

Probability Machine Learning

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Exercise1: To evaluate a new test to detect Hansen's disease, a group of 5 percent of those known to have Hansen's disease were tested. The test found Hansen's disease in 98 percent of people with the disease and 3 percent of those who don't. What is the probability that someone who tests positive for Hansen's disease on this new test actually has the disease?

SOLVED:

A: known to have Hansen disease

B: known to have no Hansen disease

H: Positive Hansen disease result

$P(A) = 0.05$

$P(B) = 0.95$

$P(H|A) = 0.98$

$P(H|B) = 0.03$

So the probability that someone who tests positive for Hansen' disease on this new test actually has the disease is:

$$P(A|H) = \frac{P(H|A) \times P(A)}{P(H)}$$

$$P(A|H) = \frac{P(H|A) \times P(A)}{P(H|A) \times P(A) + P(H|B) \times P(B)}$$

$$P(A|H) = \frac{0.98 \times 0.05}{0.98 \times 0.05 + 0.03 \times 0.95}$$

$$P(A|H) = \frac{0.049}{0.0775} \approx 0.632$$

Exercise2: Proof the following distributions are normalized then calculate the mean and standard deviation of these distribution:

1. Univariate normal distribution.
2. (Optional) Multivariate normal distribution.

SOLVED:

1. Univariate normal distribution:

+) Univariate normal distribution have formula:

$$f(x) =$$

$$\text{Proof the Univariate is normalize} \iff \int_{-\infty}^{+\infty} f(x) dx = 1$$

$$\iff \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma^2} \times e^{\frac{(x-\mu)}{-2\sigma^2}} dx = 1$$

$$\iff \frac{1}{\sqrt{2\pi\sigma^2}} \times \int_{-\infty}^{+\infty} e^{\frac{(x-\mu)}{-2\sigma^2}} dx = 1(*)$$

Making the substitution

$$y = \frac{(x - \mu)}{\sigma}$$

, we have:

$$\frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{+\infty} e^{\frac{-y^2}{2}} dy$$

let

$$I = \int_{-\infty}^{+\infty} e^{\frac{-y^2}{2}} dy$$

. Then:

$$I^2 = \int_{-\infty}^{+\infty} e^{\frac{-y^2}{2}} dy \times \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\frac{-(y^2 + x^2)}{2}} dy dx$$

Let

$$x = r \cos \phi, y = r \sin \phi \text{ and } dy dx = r d\theta dr$$

We Have:

$$I^2 = \int_0^{+\infty} \int_0^{2\pi} e^{\frac{-r^2}{2}} r d\phi dr$$

$$\begin{aligned}
&= 2\pi \int_0^{+\infty} r e^{-\frac{r^2}{2}} dr \\
&= -2\pi e^{-\frac{r^2}{2}} \Big|_0^{+\infty} \\
&= 2\pi i
\end{aligned}$$

$$\rightarrow I = \sqrt{2\pi}$$

Substituting in (*) we have :

$$\Leftrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \times \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1(Proofed)$$

Calculatuing Mean: First: We let

$$Z = \frac{(X - \mu)}{\sigma}$$

We have:

$$\begin{aligned}
E[Z] &= \int_{-\infty}^{+\infty} x f_Z(x) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{-\frac{x^2}{2}} dx \\
&= \frac{-1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{+\infty} \\
&= 0
\end{aligned}$$

Because :

$$X = \mu + \sigma Z$$

So:

$$\begin{aligned}
E(X) &= E(\mu) + E(\sigma Z) \\
\Leftrightarrow E(X) &= \mu + E(Z)E(\sigma)
\end{aligned}$$

But: $E(Z) = 0$

$$\rightarrow E(X) = \mu$$

Calculating Variance:

$$Var(Z) = E(Z^2) - ((E(Z))^2 = E(Z^2))$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{\frac{-x^2}{2}} dx$$

We let:

$$u = x, dv = x e^{\frac{-x^2}{2}}$$

$$Var(Z) = \frac{1}{\sqrt{2\pi}} \left(-x e^{\frac{-x^2}{2}} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx$$

$$= 1$$

But:

$$X = \mu + \sigma Z$$

$$\iff Var(X) = Var(\mu) + \sigma^2 Var(Z)$$

$$\rightarrow Var(X) = 0 + \sigma^2 1$$

$$\rightarrow Var(X) = \sigma^2$$