Discrete and continuous probability distributions

Discrete

Bernoulli

$X \sim \text{Bernoulli}(p)$, probability of success in one Bernoulli trial.

$$P[X = k] = p[X = 1] + (1 - p)[X = 0]$$
$$E[X] = 1 \times p + 0 \times (1 - p) = p$$
$$\text{Var}(X) = p(1 - p)$$

Binomial

$X \sim \text{Binomial}(n, p)$, number of success in n independent Bernoulli trials with probability p.

$$P[X = k] = \tbinom{n}{k} p^k (1 - p)^{n-k} \qquad \forall k \in \{0, 1, \dots, n\}$$

$$\begin{aligned} E[X] &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1 - p)^{n-k} \\ &= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1 - p)^{n-k} \\ &= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1 - p)^{n-k} \\ &= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1 - p)^{n-k-1} \\ &= np \end{aligned}$$

$$\text{Var}(X) = np(1 - p)$$

Geometric

$X \sim \text{Geometric}(p)$, number of Bernoulli trials until one success (counting the one that succeed). p must be nonzero.

$$P[X = k] = p(1 - p)^{k-1} \qquad \forall k \in \{1, 2, \dots\}$$

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kp(1-p)^{k-1} \\
&= p\sum_{k=0}^{\infty} k(1-p)^{k-1} \\
&= p\sum_{k=0}^{\infty} -\frac{d}{dp}(1-p)^{k} \\
&= -p \times \frac{d}{dp}\sum_{k=0}^{\infty}(1-p)^{k} \\
&= -p \times \frac{d}{dp}\frac{1}{p} \\
&= p \times \frac{1}{p^2} = \frac{1}{p}
\end{aligned}
$$

$$\mathrm{Var}(X) = \tfrac{1-p}{p^2}$$

Poisson

$X \sim \mathrm{Poisson}(\lambda)$,

$$P[X = k] = \tfrac{\lambda^k e^{-\lambda}}{k!} \qquad \forall k \in \{0, 1, \ldots\}$$

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\lambda\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= e^{-\lambda}\lambda\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda
\end{aligned}
$$

$$\mathrm{Var}(X) = \lambda$$

Continuous

Uniform

$X \sim \text{Uniform}(a, b)$,

$$f(x) = \tfrac{1}{b-a} \qquad \forall x \in [a, b]$$

$$\begin{aligned} E[X] &= \int_a^b \frac{x}{b-a} dx \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

$$\text{Var}(X) = \tfrac{(b-a)^2}{12}$$

Exponential

$X \sim \text{Exponential}(\lambda)$,

$$f(x) = \lambda e^{-\lambda x} \qquad \forall x \in [0, +\infty)$$

$$\begin{aligned} E[X] &= \int_0^{+\infty} x \lambda e^{-\lambda x} dx \\ &= -x e^{-\lambda x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} dx \\ &= 0 + \frac{-1}{\lambda}(0 - 1) \\ &= \frac{1}{\lambda} \end{aligned}$$

$$\text{Var}(X) = \tfrac{1}{\lambda^2}$$

Normal

Theorem 1.

$$\iint_A f(x, y)\, dx\, dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Lemma 1.

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof. Let $I = \int_{-\infty}^{+\infty} e^{-x^2} dx$, then

$$\begin{aligned}
I^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} \, dx \, dy \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \, dx \, dy \\
&= \int_{0}^{2\pi} \int_{0}^{+\infty} e^{-r^2} r \, dr \, d\theta && \text{by Theorem 1} \\
&= 2\pi \int_{0}^{+\infty} e^{-r^2} r \, dr \\
&= \pi \int_{0}^{+\infty} e^{-u} \, du \\
&= \pi
\end{aligned}$$

□

$X \sim \text{Normal}(\sigma, \mu)$,

$$f(x) = \tfrac{1}{\sigma\sqrt{2\pi}} \exp\left(\tfrac{-1}{2}\left(\tfrac{x-\mu}{\sigma}\right)^2\right) \qquad \forall x \in (-\infty, +\infty)$$

$$\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x f(x) dx \\
&= \int_{-\infty}^{+\infty} \frac{u}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{u}{\sigma}\right)^2\right) du + \int_{-\infty}^{+\infty} \frac{\mu}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{u}{\sigma}\right)^2\right) du \\
&= 0 + \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} dv && \text{odd function} \\
&= \mu && \text{by Lemma 1}
\end{aligned}$$

$$\text{Var}(X) = \sigma^2$$