Support vector machines I: Maximum-margin linear classifiers

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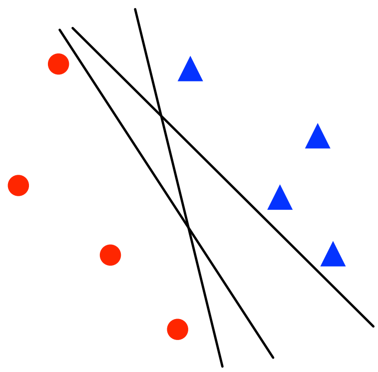
Topics we'll cover

1. The margin of a linear classifier
2. Maximizing the margin
3. A convex optimization problem
4. Support vectors

Improving upon the Perceptron

For a linearly separable data set, there are in general many possible separating hyperplanes, and Perceptron is guaranteed to find one of them.



Is there a better, more systematic choice of separator?
The one with the most buffer around it, for instance?

The learning problem

Given: training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.

Find: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $y^{(i)}(w \cdot x^{(i)} + b) > 0$ for all i.

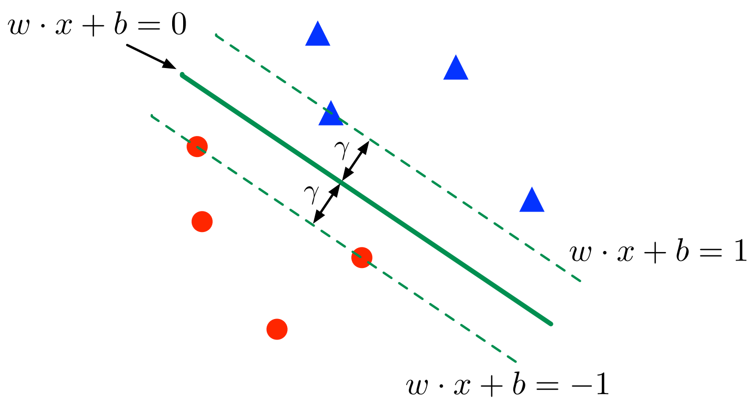
By scaling w, b, can equivalently ask for

$$y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i$$

Maximizing the margin

Given: training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.
Find: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

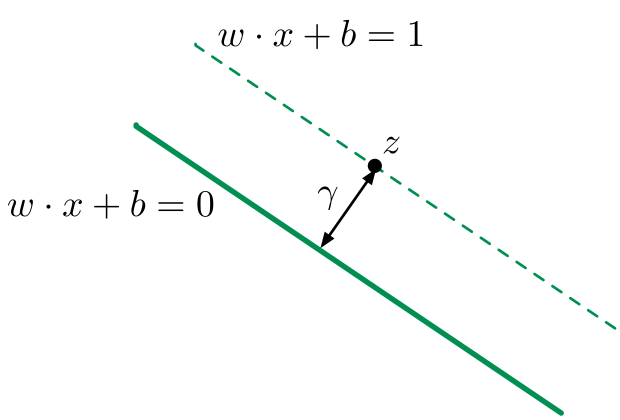
$$y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i.$$



Maximize the **margin** γ.

A formula for the margin

Close-up of a point z on the positive boundary.



A quick calculation shows that $\gamma = 1/\|w\|$.

In short: to maximize the margin, minimize $\|w\|$.

Maximum-margin linear classifier

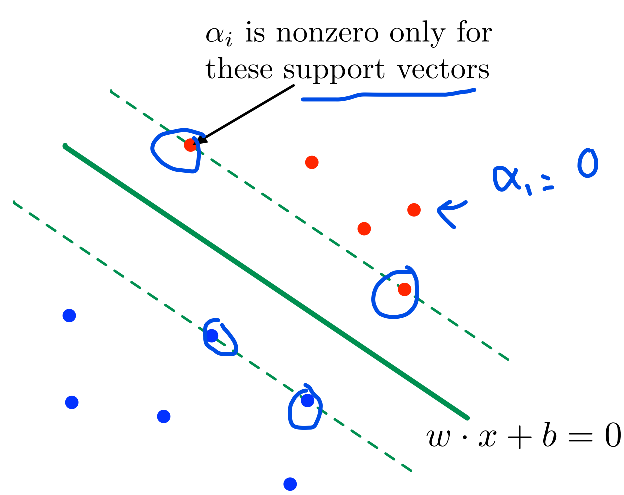
- Given $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$

$$\boxed{\begin{array}{l} \displaystyle\min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \quad \|w\|^2 \\ \text{s.t.: } y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i = 1, 2, \ldots, n \end{array}}$$

- This is a **convex optimization problem**:
  - Convex objective function
  - Linear constraints
- This means that:
  - the optimal solution can be found efficiently
  - **duality** gives us information about the solution

Support vectors

Support vectors: training points right on the margin, i.e. $y^{(i)}(w \cdot x^{(i)} + b) = 1$.



$w = \sum_{i=1}^{n} \alpha_i y^{(i)} x^{(i)}$ is a function of just the support vectors.

$\alpha_i \geq 0$

Small example: Iris data set

Fisher's **iris** data

150 data points from three classes:

- iris setosa
- iris versicolor
- iris virginica

Four measurements: petal width/length, sepal width/length

Small example: Iris data set

Two features: sepal width, petal width.
Two classes: setosa (red circles), versicolor (black triangles)