Root finding via Product-Difference algorithm

- QMOM and DQMOM both rely on Gaussian quadrature to provide closure condition. An integral approximated by N-point quadrature is of the form, for $k \geq 1$:

$$k \int L^{k-1} \boldsymbol{f(L)} \boldsymbol{n(L)dL} = k \sum_{i=1}^{N} L_i^{k-1} \boldsymbol{f(L)} w_i$$

- The foundation of quadrature-based closure is that the abscissas L_i and weights w_i can be completely described in terms of the lower-order moments of the unknown distribution function $n(L)$. The moment of the distribution is written:

$$\mu_k = \int L^k \boldsymbol{n(L)dL} = \int L^{k-1}.\boldsymbol{L}.\boldsymbol{n(L)dL} = \sum_{i=1}^{N} L_i^{k-1} \boldsymbol{L} w_i = \sum_{i=1}^{N} L_i^k w_i$$

- The first 2N moments (μ_k with $k = 0$ through $k = 2N - 1$) determined N weight and N abscissas. A naïve approach is to try to solve the non-linear equations, which is not recommended. A much better approach is to deploy an "Product-difference" algorithm which will be showed on the next paragraph.

- The birth of "Product-difference" algorithm has nothing to do with solving population balance model. The goal is to calculate the *integral of an unknown function based on known moments*. Gordon[1] begins the derivation of the PD algorithm by considering the Stieltjes[2] integral.

$$I(z) = \int_0^\infty \frac{d\psi(E)}{z + E}$$

- The integral $I(z)$ is determined based on the moment of function $\psi(E)$ (non-decreasing) which is defined as a nondecreasing distribution within the interval of integration. z is a variable that can be represented by any number, real or imaginary, but it's constant with respect to integration.

$$\mu_k = \int_0^\infty E^k d\psi(E)$$

[1] Roy G. Gordon Error Bounds in Equilibrium Statistical Mechanics

[2] T. J. Stieitjes, Recherches sur les fractions continues, Annales de la Faculte des Sciences de Toulouse

Root finding via Product-Difference algorithm

- By expanding the integrand in $I(z)$ based on finite geometric series, we receive the new integral approximation is a power series with only contain moment

$$I(z) \approx \frac{\mu_0}{z} - \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} - \frac{\mu_3}{z^4} + \cdots + (-1)^N \frac{\mu_N}{z^{N+1}}$$

- As state by Scott and Wall[1], An arbitrary power series $P(x) = \sum_{n=0}^{\infty} c_n x^n$ has a unique corresponding continued fraction of the form:

$$K(x) = c_0 + \cfrac{a_1 x^{\alpha_1}}{1 + \cfrac{a_2 x^{\alpha_2}}{1 + \cfrac{a_3 x^{\alpha_3}}{\cdots}}}$$

- Similarly, integration $I(z)$ has a corresponding continued fraction $C(z)$ since series convergence cannot be achieved in the summated form:

$$C(z) = \cfrac{\alpha_1}{z + \cfrac{\alpha_2}{1 + \cfrac{\alpha_3}{z + \cfrac{\alpha_4}{\cdots}}}}$$

- The constant α_n is defined based on the moment μ_k. Product-difference (PD) algorithm is invented for conveniently evaluating α_n term. An infinite continued fraction $C(z)$ has a mathematical meaning only as a limit.

$$C(z) = \lim_{n\to\infty} C_n(z)$$

- In actual applications, one has available only a finite number of moments, truncated fraction obtained by setting $\alpha_{n+1} = 0, \alpha_{n+2} = 0$. To proceed further, it is preferable to separate the z constant from the continuing fraction in order to isolate the component dependent on the moments. The approximant $C(z)$ is represented as its even contraction $A^e(z)$.

$$A_n(z) = C_{2n}(z)$$

Root finding via Product-Difference algorithm

$$A_{2n}(z) = \cfrac{\alpha_1}{z + \alpha_2 - \cfrac{\alpha_2\alpha_3}{z + \alpha_3 + \alpha_4 - \cfrac{\alpha_4\alpha_5}{z + \alpha_5 + \alpha_6 - \cdots}}}$$

- Multiplying the entire denominator of the right-hand side to the left, allows for the continued fraction to be represented as a composition of two matrices.

$$\left(z + \alpha_2 - \cfrac{\alpha_2\alpha_3}{z + \alpha_3 + \alpha_4 - \cfrac{\alpha_4\alpha_5}{z + \alpha_5 + \alpha_6 - \cdots}}\right) A_{2n}(z) = \alpha_1$$

- $A^e(z)$ is the first component of vector $\mathbf{x}$ which is the solution to the following set of n simultaneous linear equation. The proof of this equation can be found in Gordon's work[1] which is related to Gaussian elimination.

$$(z\mathbf{I} + \mathbf{M}) \cdot \mathbf{x} = \mathbf{e_1}\alpha_1$$

- $\mathbf{I}$ is $n \times n$ unit matrix and $\mathbf{e_1}$ is a unit vector with n rows, $\mathbf{M}$ is $n \times n$ symmetric tridiagonal matrix.

$$M \equiv \begin{pmatrix} \alpha_2 & -(\alpha_2\alpha_3)^{1/2} & 0 & 0 & 0 & \cdots \\ -(\alpha_2\alpha_3)^{1/2} & \alpha_3 + \alpha_4 & -(\alpha_4\alpha_5)^{1/2} & 0 & 0 & \cdots \\ 0 & -(\alpha_4\alpha_5)^{1/2} & \alpha_5 + \alpha_6 & -(\alpha_6\alpha_7)^{1/2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -(\alpha_{2n-2}\alpha_{2n-1})^{1/2} & \alpha_{2n-1} + \alpha_{2n} \end{pmatrix}$$

- The formal solution to these linear equations is:

$$\mathbf{x} = \alpha_1 (z\mathbf{I} + \mathbf{M})^{-1} \mathbf{e_1}$$

[1] Roy G. Gordon Error Bounds in Equilibrium Statistical Mechanics

Root finding via Product-Difference algorithm

- The formal solution to these linear equations is:

$$\mathbf{x} = \alpha_1 (z\mathbf{I} + \mathbf{M})^{-1}\mathbf{e_1}$$

- Perform eigendecomposition on matrix $\mathbf{M}$, we have the formular below. In which, U is square n×n matrix whose ith column is the eigenvector q_i of $\mathbf{M}$, $\mathbf{\Lambda}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, $\mathbf{\Lambda_{ii}} = \lambda_i$

$$\mathbf{M} = \mathbf{U\Lambda U^{-1}} \Rightarrow \mathbf{\Lambda} = \mathbf{U^{-1}MU}$$

- The linear equation becomes:

$$\mathbf{x} = \alpha_1 \mathbf{UU^{-1}}(z\mathbf{I} + \mathbf{M})^{-1}\mathbf{UU^{-1}e_1}$$

(*)

$$x_1 = \sum_j \alpha_1 \left(z + \lambda_j\right)^{-1} \mathrm{u}_{1j}^2$$

- In which, u_{1j} is the 1st element of the vector column jth of matrix $\mathbf{U}$

$$\mathrm{u}_{1j} = (\mathbf{U})_{1j}$$

$$A_n(z) = x_1 = \alpha_1 \sum_{j=1}^{n} \frac{\mathrm{u}_{1j}^2}{(z + \lambda_j)} = \sum_{j=1}^{n} \frac{w_i}{(z + \lambda_j)}$$

- We obtained the relation:

$$A_n(z) = I(z) = \int_0^\infty \frac{d\psi(E)}{z + E} = \sum_{j=1}^{n} \frac{w_i}{(z + \lambda_j)}$$

- The transformation step (*) is tricky, so details will be presented on the next slide.

Transformation step (*)

$$\mathbf{x} = \alpha_1 \mathbf{U}\mathbf{U}^{-1}(z\mathbf{I} + \mathbf{M})^{-1}\mathbf{U}\mathbf{U}^{-1}\mathbf{e_1}$$

- Set matrix $\mathbf{Q} = (z\mathbf{I} + \mathbf{M})$, we obtained:

$$\mathbf{x} = \alpha_1 \mathbf{U}\mathbf{U}^{-1}\mathbf{Q}^{-1}\mathbf{U}\mathbf{U}^{-1}\mathbf{e_1}$$
$$\mathbf{x} = \alpha_1 \mathbf{U}\boldsymbol{\Phi}\mathbf{U}^{-1}\mathbf{e_1} \;(**)$$

- In which, $\boldsymbol{\Phi}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues of matrix $\mathbf{Q}^{-1}$. Furthermore, the eigenvalues of the inverse matrix are equal to the inverse of the eigenvalues of the original matrix (easy to proof). Set φ_i is eigenvalues i^th of matrix **Q,** eigenvalues of matrix $\mathbf{Q}^{-1}$ a.k.a $(z\mathbf{I} + \mathbf{M})$ is φ_i^{-1}.
- Thanks for this Wikipedia pages [Commuting matrices – Wikipedia], we have two important pieces of the puzzle:
  - If **A** and **B** commute, they have a common eigenvector. If **A** has distinct eigenvalues, and **A** and **B** **commute**, then **A**'s eigenvectors are **B**'s eigenvectors. $\Rightarrow$ eigenvalue of $\mathbf{A} + \mathbf{B}$ = eigenvalue of $\mathbf{A}$ + eigenvalue of $\mathbf{B}$
  - The identity matrix commutes with all matrices. $\Rightarrow$ $z\mathbf{I}$ **commutes** with $\mathbf{M}$
  - **Therefore** eigenvalue of $\mathbf{Q}$ = eigenvalue of z**I** + eigenvalue of **M**
- Let's demonstrate them !!!
- For the sake of simplicity, matrix **M** is 3×3 tridiagonal matrix with the form below:

$$\mathbf{M} \equiv \begin{bmatrix} a_1 & b_1 & \\ b_1 & a_2 & b_2 \\ & b_2 & a_3 \end{bmatrix}$$

- Perform eigendecomposition on matrix **M**

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} \equiv \begin{bmatrix} a_1 & b_1 & \\ b_1 & a_2 & b_2 \\ & b_2 & a_3 \end{bmatrix} = \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix}^{-1}$$

Transformation step (*)

- Because $\mathbf{M}$ is asymmetry, $\mathbf{U}$ is guaranteed to be an orthogonal matrix, therefore $\mathbf{U^{-1}} = \mathbf{U^T}$

$$\mathbf{M} = \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{21} & \mathrm{u}_{31} \\ \mathrm{u}_{12} & \mathrm{u}_{22} & \mathrm{u}_{32} \\ \mathrm{u}_{13} & u_{23} & \mathrm{u}_{33} \end{bmatrix}$$

- Perform eigendecomposition on matrix $\mathrm{z}\mathbf{I}$, since it commute with $\mathbf{M}$, it should have the same $\mathbf{U}$

$$\mathrm{z}\mathbf{I} \equiv \mathrm{z} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \mathrm{z}\mathbf{UIU^{-1}} = \mathrm{z} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{21} & \mathrm{u}_{31} \\ \mathrm{u}_{12} & \mathrm{u}_{22} & \mathrm{u}_{32} \\ \mathrm{u}_{13} & u_{23} & \mathrm{u}_{33} \end{bmatrix}$$

- Perform eigendecomposition on matrix $\mathbf{Q} = \mathrm{z}\mathbf{I} + \mathbf{M}$

$$\boldsymbol{Q} = \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} z+\lambda_1 & & \\ & z+\lambda_2 & \\ & & z+\lambda_3 \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{21} & \mathrm{u}_{31} \\ \mathrm{u}_{12} & \mathrm{u}_{22} & \mathrm{u}_{32} \\ \mathrm{u}_{13} & u_{23} & \mathrm{u}_{33} \end{bmatrix}$$

- Perform eigendecomposition on matrix $\mathbf{Q^{-1}} = (\mathrm{z}\mathbf{I} + \mathbf{M})^{-1}$

$$\mathbf{Q^{-1}} = \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} \frac{1}{z+\lambda_1} & & \\ & \frac{1}{z+\lambda_2} & \\ & & \frac{1}{z+\lambda_3} \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{21} & \mathrm{u}_{31} \\ \mathrm{u}_{12} & \mathrm{u}_{22} & \mathrm{u}_{32} \\ \mathrm{u}_{13} & u_{23} & \mathrm{u}_{33} \end{bmatrix}$$

Transformation step (*)

- It's obvious that, eigenvector of matrix $\mathbf{Q}^{-\mathbf{1}}$ is:

$$\mathbf{\Phi} = \begin{bmatrix} (z+\lambda_1)^{-1} & & \\ & (z+\lambda_3)^{-1} & \\ & & (z+\lambda_3)^{-1} \end{bmatrix}$$

- Plug it back to the equation (**) $\mathbf{x} = \alpha_1 \mathbf{U\Phi U^{-1} e_1}$, the equation is written in full matrix form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} (z+\lambda_1)^{-1} & & \\ & (z+\lambda_3)^{-1} & \\ & & (z+\lambda_3)^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{21} & \mathrm{u}_{31} \\ \mathrm{u}_{12} & \mathrm{u}_{22} & \mathrm{u}_{32} \\ \mathrm{u}_{13} & u_{23} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \begin{bmatrix} (z+\lambda_1)^{-1} & & \\ & (z+\lambda_3)^{-1} & \\ & & (z+\lambda_3)^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} \\ \mathrm{u}_{12} \\ \mathrm{u}_{13} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{array}{c} \text{1}^{\text{st}}\text{ eigenvector} \\ \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix} \end{array} \begin{bmatrix} \mathrm{u}_{11}(z+\lambda_1)^{-1} \\ \mathrm{u}_{12}(z+\lambda_3)^{-1} \\ \mathrm{u}_{13}(z+\lambda_3)^{-1} \end{bmatrix}$$

- We are only interesting in x_1, only the first row of the matrix $\mathbf{U}$ is performed the matrix multiplication.

$$x_1 = \mathrm{u}_{11}^2(z+\lambda_1)^{-1} + \mathrm{u}_{12}^2(z+\lambda_1)^{-1} + \mathrm{u}_{13}^2(z+\lambda_1)^{-1}$$

- Let's generalize:

$$x_1 = \sum_j \alpha_1 \left(z+\lambda_j\right)^{-1} \mathrm{u}_{1j}^2$$

Root finding via Product-Difference algorithm

- Follow the slide that was mentioned before, we already obtained:

$$A_n(z) = I(z) = \int_0^\infty \frac{d\psi(E)}{z+E} = \sum_{j=1}^{n} \frac{w_i}{(z+\lambda_j)}$$

- Then we can obtain the generalization in which $(z+E)^{-1}$ is $f(E)$

$$\int_0^\infty f(E)d\psi(E) = \sum_{j=1}^{n} f(\lambda_j)w_j$$

- Let's go back to the moment equation

$$\mu_k = \int_0^\infty E^k d\psi(E) = \sum_{j=1}^{n} \lambda_j^k w_j$$

- The expression above magically becomes a powerful tool to overcome the closure problem of MOM. In the expression, abscissa λ_j is the eigenvalue jth of matrix **M,** which is constructed based on α_n, and α_n is determined in term of μ_k. Weight w_j is the product between $\alpha_1 = \mu_0$ and the square of the first component of eigenvector jth, $U_{1,j}^2$.

- The next slide is the detail procedure to construct the matrix **M**

Product-Difference algorithm

- Problem: Find determined N weight and N abscissas of N-point Gaussian quadrature. We need 2N moments from μ_0 to μ_{2N-1}
- **Part A:** Construct a square matrix **P** with size 2N+1
  - The first column is 0, except $\mathbf{P}(1,1)$
  - The second column contains the moments with alternating sign: $\mathbf{P}(i,2) = (-1)^{i-1}\mu_{i-1}$
  - The element from the column 3 is determined based on the product-difference recursion relation:

$$\mathbf{P}(i,j) = \mathbf{P}(1,j-1)\,\mathbf{P}(i+1,j-2) - \mathbf{P}(1,j-2)\,\mathbf{P}(i+1,j-1)$$

  - The values of the elements in the N^{th} column are calculated based on the elements in the $N-1$ and $N-2$ columns

$$\mathbf{P} \equiv \begin{bmatrix} 1 & \mu_0 & \mu_1 & & & & \\ 0 & -\mu_1 & -\mu_2 & & & & \\ 0 & +\mu_2 & \mu_3 & & & & \\ 0 & -\mu_3 & -\mu_4 & & & & \\ 0 & +\mu_4 & \mu_5 & & & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & (-1)^{i-1}\mu_{i-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{P} \equiv \begin{bmatrix} 1 & \mu_0 & \mu_1 & -\mu_1^2 + \mu_0\mu_2 & & & \\ 0 & -\mu_1 & -\mu_2 & \mu_1\mu_2 - \mu_0\mu_3 & & & \\ 0 & +\mu_2 & \mu_3 & -\mu_1\mu_3 + \mu_0\mu_4 & & & \\ 0 & -\mu_3 & -\mu_4 & \mu_1\mu_4 - \mu_0\mu_5 & & & \\ 0 & +\mu_4 & \mu_5 & -\mu_1\mu_5 + \mu_0\mu_6 & & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & (-1)^{i-1}\mu_{i-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\mu_0 \times 0 - 1 \times \mu_4} \qquad \boxed{\mu_1 \times (-\mu_3) - \mu_0 \times (-\mu_4)}$$

Product-Difference algorithm

- Problem: Find determined N weight and N abscissas of N-point Gaussian quadrature. We need 2N moments from μ_0 to μ_{2N-1}
- **Part B:** Construct a tridiagonal matrix **M** with size $\mathrm{N} \times \mathrm{N}$
  - Calculate the coefficient α_n based on **P** (α_n is the coefficient from continues fraction $A_n(z)$)

$$\alpha_1 = 0 \text{ (follow the work from Marchisio[1])}$$

$$\alpha_i = \frac{\mathrm{P}(1, \mathrm{i}+1)}{\mathrm{P}(1, \mathrm{i})\mathrm{P}(1, \mathrm{i}-1)}, \qquad \mathrm{i} \in 2, \dots, 2\mathrm{N}$$

  - There are 2N coefficient α_n are obtained from 2N+1 elements of the first row of matrix **P**.
  - The Jacobian **M** was then constructed with the form:

$$\mathbf{M} = \begin{pmatrix} \mathrm{a}_1 & \mathrm{b}_1 & 0 & 0 \\ \mathrm{b}_1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \mathrm{b}_{\mathrm{i}-1} \\ 0 & 0 & \mathrm{b}_{\mathrm{i}-1} & \mathrm{a}_\mathrm{i} \end{pmatrix}$$

  - In which:

$$a_i = \alpha_{2i} + \alpha_{2i-1}, \qquad \mathrm{i} \in 1, \dots, 2\mathrm{N}-1$$

$$b_i = -\sqrt{\alpha_{2i+1}\alpha_{2i-1}}, \qquad \mathrm{i} \in 1, \dots, 2\mathrm{N}-2$$

  - In **MATLAB**: $[\mathbf{\Lambda}, \mathbf{Q}] = \mathrm{eig}(\mathbf{M})$, diagonal matrix $\mathbf{\Lambda}$ of eigenvalues and matrix $\mathbf{Q}$ whose columns are the corresponding right eigenvectors
    - Abscissa $\lambda_i = \mathbf{\Lambda}(\mathrm{i}, \mathrm{i})$
    - Weight $w_i = \mu_0 \mathbf{Q}(1, \mathrm{i})$
    - $\mu_0 = 1$ (assuming $n(L)$ is a normalize distribution)

[1] Daniele L. Marchisio - Quadrature method of moments for aggregation–breakage processes