

ME 3001 Lecture - Eigenvalues and Eigenvectors

The Magic Numbers!

- **What is an Eigenvector? Eigenvalue?**

In linear algebra, an eigenvector or characteristic vector of a linear transformation is a non-zero vector whose direction does not change when that linear transformation is applied to it. More formally, if T is a linear transformation from a vector space V over a field F into itself and v is a vector in V that is not the zero vector, then v is an eigenvector of T if $T(v)$ is a scalar multiple of v . This condition can be written as the equation

$$T(\mathbf{v}) = \lambda \mathbf{v}$$

where λ is a scalar in the field F , known as the eigenvalue, characteristic value, or characteristic root associated with the eigenvector v .

If the vector space V is finite-dimensional, then the linear transformation T can be represented as a square matrix A , and the vector v by a column vector, rendering the above mapping as a matrix multiplication on the left hand side and a scaling of the column vector on the right hand side in the equation.

$$[A]\mathbf{v} = \lambda \mathbf{v}$$

- **That makes sense right?**

• **The Standard Form of the Eigenvalue problem.**

The Equations

$$\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn} - \lambda)x_n &= 0\end{aligned}$$

The Matrix Form

$$\begin{pmatrix} (a_{11} - \lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22} - \lambda) & \dots & a_{2n} \\ & \cdot & & \\ & \cdot & & \\ a_{n1} & a_{n2} & \dots & (a_{nn} - \lambda) \end{pmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \cdot & & \\ & \cdot & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ & \cdot & & \\ 0 & 0 & \dots & 1 \end{bmatrix} \right) \times \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \cdot & & \\ & \cdot & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

- Let us look at the second matrix form more closely

$$([A] - \lambda[I])\{x\} = \{0\}$$

First we need to realize that this matrix system is *Homogenous*. This follows a different rule regarding the existence of a solution.

A *Homogeneous* system has a non-trivial solution if and only if the determinant of the coefficient matrix is zero.

$$|[A]| = 0$$

Therefore, a solution to $([A] - \lambda[I])\{x\} = \{0\}$ exists if...

$$|[A] - \lambda[I]| = 0$$

This leads to a long n^{th} order polynomial in terms of λ . This will have n roots which may be real or complex.

• 3x3 Example - A Vector in 3D

For this we need the Determinant of a 3x3.

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei + bfg + cdh - ceg - bdi - afh.\end{aligned}$$

And for the Eigenvalue problem the determinant becomes...

$$(a - \lambda)(e - \lambda)(i - \lambda) + bfg + cdh - c(e - \lambda)g - bd(i - \lambda) - (a - \lambda)fh$$

Lets add some numbers.

- This concept can be visualized graphically!