Lecture Module - The Laplace Transform

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Module 2 - The Laplace Transform

Module 2 - The Laplace Transform

- Topic 1 - The Laplace Transform
- Topic 2 - Laplace Transforms Method
- Topic 3 - Partial Fraction Decomposition

Topic 1 - The Laplace Transform

- An Integral Transform
- Laplace Transform of A Derivative
- Properties of an Integral

An Integral Transform

The Laplace Transform is an Integral Transform
Given a function $x(t)$ in the time domain where $t \geq 0$, the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$$

And its inverse is similarly defined as:

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

The Laplace Domain variable s is a complex number:

$$s = \sigma + j\omega$$

An Integral Transform

An Integral Transform

Laplace Transform of A Derivative

It is useful to find the laplace transform of the derivative of a function:

$$\mathcal{L}\{\frac{d}{dt}(x(t))\} = \mathcal{L}\{\dot{x}(t)\} = s\mathcal{L}\{x(t)\} - x(t=0)$$

$$= sX(s) - x(t=0)$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$$

$$\mathcal{L}\{\frac{d^2}{dt^2}(x(t))\} = \mathcal{L}\{\ddot{x}(t)\} = s^2\mathcal{L}\{x(t)\} - sx(t=0) - \dot{x}(t=0)$$

$$= s^2X(s) - sx(t=0) - \dot{x}(t=0)$$

$$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx_0 - \dot{x}_0$$

Laplace Transform of A Derivative

Properties of an Integral

Also, remember that the transform inherits the properties of an integral.

$$\int [x(t) + y(t)]\, dt = \int x(t)dt + \int y(t)dt$$

$$\int Kx(t)dt = K \int x(t)dt \quad (K \;\; is \;\; constant)$$

Therefore these properties can be used with the Laplace transform.

Table of Transform Pairs

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$

Table of Transform Pairs

17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^nf(t),\quad n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Topic 2 - Laplace Transforms Method

- Step 1 - Apply Laplace Transform
- Step 2 - Solve for $X(s)$
- Step 3 - Rearrange to Find Invertable Form

Step 1 - Apply Laplace Transform

Example:
Solve the first order differential equation using the Laplace Transforms Method with the initial condition given.

$$4\dot{x} = sin\left(t\right) \qquad with \qquad x\left(t = 0\right) = x_0$$

Apply the Laplace Transform to both sides of the differential equation.

$$4\left(sX\left(s\right) - x_0\right) = \frac{1}{s^2 + 1}$$

Step 2 - Solve for $X(s)$

This step can seem open ended...

$$X(s) = \frac{1}{4s\left(s^2+1\right)} + \frac{x_0}{s}$$

Step 2 - Solve for $X(s)$

Step 3 - Rearrange to Find Invertable Form

Write $X(s)$ in a form that can be inverted using the table of Laplace transform pairs. This typically involves partial fraction decomposition.

$$\frac{1}{4s\left(s^2+1\right)} = \frac{1/4}{s\left(s^2+1\right)} = \frac{a}{s} + \frac{bs+c}{s^2+1}$$

Mulitply through by the denominator $4s\left(s^2+1\right)$:

$$1 = 4as\left(s^2+1\right) + 4s\left(bs+c\right) = 4\left(a+b\right)s^2 + 4cs + 4a$$

Solve for the coefficients by *equating coefficients*.

$$(a+b) = 0 \quad c = 0 \quad a = \frac{1}{4} \implies a = \frac{1}{4} \quad b = -\frac{1}{4} \quad c = 0$$

Step 2 - Solve for $X(s)$

Step 3 - Rearrange to Find Invertable Form

Step 4 - Invert for Final Answer

Substitute the coefficients into $X(s)$,

$$X(s) = \frac{x_0}{s} + \frac{1}{4s} - \frac{s}{4\left(s^2+1\right)}$$

and use the inverse transform to solve for $x(t)$. Use the Table.

$$\mathcal{L}^{-1}\left(X\left(s\right)\right) = x(t) =$$

$$= x_0 + \frac{1}{4} - \frac{1}{4}cos\left(t\right) = x_0 + \frac{1}{4}\left(1 - cos\left(t\right)\right)$$

This method works for complex problems but it can get messy...

Step 4 - Invert for Final Answer

Table of Laplace Transforms

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1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
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Step 4 - Invert for Final Answer

Table of Laplace Transforms

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1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
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Topic 3 - Partial Fraction Decomposition

- General Polynomial Form
- Case 1 - Distinct Roots
- Case 2 - Repeated Roots
- Special Case - Complex Roots

The Laplace Transform is an Integral Transform:
Given a function $x(t)$ in the time domain where $t \geq 0$, the Laplace Transform is defined as follows:

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Partial Fraction Expansion leads to a general form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \qquad n \geq m}$$

General Polynomial Form

Case 1 - Distinct Roots: n roots are real and distinct

The general form is factored:

$$X(s) = \frac{N(s)}{(s+r_1)(s+r_2)...(s+r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s+r_1)} + \frac{C_2}{(s+r_2)} + ... + \frac{C_n}{(s+r_n)}$$

Where:

$$C_i = \lim_{s \to -r_i} \{X(s)(s+r_i)\}$$

And this leads to a solution:

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + ... + C_n e^{-r_n t}$$

Case 2 - Repeated Roots: p number of roots have the same value ($s = -r$) and remaining roots are distinct and real distinct

$$X(s) = \frac{N(s)}{(s+r_1)^p(s+r_{p+1})(s+r_{p+2})...(s+r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + ...$$

$$+\frac{C_p}{(s+r_1)} + \frac{C_{p+1}}{(s+r_{p+1})} + ... + \frac{C_n}{(s+r_n)}$$

Case 2 - Repeated Roots

Coefficients for the repeated root are:

$$C_1 = \lim_{s \to -r_i} \{X(s)(s + r_i)^p\}$$

$$C_2 = \lim_{s \to -r_i} \{\frac{d}{ds} X(s)(s + r_i)^p\}$$

$$C_i = \lim_{s \to -r_i} \{\frac{1}{(i-1)!} \frac{d^{(i-1)}}{ds^{(i-1)}} X(s)(s + r_i)^p\}$$

Coefficients for the distinct roots are the same as in Case 1:
And this leads to a solution:

$$x(t) = C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \ldots$$

$$\ldots + C_p e^{-r_1 t} + C_{p+1} e^{-r_{p+1} t} \ldots + C_n e^{-r_n t}$$

Special Case - Complex Roots: the roots are distinct $\Longrightarrow$ Case 1

Example:

$$X(s) = \left[\frac{3s+7}{(4s^2+24s+136)}\right] = \left[\frac{3s+7}{4\left(s^2+6s+34\right)}\right]$$

The solution can be found by forming two perfect squares in the denominator.

$$X(s) = \frac{1}{4}\left[\frac{3s+7}{(s+3)^2+5^2}\right]$$

Special Case - Complex Roots

Now this can be expanded into the following terms which can be found in the table!

$$X(s) = \frac{1}{4}\left[C_1 \frac{5}{(s+3)^2 + 5^2} + C_2 \frac{s+3}{(s+3)^2 + 5^2}\right]$$

Multiply by the denominator and solve for C_1 and C_2.

$$3s + 7 = 5C_1 + C_2\,(s+3) = 5C_1 + c_2 s + 3C_2 \implies C_2 = 3,\, C_1 = -$$

Finally substitute and invert using the table.

$$X(s) = \frac{1}{4}\left[-\frac{2}{5}\frac{5}{(s+3)^2+s^2} + 3\frac{s+3}{(s+3)^2+s^2}\right]$$

Write the final answer in the time domain.

$$x(t) = -\frac{1}{10}e^{-3t}sin\,(5t) + \frac{3}{4}e^{-3t}cos\,(5t)$$

Special Case - Complex Roots