DDA 4010 Assignment 3 赵松霖 120090346

Problem 1:

(a) school 1:

```
> un
[1] 9.292308
[1] "CI for mean:"
> ybar + sqrt(s2) / sqrt(n) * qt(c(0.025, 0.975), n-1)
[1]  7.860184 11.067816
> print("CI for sd:")
[1] "CI for sd:"
> sqrt(1 / qgamma(c(0.975, 0.025), vn/2, sigmasqn * vn/2))
[1] 3.002789 5.169623
```

school 2:

```
> un
[1] 6.94875
> # 95% CI for mean and standard deviation
> print("CI for mean:")
[1] "CI for mean:"
> ybar + sqrt(s2) / sqrt(n) * qt(c(0.025, 0.975), n-1)
[1] 5.094113 8.972844
> print("CI for sd:")
[1] "CI for sd:"
> sqrt(1 / qgamma(c(0.975, 0.025), vn/2, sigmasqn * vn/2))
[1] 3.343751 5.885496
```

school 3:

```
> un
[1] 7.812381
> # 95% CI for mean and standard deviation
> print("CI for mean:")
[1] "CI for mean:"
> ybar + sqrt(s2) / sqrt(n) * qt(c(0.025, 0.975), n-1)
[1] 6.18302 9.72298
> print("CI for sd:")
[1] "CI for sd:"
> sqrt(1 / qgamma(c(0.975, 0.025), vn/2, sigmasqn * vn/2))
[1] 2.798522 5.121435
```

(b)

```
> mean(u_1.post < u_2.post & u_2.post < u_3.post)
[1] 0.00564
```
$P(\theta_1<\theta_2<\theta_3)$

```
> mean(u_1.post < u_3.post & u_3.post < u_2.post)
[1] 0.0038
```
$P(\theta_1<\theta_3<\theta_2)$

```
> mean(u_2.post < u_1.post & u_1.post < u_3.post)
[1] 0.0861
```
$P(\theta_2<\theta_1<\theta_3)$

```
> mean(u_2.post < u_3.post & u_3.post < u_1.post)
[1] 0.67009
```
$P(\theta_2<\theta_3<\theta_1)$

```
> mean(u_3.post < u_2.post & u_2.post < u_1.post)
[1] 0.2188
```
$P(\theta_3<\theta_2<\theta_1)$

```
> mean(u_3.post < u_1.post & u_1.post < u_2.post)
[1] 0.01557
```
$P(\theta_3<\theta_1<\theta_2)$

(c)

```
> mean(Y1_pred < Y2_pred & Y2_pred < Y3_pred)
[1] 0.1074
```
$P(\tilde{Y}_1<\tilde{Y}_2<\tilde{Y}_3)$

```
> mean(Y1_pred < Y3_pred & Y3_pred < Y2_pred)
[1] 0.1059
```
$P(\tilde{Y}_1<\tilde{Y}_3<\tilde{Y}_2)$

```
> mean(Y2_pred < Y1_pred & Y1_pred < Y3_pred)
[1] 0.1825
```
$P(\tilde{Y}_2<\tilde{Y}_1<\tilde{Y}_3)$

```
> mean(Y2_pred < Y3_pred & Y3_pred < Y1_pred)
[1] 0.2647
```
$P(\tilde{Y}_2<\tilde{Y}_3<\tilde{Y}_1)$

```
> mean(Y3_pred < Y2_pred & Y2_pred < Y1_pred)
[1] 0.2007
```
$P(\tilde{Y}_3<\tilde{Y}_2<\tilde{Y}_1)$

```
> mean(Y3_pred < Y1_pred & Y1_pred < Y2_pred)
[1] 0.1388
```
$P(\tilde{Y}_3<\tilde{Y}_1<\tilde{Y}_2)$

(d)

```
> mean(u_1.post > u_2.post & u_1.post > u_3.post)
[1] 0.88889
```
$P(\theta_1>\theta_2 \text{ and } \theta_1>\theta_3)$

```
> mean(Y1_pred > Y2_pred & Y1_pred > Y3_pred)
[1] 0.4654
```
$P(\tilde{Y}_1>\tilde{Y}_2 \text{ and } \tilde{Y}_1>\tilde{Y}_3)$

Problem 2 :

(a) $p(Y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\log p(Y|\mu,\sigma^2) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$

$\frac{\partial^2 \log p(Y|\mu,\sigma^2)}{\partial \mu^2} = \frac{\partial}{\partial\mu}\left(\frac{2(x-\mu)}{2\sigma^2}\right) = -\frac{1}{\sigma^2}$

$\frac{\partial^2 \log p(Y|\mu,\sigma^2)}{\partial(\sigma^2)^2} = \frac{\partial}{\partial\sigma^2}\left(-\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4}\right) = \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}$

$\frac{\partial^2 \log p(Y|\mu,\sigma^2)}{\partial\mu\, \partial\sigma^2} = \frac{\partial}{\partial\mu}\left(-\frac{1}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4}\right) = -\frac{x-\mu}{\sigma^4}$

$\frac{\partial^2 \log p(Y|\mu,\sigma^2)}{\partial\sigma^2\, \partial\mu} = \frac{\partial}{\partial\sigma^2}\left(\frac{x-\mu}{\sigma^2}\right) = \frac{-(x-\mu)}{\sigma^4}$

$\therefore I(\phi) = -E\begin{pmatrix} -\frac{1}{\sigma^2} & -\frac{x-\mu}{\sigma^4} \\ \frac{-(x-\mu)}{\sigma^4} & \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$ by $E(x-\mu)^2 = \sigma^2$

$\therefore p_J(\phi) \propto \sqrt{|I(\phi)|} = \sigma^{-3} = (\sigma^2)^{-\frac{3}{2}}$

(b) let $\theta|\sigma^2, y \sim N(a, b^2)$

$p(\theta|\sigma^2,y) \propto e^{-\frac{1}{2}\left(\frac{\theta-a}{b}\right)^2}$

let $\sigma^2|y \sim$ inverse-gamma $IG(c,d)$

$p(\sigma^2|y) \propto \frac{1}{(\sigma^2)^{c+1}} e^{-\frac{d}{\sigma^2}}$

$\therefore p(\theta,\sigma^2|y) \propto e^{-\frac{1}{2}\left(\frac{\theta-a}{b}\right)^2} \frac{1}{(\sigma^2)^{c+1}} e^{-\frac{d}{\sigma^2}}$

$\propto \frac{1}{(\sigma^2)^{c+1}} e^{-\left[\frac{(\theta-a)^2}{2b^2} + \frac{d}{\sigma^2}\right]}$

Since $p_J(y|\theta,\sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2}$

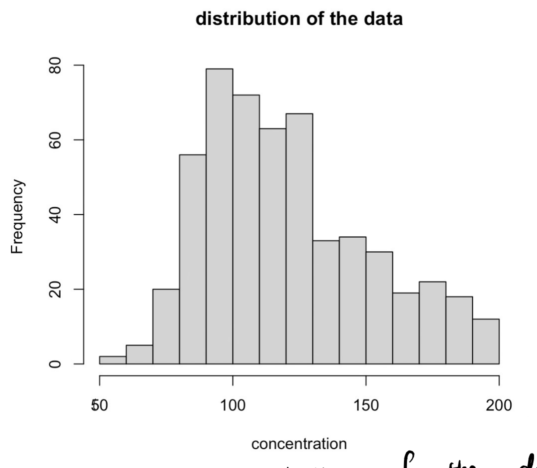
$p_J(y|\theta,\sigma^2)\, p_J(\theta,\sigma^2) \propto (\sigma^2)^{-\frac{n+3}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2}$, which has the same form as $p(\theta,\sigma^2|y)$

$\therefore p_J(\theta,\sigma^2|y) \propto p_J(y|\theta,\sigma^2)\, p_J(\theta,\sigma^2)$

$\therefore p_J(\theta,\sigma^2|y)$ is a posterior density.

Problem 3

(a)



the empirical distribution of the data deviates from the normal distribution: the distribution is not symmetric about the mean, and it's not like a bell-shaped curve.

(b) $P(y \mid x, p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2) = \prod_{i=1}^{n} P(y_i \mid x_i, p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2) = \prod_{i=1} \text{dnorm}(y_i, \theta_1, \sigma_1^2)^{x_i} \; \text{dnorm}(y_i, \theta_2, \sigma_2^2)^{1-x_i}$

$$P(x_i = 1 \mid y_i, p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2) = \frac{P(y_i \mid x_i = 1)\, P(x_i = 1)}{P(y_i)}$$

$$= \frac{p \cdot \text{dnorm}(y_i, \theta_1, \sigma_1^2)}{p\, \text{dnorm}(y_i, \theta_1, \sigma_1^2) + (1-p)\, \text{dnorm}(y_i, \theta_2, \sigma_2^2)}$$

① $\therefore\ x_i \mid y_i, p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2 \sim \text{bernoulli}\left(\frac{p \cdot \text{dnorm}(y_i, \theta_1, \sigma_1^2)}{p\, \text{dnorm}(y_i, \theta_1, \sigma_1^2) + (1-p)\, \text{dnorm}(y_i, \theta_2, \sigma_2^2)}\right)$

② let n_1 be # of 1 in $\vec{x}$ and n_2 be # of 0 in $\vec{x}$.

$$P(p \mid x, y, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2) \propto p(p)\, P(x \mid p)$$

$$\propto \text{dbeta}(p, a, b) \cdot p^{n_1} (1-p)^{n_2}$$

$$\propto p^{a-1} (1-p)^{b-1} p^{n_1} (1-p)^{n_2}$$

$$\propto p^{n_1 + a - 1} (1-p)^{n_2 + b - 1}$$

$\therefore\ p \mid x, y, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2 \sim \text{beta}(a + n_1, b + n_2)$

③ $p(\theta_1 \mid x, y, \theta_2, \sigma_1^2, \sigma_2^2) \propto p(\theta_1)\, p(y \mid x)$

$$\propto \text{dnorm}(\theta_1, \mu_0, \tau_0^2) \prod_{i=1}^{n} \left(\text{dnorm}(y_i, \theta_1, \sigma_1^2)^{x_i} \; \text{dnorm}(y_i, \theta_2, \sigma_2^2)^{1-x_i}\right)$$

$$\propto \text{dnorm}(\theta_1, \mu_0, \tau_0^2) \cdot \prod_{i=1}^{n} \text{dnorm}(y_i, \theta_1, \sigma_1^2)^{x_i}$$

$\therefore\ \theta_1 \mid x, y, \theta_2, \sigma_1^2, \sigma_2^2 \sim \text{Normal}(\mu_n, \tau_n^2)$

$$\mu_n^{(1)} = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n_1}{\sigma_1^2}\bar{y}_1}{\frac{1}{\tau_0^2} + \frac{n_1}{\sigma_1^2}} \qquad \tau_n^{(1)2} = \frac{1}{\frac{1}{\tau_0^2} + \frac{n_1}{\sigma_1^2}}$$

④ similarly: $\theta_2 \mid x, y, \theta_1, \sigma_1^2, \sigma_2^2 \sim \text{Normal}(\mu_n, \tau_n^2)$

$$\mu_n^{(2)} = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n_2}{\sigma_2^2}\bar{y}_2}{\frac{1}{\tau_0^2} + \frac{n_2}{\sigma_2^2}}$$

$$\tau_n^{(2)2} = \frac{1}{\frac{1}{\tau_0^2} + \frac{n_2}{\sigma_2^2}}$$

⑤ $p(\sigma_1^2 \mid x, y, \theta_1, \theta_2, \sigma_2^2) \propto p(\sigma_1^2)\, p(y \mid x)$

$$\propto \text{dinvgamma}\left(\sigma_1^2, \frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\right) \prod_{i=1}^{n_1} \text{dnorm}(y_i, \theta_1, \sigma_1^2)$$

$\therefore\ \sigma_1^2 \mid x, y, \theta_1, \theta_2, \sigma_2^2 \sim \text{invgamma}\left(\frac{\nu_n^{(1)}}{2}, \frac{\nu_n^{(1)}}{2}\sigma_n^{(1)2}\right)$

$$\nu_n^{(1)} = \nu_0 + n_1$$

$$\sigma_n^{(1)2} = \frac{1}{\nu_n^{(1)}}\left(\nu_0 \sigma_0^2 + \sum_{i=1}^{n_1} (y_i - \theta_1)^2\right)$$

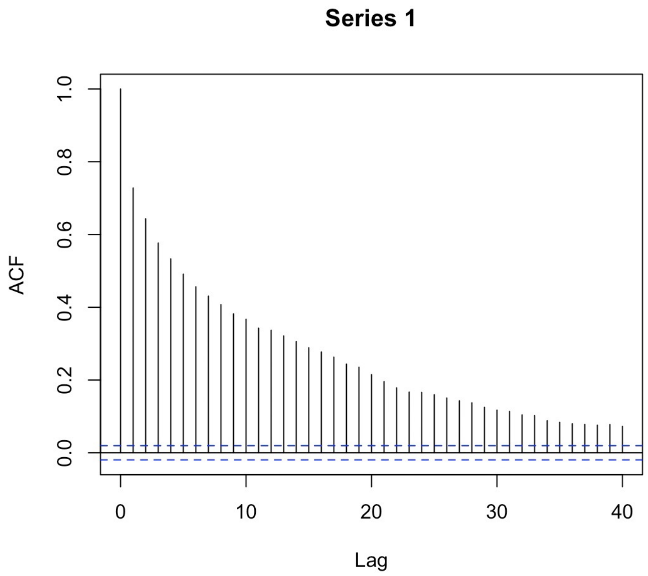
⑥ similarly,

$\sigma_2^2 \mid x, y, \theta_1, \theta_2, \sigma_1^2 \sim \text{invgamma}\left(\frac{\nu_n^{(2)}}{2}, \frac{\nu_n^{(2)}}{2}\sigma_n^{(2)2}\right)$

$$\nu_n^{(2)} = \nu_0 + n_2$$

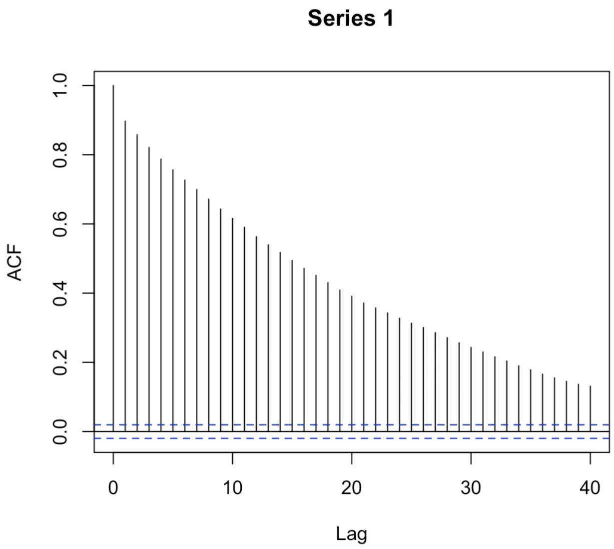
$$\sigma_n^{(2)2} = \frac{1}{\nu_n^{(2)}}\left(\nu_0 \sigma_0^2 + \sum_{i=1}^{n_2} (y_i - \theta_2)^2\right)$$

(c) auto correlation function of $\theta_{(1)}^{(s)}$;



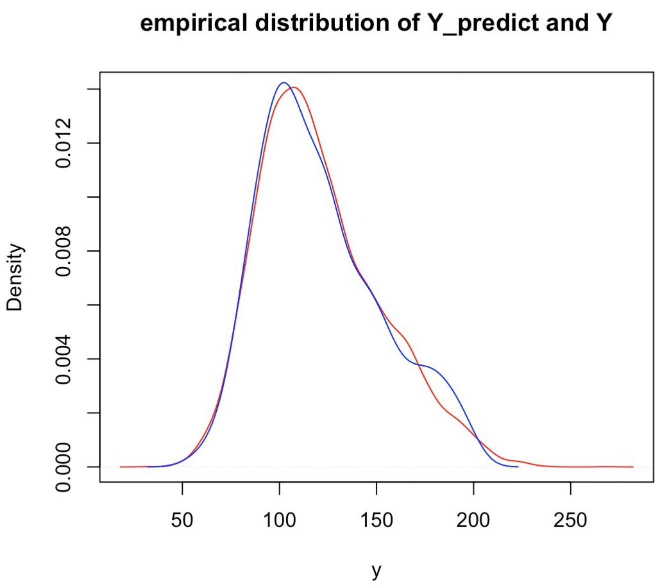

```
> effectiveSize(THETA_min)
    var1
423.6172
```

auto correlation function of $\theta_{(2)}^{(s)}$;




```
> effectiveSize(THETA_max)
    var1
226.4047
```

(d)



this model fits the original data well, since two density curves are similar.