

# Euclid summer school: Parameter estimation for a polynomial model

August 22, 2022

## 1 Analytical solution

1)

$$\begin{aligned}\mathcal{L}(\{d_i\}|\{a_k\}) &= \frac{1}{(2\pi)^{N/2}\sqrt{\det \Sigma}} \exp - \frac{1}{2} \left[ \sum_{ij} \left( d_i - \sum_{k=0}^M a_k (z_i - z_c)^k \right) \Sigma_{ij}^{-1} \left( d_j - \sum_{\ell=0}^M a_\ell (z_j - z_c)^\ell \right) \right] \\ &= \frac{1}{(2\pi)^{N/2}\sqrt{\det \Sigma}} \exp - \frac{1}{2} \left[ (\mathbf{d} - \mathcal{P}\mathbf{a})^T \Sigma^{-1} (\mathbf{d} - \mathcal{P}\mathbf{a}) \right]\end{aligned}\quad (1)$$

5)

$$\begin{aligned}\ln \mathcal{L}(\mathbf{d}|\mathbf{a}) &= -\frac{N}{2} \ln(2\pi) - \frac{1}{2} (\det(\Sigma)) - \frac{1}{2} [(\mathbf{d} - \mathcal{P}\mathbf{a})^T \Sigma^{-1} (\mathbf{d} - \mathcal{P}\mathbf{a})] \\ \frac{\partial \ln \mathcal{L}(\mathbf{d}|\mathbf{a})}{\partial a_\gamma} &= -\frac{1}{2} \sum_{ij} \left[ \left( \sum_{\alpha} \mathcal{P}_{i\alpha} \delta_{\alpha\gamma} \right) (\Sigma^{-1})_{ij} (\mathbf{d}_j - \sum_{\beta} \mathcal{P}_{j\beta} \mathbf{a}_\beta) + (\mathbf{d}_i - \sum_{\alpha} \mathcal{P}_{i\alpha} \mathbf{a}_\alpha) (\Sigma^{-1})_{ij} \sum_{\beta} \mathcal{P}_{j\beta} \delta_{\beta\gamma} \right] \\ &= -\frac{1}{2} \sum_{ij} \left[ (\mathcal{P}_{i\gamma}) (\Sigma^{-1})_{ij} (\mathbf{d}_j - \sum_{\beta} \mathcal{P}_{j\beta} \mathbf{a}_\beta) + (\mathbf{d}_i - \sum_{\alpha} \mathcal{P}_{i\alpha} \mathbf{a}_\alpha) (\Sigma^{-1})_{ij} \mathcal{P}_{j\gamma} \right] \\ &= -\sum_{ij} (\mathcal{P}_{i\gamma}) (\Sigma^{-1})_{ij} (\mathbf{d}_j - \sum_{\beta} \mathcal{P}_{j\beta} \mathbf{a}_\beta)\end{aligned}\quad (2)$$

Where we used the fact that the covariance matrix is symmetric

$$\nabla_{\mathbf{a}} \ln \mathcal{L}(\mathbf{d}|\mathbf{a}) = -\mathcal{P}^T \Sigma^{-1} (\mathbf{d} - \mathcal{P}\mathbf{a}) = 0 \quad (3)$$

$$\mathcal{P}^T \Sigma^{-1} \mathcal{P} \hat{\mathbf{a}} = \mathcal{P}^T \Sigma^{-1} \mathbf{d}$$

$$\hat{\mathbf{a}} = (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1} \mathcal{P}^T \Sigma^{-1} \mathbf{d} \quad (4)$$

6)

$$\langle \mathbf{a} \rangle = (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1} \mathcal{P}^T \Sigma^{-1} \langle \mathbf{d} \rangle = (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1} \mathcal{P}^T \Sigma^{-1} \langle \mathcal{P}\mathbf{a} + \mathbf{n} \rangle = \mathbf{a} \quad (5)$$

since  $\langle \mathbf{n} \rangle = 0$

$$\begin{aligned}\langle \mathbf{a}\mathbf{a}^T \rangle &= (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1} \mathcal{P}^T \Sigma^{-1} \langle \mathbf{d}\mathbf{d}^T \rangle \Sigma^{-1} \mathcal{P} (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1} \\ &= \mathbf{a}\mathbf{a}^T + (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1} \\ \text{Cov}(\mathbf{a}) &= (\mathcal{P}^T \Sigma^{-1} \mathcal{P})^{-1}\end{aligned}\quad (6)$$