Derivation of the CRLB

Potability distribution

We assume a continues, mono-exponential probability distribution *p* with a constant background. The probability distribution is normalised over the range 0 to *T*. Therefore, the expectation value is given by *n p*.

The following variables are used:

t: time

τ: fluorescence lifetime

T: pulse repetition period/measurement window

b: background fraction (*b* = 0, no background; *b* = 1 only background)

n: number of photons in decay

In[1]:=
```
pAssumptions = { t ≥ 0 , T > 0, τ > 0, 1 > b ≥ 0, n > 0, χ > 0, {t, T, τ, b, n, χ} ∈ Reals};
p = Simplify[ (1 - b) * PDF[ExponentialDistribution[1 / τ], t] /
      CDF[ExponentialDistribution[1 / τ], T] + b / T, pAssumptions]
```

Out[2]=
$$\frac{b - b\,e^{-\frac{T}{\tau}} - \frac{(-1+b)\,e^{-\frac{t}{\tau}}\,T}{\tau}}{T - e^{-\frac{T}{\tau}}\,T}$$

Check normalization:

In[3]:=
$$\int_0^T p\, dt$$

Out[3]= 1

CRLB for a known background

In this case only the lifetime needs to be estimated ($\theta = \{\tau\}$) and the Fisher matrix *f* has just one entry.

In[4]:=
```
fττ = Integrate[(∂τ Log[n * p])^2 * (n * p), {t, 0, T}, Assumptions → pAssumptions]
```

Out[4]=

$$
\begin{aligned}
&\frac{1}{\left(-1+e^{T/\tau}\right)^{2} T\,\tau^{4}}\\
&n \Bigg(-b\,T^{3}-e^{T/\tau}\,T^{3}+b\,e^{T/\tau}\,T^{3}-2\,b\,T^{2}\,\tau+2\,b\,e^{T/\tau}\,T^{2}\,\tau+T\,\tau^{2}-2\,b\,T\,\tau^{2}-2\,e^{T/\tau}\,T\,\tau^{2}+4\,b\,e^{T/\tau}\,T\,\tau^{2}+\\
&\quad e^{\frac{2T}{\tau}}\,T\,\tau^{2}-2\,b\,e^{\frac{2T}{\tau}}\,T\,\tau^{2}+b\,T^{2}\,\tau\,\mathrm{Log}\left[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-b\,T+b\left(-1+e^{T/\tau}\right)\tau}\right]-\\
&\quad b\,e^{\frac{2T}{\tau}}\,T^{2}\,\tau\,\mathrm{Log}\left[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-b\,T+b\left(-1+e^{T/\tau}\right)\tau}\right]+2\,b\,T\,\tau^{2}\,\mathrm{Log}\left[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-b\,T+b\left(-1+e^{T/\tau}\right)\tau}\right]-\\
&\quad 4\,b\,e^{T/\tau}\,T\,\tau^{2}\,\mathrm{Log}\left[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-b\,T+b\left(-1+e^{T/\tau}\right)\tau}\right]+2\,b\,e^{\frac{2T}{\tau}}\,T\,\tau^{2}\,\mathrm{Log}\left[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-b\,T+b\left(-1+e^{T/\tau}\right)\tau}\right]+\\
&\quad b\,T^{2}\,\tau\,\mathrm{Log}\left[e^{T/\tau}\left(T-b\,T+b\left(-1+e^{T/\tau}\right)\tau\right)\right]+2\,b\,T\,\tau^{2}\,\mathrm{Log}\left[e^{T/\tau}\left(T-b\,T+b\left(-1+e^{T/\tau}\right)\tau\right)\right]-\\
&\quad 2\,b\,e^{T/\tau}\,T\,\tau^{2}\,\mathrm{Log}\left[e^{T/\tau}\left(T-b\,T+b\left(-1+e^{T/\tau}\right)\tau\right)\right]+b\,\tau^{3}\,\mathrm{Log}\left[e^{T/\tau}\left(T-b\,T+b\left(-1+e^{T/\tau}\right)\tau\right)\right]-\\
&\quad 2\,b\,e^{T/\tau}\,\tau^{3}\,\mathrm{Log}\left[e^{T/\tau}\left(T-b\,T+b\left(-1+e^{T/\tau}\right)\tau\right)\right]+b\,e^{\frac{2T}{\tau}}\,\tau^{3}\,\mathrm{Log}\left[e^{T/\tau}\left(T-b\,T+b\left(-1+e^{T/\tau}\right)\tau\right)\right]-\\
&\quad b\,T^{2}\,\tau\,\mathrm{Log}\left[-b\,\tau+e^{T/\tau}\left(T-b\,T+b\,\tau\right)\right]-2\,b\,T\,\tau^{2}\,\mathrm{Log}\left[-b\,\tau+e^{T/\tau}\left(T-b\,T+b\,\tau\right)\right]+\\
&\quad 2\,b\,e^{T/\tau}\,T\,\tau^{2}\,\mathrm{Log}\left[-b\,\tau+e^{T/\tau}\left(T-b\,T+b\,\tau\right)\right]-b\,\tau^{3}\,\mathrm{Log}\left[-b\,\tau+e^{T/\tau}\left(T-b\,T+b\,\tau\right)\right]+\\
&\quad 2\,b\,e^{T/\tau}\,\tau^{3}\,\mathrm{Log}\left[-b\,\tau+e^{T/\tau}\left(T-b\,T+b\,\tau\right)\right]-b\,e^{\frac{2T}{\tau}}\,\tau^{3}\,\mathrm{Log}\left[-b\,\tau+e^{T/\tau}\left(T-b\,T+b\,\tau\right)\right]+\\
&\quad 2\,b\left(-1+e^{T/\tau}\right)\tau^{2}\left(T+\tau-e^{T/\tau}\,\tau\right)\mathrm{PolyLog}\left[2,\frac{(-1+b)\,e^{T/\tau}\,T}{b\left(-1+e^{T/\tau}\right)\tau}\right]-\\
&\quad 2\,b\left(-1+e^{T/\tau}\right)\tau^{2}\left(e^{T/\tau}\,(T-\tau)+\tau\right)\mathrm{PolyLog}\left[2,\frac{T-b\,T}{b\,\tau-b\,e^{T/\tau}\,\tau}\right]+\\
&\quad 2\,b\,\tau^{3}\,\mathrm{PolyLog}\left[3,\frac{(-1+b)\,e^{T/\tau}\,T}{b\left(-1+e^{T/\tau}\right)\tau}\right]-4\,b\,e^{T/\tau}\,\tau^{3}\,\mathrm{PolyLog}\left[3,\frac{(-1+b)\,e^{T/\tau}\,T}{b\left(-1+e^{T/\tau}\right)\tau}\right]+\\
&\quad 2\,b\,e^{\frac{2T}{\tau}}\,\tau^{3}\,\mathrm{PolyLog}\left[3,\frac{(-1+b)\,e^{T/\tau}\,T}{b\left(-1+e^{T/\tau}\right)\tau}\right]-2\,b\,\tau^{3}\,\mathrm{PolyLog}\left[3,\frac{T-b\,T}{b\,\tau-b\,e^{T/\tau}\,\tau}\right]+\\
&\quad 4\,b\,e^{T/\tau}\,\tau^{3}\,\mathrm{PolyLog}\left[3,\frac{T-b\,T}{b\,\tau-b\,e^{T/\tau}\,\tau}\right]-2\,b\,e^{\frac{2T}{\tau}}\,\tau^{3}\,\mathrm{PolyLog}\left[3,\frac{T-b\,T}{b\,\tau-b\,e^{T/\tau}\,\tau}\right]\Bigg)
\end{aligned}
$$

Mathematica needs some help with the simplification : We need to reduce all Log expressions and collect the τ to enable cancelling them.
For efficient numerical evaluation, Powers are expanded again to avoid errors by to large arguments in the Log. The terms with PolyLog are grouped to minimise calls to PolyLog.

In[5]:=
```
SimplifyCRLB[exprs_] := Assuming[pAssumptions,
  Simplify[
   Simplify[exprs //. {Log[a_] + Log[b_] :> Log[a * b], x_ * Log[a_] :> Log[a^x],
       a_ * τ + b_ * τ :> (a + b) * τ}] (* Cancel τ in the log terms *)
    //. {Log[a_ * b_^x_] :> Log[a] + x * Log[b],
     a_ * PolyLog[x_, c_] + b_ * PolyLog[x_, c_] :> (a + b) * PolyLog[x, c]}]
   (* Optimise for numerical evaluation *)
 ]
```

We substitute the repetition period with a relative repetition period: $\chi = T/\tau$

The CRLB (σ_τ^2) is given the inverse Fisher matrix:

In[6]:=
```
στSQ = SimplifyCRLB[1 / fττ //. {T → χ τ}]
```

Out[6]=
$$\left((-1+e^{\chi})^2\,\tau^2\,\chi\right) \Big/ \Bigg(n\Bigg(\chi - 2\,b\,\chi - 2\,e^{\chi}\,\chi + 4\,b\,e^{\chi}\,\chi + e^{2\chi}\,\chi - 2\,b\,e^{2\chi}\,\chi - 2\,b\,\chi^2 + 2\,b\,e^{\chi}\,\chi^2 - b\,\chi^3 - e^{\chi}\,\chi^3 + b\,e^{\chi}\,\chi^3 - b\,(-1+e^{\chi})\,\chi\,(2+e^{\chi}\,(-2+\chi)+\chi)\,(\mathrm{Log}[b\,(-1+e^{\chi})] - \mathrm{Log}[b\,(-1+e^{\chi}-\chi)+\chi]) + b\,(1-e^{\chi}+\chi)^2\,\mathrm{Log}\left[\frac{e^{\chi}\,(b\,(-1+e^{\chi}-\chi)+\chi)}{-b+e^{\chi}\,(b+\chi-b\,\chi)}\right] - 2\,b\,(-1+e^{\chi})\,(1+e^{\chi}\,(-1+\chi))\,\mathrm{PolyLog}\left[2,\frac{(-1+b)\,\chi}{b\,(-1+e^{\chi})}\right] - 2\,b\,(-1+e^{\chi})\,(-1+e^{\chi}-\chi)\,\mathrm{PolyLog}\left[2,\frac{(-1+b)\,e^{\chi}\,\chi}{b\,(-1+e^{\chi})}\right] - 2\,b\,(-1+e^{\chi})^2\,\mathrm{PolyLog}\left[3,\frac{(-1+b)\,\chi}{b\,(-1+e^{\chi})}\right] + 2\,b\,(-1+e^{\chi})^2\,\mathrm{PolyLog}\left[3,\frac{(-1+b)\,e^{\chi}\,\chi}{b\,(-1+e^{\chi})}\right]\Bigg)\Bigg)$$

Limit for zero background:

In[7]:=
```
Assuming[pAssumptions, FullSimplify[1 / Limit[1 / στSQ, b → 0, Direction → "FromAbove"]]]
(* The limits of 1/στSQ are somehow much faster to compute. *)
```

Out[7]=
$$-\frac{2\,\tau^2\,(-1+\mathrm{Cosh}[\chi])}{n\,(2+\chi^2-2\,\mathrm{Cosh}[\chi])}$$

Limit for infinite pulse period:

In[8]:=
```
1 / Limit[1 / στSQ, χ → ∞, Assumptions → pAssumptions, Direction → "FromBelow"]
```

Out[8]=
$$\frac{\tau^2}{n - b\,n}$$

CRLB for an unknown background

Now lifetime and background need to be estimated ($\theta = \{\tau, b\}$). The Fisher matrix has the form $f = \begin{pmatrix} f_{\tau\tau} & f_{\tau b} \\ f_{\tau b} & f_{bb} \end{pmatrix}$.

Calculate $f_{\tau b}$ and f_{bb}, and assemble Fisher matrix.

In[9]:=
```
fτb =
  Integrate[(∂τ Log[n * p]) * (∂b Log[n * p]) * (n * p), {t, 0, T}, Assumptions → pAssumptions]
```

Out[9]=
$$-\frac{1}{2\left(-1+b\right)\left(-1+e^{T/\tau}\right)T\tau^{2}}\,n\Bigg(2T^{2}-bT^{2}+b\,e^{T/\tau}T^{2}+2T\tau-2e^{T/\tau}T\tau-2T\tau\,\mathrm{Log}\Big[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-bT+b\left(-1+e^{T/\tau}\right)\tau}\Big]+$$
$$2bT\tau\,\mathrm{Log}\Big[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-bT+b\left(-1+e^{T/\tau}\right)\tau}\Big]+2e^{T/\tau}T\tau\,\mathrm{Log}\Big[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-bT+b\left(-1+e^{T/\tau}\right)\tau}\Big]-$$
$$2b\,e^{T/\tau}T\tau\,\mathrm{Log}\Big[\frac{b\left(-1+e^{T/\tau}\right)\tau}{T-bT+b\left(-1+e^{T/\tau}\right)\tau}\Big]-2T\tau\,\mathrm{Log}\Big[e^{T/\tau}\left(T-bT+b\left(-1+e^{T/\tau}\right)\tau\right)\Big]-$$
$$2\tau^{2}\,\mathrm{Log}\Big[e^{T/\tau}\left(T-bT+b\left(-1+e^{T/\tau}\right)\tau\right)\Big]+2e^{T/\tau}\tau^{2}\,\mathrm{Log}\Big[e^{T/\tau}\left(T-bT+b\left(-1+e^{T/\tau}\right)\tau\right)\Big]+$$
$$2bT\tau\,\mathrm{Log}\Big[1-\frac{b\left(-1+e^{T/\tau}\right)\tau}{\left(-1+b\right)T}\Big]-2b\,e^{T/\tau}T\tau\,\mathrm{Log}\Big[1-\frac{b\left(-1+e^{T/\tau}\right)\tau}{\left(-1+b\right)T}\Big]+$$
$$2T\tau\,\mathrm{Log}\Big[-b\tau+e^{T/\tau}\left(T-bT+b\tau\right)\Big]+2\tau^{2}\,\mathrm{Log}\Big[-b\tau+e^{T/\tau}\left(T-bT+b\tau\right)\Big]-$$
$$2e^{T/\tau}\tau^{2}\,\mathrm{Log}\Big[-b\tau+e^{T/\tau}\left(T-bT+b\tau\right)\Big]+2\left(-1+b\right)\left(-1+e^{T/\tau}\right)\tau^{2}$$
$$\mathrm{PolyLog}\Big[2,\frac{\left(-1+b\right)e^{T/\tau}T}{b\left(-1+e^{T/\tau}\right)\tau}\Big]-2b\left(-1+e^{T/\tau}\right)\tau^{2}\,\mathrm{PolyLog}\Big[2,\frac{b\left(-1+e^{T/\tau}\right)\tau}{\left(-1+b\right)T}\Big]-$$
$$2b\tau^{2}\,\mathrm{PolyLog}\Big[2,-\frac{b\left(1-e^{-\frac{T}{\tau}}\right)\tau}{T-bT}\Big]+2b\,e^{T/\tau}\tau^{2}\,\mathrm{PolyLog}\Big[2,-\frac{b\left(1-e^{-\frac{T}{\tau}}\right)\tau}{T-bT}\Big]-$$
$$2\tau^{2}\,\mathrm{PolyLog}\Big[2,\frac{T-bT}{b\tau-b\,e^{T/\tau}\tau}\Big]+2b\tau^{2}\,\mathrm{PolyLog}\Big[2,\frac{T-bT}{b\tau-b\,e^{T/\tau}\tau}\Big]+$$
$$2e^{T/\tau}\tau^{2}\,\mathrm{PolyLog}\Big[2,\frac{T-bT}{b\tau-b\,e^{T/\tau}\tau}\Big]-2b\,e^{T/\tau}\tau^{2}\,\mathrm{PolyLog}\Big[2,\frac{T-bT}{b\tau-b\,e^{T/\tau}\tau}\Big]\Bigg)$$

In[10]:=
```
fbb = Integrate[(∂b Log[n * p])^2 * (n * p), {t, 0, T}, Assumptions → pAssumptions]
```

Out[10]=
$$-\frac{n\left(\left(-1+b\right)T-\tau\,\mathrm{Log}\left[T-bT+b\left(-1+e^{T/\tau}\right)\tau\right]+\tau\,\mathrm{Log}\left[-b\tau+e^{T/\tau}\left(T-bT+b\tau\right)\right]\right)}{\left(-1+b\right)^{2}bT}$$

In[11]:=
```
f = {{fττ, fτb}, {fτb, fbb}};
```

The CRLB (σ_τ^2) is given the first diagonal element of the inverse Fisher matrix:

In[12]:=
```
στSQ = SimplifyCRLB[First@First@Inverse[f //. {T → χ τ}]]
```

Out[12]=

$$\begin{aligned}
&\Big(4\left(-1+e^{\chi}\right)^2\tau^2\chi\left(\left(-1+b\right)\chi-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]+\mathrm{Log}\left[-b+e^{\chi}\left(b+\chi-b\chi\right)\right]\right)\Big)\Big/\\
&\Bigg(n\Bigg(b\Bigg(2\chi-2e^{\chi}\chi+2\chi^2-b\chi^2+b\,e^{\chi}\chi^2-2b\left(-1+e^{\chi}\right)\chi\,\mathrm{Log}\left[1-\frac{b\left(-1+e^{\chi}\right)}{\left(-1+b\right)\chi}\right]-\\
&\quad 2\left(-1+b\right)\left(-1+e^{\chi}\right)\chi\left(\mathrm{Log}\left[b\left(-1+e^{\chi}\right)\right]-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]\right)+2\left(-1+e^{\chi}-\chi\right)\\
&\quad \mathrm{Log}\left[\frac{e^{\chi}\left(b\left(-1+e^{\chi}-\chi\right)+\chi\right)}{-b+e^{\chi}\left(b+\chi-b\chi\right)}\right]-2b\left(-1+e^{\chi}\right)\mathrm{PolyLog}\left[2,\frac{b\left(-1+e^{\chi}\right)}{\left(-1+b\right)\chi}\right]-\\
&\quad 2\left(-1+b\right)\left(-1+e^{\chi}\right)\mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]+2\left(-1+b\right)\left(-1+e^{\chi}\right)\\
&\quad \mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]+2b\left(-1+e^{\chi}\right)\mathrm{PolyLog}\left[2,-\frac{b-b\,e^{-\chi}}{\chi-b\chi}\right]\Bigg)^2+\\
&\quad 4\left(\left(-1+b\right)\chi-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]+\mathrm{Log}\left[-b+e^{\chi}\left(b+\chi-b\chi\right)\right]\right)\\
&\quad\Big(\chi-2b\chi-2e^{\chi}\chi+4b\,e^{\chi}\chi+e^{2\chi}\chi-2b\,e^{2\chi}\chi-2b\chi^2+2b\,e^{\chi}\chi^2-b\chi^3-e^{\chi}\chi^3+b\,e^{\chi}\chi^3-\\
&\quad b\left(-1+e^{\chi}\right)\chi\left(2+e^{\chi}\left(-2+\chi\right)+\chi\right)\left(\mathrm{Log}\left[b\left(-1+e^{\chi}\right)\right]-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]\right)+\\
&\quad b\left(1-e^{\chi}+\chi\right)^2\mathrm{Log}\left[\frac{e^{\chi}\left(b\left(-1+e^{\chi}-\chi\right)+\chi\right)}{-b+e^{\chi}\left(b+\chi-b\chi\right)}\right]-2b\left(-1+e^{\chi}\right)\left(1+e^{\chi}\left(-1+\chi\right)\right)\\
&\quad \mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]-2b\left(-1+e^{\chi}\right)\left(-1+e^{\chi}-\chi\right)\mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]-\\
&\quad 2b\left(-1+e^{\chi}\right)^2\mathrm{PolyLog}\left[3,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]+2b\left(-1+e^{\chi}\right)^2\mathrm{PolyLog}\left[3,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]\Big)\Bigg)
\end{aligned}$$

Limit for zero background:

In[13]:=
```
Assuming[pAssumptions, FullSimplify[1 / Limit[1 / στSQ, b → 0, Direction → "FromAbove"]]]
(* The limits of 1/στSQ are somehow much faster to compute. *)
```

Out[13]=

$$\frac{4\,\tau^2\left(2+\chi^2-2\,\mathrm{Cosh}[\chi]\right)\mathrm{Sinh}\left[\frac{\chi}{2}\right]}{n\left(-4\chi^3\,\mathrm{Cosh}\left[\frac{\chi}{2}\right]+\left(12+12\chi^2+\chi^4\right)\mathrm{Sinh}\left[\frac{\chi}{2}\right]-4\,\mathrm{Sinh}\left[\frac{3\chi}{2}\right]\right)}$$

Limit for infinite pulse period:

In[14]:=
```
1 / Limit[1 / στSQ, χ → ∞, Assumptions → pAssumptions, Direction → "FromBelow"]
```

Out[14]=

$$\frac{\tau^2}{n-b\,n}$$

CRLB of the background estimation

In[15]:=
```
σbSQ = SimplifyCRLB[Last@Last@Inverse[f //. {T → χ τ}]]
```

Out[15]=

$$
\begin{aligned}
-\Bigg(\Bigg(&4\left(-1+b\right)^{2}\chi\Big(\chi-2b\chi-2e^{\chi}\chi+4be^{\chi}\chi+e^{2\chi}\chi-2be^{2\chi}\chi-2b\chi^{2}+2be^{\chi}\chi^{2}-b\chi^{3}-e^{\chi}\chi^{3}+\\
&be^{\chi}\chi^{3}-b\left(-1+e^{\chi}\right)\chi\left(2+e^{\chi}\left(-2+\chi\right)+\chi\right)\left(\mathrm{Log}\left[b\left(-1+e^{\chi}\right)\right]-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]\right)+\\
&b\left(1-e^{\chi}+\chi\right)^{2}\mathrm{Log}\left[\frac{e^{\chi}\left(b\left(-1+e^{\chi}-\chi\right)+\chi\right)}{-b+e^{\chi}\left(b+\chi-b\chi\right)}\right]-2b\left(-1+e^{\chi}\right)\left(1+e^{\chi}\left(-1+\chi\right)\right)\\
&\mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]-2b\left(-1+e^{\chi}\right)\left(-1+e^{\chi}-\chi\right)\mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]-\\
&2b\left(-1+e^{\chi}\right)^{2}\mathrm{PolyLog}\left[3,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]+2b\left(-1+e^{\chi}\right)^{2}\mathrm{PolyLog}\left[3,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]\Big)\Bigg)\Bigg/\\
&\Bigg(n\Bigg(\Bigg(2\chi-2e^{\chi}\chi+2\chi^{2}-b\chi^{2}+be^{\chi}\chi^{2}-2b\left(-1+e^{\chi}\right)\chi\,\mathrm{Log}\left[1-\frac{b\left(-1+e^{\chi}\right)}{\left(-1+b\right)\chi}\right]-\\
&\quad 2\left(-1+b\right)\left(-1+e^{\chi}\right)\chi\left(\mathrm{Log}\left[b\left(-1+e^{\chi}\right)\right]-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]\right)+2\left(-1+e^{\chi}-\chi\right)\\
&\quad \mathrm{Log}\left[\frac{e^{\chi}\left(b\left(-1+e^{\chi}-\chi\right)+\chi\right)}{-b+e^{\chi}\left(b+\chi-b\chi\right)}\right]-2b\left(-1+e^{\chi}\right)\mathrm{PolyLog}\left[2,\frac{b\left(-1+e^{\chi}\right)}{\left(-1+b\right)\chi}\right]-\\
&\quad 2\left(-1+b\right)\left(-1+e^{\chi}\right)\mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]+2\left(-1+b\right)\left(-1+e^{\chi}\right)\\
&\quad \mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]+2b\left(-1+e^{\chi}\right)\mathrm{PolyLog}\left[2,-\frac{b-be^{-\chi}}{\chi-b\chi}\right]\Bigg)^{2}+\\
&\quad \frac{1}{b}4\left(\left(-1+b\right)\chi-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]+\mathrm{Log}\left[-b+e^{\chi}\left(b+\chi-b\chi\right)\right]\right)\\
&\quad \Big(\chi-2b\chi-2e^{\chi}\chi+4be^{\chi}\chi+e^{2\chi}\chi-2be^{2\chi}\chi-2b\chi^{2}+2be^{\chi}\chi^{2}-b\chi^{3}-e^{\chi}\chi^{3}+be^{\chi}\chi^{3}-\\
&\quad b\left(-1+e^{\chi}\right)\chi\left(2+e^{\chi}\left(-2+\chi\right)+\chi\right)\left(\mathrm{Log}\left[b\left(-1+e^{\chi}\right)\right]-\mathrm{Log}\left[b\left(-1+e^{\chi}-\chi\right)+\chi\right]\right)+\\
&\quad b\left(1-e^{\chi}+\chi\right)^{2}\mathrm{Log}\left[\frac{e^{\chi}\left(b\left(-1+e^{\chi}-\chi\right)+\chi\right)}{-b+e^{\chi}\left(b+\chi-b\chi\right)}\right]-2b\left(-1+e^{\chi}\right)\left(1+e^{\chi}\left(-1+\chi\right)\right)\mathrm{PolyLog}\Big[\\
&\quad 2,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\Big]-2b\left(-1+e^{\chi}\right)\left(-1+e^{\chi}-\chi\right)\mathrm{PolyLog}\left[2,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]-2b\\
&\quad \left(-1+e^{\chi}\right)^{2}\mathrm{PolyLog}\left[3,\frac{\left(-1+b\right)\chi}{b\left(-1+e^{\chi}\right)}\right]+2b\left(-1+e^{\chi}\right)^{2}\mathrm{PolyLog}\left[3,\frac{\left(-1+b\right)e^{\chi}\chi}{b\left(-1+e^{\chi}\right)}\right]\Big)\Bigg)\Bigg)
\end{aligned}
$$

Limit for zero background:

In[16]:=
```
Assuming[pAssumptions, FullSimplify[1 / Limit[1 / σbSQ, b → 0, Direction → "FromAbove"]]]
```

Out[16]=

$$
-\frac{2\chi^{2}\left(2+\chi^{2}-2\,\mathrm{Cosh}[\chi]\right)}{n\left(12+12\chi^{2}+\chi^{4}-\left(16+12\chi^{2}+\chi^{4}\right)\mathrm{Cosh}[\chi]+4\,\mathrm{Cosh}[2\chi]+4\chi^{3}\,\mathrm{Sinh}[\chi]\right)}
$$

Limit for infinite pulse period:

In[17]:=
```
Limit[σbSQ, {χ → ∞}, Assumptions → pAssumptions]
```

Out[17]=

$$
-\frac{\left(-1+b\right)b}{n}
$$