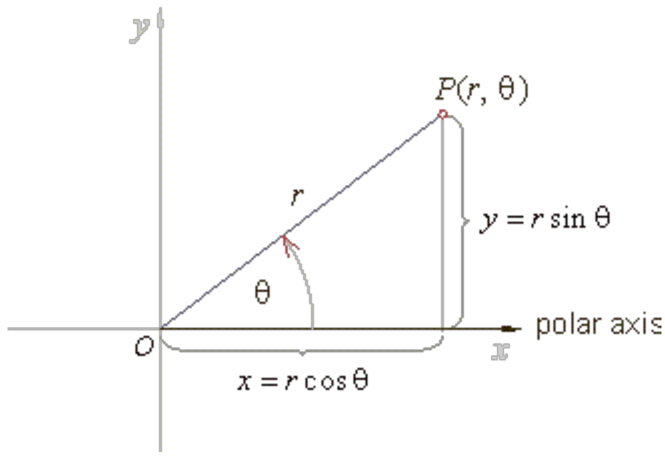
Polar coordinate system

The polar coordinate system is a two-dimensional coordinate system in which each point P on a plane is determined by the length of its position vector r and the angle q between it and the positive direction of the x-axis, where $0 \le r < +$ oo and $0 \le$ q < 2p.



Polar and Cartesian coordinates relations

$$\left.\begin{matrix} y = r\sin\theta \\ x = r\cos\theta \end{matrix}\right\} \quad \Rightarrow \quad P(x, y) = P(r\cos\theta, r\sin\theta)$$

$$\left.\begin{matrix} r = \sqrt{x^2 + y^2}, \quad \tan\theta = \dfrac{y}{x} \\ \theta = \arctan\left(\dfrac{y}{x}\right) \end{matrix}\right\} \quad \Rightarrow \quad P(r, \theta) = P\left(\sqrt{x^2 + y^2},\ \arctan\left(\dfrac{y}{x}\right)\right)$$

Since the inverse tangent function (arctan or $\tan^{-1}$) returns values in the range -p/2 < q < p/2, then

for points lying in the 2nd or 3rd quadrant and for points lying in the 4th quadrant

$$\theta = \arctan\left(\frac{y}{x}\right) + \pi, \qquad \theta = \arctan\left(\frac{y}{x}\right) + 2\pi.$$

Example: Convert Cartesian coordinates (-1, -Ö3) to polar coordinates.

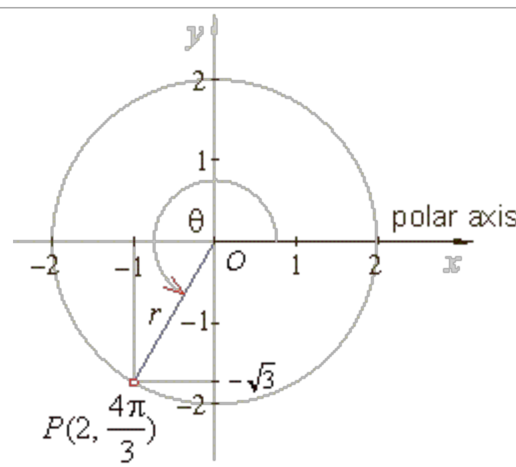
Solution: $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2,$

and since the point lies in the 3rd quadrant, then

$$\theta = \arctan\left(\frac{y}{x}\right) + \pi,$$

$$\theta = \arctan\left(\frac{-\sqrt{3}}{-1}\right) + \pi = \arctan\sqrt{3} + \pi,$$

$$\theta = \frac{\pi}{3} + \pi = \frac{4}{3}\pi.$$



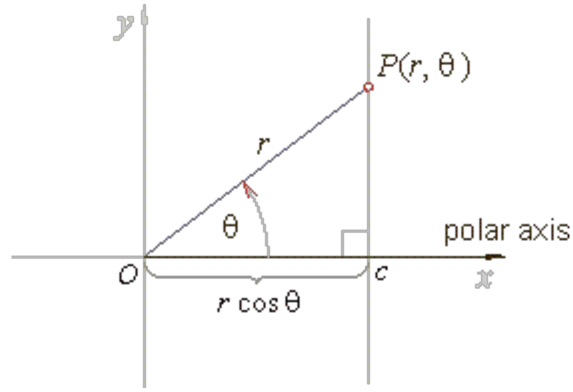
:: Equation of a line in polar form

Lines parallel to the axes, horizontal and vertical lines

Lines parallel to the *y*-axis

A vertical line, $x = c$ is represented by the equation

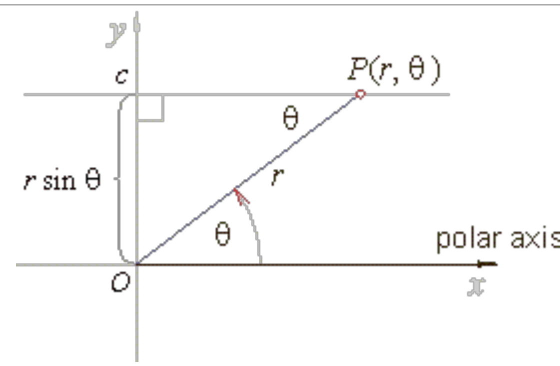
$$r\cos\theta = c \quad \text{or} \quad r = \frac{c}{\cos\theta}.$$



Lines parallel to the *x*-axis

A horizontal line, $y = c$ is represented by the equation

$$r\sin\theta = c \quad \text{or} \quad r = \frac{c}{\sin\theta}.$$



Lines running through the origin or pole (radial lines)

The equation of a line through the origin or pole that makes an angle α with the positive *x*-axis

$$y = mx, \text{ where } m = \tan\alpha \quad \text{therefore,} \quad y = \tan\alpha \cdot x \;\Big|\div x$$

$$\frac{y}{x} = \tan\alpha \;\Big|\; arc\tan$$

$$arc\tan\left(\frac{y}{x}\right) = arc\tan(\tan\alpha),$$

$$\text{since } \frac{y}{x} = \tan\theta \quad \text{then,} \quad arc\tan(\tan\theta) = arc\tan(\tan\alpha) \quad \text{or} \quad \theta = \alpha.$$

Thus, the equation of a line through the origin is represented by the equation $\theta = \alpha$ in polar coordinates.

Polar equation of a line

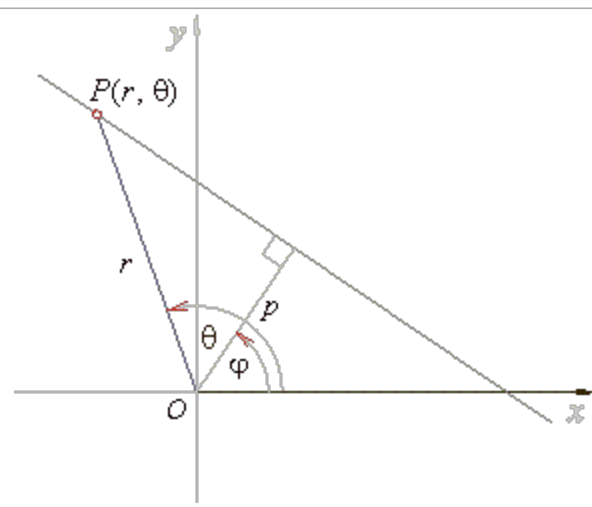
As

$$\cos(\theta - \varphi) = \frac{p}{r}$$

the polar equation of a line

$$r = \frac{p}{\cos(\theta - \varphi)}$$

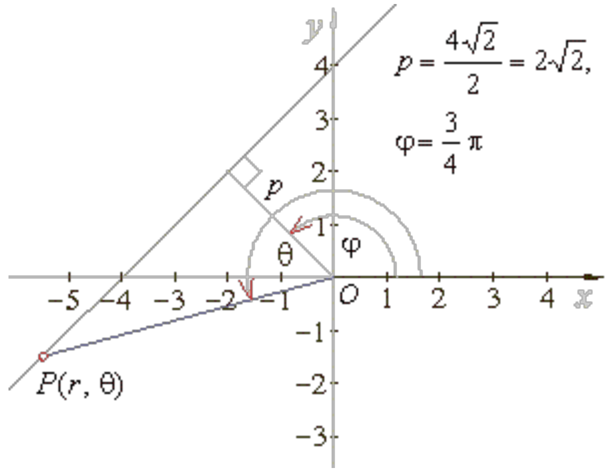
where p is the distance of the line from the pole O and φ is the angle that the segment p makes with the polar axis.



Example: Write polar equation of the line passing through points $(-4, 0)$ and $(0, 4)$.

Solution: Using polar equation of a line

$$r = \frac{p}{\cos(\theta - \varphi)}, \qquad r = \frac{2\sqrt{2}}{\cos\left(\theta - \frac{3}{4}\pi\right)}.$$



Proof, using Cartesian to polar conversion formulas.

The intercept form of the line $\frac{x}{a} + \frac{y}{b} = 1$ or

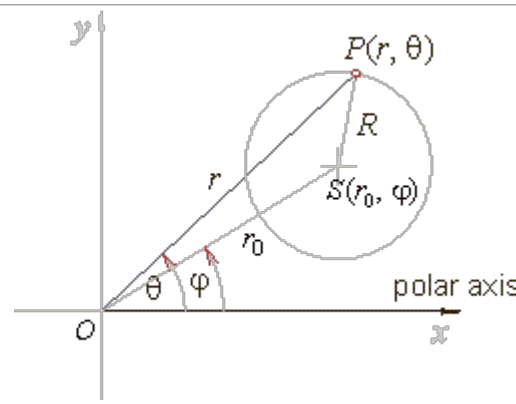
$$\frac{x}{-4} + \frac{y}{4} = 1 \quad \Big| \cdot 4$$

$$-x + y = 4, \qquad y = x + 4$$

Therefore, $-r\cos\theta + r\sin\theta = 4, \quad r(\sin\theta - \cos\theta) = 4 \quad \text{or} \quad r = \frac{4}{\sin\theta - \cos\theta} = \frac{2\sqrt{2}}{\cos\left(\theta - \frac{3}{4}\pi\right)}.$

:: Equation of a circle in polar form

General equation of a circle in polar coordinates

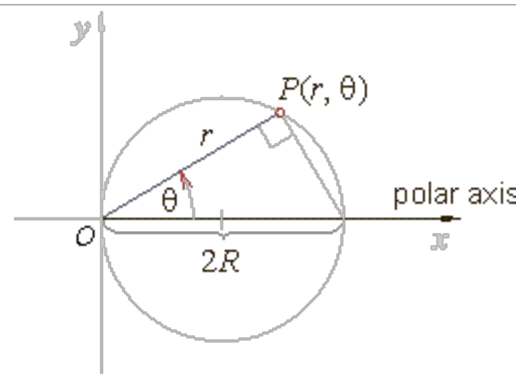


The general equation of a circle with a center at (r_0, φ) and radius R.

Using the law of cosine,

$$r^2 + r_0^2 - 2rr_0\cos(\theta - \varphi) = R^2$$

Polar equation of a circle with a center on the polar axis running through the pole



Polar equation of a circle with radius R and a center on the polar axis running through the pole O (origin).

Since $\cos\theta = \frac{r}{2R}$ then,

$$r = 2R\cos\theta$$

Polar equation of a circle with a center at the pole

Since, $r^2 = x^2 + y^2$ and $x^2 + y^2 = R^2$ then $r = R$

is polar equation of a circle with radius R and a center at the pole (origin).

Example: Convert the polar equation of a circle $r = -4\cos\theta$ into Cartesian coordinates.

Solution: As, $r = -4\cos\theta$ then $r^2 = -4r\cos\theta$,

and by using polar to Cartesian conversion formulas, $r^2 = x^2 + y^2$ and $x = r\cos\theta$

obtained is $x^2 + y^2 = -4x$

$$x^2 + 4x + y^2 = 0$$

or $$(x + 2)^2 + y^2 = 4$$

the equation of a circle with radius $R = 2$ and the center at $(-2, 0)$.