is weaker.

Corollary 22. *The reduced suspension operation defines a functor* $\uparrow : \mathcal{S}_{\mathsf{MCaTT}} \to \mathcal{S}_{\mathsf{CaTT}}$.

Proof. We have proved in Lemma 17 and Proposition 14 that this operation is well defined, and that it preserves the composition of substitutions, so it suffices to prove that it preserves the identity. We proceed by induction on the length of the context

- For the empty context $\varnothing$, we have $\mathrm{id}_\varnothing = \langle\rangle$ along with $\uparrow\varnothing = (\bullet : \star)$ and $\uparrow\langle\rangle = \langle \bullet \mapsto \bullet \rangle$, which is the identity of $\uparrow\varnothing$.
- For a context of the form $(\Gamma, x : \mathbf{1})$, we have $\mathrm{id}_{(\Gamma, x:\mathbf{1})} = \langle \mathrm{id}_\Gamma, \bullet \mapsto \bullet \rangle$. Hence $\uparrow\mathrm{id}_{(\Gamma,x:\mathbf{1})} = \uparrow\mathrm{id}_\Gamma$, and by induction $\uparrow\mathrm{id}_\Gamma = \mathrm{id}_{\uparrow\Gamma}$. We conclude using the fact that $\uparrow(\Gamma, x : \mathbf{1}) = \uparrow\Gamma$.
- For a context of the form $(\Gamma, x : A)$ with $A \neq \mathbf{1}$, we have $\uparrow\mathrm{id}_{(\Gamma,x:A)} = \langle \uparrow\mathrm{id}_\Gamma, x \mapsto x \rangle$, and by induction, we have $\uparrow\mathrm{id}_\Gamma = \mathrm{id}_{\uparrow\Gamma}$, which lets us conclude that $\uparrow\mathrm{id}_{(\Gamma,x:A)} = \mathrm{id}_{\uparrow(\Gamma,x:A)}$. $\square$

Note that in fact Lemma 17 and Proposition 14 provide more structure that this, they show that this functor is almost a morphism of category with families. In fact it defines a morphism in **Cat/Fam**, which respects the structures of categories with families of $\mathcal{S}_{\mathsf{MCaTT}}$ and $\mathcal{S}_{\mathsf{CaTT}}$. This functor however fails to be a morphism of categories with families as it does not preserve the terminal object, since the context $\varnothing$ is sent onto $(\bullet : \star)$ which is not terminal. As this result is complicated to state in a rigorous way, we later use the alternate characterization of preservation of pullbacks along display maps and terminal object for the models to make it precise.

2.5 Interaction between desuspension and reduced suspension

We have defined the theory MCaTT together with two translations, the desuspension and the reduced suspension, that define functors between their syntactic categories. We now study how these translations relate to each other syntactically and translate this result categorically as a relation between the two functors. In order to understand the interaction between the desuspension and the reduced suspension, we first show the following result, describing the action of the desuspension on the substitution $\bullet_\Gamma$.

Lemma 23. *For every context* $\Gamma \vdash$ *in the theory* CaTT*, we have the definitional equality* $\downarrow\Gamma \vdash \downarrow\bullet_\Gamma \equiv \mathrm{id}_{\downarrow\Gamma} : \downarrow\Gamma$.

Proof. We prove this by induction on the context Γ

- For the empty context $\varnothing$, we have $\downarrow\bullet_\Gamma = \langle\rangle$ and moreover $\downarrow\varnothing = \varnothing$, hence $\mathrm{id}_{\downarrow\varnothing} = \langle\rangle$.
- For a context of the form $(\Gamma, x : \star)$, we have $\downarrow\bullet_{\Gamma,x:\star} = \langle\downarrow\bullet_\Gamma, x \mapsto \downarrow\bullet\rangle$. On the other hand, we have that $\downarrow(\Gamma, x : \star) = (\downarrow\Gamma, x : \mathbf{1})$, hence $\mathrm{id}_{\downarrow(\Gamma,x:\star)} = \langle \mathrm{id}_{\downarrow\Gamma}, x \mapsto x\rangle$. By induction, we have that $\downarrow\Gamma \vdash \downarrow\bullet_\Gamma \equiv \mathrm{id}_{\downarrow\Gamma} : \downarrow\Gamma$, and moreover, by the rule $(\eta_{\mathbf{1}})$, we have $(\downarrow\Gamma, x : \mathbf{1}) \vdash \downarrow\bullet \equiv x : \mathbf{1}$. This proves the definitional equality between the substitutions.
- For a context of the form $(\Gamma, x : A)$ where A is distinct from $\star$, we have $\bullet_{(\Gamma,x:A)} = \langle\bullet_\Gamma, x \mapsto x\rangle$, and thus $\downarrow\bullet_{(\Gamma,x:A)} = \langle\downarrow\bullet_\Gamma, x \mapsto x\rangle$. Moreover, $\downarrow(\Gamma, x : A) = (\downarrow\Gamma, x : \downarrow A)$, and thus $\mathrm{id}_{\downarrow(\Gamma,x:A)} = \langle \mathrm{id}_\Gamma, x \mapsto x\rangle$. We conclude by induction using the fact that $\downarrow\Gamma \vdash \downarrow\bullet_\Gamma \equiv \mathrm{id}_{\downarrow\Gamma} : \downarrow\Gamma$. □

Desuspension of the reduced suspension. We have the following result

Proposition 24. *There is a natural isomorphism* $\downarrow \circ \uparrow \simeq \mathrm{id}_{\mathcal{S}_{\mathsf{MCaTT}}}$.

Proof. We show by mutual induction the following properties of the theory MCaTT

- For all context $\Gamma \vdash$, we have $\downarrow\uparrow\Gamma \simeq \Gamma$
- For all type $\Gamma \vdash A$, we have $\downarrow\uparrow\Gamma \vdash \downarrow\uparrow A \equiv A$
- For all term $\Gamma \vdash t : A$, we have $\downarrow\uparrow\Gamma \vdash \downarrow\uparrow t \equiv t : \downarrow\uparrow A$
- For all substitution $\Delta \vdash \gamma : \Gamma$, we have a commutative square

$$\begin{tikzcd} \Delta \arrow[r, "\gamma"] \arrow[d, "\wr"'] & \Gamma \arrow[d, "\wr"] \\ \downarrow\uparrow\Delta \arrow[r, "\downarrow\uparrow\gamma"'] & \downarrow\uparrow\Gamma \end{tikzcd}$$

Induction for contexts:

- For the empty context $\varnothing \vdash$, we have $\uparrow\varnothing = (\bullet : \star)$, and thus $\downarrow\uparrow\varnothing = \varnothing$.
- For a context of the form $(\Gamma, x : \mathbf{1}) \vdash$, we have $\uparrow(\Gamma, x : \mathbf{1}) = \uparrow\Gamma$, hence $\downarrow\uparrow(\Gamma, x : \mathbf{1}) = \downarrow\uparrow\Gamma$. By the induction case on contexts, we have that $\downarrow\uparrow\Gamma \simeq \Gamma$, and moreover $(\Gamma, x : \mathbf{1}) \simeq \Gamma$ by Lemma 11.

- For a context of the form $(\Gamma, x : A) \vdash$ with $A \neq \mathbf{1}$, we have $\uparrow(\Gamma, x : A) = (\uparrow\Gamma, x : \uparrow A)$. Since $\uparrow A$ is necessarily distinct from $\star$, we have $\downarrow\uparrow(\Gamma, x : A) = (\downarrow\uparrow\Gamma, x : \downarrow\uparrow A)$, and the induction case for context shows that $\downarrow\uparrow\Gamma \simeq \Gamma$ and the induction case for types shows $\downarrow\uparrow\Gamma \vdash \downarrow\uparrow A \equiv A$. This proves that $\downarrow\uparrow(\Gamma, x : A) \simeq (\Gamma, x : A)$.

Induction for types:

- For the type $\Gamma \vdash \mathbf{1}$, we have $\uparrow\mathbf{1} = \star$ and thus $\downarrow\uparrow\mathbf{1} = \mathbf{1}$.
- For a type of the form $\Gamma \vdash t \underset{A}{\to} u$, we have $\uparrow t \underset{A}{\to} u = \uparrow t \underset{\uparrow A}{\longrightarrow} \uparrow u$, and since $\uparrow A$ is distinct from $\star$, this implies that $\downarrow\uparrow t \underset{A}{\to} u = \downarrow\uparrow t \underset{\downarrow\uparrow A}{\longrightarrow} \downarrow\uparrow u$. The induction case for types then shows that $\downarrow\uparrow\Gamma \vdash \downarrow\uparrow A \equiv A$, and the induction case for terms shows $\downarrow\uparrow\Gamma \vdash \downarrow\uparrow t \equiv t : A$ and $\downarrow\uparrow u \equiv u : A$, which lets us conclude.

Induction for terms:

- For the term $\Gamma \vdash () : \mathbf{1}$, we have $\Gamma \vdash \downarrow\uparrow() : \mathbf{1}$, and the rule $(\eta_{\mathbf{1}})$ gives the desired definitional equality.
- For a variable $\Gamma \vdash x : A$ with $A \bullet_{\mathbf{1}}$ for the term to be in normal form, we have $\uparrow x = x$ and thus $\downarrow\uparrow x = x$.
- For a term of the form $\mathsf{mop}_{\Gamma,A}[\gamma]$, we have $\uparrow\mathsf{mop}_{\Gamma,A}[\gamma] = \mathsf{op}_{\Gamma,A}[\bullet_\Gamma \circ \uparrow\gamma]$, and thus we have the following equalities

$$\begin{aligned}
\downarrow\uparrow\mathsf{mop}_{\Gamma,A}[\gamma] &= \mathsf{mop}_{\Gamma,A}[\downarrow(\bullet_\Gamma \circ \uparrow\gamma)] \\
&= \mathsf{mop}_{\Gamma,A}[\downarrow\bullet_\Gamma \circ \downarrow\uparrow\gamma] && \text{by Lemma 13} \\
&\equiv \mathsf{mop}_{\Gamma,A}[\downarrow\uparrow\gamma] && \text{by Lemma 23} \\
&\equiv \mathsf{mop}_{\Gamma,A}[\gamma] && \text{by the induction case for substitution}
\end{aligned}$$

- The case for a term of the form $\mathsf{mop}_{\Gamma,A}[\gamma]$ follows the exact same steps.

Induction for substitutions:

- For the empty substitution $\langle\rangle$, we have $\uparrow\langle\rangle = \langle\bullet \mapsto \bullet\rangle$, and thus $\downarrow\uparrow\langle\rangle = \langle\rangle$.

- For a substitution of the form $\Delta \vdash \langle \Gamma, x \mapsto t\rangle : (\Gamma, x : \mathbf{1})$, we have $\upharpoonleft\langle \gamma, x \mapsto t\rangle = \upharpoonleft\gamma$ and thus $\downharpoonleft\upharpoonleft\langle \gamma, x \mapsto t\rangle = \downharpoonleft\upharpoonleft\gamma$. The induction case for substitutions then gives the result.
- For a substitution of the form $\Delta \vdash \langle \gamma, x \mapsto t\rangle : (\Gamma, x : A)$ with $A \neq \mathbf{1}$, we have $\upharpoonleft\langle \gamma, x \mapsto t\rangle = \langle \upharpoonleft\gamma, x \mapsto \upharpoonleft t\rangle$, and thus $\downharpoonleft\upharpoonleft\langle \gamma, x \mapsto t\rangle = \langle \downharpoonleft\upharpoonleft\gamma, x \mapsto \downharpoonleft\upharpoonleft t\rangle$. Then induction case for terms gives $\downharpoonleft\upharpoonleft t = t$, and the induction case for substitutions provides the commutative square

$$\begin{tikzcd} \Delta \arrow[r, "\gamma"] \arrow[d, "\wr"', no head] & \Gamma \arrow[d, "\wr", no head] \\ \downharpoonleft\upharpoonleft\Delta \arrow[r, "\downharpoonleft\upharpoonleft\gamma"'] & \downharpoonleft\upharpoonleft\Gamma \end{tikzcd}$$

This proves, by extending both substitutions by terms that are definitionally equal, that we have the commutative square

$$\begin{tikzcd} \Delta \arrow[r, "\langle \gamma, x \mapsto t\rangle"] \arrow[d, "\wr"', no head] & (\Gamma, x : A) \arrow[d, "\wr", no head] \\ \downharpoonleft\upharpoonleft\Delta \arrow[r, "\langle \downharpoonleft\upharpoonleft\gamma, x \mapsto \downharpoonleft\upharpoonleft t\rangle"'] & \downharpoonleft\upharpoonleft(\Gamma, x : A) \end{tikzcd}$$

$\square$

A natural transformation. In order to study the reduced suspension, we have introduced the family of substitutions $\bullet_\Gamma$ for all contexts Γ, and we have proved in Lemma 20, that it defines a family of morphisms $\bullet_\Gamma : \upharpoonleft\downharpoonleft\Gamma \to \Gamma$. Moreover, the equality $\gamma \circ \bullet_\Delta = \bullet_\Gamma \circ \upharpoonleft\downharpoonleft\gamma$ that we proved in Lemma 19 for all substitution γ can be expressed by the commutation of the following diagram

$$\begin{tikzcd} \upharpoonleft\downharpoonleft\Delta \arrow[r, "\bullet_\Delta"] \arrow[d, "\upharpoonleft\downharpoonleft\gamma"'] & \Delta \arrow[d, "\gamma"] \\ \upharpoonleft\downharpoonleft\Gamma \arrow[r, "\bullet_\Gamma"'] & \Gamma \end{tikzcd}$$

So we have in fact already proven the following proposition, which is simply a categorical reformulation of our previous fact

Proposition 25. *The family of morphisms $\bullet_\Gamma : \upharpoonleft\downharpoonleft\Gamma \to \Gamma$ define a natural transformation $\upharpoonleft \circ \downharpoonleft \Rightarrow \mathrm{id}_{\mathcal{S}_{\mathsf{CaTT}}}$.*

Adjunction between desuspension and reduced suspension. Combining Propositions 24 and 25, we have in fact proved the following categorical result about the functors induced by desuspension and the reduced suspension.

Theorem 26. *The functor $\uparrow$ is left adjoint to $\downarrow$, the counit is given by the family $\bullet_\Gamma$ and the unit is an isomorphism. This adjunction is thus coreflective.*

This adjunction is to be understood as an analogue in the world of categories of the topological adjunction between the reduced suspension and the loop space. In our terminology, the desuspension corresponds to the loop space, and the reduced suspension corresponds to the reduced suspension.

Remark 27. The coreflective adjunction exhibits $\mathcal{S}_{\mathsf{MCaTT}}$ as isomorphic to a coreflective subcategory of $\mathcal{S}_{\mathsf{CaTT}}$. Moreover the essential image of $\uparrow$ is exactly the category $\mathcal{S}_{\mathsf{CaTT},\bullet}$. So a way to understand our type theory MCaTT is that it achieves a structure of category with families on a category which is equivalent to $\mathcal{S}_{\mathsf{CaTT},\bullet}$, in such a way that that this structure coincide with the structure on CaTT under the equivalence between $\mathcal{S}_{\mathsf{MCaTT}}$ and $\mathcal{S}_{\mathsf{CaTT},\bullet}$.

2.6 Models of the type theory MCaTT.

The definition of the desuspension and reduced suspension operations that we have defined relate very tightly the models of the theory MCaTT and the models of the theory CaTT.

Desuspension acting on the models of MCaTT. Given a model $F : \mathsf{MCaTT} \to \mathbf{Set}$, we define a functor $\downarrow^* F : \mathsf{CaTT} \to \mathbf{Set}$ by precomposing with the functor $\downarrow$, hence we have $\downarrow^* F(\Gamma) = F(\downarrow\Gamma)$ and $\downarrow^* F(\gamma) = F(\downarrow\gamma)$. By Corollary 15, $\downarrow$ is naturally a morphism of categories with families, and since such morphism compose, it follows that $\downarrow^* F$ is a morphism of categories with families, hence it defines a model of CaTT. Thus the reduced suspension induces a functor $\downarrow^* : \mathbf{Mod}\,(\mathcal{S}_{\mathsf{MCaTT}}) \to \mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})$. Moreover, we have that

$$\begin{aligned}\downarrow^* F(x : \star) &= F(\downarrow(x : \star)) \\ &= F(x : \mathbf{1})\end{aligned}$$

Since by Lemma 9, the context $(x : \mathbf{1})$ is terminal and by definition of a morphism of categories with families, F preserves the terminal object, it follows that $\downarrow^* F(x : \star)$ is terminal in $\mathbf{Set}$, hence $\downarrow^* F$ is an object of $\mathbf{Mod}_\bullet\,(\mathcal{S}_{\mathsf{CaTT}})$. By corestrcition, this shows that the desuspension induces a functor $\downarrow^* : \mathbf{Mod}\,(\mathsf{MCaTT}) \to \mathbf{Mod}_\bullet\,(\mathcal{S}_{\mathsf{CaTT}})$.

Reduced suspension acting on the models of CaTT. We define a similar construction for the reduced suspension. However, since the reduced suspension does not define an image for every term and does not preserve the terminal object, the construction is slightly more involved. We first consider start by showing the following result

Lemma 28. *Given a model $F : \mathit{CaTT} \to \mathbf{Set}$ of the type theory CaTT, the functor $\uparrow^* F$ preserves the pullbacks along display maps of $\mathcal{S}_{\mathit{MCaTT}}$.*

Proof. We consider a pullback along a display map, which is of the form

$$\begin{tikzcd} (\Delta, x : A[\gamma]) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\Gamma, x : A) \arrow[d, "\pi"] \\ \Delta \arrow[r, "\gamma"'] & \Gamma \end{tikzcd}$$

We first consider the case $A = \mathbf{1}$. In this case, the image by $\uparrow^* F$ of the above square writes as

$$\begin{tikzcd} \uparrow^* F(\Delta) \arrow[r, "\uparrow^* F(\gamma)"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \uparrow^* F(\Gamma) \arrow[d, "\mathrm{id}"] \\ \uparrow^* F(\Delta) \arrow[r, "\uparrow^* F(\gamma)"'] & \uparrow^* F(\Gamma) \end{tikzcd}$$

which is a pullback. We now consider the case $A \neq \mathbf{1}$, then the equality $\uparrow(A[\gamma]) = \uparrow A[\uparrow\gamma]$ given by Lemma 17 along with Lemma 3 shows that we have the following pullback square in $\mathcal{S}_{\mathsf{CaTT}}$

$$\begin{tikzcd} \uparrow(\Delta, x : A[\gamma]) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \uparrow(\Gamma, x : A) \arrow[d, "\pi"] \\ \uparrow\Delta \arrow[r, "\uparrow\gamma"'] & \uparrow\Gamma \end{tikzcd}$$

and this F preserves pullbacks, this shows that the image of the initial square by $\uparrow^* F$ preserves the pullbacks.

$$\begin{tikzcd}
\uparrow^* F(\Delta, x : A[\gamma]) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \uparrow^* F(\Gamma, x : A) \arrow[d, "\pi"] \\
\uparrow^* F(\Delta) \arrow[r, "\gamma"'] & \uparrow^* F(\Gamma)
\end{tikzcd}$$

□

Consider a model $F : \mathsf{CaTT} \to \mathbf{Set}$ of CaTT, then combining Lemma 5 with Lemma 28 shows that $\uparrow^* F$ defines a model of $\mathcal{S}_{\mathsf{MCaTT}}$ if and only if it preserves the terminal object, and if so, the model it defines is unique. Note that by definition we have

$$\begin{aligned}
\uparrow^* F(\varnothing) &= F(\uparrow \varnothing) \\
&= F(\bullet : \star)
\end{aligned}$$

So $\uparrow^* F$ preserves the terminal object if and only if F sends the context D^0 onto a singleton, that is, if F is an object of $\mathbf{Mod}_\bullet\left(\mathcal{S}_{\mathsf{CaTT}}\right)$. This shows that $\uparrow^*$ defines a functor

$$\uparrow^* : \mathbf{Mod}_\bullet\left(\mathcal{S}_{\mathsf{CaTT}}\right) \to \mathbf{Mod}\left(\mathcal{S}_{\mathsf{MCaTT}}\right)$$

Models of the category $\mathcal{S}_{\mathsf{MCaTT}}$. We have now defined a pair of functors $\downarrow^*$ and $\uparrow^*$ between the categories $\mathbf{Mod}\left(\mathcal{S}_{\mathsf{MCaTT}}\right)$ and $\mathbf{Mod}_\bullet\left(\mathcal{S}_{\mathsf{CaTT}}\right)$.

Theorem 29. *The functors $\downarrow^*$ and $\uparrow^*$ define an equivalence of categories between $\mathbf{Mod}\left(\mathcal{S}_{\mathsf{MCaTT}}\right)$ and $\mathbf{Mod}_\bullet\left(\mathcal{S}_{\mathsf{CaTT}}\right)$*

Proof. First, Proposition 24 implies that

$$\uparrow^* \circ \downarrow^* = \left(\downarrow \circ \uparrow\right)^* \simeq \mathrm{id}_{\mathcal{S}_{\mathsf{MCaTT}}}$$

and Proposition 25 shows that there is a natural transformation, obtained by whiskering

$$\downarrow^* \circ \uparrow^* = \left(\uparrow \circ \downarrow\right)^* \Rightarrow \mathrm{id}_{\mathbf{Mod}_\bullet\left(\mathcal{S}_{\mathsf{CaTT}}\right)}$$

So it suffices to show that this natural transformation is a natural isomorphism, that is for any

$F \in \mathbf{Mod}_{\bullet}(\mathcal{S}_{\mathsf{CaTT}})$ and all context Γ in CaTT

$$F(\bullet_\Gamma) : F(\uparrow\downarrow\Gamma) \to F(\Gamma)$$

is an isomorphism. We prove this property by induction on the context Γ.

- For the empty context $\varnothing$, we necessarily have that $F(\varnothing) = \{\bullet\}$, and since $\uparrow\downarrow\varnothing = D^0$ and $F \in \mathbf{Mod}_{\bullet}(\mathcal{S}_{\mathsf{CaTT}})$, we also have that $F(\uparrow\downarrow\varnothing) = \{\bullet\}$. Hence $F(\bullet_\varnothing)$ is the unique map between the singleton to itself, which is an isomorphism.

- For a context of the form $(\Gamma, x : \star)$, it is obtained as a (trivial) pullback, and the map $\bullet_{(\Gamma,x:\star)}$ is obtained by universal property of the pullback, as follows

$$\begin{tikzcd}
\uparrow\downarrow(\Gamma, x:\star) \arrow[dr, "\bullet_{(\Gamma,x:\star)}"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "\bullet_\Gamma"'] & & \\
& (\Gamma, x : A) \arrow[r, "x"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^0 \arrow[d, "\pi"] \\
& \Gamma \arrow[r, "\star"'] & \varnothing
\end{tikzcd}$$

  Taking the image by F on this pullback yields another pullback in **Set** (since F is a model of CaTT), as follows

$$\begin{tikzcd}
F\left(\uparrow\downarrow(\Gamma, x:\star)\right) \arrow[dr, "F\bullet_{(\Gamma,x:\star)}"] \arrow[drr, bend left, "!"] \arrow[ddr, bend right, "F\bullet_\Gamma"'] & & \\
& F(\Gamma, x : A) \arrow[r, "!"] \arrow[d, "F\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{\bullet\} \arrow[d, "!"] \\
& F\Gamma \arrow[r, "!"'] & \{\bullet\}
\end{tikzcd}$$

  Since the square is a pullback, $F\pi$ is an isomorphism, and since by induction $F\bullet_\Gamma$ is also an isomorphism, necessarily $F\bullet_{\Gamma,x:\star}$ is an isomorphism.

- For a context of the form $(\Gamma, x : A)$ where A is a type distinct from $\star$, the context is obtained as a pullback, the context $\uparrow\downarrow(\Gamma, x : A)$ is also a pullback and the substitution $\bullet_{(\Gamma,x:A)}$ is obtained by universal property as described in the following diagram which has the shape of a cube whose faces are commutative and whose front and back face are pullbacks

$$\begin{tikzcd}
\uparrow\downarrow(\Gamma, x : A) \arrow[rr, "x"] \arrow[dr, "\bullet_{(\Gamma, x:A)}"] \arrow[dd, "\pi"] & & D^{n+1} \arrow[dr, equals] \arrow[dd, "\pi"] & \\
& (\Gamma, x : A) \arrow[rr, "x"] \arrow[dd, "\pi"] & & D^{n+1} \arrow[dd, "\pi"] \\
\uparrow\downarrow\Gamma \arrow[rr, "\uparrow\downarrow A"] \arrow[dr, "\bullet_\Gamma"'] & & S^n \arrow[dr, equals] & \\
& \Gamma \arrow[rr, "A"'] & & S^n
\end{tikzcd}$$

Taking the image by F of this diagram yields another cube whose faces are all commutative square, and whose front and back square are again pullback squares, since as a model, F preserves the pullbacks along the display maps (in the following figure, we have left implicit most of the arrows, they are simply the image by F of the ones of the previous figure).

$$\begin{tikzcd}
F(\uparrow\downarrow(\Gamma, x : A)) \arrow[rr, "x"] \arrow[dr, "F\bullet_{(\Gamma, x:A)}"] \arrow[dd] & & FD^{n+1} \arrow[dr, equals] \arrow[dd] & \\
& F(\Gamma, x : A) \arrow[rr] \arrow[dd] & & FD^{n+1} \arrow[dd] \\
F(\uparrow\downarrow\Gamma) \arrow[rr] \arrow[dr, "\sim", "F\bullet_\Gamma"'] & & FS^n \arrow[dr, equals] & \\
& F\Gamma \arrow[rr, "FA"'] & & FS^n
\end{tikzcd}$$

By induction, the map $F(\bullet_\Gamma)$ is an isomorphism, making the span defining $F(\Gamma, x : A)$ and the span defining $F(\uparrow\downarrow(\Gamma, x : A))$ isomorphic. This proves that the map $F(\bullet_{(\Gamma,x:A)})$ is also an isomorphism, by uniqueness of the pullback up to isomorphism.

$\square$

Reflective localization. We now give an reformulation of the construction we have presented. First note that the opposite of a category with families C embeds inside its category of models via a Yoneda embedding

$$\begin{array}{rcl} C^{\mathrm{op}} & \hookrightarrow & \mathbf{Mod}\,(C) \\ \Gamma & \mapsto & C(\Gamma, \_) \end{array}$$

Indeed, the functor $C(\Gamma, \_)$ preserves the pullbacks along display maps (and in fact all limits) by continuity of the Hom-functor. This lets us see the objects $\varnothing$ and D^0 as particular objects of $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})$, with $\varnothing$ being the initial object. We then consider the unique map $s : \varnothing \to D^0$ in the category $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})$. Then the category $\mathbf{Mod}_\bullet\,(\mathcal{S}_{\mathsf{CaTT}})$ is the category of all the s-local objects, i.e., all objects F such that the map

$$\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})\,(s, F) : \mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})\,(\varnothing, F) \to \mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})\,(D^0, F)$$

is a natural isomorphism. Indeed, since $\varnothing$ is an initial objects, s-local objects are exactly those such that $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})\,(D^0, F)$ is a singleton, which reformulates as $F(D^0)$ being a singleton by the Yoneda lemma. This exhibits $\mathbf{Mod}_\bullet\,(\mathcal{S}_{\mathsf{CaTT}})$ as a reflective localization of the category $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})$ as s. Theorem 29 then shows that $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{MCaTT}})$ is equivalent to $\mathbf{Mod}_\bullet\,(\mathcal{S}_{\mathsf{CaTT}})$, and thus is also a reflective localization of $\mathbf{Mod}\,(\mathcal{S}_{\mathsf{CaTT}})$ at s. This localization lifts the coreflective adjunction between $\mathcal{S}_{\mathsf{CaTT}}$ and $\mathcal{S}_{\mathsf{MCaTT}}$ realized by the desuspension and the reduced suspension, that we have stated in Theorem 26.

2.7 Interpretation

Assuming the fact that the models of CaTT indeed are equivalent to weak ω-category (we refer the reader to our article [8] - in preparation, for an in-depth exploration of this fact), Theorem 29 can be rephrased as showing that the models of MCaTT are the weak ω-categories with one object. The phrasing of this sentence may seem surprising, as we speculate that the correct notion to compare weak ω-categories is a notion of weak equivalence, and that the property of having a single object is not invariant under those weak equivalence. This is analogous, in the 1-categorical case, to the fact that the number of objects of a category is not invariant under equivalence of categories. We believe however, that taking in account the weak equivalences can be achieved with a model structure or a similar structure. Then we speculate that the above equivalence lifts to an equivalence between the corresponding localization, thus providing a more satisfying equivalence, between the models of MCaTT and the weak ω-categories that are weakly equivalent to weak ω-categories with a single object. We further conjecture that those should be the weak ω-categories whose groupoidal core is 0-connected. Investigation in this direction to straighten and prove those claims are left for further work.

3 Conclusion

We have developed the type theory MCaTT whose models are monoidal weak ω-categories. To this end, we rely on an existing type theory describing weak ω-categories and index the rule of our theory with the rule of the existing theory. We were then able to prove correctness results, using syntactic translations between the two theories and lifting the results as a correspondence between their models.

A forgetful functor. There is another translation from $\mathcal{S}_{\mathsf{CaTT}}$ to $\mathcal{S}_{\mathsf{MCaTT}}$ that we have not presented here. Given a ps-context Γ, we can produce its *suspension* $\Sigma\Gamma$, obtained by formally adding two objects and lifting all the dimensions by 1, in such a way that an object in Γ becomes a 1-cell between the two new objects in $\Sigma\Gamma$. We can then leverage this operation to define a translation, which gives a morphism of categories with families $u : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{S}_{\mathsf{MCaTT}}$. The induced functor on the categories of models

$$u^* : \mathbf{Mod}\left(\mathcal{S}_{\mathsf{MCaTT}}\right) \to \mathbf{Mod}\left(\mathcal{S}_{\mathsf{CaTT}}\right)$$

is the forgetful functor, which, given a monoidal weak ω-category, forgets the monoidal product and gives the underlying weak ω-category. Importantly, in this case there is no shift of dimension; the objects of the ω-category are the objects of the monoidal ω-category, but without the ability to be composed. The existence of this functor is closely related to the well-definedness of an operation of *suspension* in the theory CaTT[9].

Alternate presentations of MCaTT. We have worked out different presentations for the theory MCaTT, all of them being centered around the idea of enforcing the constraint of having a unique object of dimension -1. We give in [7] two other presentations. The first one is very similar to the one that we have given here, but the difference is that we do not introduce a type $\mathbf{1}$ in MCaTT. The price to pay is that the desuspension becomes a partial operation, since there is no type to send the type $\star$ to, but this gets compensated by the fact that the theory can be expressed without definitional equalities and thus there is no need to carefully work with terms in normal forms. Both these presentation are relatively close, and none of them really surpasses the other. The second presentation that we give in [7] encodes all the properties with lists of contexts. This presentation is more involved and significantly harder to study, however it gives a syntax that is a bit more concise. It also provides a framework which is more independent of CaTT, and for which the conditions for constructing the derivation trees are more straightforwardly verified. For these reasons, we believe that the latter presentation is to be preferred for a potential future implementation of the theory MCaTT.

Monoidal closed higher categories. It would be valuable to connect the result we have presented with recent work of Finster for integrating the theory of monoidal higher categories in the tool Opetopic [13], although it necessitates to establish a connection between globular shapes and opetopic shapes.

References

[1] Thorsten Altenkirch and Ondrej Rypacek. A syntactical approach to weak omega-groupoids. In *Computer Science Logic (CSL'12)-26th International Workshop/21st Annual Conference of the EACSL*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2012.

[2] Dimitri Ara. *Sur les ∞-groupoïdes de Grothendieck et une variante ∞-catégorique*. PhD thesis, Ph. D. thesis, Université Paris 7, 2010.

[3] John Baez. Lectures on n-categories and cohomology. Notes by M. Shulman.

[4] John C Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995.

[5] Michael A Batanin. Monoidal globular categories as a natural environment for the theory of weakn-categories. *Advances in Mathematics*, 136(1):39–103, 1998.

[6] Thibaut Benjamin. Catt formalization. `https://github.com/ThiBen/catt-formalization/`.

[7] Thibaut Benjamin. A type theoretic approach to weak ω-categories and related higher structures. PhD thesis, In preparation, 2020.

[8] Thibaut Benjamin, Eric Finster, and Samuel Mimram. Globular weak ω-categories as models of a type theory. In preparation, 2019.

[9] Thibaut Benjamin and Samuel Mimram. Suspension et Fonctorialité: Deux Opérations Implicites Utiles en CaTT. In *Journées Francophones des Langages Applicatifs*, 2019.

[10] Guillaume Brunerie. On the homotopy groups of spheres in homotopy type theory. *arXiv preprint arXiv:1606.05916*, 2016.

[11] Eugenia Cheng and Aaron Lauda. Higher-dimensional categories: an illustrated guide book. *Preprint*, 2004.

[12] Peter Dybjer. Internal Type Theory. In *Types for Proofs and Programs. TYPES 1995*, pages 120–134. Springer, Berlin, Heidelberg, 1996.

[13] Eric Finster. Opetopic. `https://github.com/ericfinster/opetopic`.

[14] Eric Finster and Samuel Mimram. A Type-Theoretical Definition of Weak ω-Categories. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12, 2017.

[15] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare kategorien*, volume 221. Springer-Verlag, 2006.

[16] Alexander Grothendieck. *Pursuing stacks.* unpublished manuscript, 1983.

[17] Tom Leinster. A survey of definitions of n-category. *Theory and applications of Categories*, 10(1):1–70, 2002.

[18] Tom Leinster. *Higher operads, higher categories*, volume 298. Cambridge University Press, 2004.

[19] Peter LeFanu Lumsdaine. Weak ω-categories from intensional type theory. In *International Conference on Typed Lambda Calculi and Applications*, pages 172–187. Springer, 2009.

[20] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[21] Georges Maltsiniotis. Grothendieck ∞-groupoids, and still another definition of ∞-categories. Preprint `arXiv:1009.2331`, 2010.

[22] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[23] Benno Van Den Berg and Richard Garner. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society*, 102(2):370–394, 2011.