Numerical Integration - Lecture 2

ME3001 - Mechanical Engineering Analysis

April 16, 2020

Euler's Method for Higher-Order Models

Lecture 2 - Euler's Method for Higher-Order Models:

- Review Euler's Method
- A More Exciting Model
- Equation Decomposition
- MATLAB Solution

Forward Integration

Last time we solved a **first order** ODE with Euler's method.

ODE and IC: $m\dot{v} + cv = 0 \qquad v(t=0) = v_0$

Slope Function:

$$v(t_{i+1}) = v(t_i) + f(t_i, v_i)\Delta t$$

$$v(\quad) = v(\quad) + f(\quad,\quad)\Delta t$$

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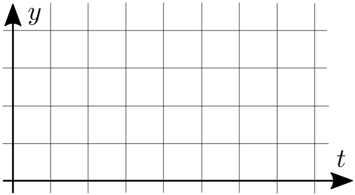


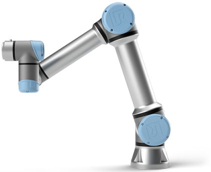
Euler's Method in MATLAB

```
% approximate with Euler's forward integration
v_eu(1)=v0;
for j=1:length(time)-1
    v_eu(j+1)=v_eu(j)+(f(time(j),v_eu(j),m,c))*dt;
end

% If this is an 'Inline Definition' of the function
% it MUST go at the bottom of the script
function [dvdt]=f(t,v,M,C)
    dvdt=-C/M*v;
end
```

A More Exciting Model

- First and second order linear models are frequently used in science and engineering
- However the world is ________ - ____________________
- Many exciting and important engineering problems invlove **more complex models** involving rotational motion.



Non-Linear Swinging Pendulum

An example of a non-linear system is an *inverted pendulum metronome*.

How will this system behave?

$$I_o\ddot{\theta} + k_T - (m \cdot g \cdot l)\, sin(\theta) = 0$$

Finding an analytical solution is **very involved** and only mathematicians have time for all that...but you can look here

As we have seen using Euler's method is not hard, but we have to setup the problem correctly. This is a reoccuring theme!

$$I_o\ddot{\theta} + k_T - (m \cdot g \cdot l)\, sin(\theta) = 0$$

To solve a second order system with an integration method like Euler's you must write the **slope function(s)**.

There are two derivatives so there are two ________ __________.

x2 First Order from x1 Second Order

One second order ODE can be **decomposed** into *two* first order ODEs through a simple change of variables. This step can be confusing, but remember it is just an algebraic substitution!

$$I_o\ddot{\theta} + k_T - (m \cdot g \cdot l)\, sin(\theta) = 0$$

The Slope Function

The differential equation must be written as a function describing the first derivative or **the slope** of the dependent variable.

$$f(x, y) = \frac{rise}{run} = \frac{dy}{dx} \neq y(x)$$

or with subscript notation shown below

$$f(x_i, y_i)$$

Careful: The first argument x is not always used and is often left out. However it is an important placeholder (ODE45()) and shows this method can be used for *non-linear* equations with generalized input functions.

Forward Integration

Using this concept to solve the initial value problem is called **Euler's forward integration** or **Euler's Method**. Most of the time, the independent variable is ________________.

Compute the values of the solution one-by-one **forward in time**.

$$\underline{y(t_{i+1}) = y(t_i) + f(t_i, y_i)\Delta t}$$

$$y(\quad\quad) = y(\quad\quad) + f(\quad\quad, \quad\quad)\Delta t$$

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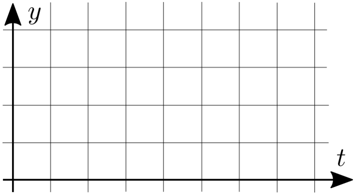

The Previous Example - Radio Flyer

If this is a valid technique we should be able to solve the problem we solved in the previous lecture. Ferrari anyone? Let's do a Radio Flyer instead.

$$m\dot{v} + cv = 0 \quad with \quad v(t=0) = v_0$$

$$\Longrightarrow \qquad v(t) = v_0 e^{-\frac{c}{m}t}$$

The Problem Statement

This method is not difficult *if* we setup the problem correctly. Read the problem statement carefully.

Approximate a solution to the differential equation using Euler's Method. Graph the solution from 0 to 10 seconds and use a stepsize of $\Delta t = 1.0,\ 1.0,$ and 1.0 seconds.

$$m\dot{v} + cv = 0 \quad with \quad v(t=0) = v_0$$

$m = 100(kg),\ \ c = 0.5(\frac{n-m}{s}), v_0 = 5.0(\frac{m}{s})$

Breakdown The Problem Statement

ODE: $m\dot{v} + cv = 0$

Initial Condition: $v(t=0) = v_0$

Parameters: $m = 100(kg), \;\; c = 0.5(\frac{n-m}{s}), v_0 = 5(\frac{m}{s})$

Strategy: Euler's Method, $\Delta t = 1.0, \;\; 0.1, \;\; \text{and} \;\; 0.01(s)$

Look at the formula we derived. What goes where?

$y_{i+1} = y_i + f(x_i, y_i)\Delta x$

Execute Euler's Method

First, write the **slope function**.

$f(t, y(t)) = f(t, y) =$

Then, start with the initial condition and compute the values of the solution *one by one, forward in time*.

$\underline{v(t_{i+1}) = v(t_i) + f(v(t_i))\Delta t}$

$v(\quad) = v(\quad) + f(\quad)\Delta t$

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This method is not suitable for manual computation.

Part 1 - Setup and Analytical Solution

```
% ME 3001 - Mechanical Engineering Analysis
% Tristan Hill - Spring 2020
% Numerical Integration - Lecture 1
clear variables;close all;clc

% define the constant parameters
m=100;c=1.5;v0=2.0;
dt=1.0;tstop=60;

% create an array of time values
time=0:dt:tstop;
% compute solution from derived equation
v_exact=v0*exp(-c/m*time);
```

Part 2 - Euler's Method

```
% approximate with Euler's forward integration
v_eu(1)=v0;
for j=1:length(time)-1
    v_eu(j+1)=v_eu(j)+(f(time(j),v_eu(j),m,c))*dt;
end

% If this is an 'Inline Definition' of the function
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function [dvdt]=f(t,v,M,C)
    dvdt=-C/M*v;
end
```

Part 3 - Graph the Solutions

```
% plot the results of both methods
figure(1);hold on
plot(time,v_exact,'r-','LineWidth',2)
plot(time,v_eu,'b*')

% add some labels
title('Radio Flyer: mdv/dt+cv=0, v(t=0)=v0')
legend('Exact','Euler''s')
xlabel('Time (s)')
ylabel('Velocity')
grid on
```

Do you believe the results?