

# Linear algebra IV

## Positive semidefinite matrices

### Topics we'll cover

- ① Positive semidefinite matrices
- ② Properties of PSD matrices
- ③ Checking if a matrix is PSD
- ④ A hierarchy of square matrices

## When is a square matrix “positive”?

- A superficial notion: when all its entries are positive
- A deeper notion: **when the quadratic function defined by it is always positive**

Example:  $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2 \geq 0$$

## Positive semidefinite matrices

Recall: every **square** matrix  $M$  encodes a **quadratic function**:

$$x \mapsto x^T M x = \sum_{i,j=1}^d M_{ij} x_i x_j$$

( $M$  is a  $d \times d$  matrix and  $x$  is a vector in  $\mathbb{R}^d$ )

A symmetric matrix  $M$  is **positive semidefinite (psd)** if:

$$x^T M x \geq 0 \text{ for all vectors } x$$

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We saw that  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  is PSD. What about  $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ ?

A symmetric matrix  $M$  is **positive semidefinite (psd)** if:

$$x^T M x \geq 0 \text{ for all vectors } x$$

When is a diagonal matrix PSD?

A symmetric matrix  $M$  is **positive semidefinite (psd)** if:

$$x^T M x \geq 0 \text{ for all vectors } x$$

If  $M$  is PSD, must  $cM$  be PSD for a constant  $c$ ?

A symmetric matrix  $M$  is **positive semidefinite (psd)** if:

$$x^T M x \geq 0 \text{ for all vectors } x$$

If  $M, N$  are of the same size and PSD, must  $M + N$  be PSD?

## Checking if a matrix is PSD

A matrix  $M$  is PSD if and only if it can be written as  $M = UU^T$  for some matrix  $U$ .

Quick check: say  $U \in \mathbb{R}^{r \times d}$  and  $M = UU^T$ .

- ①  $M$  is square.
- ②  $M$  is symmetric.
- ③ Pick any  $x \in \mathbb{R}^r$ . Then

$$\begin{aligned}x^T M x &= x^T U U^T x = (x^T U)(U^T x) \\&= (U^T x)^T (U^T x) \\&= \|U^T x\|^2 \geq 0.\end{aligned}$$

Another useful fact: any covariance matrix is PSD.

## A hierarchy of square matrices

