Lecture Module - Time Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Time Response

Lecture Module - Time Response

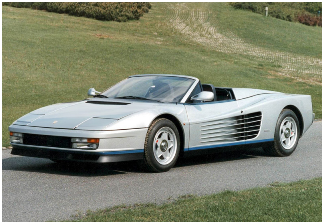
- Topic 1 - First Order Time Response
- Topic 2 - Second Order Time Response
- Topic 3 - Response and Root Location
- Topic 4 - Specification of The Step Response

Topic 1 - First Order Time Response

- Second Order Free Response
- Second Order Forced Response
- Response and System Stability
- Equation Components

Model and EOM

Consider the model of the moving mass we derived.



The EOM is:

$$m\dot{v} + cv = 0$$

Solution with Laplace Transforms Method

$$\mathcal{L}\{m\dot{v} + cv = 0\} \implies m[sV(s) - v(0)] + cV(s) = 0$$

$$(ms + c)V(s) = \frac{mv(0)}{(ms+c)} = \frac{V(0)}{s+\frac{c}{m}}$$

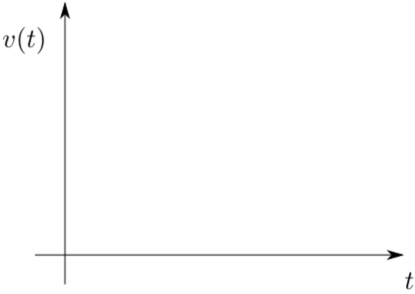
We can find the expected result from the table.

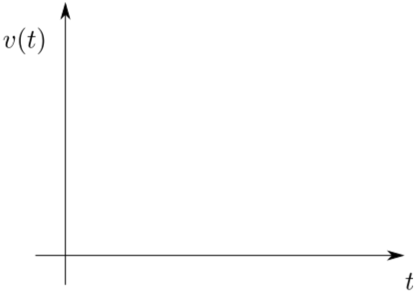
$$v(t) = v(0)e^{-\frac{c}{m}t} = v(0)e^{-\frac{t}{\tau}} \qquad \text{with} \qquad \tau =$$

Sketch Response Equation

Sketch the System Response in the time Domain.

$$v(t) = v(0)e^{-\frac{t}{\tau}}$$





Is this a stable system? What does that mean?

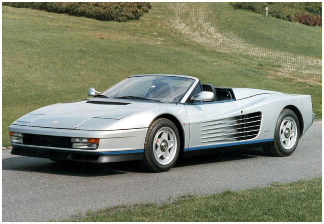
Stability in Dynamic Systems

A dynamic system is stable if ...

Second Order Forced Response

Step Input Function

Consider the model subject to a Step Input, $f(t)$.



$$m\dot{v} + cv = f(t)$$

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

Solution with Laplace Transforms Method - Step 1

The method of Laplace Transforms is shown.

$$\mathcal{L}\{m\dot{v} + cv = F\} \implies m[sV(s) - v(0)] + cV(s) = \frac{F}{c}$$

$$(ms + c)V(s) = \frac{F}{s} + mv(0)$$

Solve for $V(s)$.

$$V(s) = \frac{F}{s(ms+c)} + \frac{mv(0)}{ms+c}$$

Solution with Laplace Transforms Method - Step 2

Expand $V(s)$as a partial fraction.

$$V(s) = \frac{F}{s(ms+c)} + \frac{mv(0)}{ms+c} \implies \frac{F}{s(ms+c)} = \frac{a}{s} + \frac{b}{ms+c}$$

'Cover up' to find the coefficients.

$$a = \frac{F}{m\times 0+c} \qquad \text{and} \qquad b = \frac{F}{\frac{-c}{m}} = \frac{-Fm}{c}$$

This leads to a form that can be inverted with the table.

$$V(s) = \frac{F}{c}\left\{\frac{1}{s} - \frac{1}{s+\frac{c}{m}}\right\} + \frac{v(0)}{s+\frac{c}{m}}$$

Solution with Laplace Transforms Method - Step 3

Can you find these terms in the Table of Laplace Transforms?

$$V(s) = \frac{F}{c}\{\frac{1}{s} - \frac{1}{s+\frac{c}{m}}\} + \frac{v(0)}{s+\frac{c}{m}}$$

The inverse Laplace transform of $V(s)$ gives the time response.

$$v(t) = \frac{F}{C}\{1 - e^{-\frac{t}{\tau}}\} + v(0)e^{-\frac{t}{\tau}} = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$

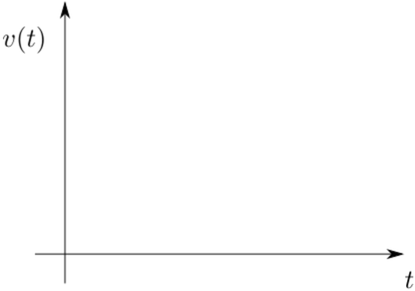
Response and System Stability

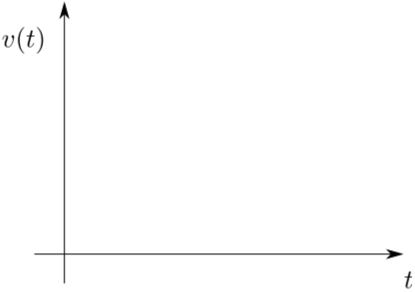
Response and System Stability

Sketch Response Equation

Sketch the System Response in the time Domain.

$$v(t) = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$





Is this a stable system?

Components of the Response

In these forms we can see the different components of the response.

$$v(t) = \frac{F}{C}\{1 - e^{-\frac{t}{\tau}}\} + v(0)e^{-\frac{t}{\tau}} = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$

- Forced Response
- Free Response
- Transient Response
- Steady-State Response

References

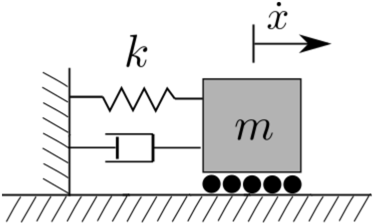
- System Dynamics, Palm III, Third Edition - Section 8.1 - Response of First Order Systems - pg. 475

Topic 2 - Second Order Time Response

- Damped Natural Frequency and Damping Ratio
- The Overdamped Case
- The Critically Damped Case
- The Underdamped Case

Mass Spring Model

Consider the mass-spring system without damping.



The EOM is:

$m\ddot{x} + kx = 0$ with

$x(t = 0) = x_0$, and $v(t = 0) = v_0$

Solution with Laplace Transforms Method

Solve for $x(t)$ with a method of your choice.

$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

Phase Shift

The solution is commonly written as a single oscillating term with a **phase shift** ϕ.

$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

Is equivalent to:

$$x(t) = Acos(\omega_n t - \phi) \;\; A = \sqrt{x_0^2 + [\frac{v_0}{\omega_n}]^2} \;\; \phi = tan^{-1}(\frac{v_0}{x_0 \omega_n})$$

Sine could be used instead.

$$x(t) = Asin(\omega_n t + \phi) \;\; A = \sqrt{x_0^2 + [\frac{v_0}{\omega_n}]^2} \;\; \phi = tan^{-1}(\frac{x(0)\omega_n}{v_0})$$

Sketch of Free Response

Sketch the **free** response in the time domain.

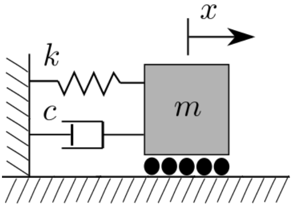
$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t)$$

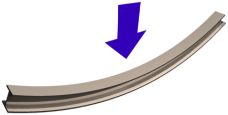
$$= Acos(\omega_n t - \phi) \;\; with \;\; \phi = tan^{-1}(\frac{v_0}{x_0 \omega_n})$$

Is this a stable system? What does the phase shift ϕ represent?

Second Order System with Damping

Now, consider the mass-spring system with damping present.





The EOM is:

$m\ddot{x} + c\dot{x} + kx = 0$ with

$x(t = 0) = x_0, \quad$ and $\; v(t = 0) = v_0$

Solution with Trial Solution Method

The trial solution method is used to derive the response equation in terms of the system variables and parameters.

$m\ddot{x} + c\dot{x} + kx = 0 \implies (mr^2 + cr + k)Ae^{rt} = 0$

You can see the *characteristic equation* becomes:

$(mr^2 + cr + k) = 0$

Solve for the roots. In system dynamics they are called $s_{1,2}$

$r_{1,2} = s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}$

The Roots of the System

The roots of the system determine the behavior.

$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

The **discriminant** $c^2 - 4mk$ drives the *case* in the trial solution method.

- IF __________ $\Longrightarrow$ Case 1: Distinct and Real
- IF __________ $\Longrightarrow$ Case 2: Repeated and Real
- IF __________ $\Longrightarrow$ Case 3: Complex Conjugate Pair

Damping Cases and the Critical Damping Value

In a system with known mass and spring constant, the damping value determines the behavior. The damping value that causes the *discriminant* to equal zero (case 2) is the **critical damping value**.

$$c^2 - 4mk = 0 \implies c = \sqrt{4mk} = 2\sqrt{mk}$$

$$c_{critical} = 2\sqrt{mk}$$

The Damping Ratio

The damping ratio ζ is the ratio of damping c to the critical damping value $c_{critical}$.

$$\zeta = \frac{c}{c_{critical}} = \frac{c}{2\sqrt{mk}}$$

Re-write the roots with this new quantity.

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

We define a new quantity, **damped natural frequency**.

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Now re-write the roots again in terms of ζ and ω_d.

$$s_{1,2} = \zeta\omega_n \pm j\omega_d$$

Damped Natural Frequency and Damping Ratio

The behavior of the system depends on the damping ratio.

Case 1	$c > 2\sqrt{mk}$	**Overdamped**	$\zeta > 1$
Case 2	$c = 2\sqrt{mk} = c_{critical}$	**Critically Damped**	$\zeta = 1$
Case 3	$c < 2\sqrt{mk}$	**Underdamped**	$\zeta < 1$

The Overdamped Case

The roots are real and distinct and the system *does not* oscillate.

Overdamped $c > 2\sqrt{mk} \implies \zeta > 1$

$$s_{1,2} = -\zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$x(t) = C_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\,)t} + C_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\,)t}$$

$$x(t) = e^{-\zeta\omega_n}\{C_1 e^{\omega_n\sqrt{\zeta^2-1}t} + C_2 e^{\omega_n\sqrt{\zeta^2-1}t}\}$$

$$C_1 = \frac{-v_0+(-\zeta+\sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}} \quad C_2 = \frac{-v_0+(\zeta+\sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

The Critically Damped Case

The roots are real and repeated and the system *does not* oscillate.

Critically Damped $c = 2\sqrt{mk} \implies \zeta = 1$

$$s_{1,2} = \frac{-c}{2m} = \zeta\omega_n = -\omega_n$$

$$x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

$$\dot{x} = -\omega_n C_1 e^{-\omega_n t} + C_2 t(-\omega_n e^{-\omega_n t}) + e^{-\omega_n t}(C_2)$$

$$x(t=0) = x_0 \implies C_1 = x_0$$

$$v(t=0) = v_0 \Rightarrow v_0 = -\omega_n C_1 + 0 + (1) C_2 \implies C_2 = v_0 + \omega_n x_0$$

The Underdamped Case

Take the derivative and solve for the second unknown.

$$x(t) = e^{-\zeta\omega_n t}\{Acos(\omega_d t) + Bsin(\omega_d t)\}$$

$$\dot{x}(t) = e^{-\zeta\omega_n t}(-\omega_d Asin(\omega_d t)) + Acos(\omega_d t)(-\zeta\omega_n e^{-\zeta\omega_n t})$$

$$+e^{-\zeta\omega_n t}(\omega_d Bcos(\omega_d t)) + Bsin(\omega_d t)e^{-\zeta\omega_n t}$$

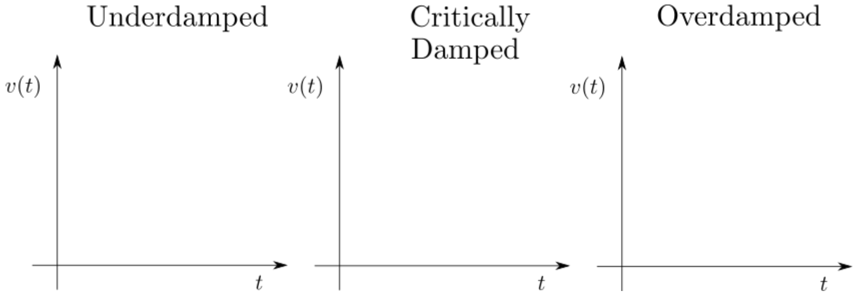
$$\dot{x}(t=0) = A(1)(-\zeta\omega_n(1)) + (1)\omega_d B(1) \implies B = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

Finally we get to the reponse equation.

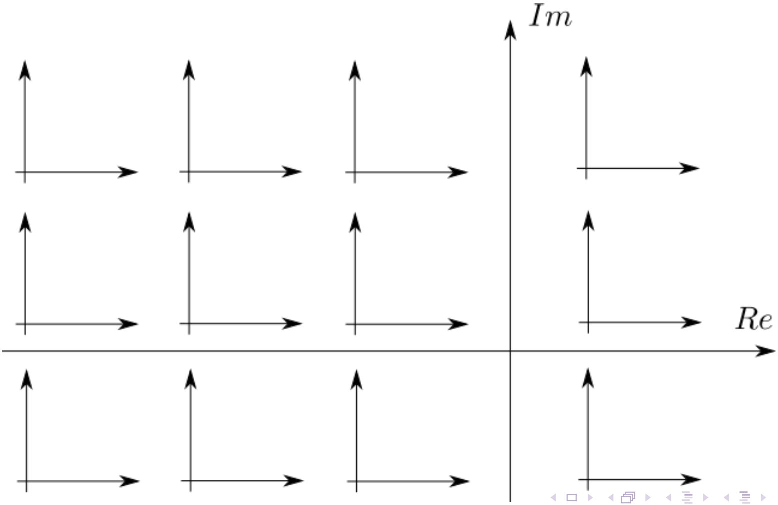
$$x(t) = e^{-\zeta\omega_n t}\{x_0 cos(\omega_d t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_d} sin(\omega_d t)\}$$

Response of the Three Different Cases

Each of the three cases behaves in a *characteristic* way.



Affects of Damping Ratio and Damped Natural Frequency

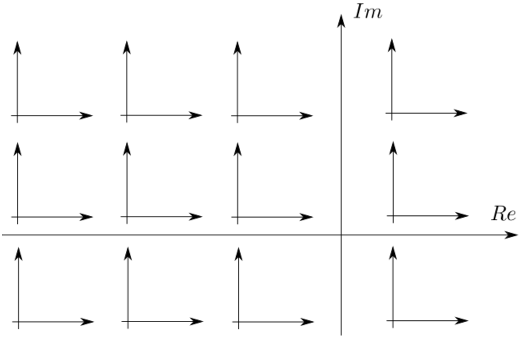


Topic 3 - Response and Root Location

- The Complex Plane
- Along a Vertical Line
- Along a Horizontal Line
- Along a Diagonal Line

Affect of Root Location on Response

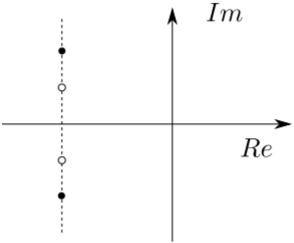
The *location* of the root in the *complex plane* shows the affects of the roots on the system behviour.



The Complex Plane

Along a Vertical Line

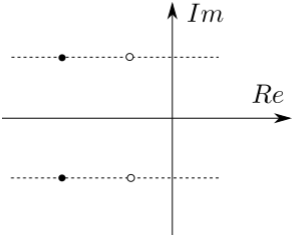
As the root moves along a vertical line...



Along a Vertical Line

Along a Horizontal Line

As the root moves along a horizontal line...

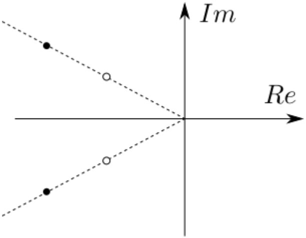


Along a Horizontal Line

Along a Horizontal Line

Along a Diagonal Line

As the root moves along a diagonal line...



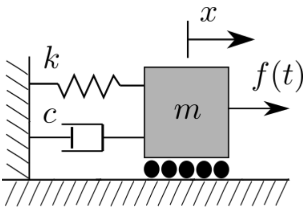
What about the angle of this line?

Topic 3 - Specification of The Step Response

- The Unit Step Response
- Rise, Peak, and Settling Time
- Maximum Overshoot and The Damping Ratio
- System Identificatio

The Mass Spring Damper

Now, consider the mass-spring system with damping present subject to **step** input. This models instantly turning on the input force $f(t)$.



Heavyside's Step Function

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

The EOM is:

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad with \;\; x(t=0) = x_0 \quad and \quad v(t=0) =$$

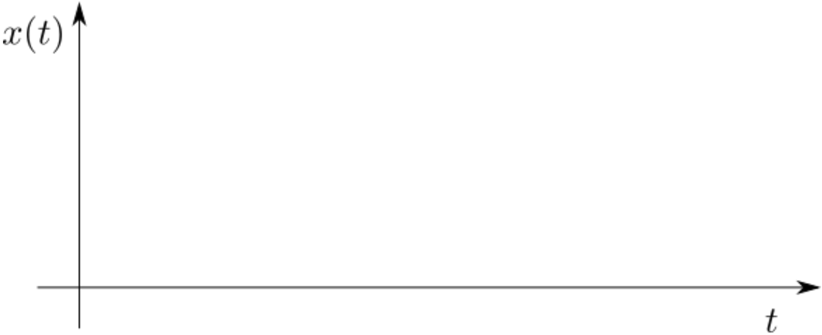
Unit Step Response

The **unit step response** is a special case of the *forced response* in which $f(t)$ is the step function of unit magnitude (F=1).

Overdamped	$x(t) = \frac{1}{k}(\frac{r_2}{r_1-r_2}e^{-r_1 t} - \frac{r_1}{r_1-r_2}e^{-r_2 t} + 1$
	$r_{1,2} = -s_{1,2}$
Critically Damped	$x(t) = \frac{1}{K}[(-1 - \omega_n t)e^{-\omega_n t} + 1]$
Underdamped	$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} sin(\omega_d t + \phi + 1\right]$
	$\phi = tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$

Rise, Peak, and Settling Time

We are going to derive several quatities that describes the response of an underdamped system.



Rise Time

The **rise time** is the time at which the response first equals the steady state value.

$$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$$

Set the *transient term* to zero and solve for t.

$$e^{-\zeta\omega_n t} sin(\omega_d t + \phi) = 0 \implies sin(\omega_d t + \phi) = 0 \implies \omega_d t + \phi = 2\pi$$

$$t_{rise} = t_r = \frac{2\pi - \phi}{\omega_d}$$

Rise, Peak, and Settling Time

The **peak time** is the time at which the response equals the maximum value. Find the derivative of the response equation and set it equal to zero.

$$\dot{x}(t) =$$

$$\left(\frac{1}{K}\frac{1}{\sqrt{1-\zeta^2}}\right)\left[e^{-\zeta\omega_n t}(\omega_d cos(\omega_d t + \phi)) + sin(\omega_d t + \phi)(-\zeta\omega_n e^{-\zeta\omega\right.$$

$$sin(\omega_d)t = 0 \implies \omega_d t = \pi \implies \quad t_{peak} = t_p = \frac{\pi}{\omega_d}$$

Rise, Peak, and Settling Time

The **settling time** is the time at which the response decays to a certain percentage of the steady state value.

It can be esitmated as:

$$t_{settling} = t_s = -\frac{ln(tolerance)}{\zeta\omega_n}$$

$$2\% \implies tolerance = 0.02$$

$$5\% \implies tolerance = 0.05$$

Rise, Peak, and Settling Time

The **maximum overshoot** is the response beyond the steady state value.

$$M_p = x(t_p) - x_{ss} \implies M_p = \frac{1}{k} e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

This is often expressed as a percentage.

$$M_{\%} = \frac{x(t_p) - x_{ss}}{x_{ss}} 100 = 100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Damping Ratio from Maximum Overshoot

The *damping ratio* can be determined from the maximum overshoot!

$$M_{\%} = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Solve for ζ.

$$\zeta = \frac{R}{\sqrt{\pi^2 + R^2}} \quad with \quad R = ln\left(\frac{100}{M\%}\right)$$

Damping Ratio from Log Decrement

The logarithmic decrement is the natural log of the ratio of the amplitudes of any two successive peaks:

$$\delta = \frac{1}{n} ln \frac{x(t)}{x(t+nT)}$$

x(t) is the overshoot (amplitude - final value) at time t and x(t + nT) is the overshoot of the peak n periods away.

The damping ratio is then found from the logarithmic decrement by:

$$\zeta = \frac{1}{\sqrt{1+\left(\frac{2\pi}{\delta}\right)^2}}$$

Damping Ratio from Log Decrement

What is the significance of all of this?

Why do we care about all of these new parameters?

System Identificatio

System Identificatio

System Identificatio

System Identificatio