Homework 2 – Tandon Bridge

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Question 5

A. Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e

   b.

$$\begin{array}{c} p \rightarrow (q \land r) \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

1	$\neg q$	Hypothesis
2	$\neg q \lor \neg r$	Addition, 1
3	$\neg(q \land r)$	De Morgan's Law, 2
4	$p \rightarrow (q \land r)$	Hypothesis
5	$\neg p$	Modus tollens, 3, 4

   e.

$$\begin{array}{c} p \lor q \\ \neg p \lor r \\ \underline{\neg q} \\ \therefore r \end{array}$$

1	p ∨ q	Hypothesis
2	¬p ∨ r	Hypothesis
3	q ∨ r	Resolution, 1, 2
4	¬q	Hypothesis
5	r	Disjunctive syllogism, 3, 4

2. Exercise 1.12.3, section c

   c.

$$\begin{array}{c} p \lor q \\ \underline{\neg p} \\ \therefore q \end{array}$$

1	p ∨ q	Hypothesis
2	¬p → q	Conditional Laws, 1
3	¬p	Hypothesis
4	q	Modus ponens, 2, 3

3. Exercise 1.12.5, section c, d

   c.

j: I get a new job
c: I will buy a new car
h: I will buy a new house

$$(c \wedge h) \rightarrow j$$
$$\underline{\neg j}$$
$$\therefore \neg c$$

c	h	j	$\neg c$	$\neg j$	$(c \wedge h) \rightarrow j$
T	T	T	F	F	T
T	F	T	F	F	T
T	T	F	F	T	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	T	F	T
F	T	F	T	T	T
F	F	F	T	T	T

The argument is invalid as there is a case where the hypotheses, $(c \wedge h) \rightarrow j$ and $\neg j$ are true, and the conclusion, $\neg c$, is false.

d.

j: I get a new job

c: I will buy a new car

h: I will buy a new house

$$(c \wedge h) \rightarrow j$$
$$\neg j$$
$$\underline{h}$$
$$\therefore \neg c$$

c	h	j	$\neg c$	$\neg j$	$c \wedge h$	$(c \wedge h) \rightarrow j$
T	T	T	F	F	T	T
T	F	T	F	F	F	T
T	T	F	F	T	T	F
T	F	F	F	T	F	T
F	T	T	T	F	F	T
F	F	T	T	F	F	T
F	T	F	T	T	F	T
F	F	F	T	T	F	T

The argument is valid as there is a case where the hypotheses, $(c \wedge h) \rightarrow j$, $\neg j$, and h are true, and the conclusion, $\neg c$, is true.

1.	$(c \wedge h) \rightarrow j$	Hypothesis
2.	$\neg j$	Hypothesis
3.	$\neg(c \wedge h)$	Modus tollens, 1, 2

4.	$\neg c \lor \neg h$	De Morgan's Laws, 3
5.	h	Hypothesis
6.	$\neg c$	Disjunctive syllogism, 4, 5

B. Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

   b.

$$\exists x \big(P(x) \lor Q(x)\big)$$

$$\underline{\exists x \neg Q(x)}$$

$$\therefore \exists x P(\mathrm{x})$$

	P	Q	P ∨ Q	¬Q
a	F	T	T	F
b	F	F	F	T

The argument above is invalid. Under the values of P and Q given in the table, ∃x (P(x) ∨ Q(x)) is true as Q(a) is True. Additionally, ∃x ¬Q(x) is True as ¬Q(b) = True. However, ∃x P(x) is False as neither P(a) nor P(b) is True.

2. Exercise 1.13.5, section d, e

   d.

   M(x): Student missed class

   D(x): Student had detention

$$\forall x \big(M(x) \rightarrow D(x)\big)$$

$$Penelope\ is\ a\ student\ in\ the\ class$$

$$\underline{\neg M(Penelope)}$$

$$\therefore \neg D(Penelope)$$

$M(x)$	$D(x)$	$\neg M(x)$	$\neg D(x)$	$M(x) \rightarrow D(x)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The argument is invalid as if all of the hypotheses, $\forall x(M(x) \to D(x))$, $Penelope\ is\ a\ student\ in\ the\ class$, and $\neg M(Penelope)$, the conclusion, $\neg D(Penelope)$, can be false.

e.

M(x): Student missed class

D(x): Student had detention

A(x): Student had an A in the class

$$\forall x\left(\left(M(x) \vee D(x)\right) \to \neg A(x)\right)$$

$$Penelope\ is\ a\ student\ in\ the\ class$$

$$\underline{A(Penelope)}$$

$$\neg D(Penelope)$$

$M(x)$	$D(x)$	$A(x)$	$\neg D(x)$	$\neg A(x)$	$M(x) \vee D(x)$	$\left(M(x) \vee D(x)\right) \to \neg A(x)$
T	T	T	F	F	T	F
T	F	T	T	F	T	F
T	T	F	F	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	F
F	F	T	T	F	F	T
F	T	F	F	T	T	T
F	F	F	T	T	F	T

The argument is valid as there is no case where the hypotheses, $\forall x\left(\left(M(x) \vee D(x)\right) \to \neg A(x)\right)$, $Penelope\ is\ a\ student\ in\ the\ class$, and $A(Penelope)$, are true and the conclusion, $\neg D(x)$ is false.

1.	$\forall x\left(\left(M(x) \vee D(x)\right) \to \neg A(x)\right)$	Hypothesis
2.	$Penelope\ is\ a\ student\ in\ the\ class$	Hypothesis
3.	$\left(M(Penelope) \vee D(Penelope)\right) \to \neg A(Penelope)$	Universal instantiation, 1, 2
4.	$\neg\left(M(Penelope) \vee D(Penelope)\right) \vee \neg A(Penelope)$	Conditional Laws, 3
5.	$\left(\neg M(Penelope) \wedge \neg D(Penelope)\right) \vee \neg A(Penelope)$	De Morgan's Laws, 4
6.	$A(Penelope)$	Hypothesis
7.	$\neg M(Penelope) \wedge \neg D(Penelope)$	Disjunctive syllogism, 5, 6
8.	$\neg D(Penelope)$	Simplification, 7

Question 6

Exercise 2.4.1

d.

Theorem: The product of two odd integers is an odd integer.

Proof:

Let x and y be odd integers. We shall prove that the product of xy is equal to an odd integer.

Since x is odd, there is an integer k such that $x = 2k + 1$. Since y is odd, there is an integer j such that $y = 2j + 1$.

$xy = (2k + 1)(2j + 1)$

$xy = 4kj + 2k + 2j + 1$

$xy = 2(2kj + k + j) + 1$

Since k and j are integers, then kj is also an integer

Since $xy = 2m + 1$, where $m = 2kj + k + j$ is an integer, xy is odd. ■

Exercise 2.4.3

b.

Theorem: If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$

Proof:

Assume x is a real number and $x \leq 3$. We shall prove that $12 - 7x + x^2 \geq 0$.

Since $x \leq 3$, then $x - 3 \leq 0$

Since $x - 3 \leq 0$, $x - 4 < 0$

Since $x - 3$ and $x - 4$ are either less than or equal to zero or less than zero, respectively, they are both negative numbers or one is a negative number and the other is zero.

Since when two negative numbers are multiplied together it makes a positive number and since when we multiply any number by zero it makes zero, if we multiply $(x - 4)(x - 3)$ together, it will be ≥ 0

Since $(x - 4)(x - 3) \geq 0$, we can multiply it out to $12 - 7x + x^2 \geq 0$

Therefore, if x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$. ■

Question 7

Exercise 2.5.1, section d

d.

Theorem: For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Contrapositive: For every integer n, if n is even, then $n^2 - 2n + 7$ is odd.

Proof:

Let n be an integer, such that n is even

We will prove that $n^2 - 2n + 7$ is odd

Since n is even, $n = 2k$

Plugging $n = 2k$ into $n^2 - 2n + 7$ results in $(2k)^2 - 2(2k) + 7$

$(2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7 = 2(2k^2 - 2k + 3) + 1$

Therefore, $n^2 - 2n + 7$ is odd as $2m + 1$ signifies an odd number and $m = 2k^2 - 2k + 3$ given that n is even. ■

Exercise 2.5.4, sections a, b

a.

Theorem: For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Contrapositive: For every pair of real numbers x and y, if $x > y$, then $x^3 + xy^2 > x^2y + y^3$.

Proof:

Assume that $x > y$, where x and y are real numbers.

We will prove that $x^3 + xy^2 > x^2y + y^3$.

Since $x > y$, $x(x^2 + y^2) > y(x^2 + y^2)$

This means $x^3 + xy^2 > x^2y + y^3$

Therefore, we can conclude that $x^3 + xy^2 > x^2y + y^3$. ■

b.

Theorem: For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Contrapositive: For every pair of real numbers x and y, if $x \leq 10$ and $y \leq 10$ then $x + y \leq 20$.

Proof: Assume that $x \leq 10$ and $y \leq 10$, where x and y are real numbers.

We will prove that $x + y \leq 20$.

Since $y \leq 10, x + y \leq x + 10$

$x + y - 10 \leq x$ and since $x \leq 10$

$x + y - 10 \leq 10$

Therefore, this implies that $x + y \leq 20$, which makes the contrapositive and the theorem true. ■

Exercise 2.5.5, section c

c.

Theorem: For every real number x where $x \neq 0$, if x is irrational, then $\frac{1}{x}$ is also irrational.

Contrapositive: For every real number x where $x \neq 0$, if $\frac{1}{x}$ is also rational, then x is rational

Proof:

Assume $\frac{1}{x}$ is rational, where x is a non-zero real number.

We will prove that x is rational.

There is a number b, where $b = \frac{1}{x}$

We will plug b into $\frac{1}{x}$

$\frac{1}{b} = \frac{1}{\frac{1}{x}} = x$

Therefore, x is also a rational number. ■

Question 8

Exercise 2.6.6

c.

Theorem: The average of three real numbers—x, y, and z—is greater than or equal to at least one of the numbers.

Contradiction: The average of three real numbers—x, y, and z—is less than or equal to all three numbers.

Proof:

Assume $\frac{x+y+z}{3} < x$, $\frac{x+y+z}{3} < y$, and $\frac{x+y+z}{3} < z$, where x, y, and z are all real numbers.

Since $\frac{x+y+z}{3} < x$, y, and z, $3\left(\frac{x+y+z}{3}\right) < x + y + z$.

$x + y + z < x + y + z$, which cannot be.

Therefore, since $x + y + z$ cannot be less than itself, the statement that x, y, and z can all three be larger than the average of x, y, and z is false. ■

d.

Theorem: There is no smallest integer.

Contradiction: There is a smallest integer.

Proof.

Assume there is a smallest integer r, where $\neg\exists s(s \leq r)$

If we divide r by 2, we get $\frac{r}{2}$

$\frac{r}{2}$ would be smaller than r, and would fulfill the existence of s. This can't be true, because we already established that r was the smallest integer. Thus, we have established a contradiction and we must conclude that the assumption that there exists a smallest integer, is false. ■

Question 9

Exercise 2.7.2, section b

Theorem: If integers x and y have the same parity, then $x + y$ is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are both either even or both odd.

Proof:

Case 1: x and y are both even. Since x is even, $x = 2k$ for some integer k. Since y is even, $y = 2j$ for some integer j. $x + y = 2k + 2j = 2(\mathrm{k} + \mathrm{j})$. Since $x + y$ is equivalent to 2 times an integer, we know that it must be even.

Case 2: x and y are both odd. Since x is odd, $x = 2k + 1$ for some integer k. Since y is odd, $y = 2j + 1$ for some integer j. $x + y = 2k + 1 + 2j + 1 = 2k + 2j + 2 = 2(k + j + 1)$. Since $x + y$ is equivalent to 2 times an integer, we know that it must be even. ■