Introduction to Game Theory

Content

Basics

Definitions

Game	An action situation where there are two or more mutually aware players and the actions for each depends on the acting of all > Strategic Interaction
Decision	an action situation in a passive environment where a person can choose without consern for the reactions or responses of others > No Interactions
imperfect Information	game has either external or strategic uncertainity. At same point of the game one player does not know the full history of the game
Incomplete Information	One player knows more than another (asymetric informations)
strategy (in an extensive form game)	complete plan of action that tells what a player does at any information set in which he might be called to decide > given a strategy, a final node can be determined
actions	what a player can do when he has to move
information set	collection of decision nodes which cannot be distinguished by the player with the move
move	A move is a single action to be taken by a player at a node controlled by him.

Game Types

sequential game	
simultaneuos game	

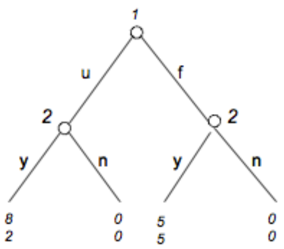
zero-sum game	
repeated game	

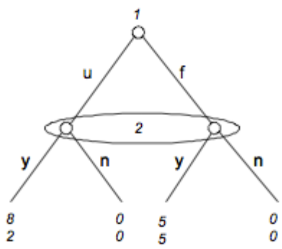
Game Forms

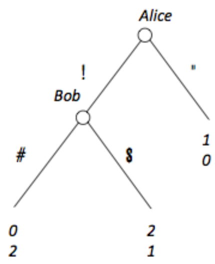
The extensive Form

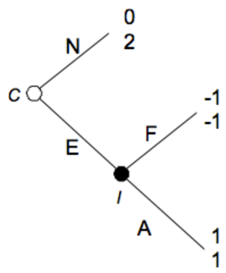
Sequential move games

Game Trees









Game consists of

- Players
- Order of moves
- Actions (What players can do when they have to move)
- Information sets (What each player knows when he has to move)
- Payoffs received by each player for each combination of moves that could be chosen by the players
- The probability distribution over exogenous events. New player: Nature

Backward induction

- go the end of the tree.
- tick the branch at each node that gives the highest payoff to the player who has the move.
- continue until you reach the root.
- each time we do this we take into account that we will go only on ticked branches.
- Putting these together yields the paths of the play that will result in the
- backward induction solution.

This way we get read of some equilibria which involve non credible threats. Remember the challenger-incumbent game!

The Normal Form

Simultaneous move games

Game consists of

- a set of players I = {1, ..., n} indexed by i
- a set of strategies Si for every player i.
- for each player i, preferences over the set of action profiles. These preferences are represented by the utility function ui

Strategies

Pure Strategies: one specific action (strategy) for sure

Mixed Strategies: the player randomizes and chooses different actions with positive probability. For instance tossing a coin, or rolling a die, or looking at the stock market (if the Dow Jones is up then A if down B).

Set of Strategies: discrete (go to the cinema or not) or continuous (supply, price)

TALIA chooses:

Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don't	6, 3, 3	4, 4, 1

Don't Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don't	4, 1, 4	2, 2, 2

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

Nash Equilibrium

A Nash equilibrium is a list of strategies, one for each player, such that no player can get a better payoff by switching to some other strategy that is available to her while all the other players adhere to the strategies specified for them in the list.

Mixed Strategy Nash Equilibrium

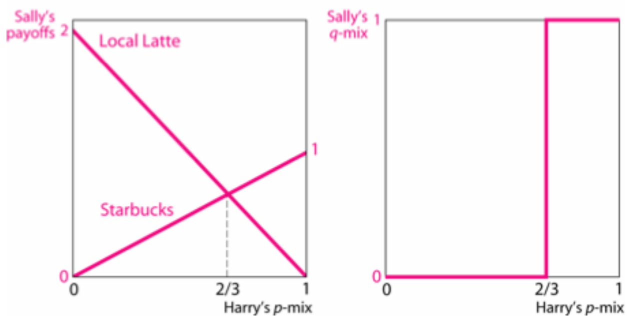
A mixed-strategy profile $\sigma* = \{\sigma 1*, \sigma 2*, ..., \sigma n*\}$ in a finite normal-form game is a mixed Nash equilibrium if and only if, for each player i

- the expected payoff to every action to which $\sigma i*$ assigns positive probability is the same (given all other players' strategies as given)

- the expected payoff to every action to which $\sigma i*$ assigns zero probability cannot be larger than the payoff of any action to which $\sigma i*$ assigns positive probability

- each player's expected payoff in an equilibrium is the expected payoff to any of the actions that is used with positive probability. Simplifies the calculation

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1	0
	Local Latte	0	2
	p-mix	p	$2(1-p)$



Subgames

Definition

A subgame
- begins at a decision node x that is a singleton (this is always the case in games with perfect information)
- includes all decision and terminal nodes following x in the tree
- If the game is of imperfect information, it cannot cut any information set, namely if a subgame contains any part of an information set, then it contains all the nodes in that information set

Subgame Perfect Equilibrium

A Nash equilibrium is subgame perfect if the players' strategies constitute a Nash equilibrium in every subgame.

Repeated Games

Definition

Let G be a stage game. Repeated T times with t = {0, 1, ... , T-1}. Players i = {1,2, ... , n}

Players observe the outcome of previous repetitions (game with observed actions).

$G = \{N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N}\}$
$u_i : A_1 \times .. \times A_n \rightarrow R$

a_i pure strategy of the stage game (called now actions).

H_t set of all possible histories h_t up to t – 1.

h_t A possible history (all players chose C in the PD in all previous periods)

s_i pure strategy in the repeated game maps possible period t histories $h_t \in H_t$ to actions $a_i \in A_i : s_i(h_t)=a_i^t$.

Example

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Histories after the first round:
$H_1 = \{(C,C),(C,D),(D,C),(D,D)\} = \{h_1^1, h_1^2, h_1^3, h_1^4\}$

Strategies for T=2:
start with C, play C if 2 played C, play D if he played D: {C,{C,D,C,D}}
other strategies: {C,{C,C,C,C}}, {C,{C,D,D,D}}, ...

Incomplete Information

Static Game

Definition

- The set of players I.

- For each player i:
    - a set of possible types Ti (in the example the girl has two types (loving and leaving), the boy only one)
    - set of actions Ai (Bach and Schubert)

- payoffs which depend on the actions taken by all players and types: $\forall i \in I, u_i : A \times T \rightarrow R$ (in the example the boys payoffs are independent of the girl's type. Not necessarily the case)

- Probability measure on types p : T → [0, 1] (in the example loving with probability 1/2)

Example Strategy:
S_g = {(B,B),(B,S),(S,B),(S,S))}, where (B,S) means B (if loving), S (if leaving).

Bayesian Nash Equilibrium

In a static Bayesian Game the strategies $s^* = (s_1^*, ..., s_n^*)$ form a (pure-)strategy Bayesian Nash equilibrium if for each player i and for each of i's types $t_i \in T_i$, no player wants to change her strategy, even if the change involves only one action by one type.

Example

G,B	B	S
B,B	1.5,1	1.5,0
B,S	2,0.5	0,1.5
S,B	0,0.5	2,1.5
S,S	0.5,0	0.5,3

$$S_g = \{(B,B),(B,S),(S,B),(S,S)\}$$
$$S_b = \{B,S\}$$

Signaling Game

Definition

- 2 players: I={1,2}
- Player 1 is the sender, player 2 is the receiver.
- Player 1 has private information about his type. (He knows the type ti ∊ Ti he is).
- Player 2's type is common knowledge.
- It is common knowledge that player 2 has prior beliefs p(t1) about player 1's type.

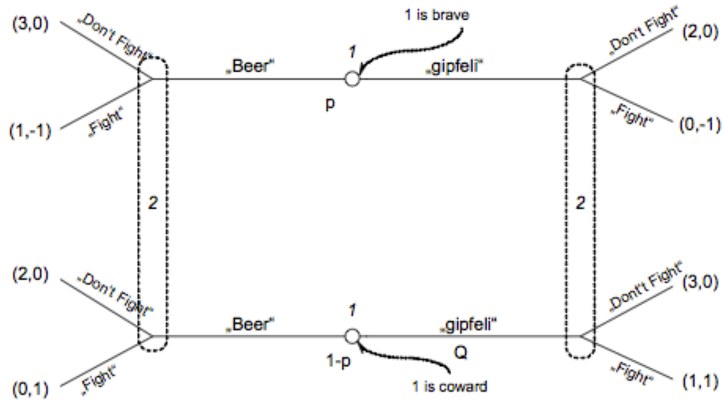
Timing

- 1 chooses an action which is observed by 2 (the breakfast). This action is typically called message.
- 2, after observing 1, updates his beliefs p(t1/observed action) and chooses an action (fight or not).
- The game ends.

On / Off the equilibrium path

Those information set which are not reached under the equilibrium strategies are said to be "off" the equilibrium path.

Example



$$\text{Strategies} \; : \; \{(B,B),(DF,F)\}$$
$$\text{Beliefs} \; : \; \Pr(brave/Beer) = p \text{ and } \Pr(brave/gipfeli) < 1/2$$

Spence Model

Education can serve as a reliable signal of the worker's type if the ease of acquiring an education is correlated with worker productivity.
When type H individuals are able to obtain education at lower cost than are type L individuals, a separating equilibrium exists.

The model
- A worker knows his ability or productivity θ.
- $\theta \in \{\theta^L, \theta^H\}$ with probabilities p^L and p^H . We assume $\theta^L = 100$ and $\theta^H = 150$.
- The firm does not know θ.
- The worker sends a signal: the number of courses taken n.
- Those with high productive take courses with less effort cost: $c^H = 6$ vs $c^L = 9$.
- The firm observes the education and offers wages w(n).
- There is competition between employers: the worker is paid his expected productivity.

Pooling Equilibria	Both worker types choose the same number of courses
Separating Equilibria	Self Selection: workers with θ^H take more courses than those with θ^L

Evolutionary Game Theory

- In most of evolutionary game theory, the units of selection are strategies in a game. Those strategies which do "well" will grow relative to those which perform badly.
- New equilibrium idea: Evolutionary stability. In equilibrium there is no evolution since all the strategies are doing equally well, what happens if there are some mutants, some agents play something different? Will they spread? Will they disappear? The focus is on robustness to mutations.

Evolutionarily stable strategy

x is ESS if:

$$u(x,x) \geq u(x',x) \ \forall x'$$

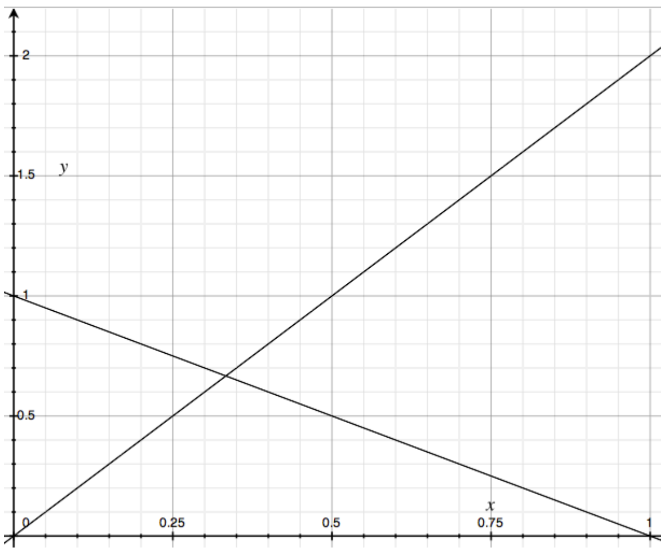
$$u(x',x) = u(x,x) \Rightarrow u(x',x') < u(x,x')$$

Note that an ESS is a NE with an additional requirement. The first condition is that of a NE (a best reply to itself). The second says that **it has to be a better reply to any alternative** best reply than the alternative itself.

Example

	A	B
A	2, 2	0, 0
B	0, 0	1, 1

compare the different utility functions:



2 Types A & B, e = share of type A

u(A, eA + (1-e)B) = 2x
u(B, eA + (1-e)B) = 1 - x

if uA > uB -> e increases and vice versa:

I-<-<-<- 1 / 3 ->->->->->->- 1 -> e

Bargaining

Nash axiomatic model

- Set of bargainers N . I will restrict attention to N = {1, 2}.

- The bargaining set: A.
  An element a ∊ A is pair (a1, a2) which says what each player gets.
  We can define utilities on the agreements in A and get a new set
  U = {(u1(a1), u2(a2)) ∊ R2 : (a1, a2) ∊ A}. (Plot examples)

- The disagreement event d = (d1, d2).
  If they fail to reach an agreement, the disagreement event d occurs (ex. get nothing).
  We assume that the bargainers can agree to disagree: d ∊ A.
  We also attach utilities to the disagreement: (u1(d1), u2(d2)).

Cake Example

Share a cake between two players:

$$A = \{(x, y) \in R^2 : x + y \leq 1 \text{ and} x, y \geq 0 \text{ for } i = 1, 2\}$$

$$U = \{(u_1(x), u_2(y)) \in R^2 : (x, y) \in A\}$$

$$d = (0, 0), \quad u_1(0) = u_1(0) = 0$$

The Nash axioms

I. Invariance to equivalent utility representations
   positive affine transformation of the utility function doesn‘t change the final solution

II. Pareto Efficiency
   Players won‘t agree if there is another option in which both are better off

III. Symmetry
   In a symmetric game u1(d1) = u2(d2), and (v1, v2)∊ U if and only if (v2, v1) ∊ U, then both players should be treated equally.

IV. Independence
   Options which are not selected should not change the solution if they are no longer available.

$$\max_{a \in A} \quad (u(a_1) - u(d_1)) \cdot (u_2(a_2) - u(d_2))$$

$$\text{s.t. } (u(a_1), u_2(a_2)) \geq (u(d_1), u(d_2))$$

Rubenstein‘s alternating offers model

Players 1 and 2 play the following game: A cake of size one has to be divided. For this, in a starting round, 1 makes a proposal of x that 2 can accept or turn down. If he accepts the cake is split according to (1 – x , x). If he declines the game moves one round forward and it is now 2 who proposes a share y. 1 can accept the proposal and the cake is split according to (y , 1 – y), or he can reject in which case the game moves one round further.

Time is valuable such that seen from the starting round, any payoff z in round n is worth $\delta_i^n z$ with $\delta i \in (0,1)$.

It can be shown that for T infinite, there is a unique subgame perfect equilibrium in which : Agreement is reached immediately, 1 gets
$1/(1 + \delta)$, and 2 gets $\delta/(1 + \delta)$.