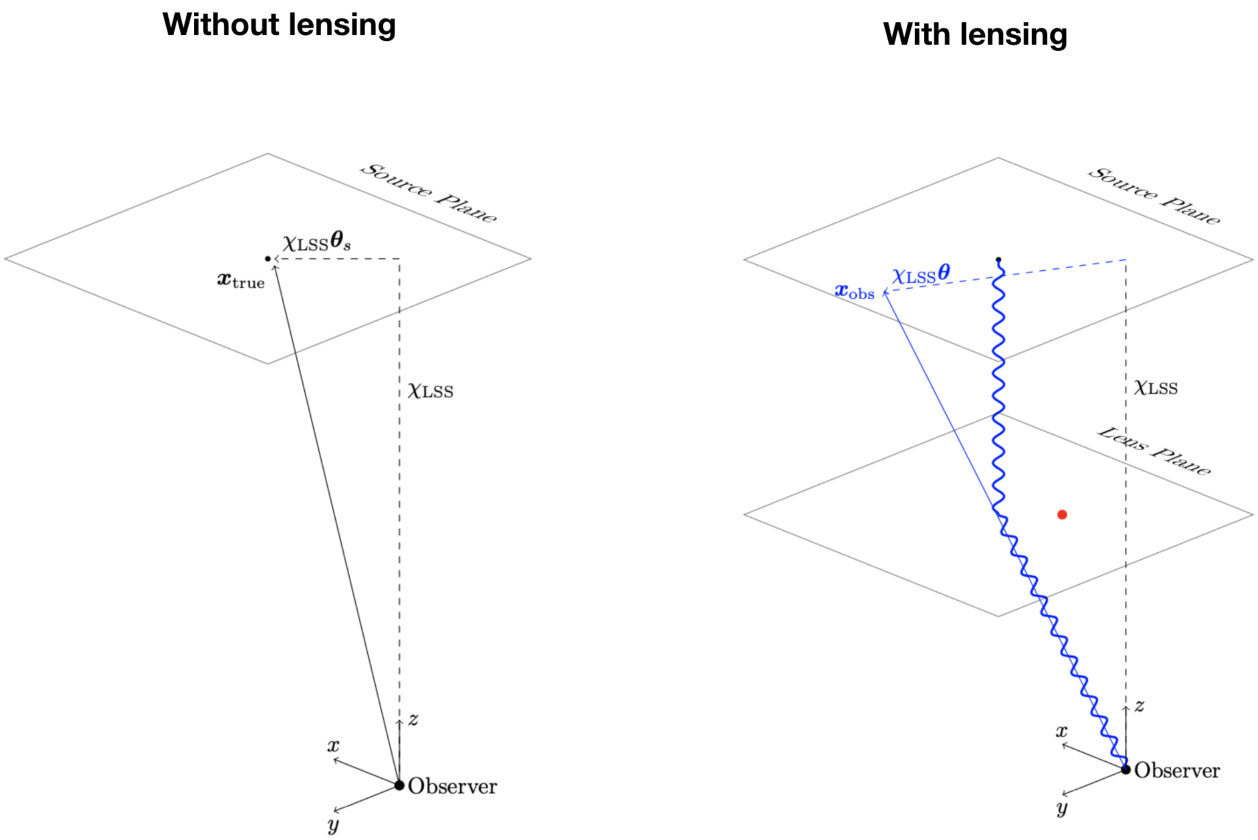

Figure 1: Path of a light ray from the last scattering surface to us. (left) assuming no lensing, (right) the light ray is lensed by a mass in the lens plane.

1 The comoving distance

The homogeneous FLRW universe is described by the following metric

$$ds^2 = -c^2dt^2 + a^2\delta_{ij}dx^idx^j \tag{1}$$

The metric describes an expanding Universe with scale factor a, a serves to stretch the fixed grid defined by the coordinate x^i, we say that the coordinate x^i are comoving, they are not affected by the Universe dynamics. Let's consider the path of a radial light ray, we use χ here as a radial comoving coordinate

$$ds^2 = -c^2dt^2 + a^2d\chi^2 = 0 \tag{2}$$

If an object that emit light at t_e and with receive it at t_0, and if we assume that we are at the center of the coordinate system, the object is at comoving coordinate

$$\chi = \int_{t_e}^{t_0} \frac{cdt}{a(t)} \tag{3}$$

note that a(t) is an increasing function of time, and $t_0 > t$ so χ here is positive.

The comoving distance can be expressed in term of scale factor

$$\chi = \int_{t_e}^{t_0} \frac{cdt}{a} = \int_{a_e}^{1} \frac{cda}{a}\frac{dt}{da} = \int_{a_e}^{1} \frac{cda}{H(a)a^2} \tag{4}$$

using $H = \frac{da/dt}{a}$, and $a(t_0) = 1$ or as a function of redshift using $a = \frac{1}{1+z}$

$$\frac{da}{dz} = -\left(\frac{1}{1+z}\right)^2 = -a^2 \tag{5}$$

$$\chi = \int_{a_e}^{1} \frac{cda}{H(a)a^2} = \int_{0}^{z_e} \frac{cdz}{H(z)} \tag{6}$$

2 The lensing potential

The true position of the source is at location

$$\boldsymbol{x}_{\text{true}} = \chi_{\text{LSS}} \begin{pmatrix} \theta_S^1 \\ \theta_S^2 \\ 1 \end{pmatrix} \tag{7}$$

Where θ_S^1, θ_S^2 are small quantities in our reference frame, its apparent location is $\boldsymbol{x}_{\text{obs}} = \chi_{\text{LSS}} \begin{pmatrix} \theta^1 \\ \theta^2 \\ 1 \end{pmatrix}$, we need to relate θ_S^i to θ^i. We will work in the perturbed FLRW metric in Newtonian gauge

$$ds^2 = -(1 + 2\psi(\boldsymbol{x}, t))dt^2 + a^2(t)\delta_{ij}[1 + 2\phi(\boldsymbol{x}, t)]dx^i dx^j \tag{8}$$

which has the following non zero Christoffel symbol

$$\begin{aligned} \Gamma^0_{00} &= \dot{\psi} && (9) \\ \Gamma^i_{j0} &= \delta^i_j(H + \dot{\phi}) && (10) \\ \Gamma^i_{00} &= \partial^i \psi && (11) \\ \Gamma^i_{jk} &= (\delta^i_j \partial_k - \delta^i_k \partial_j - \delta^{im}\delta_{jk}\partial_m)\phi && (12) \end{aligned}$$

we will study the path of photon, we have the following condition for their four-momentum $P^\mu = \frac{dx^\mu}{d\lambda}$: $P^\mu P_\mu = 0$, defining $p = g_{ij}P^i P^j$

$$\begin{aligned} -(1 + 2\psi(\boldsymbol{x}, t))(P^0)^2 + p^2 = 0 && (13) \\ \frac{dt}{d\lambda} = P^0 \approx p(1 - \psi) && (14) \\ && (15) \end{aligned}$$

The geodesic equation is

$$\frac{d^2 x^\rho}{d\lambda^2} = -\Gamma^\rho_{\mu,\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{16}$$

with 1 and 2 components

$$\frac{d^2(\chi\theta^i)}{d\lambda^2} = -\Gamma^i_{\mu,\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{17}$$

let's first expand the left hand side,

$$\begin{aligned} \frac{d^2(\chi\theta^i)}{d\lambda^2} &= \frac{d}{d\lambda}\left(\frac{d(\chi\theta^i)}{d\lambda}\right) && (18) \\ &= \frac{d}{d\lambda}\left(\frac{d(\chi\theta^i)}{d\chi}\frac{d\chi}{dt}\frac{dt}{d\lambda}\right) && (19) \\ &= \frac{d\chi}{dt}\frac{dt}{d\lambda}\frac{d}{d\chi}\left(\frac{d(\chi\theta^i)}{d\chi}\frac{d\chi}{dt}\frac{dt}{d\lambda}\right) && (20) \end{aligned}$$

Now let's expand the right hand side

$$-\Gamma^i_{\mu,\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\Gamma^i_{\mu,\nu}\frac{dx^\mu}{d\chi}\frac{dx^\nu}{d\chi}\left(\frac{d\chi}{dt}\right)^2\left(\frac{dt}{d\lambda}\right)^2 \tag{21}$$

putting this together we have

$$\frac{d}{d\chi}\left(\frac{d(\chi\theta^i)}{d\chi}\frac{d\chi}{dt}\frac{dt}{d\lambda}\right) = -\Gamma^i_{\mu,\nu}\frac{dx^\mu}{d\chi}\frac{dx^\nu}{d\chi}\frac{d\chi}{dt}\frac{dt}{d\lambda} \tag{22}$$

$$\frac{d}{d\chi}\left(\frac{d(\chi\theta^i)}{d\chi}\frac{d\chi}{dt}\frac{dt}{d\lambda}\right) = -\left[\Gamma^i_{0,0}\left(\frac{dx^0}{d\chi}\right)^2 + 2\Gamma^i_{0,j}\frac{dx^0}{d\chi}\frac{dx^j}{d\chi} + \Gamma^i_{j,k}\frac{dx^j}{d\chi}\frac{dx^k}{d\chi}\right]\frac{d\chi}{dt}\frac{dt}{d\lambda} \tag{23}$$

at zero order, we have $d\chi = -d\eta = -\frac{dt}{a}$ and $\frac{dt}{d\lambda} = p$, we will also use that $\frac{d(\chi\theta^i)}{d\chi}$ is a small quantity,

$$\frac{d(\chi\theta^i)}{d\chi} = \theta^i + \chi\frac{d\theta^i}{d\chi} \tag{24}$$

θ^i is small and $\frac{d\theta^i}{d\chi}$ is of order of the gravitational potential perturbation (since it's zero with no grav lensing effect).

$$\frac{d}{d\chi}\left(-\frac{p}{a}\frac{d(\chi\theta^i)}{d\chi}\right) = \frac{p}{a}\left[a^2\partial^i\psi - 2aH\frac{d(\chi\theta^i)}{d\chi} + \Gamma^i_{j,k}\frac{dx^j}{d\chi}\frac{dx^k}{d\chi}\right] \tag{25}$$

$$\Gamma^i_{j,k}\frac{dx^j}{d\chi}\frac{dx^k}{d\chi} = \Gamma^i_{3,3} = -\delta^{im}\partial_m\phi \tag{26}$$

because the $\frac{dx^i}{d\chi}$ for $i \in (1,2)$ is first order in perturbation while $\frac{dx^3}{d\chi} = 1$

$$\frac{d}{d\chi}\left(-\frac{p}{a}\frac{d(\chi\theta^i)}{d\chi}\right) = \frac{p}{a}\left[a^2\partial^i\psi - 2aH\frac{d(\chi\theta^i)}{d\chi} - \delta^{im}\partial_m\phi\right] \tag{27}$$

We know that at 0th order, $p = \frac{C}{a}$ with C = cst

$$\begin{aligned}
\frac{d}{d\chi}\left(\frac{1}{a^2}\frac{d(\chi\theta^i)}{d\chi}\right) &= -\frac{1}{a^2}\left[a^2\partial^i\psi - 2aH\frac{d(\chi\theta^i)}{d\chi} - \delta^{im}\partial_m\phi\right]\\
\frac{d}{d\chi}\left(\frac{1}{a^2}\frac{d(\chi\theta^i)}{d\chi}\right) &= -\left[\partial^i\psi - \frac{2}{a}H\frac{d(\chi\theta^i)}{d\chi} - \partial^i\phi\right]\\
\frac{d}{d\chi}\left(\frac{1}{a^2}\frac{d(\chi\theta^i)}{d\chi}\right) &= -\left[\partial^i(\psi-\phi) - \frac{2}{a}H\frac{d(\chi\theta^i)}{d\chi}\right] \qquad (28)
\end{aligned}$$

Where I use the homogenous FLRW metric to raise the index in the third term of the right hand side. We can expand the left hand side

$$\begin{aligned}
\frac{d}{d\chi}\left(\frac{1}{a^2}\frac{d(\chi\theta^i)}{d\chi}\right) &= \frac{1}{a^2}\frac{d^2(\chi\theta^i)}{d\chi^2} + \frac{da^{-2}}{d\chi}\frac{d(\chi\theta^i)}{d\chi}\\
\frac{d}{d\chi}\left(\frac{1}{a^2}\frac{d(\chi\theta^i)}{d\chi}\right) &= \frac{1}{a^2}\frac{d^2(\chi\theta^i)}{d\chi^2} - \frac{2}{a^3}\frac{da}{dt}\frac{dt}{d\chi}\frac{d(\chi\theta^i)}{d\chi}\\
\frac{d}{d\chi}\left(\frac{1}{a^2}\frac{d(\chi\theta^i)}{d\chi}\right) &= \frac{1}{a^2}\frac{d^2(\chi\theta^i)}{d\chi^2} + \frac{2H}{a}\frac{d(\chi\theta^i)}{d\chi} \qquad (29)
\end{aligned}$$

Finally

$$\begin{aligned}
\frac{d^2(\chi\theta^i)}{d\chi^2} &= -a^2\left[\partial^i(\psi-\phi)\right] \qquad (30)\\
&= -\delta^{ij}\partial_j(\psi-\phi) \qquad (31)
\end{aligned}$$

We can integrate this equation with respect to χ

$$\frac{d(\chi\theta^i)}{d\chi} = -\delta^{ij}\int_0^\chi \partial_j[\psi(\boldsymbol{x}(\boldsymbol{\theta},\chi_1),\eta_0-\chi_1) - \phi(\boldsymbol{x}(\boldsymbol{\theta},\chi_1),\eta_0-\chi_1)]d\chi_1 + C_1^i \tag{32}$$

$$\tag{33}$$

Figure 2: Two different way of doing the integration of $f(\chi_1, \chi_2)$ over the triangle area, the inner integral is shown in lines. On the left $\int_0^{\chi_{\text{LSS}}} d\chi_2 \int_0^{\chi_2} d\chi_1 f(\chi_1, \chi_2)$, on the right $\int_0^{\chi_{\text{LSS}}} d\chi_1 \int_{\chi_1}^{\chi_{\text{LSS}}} d\chi_2 f(\chi_1, \chi_2)$

We can then integrate again from 0 to the last scattering surface

$$\chi_{\text{LSS}}\theta^i(\chi_{\text{LSS}}) = -\delta^{ij} \int_0^{\chi_{\text{LSS}}} d\chi_2 \int_0^{\chi_2} \partial_j[\psi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1) - \phi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1)]d\chi_1 + C_1^i \chi_{\text{LSS}}$$

nothing that $\theta^i(\chi_{\text{LSS}}) = \theta_S^i$

$$\theta_S^i = -\frac{\delta^{ij}}{\chi_{\text{LSS}}} \int_0^{\chi_{\text{LSS}}} d\chi_2 \int_0^{\chi_2} \partial_j[\psi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1) - \phi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1)]d\chi_1 + C_1^i \tag{34}$$

in the absence of gravitational potential $\theta_S^i = \theta^i$ so $C_1^i = \theta^i$.

$$\theta_S^i = \theta^i - \frac{\delta^{ij}}{\chi_{\text{LSS}}} \int_0^{\chi_{\text{LSS}}} d\chi_2 \int_0^{\chi_2} \partial_j[\psi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1) - \phi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1)]d\chi_1 \tag{35}$$

As shown on Fig 2, we can invert the integration order to make the inner integral trivial

$$\theta_S^i = \theta^i - \frac{\delta^{ij}}{\chi_{\text{LSS}}} \int_0^{\chi_{\text{LSS}}} d\chi_1 \int_{\chi_1}^{\chi_{\text{LSS}}} \partial_j[\psi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1) - \phi(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1)]d\chi_2 \tag{36}$$

$$\theta_S^i = \theta^i - \delta^{ij} \int_0^{\chi_{\text{LSS}}} d\chi_1 \partial_j \left[\psi\left(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1\right) - \phi\left(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1\right)\right] \left(1 - \frac{\chi_1}{\chi_{\text{LSS}}}\right) \tag{37}$$

assuming $\psi = -\phi$

$$\theta_S^i = \theta^i + 2\delta^{ij} \int_0^{\chi_{\text{LSS}}} d\chi_1 \partial_j \phi\left(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1\right) \left(1 - \frac{\chi_1}{\chi_{\text{LSS}}}\right) \tag{38}$$

this is the formula we were looking for.

$$\theta_S^i = \theta^i + \Delta\theta^i \tag{39}$$

$$\Delta\theta^i = 2\delta^{ij} \int_0^{\chi_{\text{LSS}}} d\chi_1 \partial_j \phi\left(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1\right) \left(1 - \frac{\chi_1}{\chi_{\text{LSS}}}\right) \tag{40}$$

We use $\frac{\partial}{\partial x^j} = \frac{1}{\chi_1}\frac{\partial}{\partial \theta^j}$

$$\Delta\theta^i = 2\delta^{ij} \int_0^{\chi_{\text{LSS}}} \frac{d\chi_1}{\chi_1} \frac{\partial \phi}{\partial \theta^j} \left(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1\right) \left(1 - \frac{\chi_1}{\chi_{\text{LSS}}}\right) \tag{41}$$

$$= \delta^{ij} \frac{\partial}{\partial \theta^j} \phi^L(\boldsymbol{\theta}) \tag{42}$$

$$\phi^L(\boldsymbol{\theta}) = 2 \int_0^{\chi_{\text{LSS}}} \frac{d\chi_1}{\chi_1} \phi\left(\boldsymbol{x}(\boldsymbol{\theta}, \chi_1), \eta_0 - \chi_1\right) \left(1 - \frac{\chi_1}{\chi_{\text{LSS}}}\right) \tag{43}$$

3 The lensing power spectrum

Let's rewrite: $\phi\left(\boldsymbol{x}(\boldsymbol{\theta},\chi_1),\eta_0-\chi_1\right)=\phi\left(\chi_1\hat{\boldsymbol{n}},\eta(\chi_1)\right)$ with $\eta(\chi_1)=\eta_0-\chi_1$ and $\hat{\boldsymbol{n}}$ the line of sight. In order to get to the lensing power spectrum we will first Fourier transform the gravitational potential term

$$\phi\left(\chi_1\hat{\boldsymbol{n}},\eta(\chi_1)\right)=\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\hat{\boldsymbol{n}}\chi_1}\phi\left(\boldsymbol{k},\eta(\chi_1)\right) \tag{44}$$

the lensing potential become

$$\phi^L(\hat{\boldsymbol{n}}) = 2\int_0^{\chi_{\rm LSS}}\frac{d\chi_1}{\chi_1}\left(1-\frac{\chi_1}{\chi_{\rm LSS}}\right)\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\hat{\boldsymbol{n}}\chi_1}\phi\left(\boldsymbol{k},\eta(\chi_1)\right) \tag{45}$$

More torture, let's expand the Fourier coefficient in spherical harmonics

$$e^{i\boldsymbol{k}\hat{\boldsymbol{n}}\chi_1}=4\pi\sum_{\ell=0}^{\infty}i^\ell j_\ell(k\chi_1)\sum_{m=-\ell}^{\ell}Y^*_{\ell m}(\hat{\boldsymbol{k}})Y_{\ell m}(\hat{\boldsymbol{n}}) \tag{46}$$

so that

$$\begin{aligned}\phi^L(\hat{\boldsymbol{n}}) &= \sum_{\ell,m}8\pi i^\ell Y_{\ell m}(\hat{n})\int\frac{d^3k}{(2\pi)^3}Y^*_{\ell m}(\hat{\boldsymbol{k}})\int_0^{\chi_{\rm LSS}}\frac{d\chi_1}{\chi_1}\left(1-\frac{\chi_1}{\chi_{\rm LSS}}\right)j_\ell(k\chi_1)\phi\left(\boldsymbol{k},\eta(\chi_1)\right) &(47)\\ &= \sum_{\ell,m}\phi^L_{\ell m}Y_{\ell m}(\hat{n}) &(48)\end{aligned}$$

with

$$\phi^L_{\ell m}=8\pi i^\ell\int\frac{d^3k}{(2\pi)^3}Y^*_{\ell m}(\hat{\boldsymbol{k}})\int_0^{\chi_{\rm LSS}}\frac{d\chi_1}{\chi_1}\left(1-\frac{\chi_1}{\chi_{\rm LSS}}\right)j_\ell(k\chi_1)\phi\left(\boldsymbol{k},\eta(\chi_1)\right) \tag{49}$$

the lensing power spectrum is given by

$$\begin{aligned}\langle\phi^L_{\ell m}\phi^{L*}_{\ell m}\rangle = (8\pi)^2\int\frac{d^3k}{(2\pi)^3}Y^*_{\ell m}(\hat{\boldsymbol{k}})\frac{d^3k'}{(2\pi)^3}Y_{\ell m}(\hat{\boldsymbol{k}}')\\ \int_0^{\chi_{\rm LSS}}\frac{d\chi_1}{\chi_1}\left(1-\frac{\chi_1}{\chi_{\rm LSS}}\right)j_\ell(k\chi_1)\int_0^{\chi_{\rm LSS}}\frac{d\chi_2}{\chi_2}\left(1-\frac{\chi_2}{\chi_{\rm LSS}}\right)j_\ell(k'\chi_2)\langle\phi\left(\boldsymbol{k},\eta(\chi_1)\right)\phi\left(\boldsymbol{k}',\eta(\chi_2)\right)\rangle\end{aligned} \tag{50}$$

assuming homogeneity

$$\langle\phi\left(\boldsymbol{k},\eta(\chi_1)\right)\phi\left(\boldsymbol{k}',\eta(\chi_2)\right)\rangle=P_\phi(k,\eta(\chi_1),\eta(\chi_2))\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{51}$$

we get

$$\langle\phi^L_{\ell m}\phi^{L*}_{\ell m}\rangle=\frac{8}{\pi}\int k^2dk\int_0^{\chi_{\rm LSS}}\frac{d\chi_1}{\chi_1}\left(1-\frac{\chi_1}{\chi_{\rm LSS}}\right)j_\ell(k\chi_1)\int_0^{\chi_{\rm LSS}}\frac{d\chi_2}{\chi_2}\left(1-\frac{\chi_2}{\chi_{\rm LSS}}\right)j_\ell(k\chi_2)P_\phi(k,\eta(\chi_1),\eta(\chi_2)) \tag{52}$$

The final step is to use the so-called Limber approximation, we have the following formula for the bessel function

$$\frac{2}{\pi}\int k^2dkj_\ell(k\chi_1)j_\ell(k\chi_2)=\frac{1}{\chi_1^2}\delta(\chi_1-\chi_2) \tag{53}$$

and the fact that the product of spherical bessel function is sharply peaked at $k\chi_1\sim k\chi_2\sim\sqrt{\ell(\ell+1)}\sim\ell+\frac{1}{2}$ for high ℓ.

This allow us to kill one integral and evaluate the power spectrum only at the peak

$$\begin{aligned}C^L_\ell=\langle\phi^L_{\ell m}\phi^{L*}_{\ell m}\rangle &= 4\int\frac{d\chi_1}{\chi_1^4}\left(1-\frac{\chi_1}{\chi_{\rm LSS}}\right)^2P_\phi\left(k=\frac{\ell+1/2}{\chi_1},\eta(\chi_1)\right) &(54)\\ &= 4\int d\chi_1\left(\frac{\chi_{\rm LSS}-\chi_1}{\chi_1^2\chi_{\rm LSS}}\right)^2P_\phi\left(k=\frac{\ell+1/2}{\chi_1},z(\chi_1)\right) &(55)\end{aligned}$$