(Ch.9) System Response in the Frequency Domain

The term "frequency response" is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to harmonic inputs, such as sines and cosines. The forcing is, therefore, of the form:

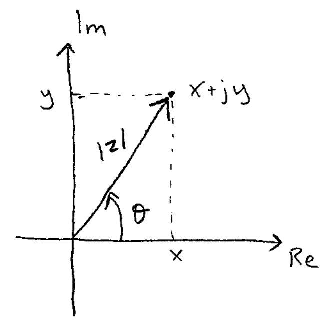
$$f(t) = A\sin\omega t$$

Many systems exhibit sinusoidal forcing: IC engines, rotating machinery, reciprocating pumps, etc. Additionally, Fourier series shows that any periodic function can be represented by a sum of a constant plus a series of sine and cosine terms.

Complex Number Review

Rectangular Representation

$$Z = x + jy$$



Polar Representation

$$Z = |Z|\angle\theta$$

Exponential Representation

$$Z = |z|e^{j\theta} = |z|(\cos\theta + j\sin\theta)$$

where $|z| = \sqrt{x^2+y^2}$, $\theta = \angle z = \tan^{-1}\left(\frac{y}{x}\right)$

Complex Algebra: let $z_1 = x_1 + jy_1$ $\quad z_2 = x_2 + jy_2$

$$Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$Z_1 Z_2 = |z_1||z_2|\angle(\theta_1+\theta_2) = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$$

$$\frac{Z_1}{Z_2} = \frac{|Z_1|}{|Z_2|}\angle(\theta_1 - \theta_2) = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\,\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}$$

Frequency Response Concept

Any linear, time-invariant (LTI) system has a transfer function, $T(s)$, that maps the input/output relationship. Under sinusoidal excitation (input) with frequency ω, if the system is <u>stable</u>, the transients will eventually dissapear leaving the s.s. response with the same frequency as the input (ω) but with different amplitude and shifted in time w.r.t. the input.

(9.1) Frequency Response of 1st Order Systems

Consider our 1st order mass damper system whos EOM was

$$m\dot{v} + cv = f(t)$$

The time constant was $\tau = \frac{m}{c}$, so we can write this system as

In order to understand the response of the system to a sinusoidal input, let's first solve the system using Laplace Transforms. We also need to assume zero ICs.

$$m\dot{v} + c\dot{v} = A\sin\omega t$$

$$msV(s) + cV(s) = \frac{A\omega}{s^2+\omega^2} \quad \swarrow \text{entry \# 8}$$

$$V(s) = \frac{A\omega}{(s^2+\omega^2)(ms+c)} = \underbrace{\frac{C_1}{ms+c}}_{\text{Transient } (e^{-at}) \text{ term}} + \underbrace{\frac{C_2\omega}{(s^2+\omega^2)} + \frac{C_3 s}{(s^2+\omega^2)}}_{\text{Steady-state sinusoidal term}}$$

entry # 6   entry # 8   entry # 9

Solving for C_1, C_2, C_3 will give:

$$C_1 = \frac{Am^2\omega}{m^2\omega^2+c^2} \qquad C_2 = \frac{Ac}{m^2\omega^2+c^2} \qquad C_3 = \frac{-Am\omega}{m^2\omega^2+c^2}$$

Substituting and inverting gives:

$$v(t) = \frac{Am\omega}{m^2\omega^2+c^2}\left(\underbrace{e^{-\frac{ct}{m}}}_{\text{Transient}} + \underbrace{\frac{c}{m\omega}\sin\omega t - \cos\omega t}_{\text{steady-state}}\right)$$

The steady-state response is:

$$v_{ss}(t) = \frac{A}{m^2\omega^2+c^2}\left(c\sin\omega t - m\omega\cos\omega t\right) = \frac{A}{\sqrt{m^2\omega^2+c^2}}\sin(\omega t+\phi), \text{ where } \phi = -\tan^{-1}\left(\frac{m\omega}{c}\right)$$

So we can see that the system responds at the same frequency as the input, but with a different amplitude and a phase shift. The ratio of the response amplitude to the input amplitude can be defined as the amplitude ratio, M:

$$M = \frac{\frac{A}{\sqrt{m^2\omega^2+c^2}}}{A} = \frac{1}{\sqrt{m^2\omega^2+c^2}}$$

Again, the phase shift is $\phi = -\tan^{-1}\left(\frac{m\omega}{c}\right)$

This was the "hard way" of finding the amplitude ratio and phase shift. Now, for the "easy way", let's start by forming the transfer function:

$$T(s) = \frac{V(s)}{F(s)} = \frac{1}{ms+c}$$

substitute $s = j\omega$ into the TF:

$$T(j\omega) = \frac{1}{j\omega m + c}$$

The magnitude of this complex number is:

$$|T(j\omega)| = \frac{|1|}{|jm\omega + c|} = \frac{1}{\sqrt{m^2\omega^2 + c^2}} = M \,!$$

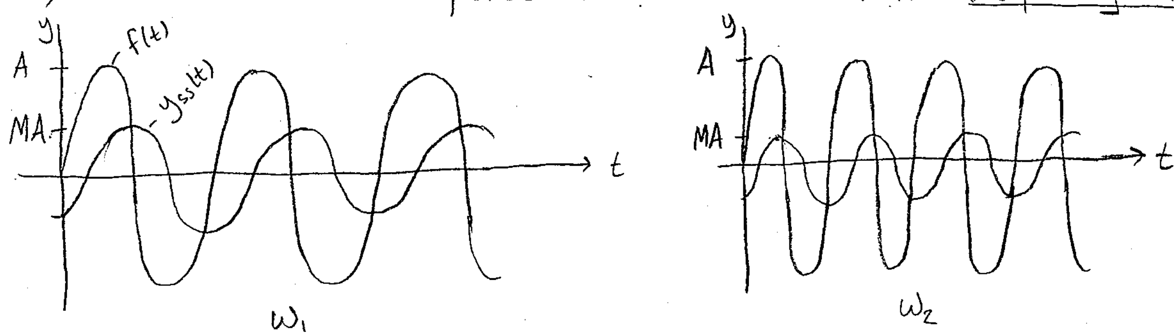
and the angle is

$$\angle T(j\omega) = \angle 1 - \angle(jm\omega + c) = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{m\omega}{c}\right) = -\tan^{-1}\left(\frac{m\omega}{c}\right) = \phi \,!$$

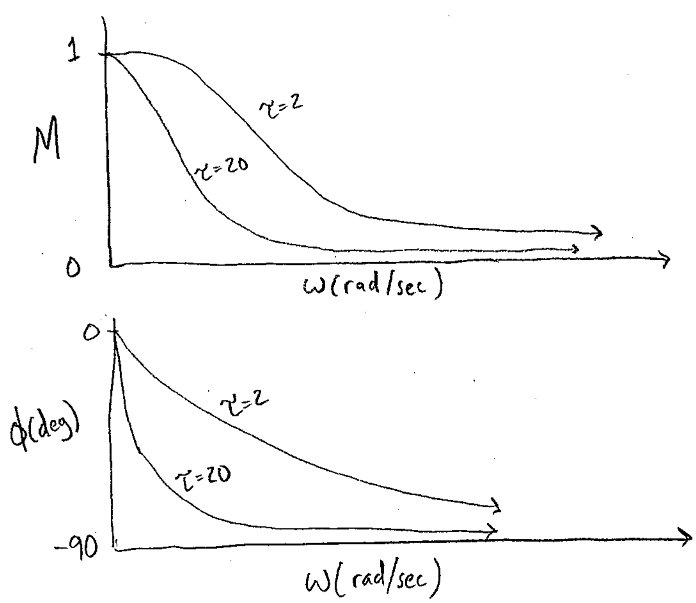
So, the steady-state response can be found by substituting $s = j\omega$ into the TF and solving for magnitude and phase. Much easier! The solution can be written as:

$$v_{ss}(t) = A\,|T(j\omega)|\sin(\omega t + \angle T(j\omega)) = \underline{MA\sin(\omega t + \phi)}$$

Now, what does the response look like? It is <u>frequency dependent</u>.



For 1^{st}-order systems, the steady-state output magnitude decreases with increasing frequency. Also, the larger τ is (recall $\tau = \frac{m}{c}$), the faster it decreases with frequency and the larger the phase shift is. Graphically:



We typically plot these diagrams on logarithmic scales (turns out we can add or subtract the magnitude plots of simpler TFs to represent a more complicated TF). On log scales, these plots are called <u>Bode plots</u> (or Frequency Response plots)

Basic logarithm properties:

$$\log(xy) = \log x + \log y \qquad \log\left(\frac{x}{y}\right) = \log x - \log y \qquad \log x^n = n \log x$$

Also, we use decibel units for magnitude and degrees for phase on Bode plots

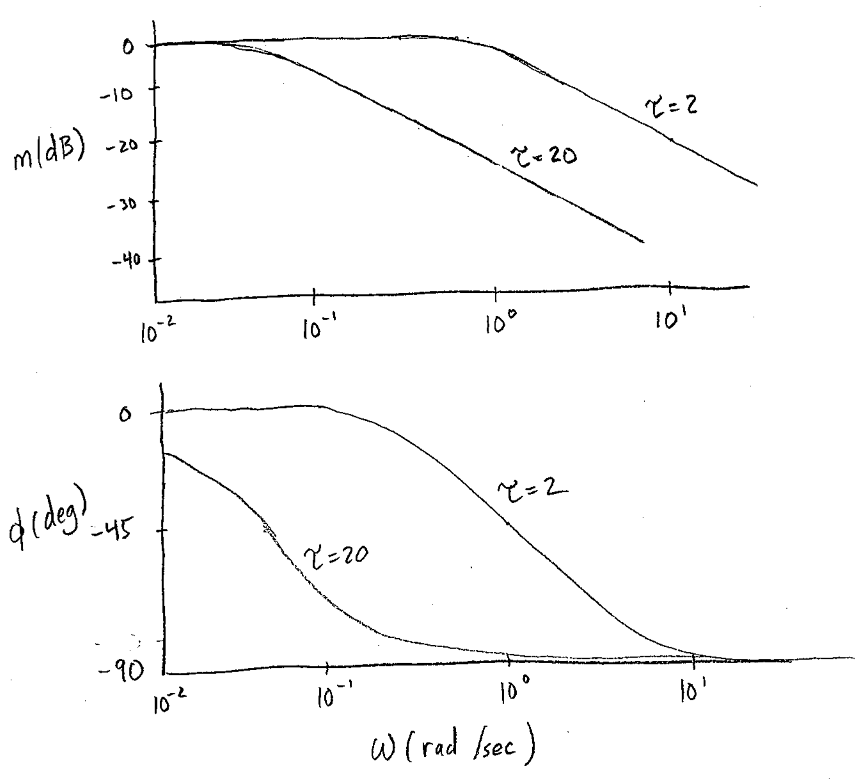
$$m(\text{dB}) = 10 \log M^2 = 20 \log M$$ note: $M = 10^{m/20}$ (can be used to convert)

So, for our example problem:

$$m(\text{dB}) = 20 \log \frac{1}{\sqrt{m^2\omega^2 + c^2}} = 20\left(\log 1 - \log\sqrt{m^2\omega^2+c^2}\right) = 20 \log 1 - 20 \log (m^2\omega^2 + c^2)^{1/2}$$

$$= -10 \log (m^2\omega^2 + c^2)$$

Now, let's look at the Bode plot



- $m=0$ corresponds to $M=1$, where the output amplitude equals the input magnitude
- $m>0$ corresponds to $M>1$, where output > input (amplification)
- $m<0$ corresponds to $M<1$, where output < input (attenuation)

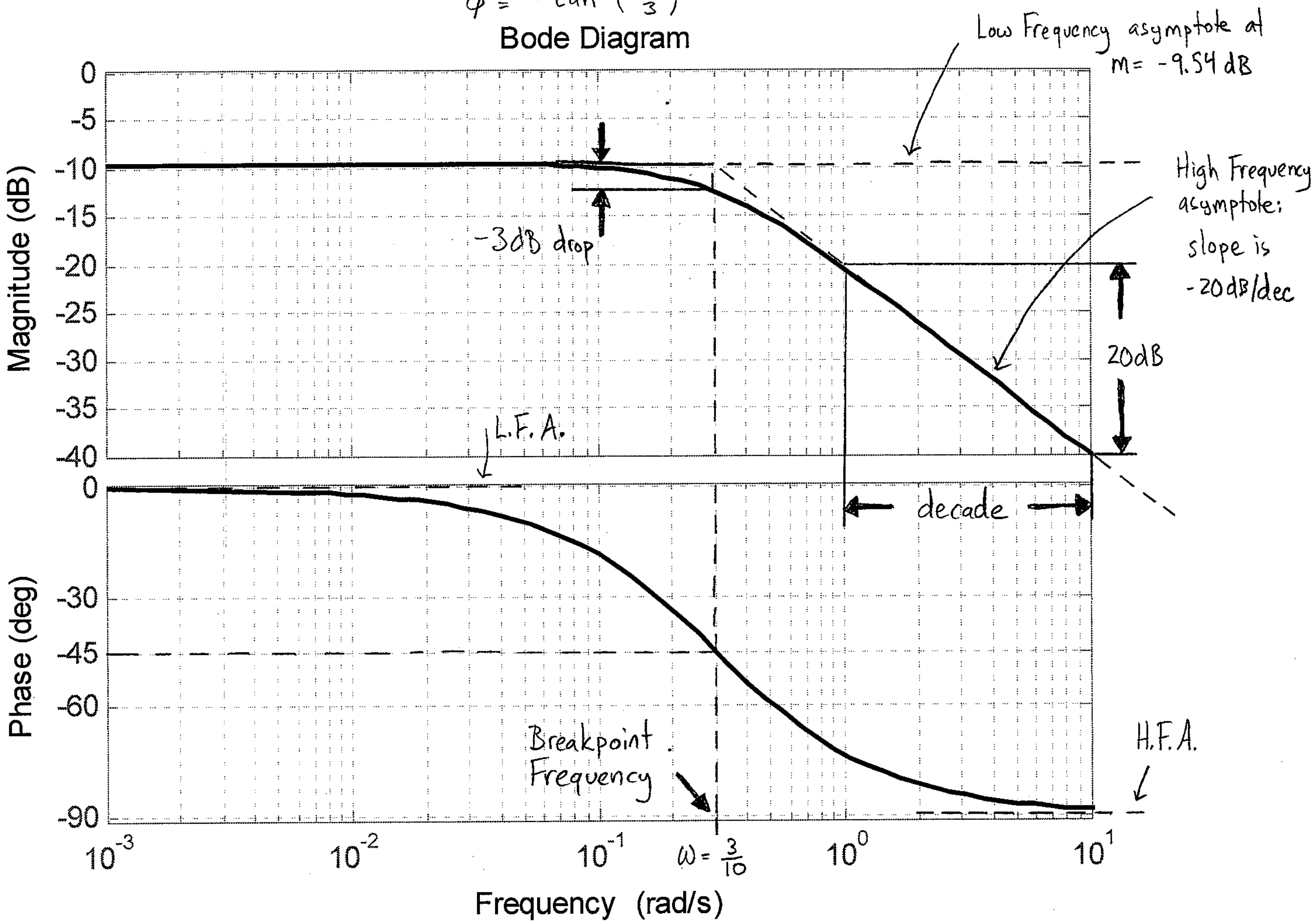
★ 1st-order systems: $m \leq 0$, so we never have amplification!

Hand Sketching 1st-Order Bode Plots

$$\tau = \frac{10}{3}$$

Bode Plots for $T(s) = \dfrac{1}{10s+3}$ $\Rightarrow$ $M = |\tau(j\omega)| = \dfrac{1}{\sqrt{10^2\omega^2 + 3^2}}$ $\Rightarrow$ $m = -10\log(10^2\omega^2 + 3^2)$

$$\phi = -\tan^{-1}\left(\frac{10\omega}{3}\right)$$



To sketch m vs. ω, we can approximate m(ω) in three frequency ranges separated by the value $1/\tau$ $1/\tau$!

- for $\omega \ll \frac{3}{10}$, $(10^2\omega^2 + 3^2) \approx 3^2 = 9 \Rightarrow m \approx -10\log(9) = -9.54$ dB (low frequency asymptote)
- for $\omega = \frac{3}{10}$, $(10^2\omega^2 + 3^2) = 18 \Rightarrow m = -10\log(18) = -12.55$ dB
  - ★ so, at $\omega = 1/\tau$, $m(\omega)$ is 3.01 dB below the low frequency asymptote. This will always be the case for all 1st-Order systems
  - ★ $\omega = 1/\tau$ is called the "breakpoint frequency"
- for $\omega \gg \frac{3}{10}$, $(10^2\omega^2 + 3^2) \approx 10^2\omega^2 \Rightarrow m \approx -10\log(10^2\omega^2) = -20\log(10\omega) = -20\log(10) - 20\log(\omega)$
  - ★ This gives a straight line vs. $\log\omega$ called the high frequency asymptote. Slope = -20 dB/decade

1st order systems always have -20 dB/dec slope on high-freq asymptote

To sketch ϕ vs. ω:

- for $\omega \ll 3/10$, $\phi \approx -\tan^{-1}(0) = 0°$
- for $\omega = 3/10$, $\phi = -\tan^{-1}(1) = -45°$
- for $\omega \gg 3/10$, $\phi \approx -\tan^{-1}(\infty) = -90°$

(9.2) Frequency Response of 2^{nd} Order Systems

When working in the log scale, a complex transfer function can be analyzed easily because factors in the numerator simply add and factors in the denomenator subtract. Graphically, we just add or subtract the contribution of each term to plot the overall system transfer function. To see this; consider

$$T(s) = K\frac{N_1(s)\,N_2(s)\ldots}{D_1(s)\,D_2(s)\ldots}$$

Substitute $s = j\omega$:

$$T(j\omega) = K\frac{N_1(j\omega)\,N_2(j\omega)\ldots}{D_1(j\omega)\,D_2(j\omega)\ldots}$$

Solve for the magnitude:

$$|T(j\omega)| = \frac{|K|\,|N_1(j\omega)|\,|N_2(j\omega)|\ \ldots}{|D_1(j\omega)|\,|D_2(j\omega)|\ \ldots} = M$$

In decibel units:

$$\begin{aligned} m(\omega) &= 20\log|T(j\omega)| \\ &= 20\log|K| + 20\log|N_1(j\omega)| + 20\log|N_2(j\omega)| + \ldots \\ &\quad - 20\log|D_1(j\omega)| - 20\log|D_2(j\omega)| - \ldots \end{aligned}$$

To plot this, we simply plot each component and add (or subtract) them together

What about the phase?

$$\begin{aligned} \phi(\omega) &= \angle T(j\omega) \\ &= \angle K + \angle N_1(j\omega) + \angle N_2(j\omega) + \ldots \\ &\quad - \angle D_1(j\omega) - \angle D_2(j\omega) - \ldots \end{aligned}$$

Similarly, we add and subtract each phase component to form the plot.

Consider our standard mass-spring-damper system with sinusoidal excitation

$$m\ddot{x} + c\dot{x} + Kx = f(t) \qquad f(t) = A\sin\omega t$$

The transfer function is

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + K}$$

If the system is <u>overdamped</u>, both roots are real & distinct. Write $T(s)$ as:

$$T(s) = \frac{1/K}{\left(\frac{m}{K}\right)s^2 + \left(\frac{c}{K}\right)s + 1} = \frac{1/K}{(\tau_1 s+1)(\tau_2 s+1)} \qquad \tau_1, \tau_2 = \text{time constants of roots}$$

Substitute $s = j\omega$

$$T(j\omega) = \frac{1/K}{(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

Solve for amplitude ratio

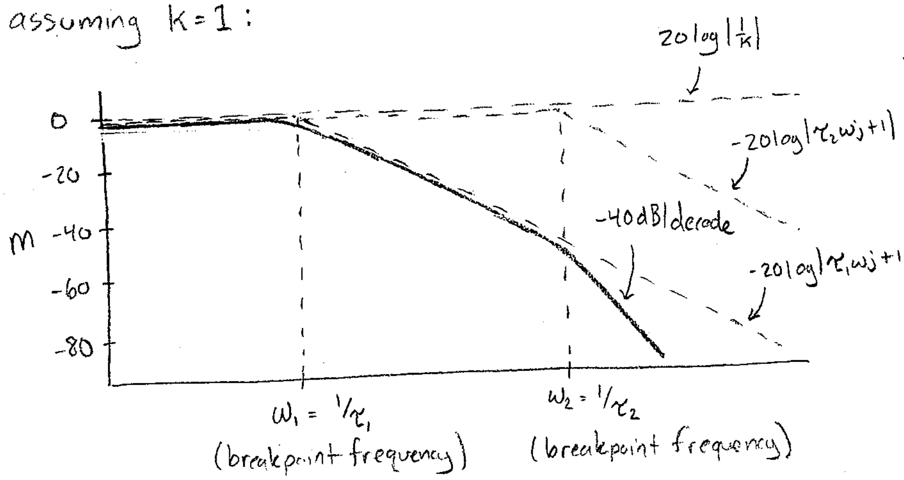
$$M(\omega) = |T(j\omega)| = \frac{|1/K|}{|\tau_1 j\omega + 1|\,|\tau_2 j\omega + 1|}$$

$$m(\omega) = 20\log M(\omega) = 20\log\left|\tfrac{1}{K}\right| - 20\log|\tau_1\omega j + 1| - 20\log|\tau_2\omega j + 1|$$

Solve for the phase angle

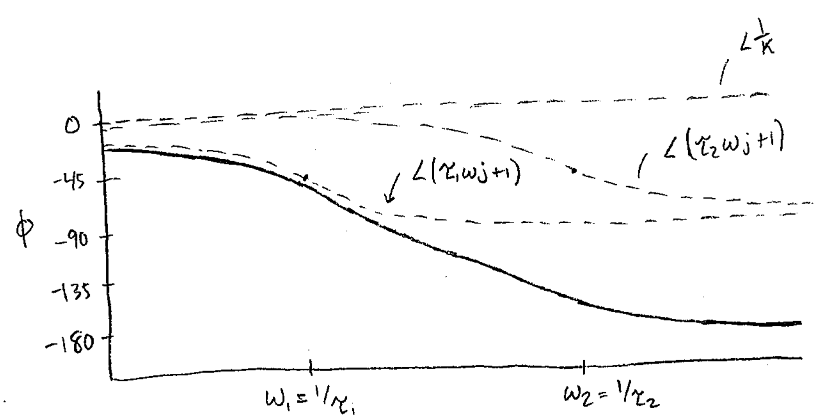
$$\phi(\omega) = \angle\tfrac{1}{K} - \angle(\tau_1\omega j + 1) - \angle(\tau_2\omega j + 1)$$

So, the magnitude plot consists of a constant term, $20\log\left|\frac{1}{K}\right|$, minus the sum of two 1st-order terms. We saw before that a first order term yields a -20 dB/decade slope. Let's sketch this assuming $K=1$:



1) Sketch each component
   a) $20\log\left|\frac{1}{1}\right| = 0$
   b) $-20\log|\tau_1\omega j + 1|$: first order term, has low freq asymptote $= 0$, high freq asymptote $= -20$ dB/decade, breakpoint $= \omega = \frac{1}{\tau_1}$
   c) $-20\log|\tau_2\omega j + 1|$: first order term, low freq asymptote $= 0$, high freq asymptote $= -20$ dB/decade, breakpoint $= \omega = \frac{1}{\tau_2}$

2) Draw Composite sketch
   - Note: the effects of each first order term kick in after their breakpoints. They also add, so for $\omega > 1/\tau_2$, the slope is -40 dB/decade.



1) Sketch each component
   a) $\angle\frac{1}{K} = \angle\frac{1}{1} = \angle 1 = 0$
      constant at $\phi = 0$
   b) $\angle(\tau_1\omega j + 1) = -\tan^{-1}(\omega\tau_1)$:
      0° for $\omega \ll 1/\tau_1$, 45° for $\omega = 1/\tau_1$,
      90° for $\omega \gg 1/\tau_1$
   c) $\angle(\tau_2\omega j + 1) = -\tan^{-1}(\omega\tau_2)$:
      0° for $\omega \gg 1/\tau_2$, 45° for $\omega = 1/\tau_2$,
      90° for $\omega \gg 1/\tau_2$

If the system is <u>underdamped</u>, there are two complex conjugate roots. Write $T(s)$ as:

$$T(s) = \frac{K\,X(s)}{F(s)} = \frac{1}{\left(\frac{m}{K}\right)s^2 + \left(\frac{c}{K}\right)s + 1} = \frac{1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

factor out K

recall $\omega_n = \sqrt{\frac{K}{m}}$, $\zeta = \frac{c}{2\sqrt{mK}}$

Now, the roots are only complex if $\zeta < 1$, so we have

$$T(s) = \frac{K\,X(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

★ By factoring out K, we form the ratio of output displacement $X(s)$ to input <u>displacement</u>, $\frac{F(s)}{K}$. Recall $F = KX \Rightarrow X = \frac{F}{K}$. It also allows us to define the TF in terms of ζ + ω_n as follows:

Substituting $s = j\omega$ and multiplying by $\frac{1/\omega_n^2}{1/\omega_n^2}$ gives

$$T(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + (2\zeta/\omega_n)j\omega + 1} = \frac{1}{1 - (\omega/\omega_n)^2 + (2\zeta\omega/\omega_n)j}$$

To simplify this expression, we can define the frequency ratio, r as:

$$r = \frac{\omega}{\omega_n}$$

Substituting

$$T(r) = \frac{1}{1 - r^2 + 2\zeta r j}$$

The amplitude ratio is then

$$M = |T(r)| = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Rightarrow m = 20\log M = -10\log\left[(1-r^2)^2 + (2\zeta r)^2\right]$$

And the phase is

$$\phi = \angle 1 - \angle(1 - r^2 + 2\zeta r j) \Rightarrow \phi = -\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

To hand sketch m vs. ω, we approximate m in three frequency ranges

- For $r \ll 1$ ($\omega \ll \omega_n$): $m = -10\log(1) = 0$ (low frequency asymptote)
- For $r \gg 1$ ($\omega \gg \omega_n$): $m = -10\log(r^4 + 4\zeta^2 r^2)$
  $\approx -10\log r^4$
  $= -40\log r$ ⇒ This gives a straight line with a slope of -40 dB/decade. (high frequency asymptote).
- For $r = 1$ ($\omega = \omega_n$), we need to consider the phenomenon known as resonance

Resonance

For 2nd order underdamped systems, the response near the breakpoint frequency (natural frequency) depends highly on the damping of the system.

Consider the fact that M is maximum when its denomenator is a minimum. Taking the derivative of the denomenator equal to 0 gives:

$$M_{MAX} \text{ occurs at } r = \sqrt{1-2\zeta^2} \Rightarrow \omega = \omega_n\sqrt{1-2\zeta^2}$$

This frequency is the resonance frequency, ω_r. Note, this peak only exists if the radical is positive $\Rightarrow$ $0 \le \zeta \le 0.707$, so

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \qquad 0 \le \zeta \le 0.707$$

The peak value at resonance is found by substituting this back into M to get

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad 0 \le \zeta \le 0.707 \qquad \text{in decibels: } m_r = -20\log\left(2\zeta\sqrt{1-\zeta^2}\right)$$

So if the damping ratio is above 0.707, we don't get a peak. The phase at the resonance frequency is

$$\phi_r = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

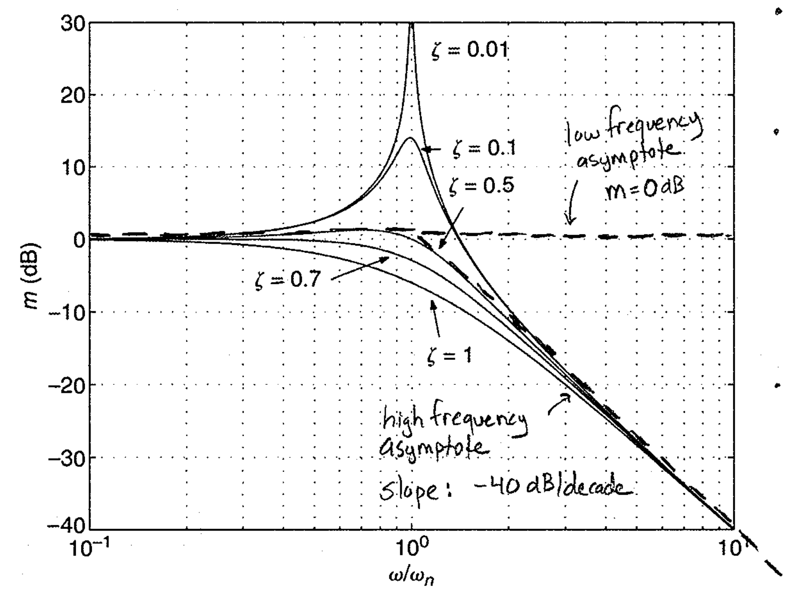
★ Note: because we multiplied the TF by K, we must divide the expressions for M and M_r by K in order to get the amplitude ratio between input force, $f(t)$ and output displacement $X_{ss}(t)$:

$$M = \frac{1}{K\sqrt{(1-r^2)^2+(2\zeta r)^2}} \Rightarrow m = -20\log(K) - 10\log\left[(1-r^2)^2+(2\zeta r)^2\right]$$

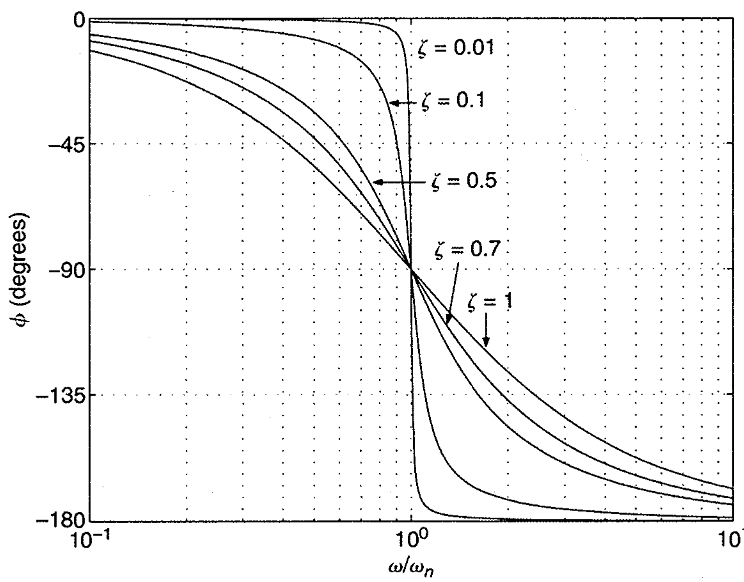
$$M_r = \frac{1}{K2\zeta\sqrt{1-\zeta^2}} \Rightarrow m_r = -20\log(K) - 20\log\left(2\zeta\sqrt{1-\zeta^2}\right)$$

Hand Sketching 2^{nd} Order Bode Plots

Bode Plots for $T(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, $\zeta \le 1$ (underdamped)



(a)



(b)

Amplitude Ratio

Approximate m in three frequency ranges

- For $r \ll 1$ $(\omega \ll \omega_n)$, $m \approx -10 \log(1) = 0$ (low frequency asymptote)
- For $r \gg 1$ $(\omega \gg \omega_n)$, $m \approx -10 \log(r^4 + 4\zeta^2 r^2)$
  $\approx -10 \log r^4$
  $= -40 \log r$

  This gives a straight line with a slope of -40 dB/decade for the high frequency asymptote.
- For $r = 1$ $(\omega = \omega_n)$, we need to consider the phenomenon known as resonance. The resonance frequency is near ω_n and is given by:

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad 0 \le \zeta \le 0.707$$

The peak value at the resonance frequency is

$$m_r = -20 \log\left(2\zeta\sqrt{1-\zeta^2}\right) \qquad 0 \le \zeta \le 0.707$$

So for $\zeta \le 0.707$, one finds ω_r and m_r and adds the point to the plot. To either side of ω_r, the signal decays back to the low and high frequency asymptotes. For smaller values of the damping ratio, the peak is sharper.

We can also add the phase value at the resonance frequency to the phase plot by using the relation

$$\phi_r = -\tan^{-1}\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$$

Phase

Approximate the phase in three frequency ranges

- For $r \ll 1$ $(\omega \ll \omega_n)$, $\phi \approx -\tan^{-1}\left(\frac{0}{1}\right) = 0°$
- For $r \gg 1$ $(\omega \gg \omega_n)$, $\phi \approx -\tan^{-1}\left(-\frac{r}{r^2}\right) = -\tan^{-1}\left(-\frac{1}{r}\right) = -180°$ (2^{nd} order terms result in a 180° phase shift)
- For $r = 1$ $(\omega = \omega_n)$, $\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -90°$

Note: the smaller the damping ratio, the steeper the slope through -90° at the breakpoint frequency (sharper curve).

Example Problem: P9.11

The model of a certain mass-spring-damper system is:

$$13\ddot{x} + 2\dot{x} + KX = 10\sin\omega t$$

Determine the value of K required so that the maximum response occurs at $\omega = 4$ rad/sec. Obtain the steady-state response at that frequency. Sketch the Bode Plot of the system.

From the problem statement, we know the system oscillates. The resonance frequency, therefore, is

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Solving for ω_n + ζ in terms of K:

$$\omega_n = \sqrt{\frac{K}{13}} \quad , \quad \zeta = \frac{2}{2\sqrt{13 \cdot K}} = \frac{1}{\sqrt{13K}}$$

Substituting and setting $\omega_r = 4$

$$4 = \sqrt{\frac{K}{13}}\sqrt{1-\frac{2}{13K}} = \sqrt{\frac{1}{13}}\sqrt{K-\frac{2}{13}} = \frac{1}{13}\sqrt{13K-2} \Rightarrow \boxed{K = 208.15}$$

The steady-state response magnitude and phase at resonance are

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \phi = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

★ But, recall in deriving this formula for Mr, we factored out K when forming the TF. To use this formula to scale the input force to output displacement, we must divide by K.

$$M_r = \frac{1}{K}\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Substituting + solving for Mr:

$$M_r = \frac{1}{208.15}\frac{1}{2\frac{1}{52.02}\sqrt{1-\left(\frac{1}{52.02}\right)^2}} = 0.125$$

Solving for ϕ:

$$\phi = -\tan^{-1}\left(\frac{\sqrt{1-2\left(\frac{1}{52.02}\right)^2}}{\frac{1}{52.02}}\right) = -\tan^{-1}(52) = -1.55 \text{ rad}$$

So the steady-state response is

$$X_{ss}(t) = 10 \cdot (0.125)\sin(4t - 1.55\text{ rad}) = \boxed{1.25\sin(4t-1.55\text{ rad})}$$

To sketch the Bode plot, we can use the general formulas for m + ϕ to estimate the response in three regions.

The magnitude and phase are

$$m = -10 \log\left[(1-r^2)^2 + (2\zeta r)^2\right] \quad , \quad \phi = -\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

Approximating m for three frequency ranges:

- For $r \ll 1$ $(\omega \ll \omega_n)$, $m = -10 \log(1) = 0$
- For $r \gg 1$ $(\omega \gg \omega_n)$, $m = -10 \log(r^4 + 4\zeta^2 r^2)$
$$\approx -10 \log(r^4)$$
$$= -40 \log(r)$$

- Resonance: we know from the problem statement that $\omega_r = 4$. In terms of r, this corresponds to:

$$r_r = \frac{\omega_r}{\omega_n} = \frac{4}{\sqrt{\frac{208.15}{13}}} = 0.9996 \approx 1.$$

The peak response in dBs at resonance is

$$m_r = 20 \log(M_r) = 20 \log\left(\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right)$$

note: this is the original form of M_r so that m gives the amplitude ratio of the output displacement to input displacement.

$$= 20 \log\left(\frac{1}{2\frac{1}{52.02}\sqrt{1-\left(\frac{1}{52.02}\right)^2}}\right) = 28.30$$

For the phase,

- For $r \ll 1$ $(\omega \ll \omega_n)$, $\phi = -\tan^{-1}\left(\frac{0}{1}\right) = 0°$
- For $r \gg 1$ $(\omega \gg \omega_n)$, $\phi = -\tan^{-1}\left(-\frac{r}{r^2}\right) = -\tan^{-1}\left(-\frac{1}{r^2}\right) \Rightarrow -180°$
- For $r = 1$ $(\omega = \omega_n)$, $\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) \Rightarrow -90°$

Bode Plot: