Discrete and continuous probability distributions

Discrete

Bernoulli

$X \sim \text{Bernoulli}(p)$, probability of success in one Bernoulli trial.

$$\begin{gathered} P[X = k] = p[X = 1] + (1 - p)[X = 0] \\ E[X] = 1 \times p + 0 \times (1 - p) = p \\ \text{Var}(X) = p(1 - p) \end{gathered}$$

Binomial

$X \sim \text{Binomial}(n, p)$, number of success in n independent Bernoulli trials with probability p.

$$P[X = k] = \tbinom{n}{k} p^k (1 - p)^{n-k} \qquad \forall k \in \{0, 1, \dots, n\}$$

$$\begin{aligned} E[X] &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1 - p)^{n-k} \\ &= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1 - p)^{n-k} \\ &= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1 - p)^{n-k} \\ &= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1 - p)^{n-k-1} \\ &= np \end{aligned}$$

$$\text{Var}(X) = np(1 - p)$$

Geometric

$X \sim \text{Geometric}(p)$, number of Bernoulli trials until one success (counting the one that succeed). p must be nonzero.

$$P[X = k] = p(1 - p)^{k-1} \qquad \forall k \in \{1, 2, \dots\}$$

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kp(1-p)^{k-1} \\
&= p\sum_{k=0}^{\infty} k(1-p)^{k-1} \\
&= p\sum_{k=0}^{\infty} -\frac{d}{dp}(1-p)^{k} \\
&= -p \times \frac{d}{dp}\sum_{k=0}^{\infty}(1-p)^{k} \\
&= -p \times \frac{d}{dp}\frac{1}{p} \\
&= p \times \frac{1}{p^2} = \frac{1}{p}
\end{aligned}$$

$$\mathrm{Var}(X) = \tfrac{1-p}{p^2}$$

Poisson

$X \sim \mathrm{Poisson}(\lambda)$,

$$P[X=k] = \tfrac{\lambda^k e^{-\lambda}}{k!} \qquad \forall k \in \{0, 1, \ldots\}$$

$$\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\lambda\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= e^{-\lambda}\lambda\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda
\end{aligned}$$

$$\mathrm{Var}(X) = \lambda$$