Time Response - Lecture 3

ME3050 - Dynamics Modeling and Controls

March 25, 2020

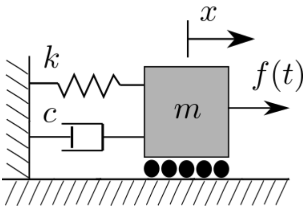
Step Response of Second Order Systems

Lecture 3 - Step Response of Second Order Systems

- The Step Input
- Response Equations for Three Cases
- Description and Specification of The Step Response
- Affect of Root Location on Response

The Step Input

Now, consider the mass-spring system with damping present subject to **step** input. This models instantly turning on the input force $f(t)$.



Heavyside's Step Function

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

The EOM is:

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{with} \quad x(t=0) = x_0 \quad \text{and} \quad v(t=0) = v_0$$

Complementary Part of Solution

The free response equations derived in the previous lecture represent the *comlpementary part* of the solution. We also need to find the *particular part* to see the step response of the system.

Once again, the solution takes one of three forms. The response equations for each are in your reference handout.

The Overdamped Case

<u>Overdamped</u> $\quad c > 2\sqrt{mk} \quad \Longrightarrow \quad \zeta > 1$

roots: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$x(t) = C_1 e^{-s_1 t} + C_2^{-s_2 t} + \frac{F}{K}$

The *unit step response* is found with zero initial conditions $x_0 = 0 \quad v_0 = 0$ and a unit step input $F = 1$.

$x(t) = \frac{1}{K}(\frac{s_2}{s_1 - s_2} e^{-s_1 t} - \frac{s_1}{s_1 - s_2} e^{-s_2 t} + 1)$

The Critically Damped Case

Critically Damped $\quad c = 2\sqrt{mk} \quad \Longrightarrow \quad \zeta = 1$

roots: $s_{1,2} = \frac{-c}{2m} = \zeta\omega_n = -\omega_n$

$$x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} + \frac{F}{K} = (C_1 + C_2 t)e^{-\omega_n t} + \frac{F}{K}$$

The *unit step response* is found with zero initial conditions $x_0 = 0 \quad v_0 = 0$ and a unit step input $F = 1$.

$$x(t) = \frac{1}{K}[(-1 - \omega_n t)e^{-\omega_n t} + 1]$$

The Underdamped Case

Underdamped $\quad c < 2\sqrt{mk} \quad \Longrightarrow \quad \zeta < 1$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

$$x(t) = C_1 e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + \frac{F}{K}$$

The *unit step response* is found with zero initial conditions $x_0 = 0 \quad v_0 = 0$ and a unit step input $F = 1$.

$$x(t) = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$$

$$\phi = tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$$

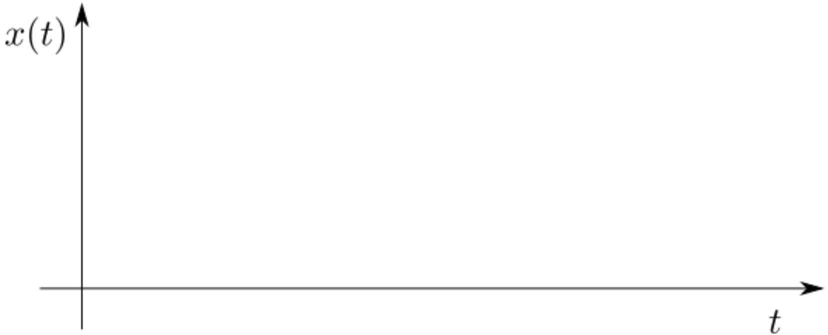
Unit Step Response

The **unit step response** is a special case of the *forced response* in which $f(t)$ is the step function of magnitude 1.

Overdamped	$x(t) = \frac{1}{K}(\frac{s_2}{s_1 - s_2}e^{-s_1 t} - \frac{s_1}{s_1 - s_2}e^{-s_2 t} + 1)$
Critically Damped	$x(t) = \frac{1}{K}[(-1 - \omega_n t)e^{-\omega_n t} + 1]$
Underdamped	$x(t) = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$ $\phi = tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$

Description and Specification of System Response

We are going to derive several quatities that describes the response of an underdamped system.



Rise Time

The **rise time** is the time at which the response first equals the steady state value.

$$x(t) = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$$

Set the *transient term* to zero and solve for t.

$$e^{-\zeta\omega_n t} sin(\omega_d t + \phi) = 0 \implies sin(\omega_d t + \phi) = 0 \implies \omega_d t + \phi = 2\pi$$

$$t_{rise} = t_r = \frac{2\pi - \phi}{\omega_d}$$

Peak Time

The **peak time** is the time at which the response equals the maximum value. Find the derivative of the response equation and set it equal to zero.

$\dot{x}(t) =$

$$\left(\frac{1}{K}\frac{1}{\sqrt{1-\zeta^2}}\right)\left[e^{-\zeta\omega_n t}(\omega_d cos(\omega_d t+\phi)) + sin(\omega_d t+\phi)(-\zeta\omega_n e^{-\zeta\omega_n t})\right]$$

$$sin(\omega_d)t = 0 \implies \omega_d t = \pi \implies \mathbf{t_{peak} = t_p} = \frac{\pi}{\omega_d}$$

Settling Time

The **settling time** is the time at which the response decays to a certain percentage of the steady state value.

It can be esitmated as:

$$t_{settling} = t_s = -\frac{ln(tolerance)}{\zeta\omega_n}$$

$2\% \implies tolerance = 0.02$

$5\% \implies tolerance = 0.05$

Maximum Overshoot

The **maximum overshoot** is the response beyond the steady state value.

$$M_p = x(t_p) - x_{ss} \implies M_p = \frac{1}{K} e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

This is often expressed as a percentage.

$$M_{\%} = \frac{x(t_p) - x_{ss}}{x_{ss}} 100 = 100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Damping Ratio from Maximum Overshoot

The *damping ratio* can be determined from the maximum overshoot!

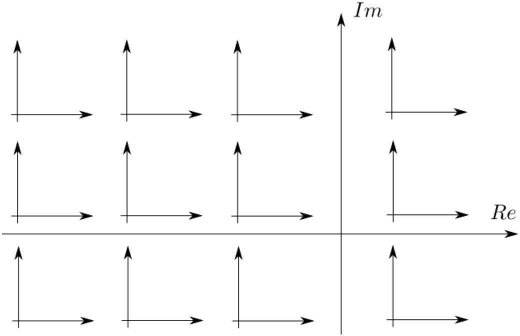
$$M_{\%} = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Solve for ζ.

$$\zeta = \frac{R}{\sqrt{\pi^2+R^2}} \qquad \text{with} \qquad R = ln\left(\frac{100}{M\%}\right)$$

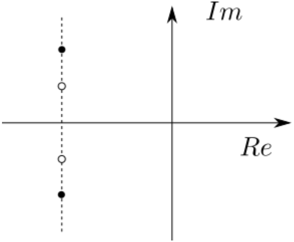
Affect of Root Location on Response

The *location* of the root in the *complex plane* shows the affects of the roots on the system behviour.



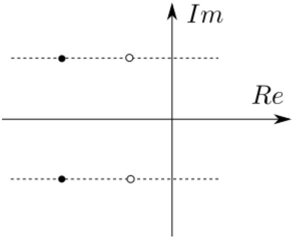
Along a Vertical Line

As the root moves along a vertical line...



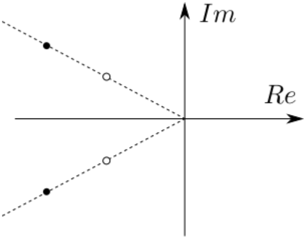
Along a Horizontal Line

As the root moves along a horizontal line...



Along a Diagonal Line

As the root moves along a diagonal line...



References

- System Dynamics, Palm III, Third Edition - Section 8.3 - Step Response of Second Order Systems