Entanglement and the Conceptual Basis of Quantum Mechanics
Part-II

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(Continued from Part I)

4. Impossibility of Hidden Variable

Not long after Schrodinger's discovery in 1926 of wave mechanics, Louis de Broglie found an equation of particle motion equivalent to the guiding wave equation for a scalar wave function but he very quickly abandoned this so-called pilot-wave approach. The pilot-wave approach was indeed initiated even before the discovery of quantum mechanics itself by Einstein who hoped that the interference phenomena involving particle-like photons could be explained if the motions of the photons were somehow guided by the electromagnetic field, which would thus play the role of what he called a *fűhrungsfeld* or guiding field.

This idea was rediscovered in 1952 by David Bohm who proceeded from the assumption that the wave function obeying Schrodinger equation does not provide a complete description or representation of the quantum system. According to him, the wave function governs the motion of the fundamental variables of the theory - the position co-ordinates of the particles. Thus quantum mechanics is here fundamentally about the behaviour of particles and the particles are described by their position co-ordinates which are the primary quantities while the wave function is secondary.

The state of a system of N particles is described by the wave function $\Psi(q_1,q_2, \ldots \ldots ,q_N)$which is a complex valued function on the space of possible configurations $q = q_i,q_2,\ldots\ldots,q_N$ of the system. The actual configuration is however defined by the actual position $Q \equiv Q_1,Q_2, \ldots\ldots ,Q_N$ of the particles and the theory is defined by two equations:

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$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \qquad \frac{dQ_i}{dt} = \hbar \operatorname{Im}(\Psi^* \partial_i \Psi / \Psi^* \Psi)(Q_1, Q_2, \ldots\ldots, Q_N)$$

where *H* is the Hamiltonian of the system. These two equations completely define Bohmian mechanics and this theory is deterministic accounting for all the phenomenon of non-relativistic quantum mechanics, from interference effects to spectral lines.

In his 1952 paper, Bohm arrived at this theory by writing the wave function in the polar form

$$\Psi = \mathrm{Re}^{i\frac{S}{\hbar}}$$

S and R are real, and rewriting the Schrodinger equation in terms of these new variables we obtain a pair of coupled evolution equations: the continuity equation for $\rho = R^2$,

$$\frac{\partial R}{\partial t} = -\frac{1}{2m}\left[R\nabla^2 S + 2\nabla R.\nabla S\right]$$

and the modified Hamiltonian-Jacobi equation

$$\frac{\partial S}{\partial t} = -\left[\frac{1}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right]$$

The quantum potential U is defined as

$$U = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

If we define a velocity **v** by

$$\mathbf{V} = -\frac{1}{m}\nabla S$$

then we easily find that the momentum **p** = m**v** satisfies

$$\frac{dp}{dt} = -\nabla(V + U)$$

so that the potential energy includes now the classical potential energy *V* as well as the quantum potential energy *U*.

The quantum potential introduces highly non-classical, non-local effects. The size of the quantum potential provides a measure of the deviation of Bohmian mechanics from its classical approximation. The quantum potential can be used to develop approximation schemes for solutions to the Schrodinger equation (Nemkh nad Frederick, Chem. Phys. Lett. 332,145, 2000).

Bohmian mechanics is manifestly non-local. The velocity as expressed in the guiding equation of any one of the particles of a many particle system will typically depend upon the position of the other, possibly distant particles, whenever the wave function of the system is entangled i.e. not a product of single particle wave functions. This is true for example for the EPR-Bohm wave function describing a pair of spin ½ particles in the singlet state. In this way, Bohmian mechanics makes explicit the most dramatic feature of quantum theory - *non-locality*. As Bell has stressed,

> That the guiding wave, in the general case, propagates not in ordinary three-space but in a multi-dimensional configuration space is the origin of the notorious "non-locality" of quantum mechanics. It is a merit of the de Broglie- Bohm version to bring this out so explicitly that it cannot be ignored.

5. General Criterion for the Entanglement of Two Indistinguishable Particles

Recently Ghirardi and Marinatto have analyzed in great detail the problem of entanglement dealing with systems of two or more distinguishable and identical particles (J. Stat. Phys.105,49,2002; Fortschr. Phys. 51, 379 2003, Phys. Rev. A 70,02109,2004). In these papers the non-occurrence of entanglement has been related to the possibility of attributing complete sets of properties to both constituents of the composite system. In this way an unambiguous criterion for deciding whether a given state vector is entangled or not has been given and it has been stressed that non-entangled states involving identical constituents can actually occur.

Several theorems have been proved to show that the correct way to identify the
entanglement is to relate it to the impossibility of attributing precise properties to the constituents of a two particles system $S = S_1 + S_2$.

Definition 1: The system S_1 described by the pure density operator $\rho = |\Psi(1,2)\rangle \times \langle\Psi(1,2)|$ is non-entangled with S_2 if there exists a projection operator $\mathbf{P}^{(1)}$ onto a one-dimensional manifold of H_1 such that

$$Tr^{(1+2)}[\mathbf{p}^{(1)} \otimes I^{(2)}\rho] = 1$$

Theorem 1. If a composite system described by the pure density operator ρ of $H = H_1 \otimes H_2$ each of the following conditions is necessary and sufficient in order that S_1 is non-entangled with S_2;

a) There exists a projection operator $\mathbf{P}^{(1)}$ onto a one-dimensional manifold of H_1 such that $Tr^{(1+2)}[\mathbf{p}^{(1)} \otimes I^{(2)}\rho] = 1$.

b) The reduced statistical operator $\rho^{(1)} = Tr^{(2)}[\rho]$ of S_1 is a projection operator onto a one-dimensional manifold of H_1.

c) The state vector $|\Psi(1,2)\rangle$ is factorizable i.e. there exist a state $|\varphi(l)\rangle \in H_1$ and a state $|\xi(2)\rangle \in H_2$ such that $|\Psi(1,2)\rangle = |\varphi(l)\rangle \otimes |\xi(2)\rangle$.

Theorem 2: A necessary and sufficient condition for the projection operator $P^{(1)}_{M1}$ onto the linear manifold M_1 of H_1 is to satisfy the two following conditions:

a) $Tr^{(1)}[\ P^{(1)}_{M1}\ \rho^{(1)} = l$

b) There is no projection operator $\mathbf{P}$ of H_1 satisfying the conditions $\mathbf{P}^{(1)} < \mathbf{P}^{(1)}_{M1}$ (i.e. it projects on a proper sub manifold N_1 of H_1)and $Tr^{(1)}[\mathbf{P}^{(1)}\rho^{(1)}] = 1$ is that the range $R[\rho^{(1)}]$of the reduced statistical operator $\rho^{(1)}$ coincides with M_1.

Theorem 3: Subsystem S_1 is non-entangled with subsystem S_2 if given the pure state $|\ \psi(l,2)\rangle$ of the composite system, the following equation holds for any pair of observable $A(l)$ of H_1 and $B(2)$ of H_2 such that $|\ \psi(l,2)\rangle$ belongs to their domains,

$$\langle\psi(l,2)|A(l) \otimes B(2)|\psi(l,2)\rangle = \langle\psi(l,2)|A(l) \times I^{(2)}|\psi(l,2)\rangle\langle\psi(l,2)|I^{(1)} \otimes B^{(2)}|\ \psi(l,2)\rangle$$

The proofs of these theorems are fairly straightforward and given in the quoted papers.

The problem of attributing properties to and the analysis of entanglement of composite systems whose constituents are identical is a quite delicate one. This is because the "principle of individuality" of physical systems has a long history in philosophy and even Leibniz claimed that, "there are in Nature no two exactly similar entities in which one cannot find an internal difference". This situation however is quite different in quantum mechanics because it does not allow the idea of particle trajectories. This implies that even if we label one of the two identical particles as 1 at a certain time and the other as 2, we will not be able even in principle to claim with respect to a subsequent act of detection that the particle which has been detected is the one labeled 1 or 2. This has led some philosophers of science to the conclusion that quantum particles cannot be regarded as individuals in any of the traditional meaning of such a term. However we simply accept the idea that when we are dealing with assemblies of identical quantum systems it is simply meaningless to try to "name" them in any way. From this quantum mechanical point of view, we believe that we are justified in considering the constituents of a system of identical particles as possessing objectively definite properties, which can be discussed in a mathematical consistent way. Correspondingly we can formulate in a precise way the idea that identical particles are non-entangled.

Definition 2: The identical constituents S_1 and S_2 of a composite quantum system
$S = S_1 + S_2$ are non-entangled when both constituents possess a complete set of properties.

Definition 3: Given a composite quantum system $S = S_1 + S_2$ of two identical particles described by the pure density operator ρ, we will say that one of the constituents has a complete set of properties if there exists a one dimensional projection operator P defined on the Hilbert space of each of the subsystems such that

$$Tr^{(1+2)}[E(1,2)\rho] = 1$$

where $E(l,2) = P^{(1)} \times I^{(2)} + I^{(1)} ® P^{(2)} - P^{(1)} ® P^{(2)}$.

This gives the probability of finding at least one of the two identical particles in the state onto which the one dimensional operator P projects. Now the following theorem relates the fact that one constituent possesses a complete set of properties to the explicit from of the state vector.

Theorem 4: One of the identical constituents of a composite quantum system $S = S_1 + S_2$ described by the pure normalized state $|\psi(1,2)>$ has a complete set of properties if $|\psi(1,2)>$ is obtained by symmtrizing or antisymmetrizing a factorized state.

Ghirardi, Marinotto and Weber who have dealt separately with the cases of identical fermions and identical bosons have given the proof of this theorem. They have based their proof on the Slater and Schmidt numbers of the Schmidt decomposition of the state vector of the composite system. This is related to the so-called von Neumann entropy of the one particle reduced density operator and this supplies us with a consistent criterion for detecting entanglement. To quote these authors,

> In particular the consideration of the von Neumann entropy is particularly useful in deciding whether the correlations of the considered states are simply due to the indistinguishability of the particles involved or are a genuine manifestation of the
> entanglement.

Conclusion

Many of us do not wish to say what David Mannin has said, "So farewell, elements of reality", nor do we agree with A. D. Aczel who has an excellent popular book on Entanglement that "even the greatest physicist of the twentieth century, Albert Einstein, was fooled ...into believing that quantum mechanics was incomplete". I find it difficult to believe, as Einstein did not believe, that moon is not there, “when nobody looks". Let me therefore conclude by quoting S. Goldstein,

> Any one who has engaged in arguments with colleagues about the foundations of quantum mechanics, whatever his position, will likely agree with the following observation of Tolstoy:
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>> ‘I know that most men, including those at ease with problems of the highest complexity, can seldom accept even the simplest and most obvious truth if it be such as would oblige them to admit the falsity of conclusions which they have delighted in explaining to colleagues, which they have proudly taught to others and which they have woven, thread by thread, into the fabric of their lives’.

This seems to be a fair warning to all of us.
(End)