

Module 10 - The Laplace Transform

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 1 - Definition of the Laplace Transform

Topic 1 - Definition of the Laplace Transform

- An Integral Transform
- Laplace Transform of A Derivative
- Properties of an Integral
- Table of Transform Pairs

An Integral Transform

The Laplace Transform is an Integral Transform

Given a function $x(t)$ in the time domain where $t \geq 0$, the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st} dt$$

And its inverse is similarly defined as:

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

The Laplace Domain variable s is a complex number:

$$s = \sigma + j\omega$$

Laplace Transform of A Derivative

It is useful to find the laplace transform of the derivative of a function:

$$\begin{aligned}\mathcal{L}\left\{\frac{d}{dt}(x(t))\right\} &= \mathcal{L}\{\dot{x}(t)\} = s\mathcal{L}\{x(t)\} - x(t=0) \\ &= sX(s) - x(t=0) \\ \mathcal{L}\{\dot{x}(t)\} &= sX(s) - x_0\end{aligned}$$

$$\begin{aligned}\mathcal{L}\left\{\frac{d^2}{dt^2}(x(t))\right\} &= \mathcal{L}\{\ddot{x}(t)\} = s^2\mathcal{L}\{x(t)\} - sx(t=0) - \dot{x}(t=0) \\ &= s^2X(s) - sx(t=0) - \dot{x}(t=0) \\ \mathcal{L}\{\ddot{x}(t)\} &= s^2X(s) - sx_0 - \dot{x}_0\end{aligned}$$

Properties of an Integral

Also, remember that the transform inherits the properties of an integral.

$$\int [x(t) + y(t)] dt = \int x(t)dt + \int y(t)dt$$
$$\int Kx(t)dt = K \int x(t)dt \quad (K \text{ is constant})$$

Therefore these properties can be used with the Laplace transform.

Table of Transform Pairs

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. \sqrt{t}	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$

Table of Transform Pairs

17. $\sinh(at)$	$\frac{a}{s^2 - a^2}$	18. $\cosh(at)$	$\frac{s}{s^2 - a^2}$
19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2 - b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2 - b^2}$
23. $t^n e^{at}, \quad n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-cs}
27. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	28. $u_c(t) g(t)$	$e^{-cs} \mathcal{L}\{g(t+c)\}$
29. $e^{ct} f(t)$	$F(s-c)$	30. $t^n f(t), \quad n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t} f(t)$	$\int_s^\infty F(u) du$	32. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s) G(s)$	34. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
37. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$		