Gaussian Distribution

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I. Multivariate Gaussian Distribution:
Proof that Multivariate Gaussian Distribution is normalize:
First, we have the PDF of the Gaussian Distribution is:

$$p(x \mid \mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{D}{2}} \mid \Sigma \mid^{\frac{1}{2}}} \times e^{\frac{-1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

The Gaussian Distribution is normalize

$$\Longleftrightarrow \int_{-\infty}^{+\infty} p(x \mid \mu, \sigma^2) = 1$$

where μ is a D-dimensional mean vector, Σ is a D x D covariance matrix, and $\mid \Sigma \mid$ denotes the determinant of Σ

Set

$$\Delta^2 = \frac{-1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$$
$$= \frac{-1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu + constant$$

Consider eigenvalues and eigenvvectors of Σ we have:

$$\Sigma u_i = \lambda_i u_i, i = 1, ..., D$$

Because Σ is a real, symmetric matrix, its eigenvalues will be real and its eigenvectors form an orthnormal set.
Proof:
1. its eigenvalues will be real
Example:
$\begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_2^2 \end{pmatrix} \Longleftrightarrow$ The equation to find the eigenvalues is :

$$(\sigma_1^2 - \lambda) \times (\sigma_2^2 - \lambda) - (\sigma_{1,2})^2 = 0$$

$$\Longleftrightarrow (\sigma_1^2 - \lambda) \times (\sigma_2^2 - \lambda) = (\sigma_{1,2})^2$$

$$\Longrightarrow \lambda \text{ must be a real number}$$

With $\lambda = \lambda_1$:

$\begin{pmatrix} \sigma_1^2 - \lambda_1 & cov(\sigma_{1,2}) \\ cov(\sigma_{1,2}) & \sigma_2^2 - \lambda_1 \end{pmatrix}$

$$(\sigma_1^2 - \lambda_1)x_1 + (\sigma_{1,2})x_2 = 0 \tag{1}$$
$$(\sigma_{1,2})x_1 + (\sigma_2^2 - \lambda_1)x_2 = 0 \tag{2}$$

From (1) we have:

$$x_1 = \frac{-y \times cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_1}$$
$$x_2 = x_2$$

So the eigenvector in this case is: $\begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_1} \\ 1 \end{pmatrix}$

With $\lambda = \lambda_2$:

$\begin{pmatrix} \sigma_1^2 - \lambda_2 & cov(\sigma_{1,2}) \\ cov(\sigma_{1,2}) & \sigma_2^2 - \lambda_2 \end{pmatrix}$

$$(\sigma_1^2 - \lambda_2)x_1 + (\sigma_{1,2})x_2 = 0 \tag{3}$$
$$(\sigma_{1,2})x_1 + (\sigma_2^2 - \lambda_2)x_2 = 0 \tag{4}$$

From (3) we have:

$$x_1 = \frac{-y \times cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2}$$
$$x_2 = x_2$$

So the eigenvector in this case is: $\begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2} \\ 1 \end{pmatrix}$

And: $\begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2} \\ 1 \end{pmatrix}^{T} \times \begin{pmatrix} \dfrac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2} \\ 1 \end{pmatrix} = 1$

So its eigenvectors form an orthnormal set.

$$\Sigma = \Sigma_{i=1}^{D} \lambda_i u_i (u_i)^T \longrightarrow \Sigma^{-1} = \Sigma_{i=1}^{D} \frac{1}{\lambda i} u_i u_i^T$$

So that:

$$\Delta^2 = \frac{-1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$$

$$= \sum_{i=1}^{D} \frac{1}{\lambda_i}(x-\mu)^T u_i u_i^T (x-\mu)$$

Let:

$$y_i = u_i^T (x-\mu)$$

$$\longrightarrow \Delta^2 = \sum_{i=1}^{D} \frac{y_i^2}{\lambda_i}$$

$$\mid \Sigma \mid^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$$

Now,we have:

$$p(x \mid \mu, \sigma^2) = p(y) = \prod_{j=1}^{D} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\frac{-(y_j)^2}{2\lambda_j}}$$

$$\Longleftrightarrow \int_{-\infty}^{+\infty} p(y)dy = \prod_{j=1}^{D} \int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\frac{-(y_j)^2}{2\lambda_j}}$$

But in the last homework we have proofed that:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-y^2}{2}} dy = 1$$

So:

$$\int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\frac{-(y_j)^2}{2\lambda_j}} = 1$$

$$\Longleftrightarrow \prod_{j=1}^{D} \int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} e^{\frac{-(y_j)^2}{2\lambda_j}} = 1$$