DDA 4010 Assignment 2 赵柏霖 120090346

1.

<1> Since a is known,

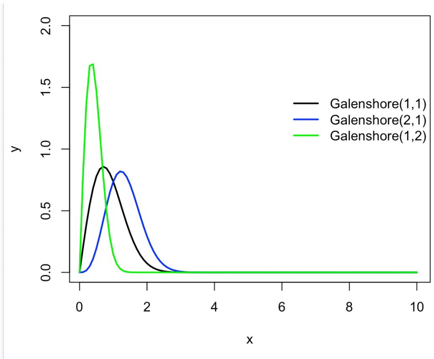
$$p(\theta|y,a) \propto p(y_1,\cdots,y_n|\theta,a)\, p(\theta|a)$$
$$\propto \theta^{2an} e^{-\theta^2 \sum_{i=1}^n y_i^2} \cdot p(\theta|a)$$

Suppose $p(\theta|a) \sim \text{Galenshore}(b,\lambda)$

$$\therefore p(\theta|y,a) \propto \theta^{2an} e^{-\theta^2 \sum_{i=1}^n y_i^2}\, \theta^{2b-1} e^{-\lambda^2\theta^2}$$
$$= \theta^{2an+2b-1} e^{-(\lambda^2+\sum_{i=1}^n y_i^2)\theta^2}$$

$\therefore p(\theta|y,a) \sim \text{Galanshore}\left(an+b, \sqrt{\lambda^2+\sum_{i=1}^n y_i^2}\right)$

$\therefore$ a class of conjugate prior densities for θ is Galenshore(b,λ)



<2> Let the prior be : $p(\theta|a) = \text{Galenshore}(b,\lambda)$

$$p(\theta|y_1,\cdots,y_n,a) \propto p(y_1,\cdots,y_n|\theta,a)\, p(\theta|a)$$
$$\propto \prod_{i=1}^n (\theta^{2a} e^{-\theta^2 y_i^2})\, \frac{2}{\Gamma(b)} \lambda^{2b} \theta^{2b-1} e^{-\lambda^2\theta^2}$$
$$\propto \theta^{2an} e^{-\theta^2(\sum_{i=1}^n y_i^2)}\, \theta^{2b-1} e^{-\lambda^2\theta^2}$$
$$= \theta^{2(an+b)-1} e^{-(\sum_{i=1}^n y_i^2+\lambda^2)\theta^2}$$
$$\sim \text{Galenshore}\left(an+b, \sqrt{\lambda^2+\sum_{i=1}^n y_i^2}\right)$$

<3>
$$\frac{p(\theta_a|Y_1,\cdots,Y_n)}{p(\theta_b|Y_1,\cdots,Y_n)} = \frac{\theta_a^{2(an+b)-1} e^{-(\sum_{i=1}^n y_i^2+\lambda^2)\theta_a^2}}{\theta_b^{2(an+b)-1} e^{-(\sum_{i=1}^n y_i^2+\lambda^2)\theta_b^2}}$$
$$= \left(\frac{\theta_a}{\theta_b}\right)^{2(an+b)-1} e^{(\sum_{i=1}^n y_i^2+\lambda^2)(\theta_b^2-\theta_a^2)}$$

$\therefore$ The sufficient statistic is $\sum_{i=1}^n y_i^2$

<4>
$$E(\theta|Y_1,\cdots,Y_n) = \frac{\Gamma(an+b+\frac{1}{2})}{\sqrt{\lambda^2+\sum_{i=1}^n y_i^2}\;\Gamma(an+b)}$$

<5>
$$p(\tilde{y}|Y_1,\cdots,Y_n) = \int_0^\infty p(\tilde{y}|\theta,y_1,\cdots,y_n)\, p(\theta|y_1,\cdots,y_n)\, d\theta$$
$$= \int_0^\infty \frac{2}{\Gamma(a)} \theta^{2a} \tilde{y}^{2a-1} e^{-\theta^2\tilde{y}^2} \frac{2}{\Gamma(an+b)} (\lambda^2+\Sigma y_i^2)^{an+b}\, \theta^{2an+2b-1} e^{-(a^2+\Sigma y_i^2)\theta^2}\, d\theta$$
$$= \frac{2}{\Gamma(a)} \frac{2}{\Gamma(an+b)} \tilde{y}^{2a-1} (\lambda^2+\Sigma y_i^2)^{an+b} \int_0^\infty \theta^{2an+2b+2a-1} e^{-(\lambda^2+\Sigma y_i^2+\tilde{y}^2)\theta^2}\, d\theta$$
$$= \frac{2}{\Gamma(a)} \frac{2}{\Gamma(an+b)} \tilde{y}^{2a-1} (\lambda^2+\Sigma y_i^2)^{an+b} \frac{1}{\frac{2}{\Gamma(an+b+a)} (\lambda^2+\Sigma y_i^2+\tilde{y}^2)^{an+b+a}}$$

$$= 2 \frac{\Gamma(an+b+a)}{\Gamma(a)\Gamma(an+b)} \tilde{y}^{2a-1} \frac{(\lambda^2+\Sigma y_i^2)^{an+b}}{(\lambda^2+\Sigma y_i^2+\tilde{y}^2)^{an+b+a}}$$

2.

<1> $f(y|\theta) = \frac{e^{-\theta}\theta^y}{y!}$

$\log f(y|\theta) = -\theta + y\log\theta - \log y!$

$\frac{\partial \log f(y|\theta)}{d\theta} = \frac{y}{\theta} - 1$

$\frac{\partial^2 \log f(y|\theta)}{\partial\theta^2} = -\frac{y}{\theta^2}$

$\therefore I(\theta) = -E(-\frac{y}{\theta^2}) = \frac{E(Y)}{\theta^2} = \frac{1}{\theta}$

$\therefore$ Jeffreys' prior: $p(\theta) \propto \sqrt{\frac{1}{\theta}}$

$p(\theta)$ is not a Gamma distribution

Jeffrey's prior does not always produce an actual probability density for θ.

<2> $f(\theta, y) \propto \sqrt{\frac{1}{\theta}} \frac{e^{-\theta}\theta^y}{y!} = \frac{\theta^{y-\frac{1}{2}} e^{-\theta}}{y!}$

$\propto \theta^{y+\frac{1}{2}-1} e^{-\theta}$

$\therefore f(\theta, y)$ is proportional to Gamma $(y+\frac{1}{2}, 1)$

$p(\theta|y) = \frac{f(\theta,y)}{f(y)} = \frac{f(\theta,y)}{\int f(\theta,y)d\theta} \propto \frac{\theta^{y-\frac{1}{2}} e^{-\theta}}{\int \theta^{y+\frac{1}{2}-1} e^{-\theta} d\theta}$

$= \frac{1}{\Gamma(y+\frac{1}{2})} \theta^{y+\frac{1}{2}-1} e^{-\theta}$

$\therefore$ We can think of $\frac{f(\theta,y)}{\int f(\theta,y)d\theta}$ as posterior density of θ given y.

3.

<1> priors: $\theta_A \sim$ gamma$(120, 10)$

$\theta_B \sim$ gamma$(12, 1)$

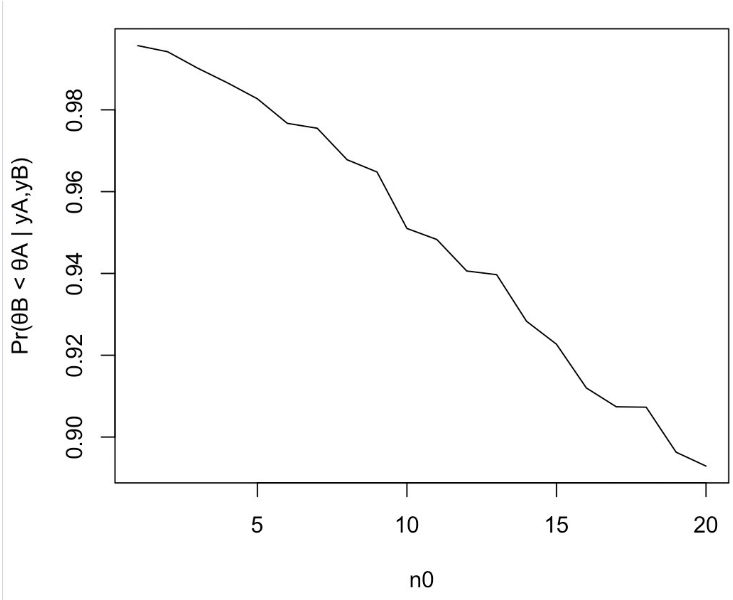
$\therefore \theta_A | y_A \sim$ gamma$(120 + \Sigma Y_{Ai}, 10 + 10)$

$\theta_B | y_B \sim$ gamma$(12 + \Sigma Y_{Bi}, 1 + 13)$

```
> set.seed(1)
> yA = c(12,9,12,14,13,13,15,8,15,6)
> yB = c(11,11,10,9,9,8,7,10,6,8,8,9,7)
> nA = 10
> nB = 13
> aA = 120
> bA = 10
> aB = 12
> bB = 1
> thetaA.posterior = rgamma(1000, aA+sum(yA), bA+nA)
> thetaB.posterior = rgamma(1000, aB+sum(yB), bB+nB)
> mean(thetaA.posterior > thetaB.posterior)
[1] 0.996
```

$\therefore Pr(\theta_B < \theta_A | y_A, y_B) = 0.996$

<2>

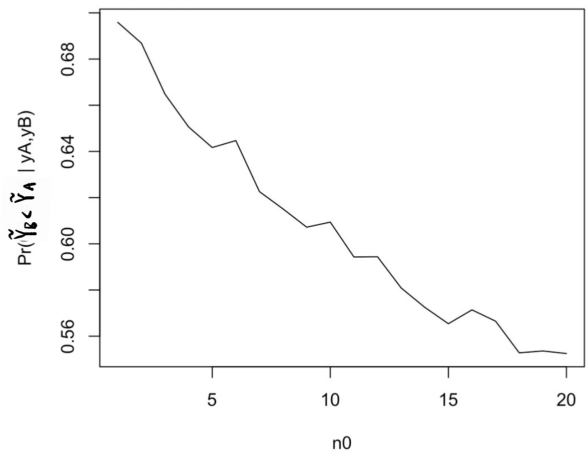


Generally, $\{\theta_B < \theta_A\}$ are insensitive to the prior distribution on θ_B.

<3> Repeat (a):

$p(\tilde{Y}_B < \tilde{Y}_A | y_A, y_B) = 0.6947$

Repeat (b):



4.

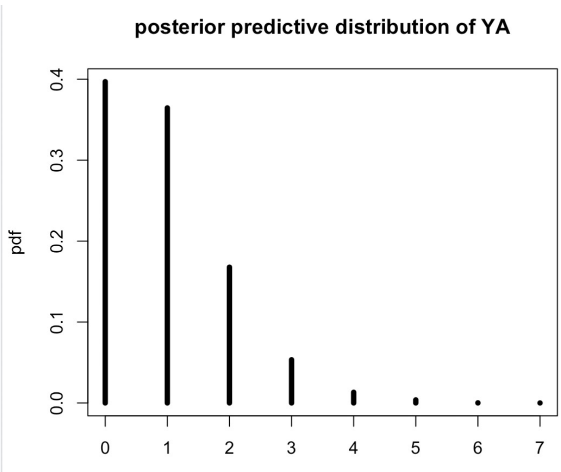
<1> $p(y|\theta) \sim \text{Poisson}(\theta)$

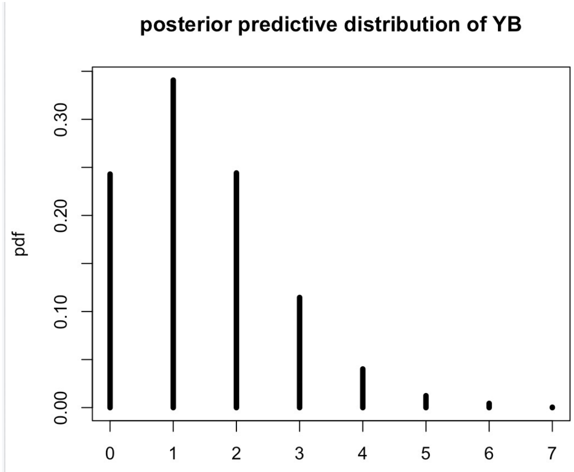
$\theta_A \sim \text{Gamma}(2, 1)$

$\therefore \ \theta_A | y_1, \cdots, y_n \sim \text{Gamma}(2 + \sum_{i=1}^{n} y_{Ai}, 1+n)$

$\theta_B \sim \text{Gamma}(2, 1)$

$\theta_B | y_1, \cdots, y_m \sim \text{Gamma}(2 + \sum_{i=1}^{m} y_{Bi}, 1+m)$





<2>

<3>

```
> quantile(theta_B-theta_A, c(0.025, 0.975))
     2.5%     97.5% 
0.1454462 0.7337033 
> quantile(YB_predict-YA_predict, c(0.025, 0.975))
  2.5%  97.5% 
-2.025  4.000 
```

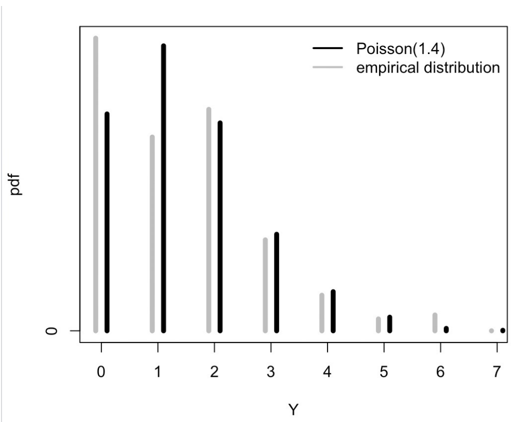
95% CI for $\theta_B - \theta_A$: $(0.1454462, 0.7337033)$

95% CI for $\tilde{Y}_B - \tilde{Y}_A$: $(-2.025, 4)$

On average, men without bachelor's degree has more children than who with bachelor's degree.

For a random new sample from class A and B, there is 95% probability that $\tilde{Y}_B - \tilde{Y}_A \in (-2.025, 4)$

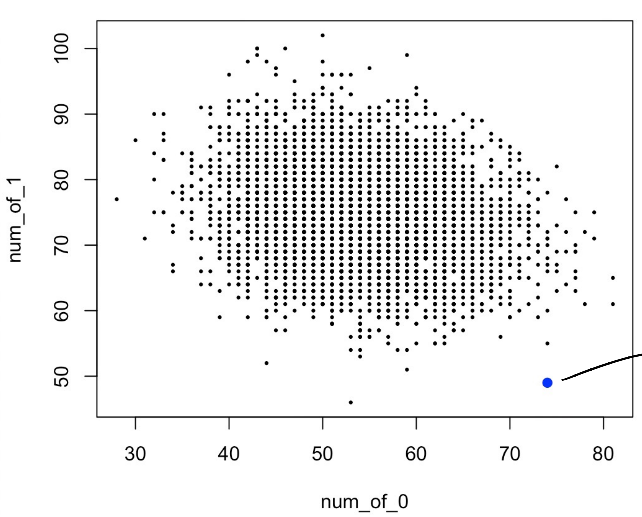
<4>



The poisson model is not a good fit for data in group B. Since it's possible that the poisson model is not the true model of the data. But the mean of the poisson model is a good fit for data.

```
> mean(data_nobach)
[1] 1.399083
```

< 5 >



(#0, #1) in the observed dataset

c. Poisson model is inadquate for the observed data, A more complex model is needed, Since the blue dot is far away from the data that we simulated from the posterior predictive distribution.