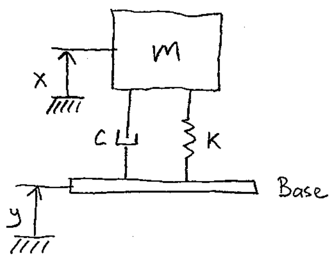
(13.1) Base Excitation

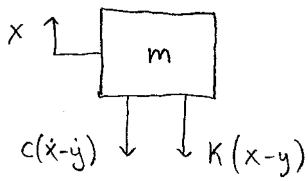
A very common vibration input to a system is motion of its base support. This is called base excitation. Consider a car traveling down a bumpy road. The road can be thought of as a moving base. The car's suspension is designed to minimize the motion and force transmitted to the passenger compartment; it is a vibration isolation system.

In vibrations, we may want to decrease the force transmitted from an object to its base (force transmissibility) or decrease the motion of the object from the excitation of the base (displacement transmissibility).

Consider the base excited mass-spring-damper



FBD:



Newton's Method

$$\Sigma F_x = m\ddot{x} = -c(\dot{x}-\dot{y}) - k(x-y)$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

The transfer function is:

$$ms^2 X(s) + cs X(s) + k X(s) = cs Y(s) + k Y(s) \Rightarrow T(s) = \frac{X(s)}{Y(s)} = \frac{cs+k}{ms^2+cs+k}$$

↑ Displacement Transmissibility

Dividing numerator + denomenator by m, we can write as:

$$\frac{X(s)}{Y(s)} = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Substitute $s = j\omega$

$$\frac{X(j\omega)}{Y(j\omega)} = \frac{2\zeta\omega_n\omega j + \omega_n^2}{-\omega^2 + 2\zeta\omega_n\omega j + \omega_n^2}$$

Dividing numerator + denomenator by ω_n, we can write as

$$\frac{X(j\omega)}{Y(j\omega)} = \frac{2\zeta r j + 1}{1 - r^2 + 2\zeta r j}$$

The displacement transmissibility magnitude and phase are

$$\left|\frac{X(j\omega)}{Y(j\omega)}\right| = \frac{X}{Y} = \sqrt{\frac{(2\zeta r)^2 + 1}{(1-r^2)^2 + (2\zeta r)^2}} = \sqrt{\frac{4\zeta^2 r^2 + 1}{(1-r^2)^2 + 4\zeta^2 r^2}}$$

This can be used to calculate the SS response amplitude, X, caused by a sinusoidal input with amplitude Y.

$$\phi = \angle(2\zeta rj + 1) - \angle(1 - r^2 + 2\zeta rj) = \tan^{-1}\left(\frac{2\zeta r}{1}\right) - \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

Now consider solving for the transfer function of the base displacement to output <u>force</u>. First, recognize that the force transmitted to the mass is

$$f_t = m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x)$$

Laplace Transform:

$$F_t(s) = c(sY(s) - sX(s)) + k(Y(s) - X(s)) = (cs + k)[Y(s) - X(s)]$$

Substituting our previous expression for $X(s)$ to eliminate the $X(s)$ term:

$$F_t(s) = (cs+k)\left[Y(s) - \frac{cs+k}{ms^2 + cs + k} Y(s)\right] = (cs+k)\frac{ms^2}{ms^2+cs+k} Y(s)$$

So the output force to base displacement is

$$\frac{F_t(s)}{Y(s)} = (cs+k)\frac{ms^2}{ms^2+cs+k}$$

But we typically divide by k to form a dimensionless quantity

$$\frac{F_t(s)}{kY(s)} = \frac{cs+k}{k}\,\frac{ms^2}{ms^2+cs+k}$$

↑
Force Transmissibility

Dividing numerator + denomenator by m

$$\frac{F_t(s)}{kY(s)} = \frac{2\zeta\omega_n s + \omega_n^2}{\omega_n^2}\,\frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Substituting $s = j\omega$

$$\frac{F_t(j\omega)}{kY(j\omega)} = \frac{2\zeta\omega_n\omega j + \omega_n^2}{\omega_n^2}\,\frac{-\omega^2}{-\omega^2 + 2\zeta\omega_n\omega j + \omega_n^2}$$

Dividing numerator + denomenator by ω_n, we get

$$\frac{F_t(j\omega)}{kY(j\omega)} = (2\zeta rj + 1)\frac{-r^2}{1 - r^2 + 2\zeta rj}$$

The force transmissibility magnitude and phase are

$$\left|\frac{F_t(j\omega)}{K\,Y(j\omega)}\right| = \frac{F_t}{KY} = r^2\sqrt{\frac{4\zeta^2r^2+1}{(1-r^2)^2+4\zeta^2r^2}}$$

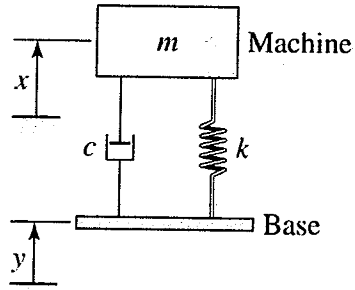
This can be used to calculate the ss amplitude of the force transmitted to the mass due to a sinusoidal input with amplitude Y.

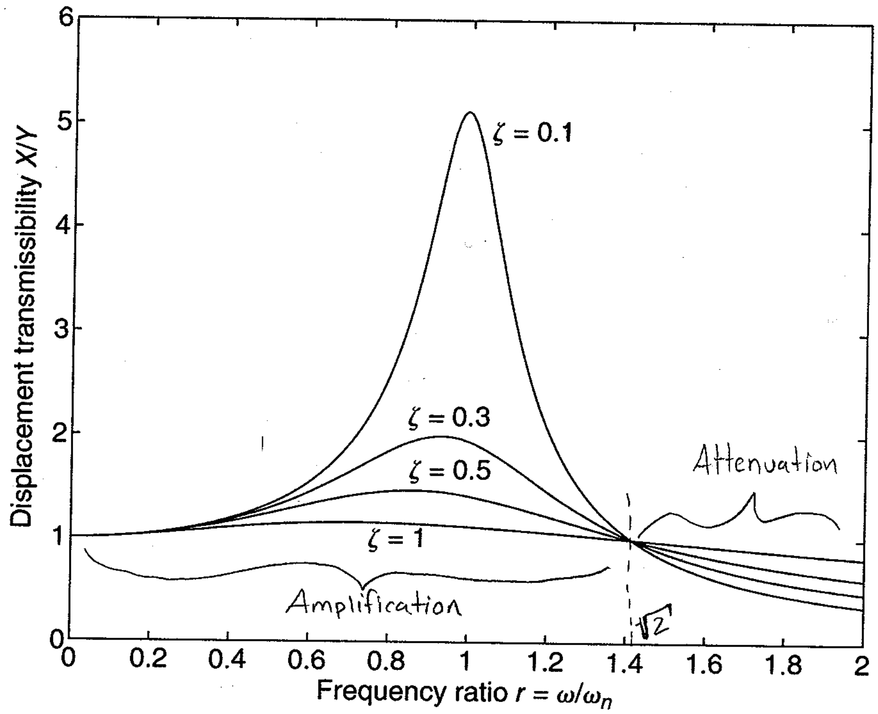
$$\phi = \angle\left(-2\zeta r^3 j - r^2\right) - \angle\left(1-r^2+2\zeta r j\right) = \tan^{-1}(2\zeta r) - \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$ (same as disp. trans.)

Note, the force + displacement transmissibilities are related by

$$\frac{F_t}{KY} = r^2\frac{X}{Y}$$

Displacement and Force Transmissibility

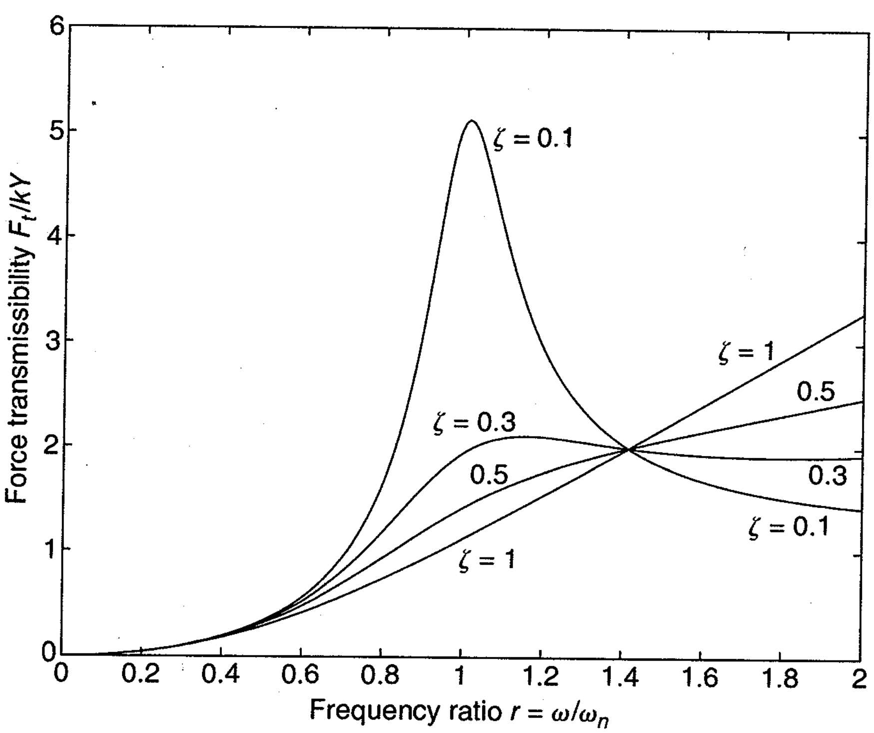




Displacement Transmissibility

$$\frac{X}{Y} = \sqrt{\frac{4\zeta^2 r^2 + 1}{(1-r^2)^2 + 4\zeta^2 r^2}}$$

- Maximum base motion is transfered to mass around $r = 1$ (at resonance)
- Below $r = \sqrt{2}$, the base motion is amplified
- Above $r = \sqrt{2}$, the base motion is attenuated
- As ζ decreases, the potential amplification increases
- As r increases beyond $\sqrt{2}$, the displacement transmissibility decreases

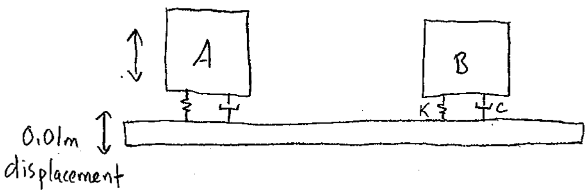


Force Transmissibility

$$\frac{F_t}{KY} = r^2 \sqrt{\frac{4\zeta^2 r^2 + 1}{(1-r^2)^2 + 4\zeta^2 r^2}}$$

- For small values of ζ, force transmissibility decreases above $r = \sqrt{2}$
- For large values of ζ, force transmissibility increases with increasing r.
- For small values of ζ, a peak in force transmissibility is found near $r = 1$.

Base Excitation Example p. 13.3



Machine A causes floor to vibrate with amplitude of 0.01m. For machine B, mass = 1500 kg, stiffness = 2×10^4 N/m, damping ratio = $\zeta = 0.04$. Find max force transmitted to machine B @ resonance.

The frequency ratio at resonance for a 2nd order system is

$$r = \frac{\omega_r}{\omega_n} = \frac{\omega_n\sqrt{1-2\zeta^2}}{\omega_n} = \sqrt{1-2\zeta^2} \Rightarrow r = \sqrt{1-2(0.04)^2} = 0.998$$

The force transmissibility is given by

$$\frac{F_t}{KY} = r^2\sqrt{\frac{4\zeta^2 r^2+1}{(1-r^2)^2+4\zeta^2r^2}} = 0.998^2\sqrt{\frac{4(0.04)^2(0.998)^2+1}{(1-0.998^2)^2+4(0.04)^2(0.998)^2}} = 12.499$$

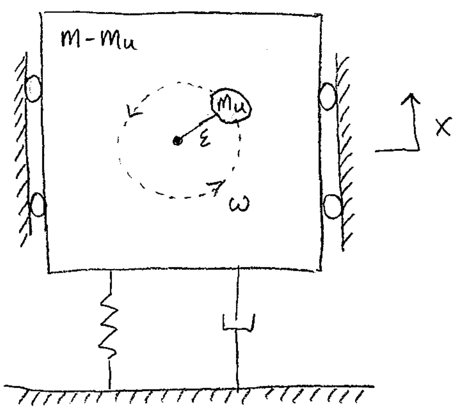
So the transmitted force is

$$F_t = KY(12.499) = 20{,}000(0.01)(12.499) = \boxed{2500\,N}$$

(13.2) Rotating Unbalance

A common cause of sinusoidal excitation is the unbalance in rotating machinery caused when the center of mass of the rotating part does not coincide with the center of rotation. This exists to some degree in every rotating machine.

Consider a machine of total mass, m, with an unbalanced mass, m_u, that is lumped at its center of mass, which is a distance, ϵ. (epsilon), away from the center of rotation. The motion of the machine is restricted in the vertical direction.



- Total mass = m, unbalanced mass = m_u, so "main" mass = $m-m_u$.
- ϵ is called the eccentricity
- Assume constant speed of rotation, ω.

To find the force that the unbalanced mass exerts on the main mass, realize that an object rotating at a constant speed has a radial acceleration:

$$a = \omega^2 r$$

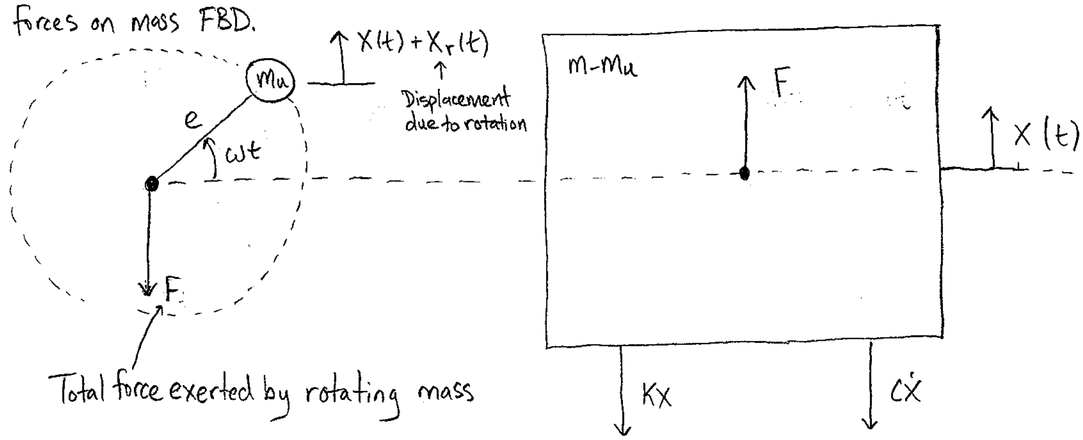
In this case, the unbalanced mass produces a radial acceleration towards the center of rotation equal to:

$$a = \epsilon\omega^2$$

Using $F=ma$, the magnitude of the force caused by this radial acceleration is

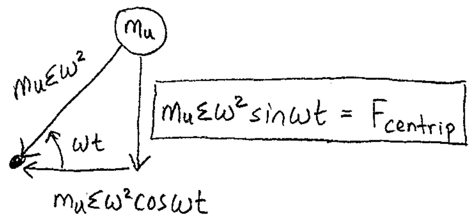
$$F = m_u \epsilon \omega^2$$

To understand the magnitude and direction of the force acting on the machine, it is useful to draw FBDs. Because the mass's motion is restricted to vertical, only draw vertical forces on mass FBD.



First, find the total force exerted by the rotating mass. There are two components acting: a component due to the mass times acceleration of the machine, plus the centripetal force.

- "Mass times acceleration" $F_{ma} = m_u \ddot{x}$
- "Centripetal force" Here, we are only interested in the vertal component since the motion of the mass is restricted.



So, the total force is:

$$F = F_{ma} + F_{centrip} = m_u \ddot{x} + m_u \varepsilon \omega^2 \sin \omega t$$

Now, using the FBD of the machine, we can derive the equation of motion

$$\Sigma F_x = (m - m_u)\ddot{x} = m_u \ddot{x} + m_u \varepsilon \omega^2 \sin \omega t - c\dot{x} - kx$$

$$m\ddot{x} + c\dot{x} + kx = m_u \varepsilon \omega^2 \sin \omega t$$

We can rewrite as

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad F_0 = m_u \varepsilon \omega^2$$

This matches our standard mass-spring-damper system w/ sinusoidal excitation: $m\ddot{x} + c\dot{x} + kx = f(t)$
We already solved this and found the transfer function as: $f(t) = A \sin \omega t$

$$T(r) = \frac{k X(r)}{F_0(r)} = \frac{1}{1 - r^2 + 2\zeta r j}$$

In the case of rotating unbalance, we want to solve for the magnitude of displacement of the machine due to the unbalance. i.e. Solve for X. So,

$$X(r) = \frac{\frac{F_0(r)}{k}}{1 - r^2 + 2\zeta r j}$$

Note: here, $F_0(r)$ is just the magnitude of the force: $F_0(r) = m_u \varepsilon \omega^2$

$$X(r) = \frac{\frac{m_u \varepsilon \omega^2}{k}}{1 - r^2 + 2\zeta r j}$$

To rewrite ω in numerator as r, multiply by $\frac{m}{m}$

$$X(r) = \frac{\frac{m}{k} m_u \varepsilon \omega^2}{m(1 - r^2 + 2\zeta r j)} = \frac{m_u \varepsilon}{m} \frac{r^2}{1 - r^2 + 2\zeta r j}$$

Solving for the magnitude, we have

$$|X(r)| = \boxed{X = \frac{m_u \varepsilon}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

This is the motion of the machine due to the rotating unbalance

We may also be interested in the force transmitted to the base by the rotating unbalance. The force transmitted to the base through the spring + damper is:

$$f_t = Kx + c\dot{x} \Rightarrow F_t(s) = (K + Cs)X(s) \Rightarrow F_t(r) = (K + Cj\omega)X(r)$$

Substituting one of our previous expressions for $X(r)$ $\left(X(r) = \frac{F_0(r)/K}{1-r^2+2\zeta rj} \right)$ into the expression for $F_t(r)$ gives

$$F_t(r) = \frac{F_0(r)\left(1 + \frac{c}{K}\omega j\right)}{1-r^2+2\zeta rj}$$

Realizing $\frac{c}{K} = \frac{2\zeta}{\omega_n}$ and solving for $\frac{F_t(r)}{F_0(r)}$ gives

$$\frac{F_t(r)}{F_0(r)} = \frac{1+2\zeta rj}{1-r^2+2\zeta rj}$$

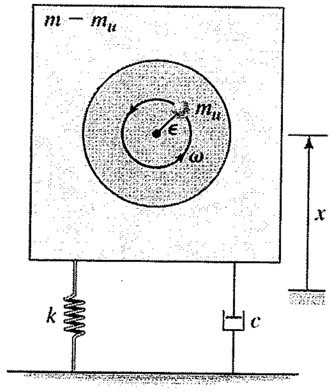
Solving for the magnitude, we can define the Transmissibility Ratio:

$$\left|\frac{F_t(r)}{F_0(r)}\right| = \frac{F_t}{F_0} = \boxed{T_r = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}}$$

This gives the ratio of transmitted force to applied force from rotating unbalance.

★ Note: Transmissibility Ratio has same expression as Displacement Transmissibility!

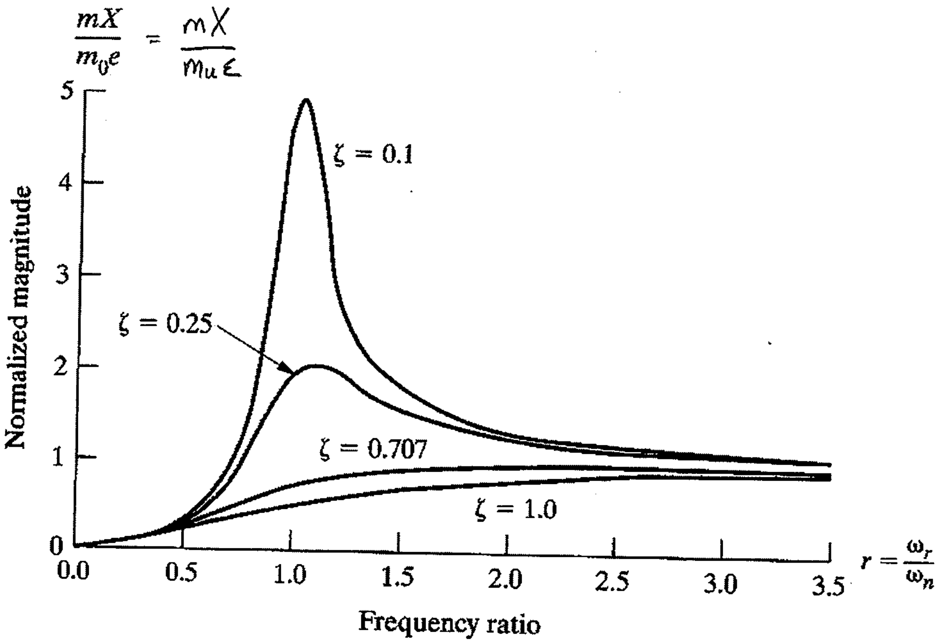
Rotating Unbalance
Magnitude and Transmissibility Ratio



Normalized Magnitude

$$\frac{mX}{m_u \varepsilon} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

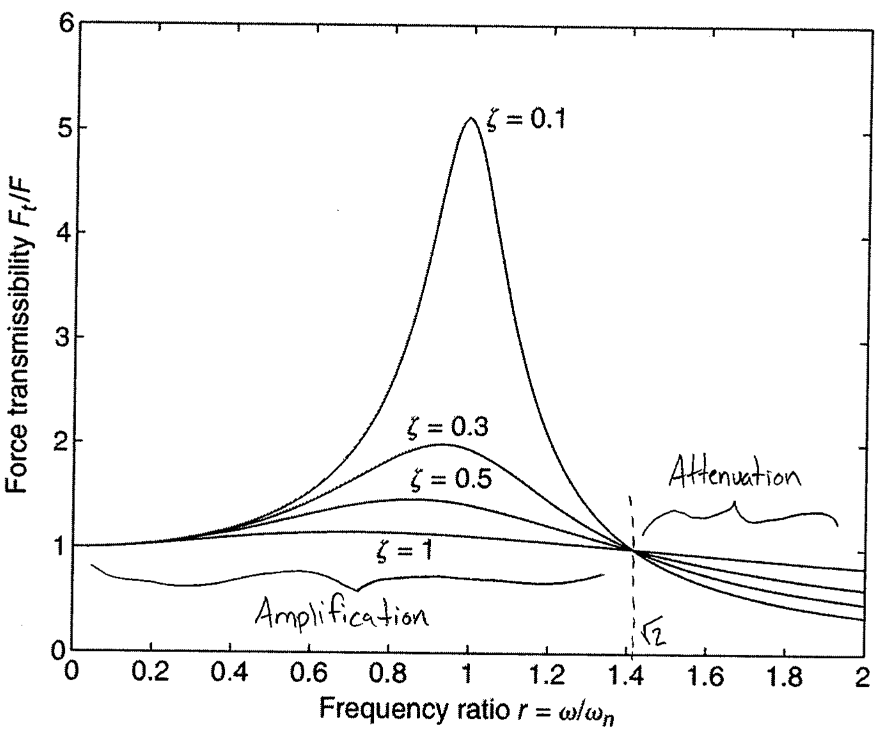
- When plotting magnitude, we nondimensionalize by forming $\frac{mX}{m_u \varepsilon}$
- Maximum displacement occurs near $r=1$
- Large displacements are only observed for systems with small damping
- Effect of unbalanced mass is minimal at large frequencies.



Transmissibility Ratio

$$T_r = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$$

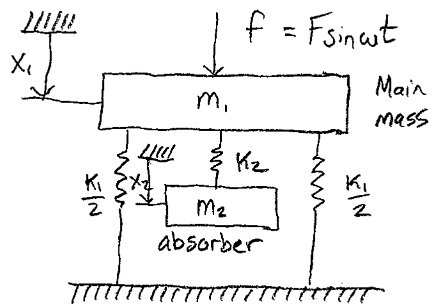
- Same as displacement transmissibility



(13.3) Vibration Absorbers

Vibration absorbers are used to reduce the vibration amplitude of a system with constant disturbance frequency (forcing input / operational frequency). Machines such as saws, sanders, and those powered by ac motors that operate at a fixed frequency are common examples of systems that may use vibration absorbers. Other examples include exhaust pipes / rear ends of vehicles, and large skyscrapers. A vibration absorber is a device consisting of an additional mass and stiffness element attached to the main system. With proper design, the absorber (often called a tuned mass damper) moves with large amplitud while the main system remains motionless (or with little motion). The disturbance energy is transferred to the absorber.

Consider the simple vibration absorber:



EOMs:

Main Mass : $m_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2x_2 = f$

Absorber : $m_2\ddot{x}_2 - K_2x_1 + K_2x_2 = 0$

Find the Transfer Function for each mass using Laplace Transforms

Main Mass : $(m_1s^2 + K_1 + K_2)X_1(s) - K_2X_2(s) = F(s)$

Absorber: $-K_2X_1(s) + (m_2s^2 + K_2)X_2(s) = 0$

Solve these two simultaneous equations for both Transfer Functions ; $T_1 = \frac{X_1(s)}{F(s)}$, $T_2 = \frac{X_2(s)}{F(s)}$

$$T_1(s) = \frac{X_1(s)}{F(s)} = \frac{m_2s^2 + K_2}{(m_1s^2+K_1+K_2)(m_2s^2+K_2) - K_2^2}$$

$$T_2(s) = \frac{X_2(s)}{F(s)} = \frac{K_2}{(m_1s^2+K_1+K_2)(m_2s^2+K_2) - K_2^2}$$

Substitute $s = j\omega$

$$T_1(j\omega) = \frac{X_1(j\omega)}{F(j\omega)} = \frac{K_2 - m_2\omega^2}{(K_1+K_2-m_1\omega^2)(K_2-m_2\omega^2) - K_2^2}$$

$$T_2(j\omega) = \frac{X_2(j\omega)}{F(j\omega)} = \frac{K_2}{(K_1+K_2-m_1\omega^2)(K_2-m_2\omega^2) - K_2^2}$$

Our goal is to select $m_2 + K_2$ such that the motion of the main mass is zero, i.e. $X_1 = 0$. If the applied force is $f = F\sin\omega t$, then the relationship $X_1(s) = T_1(s)F(s) \Rightarrow X_1(j\omega) = T_1(j\omega)F(j\omega)$ shows that the steady-state response of the main mass is

$$x_1(t) = X_1 \sin(\omega t + \phi_1)$$

where

$$X_1 = |T_1(j\omega)| F$$

so, for $X_1 = 0$, we must set $|T_1(j\omega)| = 0$. This is accomplished by setting

$$K_2 - m_2\omega^2 = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{K_2}{m_2}} = \omega_{n2}$$

Therefore, the main mass, m_1, will be motionless if we select an absorber having the same natural frequency, ω_{n2}, as the excitation frequency, ω. By doing this, we say the absorber is "tuned" to the input frequency.

To see the motion of the absorber, substitute $\omega = \sqrt{\frac{K_2}{m_2}}$ into the TF for the absorber and solve for X_2:

$$X_2(j\omega) = \frac{F(j\omega)K_2}{\left(K_1 + K_2 - m_1\frac{K_2}{m_2}\right)\left(K_2 - \cancel{m_2\frac{K_2}{m_2}}^{\,0}\right) - K_2^2} = \frac{F(j\omega)}{-K_2}$$

The magnitude of motion is:

$$X_2 = |X_2(j\omega)| = \frac{F}{K_2}$$

The absorber's motion at steady-state is (note for $\omega = \sqrt{\frac{K_2}{m_2}}$, $\phi_2 = \pi$)

$$x_2(t) = \frac{F}{K_2}\sin(\omega t + \pi) = -\frac{F}{K_2}\sin\omega t$$

From this steady-state response, note that the absorber's spring force acting on the main mass is

$$F_{absorber} = K_2(X_2 - X_1) \quad \text{since } X_1 = 0 \text{ and } X_2 = \frac{-F}{K_2}\sin\omega t, \quad F_{absorber} = -F\sin\omega t$$

So it is equal + opposite to the applied force. The net force on the main mass is, therefore, zero, hence it does not move.

Note that the allowable clearance for the absorber motion, X_2, puts a limit on the allowable range of the absorber's stiffness, which is governed by the relation

$$X_2 = \frac{F}{K_2}$$

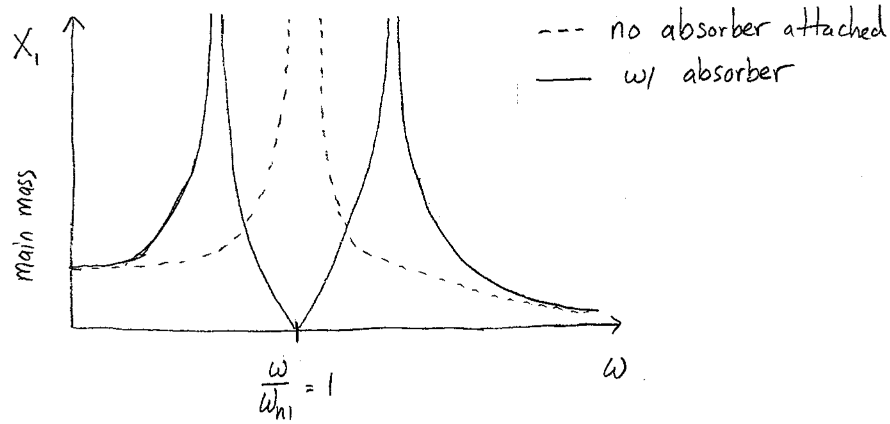
This leads to our first design equation, to select the stiffness of the tuned mass damper

$$\boxed{K_2 = \frac{F}{X_2}}$$

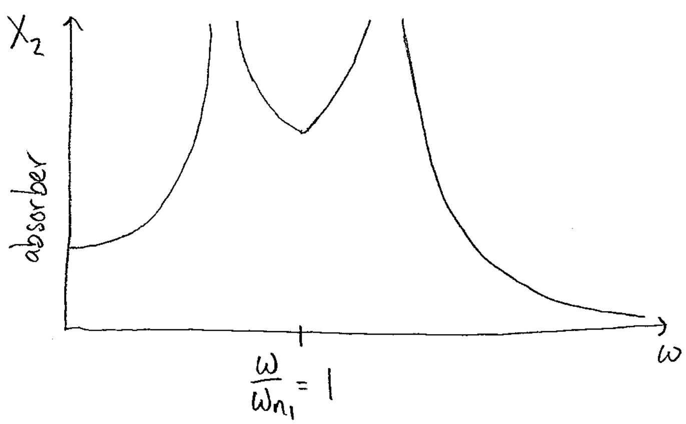
In order to select the mass of the absorber, note that we set $\sqrt{\frac{K_2}{m_2}} = \omega$, which leads to our second design equation

$$\boxed{m_2 = \frac{K_2}{\omega^2}}$$

Displacement vs. Frequency Plot



- Responses @ resonance ⇒ ∞ b/c no damping
- Addition of absorber: 1DOF ⇒ 2DOF, so two resonance frequencies
- At $\omega = \omega_{n_1} = \sqrt{\frac{K_1}{M_1}}$, main mass motion $= 0$, but at new resonance frequencies, motion is large



- Absorber motion at $\omega = \omega_{n_1}$ is large, described by $X_2 = \frac{F}{K_2}$
- Absorber motion at new resonance frequencies also large.

Notes: This analysis assumes no damping. Damping complicates the mathematics, but every real system has some damping.

While a tuned mass damper can reduce motion at the operating frequency, during start-up the system may pass through the lower of the new resonance frequencies, which can cause problems if it doesn't pass through quickly or if the damping is very small.