Philosophy 31: Symbolic Logic

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Contents

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Logical Language

- in logic, an **argument** is a set of sentences:
  - one of which is designated as a **conclusion**
  - the other sentences are called **premises**
- ex. argument:
  1. All dogs are animals.
  2. Some animals are pigs.
  3. Therefore some dogs are pigs.
  - (1) and (2) are premises, (3) is the conclusion of the argument:
    * argument is often signaled by *indicator* words eg. "therefore", "hence", "thus"
    * while premises are signaled by "because", "since", etc.
- an argument is **valid** if it is impossible for the premises to all be true and the conclusion false:
  - ie. if the premises are all true, then the conclusion must be true
    * the example argument is a *valid* argument!
  - at least, in deductive logic
    * in inductive reasoning, premises *lend support* to a conclusion
  - note that the validity of an argument has to do with its inherent logical structure
    * can replace the parts of the argument uniformly and the argument would remain valid
- an argument is **sound** if it is valid and all of its premises are true:
  - the example argument is valid, but not sound
  - note that a sound argument *must* be valid
- ex. argument:
  1. If John eats pizza he will get thirsty.
  2. If John eats pasta he will get thirsty.
  - valid argument
- ex. argument:
  1. If John eats pizza he will get thirsty.
  2. John got thirsty ∴ John ate pizza.
  - *invalid* argument
- ex. argument:
  1. If John eats pizza he will get thirsty.
  2. If John does not eat pizza he will be hungry.

    3. If John will not get sick he will not be hungry.
    4. ∴ John will get sick.
    - valid argument!
- a set of sentences **implies** a given sentence if and only if the truth of the given sentence is *gauranteed* by the truth of all members of the set
    - ie. sentence A implies another sentence B if and only if A's truth guarantees B's
    - eg.
        1. Peter likes pizza and Patsy likes pasta.
        2. Patsy likes pasta.
        * sentence (1) implies sentence (2)
- sentence A is **equivalent** to sentence B if and only if A and B always agree in truth value
    - eg.
        1. No dogs are cats.
        2. No cats are dogs.
        * equivalent sentence
    - eg.
        1. Either Peter likes pizza or Patsy likes pasta.
        2. Peter likes pizza.
        * not equivalent

Symbolic Language

- to analyze the validity of an argument:
    1. *extract* the logical structure of an argument
        - by translating into a representative **symbolic language**
    2. *generally* analyze that structure

Vocabulary

- vocabulary:
    - sentence letters are $P, Q, R, \ldots, Z$
        * with or without subscripts
    - sentential connectives:
        * $\wedge$ ie. "and", $\vee$ ie. "or", $\rightarrow$ ie. "if-then", $\leftrightarrow$ ie. "if and only if"
        * $\sim$ ie. "it is not the case that"
    - punctuation is parantheses
- ex. Peter loves pizza.
    - this is an **atomic** sentence that cannot be broken up further
- ex. Peter loves pizza because Patsy does.

  - this is a **compound** sentence since it is connected by the **binary connective** “because”
- in logic, we are only concerned with **truth functional connectives**:
  - a connective is truth functional if and only if the truth values of the joined sentences always completely determine the truth value of the compound sentence
  - eg. “and”, “or”, “then”, “if and only if” are binary truth functional connectives
  - eg. “it is not the case” ie. a negation operator is a *unary* truth functional connective
- ex. Peter loves pizza because Patsy makes pizza.
  - if we know the truth value of each component, can we determine the truth value of the compound sentence?
  - no, “because” is *not* a truth functional connective

Grammar

- in a **metalanguage**, sentences talk about a language itself
  - eg. “John” is tall vs. John is tall.
- recursive symbolic language grammar rules:
  1. sentence letter is a symbolic sentence
  2. a symbolic sentence preceded by a $\sim$ is a symbolic sentence
  3. if a binary connective is placed between two symbolic sentences and enclosed in parentheses,the result is a symbolic sentence

  - eg. $P, Q, (P \to Q), {\sim}P, ({\sim}P \leftrightarrow (P \to Q))$ are all symbolic sentences
  - informal conventions:
    1. outermost parentheses may be omitted
    2. conditionals and biconditionals are assumed to *outrank* conjunctions and disjunctions:
       * thus parentheses may be omitted around conjunctions and disjunctions when there is no ambiguity
       * eg. a reduction like $P \vee Q \wedge R$ is ambiguious
    3. allow brackets and braces
    * by convention, we restore parentheses to the left when we have a string of the same connectives
- an **atomic sentence** is a symbolic sentence containing no connectives
- a **molecular sentence** is a symbolic sentence with one more connectives
- a **negation** is any sentence of the form ${\sim}\square$
- a **conditional** is any sentence of the form $(\square \to \bigcirc)$:

  - $\square$ is the **antecedent**
  - $\bigcirc$ is the **consequent**

- a **conjunction** is any sentence of the form $(\square \land \bigcirc)$
- a **disjunction** is any sentence of the form $(\square \lor \bigcirc)$
- a **biconditional** is any sentence of the form $(\square \leftrightarrow \bigcirc)$
- the **scope** of a connective is the connective itself with the components and grouping indicators it links together:
  - ie. what the connective applies to
  - eg. in $((P \land {\sim}Q) \to R)$, $\sim$ has a scope of ${\sim}Q$, $\land$ has a scope of $(P \land {\sim}Q)$, $\to$ has a scope of the entire formula ie. sentence
  - the **main connective** is the connective occurence with the largest scope
    - always ranges over entire formula

Symbolization

Table 1: Truth Table for $\sim P$

$\square$	$\sim\square$
T	F
F	T

Table 2: Truth Table for Binary Connectives

$\square$	$\bigcirc$	$\sim\square$	$\square \rightarrow \bigcirc$	$\square \wedge \bigcirc$	$\square \vee \bigcirc$	$\square \leftrightarrow \bigcirc$
T	T	F	T	T	T	T
T	F	F	F	F	T	F
F	T	T	T	F	T	F
F	F	T	T	F	F	T

- to perform **symbolization** ie. convert English sentences to their symbolic equivalents:
  1. make a scheme of abbreviation
     - in doing so, make all the sentences positive
  2. rewrite the sentences replacing atomic components with sentence letters
  3. group
  4. replace the connectives with symbols
  - watch for stylistic variants and hidden negations in sentences
- ex. If Herbie eats pizza, then Herbie gets sick.
  - equivalent sentences:
    * Provided that Herbie eats pizza, he will get sick.
    * Herbie gets sick if he eats pizza.
  - other equivalent stylistic variants:
    * "if", "provided that", "given that", "in case", "in which case", "assuming that", "on the condition that"
  - P : Herbie eats pizza, Q : Herbie gets sick
  - If P, then Q.
  - $P \rightarrow Q$
- ex. The patient will live only if we operate.
  - P : the patient lives, Q : we operate
  - P only if Q.
  - $P \rightarrow Q$
- ex. Bruce likes Budweiser, Miller, and Heineken.

  - P : Bruce likes Budweiser, Q : Bruce likes Miller, R : Bruce likes Heineken
  - this is known as **telescoping** of conjunctions and disjunctions
  - $(P \wedge Q) \wedge R$
- ex. Peter likes pizza but not pasta.
  - one of several other stylistic variants for “and”:
    * “but”, “although”, “as well”, *commas*
  - $P \wedge \sim Q$
- ex. Peter brings his lunch unless the cafeteria is open.
  - ex. The patient will die unless we operate.
  - “unless” is a stylistic variant for “or”
  - “or” and its variants is considered *inclusive* unless otherwise specified
  - $P \vee Q$
- ex. Neither Peter nor Patsy came to the party.
  - $\ P \wedge\ Q = \ (P \vee Q)$
- ex. If Herbie eats pizza at night or drinks cheap beer, then only if his girlfriend stays with him, will he have nightmares.
  - P : Herbie eats pizza, Q : Herbie drinks beer, R : girlfriend stays, S : Herbie has nightmares
  - $(P \vee Q) \rightarrow (S \rightarrow R)$
- more variations:
  - ex. If Herbie’s girlfriend stays with him, then he will not have nightmares although he drinks cheap beer.
    * $R \rightarrow (\sim S \wedge Q)$
  - ex. If Herbie eats pizza at night, he will have nightmares unless he drinks cheap beer.
    * $P \rightarrow (S \vee Q)$
- ex. You may have ice cream or cotton candy.
  - considered inclusive by default
  - ex. You may have ice cream or cotton candy, but not both.
    * forces an exclusive-or
    * $(P \vee Q) \wedge \sim(P \wedge Q) = P \leftrightarrow\ Q$
- ex. Ruth studies hard unless she’s tired, in which case she doesn’t.
  - “unless” indicates “or”, the comma indicates an “and”, and the “in which case” indicates an “if-then”
  - $(S \vee T) \wedge (T \rightarrow \sim S)$
- ex. If Alfred and Mary are playing dice together, it is the first throw of the game, and Mary is throwing the dice, then she wins the game on the first throw if and only if she throws 7 or 11.
  - P : Alfred plays, Q : Mary plans, R : Alfred and Mary play, S : is first throw
  - T : Mary is throwing, U : Mary wins on first throw, V : Mary throws 7 or 11, W : Mary throws 7, X : Mary throws 11

- $R \wedge S \wedge T \rightarrow (U \leftrightarrow W \vee X)$
  - break down atomic structure when possible
  - note that R is not equivalent to $P \wedge Q$
- ex. Assuming either that logic is difficult or that the text is not readable, Alfred will pass only if he concentrates.
  - P : logic is difficult, Q : text is readable, R : Alfred will pass, Q : Alfred concentrates
  - $(P \vee \sim Q) \rightarrow (R \rightarrow S)$
- ex. Assuming the professor is a Communist, he will sign the oath; but if he is an idealist, he will neither sign the oath nor speak to those who do.
  - $(P \rightarrow Q) \wedge (R \rightarrow \sim(Q \vee S))$
  - ex. Among USC, UCLA, Oregon, and Arizona, exactly two will be in contention in November.
    - need to list out all the possible combinations
    - note that to specify two, we need to "and" those while negating the remaining ones
  - symbolizing an entire argument with truth values in 3:
    - If Herbie eats pizza, then he will get sick.
    - If Herbie does not eat pizza, then he will be hungry.
    - If Herbie will not get sick, then he will not be hungry.
    - Therefore Herbie will get sick.
    - P : Herbie eats pizza, Q : Herbie gets sick, R : Herbie will be hungry
    - note that it is impossible for all the premises to be true and the conclusion false:
      - thus this argument is valid
      - if there were a case where all the premises are true and the conlusion false, the argument would be invalid

Table 3: Truth Values of an Example Argument

P	Q	R	$P \rightarrow Q$	$\sim Q \rightarrow R$	$\sim Q \rightarrow \sim R$	$\therefore Q$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	F	F
F	F	F	T	F	T	F

- categorizing sentences:

- a **tautology** is a symbolic sentence that is always true
- an **impossible** symbolic sentence is always false
- otherwise, the sentence is **contingent**
- if two sentences have the same truth table, they are **logically equivalent**
  * putting a biconditional between two logically equivalent sentences makes a tautology

Derivations

- in a valid argument. if the premises are true. then the conclusion must be true:
  - instead of using large truth tables, we can use **natural deduction** to show that arguments are valid
  - uses **derivations** ie. proofs that are composed of lines, each of which is justified by a rule in our system
    * attempts to prove a conclusion
- a derivation has three columns:
  1. line numbers
  2. formula
  3. justification
- three types of proofs:
  1. direct: "Show: P. ... P."
  2. conditional: "Show: P → Q. P. ... Q."
     - hinges on an assumption P to prove Q
  3. indirect: "Show: P. ~P. ... {contradiction} P."
     - finds an impossibility to show an assumption is wrong
- ex. A direct derivation:
  1. ~~Show~~: Mustard is the murderer.
  2. Scarlet was in the billiard room.
  3. If Scarlet was in the billiard room, then the rope isn't in the study.
  4. So the rope isn't in the study.
  5. Either the rope is in the study or Plum didn't do it.
  6. So Plum didn't do it.
  7. If Plum didn't do it, then Mustard is the murderer.
  8. So Mustard is the murderer.
- ex. A direct derivation for the following symbolic argument:
  - $S, T \vee \sim P, \sim P \rightarrow R, S \rightarrow \sim T, \therefore R$
  1. ~~Show~~: R
  2. S | Premise
  3. $S \rightarrow \sim T$ | Premise
  4. $\sim T$ | 2, 3
  5. $T \vee \sim P$ | Premise
  6. $\sim P$ | 4, 5
  7. $\sim P \rightarrow R$ | Premise
  8. R | 6, 7
  9. 8 is what we want to show.
- ex. A conditional derivation for the following symbolic argument:
  - $Q \rightarrow S, S \rightarrow R, R \rightarrow T, T \rightarrow P, \therefore Q \rightarrow P$

1. ~~Show~~: $Q \to P$
2. Q | Assume
3. $Q \to S$ | Premise
4. S | 2, 3
5. $S \to R$ | Premise
6. R | 4, 5
7. $R \to T$ | Premise
8. T | 6, 7
9. $T \to P$ | Premise
10. P | 8, 9
11. 10, consequent follows

- ex. An indirect derivation for the following symbolic argument:
  - $\sim P \to Q, Q \to R, R \to \sim S, S \vee \sim Q, \therefore P$

1. ~~Show~~: P
2. $\sim P$ | Assumption
3. $\sim P \to Q$ | Premise
4. Q | 2, 3
5. $Q \to R$ | Premise
6. R | 4, 5
7. $R \to \sim S$ | Premise
8. $\sim S$ | 6, 7
9. $S \vee \sim Q$ | Premise
10. $\sim Q$ | 8, 9
11. 4 and 10 contradict

Inference Rules

- 10 **inference rules** will be used to justify lines in our derivations

1. Repetition (R): $\square, \ \therefore \ \square$
   - ie. we can repeat a line in the derivation
2. Double Negation (DN): $\square, \ \therefore \ \sim\sim\square$
   - alternatively, $\sim\sim\square, \ \therefore \ \square$
3. Modus Ponens (MP): $\square \to \bigcirc, \ \square, \ \therefore \ \bigcirc$
   - ie. "method of putting"
4. Modus Tolens (MT): $\square \to \bigcirc, \ \sim\bigcirc, \ \therefore \ \sim\square$
   - ie. "denying the consequence"
5. Simplification (S): $\square \wedge \bigcirc, \ \therefore \ \square$

- alternatively, $\square \wedge \bigcirc, \quad \therefore \bigcirc$

6. Adjunction (Adj): $\square, \quad \bigcirc, \quad \therefore \square \wedge \bigcirc$
7. Modus Tolendo Ponens (MTP): $\square \vee \bigcirc, \quad \sim\square, \quad \therefore \bigcirc$
   - ie. "method of putting by taking away"
   - alternatively, $\square \vee \bigcirc, \quad \sim\bigcirc, \quad \therefore \square$
8. Addition (Add): $\square, \quad \therefore \square \vee \bigcirc$
   - alternatively, $\bigcirc, \quad \therefore \square \vee \bigcirc$
9. Conditional Biconditional (CB): $\square \to \bigcirc, \quad \bigcirc \to \square, \quad \therefore \square \leftrightarrow \bigcirc$
10. Biconditional Conditional (BC): $\square \leftrightarrow \bigcirc, \quad \therefore \square \to \bigcirc, \quad \therefore \bigcirc \to \square$

Formal Derivation Rules

- a **derivation** is a sequence of lines that is built up in order, consisting of any of the following provisions:
  - a **show line** consists of the word "Show" followed by a symbolic sentence
    - need no justifications and can be introduced at any step
  - a **premise** is a symbolic sentence from the given set, justified with the notation "PR"
  - at any step, a line may be introduced if it follows by a rule from sentences on the previous available lines:
    - justified by citing the numbers of previous lines and the rule name
    - an available line is not preceded by a "Show" and not boxed
  - in a **direct derivation**, when a line whose sentence is the same as the closest uncancelled show line:
    - write "DD" following the justification for that line
    - draw a line through "Show"
    - draw a box around all lines below the show line, including the current line
  - as an **assumption for conditional derivation**, when a show line with a conditional sentence is introduced, the following line can be introduced with the antecedent of the conditional and justification "ASS CD"
  - in a **conditional derivation**, when a line whose sentence is the same as the consequent of closest uncancelled show line:
    - write "CD" following the justification for that line
    - draw a line through "Show"
    - draw a box around all lines below the show line, including the current line

- as an **assumption for indirect derivation**, when a show line is introduced, the following line can be introduced with the negation of the sentence on the show line and justification "ASS ID"
- in an **indirect derivation**, when a line whose sentence is the negation of a previous available line is introduced:
  * write "DD" following the justification for that line, with the line number of the contradicted sentence
  * draw a line through "Show"
  * draw a box around all lines below the show line, including the current line

Derivation Strategies

---

1. to get started:
   - to show a conditional, assume the antecedent and show the consequent
   - to show a conjunction, first show one conjunct, then the other, then use Adj. to put them together for a direct derivation
   - to show a biconditional, first show the conditional in one direction and then the other to use CB to put them together for a direct derivation
   - to show anything else, begin an indirect derivation by assuming its negation
2. look for ways to break down available lines using MP, MT, S, MTP, BC
   - may need to use DN on an avialable line first
3. if one of the lines is the negation of the conditional, show the conditional itself to generate a contradiction
4. if one line is the negation of a disjunction, see if you can use Add. with another line to get the disjunction itself to generate a contradiction
5. look for remaining conditionals and disjunctions that have not yet been broken:
   - for any conditionals, show either:
     - the antecedent to use MP to break it down
     - the negation of the consequent to use MT to break it down
   - for any disjunction, show the negation of one of the disjuncts in order to use MTP to break it down

Examples

- ex. Derive $\sim P,\ \ Q \rightarrow P\ \therefore\ \sim Q$.

```
Show ~Q
~P          PR
Q→P         PR
~Q          2 3 MT
            4 DD
```

- ex. Derive $\sim\sim(P \rightarrow Q),\ \ P\ \therefore\ Q$.

```
Show Q
~~(P→Q)     PR
P           PR
P→Q         2 DN
Q           3 4 MP
            5 DD
```

- ex. Derive $P,\ \ R \rightarrow \sim Q,\ \ P \rightarrow Q\ \therefore\ \sim R$.

```
Show ~R
P           PR
R→~Q        PR
P→Q         PR
Q           2 4 MP
~~Q         5 DN
~R          3 5 MT
            7 DD
```

- ex. Derive $P \rightarrow (Q \rightarrow R),\ \ (Q \rightarrow R) \rightarrow S,\ \ \sim S\ \therefore\ \sim P$.

```
Show ~P
P→(Q→R)     PR
(Q→R)→S     PR
~S          PR
~(Q→R)      3 4 MT
~P          2 5 MT
            6 DD
```

- ex. Derive $P \rightarrow Q,\ \ Q \rightarrow R\ \therefore\ P \rightarrow R$.

```
Show P→R
P           ASS CD
Show R
~R          ASS ID
P→Q         PR
Q→R         PR
~Q          4 6 MT
```

```
~P          5 7 MT
P           2 R     # need to repeat line so it becomes available for deriv. rule
            8 9 id
            3 CD
```

Alternatively:

```
Show P→R
P           ASS CD
Show R
~R          ASS ID
P→Q         PR
Q→R         PR
Q           2 5 MP
R           6 7 MP
            8 DD    # mixed derivation since we began with an indirect assumption
            3 DD
```

- ex. Derive $\sim P \to W \ \therefore \ (R \to \sim W) \to (R \to P)$.

```
Show (R→~W)→(R→P)
R→~W        ASS CD
Show R→P
R           ASS CD
Show P
~P          ASS ID
~P → W      PR
W           6 7 MP
~W          2 4 MP
            8 9 ID
            5 CD
            3 CD
```

- ex. Derive $(R \to S) \to P, \ \sim S \to Q \ \therefore \ \sim P \to Q$.

```
Show ~P→Q
~P          ASS CD
Show Q
~Q          ASS ID
(R→S)→P     PR
~S→Q        PR
~~Q         4 6 MT
~(R→S)      2 5 MT
Show R→S
S           7 DN
```

```
            10 CD
            8 9 ID
            3 CD
```

- ex. Derive $P \to (Q \to R),\ \ P \to ({\sim}Q \to R),\ \ {\sim}P \to (Q \to R),\ \ {\sim}P \to ({\sim}Q \to R)\ \therefore\ R.$

```
Show R
~R           ASS ID
P→(Q→R)      PR
P→(~Q→R)     PR
~P→(Q→R)     PR
~P→(~Q→R)    PR
Show P # cannot break down further, try assuming ant. of a remaining conditional
~P           ASS ID
Q→R          5 8 MP
~Q→R         6 8 MP
~Q           2 9 MT
~~Q          2 10 MT
             11 12 ID
Q→R          3 7 MP
~Q→R         4 7 MP
~Q           2 14 MT
~~Q          2 15 MT
             16 17 ID
```

- ex. Derive $(P \to Q) \to (T \to R),\ \ U \to {\sim}R,\ \ {\sim}(S \to P)\ \therefore\ U \to {\sim}T.$

```
Show U→~T
U            ASS CD
Show ~T
T            ASS ID
(P→Q)→(T→R) PR
U→~R         PR
~(S→P)       PR
~R           2 6 MP
Show S→P
S            ASS D
Show P
~P           ASS ID
Show P→Q
P            ASS CD
~P           12 R
             14 15 ID
```

```
T→R         5 13 MP
R           4 17 MP
~R          8 R
            18 19 ID
            11 CD
            7 9 ID
            3 CD
```

- ex. Derive $\therefore\ (P \wedge Q \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R))$.

```
Show (P^Q→R)→(P→(Q→R))
P^Q→R       ASS CD
Show P→(Q→R)
P           ASS CD
Show Q→R
Q           ASS CD
Show R
~R          ASS ID
P^Q         4 6 ADJ
R           2 9 MP
            8 10 ID
            7 CD
            5 CD
            3 CD
```

- ex. Derive $\sim(R \rightarrow T)\ \therefore\ \sim Q \vee T \rightarrow R \wedge \sim Q$.

```
Show ~Q+T→R^~Q
~Q+T        ASS CD
Show R^~Q
~(R→T)      PR
Show R
~R          ASS ID
Show R→T
R           ASS CD
~R          6 R
            8 9 ID
            9 CD
~(R→T)      4 R
            7 15 ID
Show ~Q
~~Q         ASS ID
T           2 18 MTP
Show R→T
```

```
T             19 R
              21 CD
~(R→T)        4 R
              20 23 ID
R^~Q          5 17 ADJ
              25 DD
              3 CD
```

- ex. Derive $(P \vee Q) \wedge {\sim}R,\ \ {\sim}R \to (S \wedge {\sim}P),\ \ Q \to (P \vee T)\ \therefore\ {\sim}T \to U$

```
Show ~T→U
~T            ASS CD
Show U
~U            ASS ID
(P+Q)^~R      PR
~R→S^~P       PR
Q→P+T         PR
P+Q           5 S
~R            5 S
S^~P          6 9 MP
S             10 S
~P            10 S
Q             8 12 MTP
P+T           7 13 MP
P             2 14 MTP
              12 15 ID
              3 CD
```

- ex. Derive $(P \to Q) \wedge (R \to P),\ \ (P \vee R) \wedge {\sim}(Q \wedge R)\ \therefore\ (P \wedge Q) \wedge {\sim}R.$

```
Show P^Q^~R
(P→Q)^(R→P)     PR
(P+R)^~(Q^R)    PR
Show P^Q
Show P
~P              ASS ID
P→Q             2 SL
R→P             2 SR
P+R             3 SL
~(Q^R)          3 SR
~R              6 8 MT
P               9 11 MTP
                6 12 ID
Show Q
```

```
~Q              ASS ID
P->Q            2 SL
~P              15 16 MT
P               5 R
                17 18 ID
P^Q             5 14 ADJ
                20 DD
Show ~R
R               ASS ID
~(Q^R)          3 SR
Q               4 SR
Q^R             25 23 ADJ
                24 26 ID
P^Q^~R          4 22 ADJ
                28 DD
```

- ex. Derive $(P \leftrightarrow {\sim}Q) \rightarrow R \therefore {\sim}R \land P \rightarrow Q$.

```
Show ~R^P->Q
~R^P        ASS CD
Show Q
~Q          ASS ID
(P<->~Q)->R PR
~R          2 SL
P           2 SR
Show P<->~Q
Show P->~Q
~Q          4 R
            10 CD
Show ~Q->P
P           7 R
            13 CD
P<->~Q      9 12 CB
            15 dd
R           5 8 MP
            6 17 ID
            3 CD
```

- ex. Derive ${\sim}Q \rightarrow R,\ \ Q \leftrightarrow {\sim}(Q \land R) \therefore {\sim}R \leftrightarrow Q$.

```
Show ~R<->Q
~Q->R       PR
Q<->~(Q^R)  PR
Show ~R->Q
```

```
~R          ASS CD
Show Q
~Q          ASS ID
R           2 7 MP
~R          5 R
            8 9 ID
            6 CD
Show Q->~R
Q           ASS CD
Show ~R
Q->~(Q^R)   3 BC
~(Q^R)      13 16 MP
Q^R         13 15 ADJ
            17 18 ID
            14 cd
~R<->Q      4 12 CB
            21 DD
```

Theorems

- some useful theorems:
  - $\therefore\ P \to P$
  - $\therefore\ Q \to (P \to Q)$
    - ie. if the consequent is true, the conditional is true
  - $\therefore\ \sim\sim P \to P$
    - together with $\therefore\ P \to \sim\sim P$
  - $\therefore\ \sim P \to (P \to Q)$
    - ie. if antecedent is false, then the conditional is true
  - $\therefore\ P \wedge Q \leftrightarrow Q \wedge P$ AKA the commutivity of conjunction
  - $\therefore\ (P \to Q) \wedge (Q \to R) \to (P \to R)$ AKA the ssociativity of conditional
  - $\therefore\ \sim(P \wedge \sim P)$ AKA law of noncontradiction
  - $\therefore\ \sim(P \to Q) \leftrightarrow P \wedge \sim Q$ AKA negation of conditional (NC):
    - the negation of a conditional is logically equivalent to antecedent and the negation of the consequent
      - can be seen from the truth table of a conditional
    - very powerful, gives a new derivation rule called NC
  - $\therefore\ P \vee Q \leftrightarrow (\sim P \to Q)$ AKA the conditional disjunction (CDJ) rule
    - similar to MTP, shows logical equivalence between a disjunction and a similar conditional
  - $\therefore\ P \vee \sim P$ AKA the law of the excluded middle

  - $\therefore\ \sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$ AKA DeMorgan's laws (DM)
    - together with $\therefore\ \sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$
  - $\therefore\ \sim(P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow \sim Q)$ AKA negation of biconditonal (NB)
- gives us more tools to break down lines:
  - for negations of conditionals, use NC
  - for negations of conjunctions and disjunctions, use DM
  - for negations of biconditonalsm use NB
- ex. Derive $\sim(P \to Q),\ \ \sim Q \to R\ \therefore\ R$ using theorems:

```
Show R
~R          ASS ID
~(P->Q)     PR
~Q->R       PR
~~Q         2 4 MT
P^~Q        3 NC
~Q          6 SR
            5 7 ID
```

- ex. Prove one of DeMorgan's laws, $\therefore\ \sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$:

```
Show ~(P+Q)<->~P^~Q
Show ~(P+Q)->~P^~Q
~(P+Q)      ASS CD
Show ~P^~Q
Show ~P
P           ASS ID
P+Q         6 ADD
~(P+Q)      3 R
            7 8 ID
Show ~Q
Q           ASS ID
P+Q         11 ADD
~(P+Q)      3 R
            12 13 ID
~P^~Q       5 10 ADJ
            15 DD
            4 CD
Show ~P^~Q->~(P+Q)
~P^~Q       ASS CD
Show ~(P+Q)
P+Q         ASS ID
~P          19 SL
Q           21 22 MTP
~Q          19 SR
```

```
            23 24 ID
            20 CD
~(P+Q)<->~P^~Q     2 18 CB
            27 DD
```

- ex. Derive $S \vee R \to Q,\ \ {\sim}(P \vee {\sim}S)\ \therefore\ {\sim}(P \leftrightarrow Q)$ using DeMorgan's:

```
Show ~(P<->Q)
P<->Q       ASS ID
S+R->Q      PR
~(P+~S)     PR
~P^~~S      4 DM
~P          5 SL
~~S         5 SR # could alternatively use ADD to build S+R
Q->P        2 BC
~Q          6 8 MT
~(S+R)      3 9 MT
~S^~R       10 DM
~S          11 SL
            7 12 ID
```

- ex. Derive $\therefore\ P \vee {\sim}P$ using a CDJ theorem:

```
Show P+~P
Show ~P->~P # CDJ proof
~P          ASS CD
            3 DD
P+~P        2 CDJ
            9 DD

Show P+~P   # alternatively, using DM
~(P+~P)     ASS ID
~P^~~P      2 DM
~P          3 SL
~~P         3 SR
            4 5 ID
```

- ex. Derive $R \vee Q,\ \ Q \leftrightarrow (Q \to {\sim}R)\ \therefore\ {\sim}R \leftrightarrow Q$:

```
Show ~R<->Q
R+Q         PR
Q<->(Q->~R) PR
Show ~R->Q
~R          ASS CD
Show Q
```

```
~Q          ASS ID
R           2 7 MTP
~R          5 R
            8 9 ID
            6 CD
Show Q->~R
Q           ASS CD
Show ~R
R           ASS ID
Q->(Q->~R)  3 BC
Q->~R       13 16 MP
~R          13 17 MP
            18 DD
            14 CD
~R<->Q      4 12 CB
            21 DD
```

- ex. Derive $\sim R \vee (P \leftrightarrow \sim S),\ \ \sim(\sim R \to P) \to S\ \therefore\ P \vee S$:

```
Show P+S
~(P+S)      ASS ID
~R+(P<->~S) PR
~(~R->P)->S PR
~P^~S       2 DM
~P          5 SL
~S          5 SR
~~(~R->P)   4 7 MT
~R->P       8 DN
~~R         6 9 MT
P<->~S      3 10 MTP
~S->P       11 BC
P           7 12 MP
            6 13 ID
```

- ex. Derive $\therefore\ (P \wedge R \to Q \vee S) \to (P \to Q) \vee (R \to S)$:

```
Show (P^R->Q+S)->(P->Q)+(R->S)
P^R->Q+S    ASS CD
Show (P->Q)+(R->S)
~((P->Q)+(R->S))    ASS ID
~(P->Q)^~(R->S)     4 DM
~(P->Q)     5 SL
~(R->S)     5 SR
P^~Q        6 NC
```

```
R^~S         7 NC
P            8 SL
~Q           8 SR
R            9 SL
~S           9 SR
P^R          10 12 ADJ
Q+S          2 14 MP
S            11 15 MTP
             13 16 ID
             3 CD
```

- ex. Derive $(P \wedge Q) \to ((R \vee S) \wedge {\sim}(R \wedge S)),\ \ S \to (R \wedge Q) \vee ({\sim}R \wedge {\sim}Q) \vee {\sim}P,\ \ R \wedge Q \to S\ \therefore\ P \to {\sim}Q$:

```
Show P→~Q
P            ASS CD
Show ~Q
Q            ASS ID
P^Q→(R+S)^~(R^S)   PR
S→(R^Q)+(~R^~Q)+~P PR
R^Q→S              PR
P^Q          2 4 ADJ
(R+S)^~(R^S)       5 8 MP
R+S          9 SL
~(R^S)       9 SR
~R+~S        11 DM
Show S
~S           ID
~(R^Q)       7 14 MT
~R+~Q        15 DM
~~Q          4 DN
~R           16 17 MTP
S            10 18 MTP
             14 19 ID
(R^Q)+(~R^~Q)+~P   6 13 MP
~~P          2 DN
(R^Q)+(~R^~Q)      21 22 MTP
Show ~(R^Q)
R^Q          ASS ID
R            25 SL
~~R          16 DN
~S           12 27 MTP
S            13 R
```

```
             28 29 ID
~R^~Q        23 24 MTP
~Q           31 SR
             4 32 ID
             3 CD
```