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A comprehensive analytical solution of the nonlinear pendulum

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Abstract

In this paper, an analytical solution for the differential equation of the simple but nonlinear pendulum is derived. This solution is valid for any time and is not limited to any special initial instance or initial values. Moreover, this solution holds if the pendulum swings over or not. The method of approach is based on JACOBI elliptic functions and starts with the solution of a pendulum that swings over. Due to a meticulous sign correction term, this solution is also valid if the pendulum does not swing over.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The simple gravity pendulum is a famous case study in classical mechanics that leads to a nonlinear differential equation of second order. A solution of this differential equation based on so-called JACOBI elliptic integrals has been known for more than 100 years [1]. Of course, there exists a vast number of papers and textbooks dealing with pendulums. But due to the great number of publications they cannot be cited here. Instead, the reader is referred to [2], where a collection of several hundred references on this topic can be found. Although the analytical solution of the nonlinear pendulum is an age-old problem, it is still an issue of international projects [3].

In recent years, some effort has been made to solve this differential equation by means of approximations, JACOBI elliptic functions and hypergeometric functions; see e.g. [4–17]. Whereas most papers are limited to a pendulum that does not swing over, one can find in [16] and [17] also a solution for a pendulum that swings over. Whereas the first reference comprises an in-depth treatment of the motion of a pendulum, the latter solely states the solution of the associated differential equation. However, to the best knowledge of the author, none of the references entail *one* formula for the motion of a pendulum that swings over or not.

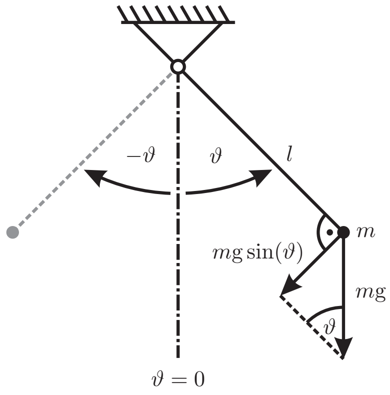

Figure 1. Nonlinear pendulum.

The aim of this paper is to deduce such a formulation, where one can just insert the initial conditions. As is shown, an appropriate sign reversal is only needed for this purpose. To derive this formulation, we solve the underlying nonlinear differential equation, which is not limited to describe the motion of a pendulum [16]. For the sake of generality, three different but equivalent formulations of this differential equation are presented in the next section. Section 3 deals with an energy consideration of the nonlinear pendulum. The main part is section 4, where a general solution of the underlying differential equation is derived. Although this solution is premised on a pendulum that swings over, it is not restricted to this case, as is shown in section 5. In addition, the solution is discussed for some limiting cases and, in section 7, a conclusion is drawn. The appendix provides an introduction to JACOBI elliptic functions and some of their basic relations, which may be helpful to comprehend the calculations.

2. Mathematical model of the nonlinear pendulum

For a statement of the problem, let us examine the pendulum shown in figure 1. It consists of a mass m that is attached to the end of a frictionless pivoted massless rod with length l. The motion of the pendulum is measured in dependence of angle ϑ and its angular velocity $\dot{\vartheta}$. Under the influence of gravity, the pendulum obeys the conservation of angular momentum

$$ml^2\ddot{\vartheta} + ml\mathrm{g}\sin(\vartheta) = 0, \tag{1a}$$

where $\mathrm{g} \approx 9.81$ m s^{-2} is the gravitational constant. At an initial instance t_0, we assume that the bob falls from an angle ϑ_0 with angular velocity $\dot{\vartheta}_0$:

$$\vartheta(t_0) = \vartheta_0 \qquad \text{and} \qquad \dot{\vartheta}(t_0) = \dot{\vartheta}_0. \tag{1b}$$

Note that the initial values are not restricted in any way, but in view of figure 1 it is reasonable to limit the absolute value of the initial angle to π.

A division of the homogeneous differential equation by the moment of inertia ml^2 leads to the well-known initial value problem

$$\ddot{\vartheta} + \Omega^2\sin(\vartheta) = 0, \qquad \text{with} \qquad \vartheta(t_0) = \vartheta_0 \qquad \text{and} \qquad \dot{\vartheta}(t_0) = \dot{\vartheta}_0, \tag{2}$$

where Ω is a positive constant with $\Omega^2 = g/l$. It is not hard to rewrite this differential equation of second order in the form of a system of two differential equations of first order:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}(\vartheta)\boldsymbol{z}(t), \qquad \text{with} \qquad \boldsymbol{z}(t_0) = \boldsymbol{z}_0, \tag{3a}$$

where the state vector $\boldsymbol{z}$ and the system matrix $\boldsymbol{A}$ are given by

$$\boldsymbol{z}(t) = \begin{bmatrix} 2\Omega\sin(\vartheta(t)/2) \\ \dot{\vartheta}(t) \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{A}(\vartheta) = \Omega\cos(\vartheta/2)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3b}$$

respectively. Furthermore, we can exploit the special symmetry of the system matrix to reformulate the differential equation of the pendulum as a complex differential equation of first order:

$$\dot{z}(t) = a(\vartheta)z(t), \qquad \text{with} \qquad z(t_0) = z_0. \tag{4a}$$

Here, $a(\vartheta)$ is a complex scalar, and the real and imaginary parts of $z(t)$ are the first and second element of the state vector, respectively [17]:

$$z(t) = 2\Omega\sin(\vartheta(t)/2) + \mathrm{j}\dot{\vartheta}(t) \qquad \text{and} \qquad a(\vartheta) = -\mathrm{j}\Omega\cos(\vartheta/2). \tag{4b}$$

In what follows, expressions for the angle and the angular velocity of the pendulum are deduced. To this end, we have to solve one of the three equivalent differential equations (2), (3a), and (4a), from which we prefer to solve the initial value problem stated in (2). As it turns out, the solution can be formulated in dependence of so-called JACOBI elliptic functions. Those readers who are not familiar with these functions are referred to the appendix, where some basic definitions and relations are given as far as they are required for solving the differential equation of the pendulum.

3. Energy considerations

At the initial instance, the energy of the pendulum is the sum of the kinetic and potential energy:

$$E_0 = \dot{\vartheta}_0^2 + E_\mathrm{p}\sin^2(\vartheta_0/2), \qquad \text{with} \qquad E_\mathrm{p} = 4\Omega^2, \tag{5}$$

which has been normalized by $ml^2/2$. Because of the assumed frictionless movement, the pendulum is a lossless system and its initial energy is preserved for all instances:

$$E_0 = \dot{\vartheta}^2(t) + E_\mathrm{p}\sin^2(\vartheta(t)/2). \tag{6}$$

Here, E_p is the maximum possible (normalized) potential energy of the pendulum, being attained when the mass is at its highest point perpendicular above the pivot. It is instructive to plot the energy with respect to its angle and its angular velocity as shown in the phase portrait in figure 2. The pendulum can only swing over if its initial energy E_0 is greater than this maximum potential energy E_p. Under this condition, it has enough energy for swinging over; otherwise, the pendulum swings back and forth.

4. Pendulum with sufficient energy for swinging over

In order to determine the angle of a pendulum with enough energy for swinging over, it is beneficial to calculate the angular velocity from the energy balance and to separate the variables [1]. For this purpose, we can make use of (6), from which we find the square of the angular velocity

$$\dot{\vartheta}^2 = E_0 - E_\mathrm{p}\sin^2(\vartheta/2) \geqslant E_0 - E_\mathrm{p} > 0. \tag{7}$$

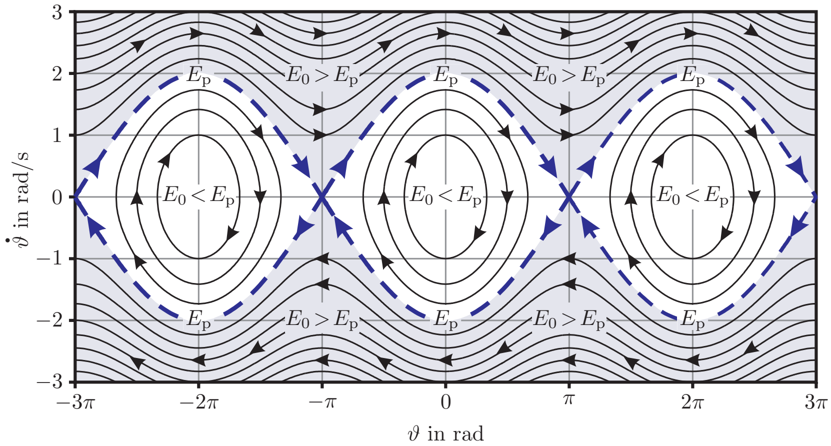

Figure 2. Phase portrait of the nonlinear pendulum.

The square root of this expression provides us with the absolute value of the angular velocity. But for a pendulum that swings periodically over, the sign of the angular velocity is constant and determined by its initial angular velocity:

$$\mathrm{sgn}(\dot\vartheta) = \mathrm{sgn}(\dot\vartheta_0), \qquad \text{with} \qquad \mathrm{sgn}(\zeta) = \begin{cases} 1 & \text{for} \quad \zeta \geqslant 0 \\ -1 & \text{for} \quad \zeta < 0 \end{cases}. \tag{8}$$

Hence, we can supplement the absolute value by this sign and get the expression

$$\dot\vartheta = \mathrm{sgn}(\dot\vartheta_0)\sqrt{E_0 - E_\mathrm{p}\sin^2(\vartheta/2)}, \tag{9}$$

which allows for a separation of the variables:

$$\frac{\mathrm{d}\vartheta}{\sqrt{1 - \varkappa^2\sin^2(\vartheta/2)}} = \mathrm{sgn}(\dot\vartheta_0)2k\Omega\,\mathrm{d}t. \tag{10}$$

The integration from t_0 to t in combination with the substitution $\vartheta = 2\theta$ shows that

$$\mathrm{F}(\vartheta/2|\varkappa) - \mathrm{F}(\vartheta_0/2|\varkappa) = \mathrm{sgn}(\dot\vartheta_0)k\Omega[t - t_0] \tag{11}$$

is the elliptic integral of the first kind with modulus

$$\varkappa = 1/k\,, \qquad \text{with} \qquad k = \sqrt{E_0/E_\mathrm{p}}\,, \tag{12}$$

cf (A.1). Since the absolute value of the initial angle is bounded by π, we have the equivalence

$$k_0 = \sin(\vartheta_0/2) \quad \Longleftrightarrow \quad \vartheta_0/2 = \arcsin(k_0), \tag{13}$$

where k_0^2 is the ratio of the initial potential energy to the maximum possible potential energy. Due to the codomain of k_0, a formulation in dependence of the inverse sn function is not difficult:

$$\mathrm{F}(\vartheta_0/2|\varkappa) = \mathrm{F}(\arcsin(k_0)|\varkappa) = \mathrm{sn}^{-1}(k_0|\varkappa). \tag{14}$$

With this new quantity, (11) takes the form

$$\mathrm{F}(\vartheta/2|\varkappa) = \theta(t) \tag{15a}$$

in which the linear function

$$\theta(t) = \mathrm{sgn}(\dot\vartheta_0)k\Omega[t - t_0] + \mathrm{sn}^{-1}(k_0|\varkappa) \tag{15b}$$

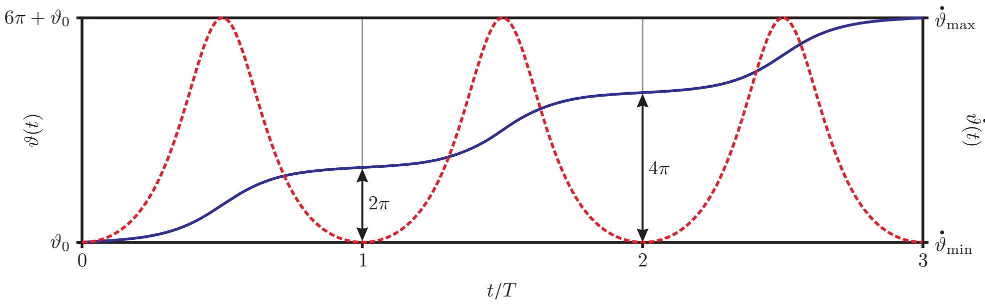

Figure 3. Angle and angular velocity (dashed line) of a pendulum with sufficient energy for swinging over: $\Omega = 1/\mathrm{s}$, $t_0 = 0$, $\vartheta_0 = \pi\, 179.9/180$, and $\dot\vartheta_0 = 0,1667/\mathrm{s}$.

can uniquely be determined:

$$\theta(t_0) = \mathrm{sn}^{-1}(k_0 |\, \varkappa) \qquad \text{and} \qquad \dot\theta(t) = \mathrm{sgn}(\dot\vartheta_0) k\Omega. \tag{15c}$$

Before we solve (15*a*) for the angle, let us compute the period T of the pendulum, i.e. the time that the pendulum needs for a complete cycle. During this time, the angle increases or decreases by 2π depending on the cycle orientation:

$$\mathrm{F}\left(\frac{\vartheta + \mathrm{sgn}(\dot\vartheta_0)2\pi}{2}\,\middle|\, \varkappa\right) = \mathrm{F}\left(\frac{\vartheta}{2}\,\middle|\,\varkappa\right) + \mathrm{sgn}(\dot\vartheta_0)2\mathrm{K}(k). \tag{16}$$

From (A.5) it follows that

$$\theta(t+T) - \theta(t) = \mathrm{sgn}(\dot\vartheta_0)2\mathrm{K}(\varkappa) = \mathrm{sgn}(\dot\vartheta_0)k\Omega T, \tag{17}$$

with the (primitive) period

$$T = 2\varkappa \mathrm{K}(\varkappa)/\Omega. \tag{18}$$

In order to solve (15*a*) for ϑ and to express ϑ in dependence of JACOBI elliptic functions, special diligence is required. An application of the sn function delivers $\sin(\vartheta/2)$ only if the sign on the right-hand side is corrected[1]:

$$\sin(\vartheta/2) = \mathrm{sn}\,(\theta(t) |\, \varkappa)\, \mathrm{sgn}\,(\mathrm{cn}\,(\theta(t) |\, \varkappa)). \tag{19}$$

Note that this correction owes only to how we count the angle. The product on the right-hand side of this equation is the periodically repeated, highlighted part of the sn function in figure A1. The angular velocity can be found by inserting (19) in (9) and considering definition (12):

$$\dot\vartheta(t) = \mathrm{sgn}(\dot\vartheta_0)\sqrt{E_0}\sqrt{1 - \varkappa^2 \mathrm{sn}^2(\theta(t)|\varkappa)}. \tag{20}$$

With the aid of (A.17), this formula can be further simplified and describes together with (15*a*) and (19) the motion of a pendulum that swings over:

$$\begin{aligned} \theta(t) &= \mathrm{sgn}(\dot\vartheta_0)k\Omega[t - t_0] + \mathrm{sn}^{-1}(k_0 |\, \varkappa), \\ \vartheta(t) &= 2 \arcsin\,(\mathrm{sn}\,(\theta(t) |\, \varkappa))\, \mathrm{sgn}\,(\mathrm{cn}\,(\theta(t) |\, \varkappa)), \\ \dot\vartheta(t) &= \mathrm{sgn}(\dot\vartheta_0)\sqrt{E_0}\mathrm{dn}\,(\theta(t) |\, \varkappa)\,. \end{aligned} \tag{21}$$

The associated plots of the angle and angular velocity are depicted in figure 3, where the resets of the angle after each swing over have been omitted.

[1] Of course, instead of using the cn function, the much simpler cosine function can be used for commutating the sign.

4.1. Discussion of the solution

If the initial angular velocity is large enough to neglect the potential energy,

$$E_0 \approx \dot{\vartheta}_0^2, \tag{22}$$

the modulus $\varkappa$ is approximately equal to zero; see (5) and (12). In this case, the JACOBI elliptic functions degenerate to trigonometric functions in accordance with (A.22), and the inverse sn function can be approximated by the arcsin function. Instead of (21), we obtain the approximations

$$\begin{aligned} \theta(t) &\approx \operatorname{sgn}(\dot{\vartheta}_0)k\Omega[t - t_0] + \vartheta_0/2, \\ \vartheta(t) &\approx 2\arcsin(\sin(\theta(t)))\operatorname{sgn}(\cos(\theta(t))), \\ \dot{\vartheta}(t) &\approx \operatorname{sgn}(\dot{\vartheta}_0)\sqrt{E_0}. \end{aligned} \tag{23}$$

The latter equation shows that the angular velocity is approximately constant. In view of (22), we can thus replace $\operatorname{sgn}(\dot{\vartheta}_0)\sqrt{E_0}$ by $\dot{\vartheta}_0$. Furthermore, the angle given in (23) is sawtooth shaped and equals $2\theta(t)$ after the removal of the 2π jumps. With these considerations, the approximations of the corrected angle and angular velocity read

$$\vartheta(t) \approx \dot{\vartheta}_0[t - t_0] + \vartheta_0 \qquad \text{and} \qquad \dot{\vartheta}(t) \approx \dot{\vartheta}_0. \tag{24}$$

This solution is plausible, because it describes the motion of a pendulum in the absence of gravitation, which is obviously true if the potential energy can be neglected in comparison to the kinetic energy.

On the other hand, if the energy is barely sufficient that the pendulum swings over, the modulus is slightly less than 1 and the JACOBI elliptic functions can be replaced by hyperbolic functions. In this situation, the mass can poise arbitrary long at its highest point as can be seen from the period, cf (A.25):

$$T = 2\varkappa\mathrm{K}(\varkappa)/\Omega, \qquad \text{with} \qquad \lim_{\varkappa\to 1} \varkappa\mathrm{K}(\varkappa) \to \infty. \tag{25}$$

5. Pendulum with insufficient energy for swinging over

Clearly, if the energy is less than the maximum possible potential energy, the pendulum does not swing over. On the evidence of (7), we recognize that the former method of approach has to be modified now, because the square of the angular velocity is negative otherwise. The angle could be determined in a similar manner but we would be faced with a bulky computation; see e.g. [1] and [14]. In fact, this effort is not necessary, because (21) holds also for a pendulum with insufficient energy for swinging over. That may be surprising, but the only difference is a modulus greater than 1. In order to find a formulation with JACOBI elliptic functions having a modulus between 0 and 1, we transform the solution to its reciprocal modulus k according to (12). A consequent application of (A.29*c*) and (A.30) to the solution given in (21) yields

$$\begin{aligned} \theta(t) &= \operatorname{sgn}(\dot{\vartheta}_0)k\Omega[t - t_0] + k\,\mathrm{sn}^{-1}(\varkappa k_0|k), \\ \vartheta(t) &= 2\arcsin(k\,\mathrm{sn}(\varkappa\theta(t)|k))\operatorname{sgn}(\mathrm{dn}(\varkappa\theta(t)|k)), \\ \dot{\vartheta}(t) &= \operatorname{sgn}(\dot{\vartheta}_0)\sqrt{E_0}\,\mathrm{cn}(\varkappa\theta(t)|k). \end{aligned} \tag{26}$$

Here, the sgn function can be dropped because its argument is a dn function which is non-negative for $0 \leqslant k \leqslant 1$ as is shown in figure A1. Additionally, the first argument of all JACOBI elliptic functions contains $\varkappa\theta(t)$ so that it is advised to use $\theta(t)$ instead:

$$\begin{aligned} \theta(t) &= \operatorname{sgn}(\dot{\vartheta}_0)\Omega[t - t_0] + \mathrm{sn}^{-1}(\varkappa k_0|k), \\ \vartheta(t) &= 2\arcsin(k\,\mathrm{sn}(\theta(t)|k)), \\ \dot{\vartheta}(t) &= \operatorname{sgn}(\dot{\vartheta}_0)\sqrt{E_0}\,\mathrm{cn}(\theta(t)|k). \end{aligned} \tag{27}$$

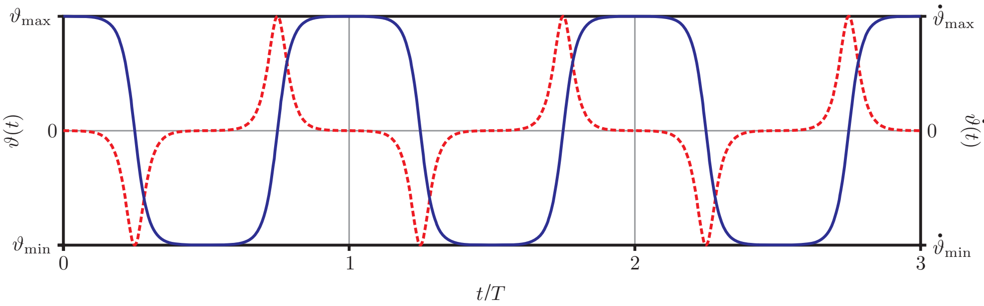

Figure 4. Angle and angular velocity (dashed line) of a pendulum with insufficient energy for swinging over: $\Omega = 1/\mathrm{s}$, $t_0 = 0$, $\vartheta_0 = \pi\,179.9/180$, and $\dot\vartheta_0 = 0$.

Figure 4 shows an example for the solution of the angle and angular velocity. As has been stated above, sn and cn are periodic functions having a real period equal to four times the complete elliptic integral of first kind. Therefore, the angle and the angular velocity are also periodic functions:

$$\vartheta(t) = \vartheta(t - T), \qquad \dot\vartheta(t) = \dot\vartheta(t - T), \qquad T = 4K(k)/\Omega. \tag{28}$$

In comparison to (18), the period seems to be doubled, but this is plausible because the pendulum has to swing back at first before the next period starts again.

If the pendulum is solely deflected by ϑ_0 without any initial angular velocity, then we have $k = k_0$; see (5), (12), and (13). For these special initial values and the initial instance $t_0 = 0$, (27) becomes

$$\begin{aligned} \theta(t) &= \Omega t + K(k), \\ \vartheta(t) &= 2\arcsin(k\,\mathrm{sn}\,(\theta(t)|\,k)), \\ \dot\vartheta(t) &= \sqrt{E_0}\mathrm{cn}\,(\theta(t)|\,k), \end{aligned} \tag{29}$$

which is the known solution in the literature [14].

5.1. Discussion of the solution

In the event of small deflections, the energy E_0 of the pendulum is small compared to the maximum possible potential energy E_p. This leads to a modulus close to zero, for which the JACOBI elliptic functions can be replaced by trigonometric functions; see (A.22). If the deflections are small for all times, then the same holds true for the initial angle ϑ_0, and we can write

$$k_0 = \sin(\vartheta_0/2) \approx \vartheta_0/2. \tag{30}$$

Moreover, we have to bear in mind that in (27) the argument of the arcsin function is small and its function value is approximately equal to its argument. Taking this into account, we find an approximation for (27):

$$\begin{aligned} \theta(t) &\approx \mathrm{sgn}(\dot\vartheta_0)\Omega[t - t_0] + \arcsin(\vartheta_0/[2k]), \\ \vartheta(t) &\approx 2k\sin(\theta(t)), \\ \dot\vartheta(t) &\approx \mathrm{sgn}(\dot\vartheta_0)\sqrt{E_0}\cos(\theta(t)). \end{aligned} \tag{31}$$

This is the well-known solution of the linearized differential equation

$$\ddot\vartheta + \Omega^2\vartheta = 0, \qquad \text{with} \qquad \vartheta(t_0) = \vartheta_0 \qquad \text{and} \qquad \dot\vartheta(t_0) = \dot\vartheta_0. \tag{32}$$

As we expect, the angle and the angular velocity have the period

$$T = 4K(k)/\Omega, \qquad \text{with} \qquad 4K(0) = 2\pi. \tag{33}$$

In contrast, when the initial energy is such that the pendulum can almost swing over, we have $k \approx 1$, and the JACOBI elliptic functions can be replaced by hyperbolic functions; see (A.23). The closer the mass gets to the point swing over, the longer of the period. In an extreme case, the period tends towards infinity:

$$T = 4K(k)/\Omega, \qquad \text{with} \qquad \lim_{k\to 1} K(k) \to \infty, \tag{34}$$

and the mass poises for an arbitrary long time at the highest position.

6. Summary

The essence of this paper is the derivation of a formula for the motion of the simple gravity pendulum. This formula is not limited in any way, i.e. it is valid for arbitrary initial values at any initial instance. The idea for the derivation was to start with a pendulum that has sufficient energy to swing over. In this case, the computation of the angle requires a proper sign reversal, which is not necessary if the pendulum does not swing over. Therefore, (21) is the general solution, while (27) is not. A formulation of the solution in dependence of JACOBI elliptic functions has the advantage of an efficient numerical computation by means of LANDEN's transformation [20].

Appendix A. Jacobi elliptic functions

A.1. The elliptic integral of the first kind and its inverse function

As a starting point, let us examine the definition of the (incomplete) elliptic integral of the first kind [18]:

$$F(\Phi|k) = \int_0^{\Phi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}. \tag{A.1}$$

Here, the JACOBI amplitude Φ and the modulus k are limited to $0 \leqslant \Phi \leqslant \pi/2$ and $0 \leqslant k \leqslant 1$, respectively. This integral cannot be solved analytically, but for some special values there exist known exceptions:

$$F(\Phi|0) = \Phi \qquad \text{and} \qquad F\,(\Phi|\,1) = \ln\left|\tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)\right|. \tag{A.2}$$

Of particular interest are some limiting values of the JACOBI amplitude. While the elliptic integral of first kind vanishes for $\Phi = 0$, it yields the complete elliptic integral of the first kind for $\Phi = \pi/2$:

$$F(0|k) = 0 \qquad \text{and} \qquad F\,(\pi/2|\,k) = K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}. \tag{A.3}$$

Since a sign inversion of the upper bound in (A.1) causes a negation of the value of the integral,

$$F\,(\Phi|\,k) = -F\,(-\Phi|\,k) \quad \text{for} \quad -\pi/2 \leqslant \Phi \leqslant 0, \tag{A.4}$$

and $F\,(\Phi|\,k)$ is an odd function with respect to Φ, the codomain in (A.1) can immediately be extended to $-\pi/2 \leqslant \Phi \leqslant \pi/2$. On the one hand, the integrand in (A.1) is π-periodic with respect to the integration variable whereby $K(k)$ is its zeroth FOURIER coefficient apart from a scaling with $2/\pi$. On the other hand, this integrand is even with respect to Φ. Hence, if we add ν half-periods to the upper bound, the value of the integral increases by ν times the period times half the zeroth FOURIER coefficient:

$$F(\Phi + \nu\frac{\pi}{2}|k) = F(\Phi|k) + \nu K(k) \quad \text{for} \quad -\pi/2 \leqslant \Phi \leqslant \pi/2 \tag{A.5}$$

and $\nu \in \mathbb{Z}$. This equality is used here in order to extend the elliptic integral of the first kind to an arbitrary real JACOBI amplitude[2]. Now, the JACOBI amplitude is introduced as the inverse function of the elliptic integral of the first kind:

$$\mathrm{am}(\zeta|k) = \Phi, \qquad \text{with} \qquad \zeta = F(\Phi|k) \tag{A.6}$$

for $-\pi/2 \leqslant \Phi \leqslant \pi/2$ and $-K(k) \leqslant \zeta \leqslant K(k)$. A sign reversal of the real variables ζ and Φ in combination with (A.4) shows that the am function is odd within the given interval. This can also be verified via (A.2), where the elliptic integral of the first kind is explicitly given and we can determine its inverse function:

$$\mathrm{am}(\zeta|0) = \zeta \qquad \text{and} \qquad \mathrm{am}(\zeta|1) = 2\arctan(\mathrm{e}^{\zeta}) - \pi/2. \tag{A.7}$$

In contrast to this, (A.3) provides the special values of the am function for an arbitrary modulus:

$$\mathrm{am}(0|k) = 0 \qquad \text{and} \qquad \mathrm{am}(K(k)|k) = \pi/2. \tag{A.8}$$

If we finally accept (A.5) as an extension of the elliptic integral of the first kind, the am function can be extended with (A.6) to arbitrary real values of ζ:

$$\mathrm{am}(\zeta + \nu K(k)|k) = \mathrm{am}(\zeta|k) + \nu\frac{\pi}{2} \quad \text{for} \quad -K(k) \leqslant \zeta \leqslant K(k) \tag{A.9}$$

and $\nu \in \mathbb{Z}$. To conclude, under the conventions of (A.5) and (A.9), the relation in (A.6) holds for all real Φ and ζ.

A.2. The functions cn, sn, *and* dn

For the introduction of some JACOBI elliptic functions, let us define a generalized exponential function

$$\mathrm{en}(\zeta|k) = \exp\left(\mathrm{j}\,\mathrm{am}(\zeta|k)\right). \tag{A.10}$$

Here, which in light of (A.8) has the special values

$$\mathrm{en}(0|k) = 1 \qquad \text{and} \qquad \mathrm{en}(K(k)|k) = \mathrm{j}. \tag{A.11}$$

This function has the advantage of being identical for the addressed different definitions of the am function. Equations (A.10) and (A.9) imply the relation

$$\mathrm{en}(\zeta + \nu K(k)|k) = \mathrm{j}^{\nu}\mathrm{en}(\zeta|k) \quad \text{for} \quad \nu \in \mathbb{Z}, \tag{A.12}$$

from which the $4K(k)$ periodicity of the en function can be taken. This periodicity is conveyed to its real and imaginary parts, being the elliptic functions

$$\mathrm{cn}(\zeta|k) = \mathrm{Re}\{\mathrm{en}(\zeta|k)\} = \cos\left(\mathrm{am}(\zeta|k)\right), \tag{A.13}$$

$$\mathrm{sn}(\zeta|k) = \mathrm{Im}\{\mathrm{en}(\zeta|k)\} = \sin\left(\mathrm{am}(\zeta|k)\right) \tag{A.14}$$

known as the cosine and sine amplitude, respectively. Since the am function in (A.10) is real, we attain the identity

$$|\mathrm{en}(\zeta|k)|^2 = 1 \quad \Longleftrightarrow \quad \mathrm{cn}^2(\zeta|k) + \mathrm{sn}^2(\zeta|k) = 1. \tag{A.15}$$

[2] It has to be emphasized that there exist different extensions in the literature, which can easily lead to confusion; see e.g. [18] and [19].

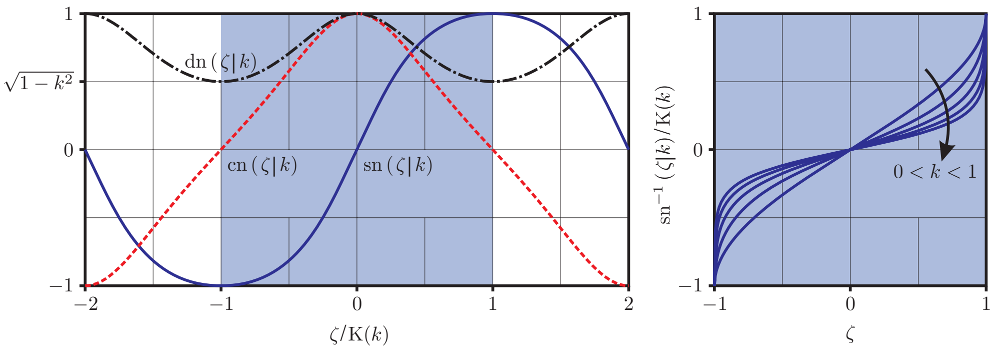

Figure A1. JACOBI elliptic functions sn, cn, dn, and sn^{-1} for $0 < k < 1$.

Besides these two elliptic functions the dn function is also needed, which is the partial derivative of the JACOBI amplitude with respect to its first argument [18]:

$$\mathrm{dn}(\zeta|k) = \mathcal{D}_\zeta \{\mathrm{am}(\zeta|k)\}. \tag{A.16}$$

The partial derivative of the identity $\zeta = F(\mathrm{am}(\zeta|k)|k)$ with respect to ζ obviously yields the value 1. Considering the upper bound of the elliptic integral, an application of the chain rule produces the identity

$$\mathrm{dn}(\zeta|k) = \sqrt{1 - k^2\mathrm{sn}^2(\zeta|k)} \quad \text{for} \quad 0 \leqslant k \leqslant 1. \tag{A.17}$$

Without proof, we mention that this equation for $k > 1$ reads, see [18],

$$|\mathrm{dn}(\zeta|k)| = \sqrt{1 - k^2\mathrm{sn}^2(\zeta|k)}. \tag{A.18}$$

In order to illustrate the elliptic functions introduced here, the sn, cn, and dn are plotted on the left-hand side of figure A1 for $0 < k < 1$. As can be seen, these elliptic functions are bounded by

$$\mathrm{cn}^2(\zeta|k) \leqslant 1, \quad \mathrm{sn}^2(\zeta|k) \leqslant 1, \quad \text{and} \quad \mathrm{dn}(\zeta|k) \geqslant \sqrt{1-k^2}, \tag{A.19}$$

which are the implications of equations (A.15) and (A.17). On the right-hand side of this figure, one can find the principal value of the inverse sn function for different values of the modulus. This function is denoted by sn^{-1} and is achieved by mirroring the sn function on the bisecting line, cf figure A1 (left and right). Evidently, the inverse sn function has the properties

$$-K(k) \leqslant \mathrm{sn}^{-1}(\zeta|k) \leqslant \mathrm{sn}^{-1}(1|k) = K(k) \quad \text{for} \quad -1 \leqslant \zeta \leqslant 1. \tag{A.20}$$

A.3. Some limiting values

The plots of the cn and sn functions let us conjecture a certain relationship to the according trigonometric functions. In order to find this connection, we consider the limiting cases where the modulus k equals 0 or 1. From definition (A.10) and equation (A.7), we obtain the expressions

$$\mathrm{en}(\zeta|0) = \exp(\mathrm{j}\zeta) \qquad \text{and} \qquad \mathrm{en}(\zeta|1) = \frac{1 + \mathrm{j}\sinh(\zeta)}{\cosh(\zeta)}. \tag{A.21}$$

Separating these equations into their real and imaginary parts yields the limiting cases of the cn and sn functions:

$$\mathrm{cn}(\zeta|0) = \cos(\zeta) \qquad \text{and} \qquad \mathrm{sn}(\zeta|0) = \sin(\zeta), \tag{A.22}$$

$$\mathrm{cn}(\zeta|1) = \mathrm{sech}(\zeta) \qquad \text{and} \qquad \mathrm{sn}(\zeta|1) = \tanh(\zeta). \tag{A.23}$$

By inserting these results into (A.17), we get

$$\mathrm{dn}(\zeta|0) = 1 \qquad \text{and} \qquad \mathrm{dn}(\zeta|1) = \mathrm{sech}(\zeta). \tag{A.24}$$

Consequently, the elliptic functions degenerate to trigonometric and hyperbolic functions if the modulus equals 0 or 1, respectively. The real periodicity of the sn and cn functions is $4K(k)$, which leads to consistent results when (A.2) is evaluated for $\Phi = \pi/2$:

$$K(0) = \frac{\pi}{2} \qquad \text{and} \qquad \lim_{k\to 1} K(k) \to \infty. \tag{A.25}$$

A.4. Some derivatives

Additional relations can be deduced from the partial derivative of the en function with respect to its first argument. For this purpose, we start from (A.10) and apply the chain rule, where the inner derivation can be taken from (A.16):

$$\mathcal{D}_\zeta\,\{\mathrm{en}(\zeta|k)\} = \mathrm{j}\mathrm{dn}(\zeta|k)\mathrm{en}(\zeta|k). \tag{A.26}$$

The real and imaginary parts of this equation deliver the differentiation rules of the cn and sn functions, respectively:

$$\mathcal{D}_\zeta\,\{\mathrm{cn}(\zeta|k)\} = -\mathrm{dn}(\zeta|k)\mathrm{sn}(\zeta|k), \tag{A.27a}$$

$$\mathcal{D}_\zeta\,\{\mathrm{sn}(\zeta|k)\} = \mathrm{dn}(\zeta|k)\mathrm{cn}(\zeta|k). \tag{A.27b}$$

A.5. Reciprocal modulus transformation

Numerical programs normally offer JACOBI elliptic functions with a modulus between 0 and 1. Since we subsequently require JACOBI elliptic functions with a modulus greater than 1, we have to deal with transforming an elliptic function to its reciprocal modulus

$$\varkappa = 1/k\,, \qquad \text{with} \qquad \varkappa < 1. \tag{A.28}$$

To transform the elliptic functions accordingly, one can make use of the formulas [18]

$$\mathrm{sn}(\zeta|k) = \varkappa\,\mathrm{sn}(k\,\zeta|\varkappa), \tag{A.29a}$$

$$\mathrm{cn}(\zeta|k) = \mathrm{dn}(k\,\zeta|\varkappa), \tag{A.29b}$$

$$\mathrm{dn}(\zeta|k) = \mathrm{cn}(k\,\zeta|\varkappa), \tag{A.29c}$$

from which the transformations of the inverse sn function and the complete elliptic integral of its first kind can be deduced:

$$\mathrm{sn}^{-1}(\zeta|k) = \varkappa\mathrm{sn}^{-1}(k\zeta|\varkappa) \qquad \text{and} \qquad K(k) = \varkappa K(\varkappa). \tag{A.30}$$

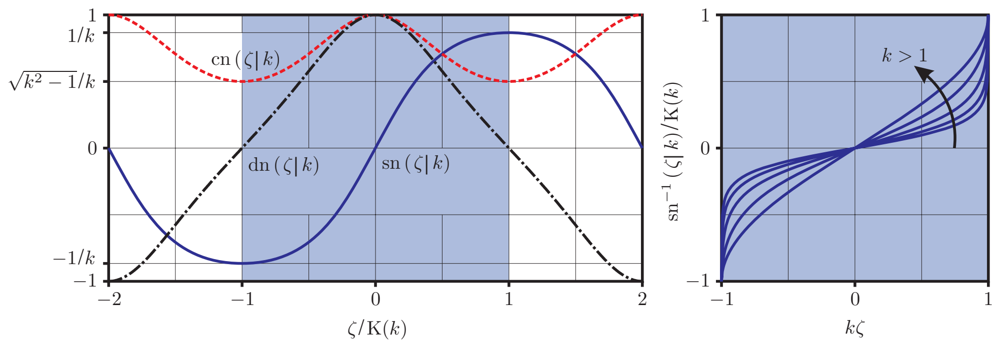

Figure A2. JACOBI elliptic functions sn, cn, dn, and sn^{-1} for $k > 1$.

In correspondence to figure A1, the plots of the elliptic functions with a modulus greater than 1 are shown in figure A2.

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