Module 10 - The Laplace Transform

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 3 - Partial Fraction Decomposition

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- General Polynomial Form
- Case 1 - Distinct Roots
- Case 2 - Repeated Roots
- Special Case - Complex Roots

General Polynomial Form

The Laplace Transform is an Integral Transform:
Given a function $x(t)$ in the time domain where $t \geq 0$,
the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}dt$$

Partial Fraction Expansion leads to a general form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad n \geq m}$$

General Polynomial Form

Case 1 - Distinct Roots: n roots are real and distinct

The general form is factored:

$$X(s) = \frac{N(s)}{(s + r_1)(s + r_2)...(s + r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s + r_1)} + \frac{C_2}{(s + r_2)} + ... + \frac{C_n}{(s + r_n)}$$

Where:

$$C_i = \lim_{s \to -r_i} \{X(s)(s + r_i)\}$$

And this leads to a solution:

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + ... + C_n e^{-r_n t}$$

Case 2 - Repeated Roots

Case 2 - Repeated Roots: p number of roots have the same value $(s = -r)$ **and remaining roots are distinct and real distinct**

$$X(s) = \frac{N(s)}{(s+r_1)^p(s+r_{p+1})(s+r_{p+2})...(s+r_n)}$$

The fraction will expand to:

$$X(s) = \frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + ...$$

$$+\frac{C_p}{(s+r_1)} + \frac{C_{p+1}}{(s+r_{p+1})} + ... + \frac{C_n}{(s+r_n)}$$

Case 2 - Repeated Roots

Coefficients for the repeated root are:

$$C_1 = \lim_{s \to -r_i} \{X(s)(s+r_i)^p\}$$

$$C_2 = \lim_{s \to -r_i} \{\frac{d}{ds} X(s)(s+r_i)^p\}$$

$$C_i = \lim_{s \to -r_i} \{\frac{1}{(i-1)!} \frac{d^{(i-1)}}{ds^{(i-1)}} X(s)(s+r_i)^p\}$$

Coefficients for the distinct roots are the same as in Case 1:
And this leads to a solution:

$$x(t) = C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + ...$$

$$... + C_p e^{-r_1 t} + C_{p+1} e^{-r_{p+1} t} ... + C_n e^{-r_n t}$$

Special Case - Complex Roots

Special Case - Complex Roots: the roots are distinct $\implies$ Case 1
Example:

$$X(s) = \left[\frac{3s+7}{(4s^2+24s+136)}\right] = \left[\frac{3s+7}{4\left(s^2+6s+34\right)}\right]$$

The solution can be found by forming two perfect squares in the denominator.

$$X(s) = \frac{1}{4}\left[\frac{3s+7}{(s+3)^2+s^2}\right]$$

Special Case - Complex Roots

Now this can be expanded into the following terms which can be found in the table!

$$X(s) = \frac{1}{4}\left[C_1 \frac{3s+7}{(s+3)^2+s^2} + C_2 \frac{3s+7}{(s+3)^2+s^2}\right]$$

Multiply by the denominator and solve for C_1 and C_2.

$$3s+7 = 5C_1 + C_2\left(s+3\right)) = 5C_1 + c_2 s + 3C_2 \implies C_2 = 3, C_1 = \frac{-2}{5}$$

Special Case - Complex Roots

Finally substitute and invert using the table.

$$X(s) = \frac{1}{4}\left[\frac{-2}{5} frac3s + 7(s+3)^2 + s^2 + 3\frac{3s+7}{(s+3)^2+s^2}\right]$$

$$x(t) = -\frac{1}{10}e^{-3t} sin\,(5t) + \frac{3}{4}e^{-3t} cos\,(5t)$$