Chapter 1
Complex Numbers & Functions
Lecture 1: Complex Numbers

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Outline

Complex numbers: Motivations

Consider quadratic equation

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$$\sqrt{\Delta} = \sqrt{(-\Delta)j^2} = j\sqrt{-\Delta}$$

Definition of complex numbers

Then, $az^2 + bz + c = 0$ has solution when $\Delta < 0$:

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are imaginary numbers, but

$$j^2 = -1, \quad \sqrt{-2}\sqrt{-8} = j\sqrt{2} \cdot j\sqrt{8} = -4, \quad j^{4n} = 1,$$

are real.

Definition of complex numbers

For example, the solution of

$$z^2 - z + 1 = 0$$

is

$$z = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}.$$

A complex number is understood as: any real, imaginary numbers, or combinations like $1 \pm j$, $1/2 \pm j\sqrt{3}/2$.

Real Part and Imaginary Part

Complex number $z = x + yj$, where $x, y \in \mathbb{R}$
x is called the real part of z, denoted by $Re(z)$
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Note. If $x = 0$, the complex number is said to be purely imaginary, and if $y = 0$ it is said to be purely real.

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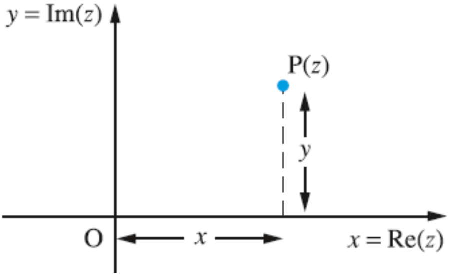

x-axis: called real axis y-axis: called imaginary axis

The arithmetic of complex numbers

(i) Equality

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Two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

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To add or subtract two complex numbers, we simply perform the operations on their corresponding real and imaginary parts. Precisely, if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

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$$\boxed{z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)},$$

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$$\boxed{z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)},$$

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If $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) = x_1 x_2 + jy_1 x_2 + jx_1 y_2 + j^2 y_1 y_2.$$

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$$\boxed{z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)}.$$

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Then, we multiply "top and bottom" by $x_2 - jy_2$:

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - iy_2)}{(x_2 + jy_2)(x_2 - jy_2)}.$$

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$$\frac{z_1}{z_2} = \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2},$$

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or

$$\boxed{\frac{z_1}{z_2} = \frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2} + j\frac{(x_2y_1-x_1y_2)}{x_2^2+y_2^2}}$$

Complex conjugate

The number

$$z^* = x - jy$$

is called the complex conjugate of the complex number

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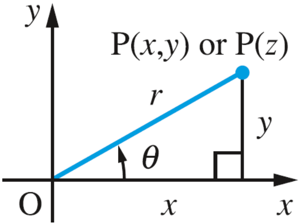
The following properties are direct:

$$\begin{aligned} z + z^* &= 2x = 2Re(z) \\ z - z^* &= 2jy = 2Im(z) \\ zz^* &= (x + jy)(x - jy) = x^2 + y^2 \end{aligned} \tag{4}$$

Modulus and argument

Complex number z in Argand diagram:



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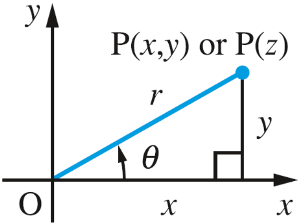

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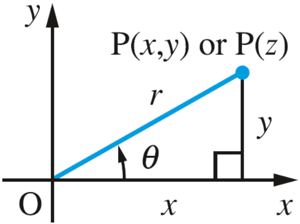
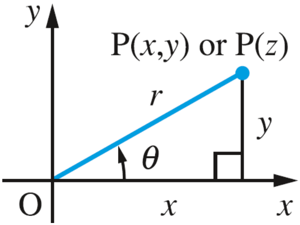


Length $r = |OP|$: called the modulus of z, denoted by *modz* or $|z|$

Angle θ between the positive Ox and OP: called the argument of z, denoted by $\arg z$

Modulus and argument



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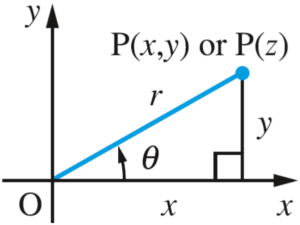

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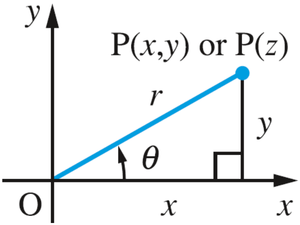

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\begin{aligned}
|z| &= r = \sqrt{x^2 + y^2} \\
\arg z &= \theta, \qquad \text{where} \quad \tan\theta = \frac{y}{x}, z \neq 0.
\end{aligned}
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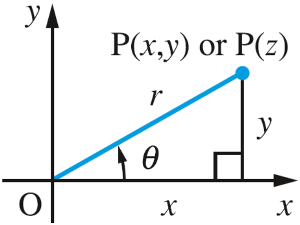
Modulus and argument



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Convention: $-\pi < \arg z \leq \pi$

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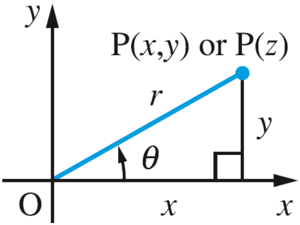

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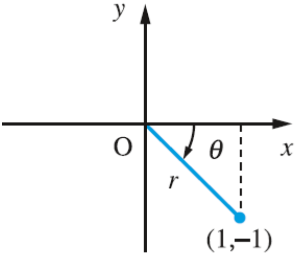
Note. Be careful to place θ in the right quadrant!

Example 1

Determine the modulus and argument of $1 - j$



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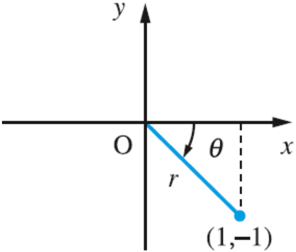

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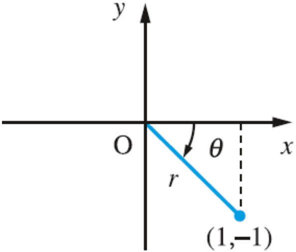

Solution.

$$|1 - j| = \sqrt{1^2 + (-1)^2} = \sqrt{2},$$

$$\arg(1 - j) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}.$$

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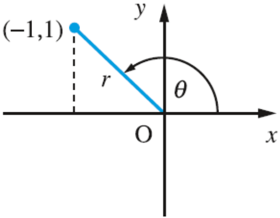
Determine the modulus and argument of $-1 + j$



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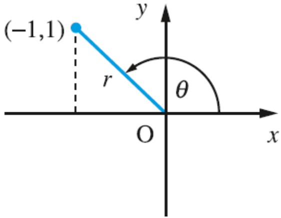

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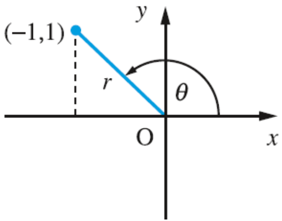

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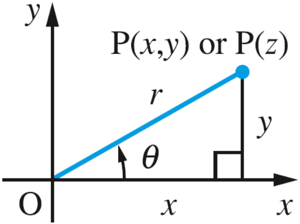
$$\arg(-1+j) = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{1}{4}\pi = \frac{3}{4}\pi.$$

Polar form of a complex number

Complex number $z = x + jy$



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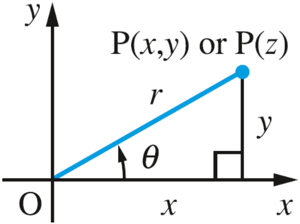

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

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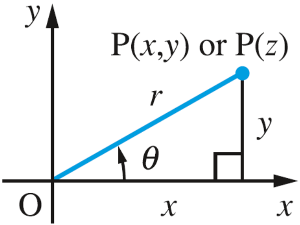

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polar form

$$z = r(\cos\theta + j\sin\theta)$$

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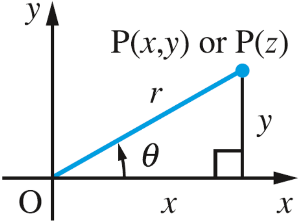

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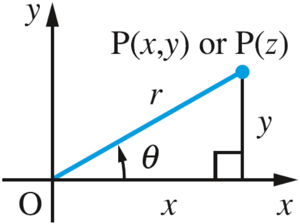

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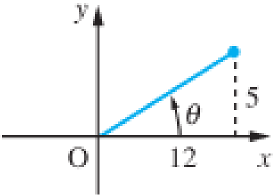

Solution:

$$|12 + j5| = 13,$$

Polar form of a complex number: Examples

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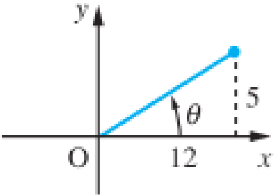

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$$|12 + j5| = 13, \quad \arg(12 + j5) = \tan^{-1}(5/12) = 0.395$$

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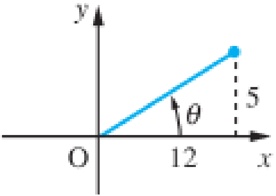

Solution:

$$|12 + j5| = 13, \quad \arg(12 + j5) = \tan^{-1}(5/12) = 0.395$$

$$12 + j5 = 13[\cos(0.395) + j\sin(0.395)]$$

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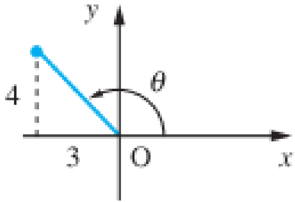

Solution:

$|-3+j4| = 5,$

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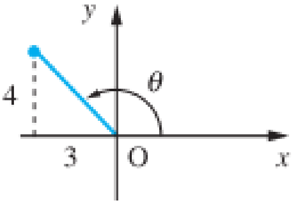

Solution:

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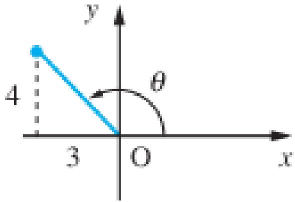

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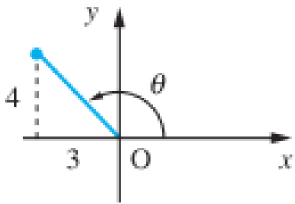

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$$-3 + j4 = 5[\cos(2.214) + j\sin(2.214)]$$

Multiplication in polar form

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Note. Make sure that $-\pi < \arg(z_1 z_2) \le \pi$.

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$$\boxed{z = r(\cos\theta + j\sin\theta) = re^{j\theta}} \tag{12}$$

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This is sometimes called its principal value.

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Homework Chapter 1

-Textbook: G. James, Modern Engineering Mathematics
Chapter 3: Complex Numbers
Exercises: 14, 16, 17, 19, 20, 21, 24, 25, 28, 31

-Textbook: G. James, Advanced Modern Engineering Mathematics
Chapter 4: Functions of a Complex Variable
Exercises: 1, 2, 4, 24, 28, 30, 38, 40, 44, 45