Lecture Module - ODE Review

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Module 2 - ODE Review

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- Topic 1 - ODE Review
- Topic 2 - Separation of Variables
- Topic 3 - The Trial Solution Method

Topic 1 - ODE Review

- Definitions and Classification
- Engineering Applications
- Example

What is a Differential Equation?

Definition:

A **differential equation** is an equation which describes a function and one or more of its ____________________ of the

____________________ ____________________

with respect to the ________________________.

Standard Form of an ODE

Ordinary Differential Equations are written in the following form.

$$a_n \frac{dy^{(n)}}{d^{(n)}x} + a_{n-1} \frac{dy^{(n-1)}}{d^{(n-1)}x} + \ldots + a_2 \frac{dy^2}{d^2x} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

The apostrophe is commonly used for the derivative.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

If time is the independent variable the equation changes slightly.

Is the differential equation ordinary or partial?

An **ordinary** differential equation has ____________ independent variable and ____________ dependent variable.

A **partial** differential equation has ______________________________ independent variable ____________ dependent variable.

What is the order of the equation?

The **order** of a differential equation is the

______________________________ ______________________________

present in the equation.

What is the degree of the equation?

The **degree** of a differential equation is the ________________

of its highest derivative, after the equation has been made rational

and integral in all of its derivatives.

Is the differential equation linear or non-linear?

An ordinary differential equation is ________________ if the following statements are true.

1. *The dependent variable and its derivatives are of the first degree.*
2. *The coefficients are constants or dependent on the independent variable.*

If either rule is broken, the equation is _____-________________.

Engineering Applications

Differential equations are used to describe physical systems in many areas of engineering. An equation that represents a physical (or theoretical) system is known as a ______________________________ ______________________________.

- Solid Mechanics
- Kinematics and Dynamics
- Heat Transfer and Thermodynamics
- Fluid Mechanics

Engineering Applications

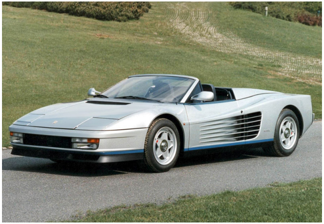
Example

Newton's Second Law

$\Sigma F = ma$

leads to an *equation of motion*.

$\dot{y} + \frac{c}{m} y = f(t)$



Topic 2 - Separation of Variables

- Review
- Separation of Variables
- Example

What is a Differential Equation? Solution?

A **differential equation** is an equation which describes a function and one or more of its ______________________ of the ______________________ ______________________ with respect to the ______________________.

The **solution** to a differential equation describes the ______________________ ______________________ as a function of the ______________________ ______________________.

Separation of Variables

Separation of Variables: **analytical** for solving differential equations

- Step 1 - Separate

- Step 2 - Integrate

- Step 2 - Solve for Unknowns

Separation of Variables

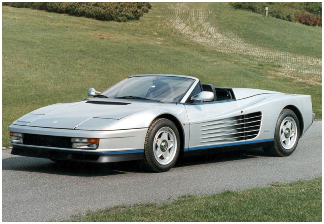
Alternative methods to find solution:

- -
- -
- -

Problem Statement

Remember our example from the previous lecture?

$$\dot{v} + \frac{c}{m}v = f(t)$$



We are going to find an **analytical solution** to this problem.

Separation of Variables

Assume the external force $f(t)$ is zero. Use separation of variables to find the solution $v(t)$.

$$\dot{v} + \frac{c}{m}v = 0$$

Solution

The solution $v(t)$ has been found. What does it mean? What do we do next?

$$v(t) =$$

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Graph of Solution

What does the solution look like?

$$v(t) = v_0 e^{-\frac{c}{m}t}$$

Topic 3 - The Trial Solution Method

- Exponential Assumption
- Complementary Solution
- Particular Solution
- Apply Initial Conditions
- Summary - 3 Cases

Trial Solution Method

Use the **trial solution method** to solve the ODE.

This is an **analytical** method that you learned in calculus but it may have been called something different. In the Zill book it is called *Homogenous Linear ... Constant Coefficients (4.3-4.4)*.

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Exponential Assumption

Complementary Solution

<u>Step 1</u> - Find the **complementary part** of the solution from the left hand side of the ODE alone (LHS=0).

$$a_2 y'' + a_1 y' + a_0 y = f \quad \rightarrow \quad a_2 y'' + a_1 y' + a_0 y = 0$$

Assume an exponential solution for the complementary part.

$$y_{complementary} = y_c(x) =$$

Substitute this solution into the ODE (LHS=0).

Particular Solution

Step 2 - Find the **particular part** of the solution from the entire equation (LHS=RHS).

$$a_2 y'' + a_1 y' + a_0 y = f$$

The *form of the particular part* follows the RHS of the ODE.

$$y_{particular} = y_p(x) =$$

Substitute this solution into the ODE above and solve for any unknown constants in $y_p(x)$.

Apply Initial Conditions

Step 3 - Now combine the **complementary** and **particular** solutions through *superposition*.

$$y(x) = y_c(x) + y_p(x) =$$

The ODE is first order and we have _____ unknown. Coincidence?

$$y(x) =$$

This **initial value problem** requires _____ intial condition.

Apply Initial Conditions

$$y(x = 0) =$$

$$y'(x = 0) =$$

Apply Initial Conditions

What does the solution look like this time?

$$y(x) =$$

Apply Initial Conditions

Summary - 3 Cases

If the differential equation is first and linear, the complementary solution takes the following form.

$$y(x) = Ae^{sx}$$

Summary - 3 Cases

If the differential equation is second order and linear, the **complementary solution** takes one of the following forms.

Case 1: $s_1, s_2 \in \mathbb{R}$, $s_1 \neq s_2$

$$y(x) = c_1 e^{s_1 x} + c_2 e^{s_2 x}$$

Case 2: $s_1, s_2 \in \mathbb{R}$, $s_1 = s_2 = s$

$$y(x) = c_1 e^{sx} + c_2 x e^{sx}$$

Case 3: $s_1, s_2 \notin \mathbb{R}$, $s_1, s_2 = \alpha \pm \beta$

$$y(x) = e^{\alpha x} \left(c_1 cos\left(\beta x\right) + c_2 sin\left(\beta x\right)\right)$$

Summary - 3 Cases

The **particular solution** takes the form of the right hand side of the equation.

Example	Form	Particular Solution
$... = 10$	Constant	$y_p = B$
$... = 12x$	Linear	$y_p = Bx + C$
$... = 20e^{2x}$	Exponential	$y_p = Be^{2x}$
$... = a\cos(\beta x)$	Sinusoidal	$B\cos(\beta x) + C\sin(\beta x)$
$... = a\sin(\beta x)$	Sinusoidal	$B\cos(\beta x) + C\sin(\beta x)$