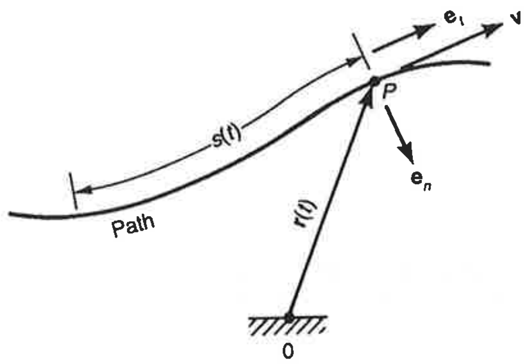


Figure 8.1

KINEMATICS OF A PARTICLE

Consider a point P that moves along a smooth path as indicated in Fig. 8.1. The position of the point may be specified by the vector $\mathbf{r}(t)$, defined to extend from an arbitrarily selected, fixed point 0 to the moving point P. The **velocity** $\mathbf{v}$ of the point is defined to be the derivative with respect to t of $\mathbf{r}(t)$, written as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{8.1}$$

Although this definition is sometimes used for evaluation [that is, by differentiating a specific expression for $\mathbf{r}(t)$], it will often be more direct to use other relationships. It follows from the above definition that the velocity vector is tangent to the path of the particle; thus, upon introduction of a unit vector $\mathbf{e}_t$, defined to be tangent to the path, the velocity can also be expressed as

$$\mathbf{v} = v\mathbf{e}_t \tag{8.2}$$

The position of P can also be specified in terms of the distance $s(t)$ traveled along the path from an arbitrarily selected reference point. Then an incremental change in position may be approximated as $\Delta\mathbf{r} \approx \Delta s\mathbf{e}_t$, in which the accuracy increases as the increments Δt and $\Delta\mathbf{r}$ approach zero. This leads to still another way of expressing the velocity as

$$\mathbf{v} = \frac{ds}{dt}\mathbf{e}_t \tag{8.3}$$

The scalar

$$v = \frac{ds}{dt} = \dot{s} \tag{8.4}$$

can be either positive or negative, depending on whether the motion is in the same or the opposite direction as that selected in the definition of $\mathbf{e}_t$.

The **acceleration** of the point is defined as the derivative of the velocity with respect to time:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{8.5}$$

A useful relationship follows from application to Eq. (8.2) of the rules for differentiating products and functions of functions:

$$\frac{d\mathbf{v}}{dt} = \dot{v}\mathbf{e}_t + \mathbf{v}\frac{ds}{dt}\frac{d\mathbf{e}_t}{ds}$$

As the direction of $\mathbf{e}_t$ varies, the square of its magnitude, $|\mathbf{e}_t|^2 = \mathbf{e}_t \bullet \mathbf{e}_t$, remains fixed and equal to 1, so that

$$\frac{d}{ds}|\mathbf{e}_t|^2 = \frac{d}{ds}(\mathbf{e}_t \bullet \mathbf{e}_t) = 2\mathbf{e}_t \bullet \frac{d\mathbf{e}_t}{ds} = 0$$

This shows that $d\mathbf{e}_t/ds$ is either zero or perpendicular to $\mathbf{e}_t$. With another unit vector $\mathbf{e}_n$ defined to be in the direction of $d\mathbf{e}_t/ds$, this vector may be expressed as

$$\frac{d\mathbf{e}_t}{ds} = \kappa \mathbf{e}_n$$

The scalar κ is called the local **curvature** of the path; its reciprocal, $\rho = 1/\kappa$, is called the local **radius of curvature** of the path. In the special case in which the path is straight, the curvature, and hence $d\mathbf{e}_t/ds$, are zero. These lead to the following expression for the **acceleration** of the point:

$$\mathbf{a} = \dot{v}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n \tag{8.6}$$

The two terms express the **tangential** and **normal** (or **centripetal**) components of acceleration.

If a driver of a car with sufficient capability "steps on the gas," a positive value of $\dot{v}$ is induced, whereas if he "steps on the brake," a negative value is induced. If the path of the car is straight (zero curvature or "infinite" radius of curvature), the entire acceleration is $\dot{v}\mathbf{e}_t$. If the car is rounding a curve, there is an additional component of acceleration directed laterally, toward the center of curvature of the path. These components are indicated in Fig. 8.2, a view of the plane of $\mathbf{e}_t$ and $d\mathbf{e}_t/ds$.

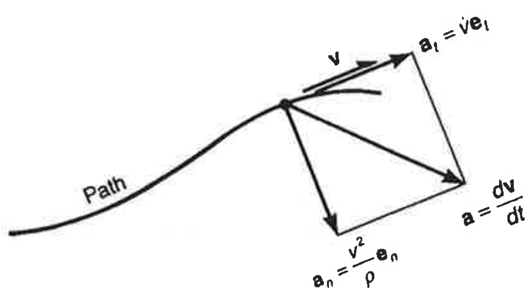


Figure 8.2

Example 8.1

At a certain instant, the velocity and acceleration of a point have the rectangular Cartesian components given by

$$\mathbf{v} = (3.5\mathbf{e}_x - 7.2\mathbf{e}_y + 9.6\mathbf{e}_z)\ \text{m/s}$$
$$\mathbf{a} = (-20\mathbf{e}_x + 20\mathbf{e}_y + 10\mathbf{e}_z)\ \text{m/s}^2$$

At this instant, what are the rate of change of speed dv/dt and the local radius of curvature of the path?

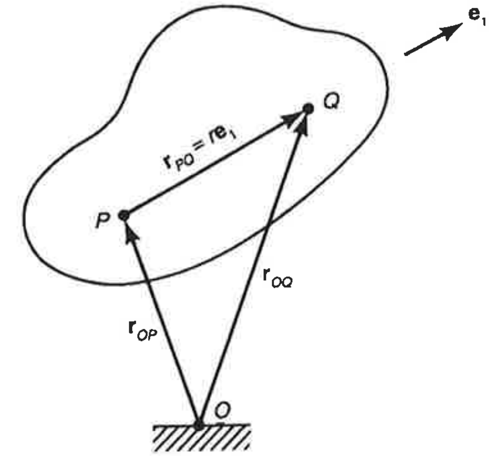


Figure 8.4

which indicates that $\dot{\mathbf{r}}_{PQ}$ is perpendicular to $\mathbf{r}_{PQ}$. Now, with a selected, fixed point designated as O, and vectors $\mathbf{r}_{OP}$ and $\mathbf{r}_{OQ}$ defined as indicated in Fig. 8.4, differentiation of the vector relationship $\mathbf{r}_{PQ} = \mathbf{r}_{OQ} - \mathbf{r}_{OP}$ leads to the relationship

$$\frac{d\mathbf{r}_{PQ}}{dt} = \mathbf{v}_Q - \mathbf{v}_P \tag{ii}$$

in which $\mathbf{v}_P$ and $\mathbf{v}_Q$ designate the velocities of P and Q, respectively. Finally, if we define $\mathbf{e}_1$ to be the unit vector in the direction of $\mathbf{r}_{PQ}$, so that

$$\mathbf{r}_{PQ} = r\mathbf{e}_1 \tag{iii}$$

then substitution of (ii) and (iii) into (i) leads to

$$2r\mathbf{e}_1 \bullet (\mathbf{v}_Q - \mathbf{v}_P) = 0$$

or

$$\mathbf{e}_1 \bullet \mathbf{v}_Q = \mathbf{e}_1 \bullet \mathbf{v}_P \tag{8.17}$$

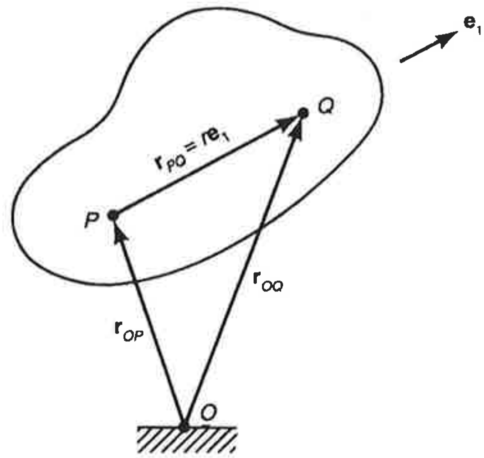
This shows that *the projections of the velocities of any two points of a rigid body onto the line connecting the two points must be equal*. This is intuitively plausible; otherwise the distance between the points would be changing. This frequently provides the most direct way of evaluating the velocities of various points within a mechanism.

Example 8.8

As the crank OQ in Exhibit 7 rotates clockwise at 200 rad/s, the piston P moves vertically. What will be the velocity of the piston at the instant when the angle θ is 50 degrees?

Solution

Since point Q must follow a circular path, its speed may be determined from Eq. (8.15): $v_Q = (0.075\text{ m})(200\text{ s}^{-1}) = 15$ m/s, with the direction of $\mathbf{v}_Q$ as indicated in the figure. Because the cylinder wall constrains the piston, its velocity is vertical. The connecting rod PQ is rigid, so the velocities of the points P and Q must



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in which $\mathbf{v}_P$ and $\mathbf{v}_Q$ designate the velocities of P and Q, respectively. Finally, if we define $\mathbf{e}_1$ to be the unit vector in the direction of $\mathbf{r}_{PQ}$, so that

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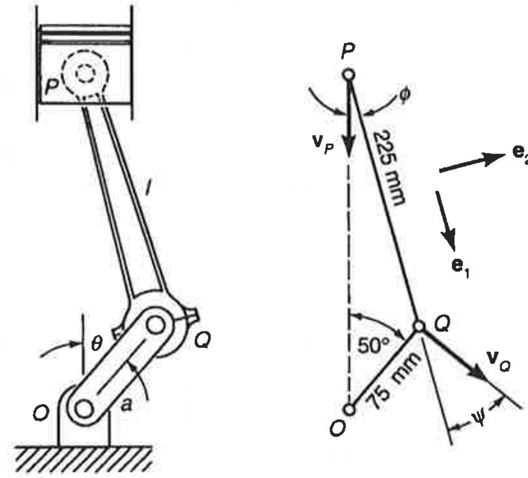

Exhibit 7

satisfy $v_P \cos\phi = v_Q \cos\psi$. The trigonometric rule of sines, applied to the triangle OPQ, gives

$$\sin\phi = \frac{a}{l}\sin\theta = \frac{75}{225}\sin 50^\circ$$

which yields $\phi = 14.8°$. The other required angle is then $\psi = 90° - \theta - \phi = 25.2°$. Once these angles are determined, the constraint equation yields the speed of the piston:

$$v_P = \frac{\cos\psi}{\cos\phi} v_Q = 14.04 \text{ m/s}$$

The Angular Velocity Vector

If a rigid body is in *plane motion*, that is, if the velocities of all points of the body lie in a fixed plane, then its orientation may be specified by the angle θ between two fixed lines, one of which passes through the body, as indicated in Fig. 8.5. The rate of change of this angle is central to the analysis of the velocities of various points of the body.

To determine this relationship, consider Fig. 8.6, which shows a position vector from the point P to point Q, both fixed in the moving body. Two configurations are shown, one at time t and another after an arbitrary change during a time increment

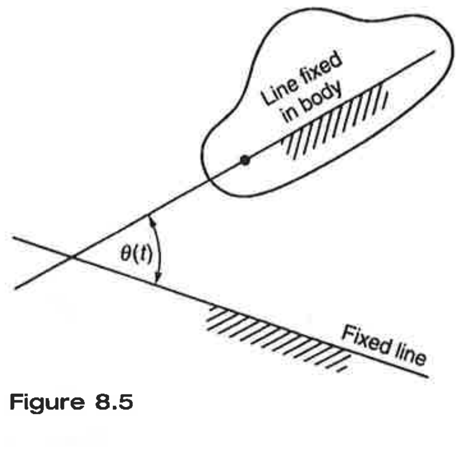

Figure 8.5

In the special case in which $\omega = 0$, this indicates that all points have the same velocity, a motion called **translation**. In the special case in which $\mathbf{v}_P = \mathbf{0}$, the motion is simply rotation about a fixed axis through P. Thus, in the general case, the two terms on the right of Eq. (8.20) can be seen to express a superposition of a translation and a rotation about P. But since P can be selected *arbitrarily*, there are as many combinations of a translation and a corresponding "center of rotation" as the analyst wishes to consider!

In all of the these cases, the angular velocity is a property of the *body's* motion, and Eq. (8.21) relates the velocities of *any* two points of a body experiencing planar motion. Dot-multiplication of each member of Eq. (8.21) with $\mathbf{e}_2$ leads to the following means of evaluating the angular velocity of a plane motion in terms of the velocities of two points:

$$\omega = \frac{\mathbf{e}_2 \bullet \mathbf{v}_Q - \mathbf{e}_2 \bullet \mathbf{v}_P}{r} \tag{8.22}$$

That is, ω will be the difference between the magnitudes of the projections of the velocities of P and Q onto the perpendicular to the line connecting P and Q, divided by the distance between P and Q.

Example 8.9

What will be the angular velocity of the connecting rod in Example 8.8, at the instant when the angle θ is 50 degrees?

Solution

Referring to Exhibit 7 for the definition of $\mathbf{e}_2$, we see that

$$\omega = \frac{v_Q \sin\psi + v_Q \sin\phi}{l}$$

$$= \frac{(15\text{ m/s})\sin 25.2° + (14.04\text{ m/s})\sin 14.8°}{0.225\text{ m}} = 44.3\text{ rad/s}$$

The positive value indicates that the rotation is counterclockwise at this instant.

Instantaneous Center of Zero Velocity

For planar motion with $\omega \neq 0$, there always exists a point C' of the body (or an imagined extension of the body) that has zero velocity. If point P of Eq. (8.21) is selected to be this special point, the equation reduces to $\mathbf{v}_Q = \mathbf{v}_{C'} + \omega \times \mathbf{r}_{C'Q} = r\omega\mathbf{e}_2$ where r is now the distance from C' to Q and $\mathbf{e}_2$ is perpendicular to the line connecting C' and Q. This latter property can be used to locate C' if the directions of the velocities of two points of the body are known.

Example 8.10

What is the location of the instantaneous center C' of the connecting rod in Examples 8 and 9? Use this to verify the previously-determined values of the angular velocity of the connecting rod and the velocity of point P.

Solution

The velocity of any point of the connecting rod must be perpendicular to the line from C' to that point. Hence C' must lie at the point of intersection of the

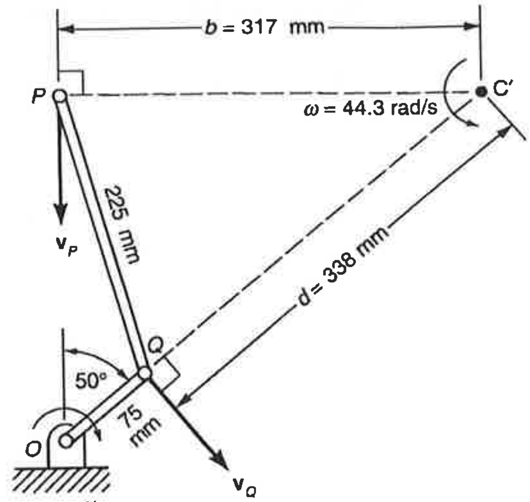

Exhibit 8

horizontal line through P and the line through Q perpendicular to $\mathbf{v}_Q$ (i.e., on the line through O and Q), as shown in Exhibit 8. The pertinent distances can be found as follows:

$$OP = (75 \text{ mm}) \cos 50° + (225 \text{ mm}) \cos 14.8° = 266 \text{ mm}$$
$$PC' = OP \tan 50° = 317 \text{ mm}$$
$$QC' = OP \sec 50° - 75 \text{ mm} = 338 \text{ mm}$$

The angular velocity of the connecting rod is then

$$\omega = \frac{v_Q}{QC'} = \frac{15 \text{ m/s}}{0.339 \text{ m}} = 44.3 \text{ rad/s}$$

and the velocity of P is then

$$v_P = PC'\omega = (0.317 \text{ m})(44.\ 3 \text{ s}^{-1}) = 14.04 \text{ m/s}$$

in agreement with values the previously obtained.

Example 8.11

Using the properties of the instantaneous center, determine the velocity of the point P on the rim of the rolling wheel in Example 8.5.

Solution

Since the wheel rolls without slipping, the point of the wheel in contact with the flat surface has zero velocity and is therefore its instantaneous center. The angular speed of the wheel is $\dot{\theta}$, and the distance from C' to P is readily determined from Exhibit 9:

$$r = 2b \sin\frac{\theta}{2}$$

The velocity of point P then has the magnitude

$$v_P = r\omega = 2b \sin\frac{\theta}{2}\dot{\theta}$$

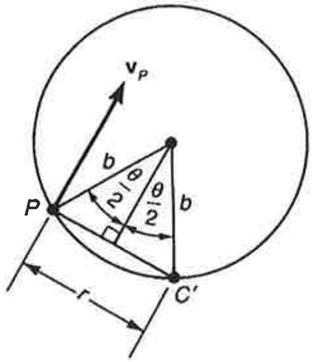

Exhibit 9

and the direction shown in Exhibit 9. This direction should be evident by inspection once it is realized that a positive $\dot{\theta}$ corresponds to clockwise rotation. The reader may find it instructive to recall the conventions for the choice of $\mathbf{e}_2$ and positive ω used in the derivation leading to Eq. (8.21) and verify the agreement. Note the simplicity of this analysis as compared with the one expressing the position of P in a rectangular Cartesian coordinate system.

Accelerations in Rigid Bodies

Formally differentiating Eq. (8.20) and substituting for $\dot{\mathbf{r}}_{PQ}$ using Eq. (8.19) leads to

$$\mathbf{a}_Q = \mathbf{a}_P + (\boldsymbol{\alpha} \times \mathbf{r}_{PQ}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{PQ}) \tag{8.23}$$

in which the vector $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is called the **angular acceleration** of the body. For planar motion, $\boldsymbol{\alpha} = \alpha \mathbf{e}_3 = \dot{\omega}\mathbf{e}_3$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2\mathbf{r}$ so that

$$\mathbf{a}_Q = \mathbf{a}_P + r\alpha\mathbf{e}_2 - r\omega^2\mathbf{e}_1 \tag{8.24}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are defined as indicated in Fig. 8.6.

Equivalent relationships, analogous to Eq. (8.17) and Eq. (8.22) for velocity, can be obtained by dot-multiplying this equation by $\mathbf{e}_1$ and by $\mathbf{e}_2$:

$$\mathbf{e}_1 \bullet \mathbf{a}_Q = \mathbf{e}_1 \bullet \mathbf{a}_P - r\omega^2 \tag{8.25}$$

$$\alpha = \frac{\mathbf{e}_2 \bullet \mathbf{a}_Q - \mathbf{e}_2 \bullet \mathbf{a}_P}{r} \tag{8.26}$$

Example 8.12

If the speed of the crank in Examples 8–10 is constant, what are the acceleration $\mathbf{a}_P$ of the piston and the angular acceleration α of the connecting rod at the instant when the angle θ is 50 degrees (Exhibit 10)?

Solution

When the crank speed is constant, the acceleration of Q is entirely centripetal, of magnitude

$$a_Q = (0.075 \text{ m})(200 \text{ s}^{-1})^2 = 3000 \text{ m/s}^2$$

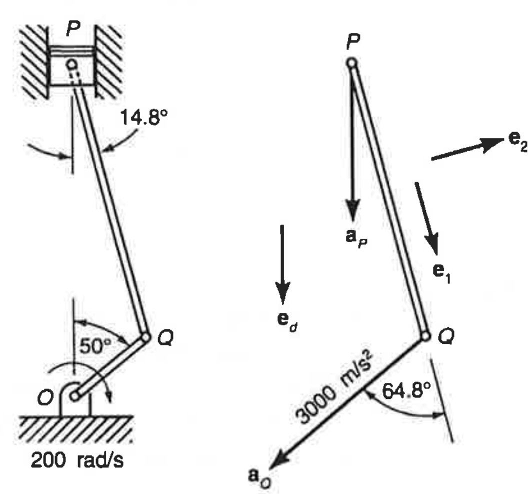

Exhibit 10

and directed toward the center of curvature O of the path of Q. The acceleration of P is vertically upward or downward. To determine the direction, we define a downward unit vector $\mathbf{e}_d$ and let $\mathbf{a}_p = a_p\mathbf{e}_d$ (see Exhibit 10). A positive value of a_P then indicates a downward acceleration and a negative value an upward acceleration. These expressions for a_Q and $\mathbf{a}_P$ are substituted into Eq. (8.25), along with the previously determined angular velocity of the rod, giving

$$(3000 \text{ m/s}^2)\cos 64.8° = a_P \cos 14.8° - (0.225 \text{ m})(44.3 \text{ s}^{-1})^2$$

which yields

$$a_P = 1779 \text{ m/s}^2$$

The angular acceleration α of the rod can then be determined from Eq. (8.26):

$$\alpha = \frac{(3000 \text{ m/s}^2)\cos 154.8° - (1779 \text{ m/s}^2)\cos 104.8°}{0.225 \text{ m}} = -10\ 050 \text{ rad/s}^2$$

The negative value indicates that the angular acceleration is clockwise; that is, the 44.3-rad/s counterclockwise angular velocity is rapidly decreasing at this instant.

NEWTON'S LAWS OF MOTION

Every element of a mechanical system must satisfy Newton's second law of motion, that is, the resultant force **f** acting on the element is related to the acceleration **a** of the element by

$$\mathbf{f} = m\mathbf{a} \tag{8.27}$$

in which m represents the mass of the element. Newton's third law requires that the force exerted on a body A by a body B is of equal magnitude and opposite direction to the force exerted on body B by body A. These laws and their logical consequences provide the basis for relating motions to the forces that cause them.

Applications to a Particle

A **particle** is an idealization of a material element in which its spatial extent is disregarded, so that the motion of all of its parts is completely characterized by the path of a geometric *point*. When the accelerations of various parts of a system differ significantly, the system is considered to be composed of a number of particles and analyzed as described in the next section.

Example 8.13

An 1800-kg aircraft in a loop maneuver follows a circular path of radius 3 km in a vertical plane. At a particular instant, its velocity is 210 m/s directed 25 degrees above the horizontal as shown in Exhibit 11. If the engine thrust is 16 kN greater than the aerodynamic drag force, what is the rate of change of the aircraft's speed, the magnitude of the aircraft's acceleration, and the aerodynamic lift force?