Module 12 - Second Order Time Response

ME3050 - Dynamics Modeling and Controls

Mechanical Engineering

Tennessee Technological University

Topic 4 - Specification of The Step Response

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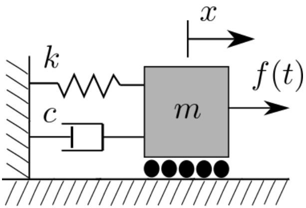
- The Unit Step Response
- Rise, Peak, Time and Settling Time
- Maximum Overshoot and The Damping Ratio

The Mass Spring Damper

Now, consider the mass-spring system with damping present subject to **step** input. This models instantly turning on the input force $f(t)$.



Heavyside's Step Function

$$f(t) = \begin{cases} 0 & t < 0 \\ F & t \geq 0 \end{cases}$$

The EOM is:

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{with} \quad x(t=0) = x_0 \quad \text{and} \quad v(t=0) = v_0$$

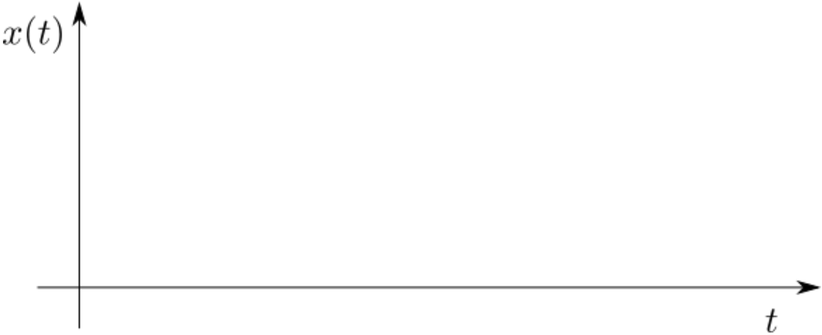
Unit Step Response

The **unit step response** is a special case of the *forced response* in which $f(t)$ is the step function of unit magnitude (F=1).

Overdamped	$x(t) = \frac{1}{K}(\frac{s_2}{s_1-s_2}e^{-s_1 t} - \frac{s_1}{s_1-s_2}e^{-s_2 t} + 1)$
Critically Damped	$x(t) = \frac{1}{K}[(-1-\omega_n t)e^{-\omega_n t} + 1]$
Underdamped	$x(t) = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$ $\phi = tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$

Description and Specification of System Response

We are going to derive several quatities that describes the response of an underdamped system.



Rise Time

The **rise time** is the time at which the response first equals the steady state value.

$$x(t) = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1\right]$$

Set the *transient term* to zero and solve for t.

$$e^{-\zeta\omega_n t} sin(\omega_d t + \phi) = 0 \implies sin(\omega_d t + \phi) = 0 \implies \omega_d t + \phi = 2\pi$$

$$t_{rise} = t_r = \frac{2\pi - \phi}{\omega_d}$$

Peak Time

The **peak time** is the time at which the response equals the maximum value. Find the derivative of the response equation and set it equal to zero.

$\dot{x}(t) =$

$$\left(\frac{1}{K}\frac{1}{\sqrt{1-\zeta^2}}\right)\left[e^{-\zeta\omega_n t}(\omega_d cos(\omega_d t+\phi)) + sin(\omega_d t+\phi)(-\zeta\omega_n e^{-\zeta\omega_n t})\right]$$

$$sin(\omega_d)t = 0 \implies \omega_d t = \pi \implies \boldsymbol{t_{peak} = t_p = \frac{\pi}{\omega_d}}$$

Settling Time

The **settling time** is the time at which the response decays to a certain percentage of the steady state value.

It can be esitmated as:

$$t_{settling} = t_s = -\frac{ln(tolerance)}{\zeta\omega_n}$$

$2\% \implies tolerance = 0.02$

$5\% \implies tolerance = 0.05$

Maximum Overshoot

The **maximum overshoot** is the response beyond the steady state value.

$$M_p = x(t_p) - x_{ss} \implies M_p = \frac{1}{K} e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

This is often expressed as a percentage.

$$M_{\%} = \frac{x(t_p) - x_{ss}}{x_{ss}} 100 = 100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Damping Ratio from Maximum Overshoot

The *damping ratio* can be determined from the maximum overshoot!

$$M_{\%} = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Solve for ζ.

$$\zeta = \frac{R}{\sqrt{\pi^2+R^2}} \quad \text{with} \quad R = ln\left(\frac{100}{M\%}\right)$$