

# Lecture Module - The Laplace Transform

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

## Module 2 - The Laplace Transform

## Module 2 - The Laplace Transform

- Topic 1 - The Laplace Transform
- Topic 2 - Laplace Transforms Method
- Topic 3 - Partial Fraction Decomposition

## Topic 1 - The Laplace Transform

- An Integral Transform
- Laplace Transform of A Derivative
- Properties of an Integral

## An Integral Transform

The Laplace Transform is an Integral Transform

Given a function  $x(t)$  in the time domain where  $t \geq 0$ ,  
the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st} dt$$

And its inverse is similarly defined as:

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

The Laplace Domain variable  $s$  is a complex number:

$$s = \sigma + j\omega$$

# An Integral Transform

# An Integral Transform

## Laplace Transform of A Derivative

It is useful to find the laplace transform of the derivative of a function:

$$\mathcal{L}\left\{\frac{d}{dt}(x(t))\right\} = \mathcal{L}\{\dot{x}(t)\} = s\mathcal{L}\{x(t)\} - x(t=0)$$

$$= sX(s) - x(t=0)$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$$

$$\mathcal{L}\left\{\frac{d^2}{dt^2}(x(t))\right\} = \mathcal{L}\{\ddot{x}(t)\} = s^2\mathcal{L}\{x(t)\} - sx(t=0) - \dot{x}(t=0)$$

$$= s^2X(s) - sx(t=0) - \dot{x}(t=0)$$

$$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx_0 - \dot{x}_0$$

# Laplace Transform of A Derivative



## Properties of an Integral

Also, remember that the transform inherits the properties of an integral.

$$\int [x(t) + y(t)] dt = \int x(t)dt + \int y(t)dt$$

$$\int Kx(t)dt = K \int x(t)dt \quad (K \text{ is constant})$$

Therefore these properties can be used with the Laplace transform.

# Table of Transform Pairs

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. $e^{at}$	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	6. $t^{n-1/2}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+1/2}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$

# Table of Transform Pairs

17. $\sinh(at)$	$\frac{a}{s^2 - a^2}$	18. $\cosh(at)$	$\frac{s}{s^2 - a^2}$
19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2 - b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2 - b^2}$
23. $t^n e^{at}, \quad n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ <a href="#">Heaviside Function</a>	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ <a href="#">Dirac Delta Function</a>	$e^{-cs}$
27. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	28. $u_c(t) g(t)$	$e^{-cs} \mathcal{L}\{g(t+c)\}$
29. $e^{ct} f(t)$	$F(s-c)$	30. $t^n f(t), \quad n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t} f(t)$	$\int_s^\infty F(u) du$	32. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$	34. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
37. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$		

## Topic 2 - Laplace Transforms Method

- Step 1 - Apply Laplace Transform
- Step 2 - Solve for  $X(s)$
- Step 3 - Rearrange to Find Invertible Form

## Step 1 - Apply Laplace Transform

Example:

Solve the first order differential equation using the Laplace Transforms Method with the initial condition given.

$$4\dot{x} = \sin(t) \quad \text{with} \quad x(t=0) = x_0$$

Apply the Laplace Transform to both sides of the differential equation.

$$4(sX(s) - x_0) = \frac{1}{s^2 + 1}$$

## Step 2 - Solve for $X(s)$

This step can seem open ended...

$$X(s) = \frac{1}{4s(s^2 + 1)} + \frac{x_0}{s}$$

## Step 2 - Solve for $X(s)$

## Step 3 - Rearrange to Find Invertible Form

Write  $X(s)$  in a form that can be inverted using the table of Laplace transform pairs. This typically involves partial fraction decomposition.

$$\frac{1}{4s(s^2 + 1)} = \frac{1/4}{s(s^2 + 1)} = \frac{a}{s} + \frac{bs + c}{s^2 + 1}$$

Multiply through by the denominator  $4s(s^2 + 1)$ :

$$1 = 4as(s^2 + 1) + 4s(bs + c) = 4(a + b)s^2 + 4cs + 4a$$

Solve for the coefficients by *equating coefficients*.

$$(a + b) = 0 \quad c = 0 \quad a = \frac{1}{4} \implies a = \frac{1}{4} \quad b = -\frac{1}{4} \quad c = 0$$



## Step 2 - Solve for $X(s)$

## Step 3 - Rearrange to Find Invertible Form

## Step 4 - Invert for Final Answer

Substitute the coefficients into  $X(s)$ ,

$$X(s) = \frac{x_0}{s} + \frac{1}{4s} - \frac{s}{4(s^2 + 1)}$$

and use the inverse transform to solve for  $x(t)$ . Use the Table.

$$\begin{aligned}\mathcal{L}^{-1}(X(s)) &= x(t) = \\ &= x_0 + \frac{1}{4} - \frac{1}{4}\cos(t) = x_0 + \frac{1}{4}(1 - \cos(t))\end{aligned}$$

This method works for complex problems but it can get messy...

## Step 4 - Invert for Final Answer

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. $e^{at}$	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
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## Step 4 - Invert for Final Answer

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
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## Topic 3 - Partial Fraction Decomposition

- General Polynomial Form
- Case 1 - Distinct Roots
- Case 2 - Repeated Roots
- Special Case - Complex Roots

## General Polynomial Form

**The Laplace Transform is an Integral Transform:**

Given a function  $x(t)$  in the time domain where  $t \geq 0$ , the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st} dt$$

Partial Fraction Expansion leads to a general form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_ms^m + b_{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad n \geq m$$

# General Polynomial Form



## Case 1 - Distinct Roots

### Case 1 - Distinct Roots: $n$ roots are real and distinct

The general form is factored:

$$X(s) = \frac{N(s)}{(s + r_1)(s + r_2) \dots (s + r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s + r_1)} + \frac{C_2}{(s + r_2)} + \dots + \frac{C_n}{(s + r_n)}$$

Where:

$$C_i = \lim_{s \rightarrow -r_i} \{X(s)(s + r_i)\}$$

And this leads to a solution:

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + \dots + C_n e^{-r_n t}$$

## Case 2 - Repeated Roots

**Case 2 - Repeated Roots:**  $p$  number of roots have the same value ( $s = -r$ ) and remaining roots are distinct and real distinct

$$X(s) = \frac{N(s)}{(s + r_1)^p (s + r_{p+1})(s + r_{p+2}) \dots (s + r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s + r_1)^p} + \frac{C_2}{(s + r_1)^{p-1}} + \dots$$
$$+ \frac{C_p}{(s + r_1)} + \frac{C_{p+1}}{(s + r_{p+1})} + \dots + \frac{C_n}{(s + r_n)}$$

## Case 2 - Repeated Roots

Coefficients for the repeated root are:

$$C_1 = \lim_{s \rightarrow -r_i} \{X(s)(s + r_i)^p\}$$

$$C_2 = \lim_{s \rightarrow -r_i} \left\{ \frac{d}{ds} X(s)(s + r_i)^p \right\}$$

$$C_i = \lim_{s \rightarrow -r_i} \left\{ \frac{1}{(i-1)!} \frac{d^{(i-1)}}{ds^{(i-1)}} X(s)(s + r_i)^p \right\}$$

Coefficients for the distinct roots are the same as in Case 1:  
And this leads to a solution:

$$x(t) = C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \dots$$

$$\dots + C_p e^{-r_1 t} + C_{p+1} e^{-r_{p+1} t} \dots + C_n e^{-r_n t}$$

## Special Case - Complex Roots

**Special Case - Complex Roots:** the roots are distinct  $\Rightarrow$   
**Case 1**

Example:

$$X(s) = \left[ \frac{3s + 7}{(4s^2 + 24s + 136)} \right] = \left[ \frac{3s + 7}{4(s^2 + 6s + 34)} \right]$$

The solution can be found by forming two perfect squares in the denominator.

$$X(s) = \frac{1}{4} \left[ \frac{3s + 7}{(s + 3)^2 + 5^2} \right]$$

## Special Case - Complex Roots

Now this can be expanded into the following terms which can be found in the table!

$$X(s) = \frac{1}{4} \left[ C_1 \frac{5}{(s+3)^2 + 5^2} + C_2 \frac{s+3}{(s+3)^2 + 5^2} \right]$$

Multiply by the denominator and solve for  $C_1$  and  $C_2$ .

$$3s + 7 = 5C_1 + C_2(s+3) = 5C_1 + C_2s + 3C_2 \implies C_2 = 3, C_1 = -$$

## Special Case - Complex Roots

Finally substitute and invert using the table.

$$X(s) = \frac{1}{4} \left[ -\frac{2}{5} \frac{5}{(s+3)^2 + s^2} + 3 \frac{s+3}{(s+3)^2 + s^2} \right]$$

Write the final answer in the time domain.

$$x(t) = -\frac{1}{10} e^{-3t} \sin(5t) + \frac{3}{4} e^{-3t} \cos(5t)$$

## Special Case - Complex Roots