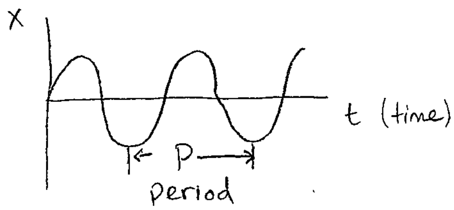
(1.2) Units

	SI	US (FPS - foot-pound-second)
Time	second (s)	second (sec)
Length	meter (m)	foot (ft)
Force	newton (N)	pound (lb)
Mass	Kilogram (kg)	slug
Energy	joule (J)	foot-pound (ft-lb)
Power	watt (W)	ft-lb/sec
Temp.	°C, °K	°F, °R

Note: $F = ma$
$W = mg$

$g = 9.81\ m/s^2$ or $32.2\ ft/s^2$

Oscillation Units



Frequency: f in cycles/second or Hz (1 Hz = 1 cps)
ω in radian/sec (angular frequency)
Conversion: $2\pi f = \omega$

Period: $P = \frac{1}{f} = \frac{2\pi}{\omega}$ (seconds)

RPM: $1\ RPM = \frac{2\pi}{60}$ rad/sec

(2.1) Differential Equations Review

System dynamics will study Ordinary Differential Equations (ODEs) in which the independent variable is time.

$$\dot{x} = \frac{dx}{dt} \qquad \ddot{x} = \frac{d^2x}{dt^2} \qquad \text{overdot notation}$$

Standard form: $2\ddot{x} + 5\dot{x} + 4x = \underbrace{5\sin(t) + 3}_{\text{Input or Forcing Function}}$

response (dependent variable)

Homogeneous vs. Nonhomogeneous:

Homogeneous if forcing function = 0 ex: $5\dot{x} + 2x = 0$

Otherwise ⇒ Nonhomogeneous

Order

The order of a DE is the order of the highest derivative

1st order: $5\dot{x} + 7x = 3\sin(t)$ 2nd order: $9\ddot{x} + 2x = 0$

Linearity

An n^{th} order ODE is linear if it is of the form:

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_2(t)\ddot{x} + a_1(t)\dot{x} + a_0(t)x = f(t)$$

Linearity (con't)

2 conditions: 1) the dependent variable (x) + its derivatives are of the first degree
2) each coefficient depends at most on the independent variable (t)

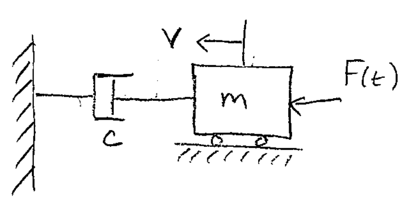
Solution Methods for ODEs

The general solution of a linear ODE is the sum of the complementary or homogeneous solution (soln of homogeneous eq.) and the particular soln (soln. of non-homogeneous eq.).

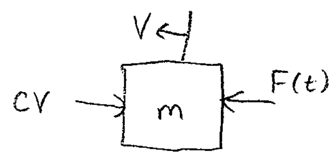
Note, the general soln. is a family of soln. curves because the complementary soln. is any linear combination of solutions of the homogeneous eq. (superposition)

First Order Systems

Mass Damper (with velocity as dependent variable)

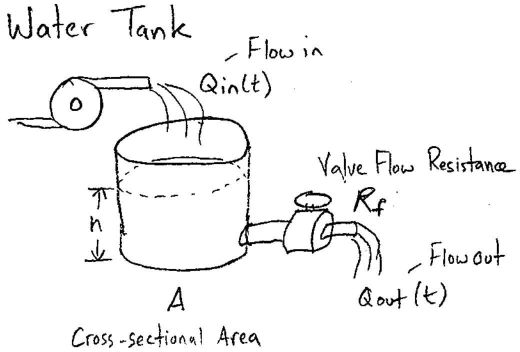


FBD:



$$\Sigma F = m\dot{V}$$
$$m\dot{V} = F(t) - cV$$
$$m\dot{V} + cV = F(t)$$

Water Tank



Slightly more complex model, but governing equation is of same form:

$$\frac{RA}{g}\frac{dh}{dt} + h = \frac{R}{\rho g} Q_{in}(t)$$

So, our general form is: $\dot{x} + ax = f(t)$ 1st order, linear ODE

Case 1) What if $f(t) = 0$? homogeneous solution

$$\dot{x} + ax = 0$$

separation of variables: $\frac{dx}{x} = -a\,dt$

$$\frac{dx}{dt} = -ax$$

integrate both sides: $\ln(x) = -at + c$

Solve for x: $e^{\ln(x)} = e^{(-at+c)}$

$$x = e^{-at} \cdot e^{c}$$
$$x = Ae^{-at}$$

define time constant: $\tau = \frac{1}{a}$

$$x = Ae^{-t/\tau}$$

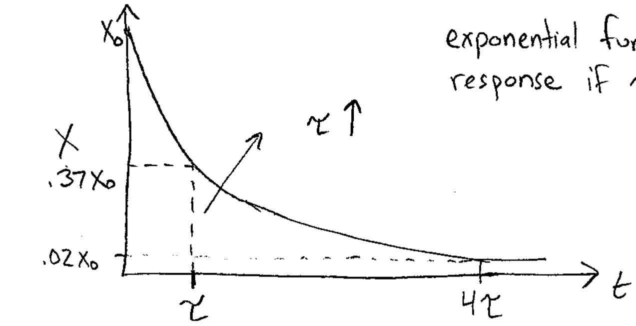
Define initial condition: @ $t=0$ $x = x_o$

$$x_o = Ae^{0} \Rightarrow x_o = A$$

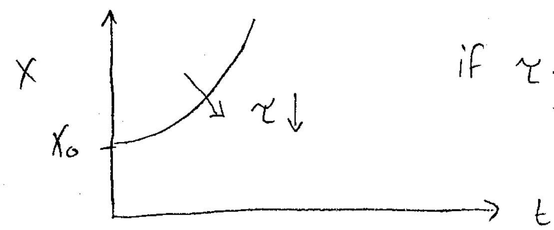
so, $x = x_o e^{-t/\tau}$

Left off here

Response: (see sec. 8.1)



exponential function response if $\underline{\tau > 0}$ $\Rightarrow$ stable system



if $\tau < 0$ $\Rightarrow$ unstable system

Case 2) What if $f(t)$ = constant (step response)

In previous example, we found homogeneous solution: $X_h = Ae^{-t/\tau}$

Now, we find particular soln, X_p:

$\dot{X}_p + aX_p = b$

b/c b = constant, X_p = constant $\Rightarrow$ $\dot{X}_p = 0$

so, $aX_p = b$

$X_p = \frac{b}{a}$

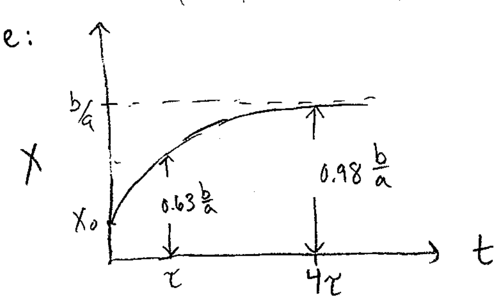
Total soln: $X = X_h + X_p \Rightarrow X = Ae^{-t/\tau} + \frac{b}{a}$

Initial conds: $@\, t=0,\ X = X_0$

$$X_0 = A + \frac{b}{a} \Rightarrow A = X_0 - \frac{b}{a}$$

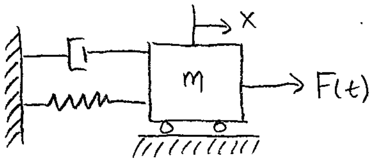
$$X = \underbrace{\left(X_0 - \frac{b}{a}\right)e^{-t/\tau}}_{\substack{\text{transient} \\ \text{(dissapears w/ time)}}} + \underbrace{\frac{b}{a}}_{\substack{\text{s.s.} \\ \text{(remains)}}} = \underbrace{X_0 e^{-t/\tau}}_{\substack{\text{free} \\ \text{(depends on ICs)}}} + \underbrace{\frac{b}{a}\left(1 - e^{-t/\tau}\right)}_{\substack{\text{forced} \\ \text{(depends on forcing func.)}}}$$

Response:



2nd Order Systems

Spring Mass Damper



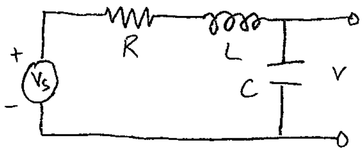
FBD:

$\Sigma F = m\ddot{x}$

$m\ddot{x} = F(t) - c\dot{x} - kx$

$m\ddot{x} + c\dot{x} + kx = F(t)$

Electrical RLC Circuit



Model: $LC\dfrac{d^2v}{dt^2} + RC\dfrac{dv}{dt} + v = V_s$

So, our general form is $a\ddot{x} + b\dot{x} + cx = f(t)$ 2nd order, linear ODE

Case 1) $f(t) = 0$ homogeneous soln.

$a\ddot{x} + b\dot{x} + cx = 0$ ☆ Use trial soln. method where we assume exponential soln.

$x = De^{st}$ $\dot{x} = sDe^{st}$ $\ddot{x} = s^2De^{st}$

- ignore trivial soln: $D=0$
- $e^{st} \neq 0$ for $t=$real

Substitute:

$as^2De^{st} + bsDe^{st} + cDe^{st} = 0 \Rightarrow (as^2 + bs + c)\cancel{De^{st}} = 0$

so,

$as^2 + bs + c = 0 \Rightarrow$ "characteristic equation"

Solve for s:

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \text{"roots"}$$

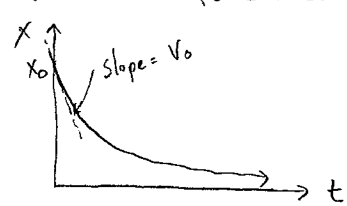
Case 1: $b^2 > 4ac \Rightarrow$ 2 distinct real roots $\Rightarrow s_1, s_2$ (see sec. 8.2)

$x = Ae^{s_1t} + Be^{s_2t}$

requires 2 ICs: $x_0, \dot{x}_0\ (v_0)$

☆ if $a,b,c < 0 \Rightarrow$ stable
however in systems we will model, this is not the case.

if $a,b,c > 0$ ☆
then $s_1, s_2 < 0$
stable response



if a, b, or $c < 0$
then s_1 or $s_2 > 0$
unstable response

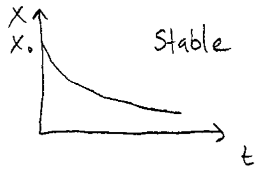
The response is typically dominated by one dominant root that leads to a dominant time constant. This is typically the term that decays slower, however, the values of A + B also have an effect.

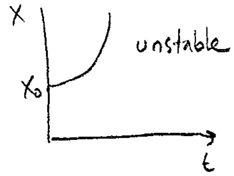
$\downarrow$
has largest τ

Case 2: $b^2 = 4ac \Rightarrow$ repeated real roots $\Rightarrow s_1, s_1$

$x = Ae^{st} + Bte^{st}$
$= (A+Bt)e^{st}$

Same stability criterion





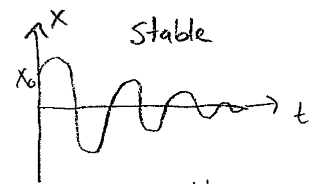
2 ICs: $x_0, \dot{x}_0$

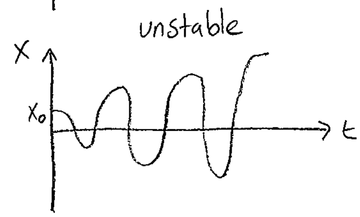
Case 3: $b^2 < 4ac \Rightarrow$ 2 complex conjugate roots $s_{1,2} = \sigma \pm j\omega$

$x = Ae^{\sigma t}\cos\omega t + Be^{\sigma t}\sin\omega t$
$= (A\cos\omega t + B\sin\omega t)e^{\sigma t}$

Same stability criterion

2 ICs: $x_0, \dot{x}_0$





(2.58 - 2.5.11) <u>Stability + Equilibrium</u>

Routh-Hurwitz condition: a, b, c all have <u>same sign</u> for stability
↳ specific for systems with characteristic eq: $ms^2 + cs + k = 0$
i.e. 2^{nd} order linear systems

Stability: free response approaches 0

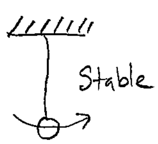
Unstable: free response approaches ∞ as $t \to \infty$

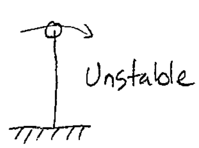
Neutral Stability: free response does not approach ∞ but also not 0.

Stability Test: linear system stable if all roots have negative real parts, neutrally stable if any root has real part $= 0$, unstable if any root has positive real part

Equilibrium: a state of no change
↳ Stable: system returns to original eq. pos. regardless of ICs
Unstable: if perturbed, system will not return to eq. pos.

Pendulum:





Examples

Ex 1 $\dot{x} + 3x = 5 \qquad x(0) = 2$

Trial solution method: assume $x = Ae^{st} \Rightarrow \dot{x} = sAe^{st}$

① Homogeneous soln:

$sAe^{st} + 3Ae^{st} = 0 \Rightarrow (s+3)Ae^{st} \Rightarrow s = -3 \Rightarrow x_h = Ae^{-3t}$

② Particular soln:

$\dot{x} + 3x = 5$ b/c forcing func. = const. try a constant soln $x = C$

$3C = 5 \quad C = \frac{5}{3} \Rightarrow x_p = \frac{5}{3}$

③ Total soln:

$x_t = x_h + x_p \Rightarrow x = Ae^{-3t} + \frac{5}{3}$

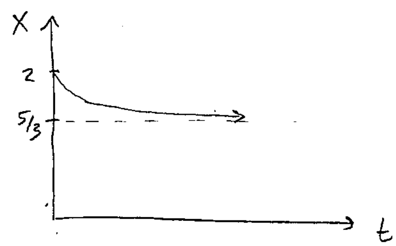
④ Apply IC:

$x(0) = A + \frac{5}{3} = 2 \Rightarrow A = \frac{1}{3}$

So:

$$\boxed{x(t) = \frac{1}{3}e^{-3t} + \frac{5}{3}}$$

Response:



Ex 2 $\ddot{x} + 5\dot{x} + 6x = 0 \qquad x(0) = 3,\ \dot{x}(0) = -8$

① Homogeneous soln:

Assume $x = Ae^{st}$

$s^2Ae^{st} + 5sAe^{st} + 6Ae^{st} = 0 \Rightarrow (s^2 + 5s + 6)Ae^{st} = 0$

Characteristic Eq: $(s^2 + 5s + 6) = 0$

Factor: $(s+2)(s+3) = 0 \Rightarrow$ roots $s_1 = -2,\ s_2 = -3$ (2 distinct real roots, $b^2 > 4ac$)

$x_h = Ae^{-2t} + Be^{-3t}$

② Particular soln:

none $x_p = 0$

③ Total soln:

$x_t = x_h + x_p = Ae^{-2t} + Be^{-3t}$

④ Apply ICs:

$x(0) = A + B = 3$

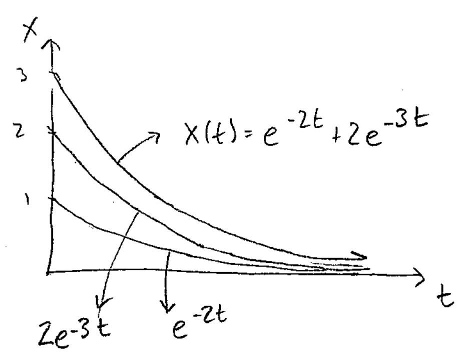
$\dot{x}(0) = -2A - 3B = -8$

$\Rightarrow A = 1,\ B = 2$

So:

$x(t) = e^{-2t} + 2e^{-3t}$

Response:



Examples

Ex 3 $\ddot{x} + 6\dot{x} + 34x = 68 \quad x(0) = 5 \quad \dot{x}(0) = 1$

① Homogeneous soln:

$\ddot{x} + 6\dot{x} + 34x = 0$

Assume $x = Ae^{st}$

$s^2Ae^{st} + 6sAe^{st} + 34Ae^{st} = 0$

Char. Eq: $(s^2 + 6s + 34) = 0$

Factor: $(s^2 + 6s + 9) + 25 = 0$

$(s+3)^2 = -25$

$s+3 = \pm j5$

$s = -3 \pm j5$ (2 complex conjugate roots, $b^2 < 4ac$)

$x_h = Ae^{-3t-j5t} + Be^{-3t+j5t}$

★ Euler's Formula:

$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{-j\theta} = \cos\theta - j\sin\theta$

$Ae^{-3t-j5t} = Ae^{-3t}e^{-j5t}$

$= Ae^{-3t}(\cos 5t - j\sin 5t)$

$Be^{-3t+j5t} = Be^{-3t}(\cos 5t + j\sin 5t)$

$x_h = e^{-3t}\left[(A+B)\cos 5t + j(B-A)\sin 5t\right]$

We can define new constants:

$c_1 = A+B, \quad c_2 = j(B-A)$

$x_h = e^{-3t}(c_1\cos 5t + c_2\sin 5t)$

② Particular soln:

$\ddot{x} + 6\dot{x} + 34x = 68$

$x_p = c$

$34c = 68 \Rightarrow c = 2 \Rightarrow x_p = 2$

③ Total soln:

$x_t = x_h + x_p = c_1e^{-3t}\cos 5t + c_2e^{-3t}\sin 5t + 2$

④ Apply ICs:

$x(0) = c_1 + 2 = 5 \Rightarrow \underline{c_1 = 3}$

$\dot{x}(t) = \left(-3c_1e^{-3t}\cos 5t - 5c_1e^{-3t}\sin 5t\right) + \left(-3c_2e^{-3t}\sin 5t + 5c_2e^{-3t}\cos 5t\right)$

$\dot{x}(0) = -3c_1 + 5c_2 = 1$

$-3(3) + 5c_2 = 1 \Rightarrow \underline{c_2 = 2}$

So:

$x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t + 2$

$= e^{-3t}(3\cos 5t + 2\sin 5t) + 2$

Note: $(3\cos 5t + 2\sin 5t) = \sqrt{3^2+2^2}\,\sin(5t + \underset{\uparrow}{\phi})$

phase shift

Response: