Linear algebra II
Linear functions and matrix products

Topics we'll cover

1. Linear functions
2. Matrix-vector products
3. Matrix-matrix products

Linear and quadratic functions

In one dimension:

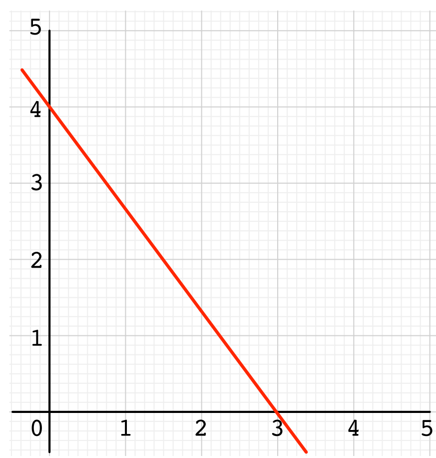
- Linear: $f(x) = 3x + 2$
- Quadratic: $f(x) = 4x^2 - 2x + 6$

In higher dimension, e.g. $x = (x_1, x_2, x_3)$:

- Linear: $3x_1 - 2x_2 + x_3 + 4$
- Quadratic: $x_1^2 - 2x_1x_3 + 6x_2^2 + 7x_1 + 9$

Linear functions and dot products

Linear separator $4x_1 + 3x_2 = 12$:



For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, linear separators are of the form:

$$w_1x_1 + w_2x_2 + \cdots + w_dx_d = c.$$

Can write as $w \cdot x = c$, for $w = (w_1, \ldots, w_d)$.

More general linear functions

A linear function from $\mathbb{R}^4$ to $\mathbb{R}$: $f(x_1, x_2, x_3, x_4) = 3x_1 - 2x_3$

$$(x_1, x_2, x_3, x_4)(3, 0, -2, 0)$$

A linear function from $\mathbb{R}^4$ to $\mathbb{R}^3$: $f(x_1, x_2, x_3, x_4) = (4x_1 - x_2, x_3, -x_1 + 6x_4)$

$$\underset{1 \times 4}{(x_1, x_2, x_3, x_4)} \quad \underset{4 \times 3}{\begin{pmatrix} (4, & 0 &) \\ (-1, & &) \\ (0, & &) \\ (0, & &) \end{pmatrix}}$$

Matrix-vector product

Product of matrix $M \in \mathbb{R}^{r \times d}$ and vector $x \in \mathbb{R}^d$:

The identity matrix

The $d \times d$ **identity matrix** I_d sends each $x \in \mathbb{R}^d$ to itself.

$$I_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Matrix-matrix product

Product of matrix $A \in \mathbb{R}^{r \times k}$ and matrix $B \in \mathbb{R}^{k \times p}$:

Matrix products

If $A \in \mathbb{R}^{r \times k}$ and $B \in \mathbb{R}^{k \times p}$, then AB is an $r \times p$ matrix with (i,j) entry

$$(AB)_{ij} \;=\; (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B) \;=\; \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

- $I_k B = B$ and $A\, I_k = A$
- Can check: $(AB)^T = B^T A^T$
- For two vectors $u, v \in \mathbb{R}^d$, what is $u^T v$?

Some special cases

For vector $x \in \mathbb{R}^d$, what are $x^T x$ and xx^T?

Associative but not commutative

- Multiplying matrices is **not commutative**: in general, $AB \neq BA$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} =$$

- But it is **associative**: $ABCD = (AB)(CD) = (A(BC))D$, etc.

  Example: if $x \in \mathbb{R}^d$ has length 2, what is $x^T x x^T x x^T x x^T x$?