(Ch.6) Electrical Systems

Many electrical systems can be modeled with the same differential equations we have used for mechanical systems. Additionally, most engineering systems contain electrical subsystems, therefore, understanding their behavior is important.

(6.1) Electrical Elements

Voltage + current typically used to describe electrical systems.

Voltage, V Volt (V) — voltage source

Charge, Q coulomb (C) = N·m/V

Current, i ampere (A) = C/s — current source

Resistance, R ohm (Ω) = V/A

Capacitance, C farad (F) = C/V

Inductance, L henry (H) = V·s/A

Battery

Ground

Terminals (input or output)

Constitutive Relations

$$i = \frac{dQ}{dt} \qquad Q = \int i\,dt$$

Resistor: $V = iR$

Capacitor: $i = C\frac{dV}{dt} \Rightarrow V = \frac{1}{C}\int i\,dt$

Inductor: $V = L\frac{di}{dt}$

$$P = iV = i^2R = \frac{V^2}{R}$$ Power, (W) watts

Physical Laws

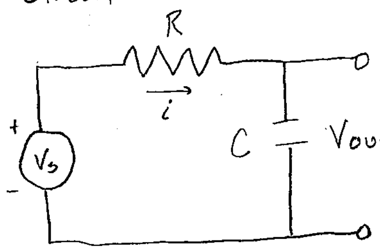
In mechanical systems, we used Newton's 2nd Law to derive EOMs. In circuits, we can use Kirchhoff's Voltage Law (KVL) + Kirchhoff's Current Law (KCL).

KVL: sum of voltages around closed circuit is zero $\quad \sum_{K=1}^{N} V_K = 0$

KCL: sum of currents flowing into a node equals sum of currents flowing out of the node. $\quad \sum_{K=1}^{N} i_K = 0$

Circuit Examples

RC Circuit



Constitutive Equation of each Component:

$$V_R = iR \quad , \quad V_C = V_{out} = \frac{1}{C}\int i\,dt$$

Apply KVL:

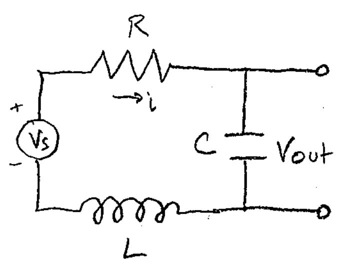
$$V_S = V_R + V_{out} = iR + \frac{1}{C}\int i\,dt$$

We can write the equation in terms f V_{out} by realizing that $\frac{1}{C}\int i\,dt$ can be left as V_{out}, and that relation (i.e. $V_{out} = \frac{1}{C}\int i\,dt$) can be differentiated to solve for i in terms of V_{out}: $i = C\frac{dV_{out}}{dt}$. Substituting gives:

$$RC\frac{dV_{out}}{dt} + V_{out} = V_S$$

Which is a classic first order ODE $(a\dot{x} + bx = f)$

RLC Circuit



Constitutive Equation of Each Component

$$V_R = iR \quad , \quad V_C = V_{out} = \frac{1}{C}\int i\,dt \quad , \quad V_L = L\frac{di}{dt}$$

Apply KVL:

$$V_S = V_R + V_{out} + V_L = iR + \frac{1}{C}\int i\,dt + L\frac{di}{dt}$$

Again, we know $V_{out} = \frac{1}{C}\int i\,dt \Rightarrow i = C\frac{dV_{out}}{dt}$. Also, we can differentiate to get $\frac{di}{dt} = C\frac{d^2V_{out}}{dt^2}$. Substituting:

$$LC\frac{d^2V_{out}}{dt} + RC\frac{dV_{out}}{dt} + V_{out} = V_S$$

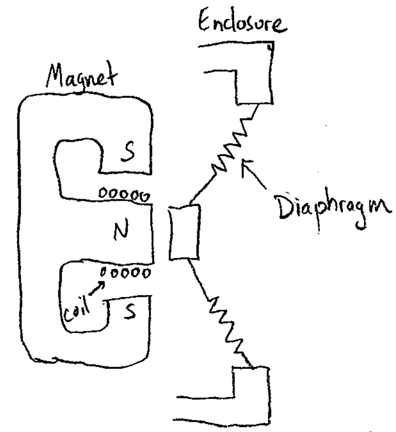
Which is a classic 2^{nd} order ODE $(a\ddot{x} + b\dot{x} + cx = f)$

What is the circuits' natural frequency?

$$\omega_n = \sqrt{\frac{1}{LC}}$$

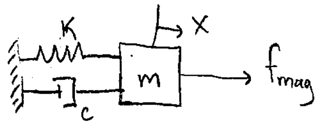
Circuit Examples (con't)

Ex. 6.7.1 : An electromagnetic speaker

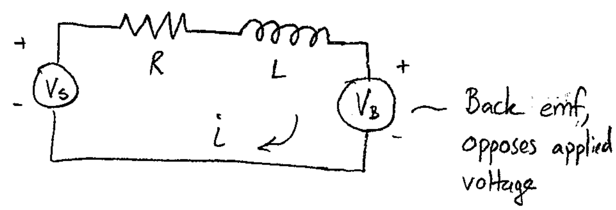


Electromechanical System: Electrical + Mechanical Parts

Mechanical



Electrical



m = mass of diaphragm + coil
K = diaphragm spring constant
c = " damping coeff.
f_{mag} = magnetic force

V_S = Source voltage
R = coil resistance
L = coil inductance
V_B = back emf

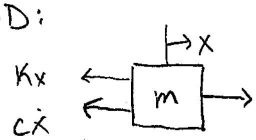
Magnetic coupling in a coil: $f = nBLi$
$= K_f i$

f = force
n = number of turns
B = flux density of magnetic field
L = length of conductor
i = current applied to coil
$(K_f = nBL)$

Back emf of conductor moving in a magnetic field: $V_B = BLv$ v = velocity
$= K_B v$ $(K_B = BL)$

Mechanical System:

FBD:



$f_{mag} = K_f i$

Newton:

$$\Sigma F = m\ddot{x} = K_f i - Kx - c\dot{x}$$

$$\boxed{m\ddot{x} + c\dot{x} + Kx = K_f i}$$

Electrical System:

Apply KVL:

$$V_S = iR + L\frac{di}{dt} + \underbrace{K_B\frac{dx}{dt}}_{\text{velocity}}$$

$$\boxed{L\frac{di}{dt} + Ri = V_S - K_B\frac{dx}{dt}}$$