ME 3001 Lecture - Eigenvalues and Eigenvectors
The Magic Numbers!

- **What is an Eigenvector? Eigenvalue?**

  In linear algebra, an eigenvector or characteristic vector of a linear transformation is a non-zero vector whose direction does not change when that linear transformation is applied to it. More formally, if T is a linear transformation from a vector space V over a field F into itself and v is a vector in V that is not the zero vector, then v is an eigenvector of T if $T(v)$ is a scalar multiple of v. This condition can be written as the equation

  $$T(\mathbf{v}) = \lambda \mathbf{v}$$

  where λ is a scalar in the field F, known as the eigenvalue, characteristic value, or characteristic root associated with the eigenvector v.

  If the vector space V is finite-dimensional, then the linear transformation T can be represented as a square matrix A, and the vector v by a column vector, rendering the above mapping as a matrix multiplication on the left hand side and a scaling of the column vector on the right hand side in the equation.

  $$[A]\mathbf{v} = \lambda \mathbf{v}$$

- **That makes sense right?**

- **The Standard Form of the Eigenvalue problem.**

The Equations

$$
\begin{aligned}
&(a_{11}-\lambda)x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0\\
&a_{21}x_1 + (a_{22}-\lambda)x_2 + ... + a_{2n}x_n = 0\\
&\qquad\qquad .\\
&\qquad\qquad .\\
&a_{n1}x_1 + a_{n2}x_2 + ... + (a_{nn}-\lambda)x_n = 0
\end{aligned}
$$

The Matrix Form

$$
\begin{pmatrix}
(a_{11}-\lambda) & a_{12} & ... & a_{1n}\\
a_{21} & (a_{22}-\lambda) & ... & a_{2n}\\
 & . & & \\
 & . & & \\
a_{n1} & a_{n2} & ... & (a_{nn}-\lambda)
\end{pmatrix}
\times
\begin{bmatrix} x_1\\ x_2\\ .\\ .\\ x_n \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ .\\ .\\ 0 \end{bmatrix}
$$

$$
\left(
\begin{bmatrix}
a_{11} & a_{12} & ... & a_{1n}\\
a_{21} & a_{22} & ... & a_{2n}\\
 & . & & \\
 & . & & \\
a_{n1} & a_{n2} & ... & a_{nn}
\end{bmatrix}
-\lambda
\begin{bmatrix}
1 & 0 & ... & 0\\
0 & 1 & ... & 0\\
0 & 0 & 1 & 0\\
 & . & & \\
0 & 0 & ... & 1
\end{bmatrix}
\right)
\times
\begin{bmatrix} x_1\\ x_2\\ .\\ .\\ x_n \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ .\\ .\\ 0 \end{bmatrix}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & ... & a_{1n}\\
a_{21} & a_{22} & ... & a_{2n}\\
 & . & & \\
 & . & & \\
a_{n1} & a_{n2} & ... & a_{nn}
\end{pmatrix}
\times
\begin{bmatrix} x_1\\ x_2\\ .\\ .\\ x_n \end{bmatrix}
= \lambda
\begin{bmatrix} x_1\\ x_2\\ .\\ .\\ x_n \end{bmatrix}
$$

- **Let us look at the second matrix form more closely**

$$([A] - \lambda[I])\{x\} = \{0\}$$

First we need to realize that this matrix system is *Homogenous*. This follows a different rule regarding the existence of a solution.

A *Homogeneous* system has a non-trivial solution if and only if the determinant of the coefficent matrix is zero.

$$|[A]| = 0$$

Therefore

$$|[A] - \lambda[I]| = 0$$

This leads to a long n^{th} order polynomial in terms of λ. This will have n roots which may be real or complex.

- 3x3 example

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei + bfg + cdh - ceg - bdi - afh.\end{aligned}$$

- **This concept can be visualized graphically!**