Lecture Module - Ordinary Differential Equations

ME3001 - Mechanical Engineering Analysis

Mechanical Engineering

Tennessee Technological University

Module 4 - Ordinary Differential Equations

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- Topic 1 - Review and Classification
- Topic 2 - Analytical Solution Techniques
- Topic 3 - Numerical Solution Techniques
- Topic 4 - ...

Topic 1 - Review and Classification

- What is a Differential Equation?
- Standard Form of an ODE
- Classification
- Examples

What is a Differential Equation?

Definition:

A **differential equation** is an equation which describes a function and one or more of its ________________________ of the

________________________ ________________________

with respect to the ________________________.

Standard Form of an ODE

Ordinary Differential Equations are written in the following form.

$$a_n \frac{dy^{(n)}}{d^{(n)}x} + a_{n-1} \frac{dy^{(n-1)}}{d^{(n-1)}x} + ... + a_2 \frac{dy^2}{d^2x} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

The apostrophe is commonly used for the derivative.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_2 y'' + a_1 y' + a_0 y = f(x)$$

If time is the independent variable the equation changes slightly.

Is the differential equation ordinary or partial?

An **ordinary** differential equation has ___________ independent variable and ___________ dependent variable.

A **partial** differential equation has ____________________________ independent variable ___________ dependent variable.

What is the order of the equation?

The **order** of a differential equation is the

________________________ ________________________

present in the equation.

What is the degree of the equation?

The **degree** of a differential equation is the ________________ of its highest derivative, after the equation has been made rational and integral in all of its derivatives.

Is the differential equation linear or non-linear?

An ordinary differential equation is ________________ if the following statements are true.

1. *The dependent variable and its derivatives are of the first degree.*
2. *The coefficients are constants or dependent on the independent variable.*

If either rule is broken, the equation is _____-________________.

Examples

Differential equations are used to describe physical systems in many areas of engineering. An equation that represents a physical (or theoretical) system is known as a ______________________________ ______________________________.

- Solid Mechanics
- Kinematics and Dynamics
- Heat Transfer and Thermodynamics
- Fluid Mechanics

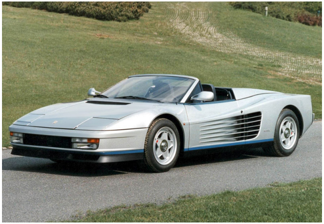
Examples

Newton's Second Law

$\Sigma F = ma$

leads to an *equation of motion*.

$\dot{y} + \frac{c}{m} y = f(t)$



Examples

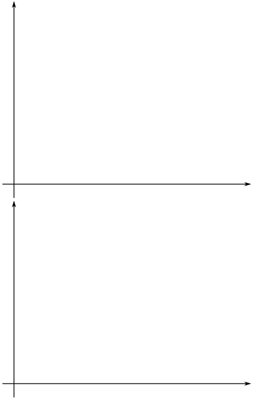
The **solution** to a differential equation describes the

____

____________________ as a function

of the ____________________

____________________.



There are many different methods for finding the solution

Topic 2 - Analytical Solution Techniques

- Analytical vs Numerical Methods
- Separation of Variables
- Trial Solution Method
- Soluition Cases

Analytical vs Numerical Methods

Analytical vs Numerical Methods

Analytical vs Numerical Methods

Separation of Variables

Separation of Variables

Trial Solution Method

Trial Solution Method

Soluition Cases

Soluition Cases

Topic 3 - Numerical Solution Techniques

- Review and Motivation
- Analytical vs Numerical Methods
- Euler's Forward Integration
- Example Problem

What is a Differential Equation? Solution?

A **differential equation** is an equation which describes a function and one or more of its ______________________ of the ______________________ ______________________ with respect to the ______________________.

The **solution** to a differential equation describes the ______________________ ______________________ as a function of the ______________________ ______________________.

Review and Motivation

Analytical vs. Numerical Solutions

Analytical

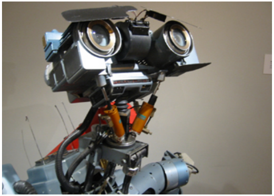
- solution to a problem that can be written in **closed form**
- solution in terms of known functions, constants, etc.
- gives an **exact answer**

Numerical

- an **approximation** to the solution of a mathematical equation
- known as **numerical integration**
- numerical integration is more than *the computation of integrals*

Which one should you choose?

? ? ? ? ?

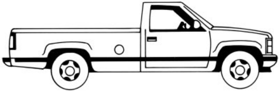


It depends on the problem. It also depends on how you intend to use the solution.

The Initial Value Problem

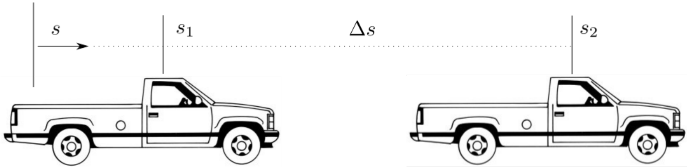
You learned about the **initial value problem** in differential equations class. Do you remember?

You have probably been thinking about this idea for much longer than that. Consider riding in a *truck* waiting to arrive at you destination...



Integrating a Rate

You may not have known it but you where **integrating** when performing these mental calculations. You can math.



The Taylor Series

James Gregory (1638-1675)

Brook Taylor (1685-1731)

Consider the Taylor Series. How does this apply to our problem?

$$y(x) \approx$$

$$y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \frac{y^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{y^{(n)}(a)}{n!}(x-a)^n$$

What does this even mean?

Euler's Method

Given a *function describing the slope* and an *initial condition*, discretized values of the solution can be approximated.

This is known as **Euler's method**.

$$y(x+\Delta x) = y(x) + \frac{dy}{dx}\Delta x = y(x) + f(x, y(x))\Delta x$$

It is commonly shown with subscript notation.

$$\boxed{y(x_{i+1}) = y(x_i) + f(x_i, y(x_i)) = y_i + f(x_i, y_i)\Delta x}$$

Careful: This is not the same as Euler's formula which is an essential trigonometric identity also used in differential equations.

The Slope Function

The differential equation must be written as a function describing the first derivative or **the slope** of the dependent variable.

$$f(x, y) = \frac{rise}{run} = \frac{dy}{dx} \neq y(x)$$

or with subscript notation shown below

$$f(x_i, y_i)$$

Careful: The first argument x is not always used and is often left out. However it is an important placeholder (ODE45()) and shows this method can be used for *non-linear* equations with generalized input functions.

Forward Integration

Using this concept to solve the initial value problem is called **Euler's forward integration** or **Euler's Method**. Most of the time, the independent variable is ________________.

Compute the values of the solution one-by-one **forward in time**.

$$\underline{y(t_{i+1}) = y(t_i) + f(t_i, y_i)\Delta t}$$

$$y(\quad\quad) = y(\quad\quad) + f(\quad\quad, \quad\quad)\Delta t$$

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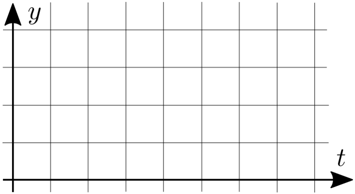

Topic 3 - ...

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The Previous Example - Radio Flyer

If this is a valid technique we should be able to solve the problem we solved in the previous lecture. Ferrari anyone? Let's do a Radio Flyer instead.

$$m\dot{v} + cv = 0 \quad with \quad v(t=0) = v_0$$

$$\Longrightarrow \qquad v(t) = v_0 e^{-\frac{c}{m}t}$$

The Problem Statement

This method is not difficult *if* we setup the problem correctly. Read the problem statement carefully.

Approximate a solution to the differential equation using Euler's Method. Graph the solution from 0 to 10 seconds and use a stepsize of $\Delta t = 1.0$, 1.0, and 1.0 seconds.

$$m\dot{v} + cv = 0 \quad with \quad v(t=0) = v_0$$

$$m = 100(kg), \;\; c = 0.5(\frac{n-m}{s}), v_0 = 5.0(\frac{m}{s})$$

Breakdown The Problem Statement

ODE: $m\dot{v} + cv = 0$

Initial Condition: $v(t = 0) = v_0$

Parameters: $m = 100(kg), \;\; c = 0.5(\frac{n-m}{s}), v_0 = 5(\frac{m}{s})$

Strategy: Euler's Method, $\Delta t = 1.0, \;\; 0.1, \;\; \text{and} \;\; 0.01(s)$

Look at the formula we derived. What goes where?

$$y_{i+1} = y_i + f(x_i, y_i)\Delta x$$

Execute Euler's Method

First, write the **slope function**.

$f(t, y(t)) = f(t, y) =$

Then, start with the initial condition and compute the values of the solution *one by one, forward in time*.

$\underline{v(t_{i+1}) = v(t_i) + f(v(t_i))\Delta t}$

$v(\quad) = v(\quad) + f(\quad)\Delta t$

$v(\quad) = v(\quad) + f(\quad)\Delta t$

$v(\quad) = v(\quad) + f(\quad)\Delta t$

$v(\quad) = v(\quad) + f(\quad)\Delta t$

This method is not suitable for manual computation.