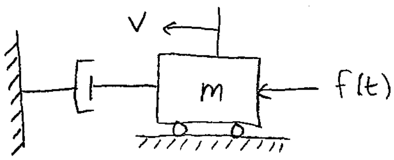
(Ch. 8) System Response in Time Domain

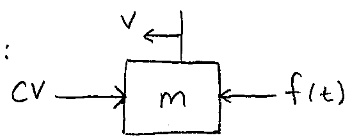
We can now combine our knowledge of modeling mechanical (or electrical / fluid / thermal) systems (Ch 3-6) with our knowledge of solving ODEs (Ch 2) to analyze systems and understand their response in the time domain to various inputs (impulse, step, ramp).

(8.1) Time Response of 1st Order Systems

Consider mass-damper system with velocity as dependent variable



FBD:



$\Sigma F = m\dot{v} = f(t) - cv$

EOM: $m\dot{v} + cv = f(t)$

1st order ODE

Free Response $(f(t) = 0)$

$$m\dot{v} + cv = 0$$

Solve using Laplace or Trial Solution method:

$$m\left[sV(s) - v(0)\right] + cV(s) = 0$$

$$(ms + c)\,V(s) = mv(0) \qquad \text{Root: } s = -c/m$$

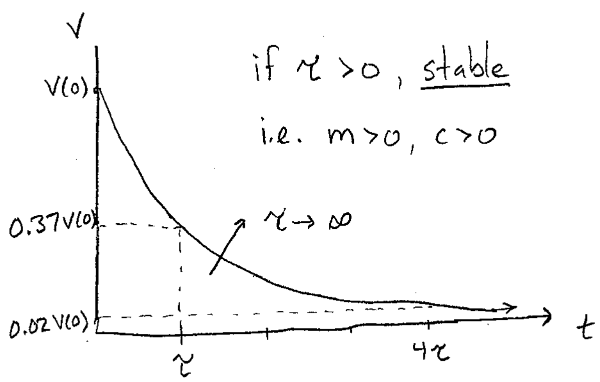
$$V(s) = \frac{mv(0)}{ms + c} = \frac{v(0)}{s + c/m} \Rightarrow \text{Matches entry \# 6 in LT table} \quad \frac{1}{s+a} = \mathcal{L}\left(e^{-at}\right)$$

So, $v(t) = v(0)\,e^{-ct/m}$

Recall from Ch. 2, we defined the time constant as τ. In this case,

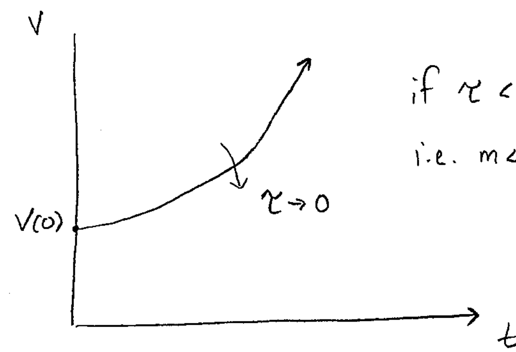
$$v(t) = v(0)\,e^{-t/\tau} \quad \text{so} \quad \tau = \frac{m}{c} \quad \text{(units of } \tau\text{: seconds)}$$

Our general response is an exponential function that starts at $v(0)$ and whos stability depends on τ. No oscillations!



if $\tau > 0$, stable

i.e. $m > 0$, $c > 0$



if $\tau < 0$, unstable

i.e. $m < 0$ or $c < 0$

Step Response ($f(t) = \text{constant} = F$)

$$m\dot{v} + cv = F$$

Laplace:

$$m\left[sV(s) - v(0)\right] + cV(s) = \frac{F}{s}$$

$$(ms + c)V(s) = \frac{F}{s} + mv(0)$$

$$V(s) = \frac{F}{s(ms+c)} + \frac{mV(0)}{ms+c} \implies$$

Partial Fraction Expansion:

$$\frac{F}{s(ms+c)} = \frac{a}{s} + \frac{b}{ms+c}$$

"cover-up"

$$a: \frac{F}{(m\cdot 0 + c)} = \frac{F}{c}$$

$$b: \frac{F}{-c/m} = \frac{-Fm}{c}$$

So,

$$V(s) = \frac{F}{c}\left(\frac{1}{s} - \frac{1}{s + c/m}\right) + \frac{V(0)}{s + c/m}$$

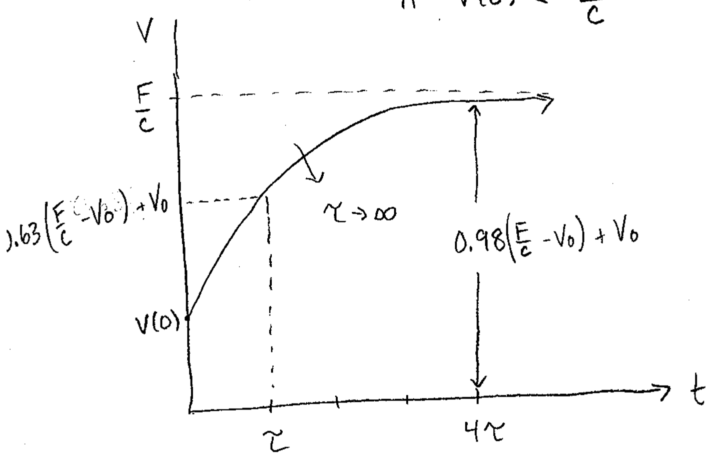
Invert using entries #6 + #2 to get

$$\tau = \frac{m}{c}$$

$$v(t) = \underbrace{\frac{F}{c}\left(1 - e^{-t/\tau}\right)}_{\text{Forced Response}} + \underbrace{V(0)e^{-t/\tau}}_{\text{Free Response}} = \underbrace{\left(V(0) - \frac{F}{c}\right)e^{-t/\tau}}_{\text{Transient Response}} + \underbrace{\frac{F}{c}}_{\text{Steady-state response}}$$

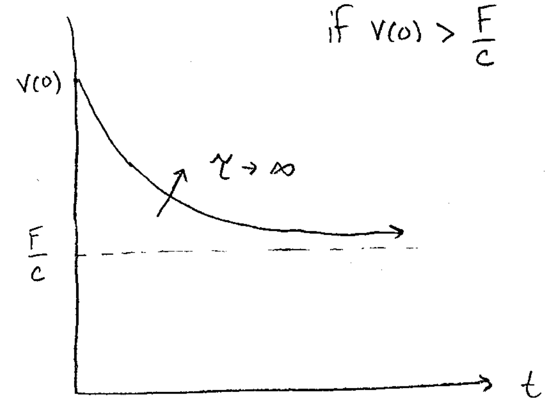
Stable Response:

if $V(0) < \frac{F}{c}$



or

if $V(0) > \frac{F}{c}$



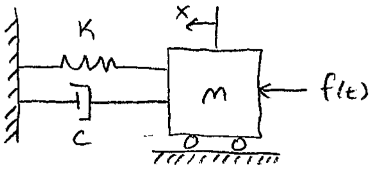
Response Time + the Time Constant

Note that as τ grows ($\tau \to \infty$), the system takes longer to approach steady state. If $\tau = m/c$, then smaller damping forces correspond to large time constants and slow responses. However, small damping also results in larger steady state response, $\frac{F}{c}$

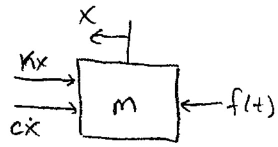
★ Think of the hydraulic door closer: large damping corresponds to faster response, and low steady state response (velocity in this case). Small damping corresponds to slower response and larger steady state response (velocity).

(8.2) Time Response of 2^{nd} Order Systems

Consider our standard mass-spring-damper system



FBD:



$\Sigma F = m\ddot{x} = f(t) - Kx - c\dot{x}$

$$\underbrace{m\ddot{x} + c\dot{x} + Kx = f(t)}_{2^{nd} \text{ order ODE}}$$

Undamped Free Response

Consider the system with no damper. We solved for the undamped free response in Ch.4

$m\ddot{x} + Kx = 0$, $x(0), \dot{x}(0)$

$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t \qquad \omega_n = \sqrt{\frac{K}{m}}$ = natural frequency

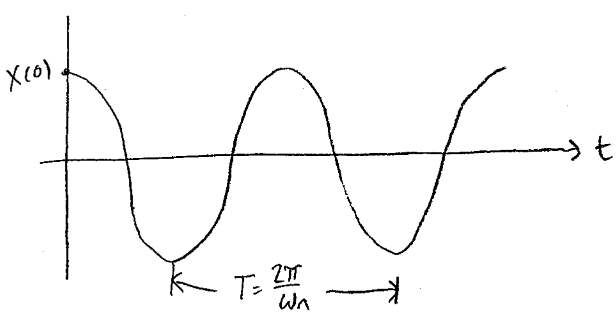
Alternatively, we have

$x(t) = A \cos(\omega_n t - \phi) \qquad A = \sqrt{x(0)^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2}, \quad \phi = \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)\,\omega_n}\right)$

<u>OR</u>

$x(t) = A \sin(\omega_n t + \phi) \qquad A = \sqrt{x(0)^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2}, \quad \phi = \tan^{-1}\left(\frac{x(0)\,\omega_n}{\dot{x}(0)}\right)$

Response:



Damped Free Response

Now consider the system including the damper

$m\ddot{x} + c\dot{x} + Kx = 0 \qquad x(0), \dot{x}(0)$

Let's solve this using Trial Solution Method

Assume a soln $x = Ce^{st}$

$ms^2Ce^{st} + csCe^{st} + KCe^{st} = 0$

$(ms^2 + cs + K)Ce^{st} = 0$

So our characteristic equation is

$$ms^2 + cs + K = 0$$

which leads to the following roots

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mK}}{2m} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}$$

Now, we know that the roots of the system dictate its behavior, so let's focus on the roots.
First, consider a system where

$$c^2 - 4mK = 0$$

From the first form above, we know this system will not oscillate. Now, solve for c:

$c = 2\sqrt{mK}$ this is the <u>critical damping value</u>.

We can also recognize that:

if $c < 2\sqrt{mK}$ the system oscillates

if $c \geq 2\sqrt{mK}$ the system does not oscillate

In order to analyze 2nd order systems, let's define another important quantity in vibrations

$\zeta = \dfrac{c}{2\sqrt{mK}}$ this is the <u>damping ratio</u>, the ratio of actual damping, c, to critical damping. Note: undefined if any root has positive real part (unstable).

Now, we can rewrite the roots as

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

And we can define another important quantity

$\omega_d = \omega_n\sqrt{1 - \zeta^2}$ this is the <u>damped natural frequency</u>

This gives the roots as

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

Now, there are three cases that we must consider

Case 1: Underdamped $c < 2\sqrt{mK}$, $\zeta < 1$ (oscillatory, complex conjugate roots)

$$x(t) = C_1 e^{(-\zeta\omega_n + j\omega_d)t} + C_2 e^{(-\zeta\omega_n - j\omega_d)t}$$

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

Use the ICs to find A + B

$$x(0) = A$$

$$\dot{x}(t) = -A\omega_d e^{-\zeta\omega_n t}\sin\omega_d t - A\zeta\omega_n e^{-\zeta\omega_n t}\cos\omega_d t + B\omega_d e^{-\zeta\omega_n t}\cos\omega_d t - B\zeta\omega_n e^{-\zeta\omega_n t}\sin\omega_d t$$

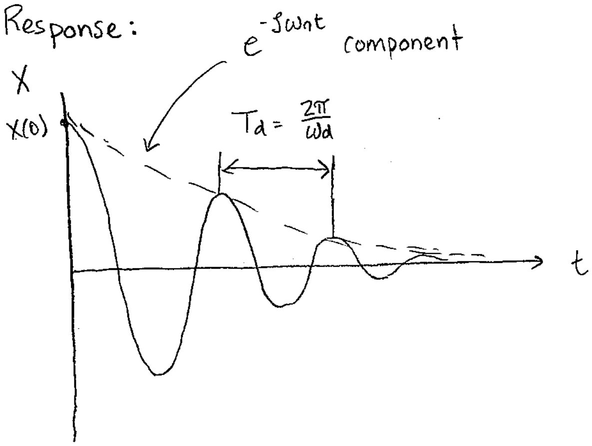
$$\dot{x}(0) = -A\zeta\omega_n + B\omega_d$$

$$\dot{x}(0) = -x(0)\zeta\omega_n + B\omega_d \Rightarrow B = \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}$$

So our total solution is

$$x(t) = e^{-\zeta\omega_n t}\left(x(0)\cos\omega_d t + \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\sin\omega_d t\right)$$

Response:



Case 2: Critically Damped $c = 2\sqrt{mk}$, $\zeta = 1$ (non-oscillatory, repeated roots)

Here, the roots are: $s_{1,2} = \frac{-c}{2m} = -\zeta\omega_n = -\omega_n$

So, we have

$$x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

Use the ICs to find C_1 + C_2:

$$x(0) = C_1$$

$$\dot{x}(t) = -C_1\omega_n e^{-\omega_n t} - \omega_n C_2 t e^{-\omega_n t} + C_2 e^{-\omega_n t}$$

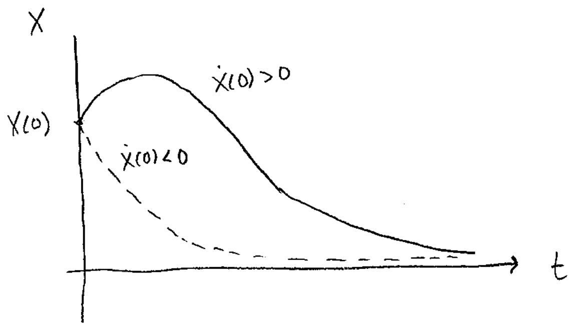
$$\dot{x}(0) = -C_1\omega_n + C_2$$

$$\dot{x}(0) = -x(0)\omega_n + C_2 \Rightarrow C_2 = \dot{x}(0) + \omega_n x(0)$$

So our total solution is

$$x(t) = x(0)e^{-\omega_n t} + (\dot{x}(0) + \omega_n x(0))te^{-\omega_n t}$$

Response:



Case 3: Overdamped $c > 2\sqrt{mk}$, $\zeta > 1$ (non-oscillatory, real, distinct roots)

Here, the roots are $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

So, we have

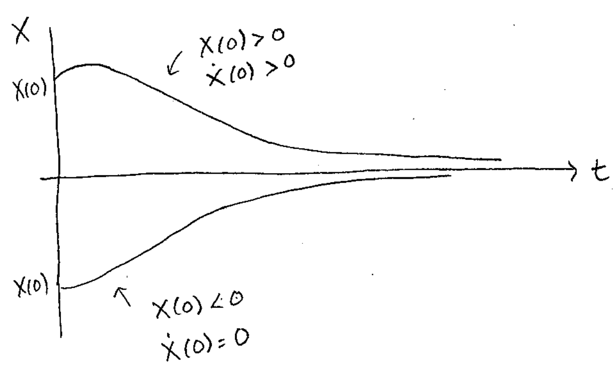
$$x(t) = C_1 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t} + C_2 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t}$$

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 e^{-\omega_n\sqrt{\zeta^2-1}\,t} + C_2 e^{\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

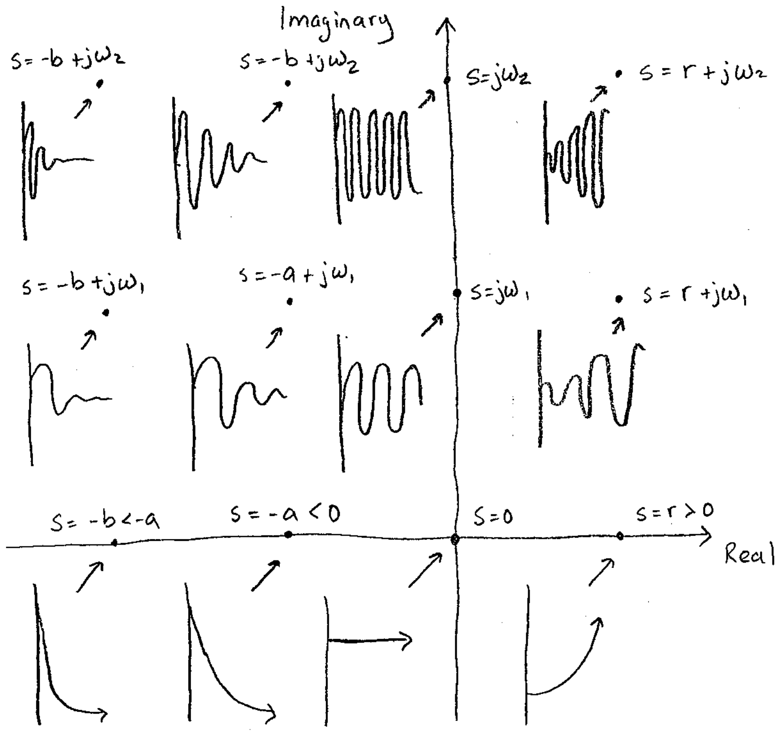
Using the ICs to solve for C_1 + C_2 (this is a bit messy), we get

$$C_1 = \frac{-\dot{x}(0) + \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}} \quad , \quad C_2 = \frac{\dot{x}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}}$$

Response:



Effect of Root Location on Response



(8.3) Step Response of 2nd Order Systems

$$m\ddot{x} + c\dot{x} + Kx = F$$

We solved the homogeneous equation when we considered the free response. Our roots are:

$$s_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_d \qquad \left(\omega_d = \omega_n\sqrt{1-\zeta^2}\right)$$

Our particular solution is solved as

$$X_p = C \Rightarrow KX_p = F \Rightarrow X_p = \frac{F}{K}$$

Again, we have three possible solutions depending on the roots: underdamped, critically damped, and overdamped. After solving for the total response (we don't have time to go through the derivations), we have:

Case 1: Underdamped $c < 2\sqrt{mK}$, $\zeta < 1$ roots: $s_{1,2} = -\zeta\omega_n \pm j\omega_d$

$$X(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{\text{Transient Response}} + \underbrace{\frac{F}{K}}_{\text{Steady-State Response}}$$

For zero ICs $(x(0) = \dot{x}(0) = 0)$ and a unitstep $(F = 1)$, we have

$$X(t) = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + 1\right] \qquad \text{where } \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$$

Case 2: Critically Damped $\quad c = 2\sqrt{mk}\,,\ \zeta = 1 \quad$ roots: $s_1 = s_2 = -\omega_n$

$$X(t) = \underbrace{(A_1 + A_2 t)e^{-\omega_n t}}_{\text{Transient Response}} + \underbrace{\frac{F}{K}}_{\text{Steady-State Response}}$$

For zero ICs and a unit step,

$$X(t) = \frac{1}{K}\left[(-1 - \omega_n t)e^{-\omega_n t} + 1\right]$$

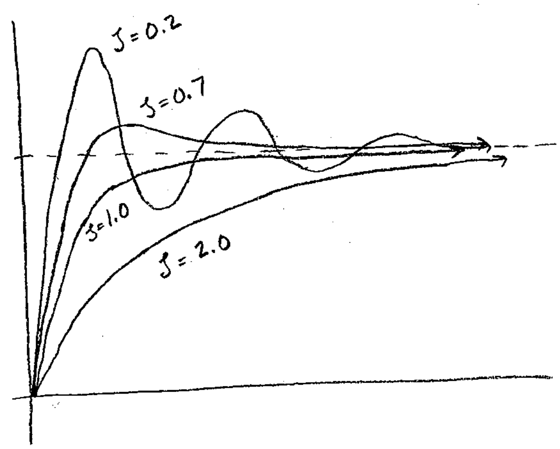
Case 3: Overdamped $\quad c > 2\sqrt{mk}\,,\ \zeta > 1 \quad$ roots: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -r_{1,2}$

$$X(t) = \underbrace{A_1 e^{-r_1 t} + A_2 e^{-r_2 t}}_{\text{Transient Response}} + \underbrace{\frac{F}{K}}_{\text{Steady-State Response}}$$

For zero ICs and a unit step

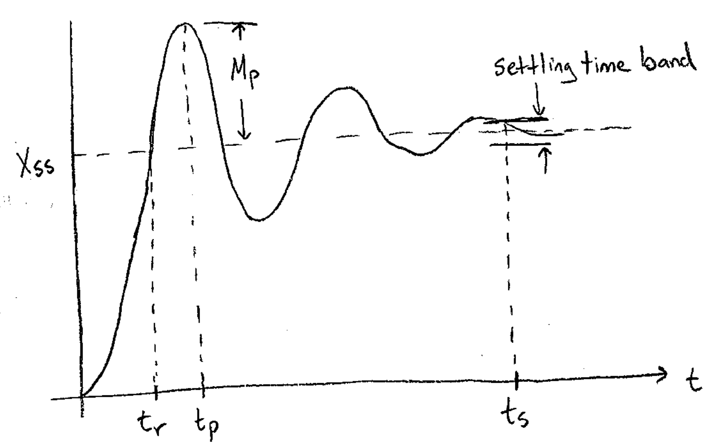
$$X(t) = \frac{1}{K}\left(\frac{r_2}{r_1 - r_2}e^{-r_1 t} - \frac{r_1}{r_1 - r_2}e^{-r_2 t} + 1\right)$$

Response:



- As ζ decreases below 1, we get more oscillations
- As ζ increased above 1, it takes longer to reach S.S.
- $\zeta = 1$ is the fastest to reach S.S. without oscillations (an underdamped system may reach the S.S. value quicker, but it will pass it and oscillate, so it doesn't settle there quicker)

Now let's consider the underdamped case further. There are several quantities we can define



M_p = Maximum Overshoot

t_r = Rise Time

t_p = Peak Time

t_s = Settling Time

Rise Time

- This is the time at which the response first reaches X_{ss}

$$X(t) = X_{ss} = \frac{1}{K} = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + 1\right]$$

This leads to

$$e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) = 0 \Rightarrow \sin(\omega_d t + \phi) = 0 \Rightarrow \omega_d t + \phi = 2\pi \Rightarrow \boxed{t_r = \frac{2\pi - \phi}{\omega_d}}$$

Peak Time

- The time at which the maximum response is reached; when the derivative is zero

$$\frac{dx}{dt} = \frac{1}{K}\left(\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t\right) = 0 \Rightarrow \sin \omega_d t = 0 \Rightarrow \omega_d t = \pi \Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Maximum Overshoot

- The maximum response beyond the steady state value. The response at t_p minus the SS response

$$M_p = X(t_p) - X_{ss} \Rightarrow M_p = \frac{1}{K} e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

- Often we express this as a percentage : $M\%$

$$\boxed{M\% = \frac{X(t_p) - X_{ss}}{X_{ss}} 100 = 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}}}$$

Settling Time

- The time required for the response to decay down to a certain percentage of the steady state.

Can be estimated as:

$$\boxed{t_s = -\frac{\ln(\text{tolerance})}{\zeta\omega_n}}$$

$2\% \Rightarrow$ tolerance $= 0.02$

$10\% \Rightarrow$ tolerance $= 0.10$

etc.

Damping Ratio from Maximum Overshoot

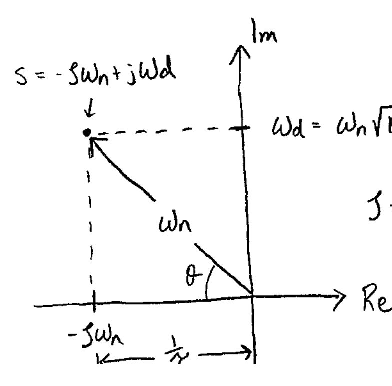
$$M\% = 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$ solve for ζ gives: $$\zeta = \frac{R}{\sqrt{\pi^2 + R^2}}$$ where $R = \ln \frac{100}{M\%}$

Effects of Root Location on Response

Our roots are : $s_{1,2} = -\zeta\omega_n \pm j\omega_d$

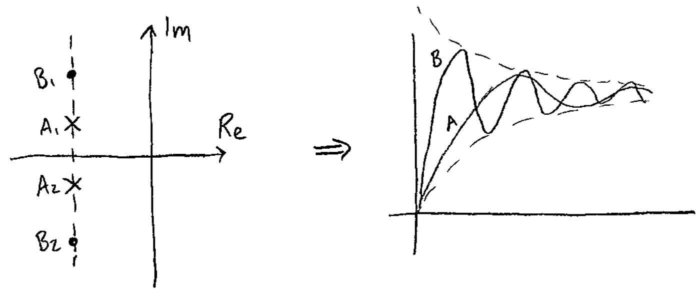
(for the underdamped case)

Graphically: The "s-plane"

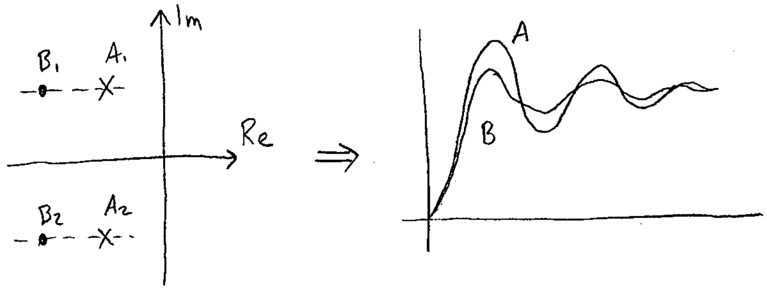


$\zeta = \cos\theta = \cos[\tan^{-1}(\tau\omega_d$

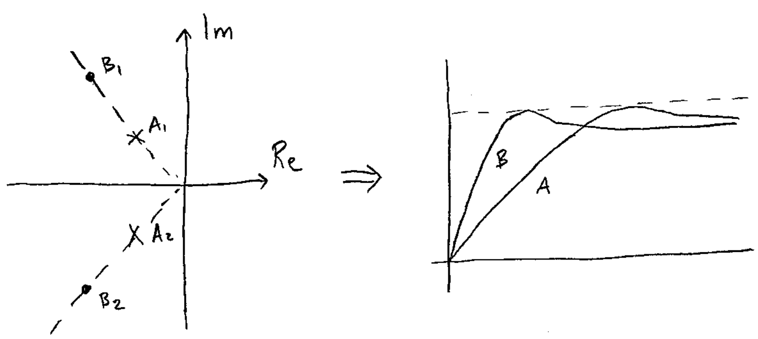
This shows us how much of the response characteristics are contained in the roots including ω_n, ζ, τ, ω_d. Let's compare a few different root locations.



- Real Part of Root: $-\zeta\omega_n = \frac{1}{\tau}$
- So this dictates the time constant.
- Signals have the same time constant, τ, and settling time: $t_s = -\frac{\ln(\text{tolerance})}{\zeta\omega_n}$



- Imaginary Part of Root: ω_d
- So this dictates the oscillation frequency.
- Signals have the same peak time, $t_p = \frac{\pi}{\omega_d}$



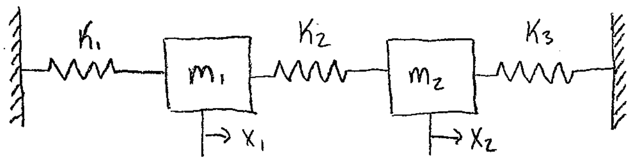
- Angle (θ): ζ
- So this dictates damping ratio
- Signals have the same overshoot, $M\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

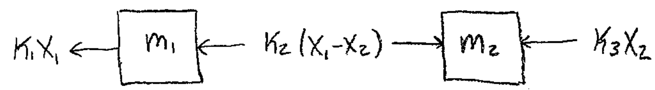
(13.4) MDOF Time Response - Natural Frequencies + Mode Shapes

Previously, we analyzed the time response of SDOF systems. MDOF systems yield systems of coupled EOMs that can result in complicated behavior. MDOF systems have multiple natural frequencies (one per DOF), and mode shapes, which describe the relative motion of each DOF.

Here we will consider a 2DOF unforced mass-spring system and find the natural frequencies and mode shapes.

FBDs (assume $x_1 > x_2$)





Newton's method:

mass 1: $\Sigma F = m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1-x_2)$

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$

mass 2: $\Sigma F = m_2\ddot{x}_2 = k_2(x_1-x_2) - k_3x_2$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = 0$$

We can write the system of equations in matrix form as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\substack{\uparrow \\ \text{mass matrix}}}{\mathbf{M}}\ \underset{\substack{\uparrow \\ \text{acceleration vector}}}{\ddot{\mathbf{x}}} + \underset{\substack{\uparrow \\ \text{stiffness matrix}}}{\mathbf{K}}\ \underset{\substack{\uparrow \\ \text{displacement vector}}}{\mathbf{x}} = \underset{\substack{\uparrow \\ \text{force vector}}}{\mathbf{0}}$$

note: could also assume exponential solns: $x = Ae^{j\omega t}$. Either way, we already know response is purely oscillatory b/c no damping.

Using an approach like the Trial Solution method, try solutions of $x_1 = A_1\sin(\omega t+\phi)$, $x_2 = A_2\sin(\omega t+\phi)$

We can define the solution as

$$\mathbf{x}(t) = \mathbf{a}\sin(\omega t+\phi) \Rightarrow \ddot{\mathbf{x}}(t) = -\omega^2\mathbf{a}\sin(\omega t+\phi) \qquad \text{note: } \mathbf{a} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Substituting gives

$$-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\sin(\omega t+\phi) + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\sin(\omega t+\phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Grouping the $\mathbf{a}\sin(\omega t+\phi)$ terms gives

Divide by $\sin(\omega t+\phi)$

$$\begin{bmatrix} -\omega^2m_1+k_1+k_2 & -k_2 \\ -k_2 & -\omega^2m_2+k_2+k_3 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\sin(\omega t+\phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -\omega^2m_1+k_1+k_2 & -k_2 \\ -k_2 & -\omega^2m_2+k_2+k_3 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The trivial solution corresponds to $A_1 = A_2 = 0$. For a nontrivial solution, the determinant of the coefficient matrix must equal 0. (this is a fundamental result of linear algebra called singularity, which dictates whether a matrix is invertable)

Recall,

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

So, we have

$$\det\begin{bmatrix} -\omega^2 m_1 + K_1 + K_2 & -K_2 \\ -K_2 & -\omega^2 m_2 + K_2 + K_3 \end{bmatrix} = (-\omega^2 m_1 + K_1 + K_2)(-\omega^2 m_2 + K_2 + K_3) - (-K_2)(-K_2) = 0$$

$$\omega^4 m_1 m_2 - \omega^2\left[m_1(K_2+K_3) + m_2(K_1+K_2)\right] + (K_1+K_2)(K_2+K_3) - K_2^2 = 0$$ characteristic equation

This is a quadratic equation in ω^2. To see this, let $\lambda = \omega^2$

$$m_1 m_2 \lambda^2 - \left[m_1(K_2+K_3) + m_2(K_1+K_2)\right]\lambda + (K_1+K_2)(K_2+K_3) - K_2^2 = 0$$

Apply quadratic formula:

$$\lambda_{1,2} = \frac{\left[m_1(K_2+K_3) + m_2(K_1+K_2)\right]}{2m_1m_2} \pm \frac{\sqrt{\left[m_1(K_2+K_3) + m_2(K_1+K_2)\right]^2 - 4m_1m_2\left[(K_1+K_2)(K_2+K_3) - K_2^2\right]}}{2m_1m_2}$$

In terms of ω, we have

$$\omega_1^2 = \lambda_1 \quad , \quad \omega_2^2 = \lambda_2$$

$$\omega_1 = \sqrt{\lambda_1} \quad , \quad \omega_2 = \sqrt{\lambda_2}$$ $\Rightarrow$ ω_1 + ω_2 are called the natural frequencies of the system. There are two; one per DOF.

Substituting back into our assumed solution, we have

$$\mathbf{x}(t) = C_1 \mathbf{a}_1 \sin(\omega_1 t + \phi_1) + C_2 \mathbf{a}_2 \sin(\omega_2 t + \phi_2)$$

Now, we still need to solve for the unknown vectors $\mathbf{a}_1$ + $\mathbf{a}_2$ (each corresponding to one natural frequency). This is done by substituting each natural frequency (ω_1, ω_2) into the following equation and solving for the ratio of $\frac{A_2}{A_1}$:

$$\begin{bmatrix} -\omega^2 m_1 + K_1 + K_2 & -K_2 \\ -K_2 & -\omega^2 m_2 + K_2 + K_3 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: when solving the matrix, you will get 2 equations that give the same ratio for each natural frequency. Although you can't get exact amplitude information (the ICs are required for this), the ratio is very important. Just pick one of the equations and solve for the ratio $\frac{A_2}{A_1}$ for each ω.

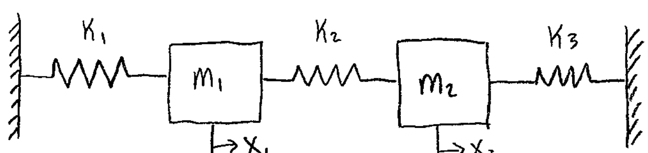
The solutions of the equation will reveal two vectors: $\begin{bmatrix} A_1^1 \\ A_2^1 \end{bmatrix}$ corresponding to ω_1 and $\begin{bmatrix} A_1^2 \\ A_2^2 \end{bmatrix}$ corresponding to ω_2. These vectors are called the mode shapes of the system. Since we don't know the exact amplitudes, we typically normalize the mode shapes by setting one of the amplitudes to 1 and using the ratio $\frac{A_2}{A_1}$ to write the 2nd amplitude. This should be clear in the example to follow.

Consider the solution we just found:

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = C_1 \begin{bmatrix} A_1^1 \\ A_2^1 \end{bmatrix} \sin(\omega_1 t + \phi_1) + C_2 \begin{bmatrix} A_1^2 \\ A_2^2 \end{bmatrix} \sin(\omega_2 t + \phi_2)$$

This shows that each mass in general oscillates at two frequencies: ω_1 + ω_2, the natural frequencies. The relative amount that each mass oscillates at each frequency is dictated by the mode shapes: $\underbrace{\begin{bmatrix} A_1^1 \\ A_2^1 \end{bmatrix}}_{\text{first mode shape}}$ and $\underbrace{\begin{bmatrix} A_1^2 \\ A_2^2 \end{bmatrix}}_{\text{second mode shape}}$. C_1, C_2, ϕ_1 + ϕ_2 are the unknown coefficients solved for using the ICs.

Example:



$m_1 = 2$, $m_2 = 5$, $K_1 = 10$, $K_2 = 40$, $K_3 = 5$

Find natural frequencies + mode shapes

From the derivation above, the assumed solution for this system can be written as:

$$\begin{bmatrix} -\omega^2 m_1 + K_1 + K_2 & -K_2 \\ -K_2 & -\omega^2 m_2 + K_2 + K_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substituting values for m's + K's and letting $\lambda = \omega^2$ gives

$$\begin{bmatrix} 50-2\lambda & -40 \\ -40 & 45-5\lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set the determinant equal to zero to find characteristic equation

$$\det \begin{bmatrix} 50-2\lambda & -40 \\ -40 & 45-5\lambda \end{bmatrix} = (50-2\lambda)(45-5\lambda) - 1600 = 0 \Rightarrow 10\lambda^2 - 340\lambda + 650 = 0$$

Solving for the roots:

$$\lambda_{1,2} = \frac{340 \pm \sqrt{(-340)^2 - 4 \cdot 10 \cdot 650}}{20} \Rightarrow \lambda_1 = 31.97 \ , \ \lambda_2 = 2.03$$

There fore,

$$\boxed{\omega_1 = \sqrt{\lambda_1} = 5.65 \ , \quad \omega_2 = \sqrt{\lambda_2} = 1.43}$$

To find the mode shapes, substitute λ_1 + λ_2 into matrix expression

Using $\lambda_1 = 31.97$:
$$\begin{bmatrix} 50-2(31.97) & -40 \\ -40 & 45-5(31.97) \end{bmatrix} \begin{bmatrix} A_1^1 \\ A_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Choose one equation to solve:

$$[50-2(31.97)]A_1^1 - 40A_2^1 = 0 \Rightarrow A_1^1 = \frac{40}{[50-2(31.97)]} A_2^1 \Rightarrow A_1^1 = -2.87 A_2^1$$

so, $\boxed{a_1 = \begin{bmatrix} 1 \\ -0.35 \end{bmatrix} \quad \text{First mode shape}}$

Using $\lambda_2 = 2.03$:
$$\begin{bmatrix} 50-2(2.03) & -40 \\ -40 & 45-5(2.03) \end{bmatrix} \begin{bmatrix} A_1^2 \\ A_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

choose one equation to solve:

$$[50-2(2.03)]A_1^2 - 40A_2^2 = 0 \Rightarrow A_1^2 = \frac{40}{[50-2(2.03)]} A_2^1 \Rightarrow A_1^2 = 0.87 A_2^2$$

so, $\boxed{a_2 = \begin{bmatrix} 1 \\ 1.15 \end{bmatrix} \quad \text{Second mode shape}}$

Substituting the natural frequencies + mode shapes into the assumed solution gives:

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -0.35 \end{bmatrix} \sin(5.65t + \phi_1) + C_2 \begin{bmatrix} 1 \\ 1.15 \end{bmatrix} \sin(1.43t + \phi_2)$$

Lastly, given ICs, we could solve for C_1, C_2, ϕ_1 + ϕ_2 to obtain the full solution.