

Lecture Module - The Laplace Transform

ME3050 - Dynamic Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 3 - Partial Fraction Decomposition

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- General Polynomial Form
- Case 1 - Distinct Roots
- Case 2 - Repeated Roots
- Special Case - Complex Roots

General Polynomial Form

The Laplace Transform is an Integral Transform:

Given a function $x(t)$ in the time domain where $t \geq 0$, the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st} dt$$

Partial Fraction Expansion leads to a general form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad n \geq m$$

General Polynomial Form

Case 1 - Distinct Roots: n roots are real and distinct

The general form is factored:

$$X(s) = \frac{N(s)}{(s + r_1)(s + r_2) \dots (s + r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s + r_1)} + \frac{C_2}{(s + r_2)} + \dots + \frac{C_n}{(s + r_n)}$$

Where:

$$C_i = \lim_{s \rightarrow -r_i} \{X(s)(s + r_i)\}$$

And this leads to a solution:

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + \dots + C_n e^{-r_n t}$$

Case 2 - Repeated Roots

Case 2 - Repeated Roots: p number of roots have the same value ($s = -r$) and remaining roots are distinct and real distinct

$$X(s) = \frac{N(s)}{(s + r_1)^p (s + r_{p+1})(s + r_{p+2}) \dots (s + r_n)}$$

The fraction will expand to:

$$\begin{aligned} X(s) = & \frac{C_1}{(s + r_1)^p} + \frac{C_2}{(s + r_1)^{p-1}} + \dots \\ & + \frac{C_p}{(s + r_1)} + \frac{C_{p+1}}{(s + r_{p+1})} + \dots + \frac{C_n}{(s + r_n)} \end{aligned}$$

Case 2 - Repeated Roots

Coefficients for the repeated root are:

$$C_1 = \lim_{s \rightarrow -r_i} \{X(s)(s + r_i)^p\}$$

$$C_2 = \lim_{s \rightarrow -r_i} \left\{ \frac{d}{ds} X(s)(s + r_i)^p \right\}$$

$$C_i = \lim_{s \rightarrow -r_i} \left\{ \frac{1}{(i-1)!} \frac{d^{(i-1)}}{ds^{(i-1)}} X(s)(s + r_i)^p \right\}$$

Coefficients for the distinct roots are the same as in Case 1:

And this leads to a solution:

$$\begin{aligned} x(t) = & C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \dots \\ & \dots + C_p e^{-r_1 t} + C_{p+1} e^{-r_{p+1} t} \dots + C_n e^{-r_n t} \end{aligned}$$

Special Case - Complex Roots

Special Case - Complex Roots: the roots are distinct \implies Case 1

Example:

$$X(s) = \left[\frac{3s + 7}{(4s^2 + 24s + 136)} \right] = \left[\frac{3s + 7}{4(s^2 + 6s + 34)} \right]$$

The solution can be found by forming two perfect squares in the denominator.

$$X(s) = \frac{1}{4} \left[\frac{3s + 7}{(s + 3)^2 + 5^2} \right]$$

Special Case - Complex Roots

Now this can be expanded into the following terms which can be found in the table!

$$X(s) = \frac{1}{4} \left[C_1 \frac{5}{(s+3)^2 + 5^2} + C_2 \frac{s+3}{(s+3)^2 + 5^2} \right]$$

Multiply by the denominator and solve for C_1 and C_2 .

$$3s + 7 = 5C_1 + C_2(s+3) = 5C_1 + c_2s + 3C_2 \implies C_2 = 3, C_1 = -\frac{2}{5}$$

Special Case - Complex Roots

Finally substitute and invert using the table.

$$X(s) = \frac{1}{4} \left[-\frac{2}{5} \frac{5}{(s+3)^2 + s^2} + 3 \frac{s+3}{(s+3)^2 + s^2} \right]$$

Write the final answer in the time domain.

$$x(t) = -\frac{1}{10} e^{-3t} \sin(5t) + \frac{3}{4} e^{-3t} \cos(5t)$$