ME 3001 Lecture - Systems of Linear Equations

Potential Problems with Gaussian Elimination

- **Remember, our motivation is to develop a General Method to solve this type of problem. Meaning I want my solution work for any problem of the standard form.**

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ & . & & \\ & . & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

$$[A] \times \vec{x} = \vec{b}$$

- **There are several errors that may occur**
  - **we obviously must follow the rules of mathematics**
  - **some problems or systems are not solvable**
  - **we want to have the least error with the least work**

- **What is 1 thing that you cannot do math???**

  - $$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & . & & \\ & . & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

  - **the** *diagonal elements* **are called** *pivot elements* **or sometimes just** *pivots*

  - **take a look back at the algorithm one more time**

    for k *from* 1 *to* n-1

    for i *from* k+1 *to* n

    fact$= a_{i,k}/a_{k,k}$

    for j *from* k *to* n

    $a_{i,j} = a_{i,j} - \text{fact} \times a_{k,j}$

    end

    $b_i = b_i - \text{fact} \times b_k$

    end

    end

  - **how can we avoid this problem?**

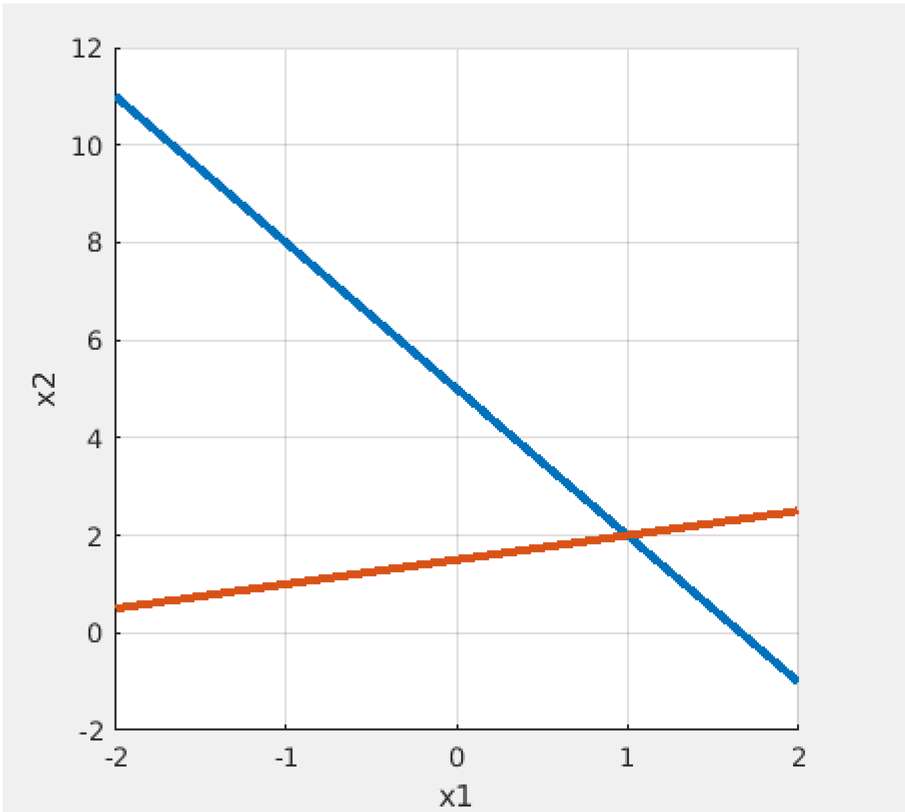
- **to avoid this issue we can use the tools we have and they are ...**
- # Elementary Row Operations
  - **Row Switching**
  - **Row Multiplication**
  - **Row Addition**
- **the act of avoiding this error is called** *pivoting*
  - **Partial Pivoting**
  - **Scaled Partial Pivoting**
  - **Complete Pivoting**
  - **Virtual Pivoting**
  - **No Pivoting - Naive Gaussian Elimination**

– **Partial Pivoting Example**

- **Some problems cannot be solved with this type of technique!**
  - non-homogeneous system is one in which ...
  - most of the time the system will be non-homogeneous
  - a non-homogeneous system has a proper solution if and only if ...

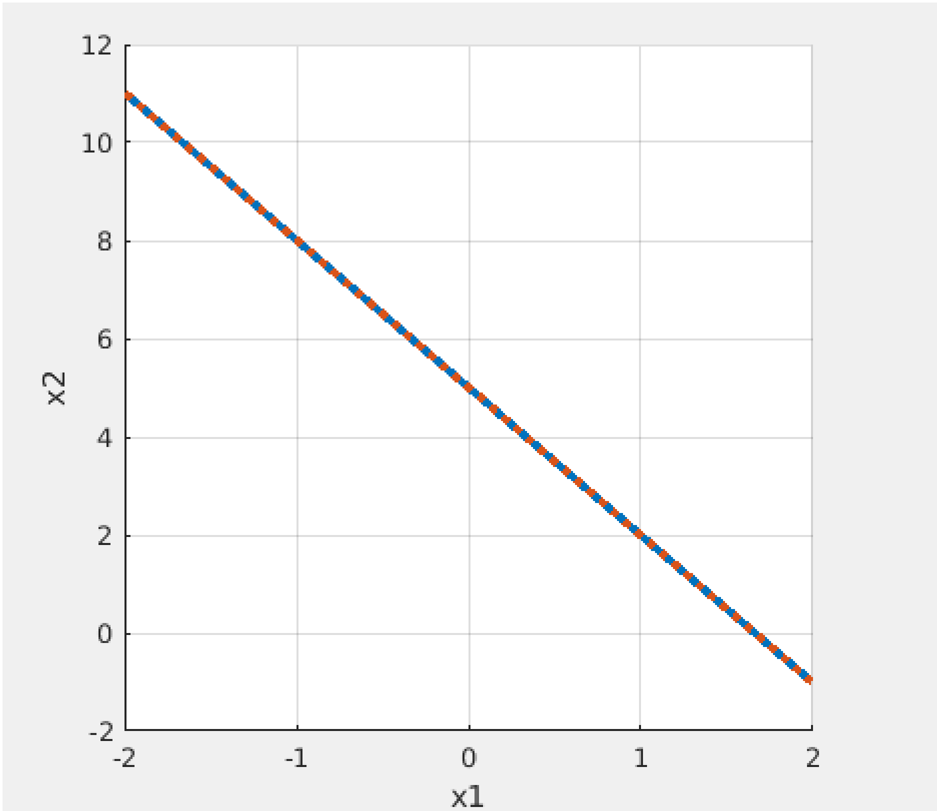
$$rank(A) = rank([A|b]) = n$$

– **Normal Case - 2 Equations - 2 Unknowns - 1 Solution**



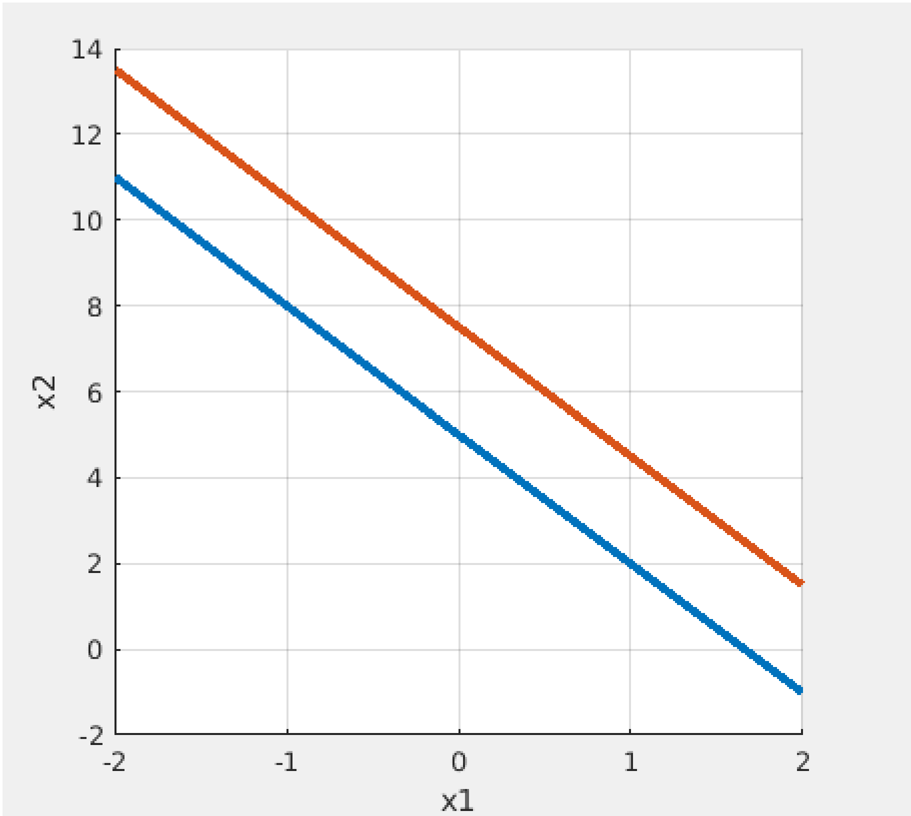
$$3x_1 + x_2 = 5$$
$$x_1 - 2x_2 = -3$$

– **Abnormal Case - 2 Equations - 2 Unknowns - ∞ Solutions**



$$3x_1 + x_2 = 5$$
$$6x_1 + 2x_2 = 10$$

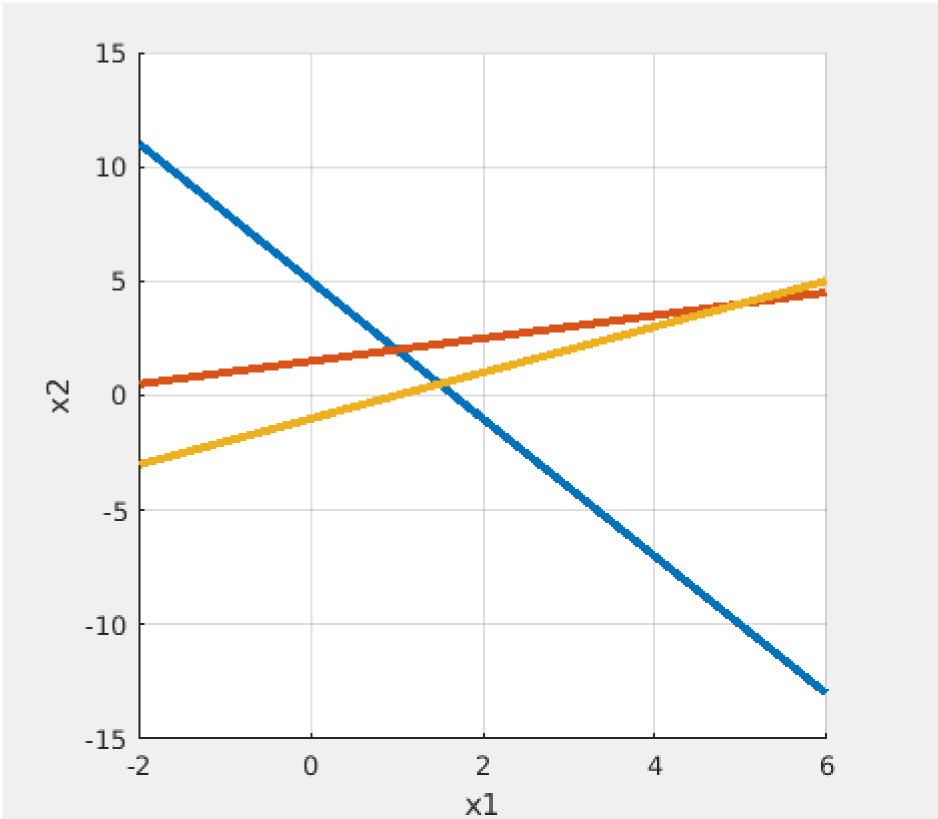
– **Abnormal Case - 2 Equations - 2 Unknowns - 0 Solutions**



$$3x_1 + x_2 = 5$$

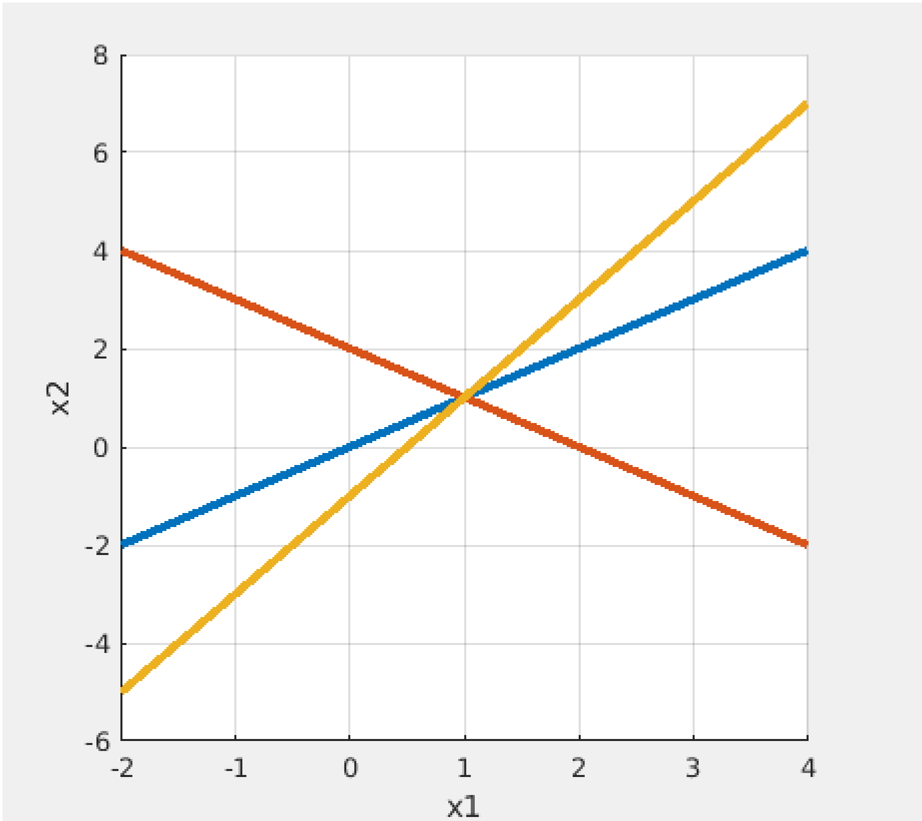
$$6x_1 + 2x_2 = 15$$

– **Abnormal Case - 3 Equations - 2 Unknowns - 0 Solutions**



$$3x_1 + x_2 = 5$$
$$x_1 - 2x_2 = -3$$
$$x_1 - x_2 = 1$$

– **Abnormal Case - 3 Equations - 2 Unknowns - 1 Solution**



$$-x_1 + x_2 = 0$$

$$x_1 + x_2 = 2$$

$$-2x_1 + x_2 = -1$$

- **We also want our answer to have as little error as possible**
  - **What causes error in the numerical methods?**
    "In software engineering and mathematics, numerical error is the combined effect of two kinds of error in a calculation. The first is caused by the finite precision of computations involving floating-point or integer values. The second usually called truncation error is the difference between the exact mathematical solution and the approximate solution obtained when simplifications are made to the mathematical equations to make them more amenable to calculation."-wikipedia
  - **2 (or 3) Major Causes**
    * **floating point computations**
    * **truncation and solution simplification**
    * **system condition**

- **The System Condition can cause problems!**

  – An ill-conditioned system can cause error.

  – A system is ill-conditioned if small changes in the coefficients on the either side of the equation create large variations in the solution.

  – Let us look at a simple 2x2 example.

$$x_1 - x_2 = 5$$
$$kx_1 - x_2 = 4$$

  – The system shown will have huge variations in the solution if $k \approx 1$

$$x_1 - x_2 = 5$$
$$(0.99)x_1 - x_2 = 4$$

  – When $k = 0.99$, this gives a solution $(x_1, x_2) = (100, 95)$

$$x_1 - x_2 = 5$$
$$(1.01)x_1 - x_2 = 4$$

  – When $k = 1.01$, this gives a solution $(x_1, x_2) = (-100, 105)$

- **REMINDER - Homework 2 is due now due on Friday !**

- **REMINDER -Exam1 is coming up!**