Frequency Response - Lecture 1

ME3050 - Dynamics Modeling and Controls

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Frequency Response of First Order Systems

Lecture 1- Frequency Response of First Order Systems

- Introduction to Chapter 9
- Review Complex Numbers
- Frequency Response of First Order Systems
- The Bode Plot

Harmonic Input Function

The term **frequency response** is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to *harmonic* input such as sines and cosines. The input (forcing) function is written below.

$$f(t) = A sin(\omega t)$$

Amplitude of the Input, A (N)

Frequency of Input, ω $(\frac{rad}{s})$

Why Study Frequency Response?

Why do we care about the way a system responds to harmonic excitation? Why is **frequency analysis** important?

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What causes **harmonic** (or sinusoidal) excitation in the real world?

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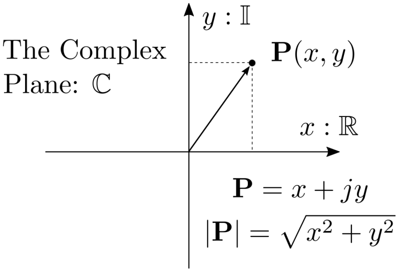
Frequency Response and Transfer Function

A linear, time-invariant (LTI) system has a **transfer function** $T(s)$ that describes the **input-output** relationship of the system. Under sinusoidal excitation (input) with frequency ω if the system is stable the transient affects in the response (output) will eventually disappear leaving the **steady state sinusoidal response** of the same frequency as the input but with a phase shift w.r.t. the input.

The Complex Plane

In an underdamped system the roots of the characteristic polynomial are complex. Before we proceed we need to review some rules of arithmetic and complex numbers.



Cartesian Representation:

$\mathbf{P} = x + jy$

Polar Representation:

$\mathbf{P} = |\mathbf{P}|\angle\theta$

Exponential Representation:

$\mathbf{P} = |\mathbf{P}|e^{j\theta} = |\mathbf{P}|\left(cos\theta + jsin\theta\right)$

Complex Number Algebra

Consider two points $\mathbf{P_1}$ and $\mathbf{P_2}$ on the complex plane.

$\mathbf{P_1} = x_1 + jy_1$ and $\mathbf{P_2} = x_2 + jy_2$

Addition: $\mathbf{P_1} + \mathbf{P_2} = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $\mathbf{P_1}\mathbf{P_2} = |\mathbf{P_1}\mathbf{P_2}|\angle(\theta_1 + \theta_2)$

Division: $\frac{\mathbf{P_1}}{\mathbf{P_2}} = (x_1 + x_2) + j(y_1 + y_2)$

Frequency Response of First Order Systems

Consider our $1^{\underline{st}}$ order mass damper system.

$m\dot{v} + cv = f(t)$ with a **time constant** $\tau = \frac{m}{c}$

The system is commonly re-written as shown below.

$$m\dot{v} + cv = f(t) \;\rightarrow\; \tau\dot{y} + y = f(t)$$

Obtain the Transfer Function

$\tau \dot{y} + y = f(t)$

Take the Laplace transform of the ODE.

$\mathcal{L}\{\tau \dot{y} + y\} = \mathcal{L}\{f(t)\}$

$\tau\left(sY(s) + y_0\right) + Y(s) = F(s)$ The initial conditions are zero.

$T(s) = \frac{Y(s)}{F(s)} = \frac{1}{\tau s+1}$ First Order Transfer Function

This considers a *generalized* input function $f(t)$ and zero ICs.

Sinusoidal Input Function

Our model is now excited by a sinusoidal input (forcing) function.

$$\tau \dot{y} + y = f(t) = Asin(\omega t)$$

Take the Laplace Transform of the equation.

$$\tau sY(s) + Y(s) = \frac{A\omega}{s^2+\omega^2}$$

Solve for $Y(s)$ and expand.

$$Y(s) = \frac{A\omega}{(s^2+\omega^2)(\tau s+1)} = \frac{C_1}{\tau s+1} + \frac{C_2 s}{(s^2+\omega^2)} + \frac{C_3\omega}{(s^2+\omega^2)}$$

Now solve for the coefficients.

$$C_1 = \frac{A\omega\tau^2}{1+\omega^2\tau^2} \quad , \quad C_2 = \frac{-A\omega\tau}{1+\omega^2\tau^2} \quad , \quad C_3 = \frac{A}{1+\omega^2\tau^2}$$

Substituting and take the inverse Laplace transform.

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau}sin\omega t\right)$$

Steady State Time Response

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau} sin\omega t\right)$$

After some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y(t) = \frac{A}{1+\omega^2\tau^2}\left(sin\omega t - \omega\tau cos\omega t\right)$$

This is re-written as a single sinusoidal term with a phase shift.

Steady State Frequency Response of First Order System

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Amplitude Ratio

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Notice that the system responds at the same frequency as the input but with a different amplitude and a phase shift. The ratio of the response amplitude to the input amplitude is called the **amplitude ratio, M**.

$$M = \frac{\frac{A}{\sqrt{1+\omega^2\tau^2}}}{A} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Fortunately we can find the **amplitude ratio** and **phase shift** directly from the transfer function. Recall the transfer function we derived.

$$T(s) = \frac{1}{\tau s+1} \qquad \text{let } s = j\omega \qquad \Longrightarrow \qquad T(j\omega) = \frac{1}{\tau j\omega+1}$$

$$|T(j\omega)| = \frac{|1|}{|\tau j\omega+1|} = \frac{1}{\sqrt{(\tau\omega)^2+1^2}} = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

Look familiar?

Amplitude Ratio and Phase Angle

$|T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = M$

$\angle T(j\omega) = \angle 1 - \angle(1 + j\omega\tau) = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{\omega\tau}{1}\right) = -tan^{-1}\left(\frac{\omega\tau}{1}\right) = \phi$

Substitute $s = j\omega$ into the transfer function and solve for the magnitude and phase angle of this complex number which represent the magnitude ratio and phase shift.

Therefore the steady state response is written as follows.

$y_{ss}(t) = A|T(j\omega)|sin(\omega t + \angle T(j\omega)) = MAsin(\omega t + \phi)$

Wasn't that fun? Can you believe we used to do that on the board?!?!

References

- System Dynamics, Palm III, Third Edition - Chapter 8 - System Response in the Time Domain