Frequency Response - Lecture 2

ME3050 - Dynamics Modeling and Controls

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The Bode Diagram

Lecture 2 - The Bode Diagram

- Review Frequency Response
- Magnitude Ratio in Decibels
- The Bode Diagram
- Graph of Frequency Response in MATLAB

Harmonic Input Function

The term **frequency response** is used to describe a system's response to a periodic input. Frequency response analysis focuses on a system's response to *harmonic* input such as sines and cosines. The input (forcing) function is written below.

$$f(t) = A sin(\omega t)$$

Amplitude of the Input,	A	(N)
Frequency of Input,	ω	$(\frac{rad}{s})$

First Order Frequency Response

The steady state response we derived is shown. Remember, after some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y_{ss}(t) = A|T(j\omega)|sin(\omega t + \angle T(j\omega)) = MAsin(\omega t + \phi)$$

The magintude ratio and phase shift can be found from $T(j\omega)$.

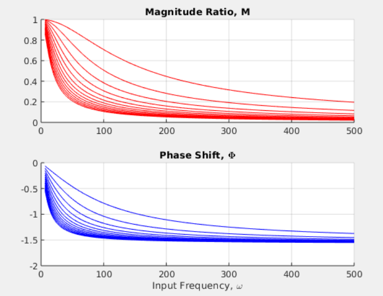
$$M(\omega) = |T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\phi(\omega) = \angle T(j\omega) = -tan^{-1}(\omega\tau)$$

Graph of Frequency Response

The amplitude of the response is determined by the input frequency.

Dependence on Input Frequency



You can see that the magnitude ratio decreases as the input frequency increases. The indivdual curves represent systems different time constants.

Magnitude Ratio on a Logarithmic Scale

These relationships are more useful shown on a logarithmic scale. Also, we make use of the properties of logorithms in our analysis.

Complex Number Algebra

Consider two points $\mathbf{P_1}$ and $\mathbf{P_2}$ on the complex plane.

$\mathbf{P_1} = x_1 + jy_1$ and $\mathbf{P_2} = x_2 + jy_2$

Addition: $$\mathbf{P_1} + \mathbf{P_2} = (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication: $$\mathbf{P_1}\mathbf{P_2} = |\mathbf{P_1}\mathbf{P_2}|\angle(\theta_1 + \theta_2)$$

Division: $$\frac{\mathbf{P_1}}{\mathbf{P_2}} = (x_1 + x_2) + j(y_1 + y_2)$$

Frequency Response of First Order Systems

Consider our $1^{\underline{st}}$ order mass damper system.

$m\dot{v} + cv = f(t)$ with a **time constant** $\tau = \frac{m}{c}$

The system is commonly re-written as shown below.

$$m\dot{v} + cv = f(t) \rightarrow \tau\dot{y} + y = f(t)$$

Obtain the Transfer Function

$$\tau \dot{y} + y = f(t)$$

Take the Laplace transform of the ODE.

$$\mathcal{L}\{\tau \dot{y} + y\} = \mathcal{L}\{f(t)\}$$

$\tau\left(sY(s) + y_0\right)) + Y(s) = F(s)$ The initial conditions are zero.

$\boxed{T(s) = \frac{Y(s)}{F(s)} = \frac{1}{\tau s+1}}$ **First Order Transfer Function**

This considers a *generalized* input function $f(t)$ and zero ICs.

Sinusoidal Input Function

Our model is now excited by a sinusoidal input (forcing) function.

$$\tau \dot{y} + y = f(t) = Asin(\omega t)$$

Take the Laplace transform. Then, solve for $Y(s)$ and expand.

$$\tau s Y(s) + Y(s) = \frac{A\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{A\omega}{(s^2+\omega^2)(\tau s+1)} = \frac{C_1}{\tau s+1} + \frac{C_2 s}{(s^2+\omega^2)} + \frac{C_3\omega}{(s^2+\omega^2)}$$

Now solve for the coefficients.

$$C_1 = \frac{A\omega\tau^2}{1+\omega^2\tau^2} \quad , \quad C_2 = \frac{-A\omega\tau}{1+\omega^2\tau^2} \quad , \quad C_3 = \frac{A}{1+\omega^2\tau^2}$$

Substituting and take the inverse Laplace transform.

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau} sin\omega t\right)$$

Steady State Time Response

$$y(t) = \frac{A\omega\tau}{1+\omega^2\tau^2}\left(e^{\frac{-t}{\tau}} - cos\omega t + \frac{1}{\omega\tau} sin\omega t\right)$$

After some amount of time passes, the transient term will disappear leaving just the sinusoidal terms.

$$y(t) = \frac{A}{1+\omega^2\tau^2}\left(sin\omega t - \omega\tau cos\omega t\right)$$

This is re-written as a single sine term with a phase shift.

Steady State Frequency Response of First Order System

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Amplitude Ratio

$$y(t) = \frac{A}{\sqrt{1+\omega^2\tau^2}} sin\left(\omega t + \phi\right) \qquad , \quad \phi = -tan^{-1}\omega\tau$$

Notice that the system responds at the same frequency as the input but with a different amplitude and a phase shift. The ratio of the response amplitude to the input amplitude is called the **amplitude ratio, M**.

$$M = \frac{\frac{A}{\sqrt{1+\omega^2\tau^2}}}{A} = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Fortunately we can find the **amplitude ratio** and **phase shift** directly from the transfer function. Recall the transfer function we derived.

$$T(s) = \frac{1}{\tau s+1} \qquad \text{let } s = j\omega \qquad \Longrightarrow \qquad T(j\omega) = \frac{1}{\tau j\omega+1}$$

$$|T(j\omega)| = \frac{|1|}{|\tau j\omega+1|} = \frac{1}{\sqrt{(\tau\omega)^2+1^2}} = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

Look familiar?

Amplitude Ratio and Phase Angle

$|T(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = M(\omega)$

$\angle T\left(j\omega\right) = \angle 1 - \angle\left(1 + j\omega\tau\right) = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{\omega\tau}{1}\right) =$
$-tan^{-1}\left(\omega\tau\right) = \phi(\omega)$

Substitute $s = j\omega$ into the transfer function and solve for the magnitude and phase angle of this complex number which represent the magnitude ratio and phase shift.

Therefore the steady state response is written as follows.

$y_{ss}\left(t\right) = A|T\left(j\omega\right)|sin\left(\omega t + \angle T\left(j\omega\right)\right) = MAsin\left(\omega t + \phi\right)$

Wasn't that fun? Can you believe we used to do that on the board?!?!

Graph of Frequency Response

What determines the amplitude of the system response?

MATLAB code

```
% ME3050 - Spring 2020 Tennessee Technological Univ.
clear variables;clc;close all

% define the system parameters
m=20;c=25;
tau=m/c;

% define the amplitude input frequency and
A=1;omega=1/2;

% calculate the magnitude ratio
M=1/sqrt(1+omega^2*tau^2);
phi=-tan(omega*tau);
```

MATLAB code

```
%consider a range of time values
dt=0.01;tstop=50;
time=0:dt:tstop;

%calculate the input and response curves
fin=A*sin(omega*time);
yout=M*A*sin(omega*time+phi);

% show the results in a figure
figure(1);hold on
plot(time,fin,'r')
plot(time,yout,'b')
```

References

- System Dynamics, Palm III, Third Edition - Chapter 9 - System Response in the Frequency Domain