(2.2) Laplace Transforms

Converts linear DEs into algebraic expressions. Reduced complexity, easy to solve

$$\mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt = X(s) \qquad t \to s \qquad s = \sigma + j\omega \text{ (complex)}$$

Handout — Table of transform pairs: Tab. 2.2.1 (p 39) Tab. 3.3.1 (p 102) 2nd ed.
" " " properties: Tab. 2.2.2 (p. 42) Tab. 3.2.2 (p 104) 2nd ed.

Transform is reversible via inverse transform $\mathcal{L}[x(t)] = X(s)$

$$\mathcal{L}^{-1}[X(s)] = x(t)$$

Examples:

$x(t) = \sin\omega t \qquad \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$ (from table)

$x(t) = e^{-at}\sin\omega t \qquad \mathcal{L}[e^{-at}\sin\omega t] = \frac{\omega}{(s+a)^2+\omega^2} = \frac{\omega}{s^2+2as+\omega^2+a^2}$ (from table)

$x(t) = c$ (const) $\qquad \mathcal{L}[c] = \frac{c}{s}$ (table)

Solving ODEs using LT

① Apply LT to equation using properties + table

ex: $4\dot{x} = \sin(t)$

$$4(sX(s) - x(0)) = \frac{1}{s^2+1}$$

② Solve for $X(s)$

$$X(s) = \frac{1}{4s(s^2+1)} + \frac{x(0)}{s}$$

③ Write $X(s)$ in a form that can be inverted

Partial Fraction Expansion

numerator is 1 order less than den.

$$\frac{1}{4s(s^2+1)} = \frac{1/4}{s(s^2+1)} = \frac{a}{s} + \frac{bs+c}{s^2+1}$$

multiply through by $4s(s^2+1)$:

$$1 = 4a(s^2+1) + 4s(bs+c) = 4(a+b)s^2 + 4cs + 4a$$

solve for coeffs:

$$(a+b) = 0 \quad c = 0 \quad a = \tfrac{1}{4} \Rightarrow \quad a = \tfrac{1}{4} \quad b = -\tfrac{1}{4} \quad c = 0$$

$$X(s) = \frac{x(0)}{s} + \frac{1}{4s} - \frac{s}{4(s^2+1)}$$

④ Solve for $x(t)$ using inverse transform

$$x(t) = x_0 + \tfrac{1}{4} - \tfrac{1}{4}\cos t = x_0 + \tfrac{1}{4}(1-\cos t)$$

Advantages: solve for homogeneous soln, particular soln + ICs simultaneously.
No integrals or derivatives involved, just algebra.

Step (3) from above can be difficult, which often leads to:

(2.4) Partial Fraction Expansion

General form of Laplace Transform: $X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$, $\underline{\underline{n \geq m}}$

Case 1) Distinct Roots: all roots real, distinct

$$X(s) = \frac{N(s)}{(s+r_1)(s+r_2)\cdots(s+r_n)} \quad \text{(factored form)}$$

Expand to:

$$X(s) = \frac{C_1}{s+r_1} + \frac{C_2}{s+r_2} + \dots + \frac{C_n}{s+r_n} \quad \text{where } C_i = X(s)(s+r_i)\Big|_{s=-r_i}$$

limit

Leads to soln:

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + \dots + C_n e^{-r_n t}$$

Ex: $X(s) = \frac{16}{s(s+2)(s+4)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+4} \Rightarrow r_1 = 0,\ r_2 = 2,\ r_3 = 4$$

$$C_1 = \lim_{s \to -0} \frac{16}{s(s+2)(s+4)}(s) = \frac{16}{8} = 2$$

$$C_2 = \lim_{s \to -2} \frac{16(s+2)}{s(s+2)(s+4)} = \frac{16}{-4} = -4$$

$$C_3 = \lim_{s \to -4} \frac{16(s+4)}{s(s+2)(s+4)} = \frac{16}{8} = 2$$

$$X(s) = \frac{2}{s} - \frac{4}{s+2} + \frac{2}{s+4} \Rightarrow x(t) = 2 - 4e^{-2t} + 2e^{-4t}$$

Case 2) Repeated Roots: where p number of roots have same value $s = -r_1$ and remaining roots are distinct + real.

$$X(s) = \frac{N(s)}{(s+r_1)^p (s+r_{p+1})(s+r_{p+2})\cdots(s+r_n)}$$

Expand to:

$$X(s) = \underbrace{\frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + \dots + \frac{C_p}{(s+r_1)}}_{\text{repeated roots}} + \underbrace{\frac{C_{p+1}}{(s+r_{p+1})} + \dots + \frac{C_n}{(s+r_n)}}_{\text{distinct, real roots}}$$

Coeff's for repeated roots:

$$C_1 = \lim_{s\to -r_1}\left[X(s)(s+r_1)^p\right], \quad C_2 = \lim_{s\to -r_1}\left\{\frac{d}{ds}\left[X(s)(s+r_1)^p\right]\right\}$$

$$C_i = \lim_{s\to -r_1}\left\{\frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}\left[X(s)(s+r_1)^p\right]\right\}$$

Coeffs for distinct roots, same as Case 1.

Leads to soln:

$$x(t) = C_1\frac{t^{p-1}}{(p-1)!}e^{-r_1 t} + C_2\frac{t^{p-2}}{(p-2)!}e^{-r_1 t} + \ldots + C_p e^{-r_1 t} + C_{p+1}e^{-r_{p+1}t} + \ldots + C_n e^{-r_n t}$$

Ex: $X(s) = \dfrac{3}{s^3 - s^2 - 8s + 12} = \dfrac{3}{(s-2)^2(s+3)}$

$$X(s) = \frac{C_1}{(s-2)^2} + \frac{C_2}{s-2} + \frac{C_3}{s+3} \Rightarrow r_1 = -2,\ r_2 = -2,\ r_3 = 3$$

$$C_1 = \lim_{s\to 2}\left[\frac{3(s-2)^2}{(s-2)^2(s+3)}\right] = \frac{3}{5}$$

$$C_2 = \lim_{s\to 2}\left\{\frac{d}{ds}\left[\frac{3}{s+3}\right]\right\} = \lim_{s\to 2}\left[\frac{-3}{(s+3)^2}\right] = -\frac{3}{25}$$

$$C_3 = \lim_{s\to -3}\left[\frac{3(s+3)}{(s-2)^2(s+3)}\right] = \frac{3}{25}$$

$$X(s) = \frac{3}{5(s-2)^2} - \frac{3}{25(s-2)} + \frac{3}{25(s+3)}$$

$$x(t) = \frac{3}{5}te^{2t} - \frac{3}{25}e^{2t} + \frac{3}{25}e^{-3t}$$

Special Case: Complex Roots: these are actually distinct roots, i.e. Case 1

$$X(s) = \frac{3s+7}{4s^2+24s+136} = \frac{3s+7}{4(s^2+6s+34)}$$

An easier soln. can be found by forming two perfect squares in the denomenator

$$X(s) = \frac{1}{4}\left[\frac{3s+7}{(s+3)^2+5^2}\right]$$

expand to

$$X(s) = \frac{1}{4}\left[C_1\frac{5}{(s+3)^2+5^2} + C_2\frac{(s+3)}{(s+3)^2+5^2}\right]$$

Note, these forms both appear in our LT table.

Solve for C_1 & C_2 by multiplying by denomenator

$$3s+7 = 5C_1 + C_2(s+3) = 5C_1 + C_2 s + 3C_2 \Rightarrow C_2 = 3,\ C_1 = -\frac{2}{5}$$

$$X(s) = \frac{1}{4}\left[-\frac{2}{5}\,\frac{5}{(s+3)^2+5^2} + 3\,\frac{(s+3)}{(s+3)^2+5^2}\right]$$

$$x(t) = -\frac{1}{10}e^{-3t}\sin 5t + \frac{3}{4}e^{-3t}\cos 5t$$

"Cover-up method" to determine coefficients

Ex: $$X(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)} = \frac{C_1}{s} + \frac{C_2}{(s+1)} + \frac{C_3}{(s+3)}$$

Solve for C_1 by covering up "s" in denomenator and setting $s=0$

$$C_1 = \frac{(0+2)(0+4)}{(0+1)(0+3)} = \frac{8}{3}$$

Repeat for $C_2 \Rightarrow$ cover up $(s+1)$ & set $s=-1$

$$C_2 = \frac{(-1+2)(-1+4)}{-1(-1+3)} = \frac{-3}{2}$$

Repeat for C_3:

$$C_3 = \frac{(-3+2)(-3+4)}{-3(-3+1)} = -\frac{1}{6}$$

$$X(s) = \frac{8/3}{s} - \frac{3/2}{(s+1)} - \frac{1/6}{(s+3)} \Rightarrow x(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

(2.6) Transfer Functions

Transfer Function (TF) is a useful way of representing a system in terms of an input/output relationship.

Ex: Consider 2^{nd} order ODE with zero ICs: $\ddot{x} + a\dot{x} + bx = f(t)$, $x(0)=0$, $\dot{x}(0)=0$

Laplace: $s^2X(s) + asX(s) + bX(s) = F(s)$

The ratio $\frac{X(s)}{F(s)}$ is the Transfer Function, $T(s)$: $$T(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2+as+b}$$

Properties of the Transfer Function

1) TF is the Laplace Transform of the forced response (hence zero ICs) divided by the LT of the input.

2) Can be used as multiplier to obtain forced response from input

$$X(s) = T(s)F(s)$$

3) The denominator is the characteristic equation, which gives useful info about the system response + stability, apart from the effects of the input + ICs.

4) The TF is equivalent to the ODE! TF ⇒ ODE, ODE ⇒ TF

Ex: Find TF given ODE: $5\ddot{x} + 30\dot{x} + 40x = 6f(t) \quad x(0)=0,\ \dot{x}(0)=0$

LT: $5[s^2X(s) - s\cancel{x(0)}^{0} - \cancel{\dot{x}(0)}^{0}] + 30[sX(s) - \cancel{x(0)}^{0}] + 40X(s) = 6F(s)$

TF: $T(s) = \dfrac{X(s)}{F(s)} = \dfrac{6}{5s^2 + 30s + 40} \longrightarrow$ characteristic polynomial

Ex: Find ODE given TF: $\dfrac{2}{s^2 + 10s + 14}$

$$T(s) = \frac{X(s)}{F(s)} = \frac{2}{s^2 + 10s + 14} \Rightarrow s^2X(s) + 10sX(s) + 14X(s) = 2F(s)$$

Inv. LT: $\ddot{x} + 10\dot{x} + 14x = 2f(t)$

Systems with Multiple Inputs:

ODE: $3\ddot{x} + 7\dot{x} + 10x = 2f(t) - 3g(t)$

LT: $3s^2X(s) + 7sX(s) + 10X(s) = 2F(s) - 3G(s)$

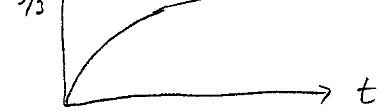
$$X(s) = \frac{2}{3s^2 + 7s + 10}F(s) - \frac{3}{3s^2 + 7s + 10}G(s)$$

★ With multiple inputs, you have multiple TFs. To find each TF, temporarily set other input = 0

2 TFs: $\dfrac{X(s)}{F(s)} = \dfrac{2}{3s^2 + 7s + 10}, \quad \dfrac{X(s)}{G(s)} = \dfrac{-3}{3s^2 + 7s + 10}$

(2.10) Transfer Functions in Matlab

Recall Ex 1 from ODEs: $\dot{x} + 3x = 5 \qquad TF = \dfrac{5}{s+3}$ Matlab: sys1 = tf([5], [1,3])
step(sys1)



Recall Ex 3 from ODEs: $\ddot{x} + 6\dot{x} + 34x = 68$

$$TF = \frac{68}{s^2+6s+34}$$

Matlab: `sys2 = tf([68], [1, 6, 34])`
`step(sys2)`