Globular weak ω-categories as models of a type theory

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Abstract

Following the conception that type theories are presentation of generalized algebraic theories, we present a type theory that correspond to the theory of weak ω-categories. We follow Maltsiniotis' definition of weak ω-categories, inspired by Grothendieck's method. In this definition, a theory, called a *cat-coherator*, is a category that encodes all the axioms for weak ω-categories, and a weak ω-category is a presheaf with certain properties over a cat-coherator. We study a type theoretic formulation of this definition introduced by Finster and Mimram, our main result is to prove that its models coincide with the notion of weak ω-categories defined by Maltsiniotis. This theorem requires the introduction of categorical tools that point towards a nerve theorem for type theories. We do not provide a account for a more general framework in which this construction still work and focus on our specific type theory along with its technical points. We also briefly present an implementation of this type theory in the form of a proof assistant that allows the user to work with weak ω-categories in a computer checked environment

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Introduction

Understanding weak ω-categories has been a long standing goal in higher category theory, and many mathematical tools have been used to this purpose. Various surveys and comparisons between these definitions have been established by Leinster [16] and Cheng and Lauda [10]. The main obstacle however both in giving a definition and in comparing these definitions relies in the very self referencing nature of the axioms.

Weak ω-groupoids are the undirected analogue to weak ω-categories, and their axioms are also self referencing, but they are much better understood, through their connection with topology. In the early 2000's it started to become clear that the iterated identity types in Martin-Löf type theory endow each type with the structure of a weak ω-groupoids [18, 24, 1]. This is one of the key observation leading to homotopy type theory [23]. Based on this observation, and following Cartmell's [9] insight that type theory could be used to formulate genralized algebraic theory, Brunerie [8] extracted out of the rule generating the identity types, a definition of weak ω-groupoids, that he could show to be equivalent to a definition proposed by Grothendieck [13].

Following Brunerie's approach, Finster and Mimram [12] a definition of weak ω-categories in the form of a type theory called CaTT. The generalization follows the lines of a generalization of Grothendieck's weak ω-groupoids to weak ω-categories, proposed by Maltsiniotis [20]. The goal of this article is to show that the type theory $\mathsf{CaTT}$is equivalent to one of the definitions proposed by Maltsiniotis. Moreover Ara [2] has proved this specific to be equivalent to a definition proposed by Leinster [17] following a method introduced by Batanin [3]. This completes this result, and establish these three definitions as three sides of the same story, expressed in different languages.

We first introduce a type theory for globular sets which serves both as a basis and as a first example to our method, since the approach we present here is only globular. We then briefly present the Grothendieck-Maltsiniotis definition of weak ω-categories in order to give motivation for the construction of the type theory CaTT. We introduce the type theory $\mathsf{CaTT}$and give some examples of derivations in this theory, as well as its properties. We then study the syntactic category of this theory and characterize it by a universal property. Finally, using this universal property, we show that its models are equivalent to the Grothendieck-Maltsiniotis definition of weak ω-groupoids.

The reader who wants to get familiarized with the type theory along the way may also experiment with the implementation [4]. Although it implements some features that we do not detail in order to make its use less cumbersome, these do not override bare-bone theory that we present. Moreover, adding these features can be understood as a meta-theoretic operation that ultimately yields to a term in the bare bone theory, and hence to not change its models [5]

1 Categorical semantics of type theory

We begin by defining the categorical setup we use to study type theory, as well as an appropriate notion of models, and computation of limits inside this setup. Note that we will not introduce any type theory just yet, but only a categorical formulation of it. We refer the reader to Section 2.2 for the presentation of what we refer to as a type theory. We write **Set** for the category of (small) sets and **SET** for the category of large sets (it contains in particular an object corresponding to the collection objects of **Set**), and we suppose that there is an inclusion from the first to the second category. We do not give full details about size issues in the following, but we have checked that those can be properly handled.

1.1 Category with families

Our chosen categorical model to study type theory is the notion of category with families introduced by Dybjer [11]. This choice is of little importance as most other models like Cartmell's categories with attributes [9] are known to be equivalent.

We write **Fam** for the category of families, where an object is a family $(A_i)_{i\in I}$ of sets A_i indexed in a set I and a morphism $f : (A_i)_{i\in I} \to (B_j)_{j\in J}$ is a pair consisting of a function $f : I \to J$ and a family of functions $(f_i : A_i \to B_{f(i)})_{i\in I}$.

Suppose given a category $\mathcal{C}$ equipped with a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Fam}$. Given an object Γ of $\mathcal{C}$, its image will be denoted

$$T\Gamma \quad = \quad \left(\mathrm{Tm}^{\Gamma}_{A}\right)_{A\in\mathrm{Ty}^{\Gamma}}$$

i.e., we write Ty^{Γ} for the index set and Tm^{Γ}_{A} for the elements of the family. Given a morphism $\sigma : \Gamma \to \Delta$ in $\mathcal{C}$ and an element $A \in \mathrm{Ty}^{\Delta}$, we write $A[\sigma]$ for the object $T\sigma(A)$ of Ty^{Γ}. Similarly, given an element $t \in \mathrm{Tm}^{\Delta}_{A}$, we write $t[\sigma]$ for the element $(T\sigma)_A(t)$ of $\mathrm{Tm}^{\Gamma}_{A[\sigma]}$. With those notations, the functoriality of T can be written as

$$\begin{aligned} A[\sigma\circ\delta] &= A[\sigma][\delta] & \qquad\qquad t[\sigma\circ\delta] &= t[\sigma][\delta] \\ A[\mathrm{id}] &= A & t[\mathrm{id}] &= t \end{aligned}$$

for composable morphisms of $\mathcal{C}$.

A *category with families* (or *CwF*) consists of a category $\mathcal{C}$ together with a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Fam}$ as above, such that $\mathcal{C}$ has a terminal object, denoted $\varnothing$, and that there is a *context comprehension* operation: given a context Γ and type $A \in \mathrm{Ty}^{\Gamma}$, there is a context (Γ, A), together with a projection morphism $\pi : (\Gamma, A) \to \Gamma$ and a term $p \in \mathrm{Tm}^{(\Gamma,A)}_{A[\pi]}$, such that for every morphism $\sigma : \Delta \to \Gamma$ in $\mathcal{C}$ together with a term $t \in \mathrm{Tm}^{\Delta}_{A[\sigma]}$, there exists a unique morphism $\langle\sigma, t\rangle : \Delta \to (\Gamma, A)$ such that $p[\langle\sigma, t\rangle] = t$:

$$\begin{tikzcd} & (\Gamma, A) \arrow[d, "\pi"] \\ \Delta \arrow[ur, dotted, "\langle\sigma,t\rangle"] \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

In a category with families, the class of *display maps* is the smallest class of morphisms containing the projection morphisms $\pi : (\Gamma, A) \to \Gamma$ and closed under composition and identities.

A *morphism* between two categories with families $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$, is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation $\phi : T \to T' \circ F$, such that F preserves the terminal object and the context comprehension operation.

A 2-*morphism* θ between two morphisms $(F, \phi) : T \to T'$ and $(F', \phi') : T \to T'$ is a natural transformation $\theta : F \to F'$ such that $T\theta \circ \phi = \phi'$.

We define a large category with families in a similar way, as a large category equipped with a functor into families of large sets indexed by a large set, and satisfying the exact same properties. Note that a category with families can be seen as a large category with families. There is a structure of category with large families on the category **Set**, where, given a set X, Ty^X is the (large) set of all function $f : Y \to X$ with codomain X and given such a function $f : Y \to X$, Tm^X_f is the (large) set of all sections of f. We define the category of *models* of a category with families $\mathcal{C}$ to be the category whose objects are the morphisms of categories with families from $\mathcal{C}$ to **Set**, and whose morphisms are the $2-$*morphisms* of categories with families.

The structure of category with families encompasses a compatibility condition between context comprehension and the action of morphisms on the type, expressed by the following lemma. In particular, it states that all pullbacks along display maps exist and that they can be explicitly computed from the given structure.

Lemma 1. *In a category with families $\mathcal{C}$, for every morphism $f : \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in \mathrm{Ty}^\Gamma$, the square*

$$\begin{tikzcd}
(\Delta, A[f]) \arrow[r, "\langle f \circ \pi', p' \rangle"] \arrow[d, "\pi'"'] & (\Gamma, A) \arrow[d, "\pi"] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd}$$

is a pullback, where $\pi' : (\Delta, A[f]) \to \Delta$ and $p' \in \mathrm{Tm}^{(\Delta, A[f])}_{A[f][\pi']}$ are obtained by context comprehension.

Proof.

$$\begin{tikzcd}
\Theta \arrow[drr, bend left, "\sigma"] \arrow[dr, bend right, "\delta"'] & & \\
& (\Delta, A[f]) \arrow[r, "\langle f \circ \pi', p' \rangle"] \arrow[d, "\pi'"] & (\Gamma, A) \arrow[d, "\pi"] \\
& \Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

Consider the term $p \in \mathrm{Tm}^{(\Gamma, A)}_{A[\pi]}$, then $p[\sigma] \in \mathrm{Tm}^{\Theta}_{A[\pi][\sigma]} = \mathrm{Tm}^{\Theta}_{A[f][\delta]}$. By context extension, we get a map $\langle \delta, p[\sigma] \rangle : \Theta \to (\Delta, A[f])$ such that $\pi' \circ \langle \delta, p[\sigma] \rangle = \delta$ and $p'[\langle \delta, p[\sigma] \rangle] = p[\sigma]$. Since moreover $p' = p[\langle f \circ \pi', p' \rangle]$, the term equality gives in fact $p[\sigma] = p[\langle f \circ \pi', p' \rangle \circ \langle \delta, p[\sigma] \rangle]$, which is a necessary condition for the upper triangle to commute, thus proving uniqueness of the map. We just have to show that this map makes the upper triangle commute. Notice that $\pi \circ \langle f \circ \pi', p' \rangle \circ \langle \delta, p[\sigma] \rangle = \pi \circ \sigma$, and $p[\sigma] = p[\langle f \circ \pi', p' \rangle \circ \langle \delta, p[\sigma] \rangle]$, by universal property of the extension for morphisms, this implies the commutativity of upper triangle. $\square$

Morphisms and pullbacks.

Lemma 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with families, together with a morphism $(F, \phi) : \mathcal{C} \to \mathcal{D}$, then for any object Γ in $\mathcal{C}$ together with an element $A \in \mathrm{Ty}^\Gamma$ and for any morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$, the following equation is satisfied*

$$F(\Delta, A[\gamma]) = (F\Delta, (\phi_\Gamma A)[F\gamma])$$

Proof. By definition of a morphism of category with families, we have $F(\Delta, A[\gamma]) = (F(\Delta), (\phi_\Delta(A[\gamma])))$, and by naturality of ϕ, the following square commutes

$$\begin{tikzcd} Ty^\Gamma \arrow[r, "\phi_\Gamma"] \arrow[d, "\_[f]"'] & Ty^{F(\Gamma)} \arrow[d, "\_[F\gamma]"] \\ Ty^\Delta \arrow[r, "\phi_\Delta"] & Ty^{F(\Delta)} \end{tikzcd}$$

This proves in particular the $\phi_\Delta(A[\gamma]) = (\phi_\Gamma A)[F\gamma]$, thus $F(\Delta, A[\gamma]) = (F\Delta, (\phi_\Gamma A)[F\gamma])$ □

Note that Lemma 1 allows to understand this result as the fact that F preserves the pullbacks along the display maps. In fact the following result shows that preserving these pullbacks is the exact condition that a functor has to satisfy in order to be a model

Lemma 3. *The models of a category with families $\mathcal{C}$ are isomorphic to the functors $\mathcal{C} \to$ **Set** that preserve the terminal object and the morphisms along the display maps.*

Proof. The Lemma 2, the underlying functor of a morphism of category with families preserves the pullbacks along the display maps, and by definition, such functor has to preserve the initial object as well. So it suffices to prove that a functor $F : \mathcal{C} \to \mathbf{Set}$ preserving the initial object and the morphisms pullbacks along display maps gives rise to a unique model. Consider such a functor F, together with an object Γ in $\mathcal{C}$ and a type $A \in \mathrm{Ty}^\Gamma$. Suppose defined ϕ such that (F, ϕ) is a model of $\mathcal{C}$, then necessarily $F(\Gamma, A) = (F\Gamma, \phi_\Gamma A) = \phi_\Gamma A$ by definition of the context comprehension in **Set**. Thus necessarily $\phi_\Gamma(A) = F(\Gamma, A)$. Consider a term $t \in \mathrm{Tm}^\Gamma_A$, then there is a morphism $\langle \mathrm{id}_\Gamma, t\rangle : \Gamma \to (\Gamma, A)$, and by definition of the category with families structure of **Set**, we then have $F(\langle \mathrm{id}_\Gamma, t\rangle) = \langle \mathrm{id}_{F\Gamma}, \phi_{\Gamma,A}(t)\rangle = t$, which proves that necessarily $\phi_{\Gamma,A}(t) = F(\langle \mathrm{id}_\Gamma, t\rangle)$. Conversely, these assignments define a natural transformation ϕ, which make (F, ϕ) into a model of F. □

This condition relies on the specific structure of category with families of **Set**, and do not subsume the notion of morphism of a category with families in general. It also justifies retrospectively not to be too precise about the size issues with **Set**, as one may as well ignore the structure of category with families on **Set** altogether, and define a model as a functor $\mathcal{C} \to \mathbf{Set}$ that preserves the terminal object and the pullback along the display maps.

1.2 Contextual categories

In order to carry some inductive constructions that we can perform on the syntax on a theory, and treat them in full generality, we will introduce the notion of *contextual categories*, due to Cartmell [9], and studied by Streicher [22] and

Voevodsky [25] under the name of *C-systems*. These are precisely the categories with families with extra structure making those inductive construction possible.

Definition 4. A contextual category is a category with families $\mathcal{C}$ together with a map ℓ associating to each object Γ of $\mathcal{C}$ a natural number $\ell(\Gamma)$ called its *length*, such that

- the terminal object $\varnothing$ is the unique object such that $\ell(\varnothing) = 0$,
- for every object Γ and type $A \in \mathrm{Ty}^{\Gamma}$, $\ell(\Gamma, A) = \ell(\Gamma) + 1$,
- for every object Γ such that $\ell(\Gamma) > 0$, there is a unique object Γ' together with a type $A \in \mathrm{Ty}^{\Gamma'}$ such that $\Gamma = (\Gamma', A)$.

Note that a contextual category is usually defined to be a category with attributes satisfying such properties. However, since categories with families and categories with attributes are equivalent, we will also refer to these as contextual categories. Also note that, the notion of contextual category is not invariant by equivalence of categories and relies of a presentation we give of a category. Their use is justified by the fact that the syntax of a type theory gives a particular presentation of a category with families, which happens to be a contextual category.

Given a contextual category $\mathcal{C}$, an object Γ whose length is strictly positive is obtained in a unique way as Γ', A, and we simply write $\pi_\Gamma : \Gamma \to (\Gamma', A)$ (or even π) instead of $\pi_{\Gamma', A}$. We also write x_Γ for the term $p_{\Gamma', A}$ in $\mathrm{Tm}^{\Gamma}_{A[\pi]}$, thought of as a variable. More generally, we declare that a term is a *variable* when it is of the form $x_\Gamma[\pi]$ where π is a display map. Note that in a contextual category, if $\pi : \Delta \to \Gamma$ is a display map, then necessarily $l(\Delta) > l(\Gamma)$. This implies that the variables of a non-empty context (Γ, A) are either $x_{(\Gamma,A)}$, or of the form $x[\pi_{(\Gamma,A)}]$ where x is a variable of Γ.

The following lemma shows that a map in a contextual category is entirely characterized by its action on variables in its target context.

Lemma 5. *Consider two maps $\gamma, \delta : \Delta \to \Gamma$, in a contextual category, such that for every variable x in Γ, $x[\gamma] = x[\delta]$ implies $\gamma = \delta$.*

Proof. We will prove this result by induction on the length of the context Γ :

- If Γ is of length 0, then necessarily, $\Gamma = \varnothing$ is the terminal object, and thus $\gamma = \delta$.
- If Γ is of length $l + 1$, then it is of the form (Γ', A), and there is a substitution $\pi : \Gamma \to \Gamma'$. Suppose that there are two substitutions $\gamma, \delta : \Delta \to \Gamma$, such that for all variables x of Γ, we have $x[\gamma] = x[\delta]$. Note that we necessarily have $\gamma = \langle \pi \circ \gamma, x_\Gamma[\gamma] \rangle$ and $\delta = \langle \pi \circ \delta, x_\Gamma[\delta] \rangle$, as it is the case for every substitutions. Then for the variable x_Γ, we have $x_\Gamma[\gamma] = x_\Gamma[\delta]$. Moreover, for every variable x of Γ', $x[\pi]$ is a variable of Γ, and thus $x[\pi][\gamma] = x[\pi][\delta]$, which proves $x[\pi \circ \gamma] = x[\pi \circ \delta]$, and y induction hypothesis, $\pi \circ \gamma = \pi \circ \delta$. We thus have proved that $\langle \pi \circ \gamma, x_\Gamma[\gamma] \rangle = \langle \pi \circ \delta, x_\Gamma[\delta] \rangle$, i.e., $\gamma = \delta$.

$\square$

2 A type theory for globular sets

We start with the introduction of a type theory for globular sets, on top of which we build the type theory for weak ω-categories later on.

2.1 The category of globular sets

The category of globes. The *category of globes* $\mathcal{G}$ is the category whose objects are the natural numbers and morphisms are generated by

$$\sigma_i, \tau_i : i \to i+1$$

subject to following *coglobular relations*:

$$\sigma_{i+1} \circ \sigma_i = \tau_{i+1} \circ \sigma_i \qquad\qquad \sigma_{i+1} \circ \tau_i = \tau_{i+1} \circ \tau_i \tag{1}$$

The category of *globular sets* $\mathbf{GSet} = \widehat{\mathcal{G}}$ is the presheaf category over the category $\mathcal{G}$. Given a globular set G, we write G_n instead of $G\,n$. Equivalently, a globular set is a family of sets $(G_n)_{n\in\mathbb{N}}$ equipped with maps $s_i, t_i : G_{i+1} \to G_i$ satisfying the *globular relations*, dual to (1)

$$s_i \circ s_{i+1} \quad = \quad s_i \circ t_{i+1} \qquad\qquad t_i \circ s_{i+1} \quad = \quad t_i \circ t_{i+1} \tag{2}$$

Given an object n, the associated representable $\mathrm{Y}(n)$ is called the *n-disk* and is usually written D^n. It can be explicitly described by

$$(D^n)_i = \begin{cases} \{*_0, *_1\} \text{ if } i < n \\ \{*\} \text{ if } i = n \\ \emptyset \text{ if } i > n \end{cases}$$

with $s(\_) = *_0$ and $t(\_) = *_1$.

The n-sphere. Given $n \in \mathbb{N}$, the *n-sphere* S^n is the globular set, equipped with an inclusion $\iota^n : S^n \hookrightarrow D^n$, defined by

- $S^{-1} = \emptyset$ is the initial object, and $\emptyset \hookrightarrow D^1$ is the unique arrow,
- S^{n+1} and ι^{n+1} are obtained by the pushout

$$\begin{tikzcd}
S^n \arrow[r, hook, "\iota_n"] \arrow[d, hook, "\iota_n"'] & D^n \arrow[d] \arrow[ddr, bend left, "\sigma_n"] & \\
D^n \arrow[r] \arrow[drr, bend right, "\tau_n"'] \arrow[ur, phantom, "\ulcorner"] & S^{n+1} \arrow[dr, dotted, "\iota_{n+1}"] & \\
 & & D^{n+1}
\end{tikzcd}$$

Finite globular sets. A globular set G is *finite* if it can be obtained as a finite colimit of representable objects. It can be shown that this is the case precisely when the set $\bigsqcup_{i\in\mathbb{N}} G_i$ is finite, because all representables themselves satisfy this property. We write **FinGSet** for the full subcategory of **GSet** whose objects are the finite presheaves. We sometimes call a finite globular set a *diagram*, and describe it using a diagrammatic notation. For instance, the diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, "h"] & z \end{tikzcd}$$

denotes the finite globular set G, whose only non-empty cell sets are

$$G_0 = \{x, y, z\} \qquad G_1 = \{f, g, h\} \qquad G_2 = \{\alpha\}$$

and whose the sources and targets are defined by

$$\begin{array}{ll} s(f) = x & t(f) = y \\ s(g) = x & t(g) = y \\ s(h) = y & t(h) = z \\ s(\alpha) = f & t(\alpha) = g \end{array}$$

Disks and spheres are finite globular sets. In small dimensions, they can be depicted as

$$D^0 = \quad \cdot \qquad\qquad S^0 = \quad \cdot \qquad \cdot$$

$$D^1 = \begin{tikzcd} \cdot \arrow[r] & \cdot \end{tikzcd} \qquad\qquad S^1 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \end{tikzcd}$$

$$D^2 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} \qquad\qquad S^2 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\;\Downarrow"] & \cdot \end{tikzcd}$$

$$D^3 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Rrightarrow\Downarrow"] & \cdot \end{tikzcd}$$

Recall that **FinGSet** is the free cocompletion of G by all finite colimits.

2.2 The theory GSeTT

The objective of this section is to describe a type theory whose models we will prove to be exactly the globular sets.

Signature. We consider a countably infinite set whose elements we call *variables*. A *term* is this theory is simply a variable (for now, we will add other terms later on). A *type* is defined inductively to be either

$$\star \qquad \text{or} \qquad t \underset{A}{\longrightarrow} u$$

where A is a type and t, u are terms. A *context* is a list

$$(x_1 : A_1, \ldots, x_n : A_n)$$

of variables $x_1, \ldots, x_n$ together with types $A_1, \ldots, A_n$, the empty context is denoted $\varnothing$. A *substitution* is a list

$$\langle x_1 : t_1, \ldots, x_n : t_n \rangle$$

of variables $x_1, \ldots, x_n$ together with terms $t_1, \ldots, t_n$. For now on, we use the following naming conventions

variables	: $x, y, \ldots$
terms	: $t, u, \ldots$
types	: $A, B, \ldots$
contexts	: $\Gamma, \Delta, \ldots$
substitutions	: $\sigma, \tau, \ldots$

Judgments. The theory will consist in four different kinds of *judgments*, for which we give the notations, along with the intuitive meaning.

$\Gamma \vdash$	: the context Γ is well-formed
$\Gamma \vdash A$	: the type A is well-formed in the context Γ
$\Gamma \vdash t : A$	: the term t has type A in context Γ
$\Gamma \vdash \sigma : \Delta$	: the substitution σ goes from the context Γ to the context Δ

Syntactic properties. Given a term t (resp. a type A, a context Γ, a substitution σ), we define the set of its *free variables* $\mathrm{Var}(t)$ (resp. $\mathrm{Var}(A)$, $\mathrm{Var}(\Gamma)$, $\mathrm{Var}(\sigma)$) by induction as follows

on terms

$$\mathrm{Var}(x) = \{x\}$$

on types

$$\mathrm{Var}(\star) = \emptyset \qquad\qquad \mathrm{Var}(t \underset{A}{\to} u) = \mathrm{Var}(A) \cup \mathrm{Var}(t) \cup \mathrm{Var}(u)$$

on contexts

$$\mathrm{Var}(\varnothing) = \emptyset \qquad\qquad \mathrm{Var}(\Gamma, x : A) = x \cup \mathrm{Var}(\Gamma)$$

on substitutions

$$\mathrm{Var}(\langle\rangle) = \emptyset \qquad\qquad \mathrm{Var}(\langle \sigma, x : t\rangle) = \mathrm{Var}(t) \cup \mathrm{Var}(\sigma)$$

Given a type A in this theory, we define its *dimension* $\dim(A)$ by the following formula. (The choice of starting at -1 is justified to give a cleaner correspondence in Lemma 10)

$$\dim(\star) = -1 \qquad\qquad \dim(t \underset{A}{\to} u) = \dim(A) + 1$$

For a context $\Gamma = (x_i : A_i)_{1 \le i \le n}$, its dimension is defined to be $\dim(\Gamma) = \max_i \dim(A_i)$, and for a term t such that the judgment $\Gamma \vdash t : A$ holds, we will define the dimension of t in the context Γ to be

$$\dim_\Gamma(t) = \dim(A) + 1$$

Given a substitution $\sigma = \langle x_i : t_i\rangle$ and a term t (resp. a type A), we define the *action of* σ, denoted $t[\sigma]$ (resp. $A[\sigma]$), by

$$x_i[\sigma] = t_i \qquad\qquad y[\sigma] = y \quad \text{if } y \notin \{x_1, \ldots, x_n\}$$

and

$$\star[\sigma] = \star \qquad\qquad (t \underset{A}{\to} u)[\sigma] = (t[\sigma]) \underset{A[\sigma]}{\longrightarrow} (u[\sigma])$$

Typing rules. Inference rules for GSeTT are given by

for terms

$$\frac{\Gamma \vdash \qquad x : A \in \Gamma}{\Gamma \vdash x : A}(\text{AX}) \qquad \frac{\Gamma, x : A \vdash \qquad \Gamma \vdash t : B}{\Gamma, x : A \vdash t : B}(\text{WK})$$

for types

$$\frac{}{\Gamma \vdash \star}(\text{OBJ}) \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\rightarrow} u}(\text{HOM})$$

for contexts

$$\frac{}{\varnothing \vdash}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE})$$

where in the rule (CE), $x \notin \mathrm{Var}(\Gamma)$; for substitutions

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \sigma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\sigma]}{\Delta \vdash \langle \sigma, x : t\rangle : (\Gamma, x : A)}(\text{SE})$$

where in the rule (SE), $x \notin \mathrm{Var}(\Gamma)$

We have defined variables as a syntactic properties of terms and types, thus independent of the judgments, but we are often more interested in the variables of a term together with its type. To express this, we write $\mathrm{Var}(t : A)$ for the union $\mathrm{Var}(t) \cup \mathrm{Var}(A)$, with the implicit convention that in the current context Γ, the judgment $\Gamma \vdash t : A$ is derivable.

Lemma 6. *The following properties can be shown*

- *If* $\Gamma \vdash A$ *then* $\Gamma \vdash$
- *If* $\Gamma \vdash t : A$ *then* $\Gamma \vdash A$
- *If* $\Delta \vdash \sigma : \Gamma$ *then* $\Delta \vdash$ *and* $\Gamma \vdash$
- *If* $\Gamma \vdash A$ *and* $\Delta \vdash \sigma : \Gamma$ *then* $\Delta \vdash A[\sigma]$
- *If* $\Gamma \vdash t : A$ *and* $\Delta \vdash \sigma : \Gamma$ *then* $\Gamma \vdash t[\sigma] : A[\sigma]$
- *If* $\Gamma \vdash x \underset{A}{\rightarrow} y$ *then* $\Gamma \vdash x : A$ *and* $\Gamma \vdash y : A$
- *If* $\Gamma \vdash A$*, then* $\mathrm{Var}(A) \subset \mathrm{Var}(\Gamma)$
- *If* $\Gamma \vdash t : A$ *then* $\mathrm{Var}(t : A) \subset \mathrm{Var}(\Gamma)$

In a type $t \underset{A}{\rightarrow} u$, the type A is the common type of t and u, and we will thus generally omit it. Similarly, when a substitution $\sigma = \langle x_i : t_i\rangle_{1\leq i\leq n}$ is such that the judgment $\Delta \vdash \sigma : \Gamma$ holds with $\Gamma = (y_i : A_i)_{1\leq i\leq m}$, then necessarily $m = n$ and $x_i = y_i$ for $1 \leq i \leq n$. For this reason, when the context Γ is given, we may leave the variables $x_1, \ldots, x_n$ implicit and write

$$\sigma = \langle t_1, \ldots, t_n\rangle = \langle t_i\rangle_{1\leq i\leq n}$$

Lemma 7. *There is at most one way to derive a judgment.*

This allows us to use the notations $\Gamma \vdash$ (resp. $\Gamma \vdash A$, $\Gamma \vdash t : A$, $\Gamma \vdash \sigma : \Delta$) to denote a context Γ (resp. a type A, a term t, a substitution σ) such that above the judgment holds, confusing the derivable judgment, the object it talks about and its derivation.

2.3 The syntactic category of GSeTT

The syntactic category. Given a context $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$, there exists a canonical substitution called the *identity substitution* $\mathrm{id}_\Gamma = \langle x_1, \ldots, x_n \rangle$ such that the following rule is admissible

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathrm{id}_\Gamma : \Gamma}$$

Moreover, given two substitutions $\Delta \vdash \sigma : \Gamma$ and $\Theta \vdash \tau : \Delta$, one can define their *composition* $\tau \circ \sigma$. If $\tau = \langle x_i : t_i \rangle_{1 \leq i \leq n}$ then we define

$$\tau \circ \sigma = \langle x_i : t_i[\sigma] \rangle_{1 \leq i \leq n}$$

with this definition, the rule following rule is admissible

$$\frac{\Gamma \vdash \sigma : \Delta \qquad \Delta \vdash \tau : \Theta}{\Gamma \vdash \tau \circ \sigma : \Theta}$$

One can also check that the following equalities hold, whenever these expressions are well-defined

$$\mathrm{id}_\Gamma \circ \sigma = \sigma \qquad\qquad \sigma \circ \mathrm{id}_\Gamma = \sigma \qquad\qquad \sigma \circ (\tau \circ \varphi) = (\sigma \circ \tau) \circ \varphi$$

This justifies that there is a category whose objects are the contexts $\Gamma \vdash$ and the morphism $\Gamma \to \Delta$ are the substitutions $\Gamma \vdash \sigma : \Delta$, called the *syntactic category* of the theory GSeTT. This category is denoted $\mathcal{S}_{\mathsf{GSeTT}}$, note that it is canonically a category with families, by choosing, for Γ an object of $\mathcal{S}_{\mathsf{GSeTT}}$, Ty^Γ to be the set of types derivable in Γ and for A such a type, Tm_A^Γ to be the set of terms of type A in Γ. Moreover, the contexts are naturally equipped with a length, as they are supported by lists, making the given presentation of the category with families $\mathcal{S}_{\mathsf{GSeTT}}$ into a C-system.

Note 8. Here we have given a presentation with named variables, but one could also give a presentation of the same type theory using unnamed variables, with for instance de Bruijn indices. This would lead to a slightly different notion of the syntactic category, which is essentially the previously defined syntactic category quotiented by the renaming (or α-conversion) between contexts. From now on, we will suppose given a presentation with unnamed variables, so that the renamings are not taken in account in the syntactic category

Disks and spheres contexts. In the category $\mathcal{S}_{\mathsf{GSeTT}}$ there are two classes of context that play an important role, the *n-disk context* D^n and the *n-sphere context* S^n. Their precise role in our theory are made clear by the Lemma 10 and by the understanding of the syntactic category provided by the Theorem 12. These contexts are defined inductively by

$$\begin{array}{lcl@{\qquad}lcl} S^{-1} & = & \varnothing & D^0 & = & (x_0 : A_0) \\ S^n & = & (D^n, x_{2n+1} : A_n) & D^{n+1} & = & (S^n, x_{2(n+1)} : A_{n+1}) \end{array}$$

where the types A_n are inductively defined by

$$\begin{array}{lcl} A_0 & = & \star \\ A_{n+1} & = & x_{2n-2} \underset{A_n}{\longrightarrow} x_{2n-1} \end{array}$$

Proposition 9. *For any integer n, the contexts D^n and S^n are well-formed, i.e., the following rules are admissible.*

$$\frac{}{D^n \vdash} \qquad\qquad \frac{}{S^n \vdash}$$

Proof. We prove the validity of these contexts by induction. First notice that $S^{-1} = \varnothing$ is well defined by the rule (EC), and that by applying successively the rules (CE) and (OBJ), D^0 is also ell defined. Then suppose that S^{k-1} and D^k are valid contexts, then the rule (AX) ensures that $D^k \vdash x_{2k} : A_k$, and by Lemma 6, this proves that $D^k \vdash A_k$, since moreover $x_{2k+1} \notin \mathrm{Var}(D^k)$, the rule (CE) applies and shows $S^k \vdash$. Moreover, the rule (AX) applies twice to show both $S^k \vdash x_{2k} : A_k$ and $S^k \vdash x_{2k+1} : A_k$, hence by the rule (HOM), this proves $S^k \vdash A_{k+1}$ and since $x_{2(k+1)} \notin S^k$, the rule (CE) applies and proves $D^{k+1} \vdash$. $\square$

Familial representability of types. The following lemma is completely central in the study of the type theory GSeTT that we conduct. It allows to understand both types and terms as special cases of substitutions, and the action of substitution then becomes pre-composition.

Lemma 10. *Let $n \in \mathbb{N} \cup \{\infty\}$, then the set of substitutions $\Gamma \vdash \gamma : S^{n-1}$ is in bijective correspondence with derivable types of dimension $n-1$. The set of substitutions $\Gamma \vdash \gamma : D^n$ is in bijective correspondence with derivable terms of dimension n.*

More explicitly, the substitution $\Gamma \vdash \gamma : S^{n-1}$, corresponds to the type $\Gamma \vdash A_{n+1}[\gamma]$, and the substitution $\Gamma \vdash \gamma : D^n$ corresponds to the term $\Gamma \vdash x_{2n}[\gamma] : A_{n+1}[\gamma]$.

Proof. We proceed by induction over the dimension n that we consider, and prove the two statements mutually

- For $n = 0$: The context $S^{-1} = \varnothing$ is terminal : there is always exactly one substitution $\Gamma \vdash \langle\rangle : \varnothing$. Similarly there is always exactly one type of dimension -1 derivable in Γ, which is the type $\star$. And this is also by definition, the type A_0

  A substitution $\Gamma \vdash \gamma : D^0 = (x_0 : \star)$ is necessarily obtained by applying the rule (SE), of the form

  $$\frac{\Gamma \vdash \gamma' : \varnothing \qquad \varnothing \vdash A_0 \qquad \Gamma \vdash t : A_0[\gamma']}{\Gamma \vdash \gamma : D^0}$$

  Note that $A_0[\gamma] = \star$, and that such a substitution is necessarily of the form $\langle t\rangle$ where $\Gamma \vdash t : \star$. Conversely, any term $\Gamma \vdash t : \star$ gives rise to a well defined substitution $\Gamma \vdash \langle t\rangle : D^0$.

- Suppose that the result holds for the sphere S^{n-1} and the disk D^n for a given n. Substitutions $\Gamma \vdash \gamma : S^n$ are exactly the ones of the form $\langle \gamma', u \rangle$ and are derived by application of the rule

$$\frac{\Gamma \vdash \gamma' : D^n \qquad D^n \vdash A_n \qquad \Gamma \vdash u : A_n[\gamma']}{\Gamma \vdash \langle \gamma', u \rangle : S^n}$$

By induction hypothesis, the judgment $\Gamma \vdash \gamma' : D^n$ is equivalent to a term $\Gamma \vdash x_{2n}[\gamma'] : A_n[\gamma']$. Thus the substitution $\Gamma : \gamma : S^n$ is equivalent to the type obtained by

$$\frac{\Gamma \vdash x_{2n}[\gamma'] : A_n \qquad \Gamma \vdash u : A_n}{\Gamma \vdash x_{2n}[\gamma'] \to u}$$

Moreover, we have that $x_{2n}[\gamma] = x_{2n}[\gamma' \circ \pi] = x_{2n}[\gamma']$, and also $x_{2n+1}[\gamma] = u$ by definition. Thus this type rewrites to

$$x_{2n}[\gamma] \to x_{2n+1}[\gamma] = A_n[\gamma]$$

Moreover, substitutions $\Gamma \vdash \gamma : D^{n+1}$ are exactly the ones of the form $\langle \gamma', u \rangle$, and are derived by the rule

$$\frac{\Gamma \vdash \gamma' : S^n \qquad S^n \vdash A_n \qquad \Gamma \vdash u : A_n[\gamma']}{\Gamma \vdash \langle \gamma', u \rangle : D^{n+1}}$$

By induction hypothesis, the judgment $\Gamma \vdash \gamma' : S^n$ is equivalent to the type $A_n[\gamma']$ in Γ, and thus a substitution γ equivalent to the data of the derivable term $\Gamma \vdash u : A_n[\gamma']$. Moreover, by definition of $\gamma = \langle \gamma', u \rangle$ we have that $u = x_{2n}[\gamma]$. Moreover, $A_n[\gamma'] = A_n[\gamma]$ Thus γ is equivalent to the data of the term

$$\Gamma \vdash x_{2n}[\gamma] : A_n[\gamma]$$

□

Note 11. Note that this proof does not rely on how the terms are constructed, so no matter what the term constructors are, this result is always true.

This lets us adopt the following notations conventions

- if $\Gamma \vdash A$ is a type, we will also denote by A the corresponding substitution $\Gamma \vdash A : S^{n-1}$. With this abuse of notations, if $\Delta \vdash \gamma : \Gamma$ is a substitution, the type $\Delta \vdash : A[\sigma]$ is identified with the substitution $\Delta \vdash A \circ \sigma : S^n$.

- if $\Gamma \vdash t : A$ is a term, we also denote $\Gamma \vdash t : D^n$ the corresponding substitution, and the term $\Delta \vdash t[\gamma] : A[\Gamma]$ correspond to the substitution $\Delta \vdash t \circ \gamma : D^n$. Moreover, if $\Gamma \vdash t : A$, then $A = \pi \circ t$. In other words, the judgments $\Gamma \vdash t : A$ correspond exactly to the commutative triangles

$$\begin{tikzcd} \Gamma \arrow[r, "t"] \arrow[dr, "A"'] & D^n \arrow[d, "\pi"] \\ & S^{n-1} \end{tikzcd}$$

The syntactic category of GSeTT. We will now characterize the syntactic category of GSeTT. This is an important step in studying the models of the theory, since understanding precisely the syntactic category gives a good insight on the functors mapping out of it. Interestingly, it will always turn out that the syntactic category is a dual to the finitely generated objects that we are studying, which makes this formulation consistent with other formalization of algebraic theories.

Theorem 12. *The category $\mathcal{S}_{\mathsf{GSeTT}}$ is equivalent to* $\mathbf{FinGSet}^{\mathrm{op}}$*, the opposite category of the finite globular sets*

Proof. We will define a functor $F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{FinGSet}^{\mathrm{op}}$. Suppose that the context Γ is written as a list $\Gamma = (x_i : A_i)$, then we define

$$(F\Gamma)_n = \{x_i, \dim(A_i) = n\} = \{\text{derivable terms of dimension } n \text{ in } \Gamma\}$$

For x of type A in Γ, with $\dim(A) = n+1$, by definition of the dimension, A is of the form $A = y \to z$, for two derivable terms y and z, with $\dim_\Gamma(y) = \dim_\Gamma(z) = n$. So $y, z \in (F\Gamma)_n$, and we define $s(x) = y$ and $t(x) = z$. The derivation rule for A implies that y and z have the same type, thus $s(y) = s(z)$ and $t(y) = t(z)$, which proves that the globular relations are satisfied, and that $F\Gamma$ is indeed a globular set.

Let $\Delta \vdash \sigma : \Gamma$ be a substitution, and write $\Gamma = (x_i : A_i)$, then the substitution σ is of the form $\sigma = \langle x_i : t_i \rangle$, where t_i is a derivable term in the context Δ, i.e., $t_i \in F\Delta$. Then define

$$\begin{array}{rccc} F\sigma & : & F\Gamma & \to & F\Delta \\ & & x_i & \mapsto & t_i \end{array}$$

This is equivalent to saying that we define $(F\sigma)x = F(x[\sigma])$. Suppose that x is of type $y \to z$ in Γ, then $x[\sigma] = y[\sigma] \to z[\sigma]$ in Δ. This means that as an element of $F\Delta$, $x[\sigma]$ satisfies $s(x[\sigma]) = y[\sigma]$ and $t(x[\sigma]) = z[\sigma]$, or in other words, $s((F\sigma)x) = (F\sigma)(sx)$ and $t((F\sigma)x) = (F\sigma)(tx)$. Hence $F\sigma$ defines a morphism of globular sets.

Consider two substitutions σ and τ such that $F\sigma = G\tau$. This implies in particular that for all variables x in Γ, $(F\sigma)x = (G\tau)x$, thus $x[\sigma] = x[\tau]$. By the Lemma 5, this proves that $\sigma = \tau$, hence F is faithful. Conversely, consider two contexts Γ and Δ where $\Delta = (x_i : A_i)_{0 \leq i \leq l}$ together with a morphism of globular sets $f : F\Gamma \to F\Delta$, then one can define the substitution $\sigma_f = \langle x_i : f(x_i) \rangle_{1 \leq i \leq l}$. We check by induction on the length l of Δ that this produce a well defined substitution σ_f such that $F(\sigma_f) = f$. If $l = 0$ then $\Delta = \varnothing$ and $\sigma_f = \langle\rangle$, then the rule (ES) gives a derivation of $\Gamma \vdash \langle\rangle : \varnothing$. If $\Delta = \Delta', x_{l+1} : A_{l+1}$, then the natural inclusion $F(\Delta') \hookrightarrow F(\Delta)$ induces by composition a map $f' : F\Delta' \to F\Gamma$. By induction hypothesis, we have $\Gamma \vdash \sigma_{f'} : \Delta'$, and since Δ is a context, we also have $\Delta \vdash A_{n+1}$. Moreover, if $A_{n+1} = \star$, then $\Gamma \vdash f(x_{n+1}) : \star$ since f preserves the dimension, and otherwise $A_{n+1} = y \to z$, and $\Gamma \vdash f(x_{n+1}) : f(y) \to f(z)$ since f is a morphism of globular sets. In both cases, this proves that $\Gamma \vdash f(x_{n+1}) : A_{n+1}[\sigma_{f'}]$. By application of the rule (SE), this proves that $\Gamma \vdash \langle \sigma_{f'}, x_{n+1} : f(n_{n+1}) \rangle : \Delta$. Since $\sigma_f = \langle \sigma_{f'}, x_{n+1} : f(x_{n+1}) \rangle$, this proves that σ_f is well defined, and by definition it satisfies $F\sigma = f$. Hence the functor F is full.

Moreover, F is essentially surjective. Indeed, considering a finite globular set X, we show by induction on the number of elements of X that we can construct a context Γ such that $F\Gamma = X$. If X is the empty globular set, then $\Gamma = \varnothing$ is well defined by the rule (EC), otherwise, if X is not empty, consider an element x of dimension maximal in X and consider the globular set Y obtained by removing this element from X. By induction the context Δ constructed from Y is well-defined. Moreover, if x is of dimension 0, then denote $A = \star$ and we have $\Delta \vdash A$, and otherwise, we have $\Delta \vdash s\ x : B$ and $\Delta \vdash tx : B$ since both $s\ x$ and tx are parallel elements in Y, and denote $A = s\ x \to tx$, this shows that $\Delta \vdash A$. In both cases, we have $\Delta \vdash A$, and the rule (CE) applies to prove that $\Delta, x : A \vdash$. Moreover $F(\Delta, x : A)$ is obtained from $F\Delta$ by adding one element x' of the same dimension as x, and such that $s\ x' = s\ x$ and $tx' = tx$ if this dimension is not 0. Since by induction $F\Delta = Y$, and hence $F(\Delta, x : A) = X$. Note that this construction requires a choice of an element of maximal dimension in X, which is arbitrary, different choices may yield to different but isomorphic contexts, which can obtained from one another by permuting the variables of the same dimension. This proves that F is essentially surjective. Given that the functor F is fully faithful and essentially surjective, it is an equivalence of categories. □

Note that the functor F can be described in the light of Lemma 10. Indeed a term of dimension n in γ is simply a substitution $\gamma \to D^n$, hence $F(\_)_n = \bigsqcup \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n)$. This allows for understanding this functor as the nerve associated to the inclusion $\mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$. We use generalizations of this constructions in more complicated scenario.

Remark 13. Under this equivalence of category, the globular set D^n corresponds exactly to the context D^n, and the globular set S^n corresponds to the globular set S^n. This justifies the choice of the same notations for the contexts and the globular sets.

Corollary 14. *The category $\mathcal{S}_{\mathsf{GSeTT}}$ is the free completion by finite limits of the category $\mathcal{G}^{\mathrm{op}}$. More precisely, for all finitely complete category $\mathcal{C}$ denote $[\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}]_{\mathrm{flim}}$ the category of functors from $\mathcal{S}_{\mathsf{GSeTT}}$ to $\mathcal{C}$ preserving the finite limits, then the inclusion functor $\mathcal{G}^{\mathrm{op}} \hookrightarrow \mathcal{S}_{\mathsf{GSeTT}}$ induces an equivalence of categories*

$$[\mathcal{G}^{\mathrm{op}}, \mathcal{C}] \simeq [\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}]_{\mathrm{flim}}$$

Proof. This is a consequence of Theorem 12, and of the fact that **FinGSet** is the free cocompletion of $\mathcal{G}$ by all finite limits. □

2.4 Models of the type theory GSeTT.

The models of the type theory GSeTT are now easily characterized using all the tools we have introduced so far and some categorical techniques.

Theorem 15. *The models of the theory GSeTT are the globular sets. More precisely there is an equivalence of categories*

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \simeq \mathbf{GSet}$$

Proof. In the theory GSeTT, there are only variables and no term constructors, hence every map in the category $\mathcal{S}_{\mathsf{GSeTT}}$ is a display map. Since a category with families has pullbacks along all the display maps, $\mathcal{S}_{\mathsf{GSeTT}}$ has all pullbacks, and since it also has a terminal object, it has all limits. Moreover, Lemma 3 shows that the models are the functors preserving the terminal object and all pullbacks, hence they are the functors preserving all finite limits.

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \simeq [\mathcal{S}_{\mathsf{GSeTT}}, \mathbf{Set}]_{\mathrm{flim}}$$

Since **Set** is finitely complete, the result is then given by Corollary 14. □

3 The Grothendieck-Maltsiniotis definition of weak ω-categories

This entire section is a quick presentation of the definition of weak ω-categories given by Maltsionitis [20], relying on the ideas for defining weak ω-groupoids introduced by Grothendieck [13]. The aim is to introduce the notions that the type theory CaTT relies on, as well as the notations we will use for these notions. For a more in-depth study of this definition, one can refer to the original article by Maltsionitis [20] or by a full account of this definition by Ara [2]

3.1 Globular extensions

Globular sums. Let $\mathcal{C}$ be a category equipped with a functor $F : \mathcal{G} \to \mathcal{C}$ (we will sometimes call such a functor a *globular structure* on $\mathcal{C}$). We denote respectively by D_n, σ_n and τ_n the images via F of $[n]$, $\boldsymbol{\sigma}_n$ and $\boldsymbol{\tau}_n$. When there is no ambiguity, we may write σ and τ, leaving the index implicit, moreover, we write also σ (resp. τ) to indicate a composite of maps of the form σ (resp. τ). In the category $\mathcal{C}$, a *globular sum* is a colimit of a diagram of the form

$$\begin{tikzcd}[column sep=small]
D_{i_1} & & D_{i_2} & & & & & D_{i_k} \\
& D_{j_1} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & D_{j_2} \arrow[ul, "\tau"] & \cdots & D_{j_{k-1}} \arrow[urr, "\sigma"'] & &
\end{tikzcd}$$

It will be useful to encode such a colimit by its *table of dimensions*

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_k \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix}$$

Dually, if a category $\mathcal{C}$ is endowed with a contravariant functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ (a *contravariant globular structure*), we will denote respectively by D_n, s_n and t_n the images by F of $[n]$, $\boldsymbol{\sigma}_n$ and $\boldsymbol{\tau}_n$. We call a *globular product* a limit of the diagram of the form

$$\begin{tikzcd}[column sep=small]
D_{i_1} \arrow[dr, "t"] & & D_{i_2} \arrow[dl, "s"] \arrow[dr, "t"] & & & & & D_{i_k} \arrow[dll, "s"] \\
& D_{j_1} & & D_{j_2} & \cdots & D_{j_{k-1}} & &
\end{tikzcd}$$

It will also be convenient to denote it by its table of dimensions, and the variance of the globular structure will distinguished between globular sums and globular products.

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_k \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix}$$

If $\mathcal{C}$ has a globular structure and $\mathcal{D}$ has a contravariant globular structure, we will say that a globular sum in $\mathcal{C}$ and a globular product in $\mathcal{D}$ are *dual* to each other if they share the same table of dimensions.

Globular extensions. A category $\mathcal{C}$ with a globular structure F is called a *globular extension* when all the globular sums exist in $\mathcal{C}$. Given two globular extensions $F : \mathcal{G} \to \mathcal{C}$ and $G : \mathcal{G} \to \mathcal{D}$, a morphism of globular extensions is a functor $H : \mathcal{C} \to \mathcal{D}$ such that $H \circ F = G$, and preserving the globular sums. Dually, a category with a contravariant globular structure that has all globular products is called a *globular coextension*, and the opposite notion of morphisms defines morphisms of coglobular extensions.

The category Θ_0. There is a universal globular extension Θ_0, which is called a *globular completion.* It is the initial object in the category of globular extensions, or equivalently it is characterized by the fact that for any globular extension $\mathcal{G} \to \mathcal{C}$, there is a unique morphism of globular extensions $\Theta_0 \to \mathcal{C}$. Note that if Θ_0 is a globular completion, then Θ_0^{op} is a *globular cocompletion*, that is for every globular coextension $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, there is a unique morphism $\Theta_0^{\mathrm{op}} \to \mathcal{C}$. The objects of the category Θ_0 are called *pasting schemes*, they are all obtained as a globular sum in Θ_0

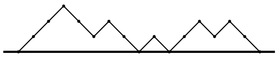
Combinatorial description of Θ_0. The pasting schemes can be defined inductively. For this we need an operation Σ which given a globular set X produces the globular set ΣX defined by $(\Sigma X)_0 = \{*_s, *_t\}$ and $(\Sigma X)_{n+1} = X_n$, with for all $x \in X_0$, $s(x) = *_s$ and $t(x) = *_t$ in ΣX. This operation is to be compared with the topological notion of suspension. This lets us formulate the following equivalent characterization of the objects of Θ_0:

Definition 16. A pasting scheme is either the globular set D^0, or it is obtained as $\Sigma X_1 \bowtie \Sigma_2 \bowtie \ldots \bowtie \Sigma X_n$, where $X_1, \ldots, X_n$ is a list of pasting schemes, and $(\Sigma Y) \bowtie (\Sigma Z)$ is obtained by taking dimension-wise the disjoint union of ΣY and ΣZ, and identifying $*_s$ from ΣY and $*_t$ from ΣZ.

- *Batanin Tree* [3] There is a bijection between the object of Θ_0 and the trees. This essentially comes down to the fact that trees satisfy the same inductive definition as pasting schemes : a tree is either a leaf or a list of trees. The correspondence between trees and pasting schemes is well known, and stronger results than a mere bijection have been proved [2, 6, 14]

- *Dyck word* : they are words on the alphabet $\{(,)\}$ which correspond to good parenthesizing of an expression, for instance (() is not a Dyck work since there is a mismatch, whereas (()(())) is a Dyck word.

- *Non decreasing parking functions* : these are non decreasing functions $f : \{0, \ldots, n\} \to \{0, \ldots, n\}$ such that $f(0) = 0$ and $f(n) = n$. We can picture these as diagrams like the following



There are a lot of other combinatorial descriptions, as these are all entities that entertain close relation with Catalan numbers, which are extremely ubiquitous.

Source and target of pasting schemes. A pasting scheme X naturally comes equipped with a source and a target, that are two distinguished sub globular sets of X which are also pasting schemes. Since the source and target are isomorphic globular sets, we will define a unique object ∂X along with the two inclusions which identify ∂X as a subobject of X

$$\sigma_X, \tau_X : \partial X \to X$$

We first define the pasting scheme ∂X to be given by the table

$$\begin{pmatrix} \overline{i_1} & & \overline{i_2} & & \cdots & & \overline{i_k} \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix} \qquad \text{where } \overline{i_k} = \begin{cases} i_k \text{ if } i_m < i \\ i-1 \text{ if } i_m = i \end{cases}$$

Note that this definition may produce tables that do not strictly fall under the scope of globular sums, as presented before, since it is possible to have the equality

$$\overline{i_m} \quad = \quad j_m \quad = \quad \overline{i_{m+1}} \quad = \quad i-1$$

However when it is the case we will chose the corresponding iterated sources and target to be the identity maps (i.e., the map iterated 0 times). We can then introduce the following rewriting rule, that does not change the colimit and thus exhibits ∂X as a pasting scheme

$$\begin{pmatrix} \cdots & i-1 & & i-1 & \cdots \\ \cdots & & i-1 & & \cdots \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} \cdots & i-1 & \cdots \\ \cdots & & \cdots \end{pmatrix}$$

Now we can define the two inclusion maps σ_X and τ_X to induced by the families

$$\overline{\sigma_{i_m}} : D_{\overline{i_m}} \longrightarrow D_{i_m} \qquad\qquad \overline{\tau_{i_m}} : D_{\overline{i_m}} \longrightarrow D_{i_m}$$

$$\overline{\sigma_{i_m}} = \begin{cases} \mathrm{id}_{D_{i_m}} & \text{if } i_m < i \\ \sigma : D_{i-1} \to D_i & \text{if } i_m = i \end{cases} \qquad \overline{\tau_{i_m}} = \begin{cases} \mathrm{id}_{D_{i_m}} & \text{if } i_m < i \\ \tau : D_{i-1} \to D_i & \text{if } i_m = i \end{cases}$$

Globular theories. Let $\mathcal{G} \to \mathcal{C}$ be a globular extension, then by universality of the globular completion, there exists a unique morphism of globular theories $F : \Theta_0 \to \mathcal{C}$. $\mathcal{G} \to \mathcal{C}$ is called a *globular theory* if the functor induced by F is faithful and is an isomorphism on the isomorphism classes of objects. Whenever it is the case, we can up to equivalence identify Θ_0 as a subcategory of $\mathcal{C}$. A *morphism of globular theory* is just a morphism of the underlying globular extensions. An arrow f of a globular theory $\mathcal{C}$ is said to be *globular* if it is in Θ_0. Dually a globular coextension $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ is called a *globular cotheory* if $\mathcal{C}^{\mathrm{op}}$ is a globular theory.

3.2 Weak ω-categories

Admissible pair of arrows. Let $\mathcal{G} \to \mathcal{C}$ be a globular extension, two arrows $f, g : D_i \to X$ in $\mathcal{C}$ are said to be *parallel* when

$$f \circ \sigma_i \quad = \quad g \circ \sigma_i \qquad\qquad f \circ \tau_i \quad = \quad g \circ \tau_i$$

If $\mathcal{C}$ is a globular theory, then an arrow f of $\mathcal{C}$ is said to be *algebraic*, when for every decomposition $f = gf'$, with g globular, then g is an identity. A pair of parallel arrows $f, g : D_i \to X$ is called an *admissible pair* if either both f and g are algebraic, or there exists a decomposition $f = \sigma_X f'$ and $g = \tau_X g'$, with f' and g' algebraic.
Dually, in a globular cotheory $\mathcal{C}$, an arrow is said to be *coalgebraic* if its opposite is algebraic in the globular theory $\mathcal{C}^{\mathrm{op}}$, and a pair of arrows is called a *coadmissible pair* if the opposite pair is admissible in $\mathcal{C}^{\mathrm{op}}$.

Cat-coherator. We will introduce here the Batanin-Leinster cat-coherator, since it is the one we will be using for our type theory. For a more general definition of cat-coherators, as well as other examples, see [20]. For the rest of this paper, we will simply say cat-coherator to refer to the Batanin-Leinster cat-coherator. The cat-coherator Θ_∞ is defined to be the colimit

$$\Theta_\infty \quad \simeq \quad \mathrm{colim}(\Theta_0 \to \Theta_1 \to \Theta_2 \to \cdots \to \Theta_n \to \cdots)$$

Where Θ_n is given by induction on n. Define E_n to be the set of all pairs of admissible arrows of Θ_n that are not in E'_n for any $n' < n$. Then we can define Θ_{n+1} to be the universal globular extension of Θ_n obtained by formally adding a lift for each pairs in E_n. In other words, for each globular extension $f : \Theta_n \to \mathcal{C}$ such that the image by f of all pairs of arrows in E_n has a lift in $\mathcal{C}$, there is an essentially unique globular extension $\tilde{f}$, which makes the following triangle commute

$$\begin{tikzcd} \Theta_n \arrow[r] \arrow[dr, "f"'] & \Theta_{n+1} \arrow[d, dotted, "\tilde{f}"] \\ & \mathcal{C} \end{tikzcd}$$

Weak ω-categories. We define a weak ω-category to be functor $F : \Theta_\infty^{\mathrm{op}} \to \mathbf{Set}$ which sends globular sums in $\Theta_\infty^{\mathrm{op}}$ to their dual for the globular structure on $\mathbf{Set}$ induced by F. The category $\omega-\mathbf{Cat}$ of weak ω-categories is the full subcategory of $\widehat{\Theta_\infty}$ whose objects are exactly the presheaves that are weak ω categories.

3.3 Identity and composition

We work out the definition of the identity 1-cell and the composition of 1-cells in weak ω-categories. They are the most basic features and we reserve more advanced examples for Section 5.2 where they are given in a type theoretic style. We refer the reader to [20, 2] for more examples in this style.

- Identity of 0-cell: Consider the pair of maps $(\mathrm{id}_{D^0}, \mathrm{id}_{D^0})$ is an admissible pair of arrows $D^0 \to D^0$, hence there exists a lift

$$\begin{tikzcd} D^1 \arrow[dr, dotted, "\iota"] & \\ D^0 \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift left, "\mathrm{id}_{D^0}"] \arrow[r, shift right, "\mathrm{id}_{D^0}"'] & D^0 \end{tikzcd}$$

For every weak ω-category $\mathcal{F} : \Theta_\infty^{\mathrm{op}} \to \mathbf{Set}$ together with an element $x \in \mathcal{F}(D^0)$, this allows us to define its *identity* 1-*cell* $i(x) \in (D^1)$ to be $i(x) = \mathcal{F}(\iota)(x)$. Moreover, by definition, $s(i(x)) = t(i(x)) = x$ as expected for the identity 1-cell on x.

- Composition of 1-cell: We consider the globular sum given as $D^1 \coprod_{D^0} D^1$. Then there are two canonical maps $\iota_1, \iota_2 : D^1 \to D^1 \coprod_{D^0} D^1$, and we consider the following admissible pair $(\iota_1\sigma, \iota_2\tau) : D^0 \to D^1 \coprod_{D^0} D^1$. This provides the lift

$$\begin{tikzcd} D^1 \arrow[dr, dotted, "c"] & \\ D^0 \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift left, "\iota_1\sigma"] \arrow[r, shift right, "\iota_2\tau"'] & D^1 \coprod_{D^0} D^1 \end{tikzcd}$$

For every weak ω-category $\mathcal{F} : \Theta_\infty^{\mathrm{op}} \to \mathbf{Set}$, a pair of composable 1-cell is the same as an element $(f, g) : \mathcal{F}(D^1 \coprod_{D^0} D^1)$, and the element $f \cdot g := \mathcal{F}(c)(f, g) \in \mathcal{F}(D^1)$ defines the composition. By definition, $s(f \cdot g) = s\ (f)$ and $t(f \cdot g) = t(g)$, as it is expected for the composition.

4 Limits and globular products in $\mathcal{S}_{\mathsf{GSeTT}}$

The aim of this section is to study the limits in the category $\mathcal{S}_{\mathsf{GSeTT}}$, and in particular the globular products. Theorem 12 shows an explicit equivalence of categories with $\mathbf{FinGSet}^{\mathrm{op}}$ which guarantees the existence of limits, and carrying out explicitly across the equivalence gives a characterization of the globular products as contexts however the one we present here is more refined.

4.1 Globular category with families.

We introduce the notion of a globular category with families in order to transfer the results about limits in $\mathcal{S}_{\mathsf{GSeTT}}$ to other categories with families along a functor. This justifies studying limits directly in the category $\mathcal{S}_{\mathsf{GSeTT}}$, which is the simpler case, without loss of generality.

A category with family $\mathcal{C}$ is called *globular* if it is equipped with a functor $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ that sends all maps in $\mathcal{G}^{\mathrm{op}}$ onto display maps. A *morphism* between two globular categories $\mathcal{C}$,$\mathcal{D}$ with families is a morphism of category $F : \mathcal{C} \to \mathcal{D}$ with families that commutes with the structural functors $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ and $\mathcal{G}^{\mathrm{op}} \to \mathcal{D}$. We call a diagram in a globular category with families $\mathcal{C}$ a $\mathcal{G}$-*diagram* if it factors through the map $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, and a limit of a $\mathcal{G}$-diagram is called a $\mathcal{G}$-*limit*

Lemma 17. *A globular category with families $\mathcal{C}$ has all finite $\mathcal{G}$-limits.*

Proof. It suffices to prove that $\mathcal{C}$ has a terminal object and all the $\mathcal{G}$-pullbacks. By definition of a category with families, $\mathcal{G}$ has a terminal object, and moreover, consider a $\mathcal{G}$-cospan in $\mathcal{C}$

$$\begin{tikzcd} & c \arrow[d] \\ c' \arrow[r] & d \end{tikzcd}$$

the map $c \to d$ is a display map in the category with families $\mathcal{C}$, hence the pullback along this map exists. ☐

Lemma 18. *A morphism of globular category with families preserves all finite $\mathcal{G}$-limits.*

Proof. It suffices to prove that such a morphism preserves terminal object and the $\mathcal{G}$-pullbacks. This holds since by definition a morphism of category with families preserves terminal object and pullbacks along display maps, and that $\mathcal{G}$-pullbacks are in particular pullbacks along display maps. ☐

These two lemmas show that the category $\mathcal{S}_{\mathsf{GSeTT}}$ that we had constructed is in fact initial, in a bi-categorical sense, that is for any globular category with families $\mathcal{C}$, there exists a essentially unique map $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$.

4.2 Ps-contexts

We give a syntactic characterization of all those contexts that correspond to globular sets that are pasting schemes. Such a context is called a *ps-context*. Let us introduce two new judgments, along with their intuitive meaning

$\Gamma \vdash_{\mathsf{ps}}$	Γ is a ps-context
$\Gamma \vdash_{\mathsf{ps}} x : A$	Γ is a partial ps-context, with dangling variable x of type A

The inference rules to derive such judgments express the idea that a ps-context is obtained by freely gluing a higher cell on the dangling cell (PSE). But this requires also requires to keep track of the dangling cell which why we need the rule (PSD). The two other rules are only used to start and end the construction.

$$\frac{}{(x:\star) \vdash_{\mathsf{ps}} x : \star}\,(\text{PSS}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f x \underset{A}{\to} y \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}\,(\text{PSE})$$

$$\frac{\Gamma \vdash f : x \underset{A}{\to} y}{\Gamma \vdash_{\mathsf{ps}} y : A}\,(\text{PSD}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}\,(\text{PS})$$

The aim is now to verify that the contexts Γ satisfying $\Gamma \vdash_{\mathsf{ps}}$ correspond exactly to those globular sets that are pasting schemes ins the syntactic category. Let us first prove the easier result that these rules effectively produces only valid contexts

Lemma 19. *The following rule is admissible*

$$\frac{\Gamma \vdash_{\mathsf{ps}}}{\Gamma \vdash}$$

Proof. We first prove by induction that the following rule is admissible

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash x : A}$$

- For a judgment obtained with the rule (PSS) $x : \star \vdash_{\mathsf{ps}} x : \star$, one can check that the judgment $x : \star \vdash x : \star$ is also derivable.

- For a judgment obtained with the rule (PSE) $\Gamma, y : A, f : x \underset{A}{\to} y$, assuming a derivation for the judgment $\Gamma \vdash_{\mathsf{ps}} x : A$. By induction, this gives a derivation for the judgment $\Gamma \vdash x : A$ and by the Lemma 6, this proves in particular that $\Gamma \vdash A$. Since moreover $y \notin \mathrm{Var}(\Gamma)$, this proves by the rule (CE) that $\Gamma, y : A \vdash$, and the rule (AX) along with an application of the rule (WK) show that $\Gamma, y : A \vdash y : A$ and $\Gamma, y : A \vdash x : A$, so the rule (ARR) applies to show $\Gamma, y : A \vdash x \underset{A}{\to} y$. A second application of the rule (CE), justified by the fact that $f \notin \mathrm{Var}(\Gamma)$ proves then that $\Gamma, y : A, f : x \underset{A}{\to} y \vdash$, and finally, applying (AX) gives a derivation for the judgment $\Gamma, y : A, f : x \underset{A}{\to} y \vdash f : x \underset{A}{\to} y$

- For a judgment obtained by application of the rule (PSD) $\Gamma \vdash_{\mathsf{ps}} y : A$, assuming that $\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y$, by induction we have a derivation for the judgment $\Gamma \vdash f : x \underset{A}{\to} y$ and by applying twice the Lemma 6, this gives a derivation for the judgment $\Gamma \vdash y : A$

Now, a judgment of the form $\Gamma \vdash_{\mathsf{ps}}$ is necessarily derived by the rule (PS), assuming $\Gamma \vdash_{\mathsf{ps}} x : \star$. By the above property, this gives a derivation for the judgment $\Gamma \vdash x : \star$, which, by the Lemma 6 gives a derivation for the judgment $\Gamma \vdash$ $\square$

4.3 PS-contexts as globular products.

Ps-contexts are globular sums.

Proposition 20. *A ps-context $\Gamma \vdash_{\mathsf{ps}}$ is a globular product in the category $\mathcal{S}_{\mathsf{GSeTT}}$*

Proof. We prove by induction that whenever $\Gamma \vdash_{\mathsf{ps}} x : A$, then Γ is the limit of a diagram

$$\begin{tikzcd}
D^{i_0} \arrow[dr] & & D^{i_1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & D^{i_m} \arrow[dl] \arrow[dr] & \\
& D^{j_0} & & D^{j_1} \quad \cdots & & D^{j_{m-1}} & & D^{\dim x}
\end{tikzcd}$$

where the canonical map $\Gamma \to D^{\dim x}$ is the term x in Γ. Note that additionally in this diagram, the map $D^{i_m} \to D^{j_m}$ is allowed to be the identity.

- For the rule (PSS): The only context which can be produced is the context $(x : \star) \simeq D^0$, which is the limit of the diagram D^0.

- For the rule (PSD): Suppose that $\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y$, and Γ is given by a limit of the prescribed form. Then since y is the target of f in Γ, there is a commutative triangle

$$\begin{tikzcd} \Gamma \arrow[d, "f"] \arrow[dr, "y"] & \\ D^{\dim f} \arrow[r, "t"'] & D^{\dim y} \end{tikzcd}$$

Hence the assumed limiting cone of Γ induces a cone over the diagram

$$\begin{tikzcd}[column sep=small] D^{i_0} \arrow[dr] & & D^{i_1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & D^{i_m} \arrow[dl] \arrow[dr] & \\ & D^{j_0} & & D^{j_1} \quad \cdots & & D^{j_{m-1}} & & D^{\dim y} \end{tikzcd}$$

where the rightmost map $\Gamma \to D^{\dim y}$ is the term y. Moreover, this cone is limiting, since the original cone we started with is limiting and the two diagrams induce same limits.

- For the rule (PSE): Suppose given a context $\Gamma \vdash_{\mathsf{ps}} x : A$ which is a limit of the prescribed form, and consider $\Gamma' = (\Gamma, y : A, f : x \to y)$ such that $\Gamma' \vdash_{\mathsf{ps}} f : x \to y$. We construct a cone of apex Γ' over the following diagram, by considering the cone of Γ and adding the term f as the rightmost arrow to $\dim f$

$$\begin{tikzcd}[column sep=small] D^{i_0} \arrow[dr] & & D^{i_1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & D^{i_m} \arrow[dl] \arrow[dr] & & D^{\dim f} \arrow[dl] \\ & D^{j_0} & & D^{j_1} \quad \cdots & & D^{j_{m-1}} & & D^{\dim x} & \end{tikzcd}$$

This defines a cone, since the source of f is x, which is the chosen arrow $\Gamma' \to D^{\dim x}$, by induction hypothesis. We show that this cone is limiting. A substitution $\Delta \to (\Gamma, y : A, f : x \to y)$ is given by a substitution $\gamma : \Delta \to \Gamma$, together with a term $\Delta \vdash t_y : A[\gamma]$ and a term $\Delta \vdash t_f : x[\gamma] \to t_y$. By induction γ induces a morphism of cone of apex Δ over the diagram of Γ, and by choosing the rightmost arrow to be t_f, this completes into a cone over the diagram of Γ' which is well defined since the source of t_f is $x[\gamma]$. The substitution $\langle \gamma, t_y, t_f \rangle$ then defines by definition a morphism of cones. Conversely, consider a morphism of cones over the diagram of Γ' of the form $\Delta \to \Gamma'$, then it induces a morphism of cones over the diagram of Γ of the form $\Delta \to \Gamma$, which by induction defines a substitution $\Delta \vdash \gamma : \Gamma$. Denote t_f the rightmost arrow of the cone of apex Δ over the diagram of Γ', and t_y to be the target of the term t_f. Then necessarily, t_y is parallel to the source of t_f, which is $x[\gamma]$, in other words, we have $\Delta \vdash t_y : A[\gamma]$. Moreover, by definition, we have $\Delta \vdash t_f : x[\gamma] \to t_y$, which proves that the morphism of cone defines a substitution $\Delta \vdash \langle \gamma, t_y, t_f \rangle : \Gamma'$. Hence the diagram we picked is a limiting diagram. Either this diagram suffices to prove the induction step, or it does not in which case the map $D^{i_m} \to D^{\dim x}$ is an identity. In this case, one can simply erase the identity, and construct the following diagram

$$\begin{tikzcd}[column sep=small]
D^{i_0} \arrow[dr] & & D^{i_1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & D^{\dim f} \arrow[dl] \\
& D^{j_0} & & D^{j_1} & \cdots & D^{j_{m-1}} &
\end{tikzcd}$$

Which is a diagram of the required form, for which Γ' is also a limit.

$\square$

Explicit computation. Given a context $\Gamma = (x_1, \dots, x_{2n+1})$ along with a derivation of $\Gamma \vdash_{\mathsf{ps}}$, we will show that Γ can be written as a globular product, by induction on the derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$. Notice that the derivation of $\Gamma \vdash_{\mathsf{ps}}$ is necessarily a sequence of applications of the rules (PSE) and (PSD), starting with an application of the rule (PSS), and ending with the rule (PS). Let $k_1, \cdots k_m$ and $l_1, \dots, l_m$ integers such that the derivation is of the form

$$\Gamma \vdash_{\mathsf{ps}} = (\text{PSS})(\text{PSE})^{k_1}(\text{PSD})^{l_1}(\text{PSE})^{k_2} \cdots (\text{PSE})^{k_m}(\text{PSD})^{l_m}(\text{PS})$$

Since the rule (PSE) increases the dimension of the dangling variable by 1 and the rule (PSD) decreases it by 1, in addition to (PSS) initiating with a dangling variable of dimension 0 and (PS) applying only with a dangling variable of dimension 0, and moreover the rule (PSE) extends the length of the context by 2, we necessarily have the following equalities

$$\sum_{i=0}^{m} l_i = \sum_{i=0}^{m} k_i = n$$

We now define two new sequences of indices $i_1, \dots, i_m$ and $j_1, \dots, j_m$ as being solutions of the following system of equation

$$\begin{cases} i_1 = k_1 \\ i_{q+1} = k_{q+1} + j_q \end{cases} \qquad j_q = i_q - l_q$$

A careful examination of the proof of Proposition 20 shows that in this case, Γ is the globular product whose dimension table is given by

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & j_1 & & j_2 & \cdots & j_{m-1} & \end{pmatrix}$$

Inverting the system. Conversely, consider a globular product Δ whose dimension table is given by

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & j_1 & & j_2 & \cdots & j_{m-1} & \end{pmatrix}$$

then by inverting the previous system, we define the numbers $k_1, \dots, k_m$ and $l_1, \dots, l_m$ as solutions of

$$\begin{cases} k_1 = i_1 \\ k_{n+1} = i_{n+1} - j_{n+1} \end{cases} \qquad \begin{cases} l_n = i_n - j_{n+1} \\ l_m = \sum_{n=1}^{m} k_n - \sum_{n=1}^{m-1} l_n \end{cases}$$

Then consider the ps-context Γ, whose derivation of $\Gamma \vdash_{\mathsf{ps}}$ is the following

$$\Gamma \vdash_{\mathsf{ps}} = (\text{PSS})(\text{PSE})^{k_1}(\text{PSD})^{l_1}(\text{PSE})^{k_2}\cdots(\text{PSE})^{k_m}(\text{PSD})^{l_m}(\text{PS})$$

This derivation is valid since $\sum k_i = \sum l_i$, moreover, it defines Γ up renaming. Then by our previous calculation, Γ is a globular product of the same dimension table as Δ, hence Δ and Γ are isomorphic. This shows that every globular product is isomorphic to a ps-context.

The category of ps-contexts. Let $\mathcal{S}_{\mathsf{PS}}$ be the full subcategory of $\mathcal{S}_{\mathsf{GSeTT}}$ whose objects are exactly the contexts Γ such that $\Gamma \vdash_{\mathsf{ps}}$ holds

Theorem 21. *The category $\mathcal{S}_{PS}$ is equivalent to Θ_0^{op}. More precisely, we have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{S}_{PS} & \hookrightarrow & \mathcal{S}_{GSeTT} \\ \wr\| & & \wr\| \\ \Theta_0^{\mathrm{op}} & \hookrightarrow & \mathbf{FinGSet}^{\mathrm{op}} \end{array}$$

Proof. The category Θ_0^{op} is obtained from $\mathcal{G}$ by formally adding all the colimits, it is the free cocompletion of $\mathcal{G}$ by globular sums. This can be explicitly constructed inside **FinGSet**, by taking the full subcategory whose objects are exactly the presheaves that are obtained as a globular sum. Since we have already proven that $\mathcal{S}_{\mathsf{GSeTT}} \simeq \mathbf{FinGSet}^{\mathrm{op}}$, it suffices to prove that under this correspondence, $\mathcal{S}_{\mathsf{PS}}$ is the full subcategory of $\mathcal{S}_{\mathsf{GSeTT}}$, whose objects exactly the contexts that correspond to globular sums, which is given by the Proposition 20 $\square$

There is in fact a slightly stronger result, since two isomorphic ps-contexts are necessarily equal up to renaming, the category $\mathcal{S}_{\mathsf{PS}}$ considered up to renaming and the category Θ^{op} are even isomorphic. Note however that ps-contexts are in normalized form, and although we have proved that every globular product is isomorphic to a ps-context, it is not true that every globular product is a ps-context. For instance, consider the two following contexts Δ and Γ

$$\Delta = (x:\star, y:\star, z:\star, f:x\to y, g:y\to z)$$
$$\Gamma = (x:\star, y:\star, f:x\to y, z:\star, g:y\to z)$$

These two contexts are isomorphic and Γ is a ps-context, hence both these contexts are globular product, however, Δ is not a ps-context.

Locally maximal variables. Let $\Gamma = (x_1, \ldots, x_n)$ be a ps-context, then a variable, then a variable x_i in Γ is said to be *locally maximal* if $\dim x_{i-1} < \dim x_i$ and $\dim x_{i+1} < \dim x_i$. These variables correspond exactly to those variables that where last added by an application of the rule (PSE) preceding an application of the rule (PSD) (with an exception if the pasting scheme only consists in a single variable).

Lemma 22. *Let $\Gamma \vdash_{\mathsf{ps}}$ be a ps-context, and $\Delta \vdash \sigma : \Gamma$, $\Delta \vdash \tau : \Gamma$ two substitutions. If for all variables of dimension locally maximal x in Γ, we have $x[\sigma] = x[\tau]$, then $\sigma = \tau$.*

Proof. This is a consequence of the decomposition of a ps-context Γ as a globular product. Indeed, the locally maximal variables correspond to the dimensions we denoted $i_0, \ldots, i_n$. An examination of the diagram shows that two cones over such a diagram sharing the same arrows going to the disks $D^{i_1} \to D^{i_n}$ are necessarily equal. $\square$

Source and targets. Ps-contexts correspond to pasting schemes, and the latter are equipped with notions of source and target, which are maps σ_X, τ_X in the category Θ_0. We provide a direct characterization of the maps corresponding map in the category $\mathcal{S}_{\mathsf{PS}}$ under the correspondence given by Theorem 21. Instead of providing the substitutions explicitly, we use the fact that σ_X, τ_X are monomorphisms of presheaves and hence epimorphisms in $\mathcal{S}_{\mathsf{GSeTT}}$, and rely on the variable names to present these substitutions in an efficient way. For instance, given the context $\Gamma = (x : \star, y : \star, f : x \to y)$, we have the following correspondence between subcontexts of Γ and epimorphisms in $\mathcal{S}_{\mathsf{GSeTT}}$

Subcontext	epimorphism
$(x : \star)$	$\Gamma \vdash \langle a : x \rangle (a : \star)$
$(y : \star)$	$\Gamma \vdash \langle a : y \rangle (a : \star)$

We define for all $i \in \mathbb{N}_{>0}$ the *i-source* of a ps-context Γ induction on the length of Γ, by setting $\partial_i^-(x : \star) = (x : \star)$ and

$$\partial_i^-(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^- \Gamma & \text{if } \dim A \ \geq i - 1 \\ \partial_i^- \Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

and similarly the *i-target* of Γ is defined by $\partial_i^+(x : \star) = (x : \star)$, and

$$\partial_i^+(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^+ \Gamma & \text{if } \dim A \ \geq i \\ \mathrm{drop}(\partial_i^+ \Gamma), y : A & \text{if } \dim A \ = i - 1 \\ \partial_i^+ \Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

where $\mathrm{drop}(\Gamma)$ is the context Γ with its last variable removed. One can check by induction on the derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$ that whenever Γ is a ps-context, both $\partial_i^- \Gamma$ and $\partial_i^+ \Gamma$ are also ps-contexts. It is straightforward in the case of the i-source, and for the i-target, it relies on the fact that whenever the drop operator is used, immediately afterwards a variable of the same type that the one that was removed is added.

We denote $\partial^-(\Gamma) = \partial^-_{\dim \Gamma - 1} \Gamma$ and $\partial^+(\Gamma) = \partial^+_{\dim \Gamma - 1} \Gamma$ and call these the *source* and *target* of Γ. Carrying explicitly the computations one can check that whenever a ps-context Γ define the pasting scheme X, the ps-contexts $\partial^-(\Gamma)$ and $\partial^+(\Gamma)$ both define the pasting scheme ∂X. Moreover the substitution that identify a variable in $\partial^-(\Gamma)$ (resp. in $\partial^+(\Gamma)$) with the same variable in Γ corresponds to the map $\sigma_x : \partial X \to X$ (resp. $\tau_x : \partial X \to X$).

Note that in the case of the ps-context $(x : \star)$ which is the only ps-context of dimension 0, the source and target are not defined.

5 Type theory for weak ω-categories

5.1 Coherences

In order to describe a type theory suitable to work with ω-category, one needs to extend the type theory for globular sets with new term constructors that will mirror the lifting that were formally added in the Grothendieck-Maltsiniotis definition of weak ω-categories.

Rules for coherences.

- operation coherences

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A \qquad \Delta \vdash \sigma : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma, t \xrightarrow[A]{} u}(\sigma) : t[\sigma] \xrightarrow[A[\sigma]]{} u[\sigma]} \text{(OP)}$$

  This rule applies under the extra assumptions that

$$\begin{cases} \mathrm{Var}(t) \cup \mathrm{Var}(A) = \mathrm{Var}(\partial^-(\Gamma)) \\ \mathrm{Var}(u) \cup \mathrm{Var}(A) = \mathrm{Var}(\partial^+(\Gamma)) \end{cases}$$

- equality cohences

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A \qquad \Delta \vdash \sigma : \Gamma}{\Gamma \vdash \mathsf{coh}_{\Gamma, t \xrightarrow[A]{} u}(\sigma) : t[\sigma] \xrightarrow[A[\sigma]]{} u[\sigma]} \text{(EQ)}$$

  This rule applies under the extra assumptions that

$$\begin{cases} \mathrm{Var}(t) \cup \mathrm{Var}(A) = \mathrm{Var}(\Gamma) \\ \mathrm{Var}(u) \cup \mathrm{Var}(A) = \mathrm{Var}(\Gamma) \end{cases}$$

Note that the rule (EQ) is slightly different than the ones presented in [12], to be made closer to the conditions of [20]. Although we have checked that changing this rule yields equivalent theories, we do not provide the proof here as it is very involved. A full account of this is given in [**?**]

Interpretation. These rules are to be understood as follows. A derivable judgment $\Gamma \vdash t : A$ such that $\mathrm{Var}(t) = \mathrm{Var}(\Gamma)$ can alternatively be pictured as a judgment $\Gamma \vdash t : A$ derived without ever using the weakening rule (WK). These can be thought of as the data of a full composition of the context Γ.

- For the rule (OP): Given a pasting scheme Γ and way to compose entirely its source and its target, this rule provides a way to compose entirely Γ.
- For the rule (EQ): Given two ways of composing entirely the pasting scheme Γ, the rule provides a cell that link them. This rule will be later proven to generate only invertible cells, and can then be thought of as: "any two ways of composing entirely a pasting scheme are weakly equivalent"

5.2 Some examples of derivations

We provide some examples of derivations that one may compute in CaTT, using the actual syntax implemented in [4]. A new coherence is introduced with the keyword `coh` and is followed by a name to identify it. Then comes a list of arguments which is the description of a ps-context followed by a column and a type. For instance the following line

```
coh id (x:*) : x -> x
```

defines a coherence called "id", which correspond to the construction $\mathsf{coh}_{(x:\star):x\to x}$. Further references to this coherence in order to produce a term have to include a substitution to the context $(x:\star)$, that is expressed as a list of arguments, for instance one may write `id y`. As will be justified by Lemma 24, we only specify the term in the substitution that correspond to locally maximal variable of the ps-context, for instance, considering the following coherence

```
coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z) : x -> z
```

one needs to write only `comp f g` instead of `comp x y f z g` when applying this coherence. Other examples of coherences one may define in CaTTinclude

- left unitality and its inverse

  ```
  coh unitl (x:*)(y:*)(f:x->y) : comp (id x) f -> f
  ```

  ```
  coh unitl- (x:*)(y:*)(f:x->y) : f -> comp (id x) f
  ```

- right unitality and its inverse

  ```
  coh unitr (x:*)(y:*)(f:x->y) : comp f (id y) -> f
  ```

  ```
  coh unitr- (x:*)(y:*)(f:x->y) : f -> comp f (id y)
  ```

- associativity and its inverse

  ```
  coh assoc (x:*)(y:*)(f:x->y)(z:*)(g:y->z)(w:*)(h:z->w) :
  comp f (comp g h) -> comp (comp f g) h
  ```

  ```
  coh assoc- (x:*)(y:*)(f:x->y)(z:*)(g:y->z)(w:*)(h:z->w) :
  comp (comp f g) h -> comp f (comp g h)
  ```

- vertical composition of 2-cells

  ```
  coh vcomp (x:*)(y:*)(f:x->y)(g:x->y)(a:f->g)(h:x->y)(b:g->h) :
  f -> h
  ```

- horizontal composition of 2-cells

  ```
  coh hcomp (x:*)(y:*)(f:x->y)(f':x->y)(a:f->f')
                 (z:*)(g:y->z)(g':y->z)(b:g->g') :
  comp f g -> comp f' g'
  ```

- left whiskering

  ```
  coh whiskl (x:*)(y:*)(f:x->y)(z:*)(g:y->z)(g':y->z)(b:g->g') :
  comp f g -> comp f g'
  ```

- right whiskering

```
coh whiskr (x:*)(y:*)(f:x->y)(f':x->y)(a:f->f')(z:*)(g:y->z) :
comp f g -> comp f' g
```

We also provide a syntax to work define terms in an arbitrary context, and not only coherences. The corresponding keyword is `let` followed with an identifier and a context, and equal and a full definition of the term in terms of previously defined term and coherences. For instance, the following term defines the squaring of an endomorphism

```
let sq (x:*)(f:x->x) = comp f f
```

5.3 Syntactic properties

From now on, we will use syntactic properties of the terms one can build in order to reason and prove various result about CaTT. Even though these properties are simple and for the most part natural to introduce, we will rely on them so strongly for all further results that it is worth dedicating some time for exploring them.

The first thing that can check about this theory is that the term constructor are nice enough, and do not break the good type theoretic property of GSeTT

Lemma 23. *All the properties cited in Lemma 6 still hold in CaTT, and every derivable judgment in CaTT has exactly one derivation.*

The syntactic category We denote $\mathcal{S}_{\mathsf{CaTT},\infty}$ the syntactic category of this theory, whose objects are the contexts Γ such that $\Gamma \vdash_{\mathsf{ps}}$ holds and maps $\Delta \to \Gamma$ are the substitutions $\Gamma \vdash \gamma : \Delta$. We also pose $\mathcal{S}_{\mathsf{PS},\infty}$ to be the full subcategory of $\mathcal{S}_{\mathsf{CaTT},\infty}$ whose objects are exactly the contexts Γ such that $\Gamma \vdash_{\mathsf{ps}}$ holds Note that $\mathcal{S}_{\mathsf{PS},\infty}$ has exactly the same objects as $\mathcal{S}_{\mathsf{PS}}$, However the morphisms are not the same. More specifically, there is a strict inclusion $\mathcal{S}_{\mathsf{PS}} \hookrightarrow \mathcal{S}_{\mathsf{PS},\infty}$ which is the identity on the objects.

Coherence depth In order to use induction arguments, we will need to introduce the notion of *depth* of a term, defined as follows:

$$\mathrm{depth}(v) = 0 \qquad\qquad \mathrm{depth}(\mathsf{coh}_{\Gamma,A}[\sigma]) = 1 + \max_{u\in\sigma} \mathrm{depth}(u)$$

This notion expresses exactly how many nested coherences are needed to write a given term. We will now consider the subcategory $\mathcal{S}_{\mathsf{CaTT},n}$, whose objects are the objects are the same as $\mathcal{S}_{\mathsf{CaTT}}$, but whose morphisms are substitution whose all terms are of depth at most n, and the full subcategory $\mathcal{S}_{\mathsf{PS},n}$ of $\mathcal{S}_{\mathsf{CaTT},n}$ whose objects are the contexts Γ such that $\Gamma \vdash_{\mathsf{ps}}$ holds.

This provides us with the following inclusions:

$$\begin{tikzcd}
\mathcal{S}_{\mathsf{GSeTT}} = \mathcal{S}_{\mathsf{CaTT},0} \arrow[r, hook] & \mathcal{S}_{\mathsf{CaTT},1} \arrow[r, hook] & \mathcal{S}_{\mathsf{CaTT},2} \arrow[r, hook] & \cdots \arrow[r, hook] & \mathcal{S}_{\mathsf{CaTT},\infty} \\
\mathcal{S}_{\mathsf{PS}} = \mathcal{S}_{\mathsf{PS},0} \arrow[r, hook] \arrow[u, hook] & \mathcal{S}_{\mathsf{PS},1} \arrow[r, hook] \arrow[u, hook] & \mathcal{S}_{\mathsf{PS},2} \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & \mathcal{S}_{\mathsf{PS},\infty} \arrow[u, hook] \\
\mathcal{G}^{\mathrm{op}} \arrow[u, hook] & & & &
\end{tikzcd}$$

Familial representability of types. While working in GSeTT we have proved various properties for which we noticed that the proof only depended on the structure of a type theory with the type formers, and it did not rely whatsoever on any assumption regarding the terms. These proposition can then transfer directly from the theory GSeTT to the CaTT freely, as the proof is exactly the same in both cases. Here we list the result that we will freely use in all the categories $\mathcal{S}_{\mathsf{CaTT},n}, (n \in \mathbb{N} \cup \{\infty\})$, as we have already proved them.

Lemma 24. *Let $n \in \mathbb{N}$, then the set of substitutions $\Gamma \vdash \gamma : \mathcal{S}^{n-1}$ is in bijective correspondence with derivable types of dimension $n-1$. The set of substitutions $\Gamma \vdash \gamma : \mathcal{D}^n$ is in bijective correspondence with derivable terms of dimension n.*

More explicitly, the substitution $\Gamma \vdash \gamma : \mathcal{S}^{n-1}$, corresponds to the type $\Gamma \vdash A_{n+1}[\gamma]$, and the substitution $\Gamma \vdash \gamma : \mathcal{D}^n$ corresponds to the term $\Gamma \vdash x_{2n}[\gamma] : A_{n+1}[\gamma]$.

This lemma is the version with coherences of Lemma 10, and it lets us use the same abuse of notations. We denote $\Gamma \vdash A : \mathcal{S}^{n-1}$ for the substitution associated to the type A, and $\Gamma \vdash t : \mathcal{D}^n$ for the substitution associated to the term t.

Globular products in $\mathcal{S}_{\mathsf{CaTT},\infty}$. We now transport the computation of ps-contexts as globular products in the category $\mathcal{S}_{\mathsf{GSeTT}}$ along the embedding functor to the category $\mathcal{S}_{\mathsf{CaTT},\infty}$.

Proposition 25. *The ps-contexts are the globular products in the category $\mathcal{S}_{\mathsf{CaTT},\infty}$.*

Proof. The inclusion functor $\mathcal{S}_{\mathsf{GSeTT}} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$ induces a structure of globular category with families on $\mathcal{S}_{\mathsf{CaTT},\infty}$, which makes this functor into a morphism of globular category with families, hence by Lemma 18 it preserves all the finite $\mathcal{G}$-limits, and it particular the globular products. $\square$

Corollary 26. *Let $\Gamma \vdash_{\mathsf{ps}}$ be a ps-context, and $\Delta \vdash \sigma : \Gamma$, $\Delta \vdash \tau : \Gamma$ two substitutions. If for all variables of dimension locally maximal x in Γ, we have $x[\sigma] = x[\tau]$, then $\sigma = \tau$.*

This justifies leaving argument implicit in the coherences. Indeed, what we call here arguments of a coherence are the terms composing a substitution from the ambient context to the ps-context defining this coherence. But by the above lemma, such a substitution is characterized by the image of the locally maximal variables in this ps-context. Hence it suffices to give these terms, and the entire substitution can be inferred from them. This is the reason why we allow ourselves to write for instance $(x : \star) \vdash \texttt{comp}\ (\texttt{id}\ x)\ (\texttt{id}\ x) : x \to x$ instead of $(x : \star) \vdash \texttt{comp}\ x\ x\ (\texttt{id}\ x)\ x (\texttt{id}\ x) : x \to x$

6 The syntactic categories associated to CaTT

6.1 The category of ps-contexts with coherences

The goal is to show that the category $\mathcal{S}_{\mathsf{PS},\infty}$ is in fact isomorphic to the category $\Theta_\infty^{\mathrm{op}}$. We will show that for all integer n, there is an isomorphism $\mathcal{S}_{\mathsf{PS},n} \simeq \Theta_n^{\mathrm{op}}$, and then take the colimit.

Coadmissible pairs substitutions. The category $\mathcal{S}_{\mathsf{CaTT},n}$, $(n \in \overline{\mathbb{N}})$ is a globular cotheory. In this category a morphism $\Gamma \vdash \gamma : \mathcal{D}^n$ is coalgebraic if for every decomposition $\gamma = \gamma' \circ \delta$ with δ a substitution in $\mathcal{S}_{\mathsf{GSeTT}}$, then δ is an identity. Note that in $\mathcal{S}_{\mathsf{GSeTT}}$ substitutions are generated by the weakenings and the renamings of all the variables (α-conversions). An identity in $\mathcal{S}_{\mathsf{GSeTT}}$ is precisely an α-conversion, thus γ is coalgebraic if it does not make any use of the weakening, or alternatively, if it contains all the variables of Δ. Note that by the lemma 24, such a substitution γ corresponds to a term $t = x_{2n}[\gamma]$ in Γ, of type $B = A_n[\gamma]$. Moreover, we have $\mathrm{Var}(\gamma) = \mathrm{Var}(t) \cup \mathrm{Var}(B)$. This proves that coalgebraic substitution are exactly the terms

$$\Gamma \vdash t : B \quad , \quad \text{with } \mathrm{Var}(\Gamma) = \mathrm{Var}(t) \cup \mathrm{Var}(B)$$

A pair of substitutions $(\gamma, \delta) : \Delta \to \Gamma$ is a coadmissible pair if either γ and δ are algebraic, or they factor as $\partial^-(\Delta) \vdash \gamma' : \Gamma$ and $\partial^+(\Delta) \vdash \delta' : \Gamma$, with γ' and δ' algebraic. Thus pairs of coadmissible substitutions are exactly the pairs of terms

$$\Gamma \vdash t : B \quad , \quad \Gamma \vdash u : B \qquad \text{with} \begin{cases} \mathrm{Var}(\Gamma)=\mathrm{Var}(t) \cup \mathrm{Var}(B) \\ \mathrm{Var}(\Gamma)=\mathrm{Var}(u) \cup \mathrm{Var}(B) \end{cases}$$

$$\text{or}$$

$$\partial^-(\Gamma) \vdash t : B \quad , \quad \partial^+(\Gamma) \vdash u : B \qquad \text{with} \begin{cases} \mathrm{Var}(\partial^-(\Gamma))=\mathrm{Var}(t) \cup \mathrm{Var}(B) \\ \mathrm{Var}(\partial^+(\Gamma))=\mathrm{Var}(u) \cup \mathrm{Var}(B) \end{cases}$$

Tower of definition. The following theorem is a consequence

Theorem 27. *There is an equivalence of categories $\Theta_n^{\mathrm{op}} \simeq \mathcal{S}_{\mathsf{PS},n}$*

Proof. The sequence Θ_n is defined to be such that for every n, $\Theta_n \to \Theta_{n+1}$ is a morphism of globular theories, and Θ_{n+1} is obtained from Θ_n by formally adding a lifting for every pair of admissible arrows that do not have a lifting in Θ_n. We will show that the dual statement exactly characterizes the extensions $\mathcal{S}_{\mathsf{PS},n} \to \mathcal{S}_{\mathsf{PS},n+1}$.

We have already proven that $\mathcal{S}_{\mathsf{PS},0} \simeq \Theta_0^{\mathrm{op}}$, thus $\mathcal{G} \to \mathcal{S}_{\mathsf{PS},0}$ is a globular cocompletion. Note that for all $n \in \mathbb{N}$, the functor $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT},n}$ induces a globular structure on $\mathcal{S}_{\mathsf{CaTT},n}$, which makes the functor into a morphism of globular categories with families. Hence $\mathcal{S}_{\mathsf{CaTT},n}$ has all globular products, and the functor $\mathcal{S}_{\mathsf{CaTT},0} \to \mathcal{S}_{CaTT,n}$ preserves these globular products. Restricting this functor to $\mathcal{S}_{\mathsf{PS},0} \to \mathcal{S}_{\mathsf{PS},n}$ shows that $\mathcal{S}_{\mathsf{PS},n}$ is a globular coextension. Moreover, the morphism $\mathcal{S}_{\mathsf{PS},n} \to \mathcal{S}_{ps,n+1}$ also preserves the globular products, and hence is a morphism of globular coextension. It is also immediate by construction that $\mathcal{S}_{\mathsf{PS},n}$ and $\mathcal{S}_{\mathsf{PS},0}$ have the same objects and that the functor $\mathcal{S}_{\mathsf{PS},0} \to \mathcal{S}_{\mathsf{PS},n}$ is faithful, and thus $\mathcal{S}_{\mathsf{PS},0} \to \mathcal{S}_{\mathsf{PS},n}$ is a globular cotheory.

A pair of coadmissible substitution from Γ correspond to two parallel terms $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$, where the coadmissibility condition corresponds exactly to the situation where either rule (OP) or rule (EQ) applies, and a lift for these substitutions correspond exactly to a term of type $t \to u$. Hence by definition $\mathcal{S}_{ps,n+1}$ is obtained from $\mathcal{S}_{ps,n}$ by formally adding a lift to each pair of coadmissible substitutions t, u such that t or u is of depth n, so define F_n to be the set of coadmissible pairs t, u such that t or u is of depth n. Consider a pair (t, u) with t or u of depth n, then for $n' < n$, the pair (t, u) do not belong to F'_n, since F'_n has pairs of the category $\mathcal{S}_{\mathsf{PS},n'}$, whose coherences are bounded by n'. Conversely, consider a pair (t,u) of admissible pairs in $\mathcal{S}_{ps,n}$ that do not belong to $F_{n'}$ for any $n' < n$. Since these $F_{n'}$ describe all admissible pairs (t,u) such that both depths of t and u are strictly lower than n, this implies that t or u is of depth n, and hence $(t,u) \in F_n$. This proves that F_n can be alternatively described as the set of coadmissible pairs in $\mathcal{S}_{\mathsf{PS},n}$ that are not in $F_{n'}$ for any $n' < n$. This is exactly the dual of the inductive definition of Θ_n, hence $\mathcal{S}_{ps,n} \simeq \Theta_n^{\mathrm{op}}$ □

Unbounded coherences. Consider now the category $\mathcal{S}_{\mathsf{PS},\infty}$, where we let the coherence depths be arbitrarily large.

Theorem 28. *There is an equivalence of categories $\mathcal{S}_{\mathsf{PS},\infty} \simeq \Theta_\infty^{\mathrm{op}}$*

Proof. By construction $\mathcal{S}_{\mathsf{PS},\infty}$ is obtained as the colimit of the inclusions of categories

$$\mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{PS},0} \to \mathcal{S}_{\mathsf{PS},1} \to \cdots \to \mathcal{S}_{\mathsf{PS},n} \to \cdots \to \mathcal{S}_{\mathsf{PS},\infty} = \mathrm{colim}_n\, \mathcal{S}_{\mathsf{PS},n}$$

and using the equivalence proven in 27, this translates to the following colimit

$$\mathcal{G}^{\mathrm{op}} \to \Theta_0^{\mathrm{op}} \to \Theta_1^{\mathrm{op}} \to \cdots \to \Theta_n^{\mathrm{op}} \to \cdots \to \mathcal{S}_{\mathsf{PS},\infty} = \mathrm{colim}_n\, \Theta_n^{\mathrm{op}}$$

□

6.2 Generating display maps

We call *generating display maps* the display maps of the form $\pi : D^n \to S^{n-1}$

Lemma 29. *A functor preserves pullbacks along all display maps if and only if it preserves the pullbacks along generating display maps.*

Proof. Suppose that F preserves all pullbacks along generating display maps, and consider a pullback of the following form

$$\begin{tikzcd}
(\Delta, y : A[\gamma]) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & (\Gamma, x : A) \arrow[d, "\pi"] \\
\Delta \arrow[r, "\gamma"] & \Gamma
\end{tikzcd}$$

Then this pullback can be composed on the left to give the following diagram

$$\begin{tikzcd}
(\Delta, y : A[\gamma]) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & (\Gamma, x : A) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & D^n \arrow[d, "\pi"] \\
\Delta \arrow[r, "\gamma"] & \Gamma \arrow[r, "A"] & S^{n-1}
\end{tikzcd}$$

where the outer square is a pullback as well. By hypothesis, F sends the right square onto a pullback and the outer square onto a pullback, since the map on the right is a generating display map. Thus F sends the left square on a pullback as well.

Since the display maps are the closure by composition of the projections maps $\pi : (\Gamma, x : A) \to \Gamma$, and F preserves the pullbacks along those, F preserves the pullbacks along all display maps. □

6.3 A characterization of substitutions

We have proved that a substitution is completely determined by its action on variables of a context, our goal is now to study the converse problem : Given a function sending variables of a context to terms of another contexts, does there exist a substitution whose action on variables is given by the specified function? Since substitution have to respect typing, the action on type cannot be completely free, it has to satisfy some conditions.

The generalized nerve functors. Note that a functor $F : \mathcal{C} \to \mathcal{D}$ always induces an associated nerve functor, or associated Yoneda embedding

$$\begin{array}{rcrcl} N_F & : & \mathcal{D}^{\mathrm{op}} & \to & \widehat{C} \\ & & d & \mapsto & \mathcal{D}(d, F\_) \end{array}$$

In the case of the theory CaTT, the categories and functors that we consider and the associated nerves are the ones denoted in the following diagram. For simplicity, we denote $V = N_D$ (presheaf of variables of a globular context), $T = N_{ID}$ (presheaf of terms of a context), and $N_0 = N_{P_0}$ and $N = N_{P_\infty}$

$$\begin{tikzcd}
 & \widehat{\mathcal{G}^{\mathrm{op}}} & & \widehat{\mathcal{S}_{\mathsf{PS},0}} & \widehat{\mathcal{S}_{\mathsf{PS},\infty}} \\
\mathcal{S}_{\mathsf{GSeTT}} \arrow[ur, bend left, "V"] \arrow[rr, "I"] & & \mathcal{S}_{\mathsf{CaTT},\infty} \arrow[ul, bend right, "T"'] \arrow[ur, bend left, "N_0"] \arrow[urr, "N"'] & & \\
 & \mathcal{G}^{\mathrm{op}} \arrow[ul, bend left, "D"] \arrow[ur, bend right, "ID"'] & & \mathcal{S}_{\mathsf{PS},0} \arrow[ul, bend left, "P_0"] & \mathcal{S}_{\mathsf{PS},\infty} \arrow[ull, "P_\infty"']
\end{tikzcd}$$

Remark that for a context Γ, Lemma 24 shows that the substitutions $\Gamma \vdash \gamma : D^n$ are exactly the terms in Γ, hence $(T\Gamma)_n$ is the set of all terms of dimension n. Moreover, call a *globular context* a context that is in $\mathcal{S}_{\mathsf{GSeTT}}$, that is a context which does not make use of any coherence. Then for all globular context Γ, $(V\Gamma)_n$ is the set of all variables of dimension n. In both cases, those sets connected by the functions of source and target. Note that in a globular contexts the source and target of a variable is again a variable, which is not the case in arbitrary contexts. This explains why V is only definable as a presheaf on globular contexts.

Substitutions to a globular context.

Lemma 30. *Let $\Delta \vdash$ be a globular context and $\Gamma \vdash$ a context. Then there is an isomorphism*

$$\mathcal{S}_{\mathsf{CaTT},\infty}(\Gamma, \Delta) \simeq \widehat{\mathcal{G}^{\mathrm{op}}}(V\Delta, T\Gamma)$$

Proof. To any substitution $\Gamma \vdash \sigma : \Delta$, we associate a natural transformation $\overline{\sigma} : V\Delta \Rightarrow T\Gamma$ defined as follows : For any dimension n, we define

$$\begin{array}{rcrcl} \overline{\sigma}_n & : & (V\Delta)_n & \to & (T\Gamma)_n \\ & & x & \mapsto & x[\sigma] \end{array}$$

In order to check that this defines a natural transformation, it suffices to check that it is compatible with the source and target inclusion of disks, i.e., that it respects the types. If x, y, z are variables in Δ such that $\Delta \vdash x : y \to z$, this amounts to saying that $\Gamma \vdash x[\sigma] : y[\sigma] \to z[\sigma]$. This comes from Lemma 23, and hence $\overline{\sigma}$ is a natural transformation. This allow us to define the map

$$\overline{} \quad : \quad \mathcal{S}_{\mathsf{CaTT},\infty}(\Gamma, \Delta) \quad \to \quad \mathrm{Nat}(V\Delta, T\Gamma)$$

The Lemma 5 states exactly that this map is injective, and we prove that it is surjective, by constructing for any natural transformation $\eta : V\Delta \Rightarrow T\Gamma$, a substitution σ such that $\overline{\sigma} = \eta$. We construct this substitution by induction on Δ. If Δ is the empty context, then $V\Delta$ is empty, so the empty substitution $\Gamma \vdash \langle\rangle : \Delta$ is a preimage. If Δ is of the form $(\Delta', x : A)$, then the natural transformation $\eta : V\Delta \Rightarrow T\Gamma$ induces a natural transformation $\eta' : V\Delta' \Rightarrow T\Gamma$. Let $\Gamma \vdash \sigma' : \Delta'$ be the preimage substitution of η', define σ to be the substitution $\Gamma \vdash \langle \sigma', x : \eta(x)\rangle : \Delta$. Since both judgments $\Gamma \vdash \sigma' : \Delta'$ and $(\Delta', x : A) \vdash$ are derivable, and the naturality of η implies that $\Gamma \vdash t : A[\sigma']$ is also derivable, an application of the rule (SE) shows that this definition yields a valid substitution $\Gamma \vdash \sigma : \Delta$. We now check that $\overline{\sigma} = \eta$. For a variable y in Δ, either y is a variable of Δ', and then $y[\sigma] = y[\sigma'] = \eta'(y) = \eta(y)$, or y is the variable x and then $x[\sigma] = \eta(x)$ □

Explicitly, a natural transformation $\eta : V\Delta \Rightarrow T\Gamma$ is the data of, for each variable x of Δ, a term $\eta(x)$ in Γ of the same dimension as x, such that if $\Delta \vdash x : y \to z$, then $\Gamma \vdash \eta(x) : \eta(y) \to \eta(z)$. This data can also be expressed in a more categorical fashion. For the globular context Γ, we consider the category $(\Delta \downarrow \mathcal{G}^{\mathrm{op}})_V$, which is the full subcategory of the coma category $(\Delta \downarrow \mathcal{G}^{\mathrm{op}})$ whose objects are those substitutions $\Delta \vdash v : D^n$ defining a variable in Δ. Then a natural transformation $V\Delta \Rightarrow T\Gamma$ is equivalent to a cone of apex Γ over the diagram $(\Delta \downarrow \mathcal{G}^{\mathrm{op}})_V \to \mathcal{G}^{\mathrm{op}} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$. Lemma 30 translates to the fact that the *canonical cone of variables* obtained by the natural transformation $V\Delta \Rightarrow T\Delta$ which associates to each variable the variable itself seen as a term is a limiting cone. A equivalent way of stating this result is by saying that the functor I the pointwise right Kan extension

$$I \;= \mathrm{Ran}_D(ID)$$

Note that all ps-context are in particularly globular, and in practice those are the only ones we are interested in. So from now on, we will only use this result in the case where the target context is a ps-context.

Algebraic natural transformations $T\Delta \Rightarrow T\Gamma$. We will now study the more general case of substitution whose targets are general contexts, and not necessarily globular. Note that the same exact result does not make sense anymore, since $V\Gamma$ is not defined in general. For instance, for the context

$$\Gamma = \texttt{(x : *) (f : id x -> id x)}$$

we would like to define $(V\Gamma)_2 = \{\texttt{f}\}$, but then its source and target are the term `id x` which is not a variable, and hence not an element of $(V\Gamma)_1$. In order to express the condition, we change the point of view, and consider natural transformation from the entire globular nerve, instead of from the presheaf of variables. Hence we establish a correspondence between the substitutions

$\Gamma \vdash \sigma : \Delta$ and the natural transformations $T\Delta \Rightarrow T\Gamma$. However, by switching from the presheaf of variables to the presheaf of terms, we have added too much freedom, and there are now such natural transformations that are ill-defined substitution; for instance, consider the contexts

$$\Gamma = \texttt{(x:*)(f:x->x)} \qquad\qquad \Delta = \texttt{(x:*)}$$

together with a natural transformation $\eta : T\Delta \Rightarrow T\Gamma$ such that $\eta(\texttt{id\ x}) = \texttt{f}$. This can never be the action of a substitution, since, since $(\texttt{id}\ x)[\gamma] = \texttt{id}\ (x[\gamma])$. To account for this, we express a compatibility condition of the natural transformations with the term constructor coh, and show that it is the only obstruction to correspond to a well-defined substitution. The compatibility condition with the constructor coh is called *algebraicity*, and is defined inductively on the coherence depth of the substitution we consider.

We first define a function $\_^* : \widehat{\mathcal{G}^{\mathrm{op}}}(T\Delta, T\Gamma) \to \widehat{\mathcal{S}_{\mathsf{PS},0}}(N_0\Delta, N_0\Gamma)$. Suppose given the natural transformation $\eta : T\Delta \Rightarrow T\Gamma$, and consider a ps-context Θ, together with a substitution $\Delta \vdash \delta : \Theta$ (i.e., δ is an element of $(N_0\Delta)_\Theta$). By Lemma 30, δ induces a natural transformation $\overline{\delta} : V\Theta \Rightarrow T\Delta$. By vertically composing this natural transformation with η, we get a natural transformation $\eta \circ \overline{\delta} : V\Theta \Rightarrow T\Gamma$. We define the substitution $\Gamma \vdash \eta^*(\delta) : \Theta$ to be the substitution associated to this natural transformation by Lemma 30. Hence it is characterized by the equation $\overline{\eta^*(\delta)} = \eta \circ \overline{\delta}$. The following lemma ensures that this definition is natural in η.

Lemma 31. *For any natural transformation $\eta \in \widehat{\mathcal{G}^{\mathrm{op}}}(T\Delta, T\Gamma)$, the family of functions η^* defines a natural transformation in $\widehat{\mathcal{S}_{\mathsf{PS},0}}(N_0\Delta, N_0\Gamma)$*

Proof. Consider a substitution $\Theta \vdash \vartheta : \Theta'$ of depth 0 between two ps-contexts (i.e., ϑ is a map in $\mathcal{S}_{\mathsf{PS},0}$), we want to show that the following square commutes

$$\begin{tikzcd}
(N_0\Delta)_\Theta \arrow[r, "\eta^*"] \arrow[d, "\vartheta\circ_"'] & (N\Gamma)_\Theta \arrow[d, "\vartheta\circ_"] \\
(N_0\Delta)_{\Theta'} \arrow[r, "\eta^*"'] & (N\Gamma)_{\Theta'}
\end{tikzcd}$$

Consider a substitution $\Delta \vdash \delta : \Theta$ and a variable x in Δ, on one hand we have

$$\begin{aligned}
\overline{\eta^*(\vartheta \circ \delta)}(x) &= \eta \circ \overline{(\vartheta \circ \delta)}(x) \\
&= \eta(x[\vartheta \circ \delta]) \\
&= \eta(x[\vartheta][\delta]) \\
&= \eta \circ \overline{\delta} \circ \overline{\vartheta}(x)
\end{aligned}$$

where the first line is by definition of η^* and the last line holds since $x[\vartheta]$ is a variable in Δ, since Δ is a ps-context hence a globular context. And on the other hand,

$$\begin{aligned}
\overline{\vartheta \circ \eta^*(\delta)}(x) &= x[\vartheta \circ \eta^*(\delta)] \\
&= x[\vartheta][\eta^*(\delta)] \\
&= \overline{\eta^*(\delta)} \circ \overline{\vartheta}(x) \\
&= \eta \circ \overline{\delta} \circ \overline{\vartheta}(x)
\end{aligned}$$

where the third line holds since $x[\vartheta]$ is a variable of Δ, and the last line holds by definition of η^*. This proves the commutation of the square, and hence that $\eta^* : N_0\Delta \Rightarrow N\Gamma$ is a well defined natural □

Definition 32. A natural transformation $\eta : T\Delta \Rightarrow T\Gamma$ is said to be *algebraic* if for all ps-context Θ together with a term t derivable in Θ and a substitution $\Delta \vdash \vartheta : \Theta$, one has the equality

$$\eta(t[\vartheta]) = t[\eta^*(\vartheta)]$$

The following lemma, aside from being technically relevant shows that restricting ourselves to algebraic natural transformation kills the additional unwanted freedom that we added while switching from the presheaf of variables to the presheaf of terms.

Lemma 33. *Two algebraic natural transformations $\eta, \eta' : T\Delta \Rightarrow T\Gamma$ are equal if and only if they coincide on all variables of Δ.*

Proof. Suppose that $\eta, \eta' : T\Delta \Rightarrow T\Gamma$ are two algebraic natural transformations that coincide on all variables of Δ. We prove that for any term t derivable in Δ, $\eta(t) = \eta'(t)$, by induction on the depth of t. If t is of depth 0, then it is a variable, and $\eta(t) = \eta'(t)$ by hypothesis. Suppose that η and η' coincide on all terms of depth at most d and consider a derivable term t of depth $d+1$ in Δ. Then t is of the form $\mathsf{coh}_{\Theta,A}[\delta]$, for a ps-context Θ and a substitution $\Delta \vdash \delta : \Theta$ of depth at most d. Since η and η' are algebraic, we have that $\eta(t) = \mathsf{coh}_{\Theta,A}[\eta^*\delta]$ and $\eta'(t) = \mathsf{coh}_{\Theta,A}[\eta'^*\delta]$, so it suffices to prove that $\eta^*\delta = \eta'^*\delta$. Note that for all variable x in Θ, we have $\overline{\eta^*\delta}(x) = \eta(x[\delta])$, and $x[\delta]$ is of depth at most d, hence by induction hypothesis, $\overline{\eta^*\delta}(x) = \eta'(x[\delta])$. This being proven for all x shows that $\overline{\eta^*\delta} = \eta' \circ \overline{Gd}$, which proves that $\eta^*\delta = \eta'^*\delta$ □

Algebraic natural transformation and nerve.

Lemma 34. *If a natural transformation $\eta \in \widehat{\mathcal{G}^{\mathrm{op}}}(T\Delta, T\Gamma)$ is algebraic, then η^* defines a natural transformation in $\widehat{\mathcal{S}_{\mathsf{PS},\infty}}(N\Delta, N\Gamma)$.*

Proof. We have already proved in Lemma 31 that η^* defines a natural transformation $N_0\Delta \Rightarrow N_0\Gamma$, so it suffices to prove that this extends to a natural transformation $N\Delta \Rightarrow N\Gamma$. Since for any ps-context Θ, we have the equality of sets $(N_0\Delta)_\Theta = (N\Delta)_\Theta$, η^* defines a family of functions $(N\Delta)_\Theta \to (N\Gamma)_\Theta$, and it suffices to check that this family is a natural transformation, that is for all substitution $\Theta \vdash \vartheta : \Theta'$ between two ps-contexts, the following square commutes

$$\begin{array}{ccc} (N\Delta)_\Theta & \xrightarrow{\eta^*} & (N\Gamma)_\Theta \\ {\scriptstyle\vartheta^*}\downarrow & & \downarrow{\scriptstyle\vartheta^*} \\ (N\Delta)_{\Theta'} & \xrightarrow[\eta^*]{} & (N\Gamma)_{\Theta'} \end{array}$$

Consider a substitution $\Delta \vdash \delta : \Theta$, and pick a variable x in $V\Theta$. On one hand we have

$$\begin{aligned} \overline{\vartheta \circ \eta^*(\delta)}(x) &= x[\vartheta \circ \eta^*(\delta)] \\ &= x[\vartheta][\eta^*(\delta)] \end{aligned}$$

and on the other hand

$$\begin{aligned}\overline{\eta^*(\vartheta \circ \delta)}(x) &= \eta \circ (\overline{\vartheta \circ \delta})(x) \\ &= \eta(x[\vartheta \circ \delta]) \\ &= \eta(x[\vartheta][\delta])\end{aligned}$$

Since η is algebraic, we get, for all variable x, the equality

$$\overline{\vartheta \circ \eta^*(\delta)}(x) = \overline{\eta^*(\vartheta \circ \delta)}(x)$$

which implies that the two natural transformation are equal, and by injectivity of $\overline{}$, this implies the following equality on substitutions

$$\vartheta \circ \eta^*(\delta) = \eta^*(\vartheta \circ \delta)$$

which is exactly the commutation of the above square. $\square$

This proof sheds a new light on the algebraicity condition, as it is exactly the condition required for a natural transformation between the presheaves of terms to lift to a transformation between the nerves which is still natural.

Lemma 35. *There is a natural isomorphism between the set of algebraic natural transformations $T\Delta \Rightarrow T\Gamma$ and the set of natural transformations $N\Delta \Rightarrow N\Gamma$*

Proof. We have already proved in Lemma 34 that an algebraic natural transformation $\eta T\Delta \Rightarrow T\Gamma$ defines a natural transformation $\eta^* : N\Delta \Rightarrow N\Gamma$. Conversely, given a natural transformation $\eta : N\Delta \Rightarrow N\Gamma$, it restricts along the inclusion functor $\mathcal{G} \hookrightarrow \mathcal{S}_{\mathsf{PS},\infty}$ to a natural transformation $\eta' : T\Delta \Rightarrow T\Gamma$. We first show that η' is an algebraic natural transformation. Note that given a ps-context Θ and a term t in Θ, together with a substitution $\Delta \vdash \delta : \Theta$, the naturality of η shows the commutation of the following square

$$\begin{tikzcd} (N\Delta)_\Theta \arrow[r, "\eta"] \arrow[d, "t[\_]"'] & (N\Gamma)_{D^n} \arrow[d, "t[\_]"] \\ (N\Delta)_\Theta \arrow[r, "\eta"'] & (N\Gamma)_{D^n} \end{tikzcd}$$

which gives the equation $t[\eta(\delta)] = \eta(t[\delta])$. Since $t[\delta]$ is a term, we have that $\eta(t[\delta]) = \eta'(t[\delta])$, so in order to show that η' is algebraic, it suffices to show that $t[\eta(\delta)] = t[\eta'^*(\delta)]$. We will show that $\eta'^*(\delta) = \eta(\delta)$. Note that for any variable x in Θ, we have that

$$\begin{aligned}\overline{\eta(\delta)}(x) &= x[\eta(\delta)] \\ &= \eta(x[\delta]) \\ &= \eta'(x[\delta]) \\ &= (\eta' \circ \overline{\delta})(x)\end{aligned}$$

So this proves the equality of natural transformations $\overline{\eta(\delta)} = \eta' \circ \overline{\delta}$. Since this equation characterizes $\eta'^*(\delta)$, it follows that $\eta(\delta) = \eta'^*(\delta)$, and hence the algebraicity of η'.

We will now show that the restriction $\eta \mapsto \eta'$ is inverse to the extension $\eta \mapsto \eta^*$. We have already proved that for all transformation $\eta : N\Delta \Rightarrow N\Gamma$, and all element δ of $N\Delta$, we have $\eta(\delta) = \eta'^*(\delta)$, so this proves that $\eta'^* = \eta$. Conversely, consider an algebraic natural transformation $\eta : T\Delta \Rightarrow T\Gamma$, we show that $(\eta^*)' = \eta$, that is, for all term t of dimension n in Δ, we have $\eta(t) = \eta^*(t)$. It suffices to show that $\overline{\eta(t)} = \overline{\eta^*(t)}$. Consider a variable x in D^n, then we have on one hand

$$\overline{\eta(t)}(x) = x[\eta(t)]$$

and on the other hand,

$$\overline{\eta^*(t)}(x) = \eta \circ \bar{t}(x) = \eta(x[t])$$

The naturality of η gives the equality between these two expressions, proving that $(\eta^*)' = \eta$. ☐

This lemma is the key point of interaction between the type theory structure and the categorical computations of the syntactic category. Let us introduce for all context Γ, the coma category $(\Gamma \downarrow \mathcal{S}_{\mathsf{PS},\infty})$, whose objects are pairs (Θ, γ), where $\Theta \vdash_{\mathsf{ps}}$ is a ps-context and $\Gamma \vdash \gamma : \Theta$ is a substitution, and whose morphisms $(\Theta_0, \gamma_0) \to (\Theta_1, \gamma_1)$ are substitutions $\Theta_1 \vdash \vartheta : \Theta_0$ such that $\vartheta\gamma_0 = \gamma_1$. Then the category $(\Gamma \downarrow \mathcal{S}_{\mathsf{PS},\infty})$ is equipped with a forgetful functor (or projection) $Q : (\Gamma \downarrow \mathcal{S}_{\mathsf{PS},\infty}) \to \mathcal{S}_{\mathsf{PS},\infty}$. We call the *canonical diagram* of Γ the composition of Q with the inclusion functor

$$(\Gamma \downarrow \mathcal{S}_{\mathsf{PS},\infty}) \to \mathcal{S}_{\mathsf{PS}\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$$

Then a natural transformation $N\Delta \Rightarrow N\Gamma$ is equivalent to a cone of apex Δ over the canonical diagram of Γ. In particular, the identity natural transformation $\mathrm{id} : N\Gamma \Rightarrow N\Gamma$ defines the *canonical cone* of Γ. These constructions can be found in [19, Lemma on p. 245] and [21, Lemma 6.3.8 on p. 202].

Similarly, the data of a natural transformation $T\Delta \Rightarrow T\Gamma$ is equivalent to a cone over the diagram $(\Gamma \downarrow \mathcal{G}^{\mathrm{op}}) \to \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT},\infty}$. Since every ps-context is a globular product of disk, it is clear that a cone over the canonical diagram of Γ can be developed in a cone over the diagram $(\Gamma \downarrow \mathcal{G}^{\mathrm{op}}) \to \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT},\infty}$, this lemma can be understood as giving the converse construction - under the condition of algebraicity, a cone over the latter diagram can be assembled into a cone over the canonical diagram of Γ.

Substitutions to arbitrary contexts.

Theorem 36. $\mathcal{S}_{\mathsf{CaTT},\infty}(\Gamma, \Delta) \simeq \mathrm{Nat}(N\Delta, N\Gamma)$

Proof. It suffices to prove that the substitutions $\Gamma \vdash \gamma : \Delta$ are naturally in bijection with the algebraic natural transformations $T\Delta \Rightarrow T\Gamma$.

To any substitution $\Gamma \vdash \gamma : \Delta$, we associate the natural transformation $\overline{\gamma}$ defined for all terms t in $T\Delta$ by $\overline{\gamma}(t) = t[\gamma]$. This family of functions defines a natural transformations, since the morphisms in $\mathcal{G}$ are the source and target, and the application of γ preserves the typing, i.e., if $\Delta \vdash t : u \to v$, then $\Gamma \vdash t[\gamma] : u[\gamma] \to v[\gamma]$. Moreover we will show that this natural transformation is algebraic. Consider a ps-context Θ together with a term t derivable in Θ and a substitution $\Delta \vdash \delta : \Theta$. Then we have that $\overline{\gamma}(t[\delta]) = t[\delta][\gamma] = t[\delta \circ \gamma]$. Moreover for all variable x, $\overline{\delta \circ \gamma}(x) = x[\delta \circ \gamma] = (\overline{\gamma} \circ \overline{\delta})(x)$, which is the defining equation for $\overline{\gamma}^*(\delta)$, hence $\delta \circ \gamma = \overline{\gamma}^*(\gamma)$, and $\overline{\gamma}(t[\delta]) = t[\overline{\gamma}^*(\delta)]$. This proves that $\overline{\gamma}$ is a algebraic natural transformation.

Conversely, given an algebraic natural transformation $\eta : T\Delta \Rightarrow T\Gamma$, we define a substitution $\Gamma \vdash \sigma_\eta : \Delta$ such that $\overline{\sigma_\eta} = \eta$ by induction over the length of Δ.

- If Δ is of length 0, then it is the terminal context $\varnothing$, and we define σ_η to be the unique substitution $\Gamma \vdash \langle\rangle : \varnothing$.

- If Δ is of length $n+1$, then $\Delta = (\Delta', x : A)$, and the projection $\Delta \vdash \pi : \Delta'$ induces a "restriction" natural transformation $T\Delta' \Rightarrow T\Gamma$. By this transformation with η, we get $\eta' : T\Delta' \Rightarrow T\Gamma$, and since η' is a restriction of η it is also algebraic. By induction this gives a substitution $\Delta' \vdash \sigma_{\eta'} : \Gamma$. We then define $\sigma_\eta \langle \sigma_\eta, \eta(x)\rangle$, and check that an application of the rule (ES) shows that σ_η is well defined. Since we have by induction $\Gamma \vdash \sigma_{\eta'} : \Delta'$, and by hypothesis $\Delta', x : A \vdash$, it suffices to check that $\Gamma \vdash \eta(x) : A[\sigma_{\eta'}]$. This is immediate if $A \; = \star$, and if $A \; = y \to z$, the naturality of η shows that $\Gamma \vdash \eta(x) = \eta(y) \to \eta(z)$, and by definition of $\sigma_{\eta'}$, we have that $A[\delta_{\eta'}] = \eta'(y) \to \eta'(z) = \eta(y) \to \eta(z)$. Hence the substitution is well defined.

  It now suffices to check that $\overline{\sigma_\eta} = \eta$, and since it is an equality between two algebraic natural transformation, it suffices by Lemma 33 to check that they have the same actions on variables. Let y be a variable in Δ, then either $y = x$, and then by definition of σ_η, we have $x[\sigma\eta] = \eta(x)$, or y is a variable in Δ', and then $y[\sigma_\eta] = y[\sigma_{\eta'}]$ and by induction, we have that $y[\sigma_{\eta'}] = \eta'(y) = \eta(y)$.

We then prove that these two maps are inverse to one another. We have already proved during the induction that $\overline{\sigma_\eta} = \eta$ for all algebraic natural transformation. We will now prove that for all substitution $\Gamma \vdash \gamma : \Delta$, we have $\sigma_{\overline{\gamma}} = \gamma$. By Lemma 5, it suffices to prove that for all variable x in Δ, we have $x[\sigma_{\overline{\gamma}}] = x[\gamma]$, but we have that

$$\begin{aligned} x[\sigma_{\overline{\gamma}}] &= \overline{\sigma_{\overline{\gamma}}}(x) \\ &= \overline{\gamma}(x) \\ &= x[\gamma] \end{aligned}$$

□

6.4 $\mathcal{S}_{\mathsf{CaTT},\infty}$ as a free completion

In Theorem 36, we have quantified precisely the data needed to define a substitution between two arbitrary contexts Γ and Δ. This is in fact a strong result categorically, and we now draw the consequences it has on the structure of the category $\mathcal{S}_{\mathsf{CaTT},\infty}$. Note that we here consider it as a category and forget the structure of category with families for now. First note that Theorem 36 is a codensity result, as it can be reformulated as the inclusion functor $\mathcal{S}_{\mathsf{PS},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$ is codense.

Canonical limits. The main application of this remark is the fact that all objects in $\mathcal{S}_{\mathsf{CaTT},\infty}$ is canonically a limit of objects $\mathcal{S}_{\mathsf{PS},\infty}$. More precisely, any context Γ is isomorphic to the following limit in $\mathcal{S}_{\mathsf{CaTT},\infty}$, that we call a *canonical limit*

$$\Gamma \simeq \lim\left((\Gamma \downarrow \mathcal{S}_{\mathsf{PS},\infty}) \to \mathcal{S}_{\mathsf{PS},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}\right)$$

The cone defining the limit is the canonical cone, corresponding to the identity natural transformation $\mathrm{id} : N\Gamma \Rightarrow N\Gamma$. Theorem 36 can also be understood in terms of Kan extensions, it then states that the identity functor $\mathrm{id}_{\mathcal{S}_{\mathsf{CaTT},\infty}}$ is the pointwise right Kan extension of $P_\infty : \mathcal{S}_{\mathsf{PS},\infty} \to \mathcal{S}_{\mathsf{CaTT},\infty}$ along itself

$$\mathrm{id}_{\mathcal{S}_{\mathsf{CaTT},\infty}} = \mathrm{Ran}_{P_\infty}(P_\infty)$$

Preservation of canonical limits. Given a category $\mathcal{C}$ equipped with a functor $F : \mathcal{S}_{\mathsf{PS},\infty} \to \mathcal{C}$, define the associated nerve functor to be

$$\begin{array}{rcrcl} N_F & : & \mathcal{C}^{\mathrm{op}} & \to & [\mathcal{S}_{\mathsf{PS},\infty}, \mathbf{Set}] \\ & & c & \mapsto & \mathcal{C}(c, F\_) \end{array}$$

Given a context Γ in $\mathcal{S}_{\mathsf{CaTT},\infty}$ and an object X in $\mathcal{C}$, a natural transformation $N\Gamma \mapsto N_F X$ is the data of a cone of apex X over the image by F of canonical diagram of Γ. Note that moreover the functoriality of F induces a natural transformation $F : N\Gamma \Rightarrow N_F\Gamma$, associating to each substitution $\Gamma \vdash \sigma : \Delta$ the morphism $F\sigma : F\Gamma \to F\Delta$. We say that F preserves the canonical limits if the cone associated to this natural transformation is a limiting cone, and we denote $[\mathcal{S}_{\mathsf{CaTT},\infty}, \mathcal{C}]_{\mathrm{canlim}}$ the category of such functors.

Proposition 37. *The functor F preserves canonical limits if and only if for all set X and all context Γ, the map*

$$\begin{array}{rcrcl} \overline{} & : & \mathcal{C}(X, F\Gamma) & \to & \mathrm{Nat}(N\Gamma, N_F X) \\ & & f & \mapsto & (\gamma \mapsto F\gamma \circ f) \end{array}$$

is an isomorphism.

Proof. Given a map $f : X \to F\Gamma$, the natural transformation $\overline{f}$ is the data of a cone of apex X over the image of the canonical cone of Γ by F. Hence the equivalence, by definition of a limit. $\square$

Free completion. The category $\mathcal{S}_{\mathsf{CaTT},\infty}$ is the free completion of the category $\mathcal{S}_{\mathsf{PS},\infty}$ in the following sense

Theorem 38. *Consider a category $\mathcal{C}$ together with a functor $F : \mathcal{S}_{PS,\infty} \to \mathcal{C}$ such that $\mathcal{C}$ has all limits of the images of all canonical diagrams of objects of $\mathcal{S}_{CaTT,\infty}$ by F. Then there exists an essentially unique functor $\tilde{F} : \mathcal{S}_{CaTT,\infty} \to C$ which preserves the canonical limits and whose restriction to $\mathcal{S}_{PS,\infty}$ is F.*

$$\begin{tikzcd} \mathcal{S}_{CaTT,\infty} \arrow[r, "\tilde{F}"] & \mathcal{C} \\ \mathcal{S}_{PS,\infty} \arrow[u, hook] \arrow[ur, "F"'] & \end{tikzcd}$$

Proof. Given the functor $F : \mathcal{S}_{\mathsf{PS},\infty} \to \mathcal{C}$, since $\mathcal{S}_{\mathsf{PS},\infty}$ is codense in $\mathcal{S}_{\mathsf{CaTT},\infty}$, an extension of F which preserves the canonical limits is necessarily given by

$$\tilde{F}(\Gamma) = \lim\left((\Gamma \downarrow \mathcal{S}_{\mathsf{PS},\infty}) \to \mathcal{S}_{\mathsf{PS},\infty} \xrightarrow{F} \mathcal{C}\right)$$

Conversely, the above functor is the pointwise right Kan extension of F along the inclusion $\mathcal{S}_{\mathsf{PS},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$, and since this inclusion is fully faithful, this pointwise right Kan extension is indeed an extension (see for instance [19, Corollary 3 on p. 239]), and by definition it preserves the canonical limits. $\square$

In practice we will always use the case where the category $\mathcal{C}$ is complete, in which case, the previous theorem simplifies to

Corollary 39. *For any complete category $\mathcal{C}$, the functor $P_\infty : \mathcal{S}_{\mathsf{PS},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$ induces a equivalence of categories*

$$[\mathcal{S}_{\mathsf{PS},\infty}, \mathcal{C}] \simeq [\mathcal{S}_{\mathsf{CaTT},\infty}, \mathcal{C}]_{\mathrm{canlim}}$$

Preservation of globular products. Since $P_\infty : \mathcal{S}_{\mathsf{PS},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$ is fully faithful and $\mathcal{S}_{\mathsf{CaTT},\infty}$ has the globular products, P_∞ preserves the globular products. So consider a functor $F : \mathcal{S}_{\mathsf{PS},\infty} \to \mathcal{C}$ such that $\mathcal{C}$ has all limits of the images of canonical diagrams by F. Suppose moreover that F preserves the globular products, then its pointwise right Kan extension $\tilde{F} : \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathcal{C}$ also preserves the globular products. Indeed, consider a context Γ in $\mathcal{S}_{\mathsf{CaTT},\infty}$ which a globular product, then it is isomorphic to a ps-context Δ, which is an object of $\mathcal{S}_{\mathsf{PS},\infty}$. Hence $\tilde{F}\Gamma$ is isomorphic to $\tilde{F}\Delta = F\Delta$. Since F preserves the globular products and Δ is a globular product over the same diagram as Γ, $\tilde{F}\Gamma$ is a globular product over the image by F of the diagram exhibiting Γ as a globular product, which is also its image by $\tilde{F}$. Conversely, for any functor $G : \mathcal{S}_{\mathsf{CaTT},\infty}$ preserving the canonical limits and the globular products, its restriction to $G \circ \iota : \mathcal{S}_{\mathsf{PS},\infty} \to \mathcal{C}$ preserves the globular products, since the fully faithful functor ι reflects them. This proves the following :

Theorem 40. *For any complete category $\mathcal{C}$, the functor $P_\infty : \mathcal{S}_{\mathsf{PS},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT},\infty}$ induces an equivalence of categories*

$$[\mathcal{S}_{\mathsf{PS},\infty}, \mathcal{C}]_{\mathrm{gprod}} \simeq [\mathcal{S}_{\mathsf{CaTT},\infty}, \mathcal{C}]_{\mathrm{gprod,canlim}}$$

6.5 Functors preserving globular products

We now suppose given a category $\mathcal{C}$ equipped with a functor $F : \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathcal{C}$ which preserves the globular products, and we reproduce the previous results for the category $\mathcal{C}$ seeing the objects contexts F.

Morphism to the image of ps-contexts.

Lemma 41. *If Γ is a ps-context, and X is an object of $\mathcal{C}$, then there is a bijection*

$$\mathbf{Set}(X, F\Gamma) \simeq \mathrm{Nat}(V\Gamma, N_F X)$$

Proof. Consider a ps-context Γ, then it can be written as a globular product. Since F preserves the globular products $F\Gamma$ is also a globular product, and a natural transformation $V\Gamma \Rightarrow N_F X$ is exactly a cone of apex X over a diagram which is equivalent to the globular product diagram of $F\Gamma$, hence the equality, by definition of a limit. $\square$

This lemma can also be formulated in terms of Kan extensions. Note that we have the following induced functors

$$\begin{tikzcd} \mathcal{S}_{\mathsf{PS},0} \arrow[r, "I_p"] & \mathcal{S}_{\mathsf{PS},\infty} \\ \mathcal{G}^{\mathrm{op}} \arrow[u, "D_p"] \arrow[ur, "I_p D_p"'] & \end{tikzcd}$$

and the previous lemmas restricted to this functor state that $I_p = \mathrm{Ran}_{D_p}(I_p D_p)$. Lemma 41 states that a functor $F : \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathcal{C}$ preserving the globular products necessarily preserves this right Kan extension, i.e., that we have $FI_p = \mathrm{Ran}\, D_p(FI_p D_p)$.

Algebraic natural transformations $T\Gamma \Rightarrow T_F X$. Suppose that there is a functor $F : \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$, which preserves the globular product, we denote T_F the nerve associated to the composite $\mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$. Similarly to the previous section, a natural transformation $N\Gamma \Rightarrow N_F X$ is redundant, and can be reduced to a natural transformation $T\Gamma \Rightarrow T_F X$, satisfying a particular algebraicity condition.

Lemma 42. *A natural transformation $\eta : T\Gamma \Rightarrow T_F X$ induces a natural transformation $\eta^* : N\Gamma \Rightarrow N_F X$.*

Proof. For a natural transformation $\eta : T\Gamma \Rightarrow T_F X$, given a ps-context Θ together with a substitution $\Gamma \vdash \gamma : \Theta$, we have the natural transformation $\overline{\gamma} : V\Theta \Rightarrow N\Gamma$, by vertically composing with η we get the natural transformation $\underline{\eta \circ \overline{\gamma}} : V\Theta \Rightarrow N_F X$, and we define $\eta^*(\gamma)$ to be the unique map such that $\underline{\eta^*(\gamma)} = \eta \circ \overline{\gamma}$. The fact that this defines a natural transformation is analogous to Lemma 31 $\square$

Definition 43. A natural transformation $\eta : T\Gamma \Rightarrow T_F X$ is *algebraic* if for ps-context Θ and for all term t in Θ, along with a substitution $\Gamma \vdash \gamma : \Theta$, the following equality is satisfied

$$\eta(t[\gamma]) = F(t) \circ (\eta^* \gamma)$$

Lemma 44. *Two algebraic natural transformation $T\Gamma \Rightarrow T_F X$ are equal if and only if they coincide on all variables.*

Proof. The proof is the same as the one of Lemma 33 $\square$

Lemma 45. *The set of algebraic natural transformations $T\Gamma \Rightarrow T_F X$ is naturally isomorphic to the set of natural transformations $N\Gamma \Rightarrow N_F X$.*

Proof. Again, the proof is essentially the same as the one of Lemma 35. $\square$

7 Models of CaTT

We now prove that the models of the type thory CaTT are equivalent to the ω-categories defined by Maltsiniotis. The crux of the argument lies in the following lemma

Lemma 46. *If $F : \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$ is a functor that preserves globular products, then F preserves canonical limits if and only if F preserves the pullbacks along display maps and the terminal object.*

Proof. Consider a functor $F : \mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$ that preserves globular product. We first reformulate our goal by applying Proposition 37 together with Lemma 45, and show that F preserves pullbacks along display maps if and only if for all context Γ and all set X, the association $f \mapsto \overline{f}$ induces a bijection between $\mathbf{Set}(X, F\Gamma)$ and the algebraic natural transformations $T\Gamma \Rightarrow T_F X$.

First we assume that F preserves the pullbacks along the display maps and the terminal object, and show that the desired map is an bijection, this by induction on the context Γ

- For the empty context $\varnothing$, it is the terminal object in the category $\mathcal{S}_{\mathsf{CaTT},\infty}$ and F preserves terminal object, hence $F\varnothing$ is the terminal object in $\mathbf{Set}$, and hence for all X, $\mathbf{Set}(X, F\varnothing)$ is a singleton. Moreover, by construction $(T\Gamma)_n$ is the set of terms of dimension n in the empty context. Since in the theory CaTT no term is derivable in the empty context, $T\varnothing$ is the empty presheaf which is initial, hence there is a unique natural transformation $T\varnothing \Rightarrow T_F X$, and it is vacuously algebraic. Hence the map $\overline{}$ is a map between two singleton sets, so it is a bijection.

- Consider a context $\Gamma = (\Gamma', x : A)$, and assume the bijection holds for Γ. Then Γ writes as the following pullback (on the left) and since F preserves pullbacks along display maps, taking image by F yields the following pullback square (on the right)

$$\begin{tikzcd} \Gamma \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n \arrow[d] \\ \Gamma' \arrow[r, "A"'] & S^{n-1} \end{tikzcd} \qquad \begin{tikzcd} F(\Gamma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(D^n) \arrow[d] \\ F(\Gamma') \arrow[r] & F(S^{n-1}) \end{tikzcd}$$

  For any set X, the continuity of the hom-functor with respect to its second variable shows that the following square is a pullback

$$\begin{tikzcd} \mathbf{Set}(X, F(\Gamma)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}(X, F(D^n)) \arrow[d] \\ \mathbf{Set}(X, F(\Gamma')) \arrow[r] & \mathbf{Set}(X, F(S^{n-1})) \end{tikzcd}$$

  An algebraic natural transformation $\eta : T\Gamma \Rightarrow T_F X$ restricts as a natural transformation $\eta' : T\Gamma' \Rightarrow T_F X$ which is again algebraic, and by induction this gives a map $\sigma_\eta G \to F\Gamma'$. Applying the last variable to η also gives a specific element $\eta(x) \in (T_F X)_n = \mathbf{Set}(X, FD^n)$. Moreover, the naturality of η shows that these two constructions fit into the following commutative triangle, which by the property of the pullback gives a unique map $\sigma\_$

$$\begin{tikzcd}
\mathrm{Nat}_{\mathrm{coh}}(T\Gamma, T_F X) \arrow[dr, dashed, "\sigma_"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathbf{Set}(X, F(\Gamma)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbf{Set}(X, F(D^n)) \arrow[d] \\
& \mathbf{Set}(X, F(\Gamma')) \arrow[r] & \mathbf{Set}(X, F(S^{n-1}))
\end{tikzcd}$$

By precomposing with the map $\overline{} : \mathbf{Set}(X, F\Gamma) \to \mathrm{Nat}_{\mathrm{coh}}(T\Gamma, T_F X)$, we have that for all map $\gamma : X \to F\Gamma$, by induction hypothesis $\sigma_{\overline{\gamma}'} = F(\pi)\gamma$, and $\overline{\gamma}(x) = x[\gamma]$, and hence by universal property of the pullback, this implies that $\sigma\_ \circ \overline{} = \mathrm{id}_{\mathbf{Set}(X, F\Gamma)}$. Conversely, for all natural algebraic transformation $\eta : T\Gamma \Rightarrow T_F X$ and all variable y in Γ, we have that $\overline{\sigma_\eta}(y) = Fy \circ \sigma_\eta$, then either y is a variable Γ' and by induction $Fy \circ \sigma_\eta = \eta'(y) = \eta(y)$, or $y = x$ and then $Fy \circ \sigma_\eta = eta(x)$ by definition of σ_η. Hence for all variable y of Γ, $\eta(y) = \overline{\sigma_\eta}(y)$, and by Lemma 44 this shows $\eta = \overline{\sigma_\eta}$ Hence $\sigma\_$ is an inverse to the map $\overline{}$, and thus the map is a bijection.

Conversely, we suppose that the map $\overline{}$ is a bijection, and show that then it also preserves pullback along display maps and terminal object. First note that for the terminal object $\varnothing$, we have already proved that there is exactly one natural transformation $T\varnothing \Rightarrow T_F X$ for all set X, and the assumed bijection then ensures that $F\varnothing$ is terminal in **Set**. So we are left to prove that F preserves pullbacks along display maps, and for this it suffices to prove that it preserves pullbacks along generating display maps. Consider such a pullback, which is of the following form (on the left), and we consider a commutative on the following form (on the right) for an arbitrary set X.

$$\begin{tikzcd}
(\Gamma, x : A) \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & D^n \arrow[d] \\
\Gamma \arrow[r, "A"'] & S^{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "t"] \arrow[d] & F(D^n) \arrow[d] \\
F\Gamma \arrow[r, "g"] & F(S^{n-1})
\end{tikzcd}$$

By the assumed bijection, the map $g : X \to F\Gamma$ corresponds exactly to an algebraic natural transformation $\overline{g} : T(\Gamma, x : A) \Rightarrow T_F X$. Under this bijection, the fact that $F(\Gamma, x : A)$ is the preserved pullback is equivalent to saying that there exists a unique algebraic natural transformation $\eta : T(\Gamma, x : A) \Rightarrow T_F X$ which coincide with $\overline{g}$ for all terms that are definable in Γ, and such that $\eta(x) = t$. By Lemma 44 the uniqueness is clear, since the requirement specifies the values on all variables of $(\Gamma, x : A)$, so it suffices to show that such a natural transformation exists. We define this natural transformation by first setting $\eta(t) = \overline{g}(t)$ for all terms in Γ, and extend it by induction on the depth of the term that are not definable in Γ.

- For a term of depth 0, it is necessarily the variable x, and we set $\eta(x) = t$. The fact that this is natural for x is equivalent to the fact that the initial square we consider commutes.

- Suppose that we have constructed η which is natural for all terms of depth at most d, and consider a term t of depth $d+1$ derivable in $(\Gamma, x : A)$. Then t is necessarily a coherence oh the form $t = \mathsf{coh}_{\Delta,B}[\gamma]$ with γ being a substitution of depth d. We then set $\eta(t) = F(\mathsf{coh}_{\Delta,B}[\mathrm{id}_\Delta]) \circ (\eta^*\gamma)$. We now check that this is natural for t : consider the source substitution $\sigma : D^n \to D^{n-1}$, we have that

$$\begin{aligned} F\sigma \circ \eta(t) &= F\sigma \circ F(\mathsf{coh}_{\Delta,B}[\mathrm{id}_\Delta]) \circ (\eta^*\gamma) \\ &= F(\sigma \circ \mathsf{coh}_{\Delta,B}[\mathrm{id}_\Delta])(\eta^*\gamma) \\ &= \eta(\sigma \circ t) \end{aligned}$$

  and similarly for the target substitution. This proves the naturality of η on the term t we constructed.

This natural transformation is algebraic by construction, hence we have proved the existence of a unique algebraic natural transformation that meets the requirements, and hence F preserves the pullbacks along the generating display maps. This shows that F preserves pullbacks along all display maps. □

Theorem 47. *The models of* CaTT *are equivalent to the weak* ω*-categories.*

Proof. The models of CaTT are the functors $\mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$ preserving the terminal object and the pullbacks along display maps 3. Moreover, those induce a morphism of globular categories with families from $\mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$, hence by Lemma 18, they also preserve the globular product, so it follows by Lemma 46 that these are exactly the functors $\mathcal{S}_{\mathsf{CaTT},\infty} \to \mathbf{Set}$ preserving the globular products and the canonical limits. Such functors are equivalent to weak ω-categories by Corollary 39 □

Further Work

The entire construction we have presented here is fairly general and we believe that it works in a much broader scope that the one introduced here. Only the arguments establishing the relation between the judgment $\Gamma \vdash_{\mathsf{ps}}$ and the pasting scheme are specific to the case of CaTT, and most other lemmas rely more on the structure given by being a type theory, than on the specific rules for CaTT. This leads us to believe that there exists a general framework in which our construction applies. Such a framework has started to be studied [15] and preliminary results show promising unification with our methods. In particular, it is known to be equivalent to monads with arities [7], and we believe our characterization of the models of CaTT amounts to a proof of the nerve theorem translated to this framework.

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