DDA 4010 – Bayesian Statistics

Exercise Sheet 1

This exercise is due on **Oct. 7th, 5:00 pm.**

Assignment A1.1 (2.2 in Textbook):
Expectations and variances: Let Y_1 and Y_2 be two independent random variables, such that $\mathrm{E}[Y_i] = \mu_i$ and $\mathrm{Var}[Y_i] = \sigma_i^2$. Using the definition of expectation and variance, compute the following quantities, where a_1 and a_2 are given constants:

- $\mathrm{E}[a_1 Y_1 + a_2 Y_2], \mathrm{Var}[a_1 Y_1 + a_2 Y_2]$;
- $\mathrm{E}[a_1 Y_1 - a_2 Y_2], \mathrm{Var}[a_1 Y_1 - a_2 Y_2]$.

Assignment A1.2 (2.5 in Textbook):
Urns: Suppose urn H is filled with 40% green balls and 60% red balls, and urn T is filled with 60% green balls and 40% red balls. Someone will flip a coin and then select a ball from urn H or urn T depending on whether the coin lands heads or tails, respectively. Let X be 1 or 0 if the coin lands heads or tails, and let Y be 1 or 0 if the ball is green or red.

- Write out the joint distribution of X and Y in a table.
- Find $\mathrm{E}[Y]$. What is the probability that the ball is green?
- Find $\mathrm{Var}[Y \mid X = 0], \mathrm{Var}[Y \mid X = 1]$ and $\mathrm{Var}[Y]$. Thinking of variance as measuring uncertainty, explain intuitively why one of these variances is larger than the others.
- Suppose you see that the ball is green. What is the probability that the coin turned up tails?

Assignment A1.3 (2.7 in Textbook):
Coherence of bets: de Finetti thought of subjective probability as follows: Your probability $p(E)$ for event E is the amount you would be willing to pay or charge in exchange for a dollar on the occurrence of E. In other words, you must be willing to - give $p(E)$ to someone, provided they give you \$1 if E occurs; - take $p(E)$ from someone, and give them \$1 if E occurs. Your probability for the event $E^c =$ "not E " is defined similarly.

- Show that it is a good idea to have $p(E) \leq 1$.
- Show that it is a good idea to have $p(E) + p(E^c) = 1$.

Assignment A1.4 (3.1 **in Textbook):**
Sample survey: Suppose we are going to sample 100 individuals from a county (of size much larger than 100) and ask each sampled person whether they support policy Z or not. Let $Y_i = 1$ if person i in the sample supports the policy, and $Y_i = 0$ otherwise.

- Assume $Y_1, \ldots, Y_{100}$ are, conditional on θ, i.i.d. binary random variables with expectation θ. Write down the joint distribution of $\Pr(Y_1 = y_1, \ldots, Y_{100} = y_{100} \mid \theta)$ in a compact form. Also write down the form of $\Pr\left(\sum Y_i = y \mid \theta\right)$.
- For the moment, suppose you believed that $\theta \in \{0.0, 0.1, \ldots, 0.9, 1.0\}$. Given that the results of the survey were $\sum_{i=1}^{100} Y_i = 57$, compute $\Pr\left(\sum Y_i = 57 \mid \theta\right)$ for each of these 11 values of θ and plot these probabilities as a function of θ.
- Now suppose you originally had no prior information to believe one of these θ-values over another, and so $\Pr(\theta = 0.0) = \Pr(\theta = 0.1) = \cdots = \Pr(\theta = 0.9) = \Pr(\theta = 1.0)$. Use Bayes' rule to compute $p\left(\theta \mid \sum_{i=1}^{n} Y_i = 57\right)$ for each θ-value. Make a plot of this posterior distribution as a function of θ.
- Now suppose you allow θ to be any value in the interval $[0, 1]$. Using the uniform prior density for θ, so that $p(\theta) = 1$, plot the posterior density $p(\theta) \times \Pr\left(\sum_{i=1}^{n} Y_i = 57 \mid \theta\right)$ as a function of θ.
- As discussed in this chapter, the posterior distribution of θ is beta $(1+57, 1 + 100 - 57)$. Plot the posterior density as a function of θ. Discuss the relationships among all of the plots you have made for this exercise.

Assignment A1.5 (3.2 **in Textbook):**
Sensitivity analysis: It is sometimes useful to express the parameters a and b in a beta distribution in terms of $\theta_0 = a/(a + b)$ and $n_0 = a + b$, so that $a = \theta_0 n_0$ and $b = (1 - \theta_0)\, n_0$. Reconsidering the sample survey data in Exercise 3.1, for each combination of $\theta_0 \in \{0.1, 0.2, \ldots, 0.9\}$ and $n_0 \in \{1, 2, 8, 16, 32\}$ find the corresponding a, b values and compute $\Pr(\theta > 0.5 \mid \sum Y_i = 57)$ using a beta (a, b) prior distribution for θ. Display the results with a contour plot, and discuss how the plot could be used to explain to someone whether or not they should believe that $\theta > 0.5$, based on the data that $\sum_{i=1}^{100} Y_i = 57$.

Assignment A1.6 (3.7 **in Textbook):**
Posterior prediction: Consider a pilot study in which $n_1 = 15$ children enrolled in special education classes were randomly selected and tested for a certain type of learning disability. In the pilot study, $y_1 = 2$ children tested positive for the disability.

- Using a uniform prior distribution, find the posterior distribution of θ, the fraction of students in special education classes who have the disability. Find the posterior mean, mode and standard deviation of θ, and plot the posterior density. Researchers would like to recruit students with the disability to participate in a long-term study, but first they need to make sure they can recruit enough students. Let $n_2 = 278$ be the number of children in special education classes in this particular school district, and let Y_2 be the number of students with the disability.
- Find $\Pr(Y_2 = y_2 \mid Y_1 = 2)$, the posterior predictive distribution of Y_2, as follows:
  i. Discuss what assumptions are needed about the joint distribution of (Y_1, Y_2) such that

the following is true:

$$\Pr(Y_2 = y_2 \mid Y_1 = 2) = \int_0^1 \Pr(Y_2 = y_2 \mid \theta)\, p(\theta \mid Y_1 = 2)\, d\theta.$$

ii. Now plug in the forms for $\Pr(Y_2 = y_2 \mid \theta)$ and $p(\theta \mid Y_1 = 2)$ in the above integral.
iii. Figure out what the above integral must be by using the calculus result discussed in Section 3.1.

- Plot the function $\Pr(Y_2 = y_2 \mid Y_1 = 2)$ as a function of y_2. Obtain the mean and standard deviation of Y_2, given $Y_1 = 2$.
- The posterior mode and the MLE (maximum likelihood estimate; see Exercise 3.14) of θ, based on data from the pilot study, are both $\hat{\theta} = 2/15$. Plot the distribution $\Pr\left(Y_2 = y_2 \mid \theta = \hat{\theta}\right)$, and find the mean and standard deviation of Y_2 given $\theta = \hat{\theta}$. Compare these results to the plots and calculations in c) and discuss any differences. Which distribution for Y_2 would you use to make predictions, and why?

Sheet 1 is due on **Oct. 7th**. Submit your solutions before **Oct. 7th**, **5:00 pm**.