

## Lecture Module - Time Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

### Time Response

## Lecture Module - Time Response

- Topic 1 - First Order Time Response
- Topic 2 - Second Order Time Response
- Topic 3 - Response and Root Location
- Topic 4 - Specification of The Step Response

## Topic 1 - First Order Time Response

- Second Order Free Response
- Second Order Forced Response
- Response and System Stability
- Equation Components

## Model and EOM

Consider the model of the moving mass we derived.



The EOM is:

$$m\dot{v} + cv = 0$$

## Solution with Laplace Transforms Method

$$\mathcal{L}\{m\dot{v} + cv = 0\} \implies m[sV(s) - v(0)] + cV(s) = 0$$

$$(ms + c)V(s) = \frac{mv(0)}{(ms+c)} = \frac{V(0)}{s + \frac{c}{m}}$$

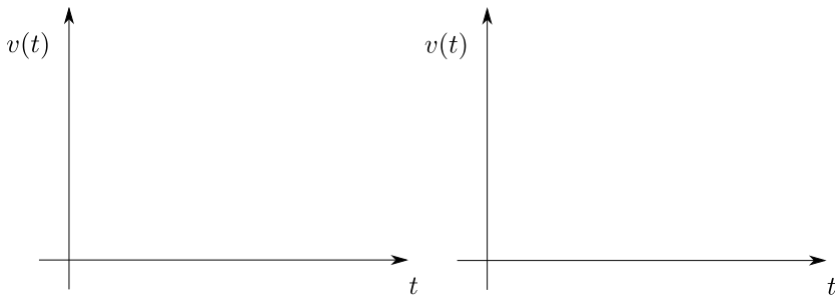
We can find the expected result from the table.

$$v(t) = v(0)e^{-\frac{c}{m}t} = v(0)e^{-\frac{t}{\tau}} \quad \text{with} \quad \tau =$$

## Sketch Response Equation

Sketch the System Response in the time Domain.

$$v(t) = v(0)e^{-\frac{t}{\tau}}$$



Is this a stable system? What does that mean?

# Stability in Dynamic Systems

A dynamic system is stable if ...

# Second Order Forced Response



## Step Input Function

Consider the model subject to a Step Input,  $f(t)$ .



$$m\dot{v} + cv = f(t)$$

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

## Solution with Laplace Transforms Method - Step 1

The method of Laplace Transforms is shown.

$$\mathcal{L}\{m\dot{v} + cv = F\} \implies m[sV(s) - v(0)] + cV(s) = \frac{F}{s}$$

$$(ms + c)V(s) = \frac{F}{s} + mv(0)$$

Solve for  $V(s)$ .

$$V(s) = \frac{F}{s(ms+c)} + \frac{mv(0)}{ms+c}$$

## Solution with Laplace Transforms Method - Step 2

Expand  $V(s)$  as a partial fraction.

$$V(s) = \frac{F}{s(ms+c)} + \frac{mv(0)}{ms+c} \implies \frac{F}{s(ms+c)} = \frac{a}{s} + \frac{b}{ms+c}$$

'Cover up' to find the coefficients.

$$a = \frac{F}{m \times 0 + c} \quad \text{and} \quad b = \frac{F}{\frac{-c}{m}} = \frac{-Fm}{c}$$

This leads to a form that can be inverted with the table.

$$V(s) = \frac{F}{c} \left\{ \frac{1}{s} - \frac{1}{s + \frac{c}{m}} \right\} + \frac{v(0)}{s + \frac{c}{m}}$$

## Solution with Laplace Transforms Method - Step 3

Can you find these terms in the Table of Laplace Transforms?

$$V(s) = \frac{F}{c} \left\{ \frac{1}{s} - \frac{1}{s + \frac{c}{m}} \right\} + \frac{v(0)}{s + \frac{c}{m}}$$

The inverse Laplace transform of  $V(s)$  gives the time response.

$$v(t) = \frac{F}{c} \{1 - e^{-\frac{t}{\tau}}\} + v(0)e^{-\frac{t}{\tau}} = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$

First Order Time Response  
Second Order Time Response  
Response and Root Location  
Specification of The Step Response

Second Order Free Response  
Second Order Forced Response  
Response and System Stability  
Equation Components

# Response and System Stability

First Order Time Response  
Second Order Time Response  
Response and Root Location  
Specification of The Step Response

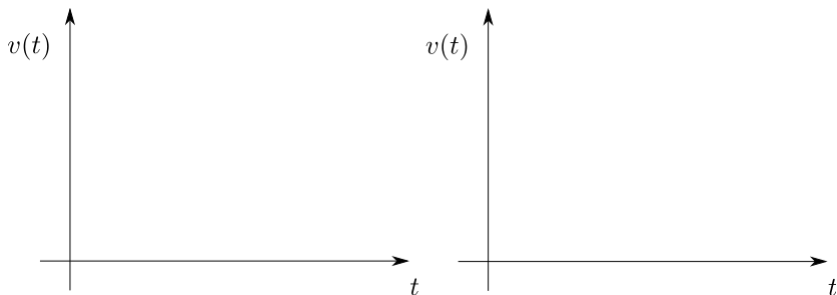
Second Order Free Response  
Second Order Forced Response  
Response and System Stability  
Equation Components

# Response and System Stability

## Sketch Response Equation

Sketch the System Response in the time Domain.

$$v(t) = \{v(0) - \frac{F}{c}\}e^{-\frac{t}{\tau}} + \frac{F}{c}$$



Is this a stable system?

## Components of the Response

In these forms we can see the different components of the response.

$$v(t) = \frac{F}{C}\{1 - e^{-\frac{t}{\tau}}\} + v(0)e^{-\frac{t}{\tau}} = \{v(0) - \frac{F}{C}\}e^{-\frac{t}{\tau}} + \frac{F}{C}$$

- Forced Response
- Free Response
- Transient Response
- Steady-State Response



## References

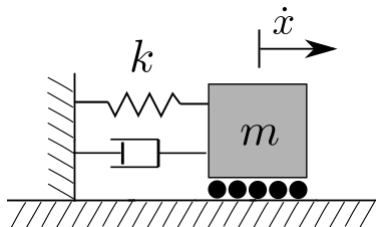
- System Dynamics, Palm III, Third Edition - Section 8.1 - Response of First Order Systems - pg. 475

## Topic 2 - Second Order Time Response

- Damped Natural Frequency and Damping Ratio
- The Overdamped Case
- The Critically Damped Case
- The Underdamped Case

## Mass Spring Model

Consider the mass-spring system without damping.



The EOM is:

$$m\ddot{x} + kx = 0 \text{ with}$$

$$x(t = 0) = x_0, \text{ and } v(t = 0) = v_0$$

## Solution with Laplace Transforms Method

Solve for  $x(t)$  with a method of your choice.

$$x(t) = \frac{v_0}{\omega_n} \sin(\omega_n t) + x_0 \cos(\omega_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

## Phase Shift

The solution is commonly written as a single oscillating term with a **phase shift**  $\phi$ .

$$x(t) = \frac{v_0}{\omega_n} \sin(\omega_n t) + x_0 \cos(\omega_n t) \text{ with } \omega_n = \sqrt{\frac{k}{m}}$$

Is equivalent to:

$$x(t) = A \cos(\omega_n t - \phi) \quad A = \sqrt{x_0^2 + \left[\frac{v_0}{\omega_n}\right]^2} \quad \phi = \tan^{-1}\left(\frac{v_0}{x_0 \omega_n}\right)$$

Sine could be used instead.

$$x(t) = A \sin(\omega_n t + \phi) \quad A = \sqrt{x_0^2 + \left[\frac{v_0}{\omega_n}\right]^2} \quad \phi = \tan^{-1}\left(\frac{x(0)\omega_n}{v_0}\right)$$

## Sketch of Free Response

Sketch the **free** response in the time domain.

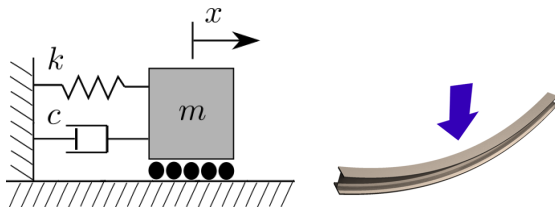
$$x(t) = \frac{v_0}{\omega_n} \sin(\omega_n t) + x_0 \cos(\omega_n t) \\ = A \cos(\omega_n t - \phi) \text{ with } \phi = \tan^{-1}\left(\frac{v_0}{x_0 \omega_n}\right)$$



Is this a stable system? What does the phase shift  $\phi$  represent?

## Second Order System with Damping

Now, consider the mass-spring system with damping present.



The EOM is:

$$m\ddot{x} + c\dot{x} + kx = 0 \text{ with}$$

$$x(t = 0) = x_0, \text{ and } v(t = 0) = v_0$$

## Solution with Trial Solution Method

The trial solution method is used to derive the response equation in terms of the system variables and parameters.

$$m\ddot{x} + c\dot{x} + kx = 0 \implies (mr^2 + cr + k)Ae^{rt} = 0$$

You can see the *characteristic equation* becomes:

$$(mr^2 + cr + k) = 0$$

Solve for the roots. In system dynamics they are called  $s_{1,2}$

$$r_{1,2} = s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$



## The Roots of the System

The roots of the system determine the behavior.

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

The **discriminant**  $c^2 - 4mk$  drives the *case* in the trial solution method.

- IF \_\_\_\_\_  $\implies$  Case 1: Distinct and Real
- IF \_\_\_\_\_  $\implies$  Case 2: Repeated and Real
- IF \_\_\_\_\_  $\implies$  Case 3: Complex Conjugate Pair

## Damping Cases and the Critical Damping Value

In a system with known mass and spring constant, the damping value determines the behavior. The damping value that causes the *discriminant* to equal zero (case 2) is the **critical damping value**.

$$c^2 - 4mk = 0 \implies c = \sqrt{4mk} = 2\sqrt{mk}$$

$$c_{critical} = 2\sqrt{mk}$$

## The Damping Ratio

The damping ratio  $\zeta$  is the ratio of damping  $c$  to the critical damping value  $c_{critical}$ .

$$\zeta = \frac{c}{c_{critical}} = \frac{c}{2\sqrt{mk}}$$

Re-write the roots with this new quantity.

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

We define a new quantity, **damped natural frequency**.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Now re-write the roots again in terms of  $\zeta$  and  $\omega_d$ .

$$s_{1,2} = \zeta\omega_n \pm j\omega_d$$

## Damped Natural Frequency and Damping Ratio

The behavior of the system depends on the damping ratio.

Case 1	$c > 2\sqrt{mk}$	<b>Overdamped</b>	$\zeta > 1$
Case 2	$c = 2\sqrt{mk} = c_{critical}$	<b>Critically Damped</b>	$\zeta = 1$
Case 3	$c < 2\sqrt{mk}$	<b>Underdamped</b>	$\zeta < 1$

## The Overdamped Case

The roots are real and distinct and the system *does not* oscillate.

Overdamped  $c > 2\sqrt{mk} \implies \zeta > 1$

$$s_{1,2} = -\zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$x(t) = C_1 e^{(-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1})t} + C_2 e^{(-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1})t}$$

$$x(t) = e^{-\zeta\omega_n t} \{ C_1 e^{\omega_n \sqrt{\zeta^2 - 1}t} + C_2 e^{-\omega_n \sqrt{\zeta^2 - 1}t} \}$$

$$C_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}} \quad C_2 = \frac{-v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

## The Critically Damped Case

The roots are real and repeated and the system *does not* oscillate.

Critically Damped  $c = 2\sqrt{mk} \implies \zeta = 1$

$$s_{1,2} = \frac{-c}{2m} = \zeta\omega_n = -\omega_n$$

$$x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

$$\dot{x} = -\omega_n C_1 e^{-\omega_n t} + C_2 t (-\omega_n e^{-\omega_n t}) + e^{-\omega_n t} (C_2)$$

$$x(t=0) = x_0 \implies C_1 = x_0$$

$$v(t=0) = v_0 \implies v_0 = -\omega_n C_1 + 0 + (1)C_2 \implies C_2 = v_0 + \omega_n x_0$$

## The Underdamped Case

Take the derivative and solve for the second unknown.

$$x(t) = e^{-\zeta\omega_n t} \{A \cos(\omega_d t) + B \sin(\omega_d t)\}$$

$$\begin{aligned} \dot{x}(t) = & e^{-\zeta\omega_n t} (-\omega_d A \sin(\omega_d t)) + A \cos(\omega_d t) (-\zeta\omega_n e^{-\zeta\omega_n t}) \\ & + e^{-\zeta\omega_n t} (\omega_d B \cos(\omega_d t)) + B \sin(\omega_d t) e^{-\zeta\omega_n t} \end{aligned}$$

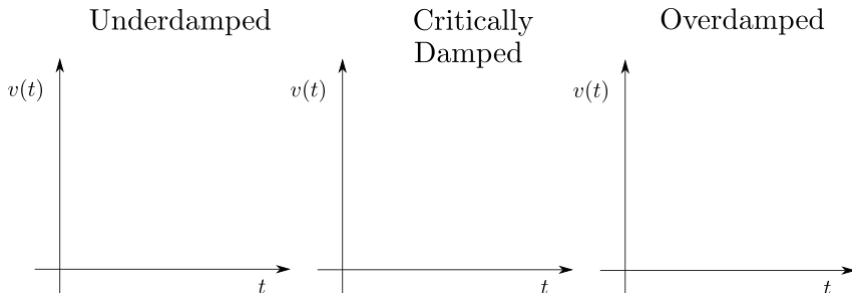
$$\dot{x}(t=0) = A(1)(-\zeta\omega_n(1)) + (1)\omega_d B(1) \implies B = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

Finally we get to the response equation.

$$x(t) = e^{-\zeta\omega_n t} \left\{ x_0 \cos(\omega_d t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_d} \sin(\omega_d t) \right\}$$

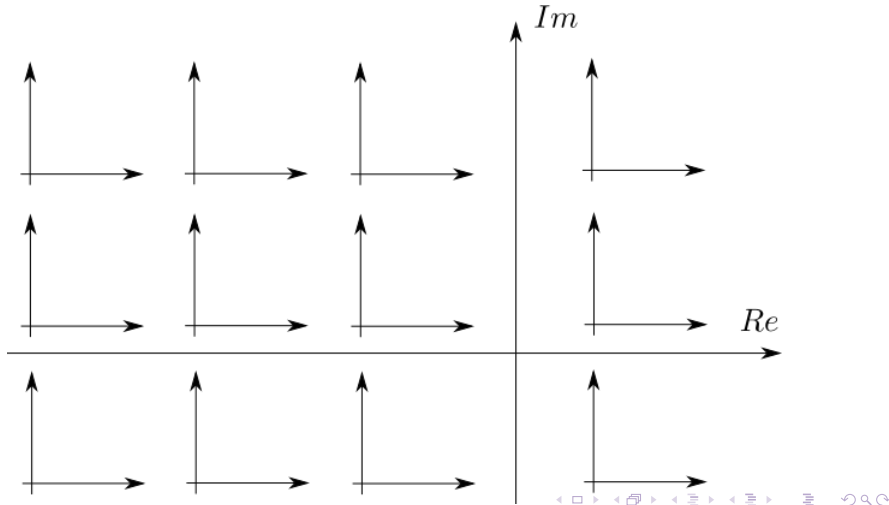
## Response of the Three Different Cases

Each of the three cases behaves in a *characteristic* way.





# Affects of Damping Ratio and Damped Natural Frequency

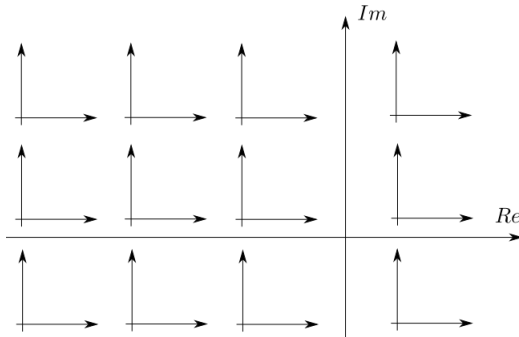


## Topic 3 - Response and Root Location

- The Complex Plane
- Along a Vertical Line
- Along a Horizontal Line
- Along a Diagonal Line

## Affect of Root Location on Response

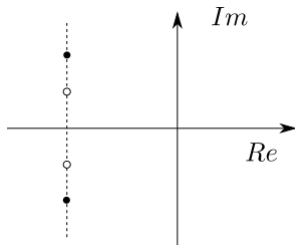
The *location* of the root in the *complex plane* shows the affects of the roots on the system behaviour.



# The Complex Plane

## Along a Vertical Line

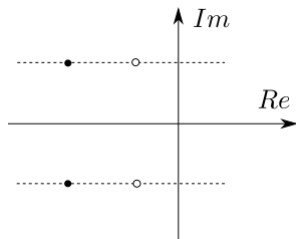
As the root moves along a vertical line...



## Along a Vertical Line

## Along a Horizontal Line

As the root moves along a horizontal line...



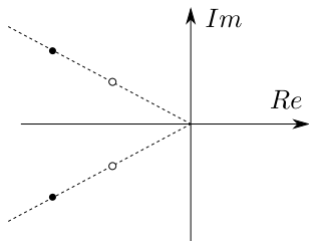
## Along a Horizontal Line



## Along a Horizontal Line

## Along a Diagonal Line

As the root moves along a diagonal line...



What about the angle of this line?

## Topic 3 - Specification of The Step Response

- The Unit Step Response
- Rise, Peak, and Settling Time
- Maximum Overshoot and The Damping Ratio
- System Identification

# The Unit Step Response

# The Unit Step Response

# Rise, Peak, and Settling Time

# Rise, Peak, and Settling Time

# Rise, Peak, and Settling Time



# Maximum Overshoot and The Damping Ratio

# Maximum Overshoot and The Damping Ratio

# Maximum Overshoot and The Damping Ratio

# Maximum Overshoot and The Damping Ratio

# Maximum Overshoot and The Damping Ratio

# System Identification

# System Identification

# System Identification



# System Identification