Lecture Module - Frequency Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Frequency Response

Lecture Module - Frequency Response

- Topic 1 - Frequency Response of First Order Systems
- Topic 2 - The Bode Diagram
- Topic 3 - Frequency Response of 2^{nd} Order Systems
- Topic 4 - Resonance

Topic 1 - Frequency Response of First Order Systems

- Frequency Input Concept
- Complex Numbers Review
- Frequency Response of First Order Systems
- Graph of Frequency Response

Frequency Input Concept

The term **frequency response** is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to *harmonic* input such as sines and cosines. The input (forcing) function is written below.

$$f(t) = Asin(\omega t)$$

Amplitude of the Input,	A	(N)
Frequency of Input,	ω	$(\frac{rad}{s})$

Frequency Input Concept

Why do we care about the way a system responds to harmonic excitation? Why is **frequency analysis** important?

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What causes **harmonic** (or sinusoidal) excitation in the real world?

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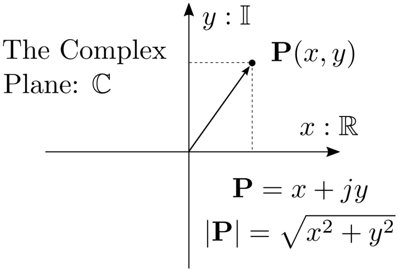
Frequency Input Concept

A linear, time-invariant (LTI) system has a **transfer function** $T(s)$ that describes the **input-output** relationship of the system. Under sinusoidal excitation (input) with frequency ω if the system is stable the transient affects in the response (output) will eventually disappear leaving the **steady state sinusoidal response** of the same frequency as the input but with a phase shift w.r.t. the input.

Complex Numbers Review

In an underdamped system the roots of the characteristic polynomial are complex. Before we proceed we need to review some rules of arithmetic and complex numbers.



Cartesian Representation:

$\text{P} = x + jy$

Polar Representation:

$\text{P} = |\text{P}|\angle\theta$

Exponential Representation:

$\text{P} = |\text{P}|e^{j\theta} = |\text{P}|\left(cos\theta + jsin\theta\right)$

Complex Number Algebra

Consider two points P_1 and P_2 on the complex plane.

$P_1 = x_1 + jy_1$ and $P_2 = x_2 + jy_2$

Addition: $$P_1 + P_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication: $$P_1 P_2 = |P_1 P_2| \angle (\theta_1 + \theta_2)$$

Division: $$\frac{P_1}{P_2} = (x_1 + x_2) + j(y_1 + y_2)$$

First Order Mass Damper

Consider our $1^{\underline{st}}$ order mass damper system.

$m\dot{v} + cv = f(t)$ with a **time constant** $\tau = \frac{m}{c}$

The system is commonly re-written as shown below.

$$m\dot{v} + cv = f(t) \quad \rightarrow \quad \tau\dot{y} + y = f(t)$$

First Order Transfer Function

$$\tau \dot{y} + y = f(t)$$

Take the Laplace transform of the ODE.

$$\mathcal{L}\{\tau \dot{y} + y\} = \mathcal{L}\{f(t)\}$$

$$\tau\left(sY(s) + y_0)\right) + Y(s) = F(s)$$ The initial conditions are zero.

$$\boxed{T(s) = \frac{Y(s)}{F(s)} = \frac{1}{\tau s+1} \qquad \text{First Order Transfer Function}}$$

This considers a *generalized* input function $f(t)$ and zero ICs.

Sinusoidal Input Function

Our model is now excited by a sinusoidal input function.

$$\tau \dot{y} + y = f(t) = Asin(\omega t)$$

Take the Laplace transform. Then, solve for $Y(s)$ and expand.

$$\tau s Y(s) + Y(s) = \frac{A\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{A\omega}{(s^2+\omega^2)(\tau s+1)} = \frac{C_1}{\tau s+1} + \frac{C_2 s}{(s^2+\omega^2)} + \frac{C_3\omega}{(s^2+\omega^2)}$$

Now solve for the coefficients.

$$C_1 = \frac{A\omega\tau^2}{1+\omega^2\tau^2} \quad , \quad C_2 = \frac{-A\omega\tau}{1+\omega^2\tau^2} \quad , \quad C_3 = \frac{A}{1+\omega^2\tau^2}$$

Substituting and take the inverse Laplace transform.

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau} sin\omega t\right)$$

Steady State Time Response

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau}sin\omega t\right)$$

After some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y(t) = \frac{A}{1+\omega^2\tau^2}\left(sin\omega t - \omega\tau cos\omega t\right)$$

This is re-written as a single sine term with a phase shift.

Steady State Frequency Response of First Order System

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}}sin\left(\omega t + \phi\right) \quad , \quad \phi = -tan^{-1}\omega\tau$$

Amplitude Ratio

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Notice that the system responds at the same frequency as the input but with a different amplitude and a phase shift. The ratio of the response amplitude to the input amplitude is called the **amplitude ratio, M**.

$$M = \frac{\frac{A}{\sqrt{1+\omega^2\tau^2}}}{A} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Fortunately we can find the **amplitude ratio** and **phase shift** directly from the transfer function. Recall the transfer function we derived.

$$T(s) = \frac{1}{\tau s+1} \qquad \text{let } s = j\omega \qquad \Longrightarrow \qquad T(j\omega) = \frac{1}{\tau j\omega+1}$$

$$|T(j\omega)| = \frac{|1|}{|\tau j\omega+1|} = \frac{1}{\sqrt{(\tau\omega)^2+1^2}} = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

Look familiar?

Phase Angle

$$|T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = M(\omega)$$

$$\angle T(j\omega) = \angle 1 - \angle (1 + j\omega\tau) = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{\omega\tau}{1}\right) =$$

$$-tan^{-1}(\omega\tau) = \phi(\omega)$$

Substitute $s = j\omega$ into the transfer function and solve for the magnitude and phase angle of this complex number which represent the magnitude ratio and phase shift.

Therefore the steady state response is written as follows.

$$y_{ss}(t) = A|T(j\omega)|sin(\omega t + \angle T(j\omega)) = MAsin(\omega t + \phi)$$

Wasn't that fun? Can you believe we used to do that on the board?!?!

Topic 2 - The Bode Diagram

- Review Frequency Response
- Amplitude ratio in Decibels
- The Bode Diagram

Review Frequency Response

Amplitude ratio in Decibels

Amplitude ratio in Decibels

The Bode Diagram

Amplitude ratio in Decibels

The Bode Diagram

Frequency Response in MATLAB

Frequency Response in MATLAB

Frequency Response in MATLAB

Topic 3 - Frequency Response of 2^{nd} Order Systems

- Review Transfer Functions
- Frequency Response of Overdamped Systems
- Frequency Response of Underdamped Systems
- MATLAB Bode Plots

Review Transfer Functions

Review Transfer Functions

Frequency Response of Overdamped Systems

Frequency Response of Underdamped Systems

Frequency Response of Underdamped Systems

MATLAB Bode Plots

MATLAB Bode Plots

MATLAB Bode Plots

MATLAB Bode Plots

Topic 3 - Resonance

- Review 2^{nd} Order Frequency Response
- The Resonance Phenomenon
- The Resonance Frequency
- MATLAB Bode Plots

Review 2^{nd} Order Frequency Response

Review 2^{nd} Order Frequency Response

The Resonance Phenomenon

MATLAB Bode Plots

MATLAB Bode Plots

The Resonance Frequency

The Resonance Frequency

The Resonance Frequency

The Resonance Frequency