(Ch 4) Mass-Spring-Damper Systems

There are three principal components in modeling lumped parameter mechanical systems: mass, spring, damper. Lumped parameter modeling: assume that components are spatially separated from one another

Mass (Inertia) Elements

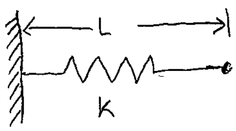
1) Translational motion: mass, m
2) Rotational motion: mass moment of inertia, I

4.1) Spring (Elastic) Elements:

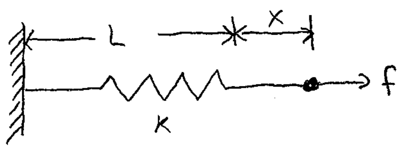
A spring is a deformable element that exerts a resistive force that is a function of displacement. It provides a restoring force.

Ideal Spring

No force Applied



Force Applied



$f = kx$ (Hooke's Law)

where k = spring constant N/m or lb/ft
x = deflection (displacement from resting position)

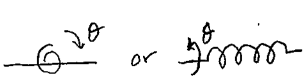
notes:
1) k is always positive
2) $f = kx$ is valid for linear springs
3) ideal spring is massless and has no internal damping
4) spring force is conservative ⇒ we can use energy method $\frac{d}{dt}(T+V)=C$

Ideal Torsional Spring

$T = k_T\,\theta$

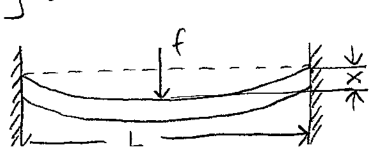
where k_T = torsional spring constant N·m/rad lb·ft/rad
θ = deflection (rotation from resting position)
T = torque exerted by spring

Symbols:



In addition to standard coil springs, a deformable structural member is also a spring. The spring constant can be found using mechanics of materials. (See Table 4.1.1)

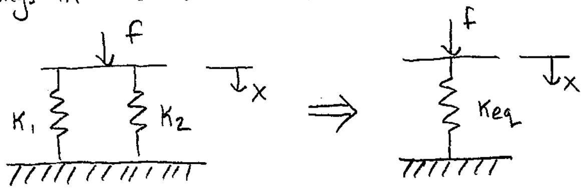
Ex:



width = w
height = h

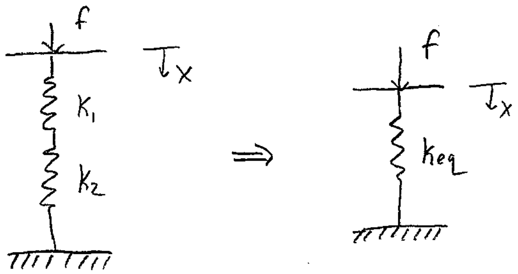
Force-Deflection Relation: $x = \frac{L^3}{16Ewh^3} f$ ⇒ Spring Constant: $k = \frac{f}{x} = \frac{16Ewh^3}{L^3}$

Springs in Series + Parallel



$K_{eq} = K_1 + K_2$ Springs in parallel add directly

General Equation: $K_{eq} = \sum_{i=1}^{N} K_i$

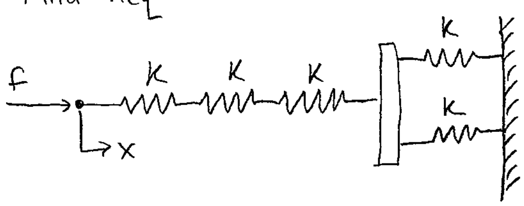


$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} \Rightarrow K_{eq} = \frac{K_1 K_2}{(K_1+K_2)}$ Springs in series add via the reciprocal

General Equation: $\frac{1}{K_{eq}} = \sum_{i=1}^{N} \frac{1}{K_i}$

Example (Ex 4.1.3)

Find K_{eq} for:

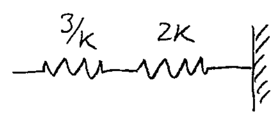


① 3 springs in series: $\frac{1}{K_{eq1}} = \frac{1}{K} + \frac{1}{K} + \frac{1}{K} = \frac{3}{K}$

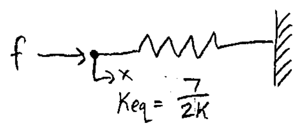
$K_{eq1} = \frac{K}{3}$

② 2 springs in parallel: $K_{eq2} = K + K = 2K$

so,

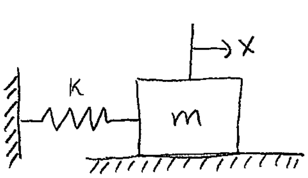


③ K_{eq1} + K_{eq2} in series: $\frac{1}{K_{eq}} = \frac{1}{K/3} + \frac{1}{2K} = \frac{7}{2K}$

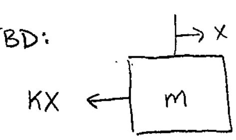


Note: Springs in parallel result in a stiffer arrangement than the springs alone, springs in series result in an arragement that is less stiff than the springs alone.

(4.2) Modeling Mass-Spring Systems



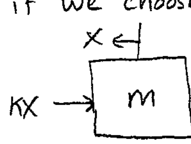
FBD:



Newton's 2^{nd} Law:

$\sum F_x = m\ddot{x} = -Kx \Rightarrow m\ddot{x} = -Kx$

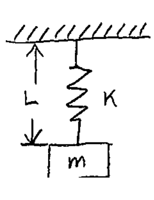
Note: if we choose x in the opposite direction, EOM is the same



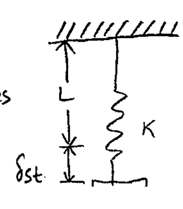
$\sum F_x = m\ddot{x} = -Kx \Rightarrow m\ddot{x} = -Kx$

Gravitational Effects

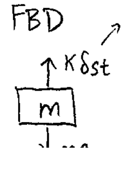
No spring stretch



Gravity stretches spring to equilibrium



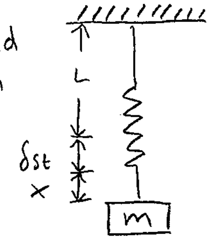
$\Rightarrow$ FBD



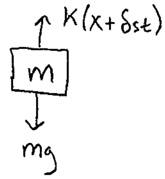
$\Rightarrow$ System is in equilibrium, so

$mg = K\delta st$

Displaced System



FBD

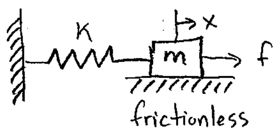


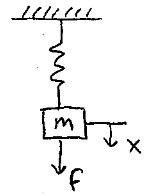
Newton

$$\Sigma F_x = m\ddot{x} = mg - K(x+\delta_{st})$$
$$m\ddot{x} = mg - Kx - K\delta_{st}$$
recall $mg = K\delta_{st}$, so
$$m\ddot{x} = -Kx$$

★ Gravity cancels with static deflection as long as x is measured from <u>equilibrium point</u>.

Solving the Equation of Motion





EOM for these typical mass-spring systems:

$$m\ddot{x} + Kx = f$$

Suppose $f=0$ and we set the mass into motion by pulling it to position $x(0)$ and releasing it with initial velocity $\dot{x}(0)$.

$m\ddot{x} + Kx = 0 \qquad x(0), \dot{x}(0)$

Trial Soln Method

$$ms^2Ce^{st} + KCe^{st} = 0 \Rightarrow (ms^2+K) = 0 \Rightarrow s = \pm j\sqrt{\tfrac{K}{m}}$$

(no forcing so $x_p = 0$) $\quad x_t = C_1 e^{-j\sqrt{\frac{K}{m}}t} + C_2 e^{j\sqrt{\frac{K}{m}}t} = A\sin\sqrt{\tfrac{K}{m}}t + B\cos\sqrt{\tfrac{K}{m}}t$

$$\dot{x}_t = \sqrt{\tfrac{K}{m}}A\cos\sqrt{\tfrac{K}{m}}t - \sqrt{\tfrac{K}{m}}B\sin\sqrt{\tfrac{K}{m}}t$$

$$x(0) = B$$

$$\dot{x}(0) = \sqrt{\tfrac{K}{m}}A \Rightarrow A = \frac{\dot{x}(0)}{\sqrt{\frac{K}{m}}}$$

$$x(t) = \frac{\dot{x}(0)}{\sqrt{\frac{K}{m}}}\sin\sqrt{\tfrac{K}{m}}t + x(0)\cos\sqrt{\tfrac{K}{m}}t$$

Now, we need to define a <u>very important</u> quantity in vibrations: <u>natural frequency</u>

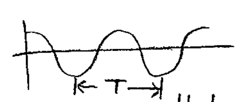
$$\boxed{\omega_n = \sqrt{\tfrac{K}{m}}}$$

This is the oscillation frequency of the unforced system (in radians!)

Rewrite x_t as

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

Notes:
- The natural frequency, ω_n, only depends on mass + stiffness, not initial conditions or forcing.
- ω_n is greater for stiffer springs and lighter masses
- The period of oscillation is $T = \frac{2\pi}{\omega_n}$



- Mass-spring system is commonly called the Harmonic Oscillator

It is common to express the response as a single sine or cosine function using trig identities

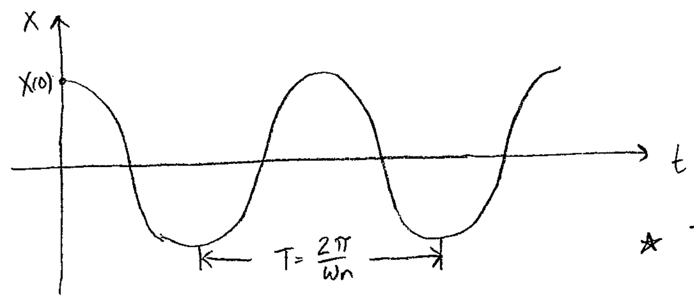
$$x(t) = A\cos(\omega_n t - \phi), \qquad A = \sqrt{x(0)^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2} \qquad \phi = \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)\,\omega_n}\right)$$

Amplitude — Phase

<u>OR</u>

$$x(t) = A\sin(\omega_n t + \phi), \qquad A = \sqrt{x(0)^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2} \qquad \phi = \tan^{-1}\left(\frac{x(0)\,\omega_n}{\dot{x}(0)}\right)$$

Response

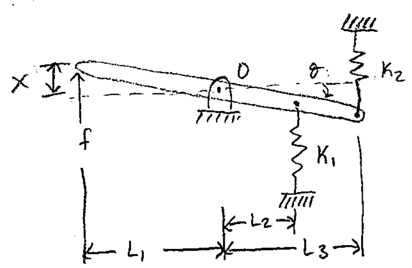


- sinusoidal, undamped, oscillates about equilibrium position, $x=0$.

★ This system ($m\ddot{x} + kx = 0$) is called Free Undamped Vibration

Example

Rigid Lever: mass $= m$, Length $= L$, Frictionless pivot, when $x=0$ & $\theta=0$, springs are at their equilibrium point

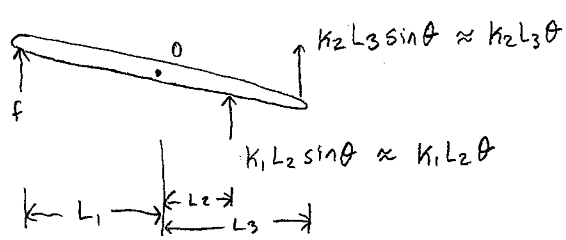


a) Assuming that θ is small, derive EOM

b) Find the natural frequency in terms of m, L's, k's

Note: small angle approximation: $\sin\theta \approx \theta$, $\cos\theta \approx 1$

<u>FBD</u>



a) Newton's Method:

$$\sum M_o = I_o\ddot{\theta} = fL_1 - k_1L_2\theta L_2 - k_2L_3\theta L_3$$

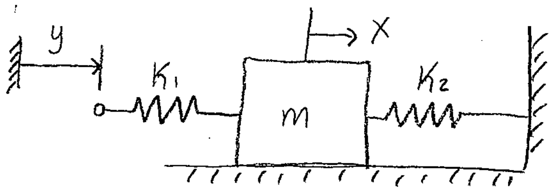
$$I_o = \frac{mL^2}{12}$$

$$\boxed{\frac{mL^2}{12}\ddot{\theta} + (k_1L_2^2 + k_2L_3^2)\theta = fL_1}$$

b) $\omega_n = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{(k_1L_2^2 + k_2L_3^2)}{mL^2/12}}}$

Displacement Input Example

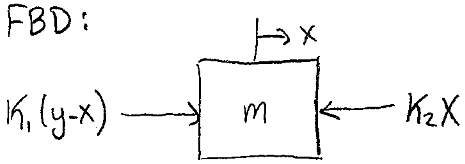
Fig. 4.2.10 (b):



★ When drawing FBD, <u>must</u> make assumption. Either assume $y > x$ or $y < x$

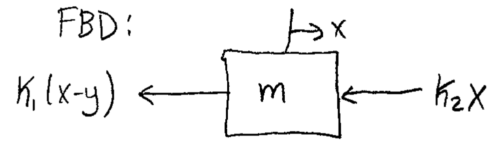
assume $y > x$

FBD:



assume $y < x$

FBD:



EOM in either case:

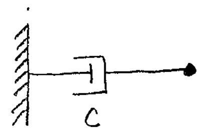
$$m\ddot{x} + (K_1 + K_2)x = K_1 y$$

(4.4) Damping Elements:

A damper is an object that resists relative velocity across it. A damper absorbs externally applied energy + dissipates it internally as heat. It removes energy from the system.

Common examples: pneumatic door closer, shock absorber in a car, mountain bike fork + rear shock.

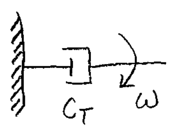
Ideal Damper (sometimes called a dashpot)



$f = cv = c\dot{x}$ where c = damping coefficient N·s/m lb-s/ft

notes:
1) c is always positive
2) ideal damper is massless and stores no internal energy
3) damping force is nonconservative ⇒ no energy method

Torsional Damper



$T = C_T \omega = C_T \dot{\theta}$ where C_T = torsional damping coefficient N·m·s/rad lb-ft-sec/rad

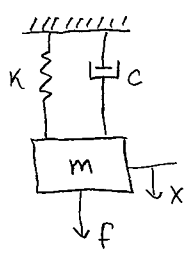
Dampers in Series + Parallel

- Same as springs

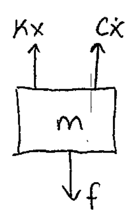
Parallel: $C_{eq} = \sum_{i=1}^{N} C_i$

Series: $\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i}$

Modeling Spring-Mass-Damper Systems



FBD:



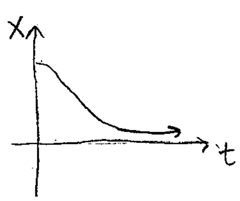
Newton's 2nd Law:

$$\sum F_x = m\ddot{x} = f - Kx - c\dot{x}$$

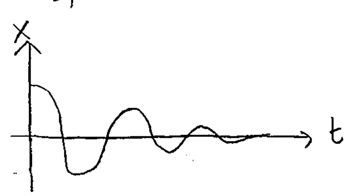
$$m\ddot{x} + c\dot{x} + Kx = f$$

We will save the details of the solution of the EOM for later in the course (Time Response, Ch. 8), but from our DE review, we know the general response is based on the roots.

Real, Distinct Roots

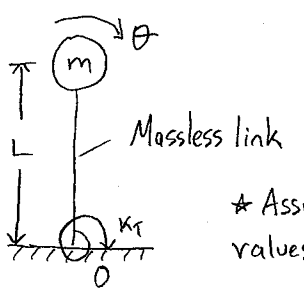


Complex Roots



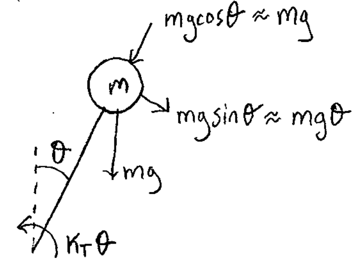
Example Problems

Ex. Metronome (Mass Spring System)



★ Assume small values of θ

FBD:



Note: small angle approximation
$\sin\theta \approx \theta$, $\cos\theta \approx 1$

Find EOM, find ω_n

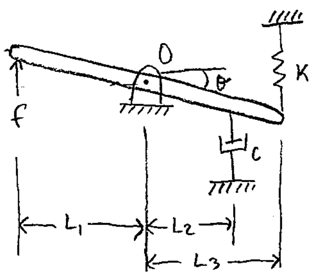
Newton's Method

$$\Sigma M_o = I_o\ddot{\theta} = mg\theta L - K_T\theta$$

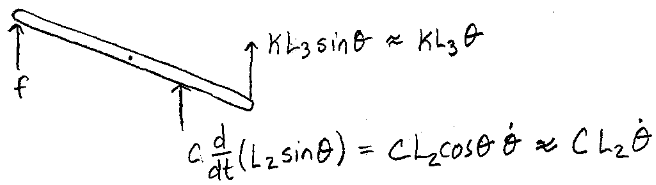
$$I_o = mL^2$$

$$mL^2\ddot{\theta} + (K_T - mgL)\theta = 0 \Rightarrow \omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{(K_T - mgL)}{mL^2}}$$

Ex:



FBD:



Rigid lever, mass = m, length = L,
Frictionless pivot, when $\theta = 0$,
spring is at equilibrium pt.

Find EOM + ω_n

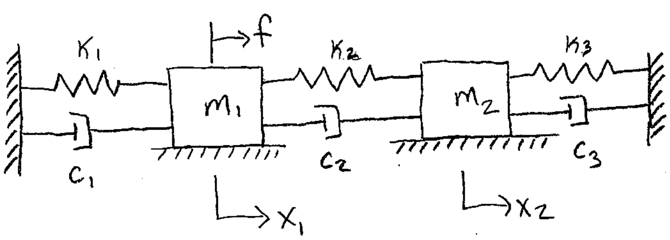
$$\Sigma M_o = I_o\ddot{\theta} = fL_1 - cL_2\dot{\theta}L_2 - KL_3\theta L_3$$

$$I_o = \frac{mL^2}{12}$$

$$\frac{mL^2}{12}\ddot{\theta} + cL_2^2\dot{\theta} + KL_3^2\theta = fL_1 \Rightarrow \omega_n = \sqrt{\frac{KL_3^2}{mL^2/12}}$$

Multiple Degrees of Freedom

Ex:



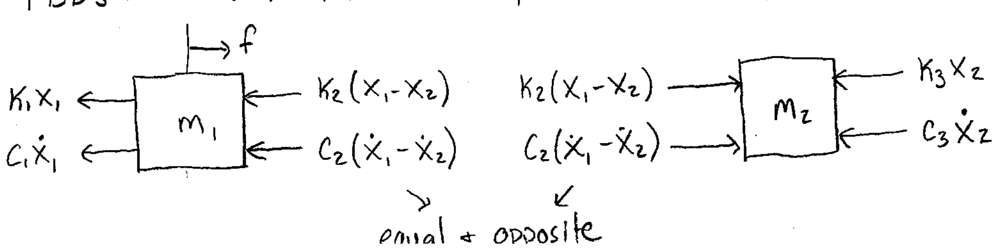
How do we handle multiple masses?

First, how many DOF? ⇒ x_1, x_2 (i.e 2) coordinates required to describe system, so 2 DOF.

How many equations needed? 2DOF ⇒ 2 EOM

FBDs:

★ Need to assume x_1 relative to x_2 ⇒ $x_1 > x_2$.



Note: doesn't matter if you displace x_1 more or less than x_2 to draw FBDs.

Newton's Method:

Mass 1: $\Sigma F_x = m_1\ddot{x}_1 = f - k_1x_1 - c_1\dot{x}_1 - k_2(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2)$

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = f$$

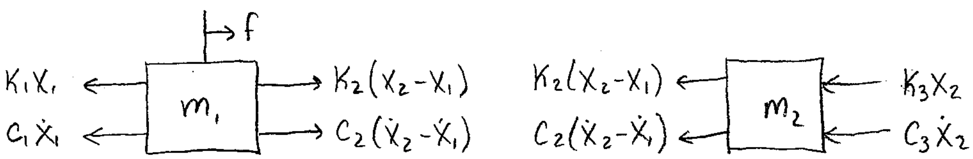
Mass 2: $\Sigma F_x = m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) - k_3x_2 - c_3\dot{x}_2$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2+c_3)\dot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = 0$$

Matrix Form: $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$

$$M\ddot{x} + C\dot{x} + Kx = F$$

What if we had assumed that x_2 displaced more than x_1?



$\Sigma F_x = m_1\ddot{x}_1 = f - k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1)$

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = f$$

Same EOMs

$\Sigma F_x = m_2\ddot{x}_2 = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - k_3x_2 - c_3\dot{x}_2$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2+c_3)\dot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = 0$$