Lecture Module - Frequency Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Frequency Response

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- Topic 1 - Frequency Response of First Order Systems
- Topic 2 - The Bode Diagram
- Topic 3 - Frequency Response of 2^{nd} Order Systems
- Topic 4 - Resonance

Topic 1 - Frequency Response of First Order Systems

- Frequency Input Concept
- Complex Numbers Review
- Frequency Response of First Order Systems
- Graph of Frequency Response

Frequency Input Concept

The term **frequency response** is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to *harmonic* input such as sines and cosines. The input (forcing) function is written below.

$$f(t) = A\sin(\omega t)$$

Amplitude of the Input,	A	(N)
Frequency of Input,	ω	$\left(\frac{rad}{s}\right)$

Frequency Input Concept

Why do we care about the way a system responds to harmonic excitation? Why is **frequency analysis** important?

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What causes **harmonic** (or sinusoidal) excitation in the real world?

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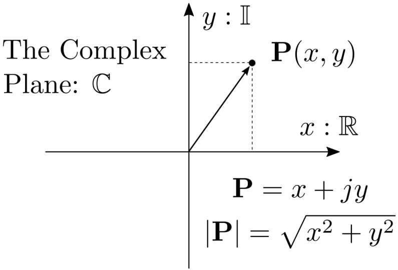
Frequency Input Concept

A linear, time-invariant (LTI) system has a **transfer function** $T(s)$ that describes the **input-output** relationship of the system. Under sinusoidal excitation (input) with frequency ω if the system is stable the transient affects in the response (output) will eventually disappear leaving the **steady state sinusoidal response** of the same frequency as the input but with a phase shift w.r.t. the input.

Complex Numbers Review

In an underdamped system the roots of the characteristic polynomial are complex. Before we proceed we need to review some rules of arithmetic and complex numbers.



Cartesian Representation:

$P = x + jy$

Polar Representation:

$P = |P|\angle\theta$

Exponential Representation:

$P = |P|e^{j\theta} = |P|\left(cos\theta + jsin\theta\right)$

Complex Number Algebra

Consider two points P_1 and P_2 on the complex plane.

$P_1 = x_1 + jy_1$ and $P_2 = x_2 + jy_2$

Addition: $$P_1 + P_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication: $$P_1 P_2 = |P_1 P_2| \angle (\theta_1 + \theta_2)$$

Division: $$\frac{P_1}{P_2} = (x_1 + x_2) + j(y_1 + y_2)$$

First Order Mass Damper

Consider our $1^{\underline{st}}$ order mass damper system.

$m\dot{v} + cv = f(t)$ with a **time constant** $\tau = \frac{m}{c}$

The system is commonly re-written as shown below.

$$m\dot{v} + cv = f(t) \quad \rightarrow \quad \tau\dot{y} + y = f(t)$$

First Order Transfer Function

$$\tau \dot{y} + y = f(t)$$

Take the Laplace transform of the ODE.

$$\mathcal{L}\{\tau \dot{y} + y\} = \mathcal{L}\{f(t)\}$$

$$\tau\left(sY(s) + y_0)\right) + Y(s) = F(s)$$ The initial conditions are zero.

$$\boxed{T(s) = \frac{Y(s)}{F(s)} = \frac{1}{\tau s+1} \qquad \text{First Order Transfer Function}}$$

This considers a *generalized* input function $f(t)$ and zero ICs.

Sinusoidal Input Function

Our model is now excited by a sinusoidal input function.

$$\tau \dot{y} + y = f(t) = Asin(\omega t)$$

Take the Laplace transform. Then, solve for $Y(s)$ and expand.

$$\tau s Y(s) + Y(s) = \frac{A\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{A\omega}{(s^2+\omega^2)(\tau s+1)} = \frac{C_1}{\tau s+1} + \frac{C_2 s}{(s^2+\omega^2)} + \frac{C_3\omega}{(s^2+\omega^2)}$$

Now solve for the coefficients.

$$C_1 = \frac{A\omega\tau^2}{1+\omega^2\tau^2} \quad , \quad C_2 = \frac{-A\omega\tau}{1+\omega^2\tau^2} \quad , \quad C_3 = \frac{A}{1+\omega^2\tau^2}$$

Substituting and take the inverse Laplace transform.

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau} sin\omega t\right)$$

Steady State Time Response

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau}sin\omega t\right)$$

After some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y(t) = \frac{A}{1+\omega^2\tau^2}\left(sin\omega t - \omega\tau cos\omega t\right)$$

This is re-written as a single sine term with a phase shift.

Steady State Frequency Response of First Order System

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}}sin\left(\omega t + \phi\right) \quad , \quad \phi = -tan^{-1}\omega\tau$$

Amplitude Ratio

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Notice that the system responds at the same frequency as the input but with a different amplitude and a phase shift. The ratio of the response amplitude to the input amplitude is called the **amplitude ratio, M**.

$$M = \frac{\frac{A}{\sqrt{1+\omega^2\tau^2}}}{A} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Fortunately we can find the **amplitude ratio** and **phase shift** directly from the transfer function. Recall the transfer function we derived.

$$T(s) = \frac{1}{\tau s+1} \qquad \text{let } s = j\omega \qquad \Longrightarrow \qquad T(j\omega) = \frac{1}{\tau j\omega+1}$$

$$|T(j\omega)| = \frac{|1|}{|\tau j\omega+1|} = \frac{1}{\sqrt{(\tau\omega)^2+1^2}} = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

Look familiar?

Phase Angle

$$|T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = M(\omega)$$

$$\angle T\left(j\omega\right) = \angle 1 - \angle\left(1 + j\omega\tau\right) = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{\omega\tau}{1}\right) =$$

$$-tan^{-1}\left(\omega\tau\right) = \phi(\omega)$$

Substitute $s = j\omega$ into the transfer function and solve for the magnitude and phase angle of this complex number which represent the magnitude ratio and phase shift.

Therefore the steady state response is written as follows.

$$y_{ss}\left(t\right) = A|T\left(j\omega\right)|sin\left(\omega t + \angle T\left(j\omega\right)\right) = MAsin\left(\omega t + \phi\right)$$

Wasn't that fun? Can you believe we used to do that on the board?!?!

Topic 2 - The Bode Diagram

- Review Frequency Response
- Amplitude ratio in Decibels
- The Bode Diagram
- Frequency Response in MATLAB

Harmonic Input Function

The term **frequency response** is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to *harmonic* input such as sines and cosines. The input (forcing) function is written below.

$$f(t) = A\sin(\omega t)$$

Amplitude of the Input, A (N)

Frequency of Input, ω $\left(\frac{rad}{s}\right)$

First Order Frequency Response

The steady state response we derived is shown. Remember, after some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y_{ss}(t) = A|T(j\omega)|sin(\omega t + \angle T(j\omega)) = MAsin(\omega t + \phi)$$

The amplitude ratio and phase shift can be found from $T(j\omega)$.

$$M(\omega) = |T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\phi(\omega) = \angle T(j\omega) = -tan^{-1}(\omega\tau)$$

Graph of Frequency Response

The amplitude of the response is determined by the input frequency.

First Order Frequency Response

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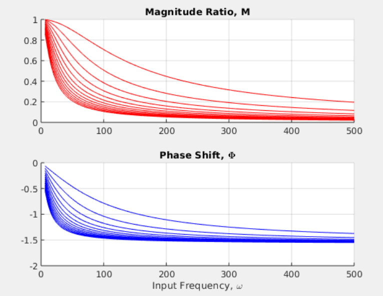
Amplitude ratio in Decibels

The amplitude of the response is determined by the input frequency.

Dependence on Input Frequency



You can see that the amplitude ratio decreases as the input frequency increases. The individual curves represent systems with different time constants.

Review Properties of Logarithms

Basic Properties of Logarithms:

Multiplication	$log(pq) = log(p) + log(q)$
Division	$log(\frac{x}{y}) = log(x) - log(y)$
Power	$log(x^n) = nlog(x)$

Units of Decibels for Magnitude:

$m(dB) = 10log(M^2) = 20log(M)$ convert back: $M = 10^{\frac{m(dB)}{20}}$

Decibel (dB), unit for expressing the ratio between two physical quantities, usually amounts of acoustic or electric power, or for measuring the relative loudness of sounds. One decibel (0.1 bel) equals 10 times the common logarithm of the power ratio. - Britannica.com

Amplitude ratio on a Logarithmic Scale

These relationships are more useful shown on a logarithmic scale. We can make use of the properties of logarithms in our analysis.

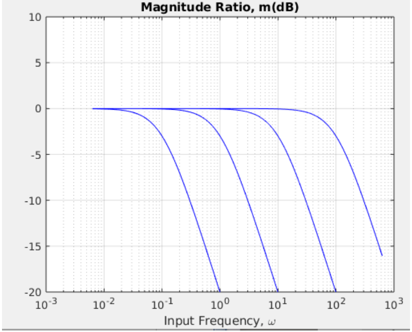
$$m\,(dB) = 20log\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right) = 20\left(log\,(1) - log\sqrt{1+\omega^2\tau^2}\right)$$

$$m\,(dB) = 20log\,(1) - 10log\,\left(1+\omega^2\tau^2\right) = -10log\,\left(1+\omega^2\tau^2\right)$$

$$m\,(dB) = -10log\,\left(1+\omega^2\tau^2\right)$$

amplitude ratio in decibels

Amplitude ratio on a Logarithmic Scale



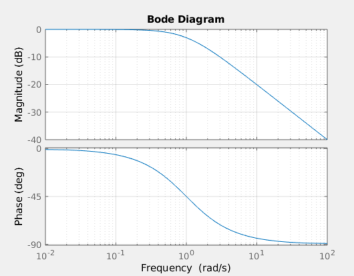
This is a Bode plot. It seems abstract but there is some very useful information shown.

Hendrik Bode (1905-1982)

The Bode Diagram

Bode Plot in MATLAB

MATLAB has a built it tool for making Bode plots.




```
figure(1)
sys=tf(1,[tau(3)
    1])
bode(sys);grid on
```

So what? What can you do with a Bode diagram?

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain

Topic 3 - Frequency Response of 2^{nd} Order Systems

- Review Transfer Functions
- Frequency Response of Overdamped Systems
- Frequency Response of Underdamped Systems
- MATLAB Bode Plots

Equivalent System Representations

The**Transfer Function** is the input-output relationship in the frequency domain and can be found from the equation of motion of the system.

$$T(s) = \frac{X(s)}{F(s)}$$

The Transfer Function is an equivalent representation of the system.

E.O.M	$\leftrightarrow$	T(s)	$\leftrightarrow$	Block Diagram

Transfer Function of 2^{nd} Order System

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad \text{with} \qquad f(t) = A sin(\omega t)$$

The transfer function can easily be found by taking the Laplace transform of the equation of motion.

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k}$$ Second Order Transfer Function

The Overdamped System

In an overdamped system, both roots are real and distinct.

The transfer function is shown below in terms of the system parameters

$T(s) = \frac{X(s)}{F(s)} = \frac{1/k}{\left(\frac{m}{k}\right)s^2+\left(\frac{c}{k}\right)s+1} = \frac{1/k}{(\tau_1 s+1)(\tau_2 s+1)}$ τ_1, τ_2 - time constants

Substitute $s = j\omega$ into the transfer function and find the amplitude ratio and phase angle.

$T(s) \to T(j\omega) = \frac{1/k}{(\tau_1 j\omega+1)(\tau_1 j\omega+1)}$

$$M(\omega) = |T(j\omega)| = \frac{|1/k|}{|\tau_1 j\omega+1||\tau_2 j\omega+1|}$$

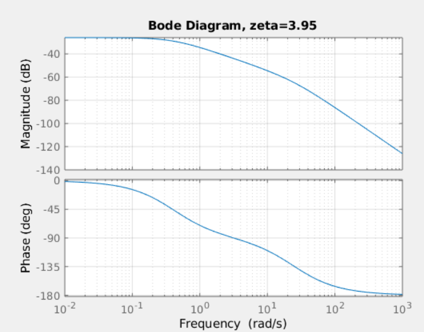
$$m(\omega) = 20logM(\omega) = 20log|1/k| - 20log|\tau_1 \omega j + 1| - 20log|\tau_2 \omega j + 1|$$

$$\phi(\omega) = \angle\frac{1}{k} - \angle(\tau_1 \omega j + 1) - \angle(\tau_2 \omega_j + 1)$$

The Bode Diagram

These three terms can be seen on the Bode diagram.

$$m(\omega) = 20logM(\omega) = 20log|1/k| - 20log|\tau_1\omega j + 1| - 20log|\tau_2\omega j + 1|$$



This shows that the magnitude ratio of the system across different regions of the input frequency.

The Underdamped System

In an underdamped system, the roots are complex conjugates.

The transfer function is shown below in terms of the system parameters

$$T(s) = \frac{X(s)}{F(s)} = \frac{1/k}{\left(\frac{m}{k}\right)s^2 + \left(\frac{c}{k}\right)s + 1} = \frac{1/k}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$T(s) = \frac{kX(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Notice we factored out k to form the ratio of output displacement $X(s)$ to input displacement $\frac{F(s)}{k}$. You can see this with Hooke's Law $F = kx \implies x = \frac{F}{k}$.

This also allows us to define the transfer function in terms of ζ and ω_n. Substitue $s = j\omega$ and mulitply the equation $\frac{1/\omega_n^2}{1/\omega_n^2}$.

$$T(s) \rightarrow T(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + \left(\frac{2\zeta}{\omega_n}\right)j\omega + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)j}$$

The Frequency Ratio

To simplify this expression we define another new quantity the frequency ratio, r as the ratio of input frequency to natural frequency of the system. The transfer function is re-written in terms of the frequency ratio.

$$r = \frac{\omega}{\omega_n} \quad \rightarrow \quad T(j\omega) \quad \rightarrow \quad T(r) = \frac{1}{1-r^2+2\zeta rj}$$

Now the amplitude ratio and phase are written in tems of r.

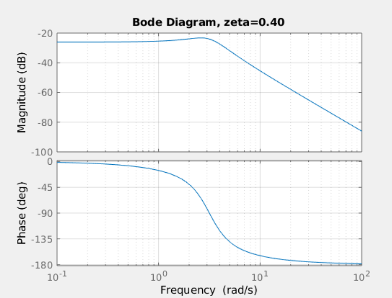
$$M = |T(r)| = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

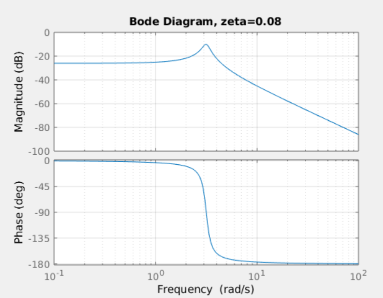
$$\implies m = 20logM = -10log\left[\left(1-r^2\right)^2 + \left(2\zeta r\right)^2\right]$$

$$\phi = \angle 1 - \angle\left(1 - r^2 + 2\zeta rj\right) \implies \phi = -tan\left(\frac{2\zeta r}{1-r^2}\right)$$

The Bode Diagram

In the underdamped second order system only two regions are present in the Bode diagram.





As the damping ratio decreases something significant happens. This Bode diagram shows something that the others before have not.

MATLAB Bode Plots

Vary the system parameter in the scripts to make the plots shown.

```
clear variables;clc;close all

% define the system parameters
m=2;c=1;k=20;
zeta=c/(2*sqrt(m*k));

% create a system object from the transfer
    function
sys=tf(1/k,[(m/k) (c/k) 1]);

% use built-in MATLAB Bode Plot
figure(1)

bode(sys);grid on
```

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain

Topic 3 - Resonance

- Review 2^{nd} Order Frequency Response
- The Resonance Phenomenon
- The Resonance Frequency
- —

Transfer Function of 2^{nd} Order System

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{with} \quad f(t) = A\sin(\omega t)$$

The transfer function can easily be found by taking the Laplace transform of the equation of motion.

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k}$$ Second Order Transfer Function

The amplitude ratio and phase angle can be found from the transfer function. Think about what M means.

Overdamped vs. Underdamped Systems

In an overdamped system, both roots are real and distinct.

$$M(\omega) = |T(j\omega)| = \frac{|1/k|}{|\tau_1 j\omega + 1||\tau_2 j\omega + 1|}$$

$$m(\omega) = 20 log M(\omega) = 20 log|1/k| - 20 log|\tau_1 \omega j + 1| - 20 log|\tau_2 \omega j + 1|$$

$$\phi(\omega) = \angle \frac{1}{k} - \angle(\tau_1 \omega j + 1) - \angle(\tau_2 \omega_j + 1)$$

In an underdamped system, the roots are complex conjugates.

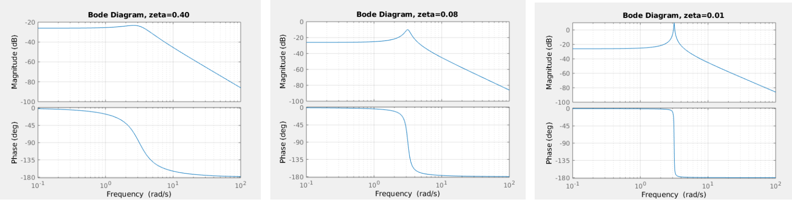
$$M(r) = |T(r)| = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{with} \quad r = \frac{\omega}{\omega_n}$$

$$\implies m = 20 log M = -10 log\left[\left(1 - r^2\right)^2 + (2\zeta r)^2\right]$$

$$\phi = \angle 1 - \angle\left(1 - r^2 + 2\zeta r j\right) \implies \phi = -tan\left(\frac{2\zeta r}{1-r^2}\right)$$

The Resonance Spike

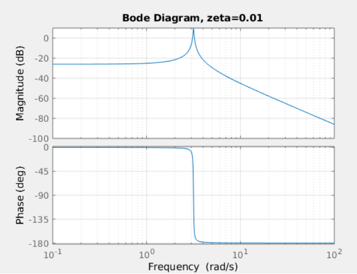
In the underdamped second order system only two regions are present separated by the point near $r = 1$.



As the damping ratio decreases something significant happens at this points. Remember, these are graphs of $m = 20logM$.

The Effects of Resonance

Consider the extreme case in the figure shown. What is the physical significance of the values of m near $r = 1$?



The Effects of Resonance

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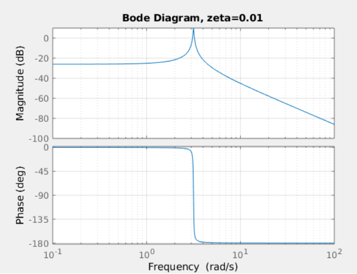
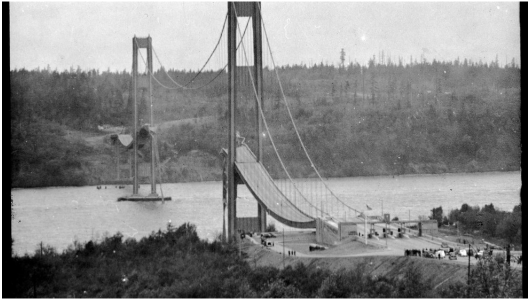


Resonance can be Destructive

The resonance peak represents a amplitude ratio greater than one meaning the output amplitude is larger that in the input amplitude. This large amplitude output displacement caused by resonance correspond to a large force in the spring members and the large forces are transmitted to the body. Large forces cause mechanical failure.



The Resonance Frequency

Where on the frequency response graphs does resonance occur? It looks like it is *near* $r = 1$.

$$M(r) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{with} \quad r = \frac{\omega}{\omega_n}$$

The value of M is maximized when the denominator is minimized. Therefore the resonance frequency is found by taking the derivative of the denominator and setting it equal to zero.

$$M_{max} \quad \text{occurs at } r = \sqrt{1 - 2\zeta^2} \implies \omega = \omega_n \sqrt{1 - 2\zeta^2}$$

An input frequency equal to the resonance frequency causes maximum output displacement.

The Resonance Frequency

The resonance event only occurs in second order systems when the radical shown is positive. This corresponds to systems with damping ratio in the range $0 < \zeta < 0.707$.

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \quad 0 < \zeta < 0.707$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{or in decibels} \qquad m_r = -20 log \left(2\zeta\sqrt{1 - \zeta^2}\right)$$

The phase at resonance can also be found.

$$\phi_r = tan^{-1}\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$$

The Resonance Frequency

It is important to note that we multiplied the transfer function by k during the derivations. Therefore if me divide the expressions for M and M_r by k to find the amplitude ratio between input force, $f(t)$ and output displacement, $x_{ss}(t)$

$$M = \frac{1}{k\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\Longrightarrow \quad m = -20log(k) - 10log\left[\left(1-r^2\right)^2 + (2\zeta r)^2\right]$$

$$M_r = \frac{1}{k2\zeta\sqrt{1-\zeta^2}}$$

$$\Longrightarrow \quad m_r = -20log\,(k) - 20log\left(2\zeta\sqrt{1-\zeta^2}\right)$$

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain