

# Reduction by stages for affine W-algebras

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## Vertex algebras

*Vertex algebras* are noncommutative and nonassociative algebras providing a rigorous mathematical framework for conformal field theories of dimension 2. They play a great role in various areas of mathematics: representation theory of infinite-dimensional Lie algebras, the Monstrous Moonshine Conjecture, the Langlands program, etc. Their axiomatic definition was given by Richard Borcherds in 1986.

### Definition

Roughly speaking, a vertex algebra  $\mathcal{V}$  is a vector space over the field of complex numbers  $\mathbf{C}$ , equipped with

- a multiplication  $\mathcal{V} \otimes_{\mathbf{C}} \mathcal{V} \rightarrow \mathcal{V}$ ,  $a \otimes b \mapsto :ab:$  (the *normally ordered product*),
- a unit vector  $\mathbf{1}$  in  $\mathcal{V}$  (the *vacuum vector*),
- a derivation operator  $\partial : \mathcal{V} \rightarrow \mathcal{V}$  (the *translation operator*),
- a  *$\lambda$ -bracket*  $[a_{\lambda}b] = \sum_{n \geq 0} a_{(n)}b \frac{1}{n!} \lambda^n$  in  $\mathcal{V}[\lambda]$  for any  $a, b$  in  $\mathcal{V}$ .

These data satisfy some axioms. For example, the  $\lambda$ -bracket controls the noncommutativity and nonassociativity of the normally ordered product.

## Examples of vertex algebras

### Virasoro vertex algebra

The *Virasoro vertex algebra*  $\mathcal{V}i\mathfrak{z}^c$  with central charge given by the complex number  $c$  is spanned (as a differential algebra) by a vector  $L$  satisfying the relation

$$[L_{\lambda}L] = \partial L + 2L\lambda + \frac{c}{2}\mathbf{1}\frac{\lambda^3}{6}.$$

### Kac-Moody vertex algebra

Let  $\mathfrak{g}$  be a simple finite-dimensional Lie algebra over  $\mathbf{C}$  and  $\kappa$  be an invariant bilinear form on  $\mathfrak{g}$ . The *Kac-Moody vertex algebra*  $\mathcal{V}^{\kappa}(\mathfrak{g})$  associated with  $\mathfrak{g}$  and  $\kappa$  is spanned by elements  $x, y$  in  $\mathfrak{g}$  with the relations

$$[x_{\lambda}y] = [x, y] + \kappa(x, y)\mathbf{1}\lambda.$$

## Affine W-algebras

Let  $f$  be a nilpotent element in  $\mathfrak{g}$ . The *affine W-algebra*  $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$  associated  $\mathfrak{g}, f$  and  $\kappa$  is a vertex algebra constructed by applying a *quantum Hamiltonian reduction* functor  $H_f$  to the Kac-Moody vertex algebra  $\mathcal{V}^{\kappa}(\mathfrak{g})$  [Feigin-Frenkel 1990 and Kac-Roan-Wakimoto 2003]:

$$\mathcal{W}^{\kappa}(\mathfrak{g}, f) := H_f(\mathcal{V}^{\kappa}(\mathfrak{g})).$$

It is finitely generated as a differential algebra, but the relations between generators are unknown in general and difficult to compute.

The affine W-algebra  $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$  only depends on the nilpotent orbit  $\mathbf{O}$  that contains  $f$ .

### Example

If  $\mathfrak{g} = \mathfrak{sl}_2$ , if  $\kappa(\bullet, \bullet) = k \text{ trace}(\bullet\bullet)$  for a complex number  $k \neq -2$ , and if  $f$  is a nonzero nilpotent element, then  $\mathcal{W}^{\kappa}(\mathfrak{sl}_2, f)$  is a Virasoro vertex algebra with central charge given by  $c_k = 1 - 6\frac{(k+1)^2}{k+2}$ .

## Reduction by stages

Let  $f_1$  and  $f_2$  be two nilpotent elements in  $\mathfrak{g}$  such that  $f_0 := f_2 - f_1$  is nilpotent. We say that *reduction by stages* holds if there exists a quantum Hamiltonian reduction functor  $H_{f_0}$  that makes the following diagram commute:

$$\begin{array}{ccc} & \mathcal{V}^{\kappa}(\mathfrak{g}) & \\ H_{f_1} \swarrow & & \searrow H_{f_2} \\ \mathcal{W}^{\kappa}(\mathfrak{g}, f_1) & \xrightarrow{H_{f_0}} & \mathcal{W}^{\kappa}(\mathfrak{g}, f_2). \end{array}$$

### Motivation

Reduction by stages can be applied to the study of representations of the affine W-algebras and to provide isomorphisms between their simple quotients.

### Example

Take  $\mathfrak{g} = \mathfrak{sl}_3$ ,  $f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $f_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . The following reduction by stages holds:

$$\begin{array}{ccc} & \mathcal{V}^{\kappa}(\mathfrak{sl}_3) & \\ H_{f_1} \swarrow & & \searrow H_{f_2} \\ \mathcal{W}^{\kappa}(\mathfrak{sl}_3, f_1) & \xrightarrow{H_{f_0}} & \mathcal{W}^{\kappa}(\mathfrak{sl}_3, f_2). \end{array}$$

## Approach: using geometry

### Associated variety

Arakawa introduced a (contravariant) functor  $\mathcal{V} \mapsto X_{\mathcal{V}}$  from the category of vertex algebras to the category of affine Poisson varieties. The variety  $X_{\mathcal{V}}$  is called the *associated variety* of  $\mathcal{V}$ .

The associated variety of  $\mathcal{V}^{\kappa}(\mathfrak{g})$  is the dual space  $\mathfrak{g}^*$  with the Kirillov-Kostant Poisson structure. The associated variety of  $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$  is the *Slodowy slice*  $S_f$  associated to  $f$ .

### Strategy [?]

Step 1. Prove the geometric reduction by stages:

$$\begin{array}{ccc} & \mathfrak{g}^* & \\ H_{f_1} \swarrow & & \searrow H_{f_2} \\ S_{f_1} & \xrightarrow{H_{f_0}} & S_{f_2}. \end{array}$$

Step 2. Build a natural map  $H_{f_0}(\mathcal{W}^{\kappa}(\mathfrak{g}, f_1)) \rightarrow \mathcal{W}^{\kappa}(\mathfrak{g}, f_2)$  by generalising ideas from [Madsen-Ragoucy 1997].

Step 3. Prove it is an isomorphism by using the canonical Li filtrations on the W-algebras and the geometric reduction by stages.

### Other approach: screening operators

Various reductions by stages were also proved using the *screening operator* description of affine W-algebras developed by Genra. See:

- [Fehily 2023 and 2024],
- [Fasquel-Nakatsuka 2023],
- [Fasquel-Fehily-Fursman-Nakatsuka 2024],
- [Fasquel-Kovalchuk-Nakatsuka 2024].

## Main results

### Theorem 1 [?]

Reduction by stages holds for Slodowy slices in the cases described in Table 1.

### Theorem 2 [?]

Reduction by stages holds for affine W-algebras in the cases described in Table 1.

In type A, the nilpotent orbit of  $\mathfrak{sl}_n$  are classified by *partitions*  $(a_1, \dots, a_{r-1}, a_r)$  of  $n$ . A *hook-type* partition is a partition of the form  $(a, 1^{n-a})$ .

Table 1. Setting for reductions by stages

$\mathfrak{g}$	$f_1$	$f_2$	Reference
type A	hook-type	hook-type	[MR97, F24]
type $A_3$	partition of 4: (2, 1 <sup>2</sup> )	partition of 4: (2, 2)	[FFFN24]
type $A_{n-1}$	partition of $n$ : ( $a_1, \dots, a_{r-1}, a_r, 1^p$ )	partition of $n$ : ( $a_1, \dots, a_{r-1}, a_r + 1, 1^{p-1}$ )	new for $n > 3$
type B	subregular	regular	[FN23]
type $C_r$	partition of $r$ : (2 <sup>2</sup> , 1 <sup>2r-4</sup> )	regular	new
type $G_2$	Bala-Carter label $\tilde{A}_1$	regular	new

## Application to the Kac-Roan-Wakimoto embedding

Let  $(a_1, \dots, a_s, a_{s+1}, \dots, a_r)$  be a partition of  $n$  associated with the nilpotent element  $f_2$  in  $\mathfrak{sl}_n$ . Set  $p := a_{s+1} + \dots + a_r$  and let  $f_1$  be a nilpotent element corresponding to the partition  $(a_1, \dots, a_s, 1^p)$  of  $n$ . Let  $f_0$  be a nilpotent element of  $\mathfrak{sl}_p$  corresponding to the partition  $(a_{s+1}, \dots, a_r)$  of  $p$ .

### Theorem [KRW03]

There is a level  $\beta$  such that there is a vertex algebra embedding  $\mathcal{V}^{\beta}(\mathfrak{sl}_p) \hookrightarrow \mathcal{W}^{\kappa}(\mathfrak{sl}_n, f_1)$ .

### Corollary [GJ, in progress]

There is a vertex algebra embedding  $\mathcal{W}^{\beta}(\mathfrak{sl}_p, f_0) \hookrightarrow \mathcal{W}^{\kappa}(\mathfrak{sl}_n, f_2)$ .

## References