# Lecture Module - Time Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

## Time Response

# Lecture Module - Time Response

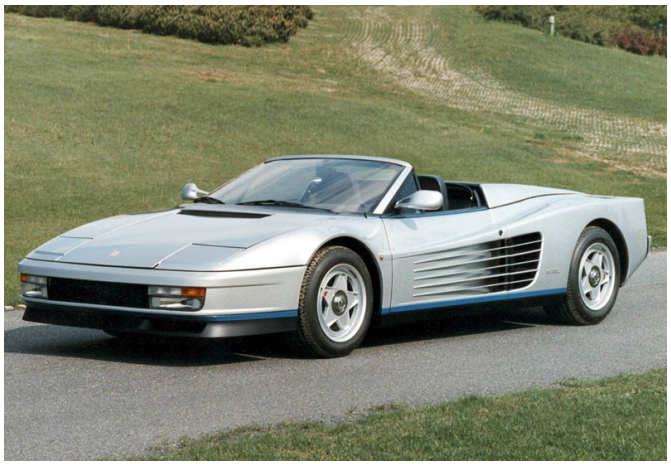
- Topic 1 - First Order Time Response
- Topic 2 - Second Order Time Response
- Topic 3 - Response and Root Location
- Topic 4 - Specification of The Step Response

# Topic 1 - First Order Time Response

- Second Order Free Response
- Second Order Forced Response
- Response and System Stability
- Equation Components

# Model and EOM

Consider the model of the moving mass we derived.



The EOM is:

$$m\dot{v} + cv = 0$$

# Solution with Laplace Transforms Method

$$\mathcal{L}\{m\dot{v} + cv = 0\} \implies m[sV(s) - v(0)] + cV(s) = 0$$

$$(ms + c)V(s) = \frac{mv(0)}{(ms+c)} = \frac{V(0)}{s+\frac{c}{m}}$$

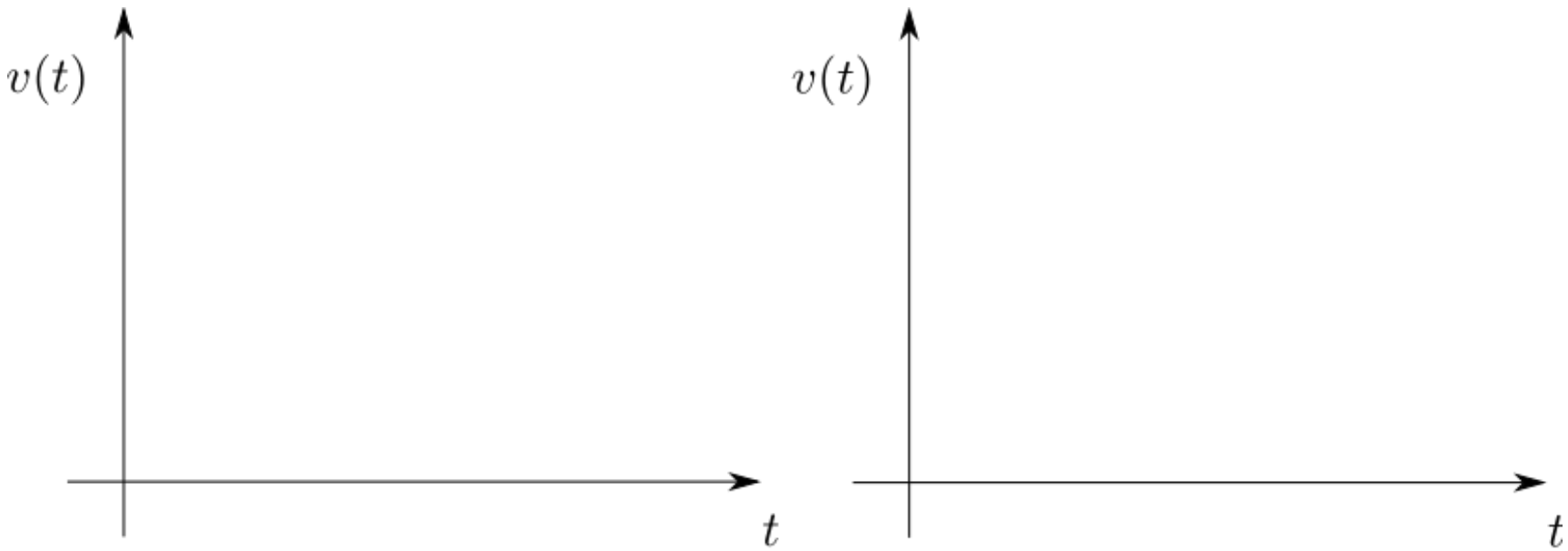
We can find the expected result from the table.

$$v(t) = v(0)e^{-\frac{c}{m}t} = v(0)e^{-\frac{t}{\tau}} \qquad \text{with} \qquad \tau =$$

# Sketch Response Equation

Sketch the System Response in the time Domain.

$$v(t) = v(0)e^{-\frac{t}{\tau}}$$




Is this a stable system? What does that mean?

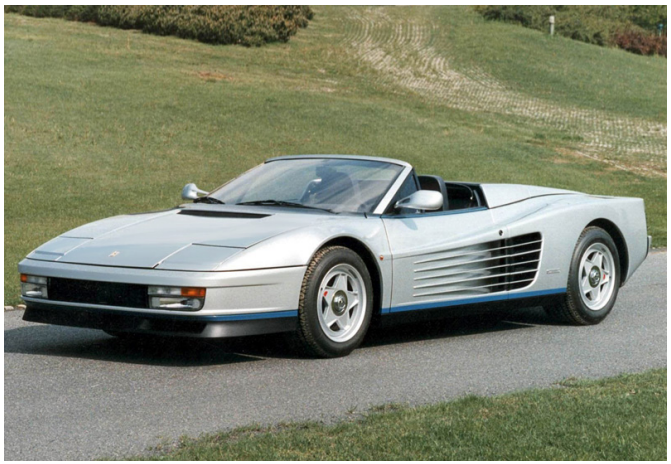
# Stability in Dynamic Systems

A dynamic system is stable if ...

# Second Order Forced Response

# Step Input Function

Consider the model subject to a Step Input, $f(t)$.



$$m\dot{v} + cv = f(t)$$

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

# Solution with Laplace Transforms Method - Step 1

The method of Laplace Transforms is shown.

$$\mathcal{L}\{m\dot{v} + cv = F\} \implies m[sV(s) - v(0)] + cV(s) = \frac{F}{c}$$

$$(ms + c)V(s) = \frac{F}{s} + mv(0)$$

Solve for $V(s)$.

$$V(s) = \frac{F}{s(ms+c)} + \frac{mv(0)}{ms+c}$$

# Solution with Laplace Transforms Method - Step 2

Expand $V(s)$as a partial fraction.

$$V(s) = \frac{F}{s(ms+c)} + \frac{mv(0)}{ms+c} \implies \frac{F}{s(ms+c)} = \frac{a}{s} + \frac{b}{ms+c}$$

'Cover up' to find the coefficients.

$$a = \frac{F}{m\times 0+c} \qquad \text{and} \qquad b = \frac{F}{\frac{-c}{m}} = \frac{-Fm}{c}$$

This leads to a form that can be inverted with the table.

$$V(s) = \frac{F}{c}\left\{\frac{1}{s} - \frac{1}{s+\frac{c}{m}}\right\} + \frac{v(0)}{s+\frac{c}{m}}$$

# Solution with Laplace Transforms Method - Step 3

Can you find these terms in the Table of Laplace Transforms?

$$V(s) = \frac{F}{c}\{\frac{1}{s} - \frac{1}{s+\frac{c}{m}}\} + \frac{v(0)}{s+\frac{c}{m}}$$

The inverse Laplace transform of $V(s)$ gives the time response.

$$v(t) = \frac{F}{C}\{1 - e^{-\frac{t}{\tau}}\} + v(0)e^{-\frac{t}{\tau}} = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$

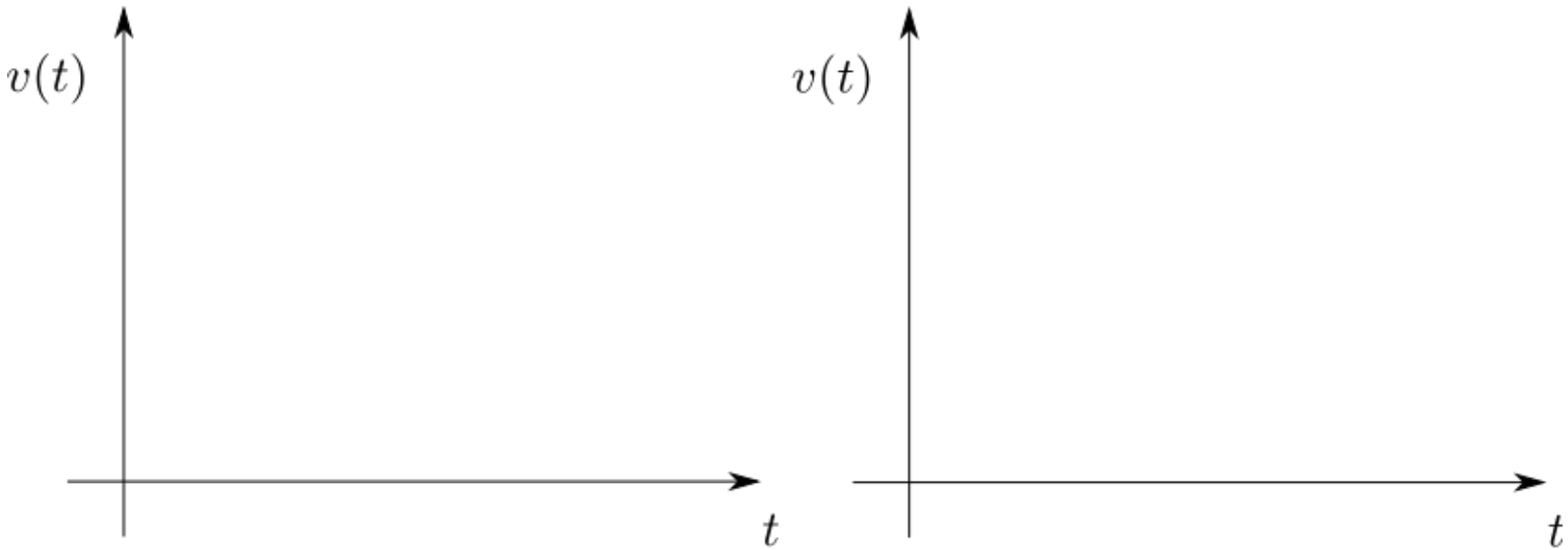
# Response and System Stability

# Response and System Stability

# Sketch Response Equation

Sketch the System Response in the time Domain.

$$v(t) = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$




Is this a stable system?

# Components of the Response

In these forms we can see the different components of the response.

$$v(t) = \frac{F}{C}\{1 - e^{-\frac{t}{\tau}}\} + v(0)e^{-\frac{t}{\tau}} = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$

- Forced Response
- Free Response
- Transient Response
- Steady-State Response

# References

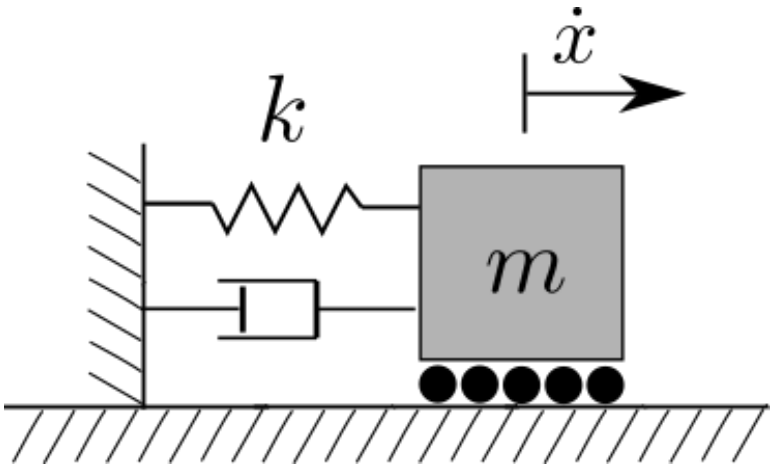
- System Dynamics, Palm III, Third Edition - Section 8.1 - Response of First Order Systems - pg. 475

# Topic 2 - Second Order Time Response

- Damped Natural Frequency and Damping Ratio
- The Overdamped Case
- The Critically Damped Case
- The Underdamped Case

# Mass Spring Model

Consider the mass-spring system without damping.




The EOM is:

$m\ddot{x} + kx = 0$ with

$x(t = 0) = x_0$, and $v(t = 0) = v_0$

# Solution with Laplace Transforms Method

Solve for $x(t)$ with a method of your choice.

$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

# Phase Shift

The solution is commonly written as a single oscillating term with a **phase shift** $\phi$.

$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

Is equivalent to:

$$x(t) = Acos(\omega_n t - \phi) \; A = \sqrt{x_0^2 + [\frac{v_0}{\omega_n}]^2} \; \phi = tan^{-1}(\frac{v_0}{x_0 \omega_n})$$

Sine could be used instead.

$$x(t) = Asin(\omega_n t + \phi) \; A = \sqrt{x_0^2 + [\frac{v_0}{\omega_n}]^2} \; \phi = tan^{-1}(\frac{x(0)\omega_n}{v_0})$$

# Sketch of Free Response

Sketch the **free** response in the time domain.

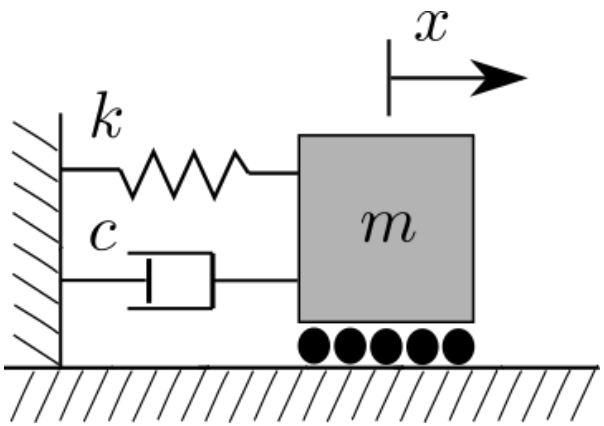
$$x(t) = \frac{v_0}{\omega_n} sin(\omega_n t) + x_0 cos(w_n t)$$

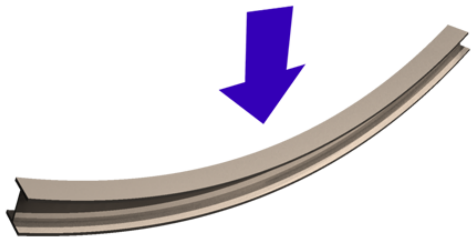
$$= Acos(\omega_n t - \phi) \ \ with \ \ \phi = tan^{-1}(\frac{v_0}{x_0 \omega_n})$$

Is this a stable system? What does the phase shift $\phi$ represent?

# Second Order System with Damping

Now, consider the mass-spring system with damping present.



The EOM is:

$m\ddot{x} + c\dot{x} + kx = 0$ with

$x(t = 0) = x_0$, and $v(t = 0) = v_0$

# Solution with Trial Solution Method

The trial solution method is used to derive the response equation in terms of the system variables and parameters.

$$m\ddot{x} + c\dot{x} + kx = 0 \implies (mr^2 + cr + k)Ae^{rt} = 0$$

You can see the *characteristic equation* becomes:

$$(mr^2 + cr + k) = 0$$

Solve for the roots. In system dynamics they are called $s_{1,2}$

$$r_{1,2} = s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}$$

# The Roots of the System

The roots of the system determine the behavior.

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

The **discriminant** $c^2 - 4mk$ drives the *case* in the trial solution method.

- IF _____ $\Longrightarrow$ Case 1: Distinct and Real
- IF _____ $\Longrightarrow$ Case 2: Repeated and Real
- IF _____ $\Longrightarrow$ Case 3: Complex Conjugate Pair

# Damping Cases and the Critical Damping Value

In a system with known mass and spring constant, the damping value determines the behavior. The damping value that causes the *discriminant* to equal zero (case 2) is the **critical damping value**.

$$c^2 - 4mk = 0 \implies c = \sqrt{4mk} = 2\sqrt{mk}$$

$$c_{critical} = 2\sqrt{mk}$$

# The Damping Ratio

The damping ratio $\zeta$ is the ratio of damping c to the critical damping value $c_{critical}$.

$$\zeta = \frac{c}{c_{critical}} = \frac{c}{2\sqrt{mk}}$$

Re-write the roots with this new quantity.

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

We define a new quantity, **damped natural frequency**.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Now re-write the roots again in terms of $\zeta$ and $\omega_d$.

$$s_{1,2} = \zeta\omega_n \pm j\omega_d$$

# Damped Natural Frequency and Damping Ratio

The behavior of the system depends on the damping ratio.

| | | | |
|---|---|---|---|
| Case 1 | $c > 2\sqrt{mk}$ | **Overdamped** | $\zeta > 1$ |
| Case 2 | $c = 2\sqrt{mk} = c_{critical}$ | **Critically Damped** | $\zeta = 1$ |
| Case 3 | $c < 2\sqrt{mk}$ | **Underdamped** | $\zeta < 1$ |

# The Overdamped Case

The roots are real and distinct and the system *does not* oscillate.

Overdamped $c > 2\sqrt{mk} \implies \zeta > 1$

$$s_{1,2} = -\zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$x(t) = C_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\,)t} + C_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\,)t}$$

$$x(t) = e^{-\zeta\omega_n}\left\{C_1 e^{\omega_n\sqrt{\zeta^2-1}\,t} + C_2 e^{\omega_n\sqrt{\zeta^2-1}\,t}\right\}$$

$$C_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}} \quad C_2 = \frac{-v_0 + (\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

# The Critically Damped Case

The roots are real and repeated and the system *does not* oscillate.

Critically Damped $c = 2\sqrt{mk} \implies \zeta = 1$

$$s_{1,2} = \frac{-c}{2m} = \zeta\omega_n = -\omega_n$$

$$x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

$$\dot{x} = -\omega_n C_1 e^{-\omega_n t} + C_2 t(-\omega_n e^{-\omega_n t}) + e^{-\omega_n t}(C_2)$$

$$x(t=0) = x_0 \implies C_1 = x_0$$

$$v(t=0) = v_0 \Rightarrow v_0 = -\omega_n C_1 + 0 + (1)C_2 \implies C_2 = v_0 + \omega_n x_0$$

# The Underdamped Case

Take the derivative and solve for the second unknown.

$$x(t) = e^{-\zeta\omega_n t}\{A\cos(\omega_d t) + B\sin(\omega_d t)\}$$

$$\dot{x}(t) = e^{-\zeta\omega_n t}(-\omega_d A\sin(\omega_d t)) + A\cos(\omega_d t)(-\zeta\omega_n e^{-\zeta\omega_n t})$$

$$+e^{-\zeta\omega_n t}(\omega_d B\cos(\omega_d t)) + B\sin(\omega_d t)e^{-\zeta\omega_n t}$$

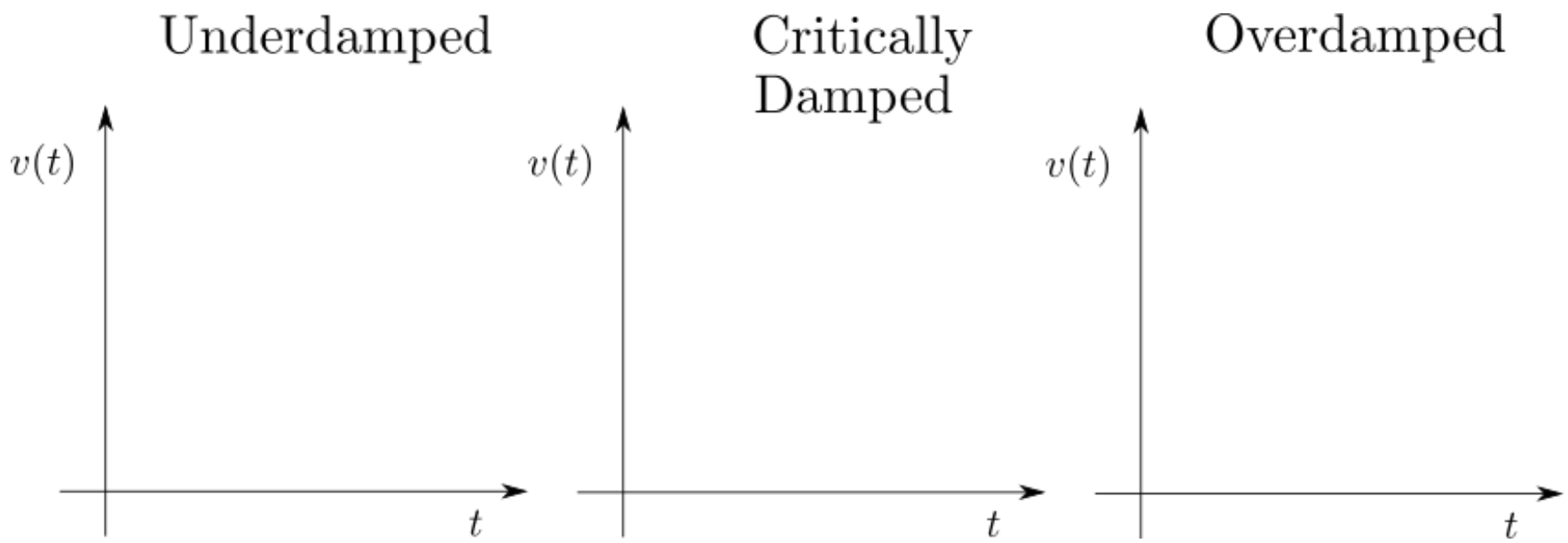
$$\dot{x}(t=0) = A(1)(-\zeta\omega_n(1)) + (1)\omega_d B(1) \implies B = \frac{v_0+\zeta\omega_n x_0}{\omega_d}$$

Finally we get to the reponse equation.

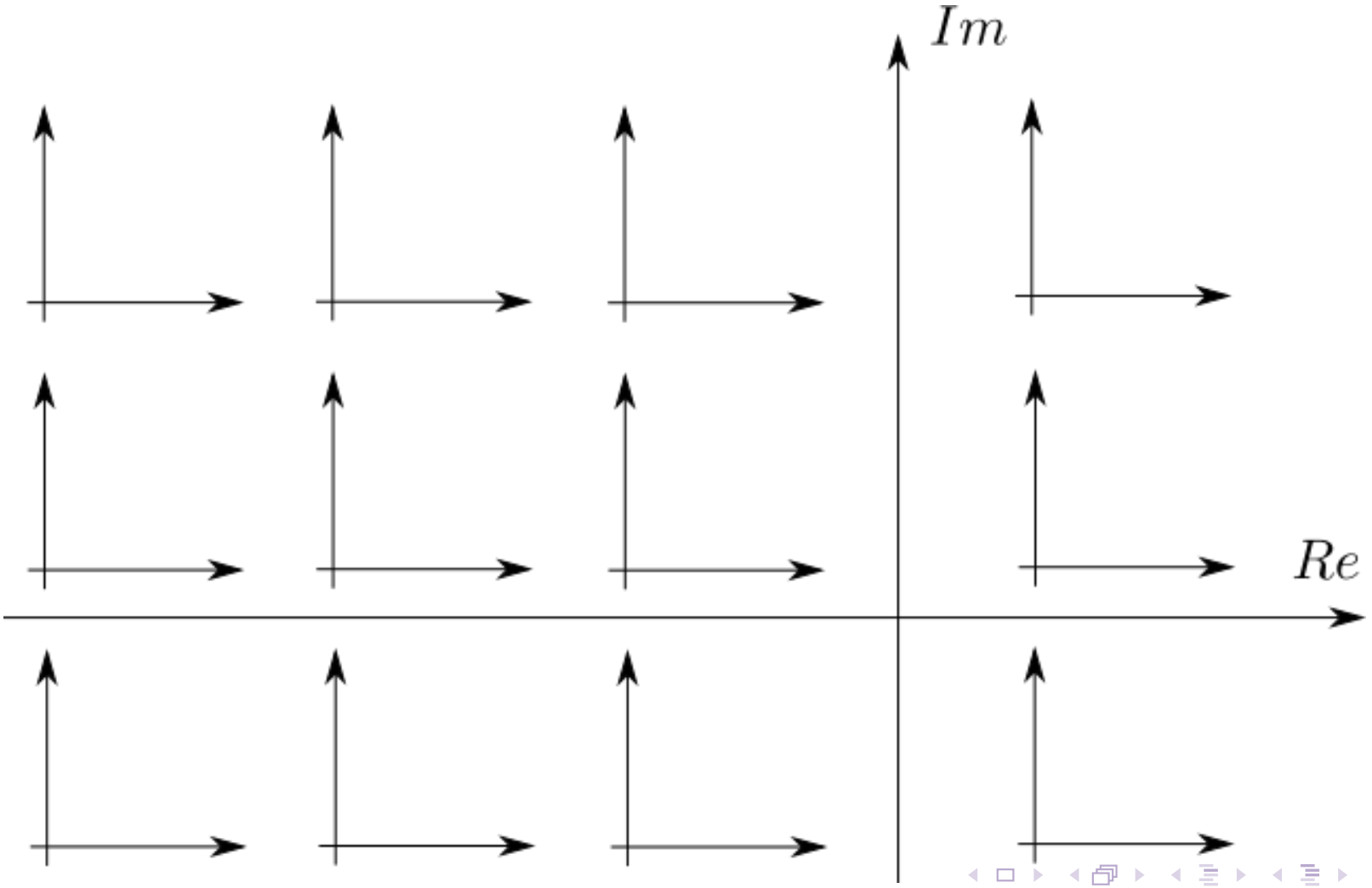
$$x(t) = e^{-\zeta\omega_n t}\{x_0\cos(\omega_d t) + \frac{v_0+\zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\}$$

# Response of the Three Different Cases

Each of the three cases behaves in a *characteristic* way.



# Affects of Damping Ratio and Damped Natural Frequency

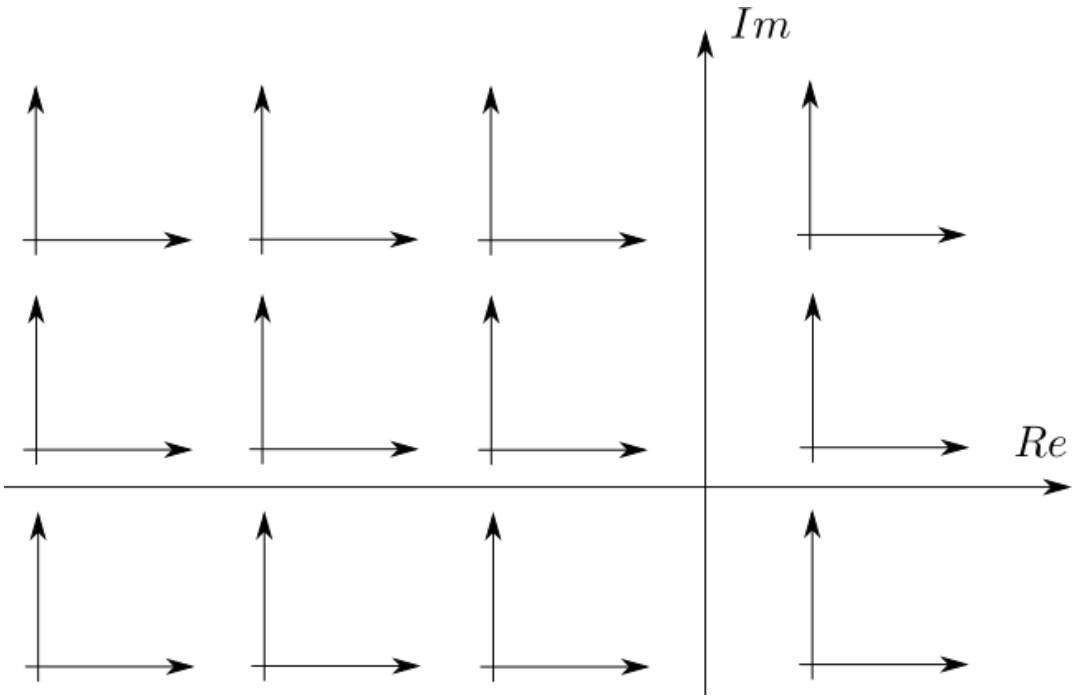


# Topic 3 - Response and Root Location

- The Complex Plane
- Along a Vertical Line
- Along a Horizontal Line
- Along a Diagonal Line

# Affect of Root Location on Response

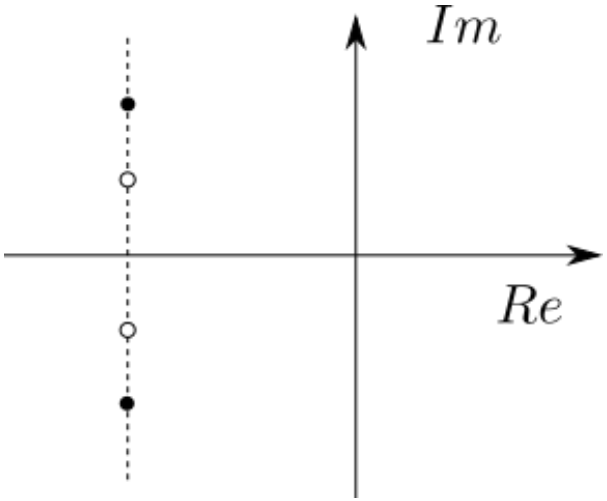
The *location* of the root in the *complex plane* shows the affects of the roots on the system behviour.



# The Complex Plane

# Along a Vertical Line

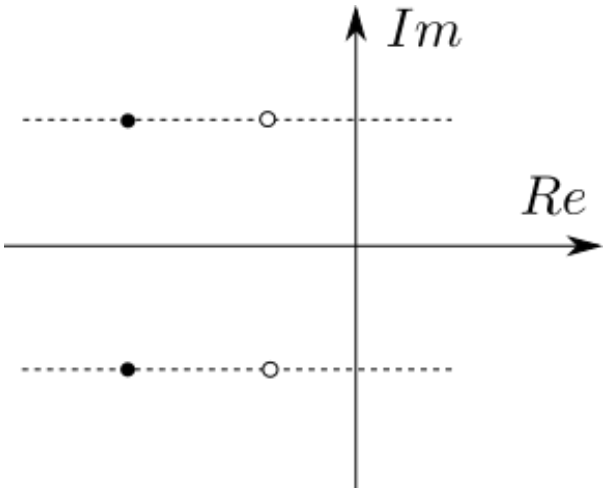
As the root moves along a vertical line...



# Along a Vertical Line

# Along a Horizontal Line

As the root moves along a horizontal line...

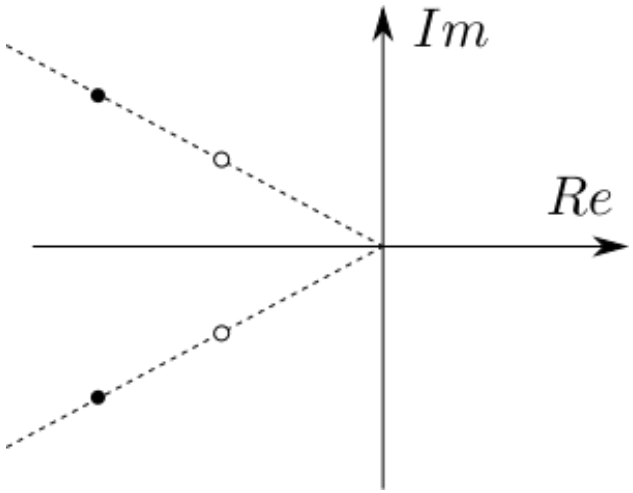


# Along a Horizontal Line

# Along a Horizontal Line

# Along a Diagonal Line

As the root moves along a diagonal line...



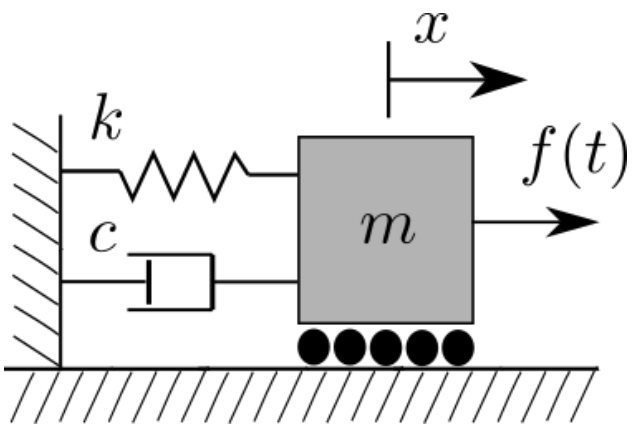

What about the angle of this line?

# Topic 3 - Specification of The Step Response

- The Unit Step Response
- Rise, Peak, and Settling Time
- Maximum Overshoot and The Damping Ratio
- System Identificatio

# The Mass Spring Damper

Now, consider the mass-spring system with damping present subject to **step** input. This models instantly turning on the input force $f(t)$.




Heavyside's Step Function

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

The EOM is:

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad with \;\; x(t=0) = x_0 \quad and \quad v(t=0) =$$

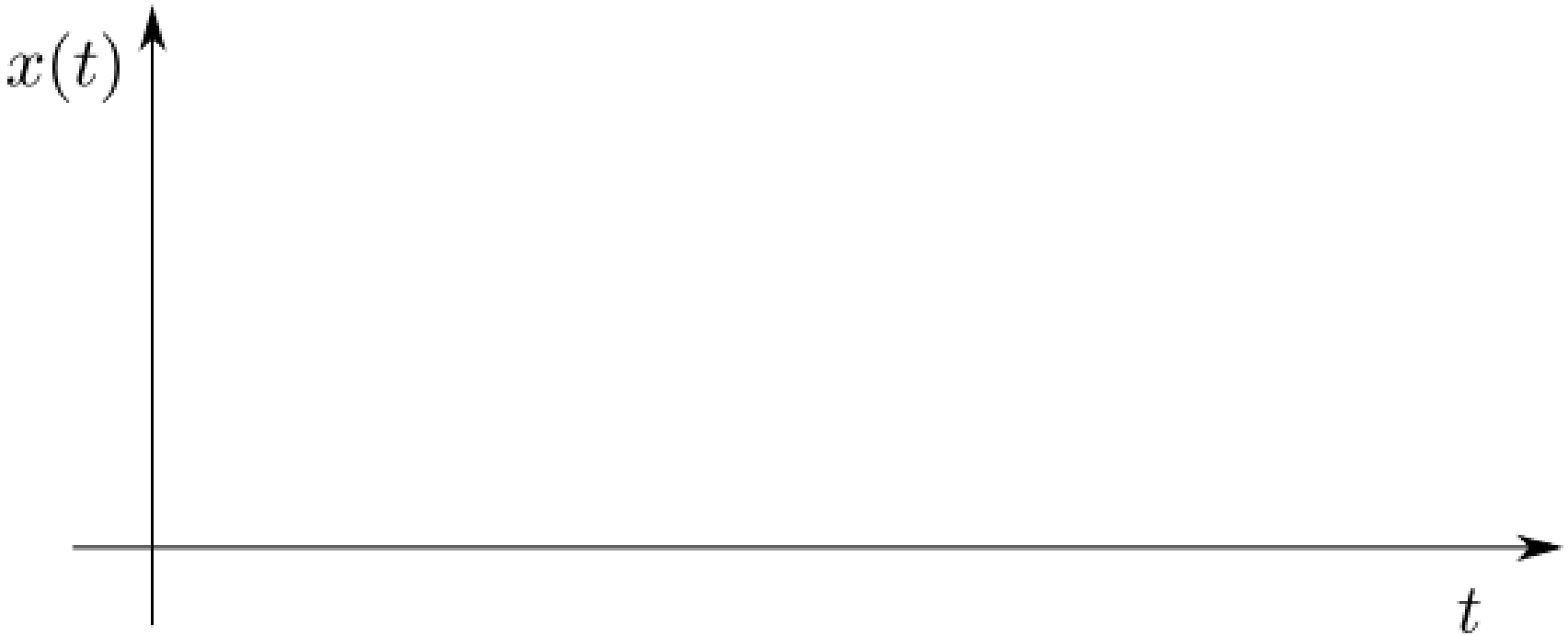
# Unit Step Response

The **unit step response** is a special case of the *forced response* in which $f(t)$ is the step function of unit magnitude (F=1).

| | |
|---|---|
| Overdamped | $x(t) = \frac{1}{k}(\frac{r_2}{r_1 - r_2}e^{-r_1 t} - \frac{r_1}{r_1 - r_2}e^{-r_2 t} + 1$ |
| | $r_{1,2} = -s_{1,2}$ |
| Critically Damped | $x(t) = \frac{1}{K}[(-1 - \omega_n t)e^{-\omega_n t} + 1]$ |
| Underdamped | $x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}sin(\omega_d t + \phi + 1\right]$ |
| | $\phi = tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$ |

# Rise, Peak, and Settling Time

We are going to derive several quatities that describes the response of an underdamped system.



# Rise Time

The **rise time** is the time at which the response first equals the steady state value.

$$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$$

Set the *transient term* to zero and solve for $t$.

$$e^{-\zeta\omega_n t} sin(\omega_d t + \phi) = 0 \implies sin(\omega_d t + \phi) = 0 \implies \omega_d t + \phi = 2\pi$$

$$t_{rise} = t_r = \frac{2\pi - \phi}{\omega_d}$$

# Rise, Peak, and Settling Time

The **peak time** is the time at which the response equals the maximum value. Find the derivative of the response equation and set it equal to zero.

$$\dot{x}(t) =$$

$$\left(\frac{1}{K}\frac{1}{\sqrt{1-\zeta^2}}\right)\left[e^{-\zeta\omega_n t}(\omega_d cos(\omega_d t+\phi))+sin(\omega_d t+\phi)(-\zeta\omega_n e^{-\zeta\omega}\right.$$

$$sin(\omega_d)t = 0 \implies \omega_d t = \pi \implies t_{peak} = t_p = \frac{\pi}{\omega_d}$$

# Rise, Peak, and Settling Time

The **settling time** is the time at which the response decays to a certain percentage of the steady state value.

It can be esitmated as:

$$t_{settling} = t_s = -\frac{ln(tolerance)}{\zeta\omega_n}$$

$$2\% \implies tolerance = 0.02$$

$$5\% \implies tolerance = 0.05$$

# Rise, Peak, and Settling Time

The **maximum overshoot** is the response beyond the steady state value.

$$M_p = x(t_p) - x_{ss} \implies M_p = \frac{1}{k} e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

This is often expressed as a percentage.

$$M_{\%} = \frac{x(t_p) - x_{ss}}{x_{ss}} 100 = 100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

# Damping Ratio from Maximum Overshoot

The *damping ratio* can be determined from the maximum overshoot!

$$M_{\%} = 100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Solve for $\zeta$.

$$\zeta = \frac{R}{\sqrt{\pi^2 + R^2}} \quad with \quad R = ln\left(\frac{100}{M\%}\right)$$

# Damping Ratio from Log Decrement

The logarithmic decrement is the natural log of the ratio of the amplitudes of any two successive peaks:

$$\delta = \frac{1}{n} ln \frac{x(t)}{x(t + nT)}$$

x(t) is the overshoot (amplitude - final value) at time t and x(t + nT) is the overshoot of the peak n periods away.

The damping ratio is then found from the logarithmic decrement by:

$$\zeta = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{\delta}\right)^2}}$$

# Damping Ratio from Log Decrement

What is the significance of all of this?

Why do we care about all of these new parameters?

# System Identificatio

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