Frequency Response - Lecture 4

ME3050 - Dynamics Modeling and Controls

April 25, 2020

Resonance

Lecture 4 - Resonance

- Review 2^{nd} Order Frequency Response
- The Resonance Phenomenon
- The Resonance Frequency
- MATLAB code for Bode Plots

Transfer Function of 2^{nd} Order System

$m\ddot{x} + c\dot{x} + kx = f(t)$ with $f(t) = Asin(\omega t)$

The transfer function can easily be found by taking the Laplace transform of the equation of motion.

$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k}$ Second Order Transfer Function

The amplitude ratio and phase angle can be found from the transfer function. Think about what M means.

Overdamped vs. Underdamped Systems

In an overdamped system, both roots are real and distinct.

$$M(\omega) = |T(j\omega)| = \frac{|1/k|}{|\tau_1 j\omega + 1||\tau_2 j\omega + 1|}$$

$$m(\omega) = 20logM(\omega) = 20log|1/k| - 20log|\tau_1\omega j + 1| - 20log|\tau_2\omega j + 1|$$

$$\phi(\omega) = \angle\frac{1}{k} - \angle(\tau_1\omega j + 1) - \angle(\tau_2\omega_j + 1)$$

In an underdamped system, the roots are complex conjugates.

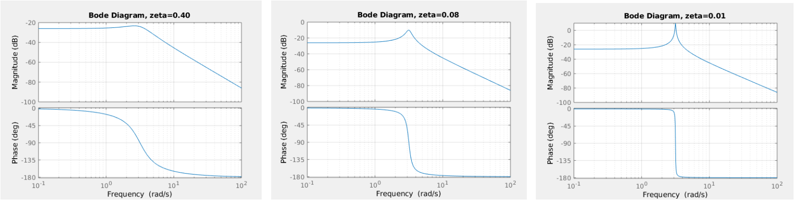
$$M(r) = |T(r)| = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{with} \quad r = \frac{\omega}{\omega_n}$$

$$\implies m = 20logM = -10log\left[\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2\right]$$

$$\phi = \angle 1 - \angle\left(1 - r^2 + 2\zeta rj\right) \implies \phi = -tan\left(\frac{2\zeta r}{1 - r^2}\right)$$

The Resonance Spike

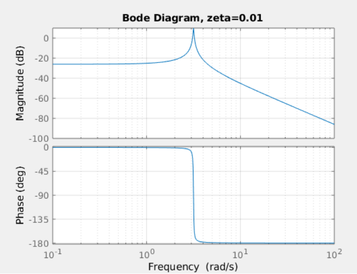
In the underdamped second order system only two regions are present separated by the point near $r = 1$.



As the damping ratio decreases something significant happens at this points. Remember, these are graphs of $m = 20 log M$.

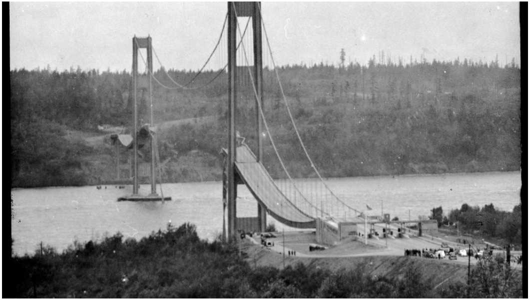
The Effects of Resonance

Consider the extreme case in the figure shown. What is the physical significance of the values of m near $r = 1$?



Resonance can be Destructive

The resonance peak represents a amplitude ratio greater than one meaning the output amplitude is larger that in the input amplitude. This large amplitude output displacement caused by resonance correspond to a large force in the spring members and the large forces are transmitted to the body. Large forces cause mechanical failure.



The Resonance Frequency

Where on the frequency response graphs does resonance occur? It looks like it is *near* $r = 1$.

$$M(r) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{with} \quad r = \frac{\omega}{\omega_n}$$

The value of M is maximized when the denominator is minimized. Therefore the resonance frequency is found by taking the derivative of the denominator and setting it equal to zero.

$$M_{max} \quad \text{occurs at } r = \sqrt{1 - 2\zeta^2} \implies \omega = \omega_n \sqrt{1 - 2\zeta^2}$$

An input frequency equal to the resonance frequency causes maximum output displacement.

The Resonance Frequency

The resonance event only occurs in second order systems when the radical shown is positive. This corresponds to systems with damping ratio in the range $0 < \zeta < 0.707$.

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \;\; 0 < \zeta < 0.707$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{or in decibels} \quad m_r = -20log\left(2\zeta\sqrt{1-\zeta^2}\right)$$

The phase at resonance can also be found.

$$\phi_r = tan^{-1}\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$$

Output Displacement to Input Force

It is important to note that we multiplied the transfer function by k during the derivations. Therefore if me divide the expressions for M and M_r by k to find the amplitude ratio between input force, $f(t)$ and output displacement, $x_{ss}(t)$

$M = \frac{1}{k\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

$$\Longrightarrow \quad m = -20log(k) - 10log\left[\left(1 - r^2\right)^2 + (2\zeta r)^2\right]$$

$M_r = \frac{1}{k2\zeta\sqrt{1-\zeta^2}}$

$$\Longrightarrow \quad m_r = -20log\left(k\right) - 20log\left(2\zeta\sqrt{1 - \zeta^2}\right)$$

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain