Module 12 - Second Order Time Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 2 - The Damping Ratio

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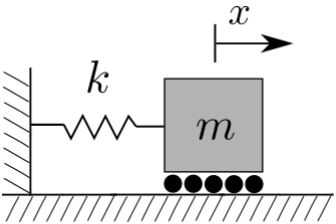
- The Damped Natural Frequency
- The Overdamped Case
- The Critically Damped Case
- The Underdamped Case

Mass Spring Model

Consider the mass-spring system without damping.



The EOM is:

$m\ddot{x} + kx = 0$ with

$x(t = 0) = x_0,$ and $v(t = 0) = v_0$

Solution with Laplace Transforms Method

Solve for $x(t)$ with a method of your choice.

$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

Phase Shift

The solution is commonly written as a single oscillating term with a **phase shift** ϕ.

$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t)$ with $\omega_n = \sqrt{\frac{k}{m}}$

Is equivalent to:

$x(t) = Acos(\omega_n t - \phi)$ $A = \sqrt{x_0^2 + [\frac{v_0}{\omega_n}]^2}$ $\phi = tan^{-1}(\frac{v_0}{x_0 \omega_n})$

Sine could be used instead.

$x(t) = Asin(\omega_n t + \phi)$ $A = \sqrt{x_0^2 + [\frac{v_0}{\omega_n}]^2}$ $\phi = tan^{-1}(\frac{x(0)\omega_n}{v_0})$

Sketch of Free Response

Sketch the **free** response in the time domain.

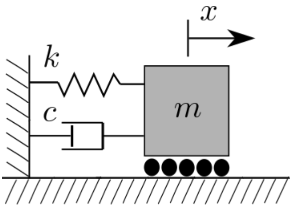
$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t)$$

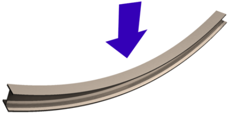
$$= Acos(\omega_n t - \phi) \;\; with \;\; \phi = tan^{-1}(\frac{v_0}{x_0 \omega_n})$$

Is this a stable system? What does the phase shift ϕ represent?

Second Order System with Damping

Now, consider the mass-spring system with damping present.





The EOM is:

$m\ddot{x} + c\dot{x} + kx = 0$ with

$x(t = 0) = x_0$, and $v(t = 0) = v_0$

Solution with Trial Solution Method

The trial solution method is used to derive the response equation in terms of the system variables and parameters.

$m\ddot{x} + c\dot{x} + kx = 0 \implies (mr^2 + cr + k)Ae^{rt} = 0$

You can see the *characteristic equation* becomes:

$(mr^2 + cr + k) = 0$

Solve for the roots. In system dynamics they are called $s_{1,2}$

$r_{1,2} = s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$

The Roots of the System

The roots of the system determine the behavior.

$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

The **discriminant** $c^2 - 4mk$ drives the *case* in the trial solution method.

- IF __________ $\Longrightarrow$ Case 1: Distinct and Real
- IF __________ $\Longrightarrow$ Case 2: Repeated and Real
- IF __________ $\Longrightarrow$ Case 3: Complex Conjugate Pair

Damping Cases and the Critical Damping Value

In a system with known mass and spring constant, the damping value determines the behavior. The damping value that causes the *discriminant* to equal zero (case 2) is the **critical damping value**.

$$c^2 - 4mk = 0 \implies c = \sqrt{4mk} = 2\sqrt{mk}$$

$$c_{critical} = 2\sqrt{mk}$$

The Damping Ratio

The damping ratio ζ is the ratio of damping c to the critical damping value $c_{critical}$.

$\zeta = \frac{c}{c_{critical}} = \frac{c}{2\sqrt{mk}}$

Re-write the roots with this new quantity.

$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

We define a new quantity, **damped natural frequency**.

$\omega_d = \omega_n\sqrt{1 - \zeta^2}$

Now re-write the roots again in terms of ζ and ω_d.

$s_{1,2} = \zeta\omega_n \pm j\omega_d$

Forms of the Response Equations

The behavior of the system depends on the damping ratio.

Case 1	$c > 2\sqrt{mk}$	**Overdamped**	$\zeta > 1$
Case 2	$c = 2\sqrt{mk} = c_{critical}$	**Critically Damped**	$\zeta = 1$
Case 3	$c < 2\sqrt{mk}$	**Underdamped**	$\zeta < 1$

The Overdamped Case

The roots are real and distinct and the system *does not* oscillate.

Overdamped $c > 2\sqrt{mk} \implies \zeta > 1$

$$s_{1,2} = -\zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$x(t) = C_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\,)t} + C_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\,)t}$$

$$x(t) = e^{-\zeta\omega_n}\{C_1 e^{\omega_n\sqrt{\zeta^2-1}t} + C_2 e^{\omega_n\sqrt{\zeta^2-1}t}\}$$

$$C_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}} \quad C_2 = \frac{-v_0 + (\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

The Critically Damped Case

The roots are real and repeated and the system *does not* oscillate.

Critically Damped $c = 2\sqrt{mk} \implies \zeta = 1$

$$s_{1,2} = \frac{-c}{2m} = \zeta\omega_n = -\omega_n$$
$$x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$
$$\dot{x} = -\omega_n C_1 e^{-\omega_n t} + C_2 t(-\omega_n e^{-\omega_n t}) + e^{-\omega_n t}(C_2)$$
$$x(t=0) = x_0 \implies C_1 = x_0$$
$$v(t=0) = v_0 \Rightarrow v_0 = -\omega_n C_1 + 0 + (1)C_2 \implies C_2 = v_0 + \omega_n x_0$$

The Underdamped Case

The roots are a complex conjugate pair and the system oscillates.

Underdamped $c < 2\sqrt{mk} \implies \zeta < 1$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

$$x(t) = C_1 e^{(\zeta\omega_n + j\omega_d)t} + C_2 e^{(\zeta\omega_n - j\omega_d)t}$$

$$x(t) = e^{-\zeta\omega_n t}\{Acos(\omega_d t) + Bsin(\omega_d t)\}$$

Use the initial position to solve for the first unknown.

$$x(t=0) = x_0 = (1)(A(1) + B(0)) \implies A = x_0$$

The Underdamped Case

Take the derivative and solve for the second unknown.

$x(t) = e^{-\zeta\omega_n t}\{Acos(\omega_d t) + Bsin(\omega_d t)\}$

$\dot{x}(t) = e^{-\zeta\omega_n t}(-\omega_d Asin(\omega_d t)) + Acos(\omega_d t)(-\zeta\omega_n e^{-\zeta\omega_n t})$

$+e^{-\zeta\omega_n t}(\omega_d Bcos(\omega_d t)) + Bsin(\omega_d t)e^{-\zeta\omega_n t}$

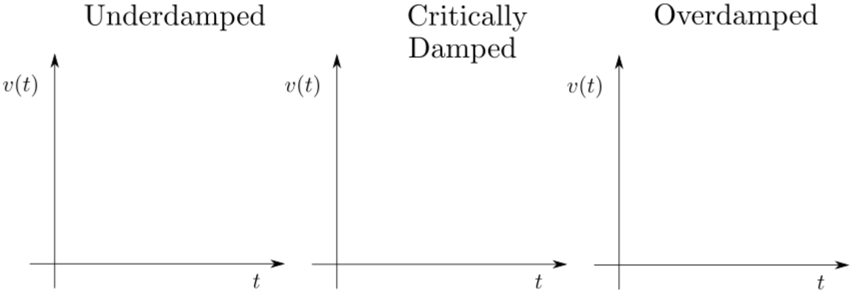
$\dot{x}(t=0) = A(1)(-\zeta\omega_n(1)) + (1)\omega_d B(1) \implies B = \frac{v_0+\zeta\omega_n x_0}{\omega_d}$

Finally we get to the reponse equation.

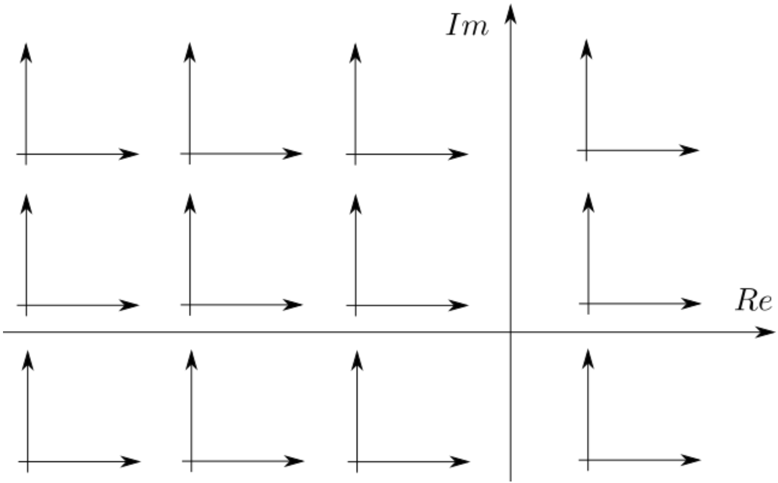
$x(t) = e^{-\zeta\omega_n t}\{x_0 cos(\omega_d t) + \frac{v_0+\zeta\omega_n x_0}{\omega_d}sin(\omega_d t)\}$

Response of the Three Different Cases

Each of the three cases behaves in a *characteristic* way.



Affects of Damping Ratio and Damped Natural Frequency



References

- System Dynamics, Palm III, Third Edition - Section 8.2 - Response of Second Order Systems - pg. 484