Euclid summer school: Parameter estimation for a polynomial model

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1 Analytical solution

1)

$$\begin{aligned}
\mathcal{L}(\{d_i\}|\{a_k\}) &= \frac{1}{(2\pi)^{N/2}\sqrt{\det\Sigma}}\exp-\frac{1}{2}\left[\sum_{ij}\left(d_i-\sum_{k=0}^{M}a_k(z_i-z_c)^k\right)\Sigma^{-1}_{ij}\left(d_j-\sum_{\ell=0}^{M}a_\ell(z_j-z_c)^\ell\right)\right] \\
&= \frac{1}{(2\pi)^{N/2}\sqrt{\det\Sigma}}\exp-\frac{1}{2}\left[(\boldsymbol{d}-\mathcal{P}\boldsymbol{a})^T\,\Sigma^{-1}\,(\boldsymbol{d}-\mathcal{P}\boldsymbol{a})\right] \qquad (1)
\end{aligned}$$

5)

$$\begin{aligned}
\ln\mathcal{L}(\boldsymbol{d}|\boldsymbol{a}) &= -\frac{N}{2}\ln(2\pi)-\frac{1}{2}(\det(\Sigma))-\frac{1}{2}\left[(\boldsymbol{d}-\mathcal{P}\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{d}-\mathcal{P}\boldsymbol{a})\right] \\
\frac{\partial\ln\mathcal{L}(\boldsymbol{d}|\boldsymbol{a})}{\partial a_\gamma} &= -\frac{1}{2}\sum_{ij}\left[(\sum_\alpha\mathcal{P}_{i\alpha}\delta_{\alpha\gamma})(\Sigma^{-1})_{ij}(\boldsymbol{d}_j-\sum_\beta\mathcal{P}_{j\beta}\boldsymbol{a}_\beta)+(\boldsymbol{d}_i-\sum_\alpha\mathcal{P}_{i\alpha}\boldsymbol{a}_\alpha)(\Sigma^{-1})_{ij}\sum_\beta\mathcal{P}_{j\beta}\delta_{\beta\gamma}\right] \\
&= -\frac{1}{2}\sum_{ij}\left[(\mathcal{P}_{i\gamma})(\Sigma^{-1})_{ij}(\boldsymbol{d}_j-\sum_\beta\mathcal{P}_{j\beta}\boldsymbol{a}_\beta)+(\boldsymbol{d}_i-\sum_\alpha\mathcal{P}_{i\alpha}\boldsymbol{a}_\alpha)(\Sigma^{-1})_{ij}\mathcal{P}_{j\gamma}\right] \\
&= -\sum_{ij}(\mathcal{P}_{i\gamma})(\Sigma^{-1})_{ij}(\boldsymbol{d}_j-\sum_\beta\mathcal{P}_{j\beta}\boldsymbol{a}_\beta) \qquad (2)
\end{aligned}$$

Where we used the fact that the covariance matrix is symmetric

$$\boldsymbol{\nabla}_{\boldsymbol{a}}\ln\mathcal{L}(\boldsymbol{d}|\boldsymbol{a}) = -\mathcal{P}^T\Sigma^{-1}(\boldsymbol{d}-\mathcal{P}\boldsymbol{a}) = 0 \qquad (3)$$

$$\mathcal{P}^T\Sigma^{-1}\mathcal{P}\hat{\boldsymbol{a}} = \mathcal{P}^T\Sigma^{-1}\boldsymbol{d}$$

$$\hat{\boldsymbol{a}} = (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\boldsymbol{d} \qquad (4)$$

6)

$$\langle\hat{\boldsymbol{a}}\rangle = (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\langle\boldsymbol{d}\rangle = (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\langle\mathcal{P}\boldsymbol{a}+\boldsymbol{n}\rangle = \boldsymbol{a} \qquad (5)$$

since $\langle\boldsymbol{n}\rangle = 0$

$$\begin{aligned}
\langle\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^T\rangle &= (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\langle\boldsymbol{d}\boldsymbol{d}^T\rangle\Sigma^{-1}\mathcal{P}(\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1} \\
&= (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\langle(\mathcal{P}\boldsymbol{a}+\boldsymbol{n})(\mathcal{P}\boldsymbol{a}+\boldsymbol{n})^T\rangle\Sigma^{-1}\mathcal{P}(\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1} \\
&= (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\mathcal{P}\boldsymbol{a}\boldsymbol{a}^T\mathcal{P}^T\Sigma^{-1}\mathcal{P}(\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1} \\
&+ (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1}\mathcal{P}^T\Sigma^{-1}\Sigma\Sigma^{-1}\mathcal{P}(\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1} \\
&= \boldsymbol{a}\boldsymbol{a}^T+(\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1} \\
\mathrm{Cov}(\hat{\boldsymbol{a}}) &= (\mathcal{P}^T\Sigma^{-1}\mathcal{P})^{-1} \qquad (6)
\end{aligned}$$