Discrete and continuous probability distributions

Discrete

Bernoulli

$X \sim \text{Bernoulli}(p)$, probability of success in one Bernoulli trial.

$$P[X = k] = p[X = 1] + (1-p)[X = 0]$$
$$E[X] = 1 \times p + 0 \times (1-p) = p$$
$$\text{Var}(X) = p(1-p)$$

Binomial

$X \sim \text{Binomial}(n, p)$, number of success in n independent Bernoulli trials with probability p.

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \qquad \forall k \in \{0, 1, \ldots, n\}$$

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1} \\
&= np
\end{aligned}
$$

$$\text{Var}(X) = np(1-p)$$

Geometric

$X \sim \text{Geometric}(p)$, number of Bernoulli trials until one success (counting the one that succeed). p must be nonzero.

$$P[X = k] = p(1-p)^{k-1} \qquad \forall k \in \{1, 2, \ldots\}$$

$$E[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1}$$

$$= p \sum_{k=0}^{\infty} k(1-p)^{k-1}$$

$$= p \sum_{k=0}^{\infty} -\frac{d}{dp}(1-p)^{k}$$

$$= -p \times \frac{d}{dp} \sum_{k=0}^{\infty} (1-p)^{k}$$

$$= -p \times \frac{d}{dp} \frac{1}{p}$$

$$= p \times \frac{1}{p^2} = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1-p}{p^2}$$

Poisson

$X \sim \text{Poisson}(\lambda),$

$$P[X = k] = \frac{\lambda^k e^{-\lambda}}{k!} \qquad \forall k \in \{0, 1, \dots\}$$

$$E[X] = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda$$

$$\text{Var}(X) = \lambda$$

Continuous

Uniform

$X \sim \text{Uniform}(a, b)$,

$$f(x) = \frac{1}{b-a} \qquad \forall x \in [a, b]$$

$$E[X] = \int_a^b \frac{x}{b-a} dx$$
$$= \frac{b^2 - a^2}{2(b-a)}$$
$$= \frac{b+a}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential

$X \sim \text{Exponential}(\lambda)$,

$$f(x) = \lambda e^{-\lambda x} \qquad \forall x \in [0, +\infty)$$

$$E[X] = \int_0^{+\infty} x \lambda e^{-\lambda x} dx$$
$$= -xe^{-\lambda x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} dx$$
$$= 0 + \frac{-1}{\lambda}(0 - 1)$$
$$= \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Normal

Theorem 1.

$$\iint_A f(x, y) \, dx \, dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv$$

Lemma 1.

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof. Let $I = \int_{-\infty}^{+\infty} e^{-x^2} dx$, then

$$I^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} \, dx \, dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \, dx \, dy$$

$$= \int_{0}^{2\pi} \int_{0}^{+\infty} e^{-r^2} r \, dr \, d\theta$$

$$= 2\pi \int_{0}^{+\infty} e^{-r^2} r \, dr$$

$$= \pi \int_{0}^{+\infty} e^{-u} \, du$$

$$= \pi$$

$\square$

$X \sim \text{Normal}(\sigma, \mu),$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \qquad \forall x \in (-\infty, +\infty)$$

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

$$= \int_{-\infty}^{+\infty} \frac{u}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{u}{\sigma}\right)^2\right) du + \int_{-\infty}^{+\infty} \frac{\mu}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{u}{\sigma}\right)^2\right) du$$

$$= 0 + \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} dv \qquad\qquad \text{odd function}$$

$$= \mu \qquad\qquad \text{by Theorem 1}$$

$$\text{Var}(X) = \sigma^2$$