(Ch. 7) Fluid + Thermal Systems

Like electrical systems, many fluid + thermal systems can be modeled with similar differential equations used for mechanical systems. Models of fluid + thermal systems are often more complicated than most mechanical + electrical systems, therefore, we will only consider a few basic examples here.

(7.1) Fluid System Background

Conservation of Mass (Main modeling principle)

$$\dot{m} = q_{mi} - q_{mo}$$

$\dot{m}\ \left(\frac{dm}{dt}\right)$ = time rate of change of mass in system (Kg/s)

q_{mi} = mass flow rate in (Kg/s)

q_{mo} = mass flow rate out (Kg/s)

Note: for incompressible fluids, conservation of mass is equivalent to conservation of volume.

Mass / Volume Flow Rate

$$q_m = \rho q_v$$

q_m = mass flow rate (Kg/s)

ρ = density (Kg/m³)

q_v = volume flow rate (m³/s)

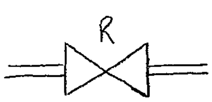
Pressure

p = pressure (N/m² or Pa)

P_a = atmospheric pressure (1.0133×10^5 Pa @ sea level)

$\rho g h$ = hydrostatic pressure, caused by weight of fluid

Fluid Resistance



- Fluid meets resistance when flowing through pipes, valves, orifices, etc.
- Mass flow rate, q_m, through a resistance is related to pressure diff, $\hat{p}$, across it.
- While typically nonlinear, the resistance can be approximated around a reference state to yield a linear relation:

$$R q_m = \hat{p} \qquad \text{where } \hat{p} = \Delta p \text{ across resistance}$$

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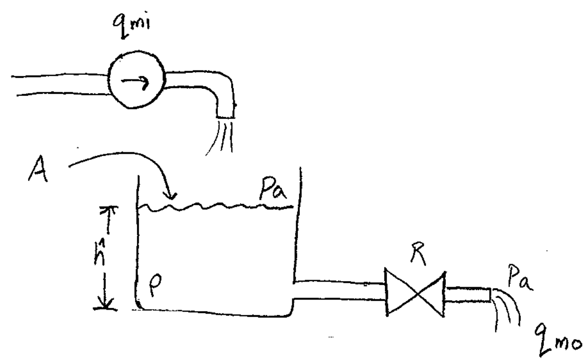
Liquid Level System



q_{mi} = mass flow rate in

q_{mo} = mass flow rate out

R = outlet resistance (linear about reference state)

h = fluid height $\Rightarrow$ $\hat{h} = h_r + h$ where h_r = reference height, h is deviation from reference

A = area of tank

p_a = atmospheric pressure

ρ = density of fluid

Develop a model of h, the deviation from reference height

Conservation of mass: $\frac{dm}{dt} = q_{mi} - q_{mo}$ mass in tank: $m = \rho A\hat{h} = \rho A(h + h_r)$

$$\frac{d(\rho A(h+h_r))}{dt} = \frac{d(\rho Ah + \rho Ah_r)}{dt} = \rho A\frac{dh}{dt} = q_{mi} - q_{mo} \quad (\text{b/c } \rho, A, h_r = \text{cst.})$$

We can find a relation between the outlet resistance + mass flow rate out

$R q_{mo} = \Delta p$, where Δp = pressure diff. across valve

$$R q_{mo} = (\rho g h + p_a) - p_a \Rightarrow q_{mo} = \frac{\rho g h}{R}$$

Substituting:

$$\rho A\frac{dh}{dt} + \frac{\rho g}{R}h = q_{mi}$$

which is a classic first order ODE $(a\dot{x} + bx = f)$

(7.6) Thermal Systems Background

Conservation of Heat Energy (Main modeling principle)

$$\dot{E}_{st} = \dot{E}_{in} - \dot{E}_{out}$$

E_{st} = stored energy in a control volume
E_{in} = energy entering control volume
E_{out} = energy exiting control volume

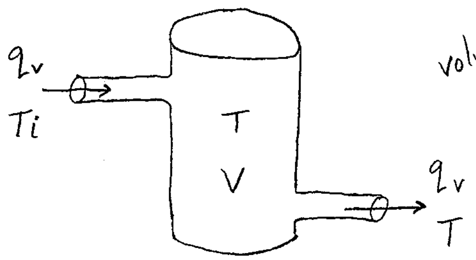
Thermal Capacitance of a mass

$$E_{st} = C\,\Delta T$$

where $C = \rho V c_p$

C = thermal capacitance (J/°C)
$\Delta T = T - T_r$ = Temperature difference (°C)
ρ = density of mass (Kg/m³)
V = volume of mass (m³)
c_p = specific heat (J/Kg°C)

Example: Temperature Dynamics of a Mixing Process



volume flow

Fluid entering a fixed insulated volume (tank) with volume, V at a given temp, T_i and rate, q_v. Fluid exiting at temp T, same rate, q_v. Fluid is well mixed in tank, so its temp is T. Fluid has specific heat, c_p and density, ρ. Develop a model for $T(t)$ with T_i as an input.

Apply conservation of energy to find model.

- Stored energy in tank: $E_{st} = \rho V c_p (T - T_r)$ where T_r is arbitrarily selected reference temperature
- Energy flowing in: $\dot{E}_{in} = \rho q_v c_p (T_i - T_r)$ note, this is a rate of energy flow b/c we know q_v, a volume flow rate
- Energy flowing out: $\dot{E}_{out} = \rho q_v c_p (T - T_r)$

Apply conservation of energy:

$$\dot{E}_{st} = \dot{E}_{in} - \dot{E}_{out} \Rightarrow \frac{d}{dt}\left[\rho V c_p (T - T_r)\right] = \rho q_v c_p (T_i - T_r) - \rho q_v c_p (T - T_r)$$

$$\rho V c_p \frac{dT}{dt} = \rho q_v c_p (T_i - T)$$

cancel ρc_p: $$V \frac{dT}{dt} = q_v (T_i - T)$$

rearrange: $$\boxed{\frac{V}{q_c}\frac{dT}{dt} + T = T_i}$$

which is a classic 1st order ODE ($a\dot{x} + bx = f$)