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Globular weak ω-categories as models of a type theory

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Abstract

We study the dependent type theory CaTT, introduced by Finster and Mimram, which presents the theory of weak ω-categories, following the idea that type theories can be considered as presentations of generalized algebraic theories. Our main contribution is a formal proof that the models of this type theory correspond precisely to weak ω-categories, as defined by Maltsiniotis, by generalizing a definition proposed by Grothendieck for weak ω-groupoids: those are defined as suitable presheaves over a cat-coherator, which is a category encoding structure expected to be found in an ω-category. This comparison is established by proving the initiality conjecture for the type theory CaTT, in a way which suggests the possible generalization to a nerve theorem for a certain class of dependent type theories

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Introduction

The notion of weak ω-category has emerged quite naturally by generalizing situations encountered in algebraic topology: it consists in an algebraic structure which comprises cells of various dimensions, which can be composed in various dimensions, and satisfy the expected laws. We are however interested in *weak* such structures here, which means that we want to encompass situations where the laws do not hold up to equality, but only up to higher-dimensional cells, which thus play the role of witnesses that those laws are satisfied. Those cells should themselves satisfy coherence laws, which should only hold up to higher cells, which should themselves satisfy coherence laws, and so on. Because of these towers of coherence cells, coming up with a suitable

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definition of weak ω-category is quite difficult. Historically, definitions of weak ω-groupoids (also called ∞-groupoids) were first proposed, such as Kan complexes [20]: those are weak ω-categories in which every cell is reversible, and are thus closer to spaces encountered in algebraic topology. Then, around the beginning of the century, various definitions for weak ω-categories have been proposed: we refer the reader to the surveys on the topic [23, 12] for a general presentation of those. The comparison between the proposals is still an ongoing research topic, and seems to be technically out of reach for now for some of them.

While originating from topology, unexpected connections were found with type theory: a series of works around 2010 revealed that the iterated identity types in Martin-Löf type theory endow each type with the structure of a weak ω-groupoid [25, 32, 1]. This key observation is in fact one of the motivations that lead to the development of homotopy type theory [31]. Based on this, and following Cartmell's insight that type theory could be used to formulate generalized algebraic theories [11], Brunerie managed to extract from the rules generating identity types, a definition of weak ω-groupoids [9], that he could show to be equivalent to a definition proposed by Grothendieck [17]. The novelty of this definition lies in the fact that it is itself formulated as a type theory.

Following Brunerie's approach, Finster and Mimram [14] gave a definition of weak ω-categories in the form of a type theory called CaTT. Their definition follows the lines of a generalization of Grothendieck's weak ω-groupoids to weak ω-categories, proposed by Maltsiniotis [26]. The goal of this article is to show that the type theory CaTT is equivalent to one of the definitions proposed by Maltsiniotis. Moreover Ara [2] has proved this specific to be equivalent to a definition proposed by Leinster [24] following a method introduced by Batanin [3]. Our result completes this circle of ideas, establishing that these three definitions are three sides of the same story, expressed in different languages.

After brief general reminders about semantics of type theory in Section 1, we introduce a type theory for globular sets in Section 2, which serves both as a basis and as a baby version of our main proof. We then briefly present the Grothendieck-Maltsiniotis definition of weak ω-categories in Section 3, in order to recall the motivations for the introduction of the type theory CaTT, which is introduced in Section 4 along with some examples of derivations in this theory, and some of its theories properties. We then study in Section 5 the syntactic category of this theory and begin relating it to the Grothendieck-Maltsiniotis definition of weak ω-categories. Finally, in Section 6, we study the models of this type theory, and show that they are equivalent to the aforementioned definition of categories. The reader who wishes to familiarize himself with the type theory along the way may also experiment with the implementation [6].

1. Categorical semantics of type theory

We begin by recalling the categorical framework we use here to study type theory, together with an associated notion of models. Note that we will not introduce any type theory just yet, but its categorical counterpart. We refer the reader to Section 2.2 for a presentation of the notion of type theory considered here.

1.1 Categories with families The categorical models of type theory considered here are categories with families, which were introduced by Dybjer [13]. This particular choice has little impact on the developments performed here since most other notions of model, such as Cartmell's categories with attributes [11], are known to be equivalent to this one.

We write **Fam** for the category of *families*, where an object is a family $(A_i)_{i \in I}$ consisting of sets A_i indexed by a set I, and a morphism $f : (A_i)_{i \in I} \to (B_j)_{j \in J}$ is a pair consisting of a function $f : I \to J$ and a family of functions $(f_i : A_i \to B_{f(i)})_{i \in I}$.

Suppose given a category $\mathcal{C}$ equipped with a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Fam}$. Given an object Γ of $\mathcal{C}$, its image under T is a family denoted

$$T\Gamma \quad = \quad \left(\mathrm{Tm}^{\Gamma}_{A}\right)_{A \in \mathrm{Ty}^{\Gamma}}$$

i.e., we write Ty^{Γ} for the indexing set and Tm^{Γ}_{A} for the elements of the family. Given a morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$ and an element $A \in \mathrm{Ty}^{\Gamma}$, we write $A[\gamma]$ for the object $\mathrm{Ty}^{\gamma}(A)$ of Ty^{Δ}. Similarly, given an element $t \in \mathrm{Tm}^{\Gamma}_{A}$, we write $t[\gamma]$ for the element $\mathrm{Tm}^{\gamma}_{A}(t)$ of $\mathrm{Tm}^{\Delta}_{A[\gamma]}$. With these notations, the functoriality of T is equivalent to the following equations

$$\begin{aligned} A[\sigma \circ \delta] &= A[\sigma][\delta] \qquad & t[\sigma \circ \delta] &= t[\sigma][\delta] \\ A[\mathrm{id}] &= A & t[\mathrm{id}] &= t \end{aligned}$$

for composable morphisms of $\mathcal{C}$.

A *category with families* consists of a category $\mathcal{C}$ together with a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Fam}$ as above, such that $\mathcal{C}$ has a terminal object, denoted $\varnothing$, and such that there is a *context comprehension* operation: given an object Γ and type $A \in \mathrm{Ty}^{\Gamma}$, there is an object (Γ, A), together with a projection morphism $\pi : (\Gamma, A) \to \Gamma$ and a term $p \in \mathrm{Tm}^{(\Gamma,A)}_{A[\pi]}$, such that for every morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$ together with a term $t \in \mathrm{Tm}^{\Delta}_{A[\gamma]}$, there exists a unique morphism $\langle \gamma, t\rangle : \Delta \to (\Gamma, A)$ such that $p[\langle \gamma, t\rangle] = t$:

$$\begin{tikzcd} & (\Gamma, A) \arrow[d, "\pi"] \\ \Delta \arrow[ur, dotted, "\langle\sigma,t\rangle"] \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

In a category with families, the class of *display maps* is the smallest class of morphisms containing the projection morphisms $\pi : (\Gamma, A) \to \Gamma$ and closed under composition and identities.

A *morphism* between two categories with families $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$, is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation $\phi : T \to T' \circ F$, such that F preserves the terminal object and the context comprehension operation on the nose. A *2-morphism* θ between two morphisms $(F, \phi) : T \to T'$ and $(F', \phi') : T \to T'$ is a natural transformation $\theta : F \to F'$ such that $T\theta \circ \phi = \phi'$.

We define a large category with families in a similar way, as a large category equipped with a functor into families of large sets indexed by a large set, and satisfying the exact same properties. Note that a category with families can be seen as a large category with families. There is a structure of category with large families on the category **Set**, where, given a set X, Ty^{X} is the (large) set of all function $f : Y \to X$ with codomain X and given such a function $f : Y \to X$, Tm^{X}_{f} is the (large) set of all sections of f. We define the category of *models* of a category with families $\mathcal{C}$ to be the category whose objects are the morphisms of categories with families from $\mathcal{C}$ to **Set**, and whose morphisms are the 2-morphisms of categories with families.

Pullbacks in a category with families. The structure of category with families enforces a compatibility condition between context comprehension and the action of morphisms on types, expressed by the following lemma: it states that all pullbacks along display maps exist and that they can be explicitly computed from the given structure.

Lemma 1. *In a category with families $\mathcal{C}$, for every morphism $f : \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in \mathrm{Ty}^\Gamma$, the square*

$$\begin{tikzcd}
(\Delta, A[f]) \arrow[r, "\langle f\circ\pi', p'\rangle"] \arrow[d, "\pi'"'] & (\Gamma, A) \arrow[d, "\pi"] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd}$$

is a pullback, where $\pi' : (\Delta, A[f]) \to \Delta$ and $p' \in \mathrm{Tm}^{(\Delta,A[f])}_{A[f][\pi']}$ are obtained by context comprehension.

Proof. Consider a diagram of the following form in $\mathcal{C}$, without the dotted arrow:

$$\begin{tikzcd}
\Theta \arrow[rrd, bend left, "\sigma"] \arrow[rd, dotted, "\langle\delta, p[\sigma]\rangle"] \arrow[rdd, bend right, "\delta"'] & & \\
& (\Delta, A[f]) \arrow[r, "\langle f\circ\pi', p'\rangle"] \arrow[d, "\pi'"] & (\Gamma, A) \arrow[d, "\pi"] \\
& \Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

Given a term $p \in \mathrm{Tm}^{(\Gamma,A)}_{A[\pi]}$, we have $p[\sigma] \in \mathrm{Tm}^{\Theta}_{A[\pi][\sigma]} = \mathrm{Tm}^{\Theta}_{A[f][\delta]}$. By context extension, we obtain a map $\langle\delta, p[\sigma]\rangle : \Theta \to (\Delta, A[f])$ such that $\pi' \circ \langle\delta, p[\sigma]\rangle = \delta$ and $p'[\langle\delta, p[\sigma]\rangle] = p[\sigma]$. Since moreover $p' = p[\langle f \circ \pi', p'\rangle]$, the previous equality amounts to $p[\sigma] = p[\langle f \circ \pi', p'\rangle \circ \langle\delta, p[\sigma]\rangle]$, which is a necessary condition for the upper triangle to commute, thus proving the uniqueness of the map. We just have to show that this map makes the upper triangle commute. Notice that $\pi \circ \langle f \circ \pi', p'\rangle \circ \langle\delta, p[\sigma]\rangle = \pi \circ \sigma$, and $p[\sigma] = p[\langle f \circ \pi', p'\rangle \circ \langle\delta, p[\sigma]\rangle]$, by universal property of the extension for morphisms, this implies the commutativity of upper triangle. □

The structure of a category with families can be thought of as a way of ensuring that the pullbacks of form of the above lemma exist, while also enforcing that they are split. This means that the choice of the pullbacks is such that taking a pullback along a composite morphism $g \circ f$ gives the same result as taking the pullback along f and then along g. In the formalism of categories with families, this means that we have $(\Gamma, A[\delta\circ\gamma]) = (\Gamma, A[\delta][\gamma])$. Since the structure of category with families provides these pullbacks, and since the morphisms of category with families preserve this structure, these morphisms also preserve these pullbacks, as witnessed by the following result.

Lemma 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with families, together with a morphism $(F, \phi) : \mathcal{C} \to \mathcal{D}$. Then, for any object Γ in $\mathcal{C}$ together with an element $A \in \mathrm{Ty}^\Gamma$, and for any morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$, the following equation is satisfied:*

$$F(\Delta, A[\gamma]) = (F\Delta, (\phi_\Gamma A)[F\gamma])$$

Proof. By definition of a morphism of category with families, we have

$$F(\Delta, A[\gamma]) = (F(\Delta), (\phi_\Delta(A[\gamma])))$$

and, by naturality of ϕ, the following square commutes

$$\begin{tikzcd}
Ty^\Gamma \arrow[r, "\phi_\Gamma"] \arrow[d, "_[f]"'] & Ty^{F(\Gamma)} \arrow[d, "_[F\gamma]"] \\
Ty^\Delta \arrow[r, "\phi_\Delta"] & Ty^{F(\Delta)}
\end{tikzcd}$$

This proves in particular the $\phi_\Delta(A[\gamma]) = (\phi_\Gamma A)[F\gamma]$, from which follows the desired equality. □

Lemma 1 allows to understand this result as the fact that F preserves pullbacks along display maps. In fact, the following result shows that preserving these pullbacks is precisely the condition that a functor from a category with families to sets has to satisfy in order to be a model of the category with families.

Lemma 3. *The models of a category with families $\mathcal{C}$ are isomorphic to the functors $\mathcal{C} \to$ **Set** that preserve the terminal object and the pullbacks along display maps.*

Proof. The Lemma 2, the underlying functor of a morphism of category with families preserves the pullbacks along display maps and, by definition, such a functor has to preserve the initial object as well. So it suffices to prove that a functor $F : \mathcal{C} \to \mathbf{Set}$ preserving the initial object and pullbacks along display maps gives rise to a unique model. Consider such a functor F, together with an object Γ in $\mathcal{C}$ and a type $A \in \mathrm{Ty}^\Gamma$. Suppose defined ϕ such that (F, ϕ) is a model of $\mathcal{C}$, then necessarily $F(\Gamma, A) = (F\Gamma, \phi_\Gamma A) = \phi_\Gamma A$ by definition of the context comprehension in **Set**. Thus necessarily $\phi_\Gamma(A) = F(\Gamma, A)$. Consider a term $t \in \mathrm{Tm}_A^\Gamma$, then there is a morphism $\langle \mathrm{id}_\Gamma, t\rangle : \Gamma \to (\Gamma, A)$, and by definition of the category with families structure of **Set**, we then have $F(\langle \mathrm{id}_\Gamma, t\rangle) = \langle \mathrm{id}_{F\Gamma}, \phi_{\Gamma,A}(t)\rangle = t$, which proves that necessarily $\phi_{\Gamma,A}(t) = F(\langle \mathrm{id}_\Gamma, t\rangle)$. Conversely, these assignments define a natural transformation ϕ, which make (F, ϕ) into a model of F. ☐

This condition relies on the specific structure of category with families of **Set**, and previous lemma does not extend as a characterization of morphisms between arbitrary categories with families. It also justifies retroactively why it is not that important to be precise about size issues with **Set**, as one may as well ignore the structure of category with families on **Set** altogether, and define a model as a functor $\mathcal{C} \to \mathbf{Set}$ that preserves the terminal object and the pullback along the display maps.

1.2 Contextual categories In order to carry on some inductive constructions on the syntax of a theory, and handle them in full generality, we also need introduce the notion of *contextual categories*, due to Cartmell [11], and studied by Streicher [30] and Voevodsky [33] under the name of *C-systems*. These equip categories with families with the extra structure required in order to perform those inductive constructions.

Definition 4. A *contextual category* consist in a category with families $\mathcal{C}$ together with a function ℓ associating to each object Γ of $\mathcal{C}$ a natural number $\ell(\Gamma)$, called its *length*, such that

- the terminal object $\varnothing$ is the unique object such that $\ell(\varnothing) = 0$,
- for every object Γ and type $A \in \mathrm{Ty}^\Gamma$, $\ell(\Gamma, A) = \ell(\Gamma) + 1$,
- for every object Γ such that $\ell(\Gamma) > 0$, there is a unique object Γ' together with a type $A \in \mathrm{Ty}^{\Gamma'}$ such that $\Gamma = (\Gamma', A)$.

Note that a contextual category is usually defined to be a category with attributes satisfying such properties. However, since categories with families and categories with attributes are equivalent, we will also refer to these as contextual categories. Also note that, the notion of contextual category is not invariant under equivalences of categories and relies on a particular presentation of a category. Its use is justified by the fact that the syntax of a type theory gives a particular presentation of a category with families, which happens to be a contextual category.

Given a contextual category $\mathcal{C}$, an object Γ whose length is strictly positive decomposes in a unique way as Γ', A, and we simply write $\pi_\Gamma : \Gamma \to (\Gamma', A)$ (or even π) instead of $\pi_{\Gamma',A}$. We also write x_Γ for the term $p_{\Gamma',A}$ in $\mathrm{Tm}^\Gamma_{A[\pi]}$, thought of as a variable. More generally, we declare that a term is a *variable* when it is of the form $x_\Gamma[\pi]$ where π is a display map. Note that in a contextual category, if $\pi : \Delta \to \Gamma$ is a display map, then necessarily $l(\Delta) > l(\Gamma)$. This implies that the variables of a non-empty context (Γ, A) are either $x_{(\Gamma,A)}$, or of the form $x[\pi_{(\Gamma,A)}]$ where x is a variable of Γ.

The following lemma shows that a map in a contextual category is entirely characterized by its action on the variables in its target context.

Lemma 5. *Consider two maps $\gamma, \delta : \Delta \to \Gamma$, in a contextual category, such that, for every variable x in Γ, $x[\gamma] = x[\delta]$. Then we have $\gamma = \delta$.*

Proof. We prove this result by induction on the length of the context Γ.

- Suppose that Γ is of length 0. Then necessarily, $\Gamma = \varnothing$ is the terminal object, and thus $\gamma = \delta$.
- Suppose that Γ is of length $l+1$. Then it is of the form (Γ', A), and there is a substitution $\pi : \Gamma \to \Gamma'$. Suppose moreover that there are two substitutions $\gamma, \delta : \Delta \to \Gamma$, such that for every variable x of Γ, we have $x[\gamma] = x[\delta]$. Note that we necessarily have $\gamma = \langle \pi \circ \gamma, x_\Gamma[\gamma] \rangle$ and $\delta = \langle \pi \circ \delta, x_\Gamma[\delta] \rangle$, as it is the case for every substitutions. Then for the variable x_Γ, we have $x_\Gamma[\gamma] = x_\Gamma[\delta]$. Moreover, for every variable x of Γ', $x[\pi]$ is a variable of Γ, and thus $x[\pi][\gamma] = x[\pi][\delta]$, which proves $x[\pi \circ \gamma] = x[\pi \circ \delta]$, and, by induction hypothesis, $\pi \circ \gamma = \pi \circ \delta$. We thus have proved that $\langle \pi \circ \gamma, x_\Gamma[\gamma] \rangle = \langle \pi \circ \delta, x_\Gamma[\delta] \rangle$, i.e., $\gamma = \delta$. $\square$

2. A type theory for globular sets

We first describe a type theory whose models are globular sets, on which we rely in order to introduce the type theory CaTT. It was previously considered by Brunerie [9] and Finster and Mimram [14], and we expand here on their work. This type theory is quite poor, as it has no term constructors (the only terms are variables): it will also serve as a simple setting in order to present the techniques and properties which will be generalized later on to the more complex type theory CaTT.

2.1 The category of globular sets

Globular sets are a generalization of graphs, which comprise not only points and arrows, but also higher dimensional cells. Similarly to graphs, the category of globular sets can be defined as a presheaf category.

The category of globes. The *category of globes* $\mathcal{G}$ is the category whose objects are the natural numbers and morphisms are generated by

$$\sigma_i, \tau_i : i \to i+1$$

subject to following *coglobular relations*:

$$\sigma_{i+1} \circ \sigma_i = \tau_{i+1} \circ \sigma_i \qquad\qquad \sigma_{i+1} \circ \tau_i = \tau_{i+1} \circ \tau_i \tag{1}$$

The category of *globular sets* $\mathbf{GSet} = \widehat{\mathcal{G}}$ is the presheaf category over the category $\mathcal{G}$. Given a globular set G, we write G_n instead of $G\,n$. Equivalently, a globular set is a family of sets $(G_n)_{n\in\mathbb{N}}$ equipped with maps $s_i, t_i : G_{i+1} \to G_i$ satisfying the *globular relations*, dual to (1):

$$s_i \circ s_{i+1} \quad = \quad s_i \circ t_{i+1} \qquad\qquad t_i \circ s_{i+1} \quad = \quad t_i \circ t_{i+1} \tag{2}$$

Often, the indices of the source and target maps can be inferred from the context and we therefore omit them and write s and t to simplify notations.

Given an object n, the associated representable $\mathrm{Y}(n)$ is called the *n-disk* and is usually written D^n. It can be explicitly described by

$$(D^n)_i = \begin{cases} \{*_0, *_1\} \text{ if } i < n \\ \{*\} \text{ if } i = n \\ \emptyset \text{ if } i > n \end{cases}$$

with $s(\_) = *_0$ and $t(\_) = *_1$.

Notation for source and target. Throughout this paper, we use the Greek lower cases σ and τ to denote the morphisms in the category $\mathcal{G}$, or to denote the image of the morphisms in $\mathcal{G}$ via a functor $F : \mathcal{G} \to \mathcal{C}$, and we use their equivalent Latin lower cases s, t to denote the image of the morphisms in $\mathcal{G}$ via a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$.

The n-sphere globular set. Given $n \in \mathbb{N}$, the *n-sphere* S^n is the globular set, equipped with an inclusion $\iota^n : S^n \hookrightarrow D^n$, defined by

- $S^{-1} = \emptyset$ is the initial object, and $\emptyset \hookrightarrow D^1$ is the unique arrow,
- S^{n+1} and ι^{n+1} are obtained by the pushout

$$\begin{tikzcd}
S^n \arrow[r, hook, "\iota_n"] \arrow[d, hook, "\iota_n"'] & D^n \arrow[d] \arrow[ddr, bend left, "\sigma_n"] & \\
D^n \arrow[r] \arrow[drr, bend right, "\tau_n"'] & S^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\iota_{n+1}"] & \\
& & D^{n+1}
\end{tikzcd}$$

This definition is well defined since, as a presheaf category, the category of globular sets is cocomplete (and the colimits are computed pointwise).

Finite globular sets. A globular set G is *finite* if it can be obtained as a finite colimit of representable objects. It can be shown that this is the case precisely when the set $\coprod_{i \in \mathbb{N}} G_i$ is finite, because all representables themselves satisfy this property. We write **FinGSet** for the full subcategory of **GSet** whose objects are the finite presheaves. We sometimes call a finite globular set a *diagram*, and describe it using a diagrammatic notation. For instance, the diagram

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, "h"] & z
\end{tikzcd}$$

denotes the finite globular set G, whose only non-empty cell sets are

$$G_0 = \{x, y, z\} \qquad G_1 = \{f, g, h\} \qquad G_2 = \{\alpha\}$$

and whose the sources and targets are defined by

$$\begin{array}{llll} s(f) = x & s(g) = x & s(h) = y & s(\alpha) = f \\ t(f) = y & t(g) = y & t(h) = z & t(\alpha) = g \end{array}$$

Disks and spheres are finite globular sets. In small dimensions, they can be depicted as

$$D^0 = \cdot \qquad\qquad S^0 = \cdot \qquad \cdot$$

$$D^1 = \begin{tikzcd} \cdot \arrow[r] & \cdot \end{tikzcd} \qquad\qquad S^1 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \end{tikzcd}$$

$$D^2 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd} \qquad\qquad S^2 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\quad\Downarrow"] & \cdot \end{tikzcd}$$

$$D^3 = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Rrightarrow\Downarrow"] & \cdot \end{tikzcd}$$

By definition, **FinGSet** is the free cocompletion of G under all finite colimits (see Section 2.4).

2.2 The theory GSeTT In this section, we introduce our notation for the type theories we consider, and describe a particular type theory describing globular sets. The precise relation between this type theory and the category of globular sets is detailed in Section 2.3 and Section 2.4.

Signature. We consider a countably infinite set whose elements are called *variables*. A *term* is this theory is simply a variable (later on, we will consider theories where terms only comprise variables). A *type* is defined inductively to be either

$$\star \qquad \text{or} \qquad t \underset{A}{\to} u$$

where A is a type and t, u are terms. A *context* is a list

$$(x_1 : A_1, \dots, x_n : A_n)$$

of variables $x_1, \dots, x_n$ together with types $A_1, \dots, A_n$, the empty context is denoted $\varnothing$. A *substitution* is a list

$$\langle x_1 \mapsto t_1, \dots, x_n \mapsto t_n \rangle$$

of variables $x_1, \dots, x_n$ together with terms $t_1, \dots, t_n$. From now on, we use the following naming conventions

variables	: $x, y, \dots$
terms	: $t, u, \dots$
types	: $A, B, \dots$
contexts	: $\Gamma, \Delta, \dots$
substitutions	: $\gamma, \delta, \dots$

Judgments. The theory will consist in four different kinds of *judgments*, for which we give the notations, along with the intuitive meaning:

$\Gamma \vdash$	: the context Γ is well-formed
$\Gamma \vdash A$	: the type A is well-formed in the context Γ
$\Gamma \vdash t : A$	: the term t has type A in context Γ
$\Delta \vdash \gamma : \Gamma$	: the substitution γ goes from the context Δ to the context Γ

Most of the time, when we refer to a context, a type, a term or a substitution, we implicitly mean such an object satisfying the adequate judgment. To emphasize this convention we add the adjective *raw* to designate an object as given by the signature, without supposing that a corresponding judgment is derivable, and we state that a property is *syntactic* when it holds for raw expressions.

Syntactic properties. Given a raw term t (resp. a raw type A, a raw context Γ, a raw substitution γ), we define the set of its *free variables* $\mathrm{Var}(t)$ (resp. $\mathrm{Var}(A)$, $\mathrm{Var}(\Gamma)$, $\mathrm{Var}(\gamma)$) by induction as follows

on terms

$$\mathrm{Var}(x) = \{x\}$$

on types

$$\mathrm{Var}(\star) = \emptyset \qquad\qquad \mathrm{Var}(t \underset{A}{\to} u) = \mathrm{Var}(A) \cup \mathrm{Var}(t) \cup \mathrm{Var}(u)$$

on contexts

$$\mathrm{Var}(\varnothing) = \emptyset \qquad\qquad \mathrm{Var}(\Gamma, x : A) = \{x\} \cup \mathrm{Var}(\Gamma)$$

on substitutions

$$\mathrm{Var}(\langle\rangle) = \emptyset \qquad\qquad \mathrm{Var}(\langle \gamma, x \mapsto t\rangle) = \mathrm{Var}(t) \cup \mathrm{Var}(\gamma)$$

Given a raw type A in this theory, we define its *dimension* $\dim(A)$ by induction by

$$\dim(\star) = -1 \qquad\qquad \dim(t \underset{A}{\to} u) = \dim(A) + 1$$

The choice of starting at -1 is dictated here by the correspondence established in Lemma 16. For a context $\Gamma = (x_i : A_i)_{1 \leq i \leq n}$, its dimension is defined to be

$$\dim(\Gamma) = \max_{1 \leq i \leq n} \dim(A_i)$$

and for a term t such that the judgment $\Gamma \vdash t : A$ holds, we will define the dimension of t in the context Γ to be

$$\dim(t) = \dim(A) + 1$$

Action of substitutions, composition, identity. Given a raw substitution γ, we define its action $t[\gamma]$ on a raw term t, its action $A[\gamma]$ on a raw type A, and is composition $\delta \circ \gamma$ with another raw substitution δ by

$$t[\langle\rangle] = t \qquad\qquad t[\langle \gamma, x \mapsto u\rangle] = \begin{cases} u & \text{if } t = x \\ t[\gamma] & \text{otherwise} \end{cases}$$

$$\star[\gamma] = \star \qquad\qquad (t \underset{A}{\to} u)[\gamma] = (t[\gamma]) \xrightarrow[(A[\gamma])]{} (u[\gamma])$$

$$\langle\rangle \circ \gamma = \langle\rangle \qquad\qquad \langle \delta, x \mapsto t\rangle \circ \gamma = \langle \delta \circ \gamma, x \mapsto t[\gamma]\rangle$$

We also define a special raw substitution associated to a raw context Γ, that we call the *identity substitution* id_Γ, by induction by

$$\mathrm{id}_\varnothing = \langle\rangle \qquad\qquad \mathrm{id}_{\Gamma, x:A} = \langle \mathrm{id}_\Gamma, x \mapsto x\rangle$$

For contexts:

$$\frac{}{\Gamma \vdash}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE}) \quad \text{when } x \notin \mathrm{Var}(\Gamma)$$

For types:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u}(\to\text{-INTRO})$$

For terms:

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR})$$

For substitutions:

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

Figure 1: Derivation rules of the theory GSeTT

Typing rules. The inference rules for the theory GSeTT are given in figure 1. We say that a context (resp. a type, term substitution) is derivable if there is a derivation tree leading to its well-formedness judgment.
We have defined the sets of free variables as a syntactic function on terms and types, thus independent of the judgments, but we are often rather interested in the variables of a typed term together. To express this, we write $\mathrm{Var}(t : A)$ for the union $\mathrm{Var}(t) \cup \mathrm{Var}(A)$, with the implicit convention that in the current context Γ, the judgment $\Gamma \vdash t : A$ is derivable.

The first few results that we mention about the theory GSeTT are proved by induction on the rules of the theory. They are quite obvious, but proving them in full details requires specifying the meta-theory that we work in much more than what have been done here. We refer the reader to [5] for details and proofs.

Lemma 6. *The following properties can be shown and are useful for later proofs*

- *if* $\Gamma \vdash A$ *then* $\Gamma \vdash$*,*
- *if* $\Gamma \vdash t : A$ *then* $\Gamma \vdash A$*,*
- *if* $\Delta \vdash \gamma : \Gamma$ *then* $\Delta \vdash$ *and* $\Gamma \vdash$*,*
- *if* $\Gamma \vdash x \underset{A}{\to} y$ *then* $\Gamma \vdash x : A$ *and* $\Gamma \vdash y : A$*,*
- *if* $\Gamma \vdash A$*, then* $\mathrm{Var}(A) \subseteq \mathrm{Var}(\Gamma)$*,*
- *if* $\Gamma \vdash t : A$ *then* $\mathrm{Var}(t : A) \subseteq \mathrm{Var}(\Gamma)$*.*

Lemma 7. *A term admits at most one type in a given context: if both* $\Gamma \vdash t : A$ *and* $\Gamma \vdash t : B$ *are derivable then* $A = B$*.*

Lemma 8. *A given judgment admits at most one derivation.*

Notational conventions. In a type $t \underset{A}{\to} u$, the type A is the common type of both t and u and is sometimes left implicit. Similarly, when a substitution $\gamma = \langle x_i \mapsto t_i \rangle_{1 \leq i \leq n}$ is such that the judgment $\Delta \vdash \gamma : \Gamma$ holds for some context $\Gamma = (y_i : A_i)_{1 \leq i \leq m}$, then necessarily $m = n$ and $x_i = y_i$ for $1 \leq i \leq n$. For this reason, when the context Γ is fixed, we may leave the variables $x_1, \ldots, x_n$ implicit and simply write

$$\gamma = \langle t_1, \ldots, t_n \rangle = \langle t_i \rangle_{1 \leq i \leq n}$$

Finally, following Lemma 8, we sometimes abusively assimilate a derivation with the judgment it derives.

2.3 The syntactic category of GSeTT Our main tool to study the semantics of a type theory is a category we associate to it, called its syntactic category. We define it in the special case of the theory GSeTT, and state some results which ensure that it is well-defined. We then study the structures present in this category, and illustrate how use those in order to study the semantics of the theory. The construction will be analogous later on in the case of the type theory CaTT, albeit more technically involved.

Admissibility of the action of the substitutions. When introducing the type theory, we have defined the actions of substitution, their compositions and the identity substitution syntactically, by induction on the signature. By induction over the rules of the theory, we can check that these operations preserves the derivability of the judgments.

Proposition 9. *The following rules are derivable*

$$\frac{\Gamma \vdash A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash A[\gamma]} \qquad\qquad \frac{\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash t[\gamma] : A[\gamma]}$$

$$\frac{\Gamma \vdash \theta : \Theta \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \theta \circ \gamma : \Theta} \qquad\qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{id}_\Gamma : \Gamma}$$

The syntactic category. The last two statements of Proposition 9 ensure that the composition of substitution and the identity substitution preserve derivability and thus can be lifted as operations on derivable objects. We keep the same notation for these operations.

Proposition 10. *The following equalities hold:*

$$\mathrm{id}_\Gamma \circ \gamma = \gamma \qquad\qquad \gamma \circ \mathrm{id}_\Delta = \gamma \qquad\qquad \gamma \circ (\delta \circ \theta) = (\gamma \circ \delta) \circ \theta$$

Note that we assume here that all the objects we manipulate are derivable, even if we leave their derivation implicit, in particular, although the second equation holds syntactically, it is not the case for the first equation which only holds for a derivable substitution $\Delta \vdash \gamma : \Gamma$, nor for the last equation which only holds for three a derivable substitutions γ, δ and θ which are composable.

The last two results of Proposition 9 as well as Proposition 10 ensure that we can build a category $\mathcal{S}_{\mathsf{GSeTT}}$, called the *syntactic category* of the theory GSeTT, whose objects are the contexts Γ such that $\Gamma \vdash$ and morphisms $\Delta \to \Gamma$ are the substitutions $\Delta \vdash \gamma : \Gamma$. The first two statements of Proposition 9 can then be read as the fact that it acts on derivable types and terms:

Proposition 11. *The composition of substitutions and the identity substitution are compatible with the action of the substitution on types and terms. More precisely, the following equations hold, for derivable objects:*

$$A[\mathrm{id}_\Gamma] = A \qquad\qquad A[\gamma \circ \delta] = A[\gamma][\delta]$$

$$t[\mathrm{id}_\Gamma] = t \qquad\qquad t[\gamma \circ \delta] = t[\gamma][\delta]$$

The Propositions 9, 10 and 12 can be summarized into the following proposition, which is crucial for studying the semantics of type theories:

Proposition 12. *The category $\mathcal{S}_{\mathsf{GSeTT}}$ carries a structure of category with families, such that, for an object Γ of $\mathcal{S}_{\mathsf{GSeTT}}$, the set Ty^{Γ} consists in the types derivable in Γ and, for A such a type, the set Tm_A^{Γ} consists in terms of type A in Γ.*

Note 13. Here we have given a presentation with named variables, but one could also give a presentation of the same type theory using unnamed variables, such as de Bruijn indices. This would lead to a slightly different notion of the syntactic category, which is essentially the previously defined syntactic category quotiented under renamings (or α-conversion) of contexts. From now on, we suppose given such a presentation with unnamed variables, so that the renamings are not explicitly taken in account in the syntactic category. Since there is no variable binders, this operation of quotienting is straightforward.

Disks and spheres contexts. In the category $\mathcal{S}_{\mathsf{GSeTT}}$, there are two classes of contexts which play an important role, the n-*disk context* D^n and the n-*sphere context* S^n. Their precise role in our theory are made clear by the Lemma 16 and by the understanding of the syntactic category provided by the Theorem 22. These contexts are defined inductively by

$$\begin{array}{lllllll} S^{-1} & = & \varnothing & \qquad & D^0 & = & (d_0 : U_0) \\ S^n & = & (D^n, d_{2n+1} : U_n) & \qquad & D^{n+1} & = & (S^n, d_{2(n+1)} : U_{n+1}) \end{array}$$

where the types U_n are inductively defined by

$$\begin{array}{lll} U_0 & = & \star \\ U_{n+1} & = & d_{2n-2} \xrightarrow[U_n]{} d_{2n-1} \end{array}$$

and where the d_i are a family of distinct variables. We reserve the notation d_i for these specific variables throughout this paper. This is simply a convenient writing convention, since ultimately we consider everything up to renaming.

Proposition 14. *For any integer n, the contexts D^n and S^n are well-formed, i.e., the following rules are admissible.*

$$\frac{}{D^n \vdash} \qquad\qquad \frac{}{S^n \vdash}$$

Proof. We prove the validity of these contexts by induction. First notice that $S^{-1} = \varnothing$ is well defined by the rule (EC), and that by applying successively the rules (CE) and (OBJ), D^0 is also well defined. Then, suppose that S^{k-1} and D^k are valid contexts. The rule (AX) ensures that $D^k \vdash d_{2k} : U_k$, and by Lemma 6, this proves that $D^k \vdash U_k$, since moreover $d_{2k+1} \notin \mathrm{Var}(D^k)$, the rule (CE) applies and shows $S^k \vdash$. Moreover, the rule (AX) applies twice to show both $S^k \vdash d_{2k} : U_k$ and $S^k \vdash d_{2k+1} : U_k$, hence by the rule (HOM), this proves $S^k \vdash U_{k+1}$ and since $d_{2(k+1)} \notin S^k$, the rule (CE) applies and proves $D^{k+1} \vdash$. ☐

We can also define two substitutions $D^{n+1} \vdash s_n : D^n$ and $D^{n+1} \vdash t_n : D^n$, with the following formulas

$$s_n = \langle d_0 \mapsto d_0, d_1 \mapsto d_1, \ldots, d_{2n-1} \mapsto d_{2n-1}, d_{2n} \mapsto d_{2n} \rangle$$
$$t_n = \langle d_0 \mapsto d_0, d_1 \mapsto d_1, \ldots, d_{2n-1} \mapsto d_{2n-1}, d_{2n} \mapsto d_{2n+1} \rangle$$

One can check that the morphisms define this way satisfy the co-globular relations (1), hence the disks objects are coglobular objects in the category $\mathcal{S}_{\mathsf{GSeTT}}$. We reformulate this fact by the following definition.

Definition 15. We define the functor $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$, sending every object n on the disk context D^n and the morphisms σ_n (resp. τ_n) on the substitution s_n (resp. t_n) in $\mathcal{S}_{\mathsf{GSeTT}}$.

Familial representability of types. The following lemma is central in our study of the type theory GSeTT. It allows to understand both types and terms as special cases of substitutions, and the action of substitution then becomes pre-composition.

Lemma 16. *For any natural number n, the map*

$$\begin{array}{rcl} \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, S^{n-1}) & \to & \left\{A \in \mathrm{Ty}^\Gamma \mid \dim(A) = n-1\right\} \\ \gamma & \mapsto & U_n[\gamma] \end{array}$$

is an isomorphism natural in Γ. Given a type A of dimension $n-1$, we denote the associated substitution

$$\chi_A : \Gamma \to S^{n-1}$$

Moreover, the maps

$$\begin{array}{rcl} (\mathcal{S}_{\mathsf{GSeTT}}/S^{n-1})(\Gamma \xrightarrow{\chi_A} S^{n-1}, D^n \xrightarrow{\pi} S^{n-1}) & \to & \mathrm{Tm}^\Gamma_A \\ \gamma & \mapsto & d_{2n}[\gamma] \end{array}$$

are also isomorphisms, natural in Γ (the source is a hom-set in the slice category of $\mathcal{S}_{\mathsf{GSeTT}}$ over S^{n-1}). Given a term $t \in \mathrm{Tm}^\Gamma_A$ of type A, we denote the associated substitution over χ_A by $\chi_t : \Gamma \to D^n$, in such a way that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\chi_t"] \arrow[dr, "\chi_A"'] & D^n \arrow[d, "\pi"] \\ & S^{n-1} \end{tikzcd}$$

Proof. We first prove that the first part of the statement implies the second one, for a given natural number n. This is a consequence of the fact that the context D^n is defined to be $(S^{n-1}, d_{2n} : U_n)$. Indeed, an object in $\mathcal{S}_{\mathsf{GSeTT}}/S^{n-1}$ is a context that comes equipped with a substitution $\gamma : \Gamma \to S^{n-1}$, and the universal property of the context comprehension operation states exactly that there is a natural isomorphism $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n) \cong \mathrm{Tm}^\Gamma_{U_n[\gamma]}$. Using the previous natural isomorphism, one can write γ as χ_A and $U_n[\gamma]$ then simplifies to A, which proves the natural isomorphism $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n) \cong \mathrm{Tm}^\Gamma_A$. We now prove by induction over the dimension n that the first part of the judgment holds.

- Case $n = 0$. The context $S^{-1} = \varnothing$ is terminal: there is always exactly one substitution $\Gamma \vdash \langle\rangle : \varnothing$. Similarly there is always exactly one type of dimension -1 derivable in Γ, which is the type $\star$, and which is the type U_0 by definition.
- Suppose that the result holds for the sphere S^{n-1}. Then, by the second part of the result that we have already proven, we get a natural isomorphism $(\mathcal{S}_{\mathsf{GSeTT}}/S^{n-1})(\Gamma, D^n) \cong \mathrm{Tm}^\Gamma_A$. Substitutions $\Gamma \vdash \gamma : S^n$ are exactly the ones of the form $\langle\gamma', u\rangle$ and are derived by the following application of the rule (SE)

$$\frac{\Gamma \vdash \gamma' : D^n \qquad D^n \vdash U_n \qquad \Gamma \vdash u : U_n[\gamma']}{\Gamma \vdash \langle\gamma', u\rangle : S^n}$$

The substitutions $\Gamma \vdash \gamma : S^n$ are thus naturally isomorphic to pairs γ', u, with $\Gamma \vdash \gamma' : D^n$ and $\Gamma \vdash u : U_n[\gamma']$. By induction, the substitutions $\Gamma \vdash \gamma' : D^n$ are of the form χ_t, for $\Gamma \vdash t : A$ a term in Γ. Then the type $U_n[\chi_t]$ rewrites as A by naturality of the previous transformation. So these substitutions are naturally isomorphic to pairs of terms of dimension n and of the same type in Γ, which are exactly the types in Γ of dimension n. □

Note 17. This proof does not rely on how the terms are constructed, so no matter what the term constructors are, this result will always hold.

We can reformulate this result in several ways. First, we can collect together all the isomorphisms $\{A \in \mathrm{Ty}^\Gamma \mid \dim A = n-1\} \cong \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, S^{n-1})$ into a single natural isomorphism

$$\mathrm{Ty}^\Gamma \cong \coprod_{n\in\mathbb{N}} \mathcal{S}_{\Gamma, S^{n-1}}$$

In other words, we have proven that the family $S^\bullet$ *familially represents* the functor Ty. We can also unravel a bit this proposition, showing that any type $\Gamma \vdash A$ corresponds uniquely to a substitution $\Gamma \vdash \chi_A : S^{n-1}$, and that any term $\Gamma \vdash t : A$ corresponds uniquely to a substitution $\Gamma \vdash \chi_t : D^n$, in such a way that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\chi_t"] \arrow[dr, "\chi_A"'] & D^n \arrow[d, "\pi"] \\ & S^{n-1} \end{tikzcd}$$

To simplify things further, we write $\mathrm{ty} : D^n \to S^{n-1}$ for the projection substitution, so that we have $\mathrm{ty} \circ \chi_t = \chi_A$. In other words, ty acts on terms by giving their associated types. The definition of the morphism ty along with Lemma 1 shows the following:

Lemma 18. *In the category $\mathcal{S}_{\mathsf{GSeTT}}$, a context of the form $(\Gamma, x : A)$ is obtained as the pullback*

$$\begin{tikzcd} (\Gamma, x : A) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner"] & D^n \arrow[d, "\mathrm{ty}"] \\ \Gamma \arrow[r, "\chi_A"'] & S^{n-1} \end{tikzcd}$$

It is straightforward using Lemma 18 to check that the sphere contexts can be obtained as iterated pullbacks of the disks contexts, dually to the way topological spheres can be obtained as pushout of topological disks.

Lemma 19. *The sphere context S^n is obtained as the following pushout*

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D^n \arrow[d, "\pi"] \\ D^n \arrow[r, "\pi"] & S^{n-1} \end{tikzcd}$$

The syntactic category of GSeTT. We now characterize the syntactic category of GSeTT. This is an important step in order to study the models of the theory, since understanding precisely the syntactic category gives good insights on the functors mapping out of it. Interestingly, in all the cases we study here, it always turn out that the syntactic category is dual to the category of finitely generated objects that we are studying, in accordance with the Gabriel-Ülmer duality [16]. In order to prove this result, we introduce a functor that we denote V (the V stands for "variable"), that we describe as follows.

Definition 20. We define a functor $V : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{FinGSet}^{\mathrm{op}}$, which to any context $\Gamma = (x_i : A_i)$, then associates

$$(V\Gamma)_n = \{x_i \mid \dim(A_i) = n\} = \{\text{derivable terms of dimension } n \text{ in } \Gamma\}$$

and to any substitution $\Delta \vdash \langle x_i : t_i\rangle : \Gamma$ associates the map

$$\begin{array}{rccc} V\gamma & : & V\Gamma & \to & V\Delta \\ & & x_i & \mapsto & t_i \end{array}$$

or equivalently, we require the equation $(V\gamma)x = V(x[\gamma])$.

Lemma 21. *The functor V is well-defined.*

Proof. For x of type A in Γ, with $\dim(A) = n+1$, by definition of the dimension, A is of the form $A = y \to z$, for two derivable terms y and z, with $\dim_\Gamma(y) = \dim_\Gamma(z) = n$. We therefore have $y, z \in (V\Gamma)_n$, and we define $s(x) = y$ and $t(x) = z$. The derivation rule for A implies that y and z have the same type, thus $s(y) = s(z)$ and $t(y) = t(z)$, which proves that the globular relations are satisfied, and that $V\Gamma$ is indeed a globular set.

Let $\Delta \vdash \gamma : \Gamma$ be a substitution, and write $\Gamma = (x_i : A_i)$, then the substitution γ is of the form $\gamma = \langle x_i : t_i\rangle$, where t_i is a derivable term in the context Δ, i.e., $t_i \in V\Delta$. Suppose that x is of type $y \to z$ in Γ, then $x[\gamma] = y[\gamma] \to z[\gamma]$ in Δ. This means that as an element of $F\Delta$, $x[\sigma]$ satisfies $s(x[\gamma]) = y[\gamma]$ and $t(x[\gamma]) = z[\gamma]$, or in other words, $s((F\gamma)x) = (F\gamma)(sx)$ and $t((F\gamma)x) = (F\gamma)(tx)$. Hence $F\gamma$ defines a morphism of globular sets. □

Theorem 22. *The functor V is part of an equivalence of categories $\mathcal{S}_{\mathsf{GSeTT}} \simeq \mathbf{FinGSet}^{\mathrm{op}}$.*

Proof. We first show that V is full and faithful. Consider two substitutions γ and δ such that $V\gamma = V\delta$. This implies in particular that for all variables x in Γ, $(V\gamma)x = (V\delta)x$, thus $x[\gamma] = x[\delta]$. By the Lemma 5, this proves that $\gamma = \delta$, hence V is faithful. Dually, consider two contexts Γ and Δ, where $\Delta = (x_i : A_i)_{0\leq i\ \leq l}$, together with a morphism of globular sets $f : V\Gamma \to V\Delta$. Then one can define the substitution $\gamma_f = \langle x_i\ : f(x_i)\rangle_{1\leq i\ \leq l}$. We check by induction on the length l of Δ that this produces a well-defined substitution γ_f such that $V(\gamma_f) = f$. If $l = 0$ then $\Delta = \varnothing$ and $\gamma_f = \langle\rangle$, then the rule (ES) gives a derivation of $\Gamma \vdash \langle\rangle : \varnothing$. If $\Delta = (\Delta', x_{l+1} : A_{l+1})$, then the natural inclusion $V\Delta' \hookrightarrow V\Delta$ induces by composition a map $f' : V\Delta' \to V\Gamma$. By induction hypothesis, we have $\Gamma \vdash \gamma_{f'} : \Delta'$, and since Δ is a context, we also have $\Delta \vdash A_{l+1}$. Moreover, if $A_{l+1} = \star$, then $\Gamma \vdash f(x_{l+1}) : \star$ since f preserves the dimension, and otherwise $A_{l+1} = y \to z$, and $\Gamma \vdash f(x_{l+1}) : f(y) \to f(z)$ since f is a morphism of globular sets. In both cases, this proves that $\Gamma \vdash f(x_{l+1}) : A_{l+1}[\gamma_{f'}]$. By application of the rule (SE), this proves that $\Gamma \vdash \langle\gamma_{f'}, x_{l+1} : f(n_{l+1})\rangle : \Delta$. Since $\gamma_f = \langle\gamma_{f'}, x_{l+1} : f(x_{l+1})\rangle$, this proves that γ_f is well defined, and by definition it satisfies $V\gamma = f$. Hence the functor V is full.

Moreover, V is essentially surjective. Indeed, considering a finite globular set X, we show by induction on the number of elements of X that we can construct a context Γ such that $V\Gamma = X$. If X is the empty globular set, then $\Gamma = \varnothing$ is well defined by the rule (EC), otherwise, if X is not empty, consider an element x of maximal dimension in X and consider the globular set Y obtained by removing this element from X. By induction, the context Δ constructed from Y is well-defined. Moreover, if x is of dimension 0, then define $A\ = \star$ and we have $\Delta \vdash A$, and otherwise, we have $\Delta \vdash sx : B$ and $\Delta \vdash tx : B$ since both sx and tx are parallel elements in Y, and define $A\ = s\ x \to tx$, this shows that $\Delta \vdash A$. In both cases, we have $\Delta \vdash A$, and the rule

(CE) applies to prove that $\Delta, x : A \vdash$. Moreover $V(\Delta, x : A)$ is obtained from $V\Delta$ by adding one element x' of the same dimension as x, and such that $sx' = sx$ and $tx' = tx$ if this dimension is not 0. Since by induction $V\Delta = Y$, we deduce $V(\Delta, x : A) = X$. This construction is not canonical, and there are in general many contexts Γ such that $V\Gamma = X$, but the fact that we can construct one shows that V is essentially surjective. Since the functor V is fully faithful and essentially surjective, it is an equivalence of categories. □

We can give an alternative description of V in the light of Lemma 16. Indeed a term of dimension n in Γ is simply a substitution $\Gamma \to D^n$, hence $V(\Gamma)_n = \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n)$. Consider the generalized nerve functor $\mathcal{S}_{\mathsf{GSeTT}}(\_, D^\bullet) : \mathcal{S}_{\mathsf{GSeTT}}^{\mathrm{op}} \to \mathbf{GSet}$ associated to the inclusion $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$. This functor can be seen as a functor $\mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{GSet}^{\mathrm{op}}$. By the previous remark, it coincides with V on objects, and hence it restricts to a functor $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^\bullet) \to \mathbf{FinGSet}$. Moreover, for any variable $\Gamma \vdash x : A$ and any substitution $\Delta \vdash \gamma : \Gamma$, we have the equalities

$$\chi_x \circ \gamma = \chi_{x[\gamma]}$$
$$V(\gamma)(x) = V(x[\gamma])$$

which show that the functors V and $\mathcal{S}_{\mathsf{GSeTT}}(\_, D^\bullet)$ coincide on morphisms. From now on, we thus identify V with the generalized nerve functor $\mathcal{S}_{\mathsf{GSeTT}}(\_, D^\bullet)$, and use this point of view in more involved situations of interest.

Remark 23. Under the equivalence of categories of Theorem 22, the globular set D^n corresponds exactly to the context D^n, and the globular set S^n corresponds to the globular set S^n. This justifies the choice of the same notations for the contexts and the globular sets.

2.4 Models of the type theory GSeTT We can use the characterization of the syntactic category of GSeTT obtained in previous section in order to study its models. This relies heavily on the fact that $\mathcal{S}_{\mathsf{GSeTT}} \simeq \mathbf{FinGSet}^{\mathrm{op}}$ is the free finite completion of the category $\mathcal{G}^{\mathrm{op}}$. We start here by proving this fact, even though this is standard material in category theory. Indeed, we later use a refinement of this construction, and the result that we prove here are useful for establishing this refined version.

Pointwise right Kan extension. Consider an object Γ in the category $\mathcal{S}_{\mathsf{GSeTT}}$, i.e., a valid context of the theory GSeTT. We can consider the coma category $\Gamma \downarrow D^\bullet$. An object of this category is a pair (n, χ_x) where n is an object of $\mathcal{G}^{\mathrm{op}}$ and χ_x is a morphism $\Gamma \to D^n$ (by Lemma 16, all such morphisms are of the form χ_x for some variable x in Γ). A morphism of between two objects (n, χ_t) and (m, χ_u) is a morphism $\alpha : m \to n$ in $\mathcal{G}$, such that $\chi_t \circ D^\alpha = \chi_u$ (i.e., , through the correspondence of Lemma 16, $t[D^\alpha] = u$ in the theory GSeTT). There is a forgetful functor $\Pi_\Gamma : \Gamma \downarrow D^\bullet \to \mathcal{G}^{\mathrm{op}}$. Then, consider a category $\mathcal{C}$ along with a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ such that for every object Γ of $\mathcal{S}_{\mathsf{GSeTT}}$ the following limit exists

$$\lim\left(\Gamma \downarrow D^\bullet \xrightarrow{\Pi_\Gamma} \mathcal{G}^{\mathrm{op}} \xrightarrow{F} \mathcal{C}\right)$$

It is then a classical result in category theory [27, th.6.2.1 and 6.3.7] that the right Kan extension $\mathrm{Ran}_{D^\bullet} F$ exists

$$\begin{tikzcd}
\mathcal{S}_{\mathsf{GSeTT}} \arrow[r, dotted, "\mathrm{Ran}_{D^\bullet} F"] & \mathcal{C} \\
\mathcal{G}^{\mathrm{op}} \arrow[u, "D^\bullet"] \arrow[ur, "F"'] \arrow[u, phantom, shift left=6, "\Rightarrow"]
\end{tikzcd}$$

and is pointwise, meaning that it is given by the formula

$$(\mathrm{Ran}_{D^\bullet} F)\Gamma = \lim\left(\Gamma \downarrow D^\bullet \xrightarrow{\Pi_\Gamma} \mathcal{G}^{\mathrm{op}} \xrightarrow{F} \mathcal{C}\right)$$

Note that all these limits are finite, since the objects in $\Gamma \downarrow D^\bullet$ are the terms in Γ in the theory GSeTT, which are the variables, and each context has a finite number of variables.

A characterization with the nerve functor. Given a category equipped with a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ (we call such a functor a *contravariant globular structure*), we define an associated nerve functor $T_F : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{G}}$ by the formula $T_F(\Gamma) = \mathcal{C}(\Gamma, F\_)$. In the case of the disk functor is $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$, we recover the functor V that we have defined in Section 2.3. In the cases of interest here, $\mathcal{C}$ is a category with family, and in this case, we will show that T_F classifies some terms. The nerve functor allows us to characterize the maps in a categor whose target is the image of the right Kan extension.

Lemma 24. *We have a natural isomorphism*

$$\mathcal{C}(\Gamma, (\mathrm{Ran}_D F)\Delta) \cong \widehat{\mathcal{G}}(V\Delta, T_F\Gamma)$$

Proof. Since $(\mathrm{Ran}_D F)\Delta$ is defined as a limit, we have a natural isomorphism between the set $\mathcal{C}(\Gamma, (\mathrm{Ran}_D F)\Delta)$ and the set of cones of apex Γ over the diagram $\Delta \downarrow D \to \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$. It is a known result of the theory of Kan extensions that the latter set is naturally isomorphic to $\widehat{\mathcal{G}}(V\Delta, T\Gamma)$, see [27, lemma 6.3.8]. Under this correspondence, one can see this equation as a compact way of stating the characterization of $(\mathrm{Ran}_{D^\bullet} F)\Delta$ as a limit, through the universal property. $\square$

Although this result is introduced here using sophisticated categorical machinery, such as Kan extensions, it reflects a simple fact about the syntax of a theory. First consider the case where the category is $\mathcal{S}_{\mathsf{GSeTT}}$ itself, then the right Kan extension $\mathrm{Ran}_{D^\bullet} D^\bullet$ turns out to be the identity functor $\mathrm{id}_{\mathcal{S}_{\mathsf{GSeTT}}}$ (this is an immediate consequence of Theorem 27 that we show later on). The above lemma then states that $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, \Delta) \cong \widehat{\mathcal{G}}(V\Delta, V\Gamma)$, that is, it states that substitutions from Γ to Δ are given by the data of a variable of Γ for every variable of Δ in a way that is compatible with the source and targets. In Section 4 we present a type theory whose syntactic category satisfies the same kind of properties, and discuss a generalization of this result to that theory in Section 6.4.

Lemma 25. *The pointwise right Kan extension* $\mathrm{Ran}_{D^\bullet} F$ *preserves limits.*

Proof. Consider a diagram $A : I \to \mathcal{S}_{\mathsf{GSeTT}}$, together with its limit $\lim A$. Then, for any object Γ in the category $\mathcal{C}$, we have the following equalities

$$\begin{aligned}
\mathcal{C}(\Gamma, (\mathrm{Ran}_D F)(\lim A)) &\cong \widehat{\mathcal{G}}(V(\lim A), T_F\Gamma) && \text{by Lemma 24}\\
&\cong \widehat{\mathcal{G}}(\mathrm{colim}(V \circ A), T_F\Gamma) && \text{since } V \text{ is an equivalence of categories}\\
&\cong \lim(\widehat{\mathcal{G}}(V \circ A, T_F\Gamma)) && \text{by contuity of the Hom functor}\\
&\cong \lim \mathcal{C}(\Gamma, \mathrm{Ran}_D(F \circ A)) && \text{by Lemma 24}\\
&\cong \mathcal{C}(\Gamma, \lim \mathrm{Ran}_D(F \circ A^{\mathrm{op}})) && \text{by contuity of the Hom functor}
\end{aligned}$$

This shows that $(\mathrm{Ran}_D F)(\lim A)$ satisfies the characterization of $\lim \mathrm{Ran}_{D^\bullet}(F \circ A)$ given in Lemma 24. $\square$

The Kan extension is an extension. Although it is not always necessarily the case in general that the Kan extension really extends the functor, it is the case for right Kan extension along the functor $D^\bullet$.

Lemma 26. *The pointwise right Kan extension* $\mathrm{Ran}_{D^\bullet} F$ *extends the functor* F*, that is, there is a natural isomorphism*

$$(\mathrm{Ran}_{D^\bullet} F)D^n \cong Fn$$

Proof. We use the characterization given by Lemma 24. For any object Γ in $\mathcal{C}$, we have $\mathcal{C}(\Gamma, (\mathrm{Ran}_{D^\bullet} F)D^n) \cong \widehat{\mathcal{G}}(VD^n, T_F\Gamma)$. Note that V is an isomorphism that sends the object D^n in $\mathcal{S}_{\mathsf{GSeTT}}$ onto the representable object D^n in $\widehat{\mathcal{G}}$. Hence, by the Yoneda lemma,

$$\begin{aligned}\mathcal{C}(\Gamma, (\mathrm{Ran}_{D^\bullet} F)D^n) &\cong (T_F\Gamma)_n \\ &\cong \mathcal{C}(\Gamma, Fn)\end{aligned}$$

Since this isomorphism is natural isomorphism in Γ, it shows that $(\mathrm{Ran}_{D^\bullet} F)D^n$ is the limit of the diagram with a single point Fn, hence $(\mathrm{Ran}_{D^\bullet} F)D^n \cong Fn$. □

The Kan extension realizes the universal completion. We have shown that the pointwise right Kan extension of a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ along $D^\bullet$ exists as soon as $\mathcal{C}$ is finitely complete, and moreover, Lemma 25 shows that it then necessarily preserves all limits, so in particular it preserves the finite limits. Hence, denoting $[\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}]_{\mathrm{flim}}$ the functors preserving the finite limits, there is a functor $\mathrm{Ran}_{D^\bullet} : [\mathcal{G}^{\mathrm{op}}, \mathcal{C}] \to [\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}]_{\mathrm{flim}}$. Standard results ensure that this functor is part of an equivalence:

Theorem 27. *Consider a finitely complete category* $\mathcal{C}$ *together with a functor* $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$*, then the following pair of functors defines an equivalence of categories*

$$\begin{array}{rcrcl} \_ \circ D^\bullet & : & [\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}]_{\mathrm{flim}} & \to & [\mathcal{G}^{\mathrm{op}}, \mathcal{C}] \\ & & F & \mapsto & F \circ D^\bullet \end{array} \qquad \begin{array}{rcrcl} \mathrm{Ran}_{D^\bullet} & : & [\mathcal{G}^{\mathrm{op}}, \mathcal{C}] & \to & [\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}]_{\mathrm{flim}} \\ & & F & \mapsto & \mathrm{Ran}_{D^\bullet} F \end{array}$$

Proof. First note that Lemma 26 shows that for any functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, we have a natural transformation $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet) \cong F$. Conversely, we show that there is a natural isomorphism $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet) \cong F$, i.e., that every functor preserving the finite limits is isomorphic to the Kan extension of its restriction. First note that by Lemma 26 shows that F and $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet)$ coincide on all the disk objects. Moreover, any object Γ is a finite limit of disk objects, by Theorem 22 along the fact that every presheaf is a colimit of representable. Since both F and $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet)$ coincide on disk objects and preserve the finite limit, they coincide on Γ, hence they are naturally isomorphic. □

Models of the theory GSeTT. The models of the type theory GSeTT are now easily characterized using all the tools we have introduced so far.

Theorem 28. *The models of the theory* GSeTT *are the globular sets. More precisely there is an equivalence of categories*

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \simeq \mathbf{GSet}$$

Proof. In the theory GSeTT, there are only variables and no term constructors, hence every map in the category $\mathcal{S}_{\mathsf{GSeTT}}$ is a display map. Since a category with families has pullbacks along display maps, $\mathcal{S}_{\mathsf{GSeTT}}$ has all pullbacks, and since it also has a terminal object, it has all finite limits. Moreover, Lemma 3 shows that the models are the functors preserving the terminal object and all pullbacks, hence they are the functors preserving all finite limits.

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \simeq [\mathcal{S}_{\mathsf{GSeTT}}, \mathbf{Set}]_{\mathrm{flim}}$$

Since **Set** is finitely complete, the result is then given by Theorem 27. ☐

Note that this proof consists in two parts:

1. we restate the models as being equivalent to the functors preserving the finite limits,
2. we use standard categorical machinery to show that these are equivalent to globular sets.

In the first step of the proof, we use the canonical structures of category with families of **Set** and of GSeTT to achieve this reformulation, whereas in the second part we work with plain categories, without making any use of the structures of category with families. Although it works in this case, this approach does not generalize easily, in particular it does not allow for studying models in other categories than **Set**. It also does not work as easily for other categories with families since the equivalence between $\mathcal{S}_{\mathsf{GSeTT}}$ and the opposite of **FinGSet** (Theorem 22) does not generalize. For this reason, we now present a generalization of this theorem, in the form of an initiality theorem for the category $\mathcal{S}_{\mathsf{GSeTT}}$. The directing idea for this result is to remove the first step of this proof and refine the second step to take in account the interaction of the category with families on both sides.

2.5 Globular categories with families We now introduce the notion of *globular categories with families*, which are particular categories with families that share a lot of structural properties with the category $\mathcal{S}_{\mathsf{GSeTT}}$.

Category with globular display maps. In order to define globular categories with families, we need to prove the following result, concerning categories equipped with a contravariant globular structure which is compatible with the structure of category with families.

Definition 29. Given a category $\mathcal{C}$ equipped with a contravariant globular structure $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, we say that a diagram $I \to \mathcal{C}$ is *globular* if it factors through F, and we call a *globular limit* the limit of a globular diagram.

Lemma 30. *Consider a category with families $\mathcal{C}$ equipped with a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ such that all the maps of $\mathcal{G}^{\mathrm{op}}$ are sent to display maps in $\mathcal{C}$. Then the category $\mathcal{C}$ has all finite globular limits, i.e., for every finite diagram $\mathcal{D} : I \to \mathcal{G}$ in the category $\mathcal{G}$, the diagram $F \circ \mathcal{D}^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{C}$ has a limit in $\mathcal{C}$. Moreover, consider two categories with families equipped with such contravariant globular structures $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ and $G : \mathcal{G}^{\mathrm{op}} \to \mathcal{D}$ along with a morphism of categories with families $f : \mathcal{C} \to \mathcal{D}$ such that $f \circ F \cong G$, then f preserves all the finite globular limits.*

Proof. Recall that $\mathcal{C}$ has all pullbacks along display maps, and since all the maps in $\mathcal{G}$ are sent to display maps, $\mathcal{C}$ has all the pullbacks along the images of the maps of $\mathcal{G}$. Since $\mathcal{C}$ has a terminal object, this implies that $\mathcal{C}$ has all the globular finite limits, which are generated by the terminal object and the globular pullbacks. Moreover, the morphism of category with families f preserves the pullbacks along the display maps, so in particular it preserves the globular pullbacks, and it also preserves the terminal object, hence it preserves all finite globular limits. ☐

In a category with families equipped with a contravariant globular structure $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ we call *disks objects* and denote D^n the images Fn of the objects n of $\mathcal{G}$. This lemma enables us define the *sphere objects* S^n_F for $n \in \mathbb{N} \cup \{-1\}$, by induction as iterated pullbacks as follows

$$S^{-1} = \varnothing \qquad\qquad \begin{tikzcd} S^n_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(n) \arrow[d, "\mathrm{ty}"] \\ F(n) \arrow[r, "\mathrm{ty}"'] & S^{n-1}_F \end{tikzcd}$$

where the map $\mathrm{ty} : F(n) \to S^{n-1}_F$ is obtained by universal property property of the pullback using the source and target morphism of $\mathcal{G}$ as follows:

$$\begin{tikzcd} F(n) \arrow[dr, "\mathrm{ty}"] \arrow[drr, bend left, "s=F\sigma"] \arrow[ddr, bend right, "t=F\tau"'] & & \\ & S^{n-1}_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(n-1) \arrow[d, "\mathrm{ty}"] \\ & F(n-1) \arrow[r, "\mathrm{ty}"'] & S^{n-2}_F \end{tikzcd}$$

Globular category with families. Using the definition of the sphere objects, we can state the definition of a globular category with families.

Definition 31. A *globular category with families* consists of a category with families $\mathcal{C}$ equipped with a functor

$$F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$$

that sends all maps to display maps, such that there is a natural transformation

$$\mathrm{ty} : \coprod_{n\in\mathbb{N}} \mathcal{C}(\Gamma, S^{n-1}_F) \to \mathrm{Ty}(\Gamma)$$

in such a way that there is an isomorphism $(S^{n-1}_F, \mathrm{ty}(\mathrm{id}_{S^{n-1}})) \cong D^n$, which makes the triangle

$$\begin{tikzcd} (S^{n-1}_F, \mathrm{ty}(\mathrm{id}_{S^{n-1}_F})) \arrow[r, "\sim"] \arrow[d, "\pi"'] & D^n \arrow[d, "\mathrm{ty}"] \\ S^{n-1}_F \arrow[r, "\mathrm{id}_{S^{n-1}_F}"'] & S^{n-1}_F \end{tikzcd}$$

commute.

We use the same notation ty for this natural transformation and for the map $F(n) \to S^{n-1}_F$ to emphasize the important connection between those given by the commutative square. We have already proved that $\mathcal{S}_{\mathsf{GSeTT}}$ is a globular categories with families, where the contravariant globular structure is given by the disk contexts, and the ty natural transformation is defined, for $\Gamma \vdash \gamma : S^{n-1}_F$, by $\mathrm{ty}(\gamma) = A_{n-1}[\gamma]$. In fact, a globular category with families can be thought of as a category with families which supports the type constructors $\star$ and $\to$ given in the theory GSeTT. There is however a subtlety in that the types $\to$ are only allowed to be constructed from on a type which is itself a an iteration of $\to$ from the type $\star$, and there may well exist types A which are not of this form and do not support the $\to$ constructor.

Morphisms of globular categories with families. We define the notion of *morphism of globular categories with families* to be a morphism of categories with families which:

- commutes (in a bicategorical sense) with the contravariant globular structures. More precisely, given two globular categories with families $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ and $G : \mathcal{G}^{\mathrm{op}} \to \mathcal{D}$, a morphism between them is a morphism of categories with families $(f, \phi) : \mathcal{C} \to \mathcal{D}$ such that the triangle

$$\begin{tikzcd} & \mathcal{G}^{\mathrm{op}} \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathcal{C} \arrow[rr, "f"'] & & \mathcal{D} \end{tikzcd}$$

  is filled by a (necessarily unique) natural isomorphism.
- respect the natural transformation ty, that is for any object Γ in $\mathcal{C}$ and any map $\gamma : \Gamma \to S_F^{n-1}$, one can consider $\mathrm{ty}(\gamma) \in \mathrm{Ty}(\Gamma)$ and use the structure of morphism of category with families to compute $\phi(\mathrm{ty}(\gamma)) \in \mathrm{Ty}(f(\Gamma))$. On the other hand, one can consider $f\gamma : f\Gamma \to f(S_F^{n-1})$. Since $f(F(n-1))$ and $G(n-1)$ are naturally isomorphic, and the sphere objects are computed from those, $S_{f\circ F}^{n-1}$ and S_G^{n-1} are naturally isomorphic. Moreover, since f is a morphism of categories with families, it preserves pullbacks along display maps, and in particular we have $f(S_F^{n-1}) \cong S_{f\circ F}^{n-1}$. This proves that $f(S_F^{n-1}) \cong S_G^{n-1}$, and hence $f(\gamma)$ can be seen as a morphism $f(\Gamma) \to S_G^{n-1}$. Under this isomorphism, one can consider $\mathrm{ty}(f(\gamma)) \in \mathrm{Ty}(f(\Gamma))$. For a morphism of globular category with families, we require these two types in $f(\Gamma)$ to agree, what summarize, under the implicit isomorphisms, as

$$\mathrm{ty}(f(\gamma)) = \phi(\mathrm{ty}(\gamma))$$

Lemma 32. *A morphism of globular categories with families preserves finite globular limits(i.e., globular limits over finite diagrams).*

Proof. A morphism of globular categories with families is by definition a morphism of category with families, hence it preserves the empty context and the context comprehension operation. This implies that it preserves both the terminal object and the pullback along display maps. Since all the globular maps are display maps, it preserves the terminal object and all pullback of globular spans. Since all globular finite limits are computed from those, it necessarily preserves all globular finite limits. □

Right Kan extension of a globular category with families. Consider a globular category with families $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$. By Lemma 32, all the diagrams in $\mathcal{C}$ factoring through F have a limit. This is in particular the case for the diagrams defining the right Kan extension $\mathrm{Ran}_{D^\bullet} F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, which thus exists and can be computed pointwise.

Lemma 33. *The pointwise right Kan extension* $\mathrm{Ran}_{D^\bullet} F$ *preserves the disks and spheres. More precisely, we always have isomorphisms*

$$(\mathrm{Ran}_{D^\bullet} F)D^n \cong Fn \qquad\qquad (\mathrm{Ran}_D F)S^n \cong S_F^n$$

Under these isomorphisms assemble into a natural isomorphism $(\mathrm{Ran}_{D^\bullet} F)\,\mathrm{ty} \cong \mathrm{ty}$.

Proof. Lemma 26 already shows that we have $(\mathrm{Ran}_{D^\bullet} F)D^n \cong Fn$. Note that the contexts S^n along with the maps $\mathrm{ty} : D^n \to S^n$ are constructed by taking limits of disk contexts by

Lemma 19. By Lemma 25, the functor $\mathrm{Ran}_{D^\bullet} F$ preserves the limits, so that their images are constructed as the corresponding limit of the images of the disks $D^\bullet$. Since we have proved that $(\mathrm{Ran}_{D^\bullet} F)D^n \cong Fn$, the image of the sphere and the ty morphisms are constructed as the corresponding limits using the objects Fn. These limits are exactly the sphere S^n_F and the map ty. ☐

Lemma 34. *The functors* $\mathrm{Ran}_{D^\bullet} F$ *can be chosen in a unique way as a morphism of globular category with families.*

Proof. Suppose that $(\mathrm{Ran}_{D^\bullet} F, \phi)$ is a morphism of globular category with families, then for every context Γ in $\mathcal{S}_{\mathsf{GSeTT}}$ and every type $\Gamma \vdash A$, we necessarily have $\phi(A) = \mathrm{ty}((\mathrm{Ran}_{D^\bullet} F)\chi_A)$, by definition of a morphism of globular category with families. Moreover, every term in the theory $\mathcal{S}_{\mathsf{GSeTT}}$ is a variable, hence it is the image of a universal term associated to a context comprehension operation, and any morphism of category with families preserves those on the nose, hence the image $\phi(t)$ is also determined for every term $\Gamma \vdash t : A$. Conversely, choosing these two associations as the definitions of ϕ on terms and types lead to a morphism of categories with families. By Lemma 18, along with Lemma 25 and Lemma 33, we have the following computation

$$\begin{aligned}
(\mathrm{Ran}_{D^\bullet} F)(\Gamma, x : A) &\cong (\mathrm{Ran}_{D^\bullet} F)(\lim \Gamma \overset{\chi_A}{\to} S^{n-1} \overset{\mathrm{ty}}{\leftarrow} D^n) \\
&\cong \lim\left((\mathrm{Ran}_{D^\bullet} F)\Gamma \xrightarrow{(\mathrm{Ran}_{D^\bullet} F)\chi_A} (\mathrm{Ran}_{D^\bullet} F)S^{n-1} \xleftarrow{(\mathrm{Ran}_{D^\bullet} F)\,\mathrm{ty}} (\mathrm{Ran}_{D^\bullet})D^n \right) \\
&\cong \lim\left((\mathrm{Ran}_{D^\bullet} F)\Gamma \xrightarrow{(\mathrm{Ran}_{D^\bullet} F)\chi_A} S^{n-1} \xleftarrow{\mathrm{ty}} Fn \right) \\
&\cong ((\mathrm{Ran}_{D^\bullet} F)\Gamma, \phi(A))
\end{aligned}$$

To construct a morphism of category with families, we require this equality to hold on the nose, and we simply have an isomorphism here. However, $\mathrm{Ran}_{D^\bullet} F$ is defined by universal property, and is subject to a choice of limits, so we can make this choice in such a way that this equality holds on the nose. We perform this choice by induction on the context Γ: for the empty context $\varnothing$, we choose $(\mathrm{Ran}_{D^\bullet} F)\varnothing$ to be the distinguished terminal object $\varnothing$ given by the category with families structures on $\mathcal{C}$, and for a context of the form $(\Gamma, x : A)$, we choose $(\mathrm{Ran}_{D^\bullet} F)(\Gamma, x : A) = ((\mathrm{Ran}_{D^\bullet} F)\Gamma, \phi(A))$. This choice is possible since every context decomposes in a unique way. ☐

From now on, when we write $\mathrm{Ran}_{D^\bullet} F$, we always assume that it is a morphism of globular categories with families, which is possible by previous lemma. Note that there is no reason why this functor should agree with F on the objects of $\mathcal{G}^{\mathrm{op}}$: even though we have $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet) \cong F$, it may happen that this equality cannot hold on the nose if we chose $\mathrm{Ran}_{D^\bullet} F$ to be a morphism of categories with families.

The initial globular category with families. The category $\mathcal{S}_{\mathsf{GSeTT}}$ is canonically equipped with a structure of globular category with families, and we show that this defines the initial object in the category of globular categories with families. This result is a particular case of a difficult problem, known as the *initiality conjecture*. Here, the theory GSeTT is extremely simple, since it has no term constructor and no equation, so that we are able to prove this initiality conjecture in a fairly elementary way, but the proof will generalize in Section 6 to the more involved theory CaTT.

Lemma 35. *Consider a morphism of globular categories with families $F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, then $F = \mathrm{Ran}_{D^\bullet}(F \circ D^\bullet)$.*

Proof. To simplify the notations, we write $F' = F \circ D^\bullet$. We first prove that F and $\mathrm{Ran}_{D^\bullet} F'$ are naturally isomorphic, by induction on the length of contexts.

- For the empty context $\varnothing$, we have that $F\varnothing = \varnothing$ since F is a morphism of categories with families. Moreover, we also have that $(\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet))\varnothing \cong \varnothing$ since $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet)$ preserves limits, and hence the terminal object.
- For a context of the form $(\Gamma, x : A)$, Lemma 18 shows that it is a limit of the form $\lim(\Gamma \to S^{n-1} \leftarrow D^n)$. Since F is a morphism of categories with families, it preserves those limits, hence

$$F(\Gamma, x : A) = \lim(F\Gamma \to FS^{n-1} \leftarrow FD^n)$$

Moreover, $\mathrm{Ran}_{D^\bullet} F'$ also preserves those limits, so we have

$$(\mathrm{Ran}_{D^\bullet} F')(\Gamma, x : A) = \lim((\mathrm{Ran}_{D^\bullet} F')\Gamma \to (\mathrm{Ran}_{D^\bullet} F')S^{n-1} \leftarrow (\mathrm{Ran}_{D^\bullet} F')D^n)$$

But since $F \circ D^\bullet$ is a contravariant globular structure, the Kan extension preserves the disk and sphere objects, and by induction, we also have that $(\mathrm{Ran}_{D^\bullet} F')\Gamma \cong F\Gamma$, hence this limit rewrites as

$$(\mathrm{Ran}_{D^\bullet} F')(\Gamma, x : A) = \lim(F\Gamma \to FS^{n-1} \leftarrow F(D^n))$$

which proves that $F(\Gamma, x : A) \cong (\mathrm{Ran}_{D^\bullet} F')(\Gamma, x : A)$.

Since this construction relies on limit computation, it is natural, and thus we have constructed a natural isomorphism $F \cong \mathrm{Ran}_{D^bullet} F'$. Recall that we have chosen $\mathrm{Ran}_{D^\bullet} F'$ in such a way that it is a morphism of globular categories with families, and there is a unique such choice. But F is also a morphism of globular categories with families which is naturally isomorphic to $\mathrm{Ran}_{D^\bullet} F'$. Hence $F = \mathrm{Ran}_{D^\bullet} F'$. ☐

Theorem 36. *The category $\mathcal{S}_{\mathsf{GSeTT}}$ is the initial globular category with families.*

Proof. We have already proved in Lemma 34 that for any globular category with families $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, there is a morphism of globular categories with families $\mathrm{Ran}_{D^\bullet} F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, so it suffices to prove that this morphism is unique. Consider two morphisms $F, G : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ that are morphisms of globular categories with families, for the same contravariant globular structure on $\mathcal{C}$, given by $F \circ D^\bullet \cong G \circ D^\bullet$. By Lemma 35, F and G are then both the right Kan extension along $D^\bullet$ of the same dual globular structures, so they are naturally isomorphic. Moreover, the both F and G are morphism of category with families, and thus the uniqueness in Lemma 34 proves that F and G are equal. ☐

This theorem is a generalization of Theorem 28, in that is characterizes the models of the theory GSeTT in any category with families $\mathcal{C}$.

Corollary 37. *The category of morphisms of categories with families $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ is equivalent to the category of globular category with families structures on $\mathcal{C}$ that coincide with the given category with families structure on $\mathcal{C}$.*

Proof. A morphism of categories with families $(F, \phi) : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ induces by precomposition with $D^\bullet$ an structure of globular category with families on $\mathcal{C}$. This structure is given by $\mathrm{ty} = F\,\mathrm{ty}$, and for every map $\chi : \Gamma \to S^{n-1}_{F \circ D^\bullet} = F(S^{n-1})$, we have $\mathrm{ty}(\chi) = \phi(\mathrm{ty}(\mathrm{ty}))[\chi]$. The functor F is a morphism of globular category with families for this structure. Conversely, taking the right Kan extension of a globular category with families along $D^\bullet$ defines a morphism of categories with families $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, and Theorem 36 proves that this association is an equivalence of categories. □

Globular category with families on Set. A globular set $G : \mathcal{G}^{\mathrm{op}} \to \mathbf{Set}$ gives rise to an essentially unique globular structure on the category with families **Set**. Indeed, in the structure of category with families of **Set**, every map is a display map, so in particular the globular maps are display maps, and the ty natural transformation is given by the pullback along the ty map, i.e., $\mathrm{ty}(\gamma) = \mathrm{ty}^*\,\gamma$:

$$\begin{tikzcd}
\Gamma \times_{S^{n-1}_G} G(n) \arrow[r] \arrow[d, "\mathrm{ty}^*\,\gamma"'] \arrow[dr, phantom, "\lrcorner"] & G(n) \arrow[d, "\mathrm{ty}"] \\
\Gamma \arrow[r, "\gamma"'] & S^{n-1}_G
\end{tikzcd}$$

By definition, this choice for the functor ty is essentially unique. Conversely, any globular category with families structure on **Set** defines a globular set by definition. This proves the following result

Proposition 38. *The category of globular sets is equivalent to the category of globular structures on the category with families* **Set**.

Note this proposition, combined with Corollary 37 shows that the **Set**-models of the theory GSeTT are equivalent to the globular sets. So this justifies the fact that Theorem 36 is indeed a generalization of Theorem 28.

3. The Grothendieck-Maltsiniotis definition of ω-categories

This entire section is a quick presentation of the definition of weak ω-categories given by Maltsionitis [26], based on the definition of weak ω-groupoid introduced by Grothendieck [17]. We introduce here the notions that the type theory CaTT relies on, as well as the notations we will use for these notions. For a more in-depth study of this definition, we refer the reader to the original article by Maltsionitis [26] or the PhD thesis of Ara for a more detailed account [2].

3.1 Pasting schemes We first introduce the notations we use, and define a subcategory of globular sets, called the *pasting schemes*. These are meant to represent composable situations in a globular set, and thus serve as the arities of the operations in ω-categories.

Globular sums. consider a category $\mathcal{C}$ with a globular structure $F : \mathcal{G} \to \mathcal{C}$ A *globular structure* on a category $\mathcal{C}$ consists in a functor $F : \mathcal{G} \to \mathcal{C}$. When given such a structure, we often denote respectively by D^n, σ_i and τ_i the images under F of n, σ_i and τ_i. When there is no ambiguity, we may write σ and τ, leaving the index implicit, moreover, we also write σ (resp. τ) to denote a composite of maps of the form σ (resp. τ). In the category $\mathcal{C}$, a *globular sum* is a colimit of a diagram of the form

$$\begin{tikzcd}[column sep=small]
D^{i_1} & & D^{i_2} & & & & D^{i_k} \\
& D^{j_1} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & D^{j_2} \arrow[ul, "\tau"'] & \cdots & D^{j_{k-1}} \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

In this diagram, we always assume that the iterated sources σ and the iterated targets τ are not identity, so that we always have the inequality $i_k > j_k < i_{k+1}$ and the diagram is canonical. Given a non-canonical diagram, one can normalize to a diagram of this form by contracting away the identity morphisms. It will be useful to encode such a colimit by its *table of dimensions*

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_k \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix}$$

Dually, if a category $\mathcal{C}$ is endowed with a contravariant functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, called a *contravariant globular structure*, we will denote respectively by D^n, s and t the images by F of n, σ and τ. We call a *globular product* a limit of the diagram of the form

$$\begin{tikzcd}[column sep=small]
D^{i_1} \arrow[dr, "t"'] & & D^{i_2} \arrow[dl, "s"] \arrow[dr, "t"'] & & & & D^{i_k} \arrow[dl, "s"] \\
& D^{j_1} & & D^{j_2} & \cdots & D^{j_{k-1}} &
\end{tikzcd}$$

It will also be convenient to denote it by its table of dimensions.

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_k \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix}$$

If $\mathcal{C}$ has a globular structure and $\mathcal{D}$ has a contravariant globular structure, we will say that a globular sum in $\mathcal{C}$ and a globular product in $\mathcal{D}$ are *dual* to each other if they share the same table of dimensions.

The category of pasting schemes. The category **GSet** is equipped with a globular structure given by the Yoneda embedding $\mathrm{Y} : \mathcal{G} \to \mathbf{GSet}$, defined by $\mathrm{Y}(n) = \mathbf{GSet}(\_, n)$. In this situation we call all the globular sets that are obtained as a globular sums the *pasting schemes*, and we denote $\mathbf{\Theta}_0$ the full subcategory of **GSet** whose objects are the pasting schemes. Note that since the globular sum diagrams are finite, the pasting schemes are finite globular sets. A few examples and counter-examples of pasting schemes are depicted in Figure 2, using the diagrammatic notation for finite globular sets.

A relation characterizing the pasting schemes. Apart from tables of dimensions, there are several ways of parametrizing pasting schemes using combinatorial structures, such as Batanin trees [3]. In fact these trees also assemble into a category, which can be proved to be equivalent to the category $\mathbf{\Theta}_0$ [2, 7, 19]. Other combinatorial descriptions of pasting schemes are also possible, such as Dyck words, or non-decreasing parking functions, as well as inductive definitions. We refer the reader to [5] for a brief presentation of these views. We focus here on a characterization due to Finster and Mimram [14], using a binary relation.

Definition 39. Consider a globular set G, we introduce the relation $\lhd$ on its set of cells to be the transitive closure of the relation generated, for every cell x of G, by

$$s(x) \lhd x \lhd t(x)$$

This relation can be used to characterize the pasting schemes among all the finite globular sets:

Theorem 40 (Finster, Mimram [14])**.** *The pasting schemes are exactly the globular sets such that $\lhd$ is total and antisymmetric, that is, when we have*

$$x \neq y \iff (x \lhd y \text{ or } y \lhd x)$$

We also say in this case that the globular set is $\lhd$-linear.

We refer the reader to the original article [14] for a proof of this theorem, and illustrate the relation $\lhd$ on a few examples, pasting schemes and non pasting schemes, in Figure 2.

globular set	relation $\lhd$	pasting scheme?
$x \xrightarrow{f} y \xrightarrow{g} z$	$x \lhd f \lhd y \lhd g \lhd z$	yes
$x \xrightarrow{f} y \xleftarrow{g} z$	$\begin{array}{ccc} x \lhd f & & \\ & \triangledown & \\ & & y \\ & \vartriangle & \\ z \lhd g & & \end{array}$	no
$\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, bend left=30, "\Downarrow\alpha"] \arrow[r, phantom, bend right=30, "\Downarrow\beta"] & y \arrow[r, "g"] & z \end{tikzcd}$	$x \lhd f \lhd \alpha \lhd f' \lhd \beta \lhd f'' \lhd y \lhd g \lhd z$	yes
$\begin{tikzcd} x \arrow[r, bend left=70, "f"] \arrow[r, bend left=15, "f'" description] \arrow[r, bend right=15, "g" description] \arrow[r, bend right=70, "g'"'] \arrow[r, phantom, bend left=45, "\Downarrow\alpha"] \arrow[r, phantom, bend right=45, "\Downarrow\beta"] & y \end{tikzcd}$	$\begin{array}{ccccc} & f \lhd \alpha \lhd f' & \\ \vartriangle & & \triangledown \\ x & & y \\ \triangledown & & \vartriangle \\ & g \lhd \beta \lhd g' & \end{array}$	no
$\begin{tikzcd} x \arrow[loop above, "f"] \end{tikzcd}$	$x \mathrel{\substack{\lhd \\ \rhd}} f$	no

Figure 2: Globular sets and the relation $\lhd$

The proof of Theorem 40 given in [14] relies in constructing a globular sum diagram associated to a $\lhd$-linear globular set. Since this association reaches all the globular sums ans is injective, this also proves the following result.

Lemma 41. *Every pasting scheme can be written as a globular sum in a unique way.*

Source and target of pasting schemes. A pasting scheme X canonically comes equipped with a source and a target, that are two distinguished sub globular sets of X which are also pasting schemes. Since the source and target are isomorphic globular sets, we will define a unique object ∂X along with the two inclusions which identify ∂X as a subobject of X in two different ways.

$$\sigma_X, \tau_X : \partial X \to X$$

	dimension table	diagram reprepsentation
X	$\begin{pmatrix} 2 & & 1 \\ & 0 & \end{pmatrix}$	$\bullet \overset{\Downarrow}{\rightrightarrows} \bullet \longrightarrow \bullet$
∂X	$\begin{pmatrix} 1 & & 1 \\ & 0 & \end{pmatrix}$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$

Figure 3: A pasting scheme and its border

When X is given by a table of dimensions as above, we define ∂X to be given by the table

$$\begin{pmatrix} \overline{i_1} & & \overline{i_2} & & \cdots & & \overline{i_k} \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix} \qquad \text{where } \overline{i_k} = \begin{cases} i_k & \text{if } i_m < i \\ i-1 & \text{if } i_m = i \end{cases}$$

Fig. 3 shows an example of a pasting scheme along with its border, represented as pasting schemes as well as as diagrams. Note that the definition of the border of a pasting scheme may produce tables that do not strictly comply with the definition of globular sums, as presented before, since it is possible to have the equality

$$\overline{i_m} = j_m = \overline{i_{m+1}} = i-1$$

However, when it is the case we will chose the corresponding iterated sources and target to be the identity maps (i.e., the map iterated 0 times). We can then normalize with the following rewriting rule, that does not change the colimit and thus exhibits ∂X as a pasting scheme

$$\begin{pmatrix} \cdots & i-1 & & i-1 & \cdots \\ \cdots & & i-1 & & \cdots \end{pmatrix} \rightsquigarrow \begin{pmatrix} \cdots & i-1 & \cdots \\ \cdots & & \cdots \end{pmatrix}$$

Now, we can define the two inclusion maps σ_X and τ_X to induced by the families

$$\overline{\sigma_{i_m}} : D_{\overline{i_m}} \longrightarrow D_{i_m} \qquad\qquad \overline{\tau_{i_m}} : D_{\overline{i_m}} \longrightarrow D_{i_m}$$

$$\overline{\sigma_{i_m}} = \begin{cases} \mathrm{id} & \text{if } i_m < i \\ \sigma_{i-1} & \text{if } i_m = i \end{cases} \qquad\qquad \overline{\tau_{i_m}} = \begin{cases} \mathrm{id} & \text{if } i_m < i \\ \tau_{i-1} & \text{if } i_m = i \end{cases}$$

Note that there is a subtlety whenever there are two or more successive cells of maximal dimension n composed in dimension $n-1$. In this case we have to renormalize the dimension of table of ∂X in order to remove multiple successive instances of $n-1$. Defining σ_X and τ_X in this case requires to handle carefully this renormalization, as illustrated in the following example:

$$\begin{tikzcd}
D^1 \arrow[dd, "\sigma"] \arrow[dr] & & & D^1 \arrow[dll] \arrow[drr] \arrow[ddl, "\sigma"] & & & D^1 \arrow[dl] \arrow[dd, "\sigma"] \\
& D^0 \arrow[dd, "\mathrm{id}"] & & & & D^0 \arrow[dd, "\mathrm{id}"] & \\
D^2 \arrow[dr] & & D^2 \arrow[dl] \arrow[dr] & & D^2 \arrow[dl] \arrow[dr] & & D^1 \arrow[dl] \\
& D^0 & & D^1 & & D^0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
D^1 \arrow[dd, "\tau"] \arrow[dr] & & & D^1 \arrow[dll] \arrow[drr] \arrow[ddr, "\tau"] & & & D^1 \arrow[dl] \arrow[dd, "\tau"] \\
& D^0 \arrow[dd, "\mathrm{id}"] & & & & D^0 \arrow[dd, "\mathrm{id}"] & \\
D^2 \arrow[dr] & & D^2 \arrow[dl] \arrow[dr] & & D^2 \arrow[dl] \arrow[dr] & & D^1 \arrow[dl] \\
& D^0 & & D^1 & & D^0 &
\end{tikzcd}$$

which exhibit the diagram on the left respectively as the source and the target of the diagram on the right:

$$\begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd} \qquad\qquad \begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2.5, "\Downarrow"] \arrow[r, phantom, shift right=2.5, "\Downarrow"] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

By convention, in the case of the pasting scheme D_0, we chose ∂D_0 to be the empty globular set, which is not a pasting scheme.

Characterization of the source and target via the relation $\triangleleft$. The notions of source and target are defined for all the pasting schemes, and are closely related to the relation $\triangleleft$ defined above. Given a pasting scheme X and two parallel cells x and y in X, we denote by $X(x, y)$ the set of cells with source x and target y. Then the relation $\triangleleft$ on the entire pasting scheme X is a preorder, and therefore also induces a preorder on the set $X(x, y)$. We thus define two sub-globular sets of the pasting scheme X of dimension n, denoted $\partial^- X$ and $\partial^+ X$ as follows

$$\begin{array}{rcc} \text{For } k < n-1 & (\partial^- X)_k = X_k & (\partial^+ X)_k = X_k \\ \text{For all } x, y \in X_{n-2} & \partial^- X(x,y) = \min X(x,y) & \partial^+ X(x,y) = \max X(x,y) \\ \text{For } k \geq n & (\partial^- X)_k = \emptyset & (\partial^+ X)_k = \emptyset \end{array}$$

Where the max and the min are respectively the maximal and minimal elements for the preorder $\triangleleft$.

Proposition 42. *The globular set $\partial^- X$ (resp. $\partial^+ X$) is the image of the source morphism $\sigma_X : \partial X \to X$ (resp. target morphism $\tau_X : \partial X \to X$).*

Proof. One can check these images, by definition morphisms σ_X and τ_X, since it removes the variables of maximal dimension n, and keeps the variables of dimension $n-2$. Proving the equality in dimension $n-1$ requires a careful handling of the subtlety that appears in the case of several successive cells dimension n composed in dimension $n-1$. $\square$

Maps in the category $\mathbf{\Theta}_0$. The caracterization of pasting schemes using the relation $\triangleleft$ allows us show that the maps in the category $\mathbf{\Theta}_0$ are very restricted.

Lemma 43. *Any map $f : X \to Y$ in the category $\mathbf{\Theta}_0$ is injective.*

Proof. A map of globular sets has to preserve the relation $\triangleleft$, since it preserves the source and target. Consider to distinct elements x and y in the pasting scheme X, then Theorem 40 proves that either $x \triangleleft y$ or $y \triangleleft x$, hence we have either $f(x) \triangleleft f(y)$ or $f(y) \triangleleft f(x)$, which by applying Theorem 40 again shows that $f(x) \neq f(y)$. $\square$

Lemma 44. *A pasting scheme has no non-trivial automorphism.*

Proof. Consider a pasting scheme X together with an automorphism $f : X \to X$. Suppose that there exists an element $x \in X$ such that $x \neq f(x)$. Then by Theorem 40, we have either $x \triangleleft f(x)$ or $f(x) \triangleleft x$: we suppose that we are in the first case, the second one being similar. Since f preserves the relation $\triangleleft$, this provides us with an infinite chain

$$x \triangleleft f(x) \triangleleft f(f(x)) \triangleleft f(f(f(x))) \triangleleft \cdots$$

which is impossible since X has only finitely many elements. The automorphism f is thus necessarily the identity. $\square$

3.2 Globular extensions and globular theories In order to define weak ω-categories, we rely on the notion of a *coherator* which is a category whose objects are the arities of the operations expected in ω-categories and the morphisms encode the algebraic that they should have. It can be thought of as an analogue of Lawvere theories in the dependently sorted case. Recall that in Lawvere theories, one requires the set of objects to be freely generated by the finite products of a single object, the coherator satisfies an analogous condition for the dependently sorted case, which is captured by the notion of *globular theory*

Globular extensions. A category $\mathcal{C}$ with a globular structure F is called a *globular extension* when all the globular sums exist in $\mathcal{C}$. Given two globular extensions $F : \mathcal{G} \to \mathcal{C}$ and $G : \mathcal{G} \to \mathcal{D}$, a morphism of globular extensions is a functor $H : \mathcal{C} \to \mathcal{D}$ such that $H \circ F = G$, which preserves globular sums. Dually, a category equipped with a contravariant globular structure and which has all globular products is called a *contravariant globular extension*, the notion of morphism is the opposite of the one for globular extensions.

The universal globular completion. There is a canonical functor $\mathcal{G} \to \mathbf{\Theta}_0$, sending an object n to the disk D^n, which is an object of $\mathbf{\Theta}_0$ as being obtained as the globular sum corresponding to the table of dimensions (n). This exhibit $\mathbf{\Theta}_0$ as the completion of $\mathcal{G}$ under globular sums: we sometimes say that it is the universal *globular completion* of $\mathcal{G}$.

Proposition 45. *The category $\mathbf{\Theta}_0$ is the universal globular extension: for any globular extension $F : \mathcal{G} \to \mathcal{C}$, there is an essentially unique morphism of globular extensions $\mathbf{\Theta}_0 \to \mathcal{C}$.*

Proof. Consider a globular extension $F : \mathcal{G} \to \mathcal{C}$ together with a morphism of globular extensions $f : \mathbf{\Theta}_0 \to \mathcal{C}$. Then an object X in $\mathbf{\Theta}_0$ decomposes as a globular sum induced by a table of dimensions

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_n \\ & j_1 & & j_2 & \cdots & j_{n-1} & \end{pmatrix}$$

By definition, $f(X)$ is the globular sum of the same diagram in $\mathcal{C}$, hence f is determined up to natural isomorphism. Conversely, we can define $f(D^n) = Fn$ and extends this definition to all the pasting schemes while preserving the globular sums since by Lemma 41 every pasting scheme is written as a globular sum in a unique way. □

Dually $\mathbf{\Theta}_0^{\mathrm{op}}$ is a *globular cocompletion*: for every contravariant globular extension $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, there is an essentially unique morphism $\mathbf{\Theta}_0^{\mathrm{op}} \to \mathcal{C}$.

The category of globular extensions. Globular extensions are characterized by the fact that they have globular sums, and the globular sums factorize through the category $\mathbf{\Theta}_0$. We can thus use the universality of the category $\mathbf{\Theta}_0$ in order to characterize the category of globular extensions as follows.

Lemma 46. *The universal property of the category $\mathbf{\Theta}_0$ induces an equivalence of categories between the category of globular categories and the full subcategory of the coslice category $\mathbf{\Theta}_0\backslash\mathbf{Cat}$ whose objects are the functors preserving the globular sums.*

Dually, there is an equivalence of categories between the category of contravariant globular extensions and the full subcategory of the coslice category $\mathbf{\Theta}_0^{\mathrm{op}}\backslash\mathbf{Cat}$, whose objects are the functors preserving the globular products.

Proof. By the universal property of $\mathbf{\Theta}_0$, a globular extension $F : \mathcal{G} \to \mathcal{C}$ induces a morphism of globular extensions $\mathbf{\Theta}_0 \to \mathcal{C}$, which is an object of the coslice $\mathbf{\Theta}_0\backslash\mathbf{Cat}$ preserving globular sums, and this assignment is functorial. Conversely, consider a functor $F : \mathbf{\Theta}_0 \to \mathcal{C}$ preserving globular sums. Then, by precomposition by the canonical functor $\mathcal{G} \to \mathbf{\Theta}_0$, it induces a globular structure on $\mathcal{C}$, and F is a morphism from of globular structures from $\mathbf{\Theta}_0$ to $\mathcal{C}$. Any globular sum diagram for this structure in $\mathcal{C}$ factorizes through F. Since $\mathbf{\Theta}_0$ has all the globular sums, this diagram has a globular sum in $\mathbf{\Theta}_0$, and since F preserves those, this diagram has a globular sum in $\mathcal{C}$, hence $\mathcal{C}$ has all the globular sums and is a globular extension. Moreover, consider a commutative triangle of the form

$$
\begin{tikzcd}
& \mathcal{G}^{\mathrm{op}} \arrow[dl, "F"'] \arrow[dr, "G"] & \\
\mathcal{C} \arrow[rr, "f"'] & & \mathcal{D}
\end{tikzcd}
$$

with F and G preserving the globular sums. Then, by the previous statement, $\mathcal{C}$ has all the globular sums, which all factor through F. Since $f \circ F = G$ preserves the globular sums, it follows that necessarily f preserves the globular sums and thus defines a morphism of globular extensions. This proves the equivalence of categories. □

Globular theories. Given a globular extension $\mathcal{G} \to \mathcal{C}$, by universality of the globular completion, there exists a unique morphism of globular completions $F : \mathbf{\Theta}_0 \to \mathcal{C}$. The functor $\mathcal{G} \to \mathcal{C}$ is called a *globular theory* if the induced functor F is faithful and bijective on the isomorphism classes of objects. Whenever it is the case, we can up to equivalence identify $\mathbf{\Theta}_0$ as a subcategory of $\mathcal{C}$. A *morphism of globular theories* is just a morphism of the underlying globular extensions. An morphism f of a globular theory $\mathcal{C}$ is said to be *globular* if it is in $\mathbf{\Theta}_0$. Dually, a contravariant globular extension $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ is called a *contravariant globular theory* if $\mathcal{C}^{\mathrm{op}}$ is a globular theory.

3.3 Weak ω-categories We have introduced the notion of globular theory, which plays the role of Lawvere theories, in the case where we have dependent sorts, the dependency are globular and the arities are given by pasting schemes. There are various such theories, and we now introduce the one we will be interested in for weak ω-categories. As it is often the case for higher structures, there is not a single theory of weak ω-categories, but several of them, called *coherators*. We introduce here one such coherator.

Admissible pair of arrows. Let $\mathcal{G} \to \mathcal{C}$ be a globular extension, two arrows $f, g : D^i \to X$ in $\mathcal{C}$ are said to be *parallel* when

$$
f \circ \sigma = g \circ \sigma \qquad\qquad f \circ \tau = g \circ \tau
$$

If $\mathcal{C}$ is a globular theory, then an arrow f of $\mathcal{C}$ is said to be *algebraic*, when for every decomposition $f = g \circ f'$, with g globular, then g is an identity. A pair of parallel arrows $f, g : D^i \to X$ is called an *admissible pair* if either both f and g are algebraic, or there exists a decomposition $f = \sigma_X \circ f'$ and $g = \tau_X \circ g'$, with f' and g' algebraic.

Definition 47. Given an admissible pair of maps $f, g : D^i \to X$, we call a *lift* a map $h : D^{i+1} \to X$

such that $h \circ \sigma = f$ and $h \circ \tau = g$

$$\begin{tikzcd} D^{i+1} \arrow[dr, "h"] & \\ D^i \arrow[u, shift left, "\sigma"] \arrow[u, shift right, "\tau"'] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

We also say that an arrow is algebraic or that a pair is admissible in a contravariant globular theory $\mathcal{C}$, to mean that it is the same in $\mathcal{C}^{\mathrm{op}}$, and a lift for an admissible is a lift in the opposite category in $\mathcal{C}^{\mathrm{op}}$.

Cat-coherator. We introduce here the Batanin-Leinster cat-coherator, which is the one we will be using for our type theory. For a more general definition of cat-coherators, as well as other examples, see [26]. For the rest of this paper, we will simply say cat-coherator to refer to the Batanin-Leinster cat-coherator. The cat-coherator $\mathbf{\Theta}_\infty$ is defined to be the colimit

$$\mathbf{\Theta}_\infty = \mathrm{colim}(\mathbf{\Theta}_0 \to \mathbf{\Theta}_1 \to \mathbf{\Theta}_2 \to \cdots \to \mathbf{\Theta}_n \to \cdots)$$

where the categories $\mathbf{\Theta}_n$ are defined by induction on n. Given $n \in \mathbb{N}$, define E_n to be the set of all pairs of admissible arrows of $\mathbf{\Theta}_n$ that are not in $E_{n'}$ for any $n' < n$. Then we can define $\mathbf{\Theta}_{n+1}$ to be the universal globular extension of $\mathbf{\Theta}_n$ obtained by formally adding a lift for each pairs in E_n. In other words $\mathbf{\Theta}_{n+1}$ is the category such that, for each globular extension $f : \mathbf{\Theta}_n \to \mathcal{C}$ for which the image by f of all pairs of arrows in E_n has a lift in $\mathcal{C}$, there is an essentially unique globular extension $\tilde{f}$, which makes the following triangle commute

$$\begin{tikzcd} \mathbf{\Theta}_n \arrow[r] \arrow[dr, "f"'] & \mathbf{\Theta}_{n+1} \arrow[d, dotted, "\tilde{f}"] \\ & \mathcal{C} \end{tikzcd}$$

Weak ω-categories. We define a *weak ω-category* to be functor $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathbf{Set}$ which sends globular sums in $\mathbf{\Theta}_\infty^{\mathrm{op}}$ to the globular product on the opposite diagram, for the globular structure on $\mathbf{Set}$ induced by F. Given $n \in \mathbb{N}$, the elements of the set FD^n are called the *n-cells* of the weak ω-category. The category ω-$\mathbf{Cat}$ of weak ω-categories is the full subcategory of $\widehat{\mathbf{\Theta}_\infty}$ whose objects are the presheaves that are weak ω-categories.

3.4 Identity and composition In order to illustrate the above definition, we show that a weak ω-category $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathbf{Set}$ has identities on 0-cells and composites of composable 1-cells. We refer the reader to [26, 2] for more examples of the same nature. More advanced examples of operations are presented in Section 4.3, where they are described in a type theoretic style.

Identities on 0-cells. The pair of maps $(\mathrm{id}_{D^0}, \mathrm{id}_{D^0})$ is admissible. Hence, there exists a lift

$$\begin{tikzcd} D^1 \arrow[dr, dotted, "\iota"] & \\ D^0 \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift left, "\mathrm{id}_{D^0}"] \arrow[r, shift right, "\mathrm{id}_{D^0}"'] & D^0 \end{tikzcd}$$

Given a 0-cell $x \in FD^0$, its *identity* 1-*cell* $i(x) \in FD^1$ is $i(x) = F\iota(x)$. Moreover, by definition, we have $s(i(x)) = t(i(x)) = x$ as expected for the identity 1-cell on x.

Composition of 1-cells. Consider the globular sum given as $D^1 \coprod_{D^0} D^1$. There are two canonical maps $\iota_1, \iota_2 : D^1 \to D^1 \coprod_{D^0} D^1$, and we consider the admissible pair

$$(\iota_1\sigma, \iota_2\tau) : D^0 \to D^1 \coprod_{D^0} D^1$$

which provides the lift

$$\begin{tikzcd} D^1 \arrow[dr, dotted, "c"] & \\ D^0 \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift left, "\iota_1\sigma"] \arrow[r, shift right, "\iota_2\tau"'] & D^1 \coprod_{D^0} D^1 \end{tikzcd}$$

A pair of composable 1-cell is the same as an element $(f, g) \in F(D^1 \coprod_{D^0} D^1)$, and the element $F(c)(f, g)$ in FD^1 defines the composition $f \cdot g$. By definition, $s(f \cdot g) = s(f)$ and $t(f \cdot g) = t(g)$, as expected for the composition.

4. Type theory for weak ω-categories

Our aim is now to extend the type theory GSeTT presented in Section 2, by adding term constructors corresponding to the algebraic structure that one need to add to globular sets in order to obtain weak ω-categories. We call the resulting theory CaTT and motivate its introduction by following the ideas of the Grothendieck-Maltsiniotis definition of weak ω-categories recalled in Section 3.

4.1 Ps-contexts We have proved in Theorem 22 that the syntactic category of the theory GSeTT is equivalent to the opposite of the category of finite globular sets. The Grothendieck-Maltsiniotis definition of weak ω-categories strongly relies on a particular class of such finite globular sets, namely the pasting schemes, obtained as globular sums. In order to translate this definition in a type theory, it is useful to transfer this notion of pasting scheme in a type theoretic framework.

Recognition algorithm. We introduce a new kind of judgment to the theory, that we denote

$$\Gamma \vdash_{\mathsf{ps}}$$

A context Γ such that the judgment $\Gamma \vdash_{\mathsf{ps}}$ is derivable is called a *ps-context*. It intuitively corresponds to a situation where the context Γ is a pasting scheme, as formally shown in Theorem 53. In order to define this judgment by induction, we also introduce an auxiliary judgment

$$\Gamma \vdash_{\mathsf{ps}} x : A$$

where the variable x is called the *dangling variable*. We require these judgments to be subject to the following inference rules:

$$\frac{}{(x : \star) \vdash_{\mathsf{ps}} x : \star}\,(\textsc{pss}) \qquad\qquad \frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}{\Gamma \vdash_{\mathsf{ps}} y : A}\,(\textsc{psd})$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\to} y \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}\,(\textsc{pse}) \quad \text{when } y, f \notin \mathrm{Var}(\Gamma) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}\,(\textsc{ps})$$

Note that every derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$ starts with the rule (PSS) and ends with the rule (PS), with an equal number of applications of the rules (PSE) and (PSD) in between.

An example of a derivation. In order to understand how a derivation of this judgment works, we have illustrated in Figure 4 the derivation of $\Gamma \vdash_{\mathsf{ps}}$ where Γ is the context

$$\Gamma = (x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2, z : \star, g : y \to z)$$

which corresponds to the globular set

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, "g"] & z \end{tikzcd}$$

We follow the step-by-step derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$, and give a graphical representation of the globular corresponding globular set being constructed, where we encircle the dangling variable on the judgment.

The rules that we have given for recognizing ps-contexts do in particular recognize usual contexts in the theory GSeTT, in other words, the following holds [14, 5]:

Proposition 48. *The following rules are admissible*

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash_{\mathsf{ps}} A} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash_{\mathsf{ps}}}{\Gamma \vdash}$$

Proof. The admissibility of the first three of these rules can be shown by mutual induction, and the admissibility of the last one is then a consequence of the former, see [5] for a detailed proof. □

The category of ps-contexts. We consider the subcategory $\mathcal{S}_{\mathsf{ps}}$ of $\mathcal{S}_{\mathsf{GSeTT}}$, whose objects are the ps-contexts, and our goal is now to study this category.

The correspondence between ps-contexts and pasting schemes. We now show that ps-contexts correspond to pasting schemes. In order to do so, we use the following useful lemma, which involves the functor $V : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{FinGSet}^{\mathrm{op}}$, introduced in Section 2 and defined by $V\Gamma = \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^\bullet)$, and the relation $\triangleleft$, recalled in Section 3.1.

Proposition 49. *For every ps-context $\Gamma \vdash_{\mathsf{ps}}$, the globular set $V\Gamma$ is $\triangleleft$-linear.*

Proof. The proof requires the introduction of subtle invariants beforehand. One can check by induction on the derivation tree, that whenever a judgment of the form $\Gamma \vdash_{\mathsf{ps}} x : A$ is derivable, there is no variable f whose type in Γ is $y \to z$ where y is an iterated target of x (i.e., there exists a sequence of terms $x = x_0, x_1, x_2, \ldots, x_n = y$ with $\Gamma \vdash x_i : a \to x_{i+1}$ for a variable a). As a consequence, in this situation, every relation of the form $x \triangleleft y$ is such that y is an iterated target of x. Using this fact, it can be shown by induction that the set $V\Gamma$ is $\triangleleft$-linear.

Now, suppose fixed a ps-context $\Gamma \vdash_{\mathsf{ps}}$. We first show that if we have a, b in Γ such that $a \neq b$ then necessarily $a \triangleleft b$ or $b \triangleleft a$, by induction on the form of the pasting scheme.

- For the pasting scheme $(x : \star)$, the statement is vacuously true since there are no two disjoint variables.
- For a ps-context of the form $\Gamma = (\Gamma', y : A, f : x \to y)$, we distinguish different cases.
  - If both a and b are in Γ' then by induction, either $a \triangleleft b$ or $b \triangleleft a$ in Γ', and thus the same holds in Γ.

$\emptyset$

(PSS)

x $$x : \star \vdash_{\mathsf{ps}} x : \star$$

(PSE)

$x \xrightarrow{f_1} y$ $$(x : \star, y : \star, f_1 : x \to y) \vdash_{\mathsf{ps}} f_1 : x \to y$$

(PSE)

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & y \end{tikzcd}$$

$$\begin{array}{l}(x : \star, y : \star, f_1 : x \to y, \\ f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} \alpha : f_1 \to f_2\end{array}$$

(PSD)

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & y \end{tikzcd}$$

$$\begin{array}{l}(x : \star, y : \star, f_1 : x \to y, \\ f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} f_2 : x \to y\end{array}$$

(PSD)

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & y \end{tikzcd}$$

$$\begin{array}{l}(x : \star, y : \star, f_1 : x \to y, \\ f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} y : \star\end{array}$$

(PSE)

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & y \arrow[r, "g"] & z \end{tikzcd}$$

$$\begin{array}{l}(x : \star, y : \star, f_1 : x \to y, \\ f_2 : x \to y, \alpha : f_1 \to f_2 \\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} g : y \to z\end{array}$$

(PSD)

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & y \arrow[r, "g"] & z \end{tikzcd}$$

$$\begin{array}{l}(x : \star, y : \star, f_1 : x \to y, \\ f_2 : x \to y, \alpha : f_1 \to f_2 \\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} z : \star\end{array}$$

(PS)

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & y \arrow[r, "g"] & z \end{tikzcd}$$

$$\Gamma \vdash_{\mathsf{ps}}$$

Figure 4: Derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$

- If a is in Γ', but not b, either $b = y$ or $b = f$, then by induction, either $a = x$, or $x \lhd x$ or $x \lhd a$. In the first two cases, since we have $x \lhd b$, the transitivity shows that $a \lhd b$. We can thus assume that $x \lhd a$. In this case, by the fact that we have proved, a is an iterated target of x. Since y is parallel to x and y is a target of f, in either case, a is also an iterated target of b, which shows that $b \lhd a$.
- If b is in Γ' but not a, the situation is symmetric to the previous case.
- If neither a nor b is in Γ', then necessarily they are f and y, and we have $f \lhd y$.

Conversely, we show that for every ps-context Γ, we never have $x \lhd x$. In order to prove this, we first note that whenever we have a relation of the form $a \lhd b$ in the ps-context $(\Gamma, y : A, f : x \to y)$ with a and b variables of Γ, we also have the same relation in Γ. Indeed, considering the chain of generating relations $a \lhd a_1 \lhd \cdots \lhd b$, it suffices to prove that there is a chain completely included in Γ. If it is not the case, that means that there are occurrences of the form $s \lhd y \lhd t$ or $x \lhd f \lhd y \lhd t$ with s the source of y and t its target (these are the only possibilities because of the fact that y can never be a source). In the first case, one can replace the occurrence with $s \lhd x \lhd t$ and in the second case, one can replace it with $s \lhd t$, in order to obtain a chain proving $a \lhd b$ in Γ. Proving that there is no variable x such that $x \lhd x$ in ps-context Γ is then a straightforward induction over the derivation tree of the judgment $\Gamma \vdash_{\mathsf{ps}}$. □

We now prove the converse, that any $\lhd$-linear globular set corresponds to a ps-context. In order to do this, we introduce the notion of *locally maximal element* of a pasting scheme as an element x such that there is no variable y such that $s(x) \lhd y \lhd x$ or $x \lhd y \lhd t(x)$. Alternatively, the locally maximal elements are the elements corresponding to the peaks in the decomposition as a globular sum.

Example 50. In the globular set

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

we have that α is maximal but f is not maximal because $f \lhd \alpha \lhd y = t(f)$.

In order to prove the result, we use the following lemma:

Lemma 51. *Consider a globular set G with two elements x, y such that $x \lhd y$ and $\dim x > \dim y$. Then $t(x) \lhd y$ or $t(x) = y$.*

Proof. Suppose that $x \lhd y$. By definition of the relation $\lhd$ there exists a sequence of elements $x = x_0 \lhd x_1 \lhd \ldots \lhd x_n = y$ such that, for every index i, $x_i = s(x_{i+1})$ or $x_{i+1} = t(x_i)$. We reason by induction on the length n of this sequence.

- If $n = 1$, then necessarily, either $y = t(x)$ or $x = s(y)$, and the condition on the dimensions implies that we have $y = t(x)$.
- Suppose that the result holds for all chains of length at most $n - 1$. Note that either $x_1 = t(x)$, or $x = s(x_1)$. The first case gives the result immediately. In the second case, the induction shows that we have a relation $t(x_1) \lhd y$, given by a chain of length less than $n-1$, so applying again the induction hypothesis proves that $t(t(x_1)) \lhd y$. And we conclude by using the fact that $t(t(x_1)) = t(s(x_1)) = t(x)$. □

Proposition 52. *For any $\lhd$-linear non-empty finite globular set G, there exists a unique ps-context Γ such that $F\Gamma = G$.*

Proof. We construct the context Γ inductively and then prove that it satisfies $\Gamma \vdash_{\mathsf{ps}} x : A$, where $V(x)$ is the greatest (for the relation $\triangleleft$) locally maximal element of G.

- If the globular set G has a unique element, then this element is necessarily of dimension 0 and we then associate the context $\Gamma = (x : \star)$, where the derivation of $\Gamma \vdash_{\mathsf{ps}} x : \star$ is given by the rule (PSS).
- If G has more than one element, write a for the greatest locally maximal element of G. We can consider the globular set G' obtained by removing a and $t(a)$ from G: indeed, by definition of locally maximal element, there is no element whose source or target is a, hence a can safely be removed. Moreover, any element x whose target is $t(a)$ satisfies $x \triangleleft t(a)$, thus either it is a or it compares to a by linearity. Since a is locally maximal, we then cannot have $a \triangleleft x$, so we necessarily have $x \triangleleft a$ and thus $x \triangleleft s(a)$. Lemma 51 then applies to show that $t(x) \triangleleft s(a)$. Since we have $t(x) = t(a)$ and also $s(a) \triangleleft t(a)$, this implies in particular that $t(a) \triangleleft t(a)$, which contradicts the linearity of G. So any element whose target is $t(a)$ is necessarily a, and since a is the greater locally maximal element, there cannot be any element whose source is $t(a)$. Hence, after removing a, one can still remove $t(a)$ safely. In fact, this analysis shows that the resulting globular set G' is still a non-empty finite $\triangleleft$-linear set, so by induction, one can construct a context Γ' such that $V(\Gamma') = G'$ and $\Gamma' \vdash_{\mathsf{ps}} x : A$, where x is the greatest locally maximal element in G'.
  - Either the greatest locally maximal element of G' is $s(a)$. In this case, we define $\Gamma = (\Gamma', t(a) : A, a : s(a) \to t(a))$, we then have $V(\Gamma) = G$ by definition, and the rule (PSE) gives a derivation of $\Gamma \vdash_{\mathsf{ps}} a : s(a) \to t(a)$.
  - Or the greatest locally maximal element of G' is b is such that $b \triangleleft s(a)$. In this case, since b is locally maximal, $s(a)$ is necessarily an iterated target of b, we denote n the number of iterations. Then applying the rule (PSD) n times gives a derivation of $\Gamma \vdash s(a) : B$. We define $\Gamma = (\Gamma', t(a) : B, a : s(a) \to t(a))$, in such a way that we have $V(\Gamma) = G$ and $\Gamma \vdash_{\mathsf{ps}} a : s(a) \to t(a)$ obtained from the previous derivation by applying the rule (PSE).

  Since a is the greatest locally maximal variable, these are the two only cases, and in both cases, we constructed a suitable preimage. □

The two previous propositions, together with Theorem 22 finally allows us to conclude:

Theorem 53. *There is an equivalence of categories*

$$\mathcal{S}_{\mathsf{ps}} \simeq \mathbf{\Theta}_0^{\mathrm{op}}$$

Proof. We have already proved, by Proposition 49, that the functor V induces a functor $\mathcal{S}_{\mathsf{ps}} \to \mathbf{\Theta}_0^{\mathrm{op}}$. Moreover, we have shown by Theorem 22 that V is fully faithful, so the restriction is also fully faithful, and Proposition 52 shows that this restriction is essentially surjective, hence it is an equivalence of categories. □

In fact one could prove a slightly more precise result in the form of a bijection between the pasting schemes and the ps-contexts up to α-equivalence (i.e., renaming of their variables).

We illustrate in Figure 5 the correspondence between the ps-contexts and the $\triangleleft$-linear contexts with our previous example of derivation, showing how we construct $\triangleleft$ to be a preorder.

Note that the notion of ps-context is not invariant under isomorphism in the category $\mathcal{S}_{\mathsf{GSeTT}}$. As an example, one can consider the two following Γ and Γ' which are isomorphic, as they only differ from the order of the variables, but the context Γ is a ps-context whereas the context Γ' is not:

$$\Gamma = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z)$$
$$\Gamma' = (x : \star, y : \star, z : \star, f : x \to y, g : y \to z)$$

Thus one can understand the notion of ps-context as a recognition algorithm for a particular representative of a context in each equivalence classes of contexts corresponding to a pasting scheme.

Uniqueness of derivation. The following results rely on a more detailed analysis of the allowed derivation trees and show that these rules enjoy good computational properties.

Proposition 54. *Given a context Γ, the derivability of the judgment $\Gamma \vdash_{\mathsf{ps}}$ is decidable, and when this judgment is derivable, it has a unique derivation.*

Proof. The proof is more subtle than it may appear at first glance, as one cannot just use a straightforward induction to prove this result. Indeed, any derivation of the judgment $\Gamma \vdash$ is obtained from a derivation of the judgment $\Gamma \vdash_{\mathsf{ps}} x : \star$, but there is no guarantee a priori that this the variable $x : \star$ is the same for all possible derivations. However, in the proof of Proposition 49 we have characterized the variable x in a judgment of the form $\Gamma \vdash_{\mathsf{ps}} x : A$ as an iterated target of the greatest locally maximal variable. This proves that whenever we have two derivations of the form $\Gamma \vdash_{\mathsf{ps}} x : A$ and $\Gamma \vdash_{\mathsf{ps}} y : B$ with $\dim A = \dim B$, then necessarily $x = y$. Moreover, Proposition 49 also characterizes the judgments $\Gamma \vdash_{\mathsf{ps}} x : A$ obtained from the rule (PSE) as those when x is locally maximal in Γ, all the other ones are obtained from the rule (PSD). These two facts together combine allow for a straightforward proof by induction on the structure of the derivation trees, that each judgment of this form has a single derivation. □

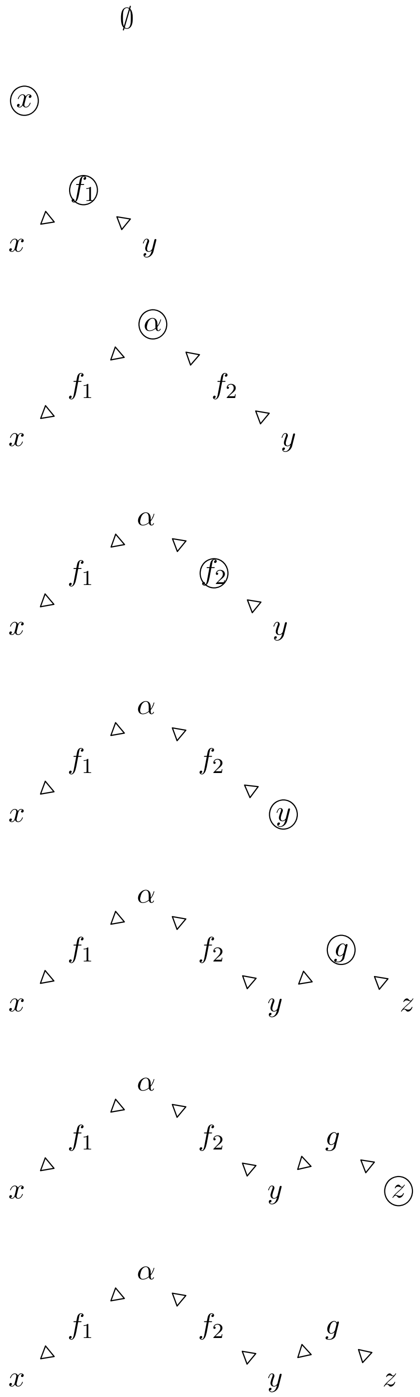
Source and target of a ps-context. The ps-contexts come equipped with a notion of source and target, which mirror the corresponding operations on pasting scheme, already presented in Section 3. Following the proofs that we have given, one could already figure out how to define these: indeed, it suffices to use the correspondence of ps-contexts and pasting schemes in order to compute the source, or the target of a pasting scheme, and then use the correspondence in the other direction to get back a ps-context. We give here a direct computation by induction on the syntax of a ps-context of this process. We define for all $i \in \mathbb{N}$ the *i-source* of a ps-context Γ induction on the length of Γ, by setting $\partial_i^-(x : \star) = (x : \star)$ and

$$\partial_i^-(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^-\Gamma & \text{if } \dim A \ \geq i-1 \\ \partial_i^-\Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

and similarly the *i-target* of Γ is defined by $\partial_i^+(x : \star) = (x : \star)$, and

$$\partial_i^+(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^+\Gamma & \text{if } \dim A \ \geq i \\ \mathrm{drop}(\partial_i^+\Gamma), y : A & \text{if } \dim A \ = i-1 \\ \partial_i^+\Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

where $\mathrm{drop}(\Gamma)$ is the context Γ with its last variable removed. One can check by induction on the derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$ that whenever Γ is a ps-context of non-zero dimension, both



$$\frac{}{x : \star \vdash_{\mathsf{ps}} x : \star}\ (\text{PSS})$$

$$\frac{}{(x : \star, y : \star, f_1 : x \to y) \vdash_{\mathsf{ps}} f_1 : x \to y}\ (\text{PSE})$$

$$\frac{}{\begin{array}{l}(x : \star, y : \star, f_1 : x \to y,\\ f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} \alpha : f_1 \to f_2\end{array}}\ (\text{PSE})$$

$$\frac{}{\begin{array}{l}(x : \star, y : \star, f_1 : x \to y,\\ f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} f_2 : x \to y\end{array}}\ (\text{PSD})$$

$$\frac{}{\begin{array}{l}(x : \star, y : \star, f_1 : x \to y,\\ f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} y : \star\end{array}}\ (\text{PSD})$$

$$\frac{}{\begin{array}{l}(x : \star, y : \star, f_1 : x \to y,\\ f_2 : x \to y, \alpha : f_1 \to f_2\\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} g : y \to z\end{array}}\ (\text{PSE})$$

$$\frac{}{\begin{array}{l}(x : \star, y : \star, f_1 : x \to y,\\ f_2 : x \to y, \alpha : f_1 \to f_2\\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} z : \star\end{array}}\ (\text{PSD})$$

$$\frac{}{\Gamma \vdash_{\mathsf{ps}}}\ (\text{PS})$$

Figure 5: $\triangleleft$-linearity of a ps-context

$\partial_i^-\Gamma$ and $\partial_i^+\Gamma$ are also ps-contexts. It is straightforward in the case of the i-source, and for the i-target, it relies on the fact that whenever the drop operator is used, immediately afterwards a variable of the same type that the one that was removed is added. We denote $\partial^-(\Gamma) = \partial^-_{\dim\Gamma-1}\Gamma$ and $\partial^+(\Gamma) = \partial^+_{\dim\Gamma-1}\Gamma$ and call these the *source* and *target* of Γ.

Lemma 55. *For every ps-context Γ, the globular set $V(\partial^-(\Gamma))$ is exactly the sub-globular set $\partial^-(V\Gamma)$ of $V\Gamma$, and similarly $V(\partial^+(\Gamma))$ is the sub-globular set $\partial^+(V\Gamma)$.*

Proof. By definition, $V(\partial^-(\Gamma))$ contains the same elements as $V(\Gamma)$ in dimension up to $\dim\Gamma - 2$, and is empty in dimensions $\dim\Gamma$ and higher. So by Proposition 42, it suffices to check that in dimension $\dim\Gamma - 1$, the globular set $V(\partial^-(\Gamma))$ contains exactly the minimal elements for the preorder $\triangleleft$ in $V(\Gamma)$, with source and target fixed. This is true by a straightforward induction. In the case of the target, it, it is similar, except one has to check that we only keep the maximal element. For the induction to work, we thus have to also show that in a derivation of the form $(\Gamma, y : A, f : x \to y) \vdash_{\mathsf{ps}}$ with $\dim A = \dim\Gamma - 2$ the last variable in the context $\partial^+\Gamma$ is the maximal element of type A in Γ. □

4.2 Operations and coherences In order to translate the Grothendieck-Maltsiniotis definition of weak ω-categories in type theory, we extend the type theory GSeTT with term constructors which correspond the operations present in those categories.

Signature of the theory. We extend the signature of the theory GSeTT with two term constructors op and coh, which correspond to the liftings that are formally added in the Grothendieck-Maltsiniotis definition of weak ω-categories. Both of these constructors take as arguments a context, a typ e and a substitution, in such a way that terms in the theory are now either variables, or of the form $\mathsf{op}_{\Gamma,A}[\gamma]$ or $\mathsf{coh}_{\Gamma,A}[\gamma]$, with Γ a context, A a type and γ a substitution. We define the set of variables of a term constructed this way as

$$\mathrm{Var}(\mathsf{op}_{\Gamma,A}[\gamma]) = \mathrm{Var}(\gamma) \qquad\qquad \mathrm{Var}(\mathsf{coh}_{\Gamma,A}[\gamma]) = \mathrm{Var}(\gamma)$$

We also need to extend the action of substitutions on terms to these new terms. This has to be defined together with the composition of substitution, as they are mutually inductive notions:

$$\begin{array}{cc}
t[\langle\rangle] = t & y[\langle\gamma, x \mapsto u\rangle] = \begin{cases} u & \text{if } y = x \\ y[\gamma] & \text{otherwise}\end{cases} \\
\mathsf{op}_{\Gamma,A}[\gamma][\delta] = \mathsf{op}[\gamma\circ\delta] & \mathsf{coh}_{\Gamma,A}[\gamma][\delta] = \mathsf{coh}_{\Gamma,A}[\gamma\circ\delta] \\
\star[\gamma] = \star & (t \underset{A}{\to} u)[\gamma] = (t[\gamma]) \xrightarrow[(A[\gamma])]{} (u[\gamma]) \\
\langle\rangle\circ\gamma = \langle\rangle & \langle\delta, x\mapsto t\rangle\circ\gamma = \langle\delta\circ\gamma, x\mapsto t[\gamma]\rangle
\end{array}$$

Rules for coherences. The introduction rules for these two term constructors are subject to two side conditions, expressing the fact that some terms uses all of the variables of a context. In order to express these conditions in a more compact way, we write $\mathrm{Var}(t : A) = \mathrm{Var}(t)\cup\mathrm{Var}(A)$ for the union of the set of variables of the term t and the set of variables of the type A. In this notation, it is always implicit that the term t is of type A in the context we are considering. The introduction rules for the term constructors op and coh are then given as follows.

For contexts:

$$\frac{}{\Gamma \vdash}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE}) \quad \text{when } x \notin \mathrm{Var}(\Gamma)$$

For types:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\longrightarrow} u}(\rightarrow\text{-INTRO})$$

For terms:

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR})$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma, t \underset{A}{\to} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{OP}) \quad \text{when } (\mathrm{C}_{\mathsf{op}})$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Gamma \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\to} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{COH}) \quad \text{when } (\mathrm{C}_{\mathsf{coh}})$$

For substitutions:

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

Figure 6: Derivation rules of the theory CaTT.

– For the constructor op, the rule is

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma, t \underset{A}{\to} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{OP})$$

subject to the side conditions

$$\mathrm{Var}(t : A) = \mathrm{Var}(\partial^-(\Gamma)) \qquad \text{and} \qquad \mathrm{Var}(u : A) = \mathrm{Var}(\partial^+(\Gamma)) \qquad (\mathrm{C}_{\mathsf{op}})$$

– For the constructor coh, the rule is

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Gamma \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\to} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{COH})$$

subject to the side conditions

$$\mathrm{Var}(t : A) = \mathrm{Var}(\Gamma) \qquad \text{and} \qquad \mathrm{Var}(u : A) = \mathrm{Var}(\Gamma) \qquad (\mathrm{C}_{\mathsf{coh}})$$

Note that the rule (COH) presented here is slightly different from the one introduced in [14]: it is equivalent but makes the presentation closer to the conditions of Maltsiniotis' definition of weak ω-categories [26]. A detailed account of the equivalence between the two presentations is given in [5, Section 3.5.1]. We give in Figure 6 a full summary of all the rules of the theory CaTT.

Interpretation. These rules are to be understood as follows. A derivable judgment $\Gamma \vdash t : A$ can be thought of as a given composite of various cells that are supposed to be known in the context Γ, and in the case of a ps-context, adding the side condition $\mathrm{Var}(t : A) = \mathrm{Var}(\Gamma)$ enforces that the composite uses all the cells of Γ. This intuition is made more formal in Section 5, where we show that the contexts of this theory are finite polygraphs for weak ω-categories. In the light of this identification, a term $\Gamma \vdash t : A$ is a cell in the free category generated by the polygraph Γ. The use of the two rules can be detailed as follows.

– Rule (OP). Given a pasting scheme Γ and a way to compose entirely its source and its target encoded as the terms $\partial^-(\Gamma) \vdash t : A$ and $\partial^+(\Gamma) \vdash u : A$ satisfying the condition (C_{op}), this rule provides a way to compose entirely Γ. The result of this composition goes from the specified composition of the source to the specified composition of the target, and is encoded as the term $\Gamma \vdash \mathsf{op}_{\Gamma, t \to u}[\mathrm{id}_\Gamma] : t \to u$.
– Rule (COH). Given two ways of composing entirely the pasting scheme Γ, encoded as a pair of terms $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$ satisfying the condition ($\mathrm{C}_{\mathsf{coh}}$), the rule provides a cell between these two compositions, encoded as the term $\Gamma \vdash \mathsf{coh}_{\Gamma, t \to u}[\mathrm{id}_\Gamma] : t \to u$. It turns out that this rule only produces invertible cells, and thus it can be reformulated as: "any two ways of composing entirely a pasting scheme are weakly equivalent", or by adopting a more topological view it expresses that the space of ways to compose a pasting is contractible.

4.3 Some examples of derivations We provide some examples of derivations that one may compute in CaTT, using the actual syntax implemented in the tool [6]. This software relies on the fact that the derivability of the judgments in the theory CaTT is dericidable and provides an algorithm to decide it. The inputs from the user are interpreted by the system as typing judgments, and the software accepts an input whenever it is able to find a derivation for the corresponding judgement. This implementation follows the convention introduced by Finster and Mimram [14] and does not distinguish between the term constructors op and coh, assuming a single term constructor with two rules that are mutually exclusive. As a result, all the new constructions are introduced with the keyword `coh`, followed by a name to identify it. Then comes a list of arguments which is the description of a ps-context followed by a column and a type. For instance the following line

```
coh id (x : *) : x -> x
```

defines a coherence called `id`, which corresponds to the construction $\mathsf{coh}_{(x:\star):x \to x}$. Note that this expression is not a complete term, as it lacks a substitution. Implicitly, we may assume that we have in fact defined the term

$$(x : \star) \vdash \mathsf{coh}_{(x:\star), x \underset{\star}{\to} x}[\mathrm{id}_{(x:\star)}] : x \underset{\star}{\to} x$$

The derivation of this judgments is then guaranteed by the software (in this example, it follows from an application of the rule (COH)). We can then use the admissibility of the action of substitutions (given by Lemma 56) to define the term $\mathsf{coh}_{(x:\star), x \to x}[\gamma]$ for any substitution. Thus further references to this coherence just have to specify the substitution γ towards the context $(x : \star)$. We encode such a substitution as a list of arguments, for instance one may write `id y` to refer to the term identity, in a context containing a variable `y` of type `*`. In general, we only specify some of the argument for instance, considering the following declaration defining composition

```
coh comp (x : *) (y : *) (f : x -> y) (z : *) (g : y -> z) : x -> z
```

one needs to write only `comp f g` instead of `comp x y f z g` when referring to it, as the terms `x`, `y` and `z` can be inferred from the data of `f` and `g`. Lemma 64 proves that it suffices to provide the terms corresponding to the locally maximal variables of the target ps-context, and the software implements an elaboration mechanism that builds a full substitution out of only these arguments. Thus it can detect automatically which argument should be left implicit and allows the user to write shorter terms. Other examples of declarations one may define in CaTT include

– left unitality and its inverse

```
coh unitl (x : *) (y : *) (f : x -> y) : comp (id x) f -> f

coh unitl- (x : *) (y : *) (f : x -> y) : f -> comp (id x) f
```

– right unitality and its inverse

```
coh unitr (x : *) (y : *) (f : x -> y) : comp f (id y) -> f

coh unitr- (x : *) (y : *) (f : x -> y) : f -> comp f (id y)
```

– associativity and its inverse

```
coh assoc (x : *) (y : *) (f : x -> y) (z : *)
          (g : y -> z) (w : *) (h : z -> w)
          : comp f (comp g h) -> comp (comp f g) h

coh assoc- (x : *) (y : *) (f : x -> y) (z : *)
           (g : y -> z) (w : *) (h : z -> w)
           : comp (comp f g) h -> comp f (comp g h)
```

– vertical composition of 2-cells

```
coh vcomp (x : *) (y : *) (f : x -> y) (g : x -> y)
          (a : f -> g) (h : x -> y) (b : g -> h) : f -> h
```

– horizontal composition of 2-cells

```
coh hcomp (x : *) (y : *) (f : x -> y) (f' : x -> y) (a : f -> f')
          (z : *) (g : y -> z) (g' : y -> z) (b : g -> g')
          : comp f g -> comp f' g'
```

– left whiskering

```
coh whiskl (x : *) (y : *) (f : x -> y) (z : *) (g : y -> z)
           (g' : y -> z) (b : g -> g') : comp f g -> comp f g'
```

– right whiskering

```
coh whiskr (x : *) (y : *) (f : x -> y) (f' : x -> y)
           (a : f -> f') (z : *) (g : y -> z) : comp f g -> comp f' g
```

We also provide a syntax to define arbitrary compositions of the above declarations in an arbitrary context. The corresponding keyword is `let` followed with an identifier and a context, the symbol `=`, and a full definition of the term using previously defined terms and declarations. For instance, the following term defines the squaring of an endomorphism

```
let sq (x : *) (f : x -> x) = comp f f
```

Note that the context associated to the keyword `coh` is necessarily a ps-context, whereas any context can be associated to the keyword `let`.

4.4 Properties of the theory CaTT In order to reason and prove results about CaTT, we will be mostly reasoning by induction on its terms. We thus first need to study some of the properties of the syntax which, even though quite simple, will prove quite useful in the following.

Preservation of the basic properties. The first thing that one can check about this theory is that the term constructor are nice enough, so that the basic properties established for GSeTT still hold for this new type theory.

Lemma 56. *All the properties of Lemma 6 still hold in* CaTT*, and every derivable judgment in* CaTT *has exactly one derivation.*

Proof. These results are proved as in the case of GSeTT, by mutual induction on the derivation trees of the various judgments. The added term constructors make things a slightly more involved than in the case with only variables, and some of the properties that could be proved on a syntactical level in the theory GSeTT only hold for derivable judgments in the theory CaTT. Apart for these technical subtleties, the generalization is straightforward. □

The syntactic category. As for the theory GSeTT, the identity substitution id_Γ associated to a context Γ is always derivable, as well as the composition of derivable substitutions, using the action of substitution on raw terms. Moreover, all the results that we have stated for the theory GSeTT still hold for the theory CaTT: it is in particular the case for Proposition 9, Proposition 10 and Proposition 12. This shows that the derivable contexts of the theory CaTT assemble into a category, whose morphisms are the derivable substitution. We write $\mathcal{S}_{\mathsf{CaTT}}$ for this category and call it the *syntactic category* of the theory CaTT. The aforementioned results imply that $\mathcal{S}_{\mathsf{CaTT}}$ is equipped with a canonical structure of category with families, where Ty^Γ is the set of derivable types in the context Γ, and Tm^Γ_A is the set of terms of type A in the context Γ.

Inclusion of $\mathcal{S}_{\mathsf{GSeTT}}$. The theory CaTT contains the theory GSeTT as a subtheory: anything that can be derived using variables (in GSeTT) only can still be derived using variables and term constructors (in CaTT). In particular, any valid context Γ in the theory GSeTT is also a valid context in the category CaTT: this is in particular the case for the disk contexts D^n and the sphere contexts S^{n-1}. Moreover, this induces a functor between the syntactic categories $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$. It is immediate that this functor is a morphism of category with families, by taking any type (resp. any term) in the theory GSeTT to the same type (resp. to the same term) in the theory CaTT.

Familial representability of types. Central in the study of GSeTT was the Lemma 16, which establishes that the family $S^\bullet$ familially represents the functor Ty. We have noted that its proof does not really depend on the extra terms which are present in the theory, and thus immediately extends to the case of CaTT:

Lemma 57. *For any natural number n, the map*

$$\begin{array}{rcl} \mathcal{S}_{\mathsf{CaTT}}(\Gamma, S^{n-1}) & \to & \left\{A \in \mathrm{Ty}^\Gamma \mid \dim(A) = n-1\right\} \\ \gamma & \mapsto & U_n[\gamma] \end{array}$$

is an isomorphism natural in Γ. Given a type A of dimension $n-1$, we denote the associated substitution

$$\chi_A : \Gamma \to S^{n-1}$$

Moreover, the maps

$$\begin{array}{rcl} (\mathcal{S}_{\mathsf{CaTT}} / S^{n-1})(\Gamma \xrightarrow{\chi_A} S^{n-1}, D^n \xrightarrow{\pi} S^{n-1}) & \to & \mathrm{Tm}^\Gamma_A \\ \gamma & \mapsto & d_{2n}[\gamma] \end{array}$$

are isomorphisms, natural in Γ (the source is a hom-set in the slice category of $\mathcal{S}_{\mathsf{CaTT}}$ over S^{n-1}). Given a term $t \in \mathrm{Tm}^\Gamma_A$ of type A, we denote the associated substitution over χ_A by $\chi_t : \Gamma \to D^n$, in such a way that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\chi_t"] \arrow[dr, "\chi_A"'] & D^n \arrow[d, "\pi"] \\ & S^{n-1} \end{tikzcd}$$

Depth of a term. In order to study the theory CaTT, we often reason by structural induction on terms. In order to justify that these inductions are well-founded, we introduce the notion of *depth* of a term and of a substitution. It is the natural number $\mathrm{depth}(t)$ (resp. $\mathrm{depth}(\gamma)$) defined by induction on the term t (resp. substitution γ) by

$$\begin{array}{rr} \mathrm{depth}(x) = 0 & \mathrm{depth}(\mathsf{coh}_{\Gamma,A}[\gamma]) = 1 + \mathrm{depth}(\gamma) \\ \mathrm{depth}(\langle\rangle) = 0 & \mathrm{depth}(\langle\gamma, t \mapsto u\rangle) = \max(\mathrm{depth}(\gamma), \mathrm{depth}(u)) \end{array}$$

Informally, the depth of a term expresses how many nested term constructors are needed to write it, and similarly for substitution. It should not be confused with the notion of "coherence depth", introduced in Section 5.

Terms in the empty context. An important property that we can prove on the theory CaTT, using induction on the depth of terms, is that the there is no way to build a term in the empty context.

Lemma 58. *In the theory CaTT there is no term derivable in the empty context.*

Proof. We prove this result by induction on the depth of the term. First note that no variable is derivable in the empty context. A term of depth $d+1$ in the empty context has to be constructed using a substitution $\varnothing \to \Delta$ of depth d, where Δ is a ps-context. Since Δ is non-empty, such a substitution has to be built out of terms that are derivable in the empty context. Since the substitution is of depth at most d, these terms are of depth at most d also, and by induction there is no such term, hence there is no such substitution. This proves that there is no term of depth $d+1$ in the context $\varnothing$. □

Variables of the characteristic substitution. Using Lemma 57, we can slightly reformulate the side conditions of the rules (OP) and (COH), which involve expressions of the form $\mathrm{Var}(t : A)$.

Lemma 59. *Consider a context $\Gamma \vdash$, together with a term $\Gamma \vdash t : A$, then the following sets of variables are equal:*

$$\mathrm{Var}(A) = \mathrm{Var}(\chi_A) \qquad\qquad \mathrm{Var}(t : A) = \mathrm{Var}(\chi_t)$$

Proof. We prove these two results by mutual induction, on the dimension of A.

- If $\dim(A) = 0$, then necessarily $A = \star$. We have $\chi_\star = \langle\rangle$ and $\mathrm{Var}(\star) = \mathrm{Var}(\langle\rangle) = \emptyset$.
- If $\dim(A) > 0$, we can write $A = t \underset{B}{\to} u$, and we have $\chi_A = \langle \chi_t, d_{2n+1} \mapsto u\rangle$ with $\dim(B) + 1 = \dim(A)$. Moreover, we have by definition

$$\begin{aligned}\mathrm{Var}(A) &= \mathrm{Var}(B) \cup \mathrm{Var}(t) \cup \mathrm{Var}(u)\\ &= \mathrm{Var}(t : B) \cup \mathrm{Var}(u)\end{aligned}$$

  and on the other hand, we have

$$\mathrm{Var}(\chi_A) = \mathrm{Var}(\chi_t) \cup \mathrm{Var}(u)$$

  The induction case for terms then shows $\mathrm{Var}(A) = \mathrm{Var}(\chi_A)$.
- For A of arbitrary dimension, we have $\chi_t = \langle \chi_A, d_{2n} \mapsto t\rangle$, and thus

$$\begin{aligned}\mathrm{Var}(t : A) &= \mathrm{Var}(A) \cup \mathrm{Var}(t)\\ \mathrm{Var}(\chi_t) &= \mathrm{Var}(\chi_A) \cup \mathrm{Var}(t)\end{aligned}$$

  The induction case for types then shows $\mathrm{Var}(t : A) = \mathrm{Var}(\chi_t)$. ◻

Globular set of variables of a term. The contexts in the theory CaTT coming from the theory GSeTT play a particular role in the theory, and we call them *globular contexts*. They are recognizable by the fact that they are built out only from variables, and as we have shown in the definition of the functor V, their variables form into a globular set. For instance, of the two following contexts, the first one is a globular context, whereas the second one is not.

$$(x : \star, y : \star, z : \star, f : x \to y, g : z \to y) \qquad\qquad (x : \star, \alpha : \mathtt{id}\ x \to \mathtt{id}\ x)$$

Lemma 60. *Consider a globular context Γ in the theory CaTT together with a derivable term $\Gamma \vdash t : A$. For every variable x in the set $\mathrm{Var}(t : A)$, its source and target also belong to this set. This equips the set $\mathrm{Var}(t : A)$ with a structure of a globular set which is a globular subset of $V\Gamma$.*

Proof. We prove this result by induction on the depth of the term t.

- Since Γ is a context in $\mathcal{S}_{\mathsf{GSeTT}}$, if the term t is of depth 0, then it is a variable $t = x$ and the map $\chi_x : \Gamma \to D^n$ defines a map in $\mathcal{S}_{\mathsf{GSeTT}}$. Then $\mathrm{Var}(\chi_x)$ is the set of elements of the image of the map $V(\chi_x) : V(D^n) \to V(\Gamma)$, so it is stable under source and target and is naturally a globular subset of $V(\Gamma)$. The result is then given by Lemma 59.
- If the term t is of depth $d+1$, it is of the form $t = \mathsf{op}_{\Delta,B}[\gamma]$ or $t = \mathsf{coh}_{\Delta,B}[\gamma]$, with γ a substitution of depth at most d. Consider a variable $x \in \mathrm{Var}(t : A)$ and denote respectively by y and z its source and target in Γ. Necessarily we have $x \in \mathrm{Var}(\gamma)$, and thus there exists a variable x' in Δ such that $x \in \mathrm{Var}(x'[\gamma])$. Then consider the variables y' and z' that are respectively the source and target of x' in Δ, in such a way that we have $\Delta \vdash x' : y' \to z'$. Then we have $\Gamma \vdash x'[\gamma] : y'[\gamma] \to z'[\gamma]$ with $x \in \mathrm{Var}(x'[\gamma] : y'[\gamma] \to z'[\gamma])$ and $x'[\gamma]$ of depth at most d. By induction this proves that $y, z \in \mathrm{Var}(x'[\gamma] : y'[\gamma] \to z'[\gamma]) \subseteq \mathrm{Var}(t)$. Hence the source and target of x belong to $\mathrm{Var}(t : A)$. ◻

5. The syntactic categories associated to CaTT

This section is dedicated to the study of the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$. We have shown in Theorem 53 that the subcategory $\mathcal{S}_{\mathsf{ps}}$ of the syntactic category $\mathcal{S}_{\mathsf{GSeTT}}$ is equivalent to the category $\mathbf{\Theta}_0$. We now show that adding the term constructors op and coh allow us to recover exactly the missing pieces of information to obtain weak ω-categories: we exhibit a subcategory $\mathcal{S}_{\mathsf{ps},\infty}$ of the category $\mathcal{S}_{\mathsf{CaTT}}$ which is equivalent to the cat-coherator $\mathbf{\Theta}_\infty^{\mathrm{op}}$.

5.1 A filtration in $\mathcal{S}_{\mathsf{CaTT}}$ We consider the full subcategory $\mathcal{S}_{\mathsf{ps},\infty}$ of $\mathcal{S}_{\mathsf{CaTT}}$, whose objects are ps-contexts. Our aim is to exhibit this category as a colimit of the form

$$\mathcal{S}_{\mathsf{ps},\infty} = \operatorname{colim}\left(\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},1} \to \mathcal{S}_{\mathsf{ps},2} \to \ldots\right)$$

that mimics the iterative construction of $\mathbf{\Theta}_\infty$ as a colimit of the $\mathbf{\Theta}_n$ in the Grothendieck-Maltsiniotis definition of weak ω-categories.

Coherence depth. We introduce the notion of *coherence depth* of a term, type or substitution in order to construct the categories $\mathcal{S}_{\mathsf{ps},n}$. It is defined inductively by

$$\begin{aligned}
\mathrm{cd}(v : A) &= \mathrm{cd}(A) & \mathrm{cd}(\mathsf{op}_{\Gamma,A}[\gamma]) &= \max(\mathrm{cd}(A)+1, \mathrm{cd}(\gamma)) \\
& & \mathrm{cd}(\mathsf{op}_{\Gamma,A}[\gamma]) &= \max(\mathrm{cd}(A)+1, \mathrm{cd}(\gamma)) \\
\mathrm{cd}(\star) &= 0 & \mathrm{cd}(t \underset{A}{\to} u) &= \max(\mathrm{cd}(A), \mathrm{cd}(t), \mathrm{cd}(u)) \\
\mathrm{cd}(\langle\rangle) &= 0 & \mathrm{cd}(\langle\gamma, x \mapsto t\rangle) &= \max(\mathrm{cd}(\gamma), \mathrm{cd}(t))
\end{aligned}$$

Note that this definition is distinct from the one of depth that we have introduced in Section 4 for reasoning on syntax.

The filtration. We define the category $\mathcal{S}_{\mathsf{ps},n}$ to be the subcategory of $\mathcal{S}_{\mathsf{ps},\infty}$ that has the same objects as the category $\mathcal{S}_{\mathsf{ps},\infty}$ and whose morphisms are substitutions of coherence depth below n. Note that in the case $n = 0$, the substitutions of coherence depth 0 are the substitutions containing only variables, and thus they are exactly the substitutions of $\mathcal{S}_{\mathsf{GSeTT}}$, i.e.,

$$\mathcal{S}_{\mathsf{ps},0} = \mathcal{S}_{\mathsf{ps}}$$

We can sum up the situation with the following diagram of inclusions

$$\begin{tikzcd}
\mathcal{S}_{\mathsf{GSeTT}} \arrow[rrrr] & & & & \mathcal{S}_{\mathsf{CaTT},\infty} \\
\mathcal{S}_{\mathsf{ps}} = \mathcal{S}_{\mathsf{ps},0} \arrow[u] \arrow[r] & \mathcal{S}_{\mathsf{ps},1} \arrow[r] & \mathcal{S}_{\mathsf{ps},2} \arrow[r] & \cdots \arrow[r] & \mathcal{S}_{\mathsf{ps},\infty} \arrow[u] \\
\mathcal{G}^{\mathrm{op}} \arrow[u] & & & &
\end{tikzcd}$$

It is straightforward from the definition that $\mathcal{S}_{\mathsf{ps},\infty}$ is the colimit of this sequence of morphisms of categories

$$\mathcal{S}_{\mathsf{ps},\infty} = \operatorname{colim}\left(\mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{ps,1} \to \mathcal{S}_{ps,2} \to \cdots\right)$$

Indeed, since all these functors are the identity on the objects, it amounts to taking the colimit of the hom-sets, which define a filtration of sets:

$$\{\Delta \vdash \gamma : \Gamma\} = \bigcup_{n\in\mathbb{N}} \{\Delta \vdash \gamma : \Gamma \mid \mathrm{cd}(\gamma) \leq n\}$$

Properties of the coherence depth. The notion of coherence depth sometimes behaves awkwardly with respect to the structure of the type theory. To illustrate this, consider the context $(x : \star, \alpha : \texttt{id}\ x \to \texttt{id}\ x)$: although the term α is of coherence depth 0, its type is of coherence depth 1. This may be a source of issue when reasoning inductively on the coherence depth, as one cannot consider all the terms and its types (an example of this issue appears in Lemma 66). However, we show below that such issues do not arise in globular contexts, which are the only ones for which we are going to consider coherence depths.

First note that the application of a substitution cannot increase the coherence depth arbitrarily:

Lemma 61. *Given a substitution γ we have*

- *for any type A,* $\mathrm{cd}(A[\gamma]) \leq \max(\mathrm{cd}(A), \mathrm{cd}(\gamma))$*,*
- *for any term t,* $\mathrm{cd}(t[\gamma]) \leq \max(\mathrm{cd}(t), \mathrm{cd}(\gamma))$*,*
- *for any substitution δ,* $\mathrm{cd}(\delta \circ \gamma) \leq \max(\mathrm{cd}(\delta), \mathrm{cd}(\gamma))$*.*

Proof. We prove this result by mutual induction on the type, term and substitution.

- For the type $\star$, we have $\star[\gamma] = \star$, and hence $\mathrm{cd}(\star[\gamma]) = 0 \leq \max(0, \mathrm{cd}(\gamma))$.
- For the type $t \underset{A}{\to} u$, we have

$$\begin{aligned}
\mathrm{cd}((t \underset{A}{\to} u)[\gamma]) &= \mathrm{cd}(t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]) \\
&= \max(\mathrm{cd}(A[\gamma]), \mathrm{cd}(t[\gamma]), \mathrm{cd}(u[\gamma])) \\
&\leq \max(\mathrm{cd}(A), \mathrm{cd}(t), \mathrm{cd}(u), \mathrm{cd}(\gamma)) && \text{by induction} \\
&\leq \max(\mathrm{cd}(t \underset{A}{\to} u), \mathrm{cd}(\gamma))
\end{aligned}$$

- For a variable x, we have $\mathrm{cd}(x[\gamma]) \leq \mathrm{cd}(\gamma)$ by definition of the coherence depth of a substitution.
- For the term $t = \mathsf{op}_{\Delta,B}[\delta]$, or for the term $t = \mathsf{coh}_{\Delta,B}[\delta]$, we have

$$\begin{aligned}
\mathrm{cd}(t[\gamma]) &= \max(\mathrm{cd}(A) + 1, \mathrm{cd}(\delta \circ \gamma)) \\
&\leq \max(\mathrm{cd}(A) + 1, \mathrm{cd}(\delta), \mathrm{cd}\,\gamma) && \text{by induction} \\
&\leq \max(\mathrm{cd}(t), \mathrm{cd}(\gamma))
\end{aligned}$$

- For the substitution $\langle\rangle$, we have $\mathrm{cd}(\langle\rangle \circ \gamma) = 0 \leq \mathrm{cd}\,\gamma$.
- For the substitution $\langle \delta, x \mapsto t \rangle$, we have

$$\begin{aligned}
\mathrm{cd}(\langle \delta, x \mapsto t \rangle \circ \gamma) &= \mathrm{cd}(\langle \delta \circ \gamma, x \mapsto t[\gamma] \rangle) \\
&= \max(\mathrm{cd}(\delta \circ \gamma), \mathrm{cd}(t[\gamma])) \\
&\leq \max(\mathrm{cd}(\delta), \mathrm{cd}(t), \mathrm{cd}(\gamma)) && \text{by induction} \\
&\leq \max(\mathrm{cd}(\langle \delta, x \mapsto t \rangle), \mathrm{cd}(\gamma))
\end{aligned}$$

From which we conclude. □

Lemma 62. *In a globular context Γ, for every derivable term $\Gamma \vdash t : A$, we have* $\mathrm{cd}(A) \leq \mathrm{cd}(t)$*.*

Proof. We distinguish between the case where t is a variable and the case where t is obtained by application of a term constructors.

– If $t = x$ is a variable, and since it is derivable in a globular context, its type is derivable in the theory GSeTT and hence is of depth 0.
– A term t is not a variable, it is either of the form $t = \mathsf{op}_{\Delta,B}[\delta]$ or $t = \mathsf{coh}_{\Delta,B}[\delta]$, and in both cases we have $\mathrm{cd}(t) = \max(\mathrm{cd}(B)+1, \mathrm{cd}(\delta))$, and the type A is obtained as $A = B[\delta]$. Lemma 61 then shows that $\mathrm{cd}(A) \leq \max(\mathrm{cd}(A), \mathrm{cd}(\delta)) \leq \mathrm{cd}(t)$. □

Corollary 63. *In a globular context Γ, for every type $\Gamma \vdash A$, we have $\mathrm{cd}(A) = \mathrm{cd}(\chi_A)$ and for every term $\Gamma \vdash t : A$, we have $\mathrm{cd}(t) = \mathrm{cd}(\chi_t)$.*

Proof. We prove these two results by mutual induction on the dimension,
– For the type $\Gamma \vdash \star$, we have $\chi_\star = \langle\rangle$, and by definition, $\mathrm{cd}(\star) = \mathrm{cd}(\langle\rangle) = 0$.
– For the type $\Gamma \vdash A$ of dimension $n \geq 0$, we can write $A = t \underset{B}{\to} u$, and we have $\chi_A = \langle \chi_t, d_{2n+1} \mapsto u\rangle$. Applying Lemma 62 shows that $\mathrm{cd}(A) = \max(\mathrm{cd}(t), \mathrm{cd}(u))$. Moreover, by definition $\mathrm{cd}(\chi_A) = \max(\mathrm{cd}(\chi_t), \mathrm{cd}(u))$. The induction case for term then shows that $\mathrm{cd}(A) = \mathrm{cd}(\chi_A)$.
– For a term $\Gamma \vdash t : A$ of dimension n, we have $\chi_t = \langle \chi_A, d_{2n} \mapsto t\rangle$, and we have by definition $\mathrm{cd}(\chi_t) = \max(\mathrm{cd}(\chi_A), \mathrm{cd}(t))$. The induction case for types together with Lemma 62 show that $\mathrm{cd}(\chi_A) = \mathrm{cd}(A) \leq \mathrm{cd}(t)$ and hence $\mathrm{cd}(\chi_t) = \mathrm{cd}(t)$. □

5.2 Globular products in the category $\mathcal{S}_{\mathsf{CaTT}}$ In order to show that $\mathcal{S}_{\mathsf{ps},\infty}$ dualizes the construction of the category $\mathbf{\Theta}_\infty$, we characterize the globular products in this category.

$\mathcal{S}_{\mathsf{CaTT}}$ as a globular category with families. The inclusion functor $I : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$ induces a structure of category with families on the category $\mathcal{S}_{\mathsf{CaTT}}$, which coincides exactly with the one given by Lemma 57. Hence for this structure, I is a morphism of globular categories with families, and thus by Lemma 32, I preserves the globular finite limits. We have thus shown,

Lemma 64. *The inclusion functor $I : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$ preserves globular products.*

Lemma 65. *The inclusion functor $I : \mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{\mathsf{ps},\infty}$ preserves globular products.*

Proof. Note that the inclusion of the full subcategory $\mathcal{S}_{\mathsf{ps},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT}}$ reflects all limits. Moreover, by Lemma 64, the composite

$$\mathcal{S}_{\mathsf{ps}} \xrightarrow{I} \mathcal{S}_{\mathsf{ps},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT}}$$

preserves the globular products. Hence the functor I also preserves the globular products. □

Reflexivity of the depth-bounded inclusion. There is a canonical functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},\infty}$, which consists in forgetting that a substitution is of bounded coherence depth. In order to understand the globular product in the categories $\mathcal{S}_{\mathsf{ps},n}$, it is useful to study the behavior of this functor with respect to globular products.

Lemma 66. *The functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},\infty}$ reflects globular products.*

Proof. Consider an object Γ which is a globular product in the category $\mathcal{S}_{\mathsf{ps},\infty}$, it suffices to show that is also a globular product in the category $\mathcal{S}_{\mathsf{ps},n}$. Any cone of apex Δ over the diagram of Γ in $\mathcal{S}_{\mathsf{ps},n}$ induces a cone over the diagram of Γ in $\mathcal{S}_{\mathsf{ps},\infty}$, which by definition of a limit defines a unique substitution $\gamma : \Delta \to \Gamma$, and it suffices to show that this substitution is in fact in $\mathcal{S}_{\mathsf{ps},n}$.

By definition, all the maps $\chi_{x[\gamma]}$ where x is a maximal variable appear in the legs of the cone of apex Δ. Since these legs are chosen in the category $\mathcal{S}_{\mathsf{ps},n}$, this shows that for every locally maximal variable $\chi_{x[\gamma]}$ is of depth at most n, and hence by Corollary 63, $x[\gamma]$ is of depth at most n. Applying Lemma 62 ensures that all the iterated sources and targets of all the $x[\Gamma]$ are of depth at most n, and since every variable of Γ is obtained as an iterated source or target of variables of dimension locally maximal in Γ, all the $x[\gamma]$ for every variable x in Γ is of depth at most n. By definition, this means that γ is of depth at most n, and hence γ is a substitution in $\mathcal{S}_{\mathsf{ps},n}$. □

Globular products in the category $\mathcal{S}_{\mathsf{ps},n}$. All the categories $\mathcal{S}_{\mathsf{ps},n}$, for $n \in \mathbb{N} \cup \{\infty\}$, have the same objects, and there are more and more morphisms when n increases. None of these categories are equipped with a structure of category with families, since they lack the possibility of extending the context by any type. However for $n = 0$ and $n = \infty$ we can exhibit them as full subcategories of categories with families. Using these structure of category with families, we could prove that the functor $\mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{\mathsf{ps},\infty}$ preserves globular products (Lemma 65), and we now use this result to study globular products in all the categories $\mathcal{S}_{\mathsf{ps},n}$.

Lemma 67. *The functors $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},n}$ preserve globular products.*

Proof. We have the commutative triangle

$$\begin{tikzcd} \mathcal{S}_{\mathsf{ps},n} \arrow[r] & \mathcal{S}_{ps,\infty} \\ \mathcal{S}_{\mathsf{ps},0} \arrow[u] \arrow[ur] & \end{tikzcd}$$

By Lemma 64, the functor $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},\infty}$ preserves the globular products and, by Lemma 66, the functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},\infty}$ reflects the globular products. This implies the the functor $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},n}$ preserves globular products. □

Lemma 68. *The categories $\mathcal{S}_{\mathsf{ps},n}$ are contravariant globular extensions, and the functors $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ are morphisms of contravariant globular extensions.*

Proof. Lemma 67 in conjunction with Lemma 46 and Theorem 53 shows that the functor $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},n}$ endows $\mathcal{S}_{\mathsf{ps},n}$ with a structure of contravariant globular extension. Moreover, Lemma 46 lifts the commutative triangle

$$\begin{tikzcd} \mathcal{S}_{\mathsf{ps},n} \arrow[r] & \mathcal{S}_{\mathsf{ps},n+1} \\ \mathcal{S}_{\mathsf{ps},0} \arrow[u] \arrow[ur] & \end{tikzcd}$$

into a morphism of contravariant globular extension $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$. □

$\mathcal{S}_{\mathsf{ps},n}$ as a contravariant globular theory. Assembling altogether the results we have proved about the categories $\mathcal{S}_{\mathsf{ps},n}$, with $n \in \mathbb{N} \cup \{\infty\}$, we have the following:

Proposition 69. *For $n \in \mathbb{N} \cup \{\infty\}$, the category $\mathcal{S}_{\mathsf{ps},n}$ is equipped with a structure of a contravariant globular theory, and the functors $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ are morphisms of contravariant globular theories.*

Proof. Lemma 68 already shows that $\mathcal{S}_{\mathsf{ps},n}$ is a contravariant globular extension, moreover note that $\mathcal{S}_{\mathsf{ps}}$ and $\mathcal{S}_{\mathsf{ps},n}$ have the same objects, but $\mathcal{S}_{\mathsf{ps},n}$ has strictly more morphisms, and the functor $\mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{\mathsf{ps},n}$ sends every object to itself and defines the inclusion of the morphisms. Hence it defines a contravariant globular theory. The same reasoning starting from Lemma 64 shows that $\mathcal{S}_{\mathsf{ps},\infty}$ is also a contravariant globular theory. By Lemma 68 the functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ is a morphism of contravariant globular extensions, and hence it is also a morphism of contravariant globular theories. $\square$

5.3 Admissible pairs of substitutions In the contravariant globular theory $\mathcal{S}_{\mathsf{ps},n}$, for $n \in \mathbb{N} \cup \{\infty\}$, we consider a morphism $\xi : \Delta \to D^n$. By Lemma 57, such a morphism can be written as $\xi = \chi_t$ for some term t. Using the notations introduced in Section 2 for defining the disk and sphere contexts, the term t in Δ can be recovered as $t = d_{2n}[\xi]$ and the type of t in Δ is $U_n[\xi]$. Note that U_n contains all the variables of D^n, excepting for the variable d_{2n}, and hence $\mathrm{Var}(d_{2n}) \cup \{U_n\} = \mathrm{Var}(D^n)$. This equality shows

$$\begin{aligned}\mathrm{Var}(\xi) &= \mathrm{Var}(d_{2n}[\xi]) \cup \mathrm{Var}(U_n[\xi]) \\ &= \mathrm{Var}(t) \cup \mathrm{Var}(A)\end{aligned}$$

Lemma 70. *Given a term $\Delta \vdash t : A$, the morphism $\chi_t : \Delta \to D^n$ is algebraic in $\mathcal{S}_{\mathsf{ps},n}$ if and only if* $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$

Proof. First suppose that $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$, and consider a factorization of the form

$$\begin{tikzcd} \Delta \arrow[r, "\gamma"'] \arrow[rr, bend left, "\chi_t"] & \Gamma \arrow[r, "\chi_u"'] & D^n \end{tikzcd}$$

with γ a globular substitution, i.e., a substitution in $\mathcal{S}_{\mathsf{ps},0}$. Then we have a term $\Gamma \vdash u : B$ such that $B[\gamma] = A$ and $u[\gamma] = t$. The condition $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$ then implies in particular that $\mathrm{Var}(\Delta) \subset \mathrm{Var}(\gamma)$. Note that under the correspondence of Theorem 22, $\mathrm{Var}(\gamma)$ is the set of elements in the image of the map $V(\gamma) : V\Gamma \to V\Delta$, and the equation $\mathrm{Var}(\Delta) \subset \mathrm{Var}(\gamma)$ then shows that the map $V(\gamma)$ is a surjective map of globular sets. By Lemma 43, any map between two pasting schemes is injective so in particular that $V(\gamma)$ is an isomorphism, and Lemma 44 shows that then it is an identity. By Theorem 22, V is an equivalence of categories, and hence γ is an identity. This proves that χ_t is a algebraic morphism. Conversely suppose that the morphism χ_t is algebraic. Lemma 60 shows that the set $\mathrm{Var}(t : A)$ can be viewed as a sub globular set of $V(\Delta)$. By the equivalence of Theorem 22 the inclusion $\mathrm{Var}(t : A) \to V(\Delta)$ provides a globular substitution $\Delta \vdash \gamma : \Gamma$. Moreover, by definition, we have $\Gamma \vdash t : A$ and $A[\gamma] = A$, $t[\gamma] = t$. Hence by algebraicity of χ_t, this shows that γ is an identity. This proves the inclusion $G \to V(\Delta)$ is the identity and thus $\mathrm{Var}(t : A) = V(\Delta)$, which by forgetting the globular set structure implies $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$. $\square$

Lemma 71. *The pairs of admissible morphisms in Γ are classified by the types $\Gamma \vdash A$ satisfying either* (C_{op}) *or* (C_{coh})*. For such a type $\Gamma \vdash A$, the terms $\Gamma \vdash t : A$ classify exactly the lifts of the corresponding admissible pair.*

Proof. The types $\Gamma \vdash A$ of non-zero dimension, with A of the form $t \underset{B}{\to} u$, classify the pairs of terms (t, u) of same type B, which are exactly the pairs of parallel maps (χ_t, χ_u). Moreover, such a pair is admissible whenever we have one of the following.

– Both χ_t and χ_u are algebraic, which by Lemma 70 translates to the two conditions $\mathrm{Var}(t : B) = \mathrm{Var}(\Gamma)$ and $\mathrm{Var}(u : B) = \mathrm{Var}(\Gamma)$: this is exactly the condition $(\mathrm{C}_{\mathsf{coh}})$.
– χ_t factors through the source inclusion of Γ as a algebraic morphism and χ_u factors through the target as a algebraic morphism. Again, by Lemma 70, these conditions translate to $\partial^-(\Gamma) \vdash t : B$ with $\mathrm{Var}(t : B) = \mathrm{Var}(\partial^-(\Gamma))$ and $\partial^+(\Gamma) \vdash u : B$ with $\mathrm{Var}(u : B) = \mathrm{Var}(\partial^+(\Gamma))$: this is the condition $(\mathrm{C}_{\mathsf{op}})$.

A lift for such a admissible pair is a map $\xi : \Gamma \to D^{\dim A+1}$, such that we have both $s(\xi) = \chi_t$ and $t(\xi) = \chi_u$. In the category $\mathcal{S}_{\mathsf{CaTT},n}$ we can encode this data as a substitution $\chi_A : \Gamma \to S^{\dim A}$:

$$\begin{tikzcd}
\Gamma \arrow[dr, "\chi_A"] \arrow[drr, bend left, "\chi_t"] \arrow[ddr, bend right, "\chi_u"'] & & \\
& S^{\dim A} \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^{\dim A} \arrow[d] \\
& D^{\dim A} \arrow[r] & S^{\dim A-1}
\end{tikzcd}$$

A lift thus amounts to a morphism $\xi : \Gamma \to D^{\dim A+1}$ in $\mathcal{S}_{\mathsf{CaTT},n}$ which makes the following triangle commute:

$$\begin{tikzcd}
\Gamma \arrow[r, "\xi"] \arrow[dr, "\chi_A"'] & D^{\dim A+1} \arrow[d, "\pi"] \\
& S^{\dim A}
\end{tikzcd}$$

By Lemma 57, these are classified by the terms $\Gamma \vdash t : A$ in the theory CaTT_n. □

5.4 Equivalence between $\mathcal{S}_{\mathsf{ps},\infty}$ and $\mathbf{\Theta}_\infty^{\mathrm{op}}$ We now prove the main theorem, that the category $\mathcal{S}_{\mathsf{ps},\infty}$ is equivalent to the opposite of the cat-coherator $\mathbf{\Theta}_\infty$. This result thus identifies the cat-coherator $\mathbf{\Theta}_\infty$ as a full subcategory of the category of the category with families $\mathcal{S}_{\mathsf{CaTT}}$. We define the set F_n to be the set of all types $\Gamma \vdash t \underset{A}{\to} u$ of coherence depth exactly n in a ps-context Γ, satisfying $(\mathrm{C}_{\mathsf{op}})$ or $(\mathrm{C}_{\mathsf{coh}})$. By Lemma 71, the family F_n can be defined inductively as the set of all pair of admissible maps in $\mathcal{S}_{\mathsf{ps},n}$ that do not belong to any $F_{n'}$ for $n < n'$.

Lemma 72. *The inclusion $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ exhibits $\mathcal{S}_{\mathsf{ps},n+1}$ as the universal coglobular extension of $\mathcal{S}_{\mathsf{ps},n}$ which has a lift for all pair of morphisms in F_n.*

Proof. By Lemma 68, this functor is a morphism of coglobular theories. Moreover consider a admissible pair $(f, g) : \Gamma \to D^n$ in F_n corresponding to a type $\Gamma \vdash A$ in the ps-context Γ, which satisfies $(\mathrm{C}_{\mathsf{op}})$ or $(\mathrm{C}_{\mathsf{coh}})$ and which is of depth n. We can derive a term t by $\Gamma \vdash \mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma] : A$ if A satisfies $(\mathrm{C}_{\mathsf{op}})$, or $\Gamma \vdash \mathsf{coh}_{\Gamma,A}[\mathrm{id}_\Gamma] : A$ if A satisfies $(\mathrm{C}_{\mathsf{coh}})$, the term t is then of coherence depth $n+1$. Hence t defines a map χ_t in the category $\mathcal{S}_{\mathsf{ps},n+1}$, which by Lemma 71 is a lift for the admissible pair (f, g). We have thus proved that $\mathcal{S}_{\mathsf{ps},n+1}$ is a contravariant globular extension which contains a lift for all pairs in F_n. We now show that this extension is universal: consider another extension $F : \mathcal{S}_{\mathsf{ps},n} \to C$ that defines a lift for all the pairs in F_n, we show that there exists a unique $\tilde{F}$ that preserves the chosen lifts and makes the following diagram commute

$$\begin{tikzcd}
\mathcal{S}_{\mathsf{ps},n} \arrow[r, "F"] \arrow[d] & C \\
\mathcal{S}_{\mathsf{ps},n+1} \arrow[ur, dotted, "\tilde{F}"'] &
\end{tikzcd}$$

Indeed, the map $\tilde{F}$ is already defined on all objects of $\mathcal{S}_{\mathsf{ps},n+1}$, and all maps of coherence depth less than n, so that it coincides with F, so it suffices to show that there is a unique extension to the maps of coherence depth $n+1$. Since all the objects in $\mathcal{S}_{\mathsf{ps},n+1}$ are globular products, it suffices to show this for the maps of the form $\Gamma \to D^n$. We can thus reformulate the condition by saying that it suffices to show that there is a unique map $\tilde{F}$ on terms, satisfying the condition $\tilde{F}(t[\gamma]) = \tilde{F}t \circ \tilde{F}\gamma$. We proceed by induction on the depth, noticing that a term of coherence depth $n+1$ cannot be a variable, hence we have already defined a unique value for $\tilde{F}$ on terms of depth 0, by our previous condition, and thus the induction is already initialized.

- For a term $\Delta \vdash \mathsf{op}_{\Gamma,A}[\gamma] : A[\gamma]$ of depth $d+1$, the value of F is uniquely determined by $\tilde{F}(\mathsf{op}_{\Gamma,A}[\gamma]) = \tilde{F}(\mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma])\tilde{F}\gamma$, and since γ is of depth d, by induction $\tilde{F}(\gamma)$ is defined, and $\tilde{F}(\mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma])$ is uniquely defined by the condition of preserving the lifts for the pairs in F_n.
- Similarly, for a term $\Delta \vdash \mathsf{coh}_{\Gamma,A}[\gamma] : A[\gamma]$ of depth $d+1$, the value of F is uniquely determined by $\tilde{F}(\mathsf{op}_{\Gamma,A}[\gamma]) = \tilde{F}(\mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma])\tilde{F}\gamma$, and since γ is of depth d, by induction $\tilde{F}(\gamma)$ is defined, and $\tilde{F}(\mathsf{coh}_{\Gamma,A}[\mathrm{id}_\Gamma])$ is uniquely defined by the condition of preserving the lifts for the pairs in F_n.

This proves that there exists a unique $\tilde{F}$ satisfying the condition, and hence $\mathcal{S}_{\mathsf{ps},n+1}$ is the universal coglobular extension obtained by adding a lift for all arrows in F_n to $\mathcal{S}_{\mathsf{ps},n}$ □

This establishes a close correspondence between the categories $\mathcal{S}_{\mathsf{ps},n}$ and $\mathbf{\Theta}_n$, and enables us to prove the following theorem.

Theorem 73. *We have an equivalence of categories*

$$\mathcal{S}_{\mathsf{ps},\infty} \simeq \mathbf{\Theta}_\infty^{\mathrm{op}}$$

Proof. By construction $\mathcal{S}_{\mathsf{ps},\infty}$ is obtained as the colimit of the inclusions of categories

$$\mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},1} \to \cdots \to \mathcal{S}_{\mathsf{ps},n} \to \cdots \to \mathcal{S}_{\mathsf{ps},\infty} = \mathrm{colim}_n\, \mathcal{S}_{\mathsf{ps},n}$$

It is therefore enough to prove that $\mathcal{S}_{\mathsf{ps},n}$ is equivalent to $\mathbf{\Theta}_n^{\mathrm{op}}$, which we do by induction.

- We have already proved that $\mathcal{S}_{\mathsf{ps},0}$ is equivalent to $\mathbf{\Theta}_0^{\mathrm{op}}$ in Theorem 53.
- Suppose that $\mathcal{S}_{\mathsf{ps},k}$ is equivalent to $\mathbf{\Theta}_k^{\mathrm{op}}$ for every $k \leq n$. Lemma 72 shows that $\mathcal{S}_{\mathsf{ps},n+1}$ is the universal contravariant globular extension that adds a lift for each pair in the set F_n. Moreover, the set F_n coincides with the set E_n defined in Section 3 and, by definition, $\mathbf{\Theta}_{n+1}$ is the universal globular extension. Hence $\mathcal{S}_{\mathsf{ps},n+1}$ and $\mathbf{\Theta}_{n+1}^{\mathrm{op}}$ satisfy the same universal property and are therefore equivalent. □

6. Models of CaTT

This section is dedicated the study of the models of the type theory CaTT using tools that generalize the ones developed in Section 2.5. In particular, we prove an initiality result analogous to Theorem 36 for the category $\mathcal{S}_{\mathsf{CaTT}}$. We then apply this result to characterize the **Set**-models of the theory and prove that they are equivalent to the weak ω-categories in the sense of Grothendieck-Maltsiniotis, presented in Section 3. We also give a detailed syntactic interpretation of the construction that we develop here, showing that although it uses abstract categorical machinery, it translates closely the intuition coming from type theory.

6.1 Cat-categories with families In the case of the category GSeTT, we have introduced the notion of globular category with families, and proved that $\mathcal{S}_{\mathsf{GSeTT}}$ is initial among them (Theorem 36), which implies that we can compute the semantics of this theory in any category with families. We further prove this result by defining the structure of a *cat-category with families*, which plays an analogue role for the theory $\mathcal{S}_{\mathsf{CaTT}}$. We denote D_P the functor $\mathcal{G}^{\mathrm{op}} \to \mathbf{\Theta}_\infty^{\mathrm{op}}$, which defines the disk objects in the category $\mathbf{\Theta}_\infty^{\mathrm{op}} \simeq \mathcal{S}_{\mathsf{ps},\infty}$. The functor D_P can also be seen as the restriction of the functor $D^\bullet$ on ps-contexts.

Definition 74. A cat-category with families is a category with families $\mathcal{C}$ equipped with a functor $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ together with a structure of globular category with families given by the functor $F \circ D_P : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, and such that F sends the globular sums to globular products in $\mathcal{C}$.

Our main example of a cat-category with families is the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$. The associated functor, that we write $P_\infty : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT}}$, is given by the inclusion $\mathcal{S}_{\mathsf{ps},\infty} \to \mathcal{S}_{\mathsf{CaTT}}$, together with the identification given by Theorem 73. In fact a cat-category with families can be thought of as a category with families which supports a type $\star$ along with all its iterated types $\to$, and for which the term constructor op and coh exist, like in the theory CaTT. From now on, we use Theorem 73 implicitly to identify the categories $\mathcal{S}_{\mathsf{ps},\infty}$ and the categories $\mathbf{\Theta}_\infty^{\mathrm{op}}$ and in particular, we may think of an object of $\mathbf{\Theta}_\infty$ as a ps-context, and of a map $\gamma \in \mathbf{\Theta}_\infty(\Gamma, \Delta)$ as a substitution $\Delta \vdash \gamma : \Gamma$ in the theory CaTT. Combining this equivalence with Lemma 57, this lets us think of maps $f : \mathbf{\Theta}_\infty(D^n, \Gamma)$ as terms in the ps-context Γ in the theory CaTT.

Morphisms of cat-categories with families. A morphism between two cat-categories with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ and $G : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{D}$ is a functor $f : \mathcal{C} \to \mathcal{D}$ inducing a morphism of globular categories with families on the induced structures and making the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] & \mathcal{D} \\ \mathbf{\Theta}_\infty^{\mathrm{op}} \arrow[u, "F"] \arrow[ur, "G"'] & \end{tikzcd}$$

commute up to natural isomorphism.

6.2 Algebraic natural transformations Consider two cat-categories with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ and $G : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{D}$, along with an object Γ in $\mathcal{C}$ and an object Δ in $\mathcal{D}$. We define a notion of *algebraic natural transformation* between $T_G\Delta$ and $T_F\Gamma$ in the category $\widehat{\mathcal{G}}$ (the functor T_F is recalled below). This can be seen as a compatibility condition, and might seem ad-hoc at first, but the reason why we are interested in such transformations will be apparent in Proposition 79, and we provide in Section 6.4 a discussion showing that from the point of view of type theory, they are actually a very natural notion to consider.

The two nerves of a cat-category with families. Suppose given a cat-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$. We write

$$\begin{aligned} N_F : \mathcal{C} &\to \widehat{\mathbf{\Theta}_\infty} \\ \Gamma &\mapsto \mathcal{C}(\Gamma, F\_) \end{aligned}$$

for the associated nerve functor. By definition, F induces a structure of globular category with families by precomposing F with the canonical functor $D_P : \mathcal{G}^{\mathrm{op}} \to \mathbf{\Theta}_\infty^{\mathrm{op}}$ which defines the disk objects within $\mathbf{\Theta}_\infty$. Recall that we write

$$\begin{aligned} T_F : \mathcal{C} &\to \widehat{\mathcal{G}} \\ \Gamma &\mapsto \mathcal{C}(\Gamma, F\_) \end{aligned}$$

for the nerve functor associated to this structure of globular category with families. Note that the definition looks the same as for N_F, but the argument is only allowed to vary in $\mathcal{G}$, viewed as a subcategory of $\mathbf{\Theta}_\infty$ In the case of the cat-category with families $\mathcal{S}_{\mathsf{CaTT}}$, we simply denote N and T these two functors.

Induced nerve transformation. Consider a natural transformation

$$\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$$

Then for every globular set X, η induces by composition, the following transformation which is natural in X:

$$\begin{aligned} \eta^\star : \widehat{\mathcal{G}}(X, T_G\Delta) &\to \widehat{\mathcal{G}}(X, T_F\Gamma) \\ \xi &\mapsto \eta \circ \xi \end{aligned}$$

In particular, consider the case where X is of the form $V\Theta$ for a ps-context $\Theta \in \mathbf{\Theta}_0$. Then Theorem 36, together with the characterisation of Kan extensions given by Lemma 24, shows that we have the two following natural isomorphisms:

$$\widehat{\mathcal{G}}(V\Theta, T_G\Delta) \cong \mathcal{D}(\Delta, G\Theta) = (N_G\Delta)_\Theta$$
$$\widehat{\mathcal{G}}(V\Theta, T_F\Delta) \cong \mathcal{C}(\Gamma, F\Theta) = (T_F\Gamma)_\Theta$$

This construction is natural in $\Theta \in \mathbf{\Theta}_0$, and hence $\eta^\star$ defines a natural transformation

$$\eta^\star : N_G\Delta \to N_F\Gamma$$

We thus have constructed a function

$$\_^\star : \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma) \to \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$$

which is natural in both Δ and Γ. Following the definitions of the isomorphisms, one can extract that for $\gamma \in \mathcal{D}(\Delta, GX)$, the transformation $\eta^\star(\gamma)$ is characterized by the fact that for every map $\chi \in \mathbf{\Theta}_0(D^n, X)$, we have

$$F\chi \circ \eta^\star\gamma = \eta(G\chi \circ \gamma)$$

A algebraic natural transformation is one that satisfies this relation not only for the maps of $\mathbf{\Theta}_0$, but also for the maps of $\mathbf{\Theta}_\infty$:

Definition 75. A natural transformation

$$\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$$

is *algebraic* if for every map $\theta \in \mathcal{D}(\Delta, G\Theta)$ and every map $\chi \in \mathbf{\Theta}_\infty(D^n, \Theta)$, we have

$$\eta(G\chi \circ \theta) = F\chi \circ \eta^\star\theta$$

We write $\widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)_{\mathrm{alg}}$ the set of algebraic natural transformations between $T_G\Delta$ and $T_F\Gamma$. Note that the algebraicity condition is a generalization of the defining equality of $\eta^\star$, but instead of holding only for maps in $\mathbf{\Theta}_0$, we require that it holds also for maps in $\mathbf{\Theta}_\infty$.

Naturality of algebraic natural transformations. We have proven that a natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$, induces a natural transformation $\eta^\star \in \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$, in fact, the following result shows that if the transformation η is algebraic, then $\eta^\star$ satisfies a stronger naturality condition.

Lemma 76. *If $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)_{\mathrm{alg}}$ is an algebraic natural transformation, then $\eta^\star$ defines a natural transformation $\eta^\star \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$.*

Proof. We have already defined, for every element γ of the presheaf $N_G\Delta$ an element $\eta^\star(\gamma)$ of the presheaf $N_F\Gamma$, and it is enough to verify that it induces a natural transformation between the presheaves $N_G\Delta$ and $N_F\Gamma$ over $\mathbf{\Theta}_\infty$. Consider an element γ of $N_G(\Delta)$, i.e., $\gamma \in \mathcal{D}(\Delta, GX)$, and recall that $\eta^\star(\gamma)$ is defined to be the transformation such that for every map $\chi \in \mathbf{\Theta}_0(D^n, X)$, we have $F\chi \circ \eta^\star\gamma = \eta(G\chi \circ \gamma)$. Given a map $f \in X \to Y$ in $\mathbf{\Theta}_\infty$, we have

$$\begin{aligned}
F\chi \circ (Ff \circ \eta^\star(\gamma)) &= F(\chi \circ f) \circ \eta^\star(\gamma) \\
&= \eta(G(\chi \circ f) \circ \gamma) && \text{by algebraicity} \\
&= \eta(G\chi \circ Gf \circ \gamma) \\
&= F\chi \circ \eta^\star(Gf \circ \gamma) && \text{by definition of } \eta^\star
\end{aligned}$$

Hence, for every variable x derivable in the context Δ in the theory $\mathcal{S}_{\mathsf{GSeTT}}$, we have

$$F\chi_x \circ (Ff \circ \eta^\star(\gamma)) = F\chi \circ \eta^\star(Gf \circ \gamma)$$

and thus we have the equality

$$Ff \circ \eta^\star(\gamma) = \eta^\star(Gf \circ \gamma)$$

This proves the commutativity of the square

$$\begin{tikzcd}
\mathcal{D}(\Delta, GY) \arrow[r, "\eta^\star_Y"] \arrow[d, "Gf\circ_"'] & \mathcal{C}(\Gamma, FY) \arrow[d, "Ff\circ_"] \\
\mathcal{D}(\Delta, GX) \arrow[r, "\eta^\star_X"'] & \mathcal{C}(\Gamma, FX)
\end{tikzcd}$$

□

For later proofs, we need to reproduce this same construction in the case where the coherence depth is bounded, to this end, we introduce a sub-presheaf $T_d\Delta$ of the presheaf $T_d\Gamma$ whose elements are all the substitutions $\Delta \to D^n$ whose coherence depth is at most d. Lemma 62 together with Corollary 63 ensures that this indeed defines a sub-presheaf. Similarly we define a sub-object $N_d\Delta$ of the presheaf $N\Delta$, whose elements are the substitutions whose depth is at most d. Even though $N_d\Delta$ is not in general a presheaf over $\mathbf{\Theta}_\infty$, Lemma 61 shows that it still defines a presheaf over $\mathbf{\Theta}_d$. The proof that we have presented just presented then also works for a natural transformation $\eta : \widehat{G}(T_d\Delta, T_F\Gamma)$ in bounded coherence depth, showing that it induces natural transformation $\eta^\star \in \widehat{\mathbf{\Theta}_d}(N_d\Delta, N_F\Gamma)$.

Algebraicity of the nerve transformations. Conversely, any given natural transformation $\eta \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$ induces a natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$. Indeed, there is an inclusion functor $D_P : \mathcal{G}^{\mathrm{op}} \to \mathbf{\Theta}_\infty^{\mathrm{op}}$, and the presheaves $T_G\Delta$ and $T_F\Gamma$ are exactly the restrictions of the presheaves $N_G\Delta$ and $N_F\Gamma$ along this inclusion.

Lemma 77. *Consider a natural transformation* $\eta \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$ *along with its restriction* $\overline{\eta} \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$*. Then the induced natural transformation* $\overline{\eta}^\star \in \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$ *coincides with* η*.*

Proof. Consider an element $\delta \in N_G\Delta$, i.e., δ is a map $\delta : \Delta \to GX$ in the category $\mathcal{D}$ for some object X of $\mathbf{\Theta}_0$. Then $\overline{\eta}^\star(\delta)$ is defined to be the unique map such that for every map $\chi : X \to D^n$ in $\mathbf{\Theta}_0^{\mathrm{op}}$ we have $F\chi \circ \overline{\eta}^\star(\delta) = \overline{\eta}(G\chi \circ \delta)$. The naturality of the transformation η ensures that $F\chi \circ \eta(\delta) = \eta(G\chi \circ \delta)$ for every map χ in the category $\mathbf{\Theta}_\infty$. In particular, this is satisfied for maps in $\mathbf{\Theta}_0$, and hence η satisfies the defining property of $\overline{\eta}^\star$, and hence $\eta = \overline{\eta}^\star$. □

Lemma 78. *For every natural transformation* $\eta \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$*, the induced natural transformation* $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$ *is algebraic.*

Proof. By Lemma 77, the algebraicity condition rewrites as $\eta(Gf \circ \theta) = Ff \circ \eta(\theta)$ for every map $\theta \in \mathcal{D}(\Delta, G\Theta)$ and every map $f \in \mathbf{\Theta}_\infty(D^n, \Theta)$. This is given by the naturality of η with respect to $\mathbf{\Theta}_\infty$. □

The equivalence. Combining the results of Lemma 76 and Lemma 78, we have proven the following result:

Proposition 79. *The two operations defined above form a natural isomorphism*

$$\widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)_{\mathrm{alg}} \cong \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$$

Proof. We have already proved in Lemmas 76 and 78 that these operations are well-defined, and moreover Lemma 77 shows that the induced transformation of a restriction is the transformation itself. So it suffices to show that restricting an induced algebraic natural transformation also yields the identity. Consider an algebraic natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$. By definition, for every object X in $\mathbf{\Theta}_0$ and for all maps $\delta \in \mathcal{D}(\Delta, GX)$ and $\chi \in \mathbf{\Theta}_0(D^n, X)$, we have the equality $\eta(G\chi \circ \delta) = F\chi \circ \eta^\star(\delta)$. In particular, taking $X = D^n$ and χ to be the identity yields $\eta(\delta) = \eta^\star(\delta)$. Hence $\eta^\star$ coincides with η on the presheaf $T_G\Delta$, and thus the induction and restriction operation are inverse operations. □

6.3 Kan extension of a cat-category with families

We now use algebraic natural transformations to define and characterize the right Kan extension of a cat-category with family $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ along the functor $P_\infty : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT}}$. Similarly to Section 2.5, this right Kan extension is the key construction to prove the initiality of the syntactic category. However, in the case of CaTT, the presence of term constructors makes the existence harder to prove, and it is the reason why we have introduced the algebraic natural transformations.

Algebraic natural transformations that agree on the variables. An important property of algebraic natural transformations, is that their value is entirely determined by their values on the variables of the theory. This is similar to substitutions, and indeed, we show later that this notion captures exactly the computation of the substitutions.

Lemma 80. *Consider a cat-category with families* $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ *along with a context* Γ *in* $\mathcal{S}_{\mathsf{CaTT}}$ *and an object* Δ *in* $\mathcal{C}$*. Two algebraic natural transformations* $\eta, \eta' \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$ *are equal if and only if we have* $\eta(\chi_x) = \eta'(\chi_x)$ *for every variable* x *in* Γ*.*

Proof. If two algebraic natural transformations are equal, then they necessarily agree on the characteristic maps of the variables, so it suffices to check the converse. Consider two algebraic natural transformations $\eta, \eta' \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$ that coincide on all the variables. We want to prove that they are equal. For this we show by induction on the depth of the term t that for any derivable term t in Γ, we have $\eta(\chi_t) = \eta'(\chi_t)$.

- Terms of depth 0 are simply variables, and for those, we have the equality by hypothesis.
- A term t of depth $d+1$ is of the form $\mathsf{op}_{\Theta,B}[\theta]$ or of the form $\mathsf{coh}_{\Theta,B}[\theta]$, with θ a substitution of depth at most d. In this case we consider the term t' to be $t' = \mathsf{op}_{\Theta,B}[\mathrm{id}_\Theta]$ or $t' = \mathsf{coh}_{\Theta,B}[\mathrm{id}_\Theta]$ respectively. This provides a factorization of the form $\chi_t = \chi_{t'} \circ \theta$. Since η and η' are algebraic, we therefore have

$$\eta(\chi_t) = F(\chi_{t'}) \circ \eta^\star(\theta) \qquad\qquad \eta'(\chi_t) = F(\chi_{t'}) \circ \eta'^\star(\theta)$$

  Moreover, for every variable y of Θ, since θ is of depth at most d, so is $y[\theta]$, and therefore, by induction, we have the following equalities:

$$\begin{aligned} \eta'^\star(\theta)(\chi_y) &= \eta'(\chi_{y[\theta]}) \\ &= \eta(\chi_{y[\theta]}) \\ &= \eta^\star(\theta)(\chi_y) \end{aligned}$$

  Thus $\eta'^\star(\theta)$ satisfies the defining property of $\eta^\star(\theta)$, and hence $\eta'^\star(\theta) = \eta^\star(\theta)$, which proves that $\eta(\chi_t) = \eta'(\chi_t)$. □

Algebraic transformation from a context comprehension. We now give a result that allows us to compute the algebraic natural transformations mapping out of the nerve of a context extension. This result is our main reason to consider algebraic transformation in the first place. In order to express it, we need a construction that we first present here. Consider a context Γ together with a derivable type $\Gamma \vdash A$ in the theory CaTT. Then, by Lemma 57, the type A is classified by a substitution $\chi_A : \Gamma \to S^{n-1}$ in the category $\mathcal{S}_{\mathsf{CaTT}}$. By Theorem 36 together with Lemma 24, this substitution gives rise to a natural transformation in $\mathcal{G}(VS^{n-1}, T\Gamma)$, and by precomposition, it induces a map $\widehat{\mathcal{G}}(T\Gamma, T_F\Delta) \to \widehat{\mathcal{G}}(VS^{n-1}, T_F\Delta)$. Applying Theorem 36 again allows us to rewrite this map as $f_A : \widehat{\mathcal{G}}(T\Gamma, T_F\Delta) \to \mathcal{D}(\Delta, FS^{n-1})$. Following the construction explicitly, one can see that for a natural transformation $\eta \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)$ the map $f_A(\eta)$ is defined by the property that for any map $\chi \in \mathcal{S}_{\mathsf{GSeTT}}(S^{n-1}, D^k)$,i.e., for any variable in the context S^{n-1}, we have the equality

$$F\chi \circ f_A(\eta) = \eta(\chi \circ \chi_A)$$

In particular, if we consider a context of the form $(\Gamma, x : A)$, then we necessarily have a derivation of the type $\Gamma \vdash A$. Moreover, the projection substitution $\pi : (\Gamma, x : A) \to \Gamma$ induces a morphism $T\pi : T\Gamma \to T(\Gamma, x : A)$ which can be thought of as a weakening, and provides a map

$$\_ \circ T\pi : \widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta) \to \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)$$

Using the fact that the following pullback square commutes

$$\begin{tikzcd} (\Gamma, x : A) \arrow[r, "\chi_x"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^n \arrow[d, "\mathrm{ty}"] \\ \Gamma \arrow[r, "\chi_A"'] & S^{n-1} \end{tikzcd}$$

one can rewrite the defining property of $f_A(\eta\circ T\pi)$ by saying that for every map $\chi \in \mathcal{S}_{\mathsf{GSeTT}}(S^{n-1}, D^k)$, we have the equality

$$F\chi \circ f_A(\eta \circ T\pi) = \eta(\chi \circ \mathrm{ty} \circ \chi_x)$$

Lemma 81. *There is an isomorphism*

$$\widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}} \cong \left\{(\eta, \chi_t) \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \times \mathcal{D}(\Delta, FD^n) \mid \mathrm{ty} \circ \chi_t = f_A(\eta)\right\}$$

This can be reformulated as saying that $\widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}}$ is obtained as a pullback of the form

$$\begin{tikzcd}
\widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (T_F\Delta)_n \arrow[d] \\
\widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \arrow[r] & \mathcal{D}(\Delta, FS^{n-1})
\end{tikzcd}$$

Proof. Consider a context $(\Gamma, x : A)$ in the theory CaTT and, in order to simplify notations, define the set

$$X = \left\{(\eta, \chi) \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \times \mathcal{D}(\Delta, FD^n) \mid \mathrm{ty} \circ \chi = f_A(\eta)\right\}$$

We consider the following map:

$$\begin{aligned}
\widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}} &\to X \\
\eta &\mapsto (\eta \circ T\pi, \eta(\chi_x))
\end{aligned}$$

We show that this map is well defined, i.e., we show that for every $\eta \in \widehat{\mathcal{G}}(TD^n, T_F\Delta)_{\mathrm{alg}}$, we have $(\eta \circ T\pi, \eta(\chi_x)) \in X$. First note that η and $\eta \circ T\pi$ act in the same way on every term on which they are both defined, but $\eta \circ T\pi$ is defined on strictly fewer terms than η. Hence, since η is algebraic, $\eta \circ T\pi$ is also necessarily so. Since, by definition, F is a cat-category with families, it provides $\mathcal{D}$ with a structure of globular category with families, and we have that $F\,\mathrm{ty} = \mathrm{ty}$. Hence, for every map $\chi \in \mathcal{S}_{\mathsf{GSeTT}}(S^{n-1}, D^k)$, we have the following equalities:

$$\begin{aligned}
F\chi \circ \mathrm{ty} \circ \eta(\chi_x) &= F(\chi \circ \mathrm{ty}) \circ \eta(\chi_x) \\
&= \eta(\chi \circ \mathrm{ty} \circ \chi_x) && \text{by naturality of } \eta
\end{aligned}$$

Hence $\mathrm{ty} \circ \eta(\chi_x)$ satisfies the defining property of $f_A(\eta\circ T\pi)$. This proves that $(\eta\circ T\pi, \eta(\chi_x)) \in X$.

We now prove that the defined map is a bijection. Consider a pair $(\eta, \chi) \in X$, then an algebraic natural transformation η' mapping onto this pair has its action on the variables determined. Indeed, a variable in $(\Gamma, x : A)$ is either the variable x, or it is a variable of Γ. For the variable x, we have, by definition of η', that $\eta'(\chi_x) = \chi$, and for a variable y in Γ, then we have the factorisation $\chi_y = \chi_y \circ \pi$ and so $\eta'(\chi_y) = \eta(\chi_y)$. By Lemma 80 this proves that the mapping is injective.

Conversely, we show that this mapping is surjective. We construct a natural transformation $\eta' \in \widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}}$ which extends the algebraic natural transformation η. First for any term t in $(\Gamma, x : A)$ which does not use the variable x, the term t is also definable in Γ, and we define $\eta'(\chi_t) = \eta(\chi_t)$. So it suffices to define the natural transformation η' on the terms in $(\Gamma, x : A)$ that contain the variable x, and to verify the naturality and algebraicity of η' on those terms. We proceed by induction on the coherence depth of the term.

- The term containing x of minimal coherence depth is necessarily the variable x itself, and in this case we define $\eta'(\chi_x) = \chi$. This assignment in natural on the variable x by definition of the set X.
- Suppose $\eta' \in \widehat{\mathcal{G}}(T_d\Gamma, T_F\Delta)$ to be defined and natural on all terms containing the variable x of coherence depth at most d, and consider a term t of depth $d+1$. Then t is necessarily of the form $t = \mathsf{op}_{\Theta,B}[\theta]$ (resp. $t = \mathsf{coh}_{\Theta,B}[\theta]$), and we define $t' = \mathsf{op}_{\Theta,B}[\mathrm{id}_\Theta]$ (resp. $t' = \mathsf{coh}_{\Theta,B}[\mathrm{id}_\Theta]$), in such a way that $\chi_t = \chi'_t \circ \theta$. Then note that θ is of coherence depth at most d, and hence defines a natural transformation in $\mathcal{G}^{\mathrm{op}}(V\Theta, T_d\Gamma)$, hence by composition with η', this provides a natural transformation in $\widehat{\mathcal{G}}(V\Theta, T_F\Delta)$ which gives a morphism $\eta'^\star(\theta) : \Delta \to F\Theta$. We then define $\eta'(\chi_t) = F(\chi_{t'}) \circ (\eta'^\star(\theta))$. We check that the transformation defined this way is natural. Consider a variable y in the context D^n, corresponding to a morphism $\chi_y : D^k \to D^n$ in the category $\mathcal{G}$, and a term t of coherence depth $d+1$ in the context $(\Gamma, x : A)$ that uses the variable x, and denote t' and θ as above. We have the equalities

$$\begin{aligned} F(\chi_y) \circ \eta'(\chi_t) &= F(\chi_y) \circ F(\chi_{t'}) \circ \eta'^\star(\theta) \\ &= F(\chi_y \circ \chi_{t'}) \circ \eta'^\star(\theta) \end{aligned}$$

  If $\chi_y \circ \chi_{t'} = \chi_z$, where z is a variable, then we have $\chi_y \circ \chi_t = \chi_z \circ \theta$, and by definition of $\eta'^\star(\theta)$, we have $F(\chi_z) \circ \eta'^\star(\theta) = \eta'(\chi_z \circ \theta)$. If $\chi_y \circ \chi_{t'} = \chi_u$ where u is not a variable, it is again of the form $\mathsf{op}_{\Xi,C}[\xi]$ (resp. $\mathsf{coh}_{\Xi,C}[\xi]$), and we denote $u' = \mathsf{op}_{\Theta',C}[\xi]$ (resp. $u' = \mathsf{coh}_{\Theta',C}[\xi]$). In this case, we have $\chi_y \circ \chi_t = \chi_{u'} \circ \xi \circ \theta$, and thus we have

$$\begin{aligned} \eta'(\chi_y \circ \chi_t) &= F(\chi_{u'}) \circ \eta'^\star(\xi \circ \theta) \\ &= F(\chi_{u'}) \circ F(\xi) \circ \eta'^\star(\theta) && \text{by naturality of } \eta'^\star \\ &= F(\chi_y \circ \chi_{t'}) \circ \eta'^\star(\theta) \end{aligned}$$

  In both cases, we have $\eta'(\chi_y \circ \chi_t) = F(\chi_y) \circ \eta'(\chi_t)$ which proves that η' is natural on χ_t.

We now prove that the natural transformation we have just defined is a preimage of the couple (η, χ), and note that by definition, we have $\eta' \circ T\pi = \eta$ and $\eta'(\chi_x) = \chi$, so it suffices to show that η' is algebraic. Consider a ps-context Θ together with a map $\theta : (\Gamma, x : A) \to \Theta$, and a map $\xi \in \mathbf{\Theta}_\infty(D^n, \Theta)$. The map ξ corresponds to a term in the ps-context Θ which is either a variable or of the form $\mathsf{op}_{\Theta',B}[\xi']$ (resp. $\mathsf{coh}_{\Theta',B}[\xi']$). If ξ defines a variable, the equality required for the algebraicity is implied by the naturality, so it suffices to verify it for the term constructors. We define $t' = \mathsf{op}_{\Theta',B}[\mathrm{id}_{\Theta'}]$ (resp. $\mathsf{coh}_{\Theta',B}[\mathrm{id}'_{\Theta'}]$), in such a way that we have $\xi = \chi_{t'} \circ \xi'$. We then have the following equalities

$$\begin{aligned} \eta'(\xi \circ \theta) &= F(t') \circ \eta'^\star(\xi' \circ \theta) \\ &= F(t') \circ F(\xi') \circ \eta'^\star(\theta) && \text{by naturality of } \eta'^\star \\ &= F(\xi) \circ \eta'^\star(\theta) \end{aligned}$$

□

Existence of the Kan extension. The previous results show that algebraic natural transformations can be built inductively following the structure of contexts, starting with the empty contexts and computed with a sequence of context comprehension operations. This lets us define and characterize the right Kan extension of any cat-category with families $\mathcal{C}$ along the functor P_∞, by proving that all the canonical diagrams of objects in $\mathcal{S}_{\mathsf{CaTT}}$ necessarily have a limit in $\mathcal{C}$.

Lemma 82. *Given a cat-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, for every context Γ in $\mathcal{S}_{\mathsf{CaTT}}$ the diagram $(\Gamma \downarrow P_\infty) \to \mathbf{\Theta}_\infty^{\mathrm{op}} \xrightarrow{F} \mathcal{C}$ has a limit.*

Proof. The limit X of such a diagram is characterized by the fact that there is a natural isomorphism

$$\mathcal{C}(\Delta, X) \cong \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_G\Delta)$$

in Δ, so that it suffices to construct an object X which satisfies this property. Proposition 79 lets us rewrite as the above isomorphism

$$\mathcal{C}(\Delta, X) \cong \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$$

We proceed by induction on Γ to show that there exists an object X satisfying this property.

- For the context $\varnothing$, an element of the presheaf $T\varnothing$ is a substitution $\varnothing \vdash \gamma : D^n$, so by Lemma 57 it is necessarily of the form χ_t where t is a term in the context $\varnothing$. Since by Lemma 58 there is no such term, this implies that there is no element in $T\varnothing$, and thus it is the empty globular set, that is initial. Moreover the only natural transformation $! : T\varnothing \to T_F\Delta$ is vacuously algebraic. Hence $\widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} = \{\bullet\}$ is a singleton. So the limit of the diagram is the terminal object in $\mathcal{C}$, which exists by definition of a category with families.
- For a context of the form $(\Gamma, x : A)$, assume that there is an object Y in $\mathcal{C}$, together with a natural isomorphism $\mathcal{C}(\Delta, Y) \cong \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$. We can apply Lemma 81, which provides the following equalities

$$\begin{aligned}
\widehat{\mathcal{G}}(T(\Gamma, A), T_F\Delta)_{\mathrm{alg}} &\cong \lim\left(\widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \longrightarrow \mathcal{C}(\Delta, FS^{n-1}) \xleftarrow{\mathcal{C}(\Delta,\mathrm{ty})} \mathcal{C}(\Delta, Fn)\right) \\
&\cong \lim\left(\mathcal{C}(\Delta, Y) \longrightarrow \mathcal{C}(\Delta, FS^{n-1}) \xleftarrow{\mathcal{C}(\Delta,\mathrm{ty})} \mathcal{C}(\Delta, Fn)\right) \\
&\cong \mathcal{C}\left(\Delta, \lim\left(Y \xrightarrow{f} FS^{n-1} \xleftarrow{\mathrm{ty}} Fn\right)\right)
\end{aligned}$$

  By definition of the structure of a globular category with families, the above limit in $\mathcal{C}$ exists and can be computed as $X = (Y, \mathrm{ty}(f))$. This choice of X by definition is an object such that $\mathcal{C}(\Delta, X) \cong \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$, hence it defines a limit for the canonical diagram associated to Γ. □

Since for every object Γ the limit of the canonical diagram associated to Γ exists, this proves that the right Kan extension $\mathrm{Ran}_{P_\infty} F$ exists and is pointwise. For every object Γ in $\mathcal{C}$ it is characterized by the fact that there is a natural isomorphism

$$\mathcal{C}(\Delta, (\mathrm{Ran}_{P_\infty} F)\Gamma) \cong \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_F\Delta)$$

Moreover, the above Lemma also shows that $(\mathrm{Ran}_{P_\infty} F)\varnothing$ is a terminal object in $\mathcal{C}$ and that we have

$$(\mathrm{Ran}_{P_\infty} F)(\Gamma, A) \cong ((\mathrm{Ran}_{P_\infty} F)\Gamma, \mathrm{ty}((\mathrm{Ran}_{P_\infty} F)\chi_A))$$

The Kan extension is an extension. We prove that right Kan extension of a cat-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ defines a functor $\mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$ which extends the functor F. Note that $N\Gamma$ is not a representable in the category $\widehat{\mathbf{\Theta}_\infty}$, but this result shows that it still enjoys the same universal property as the representable objects, with respect to all the other nerves associated to cat-category with families. The following lemma, which is analogous to the Yoneda lemma details the property of this object:

Lemma 83. *For any cat-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, and for every ps-context $\Gamma \vdash_{\mathsf{ps}}$, we have a natural isomorphism*

$$\mathbf{\Theta}_\infty(N\Gamma, N_F\Delta) \cong (N_F\Delta)_\Gamma$$

Proof. The construction is analogue in that of the Yoneda lemma in one direction. To any natural transformation $\eta \in \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_F\Delta)$ we associate the element $\eta(\mathrm{id}_\Gamma) \in (N_F\Delta)_\Gamma$ and it suffices to show that this association is a bijection.

- We first show that this association is injective, that is, if we consider two natural transformations $\eta, \eta' \in \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_F\Delta)$ such that $\eta(\mathrm{id}_\Gamma) = \eta'(\mathrm{id}_\Gamma)$ then necessarily $\eta = \eta'$. By Proposition 79, it suffices to check that η and η' coincide on all the elements of $T\Gamma$. Consider an element of $T\Gamma$, which is thus a map $\chi_t : \Gamma \to D^n$, since the induced transformation η and η' are algebraic, we have the following equalities

$$\begin{aligned}\eta(\chi_t) &= F(\chi_t) \circ \eta(\mathrm{id}_\Gamma) \\ &= F(\chi_t) \circ \eta'(\mathrm{id}_\Gamma) \\ &= \eta'(\chi_t)\end{aligned}$$

  This proves that η and η' coincide on $T\Gamma$, and hence they are equal.
- We now show that the association is surjective, that is, for every map $f : \Delta \to F\Gamma$, we have a natural transformation $\eta \in \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_F\Delta)$ such that $\eta(\mathrm{id}_\Gamma) = f$. For any substitution $\Gamma \vdash \theta : \Theta$ between the ps-context Γ and another ps-context Θ, θ defines a map $\Gamma \to \Theta$ in $\mathbf{\Theta}_\infty^{\mathrm{op}}$, thus one can consider the map $F\theta : F\Gamma \to F\Theta$. We define $\eta(\theta) = F(\theta) \circ f$. By functoriality of F and of the composition, this is a natural transformation, and it satisfies by definition $\eta(\mathrm{id}_\Gamma) = f$, which proves the surjectivity of the association. □

Lemma 84. *The right Kan extension $\mathrm{Ran}_{P_\infty} F$ preserves the $\mathbf{\Theta}_\infty^{\mathrm{op}}$-structure. More precisely, consider a ps-context $\Gamma \vdash_{\mathsf{ps}}$, which defines an object in the category $\mathbf{\Theta}_\infty$, then we have the isomorphism*

$$(\mathrm{Ran}_{P_\infty} F)\Gamma \cong F\Gamma$$

which is natural in Γ.

Proof. The object $(\mathrm{Ran}_{P_\infty} F)D^n$ is characterized by the fact that there is a natural isomorphism

$$\mathcal{C}(\Delta, (\mathrm{Ran}_{P_\infty} F)\Gamma) \cong \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_F\Delta)$$

By Lemma 83, this proves the following natural isomorphism

$$\mathcal{C}(\Delta, (\mathrm{Ran}_{P_\infty} F)\Gamma) \cong N_F(\Delta)_\Gamma = \mathcal{C}(\Delta, F\Gamma)$$

This shows that $(\mathrm{Ran}_{P_\infty} F)\Gamma \cong F\Gamma$. □

Right Kan extension as a morphism of cat-categories with families. We now prove that the right Kan extension of a cat-category with families defines in a unique way a morphism of category with families $\mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$, the same way the Kan extension of a globular category with families defines can be chosen in a unique way to be a morphism of globular category with families.

Lemma 85. *There is a unique choice that makes* $\operatorname{Ran}_{P_\infty} F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ *into a morphism of cat-categories with families.*

Proof. A morphism of cat-category (F, ϕ) with families is in particular a morphism of globular categories with families, and hence it has to satisfy $\phi(\mathrm{ty}(\gamma)) = \mathrm{ty}(F(\gamma))$. Since in the category $\mathcal{S}_{\mathsf{CaTT}}$ every type is obtained from a substitution to a sphere context, there is no choice but to pose $\phi(A) = \mathrm{ty}((\operatorname{Ran}_{P_\infty} F)\chi_A)$. Conversely, we prove that this choice defines a morphism of cat-categories with families: First note that we have already proven that $(\operatorname{Ran}_{P_\infty} F)\varnothing$ is a terminal object, so we can chose it to be the particular terminal object given by the structure of category with families on $\mathcal{C}$. Moreover, we have also proven that $(\operatorname{Ran}_{P_\infty} F)(\Gamma, A) \cong ((\operatorname{Ran}_{P_\infty} F)\Gamma, \phi(A))$, so we can chose $(\operatorname{Ran}_{P_\infty} F)(\Gamma, A)$ so that this equality is strict. This choice is made possible by the fact that the decomposition of such a context is unique. This choice hence makes $\operatorname{Ran}_{P_\infty} F$ into a morphism of category with families. In order to check that it is a cat-category with families morphism, it suffices to check that it commutes with the globular structures and with the $\mathbf{\Theta}_\infty^{\mathrm{op}}$ structure. The latter entails the former, and it is proved by Lemma 84. □

From now on, we denote $\operatorname{Ran}_{P_\infty} F$ the right Kan extension given with the correct choices that make it into a morphism of cat-category with families.

Initiality result. We prove the main theorem to study the semantics of the category $\mathcal{S}_{\mathsf{CaTT}}$: it is an analogue of Theorem 36 for cat-category with families and gives an initiality property of the syntactic category of the theory CaTT.

Lemma 86. *We consider a cat-category with families* $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ *together with a morphism of cat-categories with families* $G : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$*. Then we necessarily have* $G = \operatorname{Ran}_{P_\infty} F$*.*

Proof. We first show by induction that we have $G \cong \operatorname{Ran}_{P_\infty} F$.

- For the empty context $\varnothing$, we have already proven that both $(\operatorname{Ran}_{P_\infty} F)\varnothing$ and $G\varnothing$ are terminal, hence they are isomorphic.
- For a context of the form $(\Gamma, x : A)$, since both G and $\operatorname{Ran}_{P_\infty} F$ are morphisms of cat-category with families for the structure given by F, we have the following pullbacks in the category $\mathcal{C}$.

$$\begin{tikzcd} G(\Gamma, A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & F(D^n) \arrow[d, "\mathrm{ty}"] \\ G\Gamma \arrow[r] & S_F^{n-1} \end{tikzcd} \qquad \begin{tikzcd} (\operatorname{Ran}_{P_\infty} F)(\Gamma, A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & F(D^n) \arrow[d, "\mathrm{ty}"] \\ (\operatorname{Ran}_{P_\infty} F)\Gamma \arrow[r] & S_F^{n-1} \end{tikzcd}$$

  By induction, we have that $(\operatorname{Ran}_{P_\infty} F)\Gamma \cong G\Gamma$, hence $G(\Gamma, A)$ and $(\operatorname{Ran}_{P_\infty} F)(\Gamma, A)$ are both obtained as limit of isomorphic diagrams, hence they are isomorphic.

Since there is a unique choice that makes the right Kan extension into a morphism of cat-category with families by Lemma 85, and that both G and $\operatorname{Ran}_{P_\infty} F$ satisfy this condition, it follows that $G = \operatorname{Ran}_{P_\infty} F$. □

Theorem 87. *The category $\mathcal{S}_{\mathsf{CaTT}}$ is the initial cat-category with families*

Proof. We have already proved that if we consider a cat-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, then there is a morphism of cat-category with families constructed as $\mathrm{Ran}_{P_\infty} F : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$. Moreover, Lemma 86 shows the uniqueness of such a morphism. □

6.4 The initial cat-category with families We have proven that $\mathcal{S}_{\mathsf{CaTT}}$ is the initial cat-category with families, in Theorem 87, we now study this category and give a syntactic interpretation of the construction that we have presented in this case.

Substitutions to a globular context. We have proved with Theorem 5 that a substitution is completely determined by its action on variables of a context, we can extract from Theorem 36 a partial answer to the converse question: considering a given action on variables, is there a substitution acting this way? We already know that the action cannot be completely free, since the substitution must respect the typing, and hence the source and target. In fact, in the case where the target context is a context in $\mathcal{S}_{\mathsf{GSeTT}}$, this theorem shows that this is the only obstruction. Consider $\mathcal{S}_{\mathsf{CaTT}}$ as a globular category with families, and consider, in the category $\mathcal{S}_{\mathsf{CaTT}}$, a context Γ which comes from the theory GSeTT: Γ is constructed only from variables and term constructors do not appear in it. Then for an arbitrary context Δ, one can apply Theorem 36 to characterize the substitutions $\Delta \to \Gamma$ as being equivalent to the natural transformations $\widehat{\mathcal{G}}(V\Gamma, T\Delta)$. In other words, in this case, a substitution γ is nothing else than the data of, for every variable x in Γ, of a term t in Δ with the intent that $t = x[\gamma]$ in a way that is compatible with the source and target relations.

Substitutions to an arbitrary context. We can interpret Theorem 36 as a generalization of the previous discussion, where we characterize substitutions with an arbitrary context Γ as target (as opposed to one built from variables only). We cannot generalize it naively, by requiring that we associate a term to any variable of Γ. Indeed, if we consider the context

$$\Gamma = \texttt{(x : *) (f : id x -> id x)}$$

then the source of the variable `f` is the term `id x`, which is not itself a variable, hence the compatibility of the source and target cannot be expressed as a naturality condition. Categorically, this means that the set of variables $V\Gamma$ is not equipped with a structure of a globular set given by the source and target: in our example, we would indeed have $(V\Gamma)_2 = \{\texttt{f}\}$, but the source of this term is `id x`, which is not an element of $V(\Gamma)_1$. The solution we have chosen is to associate not only a term to any variable of Γ, but also to any term of Γ, in a way that respects the source and target, and thus we now represent substitutions $\gamma : \Delta \to \Gamma$ as natural transformations in $\mathcal{G}^{\mathrm{op}}(T\Gamma, T\Delta)$. However, this gives too much freedom, and there are such natural transformations that are ill-defined transformations. For instance, consider the contexts

$$\Gamma = \texttt{(x : *) (f : x -> x)} \qquad\qquad \Delta = \texttt{(x : *)}$$

together with a natural transformation $\eta : T\Delta \Rightarrow T\Gamma$ such that $\eta(\texttt{id x}) = \texttt{f}$. This can never be the action of a substitution, since $(\texttt{id } x)[\gamma] = \texttt{id } (x[\gamma])$. The problem is that, in this representation, we do not account for the fact that a substitution must respect the term constructors. The notion of algebraic natural transformation achieves exactly this: an algebraic natural transformation is a transformation that respecting term constructors, and Theorem 87 ensures that, by considering only the algebraic natural transformations, we recover exactly the data of the substitutions.

Codensity of the functor P_∞. Theorem 87 states that for any cat-category with families $\mathcal{C}$, the right Kan extension along P_∞ gives the unique morphism of cat-category with families from the syntactic category to $\mathcal{C}$. In particular, applying this theorem to the cat-category $\mathcal{S}_{\mathsf{CaTT}}$ with the structure given by P_∞ shows that $\mathrm{Ran}_{P_\infty} P_\infty$ is this unique morphism. Since the identity functor $\mathrm{id}_{\mathcal{S}_{\mathsf{CaTT}}}$ is also a morphism, this shows in particular that $\mathrm{id}_{\mathcal{S}_{\mathsf{CaTT}}} = \mathrm{Ran}_{P_\infty} P_\infty$: in other words, the functor P_∞ is codense. More concretely, this proves that every context in the category $\mathcal{S}_{\mathsf{CaTT}}$ is canonically obtained as a limit of ps-contexts.

Developing the limits. There is also an interesting interpretation of Proposition 79, which establishes the equivalence between algebraic natural transformations $\widehat{\mathcal{G}}(T\Gamma, T\Delta)_{\mathrm{alg}}$ and natural transformations $\widehat{\mathbf{\Theta}_\infty}(N\Gamma, N\Delta)$. Indeed, consider that the natural transformations $\widehat{\mathbf{\Theta}_\infty}(N\Gamma, N\Delta)$ are maps of cones, between a cone of apex Γ and a cone of apex Δ, over the canonical diagram of Γ. Recall that the only objects that appear in the canonical diagram of Γ are ps-contexts, which are themselves globular products of disks. Hence one can “develop” this diagram, and obtain from the above map of cones, a new map of cones, with same apex, but over a diagram only made out of disks. Proposition 79 shows that algebraic natural transformations are exactly those maps of cones between diagrams over disks that can be obtained by such an operation. This theorem can thus be seen as a way to develop a canonical limit of ps-contexts into a non-canonical limit of disks. This matches the syntactic construction of context as a succession of context comprehension, which exhibits each context as a succession of pullback of disks. In that respect, the contexts in the theory are analogous to the CW-complex in topology.

6.5 The models of the theory CaTT

We now apply Theorem 87 in order to study the **Set**-models of the theory CaTT.

Induced cat-structure. We call a cat-ctructure on a category with families $\mathcal{C}$ the data of cat-category with families on $\mathcal{C}$ that coincide with the given structure of category with families on $\mathcal{C}$. The cat-structures on a category with families form a category, whose morphisms are the morpshisms of cat-category with families. Any morphism of categories with family $F : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$ induces an essentially unique cat-structure on the category with families $\mathcal{C}$ for which F is a morphism of cat-category with families:

Lemma 88. *The morphisms of category with families $\mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$ are equivalent to the cat-structures on $\mathcal{C}$.*

Proof. A morphism of categories with families $F : \mathcal{S}_{CaTT} \to \mathcal{C}$ induces, by restriction, a structure of cat-category with families on $\mathcal{C}$, for which F is a morphism of cat-categories with families. Theorem 87 then shows that this association is an equivalence of categories. □

Cat-structure on Set. The characterization of the models of CaTT relies on a characterization of all the cat-structures on the category with families **Set**.

Lemma 89. *There is an equivalence of categories between the the cat-structures on the categort with families* **Set** *and the weak ω-categories.*

Proof. By definition, a cat-structure on the category with families **Set** defines a presheaf over $\mathbf{\Theta}_\infty$ that preserves the globular sums, and therefore is a weak ω-category. Moreover a morphism of cat-categories with families is a morphisms of presheaves over $\mathbf{\Theta}_\infty$, so this assignation is functorial. Conversely, given a presheaf $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathbf{Set}$ that sends globular sums onto globular products, there is a unique cat-structure induced by F on the category with families **Set**. Indeed, such a cat-structure is the same as a globular structure on the induced globular set $\mathcal{G}^{\mathrm{op}} \to \mathbf{Set}$, and there is a unique such globular structure by Proposition 38, thus the functor is essentially surjective. This functor is also fully faithful, as a morphism between the weak ω-categories is exactly equivalent to a morphism between the corresponding cat-structures. □

Models of the theory CaTT. With the tools that we have introduced, we can have now completely characterized the **Set**-models of the theory CaTT. This is the main result of this article and solves Conjecture 49 in [14].

Theorem 90. *There is an equivalence of categories between the models of* CaTT *and the weak ω-categories.*

Proof. This is an immediate consequence of Lemma 88. and by Lemma 89 □

Finitely generated polygraphs. Recall that the nerve functor is defined by

$$\begin{aligned} N : \mathcal{S}_{\mathsf{CaTT}} &\to \widehat{\mathbf{\Theta}_\infty} \\ \Gamma &\mapsto \mathcal{S}_{\mathsf{CaTT}}(\Gamma, \_) \end{aligned}$$

A colimit in $\mathbf{\Theta}_\infty$, which is thus a limit in $\mathcal{S}_{\mathsf{ps},\infty}$, is preserved by the functor $N\Gamma$, by continuity of the hom-functor. Hence, $N\Gamma$ defines a weak ω-category in the sense of Grothendieck-Maltsiniotis. By Theorem 90, $N\Gamma$ thus defines a model of the theory CaTT. In fact, one can describe the corresponding model, given by $\mathcal{S}_{\mathsf{CaTT}}(\Gamma, \_) : \mathcal{S}_{\mathsf{CaTT}} \to \mathbf{Set}$, which by continuity of the hom-functor preserves all the limits, and hence is a model. This shows that we have a functor $\mathcal{S}_{\mathsf{CaTT}} \to \mathbf{Mod}(\mathsf{CaTT})$, given by the coYoneda embedding, which is fully faithful and thus exhibits $\mathcal{S}_{\mathsf{CaTT}}$ as a full subcategory of the weak ω-categories. We call *finitely generated polygraphs* (or *computads*) the weak ω-categories that come from an object of $\mathcal{S}_{\mathsf{CaTT}}$, which generalize similar notions studied in higher category theory [10, 28], in particular the polygraphs play an important role in the theory of strict ω-categories as they are the cofibrant objects for the folk model structure [21]. This remark draws an analogy between our presentation of weak ω-categories and the Gabriel-Ülmer duality [16] in which the syntactic category sits inside the models as the opposite of the free finitely generated objects.

Further Work

The entire construction we have presented here is fairly general and we believe that it works in a much broader scope that the one introduced here. The only arguments specific to the case of CaTT are those establishing the relation between the judgments $\Gamma \vdash_{\mathsf{ps}}$ and the pasting schemes (Theorem 73), and most other lemmas rely on the structure given by being a type theory, rather than on the specific rules for CaTT. This leads us to believe that there exists a general framework in which our construction applies. Such a framework has started to be studied [22] and preliminary results show promising unification with our methods. In particular, the dependent type theory is known to be equivalent to monads with arities [8], and we believe our characterization of the models of CaTT amounts to a proof of the nerve theorem translated to this framework.

We believe that our work gives a promising approach to tackling the initiality conjecture, which could be solved in the aforementioned particular case of dependent type theories which entertain a close enough connection with the category CaTT. We however point out this theory is much simpler than Martin-Löf type theory and its variations, particularly since it does not have any computation rules, and we believe that the presence of those rules strongly increases the difficulty for proving this conjecture for such theories.

We believe that it would be valuable to establish a connection between our interpretation of the contexts as finitely generated polygraphs and the notion of polygraphs usually defined for strict ω-categories [21, 29]. In particular, in the strict case, polygraphs are used to characterized cofibrant objects for a model structure, and we would like to investigate whether such a model structure, or a weaker version of it in the form of a weak factorization system, would make sense for weak ω-categories [18].

The approach that we have presented in this article generalizes to other related higher structures, allowing for a type theoretic presentation of these structures. In particular, similar methods have been conducted in order to study monoidal weak ω-categories [4], cubical weak ω-categories [5] and strictly unital weak ω-categories [15]. Further work along the lines of this article includes generalizing the methods that we have presented in order to study the semantics of such theories. In the case of monoidal weak ω-categories, the theories are close enough that a transfer of the semantics can be done [4], and understanding the semantics of CaTT is enough to settle the semantics of the other theory. The theory for cubical ω-categories, can be presented in a very similar fashion to the theory CaTT and we believe that similar methods can be used to study its semantics. The theory for strictly unital ω-categories is more complicated since it contains rewriting rules. Understanding the semantics is a relevant challenge left for future work and we believe that either one could achieve it by relating this theory to CaTT or by adapting the methods that we have presented to account for the rewriting rules.

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