Frequency Response - Lecture 1

ME3050 - Dynamics Modeling and Controls

April 19, 2020

Frequency Response of First Order Systems

Lecture 1- Frequency Response of First Order Systems

- Introduction to Chapter 9

- Review Complex Numbers

- Frequency Response of First Order Systems

- Graph of Frequency Response

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

Harmonic Input Function
Why Study Frequency Response?
Frequency Response and the Transfer Function

Harmonic Input Function

The term **frequency response** is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to *harmonic* input such as sines and cosines. The input (forcing) function is written below.

$$f(t) = A\sin(\omega t)$$

Amplitude of the Input, $\quad A \quad (N)$

Frequency of Input, $\quad \omega \quad (\frac{rad}{s})$

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

Harmonic Input Function
Why Study Frequency Response?
Frequency Response and the Transfer Function

Why Study Frequency Response?

Why do we care about the way a system responds to harmonic excitation? Why is **frequency analysis** important?

-
-
-

What causes **harmonic** (or sinusoidal) excitation in the real world?

-
-
-

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

Harmonic Input Function
Why Study Frequency Response?
**Frequency Response and the Transfer Function**

Frequency Response and the Transfer Function

A linear, time-invariant (LTI) system has a **transfer function** $T(s)$ that describes the **input-output** relationship of the system. Under sinusoidal excitation (input) with frequency ω if the system is stable the transient affects in the response (output) will eventually disappear leaving the **steady state sinusoidal response** of the same frequency as the input but with a phase shift w.r.t. the input.

The Complex Plane

In an underdamped system the roots of the characteristic polynomial are complex. Before we proceed we need to review some rules of arithmetic and complex numbers.

The Complex Plane: $\mathbb{C}$
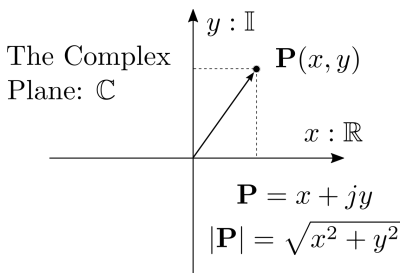
$$\mathbf{P} = x + jy$$
$$|\mathbf{P}| = \sqrt{x^2 + y^2}$$

Cartesian Representation:

$P = x + jy$

Polar Representation:

$P = |P| \angle \theta$

Exponential Representation:

$P = |P| e^{j\theta} = |P| (cos\theta + jsin\theta)$

Complex Number Algebra

Consider two points P_1 and P_2 on the complex plane.

$P_1 = x_1 + jy_1$ and $P_2 = x_2 + jy_2$

Addition: $\qquad P_1 + P_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $\qquad P_1 P_2 = |P_1 P_2| \angle (\theta_1 + \theta_2)$

Division: $\qquad \frac{P_1}{P_2} = (x_1 + x_2) + j(y_1 + y_2)$

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

First Order Mass Damper

Consider our $1^{\underline{st}}$ order mass damper system.

$m\dot{v} + cv = f(t)$ with a **time constant** $\tau = \frac{m}{c}$

The system is commonly re-written as shown below.

$m\dot{v} + cv = f(t) \;\; \rightarrow \;\; \tau\dot{y} + y = f(t)$

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

First Order Transfer Function

$\tau \dot{y} + y = f(t)$

Take the Laplace transform of the ODE.

$\mathcal{L}\{\tau \dot{y} + y\} = \mathcal{L}\{f(t)\}$

$\tau \left(sY(s) + y_0) \right) + Y(s) = F(s)$ The initial conditions are zero.

$T(s) = \frac{Y(s)}{F(s)} = \frac{1}{\tau s + 1}$ First Order Transfer Function

This considers a *generalized* input function $f(t)$ and zero ICs.

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

Sinusoidal Input Function

Our model is now excited by a sinusoidal input (forcing) function.

$\tau \dot{y} + y = f(t) = A \sin(\omega t)$

Take the Laplace transform. Then, solve for $Y(s)$ and expand.

$\tau s Y(s) + Y(s) = \frac{A\omega}{s^2 + \omega^2}$

$Y(s) = \frac{A\omega}{(s^2 + \omega^2)(\tau s + 1)} = \frac{C_1}{\tau s + 1} + \frac{C_2 s}{(s^2 + \omega^2)} + \frac{C_3 \omega}{(s^2 + \omega^2)}$

Now solve for the coefficients.

$C_1 = \frac{A\omega\tau^2}{1 + \omega^2\tau^2} \qquad, \quad C_2 = \frac{-A\omega\tau}{1 + \omega^2\tau^2} \qquad, \quad C_3 = \frac{A}{1 + \omega^2\tau^2}$

Substituting and take the inverse Laplace transform.

$y(t) = \frac{A\omega\tau}{1 + \omega^2\tau^2} \left(e^{\frac{-t}{\tau}} - \cos\omega t + \frac{1}{\omega\tau} \sin\omega t \right)$

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

Steady State Time Response

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - \cos\omega t + \frac{1}{\omega\tau}\sin\omega t\right)$$

After some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y(t) = \frac{A}{1+\omega^2\tau^2}\left(\sin\omega t - \omega\tau\cos\omega t\right)$$

This is re-written as a single sine term with a phase shift.

Steady State Frequency Response of First Order System

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}}\sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

Amplitude Ratio

$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \; \phi = -tan^{-1}\omega\tau$

Notice that the system responds at the same frequency as the input but with a different amplitude and a phase shift. The ratio of the response amplitude to the input amplitude is called the **amplitude ratio, M**.

$M = \frac{\frac{A}{\sqrt{1+\omega^2\tau^2}}}{A} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$

Fortunately we can find the **amplitude ratio** and **phase shift** directly from the transfer function. Recall the transfer function we derived.

$T(s) = \frac{1}{\tau s + 1} \qquad$ let $s = j\omega \qquad \implies \qquad T(j\omega) = \frac{1}{\tau j\omega + 1}$

$|T(j\omega)| = \frac{|1|}{|\tau j\omega + 1|} = \frac{1}{\sqrt{(\tau\omega)^2 + 1^2}} = \frac{1}{\sqrt{1+\tau^2\omega^2}} \qquad\qquad$ Look familiar?

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

Phase Angle

$|T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = M(\omega)$

$\angle T(j\omega) = \angle 1 - \angle(1 + j\omega\tau) = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{\omega\tau}{1}\right) =$
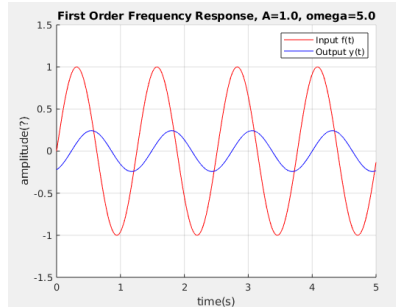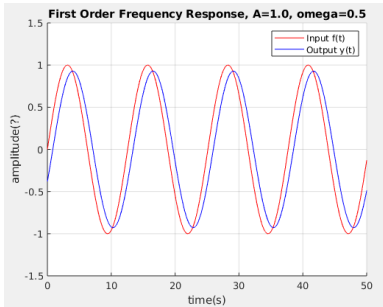
$-tan^{-1}(\omega\tau) = \phi(\omega)$

Substitute $s = j\omega$ into the transfer function and solve for the magnitude and phase angle of this complex number which represent the magnitude ratio and phase shift.

Therefore the steady state response is written as follows.

$y_{ss}(t) = A|T(j\omega)|sin(\omega t + \angle T(j\omega)) = MAsin(\omega t + \phi)$

Wasn't that fun? Can you believe we used to do that on the board?!?!

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

Graph of Frequency Response

What determines the amplitude of the system response?

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

MATLAB code

```
1   % ME3050 - Spring 2020 Tennessee Technological Univ.
2   clear variables;clc;close all
3
4   % define the system parameters
5   m=20;c=25;
6   tau=m/c;
7
8   % define the amplitude input frequency and
9   A=1;omega=1/2;
10
11  % calculate the magnitude ratio
12  M=1/sqrt(1+omega^2*tau^2);
13  phi=-tan(omega*tau);
```

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

MATLAB code

```
1   %consider a range of time values
2   dt=0.01; tstop=50;
3   time=0:dt:tstop;
4
5   %calculate the input and response curves
6   fin=A*sin(omega*time);
7   yout=M*A*sin(omega*time+phi);
8
9   % show the results in a figure
10  figure(1); hold on
11  plot(time,fin,'r')
12  plot(time,yout,'b')
```

Introduction to Chapter 9
Review Complex Numbers
Frequency Response of First Order Systems

First Order Mass Damper
First Order Transfer Function
Sinusoidal Input Function
Steady State Time Response
Amplitude Ratio and Phase Angle
Graph of Frequency Response

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain