Numerical Integration - Lecture 1

ME3001 - Mechanical Engineering Analysis

April 14, 2021

Euler's Forward Integration

Lecture 1 - Euler's Forward Integration:

- Review and Motivation
- Euler's Forward Integration
- Example Problem
- MATLAB Solution

Leonard Euler (1707-1783)

What is a Differential Equation? Solution?

A **differential equation** is an equation which describes a function and one or more of its ______________ of the ______________ ______________ with respect to the ______________.

The **solution** to a differential equation describes the ______________ ______________ as a function of the ______________ ______________.

Analytical vs. Numerical Solutions

Analytical

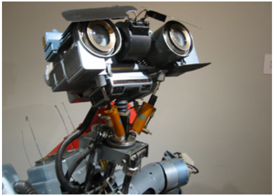
- solution to a problem that can be written in **closed form**
- solution in terms of known functions, constants, etc.
- gives an **exact answer**

Numerical

- an **approximation** to the solution of a mathematical equation
- known as **numerical integration**
- numerical integration is more than *the computation of integrals*

Which one should you choose?

? ? ? ? ?

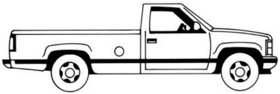


It depends on the problem. It also depends on how you intend to use the solution.

The Initial Value Problem

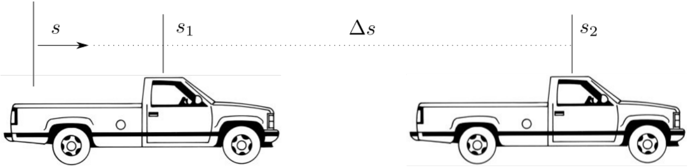
You learned about the **initial value problem** in differential equations class. Do you remember?

You have probably been thinking about this idea for much longer than that. Consider riding in a *truck* waiting to arrive at you destination...



Integrating a Rate

You may not have known it but you where **integrating** when performing these mental calculations. You can math.



The Taylor Series

James Gregory (1638-1675)

Brook Taylor (1685-1731)

Consider the Taylor Series. How does this apply to our problem?

$y(x) \approx$

$y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \frac{y^{(3)}(a)}{3!}(x-a)^3 + ... + \frac{y^{(n)}(a)}{n!}(x-a)^n$

What does this even mean?

Euler's Method

Given a *function describing the slope* and an *initial condition*, discretized values of the solution can be approximated.

This is known as **Euler's method**.

$$y(x+\Delta x) = y(x) + \frac{dy}{dx}\Delta x = y(x) + f(x, y(x))\Delta x$$

It is commonly shown with subscript notation.

$$\boxed{y(x_{i+1}) = y(x_i) + f(x_i, y(x_i)) = y_i + f(x_i, y_i)\Delta x}$$

Careful: This is not the same as Euler's formula which is an essential trigonometric identity also used in differential equations.

The Slope Function

The differential equation must be written as a function describing the first derivative or **the slope** of the dependent variable.

$$f(x, y) = \frac{rise}{run} = \frac{dy}{dx} \neq y(x)$$

or with subscript notation shown below

$$f(x_i, y_i)$$

Careful: The first argument x is not always used and is often left out. However it is an important placeholder (ODE45()) and shows this method can be used for *non-linear* equations with generalized input functions.

Forward Integration

Using this concept to solve the initial value problem is called **Euler's forward integration** or **Euler's Method**. Most of the time, the independent variable is ________________.

Compute the values of the solution one-by-one **forward in time**.

$$\underline{y(t_{i+1}) = y(t_i) + f(t_i, y_i)\Delta t}$$

$$y(\quad\quad) = y(\quad\quad) + f(\quad\quad , \quad\quad)\Delta t$$

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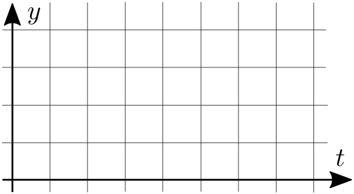

The Previous Example - Radio Flyer

If this is a valid technique we should be able to solve the problem we solved in the previous lecture. Ferrari anyone? Let's do a Radio Flyer instead.

$$m\dot{v} + cv = 0 \quad with \quad v(t=0) = v_0$$

$$\Longrightarrow \qquad v(t) = v_0 e^{-\frac{c}{m}t}$$

The Problem Statement

This method is not difficult *if* we setup the problem correctly. Read the problem statement carefully.

Approximate a solution to the differential equation using Euler's Method. Graph the solution from 0 to 10 seconds and use a stepsize of $\Delta t = 1.0,\ \ 1.0,\ $ and $\ 1.0$ seconds.

$$m\dot{v} + cv = 0 \quad with \quad v(t=0) = v_0$$

$m = 100(kg),\ \ c = 0.5(\frac{n-m}{s}), v_0 = 5.0(\frac{m}{s})$

Breakdown The Problem Statement

ODE: $m\dot{v} + cv = 0$

Initial Condition: $v(t = 0) = v_0$

Parameters: $m = 100(kg), \;\; c = 0.5(\frac{n-m}{s}), v_0 = 5(\frac{m}{s})$

Strategy: Euler's Method, $\Delta t = 1.0, \;\; 0.1, \;\; \text{and} \;\; 0.01(s)$

Look at the formula we derived. What goes where?

$$y_{i+1} = y_i + f(x_i, y_i)\Delta x$$

Execute Euler's Method

First, write the **slope function**.

$$f(t, y(t)) = f(t, y) =$$

Then, start with the initial condition and compute the values of the solution *one by one, forward in time*.

$$\underline{v(t_{i+1}) = v(t_i) + f(v(t_i))\Delta t}$$

$$v(\quad) = v(\quad) + f(\quad)\Delta t$$

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This method is not suitable for manual computation.

Part 1 - Setup and Analytical Solution

```
% ME 3001 - Mechanical Engineering Analysis
% Tristan Hill - Spring 2020
% Numerical Integration - Lecture 1
clear variables;close all;clc

% define the constant parameters
m=100;c=1.5;v0=2.0;
dt=1.0;tstop=60;

% create an array of time values
time=0:dt:tstop;
% compute solution from derived equation
v_exact=v0*exp(-c/m*time);
```

Part 2 - Euler's Method

```
% approximate with Euler's forward integration
v_eu(1)=v0;
for j=1:length(time)-1
    v_eu(j+1)=v_eu(j)+(f(time(j),v_eu(j),m,c))*dt;
end

% If this is an 'Inline Definition' of the function
% it MUST go at the bottom of the script
function [dvdt]=f(t,v,M,C)
    dvdt=-C/M*v;
end
```

Part 3 - Graph the Solutions

```
% plot the results of both methods
figure(1);hold on
plot(time,v_exact,'r-','LineWidth',2)
plot(time,v_eu,'b*')

% add some labels
title('Radio Flyer: mdv/dt+cv=0, v(t=0)=v0')
legend('Exact','Euler''s')
xlabel('Time (s)')
ylabel('Velocity')
grid on
```

Do you believe the results?