Frequency Response - Lecture 3

ME3050 - Dynamics Modeling and Controls

April 25, 2020

Frequency Response of 2nd Order Systems

Lecture 3 - Frequency Response of 2^{nd} Order Systems

- Review Transfer Functions
- Frequency Response of Overdamped Systems
- Frequency Response of Underdamped Systems
- MATLAB code for Bode Plots

Equivalent System Representations

The**Transfer Function** is the input-output relationship in the frequency domain and can be found from the equation of motion of the system.

$$T(s) = \frac{X(s)}{F(s)}$$

The Transfer Function is an equivalent representation of the system.

$$\boxed{\text{E.O.M} \quad \leftrightarrow \quad \text{T(s)} \quad \leftrightarrow \quad \text{Block Diagram}}$$

Transfer Function of 2^{nd} Order System

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad \text{with} \qquad f(t) = Asin(\omega t)$$

The transfer function can easily be found by taking the Laplace transform of the equation of motion.

$$\boxed{T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k} \qquad \text{Second Order Transfer Function}}$$

The Overdamped System

In an overdamped system, both roots are real and distinct.

The transfer function is shown below in terms of the system parameters

$$T(s) = \frac{X(s)}{F(s)} = \frac{1/k}{\left(\frac{m}{k}\right)s^2 + \left(\frac{c}{k}\right)s + 1} = \frac{1/k}{(\tau_1 s+1)(\tau_2 s+1)} \qquad \tau_1, \tau_2 \text{ - time constants}$$

Substitute $s = j\omega$ into the transfer function and find the amplitude ratio and phase angle.

$$T(s) \rightarrow T(j\omega) = \frac{1/k}{(\tau_1 j\omega+1)(\tau_1 j\omega+1)}$$

$$M(\omega) = |T(j\omega)| = \frac{|1/k|}{|\tau_1 j\omega+1||\tau_2 j\omega+1|}$$

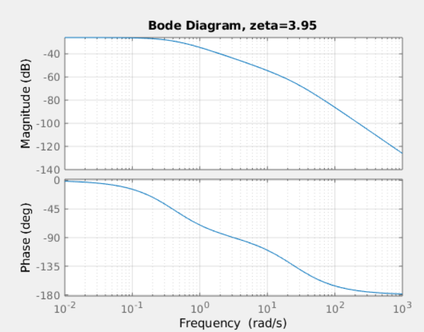
$$m(\omega) = 20logM(\omega) = 20log|1/k| - 20log|\tau_1 \omega j + 1| - 20log|\tau_2 \omega j + 1|$$

$$\phi(\omega) = \angle\frac{1}{k} - \angle(\tau_1 \omega j + 1) - \angle(\tau_2 \omega_j + 1)$$

The Bode Diagram

These three terms can be seen on the Bode diagram.

$$m(\omega) = 20 log M(\omega) = 20 log|1/k| - 20 log|\tau_1 \omega j + 1| - 20 log|\tau_2 \omega j + 1|$$



This shows that the magnitude ratio of the system across different regions of the input frequency.

The Underdamped System

In an underdamped system, the roots are complex conjugates.

The transfer function is shown below in terms of the system parameters

$$T(s) = \frac{X(s)}{F(s)} = \frac{1/k}{\left(\frac{m}{k}\right)s^2+\left(\frac{c}{k}\right)s+1} = \frac{1/k}{\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1}$$

$$T(s) = \frac{kX(s)}{F(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Notice we factored out k to form the ratio of output displacement $X(s)$ to input displacement $\frac{F(s)}{k}$. You can see this with Hooke's Law $F = kx \implies x = \frac{F}{k}$.

This also allows us to define the transfer function in terms of ζ and ω_n. Substitue $s = j\omega$ and mulitply the equation $\frac{1/\omega_n^2}{1/\omega_n^2}$.

$$T(s) \rightarrow T(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2+\left(\frac{2\zeta}{\omega_n}\right)j\omega+1} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)j}$$

The Frequency Ratio

To simplify this expression we define another new quantity the frequency ratio, r as the ratio of input frequency to natural frequency of the system. The transfer function is re-written in terms of the frequency ratio.

$$r = \frac{\omega}{\omega_n} \quad \rightarrow \quad T\left(j\omega\right) \quad \rightarrow \quad T\left(r\right) = \frac{1}{1-r^2+2\zeta rj}$$

Now the amplitude ratio and phase are written in tems of r.

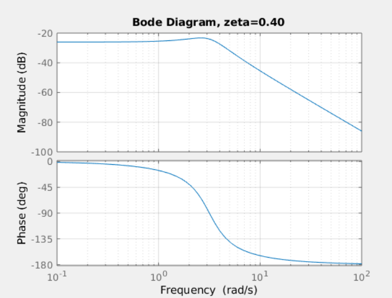
$$M = \left|T\left(r\right)\right| = \frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta rj\right)}}$$

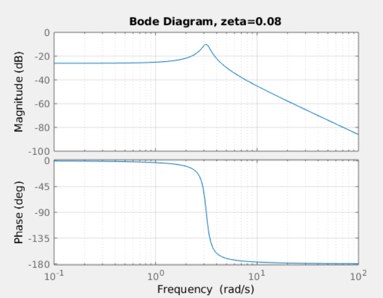
$$\implies m = 20logM = -10log\left[\left(1-r^2\right)^2+\left(2\zeta r\right)^2\right]$$

$$\phi = \angle 1 - \angle\left(1-r^2+2\zeta rj\right) \implies \phi = -tan\left(\frac{2\zeta r}{1-r^2}\right)$$

The Bode Diagram

In the underdamped second order sytem only two regions are present in the Bode diagram.

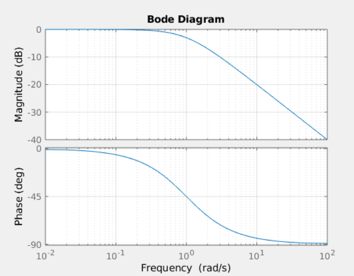




As the damping ratio decreases something significant happens. This Bode diagram shows something that the others before have not.

Bode Plot in MATLAB

MATLAB has a built in tool for making Bode plots.




```
figure(1)
sys=tf(1,[tau(3) 1])
bode(sys);grid on
```

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain