# Homework Week 5: The Chain Rule & Gradients in Physics

**Due:** Friday at 11:59pm submitted online in Canvas

**Instructions:** This assignment consists of a single, comprehensive problem based on Chapter 4 of Lang. Please submit your solutions as a single PDF file. Your work should be clear, legible, and rigorous..

## Problem 1: Orbital Mechanics and Conservation of Energy

**Context & Exposition:**
In single-variable calculus, you learned that the derivative describes a rate of change. In multivariable calculus, the **Gradient** ($\nabla f$) generalizes this concept. A profound application of the gradient is the link between the geometry of a force field and the physical laws of conservation.

Consider a satellite (or a planet) moving through space. Its motion is governed by a gravitational potential energy function, $\Phi(x, y, z)$. The force acting on the satellite is the negative gradient of this potential:

$$\mathbf{F}(X) = -\nabla\Phi(X)$$

This is what Lang (Section 4-5) calls a **Conservative Vector Field**.

In this problem, you will use the **Chain Rule** and properties of the **Gradient** to derive the Law of Conservation of Energy from scratch, and then apply it to analyze a spacecraft's trajectory. If you get stuck, read Section 4.5 of Lang.

(a) **The Geometry of Gravity (Functions of $r$):**
Let the potential energy be defined by the Newtonian gravitational potential of a massive object centered at the origin:

$$\Phi(X) = -\frac{k}{r}$$

where $k$ is a positive constant, $X = (x, y, z)$, and $r = \|X\| = \sqrt{x^2 + y^2 + z^2}$.

Using the chain rule techniques for functions depending on distance (Lang, Section 4-4), compute the gradient vector $\nabla\Phi(X)$.

- Express your result in terms of the position vector $X$ and the distance $r$.
- Explain, in plain English, the geometric relationship between the resulting force vector $\mathbf{F} = -\nabla\Phi$, the position of the satellite, and the position of the massive object. Why does this match what we expect from gravity?

(b) **The General Proof (The Chain Rule in Action):**
Let $C(t)$ be the differentiable path of a satellite of mass $m$ moving in *any* conservative field with potential $\Phi$.

Newton's Second Law states: $\mathbf{F}(C(t)) = mC''(t)$.

Define the **Total Energy** function $E(t)$ as the sum of Kinetic Energy $(K)$ and Potential Energy $(\Phi)$:

$$E(t) = \underbrace{\frac{1}{2}m \left\|C'(t)\right\|^2}_{K} + \underbrace{\Phi(C(t))}_{\Phi}$$

Prove that the total energy is constant over time (i.e., show that $\frac{dE}{dt} = 0$).
*Hint: Differentiate $E(t)$ with respect to $t$. You will need to use the Chain Rule (Lang 4-1) for the $\Phi$ term, along with the relation between potential energy and force, and Newton's law for forces. You will need the product rule/dot product rules for the $K$ term.*

(c) **Directional Derivatives and Work:**
Suppose the satellite is at a point $P$ and is moving with a velocity vector $\mathbf{v}$.

- Using the definition of the Directional Derivative (Lang 4-3), express the instantaneous rate of change of the Potential Energy $\Phi$ in the direction of motion.
- If the satellite enters a circular orbit (where the velocity $\mathbf{v}$ is always perpendicular to the position vector $X$), show that the rate of change of Potential Energy is zero.

(d) **Application Using Conservation of Energy:**
A comet of mass $m = 1000$ kg moves in a gravitational field where the potential is $\Phi(X) = -\frac{10^6}{r}$ Joules.

At time $t = 0$, the comet is at position $P_0 = (300, 400, 0)$ km and is moving with a speed of 2 km/s. Some time later, the comet passes through the point $P_1 = (100, 0, 0)$ km.

Using the Conservation Law you proved in part (b):

- Calculate the initial Total Energy $E(0)$.
- Determine the speed of the comet when it reaches point $P_1$.