Linear algebra III
Square matrices as quadratic functions

Topics we'll cover

1. Square matrices as quadratic functions
2. Special cases of square matrices: symmetric and diagonal
3. Determinant
4. Inverse

A special case

$x : d \times 1$

Recall: For vector $x \in \mathbb{R}^d$, we have $x^T x = \|x\|^2$.

What about $x^T Mx$, for arbitrary $d \times d$ matrix M?

What is $x^T Mx$ for $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$?

$1 \times d \times d \times d \quad d \times 1 = 1 \times 1$

Quadratic functions

Let M be any $d \times d$ (**square**) matrix.
For $x \in \mathbb{R}^d$, the mapping $x \mapsto x^T Mx$ is a **quadratic function** from $\mathbb{R}^d$ to $\mathbb{R}$:

$$x^T Mx = \sum_{i,j=1}^{d} M_{ij} x_i x_j.$$

What is the quadratic function associated with $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}$?

Write the quadratic function $f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2$ using matrices and vectors.

Special cases of square matrices

- **Symmetric**: $M = M^T$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 4 & 6 \end{pmatrix}$$

- **Diagonal**: $M = \text{diag}(m_1, m_2, \ldots, m_d)$

$$\text{diag}(1, 4, 7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Determinant of a square matrix

Determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|A| = ad - bc$.

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

Inverse of a square matrix

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$. Notation: A^{-1}.

Example: if $A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 1/4 \end{pmatrix}$. Check!

Inverse of a square matrix, cont'd

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$. Notation: A^{-1}.

- Not all square matrices have an inverse
- Square matrix A is invertible if and only if $|A| \neq 0$
- What is the inverse of $A = \text{diag}(a_1, \ldots, a_d)$?