Lecture Module - The Laplace Transform

ME3050 - Dynamic Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 2 - Laplace Transforms Method

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- Step 1 - Apply Laplace Transform
- Step 2 - Solve for $X(s)$
- Step 3 - Rearrange to Find Invertable Form
- Step 4 - Invert for Final Answer

Step 1 - Apply Laplace Transform

Example:
Solve the first order differential equation using the Laplace Transforms Method with the initial condition given.

$$4\dot{x} = sin\left(t\right) \qquad with \qquad x\left(t = 0\right) = x_0$$

Apply the Laplace Transform to both sides of the differential equation.

$$4\left(sX\left(s\right) - x_0\right) = \frac{1}{s^2 + 1}$$

Step 2 - Solve for $X(s)$

This step can seem open ended...

$$X(s) = \frac{1}{4s\left(s^2+1\right)} + \frac{x_0}{s}$$

Step 3 - Rearrange to Find Invertable Form

Write $X(s)$ in a form that can be inverted using the table of Laplace transform pairs. This typically involves partial fraction decomposition.

$$\frac{1}{4s\left(s^2+1\right)} = \frac{1/4}{s\left(s^2+1\right)} = \frac{a}{s} + \frac{bs+c}{s^2+1}$$

Mulitply through by the denominator $4s\left(s^2+1\right)$:

$$1 = 4as\left(s^2+1\right) + 4s\left(bs+c\right) = 4\left(a+b\right)s^2 + 4cs + 4a$$

Solve for the coefficients by *equating coefficients*.

$$\left(a+b\right) = 0 \quad c = 0 \quad a = \frac{1}{4} \implies a = \frac{1}{4} \quad b = -\frac{1}{4} \quad c = 0$$

Step 4 - Invert for Final Answer

Substitute the coefficients into $X(s)$,

$$X(s) = \frac{x_0}{s} + \frac{1}{4s} - \frac{s}{4\left(s^2+1\right)}$$

and use the inverse transform to solve for $x(t)$. Use the Table.

$$\mathcal{L}^{-1}\left(X\left(s\right)\right) = x(t) =$$

$$= x_0 + \frac{1}{4} - \frac{1}{4}cos\left(t\right) = x_0 + \frac{1}{4}\left(1 - cos\left(t\right)\right)$$

This method works for complex problems but it can get messy...

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$

17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			