Module 10 - The Laplace Transform

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 1 - Definition of the Laplace Transform

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- An Integral Transform
- Laplace Transform of A Derivative
- Properties of an Integral
- Table of Transform Pairs

An Integral Transform

The Laplace Transform is an Integral Transform
Given a function $x(t)$ in the time domain where $t \geq 0$,
the Laplace Transform is defined as follows:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}dt$$

And its inverse is similarly defined as:

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

The Laplace Domain variable s is a complex number:

$$s = \sigma + j\omega$$

Laplace Transform of A Derivative

It is useful to find the laplace transform of the derivative of a function:

$$\mathcal{L}\{\frac{d}{dt}(x(t))\} = \mathcal{L}\{\dot{x}(t)\} = s\mathcal{L}\{x(t)\} - x(t=0)$$

$$= sX(s) - x(t=0)$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$$

$$\mathcal{L}\{\frac{d^2}{dt^2}(x(t))\} = \mathcal{L}\{\ddot{x}(t)\} = s^2\mathcal{L}\{x(t)\} - sx(t=0) - \dot{x}(t=0)$$

$$= s^2X(s) - sx(t=0) - \dot{x}(t=0)$$

$$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx_0 - \dot{x}_0$$

Properties of an Integral

Also, remember that the transform inherits the properties of an integral.

$$\int [x(t) + y(t)]\, dt = \int x(t)dt + \int y(t)dt$$

$$\int Kx(t)dt = K \int x(t)dt \quad (K \;\; is \;\; constant)$$

Therefore these properties can be used with the Laplace transform.

Table of Transform Pairs

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$

Table of Transform Pairs

17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^nf(t),\quad n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Why use or learn this method?