

Reduction by stages for affine W-algebras

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Vertex algebras

Vertex algebras are noncommutative and nonassociative algebras providing a rigorous mathematical framework for conformal field theories of dimension 2. They play a great role in various areas of mathematics: representation theory of infinite-dimensional Lie algebras, the Monstrous Moonshine Conjecture, the Langlands program, etc. Their axiomatic definition was given by Richard Borcherds in 1986.

Definition

Roughly speaking, a vertex algebra \mathcal{V} is a vector space over the field of complex numbers \mathbf{C} , equipped with

- a multiplication $\mathcal{V} \otimes_{\mathbf{C}} \mathcal{V} \rightarrow \mathcal{V}$, $a \otimes b \mapsto :ab:$ (the **normally ordered product**),
- a unit vector $\mathbf{1}$ in \mathcal{V} (the **vacuum vector**),
- a derivation operator $\partial : \mathcal{V} \rightarrow \mathcal{V}$ (the **translation operator**),
- a **λ -bracket** $[a_\lambda b] = \sum_{n \geq 0} a_{(n)} b \frac{1}{n!} \lambda^n$ in $\mathcal{V}[\lambda]$ for any a, b in \mathcal{V} .

These data satisfy some axioms. For example, the λ -bracket controls the noncommutativity and nonassociativity of the normally ordered product.

Examples of vertex algebras

Virasoro vertex algebra

The **Virasoro vertex algebra** $\mathcal{V}\text{ir}^c$ with central charge given by the complex number c is spanned (as a differential algebra) by a vector L satisfying the relation

$$[L_\lambda L] = \partial L + 2L\lambda + \frac{c}{2}\mathbf{1} \frac{\lambda^3}{6}.$$

Kac-Moody vertex algebra

Let \mathfrak{g} be a simple finite-dimensional Lie algebra over \mathbf{C} and κ be an invariant bilinear form on \mathfrak{g} . The **Kac-Moody vertex algebra** $\mathcal{V}^\kappa(\mathfrak{g})$ associated with \mathfrak{g} and κ is spanned by elements x, y in \mathfrak{g} with the relations

$$[x_\lambda y] = [x, y] + \kappa(x, y)\mathbf{1} \lambda.$$

Affine W-algebras

Let f be a nilpotent element in \mathfrak{g} . The **affine W-algebra** $\mathcal{W}^\kappa(\mathfrak{g}, f)$ associated \mathfrak{g}, f and κ is a vertex algebra constructed by applying a **quantum Hamiltonian reduction** functor H_f to the Kac-Moody vertex algebra $\mathcal{V}^\kappa(\mathfrak{g})$ [Feigin-Frenkel 1990 and Kac-Roan-Wakimoto 2003]:

$$\mathcal{W}^\kappa(\mathfrak{g}, f) := H_f(\mathcal{V}^\kappa(\mathfrak{g})).$$

It is finitely generated as a differential algebra, but the relations between generators are unknown in general and difficult to compute.

The affine W-algebra $\mathcal{W}^\kappa(\mathfrak{g}, f)$ only depends on the nilpotent orbit \mathbf{O} that contains f .

Example

If $\mathfrak{g} = \mathfrak{sl}_2$, if $\kappa(\bullet, \bullet) = k \operatorname{trace}(\bullet\bullet)$ for a complex number $k \neq -2$, and if f is a nonzero nilpotent element, then $\mathcal{W}^\kappa(\mathfrak{sl}_2, f)$ is a Virasoro vertex algebra with central charge given by $c_k = 1 - 6\frac{(k+1)^2}{k+2}$.

Reduction by stages

Let f_1 and f_2 be two nilpotent elements in \mathfrak{g} such that $f_0 := f_2 - f_1$ is nilpotent. We say that **reduction by stages** holds if there exists a quantum Hamiltonian reduction functor H_{f_0} that makes the following diagram commute:

$$\begin{array}{ccc} & \mathcal{V}^\kappa(\mathfrak{g}) & \\ H_{f_1} \swarrow & & \searrow H_{f_2} \\ \mathcal{W}^\kappa(\mathfrak{g}, f_1) & \dashrightarrow & \mathcal{W}^\kappa(\mathfrak{g}, f_2). \end{array}$$

Motivation

Reduction by stages can be applied to the study of representations of the affine W-algebras and to provide isomorphisms between their simple quotients.

Example

Take $\mathfrak{g} = \mathfrak{sl}_3$, $f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The following reduction by stages holds:

$$\begin{array}{ccc} & \mathcal{V}^\kappa(\mathfrak{sl}_3) & \\ H_{f_1} \swarrow & & \searrow H_{f_2} \\ \mathcal{W}^\kappa(\mathfrak{sl}_3, f_1) & \dashrightarrow & \mathcal{W}^\kappa(\mathfrak{sl}_3, f_2). \end{array}$$

Approach: using geometry

Associated variety

Arakawa introduced a (contravariant) functor $\mathcal{V} \mapsto X_\mathcal{V}$ from the category of vertex algebras to the category of affine Poisson varieties. The variety $X_\mathcal{V}$ is called the **associated variety** of \mathcal{V} .

The associated variety of $\mathcal{V}^\kappa(\mathfrak{g})$ is the dual space \mathfrak{g}^* with the Kirillov-Kostant Poisson structure. The associated variety of $\mathcal{W}^\kappa(\mathfrak{g}, f)$ is the **Slodowy slice** S_f associated to f .

Strategy [GJ25]

$$\begin{array}{ccc} & \mathfrak{g}^* & \\ H_{f_1} \swarrow & & \searrow H_{f_2} \\ S_{f_1} & \xrightarrow{H_{f_0}} & S_{f_2}. \end{array}$$

Step 1. Prove the geometric reduction by stages:

Step 2. Build a natural map $H_{f_0}(\mathcal{W}^\kappa(\mathfrak{g}, f_1)) \rightarrow \mathcal{W}^\kappa(\mathfrak{g}, f_2)$ by generalising ideas from [Madsen-Ragoucy 1997].

Step 3. Prove it is an isomorphism by using the canonical Li filtrations on the W-algebras and the geometric reduction by stages.

Other approach: screening operators

Various reductions by stages were also proved using the **screening operator** description of affine W-algebras developed by Genra. See:

- [Fehily 2023 and 2024],
- [Fasquel-Nakatsuka 2023],
- [Fasquel-Fehily-Fursman-Nakatsuka 2024],
- [Fasquel-Kovalchuk-Nakatsuka 2024].

Main results

Theorem 1 [GJ24]

Reduction by stages holds for Slodowy slices in the cases described in Table 1.

Theorem 2 [GJ25]

Reduction by stages holds for affine W-algebras in the cases described in Table 1.

In type A, the nilpotent orbit of \mathfrak{sl}_n are classified by **partitions** $(a_1, \dots, a_{r-1}, a_r)$ of n . A **hook-type** partition is a partition of the form $(a, 1^{n-a})$.

Table 1. Setting for reductions by stages

\mathfrak{g}	f_1	f_2	Reference
type A	hook-type	hook-type	[MR97, F24]
type A_3	partition of 4: $(2, 1^2)$	partition of 4: $(2, 2)$	[FFFN24]
type A_{n-1}	partition of n : $(a_1, \dots, a_{r-1}, a_r, 1^p)$	partition of n : $(a_1, \dots, a_{r-1}, a_r + 1, 1^{p-1})$	new for $n > 3$
type B	subregular	regular	[FN23]
type C_r	partition of r : $(2^2, 1^{2r-4})$	regular	new
type G_2	Bala-Carter label \tilde{A}_1	regular	new

Application to the Kac–Roan–Wakimoto embedding

Let $(a_1, \dots, a_s, a_{s+1}, \dots, a_r)$ be a partition of n associated with the nilpotent element f_2 in \mathfrak{sl}_n . Set $p := a_{s+1} + \dots + a_r$ and let f_1 be a nilpotent element corresponding to the partition $(a_1, \dots, a_s, 1^p)$ of n . Let f_0 be a nilpotent element of \mathfrak{sl}_p corresponding to the partition (a_{s+1}, \dots, a_r) of p .

Theorem [KRW03]

There is a level β such that there is a vertex algebra embedding $\mathcal{V}^\beta(\mathfrak{sl}_p) \hookrightarrow \mathcal{W}^\kappa(\mathfrak{sl}_n, f_1)$.

Corollary [GJ, in progress]

There is a vertex algebra embedding $\mathcal{W}^\beta(\mathfrak{sl}_p, f_0) \hookrightarrow \mathcal{W}^\kappa(\mathfrak{sl}_n, f_2)$.

References

- [GJ24] Naoki Genra and Thibault Juillard. Reduction by stages for finite W-algebras. *Mathematische Zeitschrift*, 308(1):15, 2024.
- [GJ25] Naoki Genra and Thibault Juillard. Reduction by stages for affine W-algebras. 2025. Preprint.