(Ch. 10) Introduction to Feedback Control Systems

10.1

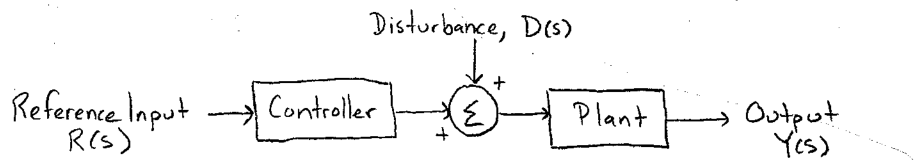
An exciting application that ties together all of the modeling and analysis we have done so far is the design of feedback control systems. In order to accurately control a system, we must model the system and understand its response to various inputs. Control systems are ubiquitous in our technological world.

Control system examples: cruise control, auto-aiming, memory seats and auto-dimming mirrors, auto-dimming cell phone screens, ovens, HVAC systems, etc.

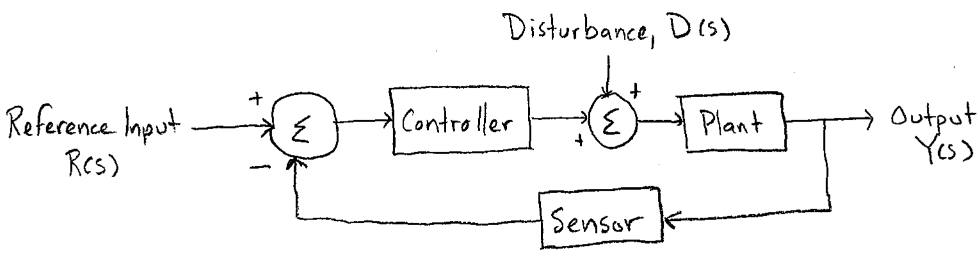
Terminology:

Consider a light switch. By itself, there is no control, the light only comes on when you turn the switch. If you put a timer on the switch, this is called "open-loop" control. The light automatically turns on, but there is no sensor to provide feedback to say if the light turned on. Now, if you use a light sensor, this is "closed-loop" feedback.

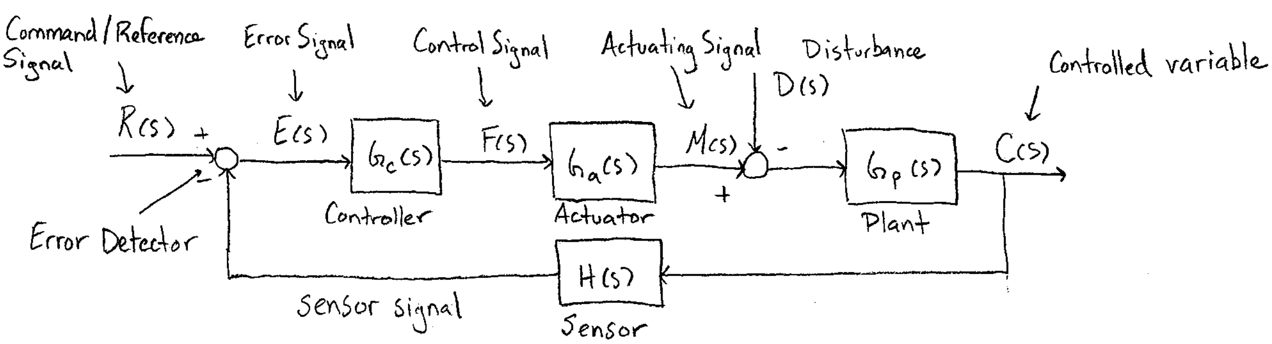
- "Open-loop" control: when a control system does not use a sensor to measure the system output being controlled in computing the control action to take. The output does not affect the control action. Example: toaster



- "Closed-loop" control: when a sensor is used to measure the output and that information is used to modify the control action. The output does affect the control action. Example: thermostat



Control system components:



- $R(s)$, Command / Reference Signal: where you want the output to be
- $C(s)$, Controlled Variable: where the output actually is
- $D(s)$, Disturbance: external variables that can't be accounted for in the plant
- Plant: the system being controlled
- Controller: the logic element that compares the command signal to the measured signal and decides what action to take
- Actuator: the device that produces physical measures (force, torque, pressure, heat, etc.) to influence the plant.
- Sensor: measures the controlled output.

★ If $H(s)C(s) \neq R(s)$, then there is error and the system needs control. The purpose of this chapter is to learn how to design $G_c(s)$ to get $C(s) = R(s)$.

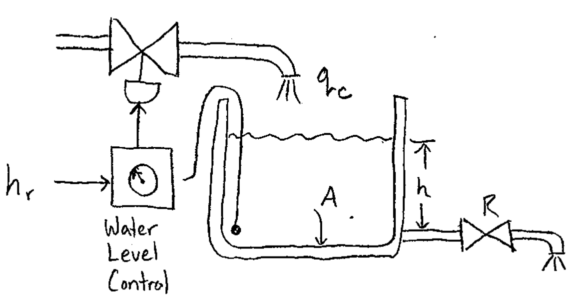
Example: Washing Machines - open loop

Select load → determines water level. Level is set using a pressure switch.

Select type; regular → delicate → sets timer.

Both open loop systems.

Consider water level. Switch is used to turn water on and off, but it does not detect the actual water height, it only detects pressure which is assumed to be proportional to the water level.



linearized model:

$$RA\dot{h} + gh = Rq_c \qquad q_c = \text{input} \quad h = \text{output} \qquad T(s) = \frac{H(s)}{Q_c(s)}$$

$$Q_c(s) \longrightarrow \boxed{\frac{R}{RAs+g}} \longrightarrow H(s)$$

Suppose we want to control the inlet flow rate, q_c, in order to control the liquid level. This can be done by controlling the inlet valve. Our block diagram is:

$$H_r(s) \longrightarrow \underbrace{\boxed{G_c(s) = \frac{R_eA_e s + g}{R_e}}}_{\text{Controller}} \longrightarrow \overbrace{\underbrace{\boxed{G_a(s) = 1}}_{\text{Actuator}}}^{\text{Valve}} \xrightarrow{Q_c(s)} \overbrace{\underbrace{\boxed{G_p(s) = \frac{R}{RAs+g}}}_{\text{Plant}}}^{\text{Tank}} \longrightarrow H(s)$$

If we can estimate the area (A_e) and resistance (R_e), then the required flow rate from the model is:

$$q_c = A_e \dot{h}_r + \frac{g}{R_e} h_r \Rightarrow G_c(s) = \frac{Q_c(s)}{H_r(s)} = \frac{R_e A_e s + g}{R_e}$$

The block diagram of the control system gives:

$$H(s) = Q_c(s)\frac{R}{RAs+g} = \frac{R_e A_e s + g}{R_e}\,\frac{R}{RAs+g}\,H_r(s)$$

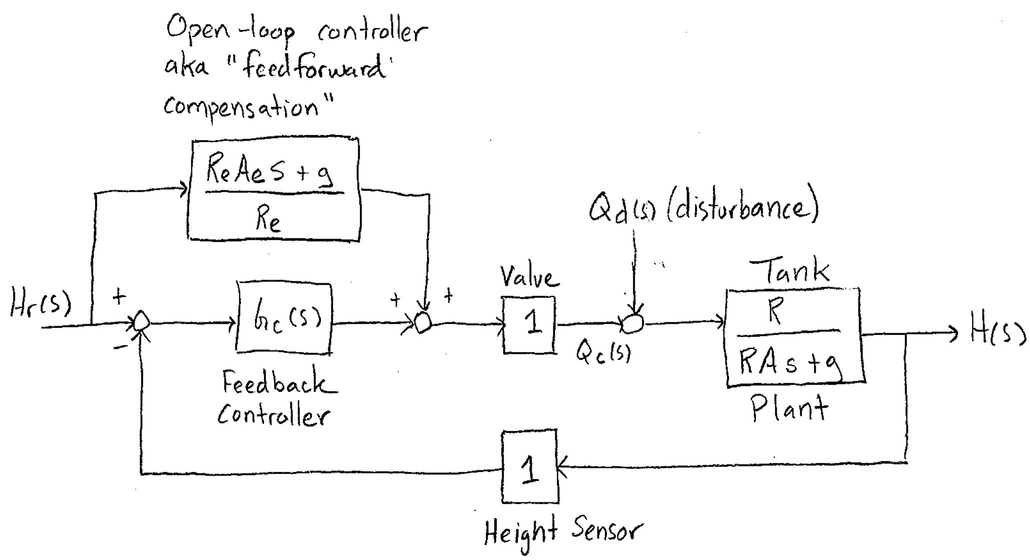
If $R_e = R$, $A_e = A$, this gives $H(s) = H_r(s)$, so the desired height is reached.

There are several problems with this open-loop approach:

- A_e + R_e are estimates whose exact value may be difficult to find
- Clogged pipes, leaky valves can change plant.
- Controller doesn't take exact derivatives ($\dot{h}_r$)
- $\frac{R}{RAs+g}$ is also an estimate.

Washing machine - closed loop

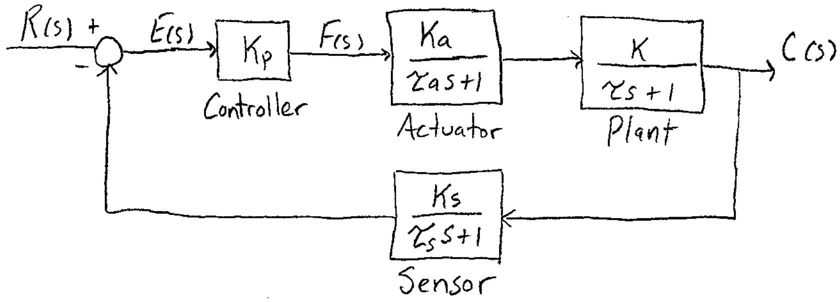
A liquid level sensor can be added to the system to provide closed-loop feedback control. The sensor controls the valve to adjust the flow.



Ch. 10 focuses on the design of $G_c(s)$. We will focus on feedback control and not consider feedforward compensation.

10.3) Modeling Control Systems

Consider washing machine, closed loop. Plant is first-order. We can model the sensor and actuator as first-order since they do take some time to respond. If the controller transfer function is constant, K_p, called "proportional gain", we can draw our system as:

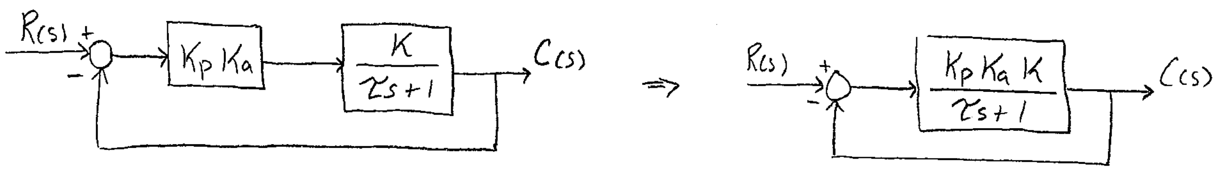


If sensor, actuator response times are small compared to plant, then:

- if $\tau_s, \tau_a << \tau \Rightarrow G_a(s) = K_a, \quad G_s(s) = K_s$

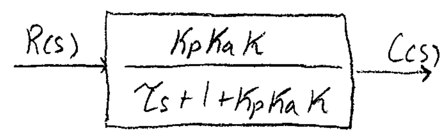
If sensor is accurate, $K_s = 1 \Rightarrow G_s(s) = 1$

This simplifies the system to:

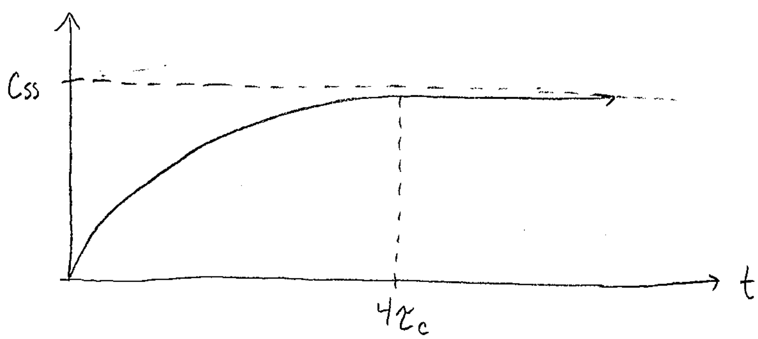


Whose transfer function is: (recall "feedback loop simplification" from block diagrams)

$$\frac{C(s)}{R(s)} = \frac{\frac{K_p K_a K}{\tau s+1}}{1 + \frac{K_p K_a K}{\tau s+1}} = \frac{K_p K_a K}{\tau s+1 + K_p K_a K} \Rightarrow$$



This is a first-order system. The step response is



1^{st} order review:

$$a\dot{y} + by = f_o$$

$$T(s) = \frac{F_o}{as+b} \Rightarrow \begin{aligned} F_o &= K_p K_a K \\ a &= \tau \\ b &= 1 + K_p K_a K \end{aligned}$$

$$\tau = \frac{a}{b} \Rightarrow \tau_c = \frac{\tau}{1+K_p K_a K}$$

$$y_{ss} = \frac{F_o}{b} \Rightarrow C_{ss} = \frac{K_p K_a K}{1+K_p K_a K}$$

10.4 PID Control (Proportional-Integral-Derivative Control)

Objectives of a controller:

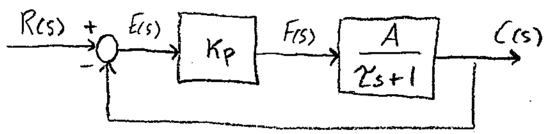
- Minimize steady-state error
- Minimize settling time
- Manipulate the transient response (e.g. minimize overshoot)

Proportional (P) Control

- Output of controller is proportional to the error between reference input, $R(s)$, and the controlled variable (output), $C(s)$.

$$G_c(s) = K_p = \text{constant} \qquad K_p \text{ is "proportional gain"}$$

Block Diagram (1st order plant)



$$\Rightarrow T(s) = \frac{C(s)}{R(s)} = \frac{\frac{K_p A}{\tau s+1}}{1+\frac{K_p A}{\tau s+1}} = \frac{K_p A}{\tau s+1+K_p A}$$

We can find the steady-state value of c using the final value theorem <u>if $T(s)$ is stable</u>.

$$C_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\,T(s)\,R(s)$$

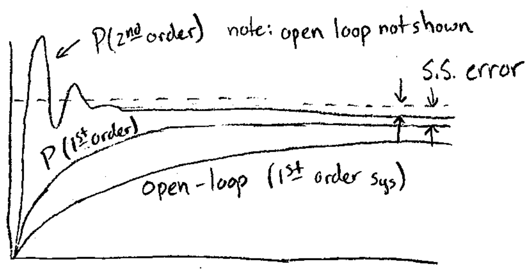
If $R(s)$ is a step input of magnitude M, then $R(s) = \frac{M}{s}$ giving

$$C_{ss} = \lim_{s\to 0} s\,T(s)\,\frac{M}{s} = M \lim_{s\to 0} T(s)$$

For our problem, we have:

$$C_{ss} = M \lim_{s\to 0} \frac{K_p A}{\tau s+1+K_p A} = M\,\frac{K_p A}{1+K_p A}$$

Response:



Comments:

- Decreases rise time as K_p increased
- More oscillations / overshoot as K_p increased (for 2nd order systems)
- Steady-state error in response

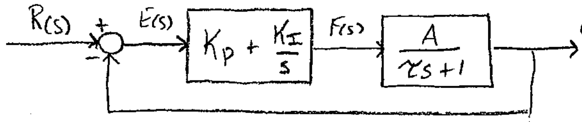
Integral (I) Control

- Output of controller is equal to integral of error between reference + controlled variable.

$$G_c(s) = \frac{K_I}{s} \qquad K_I \text{ is "integral gain"}$$

- Integral control by itself tends to make systems unstable, so we combine it with proportional to form PI control.

Block Diagram (1st order plant)



$$\Rightarrow T(s) = \frac{C(s)}{R(s)} = \frac{\frac{(K_p + K_I/s)A}{\tau s+1}}{1 + \frac{(K_p+K_I/s)A}{\tau s+1}} = \frac{(K_p + \frac{K_I}{s})A}{\tau s+1 + (K_p + \frac{K_I}{s})A}$$

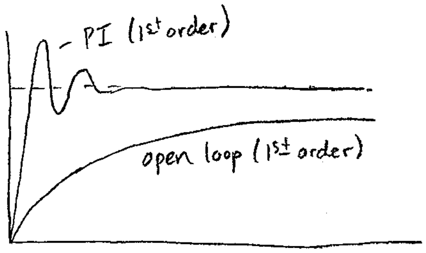
$$= \frac{(K_p s + K_I)A}{\tau s^2 + s + (K_p s + K_I)A}$$

note: integral control increases order of system by 1!

Css for step input of magnitude M using final value theorem

$$C_{ss} = M \lim_{s\to 0} T(s) = M \lim_{s \to 0} \frac{(K_p s + K_I)A}{\tau s^2 + s + (K_p s + K_I)A} = M \frac{K_I A}{K_I A} = M \Rightarrow \text{no s.s. error!}$$

Response:



Comments:

- Eliminates steady-state error
- Increases system order, creating potential for overshoot, oscillations, & instability!

Derivative (D) Control

- Output of controller is equal to derivative of error between reference & controlled variable

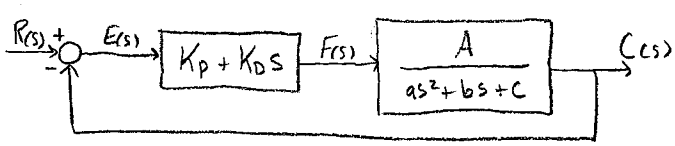
  $G_c(s) = K_D s$ $\quad$ K_D is "derivative gain"

- Should never be used alone b/c it does not produce an output if the signal is constant, even if the error is large. Can be combined with P or PI controllers to form PD + PID.

PD Control

- Not useful for 1st order systems since D doesn't eliminate S.S. error ⇒ just use P control.

Block Diagram (2nd-order plant)

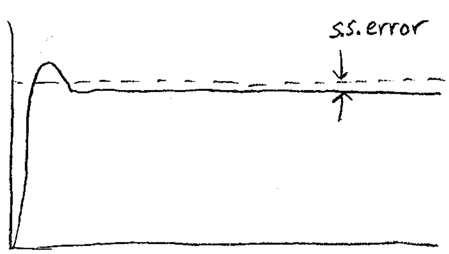


$$\Rightarrow T(s) = \frac{C(s)}{R(s)} = \frac{\frac{(K_p+K_D s)A}{as^2+bs+c}}{1+\frac{(K_p+K_D s)A}{as^2+bs+c}} = \frac{(K_p+K_D s)A}{as^2+bs+c+(K_p+K_D s)A}$$

Css for step input of magnitude M using final value theorem

$$C_{ss} = M \lim_{s\to 0} T(s) = M \lim_{s\to 0} \frac{(K_p+K_D s)A}{as^2+bs+c+(K_p+K_D s)A} = M \frac{K_p A}{c + K_p A}$$

Response:



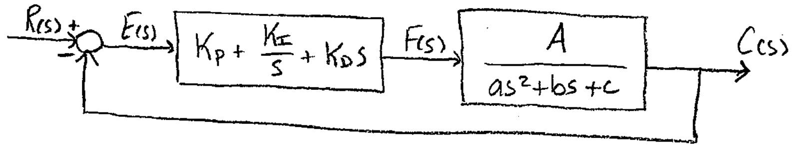
Comments:

- Reduces oscillations and overshoot, improves response time
- Does not remove S.S. error
- D term amplifies noise in measurement signals

PID Control

- Widely used in process industries, good for 1^{st} & 2^{nd} order systems

Block Diagram (2^{nd} Order Plant)



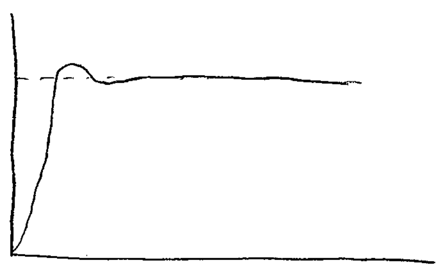
$$\Rightarrow T(s) = \frac{C(s)}{R(s)} = \frac{\frac{\left(K_P + \frac{K_I}{s} + K_D s\right)A}{as^2+bs+c}}{1 + \frac{\left(K_P + \frac{K_I}{s} + K_D s\right)A}{as^2+bs+c}}$$

$$T(s) = \frac{\left(K_P + \frac{K_I}{s} + K_D s\right)A}{as^2+bs+c + \left(K_P + \frac{K_I}{s} + K_D s\right)A} = \frac{\left(K_P s + K_I + K_D s^2\right)A}{as^3+bs^2+cs + \left(K_P s + K_I + K_D s^2\right)A}$$

C_{ss} for step input of magnitude M using final value theorem

$$C_{ss} = M \lim_{s \to 0} T(s) = M \frac{K_I A}{K_I A} = M \Rightarrow \text{so no s.s. error!}$$

Response:



Comments:

- S.S. error eliminated by I term
- D term reduces oscillations & overshoot
- High D gain can add noise to system

(10.5.5) Control System Design Procedure

Design Information

- Typically, plant is given + cannot be changed. Engineer may have to develop model.
- Typically, type of controller (electronic, mechanical, etc.) and type of actuator (electric, pneumatic, hydraulic, etc.) is given or is obvious based on system
- Step inputs are typically used to evaluate control system performance.

Design Procedure:

Step 1: Model the system, solve for $T(s) = \frac{C(s)}{R(s)}$, choose controller type, $G_c(s)$.

Step 2: Check for stability

Stable system → roots have zero or negative real parts

Routh-Hurwitz Stability Condition

- 1st-order: $a_1 s + a_0 = 0$
  Stable if $\frac{a_0}{a_1} > 0$
- 2nd-order: $a_2 s^2 + a_1 s + a_0 = 0$
  Stable if a_2, a_1, a_0 have same sign
- 3rd-order: $a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$
  Assuming $a_3 > 0$, stable if $a_2, a_1, a_0 > 0$ and $a_2 a_1 > a_3 a_0$

Step 3: Place constraints on control gains (i.e. $G_c(s)$) to ensure stability

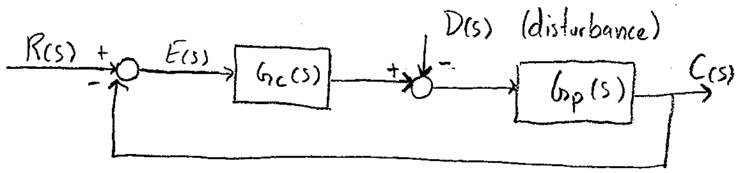
Step 4: Evaluate steady-state response (typically use final value theorem). Place additional constraints on $G_c(s)$.

Step 5: Evaluate transient response (overshoot, rise time, settling time, etc.)

Step 6: Evaluate other specifications (can actuator handle required output, etc.)

(10.6) Controlling 1^{st}-Order Plants

* Assume actuator + sensor have negligible dynamics, i.e. $G_a(s) = H(s) = 1$



First order mass-damper system: $G_p(s) = \dfrac{1}{ms+c}$

Proportional (P) Control

$G_c(s) = K_p$

Step 1: model system ✓

Solve for $T(s) = \dfrac{C(s)}{R(s)}$ and $\dfrac{C(s)}{D(s)}$ (disturbance transfer function)

set $D(s)=0$, find $\dfrac{C(s)}{R(s)} = \dfrac{\frac{K_p}{ms+c}}{1+\frac{K_p}{ms+c}} = \dfrac{K_p}{ms+c+K_p}$

set $R(s)=0$, find $\dfrac{C(s)}{D(s)}$: $C(s) = \dfrac{1}{ms+c}\left[K_p(\cancel{R(s)} - C(s)) - D(s)\right] = \dfrac{1}{ms+c}\left(-K_p C(s) - D(s)\right)$

$$C(s) = \frac{-K_p C(s)}{ms+c} - \frac{D(s)}{ms+c}$$

$$C(s)(ms+c) = -K_p C(s) - D(s)$$

$$C(s)(ms+c+K_p) = -D(s)$$

$$\frac{C(s)}{D(s)} = \frac{-1}{ms+c+K_p}$$

Choose controller type: proportional $G_c(s) = K_p$ ✓

Step 2: check stability.

- assuming $m>0$, $c>0$, then stable as long as $c+K_p>0$

Step 3: constraints on control gains for stability

- $c+K_p>0 \Rightarrow K_p > -c$

Step 4: s.s. response, step inputs for R, D $\Rightarrow$ $R(s)=\frac{1}{s}$, $D(s)=\frac{1}{s}$

- $C_{ss,r} = \lim\limits_{s\to 0} \dfrac{K_p}{ms+c+K_p} = \dfrac{K_p}{c+K_p} \neq 1$ (desired value) close if c is small.
- $C_{ss,d} = \lim\limits_{s\to 0} \dfrac{-1}{ms+c+K_p} = \dfrac{-1}{c+K_p}$ $\Rightarrow$ small if c or K_p is large

Step 5: transient response $\left(\tau = \frac{m}{c} \text{ for } 1^{st} \text{ order systems}\right)$

- Takes $\approx 4\tau$ to reach s.s. value. $\tau = \frac{m}{c+K_p}$, so it takes $\approx \frac{4m}{c+K_p}$.

Proportional Integral (PI) Control

$$G_c(s) = K_p + \frac{K_I}{s}$$

Step 1: $$\frac{C(s)}{R(s)} = \frac{K_p + \frac{K_I}{s}}{ms+c+K_p+\frac{K_I}{s}} = \frac{K_p s + K_I}{ms^2+(c+K_p)s+K_I}$$

$$\frac{C(s)}{D(s)} = \frac{-1}{ms+c+K_p+\frac{K_I}{s}} = \frac{-s}{ms^2+(c+K_p)s+K_I}$$

Step 2: 2^{nd} order sys, assume $m>0, c>0$, stable if $(c+K_p)>0$ and $K_I>0$

Step 3: $K_p > -c$, $K_I > 0$

Step 4: $$C_{ss,r} = \lim_{s\to 0} \frac{K_p s + K_I}{ms^2+(c+K_p)s+K_I} = 1 = \text{desired value}$$

$$C_{ss,d} = \lim_{s\to 0} \frac{-s}{ms^2+(c+K_p)s+K_I} = 0 \Rightarrow$$ no s.s. ✓ response to disturbance

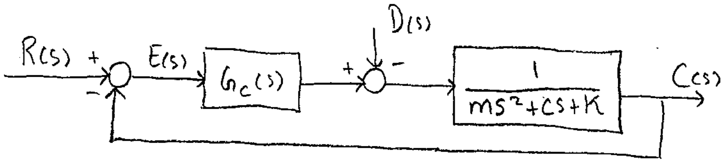
Step 5: Transient response depends on damping (overdamped, underdamped, etc.)

$$\zeta = \frac{c+K_p}{2\sqrt{mK_I}} \qquad \left(\tau = \frac{2m}{c} \text{ for } 2^{nd} \text{ order systems}\right)$$

- for $\zeta \le 1$, $\tau = \frac{2m}{c+K_p}$ $\Rightarrow$ so K_p used to define time cst., K_I for damping ratio.
- for $\zeta > 1$, must find dominant root, then $\tau_d = \frac{1}{s_d}$

These define settling time, overshoot, etc.

(10.7) Controlling 2^{nd}-Order Plants



mass-spring-damper system: $G_p(s) = \dfrac{1}{ms^2+cs+K}$

Proportional (P) Control

$G_c(s) = K_p$

Step 1: model system ✓

solve for $T(s) = \dfrac{C(s)}{R(s)} + \dfrac{C(s)}{D(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{K_p}{ms^2+cs+K}}{1+\frac{K_p}{ms^2+cs+K}} = \frac{K_p}{ms^2+cs+K+K_p}$$

$$\frac{C(s)}{D(s)} = \frac{-1}{ms^2+cs+K+K_p}$$

Step 2: assuming $m, c, K > 0$, stable if $K+K_p > 0$

Step 3: $K_p > -K$

Step 4: $C_{ss,r} = \lim_{s \to 0} T(s) = \dfrac{K_p}{K+K_p} \neq 1$ (desired value)

$$C_{ss,d} = \frac{-1}{K+K_p}$$

Step 5: Transient response depends on damping condition

$$\zeta = \frac{c}{2\sqrt{m(K+K_p)}}$$

- For $\zeta \leq 1$, $\tau = \dfrac{2m}{c}$
- for $\zeta > 1$, must find dominant root, then $\tau_d = \dfrac{1}{s_d}$

PI Control

$G_c(s) = K_p + \dfrac{K_I}{s}$

Step 1: $\dfrac{C(s)}{R(s)} = \dfrac{K_p s + K_I}{ms^3+cs^2+(K+K_p)s+K_I}$

$$\frac{C(s)}{D(s)} = \frac{-s}{ms^3+cs^2+(K+K_p)s+K_I}$$

Step 2: assuming $m, c, K > 0$, stable if $K+K_p > 0$, $K_I > 0$, $c(K+K_p) > mK_I$

Step 3: $K_P > -K$, $K_I > 0$, $K_P > \frac{mK_I}{c} - K$

Step 4: $C_{ss,r} = \frac{K_I}{K_I} = 1$ = desired value

$C_{ss,d} = 0 \Rightarrow$ no s.s. response to disturbance

Step 5: There are no convenient formulas for overshoot, τ, ζ, ω_n for 3rd-order model. A 3rd-order sys has 3 roots. If we arbitrarily pick one negative real root, then a complex conjugate pair can be selected to fit our specifications. K_P, K_I, K_D are selected to give that particular pair. See. Example 10.7.4.

PD Control

$G_c(s) = K_P + K_D s$

Step 1: $$\frac{C(s)}{R(s)} = \frac{K_P + K_D s}{ms^2 + (c+K_D)s + K + K_P}$$

$$\frac{C(s)}{D(s)} = \frac{-1}{ms^2 + (c+K_D)s + K + K_P}$$

Step 2: assuming $m, c, K > 0$, stable if $c + K_D > 0$, $K + K_P > 0$

Step 3: $K_D > -c$, $K_P > -K$

Step 4: $$C_{ss,r} = \frac{K_P}{K+K_P} \neq 1$$

$$C_{ss,d} = \frac{-1}{K+K_P}$$

select K_P to set s.s. value, select K_D to set damping / time cst.

Step 5: Transient response depends on damping

$$\zeta = \frac{c + K_D}{2\sqrt{m(K+K_P)}}$$

- For $\zeta \leq 1$, $\tau = \frac{2m}{c+K_D}$
- for $\zeta > 1$, find dominant root, $\tau_d = \frac{1}{s_d}$

PID Control

$G_c(s) = K_P + \frac{K_I}{s} + K_D s$

Step 1: $$\frac{C(s)}{R(s)} = \frac{K_D s^2 + K_P s + K_I}{ms^3 + (c+K_D)s^2 + (K+K_P)s + K_I}$$

$$\frac{C(s)}{D(s)} = \frac{-s}{ms^3 + (c+K_D)s^2 + (K+K_P)s + K_I}$$

Step 2/3: assuming $m, c, k > 0$, stable if $c + K_D > 0$, $k + K_P > 0$, $K_I > 0$,
$(c + K_D)(k + K_P) > m K_I$

Step 4: $C_{ss,r} = 1$ = desired value

$C_{ss,d} = 0 \Rightarrow$ no s.s. response to disturbance

Step 5: There are no convenient formulas for overshoot, τ, ζ, ω_n.