CHAPTER

0

Foundational Preliminaries: Answers to Within-Chapter Exercises

0A Solutions to Within-Chapter-Exercises for Part A

Exercise 0A.1

Consider the set $[0,1)$ which includes the point 0, all the points in between 0 and 1 but not the point 1. Is this set convex? What about the union of this set with the point 1? What about the union of this set with the point 1.1?

Answer: The set $[0,1)$ is convex. So is the union of this set with 1, denoted $[0,1]$ (as illustrated in the chapter). But the union with 1.1 is not convex because any line segment connecting 1.1 with a point in $[0,1)$ is not fully contained in the union of $[0,1)$ with 1.1.

Exercise 0A.2

Is the set of all rational numbers a convex set? What about the set of all non-integers? Or the set of all irrational numbers?

Answer: The set of all rational numbers is not convex because any line segment connecting two rational numbers contain irrational numbers that are not in the set of rational numbers.

Exercise 0A.3

Describe points B, C, D and E in Graph 0.2 as a pair of real numbers.

Answer: $B = (8,2)$, $C = (6,5)$, $D = (4,5)$, and $E = (7,6)$.

Exercise 0A.4

Which of the sets in Graph 0.3 is/are not convex?

Answer: Only the set in (a) is convex.

Exercise 0A.5

Can you use points A and B to arrive at the same value for the slope? What about the points D and F or the points D and C?

Answer: From A to B, we have a positive rise of 20 and a negative run of 10 — giving us a slope of $-20/10 = -2$. From D to F, we have a positive rise of 2 and a negative run of 1 — giving us a slope of $-2/1 = -2$. From D to C, we have a positive rise of 4 and a negative run of 2 — giving us a slope of $-4/2 = -2$.

Exercise 0A.6

Is the shaded set in Graph 0.4 a convex set?

Answer: Yes.

Exercise 0A.7

Suppose the blue line in Graph 0.4 had a kink in it. This kink could point "inward" (i.e. toward to origin) or "outward" (i.e. away from the origin). For which of these would the shaded area underneath the kinked line become a non-convex set?

Answer: If the kink points "inward", we get a graph like panel (b) in Graph 2.4 of the text. Connecting B and A in that graph gives us a line segment that lies fully above the set defined by the kinked lines — implying the set below is non-convex. If the kink points "outward", we get a graph like panel (a) of Graph 2.4 in the text. The set that lies underneath is then convex.

Exercise 0A.8

Check to see that the other intercepts (at B and A) are correctly labeled based on equation (0.4).

Answer: Setting $z = 0$ and $y = 0$, the equation gives us $4x = 40$ which implies $x = 10$ — the intercept at A. Setting $x = 0$ and $z = 0$, the equation gives us $2y = 40$ which implies $y = 20$ — the intercept at B.

Exercise 0A.9

Is the plane in Graph 0.5 a convex set?

Answer: Yes.

Exercise 0A.10

Given the equation (0.4) that describes the 3-dimensional plane, what is the equation that describes the magenta line segment which intersects with the plane in panel (a)?

Answer: At that slice, $z = 10$. Setting z equal to 10 in the equation $4x+2y+z = 40$, we get $4x + 2y + 10 = 40$ or $4x + 2y = 30$. Re-writing this in terms of y, we get the equation $y = 15 - 2x$, an equation with vertical intercept of 15 and slope of -2.

Exercise 0A.11

Suppose I like eating steak and will eat more steak as my income goes up. Which way will my demand curve for steak shift as my income increases? Can you think of any goods for which my demand curve might shift in the other direction as my income increases?

Answer: As income increases, the demand curve for steak will shift to the right (or "down"). For some goods, our consumption might decrease as our income increases. For instance, perhaps we buy less pasta as we get richer — implying our demand curve for pasta shifts to the left (or "up") as income increases.

Exercise 0A.12

Coffee and sugar are *complements* for me in the sense that I use sugar in my coffee. Can you guess which way my demand curve for coffee will shift as the price of sugar increases?

Answer: When the price for sugar increases, I will buy *less* sugar as I slide up the demand curve for sugar. Since sugar and coffee are complements, I will also buy less coffee — implying that my demand curve for coffee shifts to the left (or "up").

Exercise 0A.13

Ice tea and coffee are *substitutes* for me in the sense that I like both of them but will only drink a certain total amount of liquids. Can you guess which way my demand curve for coffee will shift as the price of iced tea increases?

Answer: As the price of iced tea increases, I slide up on the demand curve for iced tea, implying I will buy less iced tea. Since iced tea and coffee are substitutes for me, I will likely buy more coffee — implying my coffee consumption goes up. So my demand curve for coffee will shift to the right (or "down").

Exercise 0A.14

How would the supply curve for a firm shift if the general wage rate in the economy increases?

Answer: The supply curve would shift to the left (or "up").

Exercise 0A.15

Would you expect the supply curve for a firm that produces x to shift when the price of some other good y (that is not used in the production of x) increases?

Answer: The increase in the price of y is unrelated to the production costs for x — and so the supply curve would not shift.

Exercise 0A.16

If the supply curve depicts the supply curve for a *market* composed of many firms, we may also see shifts in the supply curve that arise from the *entry* of new firms or the *exit* of existing firms. How would the market supply curve shift as firms enter and exit?

Answer: As firms enter, the market supply curve would shift to the right (or "down"), and as firms exit, the market supply curve would shift to the left (or "up").

Exercise 0A.17

Is consumer 2's consumption also more elastic than consumer 1's when price falls?

Answer: Yes, when price falls, the green consumer 2 will increase consumption more than the magenta consumer 1.

Exercise 0A.18

Do slopes similarly change if we measure price differently — i.e. if we measure price in euros instead of dollars?

Answer: Yes, as we change the units we use on the vertical axis, slopes will change just as they do if we change units on the horizontal axis.

Exercise 0A.19

In panel (b) of Graph 0.9, the supply curves of two producers who both produce x^* at the price p^* are illustrated. Which producer is more price elastic when price increases? What about when price decreases?

Answer: The green producer 2 is more price elastic as price increases — raising output to x'' when price increases to p' while the blue producer 1 only raises output to x'. Producer 2 is similarly more price elastic when price falls.

Exercise 0A.20

How much does the consumer spend when price is \$50? How much does she spend when price increases to \$100?

Answer: When price is \$50, the consumer spends $50(700) = \$35,000$.When price is \$100, the consumer spends $100(600) = \$60,000$.

Exercise 0A.21

What is the size of the blue shaded area in panel (a)? What about the magenta area? Is the difference between the magenta and the blue area the same as the increase in spending you calculated in exercise 0A.20?

Answer: The blue shaded area has height of 50 and length of 100 — implying a size of 5,000. The magenta area has height of 50 and length of 600 — implying a size of 50(600)=30,000. The difference between the magenta and the blue areas is therefore 25,000. In the previous exercise we concluded that the consumer spends \$35,000 at the lower price and \$60,000 at the higher price — a difference of \$25,000.

Exercise 0A.22

A *price ceiling* is a government-enforced maximum legal price. In order for such a price ceiling to have an impact on the price at which goods are traded, would it have to be set above or below the equilibrium price p^*?

Answer: If the price ceiling is set above p^*, the equilibrium price p^* is legal — and so the equilibrium is undisturbed. But if the price ceiling is set below p^*, the equilibrium price p^* is no longer legal — implying that the price ceiling has an impact.

Exercise 0A.23

If a price ceiling changes the price at which goods are traded, would you expect a "shortage" or a "surplus" of goods to emerge? How would the magnitude of the shortage or surplus be related to the price elasticity of demand?

Answer: A price ceiling that has an impact is set below p^* — where the quantity demanded read off the magenta demand curve is greater than the quantity supplied (read off the blue supply curve). Thus, more is demanded than supplied at the price ceiling — implying a shortage.

Exercise 0A.24

A *price floor* is a government-enforced minimum legal price. Repeat the previous two questions for a price floor instead of a price ceiling.

Answer: If the price floor is set below p^*, it has no impact because p^* is still legal. But if the price floor is set above p^*, the equilibrium price p^* is no longer legal — implying the price floor has an impact on the market. At a price floor above p^*, the quantity demanded (read off the magenta demand curve) is less than the quantity supplied (read off the blue supply curve) — implying that more is supplied than demanded. Thus, we have a surplus.

Exercise 0A.25

Can you come to similar conclusions about decreases in market prices by looking at Graph 0.13?

Answer: If a price decrease is accompanied by a decline in market output (as in panel (a) of Graph 0.13), the price decrease must be driven by a decrease in demand. But if the price decrease is accompanied by an increase in market output (as in panel (b) of Graph 0.13), the price decrease is driven by an increase in supply.

Exercise 0A.26

Suppose that, instead of taxing the sale of x, the government *subsidized* consumer purchases of x. Thus, consumers will be paid an amount s for each good x they buy. Can you use Graph 0.15 to determine whether firms will benefit from such a consumer subsidy — and how the per-unit benefit for firms depends on the price elasticity of demand?

Answer: Firms will benefit because the equilibrium price will increase (as shown in Graph 0.15 where demand shifts up because of the consumer subsidy). But firms will benefit more the more price inelastic the supply curve — because the price increase will be larger the less elastic the supply curve is. Put differently, more of the consumer subsidy will be passed onto producers as the supply curve becomes more price inelastic.

Exercise 0A.27

If the goal of consumer subsidies is to raise economic output in a market, will the government be more likely to succeed in markets will high or low price elasticities of demand?

Answer: The government will be more successful if the market supply curve is more price elastic (as in panel (a) of Graph 0.15) than if the supply curve is more price inelastic (as in panel (b) of Graph 0.15).

0B Answers to Within-Chapter-Exercises for Part B

Exercise 0B.1

Consider the function $f(x, y, z) = xy + z$. How would you describe this function in terms of the notation of equation (0.5)? What value does the function assign to the points (0,1,2), (1,2,1) and (3,2,4)?

Answer: $f : \mathbb{R}^3 \to \mathbb{R}^1$.

Exercise 0B.2

How would you write the expression for the set of points that lie above the function in panel (a) of Graph 0.16? Which is different from expression (0.6): the necessary or the sufficient condition?

Answer: This would be written as

$$\left\{(x, y) \in \mathbb{R}^2_+ \mid y \geq x^2\right\}. \quad (0B.2)$$

The necessary condition (which comes before the vertical line in the expression) remains the same, but the sufficient condition has changed.

Exercise 0B.3

Is this set a convex set? What about the set described in expression (0.6) and the set defined in exercise 0B.2?

Answer: Yes, this is a convex set, as is the set described in equation (0.6) of the chapter. The set defined in the previous exercise, however, is non-convex.

Exercise 0B.4

Suppose that the quantity of the good x that is demanded is a function of not only p_x and I but also p_y, the price of some other good y. How would you express such a demand function in the notation of equation (0.5)?

Answer: $f : \mathbb{R}^3 \to \mathbb{R}^1$.

Exercise 0B.5

Following on exercise 0B.4, suppose the demand function took the form $f(p_x, p_y, I) = (I/2) + p_y - 2p_x$. How much of x will the consumer demand if $I = 100$, $p_x = 20$ and $p_y = 10$?

Answer: The consumer would demand $(100/2) + 10 - 2(20) = 50 + 10 - 40 = 20$.

Exercise 0B.6

Using the demand function from exercise 0B.5, derive the demand *curve* for when income is 100 and $p_y = 10$.

Answer: Substituting $I = 100$ and $p_y = 10$ into the function, we get $f(p_x) = 50 + 10 - 2p_x = 60 - 2p_x$. Inverting this, we solve the equation $x = 60 - 2p_x$ for p_x to get $p_x = 30 - 0.5x$. This gives us the equation for the demand curve that has p_x on the vertical axis and x on the horizontal — i.e. an equation with vertical intercept of 30 and slope of -0.5.

Exercise 0B.7

Verify the last sentence of the previous paragraph.

Answer: From the demand function $f(p_x, I) = (I/2) - 10p_x$, we get $\overline{g}(p_x) = (200/2) - 10p_x = 100 - 10p_x$ when we substitute in $I = 200$.

Exercise 0B.8

On a graph with p_x and I on the lower axes and x on the vertical axis, can you graph $f(p_x, I) = (I/2) - 10p_x$? Where in your graph are the slices that hold I fixed at 100 and 200? How do these slices relate to one another when graphed on a 2-dimensional graph with p_x on the horizontal and x on the vertical axis?

Answer: The graph is a 3-dimensional plane, with x (on the vertical axis) increasing as income increases and falling as price increases. Holding I fixed, we simply get downward sloping linear demand functions that have x (on the vertical axis) fall as the price increases. And as income increases, these 2-dimensional demand functions shift out.

Exercise 0B.9

Return to exercise 0B.5 and suppose that $I = 100$ and $p_y = 10$. What is $g(p_x)$? How does it change when p_y increases to 20? How does this translate to a shift in the demand *curve*?

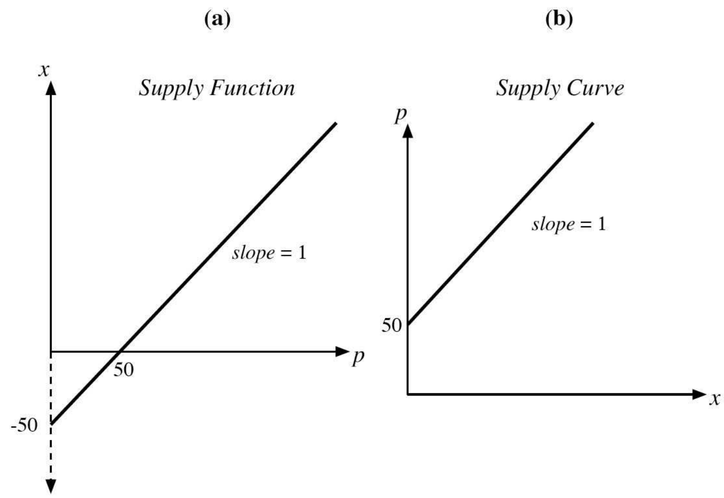
Answer: Plugging $I = 100$ and $p_y = 10$ into the function $f(p_x, p_y, I) = (I/2) + p_y - 2p_x$, we get $g(p_x) = 50 + 10 - 2p_x = 60 - 2p_x$. When p_y increases to 20, the new function becomes $g'(p_x) = 70 - 2p_x$ — which implies the slope is unchanged but the intercept shifts out. This translates to a shift of the demand curve to the right.

Exercise 0B.10

The same relationship between demand functions and demand curves exists for supply functions and supply curves. Suppose, for instance, that supply is a function of the wage rate ω and the output price p and is given by the supply function $f(\omega, p) = p - 5\omega$. Illustrate the "slice" of the supply function that holds ω fixed to

10, and then derive from it the supply curve. *(Hint: You should get two graphs analogous to Graph 0.18.)*

Answer: When ω is fixed at 10, the supply function becomes $g(p) = p - 50$ which is graphed in panel (a) of Exercise Graph 0B.10. The supply curve is the inverse — graphed in panel (b).



Exercise Graph 0B.10 : Supply Function and Supply Curve

Exercise 0B.11

What happens in your two graphs from exercise 0B.10 when ω changes to 5?

Answer: When ω is fixed at 5, the supply function becomes $g(p) = p - 25$. This shifts the supply function and curve as illustrated in Exercise Graph 0B.11.

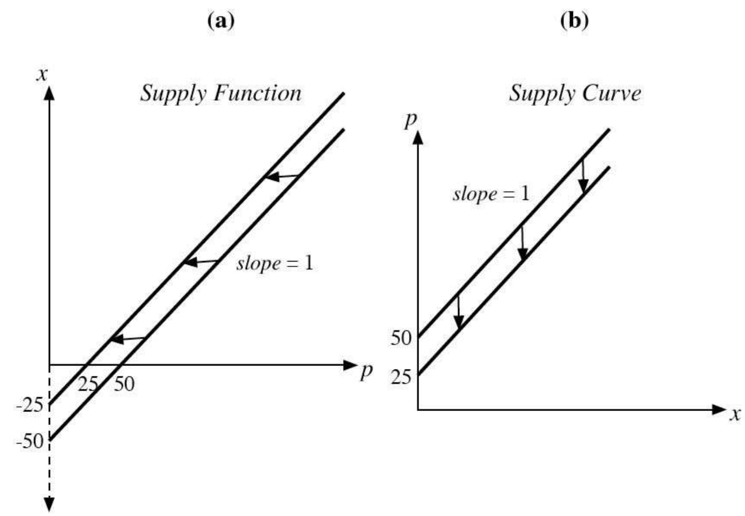
Exercise 0B.12

Can you tell how equilibrium price changes as A and B change? Can you make intuitive sense of this by linking changes in parameters to changes in Graph 0.19?

Answer: As both A and B increase, the equilibrium price rises — since both A and B appear as positive quantities in the numerator of the p^* equation. This makes intuitive sense in that an increase in A is an upward shift of the demand curve and an increase in B is an upward shift in the supply curve. Both such upward shifts will result in the intersection of D and S shifting up.

Exercise 0B.13

Can you do the same for the equilibrium quantity?



Exercise Graph 0B.11 : Shifts in Supply Function and Curve

Answer: In the equilibrium quantity equation for x^*, A enters positively but B enters negatively in the numerator. Thus, as A increases, output increases; but as B increases, output falls. This again makes intuitive sense: An increase in A is an outward shift in D — which implies a rightward shift in the intersection of S and D (and thus an increase in equilibrium output). An increase in B, on the other hand, implies an upward shift in supply which results in a leftward shift of the intersection of D and S, and thus a decrease in the equilibrium output.

Exercise 0B.14

In this Chapter 6 problem, it turns out we do not care about the value of λ. Suppose, however, we did. How would you derive it's value?

Answer: You could now plug in $x_1 = 5$ and $x_2 = 10$ into either of the first two equations and then solve for λ. In both cases, we get an approximate value of 0.03536 for λ.

Exercise 0B.15

Can you prove Rules 3 through 5 by just using the definition of exponents?

Answer: For Rule 3: x^m is the same as x multiplied m times, and x^n is the same as x multiplied n times. Thus, $x^m x^n$ must be the same as x multiplied $(m+n)$ times. For Rule 4: We are simply dividing x multiplied m times by x multiplied n times. If $m > n$, that means n of the m multiplied x's in the numerator cancel, leaving us with $(m-n)$ multiplied x's in the numerator. If $m < n$, all the x's in the numerator cancel, leaving us with $(n-m)$ multiplied x's in the denominator — i.e. $1/(x^{(n-m)})$ which can simply be written as $x^{-(n-m)}$ or $x^{(m-n)}$. For Rule 5: When we write $(x^m)^n$, we are saying that we will multiply x^m by x^m a total of n times. So we are left with x^{mn}.

Exercise 0B.16

Simplify the following: $\frac{x^{3/2}y^{1/2}}{x^{1/2}y^{-1/2}}$.

Answer: This expression reduces to xy as we subtract the exponents in the denominator from those in the numerator.

Exercise 0B.17

Simplify the following: $\left(\frac{x^{4/5}y^2}{x^{-1/2}y^4}\right)^{-2}$.

Answer: Subtracting exponents in the denominator from those in the numerator inside the parentheses, we get $\left(x^{13/10}y^{-2}\right)^{-2}$, and multiplying the exponents inside the parentheses by -2, we end up with $x^{-13/5}y^4$.

Exercise 0B.18

Solve the following quadratic equation by factoring: $2x^2+2x-12=0$.

Answer: First, we can divide both sides of the equation by 2 to get $x^2+x-6=0$. We can then factor this to get $(x+3)(x-2)=0$ — implying the solutions $x=-3$ and $x=2$.

Exercise 0B.19

An equation is still quadratic if $b=0$. Solve the following quadratic equation by factoring: $x^2-4=0$.

Answer: This is factored as $(x-2)(x+2)=0$ — giving us the solutions $x=2$ and $x=-2$.

Exercise 0B.20

Does the quadratic equation $x^2+4x+4=0$ have two solutions?

Answer: This equation factors to $(x+2)(x+2)=0$, which gives us $x=-2$ — a single solution.

Exercise 0B.21

Use the quadratic formula to verify your answer to exercises 0B.18, 0B.19 and 0B.20.

Answer: In exercise 0B.18, $a=2$, $b=2$ and $c=-12$. Thus, the quadratic formula gives the solutions

$$\frac{-2+\sqrt{2^2-4(2)(-12)}}{2(2)}=\frac{-2+\sqrt{100}}{4}=2, \tag{0B.21.i}$$

and

$$\frac{-2-\sqrt{2^2-4(2)(-12)}}{2(2)}=\frac{-2-\sqrt{100}}{4}=-3. \qquad (\text{0B.21.ii})$$

In exercise 0B.19, $a=1$, $b=0$ and $c=-4$. The quadratic formula then gives solutions

$$\frac{\sqrt{-4(1)(-4)}}{2(1)}=\frac{4}{2}=2, \qquad (\text{0B.21.iii})$$

and

$$\frac{-\sqrt{-4(1)(-4)}}{2(1)}=\frac{4}{2}=-2. \qquad (\text{0B.21.iv})$$

And in exercise 0B.20, $a=1$, $b=4$ and $c=4$. The quadratic formula then implies solutions

$$\frac{-4+\sqrt{4^2-4(1)(4)}}{2(1)}=\frac{-4}{2}=-2 \qquad (\text{0B.21.v})$$

and

$$\frac{-4-\sqrt{4^2-4(1)(4)}}{2(1)}=\frac{-4}{2}=-2. \qquad (\text{0B.21.vi})$$

Exercise 0B.22

Solve the quadratic equation $3x^2+8x+4=0$.
Answer: With $a=3$, $b=8$ and $c=4$, we get

$$\frac{-8+\sqrt{8^2-4(3)(4)}}{2(3)}=\frac{-8+\sqrt{16}}{6}=-\frac{4}{6}=-\frac{2}{3} \qquad (\text{0B.22.i})$$

and

$$\frac{-8-\sqrt{8^2-4(3)(4)}}{2(3)}=\frac{-8-\sqrt{16}}{6}=-\frac{12}{6}=-2. \qquad (\text{0B.22.ii})$$

Exercise 0B.23

Re-write the following as an expression with a single ln term: $\ln(2x)+\ln(y)-\ln(x)$.
Answer: $\ln(2x^2y)$.

Exercise 0B.24

Simplify the following: $\ln e^{(x^2+xy)}$.
Answer: $x^2 + xy$.

Exercise 0B.25

Solve the following for x: $\ln e^{(x^2+4)} = \ln e^{4x}$.
Answer: The equation can be simplified to read $x^2+4 = 4x$ and can be written in the form $x^2-4x+4 = 0$. This in turn can be factored and written as $(x-2)(x-2) = 0$ which implies $x = 2$.

Exercise 0B.26

Show that Rule 2 follows from Rule 3.
Answer: Employing Rule 3 with $n = 1$, we get $\frac{d(\gamma x)}{dx} = (1)\gamma x^{(1-1)} = \gamma$.

Exercise 0B.27

Differentiate the following with respect to x: $f(x) = 3x^3 + 2x^2 + x + 4$.
Answer: We get

$$\frac{df(x)}{dx} = 9x^2 + 4x + 1. \quad (0B.27)$$

Exercise 0B.28

Use the product rule to differentiate the function $(x+3)(2x-2)$ with respect to x.
Answer: We get

$$(1)(2x-2) + (x+3)(2) = 2x - 2 + 2x + 6 = 4x + 4. \quad (0B.28)$$

Exercise 0B.29

Multiply the function in exercise 0B.28 out and solve for its derivative with respect to x without using the product rule. Do you get the same answer?
Answer: Multiplying through we get $2x^2 + 4x - 6$. Taking the derivative, we get $4x + 4$ as in the previous exercise.

Exercise 0B.30

Use the quotient rule to solve for the first derivative of $\frac{(x+3)}{2x-2}$.
Answer: The quotient rule gets us

$$\frac{(1)(2x-2)-(x+3)(2)}{(2x-2)^2} = \frac{2x-2-2x-6}{(2x-2)^2} = -\frac{8}{(2x-2)^2} \tag{0B.30}$$

Exercise 0B.31

Multiply out $(x^3-2)^2$ and take the derivative without using the chain rule. Do you get the same result?

Answer: Multiplying out $(x^3-2)^2$, we get $(x^3-2)(x^3-2) = x^6 - 4x^3 + 4$. Taking the derivative, we get $6x^5 - 12x^2$ as in the chapter when we used the chain rule.

Exercise 0B.32

In exercise 0B.30, you used the quotient rule to evaluate the derivative of $\frac{(x+3)}{2x-2}$. This function could alternatively be written as $(x+3)(2x-2)^{-1}$. Can you combine the product rule and the chain rule to solve again for the derivative with respect to x? Is your answer the same as in exercise 0B.30?

Answer: Combining the product and chain rule, we get

$$(1)(2x-2)^{-1} + (x+3)(-1)(2x-2)^{-2}(2) = \frac{1}{(2x-2)} - \frac{2(x+3)}{(2x-2)^2} \tag{0B.32.i}$$

which can then be simplified as

$$\frac{2x-2}{(2x-2)^2} - \frac{2x+6}{(2x-2)^2} = -\frac{8}{(2x-2)^2}, \tag{0B.32.ii}$$

the same answer as we previously derived using just the quotient rule.

Exercise 0B.33

Derive the derivative of $f(x) = \ln(x^2)$ using the chain rule. Then, use Rule 1 from our logarithm section to re-write $f(x)$ in such a way that you don't have to use the chain rule. Check whether you get the same answer.

Answer: Using the chain rule, we get

$$\frac{1}{x^2}(2x) = \frac{2}{x}. \tag{0B.33.i}$$

We can also write $\ln(x^2)$ as $\ln x + \ln x$, and the derivative of this (without using the chain rule) is

$$\frac{1}{x} + \frac{1}{x} = \frac{2}{x}. \tag{0B.33.ii}$$

Exercise 0B.34

We have seen that price elasticities vary along linear demand curve. Consider now the demand curve given by the equation $x(p) = \alpha/p$. Is the same true for this demand function?

Answer: The price elasticity now is

$$\varepsilon_d = \frac{dx(p)}{dp}\frac{p}{x(p)} = \frac{-\alpha}{p^2}\left(\frac{p}{\alpha/p}\right) = -\frac{\alpha}{p^2}\left(\frac{p^2}{\alpha}\right) = -1. \qquad (\text{0B.34})$$

Thus, for the demand function $x(p) = \alpha/p$, the price elasticity of demand is constant at -1 throughout.

Exercise 0B.35

Verify that this is correct.

Answer: Applying the product and chain rules, we get

$$\frac{df(x)}{dx} = \frac{1}{2}x^{-1/2}(20-2x)^{1/2} - x^{1/2}\left(-\frac{1}{2}\right)(20-2x)^{-1/2}(-2) \qquad (\text{0B.35})$$

which simplifies to $\frac{1}{2}x^{-1/2}(20-2x)^{1/2} - x^{1/2}(20-2x)^{-1/2}$.

Exercise 0B.36

Evaluate the following statement: A derivative of zero is a necessary but not a sufficient condition for us to identify a maximum of a function.

Answer: The derivative of the function will be zero at every maximum — so the condition is a necessary condition for a maximum. But not all points at which the derivative is zero represent a maximum (because the derivative is also zero at a minimum). It is therefore not a sufficient condition for a maximum.

CHAPTER

1

An Overview of the *Study Guide* and Chapter 1

Study guides that accompany textbooks are usually written as an afterthought by someone who has little connection to the textbook that is supposed to be supported by the study guide. I think it is for this reason that most study guides are not terribly helpful — and, I am told, they are not widely used by students.

This *Study Guide* is different in that it was written by me (the author) at the same time as the textbook was being written. It is therefore completely integrated into the textbook and based entirely on material that is introduced there. To be more precise, the textbook not only provides an initial introduction to the material but also serves as a launching pad for you to engage more directly with the material. It is really only through such a more direct engagement that you will begin to internalize the various concepts – and the *Study Guide* provides some structure around which to build that engagement.

Study Guide Overview

Here are the main ways to use the *Study Guide* for Chapters 2 through 29 as you proceed:

1. Each chapter begins with a quick **bullet point overview** of the main concepts in the corresponding textbook chapter. This essentially provides a 1 minute review highlighting the big ideas.

2. **Detailed explanations to within-chapter-exercises** are then provided for every within-chapter exercise. These exercises were constructed to help you absorb the material as you read the text – and my own students have done considerably better on exams when I have provided them with answers to these questions. So I am making them generally available here.

3. The odd-numbered end-of-chapter exercises in the textbook are marked with a (†) that indicates solutions to those exercises are provided in the *Study Guide*. Each chapter of the *Study Guide* therefore contains these **selected end-of-chapter exercise solutions**.

4. Finally, each chapter has a short conclusion that offers some hopefully **helpful hints**.

Most of the *Study Guide* is, as you will see, taken up by explanations of within-chapter and end-of-chapter exercises. My hope is that you will turn to these after attempting them on your own. I think the best way to approach the within-chapter-exercises is to do them as you read the text — or as you listen to the animated graphs in the eReader corresponding to the relevant sections of the text. This way you confront the main ideas immediately — and they will stick with you much more easily.

Chapter 1 Overview

Chapter 1 differs from the other chapters in the book in that it serves primarily as a big-picture introduction to the field of microeconomics (and to the structure of the textbook). As a result, there are no within-chapter or end-of-chapter exercises. Still, there are a few points worth highlighting:

1. Economics is much broader than most people think — it is (largely) a **science** in the sense that it attempts to predict, and it does so by constructing **models** that treat individuals as trying to **do the best they can given their circumstances**.
2. Economics does not stop at analyzing individual behavior but is furthermore interested in how individual behavior results in **social consequences** that we capture in the notion of an **equilibrium**.
3. Economists use models — simplified versions of reality — to capture the essence of real-world situations. Such models are usually meant to predict in the simplest possible way — making **simplicity of models a virtue**.
4. **Positive economists** restrict themselves to using their tool kit to *predict* outcomes of different institutional incentives; **normative economists** use their tool kit to furthermore *judge* these outcomes. But the line between positive and normative economics can get blurry — as in the case of **efficiency** predictions that are often interpreted as normative prescriptions.
5. When viewing the world through the economist's lens, we will find that
    a) It isn't always (or even usually) the case that there must be a loser if there is a winner;
    b) "Good" people can behave "badly" if their incentives are sufficiently perverse; and
    c) The existence of (an economic) order does not necessary imply that the order was created by anyone — it could have emerged spontaneously.

CHAPTER

2

Choice Sets and Budget Constraints

Consumers are people who try to do the "best they can" given their "budget circumstances" or what we will call their **budget constraints**. This chapter develops a model for these budget constraints that simply specify which **bundles of goods and services** are affordable for a consumer.

Chapter Highlights

The main points of the chapter are:

1. Constraints arise from "what we bring to the table" — whether that is in the form of an **exogenous income** or an **endowment** — and from the **opportunity costs** that arise through prices.

2. Changes in "what we bring to the table" do not alter opportunity costs — and thus **shift budgets without changing slopes**.

3. Changes in prices result in changes in opportunity costs — and thus alter the **slopes of budgets**.

4. With three goods, budget constraints become planes and choice sets are 3-dimensional — or they can be treated mathematically instead of graphically.

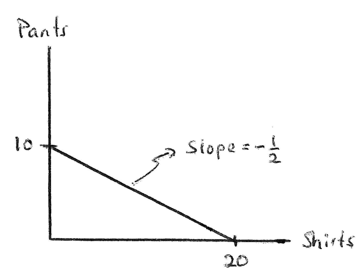
5. A **composite good** represents a way of indexing consumption other than the good of interest — and allows us to make the 2-good model more general.

2A Solutions to Within-Chapter-Exercises for Part A

Exercise 2A.1

Instead of putting pants on the horizontal axis and shirts on the vertical, put pants on the vertical and shirts on the horizontal. Show how the budget constraint looks and read from the slope what the opportunity cost of shirts (in terms of pants) and pants (in terms of shirts) is.

Answer: This is illustrated in Exercise Graph 2A.1. The slope of the budget would now be $-1/2$. Since the slope of the budget is the opportunity cost of the good on the horizontal axis in terms of the good on the vertical axis, this implies that the opportunity cost of shirts in terms of pants is 1/2. The inverse of the slope of the budget is the opportunity cost of the good on the vertical axis in terms of the good on the horizontal axis. Therefore the opportunity cost of pants in terms of shirts is 2.

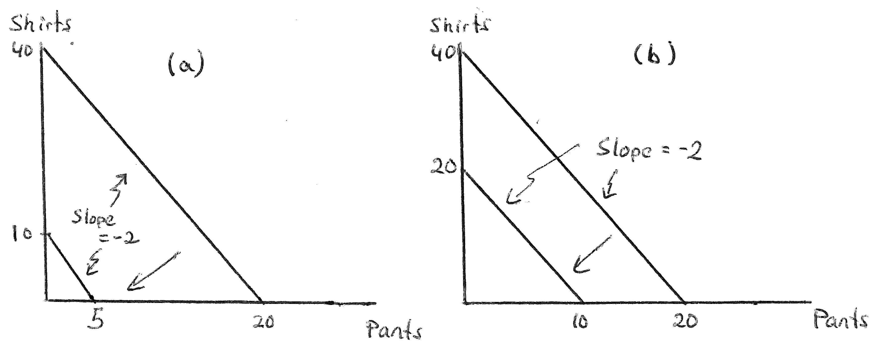


Exercise Graph 2A.1 : Graph for Within-Chapter-Exercise 2A.1

Exercise 2A.2

Demonstrate how my budget constraint would change if, on the way into the store, I had lost \$300 of the \$400 my wife had given to me. Does my opportunity cost of pants (in terms of shirts) or shirts (in terms of pants) change? What if instead the prices of pants and shirts had doubled while I was driving to the store?

Answer: The budgets would shift parallel as shown in Exercise Graph 2A.2. The slopes of the budget constraints do not change in either case — implying that the opportunity cost of one good in terms of the other does not change.

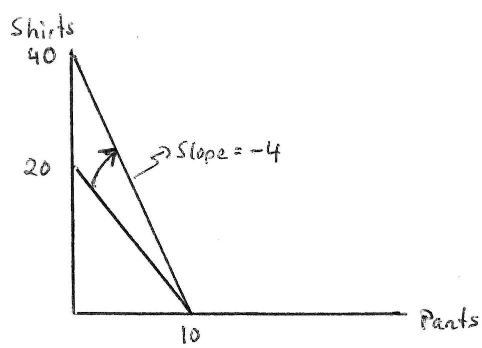


Exercise Graph 2A.2 : (a) $300 lost and (b) both prices doubled

Exercise 2A.3

How would my budget constraint change if, instead of a 50% off coupon for pants, my wife had given me a 50% off coupon for shirts? What would the opportunity cost of pants (in terms of shirts) be?

Answer: The budget constraint would change as depicted in Exercise Graph 2A.3. The new opportunity cost of pants in terms of shirts would be 4 — i.e. for every pair of pants you now buy, you would be giving up 4 (rather than 2) shirts.

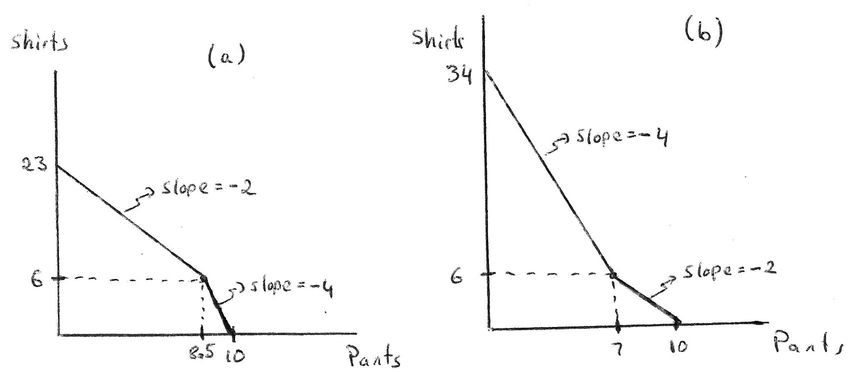


Exercise Graph 2A.3 : 50% off coupon for shirts (instead of pants)

Exercise 2A.4

Suppose that the two coupons analyzed above were for shirts instead of pants. What would the budget constraints look like?

Answer: Panel (a) of Exercise Graph 2A.4 depicts the constraint for a 50% off coupon that applies only to the first 6 shirts bought. If you buy 6 shirts with this coupon, you will have spent \$30 and will therefore have given up 1.5 pants. Thus, over this range, the opportunity cost of pants is $6/1.5 = 4$. After spending \$30 on the first 6 shirts, you could spend up to another \$170 on shirts. If you spent all of it on shirts, you could therefore afford an additional 17 shirts for a total of 23.



Exercise Graph 2A.4 : 2 coupons for shirts (instead of pants)

Panel (b) depicts the constraint for a coupon that gives 50% off for all shirts after the first 6. If you buy 6 shirts, you therefore spend \$60 (because you buy the first 6 at full price) – thus giving up the equivalent of 3 shirts. At that point, you have up to \$140 left to spend, and if you spend all of it on shirts at 50% off, you can afford to get 28 more – for a total of 34.

Exercise 2A.5

Revisit the coupons we discussed in Section 2A.3 and illustrate how these would alter the choice set when defined over pants and a composite good.

Answer: The graphs would look exactly the same as the kinked budget constraint in Graph 2.4 of the text – except that the vertical axis would be denominated in "dollars of other good consumption" with values 10 times what they are in Graph 2.4. This would also have the effect of increasing the slopes 10-fold.

Exercise 2A.6

True or False: When we model the good on the vertical axis as "dollars of consumption of other goods," the slope of the budget constraint is $-p_1$, where p_1 denotes the price of the good on the horizontal axis.

Answer: True. The slope of the budget constraint is always $-p_1/p_2$. When x_2 is a composite good denominated in dollar units, its price is $p_2 = 1$ since "1 dollar of other good consumption" by definition costs exactly 1 dollar. Thus the slope $-p_1/p_2$ simply reduces to $-p_1$.

2B Solutions to Within-Chapter-Exercises for Part B

Exercise 2B.1

What points in Graph 2.1 satisfy the necessary but not the sufficient conditions in expression (2.1)?

Answer: The points to the northeast of the blue budget line – i.e. all the non-shaded points outside the budget line. These bundles satisfy the necessary condition that $(x_1, x_2) \in \mathbb{R}^2_+$, but they do not satisfy the sufficient condition that $20x_1 + 10x_2 \leq 200$.

Exercise 2B.2

Using equation (2.5), show that the exact same change in the budget line could happen if both prices fell by half at the same time while the dollar budget remained the same. Does this make intuitive sense?

Answer: Replacing p_1 with $0.5p_1$ and p_2 with $0.5p_2$ in the equation, we get

$$x_2 = \frac{I}{0.5p_2} - \frac{0.5p_1}{0.5p_2}x_1 = \frac{2I}{p_2} - \frac{p_1}{p_2}x_1. \qquad (2B.2)$$

If the initial income is $200, this implies the budget constraint when all prices fall by half is equivalent to one with the original prices and income equal to $400. This makes intuitive sense: If all prices fall by half, then any given cash budget can buy twice as much. Thus, the simultaneous price drop is equivalent to an increase in (cash) income.

Exercise 2B.3

Using the mathematical formulation of a budget line (equation (2.5)), illustrate how the slope and intercept terms change when p_2 instead of p_1 changes. Relate this to what your intuition would tell you in a graphical model of budget lines.

Answer: When p_2 changes to p_2', the intercept changes from I/p_2 to I/p_2'. If $p_2' > p_2$, this implies that the intercept falls, while if $p_2' < p_2$ it implies that the intercept increases. This makes intuitive sense since an increase in the price of good 2 means that you can buy less of good 2 if that is all you spend your income on, and a decrease in the price of good 2 means that you can buy more of good 2 when that is all you spend your income on.

Looking at the slope term, an increase in p_2 causes $-p_1/p_2$ to fall in absolute value — implying a shallower budget line. Similarly, a decrease in p_2 causes $-p_1/p_2$ to rise in absolute value — implying a steeper budget. This also makes intuitive sense: When p_2 increases, the opportunity cost of x_1 falls (as illustrated by the shallower budget line), and when p_2 falls, the opportunity cost of x_1 increases (as illustrated by the steeper budget line).

Exercise 2B.4

Convert the two equations contained in the budget set (2.7) into a format that illustrates more clearly the intercept and slope terms (as in equation (2.5)). Then, using the numbers for prices and incomes from our example, plot the two lines on a graph. Finally, erase the portions of the lines that are not relevant given that each line applies only for some values of x_1 (as indicated in (2.7)). Compare your graph to Graph 2.4a.

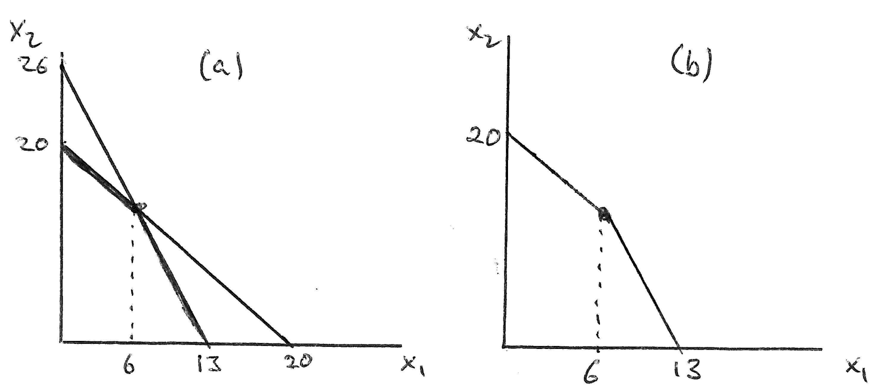
Answer: The two equations can be written as

$$x_2 = \frac{I}{p_2} - \frac{p_1}{2p_2}x_1 \text{ and } x_2 = \frac{I+3p_1}{p_2} - \frac{p_1}{p_2}x_1. \quad \text{(2B.4.i)}$$

Plugging in $I = 200$, $p_1 = 20$ and $p_2 = 10$ as in the example with pants (x_1) and shirts (x_2), this gives

$$x_2 = \frac{200}{10} - \frac{20}{2(10)}x_1 = 20 - x_1 \text{ and } x_2 = \frac{260}{10} - \frac{20}{10}x_1 = 26 - 2x_1. \quad \text{(2B.4.ii)}$$

Panel (a) in Exercise Graph 2B.4 plots these two lines, and panel (b) erases the portions that are not relevant given that the first equation applies only to values of x_1 less than or equal to 6 and the second equation applies only to values of x_1 greater than 6. The resulting graph is identical to the one we derived intuitively in the text.



Exercise Graph 2B.4 : Graphs of equations in exercise 2B.4

Exercise 2B.5

Now suppose that the 50% off coupon is applied to all pants purchased after you bought an initial 6 pants at regular price. Derive the mathematical formulation of the budget set (analogous to equation (2.7)) and then repeat the previous exercise. Compare your graph to Graph 2.4b.

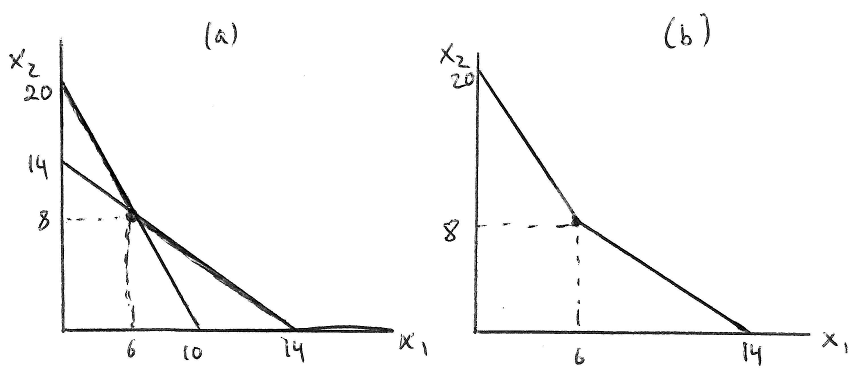
Answer: The normal budget constraint would apply to the initial range of pants (since the coupon does not kick in until 6). After that, the price of pants (p_1) falls by half. Furthermore, since we already spent \$120 to get to 6 pair of pants, we only have \$80 left — implying that the most we could buy is 8 more pants at the reduced price for a total of 14 pants. Were we to be able to buy 14 pants at a price of \$10 (which is assumed along this line segment), our total spending would be \$140 — implying that our effective income on this line segment is \$60 less than the $I = 200$ we started with. More generally, when the price falls to $0.5p_1$ after the 6th pair, the vertical intercept of the shallower budget falls to $(I - 0.5(6p_1)) = I - 3p_1$. This gives us the following definition of the budget line:

$$B(p_1, p_2, I) = \{ (x_1, x_2) \in \mathbb{R}^2_+ \mid \begin{array}{ll} p_1x_1 + p_2x_2 = I & \text{for } x_1 \leq 6 \text{ and} \\ 0.5p_1x_1 + p_2x_2 = I - 3p_1 & \text{for } x_1 > 6 \,\}. \end{array} \quad (2B.5.i)$$

Taking x_2 to one side in both of these equations, and substituting in $p_1 = 20$, $p_2 = 10$ and $I = 200$, we get

$$x_2 = \frac{200}{10} - \frac{20}{10}x_1 = 20 - 2x_1 \text{ and } x_2 = \frac{200-60}{10} - \frac{20}{2(10)}x_1 = 14 - x_1. \quad (2B.5.ii)$$

Panel (a) of Exercise Graph 2B.5 plots these two lines, and panel (b) erases the portions that are not relevant. The resulting graph is identical to the one for this coupon in the text.



Exercise Graph 2B.5 : Graphs of equations in exercise 2B.5

Exercise 2B.6

Using the equation in (2.19), derive the general equation of the budget line in terms of prices and endowments. Following steps analogous to those leading to equation (2.17), identify the intercept and slope terms. What would the budget line look like when my endowments are 10 shirts and 10 pants and when prices are $5 for pants and $10 for shirts? Relate this to both the equation you derived and to an intuitive derivation of the same budget line.

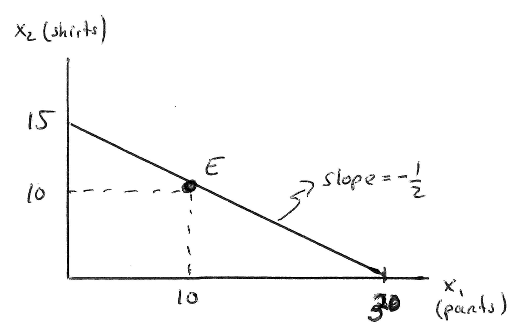
Answer: Changing the inequality to an equality and solving the equation for x_2, we get

$$x_2 = \frac{p_1 e_1 + p_2 e_2}{p_2} - \frac{p_1}{p_2} x_1. \tag{2B.6.i}$$

The slope is therefore $-p_1/p_2$ as it always is. The x_2 intercept is $(p_1e_1+p_2e_2)/p_2$ — which is just the value of my endowment divided by price. When endowments are 10 shirts and 10 pants and when prices are $p_1 = 5$ and $p_2 = 10$, the equation becomes

$$x_2 = \frac{5(10)+10(10)}{10} - \frac{5}{10} x_1 = 15 - \frac{1}{2} x_1. \tag{2B.6.ii}$$

We would intuitively derive this as follows: We would begin at the endowment point (10,10). Given the prices of pants and shirts, I could sell my 10 pants for $50 and with that I could buy 5 more shirts. Thus, the most shirts I could buy if I only bought shirts is 15 — the x_2 intercept. Since pants cost half what shirts cost, I could buy 30 pants. The resulting budget line, which is equivalent to the one derived mathematically above, is depicted in Exercise Graph 2B.6.



Exercise Graph 2B.6 : Graph of equation in exercise 2B.6

2C Solutions to Odd-Numbered End-of-Chapter Exercises

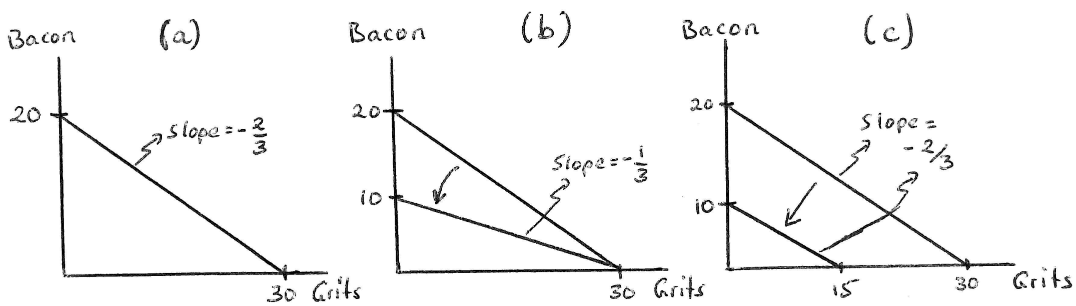
Exercise 2.1

Any good Southern breakfast includes grits (which my wife loves) and bacon (which I love). Suppose we allocate \$60 per week to consumption of grits and bacon, that grits cost \$2 per box and bacon costs \$3 per package.

A: *Use a graph with boxes of grits on the horizontal axis and packages of bacon on the vertical to answer the following:*

(a) *Illustrate my family's weekly budget constraint and choice set.*

Answer: The graph is drawn in panel (a) of Exercise Graph 2.1.



Exercise Graph 2.1 : (a) Answer to (a); (b) Answer to (c); (c) Answer to (d)

(b) *Identify the opportunity cost of bacon and grits and relate these to concepts on your graph.*

Answer: The opportunity cost of grits is equal to 2/3 of a package of bacon (which is equal to the negative slope of the budget since grits appear on the horizontal axis). The opportunity cost of a package of bacon is 3/2 of a box of grits (which is equal to the inverse of the negative slope of the budget since bacon appears on the vertical axis).

(c) *How would your graph change if a sudden appearance of a rare hog disease caused the price of bacon to rise to \$6 per package, and how does this change the opportunity cost of bacon and grits?*

Answer: This change is illustrated in panel (b) of Exercise Graph 2.1. This changes the opportunity cost of grits to 1/3 of a package of bacon, and it changes the opportunity cost of bacon to 3 boxes of grits. This makes sense: Bacon is now 3 times as expensive as grits — so you have to give up 3 boxes of grits for one package of bacon, or 1/3 of a package of bacon for 1 box of grits.

(d) *What happens in your graph if (instead of the change in (c)) the loss of my job caused us to decrease our weekly budget for Southern breakfasts from \$60 to \$30? How does this change the opportunity cost of bacon and grits?*

Answer: The change is illustrated in panel (c) of Exercise Graph 2.1. Since relative prices have not changed, opportunity costs have not changed. This is reflected in the fact that the slope stays unchanged.

B: *In the following, compare a mathematical approach to the graphical approach used in part A, using x_1 to represent boxes of grits and x_2 to represent packages of bacon:*

(a) *Write down the mathematical formulation of the budget line and choice set and identify elements in the budget equation that correspond to key features of your graph from part 2.1A(a).*

Answer: The budget equation is $p_1x_1 + p_2x_2 = I$ can also be written as

$$x_2 = \frac{I}{p_2} - \frac{p_1}{p_2}x_1. \tag{2.1.i}$$

With $I = 60$, $p_1 = 2$ and $p_2 = 3$, this becomes $x_2 = 20 - (2/3)x_1$ — an equation with intercept of 20 and slope of $-2/3$ as drawn in Exercise Graph 2.1(a).

(b) *How can you identify the opportunity cost of bacon and grits in your equation of a budget line, and how does this relate to your answer in 2.1A(b).*

Answer: The opportunity cost of x_1 (grits) is simply the negative of the slope term (in terms of units of x_2). The opportunity cost of x_2 (bacon) is the inverse of that.

(c) *Illustrate how the budget line equation changes under the scenario of 2.1A(c) and identify the change in opportunity costs.*

Answer: Substituting the new price $p_2 = 6$ into equation (2.1.i), we get $x_2 = 10 - (1/3)x_1$ — an equation with intercept of 10 and slope of $-1/3$ as depicted in panel (b) of Exercise Graph 2.1.

(d) *Repeat (c) for the scenario in 2.1A(d).*

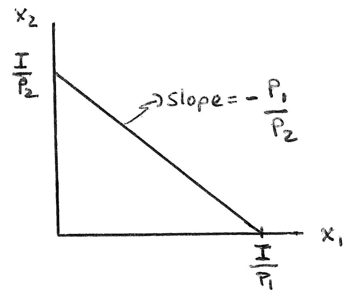
Answer: Substituting the new income $I = 30$ into equation (2.1.i) (holding prices at $p_1 = 2$ and $p_2 = 3$, we get $x_2 = 10 - (2/3)x_1$ — an equation with intercept of 10 and slope of $-2/3$ as depicted in panel (c) of Exercise Graph 2.1.

Exercise 2.3

Consider a budget for good x_1 (on the horizontal axis) and x_2 (on the vertical axis) when your economic circumstances are characterized by prices p_1 and p_2 and an exogenous income level I.

A: *Draw a budget line that represents these economic circumstances and carefully label the intercepts and slope.*

Answer: The sketch of this budget line is given in Exercise Graph 2.3.



Exercise Graph 2.3 : A budget constraint with exogenous income I

The vertical intercept is equal to how much of x_2 one could by with I if that is all one bought — which is just I/p_2. The analogous is true for x_1 on the horizontal intercept. One way to verify the slope is to recognize it is the "rise" (I/p_2) divided by the "run" (I/p_1) — which gives p_1/p_2 — and that it is negative since the budget constraint is downward sloping.

(a) *Illustrate how this line can shift parallel to itself without a change in I.*

Answer: In order for the line to shift in a parallel way, it must be that the slope $-p_1/p_2$ remains unchanged. Since we can't change I, the only values we can change are p_1 and p_2 — but since p_1/p_2 can't change, it means the only thing we can do is to multiply both prices by the same constant. So, for instance, if we multiply both prices by 2, the ratio of the new prices is $2p_1/(2p_2) = p_1/p_2$ since the 2's cancel. We therefore have not changed the slope. But we have changed the vertical intercept from I/p_2 to $I/(2p_2)$. We have therefore shifted in the line without changing its slope.

This should make intuitive sense: If our money income does not change but all prices double, then I can by half as much of everything. This is equivalent to prices staying the same and my money income dropping by half.

(b) *Illustrate how this line can rotate clockwise on its horizontal intercept without a change in p_2.*

Answer: To keep the horizontal intercept constant, we need to keep I/p_1 constant. But to rotate the line clockwise, we need to increase the vertical intercept I/p_2. Since we can't change p_2 (which would be the easiest way to do this), that leaves us only I and p_1 to change. But since we can't change I/p_1, we can only change these by multiplying them by the same constant. For instance, if we multiply both by 2, we don't change the horizontal intercept since $2I/(2p_1) = I/p_1$. But we do increase the vertical intercept from I/p_2 to $2I/p_2$. So, multiplying both I and p_1 by the same constant (greater than 1) will accomplish our goal.

This again should make intuitive sense: If you double my income and the price of good 1, I can still afford exactly as much of good 1 if that is all

I buy with my income. (Thus the unchanged horizontal intercept). But, if I only buy good 2, then a doubling of my income without a change in the price of good 2 lets me buy twice as much of good 2. The scenario is exactly the same as if p_2 had fallen by half (and I and p_1 had remained unchanged.)

B: *Write the equation of a budget line that corresponds to your graph in 2.3A.*

Answer: $p_1x_1 + p_2x_2 = I$, which can also be written as

$$x_2 = \frac{I}{p_2} - \frac{p_1}{p_2}x_1. \tag{2.3.i}$$

(a) *Use this equation to demonstrate how the change derived in 2.3A(a) can happen.*

Answer: If I replace p_1 with αp_1 and p_2 with αp_2 (where α is just a constant), I get

$$x_2 = \frac{I}{\alpha p_2} - \frac{\alpha p_1}{\alpha p_2}x_1 = \frac{(1/\alpha)I}{p_2} - \frac{p_1}{p_2}x_1. \tag{2.3.ii}$$

Thus, multiplying both prices by α is equivalent to multiplying income by $1/\alpha$ (and leaving prices unchanged).

(b) *Use the same equation to illustrate how the change derived in 2.3A(b) can happen.*

Answer: If I replace p_1 with βp_1 and I with βI, I get

$$x_2 = \frac{\beta I}{p_2} - \frac{\beta p_1}{p_2}x_1 = \frac{I}{(1/\beta)p_2} - \frac{p_1}{(1/\beta)p_2}x_1. \tag{2.3.iii}$$

Thus, this is equivalent to multiplying p_2 by $1/\beta$. So long as $\beta > 1$, it is therefore equivalent to reducing the price of good 2 (without changing the other price or income).

Exercise 2.5

Everyday Application: *Watching a Bad Movie: On one of my first dates with my wife, we went to see the movie "Spaceballs" and paid $5 per ticket.*

A: *Halfway through the movie, my wife said: "What on earth were you thinking? This movie sucks! I don't know why I let you pick movies. Let's leave."*

(a) *In trying to decide whether to stay or leave, what is the opportunity cost of staying to watch the rest of the movie?*

Answer: The opportunity cost of any activity is what we give up by undertaking that activity. The opportunity cost of staying in the movie is whatever we would choose to do with our time if we were not there. The price of the movie tickets that got us into the movie theater is NOT a part of this opportunity cost — because, whether we stay or leave, we do not get that money back.

(b) *Suppose we had read a sign on the way into theater stating "Satisfaction Guaranteed! Don't like the movie half way through — see the manager and get your money back!" How does this change your answer to part (a)?*

Answer: Now, in addition to giving up whatever it is we would be doing if we weren't watching the movie, we are also giving up the price of the movie tickets. Put differently, by staying in the movie theater, we are giving up the opportunity to get a refund — and so the cost of the tickets is a real opportunity cost of staying.

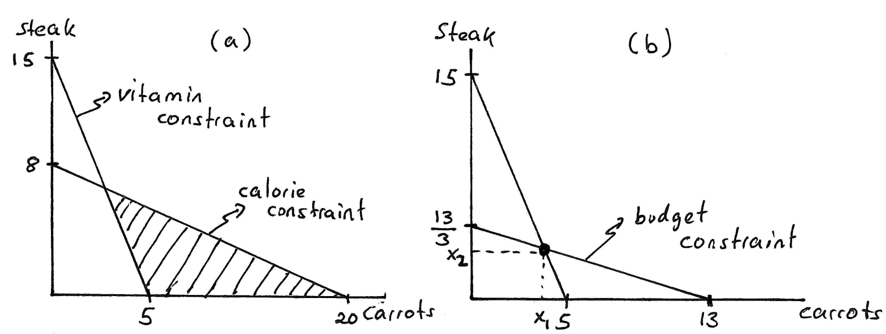
Exercise 2.7

Everyday Application: *Dieting and Nutrition*: *On a recent doctor's visit, you have been told that you must watch your calorie intake and must make sure you get enough vitamin E in your diet.*

A: *You have decided that, to make life simple, you will from now on eat only steak and carrots. A nice steak has 250 calories and 10 units of vitamins, and a serving of carrots has 100 calories and 30 units of vitamins Your doctor's instructions are that you must eat* no more *than 2000 calories and consume* at least *150 units of vitamins per day.*

(a) *In a graph with "servings of carrots" on the horizontal and steak on the vertical axis, illustrate all combinations of carrots and steaks that make up a 2000 calorie a day diet.*

Answer: This is illustrated as the "calorie constraint" in panel (a) of Exercise Graph 2.7. You can get 2000 calories only from steak if you eat 8 steaks and only from carrots if you eat 20 servings of carrots. These form the intercepts of the calorie constraint.



Exercise Graph 2.7 : (a) Calories and Vitamins; (b) Budget Constraint

(b) *On the same graph, illustrate all the combinations of carrots and steaks that provide exactly 150 units of vitamins.*

Answer: This is also illustrated in panel (a) of Exercise Graph 2.7. You can get 150 units of vitamins from steak if you eat 15 steaks only or if you eat

5 servings of carrots only. This results in the intercepts for the "vitamin constraint".

(c) *On this graph, shade in the bundles of carrots and steaks that satisfy both of your doctor's requirements.*

Answer: Your doctor wants you to eat no more than 2000 calories — which means you need to stay underneath the calorie constraint. Your doctor also wants you to get at least 150 units of vitamin E — which means you must choose a bundle *above* the vitamin constraint. This leaves you with the shaded area to choose from if you are going to satisfy both requirements.

(d) *Now suppose you can buy a serving of carrots for $2 and a steak for $6. You have $26 per day in your food budget. In your graph, illustrate your budget constraint. If you love steak and don't mind eating or not eating carrots, what bundle will you choose (assuming you take your doctor's instructions seriously)?*

Answer: With $26 you can buy 13/3 steaks if that is all you buy, or you can buy 13 servings of carrots if that is all you buy. This forms the two intercepts on your budget constraint which has a slope of $-p_1/p_2 = -1/3$ and is depicted in panel (b) of the graph. If you really like steak and don't mind eating carrots one way or another, you would want to get as much steak as possible given the constraints your doctor gave you and given your budget constraint. This leads you to consume the bundle at the intersection of the vitamin and the budget constraint in panel (b) — indicated by (x_1, x_2) in the graph. It seems from the two panels that this bundle also satisfies the calorie constraint and lies inside the shaded region.

B: *Continue with the scenario as described in part A.*

(a) *Define the line you drew in A(a) mathematically.*

Answer: This is given by $100x_1 + 250x_2 = 2000$ which can be written as

$$x_2 = 8 - \frac{2}{5}x_1. \tag{2.7.i}$$

(b) *Define the line you drew in A(b) mathematically.*

Answer: This is given by $30x_1 + 10x_2 = 150$ which can be written as

$$x_2 = 15 - 3x_1. \tag{2.7.ii}$$

(c) *In formal set notation, write down the expression that is equivalent to the shaded area in A(c).*

Answer:

$$\left\{(x_1, x_2) \in \mathbb{R}^2_+ \mid 100x_1 + 250x_2 \leq 2000 \text{ and } 30x_1 + 10x_2 \geq 150\right\} \tag{2.7.iii}$$

(d) *Derive the exact bundle you indicated on your graph in A(d).*

Answer: We would like to find the most amount of steak we can afford in the shaded region. Our budget constraint is $2x_1+6x_2=26$. Our graph suggests that this budget constraint intersects the vitamin constraint (from equation (2.7.ii)) within the shaded region (in which case that intersection gives us the most steak we can afford in the shaded region). To find this intersection, we can plug equation (2.7.ii) into the budget constraint $2x_1+6x_2=26$ to get

$$2x_1+6(15-3x_1)=26, \tag{2.7.iv}$$

and then solve for x_1 to get $x_1=4$. Plugging this back into either the budget constraint or the vitamin constraint, we can get $x_2=3$. We know this lies on the vitamin constraint as well as the budget constraint. To check to make sure it lies in the shaded region, we just have to make sure it also satisfies the doctor's orders that you consume fewer than 2000 calories. The bundle (x_1,x_2)=(4,3) results in calories of $4(100)+3(250)$ = 1150, well within doctor's orders.

Exercise 2.9

Business Application: *Pricing and Quantity Discounts*: *Businesses often give quantity discounts. Below, you will analyze how such discounts can impact choice sets.*

A: *I recently discovered that a local copy service charges our economics department \$0.05 per page (or \$5 per 100 pages) for the first 10,000 copies in any given month but then reduces the price per page to \$0.035 for each additional page up to 100,000 copies and to \$0.02 per each page beyond 100,000. Suppose our department has a monthly overall budget of \$5,000.*

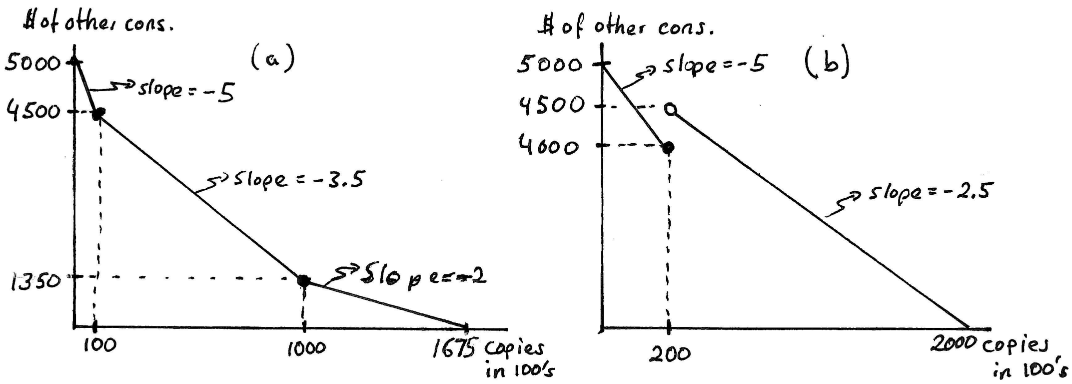
(a) *Putting "pages copied in units of 100" on the horizontal axis and "dollars spent on other goods" on the vertical, illustrate this budget constraint. Carefully label all intercepts and slopes.*

Answer: Panel (a) of Exercise Graph 2.9 traces out this budget constraint and labels the relevant slopes and kink points.

(b) *Suppose the copy service changes its pricing policy to \$0.05 per page for monthly copying up to 20,000 and \$0.025 per page for* all *pages if copying exceeds 20,000 per month. (*Hint*: Your budget line will contain a jump.)*

Answer: Panel (b) of Exercise Graph 2.9 depicts this budget. The first portion (beginning at the x_2 intercept) is relatively straightforward. The second part arises for the following reason: The problem says that, if you copy more than 2000 pages, *all* pages cost only \$0.025 per page — including the first 2000. Thus, when you copy 20,000 pages per month, you total bill is \$1,000. But when you copy 2001 pages, your total bill is \$500.025.

(c) *What is the marginal (or "additional") cost of the first page copied after 20,000 in part (b)? What is the marginal cost of the first page copied after 20,001 in part (b)?*



Exercise Graph 2.9 : (a) Constraint from 2.9A(a); (b) Constraint from 2.9A(b)

Answer: The marginal cost of the first page after 20,000 is -$499.975, and the marginal cost of the next page after that is 2.5 cents. To see the difference between these, think of the marginal cost as the increase in the total photo-copy bill for each additional page. When going from 20,000 to 20,001, the total bill falls by $499.975. When going from 20,001 to 20,002, the total bill rises by 2.5 cents.

B: *Write down the mathematical expression for choice sets for each of the scenarios in 2.9A(a) and 2.9A(b) (using x_1 to denote "pages copied in units of 100" and x_2 to denote "dollars spent on other goods").*

Answer: The choice set in (a) is

$$\begin{array}{lll}\{(x_1,x_2)\in\mathbb{R}^2_+ \mid & x_2=5000-5x_1 & \text{for } x_1\le 100 \text{ and}\\ & x_2=4850-3.5x_1 & \text{for } 100<x_1\le 1000 \text{ and}\\ & x_2=3350-2x_1 & \text{for } x_1>1000\,\}.\end{array} \tag{2.9.i}$$

The choice set in (b) is

$$\begin{array}{lll}\{(x_1,x_2)\in\mathbb{R}^2_+ \mid & x_2=5000-5x_1 & \text{for } x_1\le 200 \text{ and}\\ & x_2=5000-2.5x_1 & \text{for } x_1>200\,\}.\end{array} \tag{2.9.ii}$$

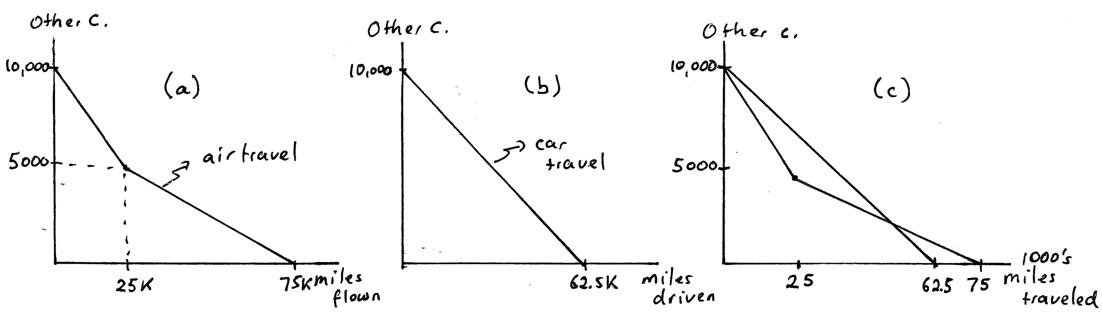
Exercise 2.11

Business Application: *Frequent Flyer Perks: Airlines offer frequent flyers different kinds of perks that we will model here as reductions in average prices per mile flown.*

A: *Suppose that an airline charges 20 cents per mile flown. However, once a customer reaches 25,000 miles in a given year, the price drops to 10 cents per mile flown for each additional mile. The alternate way to travel is to drive by car which costs 16 cents per mile.*

(a) *Consider a consumer who has a travel budget of $10,000 per year, a budget which can be spent on the cost of getting to places as well as "other consumption" while traveling. On a graph with "miles flown" on the horizontal axis and "other consumption" on the vertical, illustrate the budget constraint for someone who only considers flying (and not driving) to travel destinations.*

Answer: Panel (a) of Exercise Graph 2.11 illustrates this budget constraint.



Exercise Graph 2.11 : (a) Air travel; (b) Car travel; (c) Comparison

(b) *On a similar graph with "miles driven" on the horizontal axis, illustrate the budget constraint for someone that considers only driving (and not flying) as a means of travel.*

Answer: This is illustrated in panel (b) of the graph.

(c) *By overlaying these two budget constraints (changing the good on the horizontal axis simply to "miles traveled"), can you explain how frequent flyer perks might persuade some to fly a lot more than they otherwise would?*

Answer: Panel (c) of the graph overlays the two budget constraints. If it were not for frequent flyer miles, this consumer would never fly — because driving would be cheaper. With the frequent flyer perks, driving is cheaper initially but becomes more expensive per additional miles traveled if a traveler flies more than 25,000 miles. This particular consumer would therefore either not fly at all (and just drive), or she would fly a lot because it can only make sense to fly if she reaches the portion of the air-travel budget that crosses the car budget. (Once we learn more about how to model tastes, we will be able to say more about whether or not it makes sense for a traveler to fly under these circumstances.)

B: *Determine where the air-travel budget from A(a) intersects the car budget from A(b).*

Answer: The shallower portion of the air-travel budget (relevant for miles flown above 25,000) has equation $x_2 = 7500 - 0.1x_1$, where x_2 stands for other consumption and x_1 for miles traveled. The car budget, on the other hand, has equation $x_2 = 10000 - 0.16x_1$. To determine where they cross, we can set the two equations equal to one another and solve for x_1 — which gives $x_1 = 41,667$ miles traveled. Plugging this back into either equation gives $x_2 = \$3,333$.

Exercise 2.13

Policy Application: *Food Stamp Programs and other Types of Subsidies: The U.S. government has a food stamp program for families whose income falls below a certain poverty threshold. Food stamps have a dollar value that can be used at supermarkets for food purchases as if the stamps were cash, but the food stamps cannot be used for anything other than food.*

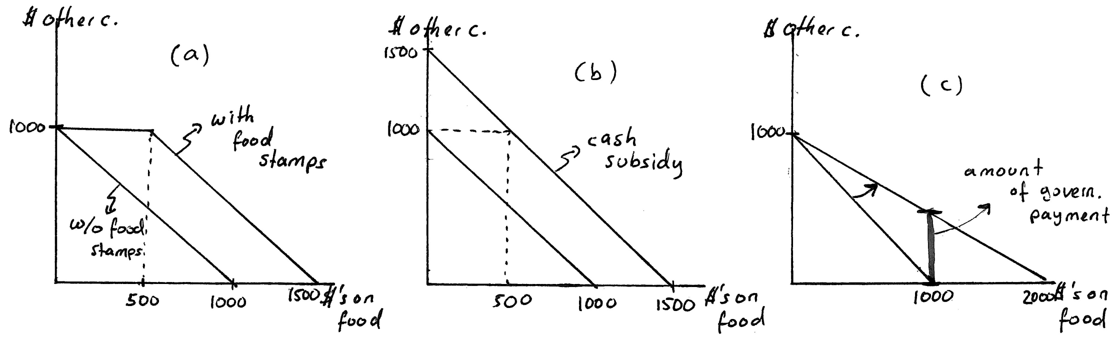
A: *Suppose the program provides $500 of food stamps per month to a particular family that has a fixed income of $1,000 per month.*

(a) *With "dollars spent on food" on the horizontal axis and "dollars spent on non-food items" on the vertical, illustrate this family's monthly budget constraint. How does the opportunity cost of food change along the budget constraint you have drawn?*

Answer: Panel (a) of Exercise Graph 2.13 illustrates the original budget — with intercept 1,000 on each axis. It then illustrates the new budget under the food stamp program. Since food stamps can only be spent on food, the "other goods" intercept does not change — owning some food stamps still only allows households to spend what they previously had on other goods. However, the family is now able to buy $1,000 in other goods even as it buys food — because it can use the food stamps on the first $500 worth of food and still have all its other income left for other consumption. Only after all the food stamps are spent — i.e. after the family has bought $500 worth of food — does the family give up other consumption when consuming additional food. As a result, the opportunity cost of food is zero until the food stamps are gone, and it is 1 after that. That is, after the food stamps are gone, the family gives up $1 in other consumption for every $1 of food it purchases.

(b) *How would this family's budget constraint differ if the government replaced the food stamp program with a cash subsidy program that simply gave this family $500 in cash instead of $500 in food stamps? Which would the family prefer, and what does your answer depend on?*

Answer: In this case, the original budget would simply shift out by $500 as depicted in panel (b). If the family consumes more than $500 of food under the food stamp program, it would not seem like anything really changes under the cash subsidy. (We can show this more formally once we introduce a model of tastes). If, on the other hand, the family consumes $500 of food under the food stamps, it may well be that it would prefer to get cash instead so that it can consume more other goods instead.



Exercise Graph 2.13 : (a) Food Stamps; (b) Cash; (c) Re-imburse half

(c) *How would the budget constraint change if the government simply agreed to reimburse the family for half its food expenses?*

Answer: In this case, the government essentially reduces the price of $1 of food to 50 cents because whenever $1 is spent on food, the government reimburses the family 50 cents. The resulting change in the family budget is then depicted in panel (c) of the graph.

(d) *If the government spends the same amount for this family on the program described in (c) as it did on the food stamp program, how much food will the family consume? Illustrate the amount the government is spending as a vertical distance between the budget lines you have drawn.*

Answer: If the government spent $500 for this family under this program, then the family will be consuming $1,000 of food and $500 in other goods. You can illustrate the $500 the government is spending as the distance between the two budget constraints at $1,000 of food consumption. The reasoning for this is as follows: On the original budget line, you can see that consuming $1,000 of food implies nothing is left over for "other consumption". When the family consumes $1,000 of food under the new program, it is able to consume $500 in other goods because of the program — so the government must have made that possible by giving $500 to the family.

B: *Write down the mathematical expression for the choice set you drew in 2.13A(a), letting x_1 represent dollars spent on food and x_2 represent dollars spent on non-food consumption. How does this expression change in 2.13A(b) and 2.13A(c)?*

Answer: The original budget constraint (prior to any program) is just $x_2 = 1000 - x_1$, and the budget constraint with the $500 cash payment in A(b) is $x_2 = 1500 - x_1$. The choice set under food stamps (depicted in panel (a)) then is

$$\{(x_1, x_2) \in \mathbb{R}^2_+ \mid \begin{array}{ll} x_2 = 1000 & \text{for } x_1 \leq 500 \text{ and} \\ x_2 = 1500 - x_1 & \text{for } x_1 > 500\ \}, \end{array} \tag{2.13.i}$$

while the choice set in panel (b) under the cash subsidy is

$$\left\{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_2 = 1500 - x_1\right\}. \tag{2.13.ii}$$

Finally, the choice set under the re-imbursement plan from A(c) is

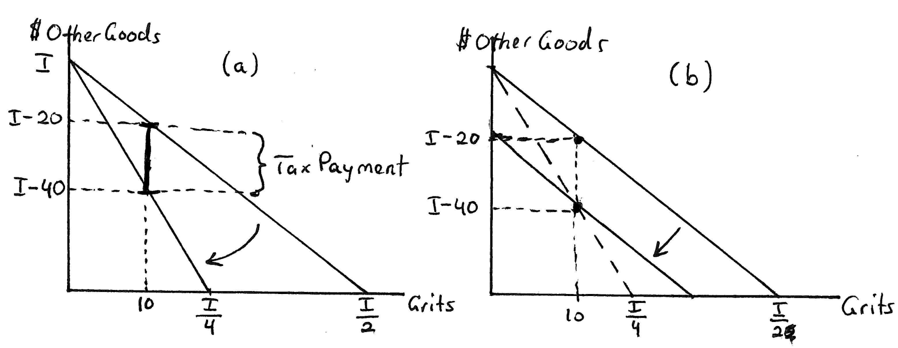
$$\left\{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_2 = 1000 - \frac{1}{2}x_1\right\}. \tag{2.13.iii}$$

Exercise 2.15

Policy Application: *Taxing Goods* versus *Lump Sum Taxes*: *I have finally convinced my local congressman that my wife's taste for grits are nuts and that the world should be protected from too much grits consumption. As a result, my congressman has agreed to sponsor new legislation to tax grits consumption which will raise the price of grits from \$2 per box to \$4 per box. We carefully observe my wife's shopping behavior and notice with pleasure that she now purchases 10 boxes of grits per month rather than her previous 15 boxes.*

A: *Putting "boxes of grits per month" on the horizontal and "dollars of other consumption" on the vertical, illustrate my wife's budget line before and after the tax is imposed. (You can simply denote income by I.)*

Answer: The tax raises the price, thus resulting in a rotation of the budget line as illustrated in panel (a) of Exercise Graph 2.15. Since no indication of an income level was given in the problem, income is simply denoted I.



Exercise Graph 2.15 : (a) Tax on Grits; (b) Lump Sum Rebate

(a) *How much tax revenue is the government collecting per month from my wife? Illustrate this as a vertical distance on your graph. (*Hint*: If you know how much she is consuming after the tax and how much in other consumption this leaves her with, and if you know how much in other consumption she would have had if she consumed that same quantity before the imposition of the tax, then the difference between these two "other consumption" quantities must be equal to how much she paid in tax.)*

Answer: When she consumes 10 boxes of grits after the tax, she pays $40 for grits. This leaves her with $(I-40)$ to spend on other goods. Had she bought 10 boxes of grits prior to the tax, she would have paid $20, leaving her with $(I-20)$. The difference between $(I-40)$ and $(I-20)$ is $20 — which is equal to the vertical distance in panel (a). You can verify that this is exactly how much she indeed must have paid — the tax is $2 per box and she bought 10 boxes, implying that she paid $2 times 10 or $20 in grits taxes.

(b) *Given that I live in the South, the grits tax turned out to be unpopular in my congressional district and has led to the defeat of my congressman. His replacement won on a pro-grits platform and has vowed to repeal the grits tax. However, new budget rules require him to include a new way to raise the same tax revenue that was yielded by the grits tax. He proposes to simply ask each grits consumer to pay exactly the amount he or she paid in grits taxes as a monthly lump sum payment. Ignoring for the moment the difficulty of gathering the necessary information for implementing this proposal, how would this change my wife's budget constraint?*

Answer: In panel (b) of Exercise Graph 2.15, the previous budget under the grits tax is illustrated as a dashed line. The grits tax changed the opportunity cost of grits — and thus the slope of the budget (as illustrated in panel (a)). The lump sum tax, on the other hand, does not alter opportunity costs but simply reduces income by $20, the amount of grits taxes my wife paid under the grits tax. This change is illustrated in panel (b).

B: *State the equations for the budget constraints you derived in A(a) and A(b), letting grits be denoted by x_1 and other consumption by x_2.*

Answer: The initial (before-tax) budget was $x_2 = I - 2x_1$ which becomes $x_2 = I - 4x_1$ after the imposition of the grits tax. The lump sum tax budget constraint, on the other hand, is $x_2 = I - 20 - 2x_1$.

Exercise 2.17

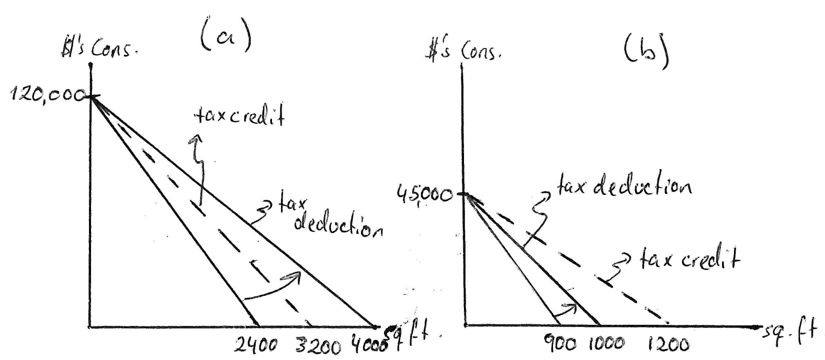
Policy Application: *Tax Deductions and Tax Credits*: *In the U.S. income tax code, a number of expenditures are "deductible". For most tax payers, the largest tax deduction comes from the portion of the income tax code that permits taxpayers to deduct home mortgage interest (on both a primary and a vacation home). This means that taxpayers who use this deduction do not have to pay income tax on the portion of their income that is spent on paying interest on their home mortgage(s). For purposes of this exercise, assume that the entire yearly price of housing is interest expense.*

A: True or False*: For someone whose marginal tax rate is 33%, this means that the government is subsidizing roughly one third of his interest/house payments.*

Answer: Consider someone who pays \$10,000 per year in mortgage interest. When this person deducts \$10,000, it means that he does not have to pay the 33% income tax on that amount. In other words, by deducting \$10,000 in mortgage interest, the person reduces his tax obligation by \$3,333.33. Thus, the government is returning 33 cents for every dollar in interest payments made — effectively causing the opportunity cost of paying \$1 in home mortgage interest to be equal to 66.67 cents. So the statement is true.

(a) *Consider a household with an income of \$200,000 who faces a tax rate of 40%, and suppose the price of a square foot of housing is \$50 per year. With square footage of housing on the horizontal axis and other consumption on the vertical, illustrate this household's budget constraint with and without tax deductibility. (Assume in this and the remaining parts of the question that the tax rate cited for a household applies to all of that household's income.)*

Answer: As just demonstrated, the tax deductibility of home mortgage interest lowers the price of owner-occupied housing, and it does so in proportion to the size of the marginal income tax rate one faces.



Exercise Graph 2.17 : Tax Deductions versus Tax Credits

Panel (a) of Exercise Graph 2.17 illustrates this graphically for the case described in this part. With a 40 percent tax rate, the household could consume as much as 0.6(200,000)=120,000 in other goods if it consumed no housing. With a price of housing of \$50 per square foot, the price falls to $(1-0.4)50 = 30$ under tax deductibility. Thus, the budget rotates out to the solid budget in panel (a) of the graph. Without deductibility, the consumer pays \$50 per square foot — which makes 120,000/50=2,400 the biggest possible house she can afford. But with deductibility, the biggest house she can afford is 120,000/30=4,000 square feet.

(b) *Repeat this for a household with income of $50,000 who faces a tax rate of 10%.*

Answer: This is illustrated in panel (b). The household could consume as much as $45,000 in other consumption after paying taxes, and the deductibility of house payments reduces the price of housing from $50 per square foot to (1 – 0.1)50 = $45 per square foot. This results in the indicated rotation of the budget from the lower to the higher solid line in the graph. The rotation is smaller in magnitude because the impact of deductibility on the after-tax price of housing is smaller. Without deductibility, the biggest affordable house is 45,000/50=900 square feet, while with deductibility the biggest possible house is 45,000/45=1,000 square feet.

(c) *An alternative way for the government to encourage home ownership would be to offer a tax* credit *instead of a tax* deduction. *A tax credit would allow all taxpayers to subtract a fraction* k *of their annual mortgage payments directly from the tax bill they would otherwise owe. (Note: Be careful — a tax credit is deducted from tax* payments *that are due, not from the taxable income.) For the households in (a) and (b), illustrate how this alters their budget if* $k = 0.25$.

Answer: This is illustrated in the two panels of Exercise Graph 2.17 — in panel (a) for the higher income household, and in panel (b) for the lower income household. By subsidizing housing through a credit rather than a deduction, the government has reduced the price of housing by the same amount (k) for everyone. In the case of deductibility, the government had made the price subsidy dependent on one's tax rate — with those facing higher tax rates also getting a higher subsidy. The price of housing how falls from $50 to (1 – 0.25)50 = $37.50 — which makes the largest affordable house for the wealthier household 120,000/37.5=3,200 square feet and, for the poorer household, 45,000/37.5=1,200 square feet. Thus, the poorer household benefits more from the credit when $k = 0.25$ while the richer household benefits more from the deduction.

(d) *Assuming that a tax deductibility program costs the same in lost tax revenues as a tax credit program, who would favor which program?*

Answer: People facing higher marginal tax rates would favor the deductibility program while people facing lower marginal tax rates would favor the tax credit.

B: *Let* x_1 *and* x_2 *represent square feet of housing and other consumption, and let the price of a square foot of housing be denoted* p.

(a) *Suppose a household faces a tax rate* t *for all income, and suppose the entire annual house payment a household makes is deductible. What is the household's budget constraint?*

Answer: The budget constraint would be $x_2 = (1-t)I - (1-t)px_1$.

(b) *Now write down the budget constraint under a tax credit as described above.*

Answer: The budget constraint would now be $x_2 = (1-t)I - (1-k)px_1$.

Conclusion: Potentially Helpful Reminders

1. When income I is exogenous, the intercepts of the budget line are I/p_1 (on the horizontal) and I/p_2 (on the vertical).

2. When income is endogenously derived from the sale of an endowment, you can calculate the person's cash budget I by simply multiplying each good's quantity in the endowment bundle by its price and adding up. (That's the value of the endowment bundle in the market). The vertical and horizontal intercepts of the budget line are then calculated just as in point 1 above.

3. The slope of budget lines — whether they emerge from exogenous incomes or endowments — is always $-p_1/p_2$, NOT $-p_2/p_1$. If good 2 is a composite good, the slope is just $-p_1$.

4. Remember that changes in the income or the endowment bundle cause parallel shifts; changes in prices cause rotations. And — if the income is exogenous, the rotation is through the intercept on the axis whose price has not changed; but if the income is endogenously derived from an endowment, the rotation is through the endowment bundle.

5. Be sure to do end-of-chapter exercises 2.6 and 2.15. Exercise 2.6 forms the basis for material introduced in Chapters 7, and Exercise 2.15 introduces a technique used repeatedly in Chapters 8 through 10.

CHAPTER

3

Choice Sets in Labor and Financial Markets

This chapter is a straightforward extension of Chapter 2 where we had shown that budget constraints can arise from someone owning an **endowment** that he can sell to generate the income needed for purchasing a different consumption bundle. That's exactly what workers and savers do: Workers own their time and can sell it to earn income, and savers own some investment that they can sell and turn into consumption. (Borrowers, on the other hand, own some future asset — such as the income they can earn in the future — that they can sell.) Once you understand how budgets can arise from stuff we own, it becomes straightforward to think about workers and savers/borrowers.

Chapter Highlights

The main points of the chapter are:

1. **Wages** and **interest rates** are prices in particular markets — and therefore give rise to the opportunity costs we face when making choices in those markets as leisure time or investments are sold.

2. When budgets arise from the sale of endowments, price increases no longer unambiguously shrink budgets nor do price decreases unambiguously increase budgets **as budget lines rotate through the endowment bundle**.

3. **Endowment bundles** are those that can always be consumed regardless of what prices (or wages or interest rates) emerge in the economy.

4. **Government policies** can change the economic incentives faced by workers and savers by changing the choice sets they face.

5. An amount \$X in the future has a **present value** less than \$X — because borrowing on that amount to consume more now entails paying interest.

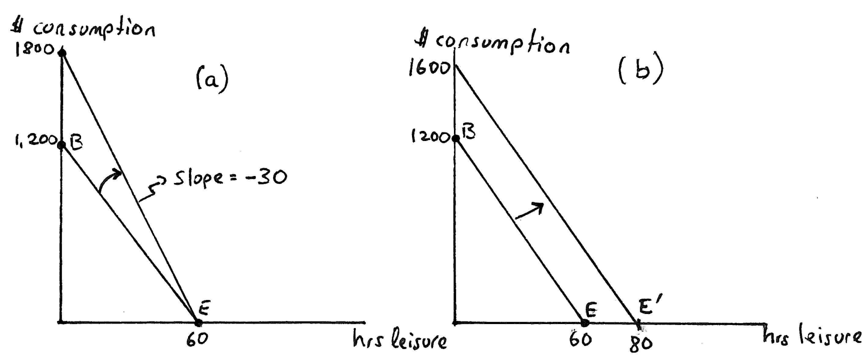
6. The 2-dimensional models of leisure/consumption and intertemporal budgets are really just **"slices" of higher dimensional budgets** along which some things are held fixed.

3A Solutions to Within-Chapter Exercises for Part A

Exercise 3A.1

Illustrate what happens to the original budget constraint if your wage increases to \$30 per hour. What if your friend instead introduces you to caffeine which allows you to sleep less and thus take up to 80 hours of leisure time per week?

Answer: If the wage goes up to \$30 per hour, you could earn as much as \$1,800 per week if you spent all 60 hours working. Thus, the consumption-intercept goes to 1,800, but the endowment point E has not changed. The resulting budget constraint (graphed in panel (a) of Exercise Graph 3A.1) therefore rotates clockwise through E and has a new slope equal to $-w = -30$. If, on the other hand, the endowment of leisure goes up to 80, the budget shifts parallel as in panel (b) of the graph. The slope remains at $-w = -20$ since the wage has not changed.



Exercise Graph 3A.1 : (a) An increase in w; (b) An increase in Leisure

Exercise 3A.2

Verify the dollar quantities on the axes in Graph 3.3a-c.

Answer: In panel (a), you have \$10,000 now, which places your endowment point on the horizontal axis at \$10,000. At a 10% interest rate, you would have earned \$1,000 in interest if you consumed nothing today and you saved everything.

That would leave you with $11,000 of consumption next year at B. When the interest rate falls to 5%, the most in interest you can earn is $500, leaving you with $10,500 next year if you consume nothing this year. This is the relevant intercept at B'.

In panel (b), you earn $11,000 next year but nothing now — thus placing your endowment point on the vertical axis at $11,000. When you borrow on next year's income in order to consume this year, the most the bank will lend you is an amount that, when paid back with interest, will be equal to what you earn next year. When the interest rate is 10%, the bank would then be willing to lend you $\$11,000/(1+0.1) = \$10,000$. If you ended up borrowing $10,000, you would then owe the bank $10,000 plus interest of $1,000 for a total of $11,000. Thus, at a 10% interest rate, the most you can consume this year is $10,000 if you are willing to not consume at all next year (bundle A). When the interest rate falls to 5%, the bank would be willing to lend you up to $11,000/(1+1.05)=$10,476.19 (bundle A').

In panel (c), you earn $5,000 now and $5,500 next year — making that your endowment point. Were you to save all your current income at a 10% interest rate, you could have $5,500 in the bank next year — which, together with your $5,500 income next year, would allow you total consumption of $11,000 (bundle B). If, on the other hand, you decide to do all your consumption this year, you can borrow $5,500/(1+0.1)=$5,000 — which, together with this year's income of $5,000, leaves you with $10,000 in consumption this year (bundle A). At a 5% interest rate, on the other hand, you would accumulate $5,000(1+0.05)=$5,250 in your savings account by saving all your current income, leaving you with a total of $10,750 (bundle B') next year when next year's income of $5,500 is added. Or you can consume all now, with the bank lending you $5,500/(1+0.05)=$5,238.10 that, together with this year's income of $5,000, lets you consume $10,238 (bundle A') now.

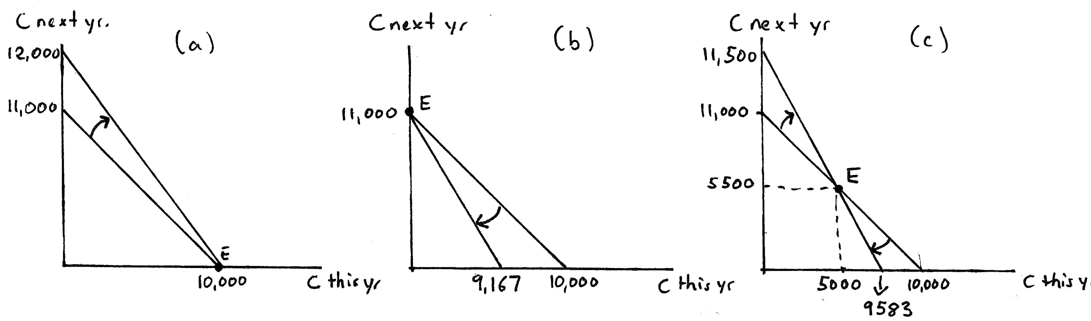
Exercise 3A.3

In each of the panels of Graph 3.3, how would the choice set change if the interest rate went to 20%?

Answer: Since the interest rate is higher, the slopes would all become steeper — with the constraints rotating through the relevant endowment bundle. These changes are depicted in panels (a) through (c) of Exercise Graph 3A.3. For the reasoning behind the intercepts, see the previous within-chapter-exercise.

Exercise 3A.4

So far, we have implicitly assumed that interest compounds yearly —- i.e. you begin to earn interest on interest only at the end of each year. Often, interest compounds more frequently. Suppose that you put $10,000 in the bank now at an annual interest rate of 10% but that interest compounds monthly rather than yearly. Your monthly interest rate is then 10/12 or 0.833%. Defining n as the number of months and using the information in the previous paragraph, how much would



Exercise Graph 3A.3 : An increase in the interest rate to 20%

you have in the bank after 1 year? Compare this to the amount we calculated you would have when interest compounds annually.

Answer: You would have $10000(1+r)^n = 10000(1.00833)^{12} = 11,047.13$. Thus, over a 1 year period, by putting \$10,000 into a savings account at 10% annual interest that compounds monthly, you get \$47.13 more in interest than when putting the same amount into a savings account at the same interest rate but compounded annually.

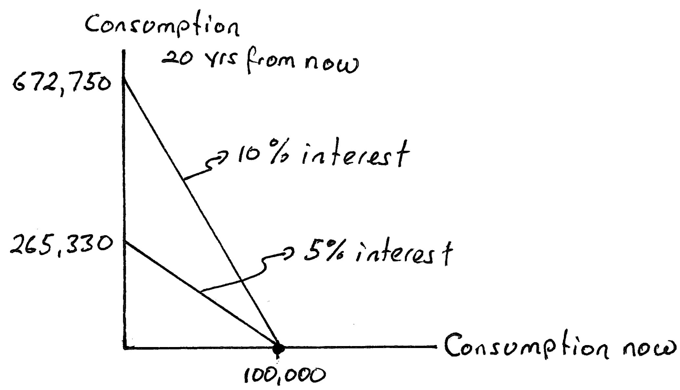
Exercise 3A.5

Suppose you just inherited \$100,000 and you are trying to choose how much of this to consume now and how much of it to save for retirement 20 years from now. Illustrate your choice set with "dollars of consumption now" and "dollars of consumption 20 years from now" assuming an interest rate of 5% (compounded annually). What happens if the interest rate suddenly jumps to 10% (compounded annually)?

Answer: Regardless of the interest rate, you can choose to consume the entire \$100,000 now (which therefore is your endowment point that lies on the horizontal axis). If you save all of it, you will collect $100000(1+r)^{20}$ where r is 0.05 when the interest rate is 5% and 0.1 when the interest rate is 10%. This results in intercepts on the vertical axis of \$265,330 if the interest rate is 5% and \$672,750 if the interest rate is 10%. This is depicted in Exercise Graph 3A.5.

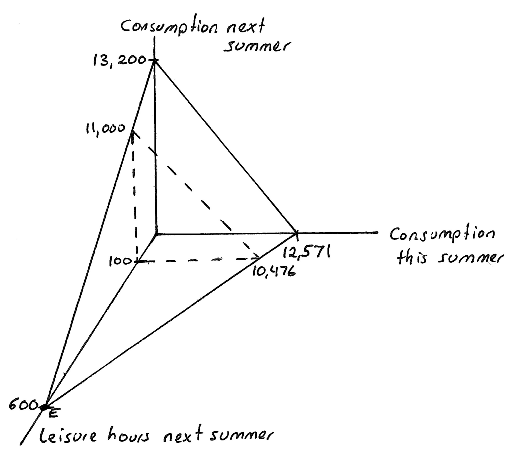
Exercise 3A.6

Draw a budget constraint similar to Graph 3.5 assuming you do not work this summer but rather next summer at a wage of \$22 per hour (with a total possible number of leisure hours of 600 next summer) and assuming that the interest rate is 5%. Where is the 5% interest rate budget line from Graph 3.3b in the graph you have just drawn?



Exercise Graph 3A.5 : Consume now or 20 years from now

Answer: The graph is depicted in Exercise Graph 3A.6, with the dashed slice equivalent to the budget line in the 2-dimensional graph earlier in the text. The endowment point is once again leisure — only this time 600 leisure hours *next* summer. At \$22 per hour, this translates into \$13,200 of consumption next summer if you choose to work all 600 hours. If you were to work all 600 hours but you wanted to borrow and consume all the resulting income now, your could consume $\$13,200/(1+0.05) = \$12,571$. The 2-dimensional graph earlier on took \$11,000 of income next summer as the starting point, thus implicitly assuming that you have chosen to work for 500 hours next summer, leaving you with 100 hours of leisure. Thus, the slice at 100 hours of leisure next summer represents the budget line in Graph 3.3b in the textbook.



Exercise Graph 3A.6 : Consuming over 2 summers but working only next summer

3B Solutions to Within-Chapter Exercises for Part B

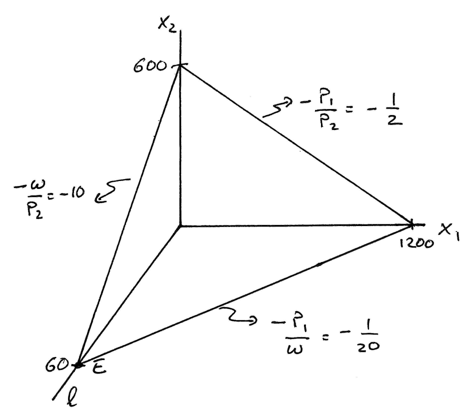
Exercise 3B.1

Graph the choice set in equation (3.5) when n=2, $p_1 = 1$, $p_2 = 2$, $w = 20$ and $L = 60$.

Answer: Using the values given in the exercise, the choice set is defined as

$$\left\{(x_1, x_2, \ell) \in \mathbb{R}^3_+ \mid x_1 + 2x_2 \leq 20(60 - \ell)\right\}. \tag{3B.1}$$

This is depicted in Exercise Graph 3B.1. The endowment point is leisure of 60 hours with no consumption. If this worker works all the time, she will earn \$1,200 given that she earns a wage of \$20 per hour. With that, she could buy as many as 1,200 units of x_1 if that was all she bought (as $p_1 = 1$), or as many as 600 units of x_2 if that is all she bought (at $p_2 = 2$).



Exercise Graph 3B.1 : Leisure and 2 goods

Exercise 3B.2

Translate the choice sets graphed in Graph 3.2 into mathematical notation defining the choice sets.

Answer: The choice set in panel (a) of the textbook Graph 3.2 is

$$\begin{aligned}\{\,(\ell, c) \in \mathbb{R}^2_+ \mid \quad & c = 1000 - 10\ell \quad \text{for } \ell \leq 20 \text{ and} \\ & c = 1200 - 20\ell \quad \text{for } \ell > 20\,\},\end{aligned} \tag{3B.2.i}$$

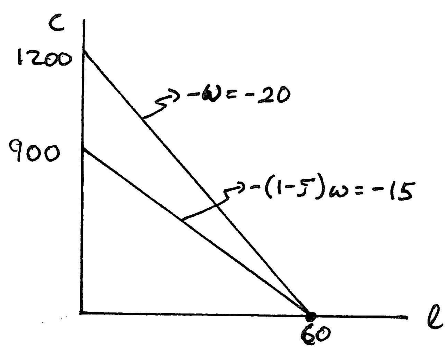
while the choice set in panel (b) is

$$\{\,(\ell,c)\in\mathbb{R}^2_+ \mid \quad \begin{array}{ll} c = 1400-30\ell & \text{for } \ell \le 20 \text{ and} \\ c = 1200-20\ell & \text{for } \ell > 20\,\}. \end{array} \tag{3B.2.ii}$$

Exercise 3B.3

Suppose $w = 20$ and $L = 60$. Graph the budget constraint in the absence of taxes. Then suppose a wage tax $t = 0.25$ is introduced. Illustrate how this changes your equation and the graph.

Answer: The budget equation is given by $c = (1-t)w(L-\ell)$ which is $c = 20(60-\ell) = 1200-20\ell$ when $t = 0$ and $c = (1-0.25)20(60-\ell) = 900-15\ell$ when $t = 0.25$. These are depicted in Exercise Graph 3B.3 below.



Exercise Graph 3B.3 : 25% tax on wages

Exercise 3B.4

How would the budget line equation change if, instead of a tax on wages, the government imposed a tax on all consumption goods such that the tax paid by consumers equaled 25% of consumption. Show how this changes the equation and the corresponding graph of the budget line.

Answer: Since all income earned through wages is by definition consumed in this model, a tax equivalent to 25% of consumption is equivalent to a 25% wage tax. So nothing would change in the equation or graph.

Exercise 3B.5

Suppose $(e_1 - c_1)$ is negative — i.e. suppose you are borrowing rather than saving in period 1. Can you still make intuitive sense of the equation?

Answer: When $(e_1 - c_1)$ is negative, you have consumed your period 1 endowment e_1 plus an amount $(c_1 - e_1)$ on top of it. The only way you could consume more than you had in period 1 is to borrow from period 2 — thus you must have borrowed the amount $(c_1 - e_1)$. One year later, you have to pay back that amount plus interest for a total of $(1+r)(c_1 - e_1)$. We therefore have to subtract that amount from e_2 to determine how much you will have left after paying back what you owe to the bank. Thus, consumption c_2 in period 2 must be no more than $e_2 - (1+r)(c_1 - e_1)$, which can be re-written as $c_2 \leq (1+r)(e_1 - c_1) + e_2$. In the case of borrowing, the quantity $(1+r)(e_1 - c_1)$ is therefore negative and equal to what you owe the bank in period 2.

Exercise 3B.6

Use the information behind each of the scenarios graphed in Graph 3.3 to plug into equation (3.8) that scenario's relevant values for e_1, e_2 and r. Then demonstrate that the budget lines graphed are consistent with the underlying mathematics of equation (3.8), and more generally, make intuitive sense of the intercept and slope terms as they appear in equation (3.8).

Answer: In panel (a) of the textbook Graph 3.3, $(e_1, e_2) = (10000, 0)$ and the interest rate is initially $r = 0.1$ and then falls to $r = 0.05$. Plugging these into the equation, we get $c_2 \leq 10000(1+r) - (1+r)c_1$. The intercept term is then 11,000 when $r = 0.1$ and 10,500 when $r = 0.05$, and the slopes are analogously either 1.10 or 1.05. This is precisely what is graphed in the textbook.

In panel (b) of the textbook Graph 3.3, $(e_1, e_2) = (0, 11000)$. The equation then becomes $c_2 \leq 11000 - (1+r)c_1$. Thus, regardless of the interest rate, the intercept is 11,000, but the slope is 1.10 when $r = 0.1$ and 1.05 when $r = 0.05$, all as depicted in the graphs in the text.

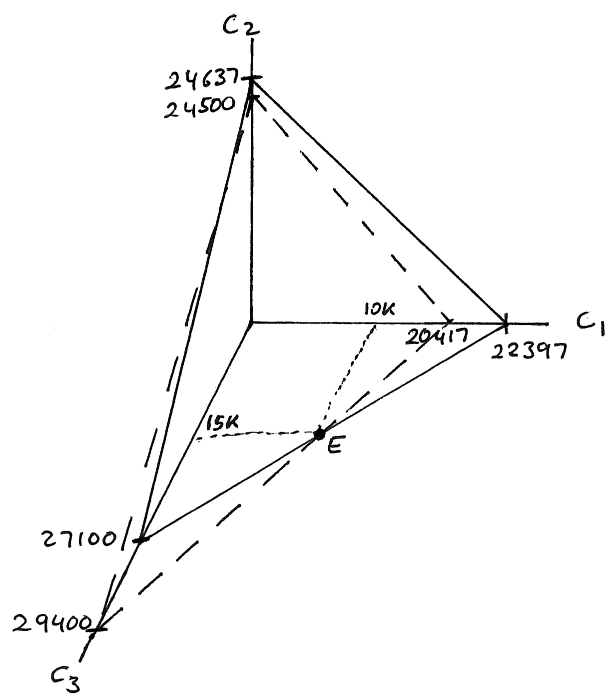
Finally, in panel (c) of the textbook Graph 3.3, $(e_1, e_2) = (5000, 5500)$. When plugged into the equation, this becomes $c_2 \leq (1+r)5000 + 5500 - (1+r)c_1$. This gives an intercept of $1.1(5000) + 5500 = 11000$ when $r = 0.10$ and $1.05(5000) + 5500 = 10750$ when $r = 0.05$, with slopes of 1.10 and 1.05 — all as depicted in the graph in the text.

For an intuitive explanation of all this, see the answer to within-chapter exercise 3A.2.

Exercise 3B.7

Suppose you expect to earn \$10,000 this summer, \$0 next summer and \$15,000 two summers from now. Using c_1, c_2, and c_3 to denote consumption over these three summers, write down your budget constraint assuming an annual (and annually compounding) interest rate of 10%. Then illustrate this constraint on a three dimensional graph with c_1, c_2, and c_3 on the three axes. How does your equation and graph change if the interest rate increases to 20%?

Answer: Exercise Graph 3B.7 depicts the constraint when the interest rate is 10% in solid lines and the constraint when the interest rate is 20% in dashed lines.



Exercise Graph 3B.7 : Income now and 2 years from now

Both constraints pass through the endowment point E. The underlying equation for each budget plane is

$$c_3 + (1+r)c_2 + (1+r)^2 c_1 = 15000 + 10000(1+r)^2. \tag{3B.7}$$

Consider first how this relates to the solid budget constraint when the interest rate is $r = 0.10$. To determine the c_3 intercept, we set $c_1 = c_2 = 0$ and get that $c_3 = 15000 + 10000(1+0.1)^2) = 27100$. To determine the c_2 intercept, we set $c_1 = c_3 = 0$ and get $(1+0.1)c_2 = 27100$ or $c_2 = 27100/1.1 = 24637$. Finally, to get the c_1 intercept, we set $c_3 = c_2 = 0$ and get $(1+0.1)^2 c_1 = 27100$ or $c_3 = 22397$. Notice that, focusing simply on the slice of the graph that holds $c_3 = 0$, we see that the c_2 intercept (24,637) divided by the c_1 intercept (22,397) is 1.1 or $(1+r)$ — which is exactly the slope we would expect for a one-period intertemporal budget constraint. The same is true for the slope on the slice that holds $c_1 = 0$. And, for the bottom slice where $c_2 = 0$, the slope is 1.21 or $(1+r)^2$ — again what we would expect for a 2-period intertemporal budget constraint.

The intercepts on the dashed budget plane can be similarly calculated, this time substituting $r = 0.2$ rather than $r = 0.1$ into the equations.

Exercise 3B.8

When $L = 600$, $w = 20$ and $r = 0.1$, show how the equation above translates directly into Graph 3.5.

Answer: Plugging in these values, we get $1.1c_1 + c_2 = 1.1(20)(600 - \ell)$ which can also be written as

$$1.1c_1 + c_2 + 22\ell = 13200. \tag{3B.8}$$

To determine the intercept on the c_1 axis, we simply set $c_2 = \ell = 0$ and get $1.1c_1 = 13200$ or $c_1 = 12000$. To determine the intercept on the c_2 axis, we set $c_1 = \ell = 0$ and get $c_2 = 13200$. Finally, we can check that the ℓ intercept is equal to the leisure endowment — by plugging $c_1 = c_2 = 0$ in to get $22\ell = 13200$ or $\ell = 600$.

Exercise 3B.9

Define mathematically a generalized version of the choice set in expression (3.18) under the assumption that you have both a leisure endowment L_1 this summer and another leisure endowment L_2 next summer. What is the value of L_2 in order for Graph 3.5 to be the correct 3-dimensional "slice" of this 4-dimensional choice set?

Answer: The budget set would then be defined as

$$\begin{aligned} B(L_1, L_2, w, r) = \{&(c_1, c_2, \ell_1, \ell_2) \in \mathbb{R}^4_+ \mid \\ &(1+r)c_1 + c_2 = (1+r)w(L_1 - \ell_1) + w(L_2 - \ell_2)\}. \end{aligned} \tag{3B.9.i}$$

When $L_2 = \ell_2$, the equation inside the set becomes

$$(1+r)c_1 + c_2 = (1+r)w(L_1 - \ell_1), \tag{3B.9.ii}$$

which is exactly the 3-dimensional choice set referred to in the text (where it was implicitly assumed that you consume all your leisure next summer and thus derive no income next summer).

3C Solutions to Odd-Numbered End-of-Chapter Exercises

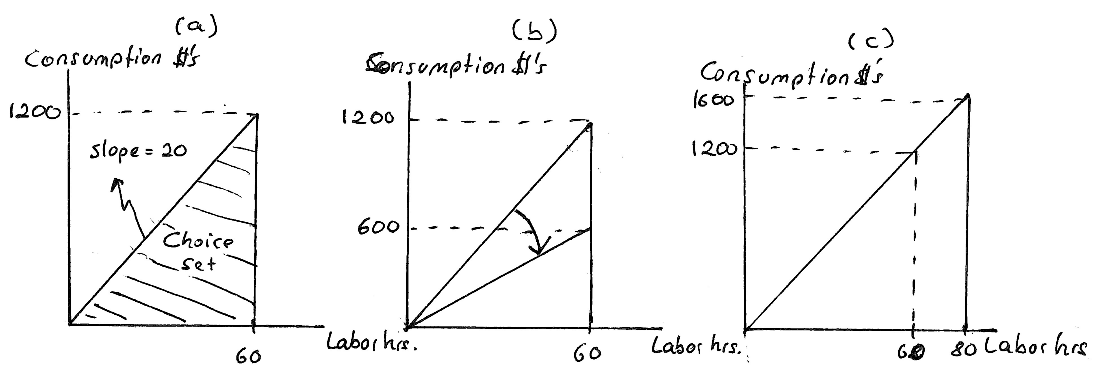
Exercise 3.1

In this chapter, we graphed budget constraints illustrating the trade-off between consumption and leisure.

A: *Suppose that your wage is $20 per hour and you have up to 60 hours per week that you could work.*

(a) *Now, instead of putting leisure hours on the horizontal axis (as we did in Graph 3.1), put labor hours on the horizontal axis (with consumption in dollars still on the vertical). What would your choice set and budget constraint look like now?*

Answer: Panel (a) of Exercise Graph 3.1 illustrates this choice set and budget constraint. It would begin at the origin (where no labor is provided and thus no income earned for consumption) and would rise by the wage rate (i.e. $20) for each hour of labor.



Exercise Graph 3.1 : Labor/Consumption Tradeoff

(b) *Where on your graph would the endowment point be?*

Answer: The endowment point is always the bundle that a consumer can consume regardless of market prices (or wages). In this case, the bundle (0,0) — i.e. no labor and no consumption, is always possible. This is equivalent to the endowment bundle (60,0) when we put leisure instead of labor on the horizontal axis.

(c) *What is the interpretation of the slope of the budget constraint you just graphed?*

Answer: The slope is equal to the wage rate (just as it is equal to the negative wage rate when leisure is graphed on the horizontal axis).

(d) *If wages fall to \$10 per hour, how does your graph change?*

Answer: It changes as in panel (b) of Exercise Graph 3.1, with a new slope of 10 rather than 20.

(e) *If instead a new caffeine drink allows you to work up to 80 rather than 60 hours per week, how would your graph change?*

Answer: Since wages have not changed, the graph would be identical up to 60 hours of work. But now 20 additional potential hours of work are possible — causing the budget constraint to extend all the way to 80 hours of labor. This is depicted in panel (c).

B: *How would you write the choice set over consumption c and labor l as a function of the wage w and leisure endowment L?*

Answer: The choice set would be

$$C(w,L) = \left\{(c,l) \in \mathbb{R}^2_+ \mid c \le wl \text{ and } l \le L\right\}. \tag{3.1.i}$$

Exercise 3.3

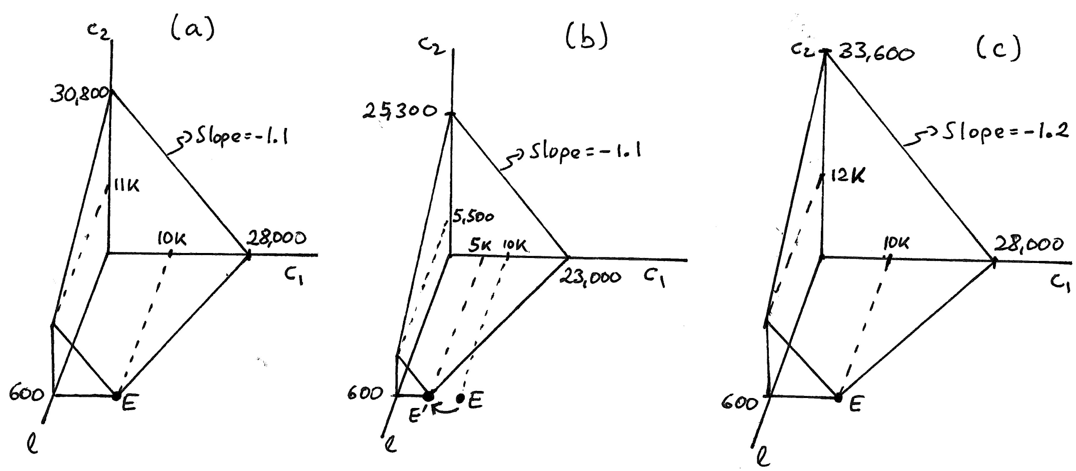
You have \$10,000 sitting in a savings account, 600 hours of leisure time this summer and an opportunity to work at a \$30 hourly wage.

A: *Next summer is the last summer before you start working for a living, and so you plan to take the whole summer off and relax. You need to decide how much to work this summer and how much to spend on consumption this summer and next summer. Any investments you make for the year will yield a 10% rate of return over the coming year.*

(a) *On a three dimensional graph with this summer's leisure (ℓ), this summer's consumption (c_1) and next summer's consumption (c_2) on the axes, illustrate your endowment point as well as your budget constraint. Carefully label your graph and indicate where the endowment point is.*

Answer: This is graphed in panel (a) of Exercise Graph 3.3.

The endowment point — the point on the budget constraint that is always available regardless of prices — is $(\ell, c_1, c_2) = (600, 10000, 0)$ where no work is done and all \$10,000 in the savings account is consumed now. No matter what the wage or the interest rate is, you can always consume this bundle. However, you can also work up to 600 hours this summer, which would earn you up to an additional \$18,000 for consumption now. So the most you could consume this summer if you worked all the time and emptied your savings account is \$28,000, the c_1 intercept. You can't consume any more than 600 hours of leisure — so 600 is the ℓ intercept. If you earned \$18,000 this summer (by working all the time and thus consuming zero leisure), and if you consumed nothing this summer, then the most you could consume next summer is \$28,000 times $(1+r)$ where $r = 0.1$ is the interest rate. This gives the c_2 intercept of \$30,800. If you don't work (i.e. $\ell = 600$) and you consume nothing this summer



Exercise Graph 3.3 : Leisure/Consumption and Intertemporal Tradeoffs Combined

(i.e. $c_1 = 0$), then you simply have your \$10,000 from your savings account plus \$1,000 in interest next summer, for a total of \$11,000 in c_2. The overall shape of the budget constraint then becomes the usual triangular shape but, because you can't buy leisure beyond 600 hours with your savings, the tip of the triangle is cut off.

(b) *How does your answer change if you suddenly realize you still need to pay \$5,000 in tuition for next year, payable immediately?*

Answer: This is graphed in panel (b). Since \$5,000 has to leave your savings account right now, this leaves you with only \$5,000 in the account and thus shifts your endowment point in. The rest of the budget constraint is derived through similar logic to what was used above. The budget constraint is again a triangular plane with the tip cut off, only now the tip that's cut off is smaller. (Had the immediately-due tuition payment been \$10,000, the cut-off tip would have disappeared.)

(c) *How does your answer change if instead the interest rate doubles to 20%?*

Answer: This is illustrated in panel (c) of Exercise Graph 3.3 where *E* is back to what it was in part (a) but the amount of consumption next summer goes up because of the higher interest rate. The reasoning for the various intercepts is similar to that above.

(d) *In (b) and (c), which slopes are different than in (a)?*

Answer: Slopes are formed by ratios of prices — so the only way that slopes can change is if a price has changed. In the scenario in (b), no price has changed. Thus, the only thing that happens is that *E* shifts in as indicated in the graph. This changes the various intercepts, but the slopes in each plane are parallel to those from panel (a). In the scenario in (c), on the other hand, the interest rate changes. The interest rate is a price that is reflected in any intertemporal budget constraint — i.e. any budget constraint that spans across time periods. In our graph in panel (c), this includes the constraint that lies in the plane that shows the tradeoff between c_1 and c_2, and the plane that shows the tradeoff between ℓ (which is leisure *this* summer) and c_2. Those slopes change as the interest rate changes, but the slope in the plane that illustrates the tradeoff between ℓ and c_1 does not — because that tradeoff happens within the same time period and thus does not involve interest payments.

B: *Derive the mathematical expression for your budget constraint in 3.3A and explain how elements of this expression relate to the slopes and intercepts you graphed.*

Answer: An intuitive way to construct this mathematical expression involves thinking about how much consumption c_2 is possible next summer. If you consume ℓ amount of leisure this summer, you will have a total of $10000 + 30(600 - \ell)$ available for consumption *this* summer — the \$10,000 in the savings account plus your earnings (at a wage of \$30) from hours that you did not consume as leisure. When we take this amount and subtract from it the consumption c_1 you actually undertake this summer, we get the amount that will be in the savings account for the year to accumulate interest. Thus, next summer, you will have $10000 + 30(600 - \ell) - c_1$ times $(1 + r)$ (where $r = 0.1$ is the interest rate); i.e.

$$c_2 = 1.1\,[10000 + (30)(600 - \ell) - c_1] = 30800 - 33\ell - 1.1c_1\,, \tag{3.3.i}$$

or, written differently,

$$1.1c_1 + c_2 + 33\ell = 30800. \tag{3.3.ii}$$

The only caveat to this when we define the budget plane is that we have to take into account that you cannot consume more than 600 hours of leisure (or negative quantities of consumption). We can incorporate this by defining the budget plane as

$$\left\{(c_1, c_2, \ell) \in \mathbb{R}^3_+ \mid 1.1c_1 + c_2 + 33\ell = 30800 \text{ and } \ell \leq 600\right\}. \tag{3.3.iii}$$

Exercise 3.5

Suppose you are a carefree 20-year old bachelor whose lifestyle is supported by expected payments from a trust fund established by a relative who has since passed away. The trust fund will pay you \$x when you turn 21 (a year from now), another \$y when you turn 25 and \$z when you turn 30. You plan to marry a rich heiress on your 30th birthday and therefore only have to support yourself for the next 10 years. The bank that maintains the trust account is willing to lend money to you at a 10% interest rate and pays 10% interest on savings. (Assume annual compounding.)

A: *Suppose* $x = y = z = 100,000$.

(a) *What is the most that you could consume this year?*

Answer: For the initial \$100,000 you get next year when you turn 21, the bank would be willing to lend you \$100,000/(1+0.1)=\$90,909.09. For the \$100,000 you get 5 years from now when you turn 25, the bank would be willing to lend you $\$100,000/(1+0.1)^5$=\$62,092.13. And the bank would lend you up to $\$100,000/(1+0.1)^{10}$=\$38,554.33 for the \$100,000 you get on your 30th birthday (10 years from now). That sums to \$191,555.55.

(b) *What is the most you could spend at your bachelor party 10 years from now if you find a way to live without eating?*

Answer: If you saved your initial \$100,000 for 9 years (until your 30th birthday around which you will have your bachelor's party), you would have accumulated $\$100,000(1+0.1)^9 = \$235,794.77$. If you save the \$100,000 you get on your 25th birthday for 5 years, you would accumulate $\$100,000(1+0.1)^5$=\$161,051. Add those two amounts to the \$100,000 you get when you are 30, and you get that you could spend a total of \$496,845.77 on your bachelor's party.

B: *Define your 10 year intertemporal budget constraint mathematically in terms of x, y and z, letting c_1 denote this year's consumption, c_2 next year's consumption, etc. Let the annual interest rate be denoted by r.*

Answer: You can think of this in the following way: First, how much will you have in wealth on your 30th birthday if you spend nothing and save everything? You will have had x in the bank for 9 years and y for 5 years — in addition to just having received z (on your 30th birthday). Thus, you will have

$$\text{Potential Wealth on 30th Birthday} = (1+r)^9 x + (1+r)^5 y + z. \tag{3.5.i}$$

Next, we can ask how much would be available for consumption in the year that starts with your 29th birthday (assuming you have to borrow whatever you intend to spend at the beginning of that year). Since we know how much wealth you would have on your 30th birthday if you did not spend anything leading up to it, we know that the most you can consume in the year before is how much you could borrow on that wealth; which means that the most you could consume in year 10, c_{10}^{max}, is andamount that would allow you to pay back $(1+r)^9 x + (1+r)^5 y + z$ one year from then — i.e.

$$(1+r)c_{10}^{max} = (1+r)^9 x + (1+r)^5 y + z. \tag{3.5.ii}$$

The most you can consume in year 9, c_9^{max}, is similarly what you could pay back on your 30th birthday (with 2 years of interest) minus what you actually consumed in year 10 (c_{10}) (plus one year interest) — i.e.

$$(1+r)^2 c_9^{max} + (1+r)c_{10} = (1+r)^9 x + (1+r)^5 y + z. \tag{3.5.iii}$$

Continuing this same logic backwards, we then get the 10-year intertemporal budget constraint

$$(1+r)^{10}c_1+(1+r)^9c_2+...+(1+r)^2c_9+(1+r)c_{10} \leq (1+r)^9x+(1+r)^5y+z. \quad (3.5.iv)$$

Exercise 3.7

Everyday Application: *Investing for Retirement: Suppose you were just told that you will receive an end-of-the-year bonus of $15,000 from your company. Suppose further that your marginal income tax rate is 33.33% — which means that you will have to pay $5,000 in income tax on this bonus. And suppose that you expect the average rate of return on an investment account you have set up with your broker to be 10% annually (and, for purposes of this example, assume interest compounds annually.)*

A: *Suppose you have decided to save all of this bonus for retirement 30 years from now.*

(a) *In a regular investment account, you will have to pay taxes on the interest you earn each year. Thus, even though you earn 10%, you have to pay a third in taxes — leaving you with an after-tax return of 6.67%. Under these circumstances, how much will you have accumulated in your account 30 years from now?*

Answer: You will have $\$10000(1+0.06667)^{30} = \$69,327$. Since you have already paid taxes on the initial bonus and on all interest income, no further taxes are due — so all $69,327 is available for consumption.

(b) *An alternative investment strategy is to place your bonus into a 401K "tax-advantaged" retirement account. The federal government has set these up to encourage greater savings for retirement. They work as follows: you do not have to pay taxes on any income that you put directly into such an account if you put it there as soon as you earn it, and you do not have to pay taxes on any interest you earn. Thus, you can put the full $15,000 bonus into the 401K account, and you can earn the full 10% return each year for the next 30 years. You do, however, have to pay taxes on any amount that you choose to withdraw after you retire. Suppose you plan to withdraw the entire accumulated balance as soon as you retire 30 years from now, and suppose that you expect you will still be paying 33.33% taxes at that time. How much will you have accumulated in your 401K account, and how much will you have after you pay taxes? Compare this to your answer to (a) — i.e. to the amount you would have at retirement if you saved outside the 401K plan.*

Answer: Your account will grow to $\$15000(1+0.1)^{30} = \$261,741$. But you still have to pay one third in taxes — leaving you with $(2/3)*(\$261,741) \approx \$174,503$. This is substantially larger than the amount of $69,327 we calculated in part (a).

(c) True or False: *By allowing individuals to defer paying taxes into the future, 401K accounts result in a higher rate of return for retirement savings.*

Answer: This is true. In both cases, you end up paying taxes on all your income — both the initial income as well as interest income. The only difference between the two investment strategies is that in one case income is taxed as it is made, in the other case it is taxed at the end when it is withdrawn for consumption. In the latter case, the investor benefits from accumulating more interest faster. In our example, for instance, you end up with $105,175 more with a 401K account than in a non-tax advantaged account.

B: *Suppose more generally that you earn an amount I now, that you face (and will face in the future) a marginal tax rate of t (expressed as a fraction between 0 and 1), that the interest rate now (and in the future) is r and that you plan to invest for n periods into the future.*

(a) *How much consumption will you be able to undertake n years from now if you first pay your income tax on the amount I, then place the remainder in a savings account whose interest income is taxed each year. (Assume you add nothing further to the savings account between now and n years from now).*

Answer: You would place $(1-t)I$ into the account, and it would earn an after-tax rate of return of $(1-t)r$. Over n years, this results in $(1-t)I(1+(1-t)r)^n$.

(b) *Now suppose you put the entire amount I into a tax-advantaged retirement account in which interest income can accumulate tax-free. Any amount that is taken out of the account is then taxed as regular income. Assume you plan to take the entire balance in the account out n years from now (but nothing before then). How much consumption can you fund from this source n years from now?*

Answer: You would then accumulate $I(1+r)^n$, but that amount would then be taxed at the end. So, what you are left with would be $(1-t)I(1+r)^n$.

(c) *Compare your answers to (a) and (b) and indicate whether you can tell which will be higher.*

Answer: In both cases, a quantity $(1-t)I$ is multiplied by another term in parentheses. In the case of no tax-advantaged treatment, this second term is $(1+(1-t)r)^n$; in the 401K case, it is $(1+r)^n$. So the latter is bigger so long as $(1+r)$ is larger than $(1+(1-t)r)$ which is equal to $(1+r-tr)$. When expressed this way, it is clear that the latter is smaller by tr — tax advantaged savings accounts always result in larger future consumption for given levels of investment.

Exercise 3.9

Business Application: *Present Value of Winning Lottery Tickets*: *The introduction to intertemporal budgeting in this chapter can be applied to thinking about the* *pricing of basic financial assets. The assets we will consider will differ in terms of when they pay income to the owner of the asset. In order to know how much such assets are worth, we have to determine their* present value — *which is equal to how much* current *consumption such an asset would allow us to undertake.*

A: *Suppose you just won the lottery and your lottery ticket is transferable to someone else you designate — i.e. you can sell your ticket. In each case below, the lottery claims that you won $100,000. Since you can sell your ticket, it is a financial asset, but depending on how exactly the holder of the ticket received the $100,000, the asset is worth different amounts. Think about what you would be willing to actually sell this asset for by considering how much* current *consumption value the asset contains — assuming the annual interest rate is 10%.*

(a) *The holder of the ticket is given a $100,000 government bond that "matures" in 10 years. This means that in 10 years the owner of this bond can cash it for $100,000.*

Answer: To know how much this lottery ticket is worth, we have to determine how much the bank would be willing to lend us for current consumption. If we could get the $100,000 one year from now, we know the bank would lend us up to $100000/1.1=$90,909.09. If we get the $100,000 two years from now, however, the most the bank would be willing to lend us is $\$100000/(1.1^2) = \$82,644.63$. And if we can only get to the $100,000 ten years from now, the most we can get for it now is $(\$100000/(1.1^{10}) = \$38,554.33$. Thus, that's the least you would be willing to sell the bond (and thus your ticket) for — and the most anyone else who faces a 10% interest rate should be willing to pay.

(b) *The holder of the ticket will be awarded $50,000 now and $50,000 ten years from now.*

Answer: The most you can borrow on an amount $50,000 ten years from now is the amount $\$50000/(1.1^{10}) = \$19,277.16$. Thus, together with the $50,000 the lottery awards you now, the most you could consume now is $69,277.16. Thus, that is the least you would be willing to sell your ticket for.

(c) *The holder of the ticket will receive 10 checks for $10,000 — one now, and one on the next 9 anniversaries of the day he/she won the lottery.*

Answer: For a check n years from now, I can borrow $\$10000/(1.1^n)$. Calculating this for each of the next 9 years, the checks will be worth $9,090.91, $8,264.46, $7,513.15, $6,830.13, $6,209.21, $5,644.74, $5,131.58, $4,665.07 and $4,240.98. Summing these and adding the value of my current $10,000 check, the total possible consumption I can undertake is then $67,590.24 — which is the current value of the ticket.

(d) *How does you answer to part (c) change if the first of 10 checks arrived 1 year from now, with the second check arriving 2 years from now, the third 3 years from now, etc.?*

Answer: The value of the *first* 9 checks would then be the same as the value of the *last* 9 checks in the previous part. But you would lose the first check from the previous part (which was worth \$10,000) to be replaced with a check 10 years from now, which is worth $\$10000/(1.1^{10}) =$ $\$3,885.43$. Thus, the ticket would be worth \$10,000−\$3,885.43=\$6,114.57 less, or \$61,445.67 instead of \$67,590.24.

(e) *The holder of the ticket gets \$100,000 the moment he/she presents the ticket.*

Answer: This ticket is, of course, the only one that's worth \$100,000 as claimed by the lottery.

B: *More generally, suppose the lottery winnings are paid out in installments of $x_1, x_2, ..., x_{10}$, with payment x_i occurring $(i-1)$ years from now. Suppose the annual interest rate is r.*

(a) *Determine a formula for how valuable such a stream of income is in present day consumption — i.e. how much present consumption could you undertake given that the bank is willing to lend you money on future income?*

Answer: The present consumption c that could be financed by such a stream of payments is

$$c = x_1 + \frac{x_2}{(1+r)} + \frac{x_3}{(1+r)^2} + \frac{x_4}{(1+r)^3} + ... + \frac{x_9}{(1+r)^8} + \frac{x_{10}}{(1+r)^9} \tag{3.9.i}$$

which can also be written as

$$c = \sum_{i=1}^{10} \frac{x_i}{(1+r)^{i-1}}. \tag{3.9.ii}$$

(b) *Check to make sure that your formula works for each of the scenarios in part A.*

Answer: Plugging in the appropriate values for each part, you should get the same answers as you did in part A.

(c) *The scenario described in part A(c) is an example of a \$10,000 payment followed by an annual "annuity" payment. Consider an annuity that promises to pay out \$10,000 every year starting 1 year from now for n years. How much would you be willing to pay for such an annuity?*

Answer: The present consumption that could be financed by such an annuity is

$$c = \frac{10000}{(1+r)} + \frac{10000}{(1+r)^2} + \frac{10000}{(1+r)^3} + ... + \frac{10000}{(1+r)^{n-1}} + \frac{10000}{(1+r)^n}, \tag{3.9.iii}$$

which can also be written as

$$c = \sum_{i=1}^{n} \frac{10000}{(1+r)^i} = 10000 \sum_{i=1}^{n} \frac{1}{(1+r)^i}. \tag{3.9.iv}$$

(d) *How does your answer change if the annuity starts with its first payment now?*

Answer: All that happens is that the expressions in the previous part get an additional \$10,000 added, which would allow us to write the second expression as

$$c=\sum_{i=0}^{n}\frac{10000}{(1+r)^i}=10000\sum_{i=0}^{n}\frac{1}{(1+r)^i}. \qquad (3.9.v)$$

(e) *What if the annuity from (c) is one that never ends? (To give the cleanest possible answer to this, you should recall from your math classes that an infinite series of* $1/(1+x)+1/(1+x)^2+1/(1+x)^3+...=1/x$*.) How much would this annuity be worth if the interest rate is 10%?*

Answer: Using our equation (3.9.iv) from part (c), we can write this as an infinite series

$$c=10000\sum_{i=1}^{\infty}\frac{1}{(1+r)^i}=\frac{10000}{r}. \qquad (3.9.vi)$$

If the interest rate is $r=0.1$, then this expression tells us the annuity would be worth \$100,000.

Exercise 3.11

Business Application: *Compound Interest over the Long Run: Uncle Vern has just come into some money — \$100,000 — and is thinking about putting this away into some investment accounts for a while.*

A: *Vern is a simple guy — so he goes to the bank and asks them what the easiest option for him is. They tell him he could put it into a savings account with a 10% interest rate (compounded annually).*

(a) *Vern quickly does some math to see how much money he'll have 1 year from now, 5 years from now, 10 years from now and 25 years from now assuming he never makes withdrawals. He doesn't know much about compounding — so he just guesses that if he leaves the money in for 1 year, he'll have 10% more; if he leaves it in 5 years at 10% per year he'll have 50% more; if he leaves it in for 10 years he'll have 100% more and if he leaves it in for 25 years he'll have 250% more. How much does he expect to have at these different times in the future?*

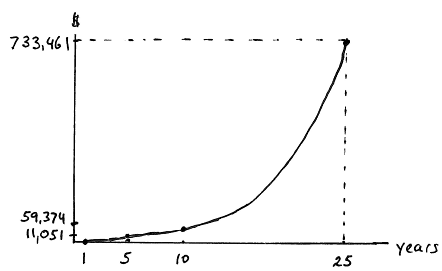
Answer: He expects to have \$110,000 1 year from now, \$150,000 five years from now, \$200,000 ten years from now and \$350,000 twenty-five years from now.

(b) *Taking the compounding of interest into account, how much will he really have?*

Answer: Using our usual formula, the actual balance n years from now is $\$100000(1.1)^n$. This gives \$110,000 one year from now, \$161,051 five years from now, \$259,374.25 ten years from now, and \$1,083,460.59 twenty five years from now.

(c) *On a graph with years on the horizontal axis and dollars on the vertical, illustrate the size of Vern's error for the different time intervals for which he calculated the size of his savings account.*

Answer: The size of the error is \$0 one year from now, \$11,051 five years from now, \$59,374.25 ten years from now and \$733,460.59 twenty-five years from now. This is graphed in Exercise Graph 3.11.



Exercise Graph 3.11 : Error from not compounding over time

(d) *True/False: Errors made by not taking the compounding of interest into account expand at an increasing rate over time.*

Answer: The statement is clearly true based on the answers above.

B: *Suppose that the annual interest rate is r.*

(a) *Assuming you will put x into an account now and leave it in for n years, derive the implicit formula Vern used when he did not take into account interest compounding.*

Answer: Letting y_n denote the amount he projected will be in the savings account n years from now, he used the formula $y_n = x(1+nr)$.

(b) *What is the correct formula that includes compounding.*

Answer: Using z_n to determine the actual amount in the savings account n years from now, the correct formula is $z_n = x(1+r)^n$

(c) Define a new function that is the difference between these. Then take the first and second derivatives with respect to n and interpret them.

Answer: The new function is $z_n - y_n = x(1+r)^n - x(1+nr)$. First, note that, when $n = 1$, this function reduces to zero — implying the difference between Vern's prediction and reality is zero 1 year from now (just as you determined earlier in the problem). The derivative of this function with respect to n is

$$\frac{\partial(z_n - y_n)}{\partial n} = x(1+r)\ln(1+r) - xr. \tag{3.11.i}$$

For any $n \geq 1$, this is clearly positive — which means the difference is increasing with time. The second derivative with respect to n is $x(1+$

$r)^n[\ln(1+r)]^2$ which is also positive. Thus the rate at which the difference increases is increasing with time. This is exactly what we illustrated in Exercise Graph 3.11.

Exercise 3.13

Business Application: *Buying Houses with Annuities*: *Annuities are streams of payments that the owner of an annuity receives for some specified period of time. The holder of an annuity can sell it to someone else who then becomes the recipient of the remaining stream of payments that are still owed.*

A: *Some people who retire and own their own home finance their retirement by selling their house for an annuity: The buyer agrees to pay $x per year for n years in exchange for becoming the owner of the house after n years.*

(a) *Suppose you have your eye on a house down the street owned by someone who recently retired. You approach the owner and offer to pay her $100,000 each year (starting next year) for 5 years in exchange for getting the house in 5 years. What is the value of the annuity you are offering her assuming the interest rate is 10%?*

Answer: The value would be

$$\frac{\$100,000}{1.1}+\frac{\$100,000}{1.1^2}+\frac{\$100,000}{1.1^3}+\frac{\$100,000}{1.1^4}+\frac{\$100,000}{1.1^5}=\$379,078.68. \qquad (3.13.i)$$

(b) *What if the interest rate is 5%?*

Answer: Now the value would be

$$\frac{\$100,000}{1.05}+\frac{\$100,000}{1.05^2}+\frac{\$100,000}{1.05^3}+\frac{\$100,000}{1.05^4}+\frac{\$100,000}{1.05^5}=\$432,947.67. \qquad (3.13.ii)$$

(c) *The house's estimated current value is $400,000 (and your real estate agent assures you that homes are appreciating at the same rate as the interest rate.) Should the owner accept your deal if the interest rate is 10%? What if it is 5%?*

Answer: Since the house appreciates at the interest rate, we can use its current value and compare it to the current value of the annuity. Given what we calculated above, accepting the annuity is a good deal for the current owner at the 5% interest rate but not at the 10% interest rate.

(d) True/False*: The value of an annuity increases as the interest rate increases.*

Answer: This is false, as we just demonstrated above. The value of an annuity increases as the interest rate falls. That should make sense — if I have a given amount to invest, I can invest it where it earns interest, or I can buy an annuity. When the interest rate falls, I will not be able to make as much by investing the money where it makes interest. So that should mean getting a fixed payment through an annuity becomes more valuable.

(e) *Suppose that, after making the second payment on the annuity, you fall in love with someone from a distant place and decide to move there. The house has appreciated in value (from its starting value of $400,000) by 10% each of the past two years. You no longer want the house and therefore would like to sell your right to the house in 3 years in exchange for having someone else make the last 3 annuity payments. How much will you be able to get paid to transfer this contract to someone else if the annual interest rate is always 10%?*

Answer: After two years, the house will be worth $\$400,000(1.1^2) = \$484,000$. The value of the annuity with 3 more payments is

$$\frac{\$100,000}{1.1} + \frac{\$100,000}{1.1^2} + \frac{\$100,000}{1.1^3} = \$248,685.20. \tag{3.13.iii}$$

Thus, you should be able to get \$484,000−\$248,685.20=\$235,314.80 for selling your contract.

B: *In some countries, retirees are able to make contracts similar to those in part A except that they are entitled to annuity payments until they die and the house only transfers to the new owner after the retiree dies.*

(a) *Suppose you offer someone whose house is valued at $400,000 an annual annuity payment (beginning next year) of $50,000. Suppose the interest rate is 10% and housing appreciates in value at the interest rate. This will turn from a good deal to a bad deal for you when the person lives n number of years. What's n? (This might be easiest to answer if you open a spreadsheet and you program it to calculate the value of annuity payments into the future.)*

Answer: If the person lives for another 16 years, the value of the annuity is \$391,185.43. If the person lives an addition year (for a total of 17 years), the value of the annuity becomes \$401,077.67. Thus $n = 17$ because it is a good deal for you as long as the person lives fewer than 17 more years.

(b) *Recalling that the sum of the infinite series* $1/(1+x) + 1/(1+x)^2 + +1/(1+x)^3 + ...$ *is* $1/x$*, what is the most you would be willing to pay in an annual annuity if you want to be absolutely certain that you are not making a bad deal?*

Answer: If you approximate a long life by infinity, then you would want to pay an annual amount no greater than an amount x that solves the equation $\$400,000 = x/0.1$. Solving this for x gives $x = \$40,000$.

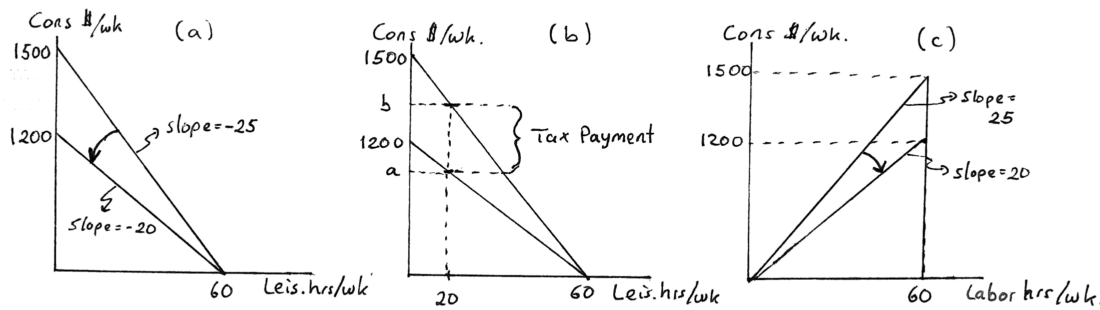
Exercise 3.15

Policy Application: *Wage Taxes and Budget Constraints*: *Suppose you have 60 hours of leisure that you could devote to work per week, and suppose that you can earn an hourly wage of $25.*

A: *Suppose the government imposes a 20% tax on all wage income.*

(a) *Illustrate your weekly budget constraint before and after the tax on a graph with weekly leisure hours on the horizontal and weekly consumption (measured in dollars) on the vertical axis. Carefully label all intercepts and slopes.*

Answer: The two budget constraints are illustrated in panel (a) of Exercise Graph 3.15. The after-tax wage is $w = 20$, 20% less than the before tax wage $w = 25$.



Exercise Graph 3.15 : Wage Taxes and Tax Payments

(b) *Suppose you decide to work 40 hours per week after the tax is imposed. How much wage tax do you pay per week? Can you illustrate this as a vertical distance in your graph? (*Hint*: Follow a method similar to that developed in end-of-chapter exercise 2.15)*

Answer: This is illustrated in panel (b) of the graph. When you work 40 hours a week, you consume 20 hours of leisure. On the after-tax budget, that leaves you with a in consumption. On the before-tax budget, it leaves you with b in consumption. The vertical difference $(b - a)$ must therefore be the total tax payment you made under the tax. Note that $a = 800$ and $b = 1000$ — with the vertical difference therefore equal to $(b - a) = 200$. This makes sense — when the worker is working for 40 hours a week, he is earning $1,000 before taxes and thus pays $200 in taxes at a 20% tax rate.

(c) *Suppose that instead of leisure hours on the horizontal axis, you put labor hours on this axis. Illustrate your budget constraints that have the same information as the ones you drew in (a). Assume again that the leisure endowment is 60 per week.*

Answer: This is illustrated in panel (c) of the graph.

B: Suppose the government imposes a tax rate t (expressed as a rate between 0 and 1) on all wage income.

(a) *Write down the mathematical equations for the budget constraints and describe how they relate to the constraints you drew in A(a).*

Answer: For every hour of labor, a worker makes w but pays tw in taxes. Thus, his after-tax wage is $(1-t)w$. He will be able to consume as much as he earns, and how much he earns depends on how much leisure he does not consume. Letting ℓ be leisure hours consumed, this implies that $c=(1-t)w(60-\ell)$. For the case where $w=25$ and $t=0.2$, this becomes $c=0.8(25)(60-\ell)=1200-20\ell$ which is the after-tax equation we graphed in panel (a). The before-tax equation has $t=0$, with $c=25(60-\ell)=1500-25\ell$.

(b) *Use your equation to verify your answer to part A(b).*

Answer: Using the after-tax equation $c=1200-20\ell$, we can plug in $\ell=20$ which is when you choose to work 40 hours per week. This gives $c=1200-20(20)=800$ which is the consumption level denoted a in panel (b) of the graph. Using the before-tax equation $c=1500-25\ell$, we get $c=1500-25(20)=1000$ which is the consumption level denoted b in panel (b) of the graph. The difference between the two consumption levels is \$200 — which is the tax payment per week. This is intuitively correct — if you work for 40 hours at a wage of \$25, you earn \$1,000 per week, and if you pay taxes of 20% on that, you will pay \$200 in taxes.

(c) *Write down the mathematical equations for the budget constraints you derived in B(a) but now make consumption a function of labor, not leisure hours. Relate this to your graph in A(c).*

Answer: Let labor hours be denoted l. Then, with a tax t and wage w, your consumption c is simply the portion of your pay check that you get to keep — which is $(1-t)wl$. Thus, $c=(1-t)wl$. When $w=25$ and $t=0.2$, this becomes $c=0.8(25)l=20l$ which is the after tax budget constraint graphed in panel (c) of Exercise Graph 3.15.

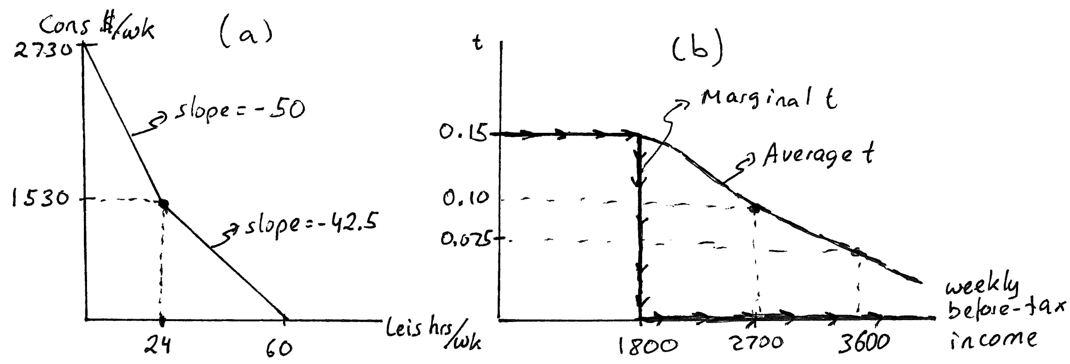
Exercise 3.17

Policy Application: *Social Security (or Payroll) Taxes*: *Social Security is funded through a payroll tax that is separate from the federal income tax. It works in a way similar to the following example: For the first \$1,800 in weekly earnings, the government charges a 15% wage tax but then charges no payroll tax for all earnings above \$1,800 per week.*

A: *Suppose that a worker has 60 hours of leisure time per week and can earn \$50 per hour.*

(a) *Draw this worker's budget constraint with weekly leisure hours on the horizontal axis and weekly consumption (in dollars) on the vertical.*

Answer: Panel (a) of Exercise Graph 3.17 traces out this budget constraint. The kink point happens at 24 hours of leisure — or 36 hours of labor. At that point, the worker earns \$1800 before taxes and pays 0.15(\$1800)=\$270 in taxes, leaving him with \$1,530 in consumption. For any lower levels of leisure (more work), the worker incurs no additional tax, causing his budget constraint to get steeper.



Exercise Graph 3.17 : Regressive Payroll Tax

(b) *Using the definitions given in exercise 3.16, what is the marginal and average tax rate for this worker assuming he works 30 hours per week? What if he works 40 hours per week? What if he works 50 hours per week?*

Answer: If he works 30 hours, his marginal and average tax rates are both 0.15 or 15%. If he works 40 or 50 hours, his marginal tax rate is zero. His before tax income at 40 hours is $2,000 and at 50 hours it is $2,500. In both cases, he pays $270 in weekly payroll taxes. Thus, his average tax rate at 40 hours of work is 270/2000=0.135 or 13.5%. His average tax rate at 50 hours of work is 270/2500=0.108 or 10.8%.

(c) *A wage tax is called* regressive *if the average tax rate falls as earnings increase. On a graph with weekly before-tax income on the horizontal axis and tax rates on the vertical, illustrate the marginal and average tax rates as income increases. Is this tax regressive?*

Answer: This is graphed in panel (b) of Exercise Graph 3.17. Taxes with declining average tax rates are regressive — so yes, this tax is regressive.

(d) True or False*: Budget constraints illustrating the tradeoffs between leisure and consumption will have no kinks if a wage tax is proportional. However, if the tax system is designed with different tax brackets for different incomes, budget constraints will have kinks that point inward when a wage tax is regressive and kinks that point outward when a wage tax is progressive.*

Answer: This is true as illustrated in this exercise and exercise 3.16.

B: *Consider the more general case of a tax that imposes a rate t on income immediately but then falls to zero for income larger than x.*

(a) *Derive the average tax rate function $a(I,t,x)$ (where I represents weekly income).*

Answer: The function is

$$a(I,t,x) = \begin{array}{ll} t & \text{if } I \le x \text{ and} \\ (tx)/I & \text{if } x < I. \end{array} \tag{3.17.i}$$

(b) *Derive the marginal tax rate function $m(I,t,x)$.*

Answer: The marginal tax rate function is

$$m(I,t,x) = \begin{array}{ll} t & \text{if } I < x \text{ and} \\ 0 & \text{if } x \ge I. \end{array} \tag{3.17.ii}$$

(c) *Does the average tax rate reach the marginal tax rate for high enough income?*

Answer: No, it merely converges to 0 as income gets large but never reaches it because everyone always continues to pay tx regardless of how high income gets.

Exercise 3.19

Policy Application: *The Earned Income Tax Credit: During the Clinton Administration, the EITC — or Earned Income Tax Credit, was expanded considerably. The program provides a wage subsidy to low income families through the tax code in a way similar to this example: Suppose, as in the previous exercise, that you can earn \$5 per hour. Under the EITC, the government supplements your first \$20 of daily earnings by 100% and the next \$15 in daily earnings by 50%. For any daily income above \$35, the government imposes a 20% tax.*

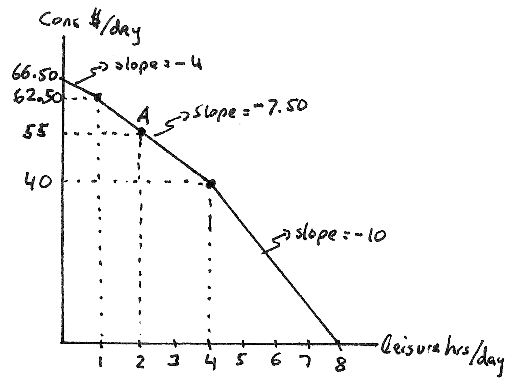
A: *Suppose you have at most 8 hours of leisure time per day.*

(a) *Illustrate your budget constraint (with daily leisure on the horizontal and daily consumption on the vertical axis) under this EITC.*

Answer: The budget constraint is graphed in Exercise Graph 3.19(1). For the first 4 hours of labor, the take-home wage is \$10 per hour because of the 100% subsidy. For the next 3 hours of labor, the take-home wage is \$7.50 because of the 50% subsidy. Finally, for any work beyond 7 hours, the take home wage is \$4 because of the 20% tax.

(b) *Suppose the government ends up paying a total of \$25 per day to a particular worker under this program and collects no tax revenue. Identify the point on the budget constraint this worker has chosen. How much is he working per day?*

Answer: The worker would work for 6 hours. At a wage of \$5, this would mean making \$30 per day. But, for the first \$20, the government adds \$20, and for the next \$10 hours, the government adds \$5 — for a total EITC supplement of \$25. Thus, the worker will have \$55 in income for other consumption. This gives us A in the graph — leisure of 2 hours per day (because of 6 hours of work) and consumption of \$55 per day.



Exercise Graph 3.19(1) : EITC Budget Constraint

(c) *Return to your graph of the same worker's budget constraint under the AFDC program in exercise 3.18. Suppose that the government paid a total of $25 in daily AFDC benefits to this worker. How much is he working?*

Answer: The worker is working at most 1 hour.

(d) *Discuss how the difference in trade-offs implicit in the EITC and AFDC programs could cause the same individual to make radically different choices in the labor market.*

Answer: Despite the government spending the same on the worker under AFDC and EITC, the worker might choose to not work much under AFDC and a lot under EITC. This is because of the implicit large tax rate imposed on the worker under AFDC but not under EITC.

B: *More generally, consider an EITC program in which the first x dollars of income are subsidized at a rate $2s$; the next x dollars are subsidized at a rate s; and any earnings above $2x$ are taxed at a rate t.*

(a) *Derive the marginal tax rate function $m(I, x, s, t)$ where I stands for labor market income.*

Answer: This function is

$$m(I,x,s,t) = \begin{array}{ll} -2s & \text{if } I < x \text{ and} \\ -s & \text{if } x \le I < 2x \text{ and} \\ t & \text{if } I \ge 2x. \end{array} \tag{3.19.i}$$

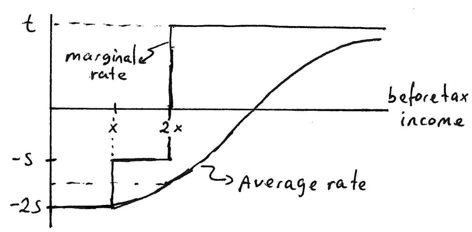
(b) *Derive the average tax rate function $a(I, x, s, t)$ where I again stands for labor market income.*

Answer: This is

$$a(I,x,s,t) = \begin{array}{ll} -2s & \text{if } I \le x \text{ and} \\ -[2sx + s(I-x)]/I & \text{if } x < I \le 2x \text{ and} \\ [-3sx + t(I-2x)]/I & \text{if } I > 2x. \end{array} \quad (3.19.ii)$$

(c) *Graph the average and marginal tax functions on a graph with before-tax income on the horizontal axis and tax rates on the vertical. Is the EITC progressive?*

Answer: This is graphed in Exercise Graph 3.19(2). Since average tax rates rise as income rises, the EITC is progressive.



Exercise Graph 3.19(2) : Average and Marginal Tax Rates under the EITC

Conclusion: Potentially Helpful Reminders

1. When illustrating either worker or intertemporal budget constraints, always be sure you know where the endowment bundle lies — because the constraint will always rotate through that bundle as wages or interest rates change.

2. If you are unsure where the endowment bundle lies, just ask yourself: Which bundle can the worker (or saver or borrower) consumer *regardless* of what wages or interest rates are?

3. When leisure is modeled on the horizontal axis, then the slope of the worker's budget is $-w$ — or minus the wage. When current consumption is modeled on the horizontal axis of an intertemporal budget constraint, the slope of the budget line is $-(1+r)$

4. If you are interested in finance applications, check out in particular the end-of-chapter exercises 3.9 through 3.14.

5. End-of-chapter exercises 3.15 takes you through illustrating tax revenue in the worker budget graph — a skill that will show up repeatedly throughout the text, beginning in Chapter 8.

CHAPTER

4

Tastes and Indifference Curves

This chapter begins a 2-chapter treatment of tastes — or what we also call "preferences". In the first of these chapters, we simply investigate the basic logic behind modeling tastes and the most fundamental assumptions we make. In the next chapter, we then turn to what specific types of tastes look like. Call me a geek — but I think it's pretty cool that we have found ways of systematically modeling something as abstract as people's tastes!

Chapter Highlights

The main points of the chapter are:

1. **Tastes have nothing to do with budgets** — they are conceptually distinct. Budgets are all about what is feasible — and they arise objectively from "what we bring to the table" and the prices we face. Tastes are all about what we desire and have nothing to do with incomes, endowments or prices.

2. While our model of tastes respects the fact that there is a **great diversity of tastes across people**, we assume that **some aspects of tastes are constant across individuals**. The most basic of these are completeness and transitivity, but the assumptions of monotonicity, convexity and continuity are also intuitively appealing in most circumstances. When we say tastes are "rational" we only mean that individuals with those tastes are capable of making decisions.

3. **Maps of indifference curves are a way of describing tastes**, with the usual shapes and orderings arising from our assumptions of monotonicity and convexity. For those covering part B, indifference curves are simply levels of utility functions — and these levels can no more cross than the elevations of mountains on geography maps can cross.

4. We typically assume that **utility cannot be measured objectively** — which is why the labels on indifference curves do not matter beyond indicating the **ordering** of what is better and what is worse.

5. The slope of an indifference curve — or the **marginal rate of substitution** — at a particular bundle tells us how an individual feels about different goods *at the margin* — i.e. how much the individual is willing to trade one good for another *given that she currently has this particular bundle*.

4A Solutions to Within-Chapter-Exercises for Part A

Exercise 4A.1

Do we know from the monotonicity assumption how E relates to D, A and B? Do we know how A relates to D?

Answer: E must be preferred to D because it contains more of everything (i.e. more pants and more shirts). Monotonicity does not tell us anything about the relationship between A and E — A has more shirts but fewer pants and E has more pants but fewer shirts. For analogous reasons, monotonicity does not tell us anything about how E and B are ranked. A has more shirts and the same number of pants as D — so we know that A is at least as good as D (and probably better).

Exercise 4A.2

What other goods are such that we would prefer to have fewer of them rather than many? How can we re-conceptualize choices over such goods so that it becomes reasonable to assume "more is better"?

Answer: Examples might include pollution, bugs in our houses, weeds in our yard and disease in our bodies. In each case, however, we can re-conceptualize the "bad" by redefining it into a "good" that we want more of. We want less pollution but more clean air and water; fewer bugs in our houses or more "bug-free" square feet of housing; fewer weeds in our yard but more square feet of weed-less grass; less disease and more health.

Exercise 4A.3

Combining the convexity and monotonicity assumptions, can you now conclude something about the relationship between the pairs E and A and E and B if you do not know how A and B are related? What if you know that I am indifferent between A and B?

Answer: We can only apply the convexity assumption if we know some pair of bundles we are indifferent between — because convexity says that, when faced with bundles we are indifferent between, we prefer averages of such bundles (or at the very least like averages just as much). So, without knowing more, I can't use monotonicity and convexity to say anything about how A and E (or B and E) are related to one another. If we know that I am indifferent between A and B, on the other hand, then I know that C is at least as good as A and B because C is the average between A and B. Since E has more of everything than C, we also know from monotonicity that E is better than C. So E is better than C which is at least as good as A and B. By transitivity, that implies that E is better than A and B.

Exercise 4A.4

Knowing that I am indifferent between A and B, can you now conclude something about how B and D are ranked by me? In order to reach this conclusion, do you have to invoke the convexity assumption?

Answer: By just invoking the monotonicity assumption, I know that A is at least as good as D since it has more of one good and the same of the other. If A is indifferent to B, I then also know (by transitivity) that B is at least as good as D. Invoking convexity won't actually allow me to say anything beyond that since indifference between A, B and D is consistent with convexity. (It is not consistent with a *strict* notion of convexity — where by "strict" we mean that averages are strictly better than (indifferent) extremes. In that case, A and B are definitely preferred to D if we are indifferent between A and B.)

Exercise 4A.5

Illustrate the area in Graph 4.2b in which F must lie — keeping in mind the monotonicity assumption.

Answer: By monotonicity, F must have less than C and must therefore lie to the southwest of C. Thus, it must have no more than 5 shirts and no more than 6 pants. But it also cannot have fewer than 4 pants because then it would contain fewer pants and shirts than A and would therefore be worse than A. And it cannot have fewer than 2 shirts because it would then have less of everything than B and could no longer be indifferent to B. F must therefore lie in the area illustrated in Exercise Graph 4A.5.

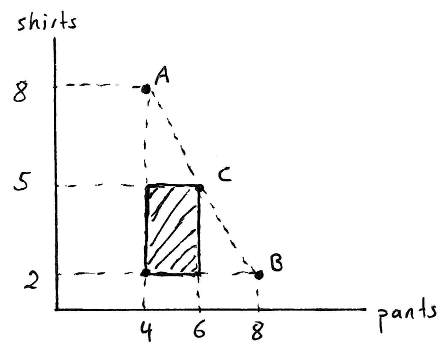
Exercise 4A.6

Suppose our tastes satisfy *weak* convexity in the sense that averages are just as good (rather than strictly better than) extremes. Where does F lie in relation to C in that case?

Answer: In that case F is the same bundle as C — because C is the average of the more extreme bundles A and B.

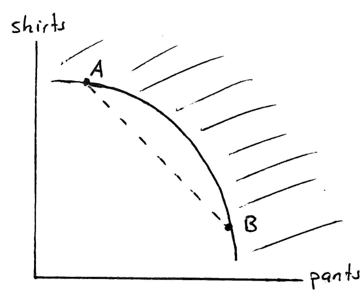
Exercise 4A.7

Suppose extremes are better than averages. What would an indifference curve look like? Would it still imply diminishing marginal rates of substitution?



Exercise Graph 4A.5 : Graph for Within-Chapter-Exercise 4A.5

Answer: The indifference curve would bend away from instead of toward the origin, as illustrated in Exercise Graph 4A.7. The shaded area to the northeast of the indifference curve would contain all the better bundles (because of monotonicity). But the line connecting A and B — which contains averages between A and B — does not lie in this "better" region. Therefore, averages are worse than extremes. The slope of this indifference curve is then shallow at A and becomes steeper as we move along the indifference curve to B. Thus, the marginal rate of substitution is no longer diminishing along the indifference curve — and the indifference curve exhibits increasing marginal rates of substitution.



Exercise Graph 4A.7 : Non-concex tastes

Exercise 4A.8

Suppose averages are just as good as extremes? Would it still imply diminishing marginal rates of substitution?

Answer: If averages are just as good as extremes, then indifference curves are straight lines. As a result, the slope would be the same along an indifference curve —

implying constant rather than diminishing marginal rates of substitution. This is the borderline case between strictly convex tastes that have diminishing marginal rates of substitution and strictly non-convex tastes that have strictly increasing marginal rates of substitution.

Exercise 4A.9

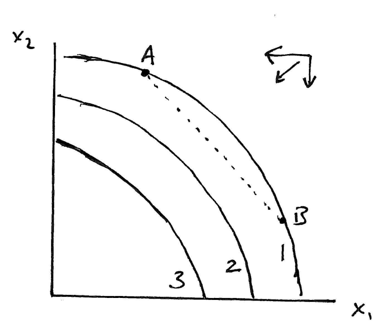
Show how you can prove the last sentence in the previous paragraph by appealing to the transitivity of tastes.

Answer: Pick any bundle that lies on the bold portion of the indifference curve to the southwest of E and call it B. As noted in the text, we know from monotonicity that E is better than B. Because A and B lie on the same indifference curve, you are indifferent between them. Thus, E is better than B which is indifferent to A. Transitivity then implies that E is better than A.

Exercise 4A.10

Suppose less is better than more and averages are better than extremes. Draw three indifference curves (with numerical labels) that would be consistent with this.

Answer: Exercise Graph 4A.10 illustrates three such curves. Since less is better, the consumer becomes better off in the direction of the arrows at the top right of the graph. Thus, if I take A and B that lie on the same indifference curve, the line connecting them (which contains averages of the two) lies fully in the region that is more preferred. Thus, averages are indeed better than extremes. Since the consumer becomes better off as she moves southwest, the numbers accompanying the indifference curves must be increasing as we approach the origin.



Exercise Graph 4A.10 : Convex tastes over “bads”

4B Solutions to Within-Chapter-Exercises in Part B

Exercise 4B.1

True or False: If only one of the statements in (4.6) is true for a given set of bundles, then that statement's "$\succsim$" can be replaced by "$\succ$".

Answer: True. If both statements are true, then the consumer is indifferent between the A and the B bundles (because that is the only way that the A bundle can be at least as good as B and the B bundle can be at least as good as A at the same time). If only one of the statements is true, then the consumer is not indifferent between the bundles. That must mean that one of the bundles is strictly preferred to the other, which means we can indeed replace "$\succsim$" with "$\succ$".

Exercise 4B.2

Does transitivity also imply that (4.8) implies (4.9) when "$\succsim$" is replaced by "$\succ$"?

Answer: Yes. If A is strictly preferred to B and B is strictly preferred to C, transitivity implies that A must be strictly preferred to C.

Exercise 4B.3

True or False: Assuming tastes are transitive, the third line in expression (4.11) is logically implied by the first and second lines.

Answer: True. Suppose we call the averaged bundle C. Then the first two lines say that the consumer being indifferent between A and B implies that she thinks C is at least as good as A. Thus, $C \succsim A \sim B$ implies by transitivity that $C \succsim B$, which is what the third line says.

Exercise 4B.4

If you were searching for the steepest possible straight route up the last 2,000 feet of Mount "Nechyba" (in Graph 4.9), from what direction would you approach the mountain?

Answer: It looks like you would approach it from the northwest (heading up the mountain toward the southeast) — because that is where the levels in the graph are closest to one another (which is where the mountain must be steepest).

Exercise 4B.5

In Political Science models, politicians are sometimes assumed to choose between bundles of spending on various issues — say military and domestic spending. Since they have to impose taxes to fund this spending, more is not necessarily better than less, and thus most politicians have some ideal bundle of domestic and military spending. How would such tastes be similar to the geographic mountain analogy?

Answer: Such tastes would be similar in that the "utility mountain" would have a peak just like geographic mountains do. This is not usually the case for our "utility mountains" because usually we make the assumption that more is better — which means we can always climb higher up a mountain without peak. (More on this in end-of-chapter exercise 4.11.)

Exercise 4B.6

How does the expression for the marginal rate of substitution change if tastes could instead be summarized by the utility function $u(x_1,x_2) = x_1^{1/4}x_2^{3/4}$

Answer: We would calculate this as

$$MRS = -\frac{(1/4)(x_1^{-3/4}x_2^{3/4})}{(3/4)(x_1^{1/4}x_2^{-1/4})} = -\frac{x_2}{3x_1}. \tag{4B.6}$$

Exercise 4B.7

Can you verify that squaring the utility function in exercise 4B.6 also does not change the underlying indifference curves?

Answer: Squaring the utility function from the previous exercise results in $v(x_1,x_2) = (u(x_1,x_2))^2 = \left(x_1^{1/4}x_2^{3/4}\right)^2 = x_1^{1/2}x_2^{3/2}$. This will give rise to the same indifference curves so long as the MRS everywhere remains unchanged. The MRS is

$$MRS = -\frac{(1/2)(x_1^{-1/2}x_2^{3/2})}{(3/2)(x_1^{1/2}x_2^{1/2})} = -\frac{x_2}{3x_1}, \tag{4B.7}$$

exactly as it was before. Thus, the shape of the indifference curves is unaffected.

Exercise 4B.8

Illustrate that the same conclusion we reached with respect to u and v representing the same indifference curves also holds when we take the square root of u — i.e. when we consider the function $w(x_1,x_2) = (x_1^{1/2}x_2^{1/2})^{1/2} = x_1^{1/4}x_2^{1/4}$.

Answer: The MRS is then

$$MRS = -\frac{(1/4)(x_1^{-3/4}x_2^{1/4})}{(1/4)(x_1^{1/4}x_2^{-3/4})} = -\frac{x_2}{x_1}, \tag{4B.8}$$

exactly as it was when we calculated the MRS for $u(x_1,x_2) = x_1^{1/2}x_2^{1/2}$ in the text.

Exercise 4B.9

Consider the utility function $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$. Take natural logs of this function and calculate the MRS of the new function. Is the natural log transformation one that can be applied to utility functions such that the new utility function represents the same underlying tastes?

Answer: Taking logs, we get: $\ln u(x_1, x_2) = \ln\left(x_1^{1/2} x_2^{1/2}\right) = (1/2)\ln x_1 + (1/2)\ln x_2$. Note that the derivative of this with respect to x_1 is $1/(2x_1)$ and the derivative with respect to x_2 is $1/(2x_2)$. The MRS is then

$$MRS = -\frac{1/(2x_1)}{1/(2x_2)} = -\frac{x_2}{x_1}, \tag{4B.9}$$

exactly as it was before the log transformation. Thus, taking logs does not change the shape of indifference curves. Logs also do not change the ordering of the labels on indifference curves. Thus, when we take the log of a utility function, the new utility function represents the same tastes.

Exercise 4B.10

Consider the utility function $u(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/2} x_3^{1/2}$. Take natural logs of this function and calculate the marginal rates of substitution of each pair of goods. Is the natural log transformation one that can be applied to utility functions of three goods such that the new utility function represents the same underlying tastes?

Answer: Taking logs, we get a new function $v(x_1, x_2, x_3) = (1/2)\ln x_1 + (1/2)\ln x_2 + (1/2)\ln x_3$. Taking any pair of good x_i and x_j (where i, and j can take values of 1, 2, and 3 but $i \neq j$), we get

$$MRS = -\frac{1/(2x_i)}{1/(2x_j)} = -\frac{x_j}{x_i}. \tag{4B.10.i}$$

If we instead work with the original utility function $u(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/2} x_3^{1/2}$, we can similarly calculate the MRS between x_i and x_j while denoting the third good as x_k:

$$MRS = -\frac{(1/2)x_i^{-1/2} x_j^{1/2} x_k^{1/2}}{(1/2)x_i^{1/2} x_j^{-1/2} x_k^{1/2}} = -\frac{x_j}{x_i}. \tag{4B.10.ii}$$

We therefore again get the same expressions for the MRS between any two goods after we take logs of the utility function as we do before. Logs are general transformations that can always be applied to a utility function (regardless of how many goods the function is over) to get a new utility function that represents the same underlying tastes.

Exercise 4B.11

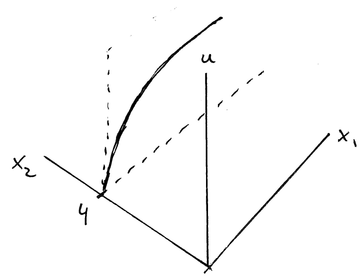
What would be the expression of the slope of the slice of the utility function $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ when x_1 is fixed at 9? What is the slope of that slice when $x_2 = 4$?

Answer: When $x_1 = 9$, the expression reduces to $(1/2)(9)^{1/2} x_2^{-1/2} = (3/2) x_2^{-1/2}$. This is the expression of the slope of the slice holding $x_1 = 9$. When $x_2 = 4$, that slope is $(3/2)(4)^{-1/2} = 3/4$.

Exercise 4B.12

Calculate $\partial u/\partial x_1$ for $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$. What does this reduce to when x_2 is fixed at 4? Where in Graph 4.12 does the slice along which this partial derivative represents the slope lie?

Answer: $\partial u/\partial x_1 = (1/2) x_1^{-1/2} x_2^{1/2}$ reduces to $x_1^{-1/2}$ when $x_2 = 4$. The relevant slice is depicted in Exercise Graph 4B.12.



Exercise Graph 4B.12 : Slice holding x_2 constant at 4

Exercise 4B.13

Calculate $\partial u/\partial x_1$ for the function $u(x_1, x_2) = 10 \ln x_1 + 5 \ln x_2$.

Answer: $\partial u/\partial x_1 = 10/x_1$.

Exercise 4B.14

Calculate $\partial u/\partial x_1$ for the function $u(x_1, x_2) = (2x_1 + 3x_2)^3$. (Remember to use the Chain Rule.)

Answer: $\partial u/\partial x_1 = 3(2x_1 + 3x_2)^2 (2) = 6(2x_1 + 3x_2)^2$.

Exercise 4B.15

Verify that equation (4.28) is correct.

Answer: The partial derivatives are

$$\frac{\partial u}{\partial x_1} = \frac{1}{2}x_1^{-1/2}x_2^{1/2} = \frac{x_2^{1/2}}{2x_1^{1/2}} \tag{4B.15.i}$$

and

$$\frac{\partial u}{\partial x_2} = \frac{1}{2}x_1^{1/2}x_2^{-1/2} = \frac{x_1^{1/2}}{2x_2^{1/2}}. \tag{4B.15.ii}$$

When substituted into the equation, it verifies what is in the text.

Exercise 4B.16

Calculate the total differential du of $u(x_1, x_2) = 10\ln x_1 + 5\ln x_2$.
Answer: This is

$$du = \frac{10}{x_1}dx_1 + \frac{5}{x_2}dx_2. \tag{4B.16}$$

4C Solutions to Odd Numbered End-of-Chapter Exercises

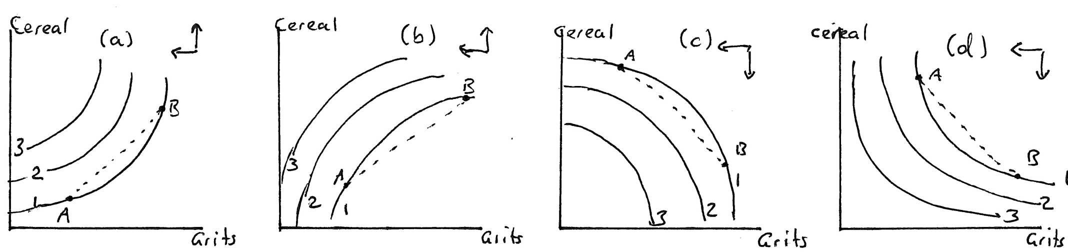
Exercise 4.1

I hate grits so much that the very idea of owning grits makes me sick. I do, on the other hand, enjoy a good breakfast of Coco Puffs Cereal.

A: *In each of the following, put boxes of grits on the horizontal axis and boxes of cereal on the vertical. Then graph three indifference curves and number them.*

(a) *Assume that my tastes satisfy the convexity and continuity assumptions and otherwise satisfy the description above.*

Answer: Panel (a) of Exercise Graph 4.1 graphs an example of such tastes. In the top right corner, arrows indicate the directions in which I become better off. As you will see in this exercise, convexity always implies that indifference curves bend toward the origin that is created by arrows such as these that indicate the directions in which a consumer becomes better off. In the graph, A and B appear on the same indifference curve — and the line connecting them lies "above" the curve in the sense that it contains only bundles to the northwest that are more preferred.



Exercise Graph 4.1 : Grits and Cereal

(b) *How would your answer change if my tastes were "non-convex" — i.e. if averages were worse than extremes.*

Answer: Panel (b) graphs an example of such tastes. I still become better off moving toward the northwest where there are fewer grits and more cereal. But now the indifference curves bend in the other direction. A and B again lie on the same indifference curve, but the line connecting them now lies "below" the indifference curve in the sense that all bundles on that line segment lie to the southeast where I become worse off.

(c) *How would your answer to (a) change if I hated both Coco Puffs and grits but we again assumed my tastes satisfy the convexity assumption.*

Answer: An example of such tastes is graphed in panel (c) of Exercise Graph 4.1. Now the arrows at the top right of the graph point down and left, with better points lying to the southwest as we move toward the origin of the graph. Convexity again implies that indifference curves bend toward the origin that is created by the arrows that indicate which direction makes us better off. Bundles A and B again lie on the same indifference curve, and the line connecting them lies "above" the indifference curve in the sense that it lies to the southwest where I become better off.

(d) *What if I hated both goods and my tastes were non-convex?*

Answer: An example of such tastes is graphed in panel (d) of the graph. As in panel (c), the consumer becomes better off moving toward the southwest. But because tastes are non-convex, the indifference curves now bend in the other direction (and away from the origin that is created by the arrows in the top right corner of the graph). A and B are once again on the same indifference curve, but the line connecting them now lies "below" the indifference curve in the sense that it lies to the northeast where the consumer becomes worse off.

B: *Now suppose you like both grits and Coco Puffs, that your tastes satisfy our five basic assumptions and that they can be represented by the utility function $u(x_1, x_2) = x_1 x_2$.*

(a) *Consider two bundles, A=(1,20) and B=(10,2). Which one do you prefer?*

Answer: You would be indifferent between the two because, when you plug these into the utility function, you get the same utility value; i.e. $u(1,20) = 1(20) = 20$ and $u(10,2) = 10(2) = 20$.

(b) *Use bundles A and B to illustrate that these tastes are in fact convex.*

Answer: Suppose I construct a new bundle C that is the average of A and B — i.e. take half of A and mix it with half of B. This would give 5.5 boxes of grits and 11 boxes of cereal; i.e. C=(5.5,11). Plugging this into the utility function, we get $u(5.5,11) = 5.5(11) = 60.5$. Thus, utility of the average is higher than utility of the extremes.

(c) *What is the MRS at bundle A? What is it at bundle B?*

Answer: The MRS for this utility functions is

$$MRS = -\frac{\partial u / \partial x_1}{\partial u / \partial x_2} = -\frac{x_2}{x_1}. \tag{4.1.i}$$

Plugging in the values for x_1 and x_2 at A and B, we then get $MRS^A = -20$ and $MRS^B = -2/10 = -1/5$.

(d) *What is the simplest possible transformation of this function that would represent tastes consistent with those described in A(d)?*

Answer: The simplest possible transformation would be to multiply the function by a negative 1. This would leave the shape of the indifference curves unchanged because the MRS would be the same. (The negative would cancel in the calculation of MRS.) But the ordering of the numbers accompanying the indifference curves would change because each number would now be multiplied by minus 1. This means that, rather than numbers going up as we move toward the northeast of the graph, numbers will go up as we go to the southwest of the graph. The indifference map would therefore look like the one we graphed in panel (d) of Exercise Graph 4.1.

(e) *Now consider tastes that are instead defined by the function $u(x_1,x_2) = x_1^2 + x_2^2$. What is the MRS of this function?*

Answer: The MRS is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{2x_1}{2x_2} = -\frac{x_1}{x_2}. \tag{4.1.ii}$$

(f) *Do these tastes have diminishing marginal rates of substitution? Are they convex?*

Answer: Notice that the MRS is the inverse of what we calculated for the Cobb-Douglas utility function x_1x_2. Consider, for instance, the bundles (1,5) and (5,1) which both lie on the same indifference curve (that gets utility 26). At (1,5), $MRS = -1/5$ while at (5,1), $MRS = -5/1 = -5$. Thus, the MRS is shallow toward the left of the indifference curve and gets steeper toward the right — we have increasing marginal rates of substitution rather than diminishing marginal rates of substitution. Put differently, these indifference curves bend away from rather than toward the origin. Since more is better, this implies that tastes are not convex.

(g) *How could you most easily turn this utility function into one that represents tastes like those described in A(c)?*

Answer: In A(c), the two goods are "bads" and tastes are convex. The tastes represented by the utility function $u(x_1,x_2) = x_1^2 + x_2^2$ in the previous part give rise to indifference curve with the shape needed for those in A(c) — but the direction of the labeling is one that assigns higher labels to bundles that contain more rather than fewer goods. By simply multiplying the function by -1, however, we reverse the labels and thus have indifference curves with the right shapes and labels increasing in the right direction. Thus, $v(x_1,x_2) = -x_1^2 - x_2^2$ would represent tastes such as those in A(c).

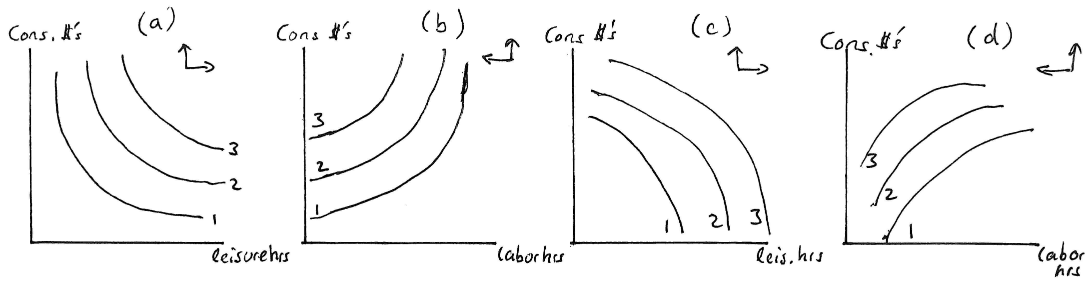
Exercise 4.3

Consider my tastes for consumption and leisure.

A: *Begin by assuming that my tastes over consumption and leisure satisfy our 5 basic assumptions.*

(a) *On a graph with leisure hours per week on the horizontal axis and consumption dollars per week on the vertical, give an example of three indifference curves (with associated utility numbers) from an indifference map that satisfies our assumptions.*

Answer: Panel (a) of Exercise Graph 4.3 graphs an example of three indifference curves that satisfy these assumptions.



Exercise Graph 4.3 : Tastes over Consumption and Labor/Leisure

(b) *Now redefine the good on the horizontal axis as "labor hours" rather than "leisure hours". How would the same tastes look in this graph?*

Answer: Panel (b) illustrates these indifference curves with the good on the horizontal axis redefined. I now become better off going to the northwest in the graph since I prefer less labor.

(c) *How would both of your graphs change if tastes over leisure and consumption were non-convex — i.e. if averages were worse than extremes.*

Answer: Panels (c) and (d) illustrate examples of indifference curves with leisure (panel (c)) and labor (panel (d)) when tastes are non-convex. The line connecting any two points on any of these indifference curves contains only bundles that lie in the "worse" region — implying that averages are worse than extremes.

B: *Suppose your tastes over consumption and leisure could be described by the utility function $u(\ell, c) = \ell^{1/2}c^{1/2}$.*

(a) *Do these tastes satisfy our 5 basic assumptions?*

Answer: Yes. The utility function is one that has been used a number of times in the chapter. It is clearly a continuous function that assigns higher value to bundles that have more consumption and leisure (i.e. it represents monotonic tastes). The *MRS* for this utility function is given by $MRS = -\ell/c$. When ℓ is low and c is high (i.e. to the left in our graph), the *MRS* is therefore large in absolute value, and when ℓ is low and c is high (i.e. to the right in our graph), the *MRS* is small in absolute value. Thus, we have indifference curves that have the property of diminishing marginal rates of substitution — which is the case only when convexity is satisfied. Thus, continuity, monotonicity and convexity are all satisfied. (And the function clearly assigns utility values to all bundles — thus representing complete tastes — and any mathematical function automatically satisfies transitivity.)

(b) *Can you find a utility function that would describe the same tastes when the second good is defined as labor hours instead of leisure hours? (*Hint*:*

Suppose your weekly endowment of leisure time is 60 hours. How do labor hours relate to leisure hours?)

Answer: Let l represent labor hours and assume that I have a total of 60 hours per week in possible leisure time. Then, since $\ell = 60 - l$ (because the leisure hours we actually consume are just those during which we do not work), we can write the utility function in terms of l instead of ℓ by replacing ℓ with $(60 - l)$. Our new function is then $v(c,l) = c^{1/2}(60-l)^{1/2}$.

(c) *What is the marginal rate of substitution for the function you just derived? How does that relate to the sign of the slopes of indifference curves you graphed in part A(b)?*

Answer: The marginal rate of substitution is

$$MRS = -\frac{\partial u/\partial l}{\partial u/\partial c} = -\frac{(-1/2)c^{1/2}(60-l)^{-1/2}}{(1/2)c^{-1/2}(60-l)^{1/2}} = \frac{c}{60-l}. \tag{4.3.i}$$

Note the minus sign that appears in the denominator (because of the Chain Rule), which cancels the minus sign in front of the fraction to give a positive MRS. This is exactly what we graphed in panel (b) of Exercise Graph 4.3 where the slope of indifference curves is positive. (The expression above also implies that slopes start shallow and become steeper as they do in our graph — see the answer to the next part for an explanation to this.)

(d) *Do the tastes represented by the utility function in part (b) satisfy our 5 basic assumptions?*

Answer: They do not because l enters negatively — which implies more l reduces utility. Thus, monotonicity is violated because of the way we have redefined the goods. The other assumptions, however, still hold. We still have a continuous function that assigns values to all bundles (i.e. we have continuity, completeness and transitivity). Also, when l and c are both low (to the left of the graph), the denominator of our MRS is large while the numerator is small — leading to a small positive number. When l and c are both high (to the right of the graph), on the other hand, the denominator becomes small while the numerator is large — leading to a large positive number. Thus, the slope of indifference curves starts small (i.e. shallow) and becomes large (i.e. steep) — precisely as depicted in panel (b) of Exercise Graph 4.3 that mapped out indifference curves under the assumption of convexity.

Exercise 4.5

In this exercise, we explore the concept of marginal rates of substitution (and, in part B, its relation to utility functions) further.

A: *Suppose I own 3 bananas and 6 apples, and you own 5 bananas and 10 apples.*

(a) *With bananas on the horizontal axis and apples on the vertical, the slope of my indifference curve at my current bundle is −2, and the slope of your* *indifference curve through your current bundle is* -1. *Assume that our tastes satisfy our usual five assumptions. Can you suggest a trade to me that would make both of us better off? (Feel free to assume we can trade fractions of apples and bananas).*

Answer: The slope of my indifference curve at my bundle tells us that I am willing to trade as many as 2 apples to get one more banana. The slope of your indifference curve at your bundle tells us that you are willing to trade apples and bananas one for one. If you offer me 1 banana in exchange for 1.5 apples, you would be better off because you would have been willing to accept as little as 1 apple for 1 banana. I would also be better off because I would be willing to give you as many as 2 apples for 1 banana — only having to give you 1.5 apples is better than that. (If you are uncomfortable with fractions of apples being traded, you could also propose giving me 2 bananas for 3 apples.)

This is only one possible example of a trade that would make us both better off. You could propose to give me 1 banana for x apples, where x can lie between 1 and 2. Since I am willing to give up as many as 2 apples for one banana, any such trade would make me better off, and since you are willing to trade them one for one, the same would be true for you.

(b) *After we engage in the trade you suggested, will our MRS's have gone up or down (in absolute value)?*

Answer: Any trade that makes both of us better off moves me in the direction of more bananas and fewer apples — which, given diminishing marginal rates of substitution, should decrease the absolute value of my *MRS*; i.e. as I get more bananas and fewer apples, I will be willing to trade fewer apples to get one more banana than I was willing to originally. You, on the other hand, are giving up bananas and getting apples, which moves you in the opposite direction toward fewer bananas and more apples. Thus, you will become less willing to trade 1 banana for 1 apple and will in future trades demand more bananas in exchange for 1 apple. Thus, in absolute value, your *MRS* will get larger.

(c) *If the values for our MRS's at our current consumption bundles were reversed, how would your answers to (a) and (b) change?*

Answer: The trades would simply go in the other direction; i.e. I would be willing to trade 1 banana for x apples so long as x is at least 1, and you would be willing to accept such a trade so long as x is no more than 2. Thus, x again lies between 1 and 2 if both of us are to be better off from the trade, only now I am giving you bananas in exchange for apples rather than the other way around.

(d) *What would have to be true about our MRS's at our current bundles in order for you not to be able to come up with a mutually beneficial trade?*

Answer: In order for us not to be able to trade in a mutually beneficial way, your *MRS* at your current bundle would have to be identical to my *MRS* at my current bundle.

(e) True or False*: If we have different tastes, then we will always be able to trade with both of us benefitting.*

Answer: This statement is generally false. What matters is not that we have different tastes (i.e. different maps of indifference curves). What matters instead is that, at our current consumption bundle, we value goods differently — that at our current bundle, our *MRS*'s are different. It is quite possible for us to have different tastes (i.e. different maps of indifference curves) but to also be at bundles where our *MRS* is the same. In that case, we would have the same tastes *at the margin* even though we have different tastes overall (i.e. different indifference maps.)

(f) True or False*: If we have the same tastes, then we will never be able to trade with both of us benefitting.*

Answer: False. People with the same tastes but different bundles of goods may well have different marginal rates of substitution at their current bundles — and this opens the possibility of trading with benefits for both sides.

B: *Consider the following five utility functions and assume that α and β are positive real numbers:*

$$\begin{aligned}
&1.\ u^A(x_1,x_2) = x_1^{\alpha}x_2^{\beta}\\
&2.\ u^B(x_1,x_2) = \alpha x_1 + \beta x_2\\
&3.\ u^C(x_1,x_2) = \alpha x_1 + \beta \ln x_2\\
&4.\ u^D(x_1,x_2) = \left(\frac{\alpha}{\beta}\right)\ln\, x_1 + \ln x_2\\
&5.\ u^E(x_1,x_2) = -\alpha \ln x_1 - \beta \ln x_2
\end{aligned} \tag{4.5}$$

(a) *Calculate the formula for MRS for each of these utility functions.*

Answer: These would be

$$\begin{aligned}
&1.\ MRS^A = -\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}} = -\frac{\alpha x_2}{\beta x_1}\\
&2.\ MRS^B = -\frac{\alpha}{\beta}\\
&3.\ MRS^C = -\frac{\alpha}{\beta/x_2} = -\frac{\alpha x_2}{\beta}\\
&4.\ MRS^D = -\frac{\alpha/(\beta x_1)}{1/x_2} = -\frac{\alpha x_2}{\beta x_1}\\
&5.\ MRS^E = -\frac{-\alpha/x_1}{-\beta/x_2} = -\frac{\alpha x_2}{\beta x_1}
\end{aligned} \tag{4.5.i}$$

(b) *Which utility functions represent tastes that have linear indifference curves?*

Answer: Linear indifference curves are indifference curves that have the same slope everywhere — i.e. indifference curves with constant rather than diminishing *MRS*. Thus, the *MRS* cannot depend on x_1 or x_2 for the indifference curve to be linear — which is the case only for $u^B(x_1, x_2)$.

(c) *Which of these utility functions represent the same underlying tastes?*

Answer: Two conditions have to be met for utility functions to represent the same tastes: (1) the indifference curves they give rise to must have the same shapes, and (2) the numbering on the indifference curves needs to have the same order (though not the same magnitude.) To check that indifference curves from two utility functions have the same shape, we have to check that the *MRS* for those utility functions are the same. This is true for u^A, u^D and u^E. To check that the ordering of the numbers associated with indifference curves goes in the same direction, we need to go back to the utility functions. In u^A, for instance, more of x_1 and/or x_2 means higher utility values. The same is true for u^D. Thus u^A and u^D represent the same underlying tastes because they give rise to the same shapes for all the indifference curves and both have increasing numbers associated with indifference curves as we move northeast in the graph of the indifference curves. But u^E is different: While it gives rise to indifference curves with the same shapes as u^A and u^D, the utility values associated with the indifference curves become increasingly negative — i.e. they decline — as we increase x_1 and/or x_2. Thus, higher numerical labels for indifference curves happen to the southwest rather than the northeast — indicating that less is better than more. So the only two utility functions in this problem that represent the same tastes are u^A and u^D.

(d) *Which of these utility functions represent tastes that do not satisfy the monotonicity assumption?*

Answer: As just discussed in the answer to B(c), u^E represents tastes for which less is better than more — because the labeling on the indifference curves gets increasingly negative as we move to the northeast (more of everything) and increasingly less negative as we move toward the origin. In all other cases, more x_1 and/or more x_2 creates greater utility as measured by the utility functions.

(e) *Which of these utility functions represent tastes that do not satisfy the convexity assumption?*

Answer: As we move to the right on an indifference curve, x_1 increases and x_2 decreases. We can then look at the formulae for *MRS* that we derived for each utility function to see what happens to the *MRS* as x_1 increases while x_2 decreases. In MRS^A, for instance, this would result in a decrease in the numerator and an increase in the denominator — i.e. we are dividing a smaller number by a larger number as x_1 increases and x_2 decreases. Thus, in absolute value, the *MRS* declines as we move to the right in our graph — which implies we have diminishing *MRS* and the usual shape for the indifference curves. Since they share the same *MRS*, the same holds for u^D and u^E. For u^C, it is similarly true that an increase in x_1 accompanied by a decrease in x_2 (i.e. a movement along the indifference curve toward the right in the graph) causes the MRS to fall — only this time x_1 plays no role and the drop is entirely due to the reduction in the numerator. For u^B, the MRS is constant — implying no change in the MRS as we move along an indifference curve to the right in the graph.

We can then conclude the following: u^B satisfies the convexity assumption but barely so — averages are the same as extremes (but not better). Furthermore, u^A, u^C and u^D all represent monotonic tastes with diminishing marginal rates of substitution along indifference curves. Thus, averages between extremes that lie on the same indifference curve will be preferred to the extremes because the averages lie to the northeast of some bundles on the indifference curves on which the extremes lie, and, since more is better, this implies the averages are better than the extremes. So u^A, u^C and u^D all satisfy the convexity assumption. That leaves only u^E which we concluded before does not satisfy the monotonicity assumption but its indifference curves look exactly like they do for u^A and u^D. If you pick any two bundles on an indifference curve, it will therefore again be true that the average of those bundles lies to the northeast of some of the bundles on that indifference curve — but now a movement to the northeast makes the individual worse off, not better off. Thus, averages are worse than extremes for the tastes represented by u^E — which implies that u^E represents tastes that are neither convex nor monotonic.

(f) *Which of these utility functions represent tastes that are not rational (i.e. that do not satisfy the completeness and transitivity assumptions)?*

Answer: Each of these is a function that satisfies the mathematical properties of functions. In each case, you can plug in any bundle (x_1, x_2) and the function will assign a utility value. Thus, any two bundles can be compared — and completeness is satisfied. Furthermore, it is mathematically not possible for a function to assign a value to bundle A that is higher than the value it assigns to a different bundle B which in turn is higher than the value assigned to a third bundle C — without it also being true that the value assigned to C is lower than the value assigned to A. Thus, transitivity is satisfied.

(g) *Which of these utility functions represent tastes that are not continuous?*

Answer: All the functions are continuous without sudden jumps — and therefore represent tastes that are similarly continuous.

(h) *Consider the following statement: "Benefits from trade emerge because we have different tastes. If individuals had the same tastes, they would not be able to benefit from trading with one another." Is this statement ever true, and if so, are there any tastes represented by the utility functions in this problem for which the statement is true?*

Answer: What we found in our answers in part A is that, in order for individuals to be able to benefit from trading, it must be the case that their indifference curves through their current consumption bundle have different slopes. It does not matter whether their indifference maps are identical. So long as they are at different current bundles that have different *MRS*'s, mutually beneficial trades are possible. You and I, for instance, might have identical tastes over apples and bananas, but I might have mostly bananas and you might have mostly apples. Then you would probably be willing to trade lots of apples for more bananas, and I'd be willing to let go of bananas pretty easily to get more apples. The only way we cannot benefit from trading with one another is if our *MRS*'s through our current bundle are the same. This might be true for some bundles when we have identical tastes (such as when we currently own the same bundle), but it is not generally true just because we have the same tastes. The only utility function from this problem for which the statement generally holds is therefore u^B, the utility function that represents tastes with the same *MRS* at all bundles. If you and I shared those tastes, then we would have the same *MRS* regardless of which bundles we currently owned — and this makes it impossible for us to become better off through trade.

The statement in this problem could be re-phrased in a way that would make it universally true for all tastes: "Benefits from trade emerge because we have different tastes *at the margin*" — that is, when we have the same willingness to trade goods off for one another around the bundle we currently consume, then we have the same *MRS* and can't trade.

Exercise 4.7

Everyday Application: *Did 9/11 Change Tastes?*: *In another textbook, the argument is made that consumer tastes over "airline miles traveled" and "other goods" changed as a result of the tragic events of September 11, 2001.*

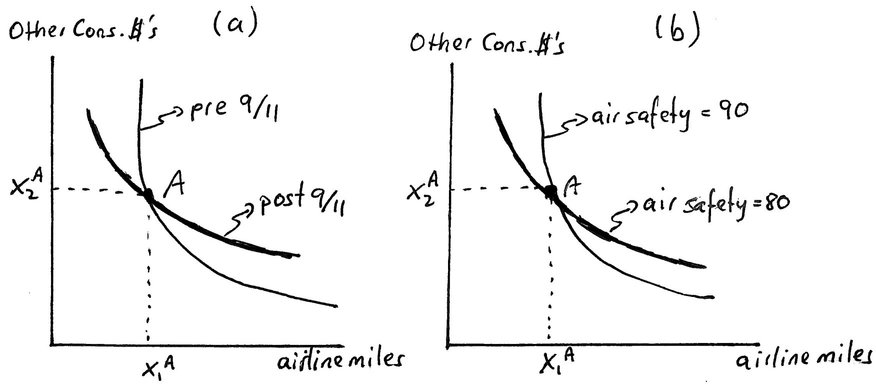
A: *Below we will see how you might think of that argument as true or false depending on how you model tastes.*

(a) *To see the reasoning behind the argument that tastes changed, draw a graph with "airline miles traveled" on the horizontal axis and "other goods" (denominated in dollars) on the vertical. Draw one indifference curve from the map of indifference curves that represent a typical consumer's tastes (and that satisfy our usual assumptions.)*

Answer: This is illustrated in panel (a) of Exercise Graph 4.7 with the indifference curve labeled "pre-9/11".

(b) *Pick a bundle on the indifference curve on your graph and denote it A. Given the perception of increased risk, what do you think happened to the typical consumer's MRS at this point after September 11, 2001?*

Answer: The *MRS* tells us how much in "dollars of other goods" a consumer is willing to give up to travel one more mile by air. After 9/11, it would stand to reason that the typical consumer would give up fewer dollars for additional air travel than before. Thus, the slope of the indiffer-



Exercise Graph 4.7 : Tastes before and after 9/11

ence curve at A should become shallower — which implies that the MRS is falling in absolute value.

(c) *For a consumer who perceives a greater risk of air travel after September 11, 2001, what is likely to be the relationship of the indifference curves from the old indifference map to the indifference curves from the new indifference map at every bundle?*

Answer: The reasoning from (b) holds not just at A but at all bundles. Thus, we would expect the new indifference map to have indifference curves with shallower slopes at every bundle.

(d) *Within the context of the model we have developed so far, does this imply that the typical consumer's tastes for air-travel have changed?*

Answer: Rationality (as we have defined it) rules out the possibility for indifference curves to cross. Thus, within the context of this model, it certainly seems that tastes must have changed.

(e) *Now suppose that we thought more comprehensively about the tastes of our consumer. In particular, suppose we add a third good that consumers care about — "air safety". Imagine a 3-dimensional graph, with "air miles traveled" on the horizontal axis and "other goods" on the vertical (as before) — and with "air safety" on the third axis coming out at you. Suppose "air safety" can be expressed as a value between 0 and 100, with 0 meaning certain death when one steps on an airplane and 100 meaning no risk at all. Suppose that before 9/11, consumers thought that air safety stood at 90. On the slice of your 3-dimensional graph that holds air safety constant at 90, illustrate the pre-9/11 indifference curve that passes through (x_1^A, x_2^A), the level of air miles traveled (x_1^A) and other goods consumed (x_2^A) before 9/11.*

Answer: This is illustrated in panel (b) of Exercise Graph 4.7 as the indifference curve labeled "air safety = 90".

(f) *Suppose the events of 9/11 cause air safety to fall to 80. Illustrate your post-9/11 indifference curve through (x_1^A, x_2^A) on the slice that holds air safety constant at 80 but draw that slice on top of the one you just drew in (e).*

Answer: This is also done in panel (b) of the graph.

(g) *Explain that, while you could argue that our tastes changed in our original model, in a bigger sense you could also argue that our tastes did not change after 9/11, only our circumstances did.*

Answer: When we explicitly include air safety as something we value as consumers, we get indifference surfaces that lie in 3 dimensions. But since we don't get to choose the level of air safety, we effectively operate on a 2-dimensional slice of that 3-dimensional indifference surface — the slice that corresponds to the current level of air safety. That slice looks just like any ordinary indifference curve in a 2-good model even though it comes from a 3-good model. When 9/11 changes the perceptions of air safety, outside circumstances are shifting us to a different portion of our 3-dimensional indifference surface — with that slice once again giving rise to indifference curves that look like the ones we ordinarily graph in a 2-good model. But when viewed from this perspective, the fact that the indifference curve that corresponds to more air safety crosses the indifference curve that corresponds to less air safety merely arises because we are graphing two different slices of a 3-dimensional surface in the same 2-dimensional space. While both curves then contain the bundle (x_1^A, x_2^A), they occur at different levels of x_3. The pre-9/11 indifference curve really goes through bundle $(x_1^A, x_2^A, 90)$ while the post-9/11 indifference curve really goes through bundle $(x_1^A, x_2^A, 80)$ — and the two therefore do not cross. Thus, when viewed from this larger perspective, tastes have not changed, only circumstances have.

B: *Suppose an average traveler's tastes can be described by the utility function $u(x_1, x_2, x_3) = x_1x_3 + x_2$, where x_1 is miles traveled by air, x_2 is "other consumption" and x_3 is an index of air safety that ranges from 0 to 100.*

(a) *Calculate the MRS of other goods for airline miles — i.e. the MRS that represents the slope of the indifference curves when x_1 is on the horizontal and x_2 is on the vertical axis.*

Answer: The MRS is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{x_3}{1} = -x_3. \tag{4.7.i}$$

(b) *What happens to the MRS when air safety (x_3) falls from 90 to 80?*

Answer: It changes from −90 to −80.

(c) *Is this consistent with your conclusions from part A? In the context of this model, have tastes changed?*

Answer: The change in the MRS as air safety falls is a decrease in absolute value — i.e. the slope of the indifference curve over x_1 and x_2 becomes shallower just as we concluded in part A. But we are representing tastes with exactly the same utility function as before — so tastes cannot have changed.

(d) *Suppose that $u(x_1,x_2,x_3) = x_1x_2x_3$ instead. Does the MRS of other consumption for air miles traveled still change as air safety changes? Is this likely to be a good model of tastes for analyzing what happened to consumer demand after 9/11?*

Answer: The MRS now is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{x_2x_3}{x_1x_3} = -\frac{x_2}{x_1}. \tag{4.7.ii}$$

Thus, the MRS for tastes represented by this utility function is unaffected by x_3 — the level of air safety. This would imply that the two indifference curves in panel (b) of Exercise Graph 4.7 would lie on top of one another. If we think consumers felt differently about air travel after 9/11 than before, then this utility function would not be a good one to choose for analyzing changes in consumer behavior.

(e) *What if $u(x_1,x_2,x_3) = x_2x_3 + x_1$?*

Answer: In this case, the MRS is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{1}{x_3}. \tag{4.7.iii}$$

This would imply that as x_3 — air safety — falls, the MRS increases in absolute value; i.e. it would mean that a decrease in air safety would make us willing to spend more on additional air travel than what we were willing to spend before. It would thus result in a steeper rather than a shallower slope for indifference curves post-9/11. It seems unlikely that a typical consumer would respond in this way to changes in air safety.

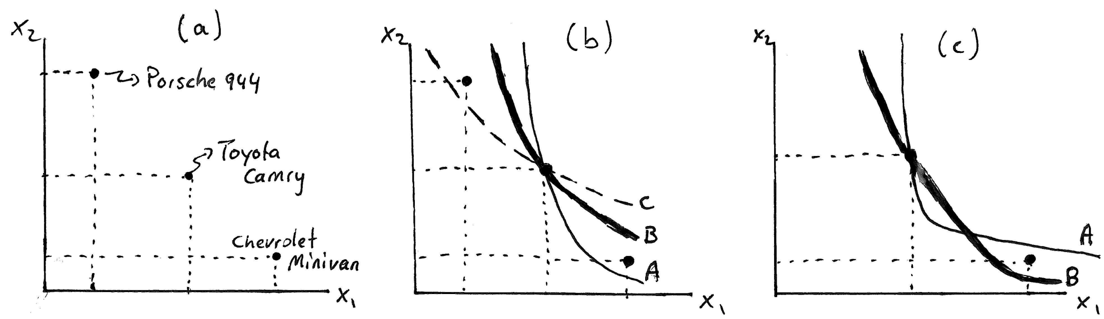
Exercise 4.9

Business Application: *Tastes for Cars and Product Characteristics: People buy all sorts of different cars depending on their income levels as well as their tastes. Industrial organization economists who study product characteristic choices (and advise firms like car manufacturers) often model consumer tastes as tastes over product characteristics (rather than as tastes over different types of products). We explore this concept below.*

A: *Suppose people cared about two different aspects of cars: the size of the interior passenger cabin and the quality of handling of the car on the road.*

(a) *Putting x_1 = "cubic feet of interior space" on the horizontal axis and x_2 = "speed at which the car can handle a curved mountain road" on the vertical, where would you generally locate the following types of cars assuming that they will fall on one line in your graph: a Chevrolet Minivan, a Porsche 944, and a Toyota Camry.*

Answer: Panel (a) of Exercise Graph 4.9 illustrates where the product characteristics of these cars would place them in a graph with interior space on the horizontal axis and speed on the vertical. Porsche's do not have much space in the interior but they handle well at high speeds. Minivans have tons of interior space but don't handle that well at high speeds. And Toyota Camrys are somewhere in between — with more space than Prosche's but not as much as minivans, and with better handling at high speeds than minivans but not as good as Porsches.



Exercise Graph 4.9 : Porsche, Toyota and Chevy

(b) *Suppose we considered three different individuals whose tastes satisfy our 5 basic assumptions, and suppose each person owns one of the three types of cars. Suppose further that each indifference curve from one person's indifference map crosses any indifference curve from another person's indifference map at most once. (When two indifference maps satisfy this condition, we often say that they satisfy the* single crossing property.*) Now suppose you know person A's MRS at the Toyota Camry is larger (in absolute value) than person B's, and person B's MRS at the Toyota Camry is larger (in absolute value) than person C's. Who owns which car?*

Answer: The indifference curves (through the Toyota Camry) for the 3 individuals are depicted in panel (b) of Exercise Graph 4.9. In order for one of these cars to be the most preferred for one and only one of the individuals, it must be that the Porsche lies above one person's indifference curve through the Camry and the minivan lies above another person's indifference curve through the Camry. If indifference curves from different indifference maps cross only once, it logically has to follow that the steepest indifference curve through the Camry lies below the minivan and the shallowest indifference curve through the Camry falls below the Porsche. Since person A's *MRS* is largest in absolute value, person A's indifference curve through the Camry has the steepest slope. By the same reasoning, person C has the shallowest slope going through the Camry. Thus, person A owns the minivan, person B owns the Camry and person C owns the Porsche.

(c) *Suppose we had not assumed the "single crossing property" in part (a). Would you have been able to answer the question "Who owns which car" assuming everything else remained the same?*

Answer: No, you would not have been able to answer the question. The ambiguity that arises when indifference curves from different indifference maps can cross more than once is depicted in panel (c) of Exercise Graph 4.9. Here, person B's (bold) indifference curve is shallower at the Camry than person A's just as described in the problem. However, person A's indifference curve takes a sharp turn at some point to the right of the Camry while person B's continues at roughly the same slope. Thus, B's indifference curve ends up below the minivan (making the minivan better for B than the Camry) while person A's indifference curve ends up above the minivan (making the Camry better for him than the minivan). Thus, once we allow multiple crossing of indifference curves from different indifference maps, it becomes ambiguous who is driving which car.

(d) *Suppose you are currently person B and you just found out that your uncle has passed away and bequeathed to you his 3 children, aged 4, 6 and 8 (and nothing else). This results in a change in how you value space and maneuverability. Is your new MRS at the Toyota Camry now larger or smaller (in absolute value)?*

Answer: You would now be wiling to sacrifice more speed and maneuverability for an increase in interior cabin space — which means the slope of your indifference curve at the Camry should get steeper. Thus, the MRS will increase in absolute value.

(e) *What are some other features of cars that might matter to consumers but that you could not fit easily into a 2-dimensional graphical model?*

Answer: You could think of many other car features: the quality of the upholstery, the shape of the seats, the color of the exterior and interior, whether there is a sun-roof, the quality of the speakers on the stereo system, the degree to which each passenger can control air temperature, the size of the engine, etc.

B: *Let x_1 denote cubic feet of interior space and let x_2 denote maneuverability as defined in part A. Suppose that the tastes of persons A, B and C can be represented by the utility functions $u^A(x_1,x_2) = x_1^{\alpha}x_2$, $u^B(x_1,x_2) = x_1^{\beta}x_2$ and $u^C(x_1,x_2) = x_1^{\gamma}x_2$ respectively.*

(a) *Calculate the MRS for each person.*

Answer: The MRS for person A is

$$MRS^A = -\frac{\partial u^A/\partial x_1}{\partial u^A/\partial x_2} = -\frac{\alpha x_1^{\alpha-1}x_2}{x_1^{\alpha}} = -\alpha\frac{x_2}{x_1}. \tag{4.9.i}$$

Similarly, $MRS^B = -\beta x_2/x_1$ and $MRS^C = -\gamma x_2/x_1$.

(b) *Assuming α, β and γ take on different values, is the "single crossing property" defined in part A(b) satisfied?*

Answer: Pick any product characteristic bundle $(\overline{x}_1, \overline{x}_2)$. Consider individual A and individual B and how their MRS's are related to one another at that bundle by dividing one MRS by the other; i.e.

$$\frac{MRS^A}{MRS^B} = \frac{-\alpha\overline{x}_2/\overline{x}_1}{-\beta\overline{x}_2/\overline{x}_1} = \frac{\alpha}{\beta}. \quad (4.9.ii)$$

Now, it does not matter what bundle $(\overline{x}_1, \overline{x}_2)$ I use, the above equation tells me that the MRS^A is always equal to α/β times the MRS^B. Thus, any indifference curve from A's indifference map can cross any indifference curve from B's indifference map only once. If that were not the case, (as in panel (c) of the graph), the relationship between the slopes of the indifference curves would have to be different at the second crossing — but we have just concluded that this relationship is the same everywhere. The same of course holds for any other pair of individuals from our group of persons A, B and C.

(c) *Given the description of the three persons in part A(b), what is the relationship between α, β and γ?*

Answer: Since A's indifference curve at any product characteristic bundle is steeper than B's and B's is steeper than C's, it must be that $\alpha > \beta > \gamma$.

(d) *How could you turn your graphical model into a mathematical model that includes factors you raised in part A(e)?*

Answer: All that's required is that the utility function includes more product characteristics. So, if we identify n different product characteristics that matter to consumers, we would model their tastes as represented by a utility function $u(x_1, x_2, ..., x_n)$ where x_i is the ith product characteristic.

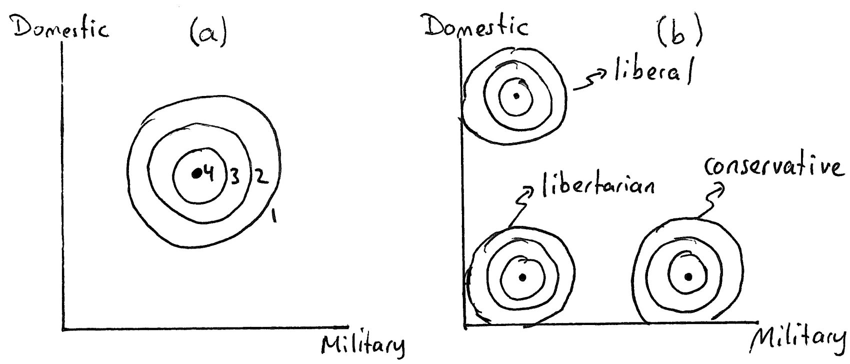
Exercise 4.11

Policy Application: *Ideology and Preferences of Politicians*: *Political scientists often assume that politicians have tastes that can be thought of in the following way: Suppose that the two issues a politician cares about are domestic spending and military spending. Put military spending on the horizontal axis and domestic spending on the vertical axis. Then each politician has some "ideal point" — some combination of military and domestic spending that makes him/her happiest.*

A: *Suppose that a politician cares only about how far the actual policy bundle is from his ideal point, not the direction in which it deviates from his ideal point.*

(a) *On a graph, pick any arbitrary "ideal point" and illustrate what 3 indifference "curves" would look like for such a politician. Put numerical labels on these to indicate which represent more preferred policy bundles.*

Answer: The first panel in Exercise Graph 4.11(1) illustrates an example of such indifference curves. The ideal point is at the center of concentric circles, with circles farther way from the ideal point representing policy bundles with less and less utility. Since distance from the ideal point is all that matters, the indifference "curves" should be circles with the ideal point at their center.



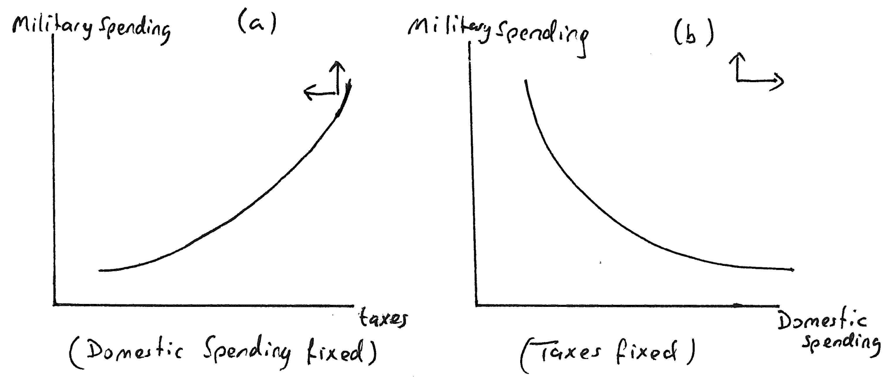
Exercise Graph 4.11(1) : Ideology and Political Tastes

(b) *On a separate graph, illustrate how tastes would be different for a political conservative (who likes a lot of military spending but is not as keen on domestic spending), a liberal (who likes domestic spending but is not as interested in military spending) and a libertarian (who does not like government spending in any direction to get very large).*

Answer: This is illustrated in the second panel of Exercise Graph 4.11(1). The politician's ideology determines the location of his ideal point, with ideal points lining up as described in the problem. Indifference "curves" will then again be concentric circles with each ideal point at the center of the circular indifference curves.

(c) *This way of graphing political preferences is a short-cut because it incorporates directly into tastes the fact that there are taxes that have to pay for government spending. Most politicians would love to spend increasingly more on everything, but they don't because of the increasing political cost of having to raise taxes to fund spending. Thus, there are really 3 goods we could be modeling: military spending, domestic spending and taxes, where a politician's tastes are monotone in the first two goods but not in the last. First, think of this as three goods over which tastes satisfy all our usual assumptions* — including monotonicity *and convexity — where we define the goods as spending on military, spending on domestic goods and the "relative absence of taxes". What would indifference "curves" for a politician look like in a 3-dimensional graph? Since it is difficult to draw this, can you describe it in words and show what a 2-dimensional slice looks like if it holds one of the goods fixed?*

Answer: The indifference "curves" in this 3-dimensional graph would be bowl-shaped, with the tip of the bowl facing the origin. Along any slice that holds one of the goods fixed, the shape would be the usual shape of an indifference curve in 2 dimensions as, for example, that depicted in panel (b) of Exercise Graph 4.11(2).



Exercise Graph 4.11(2) : Ideology and Political Tastes: Part 2

(d) *Now suppose you model the same tastes, but this time you let the third good be defined as "level of taxation" rather than "relative absence of taxes". Now monotonicity no longer holds in one dimension. Can you now graph what a slice of this 3-dimensional indifference surface would look like if it holds domestic spending fixed and has taxes on the horizontal and military spending on the vertical axis? What would a slice look like that holds taxes fixed and has domestic spending on the horizontal and military spending on the vertical axis?*

Answer: The indifference surface would still be bowl shaped but would now point toward the far end of the tax axis. The slice with military spending on the vertical and taxes on the horizontal is graphed in panel (a) of Exercise Graph 4.11(2) where the politician becomes better off with less taxes and more military spending. The slice with military spending on the vertical and domestic spending on the horizontal axis is illustrated in panel (b) — and looks like an ordinary indifference curve since taxes are fixed along the slice.

(e) *Pick a point on the slice that holds taxes fixed. How does the MRS at that point differ for a conservative from that of a liberal?*

Answer: The slope at that point would be shallower for a conservative than for a liberal because a conservative is willing to give up less military spending to get one more dollar of domestic spending. So, in absolute value, the conservative's MRS is smaller than the liberal's.

(f) *Pick a point on the slice that holds domestic spending fixed. How would the MRS at that point differ for a libertarian compared to a conservative?*

Answer: Libertarians would need to get a lot more military spending to justify one more unit of taxation while conservatives would need less.

Thus, libertarians would have a steeper slope — i.e. a higher MRS (in absolute value).

B: *Consider the following equation* $u(x_1, x_2) = P - \left((x_1 - a)^2 + (x_2 - b)^2\right)$.

(a) *Can you verify that this equation represents tastes such as those described in this problem (and graphed in part A(a))?*

Answer: Along any indifference curve, the utility level is constant. Consider one such indifference curve with utility constant at $\overline{u}$. This can then be written as

$$P - \overline{u} = (x_1 - a)^2 + (x_2 - b)^2. \tag{4.11.i}$$

which is the equation of a circle with center (a, b) and radius $(P - \overline{u})^{1/2}$. At the ideal point $(x_1, x_2) = (a, b)$, utility is at its peak P. As x_1 deviates in either direction (with $x_2 = b$), utility declines by $(x_1 - a)^2$. For instance, if x_1 deviates in either direction by 1, utility declines to $(P - 1)$, and if x_1 deviates by 2 in either direction, utility declines to $(P-4)$. The same is true for deviations of x_2 in either direction (holding $x_1 = a$). And the same holds for any deviation from (a, b) in directions that involve changes in both x_1 and x_2. Thus, utility declines from its peak in relation to a policy bundle's distance from the ideal point (a, b).

(b) *What would change in this equation as you model conservative, liberal and libertarian politicians?*

Answer: Conservatives would have $a > b$ and liberals $b > a$. Libertarians would have low values of a relative to those of conservatives and low levels of b relative to liberals.

(c) *Do these tastes satisfy the convexity property?*

Answer: Yes, they do. To see this, take any two points on an indifference circle. The line connecting those two points lies in the region of policy bundles that are better than those on the indifference circle. Thus, averages of policy bundles that the politician is indifferent between are better than extremes.

(d) *Can you think of a way to write a utility function that represents the tastes you were asked to envision in A(c) and A(d)? Let t represent the tax rate with an upper bound of 1.*

Answer: To turn the tax from a "bad" to a "good", we can define it as the "relative absence of a tax" by writing it as $(1 - t)$. We can then treat $(1 - t)$ just like any other good, writing the utility function, for instance, as $u(x_1, x_2, t) = x_1^{\alpha} x_2^{\beta} (1 - t)^{\gamma}$ where α, β and γ are just numbers on the real line.

Exercise 4.13

In this exercise, we will explore some logical relationships between families of tastes that satisfy different assumptions.

A: *Suppose we define a strong and a weak version of convexity as follows: Tastes are said to be* strongly convex *if, whenever a person with those tastes is indifferent between A and B, she strictly prefers the average of A and B (to A and B). Tastes are said to be* weakly convex *if, whenever a person with those tastes is indifferent between A and B, the average of A and B is at least as good as A and B for that person.*

(a) *Let the set of all tastes that satisfy strong convexity be denoted as SC and the set of all tastes that satisfy weak convexity as WC. Which set is contained in the other? (We would, for instance, say that "WC is contained in SC" if any taste that satisfies weak convexity also automatically satisfies strong convexity.)*

Answer: Suppose your tastes satisfy the strong convexity condition. Then you always strictly prefer averages to extremes (where the extremes are such that you are indifferent between them). That automatically means that the average between such extremes is *at least as good as* the extremes — which means that your tastes automatically satisfy weak convexity. Thus, the set SC must be fully contained within the set WC.

(b) *Consider the set of tastes that are contained in one and only one of the two sets defined above. What must be true about some indifference curves on any indifference map from this newly defined set of tastes?*

Answer: We already concluded above that all strongly convex tastes are also weakly convex. So tastes that are strongly convex cannot be in the newly defined set because they appear in both SC and WC — and we are defining our new set to contain tastes that are only in one of these sets. The newly defined set therefore contains only tastes that satisfy weak convexity but not strong convexity. The only difference between weak and strong convexity is that the former permits averages to be just as good as extremes while the latter insists that averages are strictly better than extremes. When an average is just as good as two extremes from the same indifference curve, it must be that the line connecting the extremes is all part of the same indifference curve. Thus, some indifference curves in a weakly convex indifference map that lies outside SC must have "flat spots" that are line segments.

(c) *Suppose you are told the following about 3 people: Person 1 strictly prefers bundle A to bundle B whenever A contains more of each and every good than bundle B. If only some goods are represented in greater quantity in A than in B while the remaining goods are represented in equal quantity, then A is at least as good as B for this person. Such tastes are often said to be* weakly monotonic. *Person 2 likes bundle A strictly better than B whenever at least some goods are represented in greater quantity in A than in B while others may be represented in equal quantity. Such tastes are said to be* strongly monotonic. *Finally, person 3's tastes are such that, for every bundle A, there always exists a bundle B very close to A that is strictly better than A. Such tastes are said to satisfy* local nonsatiation. *Call the set of tastes that satisfy strict monotonicity SM, the set of tastes that satisfy weak* *monotonicity WM, and the set of tastes that satisfy local non-satiation L. What is the relationship between these sets? Put differently, is any set contained in any other set?*

Answer: If your tastes satisfy strong monotonicity, it means that A is strictly preferred to B even if A and B are identical in every way except that A has more of one good than B. This means that your tastes would automatically satisfy weak monotonicity — because weak monotonicity only requires that A is at least as good under that condition and thus permits indifference between A and B unless all goods are more highly represented in A than in B. All strongly monotone tastes are weakly monotone, which means SM is fully contained in WM. Local non-satiation only requires that, for every bundle A, there exists some bundle B close to A such that B is preferred to A. If your tastes satisfy strong monotonicity, then we know such a bundle always exists: Begin at some A and then add a tiny bit of every good to A to form B. As long as we add a tiny bit to all goods, strong monotonicity says B is strictly better than A. The same works for weakly monotonic tastes. Thus, both SM and WM are fully contained in L. But there are also tastes in L such that these tastes are not in WM. Consider tastes where at some bundle A there are no bundles with more goods close to A that are preferred to A but there is a bundle with slightly fewer goods that is preferred to B. Then such tastes would satisfy local non-satiation but not weak (or strong) convexity.

(d) *What is true about tastes that fall in one and only one of these three sets?*

Answer: Since we have just concluded that SM is contained in WM which is contained in L, such tastes must satisfy local non-satiation but not weak monotonicity. Consider tastes over labor and consumption. We would generally like to expend less labor and have more consumption. Such tastes are not strongly or weakly monotonic because A is strictly less preferred to B if A contains the same amount of consumption but more labor. But they do satisfy local non-satiation because for every A, we can make the person better off through less labor or more consumption (or both).

(e) *What is true of tastes that are in one and only one of the two sets SM and WM?*

Answer: Since SM is contained in WM, such tastes must be weakly monotonic. (If they were strongly monotonic, they would be contained in both sets). Consider bundles A and B that are identical in every way except that A has more of one of the goods than B. For tastes to be weakly monotonic but not strongly monotonic, it must be that there exists such an A and B and that a person with such tastes is indifferent between A and B. (If such a person strictly preferred all such A bundles to all such B bundles, her tastes would be strongly monotonic.) Thus, tastes that fall in WM but not SM must have some indifference curves with either horizontal or vertical "flat spots".

B: *Below we will consider the logical implications of convexity for utility functions. For the following definitions,* $0 \le \alpha \le 1$. *A function* $f : \mathbb{R}^2_+ \to \mathbb{R}^1$ *is defined to be* quasiconcave *if and only if the following is true: Whenever* $f(x_1^A, x_2^A) \le f(x_1^B, x_2^B)$*, then* $f(x_1^A, x_2^A) \le f\left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right)$. *The same type of function is defined to be* concave *if and only if* $\alpha f(x_1^A, x_2^A) + (1-\alpha) f(x_1^B, x_2^B) \le f\left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right)$.

(a) True or False*: All concave functions are quasiconcave but not all quasiconcave functions are concave.*

Answer: True. Suppose we start with a concave function f. Then

$$\alpha f(x_1^A, x_2^A) + (1-\alpha) f(x_1^B, x_2^B) \le f\left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right). \tag{4.13.i}$$

Now suppose that $f(x_1^A, x_2^A) \le f(x_1^B, x_2^B)$. Then it must be true that

$$f(x_1^A, x_2^A) \le \alpha f(x_1^A, x_2^A) + (1-\alpha) f(x_1^B, x_2^B). \tag{4.13.ii}$$

But that implies that whenever $f(x_1^A, x_2^A) \le f(x_1^B, x_2^B)$, then

$$f(x_1^A, x_2^A) \le f\left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right) \tag{4.13.iii}$$

— which is the definition of a quasi-concave function. Thus, *concavity of a function implies quasi-concavity.*

But the reverse does not have to hold. Suppose that when $\alpha = 0.5$, $f(x_1^A, x_2^A) = 10$, $f(x_1^B, x_2^B) = 100$ and $f\left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right) = 20$. The condition for quasi-concavity is satisfied — so suppose f is in fact quasi-concave throughout. Notice, however, that $\alpha f(x_1^A, x_2^A) + (1-\alpha) f(x_1^B, x_2^B) = 0.5(10) + (0.5)100 = 55$. Thus,

$$20 = f\left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right) < \alpha f(x_1^A, x_2^A) + (1-\alpha) f(x_1^B, x_2^B) = 55, \tag{4.13.iv}$$

which directly violates concavity.

An example of a function that is quasi-concave but not concave is $u(x_1, x_2) = x_1^2 x_2^2$.

(b) *Demonstrate that, if u is a quasiconcave utility function, the tastes represented by u are convex.*

Answer: Tastes are convex if averages of bundles over which we are indifferent are better than those bundles. Suppose tastes are represented by u and u is quasiconcave. Pick $A = (x_1^A, x_2^A)$ and $B = (x_1^B, x_2^B)$ such that $u(x_1^A, x_2^A) = u(x_1^B, x_2^B)$. Let bundle C be some weighted average between A and B; i.e.

$$C = (x_1^C, x_2^C) = \left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right). \tag{4.13.v}$$

Then quasiconcavity of u implies that

$$u(x_1^A, x_2^A) \leq u(x_1^C, x_2^C), \tag{4.13.vi}$$

which tells us that the average bundle C is at least as good as the extreme bundles A and B (since $u(x_1^A, x_2^A) = u(x_1^B, x_2^B)$) that the individual is indifferent between. Thus, *quasiconcavity of the utility function implies convexity of underlying tastes represented by that utility function.*

(c) *Do your conclusions above imply that, if u is a concave utility function, the tastes represented by u are convex?*

Answer: Since we concluded in (a) that all concave functions are quasiconcave, and since we concluded in (b) that all quasiconcave utility functions represent tastes that satisfy convexity, it must be that all concave utility functions also represent tastes that are convex.

(d) *Demonstrate that, if tastes over two goods are convex, any utility functions that represents those tastes must be quasiconcave.*

Answer: Suppose we consider bundle $A = (x_1^A, x_2^A)$ and $B = u(x_1^B, x_2^B)$ over which an individual with convex tastes is indifferent. Any utility function that represents these tastes must therefore be such that $u(x_1^A, x_2^A) = u(x_1^B, x_2^B)$ which makes the statement

$$u(x_1^A, x_2^A) \leq u(x_1^B, x_2^B) \tag{4.13.vii}$$

also true (since the inequality is weak). Now define a weighted average C of bundles A and B; i.e.

$$C = (x_1^C, x_2^C) = \left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right). \tag{4.13.viii}$$

Convexity of tastes implies that C is at least as good as A. Thus, any utility function that represents these tastes must be such that

$$u(x_1^A, x_2^A) \leq u(x_1^C, x_2^C). \tag{4.13.ix}$$

We have therefore concluded that the utility function representing convex tastes must be such that, whenever $u(x_1^A, x_2^A) \leq u(x_1^B, x_2^B)$, then

$$u(x_1^A, x_2^A) \leq \left(\alpha x_1^A + (1-\alpha)x_1^B, \alpha x_2^A + (1-\alpha)x_2^B\right), \tag{4.13.x}$$

which is the definition of a quasiconcave function. Thus, *convexity of tastes implies quasiconcavity of any utility function that represents those tastes.*(We actually showed that this statement holds when $u^A = u^B$ — but the same reasoning holds when $u^A < u^B$.)

(e) *Do your conclusions above imply that, if tastes over two goods are convex, any utility function that represents those tastes must be concave?*

Answer: No. We have concluded that convexity of tastes implies quasiconcavity of utility functions and we have shown in (a) that there are quasiconcave utility functions that are *not* concave. So the fact that convexity is represented by quasiconcave utility functions does not imply that convexity requires concave utility functions. In fact it does not — it only requires quasiconcavity.

(f) *Do the previous conclusions imply that utility functions which are not quasiconcave represent tastes that are not convex?*

Answer: Yes. In (d) we showed that convexity *necessarily* means that utility functions will be quasiconcave. Thus, when utility functions are *not* quasiconcave, they cannot represent convex tastes. They must therefore represent non-convex tastes.

Conclusion: Potentially Helpful Reminders

1. Convexity in tastes is easy to recognize when "more is better" — but might be a bit confusing otherwise. Here is a simple trick to check whether the tastes you have drawn are convex: Use two arrows that have the same starting point and indicate which horizontal and vertical direction is "better" for the consumer. (When tastes are monotonic, these point to the right and up.) Convexity then implies that the indifference curves bend toward the corner of the arrows you have drawn. (Graph 4.3 in the answer to within-chapter exercise 4A.10 has an example of this. Another example is in Graph 4.6 in the answer to end-of-chapter exercise 4.11.)

2. It should be reasonably clear that tastes — how we *subjectively feel* about stuff — should not typically depend on prices (which only affect what we can *objectively afford*). Put differently, our *circumstances* are different from our *tastes*. But sometimes that gets a little hazy when circumstances other than the usual budget parameters matter. An example of this is given in end-of-chapter exercise 4.7 where we think of "air safety" as one of the circumstances a consumer cannot himself change.

3. Exercise 4.5 is a good exercise to prepare for some of the ideas that are coming up in Chapter 6 as well as later on in Chapter 16.

4. But the last two end-of-chapter exercises are relatively abstract and probably beyond the level of most (but not all) courses that use this text.

CHAPTER

5

Different Types of Tastes

While Chapter 4 introduced us to a general way of thinking about tastes, Chapter 5 gets much more specific and introduces particular dimensions along which we might differentiate between tastes. In particular, we differentiate tastes based on

1. The curvature of individual indifference curves — or how quickly the MRS changes *along an indifference curve*;

2. The relationships between indifference curves — or how the MRS changes *across indifference curves within an indifference map*; and

3. Whether or not indifference curves *cross horizontal or vertical axes* or whether they *converge to the axes*.

The first of these in turn determines the degree to which consumers are willing to substitute between goods (and will lead to what we call the "substitution effect" in Chapter 7) while the second of these determines how consumer behavior responds to changes in income (and will lead to what we call the "income effect" in Chapter 7). Finally, the third category of taste differences becomes important in Chapter 6 where we will see how corner versus interior optimal solutions for a consumer emerge.

Chapter Highlights

The main points of the chapter are:

1. The **degree of substitutability** or, in part B language, the **elasticity of substitution** for a consumer at a particular consumption bundle arises from the **curvature** of the indifference curve at that bundle. There may be no substitutability (as in perfect complements) or perfect substitutability (perfect substitutes) or an infinite number of cases in between these extremes.

2. **Quasilinearity** and **Homotheticity** of tastes represent special cases that describe how indifference curves from the same map relate to one another. These properties have no direct relationship to the concept of substitutability. Tastes are quasilinear in a good x if the MRS only depends on the level of x consumption (and not the level of other goods' consumption. Tastes are homothetic when the MRS depends only on the relative levels of the goods in a bundle.

3. Sometimes it is reasonable to assume that indifference curves only converge to the axes without ever crossing them; other times we assume that they cross the axes. When an indifference curve crosses an axis, it means that we can gain utility beyond what we have by not consuming even if we consume none of one of the goods. When indifference curves only converge to the axes, then some consumption of all goods is necessary in order for a consumer to experience utility above what she would experience by not consuming at all.

4. If you are reading part B of the chapter, you should begin to understand the family of **constant elasticity of substitution utility functions** — with perfect complements, perfect substitutes and Cobb-Douglas tastes as special cases. You should also be able to demonstrate whether a utility function is homothetic or quasilinear. (Most utility functions we use in this text tend to be one or the other.)

5A Solutions to Within-Chapter-Exercises for Part A

Exercise 5A.1

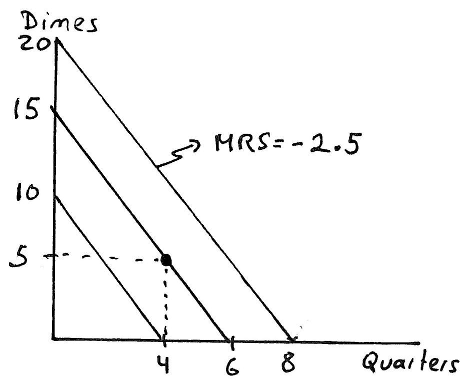
How would the graph of indifference curves change if Coke came in 8 ounce cans and Pepsi came in 4 ounce cans?

Answer: The indifference curves would then have slope of -2 instead of -1 because you would be willing to trade 2 four ounce cans of Pepsi for 1 eight ounce can of Coke. For instance, the indifference curve that contains 1 can of Coke on the horizontal axis would also contain 2 cans of Pepsi on the vertical as well as half a can of Coke and 1 can of Pepsi. All those combinations contain 16 ounces of soft drink.

Exercise 5A.2

On a graph with "quarters" on the horizontal axis and "dimes" on the vertical, what might your indifference curves look like? Use the same method we just employed to graph my indifference curves for Coke and Pepsi — by beginning with one arbitrary bundle of quarters and dimes (say 4 quarters and 5 dimes) and then asking which other bundles might be just as good.

Answer: Dimes are worth 10 cents while quarters are worth 25 cents. Thus, you are willing to trade 2.5 dimes for 1 quarter. At 4 quarters and 5 dimes, you have \$1.50. Any other combination of dimes and quarters should be equally desirable. For instance, 15 dimes also make \$1.50, as do 6 quarters. Thus, the indifference curve through the bundle (4,5) has intercept 6 on the horizontal (quarters) axis and 15 on the vertical (dimes) axis. This gives it a slope of −2.5 which is in fact the rate at which we are willing to trade dimes for quarters. This (and two other) indifference curves are depicted in Exercise Graph 5A.2.



Exercise Graph 5A.2 : Tastes over Dimes and Quarters

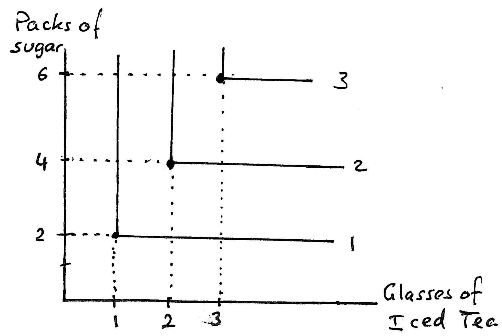
Exercise 5A.3

What would my wife's indifference curves for packs of sugar and glasses of iced tea look like if she required 2 instead of one packs of sugar for each glass of iced tea?

Answer: The corners of the indifference curves would now occur at bundles with twice as much sugar as iced tea. For instance, 1 glass of iced tea and 2 packets of sugar make a complete beverage, and no additional sugar and no additional iced tea will make her better off unless she gets more of both. This gives us the indifference curve labeled "1" in Exercise Graph 5A.3. Similarly, 2 glasses of iced tea and 4 packets of sugar make 2 complete beverages, giving the corner of the indifference curve labeled "2".

Exercise 5A.4

Suppose I told you that each of the indifference maps graphed in Graph 5.3 corresponded to my tastes for one of the following sets of goods, which pair would



Exercise Graph 5A.3 : 2 sugars for each iced tea

you think corresponds to which map? Pair 1: Levi Jeans and Wrangler Jeans; Pair 2: Pants and Shirts; Pair 3: Jeans and Dockers pants.

Answer: To answer this, we should ask which of the pairs represents goods that seem most substitutable for one another. I would think that would be Pair 1 since that includes two different types of jeans (which many of us probably can't even tell apart easily). Thus, I would think that panel (a) represents Pair 1. We could then ask which of the three pairs represent goods that are most complementary (or least substitutable). Of the remaining pairs, pants and shirts seems less substitutable than Jeans and Dockers pants. Thus Pair 2 — pants and shirts — would correspond to panel (c) where there is the least substitutability between the goods. This leaves panel (b) for Pair 3 — Jeans and Dockers pants.

Exercise 5A.5

Are my tastes over Coke and Pepsi as described in Section 5A.1 homothetic? Are my wife's tastes over iced tea and sugar homothetic? Why or why not?

Answer: Yes, both are homothetic. Homothetic tastes are tastes such that the *MRS* is the same along any ray from the origin. For perfect substitutes like Coke and Pepsi, the *MRS* is the same everywhere — which means it is certainly the same along any ray from the origin. For perfect complements like sugar and iced tea, it is easy to also see that the slope of the indifference curves does not change along any ray from the origin. Below the 45 degree line (when one pack of sugar goes with one iced tea), the indifference curve is flat along any ray from the origin; above the 45 degree line, the indifference curve is vertical along any ray from the origin. On the 45 degree line, there is no slope since this is where all the corners of the indifference curve lie. (Since the slope is technically undefined for parts of the indifference map for perfect complements, you can think of this instead as the limit of a sequence of indifference maps that graphs increasingly complementary goods — with each of the maps in the sequence having the characteristic that the *MRS* is unchanged along any ray from the origin.)

Exercise 5A.6

Are my tastes over Coke and Pepsi as described in Section 5A.1 quasilinear? Are my wife's tastes over iced tea and sugar quasilinear? Why or why not?

Answer: Tastes are quasilinear in the good on the horizontal axis if the MRS is unchanged along any vertical line emanating from the horizontal axis. (Alternatively, tastes are quasilinear in the good on the vertical axis if the MRS is unchanged along any horizontal line emanating from the vertical axis.) For perfect substitutes like Coke and Pepsi, the MRS is the same everywhere — which means it is certainly the same along any vertical or horizontal line. Thus, perfect substitutes are quasilinear in both goods. Perfect complements like tea and sugar, on the other hand, are not quasilinear in either good. Along any vertical line emanating from the horizontal axis, the indifference curve at some point changes from being horizontal to vertical. (The reverse is true for any horizontal line emanating from the vertical axis). You can also again think of the indifference maps that come closer and closer to those of perfect complements and treat perfect complements as the limiting case. For all maps that approach those of perfect complements, the slopes of indifference curves change along vertical and horizontal lines. Thus neither of the goods is quasilinear.

Exercise 5A.7

Can you explain why tastes for perfect substitutes are the only tastes that are both quasilinear and homothetic?

Answer: Quasilinearity implies that the MRS does not change along any vertical line emanating from the horizontal axis (or along any horizontal line emanating from the vertical axis). Homotheticity implies that the MRS is constant along any ray from the origin. Consider any vertical line emanating from the horizontal axis. All rays emanating from the origin pass through that line at some point. So if the MRS has to be the same along the vertical line and it has to he the same along rays from the origin, it must be that the MRS is the same everywhere. (The same is true if we instead considered a horizontal line emanating from the vertical axis when the good on the vertical axis is quasilinear). And the only tastes for which the MRS is the same everywhere are those of perfect substitutes.

Exercise 5A.8

True or False: Quasilinear goods are never essential.

Answer: The idea of an "essential" good is meant to capture the following: Is the good such that if I were to consume none of it, I might as well not consume any goods at all? The good on the horizontal axis is then *not* essential if indifference curves cross there vertical axis. (If they do cross the vertical axis, then I can consume none of the good on the horizontal axis and still get utility greater than I would if I consumed at the origin of the graph.) Quasilinear goods are typically of this type; that is they typically have indifference maps with curves crossing the axes, in which case they are not essential. I say typically, however, because there are mathematical subtleties I am neglecting. For instance, the utility function $u(x_1, x_2) = \ln x_1 + x_2$ has indifference curves that converge to the vertical axis but never cross it – but they do cross a vertical line at $x_1 = 1$ (and other vertical lines closer to $x_1 = 0$). It is therefore possible for a quasilinear good to be, strictly speaking, essential.

5B Solutions to Within-Chapter-Exercises for Part B

Exercise 5B.1

Calculate the same approximate elasticity of substitution for the indifference curve in Graph 5.7b.

Answer: The ratio (x_2/x_1) changes from $10/2 = 5$ to $8/4 = 2$. The percentage change in this ratio is therefore $-3/5 = -0.6$. The percentage change in the MRS is again 0.5. Thus, the elasticity of substitution is $(0.6/0.5) = 1.2$.

Exercise 5B.2

What numerical labels would be attached to the 3 indifference curves in Graph 5.1 by the utility function in equation (5.2)?

Answer: Each indifference curve would have the label equal to its vertical (or horizontal) intercept; i.e. 1 for the lowest, 2 for the middle and 3 for the highest indifference curve in the graph.

Exercise 5B.3

Suppose you measured coke in 8 ounce cans and Pepsi in 4 ounce cans. Draw indifference curves and find the simplest possible utility function that would give rise to those indifference curves.

Answer: Such indifference curves are drawn in Exercise Graph 5B.3 where the consumer is willing to trade 2 (4 oz) cans of Pepsi for 1 (8 oz) can of Coke — leading to slopes of −2 when Coke is graphed on the horizontal axis. You therefore get twice as much happiness from a can of Coke as from a can of Pepsi, which implies one way of representing these tastes is

$$u(x_1, x_2) = 2x_1 + x_2. \tag{5B.3.i}$$

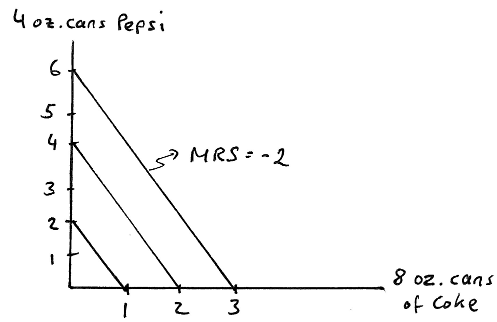
You can check that the MRS in this case is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{2}{1} = -2. \tag{5B.3.ii}$$

Exercise 5B.4

Can you use similar reasoning to determine the elasticity of substitution for the utility function you derived in exercise 5B.3?

Answer: The exact same reasoning holds for all indifference maps with linear indifference curves. Again, it is easiest to think of an indifference map that is close to linear everywhere — and then to think what happens as such an indifference



Exercise Graph 5B.3 : 8 oz Coke and 4 oz Pepsi

map approaches that of perfectly linear indifference curves. For indifference curves that are close to those with $MRS = -2$ everywhere, we can start at a bundle A with little x_1 and a lot x_2. Even a small change in the MRS will result in a large move down that indifference curve. Thus, the percentage change in the ratio of the goods (which is the numerator in the elasticity of substitution equation) is large for a small percentage change in the MRS (which is the denominator in the elasticity equation). In the limit, I can get larger and larger changes in this numerator with smaller and smaller changes in the denominator as the indifference curve gets closer and closer to being linear. Thus, in the limit the elasticity of substitution is ∞.

Exercise 5B.5

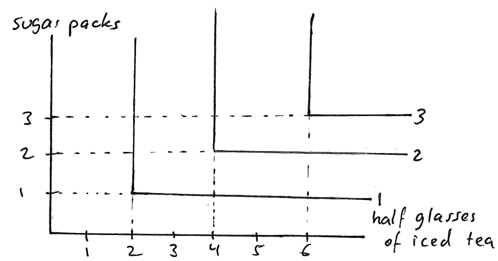
Plug the bundles $(3,1)$, $(2,1)$, $(1,1)$, $(1,2)$ and $(1,3)$ into this utility function and verify that each is shown to give the same "utility" — thus lying on the same indifference curve as plotted in Graph 5.2. What numerical labels does this indifference curve attach to each of the 3 indifference curves in Graph 5.2?

Answer: In each of these bundles, the utility function $u(x_1,x_2) = \min\{x_1,x_2\}$ picks the lower of the two quantities and assigns that number as the utility level of that bundle. Since the lower of the two values in each of these bundles is 1, each is assigned a utility value of 1. The values assigned to the three indifference curves are 1, 2 and 3 — the values of the vertical and horizontal coordinates at the corners of each indifference curve.

Exercise 5B.6

How would your graph and the corresponding utility function change if we measured iced tea in "half glasses" instead of glasses.

Answer: In that case, the perfect beverage requires 1 pack of sugar for every 2 units (half glasses) of iced tea. Any more sugar for 2 units of iced tea would add no further utility unless more tea was added as well, and more tea for 1 pack of sugar would not add more utility unless more sugar was added as well. Thus, the indifference curves representing the same tastes as before would look as in Exercise Graph 5B.6, with the corner points now lying on a ray from the origin that lies below the 45 degree line. A utility function that results in the labeling of the indifference curves that arises in this graph is $u(x_1, x_2) = \min\{0.5x_1, x_2\}$.



Exercise Graph 5B.6 : Half Glasses of tea and full packs of sugar

Exercise 5B.7

Can you determine intuitively what the elasticity of substitution is for the utility function you defined in exercise 5B.6?

Answer: It is again easiest to do this for tastes that are very close to those we graphed in Exercise Graph 5B.6 but without the sharp kink. Pick A a bit above the ray on which the corners of the indifference curves lie — with the ratio of x_1/x_2 just above 0.5. Then imagine moving to a shallower slope of the indifference curve that contains A. Because of the large curvature of the indifference curve around the ray that connects the corners of the indifference curves, even a relatively large change in the MRS will not cause us to have to slide very far along the indifference curve — implying a relatively modest change in the ratio x_1/x_2. Thus, for a large percentage change in the MRS (which is the denominator in the elasticity equation), we get a relatively small change in the ratio x_1/x_2 (which is the denominator in the elasticity equation.) As the indifference curve gets closer and closer to that of perfect complements, the percentage change in the consumption good ratio will fall for any percentage change in the MRS — and will approach 0 as the indifference curve approaches that of perfect complements. Thus, the numerator in the elasticity equation approaches zero — leaving us with an elasticity of substitution of zero in the limit.

Exercise 5B.8

Demonstrate that the functions u and v both give rise to indifference curves that exhibit the same shape by showing that the MRS for each function is the same.

Answer: The MRS of $v = x_1^{\alpha} x_2^{1-\alpha}$ is

$$MRS^v = -\frac{\partial v/\partial x_1}{\partial v/\partial x_2} = -\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha)x_1^{\alpha} x_2^{-\alpha}} = -\frac{\alpha x_2}{(1-\alpha)x_1}, \tag{5B.8.i}$$

and, since $\alpha = \gamma/(\gamma+\delta)$, this can also be written as

$$MRS^v = -\frac{\alpha x_2}{(1-\alpha)x_1} = -\frac{(\gamma/(\gamma+\delta))x_2}{(1-\gamma/(\gamma+\delta))x_1} = -\frac{(\gamma/(\gamma+\delta))x_2}{(\delta/(\gamma+\delta))x_1} = -\frac{\gamma x_2}{\delta x_1}. \tag{5B.8.ii}$$

The MRS of the function $u(x_1, x_2) = x_1^{\gamma} x_2^{\delta}$ is

$$MRS^u = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\gamma x_1^{\gamma-1} x_2^{\delta}}{\delta x_1^{\gamma} x_2^{\delta-1}} = -\frac{\gamma x_2}{\delta x_1}. \tag{5B.8.iii}$$

Thus, $MRS^v = MRS^u$, which implies the indifference curves arising from the two utility functions are identical.

Exercise 5B.9

Derive the MRS for the Cobb-Douglas utility function and use it to show what happens to the slope of indifference curves along the 45-degree line as α changes.

Answer: The MRS for the Cobb-Douglas function which is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\alpha x_1^{(\alpha-1)} x_2^{(1-\alpha)}}{(1-\alpha)x_1^{\alpha} x_2^{-\alpha}} = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{x_2}{x_1}\right). \tag{5B.9}$$

Along the 45-degree line, $x_1 = x_2$ — which implies $x_2/x_1 = 1$ and the MRS along the 45 degree line is simply $-\alpha/(1-\alpha)$. Thus, when $\alpha = 0.5$, the MRS along the 45 degree line is exactly -1. When $\alpha > 0.5$, the MRS on the 45 degree line is greater than 1 in absolute value, and when $\alpha < 0.5$, the MRS is less than 1 in absolute value along the 45 degree line.

Exercise 5B.10

What is the elasticity of substitution in each panel of Graph 5.10?

Answer: The elasticity of substitution for CES utility functions is $\sigma = 1/(1+\rho)$. Thus, the $\rho = -0.8$ in panel (a) translates to $\sigma = 5$; the $\rho = -0.2$ in panel (b) translates to $\sigma = 1.25$; and the $\rho = 2$ in panel (c) translates to $\sigma = 0.33$.

Exercise 5B.11

Can you describe what happens to the slopes of the indifference curves on the 45 degree line, above the 45 degree line and below the 45 degee line as ρ becomes large (and as the elasticity of substitution therefore becomes small)?

Answer: The slopes of the indifference curves are described by the MRS which is given by

$$MRS = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{x_2}{x_1}\right)^{\rho+1}. \qquad (5B.11)$$

First, consider bundles on the 45-degree line where $x_1 = x_2$ and thus $x_2/x_1 = 1$. In this case, the second term in the equation remains 1 as ρ gets large — and the MRS therefore stays constant at $-\alpha/(1-\alpha)$.

Next, consider a bundle above the 45 degree line — i.e. a bundle such that $x_1 < x_2$. This implies that $x_2/x_1 > 1$ — which means the second term in the MRS equation increases as ρ gets large. Thus, as ρ gets large, the slope of indifference curves above the 45-degree line become steeper (approaching vertical lines as ρ approaches infinity.)

Finally, suppose we consider a bundle below the 45 degree line — i.e. a bundle such that $x_1 > x_2$. This implies $x_2/x_1 < 1$ — which implies that the second term in the MRS equation decreases as ρ gets large. Thus, the slopes of indifference curves get shallower below the 45 degree line (approaching horizontal lines as ρ approaches infinity).

Thus, as ρ approaches infinity (and as the elasticity of substitution therefore approaches 0), the slopes of indifference curves along the 45 degree line remain unchanged while they flatten out below the 45 degree line and straighten up above the 45 degree line. In other words, as ρ gets large, the shape of the indifference curves approach those of perfect complements.

Exercise 5B.12

On the "Exploring Relationships" animation associated with Graph 5.10, develop an intuition for the role of the α parameter in CES utility functions and compare those to what emerges in Graph 5.9.

Answer: No particular answer here — the animated version should illustrate how changing α alters the shapes of indifference curves in ways that should seem familiar from our Cobb-Douglas example in the text.

Exercise 5B.13

Show that, when we normalize the exponents of the Cobb-Douglas utility function to sum to 1, the function is homogeneous of degree 1.

Answer: Using the utility function $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$,

$$u(tx_1,tx_2) = (tx_1)^{\alpha}(tx_2)^{(1-\alpha)} = t^{\alpha}x_1^{\alpha}t^{(1-\alpha)}x_2^{(1-\alpha)} = tx_1^{\alpha}x_2^{(1-\alpha)} = tu(x_1,x_2). \qquad (5B.13)$$

Exercise 5B.14

Consider the following variant of the CES function that will play an important role in producer theory: $f(x_1,x_2) = \left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-\beta/\rho}$. Show that this function is homogeneous of degree β.

Answer:

$$
\begin{aligned}
f(tx_1, tx_2) &= \left(\alpha(tx_1)^{-\rho} + (1-\alpha)(tx_2)^{-\rho}\right)^{-\beta/\rho} = \\
&= \left(t^{-\rho}(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho})\right)^{-\beta/\rho} = t^{\beta}\left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-\beta/\rho} = \\
&= t^{\beta} f(x_1, x_2). \qquad (5B.14)
\end{aligned}
$$

Exercise 5B.15

Can you demonstrate, using the definition of a homogeneous function, that it is generally possible to transform a function that is homogeneous of degree k to one that is homogeneous of degree 1 in the way suggested above?

Answer: Suppose a function $f(x_1, x_2)$ is homogeneous of degree k. Then this implies that $f(tx_1, tx_2) = t^k f(x_1, x_2)$ for any $t > 0$. Now consider the function

$$
v(x_1, x_2) = \left(f(x_1, x_2)\right)^{1/k}. \qquad (5B.15.i)
$$

Then

$$
\begin{aligned}
v(tx_1, tx_2) &= \left(f(tx_1, tx_2)\right)^{1/k} = \left(t^k f(x_1, x_2)\right)^{1/k} \\
&= t\left(f(x_1, x_2)\right)^{1/k} = t v(x_1, x_2). \qquad (5B.15.ii)
\end{aligned}
$$

Thus, $v(tx_1, tx_2) = tv(x_1, x_2)$ which is the definition of a function that is homogeneous of degree 1.

Exercise 5B.16

Use the mathematical expression for quasilinear tastes to illustrate that neither good is essential if tastes are quasilinear in one of the goods.

Answer: If tastes are quasilinear in x_1, then we can represent them by a function

$$
u(x_1, x_2) = v(x_1) + x_2. \qquad (5B.16)
$$

At the bundle (0,0), this would result in utility of $u(0,0) = v(0)$. If the consumer consumes $(x_1, 0)$ — i.e. if she consumes only x_1 but no x_2, her utility is $u(x_1, 0) = v(x_1)$ which is greater than $v(0)$ which she gets by consuming nothing. Thus, the consumer can get more utility by consuming only x_1 than she could by consuming nothing — which implies that x_2 is not essential. Similarly, if she consumes a bundle $(0, x_2)$ — i.e. if she consumes only x_2 and no x_1, she gets utility $u(0, x_2) = v(0) + x_2$ which is also greater than $u(0,0) = v(0)$. Thus, x_1 is not essential.

Exercise 5B.17

Show that both goods are essential if tastes can be represented by Cobb-Douglas utility functions.

Answer: Suppose tastes can be represented by $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$. Then the utility from consuming (0,0) is $u(0,0) = 0$. Now consider the utility from a bundle $(x_1,0)$ — i.e. a bundle with no x_2 consumption. Utility from such a bundle is $u(x_1,0) = x_1^{\alpha}(0) = 0$ — exactly what it is when the consumer doesn't consume anything at all. Thus, x_2 is essential. By similar reasoning, x_1 is essential.

Exercise 5B.18

Can you demonstrate similarly that $\sigma = 1$ for the Cobb-Douglas utility function $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$?

Answer: We know from our previous work that the MRS of a Cobb-Douglas utility function of this type is $MRS = -(\alpha x_2)/((1-\alpha)x_1)$. Taking absolute values of both sides and solving for (x_2/x_1), we get

$$\frac{x_2}{x_1} = \frac{(1-\alpha)}{\alpha}|MRS|, \tag{5B.18.i}$$

and taking logs,

$$\ln\frac{x_2}{x_1} = \ln|MRS| + \ln\frac{(1-\alpha)}{\alpha}. \tag{5B.18.ii}$$

We can then apply the elasticity formula from the appendix to get

$$\sigma = \frac{d\ln(x_2/x_1)}{d\ln|MRS|} = 1. \tag{5B.18.iii}$$

5C Solutions to Odd Numbered End-of-Chapter Exercises

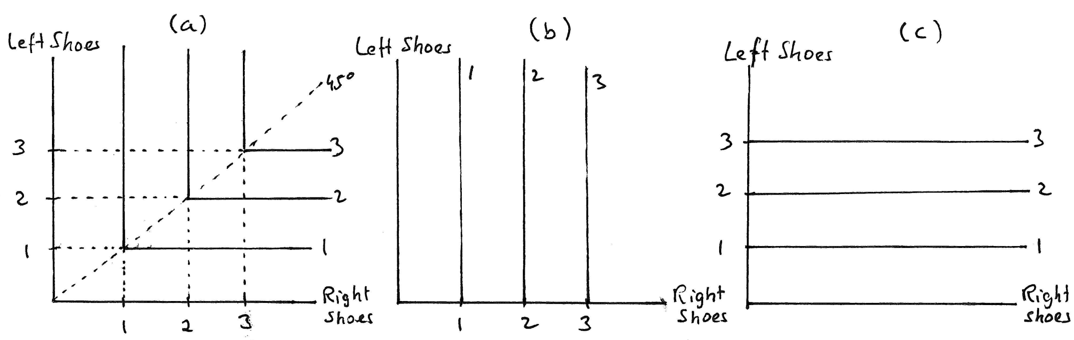
Exercise 5.1

Consider your tastes for right and left shoes.

A: *Suppose you, like most of us, are the kind of person that is rather picky about having the shoes you wear on your right foot be designed for right feet and the shoes you wear on your left foot be designed for left feet. In fact you are so picky that you would never wear a left shoe on your right foot or a right shoe on your left foot — nor would you ever choose (if you can help it) not to wear shoes on one of your feet.*

(a) *In a graph with the number of right shoes on the horizontal axis and the number of left shoes on the vertical, illustrate three indifference curves that are part of your indifference map.*

Answer: Panel (a) of Exercise Graph 5.1 illustrates the three indifference curves corresponding to the utility you get from 1 pair of shoes, 2 pair of shoes and 3 pair of shoes. Right and left shoes are perfect complements.



Exercise Graph 5.1 : Right Shoes and Left Shoes

(b) *Now suppose you hurt your left leg and have to wear a cast (which means you cannot wear shoes on your left foot) for 6 months. Illustrate how the indifference curves you have drawn would change for this period. Can you think of why goods such as left shows in this case are called* neutral goods*?*

Answer: Panel (b) of Exercise Graph 5.1 illustrates such indifference curves. For any given number of right shoes, utility would not change as you get more left shoes since you have no use for left shoes. The only way to get to higher utility is to increase right shoes. Goods like left shoes in this example are sometimes called *neutral goods* because you do not care one way or another if you have any of them.

(c) *Suppose you hurt your right foot instead. How would this change your answer to part (b).*

Answer: This is illustrated in panel (c) of Exercise Graph 5.1. Now you can only become better off by getting more left shoes, but getting more right shoes (for any level of left shoes) does nothing to change your utility.

(d) *Are any of the tastes you have graphed homothetic? Are any quasilinear?*

Answer: All 3 are homothetic — the slopes (to the extent to which these are defined) of the indifference curves in all three maps are the same along any ray from the origin. The panel (a) perfect complements case is not quasilinear because, for any quantity of right shoes, the "slope" changes from perfectly horizontal to perfectly vertical at some level of left shoes. And for any quantity of left shoes, the "slope" changes from perfectly vertical to perfectly horizontal at some level of right shoes. But the tastes in panels (b) and (c) are quasilinear in both goods — along any horizontal and vertical line, the "slope" remains the same. You can view the latter two as the limit cases of perfect substitutes. For instance, in panel (c) we could add a slight negative slope to the indifference curves, and we would then have indifference curves with the same MRS everywhere. Put differently, we'd have perfect substitutes where we are willing to trade very small numbers of left shoes for many right shoes. Then imagine a sequence of such indifference maps, with each indifference curve in the sequence having a slope that is half the slope of the previous one. Every indifference map in that sequence is similarly one of perfect substitutes with constant MRS, and the limit of that sequence is the indifference map depicted in panel (c).

(e) *In the three different tastes that you graphed, are any of the goods ever "essential"? Are any not essential?*

Answer: A good is essential if there is no way to attain utility greater than what one would attain at the origin without consuming at least some of that good. In panel (a), both goods are therefore essential — because you have to consume the goods in pairs in order to get any utility from consuming either. In panel (b), right shoes are essential but left shoes are not, and in panel (c) left shoes are essential but right shoes are not.

B: *Continue with the description of your tastes given in part A above and let x_1 represent right shoes and let x_2 represent left shoes.*

(a) *Write down a utility function that represents your tastes as illustrated in A(a). Can you think of a second utility function that also represents these tastes?*

Answer: This is just a case of perfect complements — so the utility function $u(x_1,x_2) = \min\{x_1,x_2\}$ would be one that works for representing these tastes. So would a function $v(x_1,x_2) = \alpha \min\{x_1,x_2\}$ for any $\alpha > 0$, or $w(x_1,x_2) = (\min\{x_1,x_2\})^{\beta}$ for any $\beta > 0$, or any number of other transformations that don't alter the ordering of indifference curves.

(b) *Write down a utility function that represents your tastes as graphed in A(b).*

Answer: Since only right shoes (x_1) matter, utility cannot vary with the number of left shoes (x_2). A function like $u(x_1, x_2) = x_1$ would therefore suffice.

(c) *Write down a utility function that represents your tastes as drawn in A(c).*

Answer: Since only left shoes (x_2) matter, utility cannot vary with the number of right shoes (x_1). A function like $u(x_1, x_2) = x_2$ would therefore suffice.

(d) *Can any of the tastes you have graphed in part A be represented by a utility function that is homogeneous of degree 1? If so, can they also be represented by a utility function that is not homogeneous?*

Answer: A function $u(x_1, x_2)$ is homogeneous of degree 1 if $u(tx_1, tx_2) = tu(x_1, x_2)$. The utility function $u(x_1, x_2) = \min\{x_1, x_2\}$ in our answer to part B(a), for instance, is homogeneous of degree 1 because

$$u(tx_1, tx_2) = \min\{tx_1, tx_2\} = t\min\{x_1, x_2\} = tu(x_1, x_2). \tag{5.1.i}$$

Similarly, the functions $u(x_1, x_2) = x_1$ from part B(b) and $u(x_1, x_2) = x_2$ from part B(c) are homogeneous of degree 1. Each of these three functions can be turned into a function that is not homogeneous by simply adding a constant. Adding such a constant does not change the underlying shape of indifference curves — and so it does not alter the kinds of tastes that we are modeling. But, for instance, $f(x_1, x_2) = \alpha + \min\{x_1, x_2\}$ is such that

$$f(tx_1, tx_2) = \alpha + \min\{tx_1, tx_2\} \neq t^k\alpha + t^k \min\{x_1, x_2\} = t^k f(x_1, x_2) \tag{5.1.ii}$$

for any $k > 0$.

(e) *Refer to end-of-chapter exercise 4.13 where the concepts of "strong monotonicity," "weak monotonicity" and "local non-satiation" were defined. Which of these are satisfied by the tastes you have graphed in this exercise?*

Answer: All satisfy local non-satiation because for any bundle, there is always another bundle close by that is more preferred. All satisfy weak monotonicity — because for any bundle, adding more of one of the goods is at least as good as the original bundle. But they don't satisfy strong monotonicity — because in each case there is a way to add more of one good to a bundle without making the individual strictly better off.

(f) *Refer again to end-of-chapter exercise 4.13 where the concepts of "strong convexity" and "weak convexity" were defined. Which of these are satisfied by the tastes you have graphed in this exercise?*

Answer: All satisfy weak convexity because, for any two bundles on a given indifference curve, any weighted average of the bundles (which lies on a line connecting the two bundles) is at least as good as the more extreme bundles. They do not satisfy strong convexity because in each case we can find two bundles that lie on a line segment of the indifference curves — and for those bundles, weighted averages are not strictly better than the extremes.

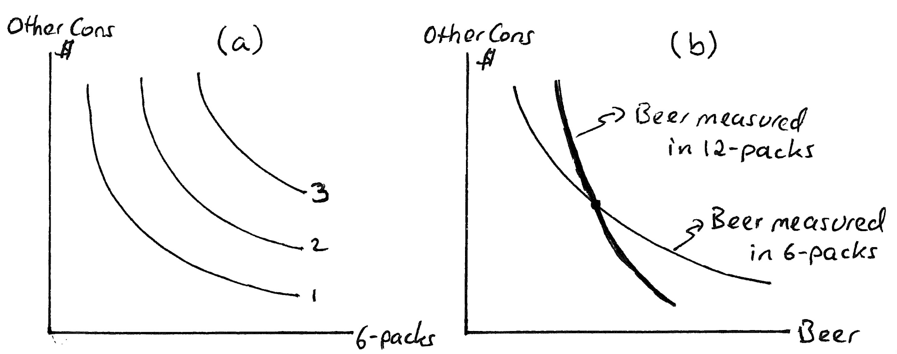
Exercise 5.3

Beer comes in six and twelve-packs. In this exercise we will see how your model of tastes for beer and other consumption might be affected by the units in which we measure beer.

A: *Suppose initially that your favorite beer is only sold in six-packs.*

(a) *On a graph with beer on the horizontal axis and other consumption (in dollars) on the vertical, depict three indifference curves that satisfy our usual five assumptions assuming that the units in which beer is measured is six-packs.*

Answer: An example of 3 such indifference curves is depicted in panel (a) of Exercise Graph 5.3



Exercise Graph 5.3 : Six and 12-packs of Beer

(b) *Now suppose the beer company eliminates six-packs and sells all its beer in twelve-packs instead. What happens to the MRS at each bundle in your graph if 1 unit of beer now represents a twelve-pack instead of a six-pack.*

Answer: At every bundle, you would now be willing to give up twice as many dollars of other consumption for one more unit of beer than you were before — because one more unit of beer is twice as much beer as it was before. Thus, the *MRS* has to be twice as large in absolute value at every consumption bundle.

(c) *In a second graph, illustrate one of the indifference curves you drew in part (a). Pick a bundle on that indifference curve and then draw the indifference curve through that bundle assuming we are measuring beer in twelve-packs instead. Which indifference curve would you rather be on?*

Answer: In panel (b) of Exercise Graph 5.3, this is illustrated — with the indifference curve that measures beer in 12-packs having twice the slope in absolute value as the indifference curve that measures beer in 6-packs. You would of course rather be on the indifference curve with beer measured in 12 packs.

(d) *Does the fact that these indifference curves cross imply that tastes for beer change when the beer company switches from 6-packs to 12-packs?*

Answer: No. The shape of indifference curves on any indifference map is determined in part by the units used to measure quantities of the goods. The two indifference maps from which the indifference curves in panel (b) arise represent the same tastes if they have the same *MRS* adjusted for the units used to measure beer — i.e. if beer is measured in units twice as large, the *MRS* at every bundle has to be twice as large in absolute value.

B: *Let x_1 represent beer and let x_2 represent dollars of other consumption. Suppose that, when x_1 is measured in units of six-packs, your tastes are captured by the utility function $u(x_1, x_2) = x_1 x_2$.*

(a) *What is the MRS of other goods for beer?*

Answer: The *MRS* is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{x_2}{x_1} \tag{5.3.i}$$

(b) *What does the MRS have to be if x_1 is measured in units of 12-packs?*

Answer: As we argued already, the *MRS* has to be twice as large in absolute value since now you would be willing to pay twice as much for one more unit of x_1 since it is measured in units twice as large.

(c) *Give a utility function that represents your tastes when x_1 is measured in 12-packs and check to make sure it has the MRS you concluded it must have.*

Answer: The utility function $v(x_1, x_2) = x_1^2 x_2$ would be one function that could represent tastes over 12-packs of beer. The *MRS* of this function is

$$MRS = -\frac{\partial v/\partial x_1}{\partial v/\partial x_2} = -\frac{2x_1 x_2}{x_1^2} = -2\frac{x_2}{x_1}, \tag{5.3.ii}$$

which is twice as large in absolute value as the *MRS* of the original utility function u.

(d) *Can you use this example to explain why it is useful to measure the substitutability between different goods using percentage terms (as in the equation for the elasticity of substitution) rather than basing it simply on the absolute value of slopes at different bundles?*

Answer: The units used to measure goods affect the way that indifference curves look, but they don't affect the underlying tastes represented by those indifference curves. If a measure of substitutability were to use the absolute value of slopes at different bundles, the choice of units would partly determine the value of our measure of substitutability. But by using percentage changes instead of absolute changed in the formula for the elasticity of substitution, the units cancel — and our measure becomes independent of the units. For instance, the utility function we derived in the previous part for the case where we measure beer in 12-packs is Cobb-Douglas just as the utility function we used to measure those same tastes when beer was measured in 6-packs. We know that all Cobb-Douglas utility functions have elasticity of substitution of 1 — and so we know we have not changed the elasticity of substitution when we altered the units used to measure one of the goods. Thus, we have defined in the elasticity of substitution a measure of substitutability that is immune to the units chosen to measure the goods on each axis.

Exercise 5.5

Everyday Application: *Personality and Tastes for Current and Future Consumption: Consider two brothers, Eddy and Larry, who, despite growing up in the same household, have grown quite different personalities.*

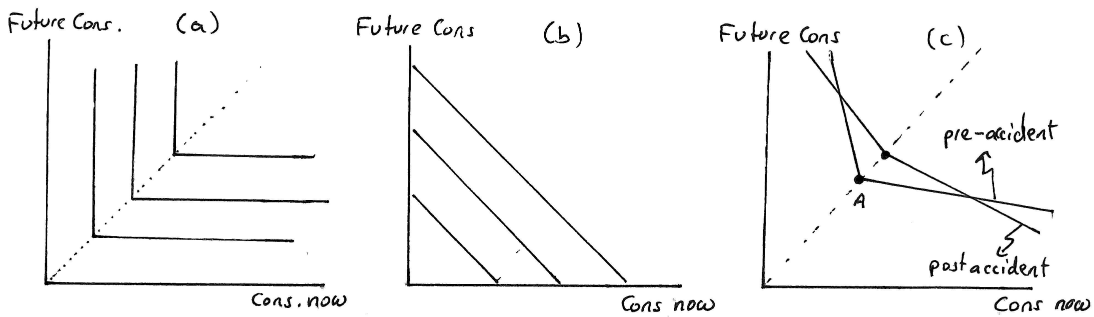
A: *Eddy is known to his friends as "steady Eddy" — he likes predictability and wants to know that he'll have what he has now again in the future. Larry, known to his friends as "crazy Larry", adapts easily to changing circumstances. One year he consumes everything around him like a drunken sailor, the next he retreats to a Buddhist monestary and finds contentment in experiencing poverty.*

(a) *Take the characterization of Eddy and Larry to its extreme (within the assumptions about tastes that we introduced in Chapter 4) and draw two indifference maps with "current consumption" on the horizontal axis and "future consumption" on the vertical — one for steady Eddy and one for crazy Larry.*

Answer: The description indicates that Eddy does not trade off consumption across time very easily while Larry does. In the extreme, that would mean that consumption now and consumption in the future are perfect complements for Eddy and perfect substitutes for Larry. (A less extreme version would have consumption now and consumption in the future be closer to perfect complements for Eddy than for Larry.) The extreme indifference maps for Eddy and Larry are drawn in panels (a) and (b) (respectively) of Exercise Graph 5.5.

(b) *Eddy and Larry have another brother named Daryl who everyone thinks is a weighted average between his brothers' extremes. Suppose he is a lot more like steady Eddy than he is like crazy Larry — i.e. he is a weighted average between the two but with more weight placed on the Eddy part of his personality. Pick a bundle A on the 45 degree line and draw a plausible indifference curve for Daryl through A. (If you take the above literally in a certain way, you would get a kink in Daryl's indifference curve.) Could his tastes be homothetic?*

Answer: His indifference curves would be flatter than Eddy's but not as flat as Larry's, and since he is more like Eddy, they would look more like Eddy's. One plausible such indifference curve — labeled "pre-accident"



Exercise Graph 5.5 : Steady Eddy, Crazy Larry and Unstable Daryl

— through a bundle A on the 45 degree line is drawn in panel (c) of the graph. The indifference curve has a kink at A because at A it is unclear what it would mean to "average" the indifference maps. A less literal interpretation of the problem might not have a kink at that point — but would have a somewhat smoother version of an indifference curve like the one graphed here. Both Eddy's and Larry's indifference maps are homothetic — and an average between their indifference maps should also be homothetic. In panel (c), the other indifference curves would contain parallel line segments emanating from the 45 degree line — and the MRS would therefore be the same along any ray from the origin. The same can easily be true of indifference maps without the sharp kink on the 45 degree line. This illustrates that homotheticity of tastes can allow for many different degrees of substitutability.

(c) *One day Daryl suffers a blow to his head — and suddenly it appears that he is more like crazy Larry than like steady Eddy; i.e. the weights in his weighted average personality have flipped. Can his tastes still be homothetic?*

Answer: Yes, they would simply have indifference curves with line segments flatter than the indifference curve through bundle A — indifference curves like the one labeled "post-accident" in panel (c) of the graph. This would continue to satisfy the homotheticity condition. This would also hold for smoother versions of the indifference curves — i.e. versions that don't have a kink point on the 45 degree line.

(d) *In end-of-chapter exercise 4.9, we defined what it means for two indifference maps to satisfy a "single crossing property". Would you expect that Daryl's pre-accident and post-accident indifference maps satisfy that property?*

Answer: No, they would not. This is easily seen in panel (c) of the graph where the pre- and post-accident indifference curve cross twice. (Note that this conclusion also is not dependent on the kink in the indifference curves.)

(e) *If you were told that either Eddy or Larry saves every month for retirement and the other smokes a lot, which brother is doing what?*

Answer: I would guess that a person who views consumption across time as not very substitutable would make sure to save so that he can consume at the same levels when he stops earning income. At the same time, someone who views consumption now and in the future substitutable might be willing to enjoy a lot of smoking now even if it decreases the quality of life later.

B: *Suppose that one of the brothers' tastes can be captured by the function* $u(x_1, x_2) = \min\{x_1, x_2\}$ *where* x_1 *represents dollars of current consumption and* x_2 *represents dollars of future consumption.*

(a) *Which brother is it?*

Answer: It's steady Eddy — since he is not willing to trade consumption across time periods and thus has indifference curves that treat consumption now and consumption in the future as perfect complements (or something close to it).

(b) *Suppose that when people say that Daryl is the weighted average of his brothers, what they mean is that his elasticity of substitution of current for future consumption lies in between those of his borthers. If Larry and Daryl have tastes that could be characterized by one (or more) of the utility functions from end-of-chapter exercise 4.5, which functions would apply to whom?*

Answer: Crazy Larry's would be perfect substitutes — which are given by utility function (2) in problem 4.5. By looking at MRS's and the ordering of indifference curves, we concluded in the answer to problem 4.5 that the utility functions (1) and (4) represented the same Cobb-Douglas tastes. Cobb-Douglas tastes are members of the family of CES tastes — which have perfect complements and perfect substitutes at the extremes. Thus, one way of thinking about Daryl being the average of his brothers would be to think of his elasticity of substitution being in some sense in between those of his brothers'. In that case, Cobb-Douglas tastes seem plausible tastes for Daryl. Note that is a different notion of what it might mean for Daryl to be the weighted average of his brothers from that which resulted in a kink point in the indifference curves in our graph.

(c) *Which of the functions in end-of-chapter exercise 4.5 are homothetic? Which are quasilinear (and in which good)?*

Answer: When the MRS depends only on the ratio of x_2 to x_1, then this means that it is the same for bundle A as it is for any bundle that multiplies the goods in A by the same constant — i.e. it is the same along any ray from the origin. Utility functions (1), (4) and (5) all have this feature and thus all represent homothetic tastes. Furthermore, utility function (2) gives rise to the same MRS everywhere — so it, too, represents tastes that are homothetic. Utility function (3), on the other hand, has MRS that depends only on x_2 — which means that, if me multiply both goods in a bundle A, we end up getting a different MRS. Thus, utility function (3) is not homothetic.

For a utility function to represent tastes that are quasilinear, the MRS has to be independent of one of the two goods. This is true for utility function (3) where the MRS depends only on x_2. Thus, for any level of x_2, the MRS is unchanged regardless of how much x_1 is in the bundle. Put differently, we can draw a horizontal line in our graph with x_2 on the vertical axis and know that the MRS along that line is constant. Thus, utility function (3) represents tastes quasilinear in x_2. Finally, utility function (2) has the same MRS everywhere and is thus quasilinear in both goods. The other utility functions have MRS varying with both goods and are therefore not quasilinear.

(d) *Despite being so different, is it possible that both steady Eddy and crazy Larry have tastes that can be represented by Cobb Douglas utility functions?*

Answer: No, that does not seem plausible since the description of steady Eddy and crazy Larry clearly indicates very different elasticities of substitution between current and future consumption — and all Cobb-Douglas preferences have elasticity of substitution of 1.

(e) *Is it possible that all their tastes could be represented by CES utility functions? Explain.*

Answer: Yes, this is possible since CES utility functions encompass functions ranging from elasticity of substitution of 0 (perfect complements) to ∞ (perfect substitutes). The essential difference between the three brothers is their elasticity of substitution between current and future consumption — and the CES family of utility functions gives the flexibility to allow that to vary completely across individuals.

Exercise 5.7

Everyday Application: *Tastes for Paperclips: Consider my tastes for paperclips and "all other goods" (denominated in dollar units).*

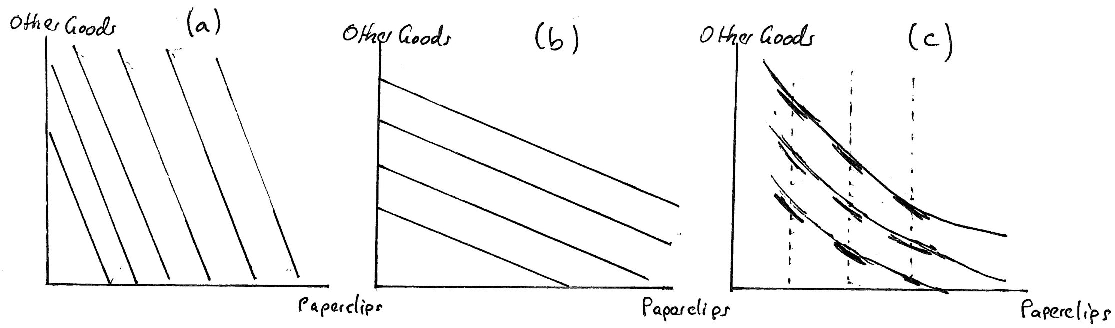
A: *Suppose that my willingness to trade paper clips for other goods does* not *depend on how many other goods I am also currently consuming.*

(a) *Does this imply that "other goods" are "essential" for me?*

Answer: This means that, plotting paperclips on the horizontal axis, the MRS is unchanged along any vertical line we draw — which in turn means that the indifference curves must cross the paperclip axis. When an indifference curve crosses the paperclip axis, it means that I am able to get utility greater than at the origin without consuming any "other good". Thus, other goods are not essential for me.

(b) *Suppose that, in addition, my willingness to trade paperclips for other goods does not depend on how many paperclips I am currently consuming. On two graphs, each with paperclips on the horizontal axis and "dollars of* *other goods" on the vertical, give two examples of what my indifference curves might look like.*

Answer: If my *MRS* does not depend on either my level of paperclip consumption or my level of other good consumption, it means that my *MRS* is the same everywhere. Thus, paperclips and other goods are perfect substitutes — and the only thing we are not sure about is at what rate I am willing to substitute one for the other. Thus, we can get different maps of indifference curves where those maps differ in terms of the *MRS* that holds everywhere within each map. Two such indifference maps are illustrated in panels (a) and (b) of Exercise Graph 5.7.



Exercise Graph 5.7 : Paperclips and other Goods

(c) *How much can the MRS vary* within *an indifference map that satisfies the conditions in part (b)? How much can it vary* between two *indifference maps that both satisfy the conditions in part (b)?*

Answer: As we just concluded, the *MRS* cannot vary *within* an indifference map under these conditions — but it can vary *across* indifference maps that satisfy these conditions.

(d) *Now suppose that the statement in (a) holds for my tastes but the statement in part (b) does not. Illustrate an indifference map that is consistent with this.*

Answer: Tastes are now simply quasilinear in paperclips — and if they are not also quasilinear in other goods, we have ruled out the case of perfect substitutes. An example of such an indifference map is illustrated in panel (c) of Exercise Graph 5.7.

(e) *How much can the MRS vary* within *an indifference map that satisfies the conditions of part (d)?*

Answer: The *MRS* can range from zero to minus infinity within the same map.

(f) *Which condition do you think is more likely to be satisfied in someone's tastes — that the willingness to trade paperclips for other goods is independent of the level of paperclip consumption or that it is independent of the level of other goods consumption?*

Answer: It seems more realistic to assume that our MRS is independent of how much in other goods we are consuming. Paperclips are a small part of our overall consumption bundle — and as we consume more of other goods (because, for instance, our income is increasing), it seems unlikely that we will change how we feel about paperclips on the margin. We have only so much need for paperclips, though — so as we get more paperclips in our consumption bundle, we are probably less willing to pay the same amount for more paperclips as we were willing to pay for the original paperclips that made it into the consumption bundle. So it seems unlikely that our MRS is independent of the level of paperclip consumption.

(g) *Are any of the indifference maps above homothetic? Are any of them quasilinear?*

Answer: The maps in panels (a) and (b) are homothetic and quasilinear, while the map in panel (c) is only quasilinear (in paperclips).

B: *Let paperclips be denoted by x_1 and other goods by x_2.*

(a) *Write down two utility functions, one for each of the indifference maps from which you graphed indifference curves in A(b).*

Answer: Consider the utility function $u(x_1, x_2) = \alpha x_1 + x_2$. When $\alpha = 1$, we have the case of perfect substitutes where the consumer is willing to always trade the goods one for one. More generally, the consumer is willing to trade α of x_2 for one x_1. Thus, when $\alpha < 1$, the consumer is willing to trade less than one unit of x_2 for one unit of x_1 — implying shallower indifference curves as in panel (b); and when $\alpha > 1$, the consumer is willing to trade more than one unit of x_2 for one unit of x_1 — implying steeper indifference curves as in panel (a). You can also see this by simply deriving the MRS as $(-\alpha)$.

(b) *Are the utility functions you wrote down homogeneous? If the answer is no, could you find utility functions that represent those same tastes and are homogeneous? If the answer is yes, could you find utility functions that are not homogeneous but still represent the same tastes?*

Answer: Yes, the utility function $u(x_1, x_2) = \alpha x_1 + x_2$ is homogeneous. In fact, it is homogeneous of degree 1 since

$$u(tx_1, tx_2) = \alpha(tx_1) + tx_2 = t(\alpha x_1 + x_2) = tu(x_1, x_2). \tag{5.7.i}$$

You can transform this into a non-homogeneous utility function by simply adding a constant — say 10 — to get $v(x_1, x_2) = \alpha x_1 + x_2 + 10$. Then

$$v(tx_1, tx_2) = \alpha(tx_1) + tx_2 + 10 \neq t(\alpha x_1 + x_2) + 10t = t^k v(x_1, x_2) \tag{5.7.ii}$$

for all $k > 0$.

(c) *Are the functions you wrote down homogeneous of degree 1? If the answer is no, could you find utility functions that are homogeneous of degree 1 and represent the same tastes? If the answer is yes, could you find utility functions that are not homogeneous of degree k and still represent the same tastes?*

Answer: Equation (5.7.i) demonstrates that the function $u(x_1, x_2) = \alpha x_1 + x_2$ is homogeneous of degree 1. You can turn this into a function that is homogeneous of degree k by taking it to the kth power; i.e. $w(x_1, x_2) = (u(x_1, x_2))^k = (\alpha x_1 + x_2)^k$. Then

$$w(tx_1, tx_2) = (\alpha(tx_1) + tx_2)^k = (t(\alpha x_1) + x_2))^k = t^k(\alpha(x_1) + x_2)^k = t^k w(x_1, x_2). \tag{5.7.iii}$$

(d) *Is there any indifference map you could have drawn when answering A(d) which can be represented by a utility function that is homogeneous? Why or why not?*

Answer: No. Homogeneous functions have the property that they give rise to homothetic indifference maps — with the MRS constant along any ray from the origin. The indifference map in A(d) is quasilinear — and the only way that it can be both quasilinear and homothetic is for the goods to be perfect substitutes. But A(d) specifically ruled out linear indifference curves.

Exercise 5.9

Everday Application: *Syllabi-Induced Tastes over Exam Grades*: *Suppose you are taking two classes, economics and physics. In both classes, only two exams are given during the semester.*

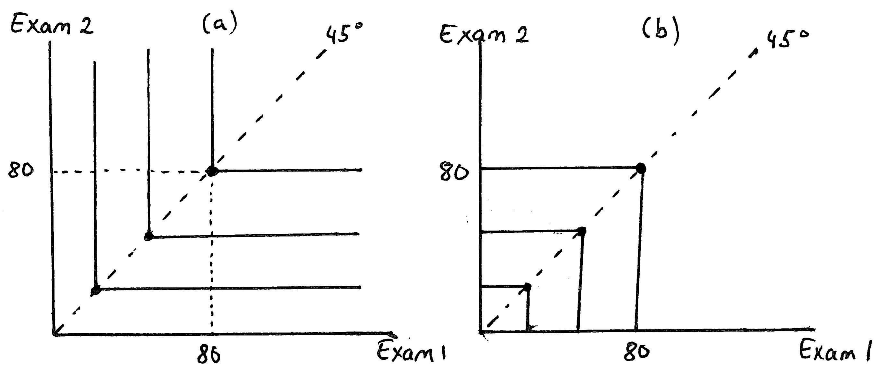
A: *Since economists are nice people, your economics professor drops the lower exam grade and bases your entire grade on the higher of the two grades. Physicists are another story. Your physics professor will do the opposite — he will drop your highest grade and base your entire class grade on your lower score.*

(a) *With the first exam grade (ranging from 0 to 100) on the horizontal axis and the second exam grade (also ranging from 0 to 100) on the vertical, illustrate your indifference curves for your physics class.*

Answer: These are illustrated in panel (a) of Exercise Graph 5.9. Since the physics professor will only count the lower score, only the lower score matters for generating utility. Thus, the two exam scores are perfect complements — the only way to get more utility is for the minimum score to increase.

(b) *Repeat this for your economics class.*

Answer: Since the economics professor is only counting the higher score, only the higher score matters for utility. Thus, if you have an 80, for instance, on your first exam, your utility will remain the same if your second exam is an 80 or less. Similarly, if your second exam is an 80, your



Exercise Graph 5.9 : Physics and Economics Exams

first exam does not matter if it is less than or equal to 80. One of your indifference curves therefore is the point (80,80) as well as the bundles (x,80) and (80,x) for all $x \leq 80$. This forms one of the indifference curves in panel (b) of Exercise Graph 5.9, with the remaining indifference curves in the graph similarly derived.

(c) *Suppose all you care about is your final grade in a class and you otherwise value all classes equally. Consider a pair of exam scores* (x_1, x_2) *and suppose you knew before registering for a class what that pair will be — and that it will be the same for the economics and the physics class. What must be true about this pair in order for you to be indifferent between registering for economics and registering for physics?*

<u>Answer</u>: The fact that you care only about your final grade and you value all classes equally means that getting an 80 in your economics class means exactly as much to you as getting an 80 in your physics class. The highest indifference curve in panel (a) of the graph illustrates all the different ways of getting an 80 in your physics class. Similarly, the highest indifference curve in panel (b) of the graph illustrates all the different ways of getting an 80 in the economics class. Thus, any of the pairs of grades on the relevant indifference curve in panel (a) is just as good as any of the pairs of grades on the relevant indifference curve in panel (b). But those indifference curves only share one pair of grades in common — the pair (80,80). The same reasoning holds for any other pair of exam grades that might make you indifferent between the two classes. Thus, it must be that $x_1 = x_2$.

B: *Consider the same scenario as the one described in part A.*

(a) *Give a utility function that could be used to represent your tastes as you described them with the indifference curves you plotted in A(a)?*

<u>Answer</u>: The simples example of such a utility function is $u(x_1, x_2) = \min\{x_1, x_2\}$.

(b) *Repeat for the tastes as you described them with the indifference curves you plotted in A(b).*

Answer: Now it is the maximum, not the minimum, grade that matters. So the corresponding utility function could take the form $u(x_1, x_2) = \max\{x_1, x_2\}$.

Exercise 5.11

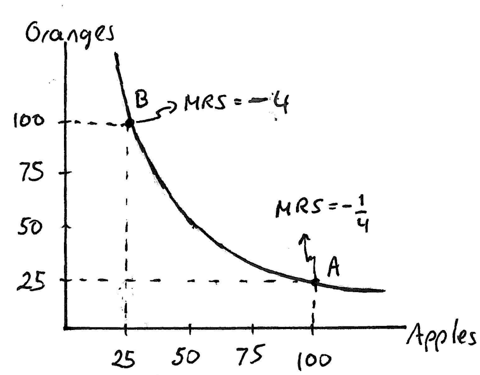
In this exercise, we are working with the concept of an elasticity of substitution. This concept was introduced in part B of the Chapter. Thus, this entire question relates to material from part B, but the A-part of the question can be done simply by knowing the formula for an elasticity of substitution while the B-part of the question requires further material from part B of the Chapter. In Section 5B.1, we defined the elasticity of substitution as

$$\sigma = \left| \frac{\%\Delta(x_2/x_1)}{\%\Delta MRS} \right|. \tag{5.11}$$

A: *Suppose you consume only apples and oranges. Last month, you consumed bundle A=(100,25) — 100 apples and 25 oranges, and you were willing to trade at most 4 apples for every orange. Two months ago, oranges were in season and you consumed B=(25,100) and were willing to trade at most 4 oranges for 1 apple. Suppose your happiness was unchanged over the past two months.*

(a) *On a graph with apples on the horizontal axis and oranges on the vertical, illustrate the indifference curve on which you have been operating these past two months and label the MRS where you know it.*

Answer: This is illustrated in Exercise Graph 5.11.



Exercise Graph 5.11 : Elasticity of Subsitution

(b) *Using the formula for elasticity of substitution, estimate your elasticity of substitution of apples for oranges.*

Answer: This is

$$\sigma = \left| \frac{((100/25) - (25/100))/(100/25)}{(-4-(-1/4))/(-4)} \right| = \left| \frac{(15/4)/4}{(15/4)/4} \right| = 1. \tag{5.11.i}$$

(c) *Suppose we know that the elasticity of substitution is in fact the same at every bundle for you and is equal to what you calculated in (b). Suppose the bundle C=(50,50) is another bundle that makes you just as happy as bundles A and B. What is the MRS at bundle C?*

Answer: Using B and C in the elasticity of substitution formula, setting σ equal to 1 and letting the MRS at C be denoted by x , we get

$$\left|\frac{((100/25)-(50/50))/(100/25)}{(-4-x)/(-4)}\right| = \left|\frac{3/4}{(4+x)/4}\right| = 1, \tag{5.11.ii}$$

and solving this for x, we get $x = -1$ — i.e. the MRS at C is equal to -1.

(d) *Consider a bundle $D = (25,25)$. If your tastes are homothetic, what is the MRS at bundle D?*

Answer: Since it, like bundle C, lies on the 45 degree line, homotheticity implies the MRS is again -1.

(e) *Suppose you are consuming 50 apples, you are willing to trade 4 apples for one orange and you are just as happy as you were when you consumed at bundle D. How many oranges are you consuming (assuming the same elasticity of substitution)?*

Answer: Let the number of oranges be denoted y. Using the bundle $(50, y)$ and $D = (25,25)$ in the elasticity formula and setting it to 1, we get

$$\left|\frac{((50/y)-(25/25))/(50/y)}{(-4-(-1))/(-4)}\right| = \left|\frac{((50/y)-1)/(50/y)}{(3/4)}\right| = 1. \tag{5.11.iii}$$

Solving this for y, we get $y = 12.5$.

(f) *Call the bundle you derived in part (e) E. If the elasticity is as it was before, at what bundle would you be just as happy as at E but would be willing to trade 4 oranges for 1 apple?*

Answer: If the elasticity is 1 from D to E and is again supposed to be 1 from D to this new bundle, there must be symmetry around the 45 degree line (as there was between A and B). At $E = (50,12.5)$, the MRS is $-1/4$, and the necessary symmetry then means that $MRS = -4$ at (12.5,50).

B: *Suppose your tastes can be summarized by the utility function* $u(x_1, x_2) = \left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho}$.

(a) *In order for these tastes to contain an indifference curve such as the one containing bundle A that you graphed in A(a), what must be the value of ρ? What about α?*

Answer: The elasticity of substitution for the CES utility function can be written as $\sigma = 1/(1+\rho)$. We already determined that the elasticity of substitution in this problem is 1. Thus, $1 = 1/(1+\rho)$ which implies $\rho = 0$. Since our graph is symmetric around the 45 degree line, it must furthermore be true that $\alpha = 0.5$ — i.e. x_1 and x_2 enter symmetrically into the utility function.

(b) *Suppose you were told that the same tastes can be represented by $u(x_1, x_2) = x_1^{\gamma} x_2^{\delta}$. In light of your answer above, is this possible? If so, what has to be true about γ and δ given the symmetry of the indifference curves on the two sides of the 45 degree line?*

Answer: Yes — it is possible because we determined that the elasticity of substitution is 1 everywhere, which is true for Cobb-Douglas utility functions of the form $u(x_1, x_2) = x_1^{\gamma} x_2^{\delta}$. The symmetry implies $\gamma = \delta$.

(c) *What exact value(s) do the exponents γ and δ take if the label on the indifference curve containing bundle A is 50? What if that label is 2,500? What if the label is 6,250,000?*

Answer: If the utility at A is 50, it means $50^{\gamma} 50^{\delta} = 50$. Since we just concluded in (a) that $\gamma = \delta$, this implies that $\gamma = \delta = 0.5$. If the utility is 2,500, then $\gamma = \delta = 1$, and if the utility is 6,250,000, $\gamma = \delta = 2$.

(d) *Verify that bundles A, B and C (as defined in part A) indeed lie on the same indifference curve when tastes are represented by the three different utility functions you implicitly derived in B(c). Which of these utility functions is homogeneous of degree 1? Which is homogeneous of degree 2? Is the third utility function also homogeneous?*

Answer: The bundles are A=(100,25), B=(25,100) and C=(50,50). The following equations hold, verifying that these must be on the same indifference curve for each of the three utility functions: $u(x_1, x_2) = x_1^{0.5} x_2^{0.5}$, $v(x_1, x_2) = x_1 x_2$ and $w(x_1, x_2) = x_1^2 x_2^2$:

$$\begin{aligned} u(100,25) &= u(25,100) = u(50,50) = 50 \\ v(100,25) &= v(25,100) = v(50,50) = 2{,}500 \\ w(100,25) &= w(25,100) = w(50,50) = 6{,}250{,}000. \end{aligned} \tag{5.11.iv}$$

The following illustrate the homogeneity properties of the three functions:

$$\begin{aligned} u(tx_1, tx_2) &= (tx_1)^{0.5}(tx_2)^{0.5} = t x_1^{0.5} x_2^{0.5} = t u(x_1, x_2) \\ v(tx_1, tx_2) &= (tx_1)(tx_2) = t^2 x_1 x_2 = t^2 v(x_1, x_2) \\ w(tx_1, tx_2) &= (tx_1)^2 (tx_2)^2 = t^4 x_1^2 x_2^2 = t^4 w(x_1, x_2). \end{aligned} \tag{5.11.v}$$

Thus, u is homogeneous of degree 1, v is homogeneous of degree 2 and w is homogeneous of degree 4.

(e) *What values do each of these utility functions assign to the indifference curve that contains bundle D?*

Answer: Recall that $D = (25, 25)$. Thus, the three utility functions assign values of $u(25,25) = 25^{0.5} 25^{0.5} = 25$; $v(25,25) = 25(25) = 625$; and $w(25,25) = 25^2(25^2) = 390,625$.

(f) True or False*: Homogeneity of degree 1 implies that a doubling of goods in a consumption basket leads to "twice" the utility as measured by the homogeneous function, whereas homogeneity greater than 1 implies that a* *doubling of goods in a consumption bundle leads to more than "twice" the utility.*

Answer: This is true. We already showed an example of this. More generally, you can see this from the definition of a function that is homogeneous of degree k; i.e. $u(tx_1,tx_2) = t^k u(x_1,x_2)$. Substituting $k = 2$, $u(2x_1,2x_2) = 2^k u(x_1,x_2)$. When $k = 1$ — i.e. when the utility function is homogeneous of degree 1, this implies $u(2x_1,2x_2) = 2u(x_1,x_2)$ — a doubling of goods leads to a doubling of utility assigned to the bundle. More generally, a doubling of goods leads to 2^k times as much utility assigned to the new bundle — and 2^k is greater than 2 when $k > 1$ (and less than 2 when $k < 1$.)

(g) *Demonstrate that the MRS is unchanged regardless of which of the three utility functions derived in B(c) is used.*

Answer: The MRS of a Cobb-Douglas utility function $u(x_1,x_2) = x_1^\gamma x_2^\delta$ is $MRS = -(\gamma x_2)/(\delta x_1)$ which reduces to $-x_2/x_1$ when $\gamma = \delta$ which is the case for all three of the utility functions. Thus, the MRS is the same for the three functions.

(h) *Can you think of representing these tastes with a utiliy function that assigns the value of 100 to the indifference curve containing bundle A and 75 to the indifference curve containing bundle D? Is the utility function you derived homogeneous?*

Answer: The function $u(x_1,x_2) = x_1^{0.5}x_2^{0.5} + 50$ would work. This function is not homogeneous (but it is homothetic).

(i) True or False: *Homothetic tastes can always be represented by functions that are homogeneous of degree k (where k is greater than zero), but even functions that are not homogeneous can represent tastes that are homothetic.*

Answer: This is true. We showed in the text that $MRS(tx_1,tx_2) = MRS(x_1,x_2)$ for homogeneous functions — thus, for homogeneous functions, the MRS is constant along any ray from the origin, the definition of homothetic tastes. At the same time, we just saw in the answer to the previous part an example of a non-homogeneous function that still represents homothetic tastes.

(j) True or False*: The marginal rate of substitution is homogeneous of degree 0 if and only if the underlying tastes are homothetic.*

Answer: For any set of homothetic tastes, the MRS is constant along rays from the origin; i.e. $MRS(tx_1,tx_2) = MRS(x_1,x_2)$. Thus, for homothetic tastes, the MRS is indeed homogeneous of degree 0. But $MRS(tx_1,tx_2) = MRS(x_1,x_2)$ defines homotheticity — so non-homothetic tastes will not have this property, which implies their MRS is not homogeneous of degree zero. The statement is therefore true.

Conclusion: Potentially Helpful Reminders

1. Keep in mind the distinction between how the *MRS* changes along an indifference curve (which tells us about substitutability) and how the *MRS* changes across the indifference map (which leads to ideas like homotheticity and quasilinearity).

2. The idea of substitutability will become critical in Chapter 7 when we introduce substitution effects (which will depend only on the shape of an indifference curve). The ideas of homotheticity and quasiliearity become important as we introduce income effects (in Chapter 7) — which will be measured across an indifference map (rather than along an indifference curve).

3. Extremes like perfect substitutes and perfect complements are useful to keep in mind because they make it easy to remember which way an indifference map looks if the goods are relatively more substitutable as opposed to relatively more complementary and vice versa.

4. Special cases like homothetic and quasilinear tastes will become useful borderline cases in Chapter 7 — with homothetic tastes being the borderline case between luxury goods and necessities, and with quasilinear tastes being the borderline case between normal and inferior goods. (These terms are defined in Chapter 7.)

CHAPTER

6

Doing the “Best” We Can

While Chapter 4 introduced us to a general way of thinking about tastes, Chapter 5 gets much more specific and introduces particular dimensions along which we might differentiate between tastes. In particular, we differentiate tastes based on

1. The curvature of individual indifference curves — or how quickly the MRS changes *along an indifference curve*;
2. The relationships between indifference curves — or how the MRS changes *across indifference curves within an indifference map*; and
3. Whether or not indifference curves *cross horizontal or vertical axes* or whether they *converge to the axes*.

The first of these in turn determines the degree to which consumers are willing to substitute between goods (and will lead to what we call the "substitution effect" in Chapter 7) while the second of these determines how consumer behavior responds to changes in income (and will lead to what we call the "income effect" in Chapter 7). Finally, the third category of taste differences becomes important in Chapter 6 where we will see how corner versus interior optimal solutions for a consumer emerge.

Chapter Highlights

The main points of the chapter are:

1. The **degree of substitutability** or, in part B language, the **elasticity of substitution** for a consumer at a particular consumption bundle arises from the **curvature** of the indifference curve at that bundle. There may be no substitutability (as in perfect complements) or perfect substitutability (perfect substitutes) or an infinite number of cases in between these extremes.

2. **Quasilinearity** and **Homotheticity** of tastes represent special cases that describe how indifference curves from the same map relate to one another. These properties have no direct relationship to the concept of substitutability. Tastes are quasilinear in a good x if the MRS only depends on the level of x consumption (and not the level of other goods' consumption. Tastes are homothetic when the MRS depends only on the relative levels of the goods in a bundle.

3. Sometimes it is reasonable to assume that indifference curves only converge to the axes without ever crossing them; other times we assume that they cross the axes. When an indifference curve crosses an axis, it means that we can gain utility beyond what we have by not consuming even if we consume none of one of the goods. When indifference curves only converge to the axes, then some consumption of all goods is necessary in order for a consumer to experience utility above what she would experience by not consuming at all.

4. If you are reading part B of the chapter, you should begin to understand the family of **constant elasticity of substitution utility functions** — with perfect complements, perfect substitutes and Cobb-Douglas tastes as special cases. You should also be able to demonstrate whether a utility function is homothetic or quasilinear. (Most utility functions we use in this text tend to be one or the other.)

6A Solutions to Within-Chapter-Exercises for Part A

Exercise 6A.1

In Chapter 2 we discussed a scenario under which my wife gives me a coupon that reduces the effective price of pants to $10 a pair. Assuming the same tastes, what would be my best bundle?

Answer: In that case, the slope of the budget constraint is $-p_1/p_2 = -1$ — so the optimal bundle would have to have $MRS = -1$ as well. In describing tastes here, we said that the MRS is equal to -1 at bundles where I have an equal number of shirts and pants — that is, along the 45 degree line. Thus, the optimal bundle would occur at the midpoint of the budget line that has intercepts of 20 on each axis — which is at the bundle (10,10) — 10 pants and 10 shirts.

Exercise 6A.2

Suppose both you and I have a bundle of 6 pants and 6 shirts, and suppose that my MRS of shirts for pants is -1 and yours is -2. Suppose further that neither one of us has access to Wal-Mart. Propose a trade that would make both of us better off.

Answer: In this case, you are willing to trade 2 shirts for 1 pair of pants whereas I am willing to trade them one for one. Assuming we can trade fractions of shirts and pants, a trade in which you give me 1.5 shirts for 1 pair of pants would make you better off (because you would have been willing to give up as many as 2 shirts for 1 pair of pants) and would also make me better off (because I would have been willing to accept as little as 1 shirt for 1 pair of pants). If we don't want to assume we can trade in fractions of goods, then the trade of 3 shirts for 2 pants would work similarly.

Exercise 6A.3

We keep using the phrase "at the margin" — as, for example, when we say that tastes for those leaving Wal-Mart will be the "same at the margin." What do economists mean by this "at the margin" phrase?

Answer: "At the margin" means approximately around the bundle that we are discussing. To say that tastes are the same "at the margin" is the same as saying that around the bundles that individuals currently have (as they leave Wal-Mart), their tastes are the same — but that's not necessarily the same as saying that tastes are the same everywhere. "At the margin" restricts our attention to just a small subset of the larger space in which tastes reside.

Exercise 6A.4

In the previous section, we argued that Wal-Mart's policy of charging the same price to all consumers insures that there are no further gains from trade for goods contained in the shopping baskets of individuals that leave Wal-Mart. The argument assumed that all consumers end up at an interior solution, not a corner solution. Can you see why the conclusion still stands when some people optimize at corner solutions where their *MRS* may be quite different from the *MRS*'s of those who optimize at interior solutions?

Answer: When everyone optimizes at an interior solution, everyone's *MRS* must be the same as everyone else's when they leave Wal-Mart — i.e. our tastes are the same at the margin, thus allowing for no further gains from trade. Now imagine that we consider shirts and pants — and someone leaves Wal-Mart with only shirts and no pants. That person, call her person A, is therefore at a corner solution — and for that corner solution to be optimal, it is almost certainly the case that this person's indifference curve is steeper than the budget constraint at the corner optimum. Thus, this person's tastes are not the same at the margin as those of the other consumers who optimized at a point where the slope of their budget constraint was equal to the slope of their indifference curve. Suppose, then, that person A's *MRS* is -4 and person B's *MRS* is -2 — with person B at an interior solution and person A at a corner solution where she buys only pants. Just looking at the *MRS*'s of the two people, we could say that a trade in which person A gives up 3 shirts in exchange for one pair of pants from person B would make both better off. After all, person B is willing to accept as few as 2 shirts for a pair of pants but would now get 3 instead, and person A is willing to give up as many as 4 shirts for a pair of pants but, under this trade, would only have to give up 3. The problem, however, is that person A has only pants — and therefore has not shirts to give up in a trade. Since person A's *MRS* is higher in absolute value than person B's (and since this has to be the case in order for person A to be at a corner solution with only pants when person B is at an interior solution), the only potential trades that benefit both are those that have shirts going from A to B — but none of those trades is possible because A is at a corner solution and therefore without shirts to give up. Thus, when A and B leave Wal-Mart, there are no further gains from trade even if one (or both) of them is at a corner solution and their tastes are not the same at the margin. Either people who leave Wal-Mart are at an interior solution — in which case they have the same tastes on the margin as everyone else who is at an interior solution and thus can't trade with each other anymore; OR people are at a corner solution and don't have the same tastes as others on the margin but can't trade with them because they already have traded away every unit of the thing they value less at the margin than others who are at an interior solution. Either way, all gains from trade are exhausted in Wal-Mart — and the distribution of goods for people leaving Wal-Mart is efficient.

Exercise 6A.5

Suppose the prices of Coke and Pepsi were the same. Illustrate that now there are many optimal bundles for someone with my kind of tastes. What would be my "best" bundle if Pepsi is cheaper than Coke?

Answer: When the prices of Coke and Pepsi are the same, then the budget constraint has the same slope as all the indifference curves. Therefore, one indifference curve lies right on top of the budget constraint and is therefore "tangent" at every point on the budget constraint. In that case, all bundles on the budget constraint are optimal bundles for the consumer. This makes intuitive sense — if Coke and Pepsi are priced the same and if I can't tell the difference between the two, it doesn't matter how I allocate my spending across Coke and Pepsi.

Exercise 6A.6

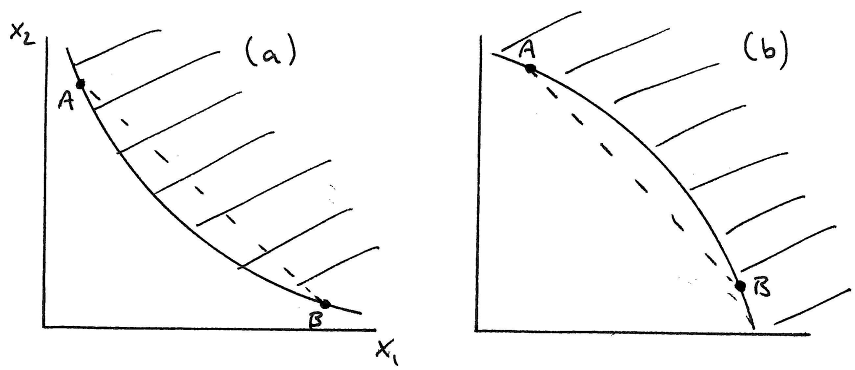
Consider a set of points that compose a solid sphere. Is this set convex? What about the set of points contained in a donut?

Answer: Any line connecting two points in a solid sphere must necessarily be entirely contained within the sphere. Thus, a solid sphere is a convex set. If I pick two points on opposite sides of a donut, on the other hand, the line connecting them will lie (at least partially) outside the donut as it passes through the hole in the middle of the donut. Thus, a donut is not a convex set.

Exercise 6A.7

We have just defined what it means for a set of points to be convex — it must be the case that any line connecting two points in the set is fully contained in the set as well. In Chapter 4, we defined tastes to be convex when "averages are better than (or at least as good as) extremes". The reason such tastes are called "convex" is because the set of bundles that is better than any given bundle is a convex set. Illustrate that this is the case with an indifference curve from an indifference map of convex tastes.

Answer: Panels (a) and (b) of Exercise Graph 6A.7 illustrate two indifference curves, one from a map in which indifference curves satisfy the convexity property, and one from a map of indifference curves that does not satisfy convexity. In both, the set of "better' bundles is shaded. Two bundles, *A* and *B*, on each indifference curve are chosen and the line connecting them is indicated. That line lies fully in the shaded "better than" set in panel (a) but fully outside the shaded "better than" set in panel (b). Thus, convexity of tastes implies convex "better than" sets for each indifference curves, while non-convexities in tastes imply non-convex "better than" sets for some indifference curves.



Exercise Graph 6A.7 : Convexity and Tastes

Exercise 6A.8

True/False: If a choice set is non-convex, there are definitely multiple "best" bundles for a consumer whose tastes satisfy the usual assumptions.

Answer: False. Non-convexities in choice sets imply that there *might* be multiple best bundles, not that there necessarily are for any given tastes of a consumer. In other words, it is easy to construct an indifference curve that only has one tangency on a non-convex budget constraint, but it is also possible to construct an indifference curve (that satisfies the convexity of tastes property) which has more than one tangency on a non-convex budget constraint.

Exercise 6A.9

True/False: If a choice set is convex, then there will be a unique "best" bundle assuming consumer tastes satisfy our usual assumptions and averages are strictly better than extremes.

Answer: This is true. A convex choice set either bends out from the origin or is a straight line with negative slope and positive intercepts. A strictly convex indifference curve, on the other hand, bends toward the origin. Thus, as we move out to higher indifference curves, there will come a point where the budge constraint (that forms the boundary of a convex choice set) contains a single point in common with the indifference curve (that forms a convex "better than" set.)

Exercise 6A.10

Suppose that the choice set is defined by linear budget constraint and tastes satisfy the usual assumptions but contain indifference curves with linear components (or "flat spots"). True/False: Then there might be multiple "best" bundles but we can be sure that the set of "best" bundles is a convex set.

Answer: True. When indifference curves have "flat spots", there is the potential that the line segment of the indifference curve (i.e. the "flat spot") has the same slope as the budget constraint and therefore each bundle on that segment is optimal (much like all bundles are optimal in the case of perfect substitutes when prices were the same for Coke and Pepsi in exercise 6A.5). The set of optimal bundles is then a line segment. Take any two points on the line segment, and it has to be the case that all points that lie on the line (between the points) connecting them also lies in the set of optimal bundles. Thus, the set of optimal bundles is itself a convex set. Of course it might also be the case that, with such indifference curves, the optimal bundle does not occur on the flat spot — and is therefore just a single point. But a set composed of a single point is trivially also a convex set.

Exercise 6A.11

True/False: When there are multiple "best" bundles due to non-convexities in tastes, the set of "best" bundles is also non-convex (assuming convex choice sets).

Answer: True. When there are non-convexities in tastes, that means that the indifference curves at some point bend away from the origin. If multiple optimal bundles arise from that, it means that these bundles will not be connected as in the previous exercise — which means that the line connecting them will contain bundles that are not optimal. Thus, the set of optimal bundles is then non-convex.

6B Solutions to Within-Chapter-Exercises for Part B

Exercise 6B.1

Solve for the optimal quantities of x_1, x_2 and x_3 in the problem defined in equation 6.11. (Hint: The problem will be considerably easier to solve if you take the logarithm the utility function (which you can do since logarithms are order preserving transformations that do not alter the shapes of indifference curves.))

Answer: Taking the hint in the problem, we can write the utility function as $v(x_1, x_2, x_3) = 0.5\ln x_1 + 0.5\ln x_2 + 0.5\ln x_3$ and the corresponding Lagrange function as

$$\mathscr{L}(x_1, x_2, x_3, \lambda) = 0.5\ln x_1 + 0.5\ln x_2 + 0.5\ln x_3 + \lambda(200 - 20x_1 - 10x_2 - 5x_3). \quad \text{(6B.1.i)}$$

Taking first order conditions with respect to each good, we get

$$\begin{aligned} 0.5x_1^{-1} &= 20\lambda \\ 0.5x_2^{-1} &= 10\lambda \\ 0.5x_3^{-1} &= 5\lambda \end{aligned} \quad \text{(6B.1.ii)}$$

Dividing the first equation by the second and solving for x_2, we get $x_2 = 2x_1$. Dividing the first equation by the third and solving for x_3 we get $x_3 = 4x_1$. Substituting these into the budget constraint (which is the fourth first order condition taken with respect to λ), we get

$$20x_1 + 10(2x_1) + 5(4x_1) = 60x_1 = 200, \quad \text{(6B.1.iii)}$$

which implies $x_1 = 3.33$. Then, using the fact that $x_2 = 2x_1$ and $x_3 = 4x_1$, we get $x_2 = 6.67$ and $x_3 = 13.33$.

Exercise 6B.2

Set up the Lagrange function for this problem and solve it to see whether you get the same solution.

Answer: The Lagrange function is

$$\mathscr{L}(x_1, x_2, \lambda) = \alpha\ln x_1 + x_2 + \lambda(200 - 20x_1 - 10x_2). \quad \text{(6B.2.i)}$$

Taking first order conditions with respect to each variable in the Lagrange function, we get

$$\begin{aligned} &\frac{\alpha}{x_1} - 20\lambda = 0 \\ &1 - 10\lambda = 0 \\ &200 - 20x_1 - 10x_2 = 0 \end{aligned} \quad \text{(6B.2.ii)}$$

The second equation implies that $\lambda = 1/10$. Substituting this into the first equation, we get $x_1 = \alpha/2$, and substituting this into the last equation, we get $x_2 = (200 - 10\alpha)/10$.

Exercise 6B.3

Demonstrate how the Lagrange method (or one of the related methods we introduced earlier in this chapter) fails even worse in the case of perfect substitutes. Can you explain what the Lagrange method is doing in this case?

Answer: Consider the utility function $u(x_1, x_2) = x_1 + x_2$. The Lagrange function would then be

$$\mathscr{L}(x_1, x_2, \lambda) = x_1 + x_2 + \lambda(I - p_1 x_1 - p_2 x_2), \tag{6B.3.i}$$

with the first two first order conditions of

$$\begin{aligned} 1 &= \lambda p_1 \\ 1 &= \lambda p_2. \end{aligned} \tag{6B.3.ii}$$

Dividing these, we would get $p_1/p_2 = 1$ or $p_1 = p_2$. But that makes no sense — the prices are taken as given by the consumer. So, suppose $p_1 = 1$ and $p_2 = 2$. The first order conditions would then give us the "result" that $p_1 = 1 = p_2 = 2$. The Lagrange method fails because, as we have seen in the intuitive section of the chapter, there generally are no interior solutions to the optimization problem for a consumer whose tastes treat the goods as perfect substitutes. Instead, the consumer simply consumes only the good that is cheaper. The only time there are interior solutions occurs when $p_1 = p_2$ (our "result" from the Lagrange method) — but in that case any bundle on the budget line is in fact optimal.

Exercise 6B.4

At what value for α will the Lagrange method correctly indicate an optimal consumption of zero shirts? Which of the panels of Graph 6.10 illustrates this?

Answer: It would have to be the case that the MRS is equal to $-p_1/p_2 = -2$ at $x_1 = 10$. The MRS for the utility function $u(x_1, x_2) = \alpha \ln x_1 + x_2$ is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\alpha/x_1}{1} = -\frac{\alpha}{x_1}. \tag{6B.4.i}$$

Thus, when α is such that $-\alpha/10 = -2$, the MRS at $x_1 = 10$ is exactly equal to the slope of the budget constraint. Solving for α we get $\alpha = 20$.

You can check that this is correct by solving the optimization problem with Lagrange function

$$\mathscr{L}(x_1, x_2, \lambda) = 20 \ln x_1 + x_2 + \lambda(200 - 20x_1 - 10x_2). \tag{6B.4.ii}$$

The first two first order conditions of this problem are

$$\frac{20}{x_1} = 20\lambda$$
$$1 = 10\lambda. \qquad (6B.4.iii)$$

These solve to give us $x_1 = 10$ and, plugging this back into the budget constraint, $x_2 = 0$. This is exactly what is illustrated in panel (a) of Graph 6.10.

Exercise 6B.5

In the previous section, we concluded that the first order conditions of the Lagrange problem may be misleading when goods are not essential. Are these conditions either necessary or sufficient in that case?

Answer: No. The conditions might not hold at the optimum (as we have seen in the case of corner solutions) — which means they are not necessary conditions for an optimum when goods are not essential. When they do hold, they might hold (as we have seen) at negative consumption levels when corner solutions are optimal — and so they are not sufficient. They are only sufficient for us to conclude we are at an optimum if they lead to positive consumption levels — in that case we would have an interior solution despite the fact that the goods are not essential.

Exercise 6B.6

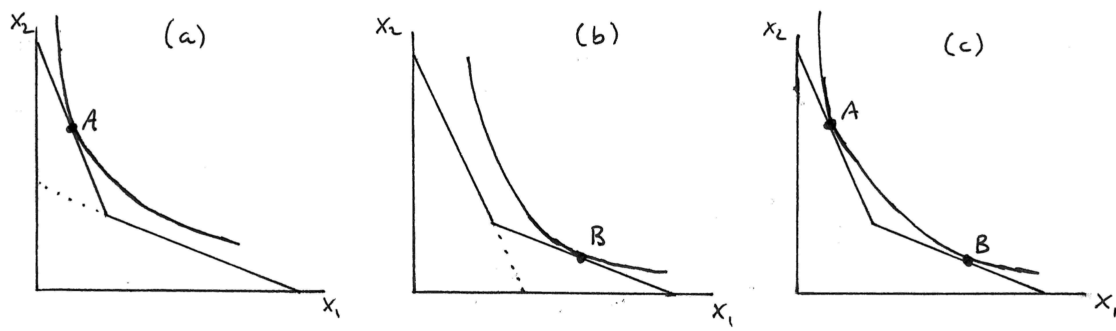
Is it necessary for the indifference curve at the kink of the budget constraint to have a kink in order for both problems in (6.26) to result in x_1=6?

Answer: No, it is not necessary so long as the kink points out rather than in. At the bundle (6,14), the indifference curve can have a slope between −2 and −1 and the kink point will in fact be optimal. (If the kink points in, however, then only an indifference curve that is also kinked at that bundle can result in this bundle being an optimum.)

Exercise 6B.7

Using the intuitions from graphical analysis similar to that in Graph 6.14, illustrate how you might go about solving for the true optimum when a choice set is non-convex due to an "inward" kink.

Answer: Essentially, there are three different possibilities, depicted in panels (a) through (c) of Exercise Graph 6B.7. In panel (a), the optimal bundle is clearly bundle *A* which in fact will be the solution to the Lagrange problem that uses the steeper budget line. The Lagrange problem that uses the shallower budget line might produce an "optimal" bundle that lies on the dashed portion of that shallower budget — in which case we know it can't be optimal given that the steeper budget contains bundles that have strictly more of everything. Alternatively, the Lagrange problem that uses the shallower budget might result in an "optimal" bundle that lies on the solid portion of that shallower budget — but when we determine the utility level at that bundle and compare it to A we would find the utility at A to be higher.



Exercise Graph 6B.7 : Optimization with an Inward Kink

In panel (b), the optimal bundle is B on the shallower portion of the budget constraint. In that case, the Lagrange problem that uses the shallower budget will find this optimal bundle. The Lagrange problem that uses the steeper budget might find an "optimal" bundle on the dashed portion of the steeper budget (in which case we would immediately know that it was not truly optimal since bundles with more of everything are in fact available) or on the solid portion. In the latter case, we we would compare the utility at that bundle to that from B and find that the utility at B is greater.

Finally, panel (c) illustrates the special case where the Lagrange problem with the steeper budget gives us A as the optimal bundle and the Lagrange problem with the shallower budget gives us B — and when we plug both of them back into the utility function, we find that they give the same utility. In that case, we have found two optimal bundles.

6C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 6.1

I have two 5-year old girls — Ellie and Jenny — at home. Suppose I begin the day by giving each girl 10 toy cars and 10 princess toys. I then ask them to plot their indifference curves that contain these endowment bundles on a graph with cars on the horizontal and princess toys on the vertical axis.

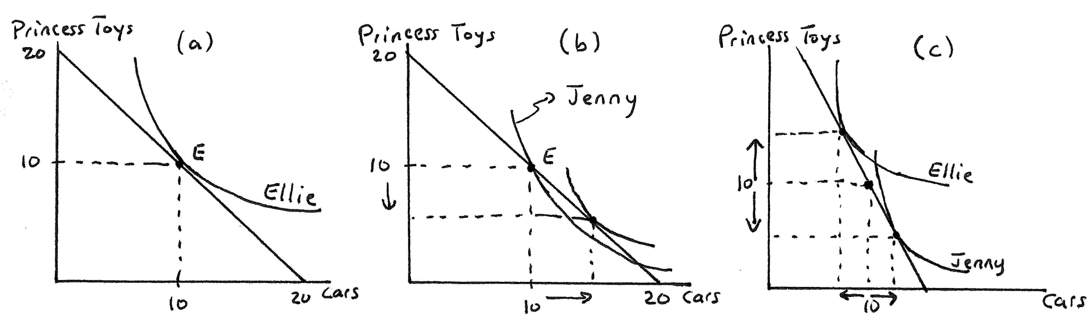
A: *Ellie's indifference curve appears to have a marginal rate of substitution of* -1 *at her endowment bundle, while Jenny's appears to have a marginal rate of substitution of* -2 *at the same bundle.*

(a) *Can you propose a trade that would make both girls better off?*

Answer: Any trade under which Jenny would give up x princess toys for 1 car and Ellie would accept x princess toys in exchange for giving up 1 car would work so long as $1 < x < 2$. This is because Jenny would be willing to give up as many as 2 princess toys for 1 car — so the trade will make her better off because she has to give up less; and Ellie would be willing to accept as little as 1 princess toy to give up 1 car — so the trade will make her better off because she gets more without giving up more.

(b) *Suppose the girls cannot figure out a trade on their own. So I open a store where they can buy and sell any toy for $1. Illustrate the budget constraint for each girl.*

Answer: The budget constraints would be the same for the two girls — because they both have the same endowment point (10,10) and both face the same prices (that result in a slope of -1). These constraints are illustrated in panels (a) and (b) of Exercise Graph 6.1, with the endowment point labeled E.



Exercise Graph 6.1 : Toy Cars and Princess Toys

(c) *Will either of the girls shop at my store? If so, what will they buy?*

Answer: We can then add Ellie's indifference curve through her endowment point in panel (a) and Jenny's indifference curve through her endowment point in panel (b). We know that Ellie's is tangent to her budget constraint because the budget constraint has a slope of -1 and the MRS described in A is also -1 at the endowment bundle E. So Ellie does not want to buy or sell anything at my store at these prices. Jenny's indifference curve at E, on the other hand, has slope -2 — and thus we know her indifference curve cuts her budget constraint at E from above. This implies that Jenny will have better points available in her choice set — with all better points lying to the right of E. Jenny will therefore want to sell princess toys and buy toy cars at my store.

(d) *Suppose I do not actually have any toys in my store and simply want my store to help the girls make trades among themselves. Suppose I fix the price at which princess toys are bought and sold to \$1. Without being specific about what the price of toy cars would have to be, illustrate, using final indifference curves for both girls on the same graph, a situation where the prices in my store result in an efficient allocation of toys.*

Answer: It would have to be that the girls have the same tastes at the margin when they leave my store. Thus, they would have to be at indifference curves that are tangent to the same budget line (because their budget goes through the same endowment bundle and has the same slope). Since Jenny likes cars more than Ellie does at their endowment points, this implies that Jenny will end up selling princess toys and buying cars while Ellie will sell car toys and buy princess toys. For the allocation of toys to be efficient, the price of cars will have to be set so that the number of cars Ellie wants to sell is exactly equal to the number of cars that Jenny wants to buy, and the number of princess toys Ellie wants to buy is exactly equal to the number of princess toys Jenny wants to sell. Thus, the arrows on each axis in panel (c) of the graph have to be the same size.

(e) *What values might the price for toy cars take to achieve the efficient trades you described in your answer to (d)?*

Answer: We concluded in (a) that mutually beneficial trades had to have terms of trades under which x princess toys are traded for 1 car, with x falling between 1 and 2. The price of toy cars must therefore be between 1 and 2 times the price of princess toys, allowing consumers to buy between 1 and 2 times as many princess toys as toy cars with any given dollar amount. Since the price of princess toys is fixed at \$1, this implies that the price of cars must lie between \$1 and \$2. You can see from panel (b) that the price of cars can't possibly be lower than \$1 because at a price of \$1 Jenny wants to buy cars and sell princess toys but Ellie is willing to do neither. Thus, the price of cars has to go up in order to induce Ellie to be willing to sell cars and to induce Jenny to demand fewer cars. At the same time, we could similarly show that the price can't be higher than \$2 — because at a price of \$2, Jenny would no longer want to trade but Ellie would definitely want to sell cars for princess toys. Depending on exactly what the indifference maps look like, some price between \$1 and \$2 will therefore be just right.

B: *Now suppose that my girls' tastes could be described by the utility function* $u(x_1,x_2)=x_1^{\alpha}x_2^{(1-\alpha)}$*, where* x_1 *represents toy cars,* x_2 *represents princess toys and* $0<\alpha<1$.

(a) *What must be the value of* α *for Ellie (given the information in part A)? What must the value be for Jenny?*

Answer: The MRS for this utility function is

$$MRS=-\frac{\partial u/\partial x_1}{\partial u/\partial x_2}=-\frac{\alpha x_1^{\alpha-1}x_2^{1-\alpha}}{(1-\alpha)x_1^{\alpha}x_2^{-\alpha}}=-\frac{\alpha x_2}{(1-\alpha)x_1}. \tag{6.1.i}$$

At the bundle (10,10), Ellie's MRS is -1 — which implies that $\alpha=0.5$ for Ellie. Similarly, for Jenny the MRS is -2 at the bundle (10,10) — which implies that $\alpha/(1-\alpha)=2$ or $\alpha=2/3$ for Jenny.

(b) *When I set all toy prices to \$1, what exactly will Ellie do? What will Jenny do?*

Answer: We can solve the general optimization problem in terms of α by writing the Lagrange function as

$$\mathcal{L}(x_1,x_2,\lambda)=x_1^{\alpha}x_2^{(1-\alpha)}+\lambda(20-x_1-x_2), \tag{6.1.ii}$$

where the 20 in the parentheses following λ is simply the value of the endowment of 10 car toys and 10 princess toys when the price of each is set to 1. The first two first order conditions of this problem are

$$\begin{aligned}\alpha x_1^{\alpha-1}x_2^{1-\alpha}&=\lambda\\(1-\alpha)x_1^{\alpha}x_2^{-\alpha}&=\lambda.\end{aligned} \tag{6.1.iii}$$

Since the right hand side of each of these is equal to λ, we can just set the left hand sides equal to each other and solve for x_2 to get

$$x_2=\frac{(1-\alpha)}{\alpha}x_1. \tag{6.1.iv}$$

Plugging this into the budget constraint $20=x_1+x_2$, we can solve for x_1 to get $x_1=20\alpha$. Plugging this back into equation (6.1.iv), we can also get $x_2=20(1-\alpha)$.

Since $\alpha=0.5$ for Ellie, this implies Ellie's optimal bundle is $(x_1,x_2)=(10,10)$ — i.e. Ellie will not trade. Since $\alpha=2/3$ for Jenny, it means Jenny's optimal bundle is $(x_1,x_2)=(13.33,6.67)$. Jenny will therefore want to trade 3.33 princess toys for 3.33 toy cars.

(c) *Given that I am fixing the price of princess toys at \$1, do I have to raise or lower the price of car toys in order for me to operate a store in which I don't keep inventory but simply facilitate trades between the girls?*

Answer: As we already concluded in part A(e), I will have to raise the price of cars to somewhere between \$1 and \$2.

(d) *Suppose I raise the price of car toys to \$1.40, and assume that it is possible to sell fractions of toys. Have I found a set of prices that allow me to keep no inventory?*

Answer: The Lagrange function written in terms of α is then

$$\mathscr{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{(1-\alpha)} + \lambda(24 - 1.4x_1 - x_2), \tag{6.1.v}$$

where 24 is the value of the endowment (10,10). The first two first order conditions are

$$\begin{aligned} \alpha x_1^{\alpha-1} x_2^{1-\alpha} &= 1.4\lambda \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} &= \lambda. \end{aligned} \tag{6.1.vi}$$

Dividing the first by the second (and thus canceling λ), we can solve for x_2 in terms of x_1 to get

$$x_2 = \frac{1.4(1-\alpha)}{\alpha} x_1. \tag{6.1.vii}$$

Substituting this into the budget constraint and solving for x_1, we get $x_1 = (24/1.4)\alpha = 17.143\alpha$. Plugging this back into equation (6.1.vii) and solving for x_2, we get $x_2 = 24(1-\alpha)$.

For Ellie, $\alpha = 0.5$ — which implies her optimal bundle will be (8.571,12). Thus, she wants to give up 1.429 of x_1 in exchange for receiving 2 of x_2. For Jenny, $\alpha = 2/3$ — which implies her optimal bundle will be (11.429,8). Jenny therefore wants to get 1.429 of x_1 in exchange for giving up 2 of x_2. The trades exactly offset each other — thus I have to keep no inventory at these prices. I am simply facilitating efficient trade between Ellie and Jenny by setting the price of cars equal to \$1.40 (while setting the price of princess toys to \$1.00.

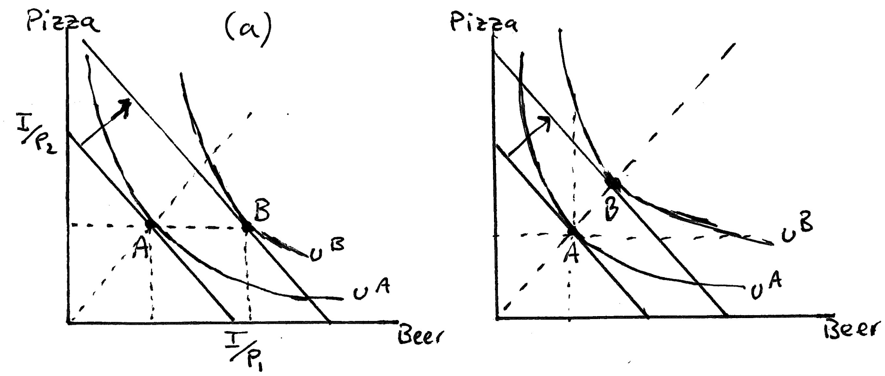
Exercise 6.3

Pizza and Beer: Sometimes we can infer something about tastes from observing only two choices under two different economic circumstances.

A: *Suppose we consume only beer and pizza (sold at prices p_1 and p_2 respectively) with an exogenously set income I.*

(a) *With the number of beers on the horizontal axis and the number of pizzas on the vertical, illustrate a budget constraint (clearly labeling intercepts and the slope) and some initial optimal (interior) bundle A.*

Answer: Panel (a) of Exercise Graph 6.3 illustrates the original budget line containing the optimal bundle A.



Exercise Graph 6.3 : Beer and Pizza

(b) *When your income goes up, I notice that you consume more beer and the same amount of pizza. Can you tell whether my tastes might be homothetic? Can you tell whether they might be quasilinear in either pizza or beer?*

Answer: The shift in income is also indicated in panel (a), with the new optimal bundle B containing more beer but the same amount of pizza. Since the two indifference curves have the same MRS along the horizontal line that holds pizza fixed at its original quantity, the tastes might indeed be quasilinear in pizza. But the tastes could not be homothetic — because, on the ray that passes through A from the origin, the MRS is greater in absolute value along the higher indifference curve than along the lower. The only way this would not be the case is if pizza and beer were perfect substitutes *and* the price of pizza is the same as the price of beer. In that case, all points on both budgets are optimal — including A initially and B after the income change. This would be the one case where tastes are both quasilinear and homothetic.

(c) *How would your answers change if I had observed you decreasing your beer consumption when income goes up?*

Answer: If I simply would have observed a decrease in your beer consumption, I could say for sure that your tastes are not quasilinear in beer (unless beer and pizza are perfect substitutes and prices happen to be such that the slopes of the budget constraints are equal to the MRS everywhere). I could similarly conclude that your tastes are not quasilinear in pizza — because, if you consume less beer with an increase in income, you must be consuming more pizza (if pizza and beer is all you consume). Finally, I could also say for sure that your tastes are not homothetic — because under homothetic tastes, consumption of all goods goes up with increases in income. Again, the one exception is the case where pizza and beer are perfect substitutes with MRS equal to the slopes of the budgets. In that case, we would again have tastes that are both quasilinear and homothetic.

(d) *How would your answers change if both beer and pizza consumption increased by the same proportion as income?*

Answer: This case is graphed in the second panel of Exercise Graph 6.3. The original bundle A and the new optimal bundle B lie on the same ray from the origin — with the indifference curves at both bundles tangent to the same slope. Thus, along this ray, the two indifference curves we know about have the same slope — which is consistent with tastes being homothetic. But the vertical and horizontal lines through A will contain bundles along u^B where the MRS differs from that at A — which implies that the tastes are not quasilinear, at least so long as we rule out the special case that the goods are perfect substitutes and the ratio of prices happens to be such that the budget lines have the same slope as the indifference curves everywhere.

B: *Suppose your tastes over beer (x_1) and pizza (x_2) can be summarize by the utility function $u(x_1,x_2) = x_1^2 x_2$ and that p_1=2, p_2=10 and weekly income I=180.*

(a) *Calculate your optimal bundle A of weekly beer and pizza consumption by simply using the fact that, at any interior solution, $MRS = -p_1/p_2$.*

Answer: Using the fact that we know $MRS = -p_1/p_2 = -2/10 = -1/5$ at the optimum, we can write

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{2x_1 x_2}{x_1^2} = -\frac{2x_2}{x_1} = -\frac{1}{5}, \tag{6.3.i}$$

and the last equality can be written as $x_2 = x_1/10$. Plugging this into the budget constraint $180 = 2x_1 + 10x_2$, we get

$$180 = 2x_1 + 10\frac{x_1}{10} = 3x_1, \tag{6.3.ii}$$

which solves to $x_1 = 60$. Plugging this back into $x_2 = x_1/10$, we also get $x_2 = 6$.

(b) *What numerical label does this utility function assign to the indifference curve that contains your optimal bundle?*

Answer: $u(60,6) = (60^2)(6) = 21,600$.

(c) *Set up the more general optimization problem where, instead of using the prices and income given above, you simply use p_1, p_2 and I. Then, derive your optimal consumption of x_1 and x_2 as a function of p_1, p_2 and I.*

Answer: The more general optimization problem is

$$\max_{x_1,x_2} u(x_1,x_2) = x_1^2 x_2 \text{ subject to } p_1x_1 + p_2x_2 = I, \tag{6.3.iii}$$

with corresponding Lagrange function

$$\mathscr{L}(x_1,x_2,\lambda) = x_1^2 x_2 + \lambda(I - p_1x_1 - p_2x_2). \tag{6.3.iv}$$

The first two first order conditions are then

$$\begin{aligned} 2x_1x_2 &= \lambda p_1 \\ x_1^2 &= \lambda p_2. \end{aligned} \tag{6.3.v}$$

Dividing the first by the second equation, we get $2x_2/x_1 = p_1/p_2$ which can be solved for x_2 to get $x_2 = (p_1x_1)/(2p_2)$. Substituting this into the budget constraint $I = p_1x_1 + p_2x_2$ (which is also the third first order condition), we get

$$I = p_1x_1 + p_2\frac{p_1x_1}{2p_2} = \frac{2p_1x_1}{2} + \frac{p_1x_1}{2} = \frac{3p_1x_1}{2}, \tag{6.3.vi}$$

and this can be solved for x_1 as $x_1 = 2I/(3p_1)$. Plugging this back into the expression $2x_2/x_1 = p_1/p_2$, we can then solve for x_2 as $x_2 = I/(3p_2)$.

(d) *Plug the values p_1=2, p_2=10 and I=180 into your answer to B(c) and verify that you get the same result you originally calculated in B(a).*

Answer: Our solution so far was $x_1 = 2I/(3p_1)$ and $x_2 = I/(3p_2)$. Plugging in the specific values for prices and income, we therefore get $x_1 = 2(180)/(3(2)) = 360/6 = 60$ and $x_2 = 180/(3(10)) = 180/30 = 6$ — 60 beers and 6 pizzas just as we concluded in B(a).

(e) *Using your answer to part B(c), verify that your tastes are homothetic.*

Answer: You can tell how consumption of each good changes with income by taking the derivative of $x_1 = 2I/(3p_1)$ and $x_2 = I/(3p_2)$ with respect to I. This gives

$$\frac{\partial x_1}{\partial I} = \frac{2}{3p_1} \text{ and } \frac{\partial x_2}{\partial I} = \frac{1}{3p_2}. \tag{6.3.vii}$$

Thus, as income increases, consumption of both goods increases linearly. Put differently, as income doubles, consumption of both goods doubles. This is true only for homothetic tastes where the MRS is the same along rays from the origin — which implies that optimal bundles lie on rays from the origin as income changes.

(f) *Which of the scenarios in A(b) through (d) could be generated by the utility function $u(x_1,x_2) = x_1^2x_2$?*

Answer: Only the last scenario in A(d) could be generated by this utility function since we know it represents homothetic tastes. The scenario in A(b) has tastes that are quasilinear in pizza, while the scenario in A(c) has beer consumption decreasing with an increase in income (which is inconsistent with what we derived before).

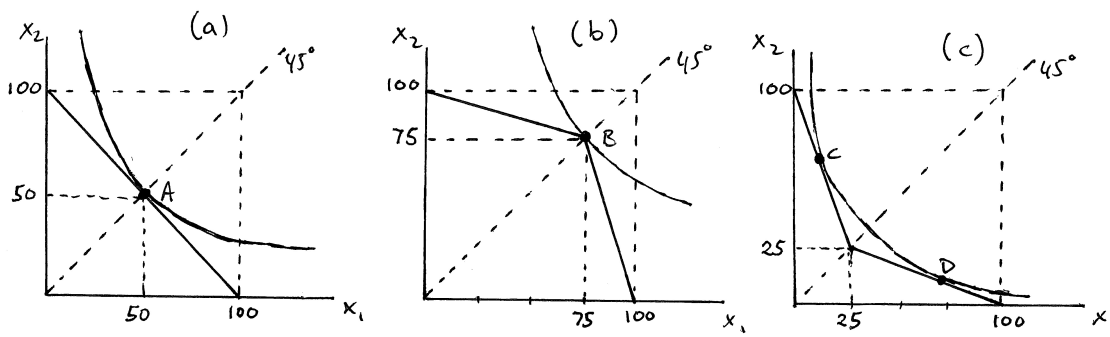
Exercise 6.5

Suppose you have an income of \$100 to spend on goods x_1 and x_2.

A: *Suppose that you have homothetic tastes that happen to have the special property that indifference curves on one side of the 45 degree line are mirror images of indifference curves on the other side of the 45 degree line.*

(a) *Illustrate your optimal consumption bundle graphically when $p_1 = 1 = p_2$.*

Answer: Panel (a) of Exercise Graph 6.5(1) illustrates the budget line in this case. Symmetry around the 45-degree line implies that the slope of indifference curves on the 45 degree line must be -1. Since the budget constraint in this case also has slope -1, the optimum must occur on the 45 degree line. This is indicated as point A in the graph.



Exercise Graph 6.5(1) : Homothetic Tastes and Optimization

(b) *Now suppose the price of the first 75 units of x_1 you buy is 1/3 while the price for any additional units beyond that is 3. The price of x_2 remains at 1 throughout. Illustrate your new budget and optimal bundle.*

Answer: This implies that the first 75 units of x_1 cost \$25, leaving you with \$75 to spend on x_2. The kink point therefore happens at the bundle (75,75). Since the price of x_1 is 3 from then on, you can buy at most 25 more units with the \$75 you have left after buying the first 75 units of x_1. The budget constraint therefore looks as it does in panel (b) of the graph. The symmetry of the indifference curves then still implies that the optimum happens on the 45 degree line at the kink point B.

(c) *Suppose instead that the price for the first 25 units of x_1 is 3 but then falls to 1/3 for all units beyond 25 (with the price of x_2 still at 1). Illustrate this budget constraint and indicate what would be optimal.*

Answer: After buying 25 units of x_1 at \$3 per unit, you have only \$25 left. Thus, the new kink point happens at (25,25). Since the resulting budget line (graphed in panel (c)) is symmetric around the 45 degree line, the symmetry of the indifference curves implies that there will be two optimal bundles (indicated by C and D). These may happen anywhere along the budget line depending on how substitutable the two goods are for one another. If the indifference curves themselves are kinked at the 45-degree line, it may even be the case that $C = D$ so long as the kink is more severe than the kink of the budget constraint (as would be the case for perfect complements).

(d) *If the homothetic tastes did not have the symmetry property, which of your answers might not change?*

Answer: Without the symmetry property, the optimal bundle in (a) would be to the left or right of the 45 degree line, and there would not be two optimal bundles at symmetric distances from the 45 degree line in panel (c). (There might still be two optimal bundles, or there might only be one.) But in panel (b), the optimum might well still occur at the kink point because many different marginal rates of substitution can be "tangent" at that kink.

B: *Suppose that your tastes can be summarized by the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$.*

(a) *Does this utility function represent tastes that have the symmetry property described in A?*

Answer: Yes. The MRS for this utility function is $-x_2/x_1$ — which is equal to -1 when $x_1 = x_2$ on the 45 degree line. We can furthermore see that the symmetry holds — if we place x_1 on the vertical instead of the horizontal axis, the MRS simply switches to $-x_1/x_2$ and thus retains the same shape as before.

(b) *Calculate the optimal consumption bundle when $p_1 = 1 = p_2$.*

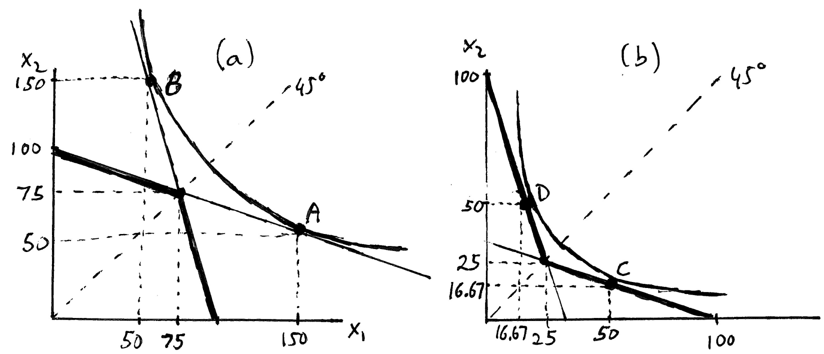
Answer: The optimum occurs where $MRS = -p_1/p_2$ which is $-x_2/x_1 = -1$. Solving for x_2 we get $x_2 = x_1$, and plugging this into the budget constraint, we get $x_1 + x_2 = x_1 + x_1 = 2x_1 = 100$ or $x_1 = 50$ (which then also implies $x_2 = 50$).

(c) *Derive the two equations that make up the budget constraint you drew in part A(b) and use the method described in the appendix to this chapter to calculate the optimal bundle under that budget constraint.*

Answer: The first segment of the budget constraint is $x_2 = 100 - (1/3)x_1$ and the second line segment is $x_2 = 300 - 3x_1$. Optimal tangencies occur where $MRS = -x_2/x_1 = -p_1/p_2$, which implies $x_2 = (p_1 x_1)/p_2$ or $x_2 = p_1 x_1$ since $p_2 = 1$.

Along the first line segment, $p_1 = 1/3$. Substituting $x_2 = p_1 x_1 = (1/3)x_1$ into $x_2 = 100 - (1/3)x_1$, we get $(1/3)x_1 = 100 - (1/3)x_1$ or $(2/3)x_1 = 100$. Solving for x_1, we get $x_1 = 150$ which lies on the portion of the budget line that is not truly part of the kinked budget. This is illustrated as bundle A in panel (a) of Exercise Graph 6.5(2).

Along the second line segment, $p_1 = 3$. Substituting $x_2 = p_1 x_1 = 3x_1$ into $x_2 = 300 - 3x_1$, we get $3x_1 = 300 - 3x_1$ or $6x_1 = 300$. Solving for x_1, we get $x_1 = 50$ which also lies on the portion of the budget line that is not truly part of the kinked budget. This is illustrated as bundle B in panel (a) of Exercise Graph 6.5(2). Note that, due to the symmetry of the indifference curves, bundles A and B lie on the same indifference curve.



Exercise Graph 6.5(2) : Homothetic Tastes and Optimization: Part 2

Since both optimization problems — i.e. the problems using both of the extended line segments as budgets — result in solutions outside the actual kinked budget, the actual optimum lies at the kink point.

(d) *Repeat for the budget constraint you drew in A(c).*

Answer: The first segment of the budget constraint is now $x_2 = 100 - 3x_1$ and the second line segment is $x_2 = 33.33 - (1/3)x_1$. Optimal tangencies occur again where $MRS = -x_2/x_1 = -p_1/p_2$, which implies $x_2 = (p_1x_1)/p_2$ or $x_2 = p_1x_1$ since $p_2 = 1$.

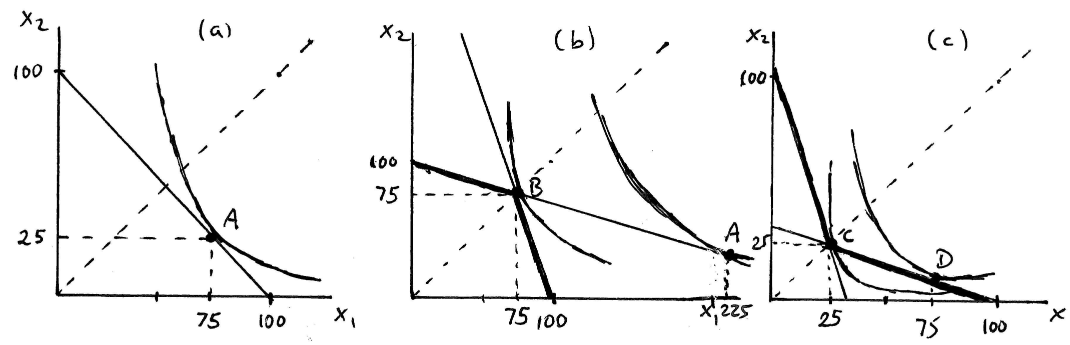
Along the first line segment, $p_1 = 3$. Substituting $x_2 = p_1x_1 = 3x_1$ into $x_2 = 100 - 3x_1$, we get $3x_1 = 100 - 3x_1$ or $6x_1 = 100$. Solving for x_1, we get $x_1 = (100/6) = 16.67$ which lies on the portion of the budget line that is in fact part of the kinked budget. This is illustrated as bundle C in panel (b) of Exercise Graph 6.5(2).

Along the second line segment, $p_1 = (1/3)$. Substituting $x_2 = p_1x_1 = (1/3)x_1$ into $x_2 = 33.33 - (1/3)x_1$, we get $(1/3)x_1 = 33.33 - (1/3)x_1$ or $(2/3)x_1 = 33.33$. Solving for x_1, we get $x_1 = 50$ which also lies on the portion of the budget line that is in fact part of the kinked budget. This is illustrated as bundle D in panel (b) of Exercise Graph 6.5(2). Note again that, due to the symmetry of the indifference curves, bundles C and D lie on the same indifference curve. Both of these bundles are therefore optimal.

(e) *Repeat (b) through (d) assuming instead $u(x_1, x_2) = x_1^{3/4}x_2^{1/4}$ and illustrate your answers in graphs.*

Answer: The MRS for this function is $MRS = -3x_2/x_1$. Thus, optimal solutions occur at $MRS = -3x_2/x_1 = -p_1/p_2$ or, equivalently, where $x_2 = p_1x_1/3p_2$ which can furthermore be simplified to $x_2 = p_1x_1/3$ since $p_2 = 1$.

When $p_1 = p_2 = 1$ as in part (b), our optimality condition reduces to $x_2 = x_1/3$. Putting this into the budget constraint, we get $x_1 + x_2 = x_1 + (x_1/3) = 100$ or $(4/3)x_1 = 100$. Solving for x_1 we then get $x_1 = 75$ which implies $x_2 = 25$. This is graphed as A in panel (a) of Exercise Graph 6.5(3).



Exercise Graph 6.5(3) : Homothetic Tastes and Optimization: Part 3

In the scenario of part (c), the first segment of the budget constraint is $x_2 = 100 - (1/3)x_1$ and the second line segment is $x_2 = 300 - 3x_1$. Substituting our optimality condition $x_2 = p_1 x_1/3$ into the first equation and letting $p_1 = 1/3$, we get $x_2 = (1/3)(x_1/3) = 100 - (1/3)x_1$ or $(1/9)x_1 = 100 - (1/3)x_1$ which solves to $x_1 = 225$ which is clearly outside the actual kinked budget and is illustrated as A in panel (b) of Exercise Graph 6.5(3). Similarly, substituting our optimality condition $x_2 = p_1 x_1/3$ into the second equation and letting $p_1 = 3$, we get $x_2 = 3x_1/3 = 300 - 3x_1$ or $x_1 = 300 - 3x_1$. Solving for x_1, we get $x_1 = 75$. This is exactly the kink point — and is therefore the optimal solution, illustrated as B in panel (b) of Exercise Graph 6.5(3).

In the scenario of part (d), the first segment of the budget constraint is $x_2 = 100 - 3x_1$ and the second line segment is $x_2 = 33.33 - (1/3)x_1$. Substituting our optimality condition $x_2 = p_1 x_1/3$ into the first equation and letting $p_1 = 3$, we get $x_2 = 3(x_1/3) = 100 - 3x_1$ or $x_1 = 100 - 3x_1$ which solves to $x_1 = 25$. This happens right at the kink point — which means it could not possibly be an optimum since the indifference curve cuts the other part of the budget constraint. This is illustrated in panel (c) of Exercise Graph 6.5(3) where the kink point is denoted C. Substituting our optimality condition $x_2 = p_1 x_1/3$ into the second equation and letting $p_1 = 1/3$, we get $x_2 = (1/3)x_1/3 = 33.33 - (1/3)x_1$ or $(1/9)x_1 = 33.33 - (1/3)x_1$. Solving for x_1, we get $x_1 = 75$. This, illustrated as D in panel (c) of Exercise Graph 6.5(3), is in fact on the actual kinked budget and is therefore the optimal bundle.

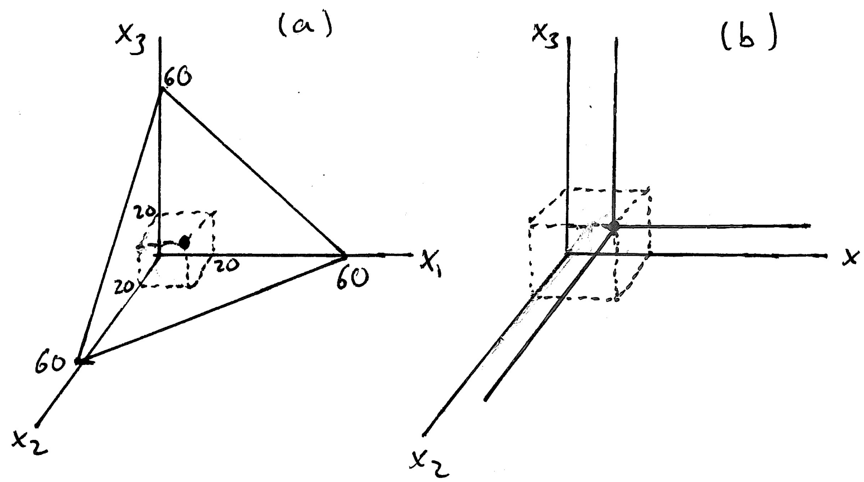
Exercise 6.7

Coffee, Milk and Sugar: Suppose there are three different goods: cups of coffee (x_1), ounces of milk (x_2) and packets of sugar (x_3).

A: *Suppose each of these goods costs 25 cents and you have an exogenous income of $15.*

(a) *Illustrate your budget constraint in three dimensions and carefully label all intercepts.*

Answer: At 25 cents a piece, I can buy as much as 60 of any one of the goods with $15 assuming I don't buy anything else. Thus, panel (a) in Exercise Graph 6.7 has intercept of 60 on each axis (which makes all the slopes equal to -1.)



Exercise Graph 6.7 : Coffee, Milk and Sugar

(b) *Suppose that the only way you get enjoyment from a cup of coffee is to have at least one ounce of milk and one packet of sugar in the coffee, the only way you get enjoyment from an ounce of milk is to have at least one cup of coffee and one packet of sugar, and the only way you get enjoyment from a packet of sugar is to have at least one cup of coffee and one ounce of milk. What is the optimal consumption bundle on your budget constraint.*

Answer: The three goods are therefore perfect complements. This would mean that you would want to consume equal amounts of all three goods — which, given the prices and income, you would do when $x_1 = x_2 = x_3 = 20$. Put differently, you would want to consume 20 perfectly balanced cups of coffee (with an ounce of milk and a packet of sugar in each).

(c) *What does your optimal indifference curve look like?*

Answer: The indifference curve has a corner along the ray from the origin on which all goods are represented in identical quantities. The rest of the indifference "curve" is composed of planes parallel to each of the planes formed by the axes in the graph but ending at the corner. Panel (b) of Exercise Graph 6.7 is an attempt to graph this. Essentially, the indifference "curve" is like three sides of a box with the corner of the box pointing toward the origin and located along the ray that holds all goods equal to one another.

(d) *If your income falls to $10 — what will be your optimal consumption bundle?*

Answer: You would still want to consume the three goods in equal amounts — which means now you could consume 2/3 of what you did before. Before, you were able to consume 20 cups of coffee (with milk and sugar). Now you can only consume 40/3=13.33 cups (with milk and sugar).

(e) *If instead of a drop in income the price of coffee goes to 50 cents, how does your optimal bundle change?*

Answer: Because the goods are perfect complements, it would still need to be the case that you buy the same quantity of each of the goods. Thus, $0.5x_1 + 0.25x_2 + 0.25x_3 = 15$ but $x_1 = x_2 = x_3$ at any optimum. Thus, letting x denote the quantity of each of the goods, $0.5x + 0.25x + 0.25x = 15$ or $x = 15$. Thus, you would drink 15 cups of coffee with milk and sugar.

(f) *Suppose your tastes are less extreme and you are willing to substitute some coffee for milk, some milk for sugar and some sugar for coffee. Suppose that the optimal consumption bundle you identified in (b) is still optimal under these less extreme tastes. Can you picture what the optimal indifference curve might look like in your picture of the budget constraint?*

Answer: The indifference "curve" would still point toward the origin but would now be more "bowl-shaped" rather than "box-shaped" since the corner on the indifference curve would become smooth.

(g) *If tastes are still homothetic (but of the less extreme variety discussed in (f)), would your answers to (d) or (e) change?*

Answer: If 20 cups of coffee with 20 ounces of milk and 20 packets of sugar is optimal under the original income of $15, and if tastes are homothetic, then the ratio of the goods will remains the same if income changes. Thus, the answer to (d) does not change — you would consume 13.33 cups of coffee with as many sugars and ounces of milk when income falls to $10. But when opportunity costs change — as in (e) where the price of a cup of coffee doubles, you will now substitute away from coffee and toward milk and sugar. Thus, you would drink fewer cups of coffee than we concluded in (e), but the coffee would be lighter (because of more milk) and sweeter (because of more sugar).

B: *Continue with the assumption of an income of $15 and prices for coffee, milk and sugar of 25 cents each.*

(a) *Write down the budget constraint.*

Answer: $0.25x_1 + 0.25x_2 + 0.25x_3 = 15$.

(b) *Write down a utility function that represents the tastes described in A(b).*
Answer: $u(x_1, x_2, x_3) = \min\{x_1, x_2, x_3\}$.

(c) *Suppose that instead your tastes are less extreme and can be represented by the utility function $u(x_1, x_2, x_3) = x_1^{\alpha} x_2^{\beta} x_3$. Calculate your optimal consumption of x_1, x_2 and x_3 when your economic circumstances are described by the prices p_1, p_2 and p_3 and income is given by I.*
Answer: It becomes notationally a bit easier to just take the log of the utility function before doing this problem. Thus, we can use the function $v(x_1, x_2, x_3) = \alpha \ln x_1 + \beta \ln x_2 + \ln x_3$. This gives us an optimization problem that can be written as

$$\max_{x_1, x_2, x_3} \alpha \ln x_1 + \beta \ln x_2 + \ln x_3 \text{ subject to } p_1 x_1 + p_2 x_2 + p_3 x_3 = I. \tag{6.7.i}$$

The Lagrange function for this problem is

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = \alpha \ln x_1 + \beta \ln x_2 + \ln x_3 + \lambda(I - p_1 x_1 + p_2 x_2 + p_3 x_3), \tag{6.7.ii}$$

which gives us first order conditions of

$$\begin{aligned} \frac{\alpha}{x_1} &= \lambda p_1 \\ \frac{\beta}{x_2} &= \lambda p_2 \\ \frac{1}{x_3} &= \lambda p_3 \\ p_1 x_1 + p_2 x_2 + p_3 x_3 &= I. \end{aligned} \tag{6.7.iii}$$

Solving the third equation for λ and substituting this into the first and second equations, we can solve for x_1 and x_2 to get

$$x_1 = \frac{\alpha p_3 x_3}{p_1} \text{ and } x_2 = \frac{\beta p_3 x_3}{p_2}. \tag{6.7.iv}$$

We can then substitute these into the final first order condition (which is equal to the budget constraint) to get

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = p_1 \frac{\alpha p_3 x_3}{p_1} + p_2 \frac{\beta p_3 x_3}{p_2} + p_3 x_3 = (\alpha + \beta + 1) p_3 x_3 = I. \tag{6.7.v}$$

Solving for x_3, and then using this to plug into equations (6.7.iv), gives

$$x_1 = \frac{\alpha I}{(\alpha + \beta + 1) p_1}, \; x_2 = \frac{\beta I}{(\alpha + \beta + 1) p_2} \text{ and } x_3 = \frac{I}{(\alpha + \beta + 1) p_3}. \tag{6.7.vi}$$

(d) *What values must α and β take in order for the optimum you identified in A(b) to remain the optimum under these less extreme tastes?*

Answer: In A(b), $p_1 = p_2 = p_3 = 0.25$ and $I = 15$. Thus, the solutions in (6.7.vi) become

$$x_1 = \frac{60\alpha}{(\alpha+\beta+1)}, \ x_2 = \frac{60\beta}{(\alpha+\beta+1)} \text{ and } x_3 = \frac{60}{(\alpha+\beta+1)}. \quad (6.7.vii)$$

In order for the solution to be $x_1 = x_2 = x_3 = 20$ as in A(b), this implies that $\alpha = \beta = 1$.

(e) *Suppose α and β are as you concluded in part B(d). How does your optimal consumption bundle under these less extreme tastes change if income falls to \$10 or if the price of coffee increases to 50 cents? Compare your answers to your answer for the more extreme tastes in A(d) and (e).*

Answer: Using $\alpha = \beta = 1$ as we have just concluded, the expressions become $x_1 = I/(3p_1)$, $x_2 = I/(3p_2)$ and $x_3 = I/(3p_3)$. Substituting $p_1 = p_2 = p_3 = 0.25$ and $I = 10$, we get $x_1 = x_2 = x_3 = 13.33$ which is identical to what we concluded in A(d) under the more extreme tastes. Substituting $p_1 = 0.50$, $p_2 = p_3 = 0.25$ and $I = 15$, on the other hand, we get $x_1 = 10$, $x_2 = 20$ and $x_3 = 20$. This differs from the answer in A(e) where no substitutability between the goods was permitted — now you end up drinking less coffee but with more milk and sugar in each cup.

(f) True or False*: Just as the usual shapes of indifference curves represent two dimensional "slices" of a 3-dimensional utility function, 3-dimensional "indifference bowls" emerge when there are three goods — and these "bowls" represent slices of a 4-dimensional utility function.*

Answer: This is true. The utility function with three goods can be plotted in 4 dimensions — one for each good and one to indicate the utility level of each bundle — but the indifference "curves" hold utility fixed and can therefore be represented in 3 dimensions. This is analogous to slicing a 3 dimensional utility function with two goods to get two dimensional indifference curves.

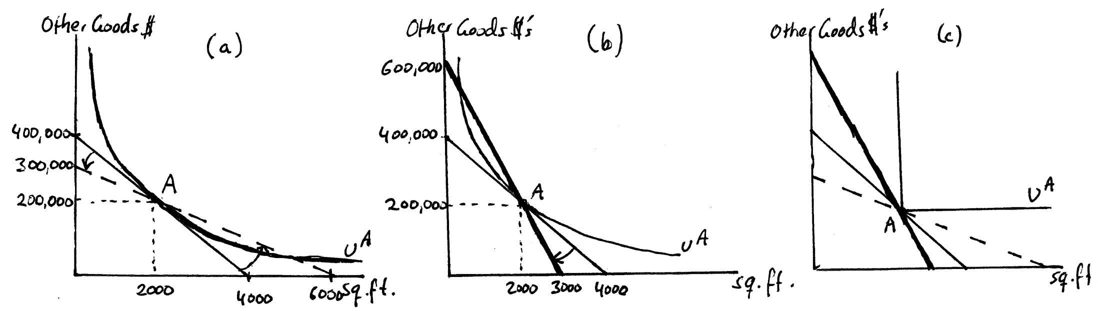
Exercise 6.9

Everyday Application: *Price Fluctuations in the Housing Market: Suppose you have \$400,000 to spend on a house and "other goods" (denominated in dollars).*

A: *The price of 1 square foot of housing is \$100 and you choose to purchase your optimally sized house at 2000 square feet. Assume throughout that you spend money on housing solely for its consumption value (and not as part of your investment strategy).*

(a) *On a graph with "square feet of housing" on the horizontal axis and "other goods" on the vertical, illustrate your budget constraint and your optimal bundle A.*

Answer: The budget constraint would have vertical intercept of \$400,000 (since this is how much other goods you can consume if you buy no housing) and horizontal intercept of 4,000 square feet of housing (since that is how much you can afford at \$100 per square foot if you spend all your money on housing.) The slope of this budget is −100. The budget is depicted as the solid line in panel (a) of Exercise Graph 6.9.



Exercise Graph 6.9 : Housing Price Fluctuations

(b) *After you bought the house, the price of housing falls to \$50 per square foot. Given that you can sell your house from bundle A if you want to, are you better or worse off?*

Answer: The (dashed) new budget line is also drawn in panel (a) of the graph. Note that it has to go through A because A is your endowment point once you have bought the 2000 square foot house. Thus, you can always choose to consume that bundle regardless of what happens to prices. But you can also sell your 2000 square foot house for \$100,000 — which would give you \$300,000 in consumption, your new vertical intercept. Or you can take that \$300,000 and spend it on a new house and thereby buy as much as a 6,000 square foot house since housing now only costs \$50 per square foot. Since your indifference curve at A is tangent to your original budget line, the new (shallower) budget line cuts that indifference curve from below at bundle A. All the new bundles that are now affordable and that lie above the original indifference curve u^A therefore lie to the right of A. You are better off at any of those bundles on the dashed line that lie above the indifference curve u^A.

(c) *Assuming you can easily buy and sell houses, will you now buy a different house? If so, is your new house smaller or larger than your initial house?*

Answer: You will buy a larger house — since all the better bundles on the dashed line in panel (a) are to the right of A and therefore include a house larger than 2000 square feet.

(d) *Does your answer to (c) differ depending on whether you assume tastes are quasilinear in housing or homothetic?*

Answer: No — in both cases you would end up better off consuming a larger house.

(e) *How does your answer to (c) change if the price of housing went up to \$200 per square foot rather than down to \$50.*

Answer: Panel (b) of Exercise Graph 6.9 illustrates this change in prices. The original budget constraint (from \$400,000 on the vertical to 4,000 square feet on the horizontal axis) with bundle A is replicated from panel (a) and illustrates the budget when the price per square foot of housing is \$100. The steeper bold line going through A illustrates the new budget line when A is the endowment point and the price of housing goes to \$200 per square foot. If you sell your 2000 square foot house at \$200 per square foot, you would get \$400,000 for it — which, added to the \$200,000 you have would give you as much as \$600,000 in consumption if you choose not to buy another house. If you do buy another house, the largest possible house at the new prices is now a 3000 square foot house. But you can always choose to stay at A — so A too is on the new budget line. The bundles on the new bold budget that also lie above the indifference curve u^A all lie to the left of A — indicating that the new house that you would purchase would be smaller than your original 2000 square foot house.

(f) *What form would tastes have to take in order for you to not sell your \$2000 square foot house when the price per square foot goes up or down?*

Answer: The indifference curve through A would have to have a kink in it, as would be the case if housing and other goods are perfect complements. This is illustrated in panel (c) of Exercise Graph 6.9 where all three budget lines are drawn, as is an indifference curve u^A that treats the two goods as perfect complements. Technically, it could also be the case that the indifference curve through A has a less severe kink at A — one where the slope to the left of A is steeper than the bold budget line and the slope to the right of A is shallower than the slope of the dashed budget line. What is important is that there is a sufficiently severe kink — with no substitutability on the margin between the goods at the kink point. If there is no kink at A — i.e. if there is any substitutability at the margin between housing and other goods at A — then the bold and dashed indifference curves must necessarily cut the indifference curve at A in the ways (though not necessarily with the magnitudes) illustrated in (a) and (b).

(g) True or False*: So long as housing and other consumption is at least somewhat substitutable, any change in the price per square foot of housing makes homeowners better off (assuming it is easy to buy and sell houses.)*

Answer: This is true, as just argued in the previous answer.

(h) True or False*: Renters are always better off when the rental price of housing goes down and worse off when it goes up.*

Answer: This is true. Renters do not have endowment points in this model as homeowners do. So changes in the rental price of housing rotate the budget line through the vertical intercept — which implies that a drop in housing prices unambiguously expands the budget set at every level of housing and an increase in housing prices unambiguously shrinks the choice set at every level of housing.

B: *Suppose your tastes for "square feet of housing" (x_1) and "other goods" (x_2) can be represented by the utility function $u(x_1,x_2)=x_1x_2$.*

(a) *Calculate your optimal housing consumption as a function of the price of housing (p_1) and your exogenous income I (assuming of course that p_2 is by definition equal to 1.)*

Answer: We want to solve the problem

$$\max_{x_1,x_2} u(x_1,x_2)=x_1x_2 \text{ subject to } p_1x_1+x_2=I. \tag{6.9.i}$$

The Lagrange function for this problem is

$$\mathscr{L}(x_1,x_2,\lambda)=x_1x_2+\lambda(I-p_1x_1-x_2), \tag{6.9.ii}$$

which give us first order conditions

$$\begin{aligned} x_2 &= \lambda p_1 \\ x_1 &= \lambda \\ p_1x_1+x_2 &= I. \end{aligned} \tag{6.9.iii}$$

Substituting the second equation into the first, we get $x_2=x_1p_1$, and substituting this into the last equation, we get $p_1x_1+p_1x_1=I$ or $x_1=I/(2p_1)$. Finally, plugging this back into $x_2=x_1p_1$, we get $x_2=I/2$.

(b) *Using your answer, verify that you will purchase a 2000 square foot house when your income is \$400,000 and the price per square foot is \$100.*

Answer: We just concluded that $x_1=I/(2p_1)$. When $p_1=100$ and $I=400,000$, this implies $x_1=400,000/(2(100))=2000$.

(c) *Now suppose the price of housing falls to \$50 per square foot and you choose to sell your 2000 square foot house. How big a house would you now buy?*

Answer: By selling your 2000 square foot house at \$50 per square foot, you would make \$100,000. Added to the \$200,000 you had left over after you bought your original 2000 square foot house, this gives you a total income of \$300,000. Plugging I=300,000 and $p_1=50$ into our equation for the optimal housing quantity $x_1=I/(2p_1)$, we get x_1=300,000/(2(50))=3000. Thus, you will buy a 3000 square foot house.

(d) *Calculate your utility (as measured by your utility function) at your initial 2000 square foot house and your new utility after you bought your new house? Did the price decline make you better off?*

Answer: Your initial consumption bundle was (2000, 200000). That gives utility

$$u(2000,200000)=2000(200000)=400,000,000. \tag{6.9.iv}$$

When price fell, you end up at the bundle (3000,150000) which gives utility

$$u(3000, 150000) = 3000(150000) = 450,000,000. \tag{6.9.v}$$

Since your utility after the price decline is higher than before, you are better off.

(e) *How would your answers to B(c) and B(d) change if, instead of falling, the price of housing had increased to \$200 per square foot?*

Answer: Again, we have already calculated that $x_1 = I/(2p_1)$ and $x_2 = I/2$. When price increases to \$200 and you already own a 2000 square foot house, you can now sell your house for \$400,000 which, added to the \$200,000 you had left over after buying your original house, gives you up to \$600,000 to spend. Treating this as your new I and plugging in the new housing price $p_1 = 200$, we then get that your new optimal bundle has $x_1 = 600000/(2(200)) = 1500$ and $x_2 = 600000/2 = 300,000$. Thus you will buy a 1500 square foot house and consume \$300,000 in other goods. This gives you utility

$$u(1500, 300000) = 1500(300000) = 450,000,000, \tag{6.9.vi}$$

which is greater than the utility you had originally and equal to the utility you received from the price decrease before. Thus, a price increase to \$200 per square foot makes you better off, exactly as much as a drop in price to \$50 per square foot. You are therefore indifferent between the price increase and the price decrease.

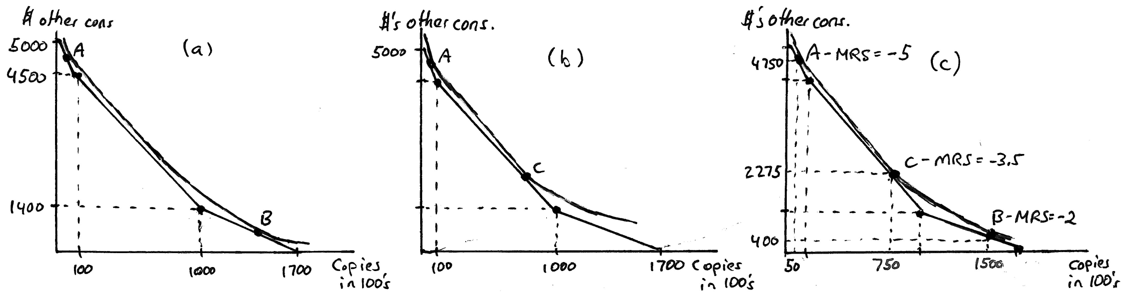
Exercise 6.11

Business Application: *Quantity Discounts and Optimal Choices*: *In end-of-chapter exercise 2.9, you illustrated my department's budget constraint between "pages copied in units of 100" and "dollars spent on other goods" given the quantity discounts our local copy service gives the department. Assume the same budget constraint as the one described in 2.9A.*

A: *In this exercise, assume that my department's tastes do not change with time (or with who happens to be department chair). When we ask below whether someone is "respecting the department's tastes" we mean whether that person is using the department's tastes to make optimal decisions for the department given the circumstances faced by the department. Assume throughout that my department's tastes are convex.*

(a) True or False*: If copies and other expenditures are very substitutable for my department, then you should observe either very little or a great deal of photocopying by our department at the local copy shop.*

Answer: This is true. Panel (a) of Exercise Graph 6.11 illustrates one possibility of this with a single indifference curve tangent at low and high numbers of photocopies (bundles A and B). If the indifference curves have more curvature, then the tangency would lie on the middle portion of the budget constraint with a single optimal quantity that lies in between what one might consider as high and low. It is of course also possible that indifference curves with very little curvature are steeper that the steepest part of the budget — leading to an extreme corner solution on one end of the budget; or that they are very shallow leading to an extreme corner solution on the other end.



Exercise Graph 6.11 : Discounts and Photocopies

(b) *Suppose that I was department chair last year and had approximately 5,000 copies per month made. This year, I am on leave and an interim chair has taken my place. He has chosen to make 150,000 copies per month. Given that our department's tastes are not changing over time, can you say that either I or the current interim chair is not respecting the department's tastes?*

Answer: No, we cannot say that with any certainty. In fact, the indifference curve in panel (a) of Exercise Graph 6.11 illustrates the case where both chairs are respecting the department's tastes despite making very different decision. The reason for this is the non-convexity in the budget set created by the discount policy of the photocopy store.

(c) *Now the interim chair has decided to go on vacation for a month — and an interim interim chair has been named for that month. He has decided to purchase 75,000 copies per month. If I was respecting the department's tastes, is this interim interim chair necessarily violating them?*

Answer: No, not necessarily. Panel (b) of the graph gives and example of an indifference curve that would make both choices, *A* and *C*, optimal from the department's perspective.

(d) *If both I and the initial interim chair were respecting the department's tastes, is the new interim interim chair necessarily violating them?*

Answer: Again, not necessarily. This is illustrated in panel (c) of Exercise Graph 6.11.

B: *Consider the decisions made by the 3 chairs as described above.*

(a) *If I and the second interim chair (i.e. the interim interim chair) both respected the department's tastes, can you approximate the elasticity of substitution of the department's tastes?*

Answer: The elasticity of substitution σ is given by

$$\sigma = \left|\frac{\%\Delta(x_2/x_1)}{\%\Delta MRS}\right| = \left|\left(\frac{(x_2^A/x_1^A)-(x_2^C/x_1^C)}{(x_2^A/x_1^A)}\right)\left(\frac{MRS^A}{MRS^A - MRS^C}\right)\right|, \quad (6.11.i)$$

where bundle A is (50,4750) and C is (750,2275) as depicted in panel (c) of the graph. We furthermore know that $MRS^A = -5$ and $MRS^C = -3.5$. Thus

$$\sigma = \left|\left(\frac{(4750/50)-(2275/750)}{4750/50}\right)\left(\frac{-5}{-5-(-3.5)}\right)\right| = 3.23. \quad (6.11.ii)$$

(b) *If the first and second interim chairs both respected the department's tastes, can you approximate the elasticity of substitution for the department?*

Answer: Now the relevant bundles are C=(750,2275) and B=(1500,400) with $MRS^C = -3.5$ and $MRS^B = -2$, which implies

$$\sigma = \left|\left(\frac{(2275/750)-(400/1500)}{2275/750}\right)\left(\frac{-3.5}{-3.5-(-2)}\right)\right| = 2.13. \quad (6.11.iii)$$

(c) *Could the underlying tastes under which all three chairs respect the department's tastes be represented by a CES utility function?*

Answer: Since we get different elasticity of substitution estimates from the different pairs of choices, tastes that rationalize all three choices given the budget constraint cannot be represented by a *constant* elasticity of substitution utility function that has the same elasticity of substitution everywhere.

Exercise 6.13

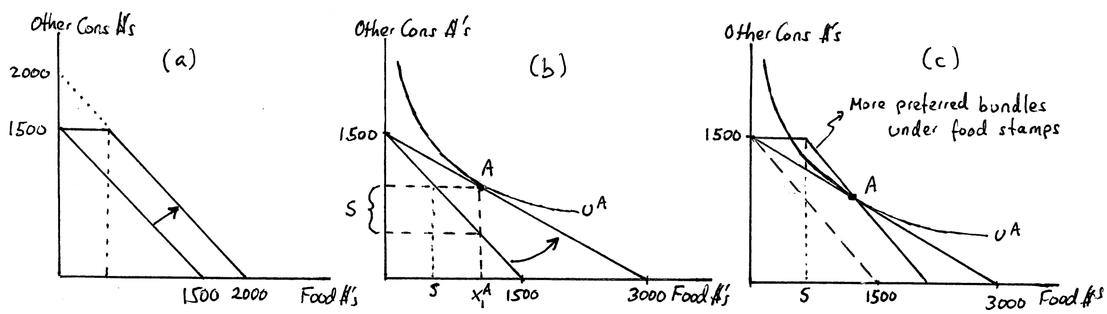
Policy Application: *Food Stamps versus Food Subsidies: In exercise 2.13, you considered the food stamp programs in the US. Under this program, poor households receive a certain quantity of "food stamps" — stamps that contain a dollar value which is accepted like cash for food purchases at grocery stores.*

A: *Consider a household with monthly income of \$1,500 and suppose that this household qualifies for food stamps in the amount of \$500.*

(a) *Illustrate this household's budget, both with and without the food stamp program, with "dollars spent on food" (on the horizontal axis) and "dollars spent on other goods" on the vertical. What has to be true for the household to be just as well off under this food stamp program as it would be if the government simply gave \$500 in cash to the household (instead of food stamps)?*

Answer: Panel (a) of Exercise Graph 6.13 illustrates these two budgets. The budget under food stamps has a flat spot at the top because the first \$500 in food consumption can be paid for through the food stamps but non-food items cannot be paid for with those stamps. As long as the household would have purchased at least \$500 in food under a budget of

$2,000 per month, the food stamp program is exactly like a cash subsidy program for this household. Put differently, so long as the indifference curve tangent to the extended outer budget in panel (a) is tangent at food consumption levels greater than $500, there is no difference between the two types of programs.



Exercise Graph 6.13 : Food Stamps, Cash and Food Subsidies

(b) *Consider the following alternate policy: Instead of food stamps, the government tells this household that it will reimburse 50% of the household's food bills. On a separate graph, illustrate the household's budget (in the absence of food stamps) with and without this alternate program.*

Answer: Panel (b) of the graph illustrates the initial budget (going from $1500 on the vertical axis to $1500 on the horizontal) and the new budget that has shallower slope because $1 of food now only costs 50 cents.

(c) *Choose an optimal bundle A on the alternate program budget line and determine how much the government is paying to this household (as a vertical distance in your graph). Call this amount S.*

Answer: This is also illustrated in panel (b) of the graph. At bundle A, the household is consuming x_1^A in food. We can then read off the vertical axis how much in other consumption the household was able to undertake at A and compare it to how much it would have been able to consume of other goods had it consumed x_1^A in food prior to the subsidy. The difference between these two amounts is S.

(d) *Now suppose the government decided to abolish the program and instead gives the same amount S in food stamps. How does this change the household's budget?*

Answer: This change is illustrated in panel (c) of the graph. In both cases, the bundle A will be available to the consumer because the government is giving S under both programs. However, under the food stamp program, the subsidy amount remains the same regardless of how much food the household consumes, whereas under the price subsidy program the amount of government transfer decreases if the household consumes less food and increases if it consumes more food. Put differently, there is no change in opportunity costs under the cash subsidy, with the price of food going back up to $1 for every $1 of food.

(e) *Will this household be happy about the change from the first alternate program to the food stamp program?*

Answer: The household prefers the food stamps to the price subsidy. You can see this in panel (c) where the indifference curve that makes A optimal under the price subsidy is tangent to the shallower (price subsidy) budget. But this means that the new food stamp budget cuts this indifference curve from above, making a set of new bundles that lie above the indifference curve u^A available to the household.

(f) *If some politicians want to increase food consumption by the poor and others just want to make the poor happier, will they differ on what policy is best?*

Answer: Yes, they will differ. The food price subsidy causes the poor to consume more food whereas the equally costly food stamp program is more preferred by poor households (i.e. makes them happier).

(g) True or False*: The less substitutable food is for other goods, the greater the difference in food consumption between equally funded cash and food subsidy programs.*

Answer: This is false. Imagine making u^A in panel (c) of our graph the shape that presumes food and other goods are perfect complements. In that case, the equally costly food stamp program, which still contains A, will no longer cut the indifference curve u^A — thus eliminating the "better" bundles on the food stamp budget that we identified in panel (c). The household would therefore consume the same amount of food under either program. Then imagine increasing the substitutability between food and other goods at point A — as you do so, more and more "better" bundles become available.

(h) *Consider a third possible alternative — giving cash instead of food stamps.* True or False*: As the food stamp program becomes more generous, the household will at some point prefer a pure cash transfer over an equally costly food stamp program.*

Answer: This is true and relates to our answer to part (a). Since food stamps can only be spent on food, they are equivalent to cash so long as the household would choose to spend at least the value of food stamps on food even if the stamps were replaced by cash. But as the food stamp program becomes more generous, it will at some point be the case that the household would in fact use the food stamps to buy non-food items if it could — and it is at that point that the household would strictly prefer the cash program over the food stamp program.

B: *Suppose this household's tastes for spending on food (x_1) and spending on other goods (x_2) can be characterized by the utility function* $u(x_1, x_2) = \alpha \ln x_1 + \ln x_2$.

(a) *Calculate the level of food and other good purchases as a function of I and the price of food p_1 (leaving the price of dollars on other goods as just 1).*

Answer: We are asked to solve the problem

$$\max_{x_1,x_2} u(x_1,x_2) = \alpha \ln x_1 + \ln x_2 \text{ subject to } p_1x_1 + x_2 = I. \qquad (6.13.i)$$

The Lagrange function for this problem is

$$\mathcal{L}(x_1,x_2,\lambda) = \alpha \ln x_1 + \ln x_2 + \lambda(I - p_1x_1 - x_2), \qquad (6.13.ii)$$

which gives rise to first order conditions, the first two of which are

$$\begin{aligned} \frac{\alpha}{x_1} &= \lambda p_1 \\ \frac{1}{x_2} &= \lambda \end{aligned} \qquad (6.13.iii)$$

Substituting the second equation into the first for λ, we get $x_2 = p_1x_1/\alpha$. And substituting this into the budget constraint (which is the third first order condition), we get $p_1x_1 + x_2 = p_1x_1 + p_1x_1/\alpha = I$ which we can solve for x_1 to get

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1}, \qquad (6.13.iv)$$

and substituting this back into $x_2 = p_1x_1/\alpha$,

$$x_2 = \frac{I}{(\alpha+1)}. \qquad (6.13.v)$$

(b) *For the household described in part A, what is the range of α that makes the \$500 food stamp program equivalent to a cash gift of \$500?*

Answer: The food stamps are equivalent to a cash gift so long as the household would have spent at least the value of the food stamps on food were it to receive the cash gift instead. Our household has income $I = 1500$ and the price of food is p_1 =\$1 in the absence of a price subsidy. To determine the value of α at which the household would buy exactly \$500 of food with a cash gift of \$500, we need to substitute \$2,000 for I and \$1 for p_1 into our equation for x_1, set it to \$500 and solve for α; i.e.

$$\frac{2000\alpha}{\alpha+1} = 500 \text{ implies } \alpha = \frac{1}{3}. \qquad (6.13.vi)$$

Thus, for $\alpha > 1/3$, the cash subsidy is equivalent to the food stamp program of \$500.

(c) *Suppose for the remainder of the problem that $\alpha = 0.5$. How much food will this household buy under the alternate policy described in A(b)?*

Answer: Under this policy, p_1 drops to 1/2 while I remains at 1500. The household will therefore buy

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(1500)}{1.5(0.5)} = 1000. \tag{6.13.vii}$$

(d) *How much does this alternate policy cost the government for this household? Call this amount S.*

Answer: If the household buys \$1,000 of food and the government reimburses half, then $S = 500$.

(e) *How much food will the household buy if the government gives S as a cash payment and abolishes the alternate food subsidy program?*

Answer: In that case, $I = 1500 + S = 1500 + 500 = 2000$ and p_1 goes back to 1. Thus,

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(2000)}{1.5(1)} = 666.67. \tag{6.13.viii}$$

(f) *Determine which policy — the price subsidy that leads to an amount S being given to the household, or the equally costly cash payment in part (e) — is preferred by the household.*

Answer: Under the price subsidy policy, the household pays \$500 to get \$1000 of food, leaving it with \$1000 in other consumption. Thus, it consumes a bundle (1000,1000). This gives utility

$$u(1000, 1000) = 0.5\ln(1000) + \ln(1000) = 10.362. \tag{6.13.ix}$$

Under the cash subsidy policy, the household gets \$500 in cash to raise its total income to \$2000 of which it spends \$666.67 on food, leaving it with \$1333.33 in other spending; i.e. under cash subsidy, the household consumes bundle (667.67,1333.33). This gives utility

$$u(666.67, 1333.33) = 0.5\ln(666.67) + \ln(1333.33) = 10.447. \tag{6.13.x}$$

The household is happier under the cash subsidy policy.

(g) *Now suppose the government considered subsidizing food more heavily. Calculate the utility that the household will receive from three* equally funded *policies: a 75% food price subsidy (i.e. a subsidy where the government pays 75% of food bills), a food stamp program and a cash gift program.*

Answer: First, consider the price subsidy program that lowers the price p_1 from 1 to 0.25 while keeping I at \$1,500. This will result in food and other good consumption of

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(1500)}{(1.5)(0.25)} = 2000 \text{ and } x_2 = \frac{I}{(\alpha+1)} = \frac{(1500)}{1.5} = 1000. \tag{6.13.xi}$$

The utility of this bundle is then

$$u(\text{price subsidy}) = u(2000, 1000) = 0.5\ln(2000) + \ln(1000) = 10.708. \tag{6.13.xii}$$

Since food consumption under the price subsidy is 2000, this implies $S = 0.75(2000) = 1500$. If $S = 1500$ is simply given as cash (where income then becomes \$3000 and p_1 goes back up to 1), this will result in food and other consumption of

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(3000)}{(1.5)(1)} = 1000 \text{ and } x_2 = \frac{I}{(\alpha+1)} = \frac{(3000)}{1.5} = 2000, \tag{6.13.xiii}$$

giving utility of

$$u(\text{cash}) = u(1000, 2000) = 0.5\ln(1000) + \ln(2000) = 11.055. \tag{6.13.xiv}$$

Finally, under the food stamp program of size $S = 1500$, the household would be forced to consume \$1500 of food rather than \$1000 of food that it would have chosen had the money been given in terms of unrestricted cash. Thus, under food stamps, the consumer would buy the bundle (1500,1500) giving utility

$$u(\text{food stamps}) = u(1500, 1500) = 0.5\ln(1500) + \ln(1500) = 10.970. \tag{6.13.xv}$$

Here the food stamp program has gotten so large that it is no longer equivalent to getting cash — and so the consumer prefers the cash to the food stamps but still prefers the food stamps to the food price subsidy.

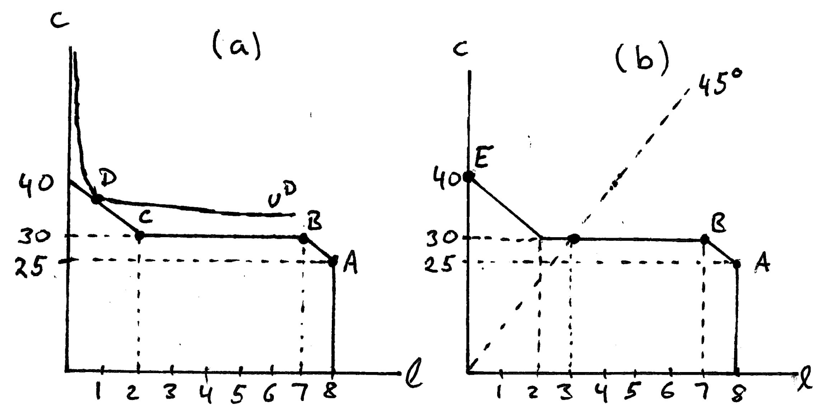
Exercise 6.15

Policy Application: *AFDC and Work Disincentives*: *Consider the AFDC program for an individual as described in end-of-chapter exercise 3.18.*

A: *Consider again an individual who can work up to 8 hours per day at a wage of \$5 per hour.*

(a) *Replicate the budget constraint you were asked to illustrate in 3.18A.*

Answer: This is done in panel (a) of Exercise Graph 6.15(1), with leisure hours on the horizontal and consumption dollars on the vertical axis.



Exercise Graph 6.15(1) : AFDC and Work Disincentives

(b) True or False*: If this person's tastes are homothetic, then he/she will work no more than 1 hour per day.*

Answer: This is false. Suppose, for instance, that leisure and consumption were perfect complements in the sense that this person wants to consume 1 hour of leisure with every \$35 of consumption. Indifference curves would then be L-shaped, with corners happening at bundles like (1,35) and (2,70). This would imply an optimal choice at (1,35) where the worker takes exactly 1 hour of leisure per day and works 7 hours per day. Such tastes are homothetic, as are less extreme tastes that allow for some (but not too much) substitutability between leisure and consumption. An example of an indifference curve u^D from a somewhat less extreme indifference map is illustrated in panel (a) of the graph — with tangency at D.

(c) *For purposes of defining a 45-degree line for this part of the question, assume that you have drawn hours on the horizontal axis 10 times as large as dollars on the vertical. This implies that the 45-degree line contains bundles like (1,10), (2,20), etc. How much would this person work if his tastes are homothetic and symmetric across this 45-degree line? (By "symmetric across the 45-degree line" I mean that the portions of the indifference curves to one side of the 45 degree line are mirror images to the portions of the indifference curves to the other side of the 45 degree line.)*

Answer: Panel (b) of the graph depicts this "45 degree line" where \$10 on the vertical axis is the same distance as 1 hour on the horizontal. In order for indifference curves to be symmetric around this line, it must be that the slope of the indifference curve for bundles on the 45 degree line is -1. But since we are measuring \$10 as geometrically equivalent to 1 hour, a slope of -1 is really a slope, or MRS of -10. If we were to draw a line from the point (0,40) to (3,30), this line would have a slope of $-10/3$. But any indifference curve has a slope of -10 on the 45 degree line — so we know that the indifference curve at (3,30) has a slope of -10 at that point and gets steeper to the left. So all indifference curves going through (3,30) or above on the 45 degree line pass above the budget constraint to the left of the 45 degree line. Thus, such "symmetric" tastes will have an optimum to the right of the 45 degree line — most likely at B but plausibly between B and A.

(d) *Suppose you knew that the individual's indifference curves were linear but you did not know the MRS. Which bundles on the budget constraint could in principle be optimal and for what ranges of the MRS?*

Answer: Bundles on the budget between A and B could be optimal, as could bundle E. In particular for MRS between 0 and $-10/7$, E would be optimal and the individual would work all the time and take no leisure. This is because indifference curves would be straight lines with sufficiently shallow slope to make the corner solution E optimal. For MRS between $-10/7$ and -5, B would be optimal. For $MRS = -5$, any bundle on the budget between B and A is optimal, with all these bundles lying on one indifference curve that is also the highest possible indifference curve for such an individual. Finally, for MRS less than -5, A becomes the optimal bundle.

(e) *Suppose you knew that, for a particular person facing this budget constraint, there are two optimal solutions. How much in AFDC payments does this person collect at each of these optimal bundles (assuming the person's tastes satisfy our usual assumptions)?*

Answer: The only way there can be exactly two optimal solutions is if one of these is B and the other lies anywhere from E to C. The person collects no AFDC between E and C but the full \$25 daily benefit at B.

B: *Suppose this worker's tastes can be summarized by the Cobb-Douglas utility function $u(\ell, c) = \ell^{1-\alpha} c^{\alpha}$ where ℓ stands for leisure and c for consumption.*

(a) *Forget for a moment the AFDC program and suppose that the budget constraint for our worker could simply be written as $c = I - 5\ell$. Calculate the optimal amount of consumption and leisure as a function of α and I.*

Answer: We need to solve the problem

$$\max_{\ell,c} u(\ell, c) = \ell^{1-\alpha} c^{\alpha} \text{ subject to } c = I - 5\ell. \tag{6.15.i}$$

Setting up the Lagrangian, taking first order conditions and solving for ℓ and c, we get

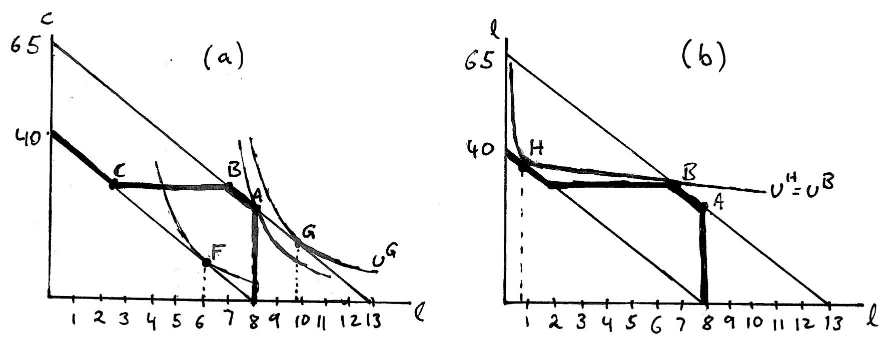
$$\ell = \frac{(1-\alpha)I}{5} \text{ and } c = \alpha I. \tag{6.15.ii}$$

(b) *On your graph of the AFDC budget constraint for this worker, there are two line segments with slope −5 — one for 0-2 hours of leisure and another for 7-8 hours of leisure. Each of these lie on a line defined by $c = I - 5\ell$ except that I is different for the two equations that contain these line segments. What are the relevant I's to identify the right equations on which these budget constraint segments lie?*

Answer: It's easy to see from the graph that I is 40 for the lower line and 65 for the higher.

(c) *Suppose $\alpha = 0.25$. If this worker were to optimize using the two budget constraints you have identified with the two different I's, how much leisure would he choose under each constraint? Can you illustrate what you find in a graph and tell from this where on the real AFDC budget constraint this worker will optimize?*

Answer: When $I = 40$, he would optimize at $\ell = (1-0.25)40/5 = 6$ and when $I = 65$, he would optimize at $\ell = (1-0.25)65/5 = 9.75$. This is illustrated in panel (a) of Exercise Graph 6.15(2) where F with 6 hours of leisure occurs on the lower budget line and G with 9.75 hours of leisure occurs on the higher.



Exercise Graph 6.15(2) : AFDC and Work Disincentives: Part 2

F cannot be optimal inside the (bold) AFDC budget because it lies inside that budget. G, on the other hand, lies outside the (bold) AFDC budget and is therefore not feasible. But we do see that the indifference curve u^G is steeper than −5 on the ray connecting the origin to the kink point A — which implies the highest possible indifference curve on the bold AFDC budget goes through that kink point. Utility at $A = (8,25)$, for instance, would be $u(8,25) = 8^{0.75}25^{0.25} = 10.63$ while utility at $B = (7,30)$ is $u(7,30) = 7^{0.75}30^{0.25} = 10.07$. Thus, the real optimum when $\alpha = 0.25$ is bundle A with no work and all leisure.

(d) *As α increases, what happens to the MRS at each bundle?*

Answer: The MRS for $u(\ell, c) = \ell^{1-\alpha}c^{\alpha}$ is

$$MRS = -\frac{\partial u/\partial \ell}{\partial u/\partial c} = -\frac{-(1-\alpha)\ell^{-\alpha}c^{\alpha}}{\alpha\ell^{1-\alpha}c^{\alpha-1}} = -\frac{(1-\alpha)c}{\alpha\ell}. \tag{6.15.iii}$$

Thus, at any bundle (ℓ, c), the MRS becomes larger in absolute value as α decreases and smaller in absolute value as α increases. Put differently, the slope of an indifference curve at any bundle becomes steeper as α gets smaller and shallower as α gets larger.

(e) *Repeat B(c) for* $\alpha = 0.3846$ *and for* $\alpha = 0.4615$. *What can you now say about this worker's choice for any* $0 < \alpha < 0.3846$*? What can you say about this worker's leisure choice if* $0.3846 < \alpha < 0.4615$*?*

Answer: When $\alpha = 0.3846$, $\ell = (1-0.3846)40/5 = 4.92$ at the lower budget line and $\ell = (1-0.3846)65/5 = 8$ on the higher budget line. The solution on the lower budget line lies inside the AFDC budget and is therefore not optimal. The solution of 8 hours of leisure on the higher budget, on the other hand, is within the AFDC budget — it is bundle A. Thus, when $\alpha = 0.3846$, the highest possible indifference curve on the AFDC budget is just tangent to the extended budget line $c = 65-5\ell$ at A. Since lower α's mean steeper indifference curves at every point, we can conclude from that that A will be optimal for all α's that lie between 0 and 0.3846. When $\alpha = 0.4615$, $\ell = (1-0.4615)40/5 = 4.31$ at the lower budget line and $\ell = (1-0.4615)65/5 = 7$ on the higher budget line. The solution on the lower budget is again inside the AFDC budget — so it cannot be optimal. The solution of 7 leisure hours on the higher budget, on the other hand, corresponds to B on the AFDC budget. Thus, when $\alpha = 0.4615$, the highest indifference curve on the AFDC budget is just tangent to the extended budget line $c = 65-5\ell$ at B. Since the slope of indifference curves becomes steeper as α falls, this implies that, for α between 0.3846 and 0.4615, the optimal leisure choice will lie in between A and B on the AFDC budget at $\ell = (1-\alpha)65/5 = 13(1-\alpha)$.

(f) *Repeat B(c) for* $\alpha = 0.9214$ *and calculate the utility associated with the resulting choice. Compare this to the utility of consuming at the kink point (7,30) and illustrate what you have found on a graph. What can you conclude about this worker's choice if* $0.4615 < \alpha < 0.9214$*?*

Answer: When $\alpha = 0.9214$, $\ell = (1-0.9214)40/5 = 0.629$ giving consumption of $w(8-\ell) = 5(8-0.629) = 36.856$. (On the higher budget line, $\ell = (1-0.9214)40/5 = 1.02$ which lies outside the AFDC budget). The bundle on the lower $c = 40-5\ell$ line, (0.629,36.856), gives utility $u(0.629, 36.856) = 0.629^{(1-0.9214)}36.856^{0.9214} = 26.76$. At B, the consumer would get utility $u(7,30) = 7^{(1-0.9214)}30^{0.9214} = 26.76$. Thus, the optimal bundle H on the budget line $c = 40-5\ell$ lies on the same indifference curve as B — as depicted in panel (b) of Exercise Graph 6.15(2). For $\alpha < 0.9214$, the indifference curve at H would be steeper and would therefore cut the AFDC budget while passing below B — and thus B is optimal for α just below 0.9214. Thus B is the optimal bundle for $0.4615 < \alpha < 0.9214$.

(g) *How much leisure will the worker take if* $0.9214 < \alpha < 1$*?*

Answer: Given that indifference curves become shallower at every bundle as α increases, we know that the indifference curve at H will be shallower for $\alpha > 0.9214$ than the one depicted in panel (b) of Exercise Graph 6.15(2). This implies that the optimal bundle for $\alpha > 0.9214$ lies to the left of H at $\ell = (1-\alpha)40/5 = 8(1-\alpha)$.

(h) *Describe in words what this tells you about what it would take for a worker to overcome the work disincentives under the AFDC program.*

Answer: The exponent α tells us how much weight a person places in his tastes on consumption rather than leisure. When α is high, consumption is valued much more than leisure — so even a small increase in consumption can justify giving up a lot of leisure. Thus, for very high α, it is possible that someone with the AFDC budget constraint will in fact work close to full time despite the work disincentives. But that person's tastes would have to be pretty extreme — he would have to place virtually no value on leisure time. For anyone that places some non-trivial value on leisure time — which implies α isn't close to 1 or, to be more precise, $\alpha < 0.9214$ — the payoff from working close to full time is simply not high enough to sacrifice that much leisure. Thus, for most values of α, the person will choose to work less than 1 hour per day.

Conclusion: Potentially Helpful Reminders

1. Keep in mind the distinction between how the MRS changes along an indifference curve (which tells us about substitutability) and how the MRS changes across the indifference map (which leads to ideas like homotheticity and quasilinearity).

2. The idea of substitutability will become critical in Chapter 7 when we introduce substitution effects (which will depend only on the shape of an indifference curve). The ideas of homotheticity and quasiliearity become important as we introduce income effects (in Chapter 7) — which will be measured across an indifference map (rather than along an indifference curve).

3. Extremes like perfect substitutes and perfect complements are useful to keep in mind because they make it easy to remember which way an indifference map looks if the goods are relatively more substitutable as opposed to relatively more complementary and vice versa.

4. Special cases like homothetic and quasilinear tastes will become useful borderline cases in Chapter 7 — with homothetic tastes being the borderline case between luxury goods and necessities, and with quasilinear tastes being the borderline case between normal and inferior goods. (These terms are defined in Chapter 7.)

CHAPTER

7

Income and Substitution Effects in Consumer Goods Markest

In Chapter 6 we showed how economic circumstances combine with tastes to result in choice or behavior. In Chapter 7 we show how consumer choices (and thus the consumer behavior we observe) change as circumstances change — i.e. as incomes and prices change. Put differently, we will now show how "people respond to incentives" in the consumer goods market.

Chapter Highlights

The main points of the chapter are:

1. There are **two ways in which economic circumstances typically change**: a change in income and a change in opportunity costs.

2. When only **income changes**, we can predict the change in behavior if we know something about **how indifference curves relate to one another** — because we jump from one indifference curve to another. Whether tastes are quasilinear or homothetic, whether goods are normal or inferior — these are statements about that relationship between indifference curves.

3. When only **opportunity costs change** and *real* income remains constant, we don't need to know anything about the relationship of indifference curves to one another — because the change in behavior occurs along a single indifference curve. Thus, **the *shape* of the relevant indifference curve is all that matters** — which is the same as saying that the degree of substitutability of the goods at the margin is all that matters.

4. **Substitution effects** arise as we slide along indifference curves because opportunity costs have changed; **income effects** arise as we jump between indifference curves because real income has changed.

5. **Price changes give rise to both of these effects**. To identify the substitution effect, we only look at the initial indifference curve and thus need to know about the substitutability of goods at the margin; to identify income effects, we have to know how indifference curves relate to one another.

6. In the calculus-based material of Part B of *Microeconomics: An Intuitive Approach with Calculus*, we show how **constrained utility maximization** gives us the choices that people make as incentives change while the **constrained expenditure minimization** problem allows us to disentangle the substitution effect from the income effect.

7A Solutions to Within-Chapter-Exercises for Part A

Exercise 7A.1

Is it also the case that whenever there is a positive income effect on our consumption of one good, there must be a negative income effect on our consumption of a different good?

Answer: No — since it is possible for our consumption of all goods to go up as income increases, the income effect could be positive for all goods.

Exercise 7A.2

Can a good be an inferior good at all income levels? (*Hint*: Consider the bundle (0,0).)

Answer: No. The reason is that, in order for a good to be inferior, it must be that you consume more of it as income falls. But, as income falls toward zero, at some point it will not be possible to consume more as income falls — because there simply won't be enough income to consume more. Thus, around the origin, no good can be inferior.

Exercise 7A.3

Are all inferior goods necessities? Are all necessities inferior goods? (*Hint*: The answer to the first is yes; the answer to the second is no.) Explain.

Answer: If you consume less of a good as income goes up, then it must be true that you spend a smaller fraction of your income on that good as income goes up. Thus, all inferior goods are necessities. At the same time, it may be the case that the fraction of your income spent on a good declines as your income goes up — but you still buy more of the good. (For instance, suppose your income goes up by 10% and you choose to consume 5% more of a good. Then the fraction of income spent on that good is declining even though you are increasing your consumption of the good as your income goes up.) Thus, necessities could be normal goods.

Exercise 7A.4

At a particular consumption bundle, can both goods (in a 2-good model) be luxuries? Can they both be necessities?

Answer: No. In a 2-good model, you will end up spending all your income as income increases. So suppose you are currently spending all your income on the two goods and your income now increases by 10%. If your consumption of both goods increases by more than 10%, then you would now be spending more than your new income. If your consumption of both goods increases by less than 10%, you would be spending less than your new income.

Exercise 7A.5

If you knew only that my brother and I had the same income (but not necessarily the same tastes), could you tell which one of us drove more miles — the one that rented or the one that took taxis?

Answer: Yes. Suppose my brother faces the intersecting budget lines — with the steeper one representing taxis and the shallower one representing rental cars. He chooses the steeper (taxi) budget line. Then we know that he must be consuming a bundle to the left of the intersection point of the two lines — because if he chose to the right of that point, he could have had more of everything on the shallower budget and thus should have chosen the shallower (rental car) budget instead. Thus, by choosing the steeper taxi budget, we know my brother consumes to the left of the intersection point. I, on the other hand, chose the shallower rental car budget. If I were to then choose a bundle to the left of the intersection point, I could have done better choosing the steeper budget because I could get more of both goods. Thus I must be consuming to the right of the intersection point. If my brother and I have the same incomes (and thus face the same taxi and rental car budgets), it therefore must be the case that my brother consumes to the left of the intersection point on the taxi budget and I consume to the right on the rental car budget. We can unambiguously say I consumed more miles driven.

Exercise 7A.6

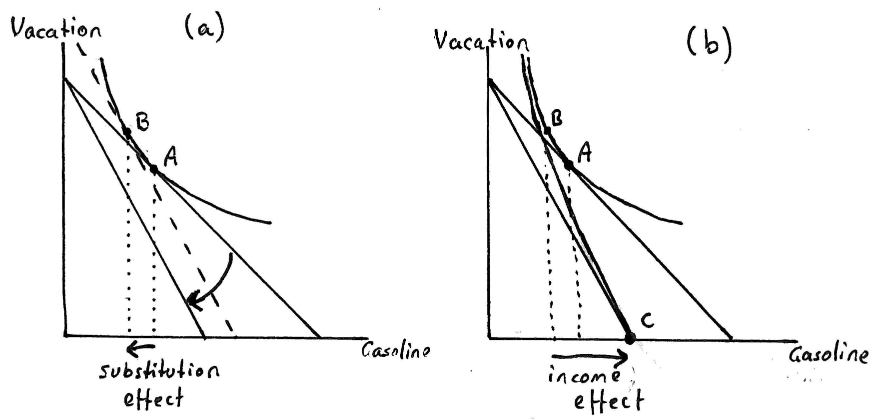
True or False: If you observed my brother and me consuming the same number of miles driven during our vacations, then our tastes must be those of perfect complements between miles driven and other consumption.

Answer: It would at a minimum have to be the case that the indifference curve at the intersection of the two budget lines has a sharp kink at that point. That kink could be such that it forms a right angle — thus creating the typical perfect complements indifference curve. But at a minimum it has to be such that the upper part of the indifference curve is steeper than the taxi budget and the lower part is shallower than the rental car budget — with a kink at the intersection.

Exercise 7A.7

Can you re-tell the Heating Gasoline-in-Midwest story in terms of income and substitution effects in a graph with "yearly gallons of gasoline consumption" on the horizontal axis and "yearly time on vacation in Florida" on the vertical?

Answer: In panel (a) of Exercise Graph 7A.7, bundle A is the original consumption bundle prior to the increase in the price of gasoline. The increase in the price of gasoline then rotates the budget clockwise. Bundle B lies on the compensated budget at the new price of gasoline — and the move from A to B is the substitution effect. As always, the substitution effect causes a decrease in consumption of the good (gasoline) that has become more expensive.



Exercise Graph 7A.7 : Gasoline and Florida Vacation Time

Panel (b) illustrates C — with no consumption of Florida vacation time. This corner solution is rationalized by an indifference curve that crosses the new budget at C — creating an income effect in the opposite direction of the substitution effect. Since the income effect is larger than the substitution effect, the consumer shifts from A before the increase in the price of gasoline to C after the price increase — with an overall increase in gasoline consumption resulting from the price increase.

Exercise 7A.8

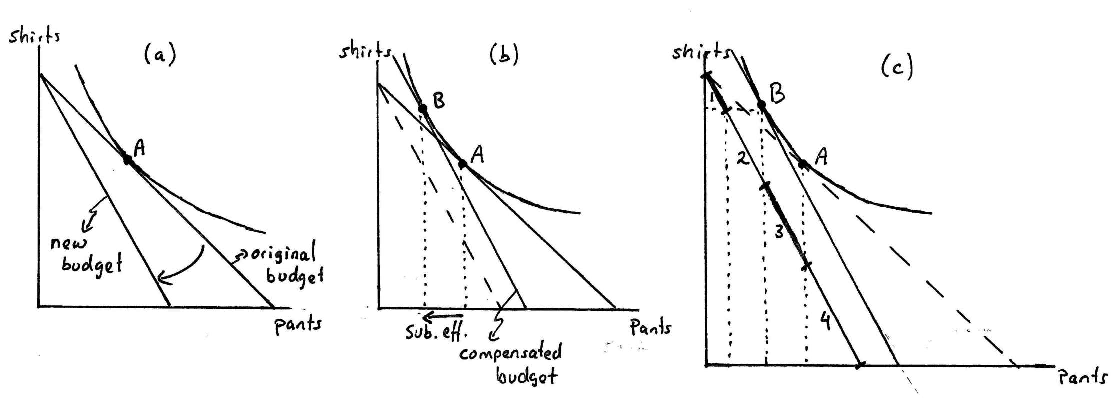
In panel (c) of Graph 7.7, where would the final optimal bundle on the magenta budget lie if tastes were nomothetic? What if they were quasilinear?

Answer: If tastes were nomothetic, the final bundle would lie where a ray from the origin through B intersects the magenta budget. If tastes were quasilinear (in pants), the final bundle would lie where the vertical ray through B intersects the magenta budget (at the borderline between pants being normal and inferior). (If tastes were quasilinear in shirts, the final bundle would lie where the horizontal ray through B intersects the magenta budget.)

Exercise 7A.9

Replicate Graph 7.7 for an increase in the price of pants (rather than a decrease).

Answer: Panel (a) of Exercise Graph 7A.9 illustrates the original consumption bundle A and the change in the budget constraint when the price of pants increases. Panel (b) illustrates the compensated budget and the resulting bundle B — with the substitution effect as the movement from A to B. As always, this effect says the consumer will consume less of what has become more expensive, more of what has become relatively cheaper. Finally, panel (c) identifies four regions (labeled 1, 2, 3 and 4) on the new (uncompensated) budget line. If the consumer ends up optimizing in region 1, her consumption of pants decreases and her consumption of shirts increases with a decline in income (relative to the compensated budget) — which implies that pants are a normal good and shirts are inferior. In region 2, the consumption of both goods declines with income — thus both pants and shirts are normal goods. In regions 3 and 4, consumption of shirts decreases and consumption of pants increases with a drop in income (from the compensated budget) — thus making shirts normal and pants inferior. In region 3, however, the consumer still buys fewer pants as the price increases (i.e. C is to the left of A) — which means pants are regular inferior; in region 4, on the other hand, pants consumption goes up with an increase in price, which makes pants a Giffen good.



Exercise Graph 7A.9 : Gasoline and Florida Vacation Time

Exercise 7A.10

Can you explain the following Venn Diagram?

Answer: The diagram illustrates that the set of all goods can be divided into two broad subsets — normal goods and inferior goods, with goods at the border between these subsets represented by quasilinear goods. The set of Giffen goods is fully contained in the subset of inferior goods — that is, every Giffen good is an inferior good, but not every inferior good is a Giffen good. (We use the term "regular inferior good" to denote the subset of inferior goods that is not Giffen.) And just as the set of all goods can be subdivided into normal and inferior goods, it can be subdivided into necessities and luxuries, with the borderline between those two subsets representing homothetic goods. The set of luxury goods is then fully contained in the set of normal goods — that is, every luxury good is a normal good but not all normal goods are luxury goods. Necessities, on the other hand, can be normal or inferior (including regular inferior and Giffen).

7B Solutions to Within-Chapter-Exercises for Part B

Exercise 7B.1

Set up my brother's constrained optimization problem and solve it to check that his optimal consumption bundle is indeed equal to this.

Answer: My brother's optimization problem is

$$\max_{x_1,x_2} u(x_1,x_2) = x_1^{0.1}x_2^{0.9} \text{ subject to } x_1+x_2 = 2000, \tag{7B.1.i}$$

which gives rise to the Lagrange function

$$\mathscr{L}(x_1,x_2,\lambda) = x_1^{0.1}x_2^{0.9} + \lambda(2000 - x_1 - x_2). \tag{7B.1.ii}$$

The first two first order conditions for this problem are then

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x_1} &= 0.1x_1^{-0.9}x_2^{0.9} - \lambda = 0, \\ \frac{\partial \mathscr{L}}{\partial x_2} &= 0.9x_1^{0.1}x_2^{-0.1} - \lambda = 0. \end{aligned} \tag{7B.1.iii}$$

Moving λ to the other side of each equation, dividing the equations by one another and solving for x_1 gives us $x_1 = x_2/9$. Substituting this into the budget constraint $x_1 + x_2 = 2000$, we get $x_2/9 + x_2 = 2000$ which solves to $x_2 = 1,800$. Plugging this back into $x_1 = x_2/9$ furthermore gives $x_1 = 200$.

Exercise 7B.2

How much did I pay in a fixed rental car fee in order for me to be indifferent in this example to taking taxis? Why is this amount larger than in the Cobb-Douglas case we calculated earlier?

Answer: At B, I am consuming 2,551 miles at a per-mile cost of \$0.2 — for a total of \$510.20. At that bundle, I am also consuming approximately \$918 in other consumption. Thus, I am spending a total of approximately $\$918 + \$510 = \$1,428$ after having paid the fixed fee for the rental car. Since I started with \$2,000, that means the rental car fee must have been $\$2000 - \$1428 = \$572$. This amount is larger than under Cobb-Douglas preferences because the implicit elasticity of substitution is now 2 rather than 1.

Exercise 7B.3

Check to see that this solution is correct.
<u>Answer</u>: The Lagrange function for this optimization problem is

$$\mathscr{L}(x_1,x_2,\lambda) = x_1^{0.5}x_2^{0.5} + \lambda(200 - p_1x_1 - 10x_2). \qquad (7B.3.i)$$

The first two first order conditions for this problem are then

$$\frac{\partial \mathscr{L}}{\partial x_1} = 0.5x_1^{-0.5}x_2^{0.5} - \lambda p_1 = 0,$$
$$\frac{\partial \mathscr{L}}{\partial x_2} = 0.5x_1^{0.5}x_2^{-0.5} - 10\lambda = 0. \qquad (7B.3.ii)$$

Moving the λ terms to the other side, dividing the equations by one another and then solving for x_1, we get $x_1 = 10x_2/p_1$. Plugging this into the budget constraint $p_1x_1 + 10x_2 = 200$ and solving for x_2, we get $x_2 = 10$, and plugging this back into $x_1 = 10x_2/p_1$, we get $x_1 = 100/p_1$.

Exercise 7B.4

Verify the above solutions to the minimization problem.
<u>Answer</u>: The Lagrange function for this optimization problem is

$$\mathscr{L}(x_1,x_2,\lambda) = 10x_1 + 10x_2 + \lambda(u^A - x_1^{0.5}x_2^{0.5}). \qquad (7B.4.i)$$

The first two first order conditions for this problem are then

$$\frac{\partial \mathscr{L}}{\partial x_1} = 10 - 0.5\lambda x_1^{-0.5}x_2^{0.5} = 0,$$
$$\frac{\partial \mathscr{L}}{\partial x_2} = 10 - 0.5\lambda x_1^{0.5}x_2^{-0.5} = 0. \qquad (7B.4.ii)$$

Solving these for x_1 in the usual way gives us $x_1 = x_2$. Plugging this into the constraint $u^A = x_1^{0.5}x_2^{0.5}$, we then get $x_2 = u^A$, and — given we concluded $x_1 = x_2$, $x_1 = u^A$. Since $u^A \approx 7.071$, this implies $x_1 = x_2 \approx 7.071$.

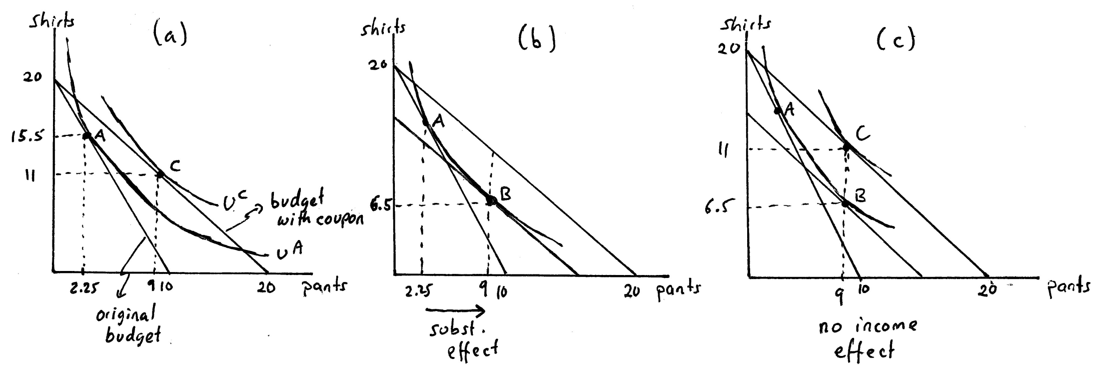
Exercise 7B.5

Notice that the ratio of my pants to shirts consumption is the same (= 1) at bundles B and C. What feature of Cobb-Douglas tastes is responsible for this result?
<u>Answer</u>: Cobb-Douglas tastes are homothetic — which implies that optimal consumption bundles lie on the same ray from the origin for all income levels (assuming no price changes).

Exercise 7B.6

Using the previous calculations, plot graphs similar to Graph 7.10 illustrating income and substitution effects when my tastes can be represented by the utility function $u(x_1,x_2) = 6x_1^{0.5} + x_2$.

Answer: This is done in Exercise Graph 7B.6. Notice again that there is no income effect relative to the good x_1 (pants) — which is because of the fact that the utility function represents tastes that are quasilinear in x_1. (Quasilinear goods have no income effects.)



Exercise Graph 7B.6 : Pants and Shirts with Quasilinear Tastes

7C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 7.1

Here, we consider some logical relationships between preferences and types of goods.

A: *Suppose you consider all the goods that you might potentially want to consume.*

(a) *Is it possible for all these goods to be be luxury goods at every consumption bundle? Is it possible for all of them to be necessities?*

Answer: Neither is possible. If they were all luxuries, then, as income increases by some percentage, consumption of each good would increase by a greater percentage. This is logically impossible. If they were all necessities, then, as income increases by some percentage, consumption of each good would increase by a lesser percentage. This implies that some income would remain unspent, which is inconsistent with optimization.

(b) *Is it possible for all goods to be inferior goods at every consumption bundle? Is it possible for all of them to be normal goods?*

Answer: The first is not possible but the second is. If all goods are inferior, then, as income falls, the consumer would increase her consumption of all goods. But that is logically impossible since income is declining. If all goods are normal goods, than consumption of all increases with increases in income and decreases with decreases in income — which is logically possible.

(c) True or False*: When tastes are homothetic, all goods are normal goods.*

Answer: True. Homothetic tastes are defined by the fact that the *MRS* remains constant along any ray from the origin. Thus, if we find a tangency of an indifference curve with a budget line, we know that, as income changes, indifference curves will always be tangent to the new budget along the ray that connects the original tangency to the origin. Thus, as income increases, consumption of all goods increases, and when income decreases, consumption of all goods decreases.

(d) True or False*: When tastes are homothetic, some goods could be luxuries while others could be necessities.*

Answer: False. We just explained that for homothetic tastes, the optimal bundles (for a given set of prices) lie on rays from the origin as income changes. Thus, as income increases by some percentage, consumption of all goods increases by the same percentage. Thus, all goods are borderline between luxuries and necessities.

(e) True or False*: When tastes are quasilinear, one of the goods is a necessity.*

Answer: True. As income changes, consumption of one of the goods does not change. Thus, as income increases, the percentage of income spent on that good decreases — making that good a necessity.

(f) True or False: *In a two good model, if the two goods are perfect complements, they must both be normal goods.*

Answer: True — since the goods are always consumed as pairs, consumption of both increases as income increases.

(g) True or False: *In a 3-good model, if two of the goods are perfect complements, they must both be normal goods.*

Answer: False. Since there is a third good, it may be that this third good is a normal good while the perfectly complementary goods are (jointly) inferior. Suppose, for instance, that rum and coke are perfect complements for someone, but that the person also has a taste for really good single malt scotch. As income goes up, he increases his consumption of single malt scotch and lowers his consumption of rum and cokes. Rum and coke would be perfect complements, but as income goes up, less of both would be consumed.

B: *In each of the following cases, suppose that a person whose tastes can be characterized by the given utility function has income I and faces prices that are all equal to 1. Illustrate mathematically how his consumption of each good changes with income and use your answer to determine whether the goods are normal or inferior, luxuries or necessities.*

(a) $u(x_1, x_2) = x_1 x_2$

Answer: In each case, we can set up the optimization problem

$$\max_{x_1, x_2} u(x_1, x_2) \text{ subject to } x_1 + x_2 = I \tag{7.1.i}$$

and solve it for x_1 and x_2 as a function of I. For the function $u(x_1, x_2) = x_1 x_2$, this gives us $x_1(I) = x_2(I) = I/2$. Thus, half of all income is spent on x_1 and half on x_2, which implies that, when income doubles, so does consumption of each of the two goods. Thus, the goods are borderline between luxuries and necessities — and they are both normal.

(b) $u(x_1, x_2) = x_1 + \ln x_2$

Answer: Solving this optimization problem again with the new utility function, we get $x_1(I) = I - 1$ and $x_2(I) = 1$. Consumption of x_2 is therefore independent of income — which means the good is borderline between normal and inferior. The fraction of income spent on x_2 declines with income — which means the good is a necessity. Good x_1, on the other hand, is a normal good — and a luxury.

(c) $u(x_1, x_2) = \ln x_1 + \ln x_2$

Answer: For this utility function, we again get $x_1(I) = x_2(I) = I/2$ as in (a). (This makes sense since the utility function here is a monotone transformation of the utility function in (a).) So the same answer as in (a) applies.

(d) $u(x_1, x_2, x_3) = 2\ln x_1 + \ln x_2 + 4\ln x_3$

Answer: We can again solve the same optimization problem, except that we now have 3 choice variables. We would write the Lagrange function as

$$\mathscr{L}(x_1, x_2, x_3, \lambda) = 2\ln x_1 + \ln x_2 + 4\ln x_3 + \lambda(I - x_1 - x_2 - x_3) \tag{7.1.ii}$$

and the first three first order conditions as

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x_1} &= \frac{2}{x_1} - \lambda = 0, \\ \frac{\partial \mathscr{L}}{\partial x_2} &= \frac{1}{x_2} - \lambda = 0, \\ \frac{\partial \mathscr{L}}{\partial x_3} &= \frac{4}{x_3} - \lambda = 0. \end{aligned} \tag{7.1.iii}$$

The first and second can be used to write $x_2 = x_1/2$, and the first and third can be combined to give us $x_3 = 2x_1$. Substituting these into the budget constraint $x_1 + x_2 + x_3 = I$ gives us $x_1 + x_1/2 + 2x_1 = I$ which solves to $x_1(I) = 2I/7$. Substituting this back into $x_2 = x_1/2$ and $x_3 = 2x_1$ then gives us $x_2(I) = I/7$ and $x_3(I) = 4I/7$. The consumption of each of the three goods is therefore a constant fraction of income — which implies all three goods are normal and borderline between luxuries and necessities.

(e) $u(x_1, x_2) = 2x_1^{0.5} + \ln x_2$

Answer: Following the same set-up, we get[1]

$$x_1(I) = \left(\frac{-1 + (1 + 4I)^{1/2}}{2}\right)^2 \text{ and } x_2(I) = \frac{-1 + (1 + 4I)^{1/2}}{2} \tag{7.1.iv}$$

As income increases, consumption of both goods therefore increases (since I enters positively into both equations). However, it does not increase at a constant rate. Taking the derivative of $x_2(I)$ with respect to I, we get

$$\frac{dx_2(I)}{dI} = \frac{1}{(1 + 4I)^{1/2}}, \tag{7.1.v}$$

which is a decreasing function of I. Thus, as income increases, the fraction devoted to consumption of x_2 decreases — making x_2 a necessity (and thus x_1 a luxury good).

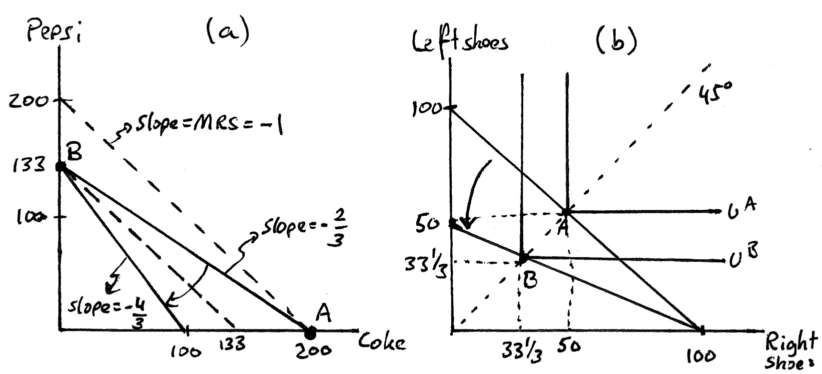
[1] Combining the first 2 first order conditions, we get $x_1 = x_2^2$, and substituting this into the budget constraint, we get $x_2^2 + x_2 - I = 0$. To solve this, we apply the quadratic formula which gives two answers for x_2. However, one of these is clearly negative.

Exercise 7.3

Consider once again my tastes for Coke and Pepsi and my tastes for right and left shoes (as described in end-of-chapter exercise 6.2).

A: *On two separate graphs — one with Coke and Pepsi on the axes, the other with right shoes and left shoes — replicate your answers to end-of-chapter exercise 6.2A(a) and (b). Label the original optimal bundles A and the new optimal bundles C.*

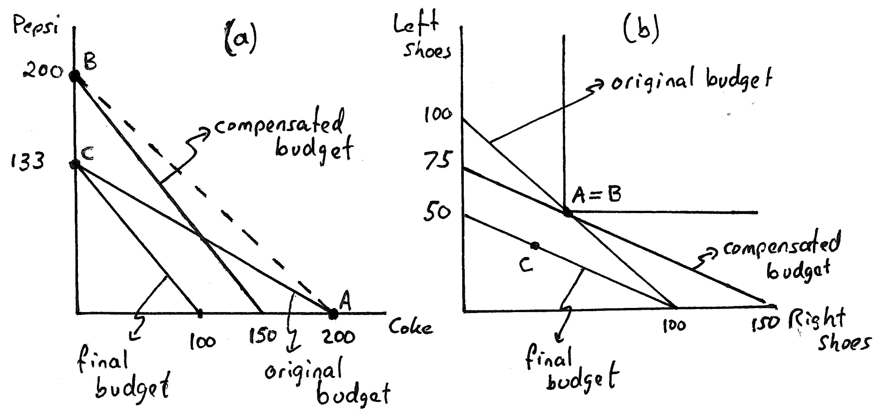
Answer: The graphs from end-of-chapter exercise 6.2A(a) and (b) are replicated in Exercise Graph 7.3(1). Note that indifference curves in panel (a) are dashed while budget lines are solid. Also, note that in this replicated graph, B is the final optimum and should now be labeled C.



Exercise Graph 7.3(1) : Replicated from End-of-Chapter exercise 6.2

(a) *In your Coke/Pepsi graph, decompose the change in behavior into income and substitution effects by drawing the compensated budget and indicating the optimal bundle B on that budget.*

Answer: In panel (a) of Exercise Graph 7.3(2), the original optimum occurs on the dashed indifference curve at bundle A while the final optimum occurs on the final budget at C. (To keep the picture uncluttered, the final indifference curve is left out.) The compensated budget has the same slope as the final budget but sufficient income to reach the original dashed indifference curve — which occurs at B. Thus, the substitution effect takes us from A to B, and the income effect to C. This should make



Exercise Graph 7.3(2) : Inc. and Subst. Effects for Perfect Substitutes and Complements

sense: For the good whose price has changed (coke), the entire change is due to the substitution effect because the goods are perfect substitutes.

(b) *Repeat (a) for your right shoes/left shoes graph.*

Answer: Panel (b) of the graph shows the analogous for perfect complements. The compensated budget has the same slope as the final budget but must be "tangent" to the original indifference curve. This happens at A — which means the usual B that includes the substitution effect lies right on top of A. Thus, there is no substitution effect — which again should make sense since there is no substitutability between the two goods.

B: *Now consider the following utility functions:* $u(x_1, x_2) = \min\{x_1, x_2\}$ *and* $u(x_1, x_2) = x_1 + x_2$.

(a) *Which of these could plausibly represent my tastes for Coke and Pepsi, and which could represent my tastes for right and left shoes?*

Answer: The first could represent tastes for right and left shoes while the second could represent tastes for Coke and Pepsi.

(b) *Use the appropriate function from above to assign utility levels to bundles A, B and C in your graph from 7.3A(a).*

Answer: The appropriate function in this case is $u(x_1, x_2) = x_1 + x_2$. The three bundles are A=(200,0), B=(0,200) and C=(0,133). Thus, the utility levels assigned to each of these bundles is $u(A) = 200 = u(B)$ and $u(C) = 133$.

(c) *Repeat this for bundles A, B and C for your graph in 7.3A(b).*

Answer: The appropriate function now is $u(x_1, x_2) = \min\{x_1, x_2\}$ and the three bundles are A=B=(50,50) and C=(33.33,33.33). The utility values associated with these bundles are $u(A) = u(B) = 50$ and $u(C) = 33.33$.

Exercise 7.5

Return to the analysis of my undying love for my wife expressed through weekly purchases of roses (as introduced in end-of-chapter exercise 6.4).

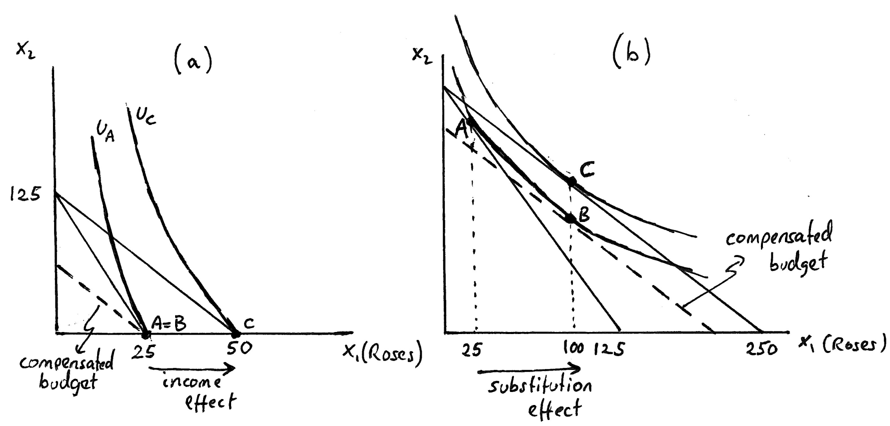
A: *Recall that initially roses cost $5 each and, with an income of $125 per week, I bought 25 roses each week. Then, when my income increased to $500 per week, I continued to buy 25 roses per week (at the same price).*

(a) *From what you observed thus far, are roses a normal or an inferior good for me? Are they a luxury or a necessity?*

Answer: As income went up, my consumption remained unchanged. This would typically indicate that the good in question is borderline normal/inferior — or quasilinear. Since the consumption at the lower income is at a corner solution, however, we cannot be certain that the good is not inferior, with the MRS at the original optimum larger in absolute value than the MRS at the new (higher income) optimum. Regardless, roses must be a necessity — whether they are borderline inferior/normal or inferior, the percentage of income spent on roses declines as income increases.

(b) *On a graph with weekly roses consumption on the horizontal and "other goods" on the vertical, illustrate my budget constraint when my weekly income is $125. Then illustrate the change in the budget constraint when income remains $125 per week and the price of roses falls to $2.50. Suppose that my optimal consumption of roses after this price change rises to 50 roses per week and illustrate this as bundle C.*

Answer: This is illustrated in panel (a) of Exercise Graph 7.5 where A is the original corner solution, C is the new corner solution and the dashed line is the compensated budget.



Exercise Graph 7.5 : Love and Roses

(c) *Illustrate the compensated budget line and use it to illustrate the income and substitution effects.*

Answer: This is also illustrated in panel (a) of the graph. In this case, there is no substitution effect (in terms of roses) and only an income effect.

(d) *Now consider the case where my income is \$500 and, when the price changes from \$5 to \$2.50, I end up consuming 100 roses per week (rather than 25). Assuming quasilinearity in roses, illustrate income and substitution effects.*

Answer: This is illustrated in panel (b) of Exercise Graph 7.5 where the dashed line is again the compensated budget line. Unlike in panel (a), the entire change in roses consumption is now due to a substitution effect rather than an income effect.

(e) True or False*: Price changes of goods that are quasilinear give rise to no income effects for the quasilinear good unless corner solutions are involved.*

Answer: This is true. We will often make the statement that income effects disappear if we assume quasilinearity of a good — because then a good is borderline normal/inferior, which implies consumption remains unchanged as income changes. This is true so long as the consumer is at an interior solution. If quasilinear tastes lead to corner solutions, then this may give rise to income effects as we see in panel (a) of the graph.

B: *Suppose again, as in 6.4B, that my tastes for roses (x_1) and other goods (x_2) can be represented by the utility function $u(x_1, x_2) = \beta x_1^{\alpha} + x_2$.*

(a) *If you have not already done so, assume that p_2 is by definition equal to 1, let $\alpha = 0.5$ and $\beta = 50$, and calculate my optimal consumption of roses and other goods as a function of p_1 and I.*

Answer: Solving the optimization problem

$$\max_{x_1,x_2} 50x_1^{0.5} + x_2 \text{ subject to } I = p_1x_1 + x_2, \tag{7.5.i}$$

we get

$$x_1 = \frac{625}{p_1^2} \text{ and } x_2 = I - \frac{625}{p_1}. \tag{7.5.ii}$$

(b) *The original scenario you graphed in 7.5A(b) contains corner solutions when my income is \$125 and the price is initially \$5 and then \$2.50. Does your answer above allow for this?*

Answer: Substituting $I = 125$ and $p_1 = 5$ into our equations (7.5.ii) for x_1 and x_2 from above, we get $x_1 = 625/(5^2) = 25$ and $x_2 = 125 - (625/5) = 0$. This is exactly the original corner solution in the scenario in part A.

Changing the price to $p_1 = 2.5$, we get $x_1 = 625/(2.5^2) = 100$ and $x_2 = 125-(625/2.5) = -125$. Given that the solution from our Lagrange method now gives us a negative consumption level for x_2, we know that the true optimum is the corner solution where all income is spent on x_1 — i.e. the bundle (50,0) just as described in the scenario in A.

At the original price, it turns out that the MRS at the corner solution is exactly equal to the slope of the budget line. At the lower price, the MRS

is large in absolute value than the budget line — which means the indifference curve cuts the budget line at the corner from above. The tangency of an indifference curve with this budget line therefore does not happen until x_2 is negative — which the Lagrange method finds but which is not economically meaningful.

(c) *Verify that the scenario in your answer to 7.5A(d) is also consistent with tastes described by this utility function — i.e. verify that A, B and C are as you described in your answer.*

Answer: Using equations (7.5.ii), we get $x_1 = 625/(5^2) = 25$ and $x_2 = 500 - (625/5) = 375$ when $p_1 = 5$ (and $I = 500$), and we get $x_1 = 625/(2.5^2) = 100$ and $x_2 = 500 - (625/2.5) = 250$ when $p_1 = 2.5$. These correspond to A and C in panel (b) of Exercise Graph 7.5.

To calculate B in the graph, we need to first find the utility level associated with the original bundle A — i.e. $u(25,375) = 50(25^{0.5}) + 375 = 625$. We then need to find what bundle the consumer would buy if she was given enough money to reach that same indifference curve at the new price; i.e. we need to solve the problem

$$\min_{x_1,x_2} 2.5x_1 + x_2 \text{ subject to } 625 = 50x_1^{0.5} + x_2. \tag{7.5.iii}$$

Solving the first order conditions, we then get $x_1 = 100$ and $x_2 = 125$ — consistent with panel (b) of the graph.

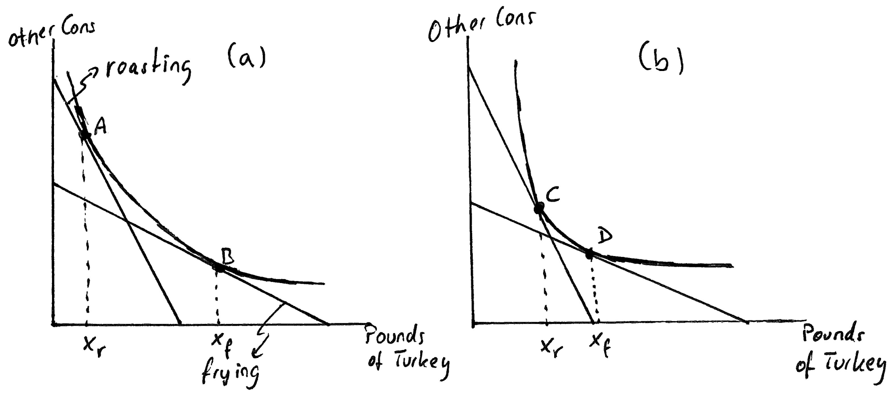
Exercise 7.7

Everyday Application: *Turkey and Thanksgiving*: *Every Thanksgiving, my wife and I debate about how we should prepare the turkey we will serve (and will then have left over). My wife likes preparing turkeys the conventional way — roasted in the oven where it has to cook at 350 degrees for 4 hours or so. I, on the other hand, like to fry turkeys in a big pot of peanut oil heated over a powerful flame outdoors. The two methods have different costs and benefits. The conventional way of cooking turkeys has very little set-up cost (since the oven is already there and just has to be turned on) but a relatively large time cost from then on. (It takes hours to cook.) The frying method, on the other hand, takes some set-up (dragging out the turkey fryer, pouring gallons of peanut oil, etc. — and then later the cleanup associated with it), but turkeys cook predictably quickly in just 3.5 minutes per pound.*

A: *As a household, we seem to be indifferent between doing it one way or another — sometimes we use the oven, sometimes we use the fryer. But we have noticed that we cook much more turkey — several turkeys, as a matter of fact, when we use the fryer than when we use the oven.*

(a) *Construct a graph with "pounds of cooked turkeys" on the horizontal and "other consumption" on the vertical. ("Other consumption" here is not denominated in dollars as normally but rather in some consumption index that takes into account the time it takes to engage in such consumption.) Think of the set-up cost for frying turkeys and the waiting cost for cooking* *them as the main costs that are relevant. Can you illustrate our family's choice of whether to fry or roast turkeys at Thanksgiving as a choice between two "budget lines"?*

Answer: This is illustrated in panel (a) of Exercise Graph 7.7(1). The set-up cost of the turkey fryer results in a lower intercept for the frying budget on the vertical axis — but the lower cost of cooking turkey results in a shallower slope.



Exercise Graph 7.7(1) : Frying versus Roasting Turkey

(b) *Can you explain the fact that we seem to eat more turkey around Thanksgiving whenever we pull out the turkey fryer as opposed to roasting the turkey in the oven?*

Answer: Since we are indifferent between frying and roasting, our optimal bundle on the two budget lines must lie on the same indifference curve. This is also illustrated in panel (a) of the graph — where it is immediately apparent that we will cook more turkey when frying than when roasting because of the lower opportunity cost.

(c) *We have some friends who also struggle each Thanksgiving with the decision of whether to fry or roast — and they, too, seem to be indifferent between the two options. But we have noticed that they only cook a little more turkey when they fry than when they roast. What is different about them?*

Answer: A possible picture for my friend's family is illustrated in panel (b) of the graph — where the indifference curve is not as flat — making the two goods less substitutable. Since the effect we are demonstrating is a pure substitution effect, it makes sense that with less substitutability between the goods, the difference in behavior is smaller for the two turkey cooking options.

B: *Suppose that, if we did not cook turkeys, we could consume 100 units of "other consumption" — but the time it takes to cook turkeys takes away from that con-*

sumption. Setting up the turkey fryer costs c units of consumption and waiting 3.5 minutes (which is how long it takes to cook 1 pound of turkey) costs 1 unit of consumption. Roasting a turkey involves no set-up cost, but it takes 5 times as long to cook per pound. Suppose that tastes can be characterized by the CES utility function $u(x_1,x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$ where x_1 is pounds of turkey and x_2 is "other consumption".

(a) *What are the two budget constraints I am facing?*

Answer: Costs are denominated in "units of consumption" — which implies that p_2, the price of consuming "other goods", is by definition 1. The price of cooking 1 pound of turkey (p_1) is then either 1 if we fry or 5 if we roast. This gives us the budget constraints

$$5x_1 + x_2 = 100 \text{ when roasting, and } x_1 + x_2 = 100 - c \text{ when fying.} \quad (7.7.i)$$

(b) *Can you calculate how much turkey someone with these tastes will roast (as a function of ρ)? How much will the same person fry? (*Hint: *Rather than solving this using the Lagrange method, use the fact that you know the MRS is equal to the slope of the budget line — and recall from chapter 5 that, for a CES utility function of this kind, $MRS = -(x_2/x_1)^{\rho+1}$.)*

Answer: At the optimum, we set the MRS equal to the ration $-p_1/p_2$. Setting MRS equal to the ratio of prices then implies

$$\left(\frac{x_2}{x_1}\right)^{\rho+1} = 5 \text{ when roasting, and } \left(\frac{x_2}{x_1}\right)^{\rho+1} = 1 \text{ when fying.} \quad (7.7.ii)$$

Solving for x_2, we get $x_2 = 5^{1/(\rho+1)}x_1$ when roasting and $x_2 = x_1$ when frying. Substituting these into the appropriate budget constraints from equation (7.7.i) and solving for x_1, we get

$$x_1 = \frac{100}{5+5^{1/(\rho+1)}} \text{ when roasting, and } x_1 = \frac{100-c}{2} \text{ when fying.} \quad (7.7.iii)$$

(c) *Suppose my family has tastes with $\rho = 0$ and my friend's with $\rho = 1$. If each of us individually roasts turkeys this Thanksgiving, how much will we each roast?*

Answer: My family will roast

$$x_1 = \frac{100}{5+5^1} = 10, \quad (7.7.iv)$$

and my friend's family will roast

$$x_1 = \frac{100}{5+5^{1/2}} = 13.82. \quad (7.7.v)$$

(d) *How much utility will each of us get (as measured by the relevant utility function)? (*Hint*: In the case where* $\rho = 0$*, the exponent* $1/\rho$ *is undefined. Use the fact that you know that when* $\rho = 0$ *the CES utility function is Cobb-Douglas.)*

Answer: To calculate utilities, we first have to calculate how much of x_2 each of us consumes. Just plugging our answers above into the first budget constraint in equation (7.7.i), we get $x_2 = 50$ for my family and $x_2 = 30.9$ for my friends. For my family, $\rho = 0$ — which means we can use the Cobb-Douglas utility function $x_1^{0.5}x_2^{0.5}$ instead of the CES functional form. Plugging $(x_1, x_2) = (10, 50)$ into $x_1^{0.5}x_2^{0.5}$ gives us utility of 22.36. For my friend's family, plugging $(x_1, x_2) = (13.82, 30.90)$ into his utility function (with $\rho = 1$), we get utility of 19.1.

(e) *Which family is happier?*

Answer: We can't know since we generally do not believe that we are measuring utility in units that can be compared across people.

(f) *If we are really indifferent between roasting and frying, what must c be for my family? What must it be for my friend's family? (*Hint*: Rather than setting up the usual minimization problem, use your answer to (b) determine c by setting utility equal to what it was for roasting).*

Answer: We know from our answer in (b) that, when frying, $x_1 = (100 - c)/2$ regardless of ρ. Plugging this into our frying budget constraint $x_1 + x_2 = 100 - c$, this implies that $x_2 = (100 - c)/2$ regardless of ρ. When $\rho = 0$, we can then plug these into the Cobb-Douglas version of the utility function and set it equal to the utility of 22.36 that we determined above my family gets when roasting turkeys; i.e.

$$\left(\frac{100-c}{2}\right)^{0.5}\left(\frac{100-c}{2}\right)^{0.5} = \left(\frac{100-c}{2}\right) = 22.36. \tag{7.7.vi}$$

Solving for c, we get $c = 55.28$. For my friend's family, we can similarly substitute $x_1 = (100-c)/2$ and $x_2 = (100-c)/2$ into his CES utility function (with $\rho = 1$) and set it equal to the utility he gets from roasting — which we calculated before to be 19.1. Thus,

$$\left[0.5\left(\frac{100-c}{2}\right)^{-1} + 0.5\left(\frac{100-c}{2}\right)^{-1}\right]^{-1} = \frac{100-c}{2} = 19.1. \tag{7.7.vii}$$

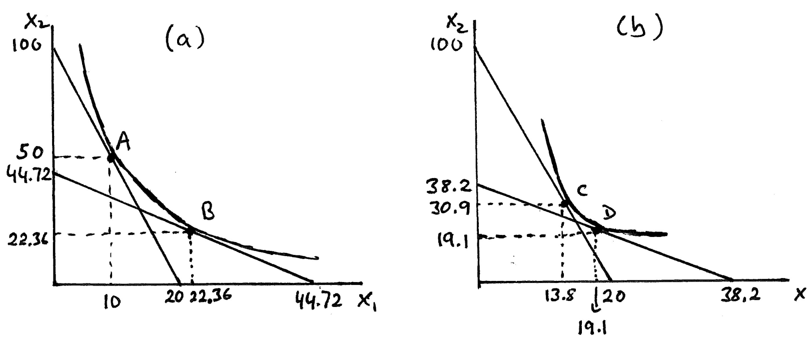
Solving for c, we get $c = 61.8$.

(g) *Given your answers so far, how much would we each have fried had we chosen to fry instead of roast (and we were truly indifferent between the two because of the different values of c we face)?*

Answer: Given that we calculated $c = 55.28$ for my family and $c = 61.8$ for my friend's, we get that $x_1 = (100 - 55.28)/2 = 22.36$ pounds for my family and $x_1 = (100 - 61.8)/2 = 19.1$ pounds for my friend's family.

(h) *Compare the size of the substitution effect you have calculated for my family and that you calculated for my friend's family and illustrate your answer in a graph with pounds of turkey on the horizontal and other consumption on the vertical. Relate the difference in the size of the substitution effect to the elasticity of substitution.*

<u>Answer</u>: My family goes from roasting 10 pounds of turkey to frying 22.23 pounds — a substitution effect of 12.36 pounds. My friend's family goes from roasting 13.82 pounds to frying 19.1 pounds — a substitution effect of 5.28 pounds. The difference, of course, is the greater substitutability that is built into my utility function with $\rho = 0$ as opposed to my friend's with $\rho = 1$. To be precise, my elasticity of substitution is 1 whereas my friend's is 0.5. The results are graphed in Exercise Graph 7.7(2).



Exercise Graph 7.7(2) : Frying versus Roasting Turkey: Part II

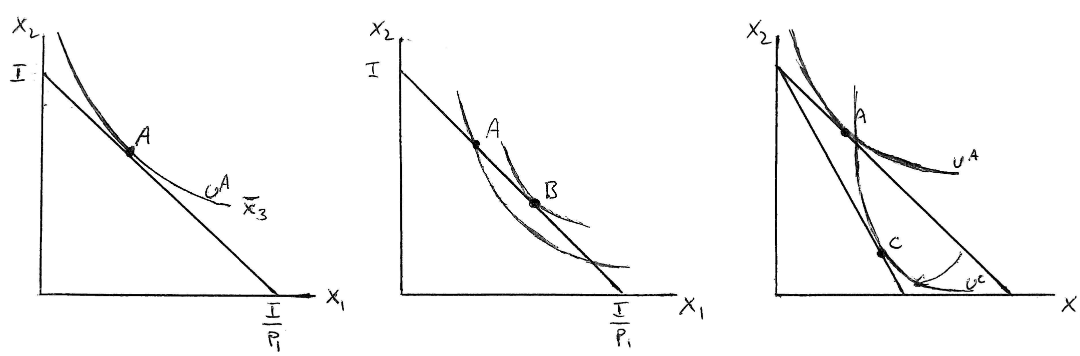
Exercise 7.9

Business Application: *Are Gucci products Giffen Goods? We defined a Giffen good as a good that consumers (with exogenous incomes) buy more of when the price increases. When students first hear about such goods, they often think of luxury goods such as expensive Gucci purses and accessories. If the marketing departments for firms like Gucci are very successful, they may find a way of associating price with "prestige" in the minds of consumers — and this may allow them to raise the price* and *sell more products. But would that make Gucci products Giffen goods? The answer, as you will see in this exercise, is no.*

A: *Suppose we model a consumer who cares about the "practical value and style of Gucci products", dollars of other consumption and the "prestige value" of being seen with Gucci products. Denote these as x_1, x_2 and x_3 respectively.*

(a) *The consumer only has to buy x_1 and x_2 — the prestige value x_3 comes with the Gucci products. Let p_1 denote the price of Gucci products and $p_2 = 1$ be the price of dollars of other consumption. Illustrate the consumer's budget constraint (assuming an exogenous income I).*

Answer: This is just like any typical budget constraint and is illustrated as part of panel (a) of Exercise Graph 7.9.



Exercise Graph 7.9 : Gucci Products and Prestige

(b) *The prestige value of Gucci purchases — x_3 — is something an individual consumer has no control over. If x_3 is fixed at a particular level $\overline{x}_3$, the consumer therefore operates on a 2-dimensional slice of her 3-dimensional indifference map over x_1, x_2 and x_3. Draw such a slice for the indifference curve that contains the consumer's optimal bundle A on the budget from part (a).*

Answer: The 2-dimensional slice of the indifference map will look exactly like our typical indifference maps over 2 goods. The optimal bundle A is illustrated as the bundle at the tangency of an indifference curve from this slice with the budget constraint from part (a).

(c) *Now suppose that Gucci manages to raise the prestige value of its products — and thus x_3 that comes with the purchase of Gucci products. For now, suppose they do this without changing p_1. This implies you will shift to a different 2-dimensional slice of your 3-dimensional indifference map. Illustrate the new 2-dimensional indifference curve that contains A. Is the new MRS at A greater or smaller in absolute value than it was before?*

Answer: This is illustrated in panel (b) of the graph. The increase in prestige implies the consumer is willing to pay more for any additional Gucci products — thus the *MRS* increases in absolute value.

(d) *Would the consumer consume more or fewer Gucci products after the increase in prestige value?*

Answer: All the bundles that lie above the indifference curve through *A* in panel (b) of the graph contain more Gucci products. The consumer will now optimize at some new bundle such as *B*.

(e) *Now suppose that Gucci manages to convince consumers that Gucci products become more desirable the more expensive they are. Put differently, the prestige value x_3 is linked to p_1, the price of the Gucci products. On a new graph, illustrate the change in the consumer's budget as a result of an increase in p_1.*

Answer: This change in the budget is no different than it would usually be — and is illustrated as part of panel (c) of Exercise Graph 7.9.

(f) *Suppose that our consumer increases her purchases of Gucci products as a result of the increase in the price p_1. Illustrate two indifference curves — one that gives rise to the original optimum A and another that gives rise to the new optimum C. Can these indifference curves cross?*

Answer: This is illustrated in panel (c) of the Graph. Since the indifference curve u^C is drawn from a different 2-dimensional slice of the 3-dimensional indifference curve over x_1, x_2 and x_3 than the indifference curve u^A, the two indifference curves can indeed cross.

(g) *Explain why, even though the behavior is consistent with what we would expect if Gucci products were a Giffen good, Gucci products are not a Giffen good in this case.*

Answer: Gucci products in this example are really bundles of 2 products — the physical product itself, and the prestige value that comes with the product. When price increases, the prestige value increases — which means we are no longer dealing with the same product as before (even though the physical characteristics of the product remain the same). Thus, while the consumer is indeed buying more Gucci products after the price increase, she is also buying more prestige that is bundled with the physical product. In terms of our 3-dimendional indifference curves, she is shifting to a different x_3 level because p_1 is higher. *Holding all else fixed*, she would not buy more Gucci products as price increases — it is only because she is buying more prestige at the higher price that it looks like she is buying more as price increases.

(h) *In a footnote in the chapter we defined the following: A good is a* Veblen good *if* preferences *for the good change as price increases — with this change in preferences possibly leading to an increase in consumption as price increases. Are Gucci products a Veblen good in this exercise?*

Answer: Yes — as price increases, tastes (i.e. the indifference map in 2 dimensions) change in the sense that we are shifting to a different slice of the true 3-D indifference surfaces. The resulting increased consumption of Gucci products as price increases is due to this "change in tastes" — or, to put it more accurately, to the change in the product that looks like a change in tastes when we graph our 2-dimensional indifference curves. This is different from Giffen behavior where the indifference map does not change with an increase in price — but consumption does.

B: *Consider the same definition of x_1, x_2 and x_3 as in part A. Suppose that the tastes for our consumer can be captured by the utility function $u(x_1,x_2,x_3) = \alpha x_3^2 \ln x_1 + x_2$.*

(a) *Set up the consumer's utility maximization problem — keeping in mind that x_3 is not a choice variable.*

Answer: The maximization problem is

$$\max_{x_1,x_2} \alpha x_3^2 \ln x_1 + x_2 \text{ subject to } p_1 x_1 + x_2 = I. \tag{7.9.i}$$

(b) *Solve for the optimal consumption of x_1 (which will be a function of the prestige value x_3).*

Answer: The Lagrange function for this problem is

$$\mathscr{L}(x_1,x_2,\lambda) = \alpha x_3^2 \ln x_1 + x_2 + \lambda(I - p x_1 - x_2). \tag{7.9.ii}$$

Solving this the usual way, we get

$$x_1 = \frac{\alpha x_3^2}{p_1} \text{ and } x_2 = I - \alpha x_3^2. \tag{7.9.iii}$$

(c) *Is x_1 normal or inferior? Is it Giffen?*

Answer: x_1 does not vary with income — thus making it quasilinear. Put differently, x_1 is borderline between normal and inferior. At the same time, x_1 falls with p_1 — implying that consumers will buy less x_1 as p_1 increases all else being equal. Thus, x_1 is not a Giffen good.

(d) *Now suppose that prestige value is a function of p_1. In particular, suppose that $x_3 = p_1$. Substitute this into your solution for x_1. Will consumption increase or decrease as p_1 increases?*

Answer: This implies that

$$x_1 = \frac{\alpha p_1^2}{p_1} = \alpha p_1. \tag{7.9.iv}$$

Thus, consumption of x_1 increases as p_1 increases.

(e) *How would you explain that x_1 is not a Giffen good despite the fact that its consumption increases as p_1 goes up?*

Answer: In order for x_1 to be a Giffen good, consumption of x_1 would have to increase with an increase in p_1 *all else remaining equal*. We showed in (b) that this is not the case — all else (including prestige) remaining constant, an increase in p_1 leads to a decrease in x_1. The only reason that x_1 increases as p_1 increases is that we allow p_1 to change the prestige value of Gucci products — and thus the very nature of those products.

Exercise 7.11

Policy Application: *Substitution Effects and Social Security Cost of Living Adjustments*: In end-of-chapter exercise 6.16, you investigated the government's practice for adjusting social security income for seniors by insuring that the average senior can always afford to buy some average bundle of goods that remains fixed. To simplify the analysis, let us again assume that the average senior consumes only two different goods.

A: *Suppose that last year our average senior optimized at the average bundle A identified by the government, and begin by assuming that we denominate the units of x_1 and x_2 such that last year $p_1 = p_2 = 1$.*

(a) *Suppose that p_1 increases. On a graph with x_1 on the horizontal and x_2 on the vertical axis, illustrate the compensated budget and the bundle B that, given your senior's tastes, would keep the senior just as well off at the new price.*

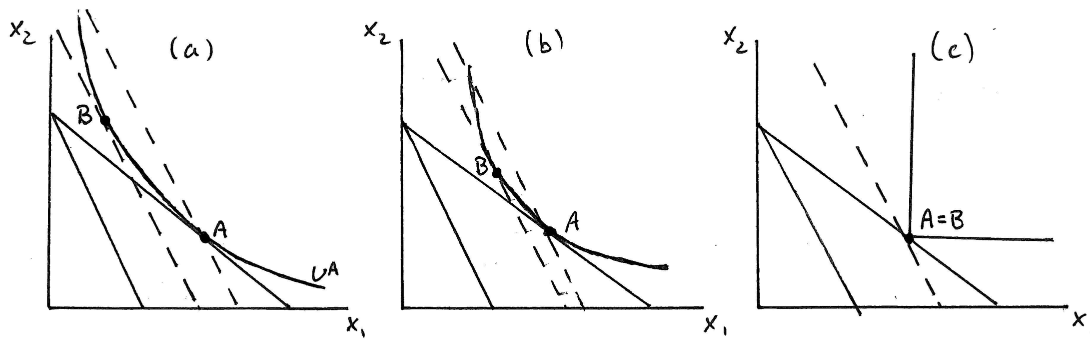
Answer: In panel (a) of Exercise Graph 7.11(1), bundle A lies on the original (solid line) budget. The price increase causes an inward rotation of that budget in the absence of compensation. To compensate the person so that he will be as happy as before, we have to raise income to the lower dashed line in the graph — the line that is tangent to B that lies on the indifference curve u^A.

(b) *In your graph, compare the level of income the senior requires to get to bundle B to the income required to get him back to bundle A.*

Answer: The income required (at the new prices) to get to A is represented by the second dashed line in panel (a) of the graph.

(c) *What determines the size of the difference in the income necessary to keep the senior just as well off when the price of good 1 increases as opposed to the income necessary for the senior to still be able to afford bundle A?*

Answer: The greater the substitutability of the two goods, the greater will be the difference between the two ways of compensating the person. This is illustrated across the three panels in Exercise Graph 7.11(1) where the degree of substitutability falls from left to right.



Exercise Graph 7.11(1) : Hicks and Slutsky Social Security Compensation

(d) *Under what condition will the two forms of compensation be identical to one another?*

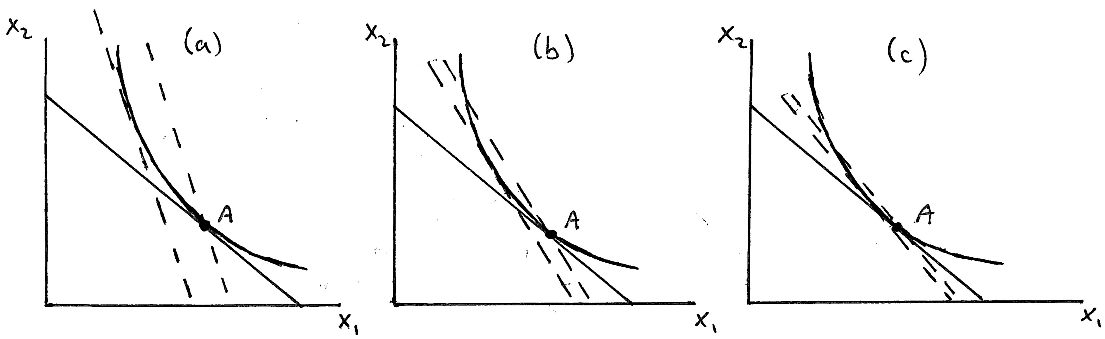
Answer: The difference between the two compensation schemes disappears entirely in panel (c) of the graph when there is no substitutability between the goods (i.e. when they are perfect complements).

(e) *You should recognize the move from A to B as a pure substitution effect as we have defined it in this chapter. Often this substitution effect is referred to as the* Hicksian substitution effect — *defined as the change in behavior when opportunity costs change but the consumer receives sufficient compensation* to remain just as happy. *Let B' be the consumption bundle the average senior would choose when compensated so as to be able to afford the original bundle A. The movement from A to B' is often called the* Slutsky substitution effect — *defined as the change in behavior when opportunity costs change but the consumer receives sufficient compensation* to be able to afford to stay at the original consumption bundle. True or False*: The government could save money by using Hicksian rather than Slutsky substitution principles to determine appropriate cost of living adjustments for social security recipients.*

Answer: The answer is true. The government in fact uses Slutsky compensation as it calculates cost of living adjustments — because it fixes a particular consumption bundle and then adjusts social security checks to make sure that seniors can still afford that bundle. For this reason, you will frequently hear proposals to adjust the way in which cost of living adjustments are calculated — with these proposals attempting to get closer to Hicksian compensation.

(f) True or False*: Hicksian and Slutsky compensation get closer to one another the smaller the price changes.*

Answer: This is true. Larger price changes result in larger substitution effects — and the difference between Hicksian and Slutsky substitution is entirely due to the substitution effect. This is illustrated in the three panels of Exercise Graph 7.11(2) where, going from left to right, the size of the price change (as evidenced in the steepness of the slope of the compensated budget) decreases.



Exercise Graph 7.11(2) : Hicks and Slutsky Social Security Compensation: Part II

B: *Now suppose that the tastes of the average senior can be captured by the Cobb-Douglas utility function $u(x_1, x_2) = x_1x_2$, where x_2 is a composite good (with price by definition equal to $p_2 = 1$). Suppose the average senior currently receives social security income I (and no other income) and with it purchases bundle (x_1^A, x_2^A).*

(a) *Determine (x_1^A, x_2^A) in terms of I and p_1.*

Answer: Solving the usual maximization problem with budget constraint $p_1x_1 + x_2 = I$, we get

$$x_1^A = \frac{I}{2p_1} \text{ and } x_2^B = \frac{I}{2}. \tag{7.11.i}$$

(b) *Suppose that p_1 is currently \$1 and I is currently \$2000. Then p_1 increases to \$2. How much will the government increase the social security check given how it is actually calculating cost of living adjustments? How will this change the senior's behavior?*

Answer: The government compensates so as to make it possible for the senior to keep affording the same bundle as before. With the values $p_1 = 1$ and $I = 2000$, $x_1^A = x_2^A = 1000$. When the price of x_1 goes to \$2, this same bundle costs $2(1000) + 1000 = \$3,000$. Thus, the government is compensating the senior by increasing the social security check by \$1,000.

With an income of \$3,000, equations (7.11.i) then tell us that the senior will consume $x_1 = 3000/(2(2)) = 750$ and $x_2 = 3000/2 = 1,500$. Thus, even though the government makes it possible for the senior to consume bundle A again after the price change, the senior will substitute away from x_1 because its opportunity cost is now higher.

(c) *How much would the government increase the social security check if it used Hicksian rather than Slutsky compensation? How would the senior's behavior change?*

Answer: If the government used Hicksian compensation, it would first need to calculate the bundle B on the original indifference curve that would make the senior just as well off at the higher price as he was at A. At A, the senior gets utility $u^A = x_1^A x_2^A = 1000(1000) = 1,000,000$. The government would then have to solve the problem

$$\min_{x_1,x_2} 2x_1 + x_2 \text{ subject to } x_1 x_2 = 1,000,000. \qquad (7.11.ii)$$

Solving the first two first order conditions, we get $x_2 = 2x_1$. Substituting this into the constraint and solving for x_1, we get $x_1 \approx 707.1$, and plugging this back into $x_2 = 2x_1$, we get $x_2 = 1414.2$. This bundle $B = (707.1, 1414.2)$ costs $2(707.1)+1414.2 = 2828.4$. Thus, under Hicksian compensation, the government would increase the senior's social security check by \$828.40 rather than \$1,000.

(d) *Can you demonstrate mathematically that Hicksian and Slutsky compensation converge to one another as the price change gets small — and diverge from each other as the price change gets large?*

Answer: We start with $p_1 = 1$ (and continue to assume $p_2 = 1$).[2] Then suppose p_1 increases to $p_1 > 1$ (or falls to $p_1 < 1$). Slutsky compensation requires that we continue to be able to purchase $A = (1000, 1000)$ — so we have to make sure the senior has income of $1000p_1 + 1000$. Since the senior starts with an income of \$2,000, this implies that Slutsky compensation is $1000p_1 + 1000 - 2000 = 1000p_1 - 1000 = 1000(p_1 - 1)$.

Hicksian compensation, on the other hand, requires we calculate the substitution effect to B as we did in the previous part for $p_1 = 2$. Setting up the same problem but letting the new price of good 1 be denoted p_1 rather than 2, we can calculate $B = (x_1^B, x_2^B) = (1000/p_1^{0.5}, 1000p_1^{0.5})$. This bundle costs

$$p_1 \frac{1000}{p_1^{0.5}} + 1000p_1^{0.5} = 2000p_1^{0.5}. \qquad (7.11.iii)$$

Given that the senior starts with \$2000, this means that Hicksian compensation must be equal to $2000p_1^{0.5} - 2000 = 2000(p_1^{0.5} - 1)$.

The difference between Slutsky compensation and Hicksian compensation, which we will call $D(p_1)$ is then

[2] We could start with any other price and change either p_1 or p_2 and the same logic will hold.

$$D(p_1) = 1000(p_1 - 1) - 2000(p_1^{0.5} - 1) = 1000p_1 - 1000 - 2000p_1^{0.5} + 2000$$
$$= 1000 + 1000p_1(1 - 2p_1^{-0.5}). \tag{7.11.iv}$$

As p_1 approaches 1, the second term in the equation goes to -1000 — making the expression go to zero; i.e. the difference between the two types of compensation goes to zero as the price increase (or decrease) gets small. In fact, it is easy to see that this difference reaches it's lowest point when $p_1 = 1$ and increases when p_1 rises above 1 as well as when p_1 falls below 1: Simply take the derivative of $D(p_1)$ which is

$$\frac{dD(p_1)}{dp_1} = 1000\left(1 - 2p_1^{-0.5}\right) + 1000p_1\left(p_1^{-1.5}\right) = 1000\left(1 - p_1^{-0.5}\right). \tag{7.11.v}$$

Then note that $dD/dp_1 < 0$ when $0 < p_1 < 1$, $dD/dp_1 = 0$ when $p_1 = 1$ and $dD/dp_1 > 0$ when $p_1 > 1$. This implies a U-shape for $D(p_1)$, with the U reaching its bottom at $p_1 = 1$ when $D(p_1) = 0$. Put into words, the difference between Slutsky and Hicks compensation is positive for any price not equal to the original price, with the difference increasing the greater the deviation in price from the original price.

(e) *We know that Cobb-Douglas utility functions are part of the CES family of utility functions — with the elasticity of substitution equal to 1. Without doing any math, can you estimate, for an increase in p_1 above 1, the range of how much Slutsky compensation can exceed Hicksian compensation with tastes that lie within the CES family? (*Hint*: Consider the extreme cases of elasticities of subsitution.)*

Answer: We know that if the two goods are perfect complements (with elasticity of substitution equal to 0), then there is no difference between the two compensation mechanisms (because, as we demonstrated in part A of the question, the difference is due entirely to the substitution effect). Thus, one end of the range of how much Slutsky compensation can exceed Hicksian compensation is zero.

The other extreme is the case of perfect substitutes. In that case, it is rational for the consumer to choose bundle A initially since the prices are identical and the indifference curve therefore lies on top of the budget line (making all bundles on the budget line optimal). But any deviation in price will result in a corner solution. Thus, if p_1 increases, the consumer can remain just as well off as she was originally by simply not consuming x_2. Thus, Hicksian compensation is zero while Slutsky compensation still aims to make bundle A affordable — i.e. Slutsky compensation is still $1000(p_1 - 1)$ as we calculated in part (d). So in this extreme case, Slutsky compensation exceeds Hicksian compensation by $1000(p_1 - 1)$.

Depending on the elasticity of substitution, Slutsky compensation may therefore exceed Hicksian compensation by as little as 0 (when the elasticity is 0) to as much as $1000(p_1 - 1)$ (when the elasticity is infinite).

Exercise 7.13

Policy Application: *Public Housing and Housing Subsidies*: *In exercise 2.14, you considered two different public housing programs in parts A(a) and (b) — one where a family is simply offered a particular apartment for a below-market rent and another where the government provides a housing price subsidy that the family can use anywhere in the private rental market.*

A: *Suppose we consider a family that earns \$1500 per month and either pays 50 cents per square foot in monthly rent for an apartment in the private market or accepts a 1500 square foot government public housing unit at the government's price of \$500 per month.*

(a) *On a graph with square feet of housing and "dollars of other consumption", illustrate two cases where the family accepts the public housing unit — one where this leads them to consume less housing than they otherwise would, another where it leads them to consume more housing than they otherwise would.*

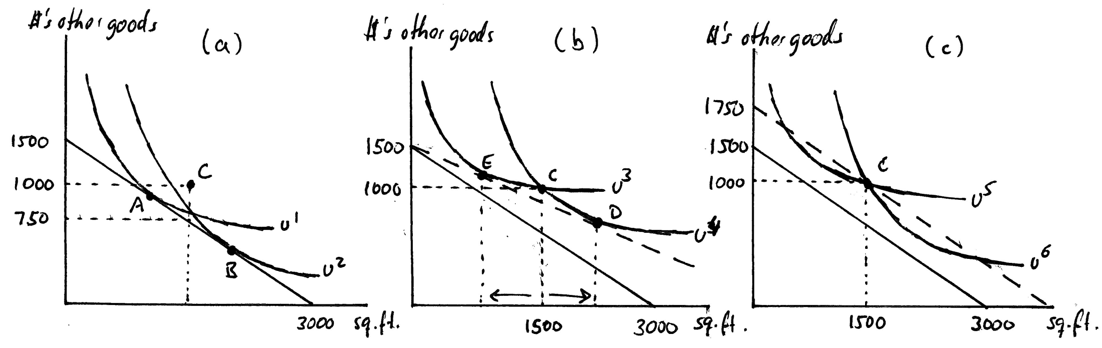
Answer: The budget constraint in the absence of public housing is drawn in panel (a) of Exercise Graph 7.13. Bundle A is optimal under tastes with indifference curve u^1 while bundle B is optimal under tastes with indifference curve u^2. (Since these indifference curves cross, they of course cannot come from the same indifference map — and thus come from different indifference maps representing different tastes.) The public housing unit permits the household to consume C — the 1500 square foot public housing unit costing \$500 (and thus leaving the household with \$1000 of other consumption). Both the household that optimizes at A and the one that optimizes at B in the absence of the public housing option will choose C if it becomes available. For household 1 this implies that public housing increases its housing consumption, but for household 2 it implies that public housing decreases housing consumption.

(b) *If we use the household's own judgment about its well-being, is it always the case that the option of public housing makes the households who choose to participate better off?*

Answer: Yes — the household would not choose the option unless it thought it is better off. In panel (a) of the graph, both households end up on higher indifference curves when choosing C.

(c) *If the policy goal behind public housing is to increase the housing consumption of the poor, is it more or less likely to succeed the less substitutable housing and other goods are?*

Answer: The less substitutable housing and other goods are, the sharper the tangency at the optimum on the original budget line. And the sharper the tangency, the less likely it is that a household can consume more than



Exercise Graph 7.13 : Public Housing and Rental Subsidies

1500 square feet of housing in the absence of public housing and still become better off at *C* in our graph. For instance, in panel (a) of the graph it is possible for *A* to be optimal and *C* to be better even if housing and other goods are perfect complements — but this is not true for *B*.

(d) *What is the government's opportunity cost of owning a public housing unit of 1500 square feet? How much does it therefore cost the government to provide the public housing unit to this family?*

Answer: The government could charge the market price of \$0.50 per square foot for the 1500 square foot public housing unit. It is therefore giving up \$750 in rent by not renting it on the open market — and it is collecting only \$500 from the public housing participant. Thus, the cost the government incurs is \$250 per month. You can also see this in panel (a) of our graph — as the vertical difference between *C* and the budget line.

(e) *Now consider instead a housing price subsidy under which the government tells qualified families that it will pay some fraction of their rental bills in the private housing market. If this rental subsidy is set so as to make the household just as well off as it was under public housing, will it lead to more or less consumption of housing than if the household chooses public housing?*

Answer: Panel (b) of Exercise Graph 7.13 illustrates that such a subsidy could lead to more or less consumption of housing.

(f) *Will giving such a rental subsidy cost more or less than providing the public housing unit? What does your answer depend on?*

Answer: It may cost more or less. If the household consumes less housing under the rental subsidy (as with indifference curve u^3), it will definitely cost less. (In the graph, the cost is the vertical difference between *E* and the original budget constraint — which must be smaller than the \$250 difference between *C* and the original constraint.) But if the rental subsidy results in more housing consumption than public housing (as with indifference curve u^D), it may cost the government more or less depending on just how much more housing is consumed.

(g) *Suppose instead that the government simply gave cash to the household. If it gave sufficient cash to make the household as well off as it is under the public housing program, would it cost the government more or less than \$250? Can you tell whether under such a cash subsidy the household consumes more or less housing than under public housing?*

Answer: It will definitely cost the government less (or at least no more) but we can't tell whether it will result in greater or lesser housing consumption. This is illustrated in panel (c) of Exercise Graph 7.13 where the dashed budget line results from the government giving \$250 in cash — the same as it spends under the public housing program. Unless the slope of the indifference curve at C just happens to be the same as the slope of the budget line, the new budget line will cut the indifference curve that contains C either from above (as in u^5) or from below (as in u^6). Either way, the household would be able to make itself better off by reaching a higher indifference curve. Thus, except for the special case where the budget line has the same slope as the indifference curve at C, it will cost the government less than \$250 to make the household as well off as it is under public housing. Put differently, there are smaller budgets with the same slope that are tangent to u^5 and u^6. But at those tangencies, housing consumption will fall below 1500 square feet in the case of u^5 and rise above 1500 in the case of u^6.

B: *Suppose that household tastes over square feet of housing (x_1) and dollars of other consumption (x_2) can be represented by $u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2$.*

(a) *Suppose that empirical studies show that we spend about a quarter of our income on housing. What does that imply about α?*

Answer: These are Cobb-Douglas tastes (equivalent to $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ which, when transformed by the natural log, turns into the one given in the problem). When the exponents of a Cobb-Douglas utility function sum to 1, the exponents denote the fraction of income spent on each good. Thus, if households spend a quarter of their income on housing, then $\alpha = 0.25$.

(b) *Consider a family with income of \$1,500 per month facing a per square foot price of $p_1 = 0.50$. For what value of α would the family not change its housing consumption when offered the 1500 square foot public housing apartment for \$500?*

Answer: 1500 square feet cost \$750 — which is half of the household's income of \$1,500. Given what we said about exponents in Cobb-Douglas utility functions representing budget shares, α would have to be 0.5 in order for the household to spend half its income on housing.

(c) *Suppose that this family has α as derived in B(a). How much of a rental price subsidy would the government have to give to this family in order to make it as well off as the family is with the public housing unit?*

Answer: With the public housing unit, the family consumes the bundle (1500, 1000) — which gives utility

$$u(1500,1000) = 0.25\ln 1500 + 0.75\ln 1000 \approx 7.009. \tag{7.13.i}$$

If you solve the maximization problem

$$\max_{x_1,x_2} \; 0.25\ln x_1 + 0.75\ln x_2 \;\text{ subject to }\; p_1x_1 + x_2 = 1500, \tag{7.13.ii}$$

you get

$$x_1 = \frac{0.25(1500)}{p_1} = \frac{375}{p_1} \;\text{ and }\; x_2 = 0.75(1500) = 1125. \tag{7.13.iii}$$

Plugging these back into the utility function, we get

$$\begin{aligned} u\left(\frac{375}{p_1}, 1125\right) &= 0.25\ln\frac{375}{p_1} + 0.75\ln 1125 \\ &= 0.25\ln 375 - 0.25\ln p_1 + 0.75\ln 1125 \approx 6.751 - 0.25\ln p_1. \end{aligned} \tag{7.13.iv}$$

In order for p_1 to be subsidized to a point where it makes the household indifferent between getting the subsidy and participating in the public housing program, this utility has to be equal to the utility of public housing (which is 7.009); i.e.

$$6.751 - 0.25\ln p_1 = 7.009. \tag{7.13.v}$$

Solving this for p_1, we get $p_1 \approx 0.356$. Thus, the subsidy that would make the household indifferent requires that the government pay a fraction of about 0.288 of rental housing (which reduces the price from 0.5 to 0.356).

(d) *How much housing will the family rent under this subsidy? How much will it cost the government to provide this subsidy?*

Answer: The household would rent

$$x_1 = \frac{0.25(1500)}{0.356} \approx 1053, \tag{7.13.vi}$$

which is less than it consumes under public housing. A house with 1053 square feet costs 1053(0.5) = 526.50 to rent — and the government under this subsidy pays 28.8% of this cost — i.e. the program costs 0.288(526.5) ≈ 151.63.

(e) *Suppose the government instead gave the family cash (without changing the price of housing). How much cash would it have to give the family in order to make it as happy?*

Answer: We already determined that the utility of participating in the public housing program is 7.009. You can find the amount of income necessary to get to that utility level in different ways. One way is to solve the minimization problem

$$\min_{x_1,x_2} \; 0.5x_1 + x_2 \; \text{ subject to } \; 0.25\ln x_1 + 0.75\ln x_2 = 7.009. \tag{7.13.vii}$$

The first two first order conditions give us $x_2 = 1.5x_1$. Substituting into the constraint, we get

$$7.009 = 0.25\ln x_1 + 0.75\ln(1.5x_1) = \ln x_1 + 0.75\ln(1.5) \approx \ln x_1 + 0.304. \tag{7.13.viii}$$

Solving for x_1, we get $x_1 \approx 816.5$, and substituting back into $x_2 = 1.5x_1$, $x_2 \approx 1224.75$. This bundle costs $0.5(816.5) + 1224.75 = 1633$. Since the household starts with \$1,500, this implies that a monthly cash payment of \$133 would make the household as well off as the public housing program (that costs \$250 per month).

(f) *If you are a policy maker whose aim is to make this household happier at the least cost to the taxpayer, how would you rank the three policies? What if your goal was to increase housing consumption by the household?*

Answer: We have calculated that the public housing policy costs \$250 per month, the rent subsidy costs approximately \$156 per month and the cash subsidy costs \$133 per month. All three policies result in the same level of household utility. So if increasing happiness at the least cost is the goal, the cash subsidy would be best, followed by the rental subsidy and then the public housing program.

We also calculated that housing consumption will be 1500 square feet under public housing, 1053 square feet under the rental subsidy and 816.5 square feet under the cash subsidy. If the goal is to increase housing consumption, the public housing program dominates the rental subsidy which dominates the cash subsidy.

Conclusion: Potentially Helpful Reminders

1. *Important Graphing Hint*: When graphing income and substitution effects, it is very helpful to draw the original indifference curve with lots of substitutability — i.e. with relatively little curvature — unless specifically told to do otherwise. If you do this, it becomes much harder to trick yourself into thinking that something which is logically impossible is actually happening in your graphs.

2. Keep in mind the following: Substitution effects always occur *along a single indifference curve* and income effects always involve *jumping from one indifference curve to another across two parallel budgets.*

3. Since concepts like homotheticity, quasilinearity, normal and inferior goods, and luxuries and necessities are definitions about how indifference curves within an indifference map relate to one another, they are relevant only for determining income effects. In fact, we can get both large and small substitution effects for any of these types of tastes and goods — with the size of the substitution effect depending on the curvature of the original indifference curve (which has no relation to whether goods are normal or inferior or homothetic, etc.).

4. In the text, we emphasize the more common of the two types of substitution effects that economists talk about — the effect that holds "real welfare" fixed and thus occurs along an indifference curve. This effect is also called the *Hicks substitution effect* and it differs from a second type of substitution effect (called the *Slutsky substitution effect*) that assumes a consumer is compensated enough to afford the original bundle (rather than to reach the original indifference curve). This second type of substitution effect is almost identical to the first, particularly for small changes in prices — and it appears in end-of-chapter exercises 7.6 and 7.11 for you to explore.

5. Often students confuse Giffen goods with a certain type of "prestige good" that people value more as it gets more expensive. That is definitely not what a Giffen good is — and you can do end-of-chapter exercise 7.9 to work through the difference between these two types of goods.

6. The math (in part B of *Microeconomics: An Intuitive Approach with Calculus*) follows straightforwardly from the graphical intuitions: Maximize utility subject to budget constraints to get what people do at bundles A and C (when income and substitution effects are combined) — but minimize the expenditure it takes to get to the original utility level at the new prices to find B (and thus the substitution effect).

CHAPTER

8

Wealth and Substitution Effects in Labor and Capital Markets

We introduced income and substitution effects in Chapter 7 and now extend those concepts to budgets that arise from endowments. The most important such budgets involve labor/leisure choices (by workers) and intertemporal consumption choices (by savers and borrowers). What we called "income effects" in Chapter 7 now become "wealth effects" — but the substitution effects remain exactly the same. In principle there is nothing new in this chapter — but it will become clearer why it will generally not work to simply try to memorize which way income (or wealth) effects and substitution effects point for different goods. A much better strategy is to understand the concepts and then be able to apply them to all possible circumstances you might encounter.

Chapter Highlights

The main points of the chapter are:

1. When you own your current consumption bundle, a **price change in either direction makes you better off** (assuming low transactions costs).

2. **The same is not true if** your budget arises from an endowment but **your optimal choice is not the endowment bundle**. If you are a net seller of a good, then an increase in the price of the good makes you better off. If you are a net buyer, an increase in the price might make you better off or worse off depending on the degree of substitutability between the goods.

3. The **substitution effect looks exactly the same way** regardless of whether the budget is exogenous or whether it arises from an endowment. The "wealth" effect, however, may point in a different direction.

4. Under reasonable assumptions about the underlying goods, wealth and substitution effects point in different directions when choices involve labor or

savings decisions. As a result **labor responses to wage changes and savings responses to interest rate changes are ambiguous**. This points to the importance of understanding the degree of substitutability between goods as one thinks about labor and savings responses to price changes.

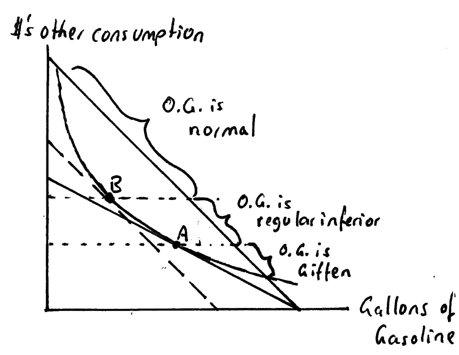
5. The underlying mathematics developed in part B of the chapter is exactly the same for deriving substitution effects, with the only difference emerging as we derive the initial and final optimal bundles where we have to use the endowment-generated budget constraint rather than the previous exogenous budget constraint (in Chapter 7).

8A Solutions to Within-Chapter-Exercises for Part A

Exercise 8A.1

Since George's situation is equivalent to a decrease in the price of other goods (with exogenous income), illustrate where on his final budget George would consume if other goods are normal, regular inferior and Giffen.

<u>Answer</u>: This is done in Exercise Graph 8A.1. Other goods are Giffen if a decrease in their price leads to less consumption than originally at *A*. They are regular inferior if they are not Giffen but an increase in income causes a decrease in consumption. The increase in income is seen from the compensated to the final budget — so if consumption falls between *A* and *B*, other goods are regular inferior. Finally, other goods are normal if an increase in income (from the compensated budget to the final budget) results in an increase in consumption (from *B*).

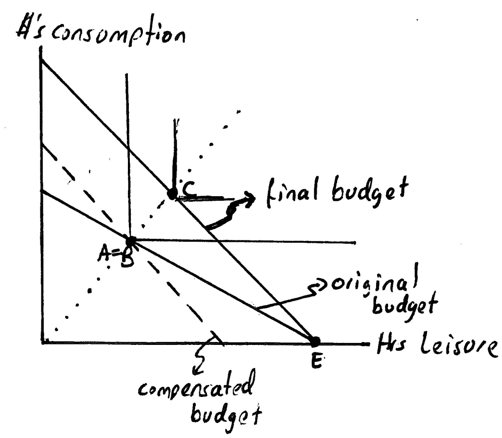


Exercise Graph 8A.1 : Type of "other goods"

Exercise 8A.2

Illustrate substitution and wealth effects — i.e. the initial bundle, the bundle that incorporates a substitution effect from a wage increase, and the final bundle chosen under the wage increase — assuming that your tastes for consumption and leisure are properly modeled as perfect complements.

Answer: This is illustrated in Exercise Graph 8A.2 where the substitution effect disappears ($A = B$) due to the perfect complementarity between consumption and leisure.



Exercise Graph 8A.2 : Wage Increase with Leisure and Consumption Perfect Complements

Exercise 8A.3

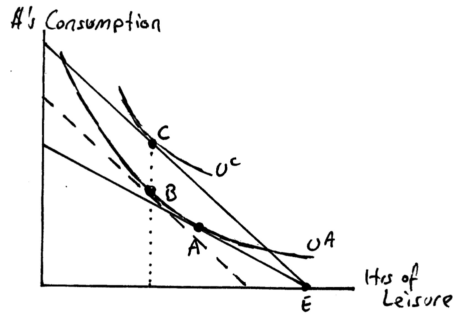
Replicate the previous exercise under the assumption that your tastes are quasilinear in leisure.

Answer: This is illustrated in Exercise Graph 8A.3 where C lies above B because of the lack of an income effect due to the quasilinearity of tastes in leisure. Thus, with respect to leisure, there is no wealth effect.

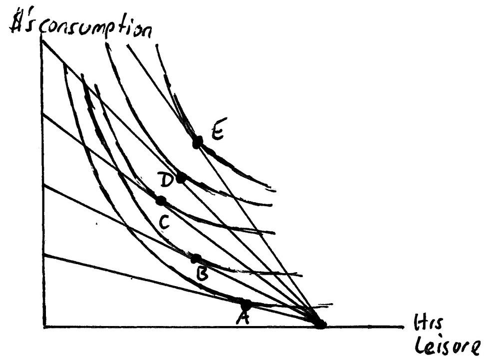
Exercise 8A.4

Illustrate a set of indifference curves that gives rise to the kind of response to wage changes as described.

Answer: This is illustrated in Exercise Graph 8A.4. At the lowest wage in the graph, A is optimal — and entails relatively little labor (and a lot of leisure). As the wage increases, B becomes optimal — with less leisure and more labor. Similarly, labor supply increases as the wage increases further and C becomes optimal. But then, as wage increases again, D is optimal — and involves less labor and more leisure than C. This continues at E where labor supply again falls as wage increases.



Exercise Graph 8A.3 : Wage Increase when Leisure is Quasilinear



Exercise Graph 8A.4 : Tastes for which labor supply initially increases and then decreases as wage goes up

Exercise 8A.5

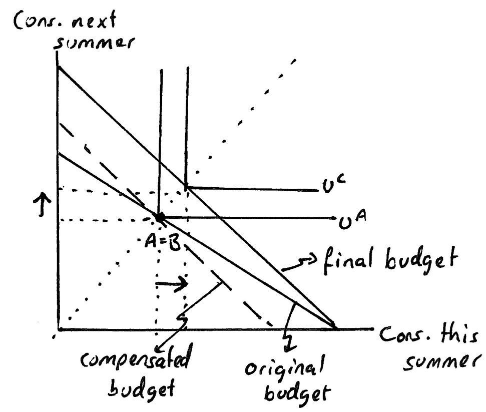
True or False: For decreases in wage taxes, substitution effects put positive pressure on tax revenues while wealth effects typically put negative pressure on revenues.

Answer: This is true. When wage taxes fall, this is (from the workers' perspective) equivalent to an increase in their (take-home) wage. An increase in the wage increases the opportunity cost of consuming leisure — which causes the substitution effect to point in the direction of less leisure, more labor. An increase in the number of hours worked would put upward pressure on tax revenues. The wealth effect of a wage increase, however, typically points in the opposite direction — at least so long as leisure is a normal good. Thus, the wealth effect of a tax decrease causes people to take more leisure and work less — which would put downward pressure on tax revenues from wage taxes.

Exercise 8A.6

Illustrate that your savings will decline with an increase in the interest rate if consumption this summer and next summer are perfect complements.

Answer: Exercise Graph 8A.6 illustrates that consumption in both periods will increase as a result of the increase in the interest rate — with the entire change due to a wealth effect (given that the substitution effect disappears with perfect complements; i.e. given *A*=*B*.) Since consumption this summer increases, the amount you are putting into your savings account decreases.



Exercise Graph 8A.6 : Increasing Interest Rate and Savings

Exercise 8A.7

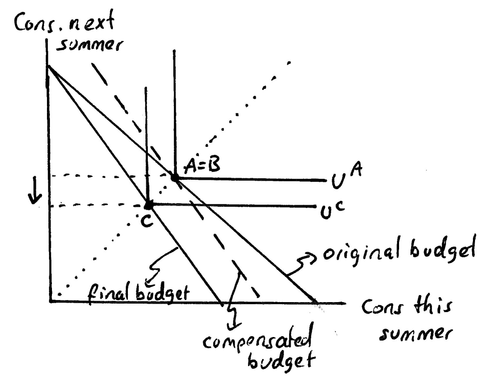
Illustrate how consumption next summer changes with an increase in the interest rate if consumption this summer and next summer are perfect complements (and all your income occurs next summer).

Answer: Exercise Graph 8A.7 illustrates that consumption next summer will unambiguously decline because the only remaining effect is the wealth effect.

Exercise 8A.8

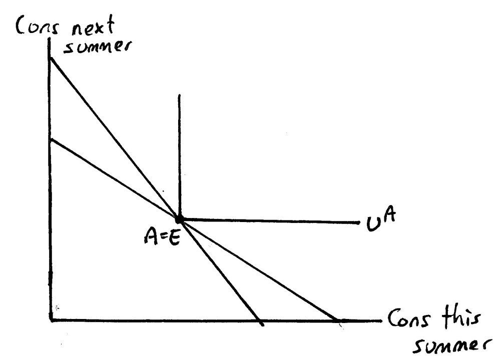
Demonstrate that the only way you will not violate Shakespeare's advice as the interest rate goes up is if consumption this summer and next are perfect complements.

Answer: If there is any curvature in the indifference curve at *A* in Exercise Graph 8A.8, then the new (steeper) budget line will cut the indifference curve from above and will thus make "better" bundles available (which will involve less consumption now; i.e. savings.) Thus, the only way we would not violate Shakespeare's advice to "neither borrow nor lend" is if there was a sufficiently large kink at *A* such that *A*



Exercise Graph 8A.7 : Increasing Interest Rate and Borrowing

once again is optimal on the new (steeper) budget. This is certainly the case when consumption now and consumption in the future are perfect complements. (It is technically also true for indifference curves with less extreme kinks which technically don't represent perfect complements.)

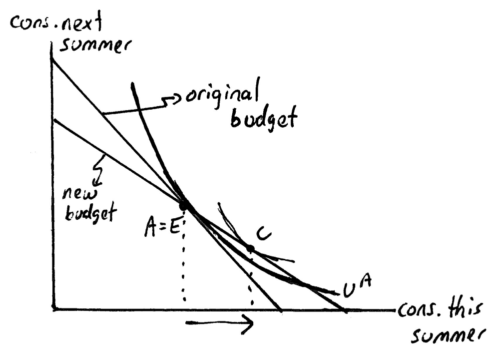


Exercise Graph 8A.8 : Sticking by Shakespeare's Advice

Exercise 8A.9

Illustrate that (unless consumption this summer and consumption next summer are perfect complements) you will violate the first part of Shakespeare's advice — not to be a borrower — if the interest rate fell instead of rose.

Answer: This is illustrated in Exercise Graph 8A.9 where the initial budget constraint is the one with steeper slope (i.e. higher interest rate), with A=E optimal where u^A is tangent to the original budget. The new budget with shallower slope (i.e. lower interest rate) then cuts the original budget at E — which implies it cuts the original indifference curve u^A from below. That in turn opens a number of new bundles that lie above u^A — all of which lie to the right of A. (One example of a possible new optimum is C). Thus, you will consume more now — which means you will borrow.



Exercise Graph 8A.9 : Violating Shakespeare in the other direction

8B Solutions to Within-Chapter-Exercises for Part B

Exercise 8B.1

With the numbers in the previous paragraph, George's income is $2,000 per week. Verify that you would get the same optimal consumption bundle if you modeled this as a constrained optimization problem in which income was exogenously set at $2,000 per week.

Answer: In that case, the budget constraint would be $2x_1 + x_2 = 2000$. We would then solve the problem

$$\max_{x_1,x_2} x_1^{0.1}x_2^{0.9} \text{ subject to } 2x_1 + x_2 = 2000. \tag{8B.1.i}$$

The Lagrange function for this problem is

$$\mathscr{L}(x_1,x_2,\lambda) = x_1^{0.1}x_2^{0.9} + \lambda(2000 - 2x_1 - x_2). \tag{8B.1.ii}$$

The first two first order conditions for this problem are then

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial x_1} &= 0.1x_1^{-0.9}x_2^{0.9} - 2\lambda = 0,\\ \frac{\partial \mathscr{L}}{\partial x_2} &= 0.9x_1^{0.1}x_2^{-0.1} - \lambda = 0.\end{aligned} \tag{8B.1.iii}$$

Taking the λ terms to the right hand side and then dividing one equation by the other (and thus eliminating λ), we can solve for x_2 in terms of x_1 to get $x_2 = 18x_1$. Plugging this back into the budget constraint, we get $2x_1 + 18x_1 = 2000$ or $x_1 = 100$. Finally, plugging this back into $x_2 = 18x_1$ gives us $x_2 = 1800$.

Exercise 8B.2

Verify that the above solutions are correct.

Answer: The Lagrange function for this problem is

$$\mathscr{L}(x_1,x_2,\lambda) = 4x_1 + x_2 + \lambda(1348 - x_1^{0.1}x_2^{0.9}). \tag{8B.2.i}$$

The first two first order conditions for this problem are then

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial x_1} &= 4 - 0.1\lambda x_1^{-0.9}x_2^{0.9} = 0,\\ \frac{\partial \mathscr{L}}{\partial x_2} &= 1 - 0.9\lambda x_1^{0.1}x_2^{-0.1} = 0.\end{aligned} \tag{8B.2.ii}$$

Adding the negative terms to both sides and then dividing the two equations by one another, we can eliminate the λ terms and can solve for $x_2 = 36x_1$. Plugging this into the constraint, we get $x_1^{0.1}(36x_1)^{0.9} = 1348$ which solves to $x_1 = 53.58$. Plugging this back into $x_2 = 36x_1$, we also get $x_2 \approx 1929$.

Exercise 8B.3

How much (negative) compensation was required to get George to be equally well off when the price of gasoline increased?

Answer: The bundle $(53.58, 1929)$ costs $4(53.58) + 1929 \approx 2143$. At a price of \$4 per gallon, George's 1000 gallons per week are worth \$4,000 per week. Since he only needs \$2,143 to remain as happy as he was when the price per gallon was \$2, the necessary compensation is approximately $-\$1,857$.

Exercise 8B.4

Solve the problem defined in equation (8.11).

Answer: The Lagrange function is

$$\mathscr{L}(c, \ell, \lambda) = c + 25\ell + \lambda(1998 - c - 400 \ln \ell). \tag{8B.4.i}$$

The first two first order conditions for this problem are then

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial c} &= 1 - \lambda = 0, \\ \frac{\partial \mathscr{L}}{\partial \ell} &= 25 - \frac{400\lambda}{\ell} = 0. \end{aligned} \tag{8B.4.ii}$$

Solving these, we get $\ell = 16$. Plugging this back into the constraint $c + 400 \ln \ell = 1998$, we get $c \approx 889$.

Exercise 8B.5

Suppose your tastes were more accurately modeled by the Cobb–Douglas utility function $u(c, \ell) = c^{0.5}\ell^{0.5}$. Determine wealth and substitution effects — and graph your answer.

Answer: First, we can determine the bundles A and C — the initial and final bundles — by solving the usual maximization problem

$$\max_{c,\ell} \ c^{0.5}\ell^{0.5} \ \text{ subject to } \ c = w(60 - \ell). \tag{8B.5.i}$$

The Lagrange function for this problem is

$$\mathscr{L}(c, \ell, \lambda) = c^{0.5}\ell^{0.5} + \lambda(60w - w\ell - c). \tag{8B.5.ii}$$

The first two first order conditions for this problem are then

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c} &= 0.5c^{-0.5}\ell^{0.5} - \lambda = 0,\\ \frac{\partial \mathcal{L}}{\partial \ell} &= 0.5c^{0.5}\ell^{-0.5} - \lambda w = 0.\end{aligned} \tag{8B.5.iii}$$

Solving these to eliminate λ, we get $c = w\ell$. Substituting into the budget constraints, $c = w(60-\ell)$, we get $w\ell = w(60-\ell)$ which solves to $\ell = 30$. Finally, plugging this back into $c = w(60-\ell)$, we get $c = 30w$.

At the initial wage of $w = 20$, the optimal bundle A is therefore $(c,\ell) = (600,30)$. At the new wage $w = 25$, the optimal bundle C is $(c,\ell) = (750,30)$.

To decompose this into substitution and wealth effects, we need to determine bundle B — the bundle we would consume if we only faced a change in opportunity costs but a change in wealth that kept us on the same indifference curve. First, we need to calculate the utility at A — which is $u^A = u(600,30) = 600^{0.5}30^{0.5} \approx 134.164$. We then solve the problem

$$\min_{c,\ell}\; c + 25\ell \;\text{ subject to }\; c^{0.5}\ell^{0.5} = 134.164. \tag{8B.5.iv}$$

The Lagrange function for this problem is

$$\mathcal{L}(c,\ell,\lambda) = c + 25\ell + \lambda(134.164 - c^{0.5}\ell^{0.5}). \tag{8B.5.v}$$

The first two first order conditions for this problem are then

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c} &= 1 - 0.5\lambda c^{-0.5}\ell^{0.5} = 0,\\ \frac{\partial \mathcal{L}}{\partial \ell} &= 25 - 0.5\lambda c^{0.5}\ell^{-0.5} = 0.\end{aligned} \tag{8B.5.vi}$$

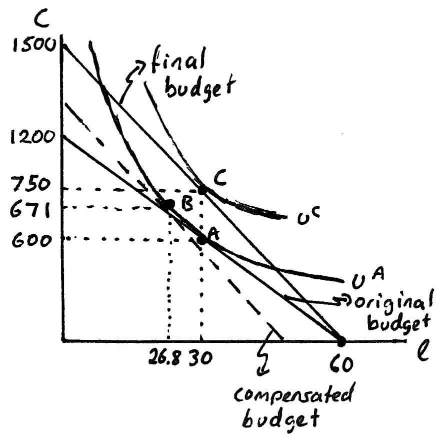
Solving these to eliminate λ, we get $c = 25\ell$. Plugging this into the constraint, we get $(25\ell)^{0.5}\ell^{0.5} = 134.164$ which reduces to $\ell \approx 26.83$. Finally, substituting back into $c = 25\ell$, we can derive $c \approx 670.82$. Thus, bundle B is $(c,\ell) = (26.83, 670.82)$ — and the movement from A to B is the substitution effect while the movement from B to C is the wealth effect. This is depicted in Exercise Graph 8B.5. In terms of leisure, the substitution and wealth effects are directly offsetting.

Exercise 8B.6

What is the equation for the Laffer Curve in Graph 8.10?

Answer: This is given by the tax rate t times the wage income which is the wage w times the amount of labor provided; i.e.

$$t\left[25\left(60 - \frac{400}{25(1-t)}\right)\right] = t\left(1500 - \frac{400}{(1-t)}\right) = 1500t - \frac{400t}{(1-t)} \tag{8B.6}$$



Exercise Graph 8B.5 : Wage Increase with Cobb-Douglas Tastes

Exercise 8B.7

Solve for the peak of the Laffer Curve (using the equation you derived in the previous exercise) and verify that it occurs at a tax rate of approximately 48.4%.

Answer: The peak occurs where the derivative of the function is equal to zero; i.e. where

$$1500 - \frac{400}{(1-t)} - \frac{400t}{(1-t)^2} = 0. \tag{8B.7.i}$$

Dividing though by 100 and multiplying by $(1-t)^2$, this turns into

$$15(1-t)^2 - 4(1-t) - 4t = 0. \tag{8B.7.ii}$$

Then, multiplying out the terms and combining like terms, we get

$$15t^2 - 30t + 11 = 0. \tag{8B.7.iii}$$

The quadratic formula tells us that any equation of the form $ax^2+bx+c=0$ has two solutions given by

$$x = \frac{-b+(b^2-4ac)^{0.5}}{2a} \text{ and } x = \frac{-b-(b^2-4ac)^{0.5}}{2a}. \tag{8B.7.iv}$$

In our equation, $x = t$, $a = 15$, $b = -30$ and $c = 11$. The two solutions given by the quadratic formula are then

$$\begin{aligned} t &= \frac{30-(30^2-4(15)(11))^{0.5}}{2(15)} \approx 0.4836 \text{ and} \\ t &= \frac{30+(30^2-4(15)(11))^{0.5}}{2(15)} \approx 1.517. \end{aligned} \tag{8B.7.v}$$

The latter occurs outside the economically relevant range of possible tax rates (which can range from 0 to 1) — which leaves us with the first solution that verifies what is graphed in the text.

Exercise 8B.8

Verify that this is indeed the solution to the problem defined in (8.15).
Answer: The Lagrange function for this problem is

$$\mathscr{L}(c_1,c_2,\lambda) = c_1^{0.5}c_2^{0.5} + \lambda(10000(1+r) - (1+r)c_1 - c_2). \tag{8B.8.i}$$

The first two first order conditions for this problem are then

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial c_1} &= 0.5c_1^{-0.5}c_2^{0.5} - \lambda(1+r) = 0,\\ \frac{\partial \mathscr{L}}{\partial c_2} &= 0.5c_1^{0.5}c_2^{-0.5} - \lambda = 0.\end{aligned} \tag{8B.8.ii}$$

These solve to $c_2 = (1+r)c_1$. Plugging this into the budget constraint, we get $10000(1+r) = (1+r)c_1 + (1+r)c_1$ which we can solve to get $c_1 = 5000$. Plugging this back into $c_2 = (1+r)c_1$, we can also solve for $c_2 = 5000(1+r)$.

Exercise 8B.9

Verify that this is indeed the solution to the problem defined in (8.17).
Answer: The Lagrange function for this problem is

$$\mathscr{L}(c_1,c_2,\lambda) = (1+r)c_1 + c_2 + \lambda\left(5244 - c_1^{0.5}c_2^{0.5}\right). \tag{8B.9.i}$$

The first two first order conditions for this problem are then

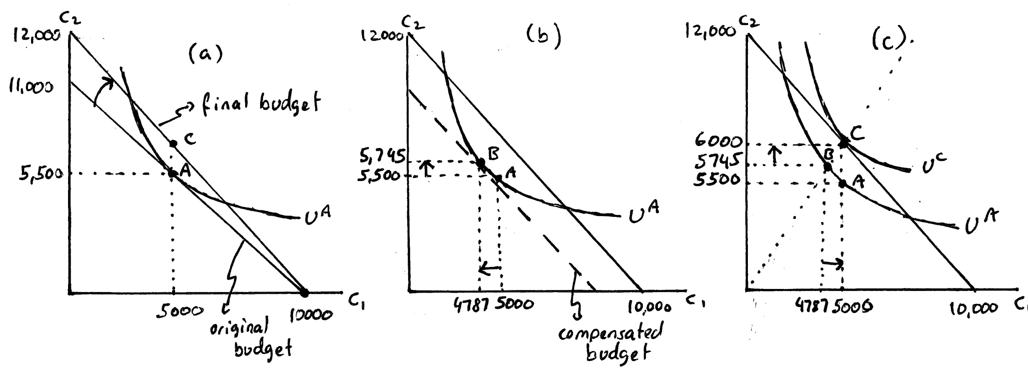
$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial c_1} &= (1+r) - 0.5\lambda c_1^{-0.5}c_2^{0.5} = 0,\\ \frac{\partial \mathscr{L}}{\partial c_2} &= 1 - 0.5\lambda c_1^{0.5}c_2^{-0.5} = 0.\end{aligned} \tag{8B.9.ii}$$

Solving these to eliminate λ, we again get $c_2 = (1+r)c_1$. Plugging this into the constraint, we get $c_1^{0.5}[(1+r)c_1]^{0.5} = 5244$ which solves to $c_1 = 5244/(1+r)^{0.5}$. Plugging this back into $c_2 = (1+r)c_1$, we also get $c_2 = 5244(1+r)^{0.5}$. When $r = 0.2$, this implies that $c_1 = 5244/(1+0.2)^{0.5} \approx 4,787.1$ and $c_2 = 5244(1+0.2)^{0.5} \approx 5,744.5$. (The answers are off slightly from what is derived in the text because we used the rounded value 5244 for utility.)

Exercise 8B.10

Using a set of graphs similar to those depicted in Graph 8.5, label the bundles that we have just calculated.

Answer: This is done in panels (a) through (c) of Exercise Graph 8B.10. Panel (a) indicates the original optimal bundle A=(5000,5500) and the change in the intertemporal budget constraint with an increase in the interest rate. The final optimum C=(5000,6000) is also anticipated in panel (a). Panel (b) focuses on the substitution effect by illustrating the compensated budget tangent to the original indifference curve at B=(4787,5745), with the arrows indicating the substitution effect relative to consumption in each period. (The effect, as always, moves us away from consumption where it has become more expensive and toward where it has become cheaper.) Finally, panel (c) illustrates (again with arrows) the wealth effect that moves us from the original indifference curve u^A to the final indifference curve u^C. Since consumption in both periods is a normal good, the wealth effect is positive for consumption in both periods — and exactly offsets the substitution effect with respect to current consumption.



Exercise Graph 8B.10 : Interest Rate Increase and Saving with Cobb-Douglas Tastes

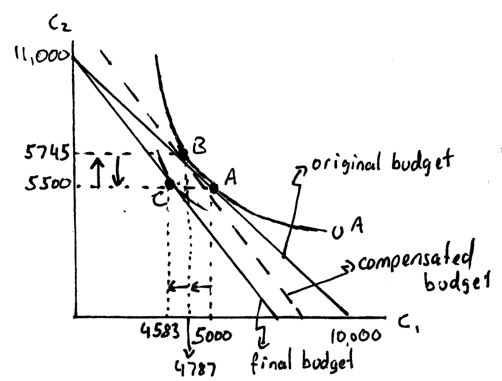
Exercise 8B.11

Illustrate what we have just calculated in a graph.

Answer: This is done in Exercise Graph 8B.11 where the wealth and substitution effects are offsetting on the vertical axis but point in the same direction on the horizontal.

Exercise 8B.12

We calculated above that consumption next summer is unchanged as the interest rate rises when tastes can be represented by the Cobb-Douglas utility function



Exercise Graph 8B.11 : Interest Rate Increase and Borrowing with Cobb-Douglas Tastes

we used. This is because this function assumes an elasticity of substitution of 1. How would this result change if the elasticity of substitution is larger or smaller than 1?

Answer: If the elasticity of substitution is greater than 1, then the substitution effect increases in magnitude and will dominate — implying that consumption next summer increases as the interest rate rises. If, on the other hand, the elasticity of substitution is less than 1, then the substitution effect decreases in magnitude and will be dominated by the wealth effect — implying that consumption next summer decreases as the interest rate rises.

Exercise 8B.13

Verify that (8.22) and (8.33) are correct.

Answer: The maximization problem is

$$\max_{c_1,c_2} c_1^{0.5}c_2^{0.5} \text{ subject to } (1+r)c_1+c_2 = 5000(1+r)+5500. \tag{8B.13.i}$$

This results in the Lagrange function

$$\mathcal{L}(c_1,c_2,\lambda) = c_1^{0.5}c_2^{0.5} + \lambda(5000(1+r)+5500-(1+r)c_1-c_2). \tag{8B.13.ii}$$

The first two first order conditions for this problem are then

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial c_1} &= 0.5c_1^{-0.5}c_2^{0.5} - \lambda(1+r) = 0,\\ \frac{\partial\mathcal{L}}{\partial c_2} &= 0.5c_1^{0.5}c_2^{-0.5} - \lambda = 0.\end{aligned} \tag{8B.13.iii}$$

As before, this solves to $c_2 = (1+r)c_1$. Substituting this into the constraint, we get $5000(1+r)+5500 = (1+r)c_1+(1+r)c_1$ which solves to $c_1 = 2500+(2750/(1+r)$

and, substituting this back into $c_2 = (1+r)c_1$, $c_2 = 2500(1+r)+2750$. At the interest rate $r = 0.1$, we therefore start at $c_1 = 2500 + (2750/1.1) = 5000$ and $c_2 = 1.1(2500) + 2750 = 5500$. When the interest rate rises to 0.2, $c_2 = 2500 + (2750/1.2) = 4791.67$ and $c_2 = 1.2(2500) + 2750 = 5750$. We therefore move from A=(5000,5500) to $C = (4792, 5750)$.

We find point B by first determining the utility at bundle A — i.e. we calculate that $u^A = 5000^{0.5}5500^{0.5} \approx 5244$. We then solve the problem

$$\min_{x_1,x_2} \; 1.2c_1 + c_2 \;\text{ subject to }\; c_1^{0.5}c_2^{0.5} = 5244. \tag{8B.13.iv}$$

After writing down the Lagrange function, we can solve the first two first order conditions as before — giving us $c_2 = 1.2c_1$. Plugging this back into the constraint $c_1^{0.5}c_2^{0.5} = 5244$, we get $c_1^{0.5}(1.2c_1)^{0.5} = 5244$ which solves to $c_1 \approx 4781.1$. Plugging this back into $c_2 = 1.2c_1$, we also get $c_2 \approx 5744.5$. (The numbers taken to two decimals are slightly different than those in the text because we rounded when we set utility equal to 5244.)

8C Solutions to Odd Numbered End-of-Chapter Exercises

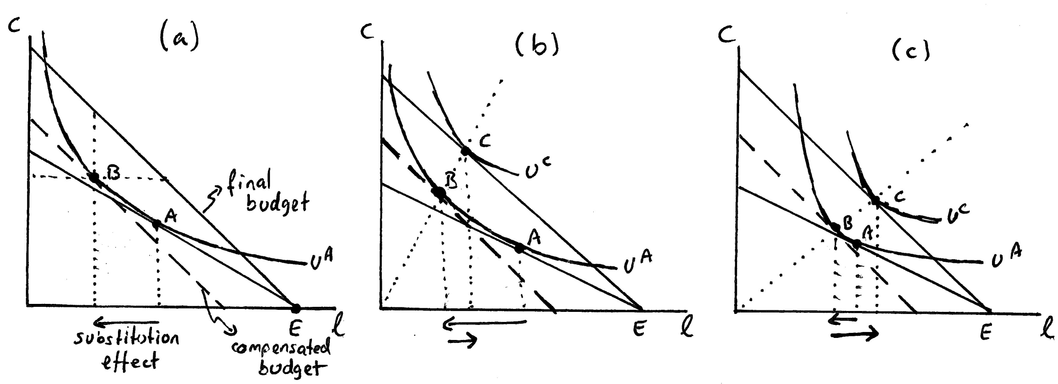
Exercise 8.1

As we have suggested in the chapter, it is often important to know whether workers will work more or less as their wage increases.

A: *In each of the following cases, can you tell whether a worker will work more or less as his wage increases?*

(a) *The worker's tastes over consumption and leisure are quasilinear in leisure.*

Answer: Panel (a) of Exercise Graph 8.1 illustrates the substitution effect for a wage increase. This effect depends only on the shape of the indifference curve that goes through the original bundle A — the more substitutable consumption and leisure are, the greater the substitution from leisure (and thus toward more labor) to consumption. If tastes are then quasilinear in leisure, we know that, as we move from the compensated to the final budget, there is no wealth effect on leisure and thus no further change in leisure (beyond the substitution effect). Thus, the worker will unambiguously work more.



Exercise Graph 8.1 : Wage Increases with Different Tastes

(b) *The worker's tastes over consumption and leisure are homothetic.*

Answer: Panels (b) and (c) of Exercise Graph 8.1 illustrate that it is ambiguous in this case whether the worker will work more or less with an increase in the wage — it depends on the size of the substitution effect. In panel (b), the indifference curve u^A is relatively flat around A — indicating a great deal of willingness on the part of the worker to substitute leisure and consumption. This gives rise to a large substitution effect. B is tangent to the (dashed) compensated budget — which is parallel to the final budget. Homotheticity then implies that, if B is optimal on the compensated budget, the optimal final bundle C lies on a ray from the origin through B. Because of the willingness to substitute between consumption and leisure, the resulting wealth effect only outweighs part of the substitution effect — leaving us with less leisure (and more labor) at the higher wage than at the original lower wage (at A). In panel (c), on the other hand, consumption and leisure are not as substitutable around A — leading to a relatively small substitution effect that is more than outweighed by a wealth effect in the opposite direction. Thus, when consumption and leisure are relatively complementary, an increase in the wage causes an increase in leisure and thus a decrease in work hours.

(c) *Leisure is a luxury good.*

Answer: We can use the same graphs as in panels (b) and (c) to again show that the answer is ambiguous. If leisure is a luxury good, then as the budget shifts out parallel, the new optimal bundle will lie to the right of the ray from the origin through the original optimum (because consumption of leisure increases faster than under homotheticity). In panel (b), that would mean C lies to the right of where it is indicated in the graph — but that still makes it plausible that the wealth effect is smaller than the substitution effect leaving us with less leisure than at A (and thus more work). In panel (c), C will again lie to the right of where it is indicated in the graph — but that implies that the wealth effect is even larger and will still outweigh the substitution effect. This will again leave us with more leisure and thus less work.

(d) *Leisure is a necessity.*

Answer: For reasons analogous to those just cited for luxury goods, the answer is still ambiguous and depends on the size of the substitution effect. This time, C will lie to the left of where it is marked in panels (b) and (c) of the graph — but that still leaves room for the ambiguity.

(e) *The worker's tastes over consumption and leisure are quasilinear in consumption.*

Answer: Going back to panel (a) of the graph, if consumption is the quasilinear good, then it will remain unchanged from the optimal bundle B on the compensated budget to the final budget. This creates a wealth effect on leisure that is opposite to the substitution effect. As drawn in panel (a), it looks like that returns us to a bundle C that will lie right above A — thus returning us to the same leisure consumption (and thus the same amount of work) as before the wage increase. But had we drawn a smaller substitution effect, the horizontal line through B would take us to the right of A on the final budget — thus causing an increase in leisure (and a decrease in work). If, on the other hand, we had made the indifference curve u^A flatter and thus had produced a larger substitution effect, the horizontal line through B would take us to the left of A on the final budget — thus causing a decrease in leisure (and thus an increase in work) from the original optimum A. As is usually the case when we have competing substitution and wealth effects, the answer is therefore again ambiguous.

B: *Suppose that tastes take the form $u(c,\ell) = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho}$.*

(a) *Set up the worker's optimization problem assuming his leisure endowment is L and his wage is w.*

Answer: The problem is

$$\max_{c,\ell} \left(0.5c^{-\rho} + 0.5\ell^{-\rho}\right)^{-1/\rho} \text{ subject to } w(L-\ell) = c. \tag{8.1.i}$$

(b) *Set up the Lagrange function corresponding to your maximization problem.*

Answer: The Lagrange function is

$$\mathcal{L}(c,\ell,\lambda) = \left(0.5c^{-\rho} + 0.5\ell^{-\rho}\right)^{-1/\rho} + \lambda(wL - w\ell - c). \tag{8.1.ii}$$

(c) *Solve for the optimal amount of leisure.*

Answer: The first two first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c} &= 0.5c^{-(\rho+1)}\left(0.5c^{-\rho} + 0.5\ell^{-\rho}\right)^{-(\rho+1)/\rho} - \lambda = 0, \\
\frac{\partial \mathcal{L}}{\partial \ell} &= 0.5\ell^{-(\rho+1)}\left(0.5c^{-\rho} + 0.5\ell^{-\rho}\right)^{-(\rho+1)/\rho} - \lambda w = 0.
\end{aligned} \tag{8.1.iii}$$

The problem simplifies quite a bit if we simply take the λ terms to the other side of each equation and then divide the second equation by the first — which gives

$$\left(\frac{c}{\ell}\right)^{(\rho+1)} = w. \tag{8.1.iv}$$

If you remember the expression of the MRS for a CES utility function from Chapter 5, you could have just skipped to this equation — which simply says the MRS is equal to the slope of the budget. The equation can then be written in terms of just $c = \ell w^{1/(\rho+1)}$. When plugged into the budget constraint $w(L-\ell) = c$, we can solve for

$$\ell = \frac{L}{1 + w^{-\rho/(\rho+1)}}. \tag{8.1.v}$$

(d) *Does leisure consumption increase or decrease as w increases? What does your answer depend on?*

Answer: We can see whether leisure increases or decreases with the wage rate by checking whether the first derivative of the equation for optimal leisure consumption from the previous exercise is positive or negative. This derivative is (after a little algebra)

$$\frac{\partial \ell}{\partial w} = \rho \left[\frac{L\left(1+w^{-\rho/(\rho+1)}\right)^{-2}}{(\rho+1)\,w^{(2\rho+1)/(\rho+1)}} \right] \tag{8.1.vi}$$

Note L and w are positive and, since ρ lies between -1 and ∞, $(\rho+1)$ is also positive. This implies that the entire term in the square brackets must be positive regardless of what value ρ takes. (The negatives in the exponents of course only affect whether the term appears in the numerator or denominator — not whether it is positive or not.) Since the bracketed term is positive, the sign of the derivative depends entirely on whether ρ is positive or negative.

If $\rho = 0$, the tastes are Cobb-Douglas with elasticity of substitution $1/(1-\rho) = 1$. In that case, $\partial\ell/\partial w = 0$ and the wage therefore does not affect leisure consumption (or labor supply). For $\rho < 0$ the elasticity of substitution is greater than 1 — and $\partial\ell/\partial w < 0$. Thus, as the elasticity of substitution rises above 1, leisure consumption declines with an increase in the wage — and work hours increase. For $\rho > 0$, on the other hand, the elasticity of substitution is less than 1 — and $\partial\ell/\partial w > 0$. Thus, as the elasticity of substitution falls below 1, leisure consumption increases with the wage — and work hours fall. (Note that the elasticity of substitution is $\sigma = 1/(1+\rho)$.)

(e) *Relate this to what you know about substitution and wealth effects in this type of problem.*

Answer: We have seen in part A of the question that the substitution effect points to less leisure (and more work) as wage increases — and, so long as leisure is a normal good, the wealth effect points in the opposite direction. For homothetic tastes (which CES tastes are), we showed that the overall effect of a wage increase on leisure consumption then depends on the substitutability of consumption and leisure. The greater the substitutability, the larger is the substitution effect — and the larger the substitution effect, the less likely it is that the wealth effect can fully offset it. We now see that for CES utility functions, the direction of the effect of a wage increase on leisure consumption depends entirely on ρ which determines the elasticity of substitution or the degree of substitutability between consumption and leisure. Elasticities below 1 make indifference curves look more like those in panel (c) of Exercise Graph 8.1 — with the wealth effect outweighing the substitution effect. Elasticities above 1, on the other hand, make the indifference curve look more like those in panel (b) where the substitution effect outweighs the wealth effect.

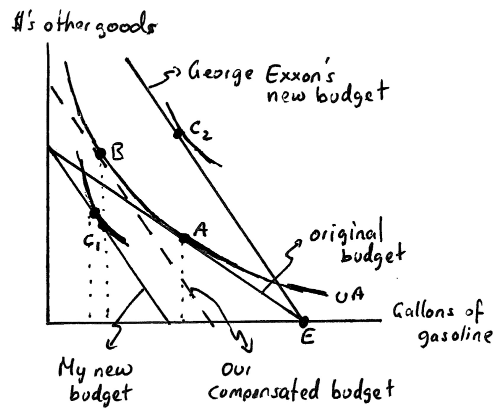
Exercise 8.3

In this chapter, we began by considering the impact of an increase in the price of gasoline on George Exxon who owns a lot of gasoline. In this exercise, assume that George and I have exactly the same tastes and that gasoline and other goods are both normal goods for us.

A: *Unlike George Exxon, however, I do not own gasoline but simply survive on an exogenous income provided to me by my generous wife.*

(a) *With gallons of gasoline on the horizontal and dollars of other goods on the vertical, graph the income and substitution effects from an increase in the price of gasoline.*

Answer: Exercise Graph 8.3(1) illustrates the original budget as the budget line tangent at the original optimum A that lies on the indifference curve u^A. Then the compensated budget line is illustrated as the dashed line, with B on u^A tangent to it. The move from A to B is the substitution effect. My final budget then lies parallel below the compensated budget — with a bundle like C_1 representing my new optimum and the move from B to C_1 representing the income effect.



Exercise Graph 8.3(1) : George Exxon and Me

(b) *Suppose George (who derives all his income from his gasoline endowment) had exactly the same budget before the price increase that I did. On the same graph, illustrate how his budget changes as a result of the price increase.*

Answer: Rather than rotating inward with a fixed point on the vertical axis, the budget rotates outward around his fixed endowment point. The new budget is indicated in the graph. It has the same slope as my new budget because we both face the same prices, but George's lies further out because he owns gasoline and thus saw his wealth increase as a result of the increase in the price of gasoline (while I saw mine decrease.)

(c) *Given that we have the same tastes, can you say whether the substitution effect is larger or smaller for George than it is for me?*

Answer: The substitution effect is exactly the same for me and George. We both start at A on the indifference curve u^A — and we both face the same compensated budget because we both face the same price increase. Thus, B is the same for George as it is for me.

(d) *Why do we call the change in behavior that is not due to the substitution effect an* income effect *in my case but a* wealth effect *in George Exxon's case?*

Answer: In my case, I simply have an exogenous income level that is not tied to any price in the economy. As a result, any price increase will make me worse off because my income can buy less. I change consumption partly because of the change in opportunity costs (giving rise to the substitution effect) but also because my "real income" has changed — thus the term income effect. The story for George is a bit different — he owns gasoline, and the value of his wealth from which he can draw income therefore depends on the price of gasoline. While an increase in the price of gasoline makes me worse off and decreases my "real income", the same increase in price actually makes George better off because it increases how much income he can draw from what he owns. Put differently, George's wealth has increased because something he owns has become more valuable — and this will impact his consumption behavior (in addition to the impact of the change in opportunity costs).

B: *In Section 8B.1, we assumed the utility function $u(x_1, x_2) = x_1^{0.1}x_2^{0.9}$ for George Exxon as well as an endowment of gasoline of 1000 gallons. We then calculated substitution and wealth effects when the price of gasoline goes up from \$2 to \$4 per gallon.*

(a) *Now consider me with my exogenous income $I = 2000$ instead. Using the same utility function we used for George in the text, derive my optimal consumption of gasoline as a function of p_1 (the price of gasoline) and p_2 (the price of other goods).*

Answer: Solving

$$\max_{x_1,x_2} x_1^{0.1}x_2^{0.9} \text{ subject to } p_1x_1 + p_2x_2 = 2000, \tag{8.3.i}$$

we can derive $x_1 = 200/p_1$ and $x_2 = 1800/p_2$.

(b) *Do I consume the same as George Exxon prior to the price increase? What about after the price increase?*

Answer: Prior to the price increase, $p_1 = 2$ — thus $x_1 = 200/2 = 100$ which is the same as we calculated in the text for George Exxon. After the price increase, I consume 200/4=50 gallons of gasoline — less than we calculated for George.

(c) *Calculate the substitution effect from this price change and compare it to what we calculated in the text for George Exxon.*

Answer: To calculate the substitution effect, we first have to know how much utility I get before the price increase. We already calculated that $x_1 = 100$, and we can similarly calculate that $x_2 = 1800/1 = 1800$ (since $p_2 = 1$). My original bundle is therefore $A = (100, 1800)$ — which gives utility $u^A = u(100, 1800) = 100^{0.1}1800^{0.9} \approx 1348$ — same as for George Exxon. Then we ask what the least is that we could spend and reach this utility level again after the price increase; i.e. we solve

$$\min_{x_1, x_2} 4x_1 + x_2 \text{ subject to } x_1^{0.1}x_2^{0.9} = 1348. \tag{8.3.ii}$$

Note that this is exactly the same problem we wrote down to determine the substitution effect for George Exxon in the text — because we are asking exactly the same question. Thus we get the same answer — $x_1 = 53.59$ and $x_2 = 1929.19$.

(d) *Suppose instead that the price of "other goods" fell from \$1 to 50 cents while the price of gasoline stayed the same at \$2. What is the change in my consumption of gasoline due to the substitution effect? Compare this to the substitution effect you calculated for the gasoline price increase above.*

Answer: We would now solve

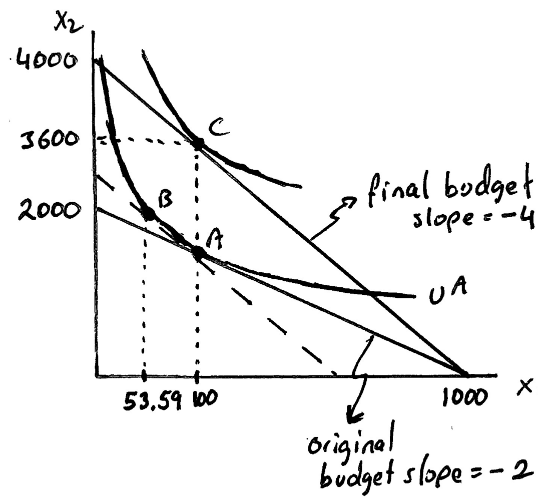
$$\min_{x_1, x_2} 2x_1 + 0.5x_2 \text{ subject to } x_1^{0.1}x_2^{0.9} = 1348. \tag{8.3.iii}$$

Note that, while this problem looks different from the problem described in (8.3.ii), it will necessarily give the same answer because the *ratio* of the prices is exactly the same, as is the indifference curve we are trying to fit the compensated budget to. Thus, the solution will be $x_1 = 53.59$ and $x_2 = 1929.19$ — i.e. the substitution effect is the same.

(e) *How much gasoline do I end up consuming? Why is this identical to the change in consumption we derived in the text for George when the price of gasoline increases? Explain intuitively using a graph.*

Answer: In B(a) we derived my optimal consumption to be $x_1 = 200/p_1$ and $x_2 = 1800/p_2$. When $p_1 = 2$ and $p_2 = 0.5$, this implies $x_1 = 100$ and $x_2 = 3600$.

The price decrease of x_2 implies that my budget will rotate outward around the horizontal intercept of my original budget (at 1000). The price increase of x_1 will cause George's budget to similarly rotate around the horizontal intercept of his original budget — which, in his case, is his endowment bundle. Thus, a decrease in p_2 caused a qualitatively similar change in my budget as an increase in p_1 does in George's budget. The two are quantitatively the same — resulting in the same final budget — if the ratio p_1/p_2 ends up being the same. Starting at $p_1 = 2$ and $p_2 = 1$, a decrease in p_2 to 0.5 causes that ratio to change from 2/1 to 2/0.5 = 4. Similarly, an increase in p_1 to 4 will cause the ratio to change to 4. Thus, George and I experience the same change in our economic circumstances under the two scenarios — which is why the change in behavior is the same (assuming we have the same tastes). This is illustrated in Exercise Graph 8.3(2).



Exercise Graph 8.3(2) : George Exxon and Me: Part II

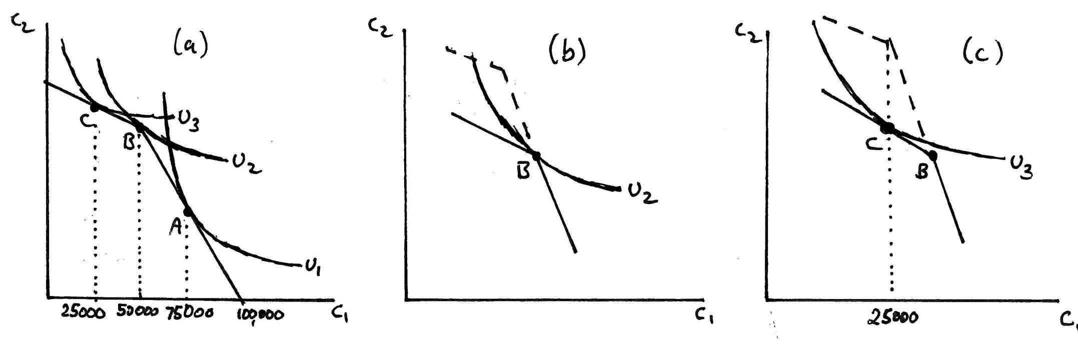
Exercise 8.5

Policy Application: *Savings Behavior and Tax Policy: Suppose you consider the savings decisions of three households - households 1, 2 and 3. Each household plans for this year's consumption and next year's consumption, and each household anticipates earning \$100,000 this year and nothing next year. The real interest rate is 10%. Assume throughout that consumption is always a normal good.*

A: *Suppose the government does not impose any tax on interest income below \$5,000 but taxes any interest income above \$5,000 at 50%.*

(a) *On a graph with "Consumption this period" (c_1) on the horizontal axis and "Consumption next period" (c_2) on the vertical, illustrate the choice set faced by each of the three households.*

Answer: Panel (a) of Exercise Graph 8.5(1) illustrates the shape of the budget constraint which has a kink at \$50,000 of consumption now (c_1) because, when consumption now is \$50,000, then savings is also \$50,000 — which, at a 10% interest rate, results in \$5,000 of interest income. This first \$5,000 of interest income is exempt — which means the slope of the lower part of the budget constraint is simply $-(1+r) = -(1+0.1) = -1.1$. At current consumption below \$50,000, however, savings are above \$50,000 — which means interest income is above \$5,000. Thus, as interest income goes above \$5,000 at \$50,000 of savings, the slope of the budget constraint becomes shallower because the government now taxes the additional interest income at 50%. To be specific, the slope goes to $-(1+0.5r) = -1.05$.



Exercise Graph 8.5(1) : Savings of 3 Households

(b) *Suppose you observe that household 1 saves $25,000, household 2 saves $50,000 and household 3 saves $75,000. Illustrate indifference curves for each household that would make these rational choices.*

Answer: Panel (a) of the graph also indicates three indifference curves that make the choices of the 3 households optimal ones — with each indifference curve labeled by the relevant household. (For instance, u_1 refers to the optimal indifference curve from household 1's indifference map, where household 1 is the household that saves $25,000 and thus consumes $75,000 now.)

(c) *Now suppose the government changes the tax system by exempting the first $7,500 rather than the first $5,000 from taxation. Thus, under the new tax, the first $7,500 in interest income is not taxed, but any interest income above $7,500 is taxed at 50%. Given what you know about each household's savings decisions before the tax change, can you tell whether each of these households will now save more? (*Note*: It is extremely difficult to draw the scenarios in this question to scale — and when not drawn to scale, the graphs can become confusing. It is easiest to simply worry about the general shapes of the budget constraints around the relevant decision points of the households that are described.)*

Answer: This policy change would extend the steep portion of the budget from $50,000 in current consumption to $25,000 in current consumption (where savings hits $75,000 and thus interest income hits $7,500). Household 1 would be unaffected by this change since the indifference curve u_1 that is tangent at A lies above any new bundle that becomes available as a result of the policy change. Thus, household 1's savings would not change.

Household 2's savings, on the other hand, would almost certainly increase. In order for B to be optimal before the policy change, this household has an indifference curve that "hangs" on the kink of the original budget constraint. That means the MRS could lie between -1.1 (which is the slope of the steep portion of the budget) and -1.05 (which is the slope of the shallower portion). If the $MRS = -1.1$ at B, then the indifference curve u_2 is tangent to the extended steep budget that runs through B after the policy change — and thus B would continue to be optimal. However, if the MRS falls anywhere from -1.05 to -1.1 at B, then the new (dashed) budget constraint will cut the indifference curve u_2 as illustrated in panel (b) of Exercise Graph 8.5(1) — thus enabling the household to choose from a set of new bundles that lie above the original indifference curve. All of these bundles are such that consumption now (c_1) falls — i.e. savings increases.

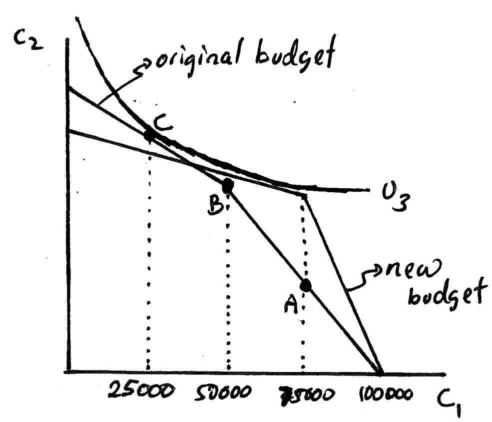
Household 3, however, will definitely not save more. Panel (c) of the graph illustrates the change for this household. The new kink point now happens right above C. If the household were to choose a bundle on the flat portion of the new (dashed) budget line, then c_1 would be an inferior good and we have assumed that consumption is always normal. (It would be inferior because, when faced with a parallel outward shift in the budget, the household would be choosing to consume less.) Thus we know that the household will choose either the kink point (and keep savings the same) or a point on the steeper portion of the new (dashed) budget — with more c_1 and thus less savings.

(d) *Instead of the tax change in part (c), suppose the government had proposed to subsidize interest income at 100% for the first \$2,500 in interest income while raising the tax on any interest income above \$2,500 to 80%. (Thus, if someone earns \$2,500 in interest, she would receive an additional \$2,500 in cash from the government. If someone earns \$3,500, on the other hand, she would receive the same \$2,500 cash subsidy but would also have to pay \$800 in a tax.) One of the three households is overheard saying:"I actually don't care whether the old policy (i.e. the policy described in part A) or this new policy goes into effect." Which of the three households could have said this, and will that household save more or less (than under the old policy) if this new policy goes into effect?*

Answer: By subsidizing savings initially, the government in effect raises the interest rate from 10% to 20% for the first \$25,000 in savings. Thus, beginning at the \$100,000 intercept on the c_1 axis, the budget constraint is twice as steep. From that point on, however, the government is in effect reducing the interest rate from 10% to 2% because of the 80% tax on interest income. Thus, beginning at \$75,000 of current (c_1) consumption and moving leftward, the budget constraint becomes shallower than it was before. It seems clear that the two budget constraints will cross at some point — the question is where. We can check, for instance, which budget gives higher consumption next period (c_2) at \$50,000 of savings where the original kink occurred. Under the original policy, you make

\$5,000 in interest when you save \$50,000 — giving you $c_2 = \$55,000$. Under the new policy, you get \$5,000 of interest (including the subsidy) for the first \$25,000 you save, you earn another \$2,500 of interest for the next \$25,000 in savings — but that is taxed at 80% to leave you with only \$500 of after-tax interest income. Thus, your total interest income (including the subsidy and subtracting out the tax) is \$5,500 — leaving you with \$55,500 in c_2. This is \$500 more than under the original policy. If you save an additional \$25,000 (for a total of \$75,000), you would earn an additional \$2,500 in interest. Under the original policy, half of that would be taxed away, leaving you with \$1,250. Under the new policy, 80% is taxed away — leaving you with only \$500 more. Thus, at \$75,000 of savings, the old policy results in greater c_2 than the new policy — \$250 more, to be exact. The old and the new budgets therefore intersect between \$75,000 and \$50,000 in savings.

The general relationship between the original and the new budget constraints is graphed in Exercise Graph 8.5(2). Households 1 and 2 must prefer the new policy since it opens up new bundles to the northeast of their original optimal bundles. Household 3, however, might be indifferent — as illustrated with the indifference curve u_3. Under the new policy, household 3 would then consume more now — and save less — if indeed it is indifferent between the policies.



Exercise Graph 8.5(2) : Savings of 3 Households: Part II

B: *Now suppose that our 3 households had tastes that can be represented by the utility function $u(c_1, c_2) = c_1^{\alpha} c_2^{(1-\alpha)}$, where c_1 is consumption now and c_2 is consumption a year from now.*

(a) *Suppose there were no tax on savings income. Write down the intertemporal budget constraint with the real interest rate denoted r and current* *income denoted I (and assume that consumer anticipate no income next period).*

Answer: The intertemporal budget constraint is

$$(1+r)c_1 + c_2 = (1+r)I. \tag{8.5.i}$$

(b) *Write down the constrained optimization problem and the accompanying Lagrange function. Then solve for c_1, current consumption, as a function of α, and solve for the implied level of savings as a function of α, I and r. Does savings depend on the interest rate?*

Answer: The maximization problem is

$$\max_{c_1,c_2} c_1^{\alpha} c_2^{(1-\alpha)} \text{ subject to } (1+r)c_1 + c_2 = (1+r)I. \tag{8.5.ii}$$

The Lagrange function for this problem is

$$\mathscr{L}(c_1, c_2, \lambda) = c_1^{\alpha} c_2^{(1-\alpha)} + \lambda((1+r)I - (1+r)c_1 - c_2). \tag{8.5.iii}$$

The first two first order conditions can be solved to yield

$$c_2 = \frac{(1-\alpha)(1+r)}{\alpha} c_1. \tag{8.5.iv}$$

Plugging this into the constraint $(1+r)c_1 + c_2 = (1+r)I$, we can solve for $c_1 = \alpha I$. Savings s is then simply c_1 subtracted from current income; i.e.

$$s = I - \alpha I = (1-\alpha)I. \tag{8.5.v}$$

Savings therefore does not depend on the interest rate.

(c) *Determine the α value for consumer 1 as described in part A.*

Answer: Consumer 1 saves 25% of her income on the portion of the budget where there is no tax — thus, it must be that $(1-\alpha) = 0.25$ or $\alpha = 0.75$.

(d) *Now suppose the initial 50% tax described in part A is introduced. Write down the budget constraint (assuming current income I and before-tax interest rate r) that is now relevant for consumers who end up saving more than \$50,000. (*Note*: Don't write down the equation for the kinked budget — write down the equation for the linear budget on which such a consumer would optimize.)*

Answer: To write down this budget, we need to know an intercept and a slope. The slope is simply $-(1+0.5r)$ since the government is taxing interest income at 50%. We can determine the c_2 intercept by calculating the total interest a consumer would earn if she saved all her income I assuming $I > 50,000$. For the first \$50,000, she would save at the untaxed interest rate of r — thus accumulating $(1+r)50000$ for next period. She would then have $(I-50000)$ left to save — on which she would earn $0.5r$ interest. In addition to accumulating $(1+r)50000$ for the first \$50,000 in savings, she would therefore accumulate $(1+0.5r)(I-50000)$ if she saved all her income. Her total possible c_2 consumption is therefore

$$(1+r)50000+(1+0.5r)(I-50000) = (1+0.5r)I+25000r. \tag{8.5.vi}$$

This, then, is the c_2 intercept. Given we already determined the slope to be $-(1+0.5r)$, the budget constraint is $c_2 = (1+0.5r)I+25000r-(1+0.5)c_1$ or

$$(1+0.5r)c_1 + c_2 = (1+0.5r)I+25000r. \tag{8.5.vii}$$

(e) *Use this budget constraint to write down the constrained optimization problem that can be solved for the optimal choice given that households save more than \$50,000. Solve for c_1 and for the implied level of savings as a function of α, I and r.*

Answer: The maximization problem is

$$\max_{c_1,c_2} c_1^{\alpha} c_2^{(1-\alpha)} \text{ subject to } (1+0.5r)c_1 + c_2 = (1+0.5r)I+25000r. \tag{8.5.viii}$$

The Lagrange function for this problem is

$$\mathcal{L}(c_1,c_2,\lambda) = c_1^{\alpha} c_2^{(1-\alpha)} + \lambda((1+0.5r)I+25000r-(1+0.5r)c_1-c_2). \tag{8.5.ix}$$

Solving this in the same way as before, we then get

$$c_1 = \alpha I + \frac{25000\alpha r}{(1+0.5r)} \tag{8.5.x}$$

and an implied savings s of

$$s = (1-\alpha)I - \frac{25000\alpha r}{(1+0.5r)}. \tag{8.5.xi}$$

(f) *What value must α take for household 3 as described in part A?*

Answer: Household 3 saves \$75,000 with income of \$100,000 and before-tax interest rate $r = 0.1$. Thus

$$75000 = (1-\alpha)100000 - \frac{25000\alpha(0.1)}{(1+0.5(0.1))} \tag{8.5.xii}$$

which solves to $\alpha \approx 0.244$.

(g) *With the values of α that you have determined for households 1 and 3, determine the impact that the tax reform described in (c) of part A would have?*

Answer: Using panels (b) and (c) of Exercise Graph 8.5(1), we concluded in part A that both households will choose to locate on the steeper portion of the budget under the new policy — i.e. on the portion defined by the constraint $(1+r)c_1 + c_2 = (1+r)I$ where $I = 100,000$ and $r = 0.1$. In B(b), we determined that savings in this case is given by $s = (1-\alpha)I$. Thus, household 1 for whom $\alpha = 0.75$ would save $(1-0.25)100,000 = 25,000$ as before. Household 3, for whom $\alpha \approx 0.244$, will save approximately $(1-0.244)100,000 = 75,600$ — but that level of savings lies to the left of the kink point of the dashed budget in panel (c). Thus, household 3 optimizes at the kink point, implying unchanged savings at $75,000.

(h) *What range of values can α take for household 2 as described in part A?*

Answer: There are several ways you could use to figure this out. One way is to note that

$$MRS = -\frac{\alpha c_2}{(1-\alpha)c_1} \tag{8.5.xiii}$$

and that this must, for household 2, lie between -1.05 and -1.10 at $(c_1, c_2) = (50000, 55000)$ in order for that kink point in the budget to be optimal. Substituting these values for c_1 and c_2 into the expression for MRS and setting it equal to these two endpoint values, we can solve

$$-\frac{55000\alpha}{50000(1-\alpha)} = -1.05 \text{ and } -\frac{55000\alpha}{50000(1-\alpha)} = -1.10 \tag{8.5.xiv}$$

to conclude that $0.488 \le \alpha \le 0.5$.

Another way to solve for this is to use our results from the previous parts. Household 2 might have a tangency with the steep portion of the budget at $(c_1, c_2) = (50000, 55000)$. We concluded in B(b) (equation (8.5.v)) that in this case, savings s is $s = (1-\alpha)I$. Thus, for household 2 to choose $50,000 in savings under the steeper portion of the budget, $50000 = (1-\alpha)100000$ which implies $\alpha = 0.5$.

Alternatively, household 2 could have a tangency with the shallow portion of the budget at $(c_1, c_2) = (50000, 55000)$. We concluded in B(3) that savings then satisfies equation (8.5.xi). Substituting $s = 50,000$, $I = 100,000$ and $r = 0.1$ into that equation, we can solve for $\alpha \approx 0.488$. Thus, again we get that $0.488 \le \alpha \le 0.5$.

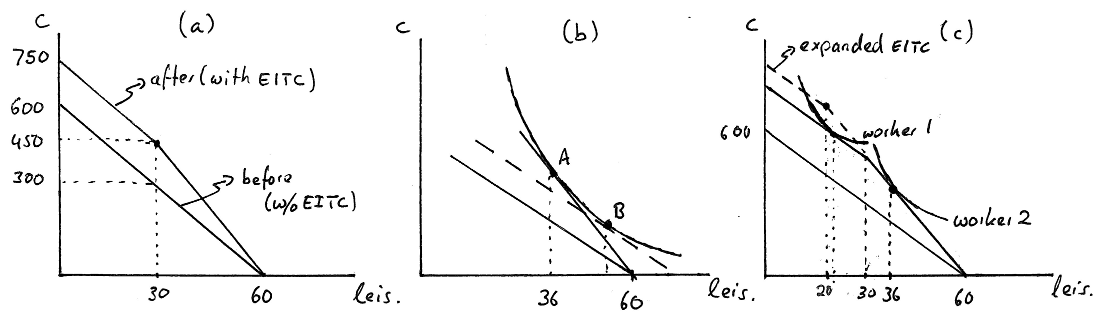
Exercise 8.7

Policy Application: *The Earned Income Tax Credit: Since the early 1970's, the U.S. government has had a program called the Earned Income Tax Credit (previously mentioned in end-of-chapter exercises in Chapter 3.) A simplified version of this program works as follows: The government subsidizes your wages by paying you 50% in addition to what your employer paid you but the subsidy applies only to the first $300 (per week) you receive from your employer. If you earn more than $300 per week, the government gives you only the subsidy for the first $300 you earned but nothing for anything additional you earn. For instance, if you earn $500 per week, the government would give you 50% of the first $300 you earned — or $150.*

A: *Suppose you consider workers 1 and 2. Both can work up to 60 hours per week at a wage of $10 per hour, and after the policy is put in place you observe that worker 1 works 39 hours per week while worker 2 works 24 hours per week. Assume throughout that Leisure is a normal good.*

(a) *Illustrate these workers' budget constraints with and without the program.*

Answer: These are illustrated in panel (a) of Exercise Graph 8.7. At a wage rate of $10 per hour, the earned income tax credit described in the problem raises the effective wage to $15 per hour for the first 30 hours of work.



Exercise Graph 8.7 : Earned Income Tax Credit

(b) *Can you tell whether the program has increased the amount that worker 1 works? Explain.*

Answer: Worker 1 works 39 hours — which means he takes 21 hours of leisure after the EITC is implemented. Removing the EITC would therefore be like a parallel shift in of the budget for this worker — and would thus produce a pure wealth (or income) effect, no substitution effect. If leisure is a normal good, that means that removing the EITC would cause the worker to reduce consumption of leisure — i.e. he would work more. It must therefore be the case that the introduction of the EITC did the opposite — it increased the worker's consumption of leisure, thus causing him to work less than he did before.

(c) *Can you tell whether worker 2 works more or less after the program than he did before? Explain.*

Answer: Worker 2 works for 24 hours per week after the introduction of the EITC — implying a leisure consumption of 36 hours per week. Thus, this worker (unlike worker 1 in the previous part) locates to the right of the kink point in the EITC budget. This implies that removing the EITC would imply an inward rotation of the budget — thus causing both a substitution and a wealth effect. This is pictured in panel (b) of the graph — where the substitution effect is the change from the bundle A to B. Removing the EITC would make consuming leisure less expensive (since now the worker would only give up \$10 for every hour of leisure rather than \$15), which is why the substitution effect says the worker will take more leisure when the EITC is removed and work less. However, from the dashed compensated budget in the graph to the no-EITC budget below, there is a decrease in wealth, and if leisure is a normal good, a decrease in wealth implies less consumption of leisure. Thus, there is a wealth effect that points in the opposite direction from the substitution effect — leaving the overall effect ambiguous. The more substitutable leisure and consumption are, the more likely it is that the removal of the EITC would cause the worker to work less. The introduction of the EITC is of course the mirror image — the more substitutable leisure and consumption are, the more likely it is that the introduction of the EITC will cause the worker to work more.

(d) *Now suppose the government expands the program by raising the cut-off from \$300 to \$400. In other words, now the government applies the subsidy to earnings up to \$400 per week. Can you tell whether worker 1 will now work more or less? What about worker 2?*

Answer: Panel (c) of the graph illustrates the initial without-EITC budget and the \$300 EITC budget as in panel (a). In addition, the dashed extension of the EITC budget represents the expanded \$400 EITC budget. This extension of the steeper EITC slope has no impact on worker 2 — worker 2 originally optimized at 36 hours of leisure, and no better bundles are made available by the expanded budget. Thus, worker 2 would do nothing differently. Worker 1, on the other hand, is affected by the change in the EITC. He initially takes 21 hours of leisure — which means the new EITC budget affects him both because it has a different slope and because it is further out. Since leisure is a normal good, we know the worker will not choose to optimize on the part of the new budget that lies to the left of the new kink point (at 20 hours of leisure) — because that would be equivalent to reducing the amount of leisure when wealth increases. So worker 1 will end up somewhere on the steeper portion of the new EITC budget — somewhere between 20 hours of leisure and 30 hours of leisure. We can't tell exactly where — there are once again offsetting wealth and substitution effects. The substitution effect says that worker 1 should now consume less leisure (i.e. work more) because leisure has become more expensive (\$15 rather than \$10 per hour). The wealth effect, on the other hand, says the worker is richer and therefore should consume more leisure (i.e. work less). Either effect may dominate. The more substitutable leisure and consumption are for the worker, the more likely it is that the worker will work more under the expanded EITC. The most he will work more, however, is 1 hour.

B: *Suppose that workers have tastes over consumption c and leisure ℓ that can be represented by the function $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$.*

(a) *Given you know which portion of the budget constraint worker 2 ends up on, can you write down the optimization problem that solves for his opti-*

mal choice? Solve the problem and determine what value α must take for worker 2 in order for him to have chosen to work 24 hours under the EITC program.

Answer: The optimization problem for worker 2 is

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } c = 15(60-\ell). \tag{8.7.i}$$

Setting up the Lagrange function and solving the first two first order conditions, we get $c = (15\alpha\ell)/(1-\alpha)$. Plugging this into the budget constraint and solving for ℓ, we get $\ell = 60(1-\alpha)$, and plugging this into $c = (15\alpha\ell)/(1-\alpha)$, we get $c = 900\alpha$.

In order for the worker to choose 24 hours of work and thus 36 hours of leisure, it must then be that $\ell = 36 = 60(1-\alpha)$. Solving for α, we get $\alpha = 24/60 = 0.4$.

(b) *Repeat the same for worker 1 — but be sure you specify the budget constraint correctly given that you know the worker is on a different portion of the EITC budget. (*Hint*: If you extend the relevant portion of the budget constraint to the leisure axis, you should find that it intersects at 75 leisure hours.)*

Answer: The optimization problem now would be

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } c = 750-10\ell. \tag{8.7.ii}$$

Going through the same steps as before, we then get $\ell = 75(1-\alpha)$ and $c = 750\alpha$. In order for this worker to choose 39 hours of work or 21 hours of leisure, it therefore has to be the case that $\ell = 21 = 75(1-\alpha)$ or $\alpha = 0.72$.

(c) *Having identified the relevant α parameters for workers 1 and 2, determine whether either of them works more or less than he would have in the absence of the program.*

Answer: In the absence of the EITC program, the workers would solve

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } c = 10(60-\ell) \tag{8.7.iii}$$

which gives $\ell = 60(1-\alpha)$ and $c = 600\alpha$. Worker 1 has $\alpha = 0.72$ — which means he takes $60(1-0.72) = 16.8$ hours of leisure without EITC and 21 hours of leisure with the EITC. Thus, worker 1 works 4.2 hours less under EITC. This is consistent with our intuitive graphs — where we concluded that the EITC has a pure wealth effect for worker 1 — causing him to work less. Worker 2 has $\alpha = 0.4$ — which means he takes $60(1-0.6) = 36$ hours of leisure before EITC and 36 hours of leisure after EITC. Thus, worker 2 does not change his work hours as a result of the EITC. This is also consistent with our graphical analysis where we found competing wealth and substitution effects for worker 2 — effects that exactly offset each other when the worker has the tastes modeled here.

(d) *Determine how each worker would respond to an increase in the EITC cut-off from \$300 to \$400.*

Answer: We already know from our intuitive analysis that nothing changes for worker 2 — he continues to operate on the steeper portion of the budget defined by the equation $c = 15(60-\ell)$ which we used in problem (8.7.i). Since the problem remains unchanged, the solution remains unchanged. For worker 1, however, the relevant budget constraint now is $c = 900-15\ell$ (rather than $750-10\ell$ as in problem (8.7.ii)). Thus, since the relevant constraint has changed, we need to solve the problem with the new constraint — which gives us $\ell = 60(1-\alpha) = 60(1-0.72) = 16.8$ and $c = 900\alpha = 900(0.72) = 648$. But this would put him on the steep budget to the right of the kink — which implies the true optimum is at the kink where $\ell = 20$. Thus, he will work 1 hour more.

(e) *For what ranges of α would a worker choose the kink-point in the original EITC budget you drew (i.e. the one with a \$300 cutoff)?*

Answer: To figure out this range, we need to determine the values of α for which 30 hours of leisure is optimal for the problems written out in equations (8.7.i) and (8.7.ii). In other words, for each of the two budget line segments, what are the values of α for which a worker would optimize at precisely the kink point. Any α between the values we get from these two exercises will be such that the kink point is optimal.

For the problem in (8.7.ii), we calculated $\ell = 60(1-\alpha)$. Setting ℓ equal to 30, we can solve for $\alpha = 0.5$. For the problem in (8.7.i), we calculated $\ell = 75(1-\alpha)$. Setting $\ell = 30$, we can solve for $\alpha = 0.6$. Thus, for $0.5 \le \alpha \le 0.6$, the kink point where the worker works for 30 hours a week is optimal.

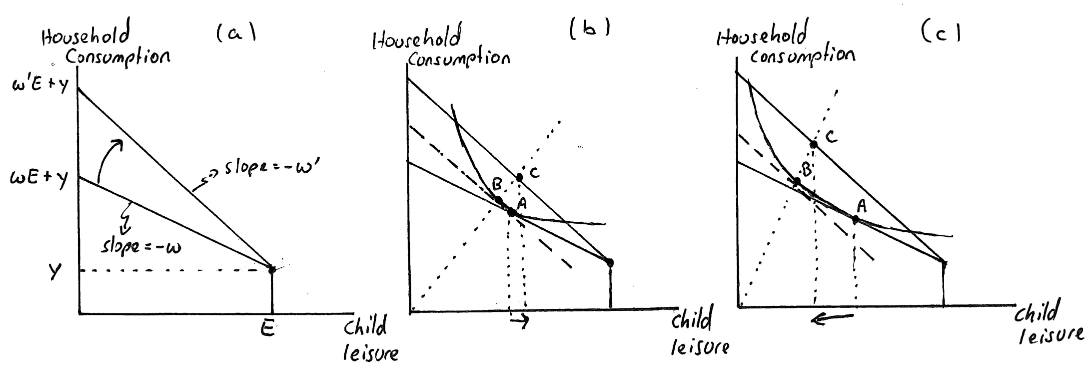
Exercise 8.9

Policy Application: *International Trade and Child Labor: The economist Jagdish Bhagwati explained in one of his public lectures that international trade causes the wage for child labor to increase in developing countries. He then discussed informally that this might lead to more child labor if parents are "bad" and less child labor if parents are "good".*

A: *Suppose that households in developing countries value two goods: "Leisure time for Children in the Household" and "Household Consumption." Assume that the adults in a household are earning \$$y$ in weekly income regardless of how many hours their children work. Assume that child wages are w per hour and that the maximum leisure time for children in a household is E hours per week.*

(a) *On a graph with "weekly leisure time for children in the household" on the horizontal axis and "weekly household consumption" on the vertical, illustrate the budget constraint for a household and label the slopes and intercepts.*

Answer: This initial budget is illustrated in panel (a) of Exercise Graph 8.9(1) where the bundle (E, y) is effectively the "endowment" bundle for the household — i.e. the bundle that does not depend on child wages.



Exercise Graph 8.9(1) : Child Labor and International Trade

(b) *Now suppose that international trade expands and, as a result, child wages increase to w'. Illustrate how this will change the household budget.*

Answer: This is also illustrated in panel (a) of the graph — the budget rotates outward around the "endowment" bundle (E, y).

(c) *Suppose that household tastes are homothetic and that households require their children to work during some but not all the time they have available. Can you tell whether children will be asked to work more or less as a result of the expansion of international trade?*

Answer: You cannot tell — it depends on the size of the substitution effect and thus on the degree of substitutability between child leisure and household consumption. We know that tastes can be homothetic with little or no substitutability between goods (as in perfect complements), and tastes can be homothetic with perfect substitutability. Of course there are lots of in between cases. In panel (b), we illustrate the case of relatively little substitutability where the substitution effect from A to B is small and outweighed by the wealth effect from B to C to result in an overall increase in leisure for children. In panel (c), on the other hand, we illustrate the case where the substitution effect outweighs the wealth effect — resulting in a decrease in leisure for children.

The substitution effect here simply says that, as child wages increase, the opportunity cost of giving leisure to children increases and households will therefore give less leisure. The wealth effect, on the other hand, says that increasing child wages make the household richer — and richer households will consume more of all normal goods, including child leisure.

(d) *In the context of the model with homothetic tastes, what distinguishes "good" parents from "bad" parents?*

Answer: Good parents are those whose tastes look more like those in panel (b) while bad parents are those whose tastes look more like panel (c). Put differently, parents become "better" in this model the more they view child leisure and household consumption as complements. This has a certain amount of intuitive appeal: Good parents are those that essentially say that they can only become better off when household consumption goes up if child welfare (i.e. child leisure) also goes up — they are complements and have to go together. Bad parents are those that view household consumption as a substitute for child welfare.

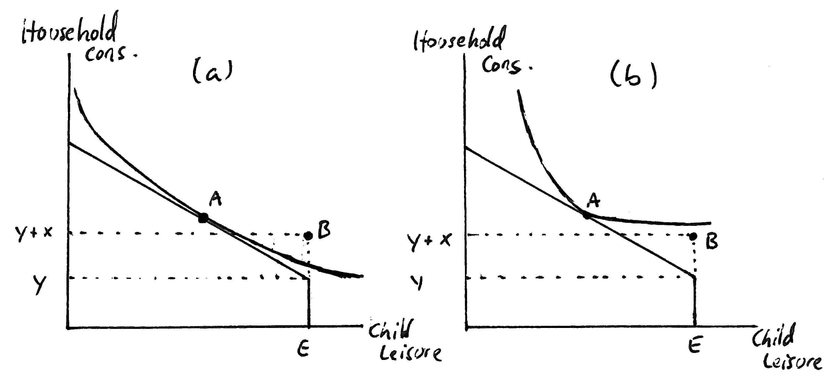
(e) *When international trade increases the wages of children, it is likely that it also increases the wages of other members of the household. Thus, in the context of our model, y — the amount brought to the household by others — would also be expected to go up. If this is so, will we observe more or less behavior that is consistent with what we have defined as "good" parent behavior?*

Answer: This would cause a parallel shift in the budget beyond the initial rotation that results from the increase child wages. Such a parallel shift gives rise to a pure wealth effect. So long as child leisure is a normal good, increases in y would therefore cause increases in consumption of all goods — including child leisure. This would strengthen the wealth effect from the increase in w and thus cause more parents to reduce the amount of work their children have to undertake. Put differently, the more y is also increased by international trade, the more substitutable child leisure and household consumption can be and still cause parents to be "good".

(f) *In some developing countries with high child labor rates, governments have instituted the following policy: If the parents agree to send a child to school instead of work, the government pays the family an amount x. (Assume the government can verify that the child is in fact sent to school and does in fact not work, and assume that the household views time at school as leisure time for the child.) How does that alter the choice set for parents? Is the policy more or less likely to succeed the more substitutable the household tastes treat child "leisure" and household consumption?*

Answer: Under this policy, the government in essence makes one additional bundle available to the household — a bundle in which the child's "leisure" or "non-work" hours are E and the household's consumption is y plus the payment x the government is providing in order for the child to attend school. This new bundle is depicted as bundle B in both panels of Exercise Graph 8.9(2). In each panel, A is the original optimal bundle before this policy was introduced. But in panel (a), the original optimal indifference curve is relatively flat and therefore passes below B while in panel (b) it is closer to the shape of perfect complements which makes it pass above B. Thus, conditional on A being the original optimum, the policy is more likely to induce the household to choose B (and thus send their child to school) the more substitutable are household consumption and child leisure.

B: *Suppose parental tastes can be captured by the utility function $u(c,\ell) = c^{0.5}\ell^{0.5}$. For simplicity, suppose further that $y = 0$.*



Exercise Graph 8.9(2) : Child Labor and International Trade: Part II

(a) *Specify the parents' constrained optimization problem and set up the appropriate Lagrange function.*

Answer: The problem to be solved is

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } c = w(E-\ell). \tag{8.9.i}$$

The Lagrange function is

$$\mathscr{L}(c,\ell,\lambda) = c^{0.5}\ell^{0.5} + \lambda(w(E-\ell)-c). \tag{8.9.ii}$$

(b) *Solve the problem you have set up to determine the level of leisure the parents will choose for their children. Does w have any impact on this decision?*

Solving the first two first order conditions, you get $c = w\ell$. Plugging this into the budget constraint, you can then solve for $\ell = E/2$ and plugging this back into $c = w\ell$, you can get $c = wE/2$. The level of leisure parents choose for their children ($\ell = E/2$) is independent of wage — so w has no impact on their decision in this case.

(c) *Explain intuitively what you have just found. Consider the CES utility function (that has the Cobb-Douglas function you just worked with as a special case). For what ranges of ρ would you expect us to be able to call parents "good" in the way that Bhagwati informally defined the term?*

Answer: For the Cobb-Douglas tastes that are modeled, the substitution effect (that causes parents to reduce their children's leisure when w increases) is exactly offset by the wealth effect (which causes parents to increase their children's leisure as w increases). We know that Cobb-Douglas tastes are CES tastes with $\rho = 0$ and elasticity of substitution of 1. As ρ falls below zero, the goods become more substitutable and as ρ

rises above zero they become more complementary. In part A we determined that parents are more likely to be "good" if they view child leisure as relatively complementary to household consumption — thus, for CES utility functions, parents are "bad" if $-1 \leq \rho < 0$ and parents are "good" if $0 < \rho \leq \infty$.

(d) *Can parents for whom household consumption is a quasilinear good ever be "good"?*

Anwer: Yes, if substitution effects are sufficiently small, such parents can be "good". This is because tastes that are quasilinear *in consumption* would only give rise to substitution effects with no wealth effect for household consumption (i.e. *on the vertical axis*). Thus, while the substitution effect points to an increase in household consumption and a decrease in child leisure, the wealth effect points to no further change in household consumption and an increase in child leisure. Put differently, while the quasilinearity of household consumption implies no wealth effect on the vertical axis, it also implies the entire wealth effect happens on the child leisure axis in the direction opposite to the substitution effect.

Be careful in this answer to pay attention to the fact that the question states that household consumption, not child leisure, is the quasilinear good. Had the question asked whether parents can be "good" if child leisure is the quasilinear good, the answer would have been an unambiguous no. This is because we would then only have a substitution effect on the horizontal axis — which implies that child leisure decreases and thus child labor increases with an increase in w.

(e) *Now suppose (with the original Cobb-Douglas tastes) that $y > 0$. If international trade pushes up the earnings of other household members — thus raising y, what happens to child leisure?*

Answer: Solving for the optimal leisure time, we get

$$\ell = \frac{wE + y}{2w}. \tag{8.9.iii}$$

The derivative of this with respect to y is positive — i.e. as y increases, so does the amount of leisure chosen for the child.

(f) *Suppose again that $y = 0$ and the government introduces the policy described in part A(f). How large does x have to be in order to cause our household to send their child to school (assuming again that the household views the child's time at school as leisure time for the child)?*

Answer: Without participating in the policy, the household consumes $c = wE/2$ and $\ell = E/2$ — and therefore gets utility $(wE/2)^{0.5}(E/2)^{0.5} = w^{0.5}E/2$. If the household participates in the policy, it's child would get leisure of E and the household consumption would be x. Thus, participating in the policy means utility of $x^{0.5}E^{0.5}$. The household will be indifferent between the two options if the utility of participating and not participating are equal; i.e. if

$$\frac{w^{0.5}E}{2} = x^{0.5}E^{0.5}. \tag{8.9.iv}$$

Solving this for x we get $x = wE/4$. For any x greater than this, the household is therefore better off choosing to send their child to school.

(g) *Using your answer to the previous part, put into words what fraction of the market value of the child's time the government has to provide in x in order for the family to choose schooling over work for their child?*

Answer: We concluded before that x has to be at least $wE/4$ in order for the household to be willing to send the child to school. The market value of the child's time endowment is wE. The amount that is required for the child to be sent to school is therefore equivalent to one quarter of the market value of the leisure time of the child.

Exercise 8.11

Policy Application: *Tax Revenues and the Laffer Curve: In this exercise, we will consider how the tax rate on wages relates to the amount of tax revenue collected.*

A: *As introduced in Section B, the* Laffer Curve *depicts the relationship between the tax rate on the horizontal axis and tax revenues on the vertical. (See the footnote in Section 8B.2.2 for background on the origins of the name of this curve.) Because people's decision on how much to work may be affected by the tax rate, deriving this relationship is not as straightforward as many think.*

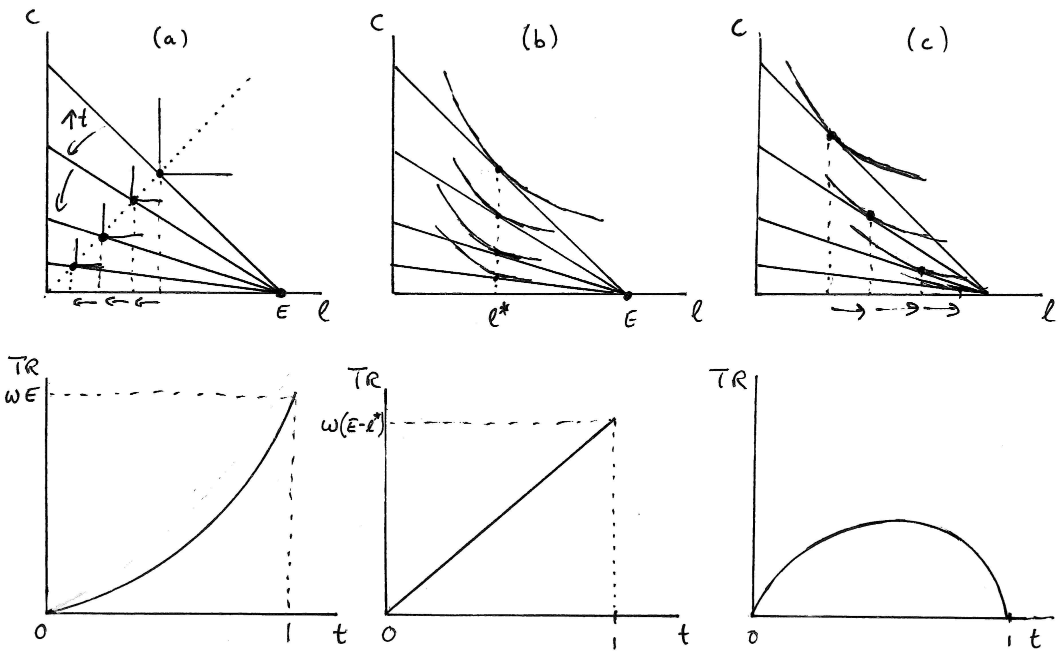
(a) *Consider first the extreme case in which leisure and consumption are perfect complements. On a graph with leisure hours on the horizontal and consumption dollars on the vertical, illustrate how increases in the tax on wages affect the consumer's optimal choice of leisure (and thus labor).*

Answer: The top portion of panel (a) of Exercise Graph 8.11(1) illustrates multiple budget lines with diminishing slopes as the tax rate on wages increases (thus causing the slope of the budget line, $-(1-t)w$, to become smaller in absolute value).

When leisure and consumption are perfect complements, all optimal bundles will always occur on a single ray from the origin. As the tax rate increases, optimal leisure consumption decreases — thus causing labor supply to increase. At the tax rate goes to 1, the entire leisure endowment E is spent on work.

(b) *Next, consider the less extreme case where a change in after-tax wages gives rise to substitution and wealth effects that exactly offset one another on the leisure axis. In which of these cases does tax revenue rise faster as the tax rate increases?*

Answer: The top portion of panel (b) of the graph illustrates this case — as the tax rate increases (and the budget becomes shallower), the optimal leisure consumption remains unchanged at ℓ^*. When wealth and substitution effects cancel each other out, as in panel (b), tax revenues increase as tax rates increase only because more money is collected for



Exercise Graph 8.11(1) : Substitution Effects and the Laffer Curve

each hour the worker works. When consumption and leisure are perfectly complementary (as in panel (a)), tax revenues rise as tax rates increase not only because more is collected for each hour that is worked but also because more hours are worked. Thus, one would expect tax revenues to rise faster under the tastes in (a) than in (b).

(c) *On a graph with the tax rate (ranging from 0 to 1) on the horizontal and tax revenues on the vertical, how does this relationship differ for tastes in (a) and (b)?*

Answer: This is depicted in the lower graphs. The graph below panel (a) illustrates that tax revenues increase at a faster rate as the tax rate increases when leisure and consumption are perfect complements. Tax revenues reach the highest point when the tax rate approaches 1 — as the worker approaches working all the time and paying all his salary in taxes.[1] The graph below panel (b) illustrates the relationship between tax rates and tax revenues when wealth and substitution effects offset each other. Tax revenues increase at the same rate (since work hours remain unaffected) — with tax revenue approaching $w(E - \ell^*)$ as the tax rate approaches 1.

[1]The problem is well defined as the tax rate increases all the way up to but not including 1. When the tax rate hits 1, it makes no difference to the worker whether he works or does not — because both leisure and consumption are "essential" goods.

(d) *Now suppose that the substitution effect outweighs the wealth effect on the leisure axis as after-tax wages change. Illustrate this and determine how it changes the relationship between tax rates and tax revenue.*

Answer: This is illustrated in panel (c) of Exercise Graph 8.11(1). Since the substitution effect (which says to consume more leisure as the after-tax wage falls) outweighs the wealth effect (that says to consume less leisure as the after-tax wage falls), increasing tax rates result in increasing leisure consumption. As tax rates increase, we therefore have two competing effects on tax revenues: On the one hand, more is collected for every hour worked. On the other hand, however, fewer hours are worked. Thus, it is quite plausible for tax rates to reach a point where additional increases in rates imply decreases in tax revenues. This relationship — which is the one that has come to be associated with the term "Laffer curve", is illustrated below panel (c).

(e) *Laffer suggested (and most economists agree) that the curve relating tax revenue (on the vertical axis) to tax rates (on the horizontal) is initially upward sloping but eventually slopes down — reaching the horizontal axis by the time the tax rate goes to 1. Which of the preferences we described in this problem can give rise to this shape?*

Answer: Only the preferences in panel (c) — those where substitution effects outweigh wealth effects on the leisure axis — can result in such a shape. Most economists agree that eventually — as tax rates approach 100 percent, tax revenue falls to zero. The only disagreement is at what point the downward sloping part of the curve begins. Part of the reason Laffer became known for this curve is that he popularized the notion that the peak of the Laffer curve may, in some instances, occur at rates considerably lower than 100 percent. To the extent that this is true, it is possible to cut tax rates and increase tax revenues. Most economists believe that, at least in the U.S., federal tax rates are now to the left of the peak on the Laffer curve — implying that tax revenues cannot be increased through tax cuts. At the same time, the top marginal tax rates were once 90% in the U.S. (in the 1960's) and 70% in 1980. It is when rates are that high (as opposed to top rates around 40% as is the case in the U.S. today) that it is considerably more likely that we are on the "wrong side of the Laffer curve."

(f) *True or False*: *If leisure is a normal good, the Laffer Curve can have an inverted U-shape only if leisure and consumption are (at least at some point) sufficiently substitutable such that the substitution effect (on leisure) outweighs the wealth effect (on leisure).*

Answer: This is true — as we have just determined. (It is plausible, though, that the effect is more like that in panel (a) for low tax rates and then increasingly becomes like panel (c) for higher and higher rates.)

B: *In Section 8B.2.2, we derived a Laffer Curve for the case where tastes were quasilinear in leisure. Now consider the case where tastes are Cobb-Douglas —*

taking the form $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$. *Assume that a worker has 60 hours of weekly leisure endowment that he can sell in the labor market for wage* w.

(a) *Suppose the worker's wages are taxed at a rate* t. *Derive his optimal leisure choice.*

Answer: We need to solve

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } c = (1-t)w(60-\ell). \tag{8.11.i}$$

Solving this in the usual way, we get $\ell = 60(1-\alpha)$.

(b) *For someone with these tastes, does the Laffer Curve take the inverted U-shape described in Section 8B.2.2. Why or why not? Which of the cases described in A does this represent?*

Answer: We just derived that $\ell = 60(1-\alpha)$ — which means that the number of weekly hours worked is equal to 60α. Thus, work hours are not impacted by the tax rate t — which means substitution and wealth effects exactly offset each other as in the case described in A(b). The relationship between tax rates t and tax revenues TR is then quite straightforward:

$$TR = w(60\alpha)t, \tag{8.11.ii}$$

which has the shape depicted in the lower graph of panel (b) in Exercise Graph 8.11(1), a straight line with intercept of zero and slope $w(60\alpha)$.

(c) *Now consider the more general CES function* $(\alpha c^{-\rho}+(1-\alpha)\ell^{-\rho})^{-1/\rho}$. *Again derive the optimal leisure consumption.*

Answer: We now need to solve the problem

$$\max_{c,\ell} \left(\alpha c^{-\rho}+(1-\alpha)\ell^{-\rho}\right)^{-1/\rho} \text{ subject to } c = (1-t)w(60-\ell). \tag{8.11.iii}$$

The usual first two first order conditions simplify to

$$c = \left(\frac{\alpha(1-t)w}{(1-\alpha)}\right)^{1/(\rho+1)}\ell. \tag{8.11.iv}$$

Substituting this into the budget constraint $c = (1-t)w(60-\ell)$, we get

$$\ell = \frac{60(1-t)w}{\left(\frac{\alpha(1-t)w}{(1-\alpha)}\right)^{1/(\rho+1)}+(1-t)w} = 60\left[\left(\frac{\alpha}{(1-\alpha)}\right)^{1/(\rho+1)}((1-t)w)^{-\rho/(\rho+1)}+1\right]^{-1}. \tag{8.11.v}$$

(d) *Does your answer simplify to what you would expect when* $\rho = 0$?

Answer When $\rho = 0$, equation (8.11.v) reduces to

$$\ell = 60\left[\left(\frac{\alpha}{(1-\alpha)}\right)^{1}((1-t)w)^{0}+1\right]^{-1} = 60\left[\left(\frac{\alpha}{(1-\alpha)}\right)+1\right]^{-1} = 60(1-\alpha). \quad (8.11.vi)$$

This is exactly what we derived for the Cobb-Douglas tastes in (a) — which makes sense since CES utility functions become Cobb-Douglas when ρ approaches 0.

(e) *Determine the range of values of ρ such that leisure consumption increases with t.*

Answer: We are interested in the change in ℓ with a change in t — i.e. we are interested in the derivative of ℓ in equation (8.11.v) with respect to t. This derivative is

$$\frac{\partial \ell}{\partial t} = -60\left[\left(\frac{\alpha}{(1-\alpha)}\right)^{\frac{1}{\rho+1}}((1-t)w)^{-\frac{\rho}{\rho+1}}+1\right]^{-2}\left(\frac{-\rho}{(\rho+1)}\right)\left(\frac{\alpha}{(1-\alpha)}\right)^{\frac{1}{\rho+1}}((1-t)w)^{-\frac{2\rho+1}{\rho+1}}(-w). \quad (8.11.vii)$$

The equation is a mess — but determining whether it is positive or negative is not too difficult. First, we can cancel the negative sign at the end with the negative sign in the middle of the expression. This gives us

$$\frac{\partial \ell}{\partial t} = -60\left[\left(\frac{\alpha}{(1-\alpha)}\right)^{\frac{1}{\rho+1}}((1-t)w)^{-\frac{\rho}{\rho+1}}+1\right]^{-2}\left(\frac{\rho}{(\rho+1)}\right)\left(\frac{\alpha}{(1-\alpha)}\right)^{\frac{1}{\rho+1}}((1-t)w)^{-\frac{2\rho+1}{\rho+1}}(w). \quad (8.11.viii)$$

None of the terms in this expression can be negative — with the exception of the middle term $(\rho/(\rho+1))$ that is negative if $-1<\rho<0$ and positive if $\rho>0$. Given the negative sign that remains at the front of the expression, we can then conclude that

$$\frac{\partial \ell}{\partial t} > 0 \text{ if and only if } -1<\rho<0, \quad (8.11.ix)$$

$$\frac{\partial \ell}{\partial t} = 0 \text{ if and only if } \rho = 0, \text{ and} \quad (8.11.x)$$

$$\frac{\partial \ell}{\partial t} < 0 \text{ if and only if } \rho > 0. \quad (8.11.xi)$$

In other words, when $-1 \le \rho < 0$, leisure consumption goes up as the tax rate increases, and when $\rho > 0$ the reverse is true. This exactly mirrors the graphs in Exercise Graph 8.11(1) — with $\rho = 0$ representing the middle panel (b), $\rho > 0$ representing the case where consumption and leisure are relatively complementary (as in panel (a)), and $-1 < \rho < 0$ representing the case where consumption and leisure is relatively substitutable (as in panel (c)).

(f) *When ρ falls in the range you have just derived, what happens to leisure consumption as t approaches 1? What does this imply for the shape of the Laffer Curve?*

Answer: We just derived that leisure consumption increases with t if and only if $-1 < \rho < 0$. Leisure consumption is given in the expression (8.11.v) which is

$$\ell = = 60\left[\left(\frac{\alpha}{(1-\alpha)}\right)^{1/(\rho+1)}((1-t)w)^{-\rho/(\rho+1)}+1\right]^{-1}. \tag{8.11.xii}$$

Note that as t approaches 1, $(1-t)w$ approaches zero. The exponent on the term $((1-t)w)$ is $-\rho/(\rho+1)$ — which is positive when $-1 < \rho < 0$.[2] Thus, the term

$$\left(\frac{\alpha}{(1-\alpha)}\right)^{1/(\rho+1)}((1-t)w)^{-\rho/(\rho+1)} \tag{8.11.xiii}$$

goes to zero as t approaches 1. This leaves us, as t goes to 1, with

$$\ell = 60[0+1]^{-1} = 60. \tag{8.11.xiv}$$

Thus, for any $-1 < \rho < 0$, leisure consumption goes to the entire leisure endowment of 60 as t approaches 1 — which means that labor supply goes to zero as t approaches 1. This further implies that tax revenues will fall to zero as the tax rate approaches 1 — as depicted in the lower portion of panel (c) in Exercise Graph 8.11(1). The Laffer curve therefore takes the inverted U-shape that we typically expect.

(g) *Suppose $\alpha = 0.25$, $w = 20$ and $\rho = -0.5$. Calculate the amount of leisure a worker would choose as a function of t. Then derive an expression for this worker's Laffer Curve and graph it.*

Answer: Plugging these values into expression (8.11.v), we get

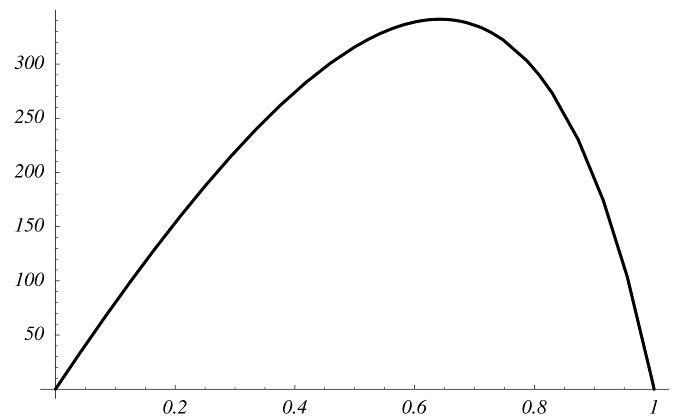
$$\ell = 60\left[\left(\frac{0.25}{0.75}\right)^{\frac{1}{0.5}}((1-t)20)^{\frac{-(-0.5)}{0.5}}+1\right]^{-1} = 60\left[\left(\frac{1}{3}\right)^{2}(1-t)20+1\right]^{-1} = \frac{540}{20(1-t)+9}. \tag{8.11.xv}$$

The government collects $TR = tw(60-\ell) = 20t(60-\ell)$ in revenue — which gives us the Laffer curve

$$TR = 20t\left(60-\frac{540}{20(1-t)+9}\right) = \frac{24000t(1-t)}{20(1-t)+9}. \tag{8.11.xvi}$$

This is plotted in Exercise Graph 8.11(2) (with t on the horizontal and TR on the vertical).

[2] This is because the denominator $(\rho+1)$ is positive, and the numerator is positive since it is a negative number multiplied by a negative sign.



Exercise Graph 8.11(2) : The Laffer Curve when $\rho = -0.5$, $\alpha = 0.25$ and $w = 20$

Conclusion: Potentially Helpful Reminders

1. If this chapter seems difficult, it is probably because you have not yet fully internalized Chapter 7. This is because the material of this chapter is conceptually almost identical to the material in the previous chapter.

2. Remember to never think about wealth effects unless you have two parallel budgets to work with. Also remember never to allow a substitution effect to move you off an indifference curve.

3. When applying definitions like normal and inferior goods, or definitions like homothetic or quasilinear tastes, always be sure you are doing so when developing the wealth effect that takes you from one budget to a parallel budget.

4. Try to make intuitive sense of substitution and wealth effects in each application. Substitution effects always point in the direction of more consumption of what's become cheaper and less consumption of what's become more expensive. Wealth effects in labor and capital markets almost always involve normal goods — and thus point in the direction of the wealth change.

5. When wealth and substitution effects point in opposite directions, your answer will typically be ambiguous: If the substitution effect is small because the goods are fairly complementary, the wealth effect will dominate; but if the substitution effect is large because the goods are fairly substitutable, then the substitution effect will dominate.

6. Homothetic tastes, for instance, can have small or large substitution effects depending on whether the indifference curves are relatively L-shaped or relatively flat. End-of-chapter exercise 8.1 illustrates this, and exercise 8.9 develops the idea in an intriguing application.

CHAPTER

9

Demand for Goods and Supply of Labor and Capital

Now we have finally arrived at the point where we talk about demand curves. These summarize the relationship between some aspect of the consumer's economic circumstances and the quantity she demands of a particular good. Although we typically think of demand curves as illustrating the relationship between quantity of x demanded and the price of x, demand curves can also illustrate the relationship between quantity and income or between quantity and the price of some other good. Each of these demand curves is just a "slice" of a more complicated demand relationship (that we call a demand function in part B) — with that slice holding all but one of the economic variables that define a consumer's economic circumstances fixed. And, just as demand curves emerge from the consumer's diagram, supply curves for labor and capital emerge from the worker's and saver's diagrams (and demand curves for capital emerge from the borrower's diagram.)

Chapter Highlights

The main points of the chapter are:

1. **Demand and supply relationships illustrate how choices depend on aspects of the economic environment** — where the economic environment potentially includes income and prices for different types of goods.

2. When we isolate the impact of one particular aspect of that economic environment, we are implicitly holding all other aspects of that environment fixed — i.e. **we are graphing a "slice" of a more complicated function** that tells us how behavior changes as all these aspects of our economic environment change.

3. While we can illustrate **these relationships** as demand and supply curves, they ultimately **emerge from the underlying choice model** we have developed and can be understood only with that framework in mind.

4. Income-demand relationships depend only on income effects — and thus the relationship of indifference curves to one another. **Price-demand relationships depend on both income and substitution effects** and thus also depend on the degree of substitutability between different goods.

5. **Labor and capital supply relationship** often involve wealth and substitution effects that point in opposite direction. This implies that, despite all goods (typically) being normal goods, the labor and capital supply curves can **slope up or down depending on which effect dominates**.

9A Solutions to Within-Chapter Exercises for Part A

Exercise 9A.1

In an earlier chapter, we mentioned that it is not possible for a good to be inferior for all income levels. Can you see in the lower panel of Graph 9.1a why this is true?

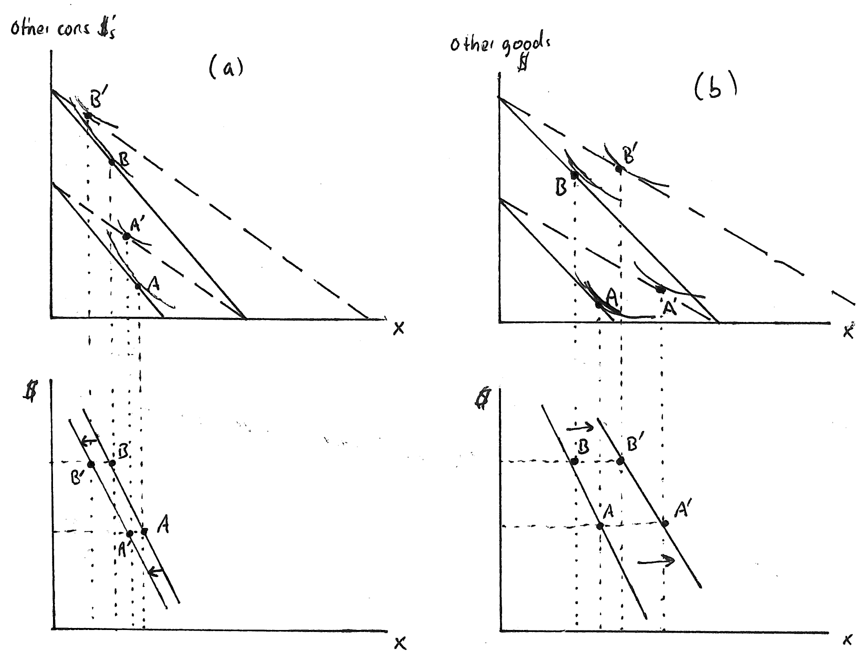
Answer: In order for pasta to be inferior for all income levels, the relationship between pasta on the horizontal axis and income on the vertical would have to continue to slope downward as income falls — eventually intersecting the pasta axis. But without any income, the consumer would in fact not be able to buy any pasta.

Exercise 9A.2

Suppose good x is an inferior good for an individual. Derive the income-demand curve as in Graph 9.1a. Then graph a decrease in the price for x for both income levels in the top panel — and show how this affects the income-demand curve in the lower panel depending on whether x is Giffen or regular inferior.

Answer: In the top portions of panels (a) and (b) of Exercise Graph 9A.2, two initial (solid) budgets are drawn representing two income levels at some initial price.

From these, points A and B are derived just as in the text. Then, two new (dashed) budgets are added — these differ from the original budgets only in that the price of x has fallen. In panel (a), the new optimal bundles A' and B' are to the left of A and B (respectively) — indicating that a decrease in the price causes a decrease in consumption of x. Thus, in panel (a), x is a Giffen good. When translated to the lower graph, A' and B' now appear to the left of A and B (respectively) — thus, the decrease in the price of x causes a leftward shift in the income-demand graph. In panel (b), on the other hand, a decrease in price causes the new optima A' and B' to lie to the right of A and B (respectively) — indicating that a decrease in the price of x causes an increase in the consumption of x. Thus, panel (b) represents the case when x is not Giffen (and is regular inferior). When translated to the



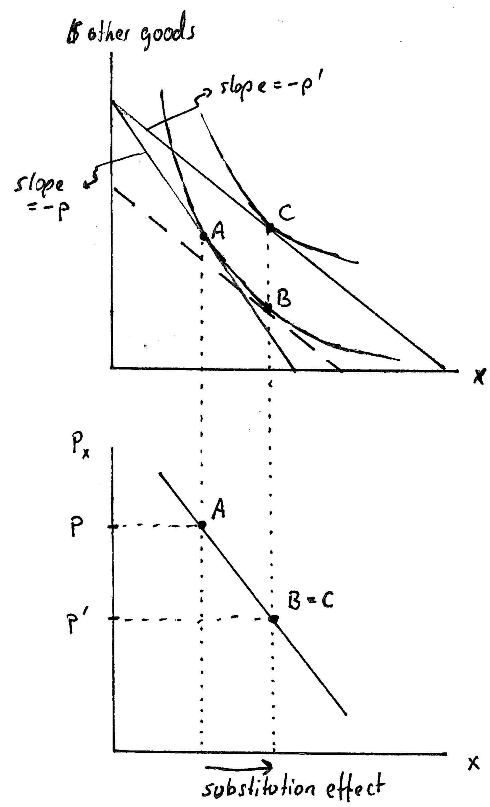
Exercise Graph 9A.2 : Shifts in Income-Demand Curves

lower graph, A' and B' now lie to the right of A and B (respectively) — i.e. a decrease in the price of x causes a rightward shift of the income-demand curve. (*Note*: The vertical axis in the lower graphs is labeled as dollars — but it is actually measuring *income*. You could — and perhaps should — therefore label the axis as *Income in Dollars.*)

Exercise 9A.3

Repeat the derivation of own-price demand curves for the case of quasilinear tastes and explain in this context again how quasilinear tastes are borderline tastes between normal and inferior goods.

Answer: This is derived in Exercise Graph 9A.3. In the case of quasilinear goods, there is no income effect from a price change (for that good) — as a result, the shape of the demand curve arises entirely from the substitution effect (with B and C coinciding on the lower graph). In the cases treated in the text, we showed that B lies to the left of C for normal goods and to the right of C for inferior (both regular and Giffen) goods. The quasilinear case is the borderline case where B lies on top of C due to the absence of an income effect. Put differently, the graphs in the text show that the income effect points in the same direction as the substitution effect for normal goods and in the opposite direction for inferior goods. For quasilinear goods, it is simply absent.



Exercise Graph 9A.3 : Demand Curve for Quasilinear Case

Exercise 9A.4

How would the own price demand curves in Graphs 9.2a through (c) change with a decrease in income? (*Hint*: Your answer for panel (a) should be different than your answers for panels (b) and (c).)

Answer: Consider first the case of x being a normal good. A normal good is one where, if income decreases, the consumer consumes less (all else being the same). In panel (a) (of the graph in the text), you can already see what happens at the lower price if income falls — B is the optimal bundle at a lower income level and C is optimal at a higher income level, with prices (i.e. the slope of the budget) the same in both cases. Thus, if income falls by the difference between these budgets, the new demand curve will go through B. The same must be true at every other price level — if income falls (and price stays at that price level), consumption falls. Thus, the entire demand curve shifts to the left with a decrease in income.

In panels (b) and (c), on the other hand, the opposite is the case. As income falls from the higher to the lower of the two parallel budgets, consumption *increases*. The new demand curve will again go through B in the lower panel, but now B lies to the right of the original demand curve. Similarly, for any other price level, if income falls, the consumer will consume more if x is inferior — thus the new demand curve lies to the right of the original.

Exercise 9A.5

What kind of good would x have to be in order for the demand curve not to shift as income changes?

Answer: The good x would have to be quasilinear — i.e. in the absence of income effects, changes in income do not change how much a consumer will buy for a given price.

Exercise 9A.6

What kind of good would x_1 have to be in order for this cross-price demand curve to slope down?

Answer: For the cross-price demand curve to slope down, point C in the graph in the text would have to lie to the right of A. Since the substitution effect pushes us in the direction of *less* x_1 as p_2 falls, the income effect would have to push us in the other direction (and outweigh the substitution effect). Thus, the income effect would have to be positive — implying that x_1 has to be a normal good.

Exercise 9A.7

In our analysis of consumer goods, we usually found that income and substitution effects point in the same direction when goods are normal. Why are wealth and substitution effects now pointing in opposite directions when leisure is a normal good?

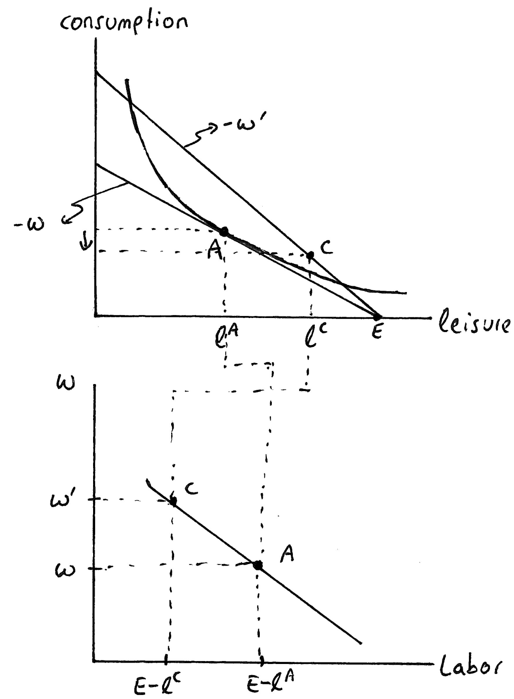
Answer: The wage is the opportunity cost of consuming leisure. But the budget in this case is not exogenous (as it was in the case of goods in the previous section). Rather, the budget arises from the fact that we *own* our leisure endowment. When the wage goes up, our leisure endowment becomes more valuable — so we have in essence become richer. This is in contrast to the price of goods in the previous section going up — there an increase in a price made us poorer. The wealth effect therefore now points in the opposite direction because the "price" increase (i.e. the wage increase) increases rather than decreases our "income".

You can also view this in strictly mechanical terms. A change in the wage rate does not change the leisure-intercept of our budget. Rather, it changes the consumption intercept. So, in strictly technical terms, an increase in the wage actually looks like a decrease in the price of consumption — the good on the vertical axis. This is analogous to what we graphed in the textbook in Graph 9.3 where we showed a cross-price demand curve where income and substitution effects operate exactly as they do in the leisure/consumption graph when wages change.

Exercise 9A.8

True or False: Leisure being an inferior good is *sufficient* but not *necessary* for labor supply to slope up.

Answer: This is true. In panel (c) of Graph 9.4 in the text, we show that when leisure is inferior, the labor supply curve must slope up. So leisure being inferior is sufficient for an upward slope of labor supply. But we also show in panel (b) that labor supply *can* slope up when leisure is normal. So it is not necessary for leisure to be inferior in order for labor supply to slope up.



Exercise Graph 9A.8 : Labor Supply when Consumption Giffen

Exercise 9A.9

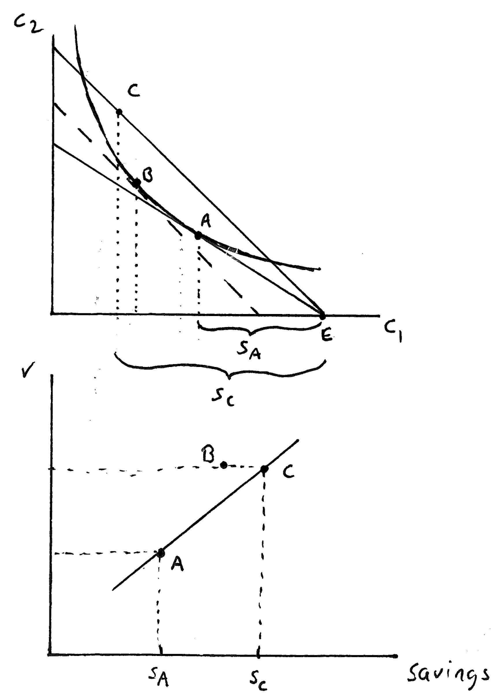
Can you tell which way the labor supply curve will slope in the unlikely event that "other consumption" is a Giffen good?

Answer: Yes — the labor supply will slope down. This is graphed in Exercise Graph 9A.8 where *C* in the top panel must lie *below A*. This is because an increase in the wage when Leisure is an endowment is technically equivalent to a decrease in the price of consumption when the budget is treated exogenously — and consumption is Giffen if such a decrease in its price causes less consumption. But if C lies below A, it must also lie to the right of A — i.e. the increase in the wage causes an increase in leisure consumption — and thus a decrease in work.

Exercise 9A.10

Would the interest rate/savings curve slope up or down if consumption this period were an inferior good?

Answer: The substitution and wealth effects would now point in the same direction for current consumption — which means that the curve would slope up. This is depicted in Exercise Graph 9A.10.



Exercise Graph 9A.10 : Consumption this Period as Inferior Good

Exercise 9A.11

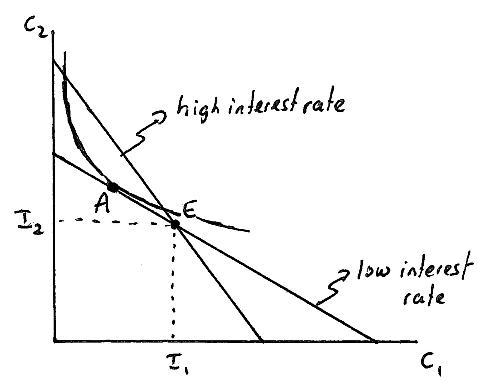
What kind of good would consumption this summer have to be in order for the interest rate/borrowing relationship to be positive in Graph 9.6?

Answer: Consumption this summer would have to be Giffen. You can see this most easily by just graphing the initial and final budget (without the compensated budget). In this graph, an increase in the interest rate is technically equivalent to an increase in the price of current consumption when income is exogenously modeled. In order for the borrowing curve to slope up, C would have to lie to the right of A — i.e. as the price of current consumption goes up, you would have to do more of it. That's the definition of a Giffen good.

Exercise 9A.12

Is it possible for someone to begin as a saver at low interest rates and switch to become a borrower as the interest rate rises?

Answer: In Exercise Graph 9A.12 we illustrate a high and low interest rate budget constraint for someone with income both now and in the future (so that it in principle it is possible for him to be either a borrower or a saver). If he is a saver under the low interest rate, his optimum looks something like bundle A (to the left of E). But if an indifference curve is tangent at A, it must necessarily lie above E — which implies no bundle to the right of E under either budget could also be optimal. Thus, it is not possible for an increase in the interest rate to induce a saver to become a borrower.

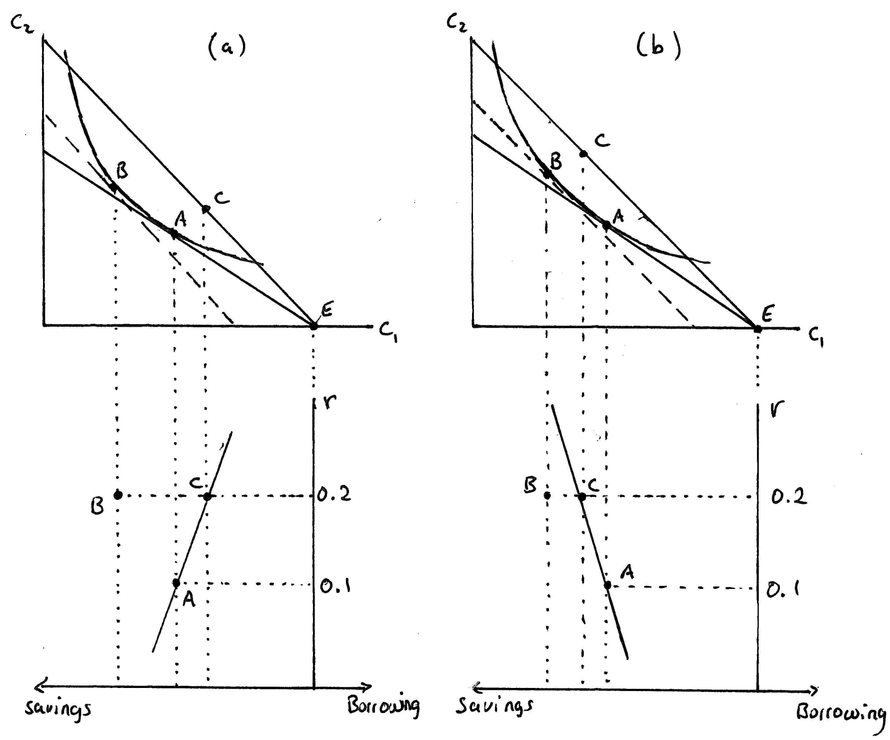


Exercise Graph 9A.12 : Not possible to switch from Saver to Borrower as r Increases

Exercise 9A.13

The technique of placing the axis below the endowment point E developed in Graph 9.7 could also be applied to the previous two graphs, Graph 9.5 and Graph 9.6. How would those graphs change?

Answer: This is illustrated in Exercise Graph 9A.13 for the case where all income is in this period and no income is expected in the next period (as in Graph 9.5 in the textbook). For the textbook Graph 9.6, nothing would change because the axis would be placed exactly as it is in the graphs in the text since the "endowment" point occurs at zero current income.



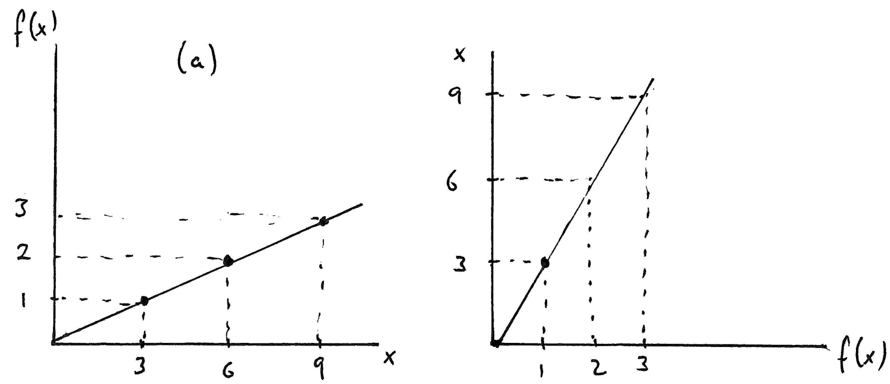
Exercise Graph 9A.13 : Savings as in Textbook Graph 9.5

9B Solutions to Within-Chapter Exercises for Part B

Exercise 9B.1

Consider the function $f(x) = x/3$. Graph this as you usually would with x on the horizontal axis and $f(x)$ on the vertical. Then graph the inverse of the function, with $f(x)$ on the horizontal and x on the vertical.

Answer: This is done in panels (a) and (b) of Exercise Graph 9B.1.



Exercise Graph 9B.1 : Direct and Inverse Graph of $f(x) = x/3$

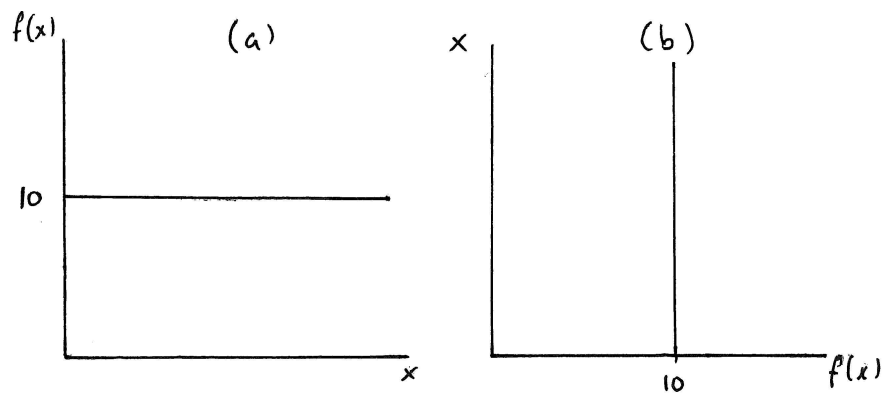
Exercise 9B.2

Repeat the previous exercise for the function $f(x) = 10$.
Answer: This is done in panels (a) and (b) of Exercise Graph 9B.2.

Exercise 9B.3

Another special case of tastes that we have emphasized throughout is the case of quasilinear tastes. Consider, for instance, the utility function $u(x_1, x_2) = 100(\ln x_1) + x_2$. Calculate the demand function for x_1 and derive some sample income–demand curves for different prices.
Answer: The Lagrange function is

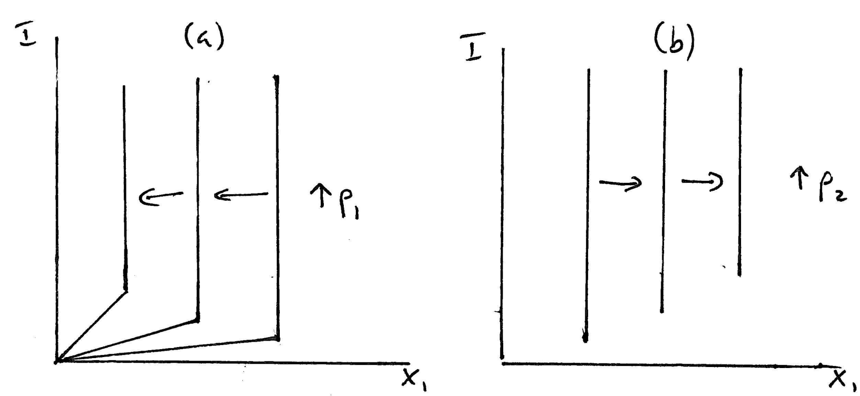
$$\mathcal{L}(x_1, x_2, \lambda) = 100(\ln x_1) + x_2 + \lambda(I - p_1 x_1 - p_2 x_2). \qquad (9B.3)$$



Exercise Graph 9B.2 : Direct and Inverse Graph of $f(x) = 10$

The first two first order conditions solve to $x_1 = 100p_2/p_1$. (Plugging this back into the budget constraint, we can also solve for $x_2 = (I - 100p_2)/p_2$.)

In panel (a) of Exercise Graph 9B.3, we graph these as p_1 changes. Note that I does not enter the demand functions for x_1 — so changes in income do not alter consumption of x_1. This should not be surprising — quasilinear goods are goods for which there are no income effects. Technically, at some point income does, however, enter — when income falls too low, we cannot afford what the demand function tells us and we end up at a corner solution. As p_1 increases, that corner solution begins to happen at lower income levels. Panel (b) illustrates changes as p_2 increases — with the portion that occurs when there is a corner solution not graphed.



Exercise Graph 9B.3 : Income Demand Graphs for Quasilinear good

Exercise 9B.4

Can you derive the same result for x_2?

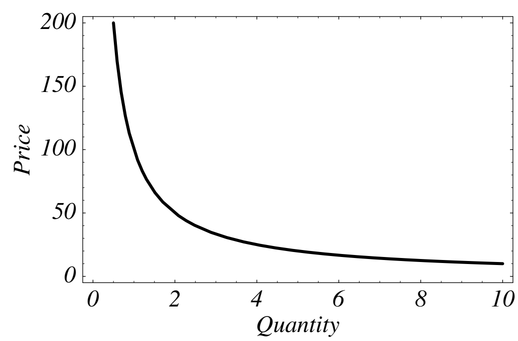
Answer: We would get

$$\frac{\partial p_2}{\partial x_2} = -\frac{(1-\alpha)I}{x_2^2} = -\frac{(1-\alpha)I}{((1-\alpha)I/p_2)^2} = -\frac{p_2^2}{(1-\alpha)I} = \left(\frac{\partial x_2}{\partial p_2}\right)^{-1}. \qquad (9B.4)$$

Exercise 9B.5

As in exercise 9B.3, consider again tastes that can be represented by the utility function $u(x_1,x_2) = 100(\ln x_1) + x_2$. Using the demand function for x_1 that you derived in the previous exercise, plot the own-price demand curve when income is 100 and when $p_2 = 1$. Then plot the demand curve again when income rises to 200. Keep in mind that you are actually plotting inverse functions as you are doing this.

Answer: The demand function we derived previously is $x_1 = 100p_2/p_1$. When $p_2 = 1$, this becomes $x_1 = 100/p_1$. The inverse is $p_1 = 100/x_1$, which is plotted in Exercise Graph 9B.5. Since I does not enter the function, changes in income do not shift it.

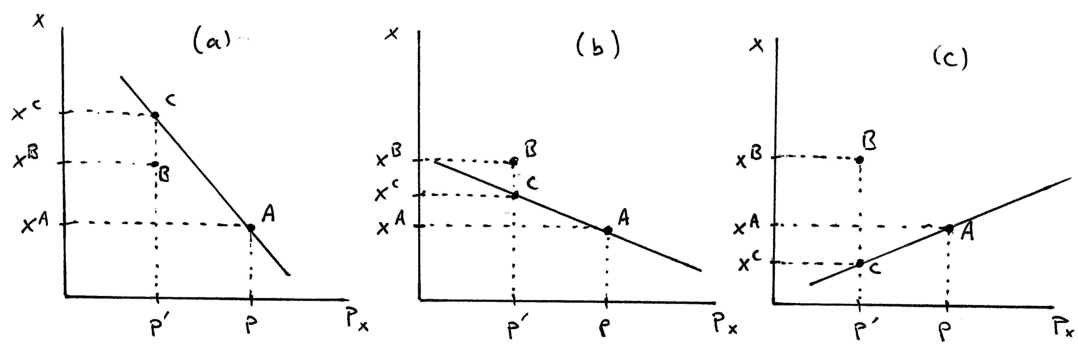


Exercise Graph 9B.5 : Own-Price Demand Curve for Quasilinear good

Exercise 9B.6

Knowing that own price demand curves are inverse slices of own price demand functions, how would the lower panels of Graph 9.2 look if you graphed slices of the actual functions (rather than the inverses) — i.e. when you put price on the horizontal and the quantities of goods on the vertical axis?

Answer:



Exercise Graph 9B.6 : Direct Demand Curves Corresponding to Graph 9.2 in the Text

Exercise 9B.7

What would the slices of the demand function (rather than the inverse slices in Graph 9.10a) look like?

Answer: They would simply be horizontal rather than vertical lines.

Exercise 9B.8

Verify that these are in fact the right demand functions for tastes represented by the CES utility function.

Answer: The Lagrange function for the maximization problem is

$$\mathscr{L}(x_1, x_2, \lambda) = (\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho})^{-1/\rho} + \lambda(I - p_1 x_1 - p_2 x_2). \tag{9B.8.i}$$

The first two first order conditions are then

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x_1} &= -\frac{1}{\rho}\left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho}\left(-\rho\alpha x_1^{-\rho-1}\right) - \lambda p_1 = 0 \\ \frac{\partial \mathscr{L}}{\partial x_2} &= -\frac{1}{\rho}\left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho}\left(-\rho(1-\alpha) x_2^{-\rho-1}\right) - \lambda p_2 = 0 \end{aligned} \tag{9B.8.ii}$$

which can also be written as

$$\begin{aligned} \left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho} \alpha x_1^{-\rho-1} &= \lambda p_1 \\ \left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho} (1-\alpha) x_2^{-\rho-1} &= \lambda p_2. \end{aligned} \tag{9B.8.iii}$$

Dividing these two equations by one another, we get (after canceling terms)

$$\frac{\alpha}{(1-\alpha)}\left(\frac{x_2}{x_1}\right)^{\rho+1} = \frac{p_1}{p_2} \tag{9B.8.iv}$$

or

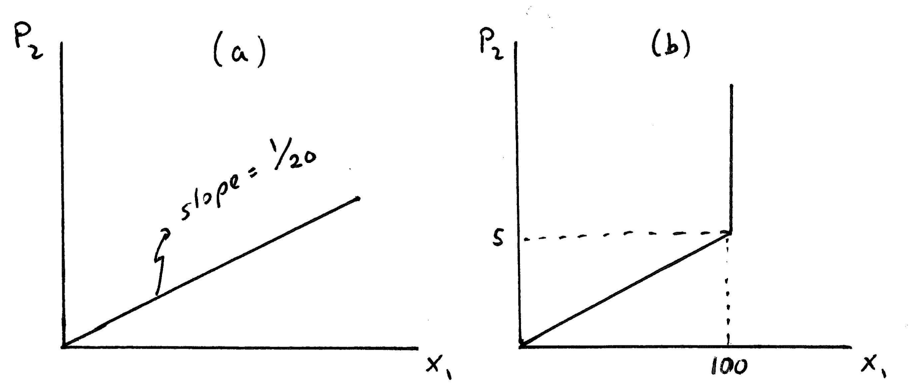
$$x_2 = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{1/(\rho+1)} x_1. \quad (9B.8.v)$$

Plugging this into the budget constraint and solving for x_1 then gives us the demand equation in the text. Plugging that back into equation (9B.8.v), we can also solve for x_2 — the demand equation given in the text.

Exercise 9B.9

In Graph 9.3, we intuitively concluded that cross-price demand curves slope up when tastes are quasilinear. Verify this for tastes that can be represented by the utility function $u(x_1, x_2) = 100(\ln x_1) + x_2$ for which you derived the demand functions in exercise 9B.3. Draw the cross-price demand curve for x_1 when income is 2,000 and $p_1 = 5$.

Answer: We determined previously that $x_1 = 100p_2/p_1$ is the demand function for x_1 in this case. The derivative of x_1 with respect to p_2 is then just $100/p_1$ which is greater than zero. Thus, the cross-price demand curve slopes up. When $p_1 = 5$, the demand function becomes simply $x_1 = 100p_2/5 = 20p_2$, and the inverse function becomes $p_2 = x_1/20$. This is graphed in panel (a) of Exercise Graph 9B.9.



Exercise Graph 9B.9 : Cross-Price Demand Curve for Quasilinear tastes

Exercise 9B.10

Suppose that income was 500 instead of 2,000 in exercise 9B.9. Determine at what point the optimization problem results in a corner solution (by calculating the demand function for x_2 and seeing when it becomes negative). Illustrate how this would change the cross price demand curve you drew in exercise 9B.9. (Hint: The change occurs in the cross price demand curve at $p_2 = 5$.)

Answer: This is illustrated in panel (b) of Exercise Graph 9B.9. In this example, you only have \$500 to spend. Your demand for x_1 is given by $x_1 = 100p_2/p_1$ which becomes $x_1 = 20p_2$ when $p_1 = 5$. Thus, when $p_2 = 5$, the demand function tells us you will buy $x_1 = 20(5) = 100$ — and at a price of $p_1 = 5$, this costs \$500 — your entire income. You therefore reach a corner solution where you buy no more of x_2 when p_2 reaches 5. For higher levels of p_2, you would then still be at the same corner solution — buying 100 units of x_1. (For the case where $I = 2{,}000$ as in the previous exercise, this corner solution is not reached until $p_2 = 20$ — which is where the kink would occur in panel (a) of the graph).

Exercise 9B.11

What function is graphed in the middle portions of each panel of Graph 9.4? What function is graphed in the bottom portion of each panel of Graph 9.4?

Answer: The function graphed in the middle panel is the inverse of the leisure demand function — or what we denoted $\ell(w,L)$ — with L held fixed. The function graphed in the lower panel is the inverse labor supply function — or what we denoted $l(w,L)$ — with L held fixed.

Exercise 9B.12

Verify these results.

Answer: The Lagrange function for this problem is

$$\mathscr{L}(c,\ell,\lambda) = c + \alpha \ln \ell + \lambda(w(L-\ell) - c). \tag{9B.12.i}$$

The first two first order conditions are

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial c} &= 1 - \lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial \ell} &= \frac{\alpha}{\ell} - \lambda w = 0. \end{aligned} \tag{9B.12.ii}$$

Solving these for ℓ we get $\ell = \alpha/w$. Plugging this back into the budget constraint $c = w(L-\ell)$, we then also get $c = wL-\alpha$. Finally, the labor supply function is simple $L-\ell$ — or $l = L - \alpha/w$.

Exercise 9B.13

Verify this leisure demand and labor supply function for the CES function that is given.

Answer: The Lagrange function for this problem is

$$\mathscr{L}(c,\ell,\lambda) = \left(\alpha c^{-\rho} + \beta \ell^{-\rho}\right)^{-1/\rho} + \lambda(w(L-\ell) - c). \tag{9B.13.i}$$

The first two first order conditions quickly solve to (after canceling a number of items) to

$$\frac{\alpha}{\beta}\left(\frac{\ell}{c}\right)^{\rho+1} = \frac{1}{w} \qquad (9B.13.ii)$$

or

$$c = \left(\frac{\alpha w}{\beta}\right)^{1/(\rho+1)} \ell. \qquad (9B.13.iii)$$

Substituting this into the budget constraint $c = w(L-\ell)$ and solving for ℓ, we get the leisure demand equation in the text. Subtracting this from L then gives us the labor supply equation.

Exercise 9B.14

Verify that these three equations are correct.
Answer: The Lagrange function for this problem is

$$\mathcal{L}(c,\ell,\lambda) = c^{\alpha}\ell^{(1-\alpha)} + \lambda((1+r)e_1 + e_2 - (1+r)c_1 - c_2). \qquad (9B.14.i)$$

The first two first order conditions solve to give us

$$c_2 = \frac{(1+r)(1-\alpha)c_1}{\alpha}. \qquad (9B.14.ii)$$

Plugging this into the budget constraint $(1+r)e_1+e_2 = (1+r)c_1+c_2$ and solving for c_1, we get the expression $c_1(r,e_1,e_2)$ in the text. Substituting this back into equation (9B.14.ii) and solving for c_2 then gives $c_2(r,e_1,e_2)$ in the text. Finally, $s(r,e_1,e_2)$ is just $c_1(r,e_1,e_2)$ subtracted from period 1 endowment e_1.

Exercise 9B.15

Consider the more general CES utility function $u(c_1,c_2) = (0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho}$ and solve for the savings supply function when you earn \$10,000 this period and nothing in the future. Then verify that you obtain the vertical relationship between savings and the interest rate when $\rho = 0$ and determine how this slope changes when $\rho > 0$ (implying relatively low elasticity of substitution) and when $\rho < 0$ (implying relatively high elasticity of substitution).
Answer: The Lagrange function is

$$\mathcal{L}(c_1,c_2,\lambda) = \left(0.5c_1^{-\rho} + 0.5c_2^{-\rho}\right)^{-1/\rho} + \lambda(10000(1+r) - (1+r)c_1 - c_2). \qquad (9B.15.i)$$

The first two first order conditions give (after some canceling of terms)

$$\left(\frac{c_2}{c_1}\right) = (1+r)^{1/(\rho+1)} \qquad (9B.15.ii)$$

or

$$c_2 = (1+r)^{1/(\rho+1)} c_1. \tag{9B.15.iii}$$

Substituting this into the budget constraint $10000(1+r) = (1+r)c_1 - c_2$ and solving for c_1, we get

$$c_1 = \frac{10,000}{1+(1+r)^{-\rho/(\rho+1)}} \tag{9B.15.iv}$$

and a savings supply function of

$$s = 10,000 - \frac{10,000}{1+(1+r)^{-\rho/(\rho+1)}}. \tag{9B.15.v}$$

To determine the relationship between the interest rate r and savings s, all we need to do is take the derivative of s with respect to r; i.e.

$$\frac{\partial s}{\partial r} = \left(\frac{-\rho}{\rho+1}\right)\left[\frac{10,000}{\left(1+(1+r)^{\rho/(\rho+1)}\right)^2}(1+r)^{\frac{-2\rho-1}{\rho+1}}\right]. \tag{9B.15.vi}$$

Note that the term in large brackets is positive. Thus, $\partial s/\partial r$ is greater than or less than zero depending on whether $-\rho/(\rho+1)$ is greater than or less than zero. Thus,

$$\begin{aligned}\frac{\partial s}{\partial r} &< 0 \text{ if and only if } \rho > 0\\ \frac{\partial s}{\partial r} &= 0 \text{ if and only if } \rho = 0\\ \frac{\partial s}{\partial r} &> 0 \text{ if and only if } \rho < 0.\end{aligned} \tag{9B.15.vii}$$

Exercise 9B.16

Using the CES utility function from exercise 9B.15, verify that the negative relationship between borrowing and the interest rate arises regardless of the value that ρ takes (whenever $e_1 = 0$ and $e_2 > 0$.)

Answer: Solving a slight variant of the problem we solved in the previous exercise, we set up a Lagrange function

$$\mathcal{L}(c_1, c_2, \lambda) = \left(0.5c_1^{-\rho} + 0.5c_2^{-\rho}\right)^{-1/\rho} + \lambda(e_2 - (1+r)c_1 - c_2). \tag{9B.16.i}$$

The first order conditions (just as in the previous problem) imply

$$c_2 = (1+r)^{1/(\rho+1)} c_1. \tag{9B.16.ii}$$

Plugging this into the budget constraint — which now is $e_2 = (1+r)c_1 + c_2$, we can solve for

$$c_1 = \frac{e_2}{1+r+(1+r)^{1/(\rho+1)}}. \tag{9B.16.iii}$$

Since we are assuming that current income e_1 is zero, any consumption in the current period must come from borrowing. Thus, c_1 as just derived gives us the amount that the consumer chooses to borrow. The derivative of c_1 with respect to the interest rate r then tells us the relationship between borrowing and r. This derivative is

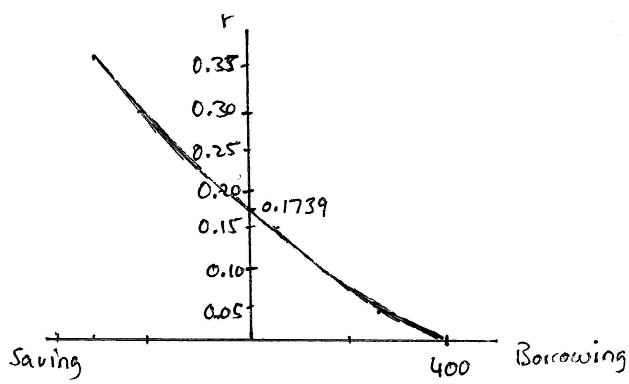
$$\frac{\partial c_1}{\partial r} = -\left[\frac{e_2}{1+r+(1+r)^{1/(\rho+1)}}\left(1+\frac{1}{\rho+1}(1+r)^{-\rho/(\rho+1)}\right)\right] \tag{9B.16.iv}$$

which is negative regardless of ρ (because the bracketed part is always positive but is preceded by a negative sign.)

Exercise 9B.17

Graph this function in a graph similar to Graph 9.7 (which is the graph of an inverse borrowing (rather than saving) function).

Answer: This is graphed in Exercise Graph 9B.17.



Exercise Graph 9B.17 : Inverse Borrowing Function

9C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 9.1

The following is intended to explore what kinds of income-demand relationships are logically possible.

A: *For each of the following, indicate whether the relationship is possible or not and explain:*

(a) *A good is a necessity and has a positive income-demand relationship.*

Answer: This is possible. A good is a necessity if, as income increases by some percentage k, the percentage increase in the consumption of the good is less than k. For instance, if income increases by 10% and consumption of the good increases by 5%, the good is a necessity. Since consumption still increases with income, the income-demand curve is still positive.

(b) *A good is a necessity and has a negative income-demand relationship.*

Answer: This, too, is possible. We just defined a necessity above — as income increases by a percentage k, the percentage increase in the consumption of the good is less than k. This leaves open the possibility that the percentage "increase" in consumption is negative — i.e. that, as income increases, consumption of the good declines. This would result in a negative income-demand relationship.

(c) *A good is a luxury and has a negative income-demand relationship.*

Answer: No, this is not possible. A luxury good is a good whose consumption increases by a greater percentage than income — thus, any time income increases, its consumption also increases. Therefore, the income-demand curve must have positive slope.

(d) *A good is quasilinear and has a negative income-demand relationship.*

Answer: Quasilinear goods are goods for which there are no income effects — thus, the income-demand relationship cannot be negative.

(e) *Tastes are homothetic and one of the goods has a negative income-demand relationship.*

Answer: If tastes are homothetic, all goods are normal goods. Thus all goods must exhibit a positive income-demand relationship.

B: *Derive the income-demand relationships for each good for the following tastes:*

(a) $u(x_1, x_2, x_3) = x_1^{\alpha} x_2^{\beta} x_3^{(1-\alpha-\beta)}$ *where α and β lie between zero and 1 and sum to less than 1.*

Answer: Cobb-Douglas utility functions in which the exponents sum to 1 have the property that the demand function for each good is simply equal to that good's exponent times income divided by that good's price.

(When the exponents do not sum to 1, the price in the denominator is multiplied by the sum of all the exponents.) You can verify this by solving the maximization problem

$$\max_{x_1,x_2,x_3} x_1^{\alpha} x_2^{\beta} x_3^{(1-\alpha-\beta)} \text{ subject to } p_1x_2 + p_2x_2 + p_3x_3 = I \tag{9.1.i}$$

and you will get

$$x_1 = \frac{\alpha I}{p_1},\ x_2 = \frac{\beta I}{p_2} \text{ and } x_3 = \frac{(1-\alpha-\beta)I}{p_3}. \tag{9.1.ii}$$

Note that this problem is somewhat easier to solve if you take the natural log of the utility function when you set up the problem; i.e. if you define the problem as

$$\max_{x_1,x_2,x_3} \alpha \ln x_1 + \beta \ln x_2 + (1-\alpha-\beta)\ln x_3 \text{ subject to } p_1x_2 + p_2x_2 + p_3x_3 = I. \tag{9.1.iii}$$

The income-demand curve we graph is then simply the inverse of these with income on the left hand side (holding prices fixed); i.e. the income demand curves are derived from

$$I = \frac{p_1x_1}{\alpha},\ I = \frac{p_2x_2}{\beta} \text{ and } I = \frac{p_3x_3}{(1-\alpha-\beta)}. \tag{9.1.iv}$$

All these have positive slopes (which is not surprising since Cobb-Douglas tastes represent normal goods.)

(b) $u(x_1,x_2) = \alpha \ln x_1 + x_2$. (Note*: To fully specify the income demand relationship in this case, you need to watch out for corner solutions.) Graph the income demand curves for x_1 and x_2 — carefully labeling slopes and intercepts.*

Answer: Solving the problem

$$\max_{x_1,x_2} \alpha \ln x_1 + x_2 \text{ subject to } p_1x_1 + p_2x_2 = I \tag{9.1.v}$$

in the usual way, we get the demand functions

$$x_1 = \frac{\alpha p_2}{p_1} \text{ and } x_2 = \frac{I-\alpha p_2}{p_2}. \tag{9.1.vi}$$

Consumption of x_1 is therefore independent of I as we expect given that x_1 is a quasilinear good. But that means that as income falls low enough, we will find ourselves at a corner solution. Put differently, the $x_1 = \alpha p_2/p_1$ quantity is not feasible for low enough incomes — in which case the consumer will simply spend all her available income on x_1. Specifically, if $I < \alpha p_2$, the consumer does not have enough income to buy $x_1 = \alpha p_2/p_1$.

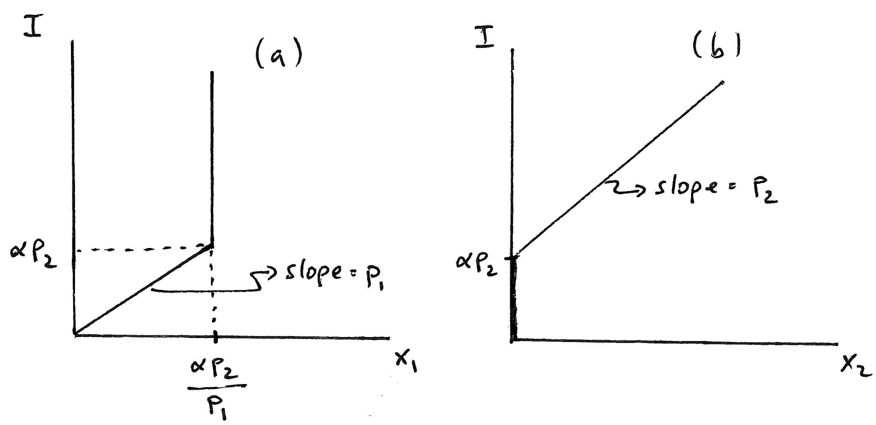
This implies that the real demand function for x_1 (taking into account corner solutions when I is sufficiently low) are

$$x_1 = \frac{I}{p_1} \text{ when } I < \alpha p_2 \text{ and}$$
$$x_1 = \frac{\alpha p_2}{p_1} \text{ when } I \geq \alpha p_2. \tag{9.1.vii}$$

Similarly, the real demand function for x_2 (taking into account corner solutions) is

$$x_2 = 0 \text{ when } I < \alpha p_2 \text{ and}$$
$$x_2 = \frac{I - \alpha p_2}{p_2} \text{ when } I \geq \alpha p_2. \tag{9.1.viii}$$

The income-demand curve is then derived from the inverse of these with I taken to the other side. The results are graphed in Exercise Graph 9.1.



Exercise Graph 9.1 : Income Demand Curves when $u(x_1, x_2) = \alpha \ln x_1 + x_2$

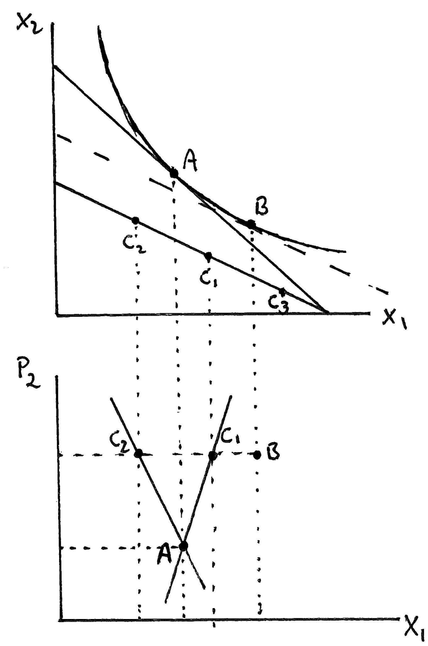
Exercise 9.3

The following is intended to explore what kinds of cross-price demand relationships are logically possible in a two-good model with exogenous income.

A: *For each of the following, indicate whether the relationship is possible or not and explain:*

(a) *A good is normal and its cross-price demand relationship is positive.*

Answer: Yes, this is possible. In Exercise Graph 9.3, we illustrate an increase in the price of good x_2, with A the original optimal bundle at the original lower price. The price increase results in a substitution effect to B — which suggests that consumption of x_1 should increase (since x_1 is now relatively cheaper than before). If x_1 is a normal good (as specified in the question), then the wealth effect will point in the opposite direction — as income declines from the compensated (dashed) budget to the final budget, consumption of x_1 will fall. If this negative wealth effect is smaller in absolute value than the positive substitution effect, we end up at a final bundle like C_1 — with consumption of x_1 having increased (from A) as a result of the increase in p_2. This gives a positive cross-price demand relationship in the lower panel of the graph.



Exercise Graph 9.3 : Cross-price Demand

(b) *A good is normal and its cross-price relationship is negative.*

Answer: This is also possible and is also illustrated in Exercise Graph 9.3. When the wealth effect outweighs the substitution effect, the final optimal bundle might be a bundle like C_2 — which implies consumption of x_1 falls as the price of x_2 increases. This then results in the negative cross-price relationship in the lower panel of the graph.

(c) *A good is inferior and its cross-price relationship is negative.*

Answer: No, this is not possible. Consider again Exercise Graph 9.3. The substitution effect does not depend on whether the good is normal or inferior — so this effect remains exactly the same, suggesting an increase in x_1 as p_2 increases. The wealth effect, however, is now different — as income falls from the compensated (dashed) budget to the new final budget, consumption of x_1 now *increases.* Thus, we end up at a final bundle to the right of B — a point like C_3 in the graph. Since substitution and wealth effects point in the same direction, we can now say unambiguously that, when x_1 is inferior, the cross-price demand relationship must be positive.

(d) *Tastes are homothetic and one of the good's cross-price relationship is negative.*

Answer: Yes, this is possible. Homothetic tastes represent tastes over normal goods. In part (b) we already showed that the cross-price demand relationship can be negative for a normal good.

(e) *Tastes are homothetic and one of the good's cross-price relationship is positive.*

Answer: Yes, this is possible. Homothetic tastes represent tastes over normal goods. In part (a) we already showed that the cross-price demand relationship can be positive for a normal good.

B: *Now consider specific tastes represented by particular utility functions.*

(a) *Suppose tastes are represented by the function $u(x_1, x_2) = \alpha \ln x_1 + x_2$. What is the shape of the cross-price demand curves for x_1 and x_2?*

Answer: We already derived the demand function for this in equation (9.1.vi) in exercise 9.1. These are

$$x_1 = \frac{\alpha p_2}{p_1} \text{ and } x_2 = \frac{I - \alpha p_2}{p_2}. \tag{9.3.i}$$

Thus, the demand for x_2 does not depend on p_1 — so the cross-price demand curve for x_2 is perfectly vertical at $x_2 = (I - \alpha p_2)/p_2$. The demand for x_1, on the other hand, does depend on p_2. Inverting the demand function to get p_2 on the left hand side, we get

$$p_2 = \frac{p_1}{\alpha} x_1. \tag{9.3.ii}$$

Thus, the cross-price demand curve is upward sloping, with slope p_1/α.

(b) *Suppose instead tastes are Cobb-Douglas. What do cross-price demand curves look like?*

Answer: Cobb-Douglas tastes can be modeled by the utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$. The demand functions that emerge from the utility maximization problem are

$$x_1 = \frac{\alpha I}{p_1} \text{ and } x_2 = \frac{(1-\alpha)I}{p_2}. \tag{9.3.iii}$$

Since the "cross-price" does not appear in either of these functions, the demand for neither good depends on the price of the other. As a result, the cross-price demand curves are perfectly vertical.

(c) *Now suppose tastes can be represented by a CES utility function. Without doing any math, can you determine for what values of ρ the cross-price demand relationship is upward sloping?*

Answer: In Exercise Graph 9.3 of part A of this exercise, we showed that cross-price demand curves for normal goods can slope up or down depending on whether the substitution effect is larger or smaller than the income effect. In particular, since the substitution effect says that the consumer will buy more of x_1 when p_2 increases, the cross-price demand curve for x_1 is more likely to slope up the greater the substitution effect. In part (b), we showed that this curve slopes neither up or down (i.e. is perfectly vertical) when tastes are Cobb-Douglas — which is the special case of CES functions where $\rho = 0$. Within the CES family of utility functions, goods become more substitutable the smaller the value of ρ — so the substitution effect becomes relatively larger as ρ falls. Thus, if $-1 < \rho < 0$, we would expect the cross-price demand curve to slope up.

(d) *Suppose tastes can be represented by the CES function $u(x_1, x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$. Verify your intuitive answer from part (c).*

Answer: The demand function for x_1 derived from utility maximization with this utility function is

$$x_1 = \frac{I}{p_1 + \left(p_1 p_2^{\rho}\right)^{1/(\rho+1)}}. \tag{9.3.iv}$$

To determine the sign on the slope of the cross-price demand curve, we take the derivative with respect to p_2; i.e.

$$\frac{\partial x_1}{\partial p_2} = -I\left(p_1 + (p_1 p_2^{\rho})^{1/(\rho+1)}\right)^{-2}\left[\frac{\rho}{\rho+1}\right]\left(p_1 p_2^{\rho}\right)^{-\rho/(\rho+1)} p_1 p_2^{\rho-1}. \tag{9.3.v}$$

Note that there is a negative sign in front of I on the right hand side of this equation — and all terms in the equation are positive except for the bracketed term $[\rho/(\rho+1)]$. (Some of the exponents are negative — but that does not make the terms themselves negative.) The bracketed term is positive for $\rho > 0$ and negative for $-1 < \rho < 0$ — which implies that $\partial x_1/\partial p_2$ is positive if and only if $-1 < \rho < 0$. This is exactly what we concluded intuitively in the previous part.

Exercise 9.5

Everyday Application: *Backward-Bending Labor Supply Curve: We have suggested in this chapter that labor economists believe that labor supply curves typically slope up when wages are low and down when wages are high. This is sometimes referred to as a* backward bending labor supply curve.

A: *Which of the following statements is inconsistent with the empirical finding of a backward bending labor supply curve?*

(a) *For the typical worker, leisure is an inferior good when wages are low and a normal good when wages are high.*

Answer: As wages increase, the substitution effect tells us that workers should work more (because taking leisure has become relatively more expensive). If leisure is an inferior good, the wealth effect also tells us that workers should work more when the wage increases. Thus, *if leisure is an inferior good*, the labor supply curve *must* slope up. If leisure is a normal good, however, the wealth effect tells us that an increase in wages should cause workers to work less. Thus, when leisure is normal, substitution and wealth effects go in the opposite direction — implying that the labor supply curve can slope up or down. Either is consistent with leisure being normal, but only an upward slope is consistent with leisure being inferior.

A backward bending labor supply curve is a labor supply curve that slopes up when wages are low and down when wages are high. If leisure is inferior when wages are low (as specified in this part of the question), this is consistent with an upward slope when wages are low. If leisure is normal when wages get high, this is consistent with either an upward or a downward slope when wages are high — and it is therefore consistent with the downward slope of the backward bending labor supply curve. Thus, the statement in this part of the question is not inconsistent with the backward bending labor supply curve.

(b) *For the typical worker, leisure is a normal good when wages are low and an inferior good when wages are high.*

Answer: (Based on the first paragraph of the answer to (a)), leisure being normal when wages are low is consistent with an upward slope of labor supply when wages are low. Leisure being inferior when wages are high, however, is inconsistent with the downward slope of the backward bending labor supply curve when wages are high. So this statement is not consistent with the backward bending labor supply behavior hypothesized by labor economists.

(c) *For the typical worker, leisure is always a normal good.*

Answer (Based on the first paragraph of the answer to (a)), leisure being a normal good is consistent with both upward and downward sloping labor supply curves. Thus, if leisure is always a normal good, it could indeed be that the labor supply curve is upward sloping for low wages and downward sloping for high wages. Thus, the statement is not inconsistent with the hypothesized backward bending labor supply curve.

(d) *For the typical worker, leisure is always an inferior good.*

Answer: The labor supply curve has to be upward sloping if leisure is inferior — but the backward bending labor supply curve hypothesizes a downward slope for high wages. Thus, leisure being always inferior is not consistent with a backward bending labor supply curve.

B: *Suppose that tastes over consumption and leisure are described by a constant elasticity of substitution utility function* $u(c,\ell) = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho}$.

(a) *Derive the labor supply curve assuming a leisure endowment L.*

Answer: From the utility maximization problem, the *leisure demand* function is

$$\ell = \frac{L}{w^{-\rho/(\rho+1)}+1}, \tag{9.5.i}$$

and the *labor supply function* $l(w)$ is then simply the leisure demand subtracted from the leisure endowment L; i.e.

$$l(w) = L - \frac{L}{w^{-\rho/(\rho+1)}+1} = \frac{w^{-\rho/(\rho+1)}L}{w^{-\rho/(\rho+1)}+1}. \tag{9.5.ii}$$

(b) *Illustrate for which values of ρ this curve is upward sloping and for which it is downward sloping.*

Answer: It is algebraically a little easier to show how the sign of the leisure demand curve (as opposed to the labor supply curve) depends on ρ — and since the labor supply curve just has the opposite slope, we can answer the question this way. The derivative of the leisure demand curve with respect to w then is

$$\frac{\partial \ell}{\partial w} = \frac{Lw^{-(2\rho+1)/(\rho+1)}}{\left(w^{-\rho/(\rho+1)}+1\right)^2}\left[\frac{\rho}{\rho+1}\right]. \tag{9.5.iii}$$

The non-bracketed term is unambiguously positive — which means that the equation is positive if and only if $\rho > 0$ and negative if and only if $-1 < \rho < 0$. Thus, the leisure demand curve slopes up for positive ρ and down for negative ρ. The opposite must then be true for labor supply.

You can show this also directly with the labor supply function by taking its derivative with respect to w. After a little algebraic manipulation, you can get

$$\frac{\partial l(w)}{\partial w} = -\left[\frac{\rho}{\rho+1}\right]\left(\frac{w^{-(2\rho+1)/(\rho+1)}L}{\left(w^{-\rho/(\rho+1)}+1\right)}\right)\left(1-\frac{w^{-\rho/(\rho+1)}}{\left(w^{-\rho/(\rho+1)}+1\right)^2}\right). \tag{9.5.iv}$$

Again, all terms except for the bracketed term are positive. (The last term in parentheses is positive because the denominator in the fraction is larger than the numerator.) Since there is a negative sign at the beginning of the right hand side of the equation, we can then conclude that the derivative is positive if and only if $-1 < \rho < 0$ and negative if and only if $\rho > 0$.

This should make intuitive sense: Substitution effects cause labor to increase with wages while wealth effects cause the opposite. Thus, the larger the substitution effect — i.e. the greater the substitutability between leisure and consumption — the more likely it is that the labor supply curve is upward sloping. And the elasticity of substitution between leisure and consumption increases as ρ falls. For this reason, the labor supply curve slopes up (and the leisure demand curve slopes down) if and only if ρ is below 0.

(c) *Is it possible for the backward bending labor supply curve to emerge from tastes captured by a CES utility function?*

Answer: No, it is not possible for a backward bending labor supply curve to emerge from any one CES utility function. Each such function has a fixed ρ — and, depending on what ρ is, the entire labor supply curve is either upward or downward sloping (or perfectly vertical in the case of $\rho = 0$.)

(d) *For practical purposes, we typically only have to worry about modeling tastes accurately at the margin — i.e. around the current bundles that consumers/workers are consuming. This is because low wage workers, for instance, may experience some increases in wages but not so much that they are suddenly high wage workers, and vice versa. If you were modeling worker behavior for a group of workers and you modeled each worker's tastes as CES over leisure and consumption, how would you assume ρ differs for low wage and high wage workers (assuming you are persuaded of the empirical validity of the backward bending labor supply curve)?*

Answer: We know from what we have done so far that the labor supply curve is upward sloping for high elasticities of substitution (i.e. $-1 < \rho < 0$) and downward sloping for low elasticities of substitution (i.e. $\rho > 0$). If we believe in backward bending labor supply curves but we only need to worry about behavior at the margin, we could therefore model low wage workers (for whom labor supply is upward sloping on the margin) with low values of ρ and high wage workers (for whom labor supply is downward sloping at the margin) with high values of ρ.

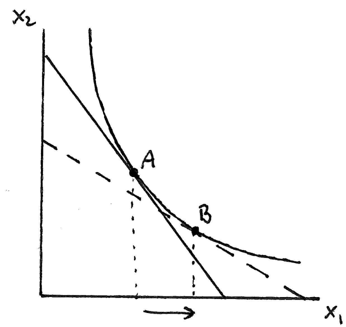
Exercise 9.7

Business Application: *Good Apples versus Bad Apples*: *People are often amazed at the quality of produce that is available in markets far away from where that produce is grown — and that it is often the case that the average quality of produce is* higher *the farther the place where the produce originates. Here we will try to explain this as the result of producers' awareness of relative demand differences resulting from substitution effects.*

A: *Suppose you own an apple orchard that produces two types of apples: High quality apples x_1 and low quality apples x_2. The market price for a pound of high quality apples is higher than that for a pound of low quality apples — i.e. $p_1 > p_2$. You sell some of your apples locally and you ship the rest to be sold in a different market. It costs you an amount c per pound of apples to get apples to that market.*

(a) *Begin with a graph of a consumer who chooses between high and low quality apples in the local store in your town. Illustrate the consumer's budget and optimal choice.*

Answer: This is illustrated with the solid budget line and indifference curve in Exercise Graph 9.7. The optimal bundle for the local consumer is A.



Exercise Graph 9.7 : Good Apples and Bad Apples

(b) *The only way you are willing to ship apples to a far-away market is if you can get as much for those apples as you can get in your town — which means you will add the per-pound transportation cost c to the price you charge for your apples. How will the slope of the budget constraint for the far-away consumer differ from that for your local consumer, and what does that imply for the opportunity cost of good apples in terms of bad apples?*

Answer: The absolute value of the slope will change from p_1/p_2 to $(p_1 + c)/(p_2+c)$. For instance, if the prices are $p_1 = 4$ and $p_2 = 2$, and if $c = 1$, the absolute value of the slope changes from $4/2 = 2$ to $(4+1)/(2+1) = 5/3$. Thus, the slope becomes shallower — implying the opportunity cost of good apples in terms of bad apples falls as c increases.

(c) *Apples represent a relatively small expenditure category for most consumers — which means that income effects are probably very small. In light of that, you may assume that the amount of income devoted to apple consumption is always an amount that gets the consumer to the same indifference curve in the "slice" of tastes that hold all goods other than x_1 and x_2 fixed. Can you determine where consumer demand for high quality apples is likely to be larger — in the home market or in the far-away market?*

Answer: The dashed budget line in the graph has the shallower slope that arises from the lower relative price of good apples in the far-away market. If the consumer changes the amount of income devoted to apple consumption so as to always end up on the same "slice" of the tastes holding all goods other than apples fixed, she will end up at bundle B — with more high quality apples and fewer low quality apples. This is a pure substitution effect — and so long as the income effect from all apples costing more is not large, the same conclusion should hold even if the consumer does not adjust her apple budget to get her all the way to the same indifference curve slice.

(d) *Explain how, in the presence of transportation costs, one would generally expect the phenomenon of finding a larger share of high quality products* *in markets that are far from the production source than in markets that are close.*

Answer: The example has general implications for all products that are consumed locally and also shipped to other markets. Since transportation costs are the same no matter what the quality of the product, the prices of both low and high quality products will have to increase by the same *absolute* amount in the far-away market. But since the high quality products have a higher price to start with, the same absolute increase in price implies a decrease in its *relative* price (i.e. relative to the low quality product). Thus, the existence of transportation costs implies that, although they increase the price of all goods, they increase the price of high quality goods disproportionately less. This results in a larger share of high quality goods being "exported" from local markets. As a result, grapes from Chile might be of average higher quality in US markets than in Chilean markets, just as apples in the hometown of the orchard may be of lower average quality than apples from that orchard in far-away markets.

B: *Suppose that we model our consumers' tastes as* $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$.

(a) *What has to be true about* α *in order for* x_1 *to be the good apples.*

Answer: It must be that the consumer gets more utility from x_1 than from the same amount of x_2 — which holds only if $\alpha > 0.5$.

(b) *Letting consumer income devoted to apple consumption be given by* I*, derive the consumer's demand for good and bad apples as a function of* p_1, p_2, I *and* c. *(Recall that* c *is the per pound transportation cost that is added to the price of apples).*

Answer: Solving the maximization problem

$$\max_{x_1,x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } (p_1+c)x_1 + (p_2+c)x_2 = I, \tag{9.7.i}$$

we get

$$x_1 = \frac{\alpha I}{(p_1+c)} \text{ and } x_2 = \frac{(1-\alpha)I}{(p_2+c)}. \tag{9.7.ii}$$

(c) *What is the ratio of demand for* x_1 *over* x_2*?*

Answer: Using the answer from above, the ratio of demand for good apples relative to bad apples is

$$\frac{x_1}{x_2} = \frac{\alpha}{(1-\alpha)} \frac{(p_2+c)}{(p_1+c)}. \tag{9.7.iii}$$

(d) *Can you tell from this in which market there will be greater relative demand for good versus bad apples — the local market or the far-away market?*

Answer: Note that, when $p_2 < p_1$, an increase in c implies an increase in $(p_2+c)/(p_1+c)$ (or conversely, as we put it in part A, a decrease in

$(p_1+c)/(p_2+c))$. Thus, x_1/x_2 — the amount of good apples relative to bad apples consumed by our consumer — increases as c increases. The far-away market will therefore have relatively greater demand for high quality apples.

(e) *In part A, we held the consumer's indifference curve in the graph fixed and argued that it is reasonable to approximate the consumer's behavior this way given that apple expenditures are typically a small fraction of a consumer's budget. Can you explain how what you just did in part B is different? Is it necessarily the case that consumers in far-away places will consume more high quality apples than consumers (with the same tastes) in local markets? Can we still conclude that far-away markets will have a higher* fraction *of high quality apples?*

Answer: In part B, we used the demand functions for the consumer — and thus held income fixed. In other words, we did not assume that, as transportation costs push up the prices of good and bad apples, the consumer adds more money to the apple budget — rather, we assumed the apple budget stays fixed. This led us to conclude that consumers will demand relatively more high quality apples than low quality apples in the far-away market. It does not necessarily mean that consumers in far-away markets (with tastes similar to local consumers) will consume a greater *absolute* number of high quality apples — but since they demand *relatively* more high quality apples, the fraction of apples that are of high quality would be expected to be larger in the far-away market.

Exercise 9.9

Policy Application: *Demand for Charities and Tax Deductibility: One of the ways in which government policy supports a variety of activities in the economy is to make contributions to those activities tax deductible. For instance, suppose you pay a marginal income tax rate t and that a fraction δ of your contributions to charity are tax deductible. Then if you give \$1 to a charity, you do not have to pay income tax on \$$\delta$ and thus you end up paying \$$\delta t$ less in taxes. Giving \$1 to charity therefore does not cost you \$1 — it only costs you \$$(1-\delta t)$.*

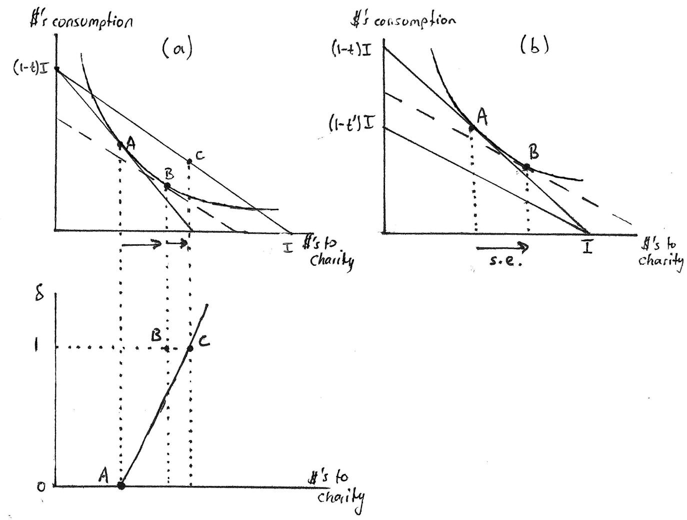
A: *In the remainder of the problem, we will refer to $\delta = 0$ as no deductibility and $\delta = 1$ as full deductibility. Assume throughout that giving to charity is a normal good.*

(a) *How much does it cost you to give \$1 to charity under no deductibility? How much does it cost under full deductibility?*

Answer: Under no deductibility, it costs you \$1 to give \$1. Under full deductibility, it costs you \$$(1-t)$ to give \$1 — because by giving \$1 to charity, you save \$t in taxes.

(b) *On a graph with "dollars given to charity" on the horizontal and "dollars spent on other consumption" on the vertical, illustrate a taxpayer's budget constraint (assuming the taxpayer pays a tax rate t on all income) under no deductibility and under full deductibility.*

Answer: This is illustrated in panel (a) of Exercise Graph 9.9 where the steeper solid line is the no-deductibility budget and the shallower solid line is the full-deductibility budget.



Exercise Graph 9.9 : Tax deductibility of Charitable Contributions

If no money is given to charity, then the consumer will be able to spend $(1-t)I$ — her after-tax income — on consumption. If, on the other hand, she gives all her income to charity under full deductibility, she has to pay no taxes — and is thus able to contribute her before tax income I.

(c) *On a separate graph, derive the relationship between δ (ranging from zero to 1 on the vertical) and charitable giving (on the horizontal).*

Answer: This is derived in the lower graph of panel (a) from the upper graph. Under no deductibility, the consumer optimizes at A. The substitution effect from the lower price for giving to charity under deductibility implies an increase in charitable giving to B — and the remaining income effect increases this further to C (given that we have assumed charitable giving is a normal good). Thus, as deductibility increases, charitably giving unambiguously increases.

(d) *Next, suppose that charitable giving is fully deductible and illustrate how the consumer's budget changes as t increases. Can you tell whether charitable giving increases or decreases as the tax rate rises?*

Answer: The change in the budget is illustrated in panel (b) of the graph. Under full deductibility, the maximum amount that a consumer can give to charity if she gives all her income remains the same as her tax rate changes — because if she gives her entire income, she owes no taxes under full deductibility. However, as t increases, she will not be able to consume as much in other consumption. The budget constraint therefore becomes shallower as t increases from t to t' — with the horizontal intercept remaining unchanged. Beginning at the lower tax rate t, the consumer optimizes at A. An increase in t makes giving to charity relatively cheaper — resulting in a substitution effect to B that implies greater charitable giving. However, there is an additional income effect — and, if charitable giving is a normal good, this effect will point in the opposite direction. Depending on which of these effects is bigger, a consumer might end up increasing or decreasing her charitable giving as her tax rate increases — the more substitutable charitable giving and personal consumption are, the more likely she is to increase her charitable giving as her tax rate increases.

(e) *Suppose that an empirical economist reports the following finding: "Increasing tax deductibility raises charitable giving, and charitable giving under full deductibility remains unchanged as the tax rate changes." Can such behavior emerge from a rationally optimizing individual?*

Answer: Yes, we have shown that it can in the answers above.

(f) *Shortly after assuming office, President Barack Obama proposed repealing the Bush tax cuts — thus raising the top income tax rate to 39.6%. At the same time, he made the controversial proposal to only allow deductions for charitable giving as if the marginal tax rate were 28%. For someone who pays the top marginal income tax under the Obama proposal, what does the proposal imply for δ? What about for someone paying a marginal tax rate of 33% or someone paying a marginal tax rate of 28%?*

Answer: The Obama proposal implies that anyone whose marginal income tax rate exceeds 28% will face a cost of 72 cents for every dollar he gives to charity; i.e. $(1-\delta t) = 0.72$. For someone who pays the top marginal tax rate, we then plug $t = 0.396$ into $(1-\delta t) = 0.72$ and solve for δ to get $\delta \approx 0.71$. When the tax rates are 33% or 28%, repeating this for $t = 0.33$ and $t = 0.28$ gives us $\delta \approx 0.85$ and $\delta = 1$. The Obama proposal therefore effectively lowers the fraction δ of charitable contributions that can be deducted by high income taxpayers.

(g) *Would you predict that the Obama proposal would reduce charitable giving?*

Answer: In part (c) we showed that as deductibility δ increases, we get unambiguously more charitable giving. By the same logic — i.e. both income and substitution effects pointing in the same direction, we conclude that charitable giving will fall as deductibility δ falls. We therefore expect the Obama proposal to result in reduced charitable giving.

(h) *Defenders of the Obama proposal point out the following: After President Ronald Reagan's 1986 Tax Reform, the top marginal income tax rate was 28% — implying that it would cost high earners 72 cents for every dollar they contribute to charity, just as it would under the Obama proposal. If that was good enough under Reagan, it should be good enough now. In what sense it the comparison right, and in what sense is it misleading?*

Answer: The first statement is absolutely correct: For high income individuals, the cost of giving $1 to charities is 72 cents under the 1986 tax reform as well as under the Obama proposal. Put differently, both proposals set the same opportunity cost for giving to charities (for high income earners) — and thus the substitution effect is the same. The difference is the income effect because the tax rates are higher under the Obama proposal than under the Reagan reform. And the income effect would predict lower charitable giving under the Obama proposal than under the terms of the 1986 tax reform.

B: *Now suppose that a taxpayer has Cobb-Douglas tastes over charitable giving* (x_1) *and other consumption* (x_2).

(a) *Derive the taxpayer's demand for charitable giving as a function of income* I, *the degree of tax deductibility* δ *and the tax rate* t.

Answer: Solving the problem

$$\max_{x_1,x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } (1-\delta t)x_1 + x_2 = (1-t)I, \tag{9.9.i}$$

we get

$$x_1 = \frac{\alpha(1-t)I}{(1-\delta t)}. \tag{9.9.ii}$$

(b) *Is this taxpayer's behavior consistent with the empirical finding by the economist in part A(e) of the question?*

Answer: Yes, it is. The first part of the empirical finding said that increasing tax deductibility will increase the consumer's charitable giving. The derivative of x_1 with respect to δ is indeed positive — thus, as δ increases (i.e. as deductibility increases), x_1 increases. The second part of the empirical finding is that, under full deductibility (i.e. when $\delta = 1$), a change in the tax rate has no effect on charitable giving. Setting δ equal to 1 in equation (9.9.ii), we get

$$x_1 = \frac{\alpha(1-t)I}{(1-t)} = \alpha I. \tag{9.9.iii}$$

Thus, under full deductibility, charitable giving is immune to the tax rate — because the income and substitution effects exactly offset each other.

Conclusion: Potentially Helpful Reminders

1. Although we do not use the B bundle that emerges from the compensated budget in this chapter, it is useful to keep in mind where it is even as we connect only A and C to derive our demand and supply relationships. This allows us to really see the role of income and wealth effects which will continue to play important roles in later chapters.

2. The only way any of our demand curves from this chapter ever pass through our point B is if the good we are modeling is quasilinear (implying no income or wealth effects for that good). The same is true for labor (or capital) supply curves if leisure (or present consumption) is quasilinear.

3. Keep in mind that the curves we have derived will usually shift when the economic variables that are held fixed along the curves are changed. For instance, an own-price demand curve will shift to the right when income rises if the underlying good is a normal good (and to the left if it is an inferior good).

4. Own-price demand curves do not shift with changes in income if the underlying good is quasilinear. (See within-chapter exercises 9A.3 through 9A.5.)

5. If you are covering the mathematical B-part of the text, you should be able to relate the concept of demand (and supply) *curves* to demand (and supply) *functions*. Two things to keep in mind: First, the *functions* typically contain multiple variables (like income and more than one price), but the *curves* only allow one of these to vary, thus holding all others implicitly fixed. It is in that sense that we say that the *curves* are slices of the *functions*. Second, the curves have price on the vertical axis and quantity on the horizontal — but the functions are in the form $x(p)$ — i.e. the slices of the functions that hold all but one variable fixed have prices on the horizontal and quantity on the vertical. As a result, we say that the *curves* are not just slices of the functions — they are *inverse slices*.

CHAPTER

10

Consumer Surplus and Dead Weight Loss

This chapter brings together all the concepts from consumer theory — and in the process illustrates the difference between (uncompensated) demand and marginal willingness to pay (or compensated demand). We already developed (uncompensated) demand in Chapter 9 where we brought our A, B and C points from the consumer diagram into a new graph that had quantity on the horizontal and price on the vertical axis. But we only used the A and C points — with B playing no real role other than allowing us to see how big the substitution effect versus the income effect was. *Compensated* demand curves arise from *compensated* budgets, and thus we now turn our attention to point B. In particular, we see that the compensated demand curve connects A and B (rather than A and C) and that this curve can also be viewed as our *marginal willingness to pay* curve. (The same distinction between compensated and uncompensated curves can be made for the supply curves that emerge from the worker and saver diagrams — but we leave that until later (Chapter 19) in the text.) We then find that it is this marginal willingness to pay curve that we can use to measure consumer surplus and changes in consumer welfare, not the uncompensated demand curve from Chapter 9. The two curves are the same only in one very special case.

Chapter Highlights

The main points of the chapter are:

1. We can quantify in dollar terms the value people place on participating in a market — and we define this as the **consumer surplus**. We can similarly quantify the value people place on either getting a lower price or having to accept a higher price.

2. To do this, we need to know the **marginal willingness to pay** for each of the goods a consumer consumes — and this is closely related to the changing MRS along the indifference curve on which the consumer operates.

3. The **marginal willingness to pay** is derived from a single indifference curve — and thus **incorporates only substitution effects**. It is the **same as the uncompensated demand curve only if there are no income effects** — only if tastes are quasilinear in the good we are modeling.

4. **Price-distorting taxes (and subsidies) are inefficient** in the consumer model **to the extent to which they give rise to substitution effects**. Therefore, the inefficiency goes away if the degree of substitutability between goods is zero.

5. The **deadweight loss** from taxes (or subsidies) can be measured as a **distance in the consumer diagram** or as **an area along the marginal willingness to pay curve**.

6. To say that a policy is **inefficient** is the same as to say that there exists in principle a way to compensate those who lose from the policy with the winnings from those who gain. That is not the same as saying that such a policy exists in practice.

7. If you are reading the B-part of the chapter: the **compensated (Hicksian) demand function** is derived from the expenditure minimization problem while the uncompensated budget is derived from the utility maximization problem. The **Slutzky equation** then illustrates the relationship of the slope of the uncompensated demand curve to the slope of the compensated demand (or marginal willingness to pay) curve. This is one of several links — summarized in our **duality** picture at the end of the chapter — between concepts emerging from the utility maximization and the expenditure minimization problems.

10A Solutions to Within-Chapter Exercises for Part A

Exercise 10A.1

As a way to review material from previous chapters, can you identify assumptions on tastes that are *sufficient* for me to know for sure that my indifference curve will be tangent to the budget line at the optimum?

Answer: This will be true so long as, in addition to the usual assumptions about tastes, we assume that both goods are “essential” as defined in Chapter 5. This implies that indifference curves never cross the axes — and thus corner solutions are not possible.

Exercise 10A.2

Demonstrate that own-price demand curves are the same as marginal willingness to pay curves for goods that can be represented by quasilinear tastes.

Answer: This is easy to see once you realize that points B and C in the lower panel of Graph 10.2 in the text will lie on top of one another when there are no income effects — i.e. when tastes are quasilinear in gasoline. C lies to the left of B when gasoline is normal, to the right when gasoline is inferior — so the only time they lie exactly on top of one another is if gasoline is borderline normal/inferior.

Exercise 10A.3

Using the graphs in Graph 9.2 of the previous chapter, determine under what condition own price demand curves are steeper and under what conditions they are shallower than marginal willingness to pay curves.

Answer: Once we realize that $MWTP$ curves connect A and B in the lower panels of the Graph 9.2 in the chapter, it is easy to see the relationship by simply connecting these and forming the $MWTP$ curves. In panel (a) where x is a normal good, $MWTP$ is steeper than own-price demand; in panel (b) where x is (regular) inferior, $MWTP$ is shallower than own-price demand.

Exercise 10A.4

What does the $MWTP$ or compensated demand curve look like if the two goods are perfect complements?

Answer: When the two goods are perfect complements, the entire change in behavior from a price change is an income effect — with no substitution effect. Since $MWTP$ curves only incorporate substitution effects, this implies that the $MWPT$ curve has to be perfectly vertical. This is illustrated in Exercise Graph 10A.4.

Exercise 10A.5

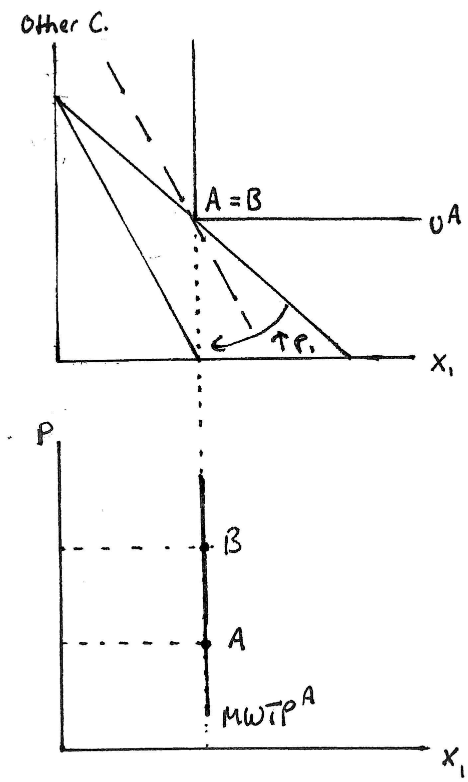
How would Graph 10.4a change if x_1 were an inferior rather than a normal good?

Answer: Bundle B would then lie to the right rather than the left of A — causing the own-price demand curve corresponding to the lower income (I^B) to lie to the *right* of the demand curve corresponding to the higher income (I^A). When a good is inferior, the demand curve therefore shifts out when income falls and in when income increases, the reverse of what is true when the good is normal.

Exercise 10A.6

How would Graph 10.4b change if x_1 were inferior rather than normal?

Answer: If x_1 is inferior, then B must again lie to the *right* of A (just as in the previous exercise). Thus, the marginal willingness to pay curve corresponding to



Exercise Graph 10A.4 : $MWTP$ for Perfect Complements

higher utility (i.e. u^A) must lie to the left of the $MWTP$ curve corresponding to lower utility (i.e. u^B).

Exercise 10A.7

On the lower panel of Graph 10.5b, where does the $MWTP$ curve corresponding to the indifference curve that contains bundle C lie?

Answer: It lies exactly on top of the $MWTP(u^A)$ curve that is depicted in the graph. Since there are no income effects, every $MWTP$ curve must lie on the uncompensated demand curve that now incorporates only substitution effects (because x_1 is quasilinear).

Exercise 10A.8

How do the upper and lower panels of Graph 10.5a change when gasoline is an inferior good?

Answer: If gasoline is an inferior good, then C lies to the right (instead of the left) of B in the top graph — which causes it to lie to the right (instead of left) of B in the lower panel. This implies that the uncompensated demand curve that connects A and C is now steeper (rather than shallower) than the compensated demand (or $MWTP$) curve that connects A and B.

Exercise 10A.9

Can you think of a scenario under which a consumer does not change her consumption of a good when it is taxed but there still exists an inefficiency from taxation?

Answer: This would require the existence of a substitution effect (which causes the inefficiency) that is exactly offset by an income effect. This is illustrated in Exercise Graph 10A.9 and occurs if a good was borderline regular inferior/Giffen. In the top panel, A represents the bundle consumed under the higher (tax-inclusive) price while bundle C represents the bundle consumed at the lower (before-tax) price. The two bundles contain the same quantity of x_1 — so behavior relative to consumption of the taxed good does not change. However, there is clearly an underlying substitution effect that is masked by an offsetting income effect. This substitution effect causes the tax revenue (labeled TR) to be less than what could have been raised by a lump sum tax that makes the consumer no worse off (labeled L). The difference between the two is the deadweight loss.

The lower panel of the graph illustrates the relevant compensated demand curve along which this deadweight loss can be measured (as described in the next section.) Note that the uncompensated demand curve would connect A and C — and would be perfectly vertical with no change in behavior.

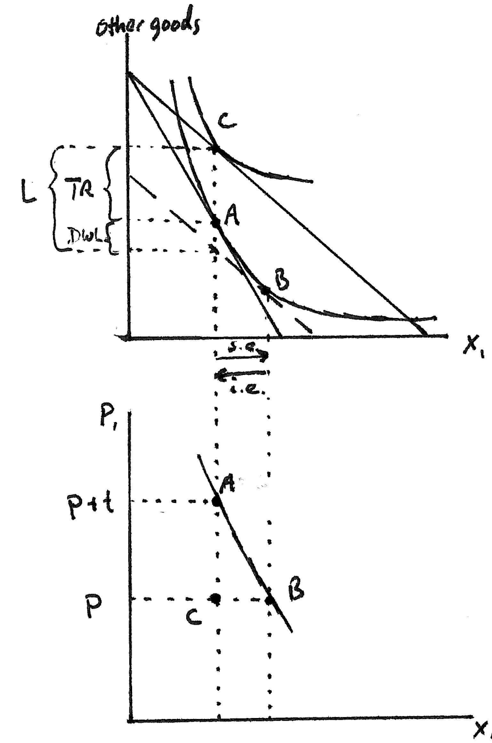
Exercise 10A.10

On a graph with consumption on the vertical axis and leisure on the horizontal, illustrate the deadweight loss of a tax on all consumption (other than the consumption of leisure).

Answer: This is illustrated in Exercise Graph 10A.10 where A is the consumption bundle after the tax is imposed and TR is the tax revenue collected from the tax. Were we to employ a non-distortionary tax instead, we could shift the before-tax budget in parallel all the way to the tangency at B and make the consumer no worse off. Through such a lump-sum tax, we could raise L in revenue — more than we raise under the distortionary tax. The difference between L and TR is the deadweight loss DWL.

Exercise 10A.11

Using Graph 10.8a, verify that the relationship between own price demand and marginal willingness to pay is as depicted in panels (a) through (c) of Graph 10.9.

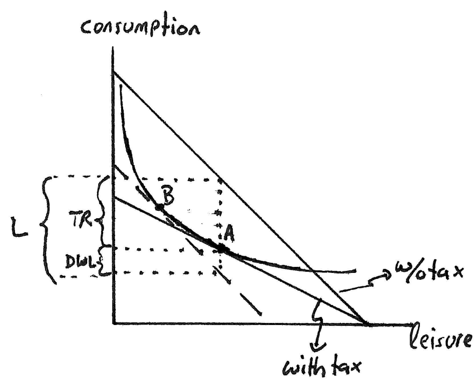


Exercise Graph 10A.9 : DWL without a Change in Behavior

Answer: In order to plot the uncompensated demand curve into the graph that has the compensated demand (or $MWTP$) curve, all we have to do is determine where C lies — i.e. where would the consumer consume in the absence of the distortionary and the lump sum tax. Starting at B, this simply means asking where the consumer would consume if she had more income (because the budget that is tangent at B is parallel to the no-tax budget). If h is a normal good, then such an increase in income implies more consumption of h — which would put C to the right of B. If h is quasilinear, then additional income does not affect consumption of h and would thus put C at the same level of h as B. Finally, if housing is inferior, an increase in income causes less consumption of h — thus causing C to lie to the left of B. This verifies the placement of C relative to B in the three panels of the graph in the text.

Exercise 10A.12

The two proposals also result in different levels of tax revenue. Which proposal actually results in higher revenue for the government? Does this strengthen



Exercise Graph 10A.10 : DWL from a tax on all Consumption

or weaken the policy proposal "to broaden the base and lower the rates"?

Answer: The tax proposal that imposes $2t$ on the single family housing market results in tax revenues equal to $2t$ times q_{p+2t}. The alternative proposal raises t times q_{p+t} in each market, or $2t$ times q_{p+t}. Thus, tax revenue under the latter proposal is a bit larger, t times $(q_{p+t} - q_{p+2t})$ to be exact. The proposal with less deadweight loss therefore also produces more revenue — which strengthens the advice to broaden the base and lower the rates.

10B Solutions to Within-Chapter Exercises for Part B

Exercise 10B.1

In Graph 10.5b we illustrated that $MWTP$ curves and own price demand curves are the same when tastes are quasilinear. Suppose tastes can be modeled with the quasilinear utility function $u(x_1,x_2) = \alpha \ln x_1 + x_2$. Verify a generalization of the intuition from Graph 10.5b — that demand functions and compensated demand functions are identical for x_1 in this case.

Answer: Solving the utility maximization problem

$$\max_{x_1,x_2} \alpha \ln x_1 + x_2 \text{ subject to } p_1x_1 + p_2x_2 = I, \tag{10B.1.i}$$

we get the (uncompensated) demand function $x_1 = \alpha p_2/p_1$. We get exactly the same function when we solve the expenditure minimization problem

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } \alpha \ln x_1 + x_2 = u. \tag{10B.1.ii}$$

Exercise 10B.2

Verify the solutions given in equations (10.8).

Answer: Solving the maximization problem

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } x_1^{\alpha}x_2^{(1-\alpha)} = u, \tag{10B.2.i}$$

we get first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \alpha x_1^{\alpha-1}x_2^{(1-\alpha)} = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - (1-\alpha)x_1^{\alpha}x_2^{-\alpha} = 0. \end{aligned} \tag{10B.2.ii}$$

Solving these for x_1, we get

$$x_1 = \frac{\alpha p_2 x_2}{(1-\alpha)p_1}. \tag{10B.2.iii}$$

Plugging this into the constraint $u = x_1^{\alpha}x_2^{(1-\alpha)}$, we get

$$u = \left(\frac{\alpha p_2 x_2}{(1-\alpha)p_1}\right)^{\alpha} x_2^{(1-\alpha)} = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{\alpha} x_2. \tag{10B.2.iv}$$

Solving this equation for x_2, we then get

$$x_2 = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u \tag{10B.2.v}$$

which is then equal to $h_2(p_1, p_2, u)$ as derived in the text. Substituting this back into (10B.2.iii), we also get

$$x_1 = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u = h_1(p_1, p_2, u). \tag{10B.2.vi}$$

Exercise 10B.3

Verify the solutions given in equations (10.8) and (10.9).

Answer: Plugging the demand functions for x_1 and x_2 into the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$, we get

$$\left(\frac{\alpha I}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)I}{p_2}\right)^{(1-\alpha)} = \frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha} p_2^{(1-\alpha)}} \tag{10B.3.i}$$

which is equal to the indirect utility function $V(p_1, p_2, I)$.

Similarly, multiplying the compensated demand functions by prices and adding them, we get

$$p_1\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u + p_2\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u = \frac{u p_1^{\alpha} p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}} \tag{10B.3.ii}$$

which is equal to the expenditure function $E(p_1, p_2, u)$.

Exercise 10B.4

Verify that (10.11) and (10.12) are true for the functions that emerge from utility maximization and expenditure minimization when tastes can be modeled by the Cobb-Douglas function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$.

Answer: Plugging the expenditure function into the (uncompensated) demand function for x_1, we get

$$\begin{aligned} x_1(p_1, p_2, E(p_1, p_2, u)) &= \frac{\alpha}{p_1}\left(\frac{u p_1^{\alpha} p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right) \\ &= \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u = h_1(p_1, p_2, u). \end{aligned} \tag{10B.4.i}$$

The same holds if we substitute the expenditure function into the (uncompensated) demand function for x_2.

Also, if we substitute the indirect utility function into the compensated demand function for x_1, we get

$$h_1(p_1,p_2,V(p_1,p_2,I)) = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)}\left(\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}}\right)$$
$$= \frac{\alpha I}{p_1} = x_1(p_1,p_2,I) \qquad (10B.4.ii)$$

and again the same holds if we substitute the indirect utility function into the compensated demand function for x_2.

Exercise 10B.5

Verify that the equations in (10.19) are correct for the Cobb-Douglas utility function $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$.

Answer: We previously derived the demand function for x_1 under these tastes as $x_1(p_1,p_2,I) = \alpha I/p_1$. From this, we can derive

$$T = tx_1(p_1+t,p_2,I) = \frac{t\alpha I}{p_1+t}. \qquad (10B.5.i)$$

We also previously derived the indirect utility and expenditure functions for these tastes as

$$V(p_1,p_2,I) = \frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}} \text{ and } E(p_1,p_2,u) = \frac{up_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \qquad (10B.5.ii)$$

The lump sum tax L is, as just derived in the text, $L = I - E(p_1,p_2,V(p_1+t,p_2,I))$. Put in terms of the expressions derived previously, this implies

$$L = I - \left(\frac{p_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p_1+t)^{\alpha}p_2^{(1-\alpha)}}\right) = I - I\left(\frac{p_1}{p_1+t}\right)^{\alpha}. \qquad (10B.5.iii)$$

Exercise 10B.6

Verify that the numbers calculated in the previous paragraph are correct.

Answer: Plugging in the appropriate values, we get

$$T = \frac{t\alpha I}{p_1+t} = \frac{(2.5)(0.25)(100,000)}{10+2.5} = 5,000 \qquad (10B.6.i)$$

and

$$L = I - I\left(\frac{p_1}{p_1+t}\right)^{\alpha} = 100,000 - 100,000\left(\frac{10}{10+2.5}\right)^{0.25} = 5,425.84. \qquad (10B.6.ii)$$

Subtracting T from L gives us a dead weight loss of approximately \$426.

Exercise 10B.7

What shape must the indifference curves have in order for the second derivative of the expenditure function with respect to price to be equal to zero (and for the "slice" of the expenditure function in panel (b) of Graph 10.14 to be equal to the blue line)?

Answer: The indifference curves would have a sharp kink — as those for perfect complements. This is because it is the substitution effect that is creating the concavity of the expenditure function slice in the graph — and that strict concavity goes away only when the substitution effect goes away. And the substitution effect only goes away if the curvature of the indifference curve goes away. Put differently, the "naive" expenditure function that says that consumer will choose the same bundle to reach the same indifference curve when prices change is literally correct if the consumer does not substitute — i.e. if the goods are perfect complements.

Exercise 10B.8

In a 2-panel graph with the top panel containing an indifference curve and the lower panel containing a compensated demand curve for x_1 derived from that indifference curve, illustrate the case when the inequality in equation (10.39) becomes an equality? (*Hint*: Remember that our graphs of compensated demand curves are graphs of the *inverse* of a slice of the compensated demand functions, with a slope of 0 turning into a slope of infinity.)

Answer: The compensated demand curve will be perfectly vertical (which is equivalent to saying $\partial h_i(p_1, p_2, u)/\partial p_i = 0$) if and only if there are no substitution effects — i.e. if the indifference curve has a sharp kink as in the case of perfect complements.

10C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 10.1

Consider a good x_1 in a model where a consumer chooses between x_1 and a composite good x_2.

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Own price demand for a good is perfectly vertical but taxing the good produces a dead weight loss.*

Answer: Panel (a) of Exercise Graph 10.1 illustrates two different prices — p_1 and p_1' — for x_1, with $p_1' > p_1$. (Thus, p_1' would be the tax-inclusive price.) The optimal consumption of x_1 is x_1^A when price is p_1' as well as when price is p_1 — thus, price has no impact on the demand for x_1. In the lower graph of panel (a), this translates into a perfectly vertical demand curve. However, there is still a substitution effect that gives rise to the deadweight loss. This deadweight loss can be either seen as the vertical distance labeled DWL in the upper graph — or as the shaded triangle in the lower graph. Thus, we have an example where the demand is vertical but there is still a deadweight loss from taxation.

(b) *Own price demand is downward sloping (not vertical) and there is no deadweight loss from taxing the good.*

Answer: In panel (b) of Exercise Graph 10.1, we again illustrate two different prices for x_1 but this time model the tastes as those for perfect complements. The demand for x_1 is higher at the lower price — where x_1^C is chosen — than at the higher price — where x_1^A is chosen. This translates into a downward sloping demand curve in the lower graph. But now there is no substitution effect — which leads to a vertical $MWTP$ curve and the disappearance of the deadweight loss triangle. Put differently, the tax revenues raised through this tax are exactly the same that we could raise through a lump sum tax that makes the consumer equally well off. We therefore have an example of a downward sloping demand curve with no deadweight loss from taxation.

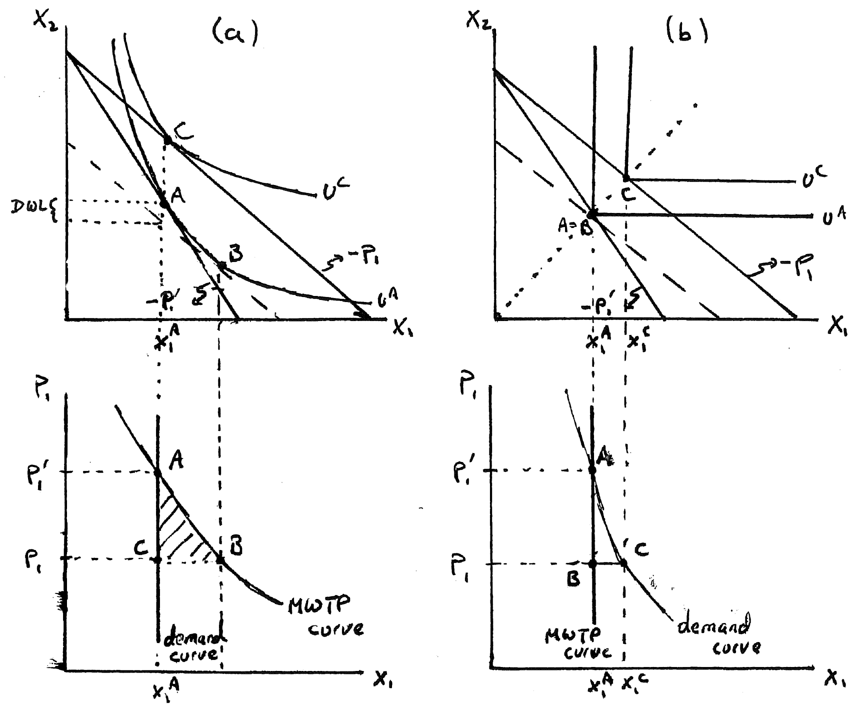
B: *Now suppose that the consumer's tastes can be summarized by the CES utility function $u(x_1, x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$.*

(a) *Are there values for ρ that would result in the scenario described in A(a)?*

Answer: To solve for the demand for x_1, we have to solve

$$\max_{x_1,x_2} (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho} \text{ subject to } I = p_1x_1 + x_2. \tag{10.1.i}$$

This gives us



Exercise Graph 10.1 : Demand and Deadweight Loss from Taxation

$$x_1 = \frac{I}{p_1 + p_1^{1/(\rho+1)}} \text{ and } x_2 = \frac{p_1^{1/(\rho+1)I}}{p_1 + p_1^{1/(\rho+1)}} = \frac{I}{p_1^{\rho/(\rho+1)} + 1}. \qquad (10.1.ii)$$

Regardless of ρ, x_1 is an inverse function of p_1, implying a downward sloping demand curve. Since there is no way for CES demands to be perfectly vertical, there are no values for ρ that would result in the scenario in A(a).

(b) *Are there values for ρ that would result in the scenario described in A(b)?*

Answer: In order for there to be no deadweight loss, there cannot be a substitution effect. The only way there is no substitution effect is if the tastes are for perfect complements — i.e. the elasticity of substitution is 0. And CES utility functions have elasticity of substitution of zero only when $\rho = \infty$. Thus, in order for the scenario to work, $\rho = \infty$. (In that case, $x_1 = I/(p_1 + 1)$ — which implies the demand for x_1 falls as price increases; i.e. demand for x_1 is downward sloping.)

(c) *Would either of the scenarios work with tastes that are quasilinear in x_1?*

Answer: No. The scenario in A(a) would not work because quasilinearity in x_1 implies no income effect — which would put C in panel (a) of Exercise Graph 10.1 directly above B (rather than above A). The substitution effect then implies that A lies to the left of C — which implies a downward sloping (and not a vertical) demand curve. The scenario in A(b) won't work because quasilinear tastes have substitution effects — and thus give rise to deadweight losses from taxation.

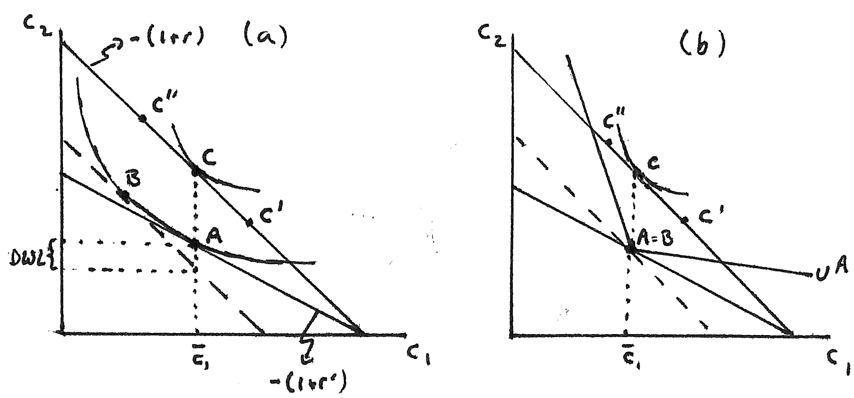
Exercise 10.3

Suppose that consumption takes place this period and next period, and consumption is always a normal good. Suppose further that income now is positive and income next period is zero.

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Savings behavior is immune to changes in the interest rate, but taxing interest income causes a dead weight loss.*

Answer: In panel (a) of Exercise Graph 10.3, we graph two budgets, one with a high interest rate r and one with a low (after-tax) interest rate r'. If tastes are such that the optimal level of consumption in period 1 (i.e. c_1) is unaffected by the interest rate, the optimal bundles A and C lie at the same level $\bar{c}_1$ (as drawn in panel (a)). Thus, a tax on interest income (which lowers the effective interest rate) causes no change in savings. At the same time, there is still a substitution effect that gives rise to a dead weight loss which is indicated as DWL on the vertical axis.



Exercise Graph 10.3 : Savings Behavior and Deadweight Loss from Interest Taxation

(b) *Savings behavior is immune to changes in the interest rate, and taxing interest income causes no dead weight loss.*

Answer: This is also in principle possible — but the indifference curve that is "tangent" at A must now have a kink sufficiently large to eliminate the substitution effect with $B = A$. This is illustrated in panel (b) of the graph. The kink cannot be as sharp as a kink for perfect complements but would need to be sufficiently sharp to eliminate the substitution effect.

(c) *Savings decreases with increases in the interest rate and there is a deadweight loss from taxation of interest.*

Answer: Savings decreases with an increase in the interest rate if A lies to the left of C — i.e. if consumption now (c_1) increases as the interest rate increases. This can be accomplished in panel (a) of Exercise Graph 10.3 by simply re-drawing the indifference curve on the higher interest rate r budget to be tangent at C'. It does not require us to change anything about the indifference curve that is tangent at A — which implies we will continue to have the same substitution effect that gives rise to the deadweight loss.

(d) *Savings increases with increases in the interest rate and there is a deadweight loss from taxation of interest.*

Answer: Savings increases with an increase in the interest rate if A lies to the right of C — i.e. if consumption now (c_1) decreases as the interest rate increases. This can be accomplished in panel (a) of Exercise Graph 10.3 by simply re-drawing the indifference curve on the higher interest rate r budget to be tangent at C''. It does not require us to change anything about the indifference curve that is tangent at A — which implies we will continue to have the same substitution effect that gives rise to the deadweight loss.

(e) *Savings decreases with an increase in the interest rate and there is no deadweight loss.*

Answer: We now have to change the optimal bundle on the high interest rate r budget in panel (b) of Exercise Graph 10.3. If savings decreases with an increase in the interest rate, A has to lie to the left of C — so that consumption now (c_1) increases as the interest rate goes up. Thus, if we change the higher indifference curve to be tangent at C', we have the desired savings response. Since we did not change the lower (kinked) indifference curve, there is still no substitution effect — and thus no deadweight loss from the tax on interest. This would also work for tastes that treat c_1 and c_2 as perfect complements.

(f) *Savings increases with an increase in the interest rate and there is no deadweight loss.*

Answer: We again have to change the optimal bundle on the high interest rate r budget in panel (b) of Exercise Graph 10.3. If savings increases with an increase in the interest rate, A has to lie to the right of C — so that consumption now (c_1) decreases as the interest rate goes up. Thus, if we change the higher indifference curve to be tangent at C'', we have the desired savings response. Since we did not change the lower (kinked) indifference curve, there is still no substitution effect — and thus no deadweight loss from the tax on interest. (This would not work for perfect complements, however — because the sharp kink required for perfect complements would not permit A to lie to the right of C.) So the statement is in fact possible — except for the fact that the problem assumed at the outset that consumption now and in the future is normal. This is not the case at C'' — which implies the statement can be true only if consumption now is an inferior good.

B: *Now suppose that tastes can be summarized by the CES utility function* $u(c_1, c_2) = (0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho}$, *where* c_1 *is consumption in the first period and* c_2 *is consumption in the second period.*

(a) *Are there values for* ρ *that would result in the scenario in A(a) and A(b)?*

Answer: In order for savings behavior to be immune to changes in the interest rate, it must be that consumption now (c_1) is immune to changes in the interest rate. We can calculate the optimal level of c_1 by solving the problem

$$\max_{c_1,c_2}(0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho} \text{ subject to } (1+r)(I-c_1) = c_2 \tag{10.3.i}$$

where I is current income. This gives us

$$c_1 = \frac{I}{1+(1+r)^{-\rho/(\rho+1)}}. \tag{10.3.ii}$$

When ρ is set to zero — i.e. when tastes become Cobb-Douglas — we then get that $c_1 = I/2$ and current consumption is therefore immune to the interest rate. This then implies that savings is immune to the interest rate if and only if $\rho = 0$. Since the elasticity of substitution is positive when $\rho = 0$, this implies there are substitution effects that result in a dead weight loss from a tax on interest. The scenario in A(a) is therefore possible, but the scenario in A(b) is not possible with these CES tastes.

(b) *Are there values for* ρ *that would result in the scenario in A(c)?*

Answer: In order for savings to decrease with an increase in the interest rate, it must be the case that consumption now (c_1) increases with an increase in the interest rate. Equation (10.3.ii) describes how c_1 changes with the interest rate — and by taking the derivative with respect to r, we can tell whether c_1 increases or decreases as r increases. This derivative is

$$\frac{\partial c_1}{\partial r} = \frac{\rho}{(\rho+1)}\left(\frac{I}{\left(1+(1+r)^{-\rho/(\rho+1)}\right)^2(1+r)^{(2\rho+1)/(\rho+1)}}\right). \tag{10.3.iii}$$

Consumption now will increase (and savings will decrease) with the interest rate if this derivative is positive. And, since the term in parenthesis is positive (given that I and r are positive), the derivative is positive if and only if $\rho > 0$. Furthermore, so long as $\rho \neq \infty$, the elasticity of substitution is positive — which implies the existence of substitution effects that create dead weight losses from taxing interest income. Therefore, the scenario in A(c) arises whenever $0 < \rho < \infty$.

(c) *Are there values for ρ that would result in the scenario in A(d)?*

Answer: Savings increases with an increase in the interest rate if and only if consumption now (c_1) falls with an increase in the interest rate. This occurs only if the derivative from equation (10.3.iii) is negative, which in turn occurs only so long as $-1 < \rho < 0$. Since the elasticity of substitution is positive for all such values of ρ, this further implies the emergence of dead weight losses from taxation of interest income. Thus, the scenario in A(d) emerges whenever $-1 < \rho < 0$.

(d) *Are there values for ρ that would result in the scenario in A(e) or A(f)?*

Answer: The only way that there are no substitution effects that give rise to dead weight losses is if the elasticity of substitution is zero — which only happens if $\rho = \infty$. As ρ approaches infinity, the optimal level of current consumption approaches to

$$c_1 = \frac{I}{1+(1+r)^{-1}}. \tag{10.3.iv}$$

The derivative of c_1 with respect to r is positive. Thus, when $\rho = \infty$ (which is necessary for there to be no deadweight loss from taxation of interest), c_1 will increase (and savings will therefore decrease) with an increase in the interest rate. Scenario A(e) therefore arises when $\rho = \infty$ and scenario A(f) is not possible under this CES specification of tastes.

Exercise 10.5

Everyday Application: *Teacher Pay and Pro-Basketball Salaries: Do we have our priorities in order? We trust our school aged children to be taught by dedicated teachers in our schools, but we pay those teachers only about \$50,000 per year. At the same time, we watch pro-basketball games as entertainment — and we pay some of the players 400 times as much!*

A: *When confronted with these facts, many people throw their hands up in the air and conclude we are just hopelessly messed up as a society — that we place more value on our entertainment than on the future of our children.*

(a) *Suppose we treat our society as a single individual. What is our marginal willingness to pay for a teacher? What is our marginal willingness to pay for a star basketball player?*

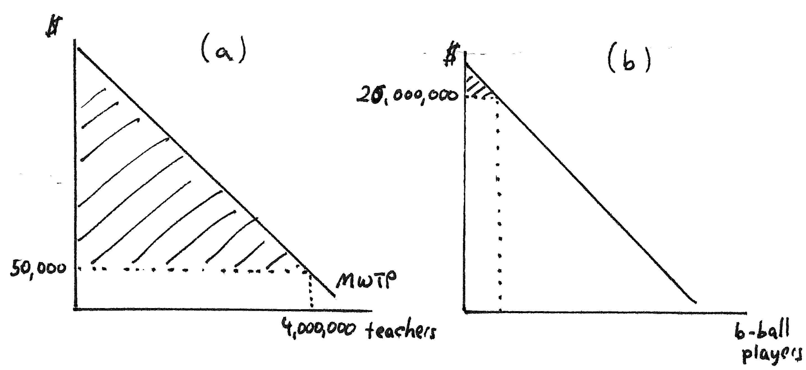
Answer: Our marginal willingness to pay for a teacher is \$50,000 per year, and our marginal willingness to pay for a star basketball player is \$20,000,000 per year.

(b) *There are about 4 million teachers that work in primary and secondary schools in the United States. What is the smallest dollar figure that could represent our total willingness to pay for teachers?*

Answer: By paying \$50,000 per year to 4 million teachers, we are paying a total of \$200 billion (i.e. \$200,000,000,000). Given we are choosing to pay this amount, that is the least that our total willingness to pay for teachers could be.

(c) *Do you think our* actual *total willingness to pay for teachers is likely to be much greater than that minimum figure? Why or why not?*

Answer: $200 billion would be our total willingness to pay if our marginal willingness to pay curve were perfectly horizontal at $50,000; i.e. if our marginal willingness to pay for the first teacher were the same as our marginal willingness to pay for each additional teacher. But that is almost certainly not the case — rather, our marginal willingness to pay for the first teacher is likely to be very high — with decreasing marginal willingness to pay for each additional teacher. Only with the 4 millionth teacher does our marginal willingness to pay reach $50,000. Since total willingness to pay is the entire area under the marginal willingness to pay curve (and not just the amount we actually pay), our total willingness to pay for our teachers is likely much higher than $200 billion per year. In panel (a) of Exercise Graph 10.5, the *additional* amount we are willing to pay for teachers (above the $200 billion we actually pay) is depicted as the large shaded area.



Exercise Graph 10.5 : Marginal and Total Willingness to Pay

(d) *For purposes of this problem, assume there are 10 star basketball players at any given time. What is the least our total willingness to pay for star basketball players could be?*

Answer: Since we are paying $20 million to each, our total willingness to pay for the 10 star basketball players is at least $200 million.

(e) *Is our actual total willingness to pay for basketball players likely to be much higher than this minimum?*

Answer: Our actual total willingness to pay is somewhat higher than that since our $MWTP$ curve for basketball players is downward sloping (just as it is for teachers). But since there are only 10 such players, the $MWTP$ for the 10th player is probably not nearly as much lower than the $MWTP$ of the first player as the $MWTP$ of the 4 millionth teacher is lower than the $MWTP$ for the first teacher. Thus, $200 million is a closer approximation to our total willingness to pay for basketball players than $200 billion is to our total willingness to pay for teachers. This can be seen graphically by comparing the shaded areas in panels (a) and (b) of Exercise Graph 10.5 — where the shaded area in each graph depicts the amount we are willing to pay *in addition* to what we had to pay.

(f) *Do the facts cited at the beginning of this question really warrant the conclusion that we place more value on our entertainment than on the future of our children?*

Answer: We concluded that our total willingness to pay for star basketball players is approximately $200 million — and our total willingness to pay for teachers is substantially larger than $200 billion. While it is true that we are willing to pay more for a star basketball player *on the margin*, the total value we place on the services from the basketball players is therefore substantially less than what we are willing to pay for the services of teachers. Put differently, not only do we actually pay more for all the teachers than we do for basketball players, but the consumer surplus we get from teachers is many orders of magnitude larger than what we get from basketball players.

(g) *Adam Smith puzzled over an analogous dilemma: He observed that people were willing to pay exorbitant amounts for diamonds but virtually nothing for water. With water essential for sustaining life and diamonds just an item that appeals to our vanity, how could we value diamonds so much more than water? This became known as the* diamond-water paradox. *Can you explain the paradox to Smith?*

Answer: Exactly the same reasoning holds as does for teachers and star basketball players. The two panels of Exercise Graph 10.5 could simply be re-labeled, with the first representing water and the second diamonds. The only difference is that the example is even more extreme — our $MWTP$ for the last gallon of water is close to zero but we consume a lot of water — causing our total willingness to pay to be represented by a very large triangle under the $MWTP$ curve. But most of us consume very few diamonds. On the margin, we value a diamond more than a gallon of water, but we place much more value on our water consumption than on our diamond consumption.

B: *Suppose our marginal willingness to pay for teachers (x_1) is given by $MWTP = A - \alpha x_1$ and our marginal willingness to pay for star basketball players (x_2) is given by $MWTP = B - \beta x_2$.*

(a) *Given the facts cited above, what is the lowest that A and B could be?*

Answer: If $MWTP$ curves were perfectly horizontal, then $A = 50,000$ and $B = 20,000,000$.

(b) *If A and B were as you just concluded, what would α and β be?*

Answer: Since $MWTP$ curves can't slope up, it would have to be that they are perfectly flat — with slope of zero. Thus, if $A = 50,000$ and $B = 20,000,000$, then $\alpha = \beta = 0$.

(c) *What would be our marginal and total willingness to pay for teachers and star basketball players?*

Answer: Under these parameters, our total willingness to pay for teachers would be \$200 billion and our total willingness to pay for star basketball players would be \$200 million.

(d) *Suppose $A = B = \$100$ million. Can you tell what α and β must be?*

Answer: In order for the marginal willingness to pay of the 4 millionth teacher to be \$50,000, the slope term α would have to be $99,950,000/4,000,000 = 24.9875$. In order for the marginal willingness to pay for the 10th basketball player to be \$20,000,000, the slope term β must be $80,000,000/10 = 8,000,000$.

(e) *Using the parameter values you just derived (with $A = B = \$100$ million), what is our total willingness to pay for teachers and star basketball players?*

Answer: The triangle in panel (a) of the graph would then be $(99,950,000(4,000,000))/2 = 199,900,000,000,000 = \$199,900$ billion. To this, we add what we actually pay for teachers — \$200 billion — and we get a total willingness to pay for teachers of \$200,100 billion or approximately \$200 *trillion*! The analogous triangle in panel (b) is $80,000,000(10)/2 = 400,000,000$. Adding what we actually pay for star basketball players, we get a total willingness to pay of \$600,000,000 or \$600 million. (That would mean we value our teachers over 300,000 times as much as our star basketball players.)

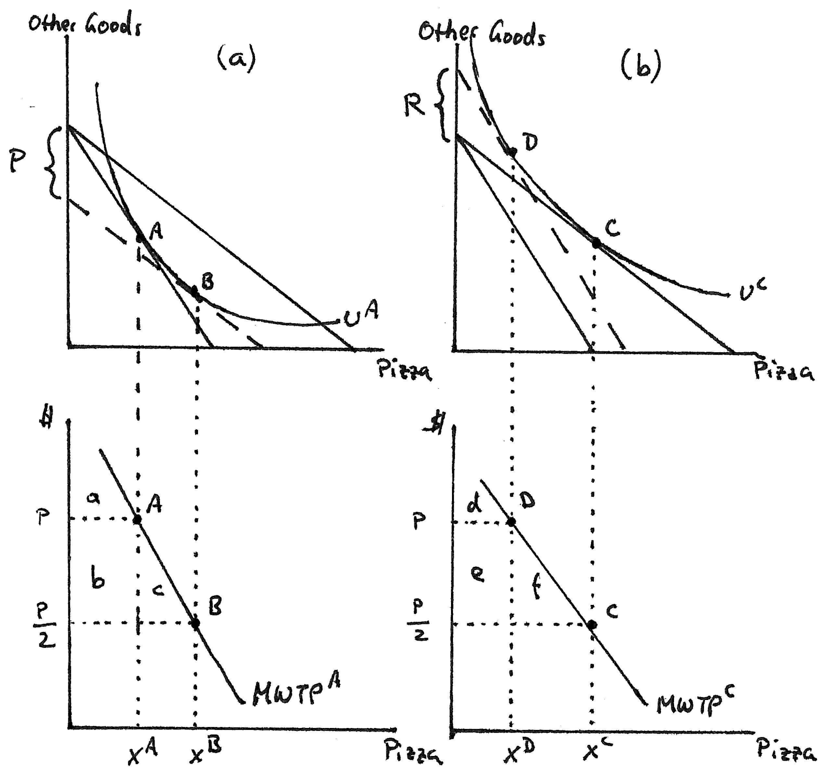
Exercise 10.7

Everyday Application: *To Trade or Not to Trade Pizza Coupons: Exploring the Difference between Willingness to Pay and Willingness to Accept: Suppose you and I are identical in every way — same exogenous income, same tastes over pizza and "other goods". The only difference between us is that I have a coupon that allows the owner of the coupon to buy as much pizza as he/she wants at 50% off.*

A: *Now suppose you approach me to see if there was any way we could make a deal under which I would sell you my coupon. Below you will explore under what conditions such a deal is possible.*

(a) *On a graph with pizza on the horizontal axis and "other goods" on the vertical, illustrate (as a vertical distance) the most you are willing to pay me for my coupon. Call this amount P.*

Answer: This is illustrated in the top graph in panel (a) of Exercise Graph 10.7.



Exercise Graph 10.7 : Trading Pizza Coupons

The steeper budget is your (without-coupon) budget, while the shallower budget is my (with coupon) budget that has half the price for pizza. Without a coupon, your optimal bundle is A — and you reach utility level u^A. Getting the coupon means getting the shallower slope for yourself — but buying it means that you are giving up income. In deciding the most you are willing to pay for a coupon, you therefore have to decide how much you are willing to shift the shallower budget in — and the most you are willing to shift it is an amount that will get you the same utility you can get without the coupon. Thus, the most you are willing to shift the coupon budget in is the amount that creates the tangency at B with u^A and the dashed budget (that has the same slope as the with-coupon budget. The vertical distance between my (with-coupon) budget and the dashed budget is the most you are willing to pay me for the coupon — a distance that can be measured anywhere between the two parallel lines. It is indicated as distance P in the graph.

(b) *On a separate but similar graph, illustrate (as a vertical distance) the least I would be willing to accept in cash to give up my coupon. Call this amount* R.

Answer: The top graph in panel (b) illustrates this. By just using the coupon, I will optimize at bundle C — and will reach the indifference curve u^C. Since I can get to that utility level without selling you the coupon, I will not be willing to make a deal that gets me less utility. By selling you the coupon, I will face a steeper budget but I will have received cash from you — i.e. I will face a budget with the steeper (no-coupon) slope but further out than your initial no-coupon budget. The least I am willing to accept for the coupon is an amount that will make me just as well off as I am with the coupon. I can determine that amount by taking your initial budget and shifting it out until it is tangent to my original optimal indifference curve u^C — which would land me at bundle D in the graph. The amount you have to give me in cash to get me to D is the vertical distance between your original (no-coupon) budget and the dashed budget in the graph. That distance, labeled R, is the least I am willing to accept for the coupon.

(c) *Below each of the graphs you have drawn in (a) and (b), illustrate the same amounts P and R (as areas) along the appropriate marginal willingness to pay curves.*

Answer: In the lower graph of panel (a), the marginal willingness to pay curve is derived from the indifference curve u^A. In the absence of a coupon, you will buy x^A in pizza at the no-coupon price p. This gives you consumer surplus of a. If you end up buying the coupon from me at the maximum price you are willing to pay (P), you will buy x^B in pizza and attain consumer surplus of $a+b+c$. Since A and B lie on the same indifference curve, you are equally happy attaining consumer surplus a without having paid me anything for the coupon or consumer surplus $a+b+c$ after paying me P for the coupon. In order for you to be truly indifferent between these two options, it must therefore be the case that $P=b+c$.

In the lower graph of panel (b), the marginal willingness to pay curve is derived from my indifference curve u^C. In the absence of selling my coupon, I buy x^C pizza — and get consumer surplus of $d+e+f$. If I sell the coupon at the lowest price R that I am willing to accept, I end up buying x^D pizza and get consumer surplus of just d. Since I am equally happy in both cases, it must be that I am indifferent between getting consumer surplus of $d+e+f$ without receiving any cash from you or getting consumer surplus d and getting R in cash. Thus, $R=e+f$.

(d) *Is P larger or smaller than R? What does your answer depend on? (*Hint*: By overlaying your lower graphs that illustrate P and R as areas along marginal willingness to pay curves, you should be able to tell whether one is bigger than the other or whether they are the same size depending on what kind of good pizza is.)*

Answer: Asking if P is larger or smaller than R is then the same as asking if $b+c$ is larger or smaller than $e+f$. Suppose first that pizza is a quasilinear good for us. Then if I transferred the indifference curve u^C onto the top graph of panel (a), the tangency C would lie vertically above B — because a move from the dashed budget in panel (a) to my original (with-coupon) budget is simply an increase in income without a price change. Such an increase in income would not change consumption of pizza when pizza is quasilinear. Similarly, if we transferred the indifference curve u^A onto panel (b), the tangency at A would lie vertically below D. This is because the move from the dashed budget in panel (b) to your (no-coupon) budget is a simple decrease in income without a price change — which causes to change in consumption of pizza when pizza is quasilinear. This implies that, in the lower graphs, A lies at exactly the same place as D and B lies at exactly the same place as C. Put differently, when pizza is a quasilinear good, $MWTP^A$ lies exactly on top of $MWTP^C$ — which implies $b+c=e+f$ or $P=R$. The most you are willing to pay me for the coupon is then exactly equal to the least I am willing to accept.

Now suppose that pizza is an inferior good. Then the same logic we just went through implies that D will lie to the left of A and C will lie to the left of B — which implies that $b+c>e+f$ or $P>R$. Thus, when pizza is an inferior good, the most you are willing to pay is greater than the least I am willing to accept for the coupon. If, on the other hand, pizza is a normal good, then the same logic implies that D lies to the right of A and C lies to the right of B — which further implies that $b+c<e+f$ of $P<R$. Thus, when pizza is a normal good for us, then the most you are willing to pay is less than the least I am willing to accept for the coupon.

(e) True or False*: You and I will be able to make a deal so long as pizza is not a normal good. Explain your answer intuitively.*

Answer: This is true. We have just concluded that when pizza is an inferior good, you are willing to pay me more than the least I am willing to accept — so there is room for us to make a deal and both become better off. When pizza is quasilinear (i.e. borderline between normal and inferior), then the least I am willing to accept is exactly the most you are willing to pay — so in principle we can make a deal but neither one of us will be better or worse off for it. But when pizza is a normal good for us, the least I am willing to accept is more than them most you are willing to pay — so there is no way we will be able to strike a deal.

Intuitively, this makes sense in the following way: We began by saying that you and I are identical in every way — except there is one way in which we are not identical: I have a coupon and you do not. Thus, I am in essence richer than you are to begin with. If pizza is a normal good, then richer people will buy more pizza than poorer people — and so the coupon has more value to richer people because they would use it more. It is for this reason that we can't make a deal if pizza is normal for us. But if pizza is an inferior good, then being richer means I will want less pizza — and so I have less use for the coupon than you do. As a result, you will be willing to pay more than the least I am willing to accept. And if pizza is quasilinear, rich and poor buy the same amount of pizza — and thus make the same use of the coupon. Thus, if pizza is quasilinear, the coupon is worth the same to us.

B: *Suppose your and my tastes can be represented by the Cobb-Douglas utility function $u(x_1,x_2) = x_1^{0.5}x_2^{0.5}$, and suppose we both have income $I = 100$. Let pizza be denoted by x_1 and "other goods" by x_2, and let the price of pizza be denoted by p. (Since "other goods" are denominated in dollars, the price of x_2 is implicitly set to 1.)*

(a) *Calculate our demand functions for pizza and other goods as a function of p.*

Answer: Solving the problem

$$\max_{x_1,x_2} x_1^{1/2}x_2^{1/2} \text{ subject to } px_1 + x_2 = 100, \tag{10.7.i}$$

we get $x_1 = 50/p$ and $x_2 = 50$.

(b) *Calculate our compensated demand for pizza (x_1) and other goods (x_2) as a function of p (ignoring for now the existence of a coupon).*

Answer: Solving the problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{1/2}x_2^{1/2}, \tag{10.7.ii}$$

we get $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2}u$.

(c) *Suppose $p = 10$ and the coupon reduces this price by half (to 5). Assume again that I have a coupon but you do not. How much utility do you and I get when we make optimal decisions?*

Answer: Using our (uncompensated) demand functions, we can calculate that your pizza consumption is $x_1 = 50/10 = 5$ while mine is $x_1 = 50/5 = 10$. This corresponds to x_A and x_D in Exercise Graph 10.7. Both of us consume 50 in other goods. Thus, your utility is $u(5,50) = 5^{1/2}50^{1/2} \approx 15.81$ and my utility is $u(10,50) = 10^{1/2}50^{1/2} \approx 22.36$.

(d) *How much pizza will you consume if you pay me the most you are willing to pay for the coupon? How much will I consume if you pay me the least I am willing to accept?*

Answer: If you pay me the most you are willing to pay, you will remain at the same utility level but will pay only half as much (i.e. \$5 instead of \$10). The compensated demand function therefore tells us that you will consume $x_1 = 15.81/(5^{1/2}) \approx 7.07$. If you pay me the least I am willing to accept, I will end up with the same utility as before but with a price that is twice as high (i.e. \$10 instead of \$5). Thus, the compensated demand function tells me that $x_1 = 22.36/(10^{1/2}) \approx 7.07$. In terms of the graphs in Exercise Graph 10.7, this implies that $x_B = 7.07 = x_D$.

(e) *Calculate the expenditure function for me and you.*

Answer: To get the expenditure function, we substitute the compensated demands back into the objective function of the minimization problem — i.e. we substitute $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2}u$ into $px_1 + x_2$ to get

$$E(p,u) = p\left(\frac{u}{p^{1/2}}\right) + p^{1/2}u = 2p^{1/2}u. \tag{10.7.iii}$$

(f) *Using your answers so far, determine R — the least I am willing to accept to give up my coupon. Then determine P — the most you are willing to pay to get a coupon.(*Hint*: Use your graphs from A(a) to determine the appropriate values to plug into the expenditure function to determine how much income I would have to have to give up my coupon. Once you have done this, you can subtract my actual income $I = 100$ to determine how much you have to give me to be willing to let go of the coupon. Then do the analogous to determine how much you'd be willing to pay, this time using your graph from A(b).)*

Answer: The budget required for me to be just as happy without the coupon (i.e. when $p = 10$) is the expenditure necessary for me to reach utility level 22.36 (which is u^C in our graph) at $p = 10$ — i.e. $E(10, 22.36) = 2(10^{1/2})22.36 \approx 141.42$. Since I started out with an income of \$100, this implies that you would have to give me approximately \$41.42 for the coupon in order for me to be just as happy; i.e. $R = 41.42$. The budget required for you to be just as happy with the coupon as you were without is the expenditure necessary to get you to utility level 15.81 (which is u^A in our graph) at the with-coupon price of 5 — i.e. $E(5, 15.81) = 2(5^{1/2})15.81 \approx 70.71$. Since you started with an income of \$100, this means you would be willing to pay me as much as $100 - 70.71 = 29.29$ to get the coupon; i.e. $P = 29.29$.

(g) *Are we able to make a deal under which I sell you my coupon? Make sense of this given what you found intuitively in part A and given what you know about Cobb-Douglas tastes.*

Answer: No, we are not able to make a deal since the most you are willing to pay me (\$29.29) is less than the least I am willing to accept (\$41.42). This is consistent with what we concluded in part A where we said that we would not be able to strike a deal if pizza is a normal good for us. Cobb-Douglas tastes are tastes over normal goods — so under the tastes represented by the utility function we have been working with, pizza is in fact a normal good.

(h) *Now suppose our tastes could instead be represented by the utility function $u(x_1, x_2) = 50\ln x_1 + x_2$. Using steps similar to what you have just done, calculate again the least I am willing to accept and the most you are willing to pay for the coupon. Explain the intuition behind your answer given what you know about quasilinear tastes.*

Answer: Solving the problem

$$\max_{x_1,x_2} 50\ln x_1 + x_2 \text{ subject to } px_1 + x_2 = 100, \quad (10.7.iv)$$

we get the (uncompensated) demands $x_1 = 50/p$ and $x_2 = 50$. Thus, both you and I consume 50 in other goods, but I consume 10 pizzas while you only consume 5 because I face a with-coupon price of \$5 per pizza while you face a without-coupon price of \$10 per pizza.

Plugging $(x_1, x_2) = (10, 50)$ into the utility function, we get my utility (equivalent to u^C in the graph) of 165.13. Plugging $(x_1, x_2) = (5, 50)$ into the utility function for you, we get your utility (equivalent to u^A in our graph) as 130.47.

Solving the minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = 50\ln x_1 + x_2, \quad (10.7.v)$$

we can derive the compensated demands $x_1 = 50/p$ and $x_2 = u - 50\ln(50/p)$. (Note that the compensated and uncompensated demands for the quasilinear good x_1 are the same — which makes sense since there are no income effects to make the two demands different.) Next, we can find the expenditure function by just plugging the compensated demands into the objective function of the minimization problem to get

$$E(p,u) = p\left(\frac{50}{p}\right) + u - 50\ln\left(\frac{50}{p}\right) = u + 50\left(1 - \ln\left(\frac{50}{p}\right)\right). \quad (10.7.vi)$$

To determine the expenditure necessary for me to get to my current utility level in the absence of the coupon (i.e. when $p = 10$ instead of $p = 5$), we calculate $E(10, 165.13) \approx 134.66$. Since I start with an income of \$100, that means the least I am willing to accept for the coupon is $R = 134.66 - 100 =$ \$34.66. To calculate the expenditure necessary to get you to your current utility level in the presence of a coupon (i.e. when $p = 5$ instead of $p = 10$), we calculate $E(5, 130.47) \approx 65.34$. Since you also start with an income of \$100, this means that the most you are willing to pay for the coupon is $P = 100 - 65.34 =$ \$34.66. Thus $P = R$ as we concluded it has to be when pizza is a quasilinear good.

(i) *Can you demonstrate, using the compensated demand functions you calculated for the two types of tastes, that the values for P and R are in fact areas under these functions (as you described in your answer to A(c)? (*Note*: This part requires you to use integral calculus.)*

<u>Answer</u>: The compensated demand function for pizza is $x_1 = 50/p$ (as calculated in the previous part). The area in the graph is the integral under this function between the with-coupon and the without-coupon prices. (Note: In our graphs, we are graphing the inverse of the compensated demand functions — which is why the area appears as an area to the left of the curve rather than an area under the curve.) Thus, the least I am willing to accept (R) is

$$R = \int_5^{10} \frac{50}{p} dp = 50 \ln p|_5^{10} = 50 \ln 10 - 50 \ln 5 = 34.66. \quad (10.7.vii)$$

Since, because of the quasilinearity of pizza, our compensated demand functions are the same (because u does not appear in the functions), the same holds for P. Thus, $P = R = 34.66$, exactly as we concluded before.

Exercise 10.9

Everyday Application: *To Take, or not to Take, the Bus*: *After you graduate, you get a job in a small city where you have taken your sister's offer of living in her apartment. Your job pays you $20 per hour and you have up to 60 hours per week available. The problem is you also have to get to work.*

A: *Your sister's place is actually pretty close to work — so you could lease a car and pay a total (including insurance and gas) of $100 per week to get to work, spending essentially no time commuting. Alternatively, you could use the city's sparse bus system — but unfortunately there is no direct bus line to your place of work and you would have to change buses a few times to get there. This would take approximately 5 hours per week.*

(a) *Now suppose that you do not consider time spent commuting as "leisure" — and you don't consider money spent on transportation as "consumption". On a graph with "leisure net of commuting time" on the horizontal axis and "consumption dollars net of commuting costs" on the vertical, illustrate your budget constraint if you choose the bus and a separate budget constraint if you choose to lease the car.*

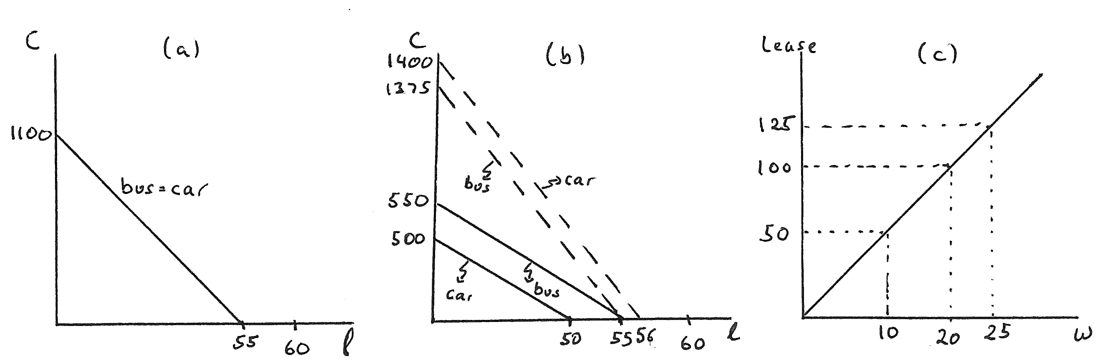
Answer: If you choose to take the bus, you reduce your leisure time from 60 to 55 hours per week — and therefore can earn up to $20(55)=$1,100 per week. If you choose to lease a car, you can work up to 60 hours per week — but you have to pay $100 no matter how much you work. If you work for 60 hours, you earn $25(60)=$1,200, but since you have to pay $100 for the lease, the most you can consume is $1,100. In order to pay for the lease, you have to work at least 5 hours. Thus, the two budget constraints are exactly the same. They are illustrated in panel (a) of Exercise Graph 10.9.

(b) *Do you prefer the bus to the car?*

Answer: Since the two choices result in the same effective budget constraint, you are indifferent between taking the bus and leasing the car.

(c) *Suppose that before you get to town you find out that a typo had been made in your offer letter and your actual wage is $10 per hour instead of $20 per hour. How does your answer change?*

Answer: If you choose the bus, the loss of leisure remains unchanged — leaving you again with 55 hours per week. At the lower wage, that allows you to consume at most $10(55)=$550 per week. If you choose the lease, it will take you 10 hours just to come up with the payment for the lease. Even if you work all 60 hours, you will therefore be left with consumption



Exercise Graph 10.9 : To Take, or not to Take, the Bus

corresponding to only 50 hours of work — or \$500. The lease budget is therefore strictly lower than the bus budget, as illustrated by the two solid budget lines in panel (b) of Exercise Graph 10.9. You would therefore choose the bus.

(d) *After a few weeks, your employer discovers just how good you are and gives you a raise to \$25 per hour. What mode of transportation do you take now?*

Answer: If you choose the bus, you again will take the same 5 hour reduction in your leisure — leaving you with at most \$25(55)=\$1,375 in consumption. If you choose the lease, you can pay for the lease with just 4 hours of work — leaving you with up to 56 hours that you can devote to generating consumption. Now the lease budget therefore strictly dominates the bus budget, as illustrated by the dashed budgets in panel (b) of the graph. You will therefore go by car.

(e) *Illustrate in a graph (not directly derived from what you have done so far) the relationship between wage on the horizontal axis and the most you'd be willing to pay for the leased car.*

Answer: We know from (a) that, when $w = 20$, a lease payment of \$100 per week makes you exactly indifferent. This is because, at a wage of \$20 per hour, it takes 5 hours to make enough money to pay for the lease — which is exactly the same number of hours as it takes to ride the bus. Thus, to get the maximum lease payment Y that you would be willing to pay at wage w, you simply multiply the number of hours required for the bus by the wage — i.e. $Y = 5w$. Put differently, the value of your time on the bus determines the maximum lease you are willing to pay. This is illustrated in panel (c) of GExercise Graph 10.9.

(f) *If the government taxes gasoline and thus increases the cost of driving a leased cars (while keeping buses running for free), predict what will happen to the demand for bus service and indicate what types of workers will be the source of the change in demand.*

Answer: Demand for bus service will increase, with the increase in demand determined by the lowest wage workers that previously chose to lease cars.

(g) *What happens if the government improves bus service by reducing the time one needs to spend to get from one place to the other?*

Answer: If the government improves bus service, a person with a wage that previously made him indifferent between the bus and leasing a car will now take the bus because the most he would be willing to pay for a lease will fall. Thus, lower wage workers will switch to the improved bus system.

B: *Now suppose your tastes were given by $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$, where c is consumption dollars net of commuting expenses and ℓ is leisure consumption net of time spent commuting. Suppose your leisure endowment is L and your wage is w.*

(a) *Derive consumption and leisure demand assuming you lease a car that costs you \$$Y$ per week which therefore implies no commuting time.*

Answer: We need to solve the problem

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } w(L-\ell) = Y + c. \tag{10.9.i}$$

This gives us

$$c = \alpha(wL - Y) \text{ and } \ell = \frac{(1-\alpha)(wL-Y}{w}. \tag{10.9.ii}$$

(b) *Next, derive your demand for consumption and leisure assuming you take the bus instead, with the bus costing no money but taking T hours per week from your leisure.*

Answer: Now we need to solve the problem

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } w(L-T-\ell) = c. \tag{10.9.iii}$$

This gives us

$$c = \alpha w(L-T) \text{ and } \ell = (1-\alpha)(L-T). \tag{10.9.iv}$$

(c) *Express the indirect utility of leasing the car as a function of Y.*

Answer: To get the indirect utility from leasing a car, we simply have to plug the optimal values for c and ℓ from equation (10.9.ii) into the utility function — which gives

$$V(Y) = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(wL-Y)}{w^{(1-\alpha)}}. \tag{10.9.v}$$

(d) *Express your indirect utility of taking the bus as a function of T.*

Answer: Now we need to take the optimal levels of c and ℓ when taking the bus (from equation (10.9.iv)) into the utility function to get

$$V(T) = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}w^{\alpha}(L-T). \tag{10.9.vi}$$

(e) *Using the indirect utility functions, determine the relationship between Y and T that would keep you indifferent between taking the bus and leasing the car. Is your answer consistent with the relationship you illustrated in A(e) and your conclusions in A(f) and A(g)?*

Answer: Setting $V(Y)$ equal to $V(T)$, i.e.

$$\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(wL-Y)}{w^{(1-\alpha)}} = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}w^{\alpha}(L-T), \tag{10.9.vii}$$

we get, after canceling some terms, $Y = wT$. This is consistent with what we found in part A where we said that $Y = 5w$ when the time to take the bus is 5 hours per week. It is also consistent with our conclusion that the lowest wage workers who previously leased cars will switch to buses if the government increases the cost of leasing a car by taxing gasoline — because as Y increases, the equation $Y = wT$ implies that w will also have to increase to keep someone indifferent between leasing a car and taking the bus. It is similarly consistent with the conclusion that a more efficient bus system — i.e. a lower T — will cause the same types of workers to switch to the bus.

(f) *Could you have skipped all these steps and derived this relationship directly from the budget constraints? Why or why not?*

Answer: Yes, you could have. The two budget constraints can be expressed as

$$c = w(L-\ell) - Y \text{ and } c = w(L-T-\ell). \tag{10.9.viii}$$

Setting these equal to each other, you get $Y = wT$. This works because the only factors operating in this problem are wealth effects — there are no substitution effects. It is analogous to what we did in part A where we simply focused on the relationship between the budget constraints to determine what the consumer would prefer — we did not have to actually draw any indifference curves.

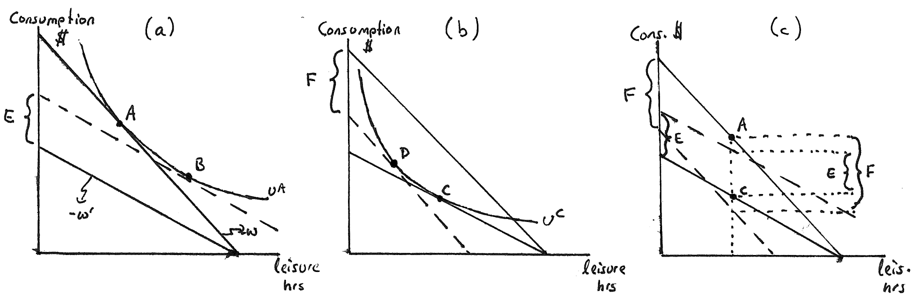
Exercise 10.11

Business Application: *Negotiating an Endorsement Deal and a Bribe*: *Suppose you are an amateur athlete and your uncle owns the cereal company "Wheaties." Your uncle offers you a job working for his company at a wage of w per hour. After looking around for other jobs, you find that the most you could make elsewhere is w', where $w' < w$. You have a weekly leisure endowment of L and can allocate any amount of that to work. Given the higher wage at Wheaties, you accept your uncle's job offer.*

A: *Then you win a gold medal in the Olympics. "Greeties", the makers of grits, ask you for an endorsement. As part of the deal, they will pay you some fixed weekly amount to appear on their boxes of grits. Unfortunately, your uncle (who hates his competitor "Greeties" with the white hot intensity of a thousand suns) will fire you if you accept the deal offered by "Greeties". Therefore, if you accept the deal, your wage falls to w'.*

(a) *On a graph with Consumption on the vertical and Leisure on the horizontal axis, graph your budget constraint before the "Greeties" offer.*

Answer: This is graphed in panel (a) of Exercise Graph 10.11(1) as the steeper of the two budget lines.



Exercise Graph 10.11(1) : Endorsement Deals and Bribes

(b) *On the same graph, illustrate your budget if you worked for someone other than your uncle prior to your success in the Olympics.*

Answer: This is illustrated as the shallower of the two budget lines in panel (a) of the graph.

(c) *Illustrate the minimum amount that "Greeties" would have to pay you (weekly) for your endorsement in order for you to accept the deal. Call this amount E.*

Answer: The optimal bundle at Wheaties is indicated as bundle A — yielding utility level u^A. You would not be willing to enter any endorsement deal that does not at least get you to that same utility level. The Greeties endorsement check must therefore be sufficient to get you to u^A. Such a deal does not change your wage at Greeties — it just causes your Greeties budget to shift out in a parallel way. If the shift is sufficient to get you to bundle B, it is the lowest possible amount you would accept. The is indicated in dollar terms on the vertical axis as E.

(d) *How does this amount E compare to the amount necessary to get you to be able to consume bundle A under a Greeties endorsement deal?*

Answer: The minimum amount E that is acceptable to you is smaller than what is necessary to get to bundle A. You can tell it is less because A lies outside the budget constraint that emerges when Greeties offers E. The difference emerges because of a substitution effect.

(e) *Now suppose that you accepted the endorsement deal from "Greeties" but, unfortunately, the check for the endorsement bounces because "Greeties" goes bankrupt. Therefore the deal is off, but your angry uncle has already fired you. Deep down inside your uncle still cares about you and will give you back your old job if you come back and ask him for it. The problem is that you have to get past his greedy secretary who has full control over who gets to see your uncle. When you get to the "Wheaties" office, she informs you that you have to commit to pay her a weekly bribe if you want access to your uncle. On a new graph, illustrate the largest possible (weekly) payment you would be willing to make. Call this F.*

<u>Answer</u>: Once the endorsement deal falls through, you will optimize at bundle C in panel (b) of Exercise Graph 10.11(1) — along the budget formed by the lower market wage w' in the absence of any lump sum payments. This would give you utility level u^C. Getting employment at Wheaties implies getting the higher wage w (and thus the steeper budget), but paying the weekly bribe means that this budget shifts inward in a parallel way. The most you would be willing to pay to get your old job back is an amount that makes you just as well off as you are without getting back to Wheaties — i.e. an amount that gets you to utility level u^C. This is equivalent to a shift in the steeper budget that gets you to the new optimal bundle D. The amount of the weekly bribe is then indicated in dollar terms on the vertical axis as F.

(f) *If your uncle's secretary just asks you for a weekly bribe that gets you to the bundle C that you would consume in the absence of returning to Wheaties, would you pay her such a bribe?*

<u>Answer</u>: Yes, you would — because the shift in the steeper budget required to get you to C is less than the shift induced by F. You can tell that this is so by the fact that C lies outside the budget that is created by the highest possible bribe F you'd be willing to pay. The reason you are willing to pay more than the amount that would make C affordable is a substitution effect.

(g) *Suppose your tastes are such that the wealth effect from a wage change is exactly offset by the substitution effect — i.e. no matter what the wage, you will always work the same amount (in the absence of receiving endorsement checks or paying bribes). In this case, can you tell whether the amount E (i.e. the minimum endorsement check) is greater than or equal to the amount F (i.e. the maximum bribe)?*

<u>Answer</u>: Yes — we will conclude that $F > E$. Here is how we can tell: If wealth and substitution effects exactly offset as described, then A (in panel (a) of Exercise Graph 10.11(1)) lies exactly above C (from panel (b) of Exercise Graph 10.11(1)) — because A is optimal at the high wage (in the absence of a bribe) and C is optimal at the low wage (in the absence of a bribe). This is illustrated in panel (c) of the graph where A and C are plotted at the same level of leisure. We then transfer the dashed lines from panel (a) and (b) — to give us the vertical distances E and F. The

key is that we concluded that the dashed line in panel (a) must lie *below* A, and the dashed line in panel (b) must lie *below* C. But it is then shown in panel (c) of the graph that this implies F — the vertical distance between the steeper lines — must be larger than E — the vertical distance between the shallower lines. You would therefore be willing to bribe more to get back your job than you required in an endorsement to give it up.

B: *Suppose that your tastes over weekly consumption c and weekly leisure ℓ can be represented by the utility function $u(c,\ell) = c^{0.5}\ell^{0.5}$ and your weekly leisure endowment is $L = 60$.*

(a) *If you accept the initial job with Wheaties, how much will you work?*

Answer: To answer this, we need to solve the problem

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } w(60-\ell) = c. \qquad (10.11.\text{i})$$

This results in

$$c = 30w \text{ and } \ell = 30. \qquad (10.11.\text{ii})$$

Thus, you will work for 30 hours at Wheaties.

(b) *Suppose you accept a deal from Greeties that pays you a weekly amount $\overline{E}$. How much will you work then? Can you tell whether this is more or less than you would work at Wheaties?*

Answer: We now have to solve the problem

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } \overline{E} + w'(60-\ell) = c. \qquad (10.11.\text{iii})$$

This solves to

$$c = 0.5\overline{E} + 30w' \text{ and } \ell = \frac{0.5\overline{E}+30w'}{w'} = \frac{0.5\overline{E}}{w'} + 30. \qquad (10.11.\text{iv})$$

You will therefore take more than the 30 hours of leisure you took at Wheaties — which implies you will work fewer hours.

(c) *Suppose that the wage w at Wheaties is \$50 per hour and the wage w' at Greeties (or any other potential employer other than Wheaties) is \$25 per hour. What is the lowest possible value for E — the weekly endorsement money from Greeties — that might get you to accept the endorsement deal?*

Answer: First, we have to figure out how much utility you can assure yourself of by just working at Wheaties — because you would not accept an endorsement deal from Greeties that results in less utility than that. At Wheaties, we calculate that $c = 30w$ and $\ell = 30$ — which implies you will consume a bundle $A = (c,\ell) = (1500, 30)$ at $w = 50$. This gives utility of $u^A = 1500^{0.5}30^{0.5} \approx 212.13$. From the results in equation (10.11.iv), we know that your consumption bundle at Greeties with an endorsement deal $\overline{E}$ results in the bundle $B = (c,\ell) = (0.5\overline{E}+750, (0.5\overline{E}+750)/25)$ when

$w' = 25$. Since E would result in the same utility at B as at A, we could then simply set the utility at the consumption bundle with the endorsement deal equal to 212.13 and solve for E — i.e. we could solve

$$(0.5E + 750)^{0.5}\left(\frac{(0.5E + 750)}{25}\right)^{0.5} = 212.13. \qquad (10.11.v)$$

This gives us $E = 612.30$.

Alternatively, we could use the expenditure function approach. To do so, we need to solve the expenditure minimization problem

$$\min_{c,\ell} w\ell + c \text{ subject to } u = c^{0.5}\ell^{0.5} \qquad (10.11.vi)$$

to solve for the compensated demands

$$c = w^{0.5}u \text{ and } \ell = w^{-0.5}u. \qquad (10.11.vii)$$

Plugging these back into the expenditure equation $w\ell + c$, we get that the expenditure necessary to get to utility level u at wage w is

$$E(w,u) = 2w^{0.5}u. \qquad (10.11.viii)$$

The value of your endowment at a wage of \$25 is 25(60)=1500. The expenditure necessary to get to utility level $u^A = 212.13$ at $w' = 25$ is $E(25, 212.13) = 2(25^{0.5})(212.13) = 2121.30$. The minimum necessary weekly endorsement check to get you to accept the Greeties offer is then the difference between these; i.e. $E = 2121.30 - 1500 = 612.30$. (Note that the E that refers to this minimum endorsement check is not the same as the *expenditure function* $E(w,u)$.)

(d) *How much will you work if you accept this endorsement deal E?*

Answer: Your optimal leisure (from the equation (10.11.iv)) is then

$$\ell = \frac{0.5\overline{E}}{w'} + 30 = \frac{0.5(612.30)}{25} + 30 \approx 42.25. \qquad (10.11.ix)$$

Thus, you will work approximately $60 - 42.25 = 17.75$ hours per week at Greeties.

(e) *Suppose you have accepted this deal but Greeties now goes out of business. What is the highest possible weekly bribe F you'd be willing to pay your uncle's secretary in order to get your job at Wheaties back?*

Answer: First, we have to calculate the utility level you would get if you do not get your old job at Wheaties back. With $E = 0$, our results from equation (10.11.iv) become identical to those from equation (10.11.ii) — $c = 30w'$ and $\ell = 30$. When $w' = 25$, this implies an optimal consumption bundle $C = (c, \ell) = (750, 30)$ — which gives utility $u^C = 750^{0.5}30^{0.5} = 150$.

We can then solve for F in one of two ways. Using the expenditure function $E(w,u) = 2w^{0.5}u$, we can calculate the minimum expenditure necessary at the Wheaties wage $w = 50$ to get to the utility level u^C — i.e. $E(50,150) = 2(50^{0.5})(150) = 2121.32$. Your leisure endowment at the Wheaties wage of $w = 50$ is worth $50(60) = 3000$. Thus, you would be willing to pay $3000 - 2121.32 = 878.68$ in a weekly bribe to get your Wheaties job back.

You can also calculate this by directly setting the utility from paying the bribe at the higher wage equal to $u^C = 150$ (in a way analogous to what we did in the previous part). This would give you the equation

$$(0.5F + 1500)^{0.5}\left(\frac{(0.5F + 1500)}{50}\right)^{0.5} = 150. \tag{10.11.x}$$

When solved for F, this gives $F = -878.68$; i.e. you would be willing to pay this amount which is the same as calculated through the expenditure function approach.

(f) *How much would you work assuming that the secretary has successfully extracted the maximum amount you are willing to pay to get your Wheaties job back?*

Answer: We can use our answer in equation (10.11.iv) since F is just like a negative endorsement deal that gets to the higher wage.

$$\begin{aligned} c &= 0.5(-F) + 30w = 0.5(-878.68) + 30(50) \approx 1060.66 \text{ and} \\ \ell &= \frac{0.5(-F)}{w} + 30 = \frac{0.5(-878.68)}{50} + 30 \approx 21.21. \end{aligned} \tag{10.11.xi}$$

Thus, you will take 21.21 hours of leisure — which means you will work for 60-21.21=38.79 hours.

(g) *Re-draw your graphs from part A but now label all the points and intercepts in accordance with your calculations. Does your prediction from A(g) about the size of the maximum bribe relative to the size of the minimum endorsement hold true?*

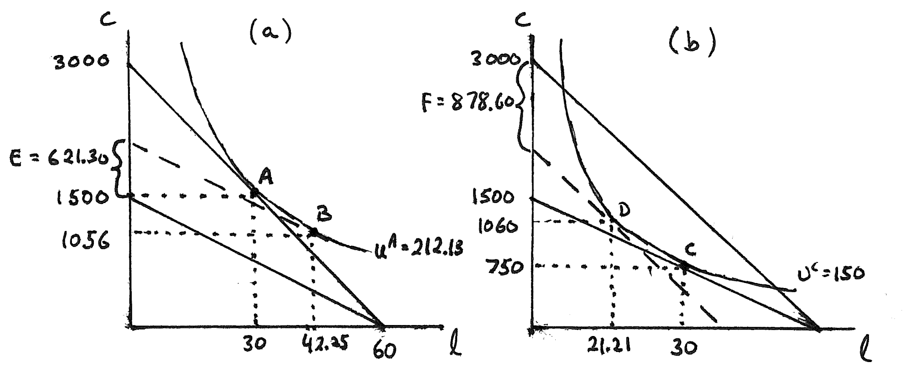
Answer: This is done in Exercise Graph 10.11(2).

The prediction from part A(g) holds true. Here we have a case where the wealth and substitution effects exactly offset one another in the absence of bribes or endorsements — which results in leisure of 30 hours at both the high wage (point A) and the low wage (point C). We predicted in part A(g) that this should lead to $F > E$ — which holds for this numerical example where $F = 1060.66 > 612.30 = E$.

Exercise 10.13

Policy Application: *Price Subsidies: Suppose the government decides to subsidize (rather than tax) consumption of grits.*

A: *Consider a consumer that consumes boxes of grits and "other goods".*



Exercise Graph 10.11(2) : Endorsement Deals and Bribes: Part 2

(a) *Begin by drawing a budget constraint (assuming some exogenous income) with grits on the horizontal axis and "other consumption" on the vertical. Then illustrate a new budget constraint with the subsidy — reflecting that each box of grits now costs the consumer less than it did before.*

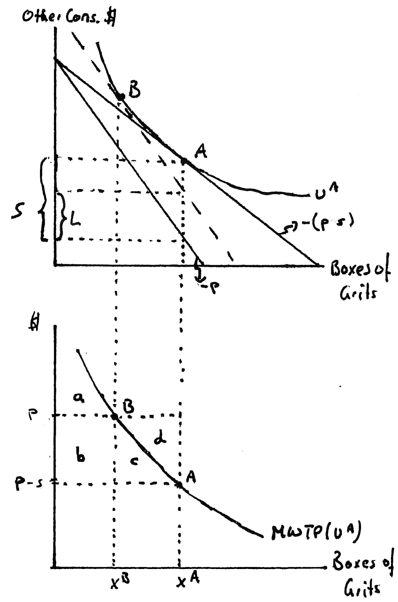
Answer: This is illustrated in the top graph of Exercise Graph 10.13(1), with $(p - s)$ indicating the price with the subsidy and p indicating the price without.

(b) *Illustrate the optimal consumption of grits with an indifference curve tangent to the after-subsidy budget. Then illustrate in your graph the amount that the government spends on the subsidy for you. Call this amount S.*

Answer: The optimal consumption bundle is illustrated as bundle A. The vertical intercept of that bundle indicates how much in other consumption the consumer is able to afford given that grits are subsidized. Had they not been subsidized, a much lower amount (read off the no-subsidy budget) would be available for other consumption. The difference is S — the amount the government paid for this consumer under the price subsidy policy.

(c) *Next, illustrate how much the government could have given you in a lump sum cash payment instead and made you just as happy as you are under the subsidy policy. Call this amount L.*

Answer: The government could have chosen not to alter the price of grits (and thus not alter the slope of the no-subsidy budget line) but instead simply shift that budget out in a parallel way by giving a cash subsidy. The amount in cash the government could have given to make the consumer just as happy as she is under the price subsidy is then an amount that creates the dashed budget which is tangent to the post-subsidy indifference curve u^A. This tangency occurs at bundle B, and the cost of this cash subsidy is simply the vertical difference between the dashed bud-



Exercise Graph 10.13(1) : Subsidizing Grits

get and the parallel no-subsidy budget. That distance can be measured anywhere (since the lines are parallel) and is indicated as the distance L.

(d) *Which is bigger — S or L?*

Answer: S is bigger than L because of the substitution effect from A to B. You don't have to give someone as much in unrestricted cash as you would have to spend in a subsidy that is restricted to the purchase of one good.

(e) *On a graph below the one you have drawn, illustrate the relevant MWTP curve and show where S and L can be found on that graph.*

Answer: The graph below the top graph derives the $MWTP$ or compensated demand curve that corresponds to utility level u^A. Under the price subsidy, the consumer consumes at A — which gives consumer surplus of $a+b+c$. The government is paying the difference between p and $(p-s)$ for each of the x^A boxes of grits the consumer buys — which means that the cost of the price subsidy is $S = b + c + d$.

Under the cash subsidy, the consumer faces the higher price p (rather than $(p-s)$) and buys x^B rather than x^A assuming she receives the cash subsidy L. This leaves her with consumer surplus of a in the grits market — but she is equally happy since both A and B lie on the same indifference curve. The only way she can be equally happy is if the cash subsidy was enough to make up for the loss in consumer surplus in the grits market — i.e. $L = b + c$.

(f) *What would your tastes have to be like in order for S to be equal to L.*

Answer: The substitution effect that creates the difference between S and L would have to disappear — which happens only if there is a sharp kink in the indifference curve at A (such as if grits and other goods are perfect complements). In that case, $A = B$ and $L = S$. In the lower graph, this implies that B lies directly above A — leading to a perfectly vertical $MWTP$ curve and the disappearance of the area d.

(g) *True or False: For almost all tastes, price subsidies are inefficient.*

Answer: This is true — so long as there aren't sharp kinks in just the right places of indifference curves — i.e. so long as goods are somewhat substitutable at the margin, $S > L$ which leaves the difference as a deadweight loss. If the substitutability goes away, so does the deadweight loss triangle d as the $MWTP$ curve becomes vertical.

B: *Suppose the consumer's tastes are Cobb-Douglas and take the form $u(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)}$ where x_1 is boxes of grits and x_2 is a composite good with price normalized to 1. The consumer's exogenous income is I.*

(a) *Suppose the government price subsidy lowers the price of grits from p to $(p - s)$. How much S will the government have to pay to fund this price subsidy for this consumer?*

Answer: We need to solve the problem

$$\max_{x_1,x_2} x_1^\alpha x_2^{(1-\alpha)} \text{ subject to } px_1 + x_2. \tag{10.13.i}$$

This solves to

$$x_1 = \frac{\alpha I}{p} \text{ and } x_2 = (1-\alpha)I. \tag{10.13.ii}$$

When the government lowers the price to $(p - s)$, demand is $x_1 = \alpha I/(p - s)$. For each box of grits, the consumer pays $(p - s)$ while the government pays s. Thus, the government's expense is

$$S = \frac{s\alpha I}{p-s}. \tag{10.13.iii}$$

(b) *How much utility does the consumer attain under this price subsidy?*

Answer: Under the price subsidy, the consumer chooses the bundle $(x_1, x_2) = (\alpha I/(p-2), (1-\alpha)I)$. Substituting this into the utility function, we get the indirect utility function

$$V(p,s) = \left(\frac{\alpha I}{p-s}\right)^\alpha ((1-\alpha)I)^{(1-\alpha)} = \frac{\alpha^\alpha (1-\alpha)^{(1-\alpha)}}{(p-s)^\alpha} I. \tag{10.13.iv}$$

(c) *How much L would the government have had to pay this consumer in cash to make the consumer equally happy as she is under the price subsidy?*

Answer: Given that we know how much utility the consumer gets under the price subsidy, we now have to ask what expenditure (in cash) would have been necessary to get to the same utility level at the non-subsidized price p. We can derive the expenditure function by solving the expenditure minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{\alpha} x_2^{(1-\alpha)}, \tag{10.13.v}$$

and then plug the compensated demands into the objective $px_1 + x_2$, or we can simply invert the indirect utility function. Either way, we get

$$E(p,u) = \frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{10.13.vi}$$

We are interested in knowing the expenditure necessary at p to get to utility level $V(p,s)$ from equation (10.13.iv); i.e. we are interested in

$$E(p,V(p,s)) = \left(\frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p-s)^{\alpha}}I\right) = \left(\frac{p}{p-s}\right)^{\alpha} I. \tag{10.13.vii}$$

This is the total expenditure necessary to get to the price-subsidy utility level $V(p,s)$. Since the consumer starts with an income I, the amount of cash L the consumer would have to get to be equally happy as under the price subsidy would therefore be

$$L = E(p,V(p,s)) - I = \left[\left(\frac{p}{p-s}\right)^{\alpha} - 1\right] I. \tag{10.13.viii}$$

(d) *What is the deadweight loss from the price subsidy?*

Answer: The deadweight loss is then just the difference between what the government spends under the price subsidy (S) and what the government could have spent in a lump sum way (L) to make the consumer just as well off. This is

$$DWL = S - L = \left[1 + \frac{s\alpha}{p-s} - \left(\frac{p}{p-s}\right)^{\alpha}\right] I. \tag{10.13.ix}$$

(e) *Suppose $I = 1000$, $p = 2$, $s = 1$ and $\alpha = 0.5$. How much grits does the consumer buy before any subsidy, under the price subsidy and under the utility-equivalent cash subsidy? What is the deadweight loss from the price subsidy?*

Answer: We have calculated that the demand for grits is $x_1 = \alpha I/p$. Thus, when the price is unsubsidized originally, the consumer buys $x_1 = 0.5(1000)/2 = 250$. The price subsidy lowers the effective price for the consumer to 1 — which implies the new quantity demanded is $x_1 = 0.5(1000)/1 = 500$. To calculate the equivalent cash subsidy, we can use equation (10.13.viii) to get

$$L = \left[\left(\frac{2}{2-1}\right)^{0.5} - 1\right](1000) \approx 414.21. \tag{10.13.x}$$

The consumer's income under the cash subsidy would therefore rise to \$1,414.21 but the price would remain at $p = 2$. The consumer's demand would therefore be $x_1 = 0.5(1414.21)/2$ which is approximately 354.

Finally, the deadweight loss is simply $(S - L)$. The cost of the price subsidy, given that the consumer will demand 500 units of x_1 and the cost of the subsidy is \$1 per unit, is \$500. The deadweight loss is therefore 500-414.21=\$85.79. You can also get this by simply plugging the relevant values into the DWL equation we calculated in equation (10.13.ix); i.e.

$$DWL = \left[1 + \frac{1(0.5)}{2-1} - \left(\frac{2}{2-1}\right)^{0.5}\right](1000) \approx 85.79. \tag{10.13.xi}$$

(f) *Continue with the values from he previous part. Can you calculate the compensated demand curve you illustrated in A(e) and verify that the area you identified as the deadweight loss is equal to what you have calculated? (*Hint*: You need to take an integral and use some of the material from the appendix to answer this.)*

<u>Answer</u>: The compensated demand curve arises from the expenditure minimization problem

$$\min_{x_1, x_2} px_1 + x_2 \ \text{ subject to } \ u = x_1^{0.5}x_2^{0.5}. \tag{10.13.xii}$$

From this, we get compensated demands

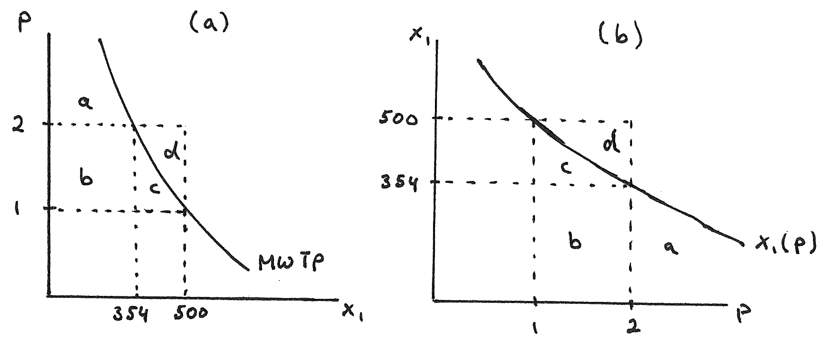
$$x_1 = \frac{u}{p^{0.5}} \ \text{ and } \ x_2 = p^{0.5}u. \tag{10.13.xiii}$$

Using the indirect utility function $V(p,s)$ from equation (10.13.iv), we can determine the utility the consumer gets under the price subsidy as

$$V(2,1) = \left(\frac{0.5^{0.5}0.5^{0.5}}{(2-1)^{0.5}}\right)(1000) = 500. \tag{10.13.xiv}$$

The appropriate compensated (or $MWTP$) curve is then $x_1 = 500/(p^{0.5})$. The inverse of this function is sketched in panel (a) of Exercise Graph 10.13(2).

It is similar to the lower graph in Exercise Graph 10.13(1) where we indicated that $S = b + c + d$, $L = b + c$ and $DWL = d$. Panel (b) of Exercise Graph 10.13(2) then simply inverts panel (a), placing x_1 on the vertical (rather than the horizontal) and p on the horizontal (rather than the vertical) axes.



Exercise Graph 10.13(2) : Subsidizing Grits: Part 2

The area $L = c + b$ is then simply the integral under the function $x_1 = 500/(p^{0.5})$ evaluated from $p = 1$ to $p = 2$. This is

$$\int_1^2 \frac{500}{p^{0.5}}dp = 2(500)p^{0.5}|_1^2 = 1000(2^{0.5} - 1) \approx 414.21. \qquad (10.13.xv)$$

This is exactly what we calculated for L in the previous part. The area $S = b + c + d$ is simply 500. Thus, the deadweight loss is $DWL = 500 - 414.21 = 85.79$, again exactly as we calculated before.

Exercise 10.15

Policy Application: *International Trade and Child Labor: Consider again the end-of-chapter problem 8.9 about the impact of international trade on child labor in the developing world.*

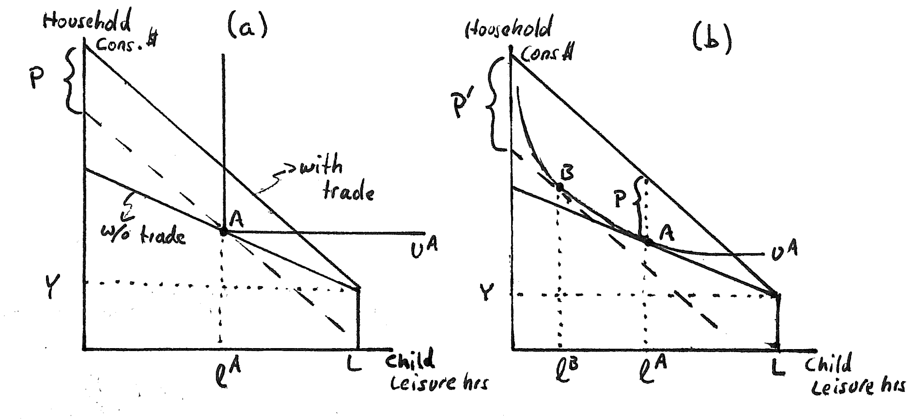
A: *Suppose again that households have non-child income Y, that children have a certain weekly time endowment L, and that child wages are w in the absence of trade and $w' > w$ with trade.*

(a) *On a graph with child leisure hours on the horizontal axis and household consumption on the vertical, illustrate the before and after trade household budget constraints.*

Answer: This is illustrated in panel (a) of Exercise Graph 10.15.

(b) *Suppose that tastes over consumption and child leisure were those of perfect complements. Illustrate in your graph how much a household would be willing to pay to permit trade — i.e. how much would a household be willing to pay to increase the child wage from w to w'?*

Answer: This is also illustrated in panel (a) of Exercise Graph 10.15. The household would be willing to give up an amount P that makes it just as well off as it would be in the absence of trade. In the absence of trade, the household optimizes at A — leaving the child with leisure of ℓ^A. Paying



Exercise Graph 10.15 : Child Leisure and Trade

a lump sum amount to get the higher wage is equivalent to shifting the after-trade (steeper) budget inward in a parallel way — and to figure out the highest payment the household is willing to make, we shift this budget until it just barely reaches the original indifference curve u^A. Given the shape of the perfect complements indifference curve, this means we shift the after-trade budget until it intersects at A. We can then read the size of the payment P as the vertical distance between the parallel lines. This distance is indicated on the vertical axis of the graph.

(c) *If the household paid the maximum it was willing to pay to cause the child wage to increase, will the child work more or less than before the wage increase?*

Answer: The child will continue to work the same amount — i.e. it will receive leisure of ℓ^A which implies that it's labor supply has not changed.

(d) *Re-draw your graph, assume that the same bundle (as at the beginning of part (b)) is optimal, but now assume that consumption and leisure are quite (though not perfectly) substitutable. Illustrate again how much the household would be willing to pay to cause the wage to increase.*

Answer: This is illustrated in panel (b) of Exercise Graph 10.15. We can now shift the after-trade budget inward until it is tangent to B — which puts it below A. The amount that the household is willing to pay is indicated as the vertical distance between the parallel lines on the vertical axis.

(e) *If the household actually had to pay this amount to get the wage to increase, will the child end up working more or less than before trade?*

Answer: The child will now work more — because it's leisure time has fallen from ℓ^A to ℓ^B — which means it's labor supply has increased by the same amount. (This is due to the emergence of the substitution effect that was absent in panel (a)).

(f) *Does your prediction of whether the child will work more or less if the household pays the maximum bribe to get the higher wage depend on how substitutable consumption and child leisure are?*

Answer: No — it does not depend on how substitutable. Even the slightest degree of substitutability in the indifference curve will cause B to lie to the left of A — which implies an increase in the child's labor supply. This goes away only if the substitutability is assumed away completely — and it never goes in the other direction (because substitution effects always point in the same direction).

(g) *Can you make a prediction about the relative size of the payment the household is willing to make to get the higher child wage as it relates to the degree of substitutability of consumption and child leisure? Are "good" parents willing to pay more or less?*

Answer: The size of the payment a household is willing to make to get the higher wages for the child increases with the degree of substitutability of household consumption with child leisure. You can see this by comparing P to P' in the graphs. In panel (b) of the graph, P is illustrated as the vertical difference between A and the after-trade budget (which is the same vertical distance found in (a) under perfect complementarity). This is unambiguously smaller than P' because B unambiguously falls to the left of A when there is some substitutability between consumption and child leisure. We concluded in the chapter 8 exercise that "good parents" (i.e. those that reduce child labor as child wages increase in the absence of a required payment to make that increase happen) are those who view consumption and child leisure as not very substitutable — parents that find it difficult to think of the household being better off if the child is not also working less. We are now finding that such "good parents" (i.e. those more like what is graphed in panel (a)) are not willing to pay as much of a bribe to get child wages to increase as "bad parents" are.

B: *Suppose that the household's tastes over consumption and leisure can be represented by the CES utility function $u(c,\ell) = (\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho})^{-1/\rho}$.*

(a) *Derive the optimal household consumption and child leisure levels assuming the household has non-child weekly income Y, the child has a weekly time endowment of L, and the child wage is w.*

Answer: We have to solve the problem

$$\max_{c,\ell} \left(\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho}\right)^{-1/\rho} \text{ subject to } Y + w(L-\ell) = c. \tag{10.15.i}$$

This gives us

$$c = \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} (Y + wL) \left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} \right]^{-1} \tag{10.15.ii}$$

and

$$\ell = (Y + wL)\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right]^{-1}. \tag{10.15.iii}$$

(b) *Verify your conclusion from end-of-chapter problem 8.9 that parents are neither "good" nor "bad" when $Y = 0$ and $\rho = 0$; i.e. parents will neither increase nor decrease child labor when w increases.*

Answer: When $Y = 0$ and $\rho = 0$, the child leisure expression becomes

$$\ell = wL\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)\right]^{-1} = wL\left[\frac{w}{(1-\alpha)}\right]^{-1} = (1-\alpha)L. \tag{10.15.iv}$$

Child leisure and thus child labor therefore do not depend on w, which implies an increase in child wages will result in neither an increase nor a decrease in child labor.

(c) *If international trade raises household income Y, what will happen to child labor in the absence of any change in child wages? Does your answer depend on how substitutable c and ℓ are?*

Answer: Since Y enters the leisure demand function positively, chid leisure will increase — i.e. child labor will fall. The answer does not qualitatively depend on the value of ρ — and thus does not depend on the elasticity of substitution.

(d) *When $\alpha = 0.5$ and $w = 1$, does your answer depend on the household elasticity of substitution between consumption and child leisure?*

Answer: When $\alpha = 0.5$ and $w = 1$, the expressions for optimal consumption and leisure reduce to

$$c = \frac{Y+L}{2} \text{ and } \ell = \frac{Y+L}{2}. \tag{10.15.v}$$

The answer therefore does not depend on the household elasticity of substitution.

(e) *How much utility will the household get when $\alpha = 0.5$ and $w = 1$?*

Answer: The utility is then given by

$$u\left(\frac{Y+L}{2}, \frac{Y+L}{2}\right) = \left[0.5\left(\frac{Y+L}{2}\right)^{-\rho} + 0.5\left(\frac{Y+L}{2}\right)^{-\rho}\right]^{-1/\rho} = \frac{Y+L}{2}. \tag{10.15.vi}$$

(f) *Derive the expenditure function for this household as a function of w and u. What does this reduce to when $\alpha = 0.5$? (Hint: You can assume $Y = 0$ for this part.)*

Answer: We now need to solve the problem

$$\min_{c,\ell}\ w\ell + c \text{ subject to } u = (\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho})^{-1/\rho}. \tag{10.15.vii}$$

This gives compensated demands

$$c=\left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)}+(1-\alpha)\right]^{1/\rho}u \tag{10.15.viii}$$

and

$$\ell=\left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)}+(1-\alpha)\right]^{1/\rho}u. \tag{10.15.ix}$$

Plugging these into the expenditure expression $w\ell+c$, we get the expenditure function

$$E(w,u)=\left[w+\left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right]\left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)}+(1-\alpha)\right]^{1/\rho}u. \tag{10.15.x}$$

When $\alpha=0.5$, this reduces to

$$E(w,u)=\left(w+w^{1/(\rho+1)}\right)\left(0.5w^{-\rho/(\rho+1)}+0.5\right)^{1/\rho}u. \tag{10.15.xi}$$

(g) *Suppose non-child income $Y=0$, child time is $L=100$, $\alpha=0.5$, $\rho=1$ and w is initially 1. Then international trade raises w to 2. How does the household respond in its allocation of child leisure?*

Answer: We calculated in (d) that, when $\alpha=0.5$ and $w=1$, consumption and leisure are both $(Y+L)/2$ regardless of ρ. When $Y=0$ and $L=100$, this implies that initial household consumption and child leisure will both be equal to 50, leaving the child to work 50 hours per week. When $\alpha=0.5$, $Y=0$, $L=100$ and $\rho=1$, the optimal household consumption from equation (10.15.ii) and the optimal child leisure from equation (10.15.iii) become

$$c=\frac{100w^{3/2}}{w+w^{1/2}} \text{ and } \ell=\frac{100w}{w+w^{1/2}}. \tag{10.15.xii}$$

Plugging in $w=2$, we then get $c\approx 82.84$ and $\ell\approx 58.58$. Household allocation of child leisure therefore increased by 8.58 hours, and child labor falls from 50 to 41.42.

(h) *Using your expenditure function, can you determine how much the household would be willing to pay to cause child wages to increase from 1 to 2? If it did in fact pay this amount, how would it change the amount of child labor?*

Answer: To determine how much the household would be willing to pay to induce an increase in child wages from 1 to 2, we first have to know household utility at the original wage $w=1$. From our answer to (e), we see that household utility when $\alpha=0.5$ and $w=1$ is independent of ρ and equal to $(Y+L)/2$ which reduces to 50 when $Y=0$ and $L=100$. The most the household would be willing to pay to cause an increase in child wages from 1 to 2 is therefore an amount that would, once the wages increase, leave the household with utility of 50.

The expenditure necessary to attain utility of 50 at $w=2$ is given by the expenditure function. In our answer to (f), we calculated this for the case when $\alpha=0.5$ in equation (10.15.xi). When $\rho=1$, this reduces to $E(w,u)=(0.5w+w^{0.5}+0.5)u$. Evaluating this at $w=2$ and $u=50$, we get $E(2,50)\approx 145.71$. The value of the household's endowment — which is just the value of the child's leisure since $Y=0$ — at $w=2$ is 2(100)=200. The most the household is willing to pay for the child wage to increase from 1 to 2 is therefore 200-145.71=54.29.

If the household did in fact pay 54.29 to cause child wages to increase from 1 to 2, we can again use equation (10.15.iii) to determine the new optimal level of child leisure. When $\alpha=0.5$ and $\rho=1$, this equation becomes

$$\ell=(Y+wL)\left(w+w^{1/2}\right)^{-1}. \tag{10.15.xiii}$$

The payment of 54.29 is then a lump sum negative amount that can be inserted in place of Y. Substituting this, and substituting $w=2$ and $L=100$, we get

$$\ell=(-54.29+2(100))\left(2+2^{1/2}\right)^{-1}\approx 42.68. \tag{10.15.xiv}$$

This implies that child labor would increase to 100-42.68=57.32 hours.

(i) *Repeat the two previous steps for the case when $\rho=-0.5$ instead of 1.*

Answer: None of our answers for $w=1$ change when ρ changes as illustrated in previous parts of the question. Thus, initially the household will again choose $c=50$, $\ell=50$ and attain utility $u=50$, with the child working 50 hours per week.

To calculate how the household decision will change when wage increases to 2, we can again use equations (10.15.ii) and (10.15.iii). When $\alpha=0.5$, $L=100$, $Y=0$ and $\rho=-0.5$, these reduce to

$$c=\frac{100w^2}{1+w} \text{ and } \ell=\frac{100}{1+w}, \tag{10.15.xv}$$

and when $w=2$, this implies $c=133.33$ and $\ell=33.33$. Thus, child labor *increases* from 50 when $w=1$ to $100-33.33=66.66$ when $w=2$.

To determine how much the household would be willing to pay to cause wages to increase in this way, we can again use the expenditure function. When $\alpha=0.5$ and $\rho=-0.5$, this reduces to

$$E(w,u)=\left(w+w^2\right)(0.5w+0.5)^{-2}u. \tag{10.15.xvi}$$

Since the household's utility at $w = 1$ is 50, we need to calculate

$$E(2,50) = \left(2+2^2\right)(0.5(2)+0.5)\,(50) = 133.33. \tag{10.15.xvii}$$

Since the value of the household endowment is 2(100)=200 when $w = 2$, the household would therefore be willing to pay up to $200-133.33 = 66.67$ in order to get the wage to increase from 1 to 2.

Finally, we can determine the amount of child labor (if the household actually paid 66.67 to cause an increase in the wage) by returning to equation (10.15.iii) which, when $\alpha = 0.5$ and $\rho = -0.5$, reduces to

$$\ell = \frac{Y + wL}{w + w^2}. \tag{10.15.xviii}$$

Since the payment of 66.67 is equivalent to a negative household income, we can set Y to -66.67. Substituting this, and letting $L = 100$, we then get $\ell = 22.22$. Thus, child labor would increase to $100 - 22.22 = 77.78$ hours per week.

(j) *Are your calculations consistent with your predictions in (f) and (g) of part A of the question?*

Answer: Yes. In (f), we predicted the following: When parents pay the most they are willing to pay to open trade and raise child wages, children will work more than they did before — and the amount they will work more increases the more substitutable consumption and child leisure are. When ρ changes from 1 to -0.5, it causes the elasticity of substitution to increase (from 0.5 to 2). We have shown that at $\rho = 1$ (when the elasticity of substitution is low), child labor increases from 50 to 57.32 when parents have to pay the maximum bribe to get the higher child wage; and at $\rho = -0.5$ (when the elasticity of substitution is high), child labor increases from 50 to 77.78.

In (g), we predicted that the size of the payment a household is willing to make to get higher child wages increases as consumption and child leisure become more substitutable. We have shown that this payment increases from 54.29 to 66.67 when ρ falls from 1 to -0.5 — i.e. when the elasticity of substitution increases from 0.5 to 2.

Conclusion: Potentially Helpful Reminders

1. The regular (or uncompensated) demand curve is always the one you want to use if you are trying to predict what consumers will actually do as a result of a price change (regardless of what causes that price change).

2. The compensated demand (or marginal willingness to pay) curve is always the one you want to use when assessing changes in consumer surplus that result from price changes.

3. Since deadweight loss is a loss of consumer surplus, it is always measured on the compensated demand curve.

4. If the underlying good is quasilinear, the compensated and uncompensated demand curves are the same curves — which implies that this is the one case where you can measure consumer surplus changes along regular (uncompensated) demand curves.

5. Since marginal willingness to pay curves arise from substitution effects, they will be steep for small substitution effects and shallow for large substitution effects.

6. For students who do the B-part of the chapter, note the multiple ways we have found to calculate the various functions in the duality picture. Note further that this allows us multiple ways of calculating things like changes in consumer surplus or deadweight losses. In particular, we can either use the expenditure function — or we can take integrals on the compensated demand curve. A number of the end-of-chapter exercises illustrate the equivalence of these two methods.

7. Again for students who do the B-part of the chapter: Be sure to understand the duality picture well. You might be given one of the functions in the picture and then asked to tackle a problem — which is hard unless you know how to get from one part of the duality picture to another. For instance, if you already have the indirect utility function and you need the expenditure function, it is much easier to simply invert the indirect utility function rather than setting up the expenditure minimization problem and solving it. But be sure not to memorize but rather try to understand why the functions are related to one another as the picture shows.

CHAPTER

11

One Input and One Output: A Short-Run Producer Model

In this chapter, we begin to look at the theory of the firm as a price taker. We do this in the simplest possible setting — where producers use a single input to produce a single output — because it is in this setting that we can develop the basics of profit maximization. As will become clearer in upcoming chapters, this one-input/one-output model can be thought of as a *short run* model of producer choice because we can think of inputs (other than the one input in the model) as being fixed in the short run. So, for instance, when we say "labor is the only input", one interpretation of this statement is that it really means "labor is the only input that can be varied in the short run" because other inputs like factory size can only be changed in the long run.

Chapter Highlights

The main points of the chapter are:

1. A **production plan** specifies a bundle of inputs and outputs just as a consumption bundle in consumer theory represents a bundle of different goods. Just as consumers choose consumption bundles that maximize utility, **producers choose production plans that maximize profit.** Both try "to do the best they can given their circumstances," but — unlike in the consumer case — what is "best" has a more concrete meaning because we can quantify profit as the difference between economic revenues and costs.

2. As a result, "indifference curves" for producers — or **isoprofit curves** — are not a matter of "tastes" but emerge from the fact that producers are indifferent between production plans that result in the same level of profit. Since the profit of any production plan depends on the input and output prices, this implies that these producer "indifference curves" emerge from prices. This is unlike the consumer case where prices have nothing to do with indifference curves.

3. The fundamental constraint the producers face is a **technological constraint** that limits which production plans are in fact technologically feasible. Unlike the consumer case where constraints are shaped by prices, the technological constraint — modeled through the **production frontier** or the **production function** — has nothing to do with prices.

4. Any profit maximizing production plan (that involves positive production) has the characteristic that the **marginal revenue product of the input** equals the marginal cost of the input represented by the input's price. From this insight we can derive both the (short run) **output supply** curve (or function) as well as the (short run) **input demand** curve (or function), each sloping in the expected direction because of the **law of diminishing marginal product**.

5. There are **two ways to think of profit maximization**: (1) As a single problem in which we equate marginal revenue products with input prices, or (2) as a two-step problem in which the firm first derives its **cost curve** (or function) and then asks where the difference between revenues and costs is maximized. The latter allows us to state the profit maximizing condition as $MC = p$ for price taking firms.

6. Finally, **economic profit** is the difference between *economic* revenues and costs. This implies that, when a firm makes zero economic profit, it is doing as well in this industry as it could be in the next best alternative industry — it does not mean that "the firm isn't making money." Thus, it makes sense for firms to produce so long as economic profit is not negative.

11A Solutions to Within-Chapter Exercises for Part A

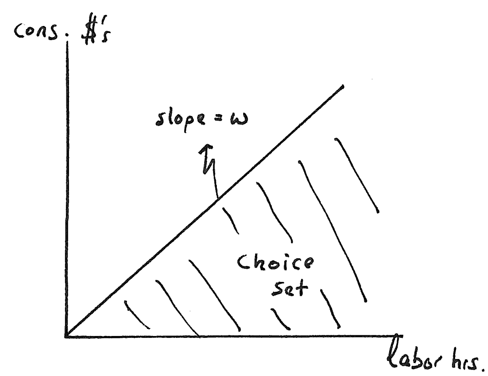
Exercise 11A.1

Can you model a worker as a "producer of consumption" and interpret his choice set within the context of the single input, single output producer model?

Answer: A worker uses the input "labor hours" to produce the output "dollars for consumption" by selling labor hours at the market wage w. This can be graphed with "labor hours" on the horizontal and "dollars of consumption" on the vertical, as in Exercise Graph 11A.1. For every input hour, the worker gets w — thus implying a slope w of the production frontier.

Exercise 11A.2

Which of the producer choice sets in Graph 11.1 is non-convex? What makes it non-convex?



Exercise Graph 11A.1 : Worker as a "Producer of Consumption"

Answer: The producer choice set in panel (b) is non-convex because you can choose two production plans (such as A' and C') such that the line connecting the two plans lies outside the set.

Exercise 11A.3

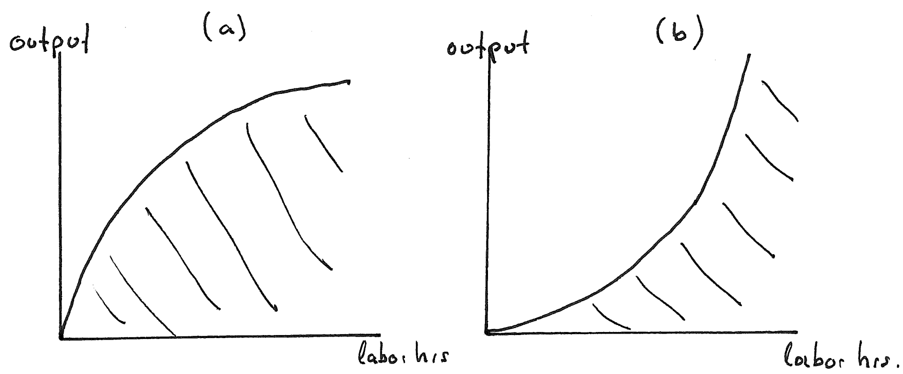
Suppose my technology was such that each additional worker hour, beginning with the second one, is less productive than the previous. Would my producer choice set be convex? What if my technology was such that each additional worker hour, beginning with the second one, is more productive than the previous.

Answer: The first of these is graphed in panel (a) of Exercise Graph 11A.3 while the second is graphed in panel (b). Each worker becoming less productive than the previous leads to the increasingly shallow slope in (a) while each worker becoming more productive than the previous leads to the increasingly steep slope in (b). The first producer choice set is convex; the second is not.

Exercise 11A.4

Under the production technology in Graph 11.1b, what is the approximate marginal benefit of hiring an additional labor hour when I already have 95 labor hours employed?

Answer: We know from Graph 11.1(b) in the text that I can produce 390 cards per day with 90 labor hours and 400 cards per day with 100 labor hours. Thus, the additional 10 labor hours increase total output by 10 cards — which gives me a marginal benefit of workers in that range (including the 95th worker) of about 1 card per worker hour. This is also reflected in the marginal product graph (Graph 11.2(b) in the text) where the marginal product for the 95th worker hour is indicated as 1.



Exercise Graph 11A.3 : Two different types of producer choice sets

Exercise 11A.5

Relate your answer from exercise 11A.4 to a point on the MP_ℓ curve plotted in Graph 11.2b.

Answer: The MP_ℓ curve tells me that the 95th worker hour results in an increase in total output of 1 card per day — which is the same as what we derived in the previous exercise from the production frontier. This is because the marginal product curve is just the slope of the production frontier, and between 90 and 100 labor hours, the slope of the production frontier is approximately 1.

Exercise 11A.6

What would the MP_ℓ curves look like for the technologies described in within-chapter exercise 11A.3?

Answer: For the first technology described in exercise 11A.3, the MP_ℓ curve would be downward sloping, and for the second it would be upward sloping.

Exercise 11A.7

True or False: The Law of Diminishing Marginal Product implies that producer choice sets in single input models must be convex beginning at some input level.

Answer: This is true. At the input level at which the law of diminishing marginal product sets in, the production frontier begins to become shallower and shallower — resulting in a choice set underneath that is convex from that point forward.

Exercise 11A.8

True or False: If the Law of Diminishing Marginal Product did not hold in the dairy industry, I could feed the entire world milk from a single cow. (*Hint*: Think of the cow as a fixed input and feed for the cow as the variable input for which you consider the marginal product in terms of milk produced per day.)

Answer: Take a given cow and begin feeding it. Each ounce of feed results in more milk per day from the cow than each previous ounce of feed. If the law of diminishing marginal product of feed does not at some point decrease, this means I can keep feeding the cow and I will get the cow to produce ever increasing amounts of milk (per day) from each additional ounce of feed. At some point, I will have given the cow so much feed that we will have produced enough milk for the world from that single cow. Obviously this is absurd, which illustrates why the assumption of diminishing marginal product of feed must hold.

Exercise 11A.9

Without knowing what prices and wages are in the economy, can you tell by looking at a single isoprofit curve whether profits for production plans along this curve are positive or negative? What has to be true about an isoprofit curve in order for profit to be zero?

Answer: The intercept of an isoprofit curve on the vertical axis is Profit divided by price. You can therefore tell whether profits along an isoprofit curve are positive by checking whether or not the isoprofit curve has positive intercept. If it has positive intercept, then profit for all the production plans along that isoprofit curve is positive (assuming price is positive); if it has negative intercept, then profit for all production plans along the curve is negative. If the isoprofit curve intersects at the origin, then profit is zero.

Exercise 11A.10

What would have to be true in order for an isoprofit curve to have a negative slope?

Answer: The slope of an isoprofit curve is w/p. In order for this slope to be negative, either wage or output price would therefore have to be negative.

Exercise 11A.11

How would the blue isoprofit curve in Graph 11.3a change if the wage rises to \$30? What if instead the output price falls to \$2?

Answer: Along the original blue isoprofit curve, wage is \$20 and output price is \$5 — resulting in a slope of $w/p = 20/5 = 4$. If the wage increases to \$30, the slope increases to $30/5 = 6$. Since the intercept is given by Profit/p (and w therefore does not enter the intercept term), the intercept remains at 40.

Now suppose that instead the price fell from \$5 to \$2. Originally, the intercept of the blue isoprofit is Profit/p=Profit/5=40 — which implies that Profit along the isoprofit curve is \$200. In order for profit to remain unchanged at the intercept when price falls to \$2, it would have to be the case that the new intercept is $200/2 = 100$. Thus, the intercept of the isoprofit would shift up from 40 to 100. The slope of the isoprofit is w/p or $20/p$ when $w = 20$. Thus, when price falls to $2, the new slope of the isoprofit curve must be $20/2 = 10$; i.e. the slope increases from 4 to 10.

Exercise 11A.12

It appears from panel (f) of Graph 11.5 that profits are smallest (i.e. most negative) when I stop hiring at 22 labor hours per day. What can you conclude about the slope of the production frontier in panel (c) of the graph at 22 daily labor hours? Explain.

Answer: At 22 daily labor hours, the slope of the production frontier must be equal to 20/5=4 which is equal to the slope of isoprofit curves ($w/p = 20/5$). Notice that the production frontier has two points with this slope — the profit maximizing point A where the highest possible isoprofit curve is obtained a second point on the initially non-convex portion of the production frontier. At both of these points, $w/p = MP_\ell$ because the slope of the isoprofit curve is (w/p) and the slope of the production frontier is MP_ℓ. Simply rearranging the terms in this equation, we can see that this is equivalent to saying that at both of these points $w = pMP_\ell$. In panel (f), we show that this holds at labor hours of 22 and 78 — so it must equivalently hold in panel (c) for those labor hours. Put differently, at both these labor hours there exists an isoprofit curve that is just tangent to the production frontier — but only the second of these tangencies — point A in panel (c) — is profit maximizing (because the other tangency lies on an isoprofit which is lower and thus has less profit associated with each of its production plans.)

Exercise 11A.13

Suppose I have already signed a contract with my former student who is providing me with the factory space, machinery and raw materials for my business, and suppose that I agreed in that contract to pay my former student $100 per month for the coming year. Is this an economic cost with respect to my decision of whether and how much to produce this year?

Answer: No, it is not an economic cost of producing for this year because I am legally obligated to pay it regardless of whether or how much I produce. Thus, my decision of whether or how much to produce has no impact on this "cost".

Exercise 11A.14

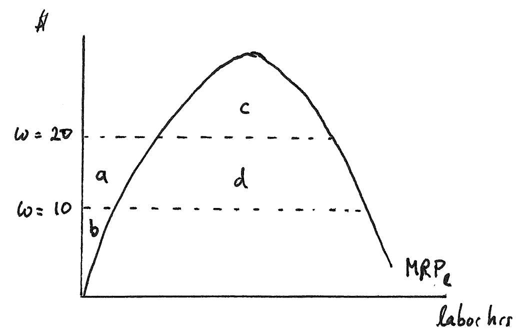
There are also production plans to the left of A where the slope of the production frontier is shallower. Why are we not considering these?

Answer: We are not considering these because all of them would lie on isoprofit curves that fall below the magenta curve that is tangent at B — and thus all of these production plans would result in less profit than B.

Exercise 11A.15

Which areas in the lower panel of Graph 11.6a add up to the $200 profit I made before wages fell? Which areas add up to the $1045 profit I make after wages fall?

Answer: The graph is replicated in Exercise Graph 11A.15 with the critical areas labeled. The $200 profit at the original wage of $20 is equal to $c-a-b$. The $1045 profit at the new wage of $10 is equal to $c+d-b$ — which is equal to the shaded green minus the shaded magenta areas in the textbook graph.



Exercise Graph 11A.15 : Changing Profit as w falls

Exercise 11A.16

Given an intercept of −100 of this isoprofit curve, what is the value of profit indicated by the shaded green minus the shaded magenta area in the lower panel of Graph 11.6b?

Answer: The shaded green minus the shaded magenta areas in panel (b) is equal to the profit at production plan C. The intercept of the isoprofit curve tangent at C in the top panel is -100 and is equal to Profit/p. Since the output price p is $5, we therefore know that Profit/5=−100 — which implies that Profit=−$500. Thus, the shaded green minus the shaded magenta areas in the lower part of panel (b) is −$500.

Exercise 11A.17

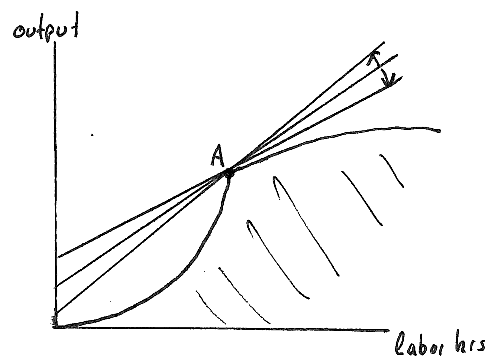
Had the increase in the market wage been less dramatic, would my best course of action still necessarily have been to shut down production?

Answer: No. For smaller increases in the wage, the slope of the isoprofit curves would increase less dramatically. For sufficiently small increases in the wage, that would result in a tangency of one of the new isoprofit curves with the production frontier such that the isoprofit curve would still have positive intercept on the vertical axis — and thus the production plan at that tangency would still give positive profit. In that case, this is the optimal production plan and the firm should not shut down.

Exercise 11A.18

What would have to be true of the production frontier in order for the original optimal production plan A to remain optimal as wages either rise somewhat or fall somewhat? (*Hint*: Consider what role kinks in the producer choice set might play.)

Answer: In Exercise Graph 11A.18, we illustrate a production frontier with a kink at A. At this kink, a number of different slopes are "tangent". Thus, beginning with wages and prices such that A is profit maximizing, we can change wages up or down within some range and still have the steeper or shallower isoprofits that result be "tangent" at A with positive intercepts on the vertical axis. In such cases, the producer would not change production plans as wages change.



Exercise Graph 11A.18 : No change in profit maximizing behavior with changes in wages

Exercise 11A.19

Why would it be economically rational for me to still stay open for business when $w = w^*$ where my profit is zero?

Answer: When economic profit is zero, this means that the producer does as well in this activity as she could in the next best alternative activity. If she spends time on her business, this means that her labor hours are being compensated — so she is indeed "making money". But she is making just as much money as she could doing the next best thing. So, when her profit is zero, she does as well as she could anywhere else — which makes it economically rational to continue producing. (It would also in this case be economically rational to do the next best thing — which would also result in zero profit.)

Exercise 11A.20

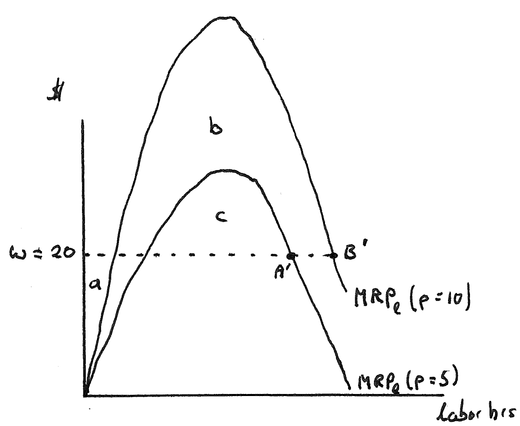
If I had signed a contract and agreed to make monthly payments for the next year to my former students who provided me with my factory space, would w^* — the highest wage at which I will still produce — be any different?

Answer: No, w^* would be no different because I have to make the monthly payments whether I produce this year or not — which means that they are not an economic cost that is relevant for my current economic decisions.

Exercise 11A.21

What areas in the lower panel of Graph 11.8 add up to my new profit? What is the dollar value of this new profit (which you can calculate from the intercept of the isoprofit curve in the top panel of the graph)?

Answer: The new profit areas on the lower panel of the graph must be read off the new (magenta) MRP_ℓ curve at the unchanged wage of $w = 20$. The graph is replicated here as Exercise Graph 11A.21 with some relevant areas labeled with lower case letters. The new profit is then $b + c - a$. The dollar value for this area can be calculated from the top panel of the graph in the text. The intercept of the new optimal (magenta) isoprofit (tangent at B) is Profit/p=209. Since $p = 10$, this implies Profit/10=209 or Profit=$2,090.



Exercise Graph 11A.21 : New Profit when p increases to $10

Exercise 11A.22

Can you tell from Graph 11.8 how the labor demand curve will change when p changes?

Answer: At the lower price $p = 5$, the labor demand curve lies on the downward sloping portion of the original (blue) $MRP_\ell(p = 5)$ curve. At the new price $p = 10$, the labor demand curve lies on the downward sloping portion of the new (magenta) $MRP_\ell(p = 10)$ curve. Thus, the labor demand curve shifts out as p increases. Similarly, it would shift in when p decreases.

Exercise 11A.23

What value would p have to take in order for isoprofits to have the same slope as when wages increased to \$30 per hour (as in Graph 11.6b)? What would be my optimal course of action in that case?

Answer: When $p = 5$ and w increases from \$20 to \$30, the slope of the isoprofit, which is always w/p, increases from 4 to 6. If w remained unchanged at \$20, the slope w/p would change to 6 if $20/p = 6$ — i.e. if $p = 10/3 = 3.33$. The optimal course of action depends only on the slope of the isoprofit — and thus, when the slope is unchanged, the optimal action remains unchanged.

Exercise 11A.24

In Graph 11.9 we implicitly held wage fixed. What happens to the supply curve when wage decreases?

Answer: In panel (a), the isoprofits for any given p become shallower as w decreases — causing the tangency with the production frontier to shift up and to the right. Thus, as w decreases, output increases for any p. This then implies that the supply curve shifts to the right.

Exercise 11A.25

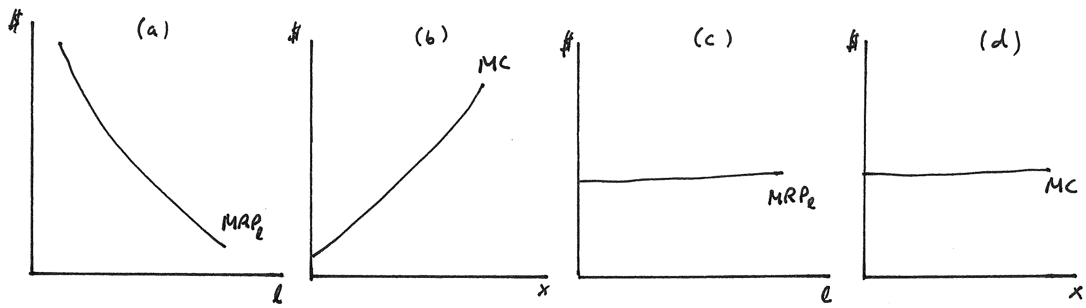
If the wage rate used to construct the panels on the right of Graph 11.10 is \$20, can you conclude what the slope of the production frontier in panel (a) at 10 units of output is? Can you conclude what labor input is required to produce 10 units of output, and then what the vertical values of the curves in panels (b) and (c) are for that level of labor input?

Answer: The slope of the production frontier at 10 units of output must be 1. This is because we know from panel (e) that the marginal cost is 20 *in terms of dollars* that are used to buy the input labor — and we know labor costs \$20 per unit. Thus the additional labor required to produce one more unit of x must be 1. Similarly, the total labor input required to produce 10 units of output must be 300/20=15. Since we just concluded that, when we are at output level of 10, it takes one additional unit of labor to produce 1 additional unit of output. Thus, at $\ell = 15$ (which is how much labor it takes to produce 10 units of x), the marginal product of labor must be 1 — which is then the vertical value of the MP_ℓ curve at $\ell = 15$. The vertical value for $\ell = 15$ in panel (c) is then just p.

Exercise 11A.26

What would be the shape of the MRP_ℓ and MC curves if the entire producer choice set was strictly convex? What would the shape be for the production frontier graphed in Graph 11.1(a)?

Answer: These are illustrated in Exercise Graph 11A.26 — with panels (a) and (b) illustrating the curves for strictly convex choice sets (where production becomes increasingly difficult throughout) and panels (c) and (d) illustrating the curves for the linear production constraint in Graph 11.1.



Exercise Graph 11A.26 : *MRP* and *MC* Curves

Exercise 11A.27

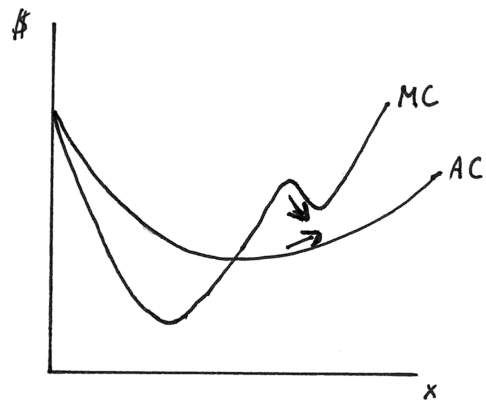
True or False: On a graph with output on the horizontal and dollars on the vertical, the "marginal revenue" curve must always be a flat line so long as the producer is a price taker.

Answer: This is true. The marginal revenue is the additional revenue the producer can get from producing one more unit of output. If the producer is a price taker, she can always sell additional units of output at the market price. Thus, the marginal revenue of producing more is always equal to the market price.

Exercise 11A.28

Can MC fall while AC rises? (*Hint*: The answer is yes.) Can you give an analogous example of marginal test grades falling while the average grade rises at the same time?

Answer: Exercise Graph 11A.28 illustrates a case where this is the case. Note that all that is required for AC to increase is for MC to *lie above* AC. It does not necessarily require MC to be increasing as well. So long as MC lies above AC, it drags up the AC even if, for some interval, the MC is actually falling. You can again think of this in terms of your grades. Suppose that, after the second exam, you have a course average of 55. Then you score a 100 on the third exam. This marginal grade of 100 raises your course average to 70. Now suppose you make a 90 on the fourth exam. Your marginal grade has fallen from 100 — but because it lies above your average (of 70) going into the exam, it will still raise your average (to 75). So, your average grade is increasing from 70 to 75 despite the fact that your marginal grade fell from 100 to 90.

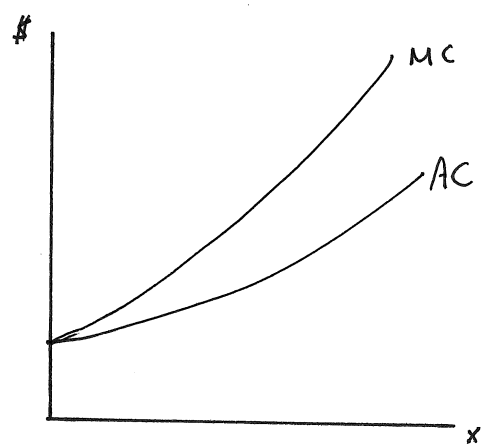


Exercise Graph 11A.28 : Falling *MC* with increasing *AC*

Exercise 11A.29

How do the marginal and average cost curves look if the producer choice set is convex?

Answer: This is illustrated in Exercise Graph 11A.29. The shapes should make intuitive sense — the curves must be upward sloping because a convex producer choice set implies it gets increasingly difficult to produce additional output starting from the first unit. Thus, it must mean that each additional unit is more expensive than the last. And, as always, *MC* and *AC* must start at the same point — and *MC* must lie above *AC* if *AC* is upward sloping.



Exercise Graph 11A.29 : *MC* and *AC* when the producer choice set is Convex

11B Solutions to Within-Chapter Exercises for Part B

Exercise 11B.1

How would this production function look differently if we did not specify that output levels off at 2α?

Answer: The production function would have negative slope beginning at $\ell = \pi/\beta$ — and would then oscillate between positive and negative slopes as labor increases.

Exercise 11B.2

Define the production function generating the production frontier in Graph 11.1a and define the corresponding producer choice set formally.

Answer: The production function has constant slope and is given by $f(\ell) = 4\ell$. The producer choice set is then simply given by

$$C(f : \mathbb{R}^1_+ \to \mathbb{R}^1_+) = \left\{(x, \ell) \in \mathbb{R}^2 \mid x \leq 4\ell\right\}. \tag{11B.2}$$

Exercise 11B.3

Given that f is really defined as in equation (11.3), how should equation (11.5) be modified to accurately reflect the marginal product of labor for labor hours above 100?

Answer: Since the marginal product of labor above 100 is zero, the full definition of the marginal product function is

$$MP_\ell = \begin{array}{ll} 6.2832\sin(0.031416\ell) & \text{for } 0 \le \ell \le 100 \text{ and} \\ 0 & \text{for } \ell > 100. \end{array} \quad (11B.3)$$

Exercise 11B.4

Derive the marginal product of labor from the production function you derived in exercise 11B.2 above. Compare this to the graphical derivation in Graph 11.2a.

Answer: Taking the derivative of the production function $f(\ell) = 4\ell$, we get $MP_\ell = 4$. This tells us that the marginal product of an additional worker is always 4 units of output — i.e. the MP curve is horizontal as derived graphically in part A of the text.

Exercise 11B.5

Check to see that the Law of Diminishing Marginal Product (of labor) is satisfied for the production function in equation (11.3).

Answer: The Law of Diminishing Marginal Product says that, at some point, the marginal product of labor decreases as more labor is hired — i.e. at some point, the derivative of MP_ℓ becomes negative. Recalling that the derivative of $(\sin x)$ is $(\cos x)$, we get that the derivative of MP_ℓ for this production function is

$$\frac{dMP_\ell}{dx} = 0.031416(6.2832)\cos(0.031416\ell) \approx 0.1974\cos(0.031416\ell) \quad (11B.5)$$

for $\ell < 100$ (since we defined the production function to flatten out at $\ell = 100$.) You can now check to see whether this derivative of the MP_ℓ is positive or negative for different values of ℓ by plugging in such different values between 0 and 100. You will find that the derivative is positive up to approximately $\ell = 50$ — indicating that the MP_ℓ curve initially slopes up (as in our graphs). However, as ℓ increases above 50, the derivative of MP_ℓ becomes negative — indicating that at approximately $\ell =$ 50, the MP_ℓ curve slopes down. Thus, the law of diminishing marginal product holds.

Exercise 11B.6

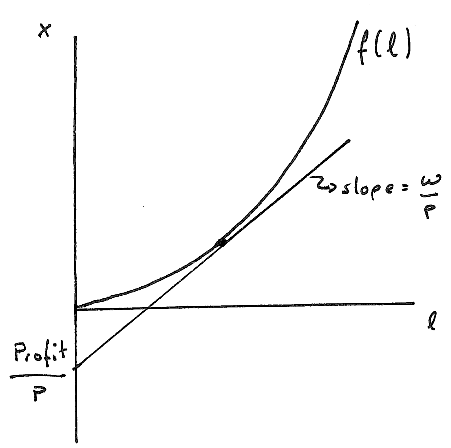
Without doing the math, can you tell if the curve $\ell(p,20)$ slopes up or down? How does it relate to $\ell(p,10)$?

Answer: We concluded in part A that, as p increases, the profit maximizing producer increases output (assuming it was optimal for her to produce to being with). With labor as the only input, this also implies that she must hire more labor. Thus, $\ell(p,20)$ is increasing in p — i.e. it slopes up. We also concluded that, as wages fall, the producer will increase output (assuming it was optimal for her to produce to begin with). Thus, when wages fall from $w = 20$ to $w = 10$, output increases and, given there is only one input available to the producer, this implies she will hire more workers. Thus, $\ell(p, 20) < \ell(p, 10)$ — which means that the labor demand curve shifts outward as w falls.

Exercise 11B.7

Consider a production function that gives rise to increasing marginal product of labor throughout (beginning with the first labor hour). *True or False*: In this case, the mathematical optimization problem will unambiguously lead to a "solution" for which profit is negative.

Answer: This is true. This is illustrated in Exercise Graph 11B.7. The shape of the production function $f(\ell)$ arises from the assumption of increasing marginal product of labor throughout. This shape implies a single tangency for isoprofits with slope w/p — and this "optimal" isoprofit must then necessarily have a negative vertical intercept. Since the vertical intercept of isoprofit curves is Profit$/p$, and since p is always assumed to be positive, it must be that the "solution" to the optimization problem gives us a production plan that results in negative profit. The true solution is therefore not the one given by the tangency.



Exercise Graph 11B.7 : Mathematical "solution" when MP_ℓ is increasing throughout

Exercise 11B.8

Consider a production function that gives rise to increasing marginal product of labor throughout (beginning with the first labor hour). *True or False*: In this case, the mathematical optimization problem will give a single solution — albeit one that minimizes rather than maximizes profit.

Answer: It is again apparent from Exercise Graph 11B.7 that, with the shape of the production function that emerges from increasing marginal product throughout, there is a single tangency for every isoprofit slope w/p. As we argued in the previous answer, this single tangency represents a production plan that gives rise to negative profit. In fact, this production plan gives the least profit (i.e. the most negative profit) of any of the possible production pans that lie on the production frontier. You can see this by simply noting that every other production plan on the frontier lies on a higher isoprofit curve — i.e. on an isoprofit parallel to the one that gives rise to the tangency but lying to the northwest. In this sense, the mathematical solution that gives the tangency production plan actually gives us the lowest possible profit conditional on producing along the production frontier (rather than inside the producer choice set.)

Exercise 11B.9

Give an example of a producer choice set and economic conditions such that infinite production would be "optimal".

Answer: The producer choice set and the economic conditions (i.e. wages and prices) depicted in Exercise Graph 11B.7 represent such an example. Pick any point on the producer choice set and draw the isoprofit curve that goes through that point. You will be able to draw a parallel isoprofit curve that lies to the northwest (i.e. with more profit) that also intersects the production frontier — with the resulting production plan using more labor and producing more output. Since you can do this for any point on the production frontier, you can keep doing it no matter how large your production already is. Thus, you do not reach a profit maximizing production plan until you produce an infinite amount of the output.

Exercise 11B.10

Do you think the scenario you outlined in the previous within-chapter exercise makes sense under the assumption of "price taking" behavior by producers?

Answer: No, it does not. The price taking assumption is based on an assumption that the market contains many small producers, none of which is large enough to have any impact on price. If any producer ever became as large as the example implies, this price taking assumption no longer makes sense — the fact that the producer keeps increasing production must eventually drive up wages (as the producer is hiring more and more scarce workers) and drive down the price that the producer can charge (as consumers won't be willing to keep paying the same price no matter how much of the good is dumped on the market). The price taking assumption is therefore inconsistent with producer choice sets that give rise to increasing marginal product throughout or even increasing marginal product for large amounts of the input.

Exercise 11B.11

What has to be true about α in order for this production function to exhibit diminishing marginal product of labor?

Answer: The marginal product of labor is

$$MP_\ell = \frac{\partial f(\ell)}{\partial \ell} = \alpha A \ell^{\alpha-1}. \tag{11B.11.i}$$

This is declining in labor if its derivative is less than 0; i.e. if

$$\frac{\partial MP_\ell}{\partial \ell} = (\alpha-1)(\alpha)A\ell^{\alpha-2} < 0 \tag{11B.11.ii}$$

which holds only if $\alpha < 1$.

Exercise 11B.12

Suppose $0 < \alpha < 1$ and $A > 0$. Holding price fixed, is the labor demand function downward sloping in the wage? Holding wage fixed, is it upward or downward sloping in price? Can you graphically illustrate why your answers hold?

Answer: To determine whether the labor demand function is downward sloping in the wage, we simply need to take the derivative of the demand function with respect to wage and check whether this derivative is negative. We then get

$$\frac{\partial \ell(p,w)}{\partial w} = \left(\frac{1}{\alpha-1}\right)\left(\frac{1}{\alpha A p}\right)^{1/\alpha-1} w^{(2-\alpha)/(\alpha-1)}. \tag{11B.12.i}$$

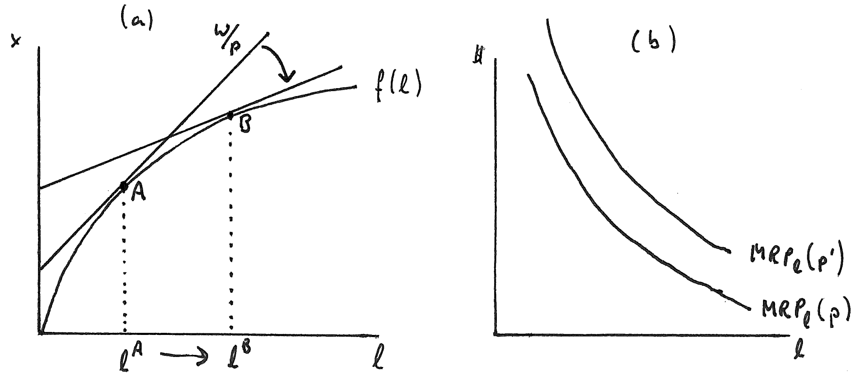
Given $0 < \alpha < 1$ and $A > 0$, the second and third terms in this equation are positive. The first term $(1/(\alpha-1)$, however, is negative. Thus, the derivative of the labor demand function with respect to w is negative — i.e. the labor demand function slopes down in w.

We can similarly determine which way the labor demand function slopes in p. Now we take the partial derivative of the labor demand with respect to p and check whether we can determine whether this derivative is positive or negative. The derivative is

$$\frac{\partial \ell(p,w)}{\partial p} = -\left(\frac{1}{\alpha-1}\right)\left(\frac{w}{\alpha A}\right)^{1/(\alpha-1)} p^{-\alpha/(\alpha-1)}. \tag{11B.12.ii}$$

As before, the second and third terms in this equation are positive (given $0 < \alpha < 1$ and $A > 0$). The first term $1/(\alpha-1)$ is negative, but it is preceded by a negative sign which makes it positive. Thus, the partial derivative of $\ell(p,w)$ with respect to p is positive — which implies that the labor demand function is increasing (or upward sloping) in p.

Both these results make intuitive sense. The first says that an increase in the cost of inputs (i.e. labor) will cause me to hire fewer workers; while the second says that an increase in the price at which I can sell the output will cause me to hire more workers. We can illustrate this graphically as we did in part A of the text. In panel



Exercise Graph 11B.12 : Changes in labor demanded as w and p change

(a) of Exercise Graph 11B.12, we illustrate the shape of the production function $f(\ell)$ which gives rise to a convex producer choice set. Suppose that w and p were initially such that the steeper isoprofit arises — giving rise to the optimal production plan A at which ℓ^A in labor hours is hired. Now suppose that w falls — causing the slope of isoprofits to fall. This results in a new profit maximizing production plan B — with ℓ^B in labor hours hired. Thus, as w falls, ℓ increases, which is the same as saying that labor demand increases as wages fall or, put differently, the labor demand function is downward sloping in w. (The labor demand function, holding p fixed, is the inverse of the labor supply curve we graph in panel (b) — but the downward slope remains as we flip the axes.) Similarly, if p increases, the isoprofit curves become shallower — again leading to an increase in labor hours hired from production plan A to B. Thus, as price increases, labor demanded goes up — i.e. the labor demand function is upward sloping in p.

The same can also be seen in the marginal revenue product graph of panel (b). Given the shape of the production function, we know that marginal product is diminishing throughout. This implies that the marginal revenue product curve is downward sloping — giving rise to the downward sloping labor demand curve and the fact that the labor demand function is downward sloping in w. At the same time, if p increases (from p to p'), then the marginal revenue product curve shifts up. Thus, for any given wage, more labor is hired as p increases — i.e. labor demand is upward sloping in p.

Exercise 11B.13

Suppose $0 < \alpha < 1$ and $A > 0$. Holding wage fixed, is the supply function upward sloping in price? Holding price fixed, is the supply function upward sloping in wage? Can you graphically illustrate why your answers hold?

Answer: To see whether the supply function is upward or downward sloping in price, we have to take the derivative of $x(p,w)$ with respect to p and check whether this derivative is positive or negative. This derivative is

$$\frac{\partial x(p,w)}{\partial p} = -\left(\frac{\alpha}{\alpha-1}\right)\left[A\left(\frac{w}{\alpha A}\right)^{\alpha/(\alpha-1)}\right]p^{(1-2\alpha)/(\alpha-1)}. \qquad (11B.13.i)$$

Since $0<\alpha<1$ and $A>0$, the bracketed term is positive, as is the last term in the equation. This leaves the first term $\alpha/(\alpha-1)$ which is negative. However, it is preceded by a negative sign — so the whole term is therefore positive. Thus, $x(p,w)$ is increasing in p — which is to say that the supply function is upward sloping in p.

We can similarly check whether the supply function is upward or downward sloping in w by considering the derivative

$$\frac{\partial x(p,w)}{\partial w} = \frac{\alpha}{\alpha-1}\left[A\left(\frac{1}{\alpha Ap}\right)^{\alpha/(\alpha-1)}\right]w^{1/(\alpha-1)}. \qquad (11B.13.ii)$$

Again, the bracketed and the last term are positive, but the first term $\alpha/(\alpha-1)$ is negative since $0<\alpha<1$. As a result, the whole derivative is negative, which implies that the output supply function is decreasing, or downward sloping, in w.

Both these results again make intuitive sense. The first says that, as output price increases, the profit maximizing supply quantity increases as well. The second says that, as input price (wage) decreases, the profit maximizing output quantity increases. This can also be seen in panel (a) of Exercise Graph 11B.12. Suppose we start at prices and wages such that A is the profit maximizing production plan. An increase in p and a decrease in w both have the effect of decreasing w/p — the slope of the isoprofit curves. Thus, both result in increased output to a new profit maximizing production plan such as B.

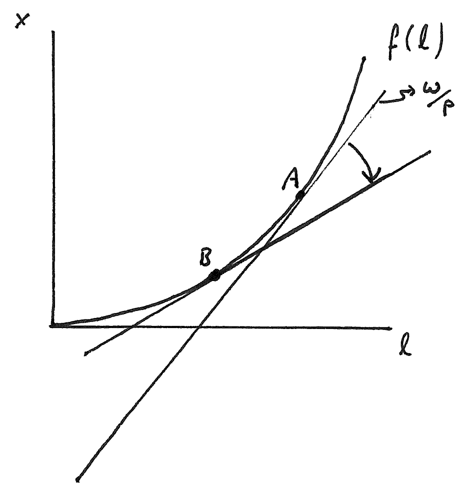
Exercise 11B.14

How do your answers to the previous two exercises change when $\alpha>1$? Can you make sense of what is going on? (*Hint*: Graph the production function and illustrate the tangencies of isoprofits for different wages and prices.)

Answer: The mathematical answers would reverse entirely — with labor demand now increasing in w and decreasing in p, and with output supply increasing in w and decreasing in p. This does not make intuitive sense. An increase in output price should cause producers to increase production and hire more labor, not the other way around; and an increase in input price (wage) should cause a decrease in production and less labor demanded. The reason that the math is now giving the wrong answer is because the "right" answer would be a corner solution. This is because, when $\alpha>1$, the production function has increasing slope — implying increasing (rather than diminishing) marginal product of labor.

To see what is going on, consider Exercise Graph 11B.14.

Here we graph a production function with the slope that $\alpha>1$ implies. Suppose we initially face wage w and price p, giving rise to the isoprofit that is tangent at A. This is the production plan that the math gives us as the "solution". The problem, of course, is that profit is negative under this plan (as indicated by the negative vertical



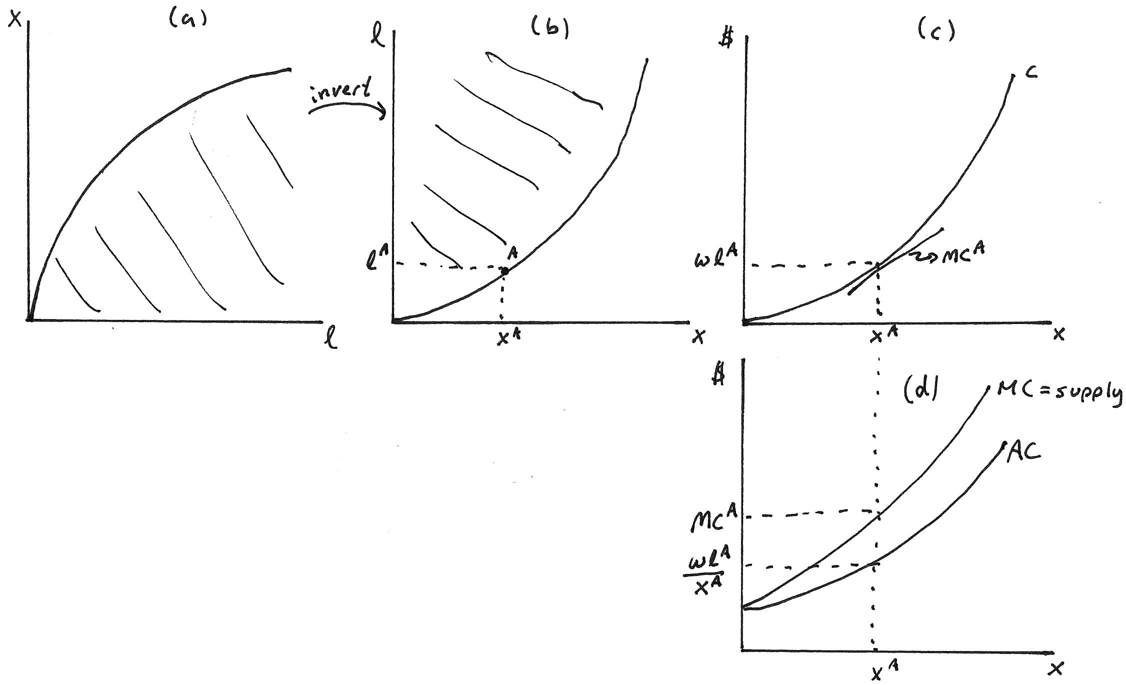
Exercise Graph 11B.14 : Nonsensical "solutions" when $\alpha > 1$

intercept of the isoprofit curve). Now suppose that w falls — which causes isoprofits to become shallower (given that their slope is w/p). The new production plan found by the mathematical optimization method we used is now B — indicating less labor input and less output supply. Thus, the math suggests that a drop in the input price (wage) results in less labor demanded and less output supplied, exactly counter to what should be true. Again, of course, profit at B is negative — so neither A nor B are truly profit maximizing production plans, which leads to the nonsensical result. Similarly, suppose p increases from the original. This similarly causes w/p to fall — making isoprofits shallower. Again we move from the "solution" A to the "solution" B — suggesting that an increase in output price causes the producer to produce less and hire fewer workers. And again the result is nonsensical because neither the original plan A nor the new plan B are actually profit maximizing. In fact, in both cases the producer should produce an infinitely large quantity.

Exercise 11B.15

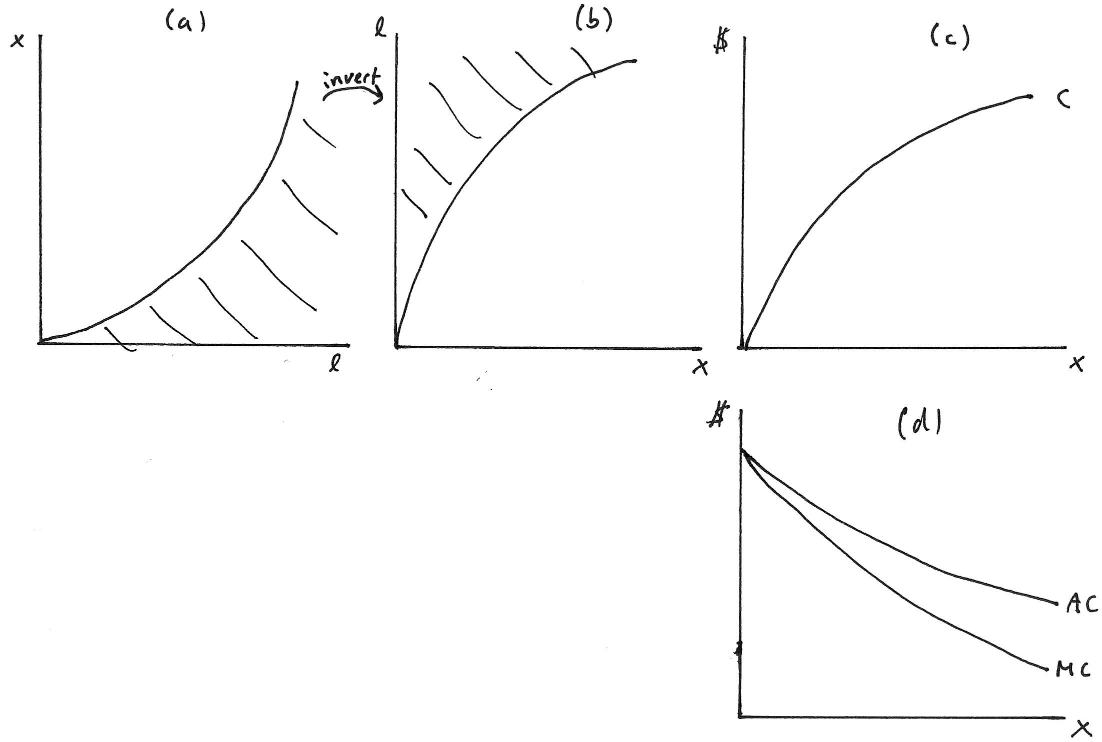
Graphically illustrate the way we have just derived the output supply function assuming α lies between 0 and 1. What changes when $\alpha > 1$?

Answer: In Exercise Graph 11B.15(1), we start with a convex production set in panel (a) (which emerges when $\alpha < 1$). This is inverted in panel (b) — which corresponds to the step where we invert the production function and write it in terms of $\ell(x)$. We then multiply this by w to get the cost function in panel (c) (graphed with w held fixed) — then take the derivative to get MC in (d) and we divide the cost function by x to get the AC in panel (d).



Exercise Graph 11B.15(1) : Solution when $\alpha > 1$

When $\alpha > 1$, the producer choice set is non-convex as illustrated in panel (a) of Exercise Graph 11B.15(2). We can again invert the production to get it into the form $\ell(x)$ that tells us how much labor is required for each level of output — then multiply by w to get the cost function that is illustrated in panel (c) with wage held fixed. Then we get the MC and AC the same way as before in panel (d). The difference is that we cannot now claim that the MC function is the supply curve — because the supply curve has to lie above AC.



Exercise Graph 11B.15(2) : Nonsensical "solutions" when $\alpha > 1$

11C Solutions to Odd Numbered End-of-Chapter Exercises

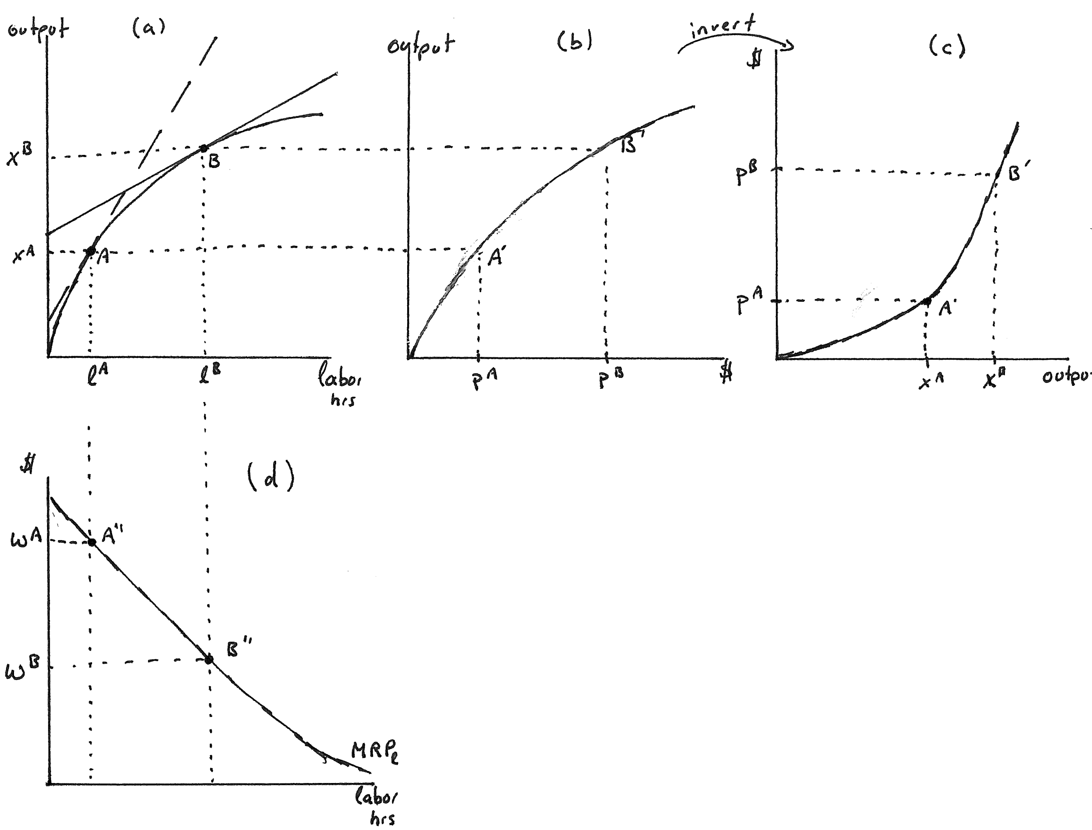
Exercise 11.1

Throughout part A of the text, we used the technology we called more "realistic" in panel (b) of Graph 11.1.

A: *Suppose now that the producer choice set was instead strictly convex everywhere.*

(a) *Illustrate what such a technology would look like in terms of a production frontier.*

Answer: This is illustrated in panel (a) of Exercise Graph 11.1(1).



Exercise Graph 11.1(1) : Production when Choice Set is Convex

(b) *Derive the output supply curve with price on the vertical and output on the horizontal axis (in graphs analogous to those in Graph 11.9) for this technology.*

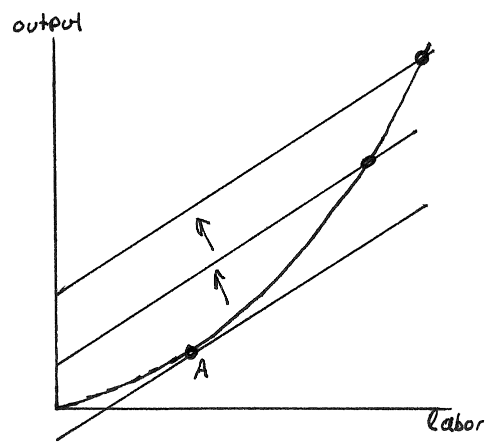
Answer: In panel (a) of Exercise Graph 11.1(1), two isoprofits corresponding to a low and a high price level are illustrated tangent to the producer choice set. The dashed one corresponds to the low output price p^A while the solid one corresponds to the higher output price p^B. The profit maximizing production plans at these two prices are then given by A and B. Panel (b) of the graph then plots the output levels x^A and x^B from these optimal production plans for the two price levels p^A and p^B on the horizontal axis. Panel (c) then simply flips the axes — giving us an upward sloping output supply curve.

(c) *Derive the labor demand curve for such a technology.*

Answer: This is done in panel (d) of Exercise Graph 11.1(1) where the marginal revenue product — which is derived from the slope of the production frontier — is downward sloping throughout (because the slope of the production frontier is becoming shallower throughout).

(d) *Now suppose the technology were instead such that the marginal product of labor is always increasing. What does this imply for the shape of the producer choice set?*

Answer: This implies that the production frontier gets steeper throughout — which in turn implies that the producer choice set is non-convex as drawn in Exercise Graph 11.1(2).



Exercise Graph 11.1(2) : Increasing Marginal Product of Labor

(e) *How much should the firm produce if it is maximizing its profits in such a case? (*Hint*: Consider corner solutions.)*

Answer: If the firm is a price taker, then it should produce an infinite amount. This is illustrated in Exercise Graph 11.1(2) where three parallel isoprofits are drawn. The lowest is tangent to the production frontier, but this is not a profit maximizing production plan because we can still go to higher isoprofits that contain production plans which are technologically feasible. In fact, we can keep going to higher and higher isoprofits that continue to intersect the production frontier — and thus can keep going up.

B: *Suppose that the production function a firm faces is* $x = f(\ell) = 100\ell^{\alpha}$.

(a) *For what values of* α *is the producer choice set strictly convex? For what values is it non-convex?*

Answer: The producer choice set is strictly convex as long as $\alpha < 1$ and non-convex if $\alpha > 1$.

(b) *Suppose* $\alpha = 0.5$. *Derive the firm's output supply and labor demand function.*

Answer: The production function in this problem takes the same form as the one used in the last part of the chapter within the text. Using the results from there, and plugging 100 in for A and 0.5 for α, we get

$$\ell(p,w) = \left(\frac{w}{50p}\right)^{-2} = \frac{2500p^2}{w^2} \quad \text{and} \quad x(p,w) = 100\left(\frac{w}{50p}\right)^{-1} = \frac{5000p}{w} \tag{11.1.i}$$

(c) *How much labor will the firm hire and how much will it produce if* $p = 10$ *and* $w = 20$?

Answer: Plugging these into our expressions for labor demand and output supply, we get $\ell = 625$ and $x = 2,500$.

(d) *How does labor demand and output supply respond to changes in* w *and* p?

Answer: The derivative of the labor demand function is negative with respect to w and positive with respect to p — which implies the firm will less labor as wage increases and more as output price increases. Similarly, the derivative of the output supply function is positive with respect to price and negative with respect to wage — which implies that supply increases in price and decreases in wage.

(e) *Suppose that* $\alpha = 1.5$. *How do your answers change?*

Answer: If we used the functions that arise from our mathematical optimization problem in the chapter, we would get the incorrect answer that

$$\ell(p,w) = \left(\frac{w}{150p}\right)^{2} \quad \text{and} \quad x(p,w) = 100\left(\frac{w}{150p}\right)^{3}. \tag{11.1.ii}$$

The answer is wrong because it is actually a local minimum, not a maximum — equivalent to the tangency at A in Exercise Graph 11.1(2). To see this, you can check to see whether profit is positive by plugging the "optimal" labor and output values into the equation $\pi = xp - w\ell$ for profit; i.e.

$$\pi = 100\left(\frac{w}{150p}\right)^3 p - w\left(\frac{w}{150p}\right)^2 = \frac{2}{3}\left(\frac{w^3}{150^2 p^2}\right) - \frac{w^3}{150^2 p^2} = -\frac{1}{3}\left(\frac{w^3}{150^2 p^2}\right) < 0. \tag{11.1.iii}$$

You can also tell that the "answer" can't possibly be right by checking how labor demand and output supply would respond to changes in w and p — both would increase with increases in wages and decreases in output prices. The correct answer, of course, is that a price taking firm under these conditions would produce an infinite amount of output. (Of course that is not realistic — such a firm would have market power and would thus not be a price taker.)

Exercise 11.3

Consider a profit maximizing firm.

A: *Explain whether the following statements are true or false:*

(a) *For price-taking, profit maximizing producers, the "constraint" is determined by the technological environment in which the producer finds herself while the "tastes" are formed by the economic environment in which the producer operates.*

Answer: The physical constraint that a producer cannot get around arises from the simple laws of physics — you can only get so much x out of the inputs you use. The more sophisticated the technology available to the producer, the more x she can squeeze out — and thus the technology creates the production constraint that tells the producer which production plans are feasible and which are not. In terms of tastes, we typically assume that producers simply care about profit — and that more profit is better than less. To form indifference curves for producers who simply care about profit, we have to find production plans that all result in the same amount of profit. And how easy it is to make profit depends on how high the output price is and how low the input prices are relative to the output price. Thus, the production plans that produce the same level of profit will differ depending on the economic environment — depending on how much the producer can get for her output and how much she has to pay for the inputs in her production plans.

(b) *Every profit maximizing producer is automatically cost-minimizing.*

Answer: This is true. If you are maximizing profit, you must be producing whatever you are producing without wasting any inputs — i.e. you must be producing the output at the minimum cost possible.

(c) *Every cost-minimizing producer is automatically profit maximizing.*

Answer: This is not true. You could be producing some arbitrary quantity without wasting any inputs — i.e. in the least cost way. But that does not mean you are producing the "right" quantity. Cost minimization is only part of profit maximization — it only takes into account input prices as they relate to the cost of production. Only when you profit maximize do you also take into account the output price and what that implies for how much you should produce in a cost minimizing way.

(d) *Price taking behavior makes sense only when marginal product diminishes at least at some point.*

Answer: This is true. If marginal product always increases, then it is getting cheaper and cheaper to produce additional units of output. And if I can sell all my output at the same price (i.e. if I am a price taker), then I should keep producing and drive down my average cost.

B: *Consider the production function $x = f(\ell) = \alpha \ln(\ell + 1)$.*

(a) *Does this production function have increasing or decreasing marginal product of labor.*

Answer: Marginal product for this production function is given by

$$MP_\ell = \frac{\alpha}{\ell - 1}. \tag{11.3.i}$$

Thus, as ℓ increases, MP_ℓ clearly decreases. You can also see this by simply taking the derivative of marginal product of labor with respect to ℓ —

$$\frac{\partial MP_\ell}{\partial \ell} = \frac{-\alpha}{(\ell - 1)^2}. \tag{11.3.ii}$$

(b) *Set up the profit maximization problem and solve for the labor input demand and output supply functions.*

Answer: The profit maximization problem is

$$\max_{\ell,x} \; px - w\ell \;\text{ subject to }\; x = \alpha \ln(\ell + 1) \tag{11.3.iii}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell} \; p\alpha \ln(\ell + 1) - w\ell. \tag{11.3.iv}$$

The first order condition for this problem is

$$\frac{p\alpha}{\ell + 1} - w = 0 \tag{11.3.v}$$

which can be solved to get the labor demand function

$$\ell(w, p) = \frac{\alpha p - w}{w}. \tag{11.3.vi}$$

Substituting this into the production function, we then get the output supply function

$$x(w,p)=\alpha\ln\left(\frac{p\alpha-w}{w}+1\right)=\alpha\ln\left(\frac{\alpha p}{w}\right). \tag{11.3.vii}$$

(c) *Recalling that* $\ln x = y$ *implies* $e^y = x$ *(where* $e \approx 2.7183$ *is the base of the natural log), invert the production function and derive from this the cost function* $c(w,x)$.

Answer: To invert the production function $x = \alpha\ln(\ell+1)$, we first note that this implies

$$e^{x/\alpha}=\ell+1 \tag{11.3.viii}$$

which can then be solved in terms of ℓ to give us the amount of labor required for each level of output; i.e.

$$\ell=e^{x/\alpha}-1. \tag{11.3.ix}$$

When multiplied by w, we then get the cost function

$$c(w,x)=w\left(e^{x/\alpha}-1\right). \tag{11.3.x}$$

(d) *Determine the marginal and average cost functions.*

Answer: The marginal cost is

$$MC(w,x)=\frac{\partial c(w,x)}{\partial x}=\frac{w}{\alpha}e^{x/\alpha}, \tag{11.3.xi}$$

and the average cost is

$$AC(w,x)=\frac{we^{x/\alpha}}{x}-\frac{w}{x}. \tag{11.3.xii}$$

(e) *Derive from this the output supply and labor demand functions. Compare them to what you derived directly from the profit maximization problem in part (b).*

Answer: To derive the output supply function, we begin by setting p equal to MC; i.e.

$$p=\frac{w}{\alpha}e^{x/\alpha}. \tag{11.3.xiii}$$

Multiplying both sides by α/w, taking natural logs and then multiplying both sides by α, we then get

$$x(p,w)=\alpha\ln\frac{\alpha p}{w}. \tag{11.3.xiv}$$

Plugging this back into equation (11.3.ix), we get the labor input demand

$$\ell(w,p)=e^{\ln(\alpha p/w)}-1=\frac{\alpha p}{w}-1=\frac{\alpha p-w}{w}. \tag{11.3.xv}$$

Note that the output supply and labor demand equations are identical to those derived directly through profit maximization earlier in the problem. The equations are correct, however, only for prices above the lowest point of AC.

(f) *In your mathematical derivations, what is required for a producer to be cost minimizing? What, in addition, is required for her to be profit maximizing?*

Answer: The only requirement for the producer to be cost minimizing is that she not waste any input — i.e. that she hire labor in accordance with equation (11.3.ix). There is no requirement imposed by cost minimization on *how much* to produce. The additional requirement imposed by profit maximization is that output quantity be chosen such that $p = MC$. Alternatively, we could phrase this same additional requirement as the requirement that $pMP_\ell = w$.

Exercise 11.5

When we discussed optimal behavior for consumers in Chapter 6, we illustrated that there may be two optimal solutions for consumers whenever there are non-convexities in either tastes or choice sets. We can now explore conditions under which multiple optimal production plans might appear in our producer model.

A: *Consider only profit maximizing firms whose tastes (or isoprofits) are shaped by prices.*

(a) *Consider first the standard production frontier that has initially increasing marginal product of labor and eventually decreasing marginal product of labor.* True or False*: If there are two points at which isoprofits are tangent to the production frontier in this model, the lower output quantity cannot possibly be part of a truly optimal production plan.*

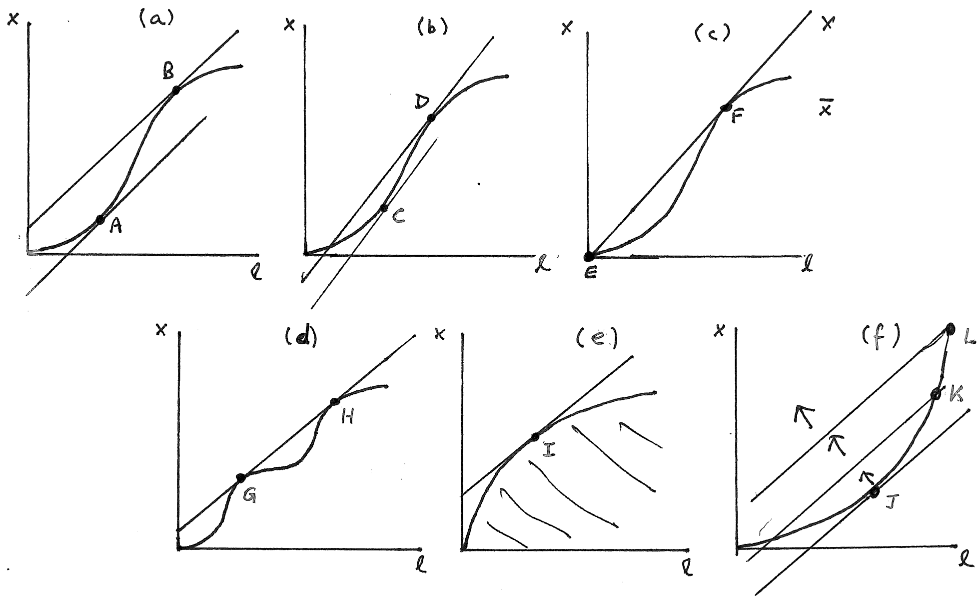
Answer: This is true. Panel (a) of Exercise Graph 11.5 illustrates this with two tangencies at A and B. Since the isoprofit tangent at A has a negative intercept, profit at the production plan A is negative.

(b) *Could it be that neither of the tangencies represents a truly optimal production plan?*

Answer: Yes. This is illustrated in panel (b) of Exercise Graph 11.5 where the two tangencies occur at production plans C and D. In this case, both isoprofit curves have negative intercepts — and both therefore involve negative profit. In this case, the profit maximizing production plan is (0,0) — i.e. no labor input is bought and no output is produced.

(c) *Illustrate a case where there are two truly optimal solutions where one of these does not occur at a tangency.*

Answer: This is illustrated in panel (c) of Exercise Graph 11.5. The production plan F occurs at a tangency and involves zero profit because the isoprofit curve intersects at the origin. Thus, profit at F is the same as profit at $E = (0,0)$ where no labor is purchased and no output is produced. Thus both E and F are truly profit maximizing production plans.



Exercise Graph 11.5 : Profit Maximizing under Different Production Sets

(d) *What would a production frontier have to look like in order for there to be two truly optimal production plans which both involve positive levels of output? (*Hint*: Consider technologies that involve multiple switches between increasing and decreasing marginal product of labor.)*

Answer: This is illustrated in panel (d) of Exercise Graph 11.5 where both G and H are tangencies that lie on the same isoprofit curve (and thus result in the same amount of positive profit).

(e) True or False*: If the producer choice set is convex, there can only be one optimal production plan.*

Answer: This is true. In panel (e) of Exercise Graph 11.5, a strictly convex production frontier is illustrated — with only I emerging as a profit maximizing production plan.

(f) *Where does the optimal production plan lie if the production frontier is such that the marginal product of labor is always increasing?*

Answer: This is illustrated in panel (f) of Exercise Graph 11.5 where the increasing marginal product of labor results in an ever steepening production frontier. The only tangency occurs at J — but J actually results in negative profit since the isoprofit has negative intercept. The producer can do better by moving to higher isoprofits that intersect the production frontier — as those that intersect at K and L. But it's always possible to go to an even higher isoprofit and mover higher on the production frontier.

Thus, the profit maximizing quantity is infinite (which, of course, does not make sense in a world of scarcity — but neither does an ever increasing marginal product of labor.)

(g) *Finally, suppose that the marginal product of labor is constant throughout. What production plans might be optimal in this case?*

Answer: This would result in a production frontier that is simply a straight line. If the ratio w/p happens to be the same as the slope of this production frontier, then every production plan on the production frontier lies on the same isoprofit curve which in turn intersects at the origin. Thus, all production plans on the production frontier yield zero profit — and all are therefore optimal. If the isoprofits are steeper than the production frontier, then all production plans on the frontier other than (0,0) result in negative profit — and only (0,0) is optimal. If, on the other hand, the isoprofits are shallower than the production frontier, the optimal output quantity is infinite for reasons similar to what we described in part (f) where we considered a production frontier with increasing marginal product of labor throughout.

B: *In the text, we used a cosine function to illustrate a production process that has initially increasing and then decreasing marginal product of labor. In many of the end-of-chapter exercises, we will instead use a function of the form $x = f(\ell) = \beta\ell^2 - \gamma\ell^3$ where β and γ are both greater than zero.*

(a) *Illustrate how the profit maximization problem results in two "solutions". (Use the quadratic formula to solve for these.)*

Answer: The profit maximization problem is then

$$\max_{\ell,x}\ px - w\ell \ \text{ subject to } \ x = \beta\ell^2 - \gamma\ell^3 \tag{11.5.i}$$

which can be written in the form of an unconstrained maximization problem by substituting the constraint into the objective to give us

$$\max_{\ell}\ p\left(\beta\ell^2 - \gamma\ell^3\right) - w\ell. \tag{11.5.ii}$$

The first order condition for this problem is then

$$-3\gamma p\ell^2 + 2\beta p\ell - w = 0. \tag{11.5.iii}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ \text{ and } \ x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.5.iv}$$

We can therefore solve equation (11.5.iii) for ℓ by applying this formula, letting $a = -3\gamma p$, $b = 2\beta p$ and $c = -w$, to give us

$$\ell = \frac{-2\beta p + \sqrt{4\beta^2 p^2 - 12\gamma p w}}{-6\gamma p} \text{ and } \ell = \frac{-2\beta p - \sqrt{4\beta^2 p^2 - 12\gamma p w}}{-6\gamma p}. \tag{11.5.v}$$

(b) *Which of your two "solutions" is unambiguously* not *the actual profit maximizing solution?*

Answer: Our two "solutions" can be rewritten as

$$\ell = \frac{\beta}{3\gamma} - \frac{\sqrt{\beta^2 p^2 - 3\gamma p w}}{3\gamma p} \text{ and } \ell = \frac{\beta}{3\gamma} + \frac{\sqrt{\beta^2 p^2 - 3\gamma p w}}{3\gamma p}, \tag{11.5.vi}$$

where the latter is clearly greater than the former. We know from our intuitive graphs that only the higher of the two production plans could be optimal — thus the first solution is definitely not a profit maximizing solution.

(c) *What else would you have to check to be sure that the other "solution" is profit maximizing?*

Answer: You would have to check that the larger of the two solutions results in positive profit.

(d) *Now consider instead a production process characterized by the equation $x = A\ell^{\alpha}$. Suppose $\alpha < 1$. Determine the profit maximizing production plan.*

Answer: The last part of the chapter sets up this optimization problem and solves for the production plan

$$\ell = \left(\frac{w}{\alpha A p}\right)^{\frac{1}{\alpha-1}} \text{ and } x = A\left(\frac{w}{\alpha A p}\right)^{\frac{\alpha}{\alpha-1}} \tag{11.5.vii}$$

(e) *What if $\alpha > 1$?*

Answer: When $\alpha > 1$, the marginal product of labor is increasing throughout — and the "solution" is not a true profit maximum because the firm should produce an infinite amount instead. You can tell that the "solution" makes no sense in this case because, when $\alpha > 1$, it would suggest that you hire more labor and produce more output as wages go up and output prices go down.

(f) *What if $\alpha = 1$?*

Answer: When $\alpha = 1$, the marginal product of labor is constant — and the production frontier is a straight line. The "solution" we calculated is not defined because the exponent $1/(\alpha - 1)$ is not defined when $\alpha = 1$. This is because there is no tangency — either all production plans on the frontier yield zero profit and all such plans are therefore profit maximizing, or the true profit maximum involves zero or an infinite level of output.

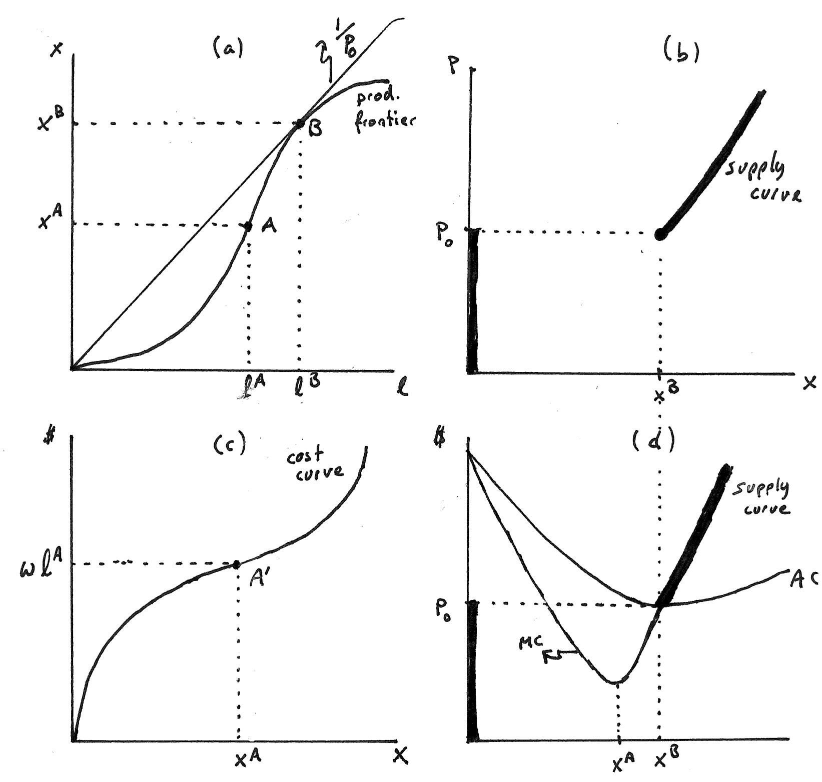
Exercise 11.7

We have shown that there are two ways in which we can think of the producer as maximizing profits: Either directly, or in a two-step process that begins with cost minimization.

A: *This exercise reviews this equivalence for the case where the production process initially has increasing marginal product of labor but eventually reaches decreasing marginal product. Assume such a production process throughout.*

(a) *Begin by plotting the production frontier with labor on the horizontal and output on the vertical axis. Identify in your graph the production plan $A = (\ell^A, x^A)$ at which increasing returns turns to decreasing returns.*

Answer: This is illustrated in panel (a) of Exercise Graph 11.7 where A lies at the point at which the production frontier stops increasing at an increasing rate and starts increasing at a decreasing rate. Put differently, at A the slope stops increasing and starts decreasing.



Exercise Graph 11.7 : 2 Ways to Maximize Profit

(b) *Suppose wage is $w = 1$. Illustrate in your graph the price p_0 at which the firm obtains zero profit by using a production plan B. Does this necessarily lie above or below A on the production frontier?*

Answer: This is also illustrated in panel (a) where B lies on an isoprofit that is tangent to the production frontier at B *and* intersects the origin (which implies zero profit). The slope of the isoprofit is $1/p_0$ since $w = 1$. It is apparent from the graph that B must lie above A on the production frontier — i.e. it must be the case that the zero-profit price results in production on the decreasing marginal product of labor portion of the production frontier.

(c) *Draw a second graph next to the one you have just drawn. With price on the vertical axis and output on the horizontal, illustrate the amount the firm produces at p_0.*

Answer: This is illustrated in panel (b) of Exercise Graph 11.7 where the point (x^B, p_0) illustrates the lowest price at which the firm produces positive output.

(d) *Suppose price rises above p_0. What changes on your graph with the production frontier — and how does that translate to points on the supply curve in your second graph?*

Answer: If price rises above p_0, the isoprofit lines become shallower — which implies the new optimal quantity lies at a tangency higher on production frontier than B. The isoprofit that is tangent at the new profit maximizing production plan also has positive intercept on the vertical axis — implying profit will be positive. Thus, output increases as p rises above p_0 — leading to an upward sloping supply curve (as illustrated in panel (b).)

(e) *What if price falls below p_0?*

Answer: If price falls below p_0, the isoprofit curves become steeper — implying tangencies to the left of B. At those tangencies, however, the intercept on the vertical axis will be negative — implying negative profit. Thus, the firm is better off not producing at all — which is why the supply curve in panel (b) is vertical at zero output level up to the price p_0.

(f) *Illustrate the cost curve on a graph below your production frontier graph. What is similar about the two graphs — and what is different — around the point that corresponds to production plan A.*

Answer: The cost curve, as illustrated in panel (c) of Exercise Graph 11.7, has the inverse shape from the production frontier — because when each additional labor unit increases production more than the last (on the increasing marginal product part of the production frontier), the cost of increasing output rises at a decreasing rate (and vice versa). Around A' — the point corresponding to A in panel (a), the cost curve switches from increasing at a decreasing rate to increasing at an increasing rate (just as the switch from increasing at an increasing rate to increasing at a decreasing rate happens on the production frontier.)

(g) *Next to your cost curve graph, illustrate the marginal and average cost curves. Which of these reaches its lowest point at the output quantity x^A? Which reaches its lowest point at x^B?*

Answer: This is illustrated in panel (d) of Exercise Graph 11.7. The marginal cost (MC) curve is the slope of the cost curve — so it reaches its lowest point at the output level x^A where the slope of the cost curve begins to get steeper. The average cost curve reaches its lowest point where the MC curve crosses it — which is also where the supply curve begins. This occurs at x^B.

(h) *Illustrate the supply curve on your graph and compare it to the one you derived in parts (c) and (d).*

Answer: The supply curve is the part of the MC curve that lies above the AC curve — with output of zero below that. This is highlighted in panel (d) of Exercise Graph 11.7 — with the resulting supply curve being identical to what we derived before in panel (b).

B: *Suppose that you face a production technology characterized by the function $x = f(\ell) = \alpha/(1+e^{-(\ell-\beta)})$.*

(a) *Assuming labor ℓ costs w and the output x can be sold at p, set up the profit maximization problem.*

Answer: The profit maximization problem is

$$\max_{\ell,x}\ px - w\ell \ \text{ subject to } \ x = \frac{\alpha}{1+e^{-(\ell-\beta)}} \tag{11.7.i}$$

which can be written as the unconstrained problem

$$\max_{\ell}\ \frac{p\alpha}{1+e^{-(\ell-\beta)}} - w\ell. \tag{11.7.ii}$$

(b) *Derive the first order condition for this problem.*

Answer: The first order condition is

$$\frac{\alpha p e^{-(\ell-\beta)}}{\left[1+e^{-(\ell-\beta)}\right]^2} = w. \tag{11.7.iii}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ into your first order condition and, using the quadratic formula, solve for y. Then, recognizing that $y = e^{-(\ell-\beta)}$ implies $\ln y = -(\ell-\beta)$, solve for the two implied labor inputs and identify which one is profit maximizing (assuming that an interior production plan is optimal).*

Answer: Substituting $y = e^{-(\ell-\beta)}$, the first order condition reduces to $\alpha py/(1+y)^2 = w$ which can be written in the form

$$wy^2 + (2w-\alpha p)y + w = 0. \tag{11.7.iv}$$

The quadratic formula then gives two solutions for y:

$$y_1 = \frac{-(2w-\alpha p)+\sqrt{(2w-\alpha p)^2-4w^2}}{2w} = \frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \tag{11.7.v}$$

and

$$y_2 = \frac{-(2w-\alpha p)-\sqrt{(2w-\alpha p)^2-4w^2}}{2w} = \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}. \tag{11.7.vi}$$

Given that $y = e^{-(\ell-\beta)}$, we can take natural logs of both sides to get $-(\ell - \beta) = \ln y$ or $\ell = \beta - \ln y$. Using the two solutions for y, we therefore get

$$\begin{aligned} \ell_1 &= \beta - \ln\left(\frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right) \quad \text{and} \\ \ell_2 &= \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right). \end{aligned} \tag{11.7.vii}$$

Since $y_1 > y_2$, we know that $\ell_1 < \ell_2$ — and thus the true profit maximizing labor input (assuming an interior production plan is profit maximizing) is given by

$$\ell(w,r,p) = \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right). \tag{11.7.viii}$$

(d) *Use your answer to solve for the supply function (assuming an interior solution is optimal).*

Answer: Plugging $\ell = \beta - \ln y_2$ into the production function, we then get

$$x = \frac{\alpha}{1+e^{-(\beta - \ln y_2 - \beta)}} = \frac{\alpha}{1+e^{\ln y_2}} = \frac{\alpha}{1+y_2}. \tag{11.7.ix}$$

Substituting in y_2 from equation (11.7.vi), this then simplifies to the supply function (assuming an interior optimum)

$$x(w,r,p) = \frac{2\alpha w}{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}. \tag{11.7.x}$$

We can also re-write this by multiplying both numerator and denominator by the term $(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p})$ to get

$$x(w,r,p) = \frac{2\alpha w\left(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}\right)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)} = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}. \tag{11.7.xi}$$

(e) *Now use the two-step method to verify your answer. Begin by solving the production function for ℓ to determine how much labor is required for each output level assuming none is wasted.*

Answer: To do the 2-step optimization, we begin by solving the production function $x = \alpha/(1+e^{-(\ell-\beta)})$ for ℓ. We can do this by first multiplying through by $(1+e^{-(\ell-\beta)})$, dividing by x and subtracting 1 from both sides to get

$$e^{-(\ell-\beta)} = \frac{\alpha}{x} - 1 = \frac{\alpha - x}{x} \tag{11.7.xii}$$

which allows us to write

$$-(\ell-\beta) = \ln\left(\frac{\alpha - x}{x}\right) \tag{11.7.xiii}$$

which then solves to

$$\ell = \beta - \ln\left(\frac{\alpha - x}{x}\right). \tag{11.7.xiv}$$

(f) *Use your answer to derive the cost function and the marginal cost function.*

Answer: The minimum cost at which the firm can produce any level of output x is then simply this inverted production function times the wage; i.e. the cost function is

$$C(w,x) = w\beta - w\ln\left(\frac{\alpha - x}{x}\right). \tag{11.7.xv}$$

From this, we can get the marginal cost function

$$MC(w,x) = \frac{\partial C}{\partial x} = \frac{\alpha w}{(\alpha - x)x}. \tag{11.7.xvi}$$

(g) *Set price equal to marginal cost and solve for the output supply function (assuming an interior solution is optimal). Can you get your answer into the same form as the supply function from your direct profit maximization problem?*

Answer: Setting $MC = p$, we can then solve for the supply function; i.e. we set $p = \alpha w/((\alpha - x)x)$, multiply through and write it in the form that allows us to once again apply the quadratic formula:

$$x^2 - \alpha x + \frac{\alpha w}{p} = 0 \quad \text{or equivalently} \quad px^2 - \alpha p x + \alpha w = 0. \tag{11.7.xvii}$$

Applying the quadratic formula, we get two "solutions":

$$x_1 = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \quad \text{and} \quad x_2 = \frac{\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \tag{11.7.xviii}$$

of which the first one is the true solution (since it is the larger of the two). The supply function (assuming an interior solution) is therefore

$$x(w,r,p) = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \tag{11.7.xix}$$

which is equivalent to the previous solution we got in equation (11.7.xi) by solving the profit maximization problem directly.

(h) *Use the supply function and your answer from part (e) to derive the labor input demand function (assuming an interior solution is optimal). Is it the same as what you derived through direct profit maximization in part (c)?*

Answer: Plugging the supply function into equation (11.7.xiv), we get

$$\ell = \beta - \ln\left[\frac{\alpha - \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}}{\frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}}\right] = \beta - \ln\left[\frac{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}\right]. \tag{11.7.xx}$$

Multiplying both the denominator and numerator within the brackets by the numerator $(\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p})$, we can then write this as

$$\begin{aligned} \ell &= \beta - \ln\left[\frac{\alpha^2 p^2 - 2\alpha p\sqrt{\alpha^2 p^2 - 4\alpha w p} + (\alpha^2 p^2 - 4\alpha w p)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)}\right] \\ &= \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right) \end{aligned} \tag{11.7.xxi}$$

which is exactly equal to the labor demand function we derived in equation (11.7.viii) through direct profit maximization.

Exercise 11.9

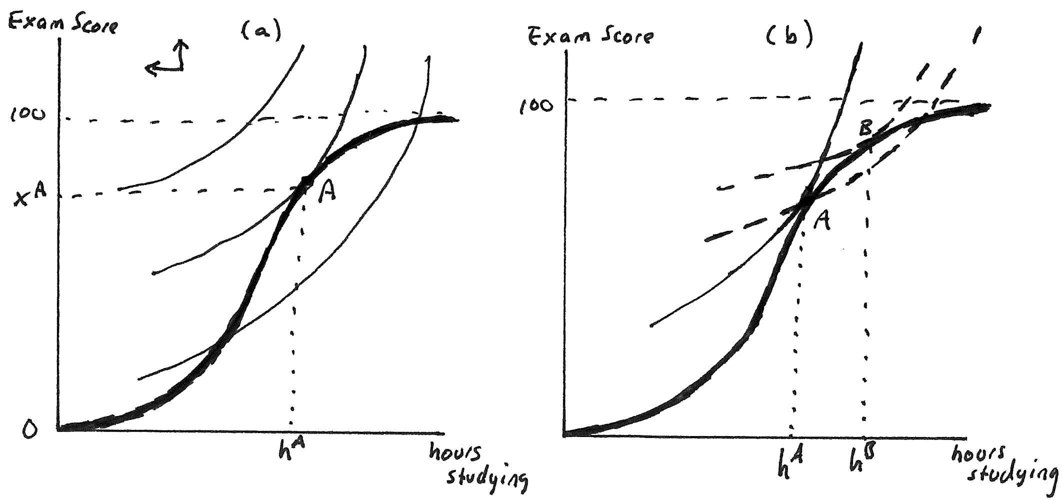
Everyday Application: *Studying for an Exam: Consider the problem you face as a student as you determine how much to study for an exam by modeling yourself as a "producer of an exam score" between 0 and 100.*

A: *Suppose that the marginal payoff to studying for the initial hours you study increases but that this marginal payoff eventually declines as you study more.*

(a) *Illustrate, on a graph with "hours studying for the exam" as an input on the horizontal axis and "exam score" (ranging from 0 to 100) as an output on the vertical axis, what your production frontier will look like.*

Answer: The production frontier is graphed as the (dark) frontier plotted in panel (a) of Exercise Graph 11.9.

(b) *Now suppose that your tastes over leisure time (i.e. non-study time) and exam scores satisfies the usual five assumptions about tastes that we outlined in Chapter 4. What will your producer tastes look like. (Be careful*



Exercise Graph 11.9 : Studying for an Exam

to recognize that the producer picture has "hours studying" and not leisure hours on the horizontal axis.)

Answer: Three indifference curves are graphed in panel (a) of Exercise Graph 11.9. You become better off as you move to the north-west in the graph — fewer hours of studying, higher exam grades.

(c) *Combining your production frontier with graphs of your indifference curves, illustrate the optimal number of hours you will study.*

Answer: Bundle A in panel (a) of the graph is your optimal "production plan" — resulting in h^A hours of studying and an exam grade of x^A.

(d) *Suppose that you and your friend differ in that your friend's marginal rate of substitution at every possible "production plan" is shallower than yours. Who will do better on the exam?*

Answer: Your friend will do better, as illustrated in panel (b) of the graph. At your optimal production plan A, your friend's indifference curve cuts the producer choice set in such a way that all the "better" plans (that lie to the north-west) involve greater effort than h^A. Your friend's optimal production plan is B.

(e) *Notice that the same model can be applied to anything we do where the amount of effort is an input and how well we perform a task is the output. As we were growing up, adults often told us: "Anything worth doing is worth doing well." Is that really true?*

Answer: Not really. Both you and your friend could have done very well on the exam — even scoring 100 — by putting more time in. But you also have other priorities in life — which is why you would prefer to devote less time to studying. It would not be optimal to devote all your time to one dimension — to doing well on this exam.

B: *Now suppose that you and your friends Larry and Daryl each face the same "production technology" $x = 3\ell^2 - 0.2\ell^3$ where x is the exam grade and ℓ is the number of hours of studying. Suppose further that each of you has tastes that can be captured by the utility function $u(\ell, x) = x - \alpha\ell$.*

(a) *Calculate your optimal hours of studying as a function of α.*

Answer: The problem that needs to be solved is

$$\max_{\ell,x} \; x - \alpha\ell \;\text{ subject to }\; x = 3\ell^2 - 0.2\ell^3 \tag{11.9.i}$$

which can be written as an unconstrained optimization problem by replacing x in the objective with the constraint — i.e.

$$\max_{\ell} \; 3\ell^2 - 0.2\ell^3 - \alpha\ell. \tag{11.9.ii}$$

Taking the first derivative and setting it to zero, we get $6\ell - 0.6\ell^2 - \alpha = 0$ or, rearranging terms slightly,

$$-0.6\ell^2 + 6\ell - \alpha = 0. \tag{11.9.iii}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \;\text{ and }\; x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.9.iv}$$

Using this to solve equation (11.9.iii) (where $a = -0.6$, $b = 6$ and $c = -\alpha$), we get

$$\ell = \frac{-6 + \sqrt{36 - 2.4\alpha}}{-1.2} \;\text{ and }\; \ell = \frac{-6 - \sqrt{36 - 2.4\alpha}}{-1.2}. \tag{11.9.v}$$

We know that these two "solutions" are tangencies between the production frontier and two indifference curves — but only the higher is a true optimum. Thus, the actual solution is simply

$$\ell = \frac{-6 - \sqrt{36 - 2.4\alpha}}{-1.2}. \tag{11.9.vi}$$

(b) *Suppose the values for α are 7, 10, and 13 for you, Larry and Daryl respectively. How much time will each of you study?*

Answer Plugging these values into the solution of equation (11.9.vi), we get that you, Larry and Daryl will study approximately 8.65, 7.87 and 6.83 hours respectively.

(c) *What exam grades will each of you get?*

Answer: Plugging these values of hours studied back into the production function, we get that you, Larry and Daryl will receive the following approximate grades respectively: 95, 88.5 and 76.2.

(d) *If each of you had 10 hours available that you could have used to study for the exam, could you each have made a 100? If so, why didn't you?*

Answer: Yes, each of you could have scored 100 by simply putting 10 hours of effort into preparing for the exam. (The answer is the same for each of you since your "production technology" is the same — just check that when you plug 10 into the production function, you actually get 100.) But you would have had to spend all your time preparing for the exam — which, given your tastes that place value on doing other things, was not worth it.

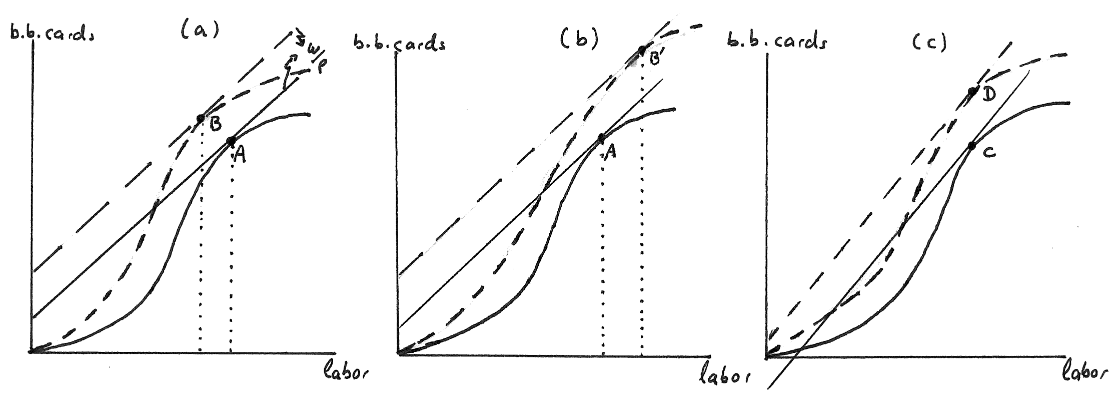
Exercise 11.11

Business Application: *Technological Change in Production*: *Suppose you and your friend Bob are in the business of producing baseball cards.*

A: *Both of you face the same production technology which has the property that the marginal product of labor initially increases for the first workers you hire but eventually decreases. You both sell your cards in a competitive market where the price of cards is p, and you hire in a competitive labor market were the wage is w.*

(a) *Illustrate your profit maximizing production plan assuming that p and w are such that you and Bob can make a positive profit.*

Answer: This is shown in the solid lines of panel (a) of Exercise Graph 11.11 where A is the profit maximizing production plan given the tangency of the (solid) isoprofit line at A.



Exercise Graph 11.11 : Technological Innovation

(b) *Now suppose you find a costless way to improve the technology of your firm in a way that unambiguously expands your producer choice set. As a result, you end up producing more than Bob (who has not found this technology). Illustrate how the new technology might have changed your production frontier.*

Answer: Two possibilities are illustrated with the dashed portions of panels (a) and (b) of Exercise Graph 11.11. In panel (a), your new profit maximizing production plan is B — which results in more baseball cards. In panel (b), your new profit maximizing production plan is B' which also results in more baseball cards.

(c) *Can you necessarily tell whether you will hire more or less labor with the new technology?*

Answer: No, it is not clear. In panel (a), you end up hiring fewer workers — while in panel (b) you end up hiring more workers.

(d) *Can you say for sure that adopting the new technology will result in more profit?*

Answer: Yes, profit increases unambiguously in both panels (as seen in the higher intercept of the dashed isoprofit curve relative to the solid one.)

(e) *Finally, suppose p falls. Illustrate how it might now be the case that Bob stops producing but you continue to stay in the business.*

Answer: A drop in p causes the slope of the isoprofits (i.e. w/p) to increase. Panel (c) then illustrates a case where Bob's "optimal" production plan C is not actually optimal because the isoprofit that is tangent at C has negative intercept and thus involves negative profit. You, on the other hand, maximize profit at D along an isoprofit with positive intercept (and thus positive profit). So Bob shuts down and you remain open.

B: *You and Bob initially face the production technology $x = 3A\ell^2 - 0.1\ell^3$, and you can sell your output for p and hire workers at a wage w.*

(a) *Derive the marginal product of labor and describe its properties.*

Answer: The marginal product of labor is calculated from the production function $x = f(\ell)$ as

$$MP_\ell = \frac{df}{d\ell} = 6A\ell - 0.3\ell^2. \tag{11.11.i}$$

This function is initially increasing but eventually decreases. To find the quantity of ℓ at which the marginal product of labor begins to decline, all we have to do is determine where the derivative of MP_ℓ is equal to zero; i.e. we need to solve

$$\frac{dMP_\ell}{d\ell} = 6A - 0.6\ell = 0 \tag{11.11.ii}$$

which solves to $\ell = 10A$. Thus, marginal product of labor increases until $\ell = 10A$ and falls thereafter. You can also check that marginal product of labor becomes negative at $\ell = 20A$.

(b) *Calculate the optimal number of baseball cards as a function of A assuming output price is given by p and the wage is $w = 20$. (Use the quadratic formula.)*

Answer: We need to solve

$$\max_{\ell,x} \; px - 20\ell \;\text{ subject to }\; x = 3A\ell^2 - 0.1\ell^3, \tag{11.11.iii}$$

which can be written as the unconstrained optimization problem

$$\max_{\ell} \; p(3A\ell^2 - 0.1\ell^3) - 20\ell. \tag{11.11.iv}$$

The first order condition for this problem is

$$-0.3p\ell^2 + 6Ap\ell - 20 = 0. \tag{11.11.v}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \;\text{ and }\; x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.11.vi}$$

We can use this formula to solve equation (11.11.v) and get the two "solutions"

$$\ell = \frac{-6Ap + \sqrt{36A^2p^2 - 24p}}{-0.6p} \;\text{ and }\; \ell = \frac{-6Ap - \sqrt{36A^2p^2 - 24p}}{-0.6p}, \tag{11.11.vii}$$

of which the latter is the true optimum (since we know from our intuitive treatment of the material that the larger of the two input amounts represents the profit maximum.)

(c) *How much will you each produce if $A = 1$ and $p = 1$, and how much profit do each of you earn?*

Answer: When $p = 1$ and $A = 1$ are plugged into the true optimum we just calculated, we get $\ell \approx 15.77$. (The "false" optimum in the first equation would come out to be $\ell \approx 4.23$.)

We can calculate the profit by first calculating the output you would produce with this much labor. Plugging $\ell = 15.77$ into the production function, we get $x \approx 353.96$. We can then calculate the profit as $\pi = px - w\ell = (1)(353.96) - 20(15.77) \approx 38.5$. (You can also verify that the second "solution" for labor input is not an optimum by calculating the profit you would get under that input level — which turns out to be -38.5.)

(d) *Now suppose you find a better technology — one that changes your production function from one where $A = 1$ to one where $A = 1.1$. How do your answers change?*

Answer: You will now hire $\ell = 18.20$, produce $x = 493.72$ and earn profit $\pi = 126.30$. (You can derive these exactly as you did for the previous case.)

(e) *Now suppose that competition in the industry intensifies and the price of baseball cards falls to $p = 0.88$. How will you and Bob change your production decisions?*

Answer: Bob still operates under $A = 1$. Plugging in $A = 1$ and $p = 0.88$ into our solution for the optimal labor input, we get $\ell \approx 14.92$. Plugging this into the production function, this would give us $x \approx 335.77$, and using these to calculate profit, we get a profit of $\pi \approx -2.99$. Since the highest profit Bob can get by producing is negative, he will no longer produce.

You, on the other hand, face a production process with $A = 1.1$. Following the same procedure, you can then calculate that your optimal production plan is $(\ell, x) \approx (17.73, 479.93)$ which gives profit of $\pi \approx 67.8$. Since profit is positive, you will indeed produce this level of output.

Exercise 11.13

Policy Application: *Determining Optimal Class Size: Public policy makers are often pressured to reduce class size in public schools in order to raise student achievement.*

A: *One way to model the production process for student achievement is to view the "teacher/student" ratio as the input. For purposes of this problem, let t be defined as the number of teachers per 1000 students; i.e. $t = 20$ means there are 20 teachers per 1,000 students. Class size in a school of 1000 students is then equal to $1000/t$.*

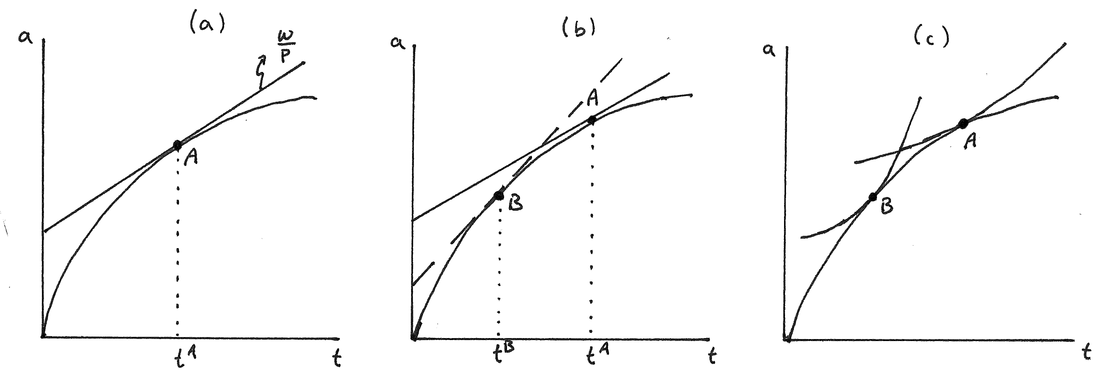
(a) *Most education scholars believe that the increase in student achievement from reducing class size is high when class size is high but diminishes as class size falls. Illustrate how this translates into a production frontier with t on the horizontal axis and average student achievement a on the vertical.*

Answer: This production frontier is pictured in panel (a) of Exercise Graph 11.13 — with large slope (or marginal product) initially that falls as t increases. Note that t is not class size — when t is small, there are few teachers per 1000 students, which implies a large class size. In our graph, class size therefore falls as t increases.

(b) *Consider a school with 1,000 students. If the annual salary of a teacher is given by w, what is the cost of raising the input t by 1 — i.e. what is the cost per unit of the input t?*

Answer: Since t is the number of teachers per 1000 students, we need to hire one more teacher to increase t by 1. Thus, the per unit cost of t is w.

(c) *Suppose a is the average score on a standardized test by students in the school, and suppose that the voting public is willing to pay p for each unit*



Exercise Graph 11.13 : Setting Class Size

increase in a. Illustrate the "production plan" that the local school board will choose if it behaves analogously to a profit maximizing firm.

Answer: This is also illustrated in panel (a) of Exercise Graph 11.13 where A is the optimal "production plan" given the willingness p of voters to get a one unit increase in achievement. The input t^A then results in class size of $1000/t^A$.

(d) *What happens to class size if teacher salaries increase?*

Answer: If teacher salaries increase, w goes up — which implies w/p increases and the isoprofits become steeper. As a result, the optimal t falls — which implies class size increases. This is illustrated in panel (b) of Exercise Graph 11.13.

(e) *How would your graph change if the voting public's willingness to pay per unit of a decreases as a increases?*

Answer: This would in effect imply that p is high when a is low but decreases as a increases. A high p implies a relatively flat slope — thus, we would get isoprofit curves that, rather than being straight lines as in panels (a) and (b) of the graph, would be shaped as depicted in panel (c). (Note: The isoprofit tangent at B represents the curve when teacher salaries are higher.)

(f) *Now suppose that you are analyzing two separate communities that fund their equally sized schools from tax contributions by voters in each school district. They face the same production technology, but the willingness to pay for marginal improvements in a is lower in community 1 than in community 2 at every production plan. How do the isoprofit maps differ for the two communities?*

Answer: At every "production plan" the isoprofits of the community with lower willingness to pay would be steeper — indicating that a greater increase in achievement would be required for every increase in t in order for the community to remain equally well off.

(g) *Illustrate how this will result in different choices of class size in the two communities.*

Answer: This can be illustrated in a graph identical to the one pictured in panel (c) of Exercise Graph 11.13 where the community with the lower willingness to pay optimizes at B while the other community optimizes at A.

(h) *Suppose that the citizens in each of the two communities described above were identical in every way except that those in community 1 have a different average income level than those in community 2. Can you hypothesize which of the two communities has greater average income?*

Answer: Community 1 is poorer. Even though they care as much about education, their willingness to pay is lower simply because they have fewer resources. (This should be familiar from consumer theory — people with identical tastes can have very different demands for goods if they have different incomes.)

(i) *Higher level governments often subsidize local government contributions to public education, particularly for poorer communities. What changes in your picture of a community's optimal class size setting when such subsidies are introduced?*

Answer: You can think of these subsidies as either reducing the teacher salaries w (since the higher level government now shares in the expense) or as raising the willingness to pay p since the voters know that they get more resources if they spend more on education. Whether viewed as a decrease in w or and increase in p, the impact on w/p is the same: w/p falls — which implies that isoprofit curves become shallower. This in turn has the impact of increasing the optimal choice of t (as illustrated in panels (b) and (c) of Exercise Graph 11.13) — which is the same as saying that class size will decrease.

B: *Suppose the production technology for average student achievement is given by $a = 100t^{0.75}$, and suppose again that we are dealing with a school that has 1000 students.*

(a) *Let w denote the annual teacher salary in thousands of dollars and let p denote the community's marginal willingness to pay for an increase in student achievement. Calculate the "profit maximizing" class size.*

Answer: We need to solve the maximization problem

$$\max_{a,t} \; pa - wt \text{ subject to } a = 100t^{0.75} \tag{11.13.i}$$

which can be written as the unconstrained optimization problem

$$\max_{t} \; a = 100t^{0.75}p - wt. \tag{11.13.ii}$$

The first order condition for this problem is $75t^{-0.25}p - w = 0$ and can be solved to get

$$t = \left(\frac{75p}{w}\right)^4. \tag{11.13.iii}$$

Since class size is $1000/t$, the optimal class size is

$$\text{Class size} = \frac{1000w^4}{(75p)^4}. \tag{11.13.iv}$$

(b) *What is the optimal class size when $w = 60$ and $p = 2$?*

Answer: Plugging these into the expression we just derived, we get an optimal class size of 25.6 students per class.

(c) *What happens to class size as teacher salaries change?*

Answer: We can simply take the derivative of our optimal class size expression with respect to w to see if it is positive or negative. Since it is clearly positive, we know that class size increases as teacher salaries increase (and decreases as teacher salaries decrease).

(d) *What happens to class size as the community's marginal willingness to pay for student achievement changes?*

Answer: This time we would take the derivative of our optimal class size expression with respect to p. Since this derivative is clearly negative, we know that class size will increase as p decreases (and decrease as p increases).

(e) *What would change if the state government subsidizes the local contribution to school spending?*

Answer: This would effectively lower w — and thus decrease class size. Alternatively you could think of it as raising the local willingness to spend money on schools — which would also have the effect of decreasing class size.

(f) *Now suppose that the community's marginal willingness to pay for additional student achievement is a function of the achievement level. In particular, suppose that $p(a) = Ba^{\beta-1}$ where $\beta \le 1$. For what values of β and B is the problem identical to the one you just solved?*

Answer: If $B = p$ and $\beta = 1$, the problem is identical to what we solved before.

(g) *Solve for the optimal t given the marginal willingness to pay of $p(a)$. What is the optimal class size when $B = 3$ and $\beta = 0.95$ (assuming again that $w = 60$.*

Answer: We would now solve the problem

$$\max_{a,t}\ p(a)a - wt = Ba^{\beta-1}a - wt = Ba^{\beta} - wt \ \text{ subject to } \ a = 100t^{0.75} \tag{11.13.v}$$

which can be written as the unconstrained maximization problem

$$\max_t B\left(100t^{0.75}\right)^{\beta} - wt. \tag{11.13.vi}$$

The first order condition for this problem is

$$0.75\beta B(100^{\beta})\, t^{0.75\beta-1} - w = 0, \tag{11.13.vii}$$

which solves to

$$t = \left(\frac{w}{0.75\beta B\left(100^{\beta}\right)}\right)^{1/(0.75\beta-1)} \tag{11.13.viii}$$

Substituting in $w = 60$, $\beta = 0.95$ and $B = 3$, we then get $t \approx 37.27$. Given that class size is $1000/t$, we get that the optimal class size under these conditions is approximately 26.83 students per class.

(h) *Under the parameter values just specified, does class size respond to changes in teacher salaries as it did before?*

Answer: Under the parameter values specified, the optimal input level t becomes

$$t = \left(\frac{w}{0.75(0.95)(3)\left(100^{0.95}\right)}\right)^{1/(0.75(0.95)-1)} \approx 169.79w^{-3.48}. \tag{11.13.ix}$$

This implies an optimal class size of

$$\text{Class Size} = \frac{1000}{t} = \frac{1000}{169.79w^{-3.48}} \approx 5.89w^{3.48}. \tag{11.13.x}$$

The derivative of class size with respect to w is then clearly positive — which implies that class size increases as teacher salaries increase.

Conclusion: Potentially Helpful Reminders

1. Remember that the one-step profit maximization method is completely equivalent to the two-step method that minimizes costs first. The former gives us the profit maximizing condition that $MRP = w$; the latter gives us the condition that $p = MC$. Since the two methods are equivalent, the two conditions imply one another in the single-input model; i.e. whenever $MRP = w$, it must be that $p = MC$ and vice versa if we are at a true optimum for the firm. (This insight will generalize to multiple inputs in the next chapter.)

2. Make sure you really understand how the 2-step method for maximizing profit is equivalent to simply maximizing profit in a single step. As the model becomes more complex when we allow for two (rather than one) input in the next chapter, we will increasingly rely on the 2-step method.

3. In the 2-step method, we begin by deriving the cost curve (or function) — and this involves no consideration of the output price p. Thus, any producer who hires inputs in competitive input markets will get the same cost curves for a given technology. In particular, it does not matter for the cost curves whether the producer is in fact a price taker in the output market — he could be a monopolist who sets price in the output market, but his cost curves would still arise in exactly the same way. This will become important in Chapters 23 and beyond where we will use the same types of cost curves to discuss markets that are not perfectly competitive (where firms are not price takers).

4. It's a good time to get used to the way we use the term *cost*. Whenever we say "cost" in this chapter and throughout this text, we mean "economic" or "opportunity" cost. If something has to be paid regardless of what action I choose, it is not a cost of choosing an action. Ever. (In Chapter 13 we will adopt the term "expense" for cases where we have to spend money regardless of what we do.)

CHAPTER

12

Production with Multiple Inputs

This chapter continues the treatment of producer theory when firms are price takers. Chapter 11 focused on the short run model in which capital is held fixed and labor is therefore the only variable input. This allowed us to introduce the ideas of profit maximization and cost minimization within the simplest possible setting. Chapter 12 now focuses on the long run model in which both capital and labor are variable. The introduction of a second input then introduces the possibility that firms will substitute between capital and labor as input prices change. It also introduces the idea of returns to scale. And we will see that the 2-step profit maximization approach that was introduced at the end of Chapter 11 — i.e. the approach that begins with costs and then adds revenues to the analysis — is much more suited to a graphical treatment than the 1-step profit maximization approach (which would require graphing in 3 dimensions.)

Chapter Highlights

The main points of the chapter are:

1. Profit maximization in the 2-input (long run) model is conceptually the same as it is for the one-input (short run) model — the profit maximizing production plans (that involve positive levels of output) again satisfying the condition that the **marginal revenue products of inputs are equal to the input prices**. The **marginal product** of each input is measured along the vertical slice of the production frontier that holds the *other* input fixed (as already developed for the marginal product of labor in Chapter 11.)

2. **Isoquants** are horizontal slices of the production frontier and are, in a technical sense, similar to indifference curves from consumer theory. Their shape indicates the degree of substitutability between capital and labor, and their slope is the (marginal) **technical rate of substitution** which is equal to the (negative) ratio of the marginal products of the inputs.

3. Unlike in consumer theory where the labeling of indifference curves had no cardinal meaning, the labeling on isoquants has a clear cardinal interpretation since production units are objectively measurable. The rate at which this labeling increases tells us whether the production frontier's slope is increasing at an increasing or decreasing rate — and thus whether the production technology is exhibiting increasing or decreasing **returns to scale**.

4. Cost minimization in the two-input model is considerably more complex than it was in the single-input model of Chapter 11 because there are now many different ways of producing any given output level without wasting inputs (i.e. in a technologically efficient way) as indicated by all input bundles on each isoquant. The **least cost way of producing** any output level then depends on input prices — and is graphically seen as the tangency between **isocosts** and isoquants.

5. For **homothetic production processes**, all cost minimizing input bundles will lie on the same ray from the origin within the isoquant graph. The vertical slice of the 3-D production frontier along that ray is then the relevant slice on which the profit maximizing production plan lies.

6. The **cost curve** is derived from the cost-minimizing input bundles on that same ray from the origin — and, analogous to what we did in Chapter 11, its shape is the inverse of the shape of the production frontier along that slice. (This shape also indicates whether the production process has increasing or decreasing returns to scale). Once we have derived the cost curve, the 2-step profit maximization proceeds exactly as it did in Chapter 11 — with output occurring where $p = MC$.

12A Solutions to Within-Chapter-Exercises for Part A

Exercise 12A.1

Suppose we are modeling all non-labor investments as capital. Is the rental rate any different depending on whether the firm uses money it already has or chooses to borrow money to make its investments?

Answer: No — for the same reason that the rental rate of photocopiers for Kinkos is the same regardless of whether Kinkos owns or rents the copiers. If the firm borrows money from another firm, it is doing so at the interest rate r which then becomes the rental rate for the financial capital it is investing. If the firm uses its own money, it is foregoing the option of lending that money to another firm at the interest rate r — and thus it again costs the firm r per dollar to invest in its own capital.

Exercise 12A.2

Explain why the vertical intercept on a three dimensional isoprofit plane is π/p (where π represents the profit associated with that isoprofit plane).

Answer: A production plan on the vertical intercept has positive x but zero ℓ and k. Profit for a production plan (ℓ, k, x) is given by $\pi = px - w\ell - rk$ — but since $\ell = k = 0$ on the vertical axis, this reduces to $\pi = px$. Put differently, when there are no input costs, profit is the same as revenue for the firm — and revenue is just price times output. Dividing both sides of $\pi = px$ by p, we get π/p — the value of the intercept of the isoprofit plane associated with profit π.

Exercise 12A.3

We have just concluded that $MP_k = r/p$ at the profit maximizing bundle. Another way to write this is that the marginal revenue product of capital $MRP_k = pMP_k$ is equal to the rental rate. Can you explain intuitively why this makes sense?

Answer: The intuition is exactly identical to the intuition developed in Chapter 11 for the condition that marginal revenue product of labor must be equal to wage at the optimum. The marginal product of capital is the additional output we get from one more unit of capital (holding fixed all other inputs). Price times the marginal product of capital is the additional *revenue* we get from one more unit of capital. Suppose we stop hiring capital when the cost of a unit of capital r is exactly equal to this marginal revenue product of capital. Since marginal product is diminishing, this means that the marginal revenue from the previous unit of capital was greater than r — and so I made money on hiring the previous unit of capital. But if I hire past the point where $MRP_k = r$, I am hiring additional units of capital for which the marginal revenue is less than what it costs me to hire those units. Thus, had I stopped hiring before $MRP_k = r$, I would have forgone the opportunity of making additional profit from hiring more capital; if, on the other hand, I hire beyond $MRP_k = r$, I am incurring losses on the additional units of capital.

Exercise 12A.4

Suppose capital is fixed in the short run but not in the long run. *True or False*: If the firm has its long run optimal level of capital k^D (in panel (f) of Graph 12.1), then it will choose ℓ^D labor in the short run. And if ℓ^A in panel (c) is not equal to ℓ^D in panel (f), it must mean that the firm does not have the long run optimal level of capital as it is making its short run labor input decision.

Answer: This is true. If the firm has capital k^D, then it is operating on the short-run slice that holds k^D fixed in panel (f). The short run isoprofit is then just a slice of the long run isoprofit plane — and is tangent at labor input level ℓ^D. If the firm chooses $\ell^A \neq \ell^D$ in the short run, then it is not operating on this slice — and thus does not have the long run profit maximizing capital level of k^D.

Exercise 12A.5

Apply the definition of an isoquant to the one-input producer model. What does the isoquant look like there? (*Hint*: Each isoquant is typically a single point.)

Answer: An isoquant for a given level of output x is the set of all input bundles that result in that level of output without wasting any input. In the one-input model, the only production plans that don't waste inputs are those that lie on the production frontier. For each level of x, we therefore have a single level of (labor) input that can produce that level of x without any input being wasted. This single labor input level is then the isoquant for producing a particular output level x.

Exercise 12A.6

Why do you think we have emphasized the concept of marginal product of an input in producer theory but not the analogous concept of marginal utility of a consumption good in consumer theory?

Answer: The marginal product of an input is the number of additional units of output that can be produced if one more unit of the input is hired. This is an objectively measurable quantity. The marginal utility of a consumption good is the additional utility that will result from consumption of one more unit of the consumption good. Since it is measured in utility terms, it is not objectively measurable (since we have no way to measure "utils" objectively).

Exercise 12A.7

Repeat this reasoning for the case where $MP_\ell = 2$ and $MP_k = 3$.

Answer: Suppose we currently produce some quantity x using ℓ units of labor and k units of capital. If $MP_\ell = 2$ and $MP_k = 3$, this implies that, at my current production plan, capital is 1.5 times as productive as labor. Suppose I want to use one less unit of capital but continue to produce the same amount as before. Then, since capital is 1.5 times as productive as labor, this would imply I would have to hire 1.5 units of labor. In other words, substituting 1 unit of capital for 1.5 units of labor leads to no change in output on the margin — which is another way of saying that my technical rate of substitution is currently $TRS = -1/1.5 = -2/3$ — which is just $-MP_\ell/MP_k$.

Exercise 12A.8

Is there a relationship analogous to equation (12.3) that exists in consumer theory and, if so why do you think we did not highlight it in our development of consumer theory?

Answer: Yes. In exactly the same way, we could derive the relationship

$$MRS = -\frac{MU_1}{MU_2} \qquad (12A.8)$$

where MU_1 and MU_2 are the marginal utility of consuming good 1 and good 2. Since marginal utility is not objectively measurable, we did not emphasize the concept. However, note that "utils" cancel on the right hand side of our equation — implying that MRS is not expressed in util terms. Thus, MRS is a meaningful and measurable concept even if MU is not.

Exercise 12A.9

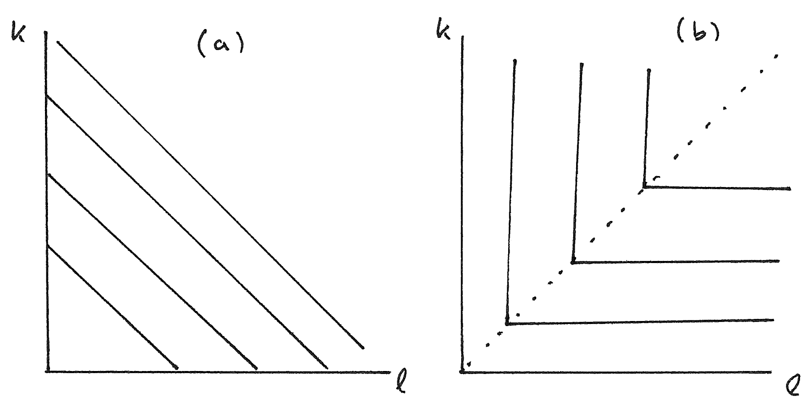
In the "old days", professors used to hand-write their academic papers and then have secretaries type them up. Once the handwritten scribbles were handed to the secretaries, there were two inputs into the production process: secretaries (labor) and typewriters (capital). If one of the production processes in Graph 12.4 represents the production for academic papers, which would it be?

Answer: There is little substitutability between secretaries and typewriters since each secretary has to be matched with one typewriter if papers are to be typed. Thus, panel (c) would come closest to representing the production for academic papers.

Exercise 12A.10

What would isoquant maps with no substitutability and perfect substitutability between inputs look like? Why are they homothetic?

Answer: These are graphed in Exercise Graph 12A.10, with panel (a) representing a production process with perfect substitutability of capital and labor and panel (b) representing perfect complementarity. These are both homothetic because the slope of the isoquants is unchanged along any ray from the origin.



Exercise Graph 12A.10 : Perfect Substitutes and Complements in Production

Exercise 12A.11

Consider a three-dimensional frontier similar to the one graphed in Graph 12.1 but with two goods on the horizontal axes and utility on the vertical. Why would we not think that vertical slices like the ones in 12.1d are meaningful in this case?

Answer: In order for those slices to be meaningful, we would have to think that we can objectively measure utility on the vertical axis. Since we have no such objective measure, the vertical slices don't have particular meaning in consumer theory. In producer theory, we measure output on the vertical axis – and since output is objectively measurable, the vertical slices are meaningful in producer theory.

Exercise 12A.12

Consider the same utility frontier described in the previous exercise. What would the horizontal slices analogous to those in Graph 12.3 be in consumer theory? Why are they meaningful when the vertical slices in Graph 12.1 are not?

Answer: These slices would be indifference curves in consumer theory. They are meaningful because they illustrate the tradeoffs that individuals are willing to make between goods – which can be stated in objective terms. Indifference curves do not require objective measures of utility as vertical slices in Graph 12.1 would.

Exercise 12A.13

Consider a real-world mountain and suppose that the shape of any horizontal slice of this mountain is a perfect (filled in) circle. I have climbed the mountain from every direction — and I have found that the climb typically starts off easy but gets harder and harder as I approach the top because the mountain gets increasingly steep. Does this mountain satisfy any of the two notions of convexity we have discussed?

Answer: A perfect (filled in) circle is a convex set. Thus, the horizontal slices of our mountain are convex sets — which means the mountain satisfies our original notion of convexity. If, however, the mountain gets steeper as we move up, vertical slices of the mountain will not be convex. Thus, our second notion of convexity does not hold.

Exercise 12A.14

Is the vertical slice described in the previous section (including all the points inside the mountain that lie on the slice) a convex set?

Answer: It is a convex set in panel (a) of Graph 12.5 and a non-convex set in panel (b) of Graph 12.5. The non-convexity in panel (b) can be seen in the fact that the line connecting A' and B' lies outside the shaded slice. (In panel (a), all lines connecting any two points in the shaded set lie fully within the set.)

Exercise 12A.15

Consider a single input production process with increasing marginal product. Is this production process increasing returns to scale? What about the production process in Graph 11.10?

Answer: Increasing marginal product in the single input model implies that we can increase the input by a factor t and thereby will raise output by a factor greater than t. Thus, the production process is increasing returns to scale. In Graph 11.10, the production process has this feature initially — but eventually becomes decreasing returns to scale.

Exercise 12A.16

True or False: Homothetic production frontiers can have increasing, decreasing or constant returns to scale.

Answer: This is true. You can take the same isoquant map and attach labels to it that would make the underlying production technology increasing, decreasing or constant returns to scale. (For increasing returns to scale, the labels would be increasing at an increasing rate; for decreasing returns to scale they would be increasing at a decreasing rate; and for constant returns to scale, they would be increasing at a constant rate.)

Exercise 12A.17

If the three panels of Graph 12.6 represented indifference curves for consumers, would there be any meaningful distinction between them? Can you see why the concept of "returns to scale" is not meaningful in consumer theory?

Answer: The distinction would not be meaningful — because the shape of the indifference curves and the ordering of the labels is the same in all three panels. Returns to scale is not meaningful in consumer theory because the statement "as I double the consumption bundle, my utility doubles" is not meaningful when we don't think we can measure utility objectively.

Exercise 12A.18

True or False: If you have decreasing marginal product of all inputs, you might have decreasing, constant, or increasing returns to scale.

Answer: This is true. The most counterintuitive case is the one where you have decreasing marginal product of all inputs but increasing returns to scale. But the fact that output increases at a diminishing rate as we add a single input at a time does not imply that output would not increase at an increasing rate if we increase all inputs simultaneously (which is increasing returns to scale.)

Exercise 12A.19

True or False: In the two-input model, decreasing returns to scale implies decreasing marginal product of all inputs.

Answer: This is true. If output increases at a diminishing rate as we add all inputs simultaneously, it must be that output increases at a diminishing rate as we add one input at a time.

Exercise 12A.20

True or False: In a two-input model, increasing returns to scale implies increasing marginal product of at least one input.

Answer: This is false. It is possible to have decreasing marginal product of all inputs and still have increasing returns to scale. If we did have increasing marginal product of one input, however, we are guaranteed to also have increasing returns to scale.

Exercise 12A.21

True or False: In the single-input model, each isoquant is composed of a single point which implies that all technologically efficient production plans are also economically efficient.

Answer: This is true. An isoquant is the set of input bundles that can produce a given output level without inputs being wasted. Since there is only one input, there is only one way to produce each output level without wasting inputs — thus the isoquant is a single point. It is technologically efficient because no input is wasted — and economically efficient because it is (by default) the least expensive of all the technologically efficient input bundles.

Exercise 12A.22

True or False: In the two input model, every economically efficient production plan must be technologically efficient but not every technologically efficient production plan is necessarily economically efficient.

Answer: This is true. In order for an input bundle to be the economically most efficient — or cheapest — way of producing an output level, it must be the case that no inputs are wastes — i.e. the input bundle must be technologically efficient for this output level. But, when there are many technologically efficient ways of producing a given level of output, some will be more expensive and some less — so they cannot all be economically efficient (i.e. cheapest).

Exercise 12A.23

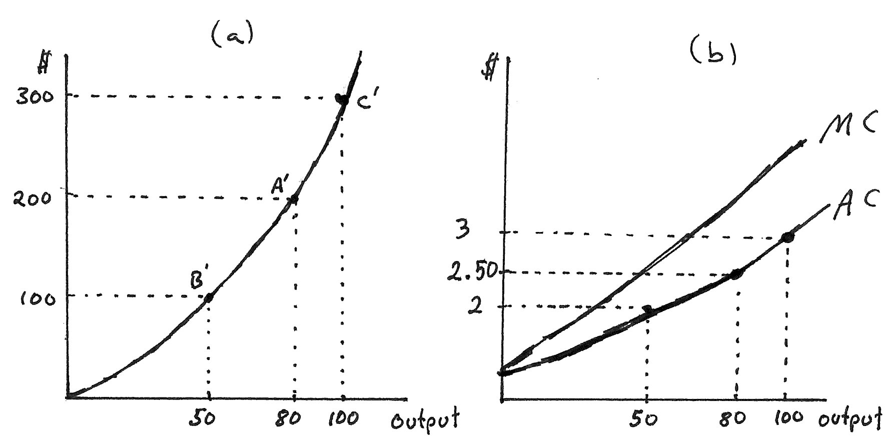
True or False: We have to know nothing about prices, wages or rental rates to determine the technologically efficient ways of producing different output levels, but we cannot generally find the economically efficient ways of producing any output level without knowing these.

Answer: This is true. Technologically efficient production just means production without wasting inputs — and we do not have to know anything about prices in the economy to know whether we are wasting inputs. Put differently, we do not have to know anything about prices to derive isoquants — they just come from the production frontier which is determined by the technology that is available to the producer. Economically efficient production means the "cheapest" way to produce — and that of course has much to do with input prices. (It does not, of course, have anything to do with the output price.)

Exercise 12A.24

Suppose the numbers associated with the isoquants in Graphs 17.7(a) and (b) had been 50, 80 and 100 instead of 50, 100 and 150. What would the total cost, *MC* and *AC* curves look like? Would this be an increasing or decreasing returns to scale production process, and how does this relate to the shape of the cost curves?

Answer: This is illustrated in Exercise Graph 12A.24. This would imply it is getting increasingly hard to produce additional units of output — i.e. the underlying technology represented by the isoquants has decreasing returns to scale. As a result, the cost of producing is increasing at an increasing rate — which causes the *MC* and *AC* curves to slope up.

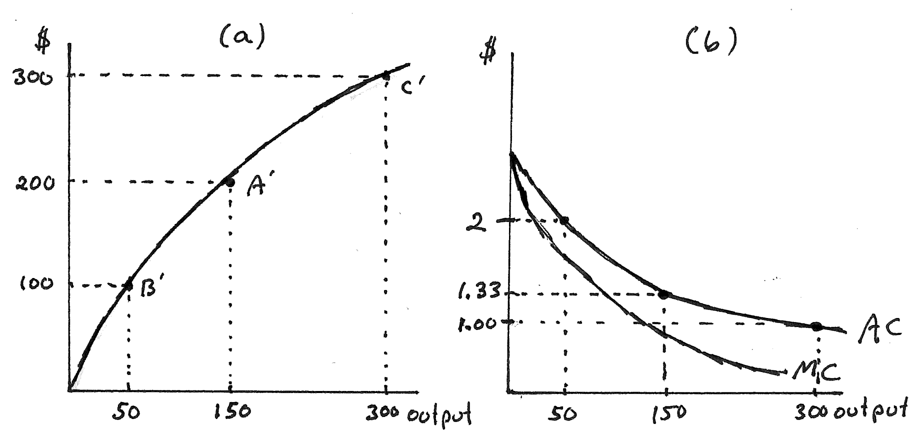


Exercise Graph 12A.24 : Decreasing Returns to Scale Cost Curves

Exercise 12A.25

How would your answer to the previous question change if the numbers associated with the isoquants were 50, 150, and 300 instead?

Answer: This is illustrated in Exercise Graph 12A.25. Production of additional goods is getting increasingly easy — which means the underlying production technology has increasing returns to scale. As a result, increased production causes costs to increase at a decreasing rate — which implies *MC* and *AC* are downward sloping.



Exercise Graph 12A.25 : Increasing Returns to Scale Cost Curves

Exercise 12A.26

If w increases, will the economically efficient production plans lie on a steeper or shallower ray from the origin? What if r increases?

Answer: If w increases, then w/r increases — which means the slope of the isocosts becomes steeper. Thus, the tangencies with isoquants will occur to the left (where the isoquants are steeper) — implying that they will occur on a ray that is steeper. If r increases, w/r falls — meaning that the isocosts get shallower. Thus, the tangencies with isoquants will occur to the right (where the isoquants are shallower) — implying that they will occur on a ray that is shallower. This should make sense — as w increases, economic efficiency will require a substitution away from labor and toward more capital, and the reverse will happen if r increases.

Exercise 12A.27

What is the shape of such a production process in the single input case? How does this compare to the shape of the vertical slice of the 3-dimensional production frontier along the ray from the origin in our graph?

Answer: The shape of such a one-input production process is the usual shape we employed in Chapter 11: On a graph with labor on the horizontal and output on the vertical, the production frontier initially increases at an increasing rate (as it becomes easier and easier to produce additional output) but eventually increases at a decreasing rate (as it becomes increasingly hard to produce additional output.) This is exactly the same shape as the slice along a ray from the origin of the 2-input production process that has initially increasing and eventual decreasing returns to scale.

Exercise 12A.28

True or False: If a producer minimizes costs, she does not necessarily maximize profits, but if she maximizes profits, she also minimizes costs. (*Hint*: Every point on the cost curve is derived from a producer minimizing the cost of producing a certain output level.)

Answer: True. Any production plan that is represented along the cost curve is cost minimizing, but only the plan where $p = MC$ is profit maximizing. But since the profit maximizing point is derived from the cost curve, it implicitly is also cost minimizing.

Exercise 12A.29

Suppose a production process begins initially with increasing returns to scale, eventually assumes constant returns to scale but never has decreasing returns. Would the MC curve ever cross the AC curve?

Answer: No, it would never cross AC. The MC and AC curves would start at the same place, with MC falling faster than AC along the increasing returns to scale portion of production. When we reach the constant returns to scale portion, MC would become flat, and AC would continue to fall at a decreasing rate as it converges (but never quite reaches) the flat MC curve.

Exercise 12A.30

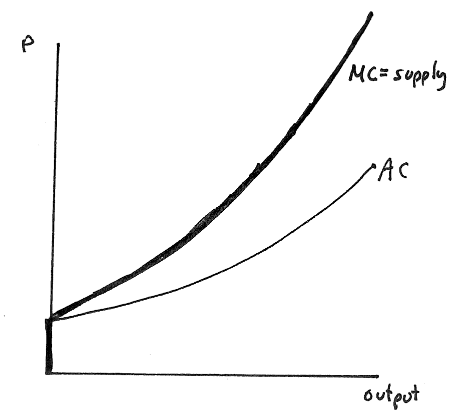
Another special case is the one graphed in Graph 12.7. What are the optimal supply choices for such a producer as the output price changes?

Answer: When $p^* = MC$, any output quantity would be optimal; when $p^* < MC$, it is optimal to produce zero (since profit would be negative); when $p^* > MC$, it would be optimal to produce an infinite amount (since you can keep making profit on each additional unit produced. Thus, the supply curve would lie on the vertical axis between $p = 0$ and $p = p^*$, horizontal at p^* and "vertical at infinity" for $p > p^*$.

Exercise 12A.31

Illustrate the output supply curve for a producer whose production frontier has decreasing returns to scale throughout (such as the case illustrated in Graph 12.1).

Answer: This is illustrated in Exercise Graph 12A.31. Decreasing returns to scale lead to a MC curve that is increasing throughout. Since it begins where AC begins, the entire MC curve lies above AC — and thus the entire MC curve is the supply curve.



Exercise Graph 12A.31 : Supply Curve with Decreasing Returns to Scale

12B Solutions to Within-Chapter-Exercises for Part B

Exercise 12B.1

Just as we can take the partial derivative of a production function with respect to one of the inputs (and call it the "marginal product of the input"), we could take the partial derivative of a utility function with respect to one of the consumption goods (and call it the "marginal utility from that good"). Why is the first of these concepts economically meaningful but the second is not?

Answer: This is because utility is not objectively measurable whereas output is. It is therefore meaningful to ask "how much additional output will one more unit of labor produce", but it is not meaningful to ask "how much additional utility will one more unit of good x yield."

Exercise 12B.2

Using the same method employed to derive the formula for MRS from a utility function, derive the formula for TRS from a production function $f(\ell, k)$.

Answer: The technical rate of substitution (TRS) is simply the change in k divided by the change in ℓ such that output remains unchanged, or

$$\frac{\Delta k}{\Delta \ell} \text{ such that } \Delta x = 0. \tag{12B.2.i}$$

Actually, what we mean by a technical rate of substitution is somewhat more precise — we are not looking for just *any* combination of changes in k and ℓ (such that Δx=0). Rather, we are looking for small changes that define the slope around a particular point. Such small changes are denoted in calculus by using "d" instead of "Δ". Thus, we can re-write (12B.2.i) as

$$\frac{dk}{d\ell} \text{ such that } dx = 0. \tag{12B.2.ii}$$

Changes in output arise from the combined change in k and ℓ, and this is expressed as the total differential (dx)

$$dx = \frac{\partial f}{\partial \ell} d\ell + \frac{\partial f}{\partial k} dk. \tag{12B.2.iii}$$

Since we are interested in changes in input bundles that result in no change in output (thus leaving us on the same isoquant), we can set expression (12B.2.iii) to zero

$$\frac{\partial f}{\partial \ell} d\ell + \frac{\partial f}{\partial k} dk = 0 \tag{12B.2.iv}$$

and then solve out for $dk/d\ell$ to get

$$\frac{dk}{d\ell} = -\frac{(\partial f/\partial \ell)}{(\partial f/\partial k)}. \tag{12B.2.v}$$

Since this expression for $dk/d\ell$ was derived from the expression $dx = 0$, it gives us the equation for small changes in k divided by small changes in ℓ such that production remains unchanged — which is precisely our definition of a technical rate substitution.

Exercise 12B.3

True or False: Producer choice sets whose frontiers are characterized by quasiconcave functions have the following property: All horizontal slices of the choice sets are convex sets.

Answer: This is true — the horizontal slices of the quasiconcave functions are isoquants that satisfy the "averages are better than extremes" property — which means the set of production plans that lie above the isoquant (and thus inside the producer choice set) is convex.

Exercise 12B.4

True or False: All quasiconcave production functions — but not all concave production functions — give rise to convex producer choice sets.

Answer: This is false. Since all concave production functions are also quasiconcave, whatever holds for quasiconcave production functions must hold for concave productions. The statement would be true of the terms "quasiconcave" and "concave" switched places.

Exercise 12B.5

True or False: Both quasiconcave and concave production functions represent production processes for which the "averages are better than extremes" property holds.

Answer: This is true. We have shown that quasiconcave production functions give rise to producer choice sets whose horizontal slices are convex sets — which in turn implies that the isoquants have the usual shape that satisfies "averages are better than extremes." And since all concave functions are also quasiconcave, the same must hold for concave production functions.

Exercise 12B.6

Verify the last statement regarding Cobb-Douglas production functions.

Answer: The Cobb-Douglas production function takes the form $f(\ell,k) = \ell^{\alpha}k^{\beta}$. When we multiply a given input bundle (ℓ,k) by some factor t, we get

$$f(t\ell, tk) = (t\ell)^\alpha (tk)^\beta = t^{(\alpha+\beta)}\ell^\alpha k^\beta = t^{(\alpha+\beta)} f(\ell, k). \tag{12B.6}$$

When $\alpha+\beta = 1$, this equation tells us that increasing the inputs be a factor of t results in an increase of output by a factor of t — which is the definition of constant returns to scale. When $\alpha+\beta < 1$, the equation tells us that such an increase in inputs will result in less than a t-fold increase in output — which is the definition of decreasing returns to scale. And when $\alpha+\beta > 1$, output increases by more that t-fold — giving us increasing returns to scale.

Exercise 12B.7

Can you give an example of a Cobb-Douglas production function that has increasing marginal product of capital and decreasing marginal product of labor? Does this production function have increasing, constant or decreasing returns to scale?

Answer: In order for the example to work, the function $f(\ell, k) = \ell^\alpha k^\beta$ would have to be such that $\beta > 1$ (to get increasing marginal product of capital) and $\alpha < 1$ (to get decreasing marginal product of labor). Since we would still have $\alpha > 0$, this implies that $\alpha+\beta > 1$ — i.e. the production function has increasing returns to scale. This should make intuitive sense: If I can increase just *one* input t-fold and get a greater than t-fold increase in output (as I can if the marginal product of capital is increasing), then I can certainly increase *both* inputs t-fold and get more than a t-fold increase in output. So — as long as we have increasing marginal product in one input, we have increasing returns to scale.

Exercise 12B.8

True or False: It is not possible for a Cobb-Douglas production process to have decreasing returns to scale and increasing marginal product of one of its inputs.

Answer: This follows immediately from our answer to the previous exercise: Increasing marginal product in the Cobb-Douglas production function implies an exponent greater than 1 — but that implies a sum of exponents greater than 1 which is in turn equivalent to increasing returns to scale. Therefore the statement is true — you cannot have decreasing returns to scale and increasing marginal product at the same time.

Exercise 12B.9

In a 3-dimensional graph with x on the vertical axis, can you use the equation (12.18) to determine the vertical intercept of an isoprofit curve $P(\pi, p, w, r)$? What about the slope when k is held fixed?

Answer: At the vertical intercept, $k = \ell = 0$ — which implies the equation simply becomes $\pi = px$ or $x = \pi/p$ which is the intercept on the vertical (x) axis. When k is held fixed at, say, $\overline{k}$, The equation becomes $\pi = px - w\ell - r\overline{k}$. Rearranging terms, we can write this as

$$x = \left(\frac{\pi + r\overline{k}}{p}\right) + \frac{w}{p}\ell. \tag{12B.9}$$

This is then an equation of the part of the production frontier that falls on the vertical slice that holds k fixed at $\overline{k}$. It has an intercept equal to the term in parenthesis, and it's slope is w/p.

Exercise 12B.10

Define profit and isoprofit curves for the case where land L is a third input and can be rented at a price r_L.

Answer: Profit is then simply

$$\pi = px - w\ell - rk - r_L L, \tag{12B.10.i}$$

and the isoprofit plane P is

$$P(\pi, p, w, r, r_L) = \left\{(x, \ell, k, L) \in \mathbb{R}^4 \mid \pi = px - w\ell - rk - r_L L\right\}. \tag{12B.10.ii}$$

Exercise 12B.11

Demonstrate that the problem as written in (12.20) gives the same answer.

Answer: Setting up the Lagrange function for this problem gives

$$\mathcal{L}(x, \ell, k, \lambda) = px - w\ell - rk + \lambda(x - f(\ell, k)), \tag{12B.11.i}$$

which results in first order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= p + \lambda = 0,\\
\frac{\partial \mathcal{L}}{\partial \ell} &= -w - \lambda \frac{\partial f(\ell, k)}{\partial \ell} = 0,\\
\frac{\partial \mathcal{L}}{\partial k} &= -r - \lambda \frac{\partial f(\ell, k)}{\partial k} = 0,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= x - f(\ell, k) = 0.
\end{aligned} \tag{12B.11.ii}$$

Solving the first of these equations for $\lambda = -p$, substituting this into the second and third equations and rearranging terms then gives

$$w = p\frac{\partial f(\ell, k)}{\partial \ell} \;\text{ and }\; r = p\frac{\partial f(\ell, k)}{\partial k}, \tag{12B.11.iii}$$

which can further be written as

$$w = pMP_\ell = MRP_\ell \;\text{ and }\; r = pMP_k = MRP_k. \tag{12B.11.iv}$$

Exercise 12B.12

Demonstrate that solving the problem as defined in equation (12.27) results in the same solution.

Answer: The Lagrange function for this problem is

$$\mathcal{L}(x,\ell,k,\lambda) = px - w\ell - rk + \lambda(x - 20\ell^{2/5}k^{2/5}). \tag{12B.12.i}$$

The first order conditions for this problem are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= p + \lambda = 0,\\
\frac{\partial \mathcal{L}}{\partial \ell} &= -w - 8\lambda\ell^{-3/5}k^{2/5} = 0,\\
\frac{\partial \mathcal{L}}{\partial k} &= -r - 8\lambda\ell^{2/5}k^{-3/5} = 0,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= x - 20\ell^{2/5}k^{2/5} = 0.
\end{aligned} \tag{12B.12.ii}$$

Plugging the $\lambda = -p$ (derived from the first equation) into the second and third equations then gives the condition that input prices are equal to marginal revenue products:

$$w = 8p\ell^{-3/5}k^{2/5} \text{ and } r = 8p\ell^{2/5}k^{-3/5}. \tag{12B.12.iii}$$

From this point forward, the problem solves out exactly as in the text. Solving the second of the two equations for k and plugging it into the first, we get the labor demand function

$$\ell(p,w,r) = \frac{(8p)^5}{r^2w^3}, \tag{12B.12.iv}$$

and plugging this in for ℓ in the second equation, we get the capital demand function

$$k(p,w,r) = \frac{(8p)^5}{w^2r^3}. \tag{12B.12.v}$$

Finally, we can derive the output supply function by plugging equations (12B.12.iv) and (12B.12.v) into the production function $f(\ell,k) = 20\ell^{2/5}k^{2/5}$ to get

$$x(p,w,r) = 20\frac{(8p)^4}{(wr)^2} = 81920\frac{p^4}{(wr)^2}. \tag{12B.12.vi}$$

Exercise 12B.13

Each panel of Graph 12.12 illustrates one of three "slices" of the respective function through the production plan ($x = 1280, \ell = 128, k = 256$). What are the other two slices for each of the three functions? Do they slope up or down?

Answer: For the supply function, the other two slices are

$$
\begin{aligned}
x(5,w,10) &= 81920\frac{5^4}{(10w)^2} = \frac{512000}{w^2} \text{ and} \\
x(5,20,r) &= 81920\frac{5^4}{(20r)^2} = \frac{128000}{r^2},
\end{aligned}
\tag{12B.13.i}
$$

both of which slope down. This makes sense: As input prices increase, less output is produced. For the labor demand function, the two other slices are

$$
\ell(p,20,10) = \frac{(8p)^5}{(10^2)(20^3)} \approx 0.0401p^5 \text{ and } \ell(5,20,r) = \frac{(8(5))^5}{20^3 r^2} = \frac{12800}{r^2}. \tag{12B.13.ii}
$$

The slope is positive for the first and negative for the second. Thus, labor demand increases as output price increases but decreases as the rental rate of capital increases.

Finally, for the capital demand function, the other two slices are

$$
k(p,20,10) = \frac{(8p)^5}{20^2(10^3)} \approx 0.082p^5 \text{ and } k(5,w,10) = \frac{(8(5))^5}{10^3 w^2} = \frac{102400}{w^2}. \tag{12B.13.iii}
$$

Again, the slope is positive for the first and negative for the second of these. Thus, capital demand increases as output price increases but decreases as wage increases.

Exercise 12B.14

Did we calculate a "conditional labor demand" function when we did cost minimization in the one-input model?

Answer: Yes, but we did not have to solve a "cost minimization" problem to do so. The only reason we need to solve a cost minimization problem now is that there are many technologically efficient production plans for each output level to choose from — and the problem allows us to determine which of these is the cheapest for a given set of input prices. In the one-input model, there was only one technologically efficient way of producing each output level — so we already knew that this was the cheapest way to produce. Thus, all we needed to do was invert the production function $x = f(\ell)$ — so that we could get the function $\ell(x)$ that told us how much labor input we needed to produce any output level. This function was then our "conditional labor demand" function — it told us, conditional on how much we want to produce, how much labor we will demand. In this case, input price was not part of the function because we knew that we would need that much labor to produce each output level no matter what the input price.

Exercise 12B.15

Why are the conditional input demand functions not a function of output price p?

Answer: Conditional input demands tell us least cost way of producing some output level x. The output price has no relevance for determining what the least cost way of producing is — it is only relevant for determining how much we should produce in order to maximize the difference between cost and revenue. Thus, only unconditional input demands are a function of output price.

Exercise 12B.16

Suppose you are determined to produce a certain output quantity $\overline{x}$. If the wage rate goes up, how will your production plan change? What if the rental rate goes up?

Answer: We can take the partial derivatives of the input demand functions with respect to wage to get

$$\frac{\partial \ell(w,r,\overline{x})}{\partial w}=\frac{-r^{1/2}}{2w^{3/2}}\left(\frac{x}{20}\right)^{5/4}<0 \text{ and } \frac{\partial k(w,r,\overline{x})}{\partial w}=\frac{1}{(wr)^{1/2}}\left(\frac{x}{20}\right)^{5/4}>0. \quad (12B.16)$$

Thus, when w increases, you will substitute away from labor and toward capital. The reverse holds if r increases (for similar reasons.)

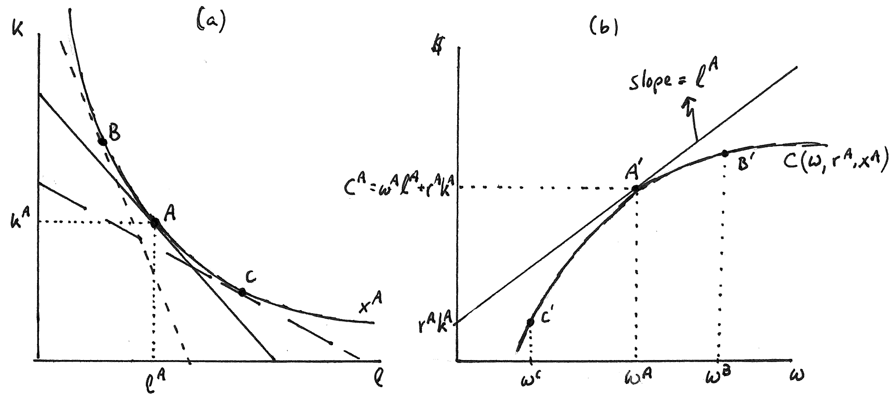
Exercise 12B.17

Can you replicate the graphical proof of the concavity of the expenditure function in the Appendix to Chapter 10 to prove that the cost function is concave in w and r?

Answer: The relevant section in the Appendix to Chapter 10 begins with "Suppose that a consumer initially consumes a bundle A when prices of x_1 and x_2 are p_1^A and p_2^A, and suppose that the consumer attains utility level u^A as a result." Let's re-write this sentence to make it apply to the producer's cost minimization problem: "Suppose that a *producer* initially *employs* a bundle A when prices of ℓ and k are w^A and r^A, and suppose that the *producer produces an output* level x^A as a result." This input bundle A is graphed in panel (a) of Exercise Graph 12B.17 where the slope of the (solid) isocost tangent to the x^A isoquant is $-w^A/r^A$.

The lowest cost at which x^A can be produced when input prices are w^A and r^A is therefore $C(w^A,r^A,x^A)=C^A=w^A\ell^A+r^Ak^A$. This is plotted in panel (b) of the graph where w is graphed on the horizontal and cost is graphed on the vertical axis. Since r^A and x^A are held fixed, we are in essence going to graph the slice of the cost function along which w varies. So far, we have plotted only one such point labeled A'.

Now suppose that w increases. If the producer does not respond by changing her input bundle, her cost will be given by the equation $C=r^Ak^A+w\ell^A$ as w changes — and this is just the equation of a line with intercept r^Ak^A and slope ℓ^A.



Exercise Graph 12B.17 : Concavity of $C(w,r,x)$ in w

This line is plotted in panel (b) of the graph and represents the costs as w changes assuming the producer naively stuck with the same input bundle (ℓ^A, k^A). But of course the producer does not do this — because she can reduce her costs by substituting away from labor and toward more capital as she slides to the new cost-minimizing input bundle B that has the new (steeper) isocost tangent to the x^A isoquant. Thus, as w increases to w^B, her costs will go up by *less* than the naive linear cost line in panel (b) suggests. The same logic implies that the producer's costs will fall by more than what is indicated by the line if w falls to w^C. This results in the cost function slice $C(w, r^A, x^A)$ taking on the concave shape in the graph. Put differently, even if the producer never substituted toward inputs that have become relatively cheaper and away from inputs that have become relatively more expensive, this slice of the cost function would be a straight line (and thus "weakly" concave). Any ability to substitute between inputs then causes the strict concavity we have derived. The same logic applies to changes in r.

Exercise 12B.18

What is the elasticity of substitution between capital and labor if the relationships in equation (12.51) hold with equality?

Answer: If these relationships hold with equality, then this implies that a cost-minimizing producer will not change her input bundle to produce a given output level as input prices change. In other words, as some inputs become relatively cheaper and others relatively more expensive, the producer does not substitute away from the more expensive to the cheaper. This can only be cost-minimizing if in fact the technology is such that substituting between inputs is not possible — which is the same as saying that the elasticity of substitution is zero.

Exercise 12B.19

Demonstrate how these indeed result from an application of the Envelope Theorem.

Answer: Substituting the constraint into the objective, we can write the profit maximization problem in an unconstrained form; i.e.

$$\max_{\ell,k} \pi = pf(\ell,k) - w\ell - rk. \tag{12B.19.i}$$

The "Lagrangian" is then simply equal to $\mathscr{L} = pf(\ell,k) - w\ell - rk$ (since there is no constraint to be multiplied by λ. The solution to the optimization problem is $\ell(w,r,p$ and $k(w,r,p)$. Substituting this solution into the objective function, we get the profit function $\pi(w,r,p)$ that tells us profit or any combination of prices (assuming the producer is profit maximizing). The envelope theorem then tells us that the derivative of this profit function with respect to a parameter (such as input and output prices) is equal to the derivative of the Lagrangian (which is just equal to the π expression in our optimization problem) with respect to that parameter *evaluated at the optimum* — i.e. evaluated at $\ell(w,r,p)$ and $k(w,r,p$. Thus,

$$\frac{\pi(w,r,p)}{\partial w} = \frac{\partial \mathscr{L}}{\partial w}\Big|_{\ell(w,r,p),k(w,r,p)} = -\ell\Big|_{\ell(w,r,p),k(w,r,p)} = -\ell(w,r,p), \tag{12B.19.ii}$$

and

$$\frac{\pi(w,r,p)}{\partial r} = \frac{\partial \mathscr{L}}{\partial r}\Big|_{\ell(w,r,p),k(w,r,p)} = -k\Big|_{\ell(w,r,p),k(w,r,p)} = -k(w,r,p). \tag{12B.19.iii}$$

Finally,

$$\begin{aligned}\frac{\pi(w,r,p)}{\partial r} &= \frac{\partial \mathscr{L}}{\partial p}\Big|_{\ell(w,r,p),k(w,r,p)} = f(\ell,k)\Big|_{\ell(w,r,p),k(w,r,p)} \\ &= f\big(\ell(w,r,p),k(w,r,p)\big) = x(w,r,p).\end{aligned} \tag{12B.19.iv}$$

Exercise 12B.20

How can you tell from panel (a) of the graph that $\pi(x^B,\ell^B) > \pi' > \pi(x^A,\ell^A)$?

Answer: The intercept of the new (magenta) isoprofit is higher than the intercept of the original (blue) isoprofit. Let the new intercept be denoted π^B/p^B and the original intercept as π^A/p^A. We know that

$$\frac{\pi^B}{p^B} > \frac{\pi^A}{p^A} \text{ and } p^B > p^A, \tag{12B.20.i}$$

which can be true only if $\pi^B > \pi^A$. Similarly,

$$\frac{\pi'}{p^B} > \frac{\pi^A}{p^A} \text{ and } p^B > p^A \text{ implies } \pi' > \pi^A. \tag{12B.20.ii}$$

Finally,

$$\frac{\pi'}{p^B} > \frac{\pi'}{p^B} \text{ implies } \pi^B > \pi'. \tag{12B.20.iii}$$

These three conclusions together imply $\pi^B > \pi' > \pi^A$.

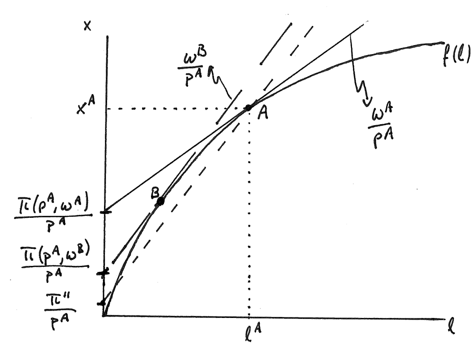
Exercise 12B.21

Use a graph similar to that in panel (a) of Graph 12.14 to motivate Graph 12.15.

Answer: This is done in Exercise Graph 12B.21 where the short run production function $f(\ell)$ is plotted with the originally optimal production plan (ℓ^A, x^A) at the original prices (w^A, p^A). An increase in the wage to w^B causes isoprofits to become steeper — with B becoming the new profit maximizing production plan. Had the producer not responded by changing her production plan, she would have operated on the steeper isoprofit that does through A rather than the one that goes through B — and would have made profit π'' instead of $\pi(p^A, w^B)$. Since the intercepts of the three isoprofits all have p^A in the denominator, it is immediate from the picture that

$$\pi(p^A, w^A) > \pi(p^A, w^B) > \pi'', \tag{12B.21}$$

exactly as in the graph of the text.



Exercise Graph 12B.21 : Deriving the convexity of the profit function in w

12C Solutions to Odd Numbered End-of-Chapter Exercises

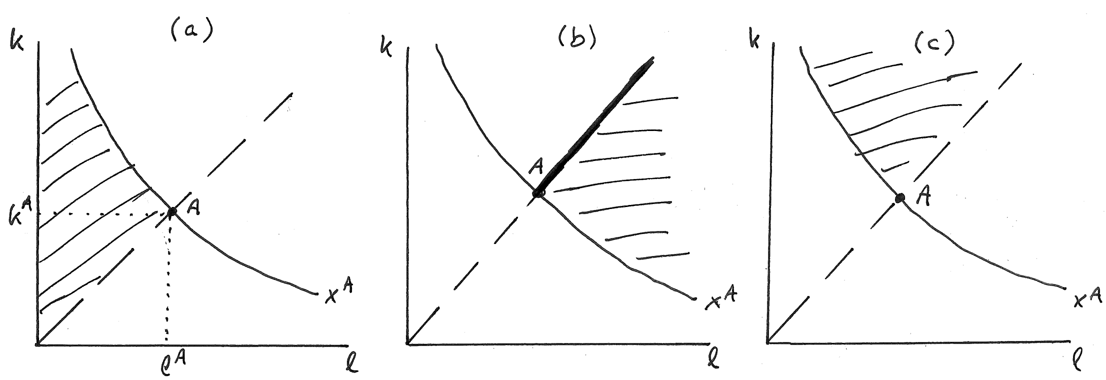
Exercise 12.1

In our development of producer theory, we have found it convenient to assume that the production technology is homothetic.

A: *In each of the following, assume that the production technology you face is indeed homothetic. Suppose further that you currently face input prices* (w^A, r^A) *and output price* p^A *— and that, at these prices, your profit maximizing production plan is* $A = (\ell^A, k^A, x^A)$*.*

(a) *On a graph with* ℓ *on the horizontal and* k *on the vertical, illustrate an isoquant through the input bundle* (ℓ^A, k^A)*. Indicate where all cost minimizing input bundles lie given the input prices* (w^A, r^A)*.*

Answer: This is depicted in panel (a) of Exercise Graph 12.1. Since the isocosts must be tangent at the profit maximizing input bundle A, homotheticity implies that all tangencies of isocosts with isoquants lie on the ray from the origin that passes through A.



Exercise Graph 12.1 : Changing Prices and Profit Maximization

(b) *Can you tell from what you know whether the shape of the production frontier exhibits increasing or decreasing returns to scale along the ray you indicated in (a)?*

Answer: You cannot tell whether the production frontier has increasing or decreasing returns to scale along the entire ray from the origin.

(c) *Can you tell whether the production frontier has increasing or decreasing returns to scale around the production plan* $A = (\ell^A, k^A, x^A)$*?*

Answer: Yes, you can tell that it must have decreasing returns to scale at A — because the isoprofit must be tangent at that point in order for A to be the profit maximizing production plan.

(d) *Now suppose that wage increases to w'. Where will your new profit maximizing production plan lie relative to the ray you identified in (a)?*

Answer: When w increases, the isocosts become steeper — which implies that they are tangent to the isoquants to the left of the ray that goes through A. Thus, the new ray on which all cost minimizing production plans lie is steeper than the ray drawn in panel (a) of Exercise Graph 12.1. Since the new profit maximizing production plan must lie on that ray (because profit maximization implies cost minimization), the new profit maximizing production plan must lie to the left of the ray that passes through A.

(e) *In light of the fact that supply curves shift to the left as input prices increase, where will your new profit maximizing input bundle lie relative to the isoquant for x^A?*

Answer: The leftward shift of supply curves as w increases implies that the profit maximizing output level falls. Thus, the new profit maximizing input bundle must lie *below* the x^A isoquant.

(f) *Combining your insights from (d) and (e), can you identify the region in which your new profit maximizing bundle will lie when wage increases to w'?*

Answer: This is illustrated as the shaded area in panel (a) of Exercise Graph 12.1. The shaded area emerges from the insight in (d) that the new profit maximizing bundle lies to the *left* of the ray through A and from the insight in (e) that it must lie *below* the isoquant for x^A.

(g) *How would your answer to (f) change if wage fell instead?*

Answer: If wage falls instead, then the isocosts become shallower — which implies that all cost minimizing bundles will now lie to the *right* of the ray through A. A drop in w will furthermore shift the output supply curve to the right — which implies that the profit maximizing production plan will involve an increase in the production of x. Thus, the new profit maximizing plan must lie to the *right* of the ray through A (because profit maximization implies cost minimization) and it must lie *above* the isoquant for x^A (because output increases). This is indicated as the shaded area in panel (b) of Exercise Graph 12.1.

(h) *Next, suppose that, instead of wage changing, the output price increases to p'. Where in your graph might your new profit maximizing production plan lie? What if p decreases?*

Answer: When output price p changes, the slopes of the isocosts (which are equal to $-w/r$) remain unchanged. Thus, all cost minimizing production plans remain on the ray through A. Since supply curves slope up, an increase in p will cause an increase in output — implying that the new profit maximizing production plan lies *above* the isoquant for x^A.

Thus, when p increases, the new profit maximizing production plan lies on the bold portion of the ray through A as indicated in panel (b) of Exercise Graph 12.1. When p decreases, on the other hand, output falls — which implies that the new profit maximizing production plan lies on the dashed portion of the ray through A in panel (b) of the graph.

(i) *Can you identify the region in your graph where the new profit maximizing plan would lie if instead the rental rate r fell?*

Answer: If r falls, the isocosts become steeper — implying the ray containing all cost minimizing production plans will be steeper than the ray through A. Thus, cost minimization implies that the new profit maximizing input bundle will lie to the *left* of the ray through A. A decrease in r further implies a shift in the supply curve to the right — which implies that output will increase. Thus, the profit maximizing input bundle must lie *above* the isoquant for x^A. This gives us the region to the *left* of the ray through A and *above* the isoquant x^A — which is equal to the shaded region in panel (c) of Exercise Graph 12.1.

B: *Consider the Cobb-Douglas production function $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ with $\alpha,\beta > 0$ and $\alpha+\beta < 1$.*

(a) *Derive the demand functions $\ell(w,r,p)$ and $k(w,r,p)$ as well as the output supply function $x(w,r,p)$.*

Answer: These result from the profit maximization problem

$$\max_{\ell,k,x} px - w\ell - rk \text{ subject to } x = A\ell^{\alpha}k^{\beta} \tag{12.1.i}$$

which can also be written as

$$\max_{\ell,k} pA\ell^{\alpha}k^{\beta} - w\ell - rk. \tag{12.1.ii}$$

Taking first order conditions and solving these, we then get input demand functions

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \tag{12.1.iii}$$

Plugging these into the production function and simplifying, we also get the output supply function

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \tag{12.1.iv}$$

(b) *Derive the conditional demand functions $\ell(w,r,x)$ and $k(w,r,x)$.*

Answer: We need to solve the cost minimization problem

$$\min_{\ell,k}\ w\ell + rk \text{ subject to } x = A\ell^{\alpha}k^{\beta}. \tag{12.1.v}$$

Setting up the Lagrangian and solving the first order conditions, we then get the conditional input demand functions

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)} \left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \text{ and } k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)} \left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.1.vi}$$

(c) *Given some initial prices (w^A, r^A, p^A), verify that all cost minimizing bundles lie on the same ray from the origin in the isoquant graph.*

Answer: Dividing the conditional input demands by one another, we get

$$\frac{k(w^A, r^A, x)}{\ell(w^A, r^A, x)} = \frac{\beta w^A}{\alpha r^A}. \tag{12.1.vii}$$

Thus, regardless of what isoquant x we try to reach, the ratio of capital to labor that minimizes the cost of reaching that isoquant is independent of x — implying that all cost minimizing input bundles lie on a ray from the origin.

(d) *If w increases, what happens to the ray on which all cost minimizing bundles lie?*

Answer: If w increases to w', the ratio of capital to labor becomes

$$\frac{\beta w'}{\alpha r^A} > \frac{\beta w^A}{\alpha r^A}; \tag{12.1.viii}$$

i.e. the ray becomes steeper as firms substitute away from labor and toward capital.

(e) *What happens to the profit maximizing input bundles?*

Answer: We see from the input demand equations in (12.1.iii) that both labor and capital demand fall as w increases. (Similarly, we see from equation (12.1.iv) that output supply falls.)

(f) *How do your answers change if w instead decreases?*

Answer: When wage falls to w'', we get that the ray on which cost minimizing bundles occur is

$$\frac{\beta w''}{\alpha r^A} < \frac{\beta w^A}{\alpha r^A}; \tag{12.1.ix}$$

i.e. the ray becomes shallower. From the input demand functions, we also see that demand for labor and capital increase — as does output (as seen in the output supply function).

(g) *If instead p increases, does the ray along which all cost minimizing bundles lie change?*

Answer: The ray along which cost minimizing bundles lie is defined by the ratio of conditional capital to conditional labor demand — which is

$$\frac{k(w,r,x)}{\ell(w,r,x)} = \frac{\beta w}{\alpha r}. \tag{12.1.x}$$

Since this does not depend on p, we can see that the ray does not depend on output price. This should make sense: Cost minimization does not take output price into account since all it asks is: "what is the least cost way of producing x?"

(h) *Where on that ray will the profit maximizing production plan lie?*

Answer: Since the ray of cost minimizing input bundles remains unchanged, we know that the new profit maximizing plan lies somewhere on that ray. From the output supply equation (12.1.iv), we see that output increases with p. Thus, the new profit maximizing production plan lies above the initial isoquant and on the same ray as the initial profit maximizing production plan.

(i) *What happens to the ray on which all cost minimizing input bundles lie if r falls? What happens to the profit maximizing input bundle?*

Answer: If r falls to r', we get

$$\frac{\beta w^A}{\alpha r'} > \frac{\beta w^A}{\alpha r^A}; \tag{12.1.xi}$$

i.e. the ray on which cost minimizing input bundles lie will be steeper as firms substitute toward capital and away from labor. From the output supply equation (12.1.iv), we can also see that a decrease in r results in an increase in output — thus, the new profit maximizing input bundle lies above the initial isoquant and to the left of the initial ray along which cost minimizing input bundles occurred.

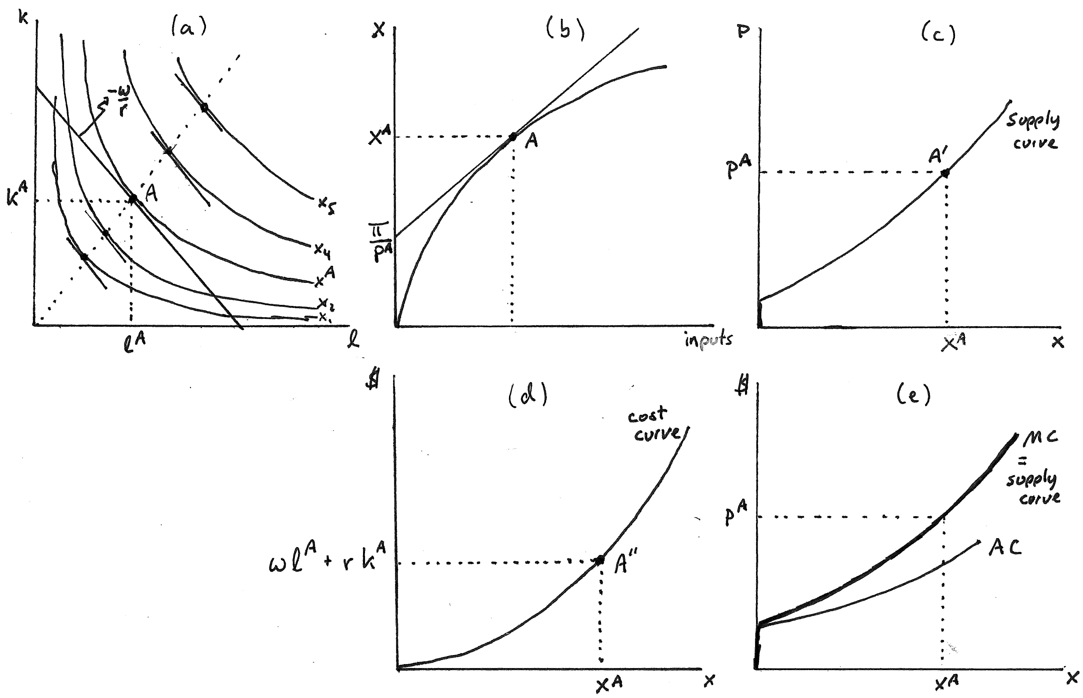
Exercise 12.3

Consider again the two ways in which we can view the producer's profit maximization problem.

A: *Suppose a homoethetic production technology involves two inputs, labor and capital, and that its producer choice set is fully convex.*

(a) *Illustrate the production frontier in an isoquant graph with labor on the horizontal axis and capital on the vertical.*

Answer: This is done in panel (a) of Exercise Graph 12.3. Since the producer choice set is convex, the horizontal slices represented by the isoquants must have the usual convex shape. In addition, the homotheticity property implies that the slopes (or TRS) of the isoquants are the same along any ray from the origin.



Exercise Graph 12.3 : 2 Ways to Derive Output Supply

(b) *Does this production process have increasing or decreasing returns to scale? How would you be able to see this on an isoquant graph like the one you have drawn?*

<u>Answer</u>: It has decreasing returns to scale — because the entire producer choice set is convex. You would only see this in an isoquant map if the isoquants are accompanied by output numbers that increase at a decreasing rate along any ray from the origin.

(c) *For a given wage w and rental rate r, show in your graph where the cost minimizing input bundles lie. What is true at each such input bundle?*

<u>Answer</u>: The input prices give us the slope of the isocost lines — which is $(-w/r)$. The isocost drawn in panel (a) is tangent at A — implying that (ℓ^A, k^A) is the cheapest input bundle that can produce the output level x^A. Since the production process is homothetic, it implies that all isoquants have the same slope along the ray from the origin through A. This further implies that all cost minimizing input bundles for the various output levels (represented by the isoquants) lie on this ray. Put differently, along this ray it is always true that $TRS = -w/r$ — the condition for cost minimization.

(d) *On a separate graph, illustrate the vertical slice (of the production frontier) that contains all these cost minimizing input bundles.*

Answer: Panel (b) of Exercise Graph 12.3 illustrates this vertical slice whose shape emerges from the decreasing returns to scale of the production process.

(e) *Assuming output can be sold at p^A, use a slice of the isoprofit plane to show the profit maximizing production plan. What, in addition to what is true at all the cost-minimizing input bundles, is true at this profit maximizing plan?*

Answer: This is also illustrated in panel (b) where the slice of the isoprofit plane is tangent at A. Since this is the profit maximizing plan, it must also be true that $p^A MP_\ell^A = w$ and $p^A MP_k^A = r$ — i.e. the marginal revenue product of each input is equal to that input's price. (Of course this automatically implies that $TRS^A = -w/r$ — which can be shown by simply dividing the two previous profit maximizing conditions by each other.)

(f) *If output price changes, would you still profit maximize on this vertical slice of the production frontier? What does the supply curve (which plots output on the horizontal and price on the vertical) look like?*

Answer: Yes, you would still produce on the same slice. This can be seen in panel (a) — a change in p changes nothing in panel (a). Thus, the cost minimizing input bundles remain unchanged, and — since profit maximization implies cost minimization — the profit maximizing plan must therefore lie on this slice. As p changes, the slope of the isoprofit line in (b) changes, becoming steeper when p falls and shallower when it rises. Thus, as p increases, output increases — and as p decreases, output decreases. This results in a shape for the supply curve as drawn in panel (c) of Exercise Graph 12.3.

(g) *Now illustrate the (total) cost curve (with output on the horizontal and dollars on the vertical axis). How is this derived from the vertical slice of the production frontier that you have drawn before?*

Answer: The vertical slice of the production frontier in panel (b) illustrates that it gets increasingly difficult to produce additional units of output as the inputs are increased in proportion to one another. This implies that the cost of increasing output will rise faster and faster as output increases — giving us the shape for the cost curve in panel (d) of Exercise Graph 12.3. This shape is essentially the inverse of the shape of the production frontier slice in (a). For the output quantity x^A, for instance, this cost is simply calculated by going back to panel (a) and checking how much of each input is required to produce x^A. We then multiply each input quantity by how much that input costs per unit to determine the total cost of producing x^A.

(h) *Derive the marginal and average cost curves and indicate where in your picture the supply curve lies.*

Answer: This is done in panel (e) of Exercise Graph 12.3 where the MC is simply the slope of the (total) cost curve from (d) — a slope that starts small (i.e. shallow) and becomes increasingly large (i.e. steeper). As always, the AC begins where the MC. Since MC is increasing throughout, this implies that AC always lies below MC. The supply curve is then, as always, the part of the MC curve that lies above the AC.

(i) *Does the supply curve you drew in part (f) look similar to the one you drew in part (h)?*

Answer: Yes — because the two methods of deriving the supply curve are equivalent.

B: *Suppose that the production technology is fully characterized by the Cobb-Douglas production function $x = f(\ell, k) = A\ell^{\alpha}k^{\beta}$ with $\alpha+\beta<1$ and A, α, and β all greater than zero.*

(a) *Set up the profit maximization problem (assuming input prices w and r and output price p). Then solve for the input demand and output supply functions. (Note: This is identical to parts B(b) and (c) of exercise 12.2 — so if you have solved it there, you can simply skip to part (b) here.)*

Answer: Derived in the usual way, the input demand functions we calculated there are

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \tag{12.3.i}$$

and the output supply function was

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.3.ii}$$

(b) *Now set up the cost minimization problem and solve for the first order conditions.*

Answer: This problem is

$$\min_{\ell,k} \; w\ell + rk \quad \text{subject to} \quad x = A\ell^{\alpha}k^{\beta}. \tag{12.3.iii}$$

The Lagrange function for this problem is

$$\mathcal{L}(\ell,k,\lambda) = w\ell + rk + \lambda\left(x - A\ell^{\alpha}k^{\beta}\right) \tag{12.3.iv}$$

giving rise to first order conditions

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\ell} &= w - \lambda A\alpha\ell^{\alpha-1}k^{\beta} = 0\\
\frac{\partial\mathcal{L}}{\partial k} &= r - \lambda A\beta\ell^{\alpha}k^{\beta-1} = 0\\
\frac{\partial\mathcal{L}}{\partial\lambda} &= x - A\ell^{\alpha}k^{\beta} = 0
\end{aligned} \tag{12.3.v}$$

(c) *Solve for the conditional labor and capital demands.*

Answer: Moving the negative terms in each of the first two first order conditions to the other side, and dividing the two conditions by each other, we get

$$\frac{w}{r}=\frac{\alpha k}{\beta \ell} \quad \text{or} \quad k=\frac{\beta w \ell}{\alpha r}. \tag{12.3.vi}$$

Substituting the latter expression for k into the third first order condition and solving for ℓ, we then get the conditional labor demand function

$$\ell(w,r,x)=\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.3.vii}$$

and substituting this back into the expression for k from equation (12.3.vi), we get the conditional capital demand function

$$k(w,r,x)=\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{12.3.viii}$$

(d) *Derive the cost function and simplify the function as much as you can. (*Hint*: You can check your answer with the cost function given for the same production process in exercise 12.4) Then derive from this the marginal and average cost functions.*

Answer: The cost function is simply the sum of the conditional input demands multiplied by the respective input prices; i.e.

$$\begin{aligned} C(w,r,x) &= w\ell(w,r,x)+rk(w,r,x) \\ &= w\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}+r\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \end{aligned} \tag{12.3.ix}$$

This can be written as

$$C(w,r,x)=\left[w\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}+r\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\right]\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.3.x}$$

which, with a little algebra, simplifies to

$$C(w,r,x)=(\alpha+\beta)\left(\frac{xw^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{12.3.xi}$$

The marginal cost function is then simply the derivative of the cost function with respect to x; i.e.

$$MC = \frac{\partial C(w,r,x)}{\partial x} = \left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \qquad (12.3.xii)$$

Finally, the average cost function is simply

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \qquad (12.3.xiii)$$

(e) *Use your answers to derive the supply function. Compare your answer to what you derived in (a).*

Answer: To derive the supply function, we set price equal to marginal cost and solve for x; i.e. we start with

$$MC = \left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)} = p. \qquad (12.3.xiv)$$

Dividing through by the term in parentheses and taking both sides to the power $(\alpha+\beta)/(1-\alpha-\beta)$, we get

$$x(w,r,p) = \left[p\left(\frac{A\alpha^{\alpha}\beta^{\beta}}{w^{\alpha} r^{\beta}}\right)^{1/(\alpha+\beta)}\right]^{(\alpha+\beta)/(1-\alpha-\beta)} = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha} r^{\beta}}\right)^{1/(1-\alpha-\beta)}, \qquad (12.3.xv)$$

exactly the same as what we derived in (a) through direct profit maximization. (This entire function is the supply function since MC lies above AC everywhere $x > 0$. You can see this by noticing from the AC and MC functions that $AC = (\alpha+\beta)MC$. Since $(\alpha+\beta) < 1$, this implies $AC < MC$ everywhere.)

(f) *Finally, derive the (unconditional) labor and capital demands. Compare your answers to those in (a).*

Answer: We now simply need to substitute $x(w,r,p)$ from above in for x in the conditional input demand equations (12.3.vii) and (12.3.viii) — and once we do that, we get back the unconditional labor and capital demands that are identical to those in the equations (12.3.i) in part (a).

Exercise 12.5

In the absence of recurring fixed costs (such as those in exercise 12.4), the U-shaped cost curves we will often graph in upcoming chapters presume some particular features of the underlying production technology when we have more than 1 input.

A: *Consider the production technology depicted in Graph 12.6 where output is on the vertical axis (that ranges from 0 to 100) and the inputs capital and labor are on the two horizontal axes. (The origin on the graph is the left-most corner).*

(a) *Suppose that output and input prices result in some optimal production plan A (that is not a corner solution). Describe in words what would be true at A relative to what we described as an isoprofit plane at the beginning of this chapter.*

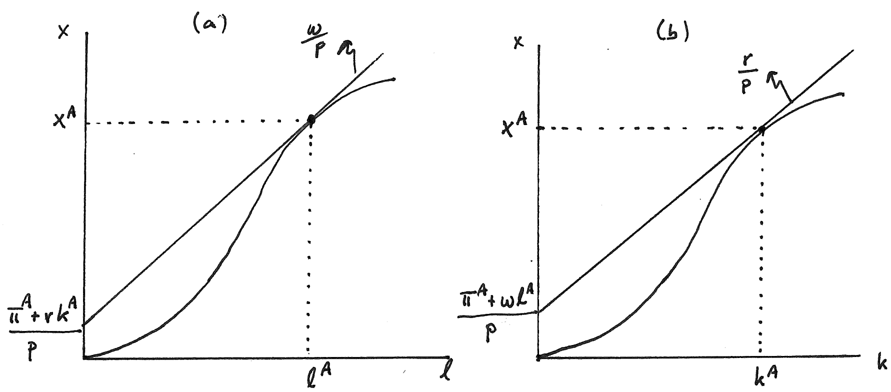
Answer: The isoprofit plane $\pi = px - w\ell - rk$ would have to be tangent to the production frontier — with no other portion of the isoprofit plane intersecting the frontier. It is like a sheet of paper tangent to a "mountain" that is initially getting steeper but eventually becomes shallower. This implies that the isoprofit plane that is tangent at A has a positive vertical intercept.

(b) *Can you tell whether this production frontier has increasing, constant or decreasing returns to scale?*

Answer: The production frontier has initially increasing but eventually decreasing returns to scale — i.e. along every horizontal ray from the origin, the slice of the production frontier has the "sigmoid" shape that we used throughout Chapter 11.

(c) *Illustrate what the slice of this graphical profit maximization problem would look like if you held capital fixed at its optimal level k^A.*

Answer: This is illustrated in panel (a) of Exercise Graph 12.5(1).



Exercise Graph 12.5(1) : Holding k^A and ℓ^A fixed

The tangency of the isoprofit plane shows up as a tangency of the line $x = [(\pi^A + rk^A)/p] + (w/p)\ell$, where the bracketed term is the vertical intercept and the (w/p) term is the slope. (This is just derived from solving the expression $\pi^A = px - w\ell - rk^A$ for x.)

(d) *How would the slice holding labor fixed at its optimal level ℓ^A differ?*

Answer: It would look similar except for re-labeling as in panel (b) of the graph.

(e) *What two conditions that have to hold at the profit maximizing production plan emerge from these pictures?*

Answer: In panels (a) and (b) of Exercise Graph 12.5(1), the slopes of the isoprofit lines are tangent to the slopes of the production frontier with one of the inputs held fixed. The slope of the production frontier at (ℓ^A, x^A) in panel (a) is the marginal product of labor at that production plan; i.e. MP_ℓ^A. And the slope of the production frontier at (k^A, x^A) in panel (b) is the marginal product of capital at that production plan; i.e. MP_k^A. Thus, the conditions that emerge are

$$MP_\ell^A = \frac{w}{p} \text{ and } MP_k^A = \frac{r}{p}. \tag{12.5.i}$$

(f) *Do you think there is another production plan on this frontier at which these conditions hold?*

Answer: Yes — this would occur on the increasing returns to scale portion of the production frontier where an isoprofit "sheet" is tangent to the lower side of the frontier. This "sheet" will, however, have a negative intercept — implying negative profit.

(g) *If output price falls, the profit maximizing production plan changes to once again meet the conditions you derived before. Might the price fall so far that no production plan satisfying these conditions is truly profit maximizing?*

Answer: A decrease in p will cause the isoprofit planes to become steeper — causing the profit maximizing production plan to slide down the production frontier as the tangent isoprofit now happens at a steeper slope. This implies that the vertical intercept also slides down — with profit falling. If the price falls too much, this intercept will become negative — implying that the true profit maximizing production plan becomes (0,0,0). Put differently, if the price falls too much, the firm is better off not producing at all rather than producing at the tangency of an isoprofit with the production frontier.

(h) *Can you tell in which direction the optimal production plan changes as output price increases?*

Answer: As output price increases, the isoprofit plane becomes shallower — which implies that the tangency with the production frontier slides up in the direction of the shallower portion of the frontier. Thus, the production plan will involve more of each input and more output.

B: *Suppose your production technology is characterized by the production function*

$$x = f(\ell, k) = \frac{\alpha}{1 + e^{-(\ell-\beta)} + e^{-(k-\gamma)}} \tag{12.5}$$

where e is the base of the natural logarithm. Given what you might have learned in one of the end-of-chapter exercises in Chapter 11 about the function $x = f(\ell) = \alpha/(1+e^{-(\ell-\beta)})$, *can you see how the shape in Graph 12.16 emerges from this extension of this function?*

Answer: Even though this question was not meant to be answered directly, the graph given in part A of the question depicts this function for the case where $\alpha = 100$ and $\beta = \gamma = 5$. The graph was generated using the software package Mathematica (as are the other machine generated graphs in some of the answers in this Chapter). As you can see, the function takes on the shape that has initially increasing and eventually diminishing slope along slices holding each input fixed (as well as along rays from the origin.) Note that ℓ and k enter symmetrically given that $\beta = \gamma$ — and the two inputs appear on the axes in the plane from which the surface emanates. The vertical axis in the graph is output x.

(a) *Set up the profit maximization problem.*

Answer: The problem is

$$\max_{x,\ell,k} px - w\ell - rk \text{ subject to } x = \frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} \tag{12.5.ii}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell,k} \frac{\alpha p}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} - w\ell - rk. \tag{12.5.iii}$$

(b) *Derive the first order conditions for this optimization problem.*

Answer: We simply take derivatives with respect to w and r and set them to zero. Thus, we get

$$\frac{\alpha p e^{-(\ell-\beta)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = w \text{ and } \frac{\alpha p e^{-(k-\gamma)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = r. \tag{12.5.iv}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ and $z = e^{-(k-\gamma)}$ into the first order conditions. Then, with the first order conditions written with w and r on the right hand sides, divide them by each other and derive from this an expression $y(z,w,r)$ and the inverse expression $z(y,w,r)$.*

Answer: These substitutions lead to the first order conditions becoming

$$\frac{\alpha p y}{(1+y+z)^2} = w \text{ and } \frac{\alpha p z}{(1+y+z)^2} = r. \tag{12.5.v}$$

Dividing the two equations by each other, we can then derive

$$y(z,w,r) = \frac{wz}{r} \text{ and } z(y,w,r) = \frac{ry}{w}. \tag{12.5.vi}$$

(d) *Substitute $y(z,w,r)$ into the first order condition that contains r. Then manipulate the resulting equation until you have it in the form az^2+bz+c (where the terms a, b and c may be functions of w, r, α and p). (*Hint*: It is helpful to multiply both sides of the equation by r.) The quadratic formula then allows you to derive two "solutions" for z. Choose the one that uses* *the negative rather than the positive sign in the quadratic formula as your "true" solution $z^*(\alpha, p, w, r)$.*

Answer: Substituting $y(z, w, r)$ into the second expression in equation (12.5.v) and multiplying both sides by the denominator, we get

$$\alpha pz = r\left(1 + \frac{wz}{r} + z\right)^2. \tag{12.5.vii}$$

Multiplying the right hand side by r lets us reduce it to

$$r^2\left(1 + \frac{wz}{r} + z\right)^2 = (r + wz + rz)^2 = (r + (w + r)z)^2. \tag{12.5.viii}$$

Thus, when we multiply both sides of equation (12.5.vii) by r, we get

$$\alpha rpz = (r + (w + r)z)^2. \tag{12.5.ix}$$

Expanding the left hand side and grouping terms, we then get

$$(w + r)^2 z^2 + [2r(w + r) - \alpha rp]z + r^2 = 0. \tag{12.5.x}$$

This is now in the form we need to apply the quadratic formula to solve for z. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$z^*(\alpha, p, w, r) = \frac{-[2r(w + r) - \alpha rp] - \sqrt{[2r(w + r) - \alpha rp]^2 - 4(w + r)^2 r^2}}{2(w + r)^2}. \tag{12.5.xi}$$

(e) *Substitute $z(y, w, r)$ into the first order condition that contains w and solve for $y^*(\alpha, p, w, r)$ in the same way you solved for $z^*(\alpha, p, w, r)$ in the previous part.*

Answer: Substituting $z(y, w, r)$ into the first expression in equation (12.5.v) and multiplying both sides by the denominator, we get

$$\alpha py = w\left(1 + y + \frac{ry}{w}\right)^2. \tag{12.5.xii}$$

Multiplying the right hand side by w lets us reduce it to

$$w^2\left(1 + y + \frac{ry}{w}\right)^2 = (w + wy + ry)^2 = (w + (w + r)y)^2. \tag{12.5.xiii}$$

Thus, when we multiply both sides of equation (12.5.xii) by w, we get

$$\alpha wpy = (w + (w + r)y)^2. \tag{12.5.xiv}$$

Expanding the left hand side and grouping terms, we then get

$$(w + r)^2 y^2 + [2w(w + r) - \alpha wp]z + w^2 = 0. \tag{12.5.xv}$$

This is now in the form we need to apply the quadratic formula to solve for y. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$y^*(\alpha,p,w,r)=\frac{-[2w(w+r)-\alpha wp]-\sqrt{[2w(w+r)-\alpha wp]^2-4(w+r)^2w^2}}{2(w+r)^2}. \tag{12.5.xvi}$$

(f) *Given the substitutions you did in part (c), you can now write $e^{-(\ell-\beta)}=y^*(\alpha,p,w,r)$ and $e^{-(k-\gamma)}=z^*(\alpha,p,w,r)$. Take natural logs of both sides to solve for labor demand $\ell(w,r,p)$ and capital demand $k(w,r,p)$ (which will be functions of the parameters α, β and γ.)*

Answer: Taking natural logs of $e^{-(\ell-\beta)}=y^*(\alpha,p,w,r)$ and $e^{-(k-\gamma)}=z^*(\alpha,p,w,r)$ gives us

$$-(\ell-\beta)=\ln y^*(\alpha,p,w,r) \text{ and } -(k-\gamma)=\ln z^*(\alpha,p,w,r) \tag{12.5.xvii}$$

which can be solved for ℓ and k to get the input demand functions:

$$\ell(w,r,p)=\beta-\ln y^*(\alpha,p,w,r) \text{ and } k(w,r,p)=\gamma-\ln z^*(\alpha,p,w,r). \tag{12.5.xviii}$$

(g) *How much labor and capital will this firm demand if $\alpha=100$, $\beta=\gamma=5=p$, $w=20=r$? (It might be easiest to type the solutions you have derived into an Excel spreadsheet in which you can set the parameters of the problem.) How much output will the firm produce? How does your answer change if r falls to $r=10$? How much profit does the firm make in the two cases.*

Answer: The firm would initially hire approximately 8.035 units of labor and capital to produce 91.23 units of output. When $r=10$, the optimal production plan would change to $(\ell,k,y)=(8.086,8.780,93.59)$ — i.e. the firm would increase production primarily by hiring more capital but also by hiring slightly more labor. Profit is 134.74 in the first case and 218.42 in the second.

(h) *Suppose you had used the other "solutions" in parts (d) and (e) — the ones that emerge from using the quadratic formula in which the square root term is added rather than subtracted. How would your answers to (g) be different — and why did we choose to ignore this "solution"?*

Answer: The solution for the initial values given in part (g) would then have been $(\ell,k,y)\approx(3.35,3.35,8.77)$ and this would change to $(\ell,k,y)\approx(2.72,3.42,6.41)$ when r falls to 10. This would be an odd outcome — with a drop in the input price r, the problem suggests that output will fall. It is wrong because profit in both cases is negative — meaning these are not profit maximizing production plans. (Profit in the first case is -90.19 and in the second -56.61.)

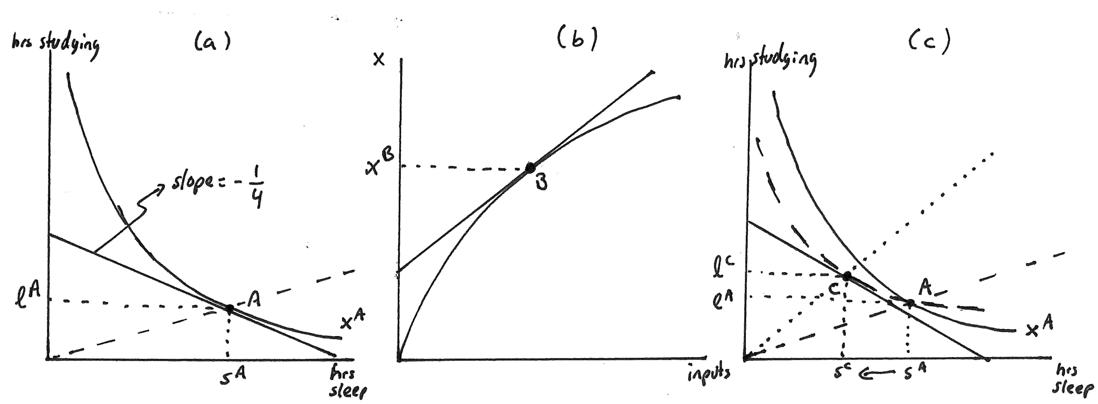
Exercise 12.7

Everyday Application: *To Study or to Sleep?: Research suggests that successful performance on exams requires preparation (i.e. studying) and rest (i.e. sleep). Neither by itself produces good exam grades — but in the right combination they maximize exam performance.*

A: *We can then model exam grades as emerging from a production process that takes hours of studying and hours of sleep as inputs. Suppose this production process is homothetic and has decreasing returns to scale.*

(a) *On a graph with hours of sleep on the horizontal axis and hours of studying on the vertical, illustrate an isoquant that represents a particular exam performance level x^A.*

Answer: This is illustrated in panel (a) of Exercise Graph 12.7 where x^A represents the isoquant with all the input bundles that can produce exam grade x^A.



Exercise Graph 12.7 : Studying and Sleeping

(b) *Suppose you are always willing to pay \$5 to get back an hour of sleep and \$20 to get back an hour of studying. Illustrate on your graph the least cost way to get to the exam grade x^A.*

Answer: This is illustrated in panel (a) of the graph with the addition of the isocost line that is tangent at A — which implies the least cost way to get exam grade x^A is to sleep s^A hours and study ℓ^A hours.

(c) *Since the production process is homothetic, where in your graph are the cost minimizing ways to get to the other exam grade isoquants?*

Answer: The cost minimizing input bundles will all lie on a ray from the origin through A — because the slopes of the isoquants are the same along any such ray, and at A the slope of the isocost is equal to the slope of the isoquant.

(d) *Using your answer to (c), can you graph a vertical slice of the production frontier that contains all the cost minimizing sleep/study input bundles?*

Answer: This is illustrated in panel (b) of Exercise Graph 12.7 where the vertical slice along the ray from the origin in panel (a) is graphed. It has a concave shape because the production process is assumed to have decreasing returns to scale.

(e) *Suppose you are willing to pay \$p for every additional point on your exam. Can you illustrate on your graph from (d) the slice of the "isoprofit" that gives you your optimal exam grade? Is this necessarily the same as the exam grade x^A from your previous graph?*

Answer: This is simply a slice of an isoprofit plane described by $\pi = px - 20\ell - 5s$, where π stands for the highest possible "profit", x is the exam grade, ℓ is the hours spent studying and s is the hours spent sleeping. It is tangent at B — with x^B being the optimal exam grade. This is not necessarily the same as x^A. We had chosen x^A arbitrarily and used it to show on what ray all cost minimizing input bundles lie. x^B lies on that ray — but does not necessarily overlap with x^A.

(f) *What would change if you placed a higher value on each exam point?*

Answer: If you place a higher value on exam grades, nothing in panel (a) will change since none of the items in that graph were derived from the knowledge of p. Neither will the production frontier slice in (b) change since the technology for producing exam grades has not changed — just the value you place on them. The only thing that changes is that the slice of the isoprofit that is tangent to the production frontier in panel (b) becomes shallower — implying that the optimal exam grade is higher.

(g) *Suppose a new caffeine/Gingseng drink comes on the market — and you find it makes you twice as productive when you study. What in your graphs will change?*

Answer: This drink has changed the production technology — so any object in your graphs that comes from the production technology will change. In particular, panel (c) illustrates the original x^A isoquant with the original cost minimizing input bundle A. If the drink makes studying twice as productive, the slope of the new isoquant must be shallower at A than it was before — resulting in the new dashed isoquant. The new cost minimizing input bundle for exam grade x^A is then C — with less sleep and more studying. Since the production technology is homothetic, this implies that all the new cost minimizing ways of getting to different exam grades will lie on the (dotted) ray from the origin through C. The vertical slice of the new production technology will then also differ.

B: *Suppose that the production technology described in part A can be captured by the production function $x = 40\ell^{0.25}s^{0.25}$ — where x is your exam grade, ℓ is the number of hours spent studying and s is the number of hours spent sleeping.*

(a) *Assume again that you'd be willing to pay \$5 to get back an hour of sleep and \$20 to get back an hour of studying. If you value each exam point at p, what is your optimal "production plan"?*

Answer: We need to solve something quite analogous to a profit maximization problem

$$\max_{\ell,s,x} px - 20\ell - 5s \quad \text{subject to} \quad x = 40\ell^{0.25}s^{0.25} \tag{12.7.i}$$

which can also be written as the unconstrained optimization problem

$$\max_{\ell,x} 40p\ell^{0.25}s^{0.25} - 20\ell - 5s. \tag{12.7.ii}$$

The two first order conditions are

$$10p\ell^{-0.75}s^{0.25} = 20 \quad \text{and} \quad 10p\ell^{0.25}s^{-0.75} = 5. \tag{12.7.iii}$$

Solving these, we get the "input demand" equations

$$\ell(p) = 0.50p^2 \quad \text{and} \quad s(p) = 2p^2. \tag{12.7.iv}$$

And plugging these into the production function, we get the exam grade "supply" function

$$x(p) = 40(0.50p^2)^{0.25}(2p^2)^{0.25} = 40p. \tag{12.7.v}$$

The optimal "production plan" therefore entails getting a grade of $40p$ by studying for $0.5p^2$ hours and sleeping $2p^2$ hours.

(b) *Can you arrive at the same answer using the Cobb-Douglas cost function (given in problem 12.4?*

Answer: Using this cost function and substituting $A = 40$, $\alpha = \beta = 0.25$, $w = 20$ and $r = 5$, we get

$$C(x) = 0.5\left(\frac{20^{0.25}(5^{0.25})x}{40(0.25^{0.25})(0.25^{0.25})}\right)^2 = 0.0125x^2. \tag{12.7.vi}$$

The marginal cost is then

$$MC(x) = \frac{\partial C(x)}{\partial x} = 0.025x. \tag{12.7.vii}$$

Setting this equal to p and solving for x, we get $x(p) = 40p$, exactly as we did before.

(c) *What is your optimal production plan when you value each exam point at \$2?*

Answer: You would study for 2 hours, sleep for 8 hours and earn an 80 on the exam.

(d) *How much would you have to value each exam point in order for you to put in the effort and sleep to get a 100 on the exam.*

Answer: You would have to value each exam point at \$2.50. You would then study for 3.125 hours, sleep for 12.5 hours and earn a 100.

(e) *What happens to your optimal production plan as the value you place on each exam point increases?*

Answer: It is easy to see from the equations (12.7.iv) and (12.7.v) that p always enters positively. As the value you place on your exam increases, you will therefore study and sleep more — and earn a higher grade.

(f) *What changes if the caffeine/Gingseng drink described in A(g) is factored into the problem?*

Answer: The underlying technology changes — which means the production function would have to change in a way that reflects this. For every 1 hour of studying, you would now get the benefit that you previously received from 2 hours of studying. Thus, the new production function would be

$$x = 40(2\ell)^{0.25}s^{0.25} \approx 47.57\ell^{0.25}s^{0.25}. \tag{12.7.viii}$$

For the previous values of sleep and study time, you can check that you would have to value an exam point by only about \$1.77 in order to make a 100 on the exam — and you would put in 2.22 hours of study time with 8.86 hours of sleep.

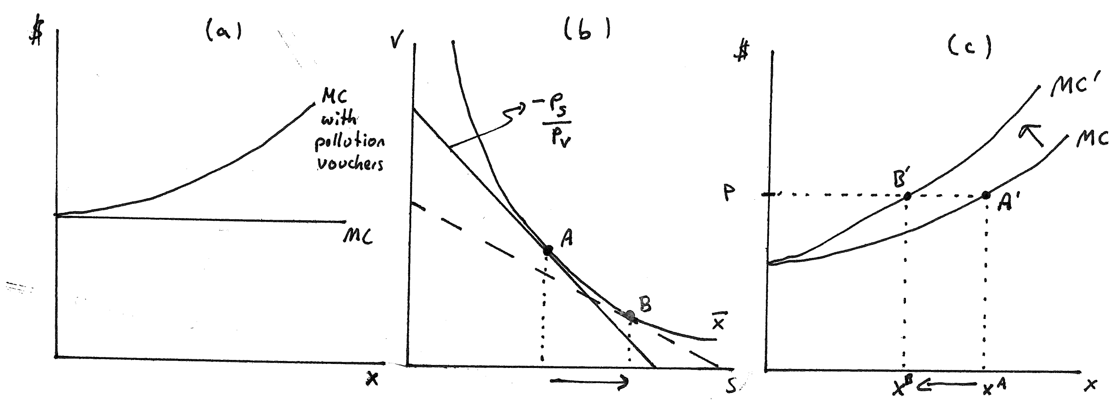
Exercise 12.9

Business and Policy Application: *Investing in Smokestack Filters under Cap-and-Trade: On their own, firms have little incentive to invest in pollution abating technologies such as smokestack filters. As a result, governments have increasingly turned to "cap-and-trade" programs. Under these programs, discussed in more detail in Chapter 21, the government puts an overall "cap" on the amount of permissible pollution and firms are permitted to pollute only to the extent to which they own sufficient numbers of pollution permits or "vouchers". If a firm does not need all of its vouchers, it can sell them at a market price p_v to firms that require more.*

A: *Suppose a firm produces x using a technology that emits pollution through smokestacks. The firm must ensure that it has sufficient pollution vouchers v to emit the level of pollution that escapes the smokestacks, but it can reduce the pollution by installing increasingly sophisticated smokestack filters s.*

(a) *Suppose that the technology for producing x requires capital and labor and, without considering pollution, has constant returns to scale. For a given set of input prices (w, r), what does the marginal cost curve look like?*

Answer: The MC curve is flat when the production technology has constant returns to scale. This is depicted in panel (a) of Exercise Graph 12.9.



Exercise Graph 12.9 : Cap-and-Trade and Smokestack Filters

(b) *Now suppose that relatively little pollution is emitted initially in the production process, but as the factory is used more intensively, pollution per unit of output increases — and thus more pollution vouchers have to be purchased per unit absent any pollution abating smokestack filters. What does this do to the marginal cost curve assuming some price p_v per pollution voucher and assuming the firm does not install smokestack filters?*

Answer: It causes the *MC* curve to be upward sloping as depicted in panel (a) of Exercise Graph 12.9.

(c) *Considering carefully the meaning of "economic cost", does your answer to (b) depend on whether the government gives the firm a certain amount of vouchers or whether the firm starts out with no vouchers and has to purchase whatever quantity is necessary for its production plan?*

Answer: It does not depend on whether the vouchers are owned by the firm or the firm has to purchase them. In both cases, the opportunity cost of using a pollution voucher to emit pollution in production is p_v. If the firm owns the voucher, it foregoes the opportunity to sell it at p_v to another firm that wishes to buy more vouchers. If the firm does not own vouchers, it must directly pay p_v per voucher.

(d) *Suppose that smokestack filters are such that initial investments in filters yield high reductions in pollution, but as additional filters are added, the marginal reduction in pollution declines. You can now think of the firm as using two additional inputs — pollution vouchers and smokestack filters — to produce output x legally. Does the overall production technology now have increasing, constant or decreasing returns to scale?*

Answer: The overall technology now has decreasing returns to scale. This is because, whether the firm uses pollution vouchers or smokestack filters or some combination of the two, it has to expend increasing resources to deal with its pollution output for any marginal increase in production.

(e) *Next, consider a graph with "smokestack filters" s on the horizontal and "pollution vouchers" v on the vertical axis. Illustrate an isoquant that shows different ways of reaching a particular output level $\overline{x}$ legally — i.e. without polluting illegally. Then illustrate the least cost way of reaching this output level (not counting the cost of labor and capital) given p_v and p_s.*

Answer: This is illustrated in panel (b) of Exercise Graph 12.9 where A is the cost minimizing bundle of smokestack filters and pollution vouchers to produce $\overline{x}$ when the prices of filters and vouchers are p_s and p_v.

(f) *If the government imposes additional limits on pollution by removing some of the pollution vouchers from the market, p_v will increase. How much will this affect the number of smokestack filters used in any given firm assuming output does not change? What does your answer depend on?*

Answer: The increase in p_v will cause isocosts to become shallower. If output does not change from $\overline{x}$, this will lead to a change in the cost minimizing bundle to B — causing the firm to use fewer vouchers and more smokestack filters. The size of the adjustment depends on the degree of substitutability between vouchers and smokestack filters in production. In other words, if it is relatively easy for the firm to install additional smokestack filters, the effect will be bigger than if it is not.

(g) *What happens to the overall marginal cost curve for the firm (including all costs of production) as p_v increases? Will output increase or decrease?*

Answer: This is illustrated in panel (c) of Exercise Graph 12.9. The marginal cost of production increases as p_v increases, rotating the MC curve from MC to MC'. For a given output price p, this implies that the profit maximizing output falls from x^A to x^B.

(h) *Can you tell whether the firm will buy more or fewer smokestack filters as p_v increases? Do you think it will produce more or less pollution?*

Answer: It is not clear whether the firm will buy more or fewer smokestack filters — because it is not clear by how much the firm will reduce its output. We know from panel (c) that the firm will produce less, and we know from panel (b) that, for the same level of output, it will buy more filters. But if the firm decreases production sufficiently much, it may end up buying fewer filters. No matter what, however, it will produce less pollution — because it produces less output with more filters for that level of output than it would have used before.

(i) True or False*: The Cap-and-Trade system reduces overall pollution by getting firms to use smokestack filters more intensively and by causing firms to reduce how much output they produce.*

Answer: This is true. As we have shown, the firm uses more smokestack filters for any given output level (panel (b) of the graph) but also produces less output (panel (c)).

B: *Suppose the cost function (not considering pollution) is given by $C(w,r,x) = 0.5w^{0.5}r^{0.5}x$, and suppose that the tradeoff between using smokestack filters* *s and pollution vouchers v to achieve legal production is given by the Cobb-Douglas production technology* $x = f(s,v) = 50s^{0.25}v^{0.25}$.

(a) *In the absence of cap-and-trade policies, does the production process have increasing, decreasing or constant returns to scale?*

Answer: The marginal cost function derived from $C(w,r,x)$ is

$$MC(w,r,x) = \frac{\partial C(w,r,x)}{\partial x} = 0.5w^{0.5}r^{0.5}. \tag{12.9.i}$$

This function is independent of x — i.e. the marginal cost is constant, which implies constant returns to scale.

(b) *Ignoring for now the cost of capital and labor, derive the cost function for producing different output levels as a function of p_s and p_v — the price of a smokestack filter and a pollution voucher.(You can derive this directly or use the fact that we know the general form of cost functions for Cobb-Douglas production functions from what is given in problem 12.4).*

Answer: Plugging in $A = 50$ and $\alpha = \beta = 0.25$ into the cost function given in problem 12.4, we get

$$C(p_s,p_v,x) = 0.5\left(\frac{xp_s^{0.25}p_v^{0.25}}{50(0.25^{0.25})(0.25^{0.25})}\right)^2 = 0.0008p_s^{0.5}p_v^{0.5}x^2. \tag{12.9.ii}$$

(c) *What is the full cost function $C(w,r,p_s,p_v)$? What is the marginal cost function?*

Answer: The cost of producing output level x is then simply the cost of labor and capital plus the cost of complying with the requirement that pollution is produced legally; i.e.

$$C(w,r,p_s,p_v) = 0.5w^{0.5}r^{0.5}x + 0.0008p_s^{0.5}p_v^{0.5}x^2. \tag{12.9.iii}$$

The marginal cost function is then

$$MC(w,r,p_s,p_v) = 0.5w^{0.5}r^{0.5} + 0.0016p_s^{0.5}p_v^{0.5}x. \tag{12.9.iv}$$

(d) *For a given output price p, derive the supply function.*

Answer: We set p equal to MC and solve for x to get

$$x(w,r,p_s,p_v,p) = \frac{p - 0.5w^{0.5}r^{0.5}}{0.0016p_s^{0.5}p_v^{0.5}}. \tag{12.9.v}$$

(e) *Using Shephard's lemma, can you derive the conditional smokestack filter demand function?*

Answer: Shephard's lemma tells us that the partial derivative of the cost function with respect to an input price is equal to the conditional input demand function for that input; i.e.

$$s(w,r,p_s,p_v,x) = \frac{\partial C(w,r,p_s,p_v)}{\partial p_s} = 0.0004\frac{p_v^{0.5}x^2}{p_s^{0.5}}. \qquad (12.9.vi)$$

(f) *Using your answers, can you derive the (unconditional) smokestack filter demand function?*

Answer: If we plug the supply function $x(w,r,p_s,p_v,p)$ into the conditional smokestack filter demand function $s(w,r,p_s,p_v,x)$, we will get the unconditional smokestack filter demand function. We then get

$$s(w,r,p_v,p_s,p) = \frac{625(p-0.5w^{0.5}r^{0.5})^2}{4p_v^{0.5}p_s^{1.5}}. \qquad (12.9.vii)$$

(g) *Use your answers to illustrate the effect of an increase in p_v on the demand for smokestack filters holding output fixed as well as the effect of an increase in p_v on the profit maximizing demand for smokestack filters.*

Answer: The derivative of the conditional demand function $s(w,r,p_s,p_v,x)$ with respect to p_v is positive — indicating that we will buy more smokestack filters conditional on producing the same quantity of output as before. The derivative of the unconditional filter demand $s(w,r,p_v,p_s,p)$ with respect to p_v, however, is negative — indicating that we will buy fewer pollution filters when we arrive at our new profit maximizing production plan. This is not because we pollute more — but rather because our supply function $x(w,r,p_s,p_v,p)$ tells us that we will produce sufficiently less such that we will need fewer overall filters even through we use more filters for the quantity that we do produce than we would have before.

Conclusion: Potentially Helpful Reminders

1. Profit maximization implies that marginal product equals input price for *ALL* inputs. Short run profit maximization therefore implies just that $MP_\ell = w$, while long run profit maximization implies that both $MP_\ell = w$ and $MP_k = r$.

2. Cost minimization implies that $TRS = -w/r$ which, since $-TRS = MP_\ell/MP_k$, is equivalent to saying $MP_\ell/MP_k = w/r$. You should be able to show that the profit maximization conditions ($MP_\ell = w$ and $MP_k = r$) imply that the cost minimization condition holds, but the reverse does not hold.

3. Profit maximization can be seen graphically as a tangency of the vertical production frontier slices that hold one input fixed with the slice of the isoprofit plane. You should understand how that tangency is equivalent to saying $MP_\ell = w$ and $MP_k = r$.

4. Cost minimization can be seen graphically as tangencies of isocosts and isoquants. You should understand how the condition $-TRS = MP_\ell/MP_k = w/r$ must logically hold at all production plans that minimize cost. You should

also understand that, when the production frontier is homothetic, *ALL* such tangencies will happen along a single ray from the origin for a given w and r. And you should understand why typically only one of those tangencies represents a *profit maximizing* production plan.

5. End-of-Chapter problem 12.1 is a good problem to practice with concepts contained in the above points — and a good problem to use for preparation for Chapter 13.

6. At the end of Chapter 11, we showed that the supply curve is the part of the marginal cost curve that lies above average cost. The same is true in this chapter when there are 2 inputs — and the same will always be true, in the short and long run, so long as we define costs correctly.

7. One of the points emphasized in end-of-chapter exercises (but only partially emphasized in the text chapter) is that U-shaped average cost curves can arise in one of two ways: (1) because of production technologies that initially exhibit increasing returns to scale but eventually turn to decreasing returns to scale; and (2) because of the existence of a recurring *fixed cost*. This idea is further developed in end-of-chapter exercise 12.4 and then in Chapter 13.

CHAPTER

13

Production Decisions in the Short and Long Run

We already saw in Chapter 12 that the single-input model in Chapter 11 was just a slice of the 2-input model with capital held fixed. We now build on this insight — illustrating how short run constraints inhibit immediate adjustments to long run production plans as underlying conditions change. We also find ways of connecting short and long run cost curves — and illustrating the difference between short run and long run profit. Throughout, we try to be very consistent in the following sense: We only call something a *cost* if it is really an economic cost — and we reserve the terms *expenditure* or *expense* for outlays that include sunk costs. This is a departure from the typical way in which textbooks treat costs — but I think it actually makes a lot that follows easier and less confusing while focusing us on what we really mean by economic costs (and economic profit).

Chapter Highlights

The main points of the chapter are:

1. Not every financial outlay by a firm is an **economic cost** for the firm. Economic costs include only opportunity costs — i.e. only those outlays that actually affect economic behavior. **If we include in "costs" only real economic costs**, then it will *always* be the case that the **supply curve is the part of the** MC **curve that lies above** AC.

2. The financial outlays on capital are *fixed* in the short run and are therefore *not* a short run cost — because they have to be paid regardless of what the firm does. These outlays become a *variable* cost in the long run because capital can be varied in the long run. There are really no such things as **fixed costs** in the short run, but there may be such costs (like recurring license fees) in the long run.

3. Firms will produce (on their short run supply curves) and *not* shut down so long as **short run profit** is not negative, and firms will produce (on their long run supply curves) and *not* exit the industry so long as **long run profit** is not negative. Both short and long run profit subtract economic costs from revenue, but what counts as an economic cost differs between the short and long run (because some financial outlays that are costs in the long run are sunk in the short run). As a result, the (short run) **shut down price is lower than the** (long run) **exit price**.

4. Output **supply responses to a change in output price are larger in the long run than in the short run** — implying that long run supply curves are shallower than short run supply curves. Similarly, **input demand curves are shallower in the long run than in the short run**.

5. **Output supply curves shift as input prices change** — to the left as they increase and to the right as they decrease. Similarly, **input demand curves shift as output prices change** — to the left as output price falls and to the right as output price increases.

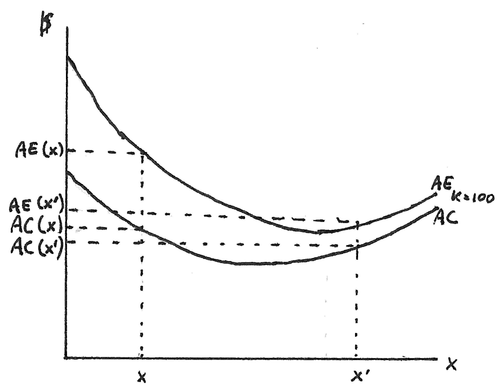
6. **Some long run economic relationships depend on the substitutability of capital and labor in production**. An increase in w may cause a long run increase or decrease in the amount of capital employed — and the long run response of output may be higher or lower than the short run response. Similarly, an increase in r may cause a long run increase or decrease in the amount of labor as well as a larger or smaller long run output response (relative to the short run response) — all depending on the relative substitutability of capital and labor.

13A Solutions to Within-Chapter-Exercises for Part A

Exercise 13A.1

Can you find similar rectangular areas that are equal to $FE_{k=100}$ for other output levels? Given that these rectangular areas have to be equal to one another, can you see why the AC and AE curves must be getting closer and closer as output rises?

Answer: The text graph illustrates the rectangular area for output level x^A, but we could similarly pick any arbitrary output level and illustrate a similar rectangular area that is equal to the fixed expenditure. For instance, in Exercise Graph 13A.1, we pick the output quantities x and x'. At x, the average expenditure is $AE(x)$ and the average cost is $AC(x)$. This implies that the fixed expenditure is the rectangle given by the distance $AE(x) - AC(x)$ times x. At x', the average expenditure is $AE(x')$ and the average cost is $AC(x')$. This implies that the fixed expenditure is the rectangle given by the distance $AE(x') - AC(x')$ times x'. As x' increases, the rectangle representing $FE_{k=100}$ gets longer — which implies that the only way for it to represent the same area as the similar rectangle for smaller output (like x) is for the rectangle to have less height. And this in turn implies that, as a simple matter of geometry, the $AE_{k=100}$ curve has to be getting closer to the $AC_{k=100}$ curve as output increases — i.e. the two curves have to converge. Of course this is simply another way of saying that the fixed expenditure is being spread across more output units on average as output increases, which means that the average fixed expenditure (per unit of output) falls with output.



Exercise Graph 13A.1 : Different Ways of Illustrating Fixed Expenditures

Exercise 13A.2

Can you explain why the MC curve intersects both the AC and the AE curves at their lowest points?

Answer: The MC curve has to intersect both the average curves at their lowest point for exactly the same reason — the only way for an average to rise is for the marginal contribution to the average to lie above the average. This is true for both curves because the only cost or expense that is added as output increases is what's included in the marginal cost curve. Once again, the analogy to grade averages used in previous chapters applies — the only way for your average course grade to rise is for your next exam grade to lie above the average.

Exercise 13A.3

Where in the graph would you locate the "marginal expenditure" curve (derived from the total expenditure curve)?

Answer: The marginal expenditure curve includes all those expenditures that increase with output. But the only expenditures that increase with output are those that are already included in the marginal cost curve. Thus, the marginal expenditure curve is no different than the marginal cost curve — and in fact would be mislabeled as an expenditure curve because it includes only true short run economic costs. You can also see this by noting that the slope of the total expenditure curve is exactly the same everywhere as the slope of the (total) short run cost curve $C_{k=100}$.

Exercise 13A.4

Can you tell from the shape of the long run (total) cost curve whether the production process is increasing, decreasing or constant returns to scale?

Answer: Since the slope is constant, the production process has constant returns to scale.

Exercise 13A.5

Verify the derivation of cost curves in panels (e) and (f) in Graph 13.2. In what sense is the relationship between short run expenditure and long run cost curves similar in this case to the case we derived in the top panels of the graph for constant returns to scale production processes?

Answer: We can use the cost minimizing input bundles A, B, and C to verify the total and average cost numbers in panels (e) and (f). With a wage rate of \$20 and a rental rate of \$10, A costs 50(20)+100(10)=\$2,000 — exactly as indicated in panel (e) in point A'. When 200 units of output cost \$2,000 to produce, then the average cost is 2000/200=\$10 — exactly as indicated in panel (f) in point A''. The same reasoning can be used to derive the points B' and B'' from B and C' and C'' from C. The result is similar to the constant returns to scale case in that every short run AE curve shares one point in common with the long run AC curve.

Exercise 13A.6

Where would you find the long run marginal cost curve in panel (f) of the Graph?

Answer: The long run marginal cost curve would have the same intercept as the long run AC curve and would intersect the AC curve at its lowest point from below.

Exercise 13A.7

A textbook author (not me!) once told his publisher to produce a graph such as panel (f) of Graph 13.2 and explained that he wanted the short run average expenditure curves corresponding to different levels of fixed capital to *each be tangent at their lowest point* to the U-shaped long run average cost curve. The graphics artist (who knew nothing about economics) came back to the author and sheepishly explained that such a graph cannot logically be drawn. What was wrong in the author's instructions?

Answer: The mistake made by the author was that he insisted the tangencies needed to occur at the lowest point of the average expenditure curves. The economics of the problem only implies that the average expenditure curves be tangent to the long run average cost curve — not that they be tangent at the lowest point of the AE curves. If the long run AC curve is U-shaped, only the average expenditure curve that is tangent at the lowest point of the AC curve has its tangency at its own lowest point. The average expenditure curves with tangencies to the left of the lowest point of AC are tangent to the left of their own lowest points, and the average expenditure curves with tangencies to the right of the lowest point of AC are tangent to the right of their own lowest point.

Exercise 13A.8

Demonstrate that the firm's (long run) profit is zero when $p = 10$.

Answer: When $p = 10$ and the firm produces where $p = MC$, it will produce precisely the quantity at the lowest point of the long run AC curve (where the MC curve crosses) — i.e. it produces 200 output units. Its total revenues are therefore \$2,000. We also know from the long run AC curve that, when the firm produces 200 units, it incurs an average cost of \$10 per unit — or a total cost of \$2,000. Thus, revenues are equal to long run costs — which implies long run profit is zero.

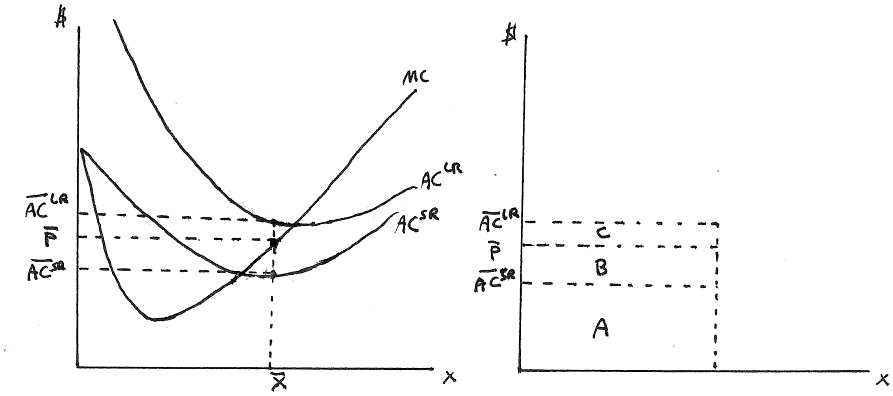
Exercise 13A.9

Can you illustrate that short run economic profits will be positive when price falls between the lowest points of the $AC_{k=100}$ and the AC^{LR} curves even though total expenditures exceed total revenues? What will long run economic profits be in that price range?

Answer: Suppose price is $\overline{p}$ as illustrated in panel (a) of Exercise Graph 13A.9 — i.e. price falls between the lowest point of the short and long run AC curves. If the firm produces, it will then produce where $\overline{p}$ equals MC — i.e. it will produce $\overline{x}$. At that level of production, it incurs a short run average cost of $\overline{AC}^{SR}$ and a long run average cost of $\overline{AC}^{LR}$ as indicated on the vertical axis. This implies a total short run cost of $\overline{AC}^{SR} * \overline{x}$ and a total long run cost of $\overline{AC}^{LR} * \overline{x}$. In panel (b) of the graph, this is illustrated as a total short run cost equal to area A and a total long run cost equal to area $A+B+C$. Total revenue, on the other hand, is $\overline{p} * \overline{x}$ — which is equal to area $A+B$. This implies that short run profit — i.e. revenue minus short run cost — is equal to $A+B-A = B$ while long run profit — i.e. revenue minus long run cost — is equal to $A+B-(A+B+C) = -C$. Thus, short run profit is positive when $\overline{x}$ is produced while long run profit is negative.

Exercise 13A.10

In many beach resorts on the east coast of the U.S., business is brisk in the summers but slow in the winters. In summers, resort rentals are sold out at high



Exercise Graph 13A.9 : Positive SR and Negative LR Profit

weekly rates, but in winters they are only partially rented at much lower rates. If you were to calculate expenses and revenues on a monthly basis, you would almost certainly find these resorts with revenues greater than expenses in the summers and expenses greater than revenues in the winters. How come these resorts don't just shut down in winters?

Answer: Monthly expenses include fixed expenses that are not economic costs in the short run but are fixed costs in the long run. It makes sense for resorts to stay open so long as they can cover at least their short run costs. If they can also recover some of the fixed expenses on top of that, all the better. By staying open in the winter, they may not recover all their fixed expenses — but if they closed, they would lose all of the fixed expenses (which they have to pay regardless of whether they are open or not).

Exercise 13A.11

Compare Graphs 13.4 and 13.5. Why is the supply curve beginning at the higher average curve in 13.5 and on the lower one in 13.4?

Answer: In Graph 13.4, the supply curve is the short run supply curve — while in Graph 13.5 it is the long run supply curve. It therefore begins at the lowest point of the short run average cost curve in the first graph while it begins at the lowest point of the long run average cost curve in the second graph.

Exercise 13A.12

Can we say for sure that the lowest point of the long run *AC* curve will shift to the right when the license fee increases?

Answer: Yes, we can be sure about that. This is because we know that the long run MC curve is unaffected by a change in the cost of the license — and that MC curves always have to cross AC curves at their lowest point.

Exercise 13A.13

True or False: $p'MP_\ell^B = w^A$.

Answer: This is true. In the short run, we can vary labor fully — which means that we can fully reach the short run profit maximizing condition that the marginal revenue product of labor has to be equal to the wage rate. (This is what we found has to be true in the short run production cases we analyzed in Chapter 11.) At the new price p', we will therefore hire labor until the marginal product of labor times p' equals the unchanged wage w^A. If B is truly the profit maximizing production plan in the short run when price increases to p', it therefore has to be the case that we have adjusted labor input to the point where $p'MP_\ell^B = w^A$.

Exercise 13A.14

If the marginal product of labor increases as additional capital is hired in the long run, can you tell whether the producer will hire additional labor (beyond ℓ^B) in the long run? Can you then identify the minimum distance above A on the ray through A the long run optimal isoquant in Graph 13.7a will lie?

Answer: If the marginal product of labor increases as the firm hires more capital in the long run, then this implies that the firm will also want to hire additional labor in the long run. As a result, its long run labor input will exceed the short run adjustment to ℓ^B — placing us onto the steeper ray and to the right of ℓ^B.

Exercise 13A.15

Can you see in panel (a) of Graph 13.8 the cost of not substituting from C to C'? Can you verify that the numbers in panel (b) are correct?

Answer: If the firm were to not substitute labor for capital, it would continue to use the input bundle C to produce 100 units of output. It would then require a "budget" equal to the dashed magenta line which has the slope indicated by the ratio of the new input prices but goes through C. Since this lies above the green "budget" that is cost-minimizing, the difference between the dashed magenta and the green "budgets" is the cost of not substituting. In panel (b), we then have three points to verify: all lie at output level of 100, but one lies on the blue, one on the (dashed) magenta, and one on the green cost curve. On the blue curve, C is derived from the original cost minimizing input bundle C in panel (a) when wage was 20 and the rental rate was 10. Since that input bundle contains 15 units of capital and 7.5 units of labor, the total cost (of producing 100 output units) is then $20 * 7.5 + 10 * 15 = \$300$ — exactly as shown in panel (b). On the (dashed) magenta curve in panel (b), we are assuming the firm did not respond to the change in the wage rate and still used input bundle C from panel (a). The cost, at the lower wage of 10, is then $10 * 7.5 + 10 * 15 = \$225$ — again exactly as shown in panel (b). Finally, the green point C' assumes that the firm has substituted labor for capital and is now using input bundle C' from panel (a). Thus, the cost of producing 100 output units is now $10 * 10.6 + 10 * 10.6 = \$212$ — as again shown in panel (b). The cost of not substituting in response to the wage change when producing 100 output units is therefore equal to \$13.

Exercise 13A.16

Are these long-run or short-run cost curves?

Answer: Since the firm is substituting labor and capital, they are long run cost curves (which allow capital to vary.)

Exercise 13A.17

Can you verify that the numbers in panel (c) are correct?

Answer: We have three points in panel (c) to verify — one on the blue, one on the green and one on the (dashed) magenta curve. On the blue curve, $r = 10$. Panel (a) shows that at that rental, the cost minimizing production plan for 100 units of output involves 15 units of capital and 7.5 units of labor. At $w = 20$, this implies a total (long run) cost of $20 * 7.5 + 10 * 15 = \$300$ — exactly as shown in panel (c). On the (dashed) magenta curve, the firm is assumed to not change its input bundle in response to r increasing from 10 to 20. Thus, we still use the input bundle from panel (a) that contains 15 units of capital and 7.5 units of labor — with a total cost of $20 * 7.5 + 20 * 15 = \$450$ — again exactly as shown in panel (c). Finally, on the green curve we assume that the firm has optimally substituted labor for the more expensive capital — switching to the input bundle that contains 10.6 units of labor and 10.6 units of capital in panel (a). This results in a total cost of $20 * 10.6 + 20 * 10.6 = \$424$.

Exercise 13A.18

Assuming the original cost minimizing input bundle remains C, which of the three curves graphed in Graph 13.8c would be different (and how would it be different) if the inputs in panel (a) of the graph were more substitutable? How would the graph change if the two inputs were perfect complements in productions?

Answer: The green curve would be different. If the inputs were more substitutable, the benefit from substituting labor for capital would be greater — implying that the green curve would lie closer to the blue and farther from the (dashed) magenta. If the inputs were perfect complements, there would be no benefit from substituting labor for capital — which implies the green curve would lie exactly on top of the (dashed) magenta curve.

Exercise 13A.19

In Graph 13.8d, we already derived conditional labor demand curves along which capital is allowed to adjust. Explain why these are *not* long run labor demand curves.

Answer: In Graph 13.8a, we showed how a decrease in the wage rate will cause producers to substitute labor for capital as they consider the economically most efficient way of producing any *fixed quantity of output* (and an increase in the wage rate will do the reverse). As a result, conditional input demand curves slope down (Graph 13.8d). This by itself does not, however, tell us how much labor demand will adjust beyond its short run adjustment when w rises or falls because *conditional* labor demand curves simply tell us how much labor a producer hires *conditional* on wanting to produce a particular level of output. As we see from Graph 13.9, for instance, an increase in w causes the new profit maximizing input bundle to fall below the original isoquant — thus causing not only a decrease in the quantity of labor demanded but also a decrease in the quantity of output supplied. The optimal output level thus changes when the wage rate changes. What we then really want to know is not how labor demand responds *conditional* on the output level remaining the same as it was before but rather how *actual* labor demand responds to changes in input prices given that the profit-maximizing output quantity changes simultaneously.

Exercise 13A.20

Can you tell from just seeing the tangency at (ℓ^A, k^A) of the isocost with the isoquant whether the production plan $A = (\ell^A, k^A, x^A)$ is profit maximizing at prices (w^A, r^A, p^A)?

Answer: No, you cannot — because p^A does not appear anywhere in the graph. We can tell that A involves the cost minimizing input bundle for producing output level x^A at input prices (w^A, r^A), but we cannot tell whether the output quantity x^A is in fact the profit maximizing output quantity. If it is, then A is in fact the profit maximizing production plan; if it is not, then (ℓ^A, k^A) is just the cost minimizing input bundle for producing output level x^A.

Exercise 13A.21

Do you see from Graph 13.9 that long run demand curves for labor (with respect to wage) must slope down, as must long run demand curves for capital (with respect to the rental rate)?

Answer: As wage increases, the graph shows us moving to the left — i.e. less labor; and as w decreases, the graph shows the reverse — i.e. more labor. Thus, the long run quantity of labor demanded increases with a drop in wage and decreases with an increase in wage — which means the long run labor demand curve must slope down. Similarly, as the rental rate increases, the graph shows us moving *down*, and as it decreases it shows us moving *up*. Since capital is measured on the vertical axis. This implies a decrease in r causes an increased use of capital while an increase in r causes less use of capital — which is the same as saying that the long run demand for capital slopes down.

Exercise 13A.22

Does Graph 13.9 tell us anything about whether the cross-price demand curve for labor (with the rental rate on the vertical axis) slopes up or down in the long run?

Answer: No, it does not. As r increases, the region of input bundles where the new profit maximizing production plan might lie contains bundles with more and less labor than ℓ^A. The same is true as r falls. Thus, based on this graph, it would seem that cross-price input demand curves could slope up or down. This is clarified in later graphs in the chapter.

Exercise 13A.23

Where in Graph 13.9 will our new production plan fall after we have made our short run labor adjustment?

Answer: It would fall somewhere along the horizontal line connecting A to k^A.

Exercise 13A.24

We know that we will decrease output in the short run as w increases because we hire fewer workers. In the case of robots and workers, do you think that we will increase or decrease output once we can hire more robots in the long run?

Answer: This will become clearer in the next section. However, note that we increase the number of robots in the long run because the MP_k has gone up — and then we increase it more in order to replace some more of our workers. This suggests that output will be higher in the long run than in the short run.

Exercise 13A.25

Suppose labor and capital were perfect complements in production. What would the analogous graph for an increase in w look like?

Answer: No matter what the ratio of input prices, production would always take place on the 45 degree line that would contain both A and C. But the perfect complementarity between capital and labor implies that the firm would do nothing to change production in the short run when w increases modestly — because capital is fixed in the short run. Thus $B = A$ — which implies the short run labor demand curve would be vertical while the long run labor demand curve would still be downward sloping (and thus again more responsive to changes in w.) The relationship between w and k would be similar to that in panel (i) of the text graph. This logic holds for modest changes in w. If w increases by a lot, then it may be the case that the firm will hire less labor in the short run.

Exercise 13A.26

Demonstrate that $MP_k^B < MP_k^A$ in panel (c) of Graph 13.10.

Answer: In panel (c), the new optimal input bundle C contains *less* capital input than the original bundle A — which implies that the producer cannot immediately switch to the long run optimum when capital is fixed in the short run. Rather, in the short run the producer switches to input bundle B which has the characteristic that the isocost containing B cuts the isoquant containing B from below — i.e. $TRS^B < -w'/r^A$ or equivalently

$$\frac{MP_\ell^B}{MP_k^B} > \frac{w'}{r^A} \text{ which implies } \frac{p^A MP_\ell^B}{p^A MP_k^B} > \frac{w'}{r^A}. \tag{13A.26}$$

In the short run, we know the firm will adjust labor until $p^A MP_\ell^B = w'$. The above equation then implies that $p^A MP_k^B < r^A$ and thus (since $p^A MP_k^A = r^A$) that $MP_k^B < MP_k^A$.

Exercise 13A.27

How is the long run response in output related to the short run response in output as w increases? What does your answer depend on? (*Hint*: You should be able to see the answer in Graph 13.10.)

Answer: The more substitutable the inputs are in production, the more likely it is that the drop in output will be less in the long run than in the short run (as in panel (a)); and the more complementary the inputs are in production, the more likely it is that the drop in output is greater in the long run than in the short run (as in panel (c).) There is also the in-between case (as in panel (b)) where the long run and short run response in output is identical.

Exercise 13A.28

Can you arrive at these conclusions intuitively using again the examples of robots and computers?

Answer: First, consider the case of robots which are relatively substitutable with workers. If the cost of robots increases, then we will substitute to workers since they are substitutable — causing a relatively small decrease in output. This is because workers and robots do similar things in production. Second, consider the case of computers and graphic artists. These are relatively complementary — each needs the other in order to increase output. Thus, if the cost of computers increases, we would not be able to substitute them easily for graphic artists — and thus, as we reduce our rentals of computers, we would need to let workers go. This causes a decrease in the number of workers (graphic artists) — and a relatively larger decrease in output.

Exercise 13A.29

In panel (a) of Graph 13.7, we determined that the firm will once again end up on the steeper ray once it can adjust capital. Call the new (long-run) input bundle at the higher output price C. Can you now tell what will determine whether C lies to the right or left of B?

Answer: From the extremes illustrated in Graph 13.12, we can tell the following: The more complementary capital and labor are in production, the more likely it will be that C lies to the right of B. It is in those cases that the marginal product of labor increases as capital increases. The more substitutable capital and labor are in production, however, the more likely it will be that C lies to the left of B. In such cases, an increase in capital decreases the marginal product of labor — which causes the firm to shift away from labor as it transitions to the long run when it can use more capital.

13B Solutions to Within-Chapter-Exercises for Part B

Exercise 13B.1

Suppose the long run production function were a function of 3 inputs — labor, capital and land, and suppose that both labor and capital were variable in the short run but land is only variable in the long run. How would we now calculate the short run cost minimizing labor and capital input levels conditional on some (short run) fixed level of land?

Answer: You would have to solve a cost minimization problem with capital and labor as choice variables. The expense associated with land would not factor into the problem. (If you did include the cost of land in the cost minimization problem, it would simply drop out as you solve for the first order conditions.)

Exercise 13B.2

Can you use the expressions above to justify the difference in the (total) cost and total expenditure curves in panel (a) of Graph 13.1 as well as the difference between AC and AE in panel (b) of that graph?

Answer: The difference between $E_{k^A}(x, w^A, r^A)$ and $C_{k^A}(x, w^A)$ is $r^A k^A$ which does not depend on x. Thus, the difference between the short run expense and cost curves is a constant amount — implying that the expenditure curve simply lies above the cost curve by that constant amount. The averages of the two expressions are

$$AE_{k^A}(x, w^A, r^A) = \frac{w^A \ell_{k^A}(x) + r^A k^A}{x} \quad \text{and} \quad AC_{k^A}(x, w^A) = \frac{w^A \ell_{k^A}(x)}{x} \tag{13B.2.i}$$

which implies that the difference between the average expenditure and cost curves is

$$AE_{k^A}(x, w^A, r^A) - AC_{k^A}(x, w^A) = \frac{r^A k^A}{x}. \tag{13B.2.ii}$$

Since $r^A k^A$ is a constant, this difference becomes smaller as x increases — leading to the fact that the short run AE curve converges to the short run AC curve.

Exercise 13B.3

Can you derive from this the relationship between long run average cost and short run average expenses as illustrated graphically in Graph 13.2?

Answer: The relationship implies that

$$\frac{E_{k^A}(x, w^A, r^A)}{x} \geq \frac{C(x, w^A, r^A)}{x} \tag{13B.3}$$

which is the same as writing $AE_{k^A}(x, w^A, r^A) \geq AC(x, w^A, r^A)$, with the expression holding with equality when $x = x^A$. This is exactly what we derived graphically: the short run average expenditure curve lies above the long run average cost curve everywhere except for the output level at which the short run fixed capital is equal to what the cost-minimizing firm would choose in the long run for that output level.

Exercise 13B.4

In the case of U-shaped average cost curves, how can you use the mathematical expressions above to argue that the short run "shut down" price is lower than the long run "exit" price?

Answer: Suppose the lowest point of the long run AC curve occurs at output level $\overline{x}$. At that point, the short run $AE_{\overline{k}}$ curve (where $\overline{k}$ is the long run optimal level of capital for producing $\overline{x}$) is tangent to the long run AC; i.e. $AC(\overline{x}) = AE_{\overline{k}}(\overline{x})$. From the fact that the difference between short run expenditures and short run costs is equal to the fixed expense on capital (in the short run), we also know that $AE_{\overline{k}}(\overline{x}) > AC_{\overline{k}}(\overline{x})$. Together, these results imply

$$AC(\overline{x}) > AC_{\overline{k}}(\overline{x}); \tag{13B.4}$$

i.e. the long run average cost curve lies above the short run AC curve at $\overline{x}$ where the long run AC attains its minimum. The exit price lies at the lowest point of AC while the shutdown price lies at the lowest point of $AC_{\overline{k}}$. We have just shown that the latter lies below the former.

Exercise 13B.5

Verify that these numbers are correct.

Answer: Substituting $(w, r, x) = (20, 10, 1280)$ into the conditional input demands, we get

$$\ell(20, 10, 1280) = \left(\frac{10}{20}\right)^{1/2} \left(\frac{1280}{20}\right)^{5/4} = \left(\frac{1}{4}\right)(64)^{5/4} = 128 \tag{13B.5.i}$$

and

$$k(20, 10, 1280) = \left(\frac{20}{10}\right)^{1/2} \left(\frac{1280}{20}\right)^{5/4} = (2)^{1/2}(64)^{5/4} = 256. \tag{13B.5.ii}$$

Multiplying these by their respective input prices and adding (or, alternatively, evaluating $C(w, r, x)$ at (20,10,1280)), we get that the cost is 20(128)+10(256)=\$5,120.

Exercise 13B.6

What is the short run cost (as opposed to expenditure) function?

Answer: The short run cost function is the same as the short run expenditure function except that it does not include the fixed expense on capital. Thus,

$$C_{k^A=256}(x,20) = 20\ell_{k^A=256}(x) = \frac{x^{5/2}}{20^{3/2}256}. \qquad (13B.6)$$

Exercise 13B.7

Verify that, when $x = 1280$, the short run expense is equal to the long run cost.
Answer: The short run expense is

$$E_{k^A=256}(1280,20,10) = \frac{1280^{5/2}}{20^{3/2}256} + 2560 = 2560 + 2560 = \$5,120. \qquad (13B.7.i)$$

The long run cost is

$$C(1280,20,10) = 0.66874(1280)^{5/4} = \$5,120. \qquad (13B.7.ii)$$

Exercise 13B.8

Does the inclusion of a fixed cost cause any change in conditional input demands? What about unconditional input demands?

Answer: In the cost minimization problem (from which conditional input demands are derived), we would now minimize $C = w\ell + rk + FC$ — but since FC enters as a constant, it would drop out as we take the derivatives to get the first order conditions that we solve for the conditional demands. Thus, the inclusion of FC has no effect on conditional input demand functions. (This should make intuitive sense: The least cost way of producing any output level still involves the same input bundle regardless of how high a license fee we have to pay to start producing). In the profit maximization problem (from which we derive the unconditional input demands), we maximize revenues minus costs — which include such fixed costs in the long run. However, when maximizing $\pi = px - w\ell - rk - FC$, we take first order conditions that are derivatives of π — and constants like FC again drop out in the process. Thus, unconditional input demands are unaffected — except to the extent to which the FC affects the exit price and thus the point at which no labor is demanded by the firm. In other words, unconditional input demand functions will remain the same as without the FC except for the fact that they are "shorter".

Exercise 13B.9

Does the inclusion of a fixed cost change either the (short-run) "shut down" price or the (long-run) "exit" price?

Answer: It cannot affect the short run shut-down price since it is not a cost in the short run. It does, however, affect the long run exit price — which is the lowest point of the long run AC curve that includes the FC.

Exercise 13B.10

Would including the fixed expense rk^A in the short run profit maximization problem (so that the objective function becomes $px - w\ell - rk^A$) make any difference as the problem is solved?

Answer: No, it would make no difference because it would drop out as we take first derivatives to get the first order conditions.

Exercise 13B.11

Equation (13.30) can also be read as "the slope of the long run output supply function is larger than the slope of the short run output supply function (with respect to price)." But the long run supply curve in Graph 13.7 appears to have a shallower (and thus smaller) slope than that of the short run supply curve. How can you reconcile what the math and the graphs seem to be telling us?

Answer: The apparent discrepancy arises from the fact that the supply curves we graph are *inverse* slices of the supply function (holding input prices fixed). Thus, when we invert the supply curve picture from part A of the chapter, steeper curves become shallower and shallower curves become steeper. The inverse supply *curves* are therefore such that the long run supply will look steeper than the short run supply — which is consistent with the partial derivatives we derived.

Exercise 13B.12

Verify that these are truly the short run output supply and input demand functions by checking to see if the short run functions give the same answers as the long run functions when (p, w, r)=(5,20,10).

Answer: Plugging in the output price of \$5 and the wage of \$20, we get

$$x_{k=256}(5,20) = 3225\left(\frac{5}{20}\right)^{2/3} \approx 1280 \;\text{ and }\; \ell_{k=256}(5,20) = 1290\left(\frac{5}{20}\right)^{5/3} \approx 128 \tag{13B.12}$$

when capital is fixed at $k = 256$.

Exercise 13B.13

Panels (a) and (b) of Graph 13.16 are analogous to panels (b) and (e) of Graph 13.8. Now calculate the relevant curves and graph them for the case that is analogous to panels (c) and (f) of Graph 13.8 where, instead of wage falling from \$20 to \$10, the rental rate of capital rises from \$10 to \$20.

Answer: The cost function is

$$C(w,r,x) = 2(wr)^{1/2}\left(\frac{x}{20}\right)^{5/4}. \tag{13B.13.i}$$

When $w = 20$ and $r = 10$, the cost function becomes $C(20,10,x) = 0.66874x^{5/4}$,

and when $w = 20$ and $r = 20$, it becomes $C(20,20,x) = 0.94574x^{5/4}$. Thus, the cost function shifts *up* by $0.277x^{5/4}$. Taking partial derivatives of these two "slices" of the cost function with respect to x, we get the marginal cost function shifting from $MC(20,10,x) = 0.83593x^{1/4}$ to $MC(20,20,x) = 1.18218x^{1/4}$ — i.e. it shifts up by $0.34625x^{1/4}$. These shifts include both the *direct* and the *indirect* effects from substitutions of labor for capital as r increases. To isolate the *direct* effect, we would have to keep the input demands equal to what they were originally when $r = 10$. We can calculate these from the conditional input demand functions

$$\ell(w,r,x) = \left(\frac{r}{w}\right)^{1/2}\left(\frac{x}{20}\right)^{5/4} \text{ and } k(w,r,x) = \left(\frac{w}{r}\right)^{1/2}\left(\frac{x}{20}\right)^{5/4}. \qquad (13B.13.ii)$$

At $w = 20$ and $r = 10$, these are $\ell(20,10,x) = 0.01672x^{5/4}$ and $k(20,10,x) = 0.03344x^{5/4}$. Isolating the direct effect assumes these remain unchanged when r increases to 20 — which would imply a "cost" function of

$$\begin{aligned}\overline{C}(20,20,x) &= 20\ell(20,10,x) + 20k(20,10,x) \\ &= 20\left(0.01672x^{5/4}\right) + 20\left(0.03344x^{5/4}\right) = 1.00311x^{5/4}\end{aligned} \qquad (13B.13.iii)$$

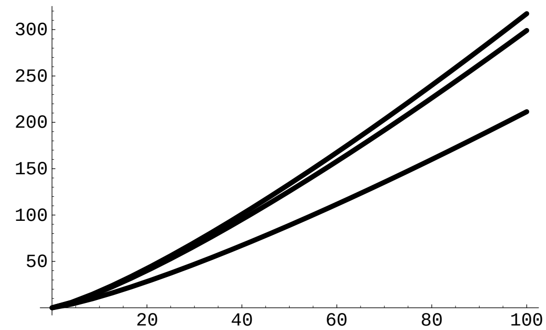
with corresponding marginal cost of $\overline{MC}(20,20,x) = 1.25389x^{1/4}$. Without taking account of the fact that firms will substitute away from capital when r increases, both the cost and marginal cost curves would therefore be higher than they actually are once substitution effects have been taken into account (just as we illustrated in Graph 13.8 of the text in panels (c) and (f)). The three cost functions we derived are then graphed in Exercise Graph 13B.13(1) with the lowest corresponding to $C(20,10,x) = 0.66874x^{5/4}$, the middle corresponding to $C(20,20,x) = 0.94574x^{5/4}$ and the highest corresponding to $\overline{C}(20,20,x)$.

Similarly, the three marginal cost curves we derived are graphed in Exercise Graph 13B.13(2), with the lowest corresponding to $MC(20,10,x) = 0.83593x^{1/4}$, the middle corresponding to $MC(20,20,x) = 1.18218x^{1/4}$ and the highest corresponding to $\overline{MC}(20,20,x) = 1.25389x^{1/4}$.

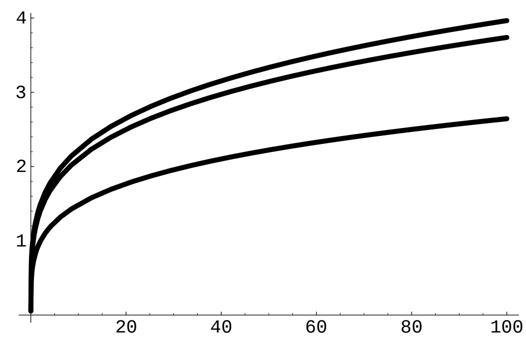
Exercise 13B.14

If the generalized CES function was used as a utility function instead of the version where A and β are set to 1, would the underlying tastes represented by that function be changed?

Answer: No, regardless of what values A and β took (so long as they are positive), the shapes of the indifference curves would be unaffected. Only their labeling would change, but the ordering would be preserved.



Exercise Graph 13B.13(1) : Costs and Substitution Effects



Exercise Graph 13B.13(2) : Marginal Costs and Substitution Effects

Exercise 13B.15

Explain why the direct effect in the table does not depend on the degree of substitutability between capital and labor in production.

Answer: The direct effect of a drop in w is the change in costs associated with just the drop in the wage and not due to any substituting behavior by the firm. Thus, the direct affect assumes the firm will continue to use the same input bundle as before to produce 5,000 units of output — in effect cutting the cost of using the same amount of labor by half when the wage drops by half. Thus, since the direct effect explicitly excludes the substitutions of capital and labor, its size does not depend on the substitutability of capital and labor.

Exercise 13B.16

Show that the short and long run input demand curves calculated for the production function $f(\ell,k)=20\ell^{2/5}k^{2/5}$ in equation (13.33) and (13.31) are downward sloping.

Answer: We simply have to show that the derivatives with respect to input prices are less than zero. For the short run labor demand curve, this derivative is

$$\frac{\partial \ell_{k=256}(p,w)}{\partial w} = -\left(\frac{2}{3}\right)\frac{2150}{p^{1/3}w^{5/3}} = -\frac{4300}{3p^{1/3}w^{5/3}} < 0. \tag{13B.16.i}$$

For the long run, the derivatives of the input demands with respect to their prices are

$$\frac{\partial \ell(p,w,r)}{\partial w} = -98304\frac{p^5}{r^2w^4} < 0 \text{ and } \frac{\partial k(p,w,r)}{\partial r} = -98304\frac{p^5}{w^2r^4} < 0. \tag{13B.16.ii}$$

Exercise 13B.17

Can you make sense of the fact that the demand for labor falls less (both in the short and long run) the more complementary labor and capital are in production?

Answer: In the short run, the firm has a fixed amount of capital that is already paid for. If capital and labor are relatively complementary in production, then using the capital implies the firm has to hold onto much of its labor in the short run. In the long run, the firm has the opportunity to substitute away from labor and into more capital — but again, if the two inputs are relatively complementary, the firm cannot employ much of such substituting behavior. If, on the other hand, capital and labor are very substitutable, then it is easier for firms to adjust labor in the short run (because it is not that needed to keep the fixed capital productively employed) as well as the long run (because it is now easier to substitute away from the more expensive labor and toward capital when the latter can be adjusted.)

Exercise 13B.18

What value of ρ — and what implied elasticity of substitution between capital and labor — corresponds to the "in between case"?

Answer: The "in between case" happens when the firm does not adjust its capital in the long run (following a change in the wage) — which occurs in the table when $\rho = -0.5$. In this case, the firm reduces its labor in the short run but makes no further adjustments (to either capital or labor) in the long run because it happens to be the case that the marginal revenue product of capital is exactly equal to the rental rate after the short run labor adjustment has been made. We learned in Chapter 5 that the elasticity of substitution for a CES function is $1/(1+\rho)$ — thus, when $\rho = -0.5$, the elasticity of substitution is 2.

Exercise 13B.19

Can you identify in Table 13.4 the relationship of the substitutability of capital and labor to the degree of short versus long run response in labor demand from an increase in output price? Is this consistent with what emerges in Graph 13.12?

Answer: As ρ approaches -1, labor and capital approach perfect substitutes in production; whereas as ρ approaches ∞, labor and capital approach perfect complements. The table shows that, the more substitutable are labor and capital, the more labor demand ($\ell_k(25,10,10)$) responds in the short run (compared to $\ell(20,10,10)$.) In the next column ($\ell(25,10,10)$), the table shows that labor falls from the short to the long run when capital and labor are relatively substitutable whereas it rises from the short to the long run when capital and labor are relatively complementary. Since the long run adjustment of capital is the same regardless of the substitutability of capital and labor (because an increase in price does not change the slopes of isocosts and thus keeps the firm maximizing along the same ray from the origin in the isocost map), this implies that the long run response of increasing capital causes the firm to reduce its labor from the short to the long run when the inputs are relatively substitutable but increase it when they are relatively complementary. This should make intuitive sense: When the inputs are relatively substitutable, the (long run) increase in capital makes labor less productive on the margin — causing the firm to let go of some of its labor. When inputs are relatively complementary, on the other hand, the (long run) increase in capital makes labor more productive on the margin — causing firms to want to hire more. In the first case, the firm does not need additional workers to work the additional machines that come on line in the long run (and can in fact replace some of them with machines) — while in the latter case the firm needs additional workers to work the new machines that come on line.

13C Solutions to Odd Numbered End-of-Chapter Exercises

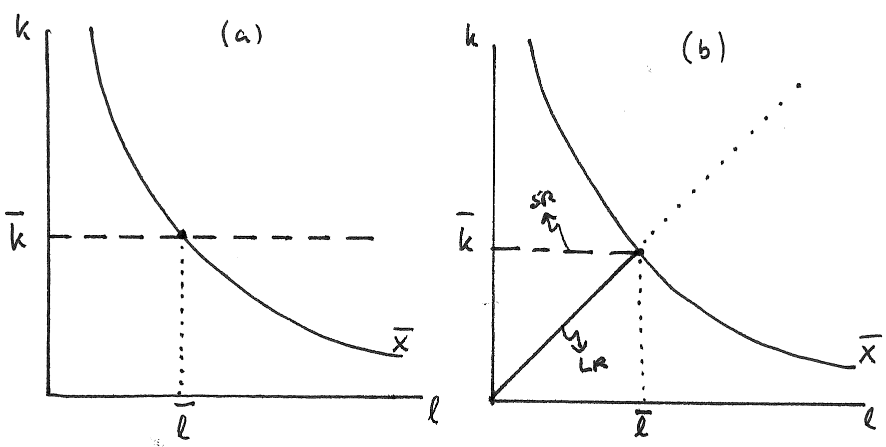
Exercise 13.1

The following problem explores the relationship between maximizing profit in the short and long run when capital is fixed in the short run.

A: *Suppose you have a homothetic production technology and you face output price p and input prices (w, r).*

(a) *On a graph with labor ℓ on the horizontal and capital k on the vertical axis, draw an isoquant and label a point on that isoquant as $(\bar{\ell}, \bar{k})$.*

Answer: This is done in both panels of Exercise Graph 13.1.



Exercise Graph 13.1 : Short Run and Long Run Profit Maximization

(b) *Suppose that the point in your graph represents a profit maximizing production plan. What has to be true at this point?*

Answer: It must be true that $pMP_k = r$ and $pMP_\ell = w$ — i.e. the marginal revenue products of the inputs must be equal to their input prices.

(c) *In your graph, illustrate the slice along which the firm must operate in the short run.*

Answer: This slice is indicated by the horizontal (dashed) line in panel (a) of Exercise Graph 13.1. Capital is fixed at $\bar{k}$ along this line.

(d) *Suppose that the production technology has decreasing returns to scale throughout. If p falls, can you illustrate all the possible points in your graph where the new profit maximizing production plan will lie in the long run? What about the short run?*

Answer: This is illustrated in panel (b) of Exercise Graph 13.1. The horizontal dashed line is a portion of the short run production slice — that portion that yields less output than the output along the isoquant $\overline{x}$. Since we know from Chapter 11 that short run production falls as price falls, any point along this dashed line can result from a decrease in p. The solid portion of the ray emanating from the origin represents the production plans that could result in the long run when firms can adjust capital. Again, we know that output will fall with a decrease in price, but in the long run capital will adjust so that the ratio of capital to labor remains constant. This is because input prices have not changed. Thus the ratio of input prices remains unchanged, which means all cost minimizing input bundles lie on this ray from the origin through the original profit maximizing bundle $(\overline{\ell}, \overline{k})$. (And, of course, it must be the case that any new long run profit maximizing production plan — while involving less output — still produces output in the least costly way.)

(e) *What condition that is satisfied in the long run will typically not be satisfied in the short run?*

Answer: Since the firm can only adjust labor in the short run, the firm will let go of labor until $pMP_\ell = w$ — i.e. until the marginal revenue product of labor is equal to the wage. However, the firm cannot adjust capital in the short run — which implies that the marginal revenue product condition will not hold for capital in the short run. In fact, since the firm will release capital in the long run, we know that it must be the case that $pMP_k < r$ at the short run profit maximum.

(f) *What qualification would you have to make to your answer in (d) if the production process had initially increasing but eventually decreasing returns to scale?*

Answer: In that case, there is a portion of the solid part of the ray (in panel (b) of the Graph) along which the firm will not profit maximize no matter how much price falls — because the firm would make a negative profit along that portion. Without knowing more, we cannot tell exactly where the solid part of the ray in the graph should end in this case — but we do know it should not extend all the way down to the origin in the graph.

B: *Consider the Cobb-Douglas production function $x = f(\ell, k) = A\ell^\alpha k^\beta$.*

(a) *For input prices (w, r) and output price p, calculate the long run input demand and output supply functions assuming $0 < \alpha, \beta \leq 1$ and $\alpha + \beta < 1$.*

Answer: Solving the usual profit maximization problem

$$\max_{\ell,k} \; pA\ell^\alpha k^\beta - w\ell - rk, \tag{13.1.i}$$

we get the input demand functions

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^\beta}{w^{(1-\beta)}r^\beta}\right)^{1/(1-\alpha-\beta)} \quad \text{and} \quad k(w,r,p) = \left(\frac{pA\alpha^\alpha\beta^{(1-\alpha)}}{w^\alpha r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \tag{13.1.ii}$$

Plugging these into the production function and simplifying, we then get the output supply function

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{13.1.iii}$$

(b) *How would your answer change if $\alpha+\beta \geq 1$?*

Answer: If $\alpha+\beta = 1$, then the production function has constant returns to scale — which implies that either all output levels are profit maximizing or only one of the corners (i.e. zero or infinity) is optimal. If $\alpha+\beta > 1$, the production function has increasing returns to scale — which implies that the firm should produce an infinite amount of output. (Of course this does not make sense in light of the fact that we are assuming price-taking behavior — i.e. we are assuming firms small enough relative to the market such that they cannot influence price.)

(c) *Suppose that capital is fixed at $\overline{k}$ in the short run. Calculate the short run input demand and output supply functions.*

Answer: We need to use the short run slice of the production function that holds $\overline{k}$ fixed — i.e. $x = \left[A\overline{k}^{\beta}\right]\ell^{\alpha}$. The short run profit maximization problem is then

$$\max_{\ell} \; p\left[A\overline{k}^{\beta}\right]\ell^{\alpha} - w\ell, \tag{13.1.iv}$$

where we do not take into account the *expense* of capital that is fixed and we treat the bracketed term as a constant. (Even if we did include it, it would drop out since only ℓ is a choice variable and the capital expense term would simply drop out as we take first order conditions). Solving this in the usual way, we get the short run labor demand function

$$\ell_{\overline{k}}(p,w) = \left(\frac{\alpha p A\overline{k}^{\beta}}{w}\right)^{1/(1-\alpha)}. \tag{13.1.v}$$

Substituting this back into the production function and simplifying, we get the short run output supply function

$$x_{\overline{k}}(p,w) = \left(A\overline{k}^{\beta}\right)^{1/(1-\alpha)}\left(\frac{\alpha p}{w}\right)^{\alpha/(1-\alpha)}. \tag{13.1.vi}$$

(d) *What has to be true about α and β for these short run functions to be correct?*

Answer: It has to be the case that the production process has decreasing returns to scale — i.e. $\alpha+\beta < 1$. Otherwise, the true solution is a corner solution that will not be picked up by our usual optimization method.

(e) *Suppose $\overline{k} = k(w,r,p)$ (where $k(w,r,p)$ is the long run capital demand function you calculated in part (a).) What is your optimal short run labor demand and output supply in that case?*

Answer: If we plug our long run capital demand function in for $\overline{k}$ in the short run labor demand function, we get, after simplifying the expression,

$$\ell_{k(w,r,p)}(p,w) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{13.1.vii}$$

Similarly, if we plug the long run capital demand function in for $\overline{k}$ in short run supply function, we get

$$x_{k(w,r,p)}(p,w) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}.\right)^{1/(1-\alpha-\beta)}. \tag{13.1.viii}$$

(f) *How do your answers compare to the long run labor demand function $\ell(w,r,p)$ and the long run supply function $x(w,r,p)$ you calculated in part (a)? Can you make intuitive sense of this?*

Answer: The short run labor demand and output supply functions we calculated are exactly equal to the long run labor demand and output supply functions calculated earlier; i.e.

$$\ell_{k(w,r,p)}(p,w) = \ell(w,r,p) \quad \text{and} \quad x_{k(w,r,p)}(p,w) = x(w,r,p). \tag{13.1.ix}$$

This should make intuitive sense: If I provide you in the short run with the optimal level of capital for you to reach the long run profit maximum at current input and output prices, then you'll choose your labor input exactly as you would in the long run. Put differently, we are "fixing" capital at exactly the long run quantity — which means there is nothing to keep you from implementing the long run profit maximizing production plan immediately.

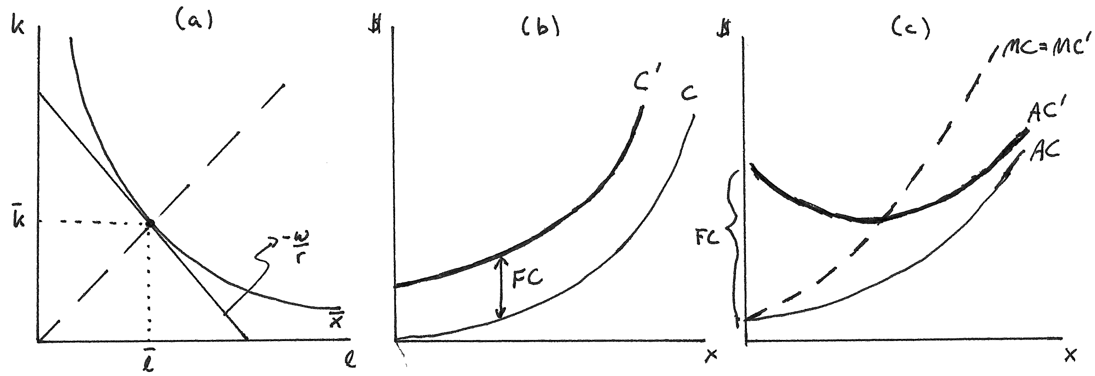
Exercise 13.3

In this exercise we add a (long run) fixed cost to the analysis.

A: *Suppose the production process for a firm is homothetic and has decreasing returns to scale.*

(a) *On a graph with labor ℓ on the horizontal and capital k on the vertical axis, draw an isoquant corresponding to output level $\overline{x}$. For some wage rate w and rental rate r, indicate the cost minimizing input bundle for producing $\overline{x}$.*

Answer: This is illustrated in panel (a) of Exercise Graph 13.3 where the input bundle $(\overline{\ell},\overline{k})$ is the cost minimizing input bundle to produce $\overline{x}$.



Exercise Graph 13.3 : Fixed Costs

(b) *Indicate in your graph the slice of the production frontier along which all cost minimizing input bundles lie for this wage and rental rate.*

Answer: Since the production process is homothetic, all tangencies of isocosts with slope $-w/r$ with isoquants for different output levels will lie on the ray emanating from the origin and passing through $(\bar{\ell}, \bar{k})$. This is also illustrated in panel (a) of Exercise Graph 13.3.

(c) *In two separate graphs, draw the (total) cost curve and the average cost curve with the marginal cost curve.*

Answer: This is illustrated in panels (b) and (c) of Exercise Graph 13.3 as C, AC and MC. Since the slice of the production frontier indicated by the dashed ray in panel (a) has decreasing returns, the shape of the cost function must be such that cost increases at an increasing rate as x goes up. The same then holds for AC, with the MC beginning at the same point as AC but lying above AC throughout.

(d) *Suppose that, in addition to paying for labor and capital, the firm has to pay a recurring fixed cost (such as a license fee). What changes in your graphs?*

Answer: Nothing changes in panel (a) — because the fact that the firm has to pay some cost to begin producing does not change how much labor and capital will be needed to reach different isoquants. The cost curve, however, shifts up to C', with C and C' parallel to each other and the difference being the FC. The marginal cost curve, however, remains unchanged since fixed costs do not enter the *additional* cost of producing output. Finally, the average cost curve moves up but, unlike the cost curve, not in a parallel fashion. It increases by the FC when $x = 1$ (because the average fixed cost is FC/x). As x increases, however, FC/x falls — which causes the new average cost curve AC' to converge to the original AC as x gets large.

(e) *What is the firm's exit price in the absence of fixed costs? What happens to that exit price when a fixed cost is added?*

Answer: In the absence of fixed costs, the firm's exit price is equal to the marginal cost of producing the first unit of output — because that is where the marginal cost curve crosses the AC curve. When fixed costs are introduced, however, the exit price rises to the lowest point of the new U-shaped average cost curve AC' where the unchanged MC curve crosses it. Thus, the exit price increases.

(f) *Does the firm's supply curve shift as we add a fixed cost?*

Answer: No, the supply curve does not shift, but it does become "shorter". It does not shift because the MC curve does not shift. It becomes "shorter" because the exit price increases. Thus, the supply curve before the introduction of the fixed cost is the entire MC curve in panel (c) of Exercise Graph 13.3, but after the FC is introduced, it shrinks to only the portion of the MC curve that lies above AC'.

(g) *Suppose that the cost minimizing input bundle for producing $\overline{x}$ that you graphed in part (a) is also the profit maximizing production plan before a fixed cost is considered. Will it still be the profit maximizing production plan after we include the fixed cost in our analysis?*

Answer: This will still be the profit maximizing production plan *if* it is optimal for the firm not to exit. In that case, price is sufficiently high relative to w and r such that it crosses MC above AC' in panel (c) of Exercise Graph 13.3. However, it may be the case that the introduction of FC implies that it is no longer profit maximizing to produce — and that a corner solution of producing nothing is optimal. This occurs if the price falls below AC'.

B: *As in exercises 13.1 and 13.2, suppose the production process is again characterized by the production function $x = f(\ell,k) = A\ell^{\alpha}k^{\beta}$ with $0 < \alpha, \beta \leq 1$ and $\alpha + \beta < 1$.*

(a) *If you have not already done so in a previous exercise, derive the (long run) cost function for this firm.*

Answer: The long run cost function (solved from the cost minimization problem) is

$$C(w,r,x) = w\ell(w,r,x) + r(w,r,x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \quad (13.3.i)$$

(b) *Now suppose that, in addition to the cost associated with inputs, the firm has to pay a recurring fixed cost of FC. Write down the cost minimization problem that includes this FC. Will the conditional input demand functions change as a result of the FC being included?*

Answer: The cost minimization problem would now be

$$\min_{\ell,k} \; w\ell + rk + FC \text{ subject to } x = A\ell^{\alpha}k^{\beta}. \quad (13.3.ii)$$

When we now write down the Lagrange function and take derivatives to get the first order conditions from which we derive the conditional input demand functions, the FC term will disappear since it enters the objective function as a constant. Thus, the conditional input demand functions will be unchanged by the addition of a FC term. This should make intuitive sense: Just because the firm has to pay something like a license fee to start producing does not mean it doesn't need exactly as much capital and labor to produce any given level of output as it did before.

(c) *Write down the new cost function and derive the marginal and average cost functions from it.*

Answer: Since the conditional input demands are no different than they were before the FC, the new cost function is the same as the one derived in (a) except that we also have to include the fixed cost; i.e. it now becomes

$$C(w,r,x) = w\ell(w,r,x) + r(w,r,x) = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} + FC. \tag{13.3.iii}$$

From this, we can derive the marginal cost function

$$MC(w,r,x) = \frac{\partial C(w,r,x)}{\partial x} = \left(\frac{w^\alpha r^\beta x^{(1-\alpha-\beta)}}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} \tag{13.3.iv}$$

and the average cost function

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x^{(1-\alpha-\beta)}}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} + \frac{FC}{x}. \tag{13.3.v}$$

Note that the marginal cost function is the same as it would be without the FC but the AC function is not.

(d) *What is the shape of the average cost curve? How does its lowest point change with changes in the FC?*

Answer: We can infer the shape of the AC curve by checking whether its derivative with respect to x is positive or negative. The derivative is

$$\frac{\partial AC(w,r,x)}{\partial x} = \left[(1-\alpha-\beta)\left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}\right] - \frac{FC}{x^2}. \tag{13.3.vi}$$

The bracketed term is positive (since $\alpha+\beta<1$) but the FC term is negative. Furthermore, as x approaches zero, the first term is smaller than the absolute value of the second term — implying an initially negative sign and thus an initially downward sloping AC curve. As x gets larger, however, the absolute value of the FC term gets smaller, with the positive bracketed term eventually outweighing the negative FC term. Thus, at some point, the slope of the AC curve becomes positive. This implies a U-shape to the AC curve. And, as FC increases, only the second term changes while the bracketed first term remains the same. Thus, the AC curve will have negative slope for a larger range of x — implying that the bottom of the U moves to the right as FC increases. We can of course also infer this from the fact that the upward sloping MC curve is unchanged as FC increases — because the MC curve must cross AC no matter how high FC gets. Thus, as FC pushes up the AC curve, it must be that its lowest point slides up along the unchanged MC curve. And, to calculate the level of output at which the AC reaches its lowest point, we can set equation (13.3.vi) to zero and solve for x to get

$$x = \left(\frac{A\alpha^\alpha \beta^\beta}{w^\alpha r^\beta}\right)\left(\frac{FC}{(1-\alpha-\beta)}\right)^{(\alpha+\beta)} \tag{13.3.vii}$$

and note again that x increases as FC increases.

(e) *Does the addition of a FC term change the (long run) marginal cost curve? Does it change the long run supply curve?*

Answer: No, the FC term does not appear in the MC equation and thus has no impact on the marginal cost curve. This, of course, makes intuitive sense — a fixed cost is paid before production even begins and therefore does not impact the cost of producing additional units of output. Since the long run supply curve is a portion of the long run MC curve, we know that the supply curve is therefore not shifted by changes in the fixed cost. However, since the supply curve is that portion of MC that lies above the AC curve, and since AC shifts up with an increase in FC, the supply curve "shrinks" as FC increases.

(f) *How would you write out the profit maximization problem for this firm including fixed costs? If you were to solve this problem, what role would the FC term play?*

Answer: The profit maximization problem would then be

$$\max_{\ell,k} \; pA\ell^\alpha k^\beta - w\ell - rk - FC. \tag{13.3.viii}$$

When we take first order conditions, we are taking derivatives of the objective function above — and thus FC disappears from the first order conditions that are used to solve the maximization problem. As a result, none of the input demand or output supply functions will change when FC is included in the profit maximization problem.

(g) *Considering not just the math but also the underlying economics, does the addition of the FC have any implications for the input demand and output supply functions?*

Answer: We see from the math that the functions produced by our optimization problem are unchanged. However, we also know that the math does not identify corner solutions. In the absence of a fixed cost, we do not have to worry about such corner solutions so long as the production function has decreasing returns to scale — but when we add a fixed cost, we know from what we did in the earlier parts of the exercise that the AC curve takes on a U-shape as a result of the addition of the FC. Thus, while the exit price before a FC term is added is zero, it is now at the lowest point of the AC curve. This implies that the functions produced by the math are correct only for sufficiently high prices and/or sufficiently low wages and rental rates. If p gets too high relative to w and r, the firm should exit and produce nothing rather than the positive amount indicated by the math. This is analogous to our conclusion that only a portion of the MC curve will be the supply curve when there is a FC while the entire MC curve is the supply curve (under decreasing returns to scale) when there is no fixed cost.

Exercise 13.5

We will often assume that a firm's long run average cost curve is U-shaped. This shape may arise for two different reasons which we explore in this exercise.

A: *Assume that the production technology uses labor ℓ and capital k as inputs, and assume throughout this problem that the firm is currently long run profit maximizing and employing a production plan that is placing it at the lowest point of its long run AC curve.*

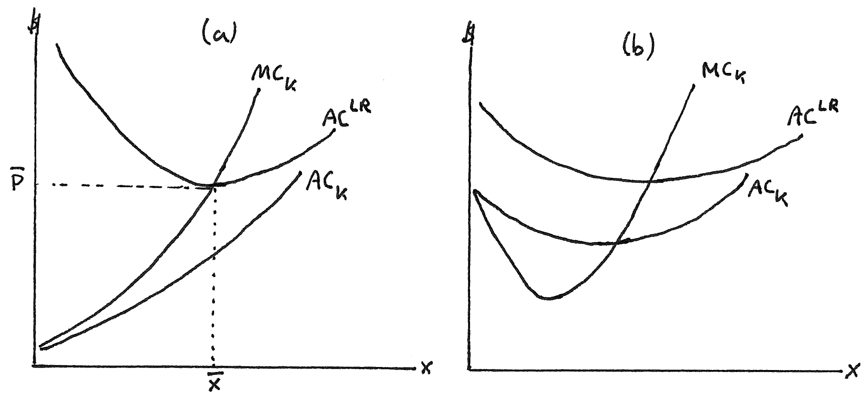
(a) *Suppose first that the technology has decreasing returns to scale but that, in order to begin producing each year, the firm has to pay a fixed license fee F. Explain why this causes the long run AC curve to be U-shaped.*

Answer: The long run AC curve is U-shaped in this case because the fixed cost F, while only an expense in the short run and therefore not included in short run AC curves, it is a real economic cost in the long run. At low levels of output, the average fixed cost F/x is large (because x is small) — causing AC to be high. As output increases, the average fixed cost F/x falls (because x gets large) — and therefore becomes a diminishing factor in the long run AC curve. Instead, the fact that the production process has decreasing returns to scale pushes AC up as x increases.

(b) *Draw a graph with the U-shaped AC curve from the production process described in part (a). Then add to this the short run MC and AC curves. Is the short run AC curve also U-shaped?*

Answer: This is illustrated in panel (a) of Exercise Graph 13.5(1).

The fact that the firm is currently profit maximizing at the lowest point of its long run AC curve implies that output price must be $\overline{p}$. The short run MC_k curve must then go through this lowest point on the AC^{LR} curve. Furthermore, since the production technology has decreasing returns to



Exercise Graph 13.5(1) : U-shaped Average Cost Curves

scale, it must be that any slice that holds capital fixed must also have decreasing marginal product of labor. Thus, the short run production function (that holds capital fixed) has decreasing returns to scale, and there are no fixed costs in the short run, only fixed expenses. This implies that the MC_k curve must be upward sloping, causing the short run AC_k curve to lie below it and be similarly upward sloping.

(c) *Next, suppose that there are no fixed costs in the long run. Instead, the production process is such that the marginal product of each input is initially increasing but eventually decreasing, and the production process as a whole has initially increasing but eventually decreasing returns to scale. (A picture of such a production process was given in Graph 12.16 in the previous chapter.) Explain why the long run AC curve is U-shaped in this case.*

Answer: The fact that the production process has initially increasing but eventually decreasing returns to scale implies that the long run average costs must initially fall but will eventually increase in the decreasing returns to scale portion of the production process. The reasoning is identical to that for the single input case with production frontiers that initially get steeper but eventually get shallower.

(d) *Draw another graph with the U-shaped AC curve. Then add the short run MC and AC curves.*

Answer: This is done in panel (b) of Exercise Graph 13.5(1). The U-shape of the short run MC_k and AC_k curves is due to the fact that, for any fixed level of capital, the short run production function has initially increasing but eventually decreasing returns to scale. That, in turn, arises from the fact that the production process has initially increasing but eventually decreasing marginal product of labor. The short run MC_k curve again intersects the lowest point of the long run AC^{LR} curve because the firm is initially profit maximizing at the lowest point of the long run AC curve.

(e) *Is it possible for short run AC curves to* not *be U-shaped if the production process has initially increasing but eventually decreasing returns to scale?*

Answer: Yes, this is possible. In order for the AC^{LR} curve to assume a U-shape (in the absence of fixed costs), the production process must have initially increasing returns to scale (that eventually turn into decreasing returns to scale). In order for the short run AC_k curve to have a U-shape, it must be that the short run production function with fixed capital initially has increasing but eventually decreasing returns to scale. But this will occur only if the marginal product of labor is initially increasing and eventually decreasing. If the marginal product of labor for the fixed capital level k is diminishing throughout, then the short run AC_k curve will be upward sloping throughout because there are no fixed costs in the short run. Since it is possible for the marginal product of each input to be decreasing but still to have increasing returns to scale of the entire production process (that varies both capital and labor), it is possible to have the U-shaped AC curve in the long run but not the short run (in the absence of fixed costs).

B: *Suppose first that the production process is Cobb-Douglas, characterized by the production function $x = f(\ell, k) = A\ell^{\alpha}k^{\beta}$ with $\alpha, \beta > 0$ and $\alpha + \beta < 1$.*

(a) *In the absence of fixed costs, you should have derived in exercise 13.2 that the long run cost function for this technology is given by*

$$C(w,r,x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.5}$$

If the firm has long run fixed costs F, what is its long run average cost function? Is the average cost curve U-shaped?

Answer: The long run average cost function is then simply $C(w,r,x)$ divided by x plus FC/x — which gives us

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} + \frac{FC}{x}. \tag{13.5.i}$$

In B(d) of exercise 13.3, we already argued that this must be U-shaped. Its derivative is

$$\frac{\partial AC(w,r,x)}{\partial x} = \left[(1-\alpha-\beta)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}\right] - \frac{FC}{x^2}. \tag{13.5.ii}$$

which, when set to zero, gives us the output level at which the long run AC curve reaches its lowest point:

$$x = \left(\frac{A\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)\left(\frac{FC}{(1-\alpha-\beta)}\right)^{(\alpha+\beta)} \tag{13.5.iii}$$

(b) *What is the short run cost curve for a fixed level of capital $\overline{k}$? Is the short run average cost curve U-shaped?*

Answer: The short run production function with fixed $\overline{k}$ is $x = f(\ell) = \left[A\overline{k}^{\beta}\right]\ell^{\alpha}$ which, when solved for ℓ, gives us the conditional labor demand of

$$\ell_{\overline{k}}(x) = \left(\frac{x}{A\overline{k}^{\beta}}\right)^{1/\alpha} \tag{13.5.iv}$$

Multiplying by wage w, we then get the short run cost function

$$C_{\overline{k}}(x) = w\left(\frac{x}{A\overline{k}^{\beta}}\right)^{1/\alpha} \tag{13.5.v}$$

from which we can derive the short run $MC_{\overline{k}}$ and $AC_{\overline{k}}$ functions

$$MC_{\overline{k}}(x) = \frac{w}{\alpha}\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha} \quad \text{and} \quad AC_{\overline{k}}(x) = w\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha}. \tag{13.5.vi}$$

These are both increasing in x — and thus the short run MC and AC curves slope up. They also converge to zero as x goes to zero. Thus, they give rise to a picture such as the on in panel (a) of Exercise Graph 13.5(1).

(c) *Now suppose that the production function is still $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ but now $\alpha+\beta > 1$. Are long run average and marginal cost curves upward or downward sloping? Are short run average cost curves upward or downward sloping? What does your answer depend on?*

Answer: The long run MC curve is downward sloping because of increasing returns to scale. The long run AC curve is similarly downward sloping (and starts above MC because of the long run fixed costs). (You can see this from the equation (13.5.i) where the exponent on x is now negative in the first term — implying that both terms decline in x.) Whether or not the short run MC and AC curves slope down depends on whether α is less than or greater than 1. If it is less than 1, then x enters the short run MC and AC functions in equation (13.5.vi) with positive exponent — implying that these costs increase with x. When $\alpha > 1$, on the other hand, x enters with negative exponent — causing the cost curves to fall with x. Thus, increasing returns to scale is consistent with both upward and downward sloping short run MC and AC curves — the key is whether the marginal product of labor increases or decreases.

(d) *Next, suppose that the production technology were given by the equation*

$$x = f(\ell, k) = \frac{\alpha}{1 + e^{-(\ell-\beta)} + e^{-(k-\gamma)}} \tag{13.5.vii}$$

where e is the base of the natural logarithm. (We first encountered this in exercises 12.5 and 12.6.) If capital is fixed at $\overline{k}$, what is the short run production function and what is the short run cost function?

Answer: The short run production function is

$$x = f_{\overline{k}}(\ell) = \frac{\alpha}{\left[1 + e^{-(\overline{k}-\gamma)}\right] + e^{-(\ell-\beta)}} \tag{13.5.viii}$$

The short run cost function is then simply this production function solved for ℓ. We can multiply both sides by the denominator of the right hand side, divide both sides by x and then subtract the bracketed term from both sides to get

$$e^{-(\ell-\beta)} = \frac{\alpha}{x} - \left[1 + e^{-(\overline{k}-\gamma)}\right] = \frac{\alpha - \left[1 + e^{-(\overline{k}-\gamma)}\right]x}{x}. \tag{13.5.ix}$$

Taking natural logs of both sides, we can then solve for the conditional short run labor demand function

$$\ell_{\overline{k}}(w, x) = \beta - \ln\left(\frac{\alpha - \left[1 + e^{-(\overline{k}-\gamma)}\right]x}{x}\right) \tag{13.5.x}$$

which, when multiplied by w, gives us the short run cost function

$$C_{\overline{k}}(w, x) = w\beta - w\ln\left(\frac{\alpha - \left[1 + e^{-(\overline{k}-\gamma)}\right]x}{x}\right). \tag{13.5.xi}$$

(e) *What is the short run marginal cost function?*

Answer: Taking the derivative of the short run cost function with respect to x, we get

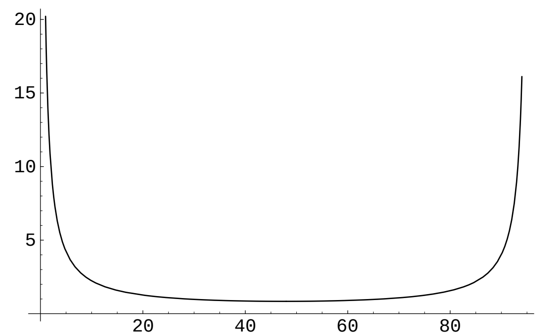
$$MC_{\overline{k}}(w, x) = \frac{w\alpha}{\left(\alpha - \left[1 + e^{-(\overline{k}-\gamma)}\right]x\right)x}. \tag{13.5.xii}$$

(f) *You should have concluded in exercise 12.6 that the long run MC function is $MC(w, r, x) = \alpha(w+r)/(x(\alpha - x))$ and demonstrated that the MC curve (and thus the long run AC curve) is U-shaped for the parameters $\alpha = 100$, $\beta = 5 = \gamma$ when $w = r = 20$. Now suppose capital is fixed at $\overline{k} = 8$. Graph the short run MC curve and use the information to conclude whether the short run AC curve is also U-shaped.*

Answer: The short run MC curve then becomes

$$MC_{\overline{k}=8} \approx \frac{2,000}{(100-1.05x)x} \tag{13.5.xiii}$$

which is plotted in Exercise Graph 13.5(2). This is obviously U-shaped — which causes the short run AC curve to be U-shaped as well.



Exercise Graph 13.5(2) : Short Run MC when $\alpha = 100$, $\beta = \gamma = 5$, $w = r = 20$ and $\overline{k} = 8$

(g) *What characteristic of the this production function is responsible for your answer in part (f)?*

Answer: The characteristic that is causing the U-shaped curves in the short run is the initially increasing marginal product of labor that causes the short run production function to display initially increasing and eventually decreasing returns to scale.

Exercise 13.7

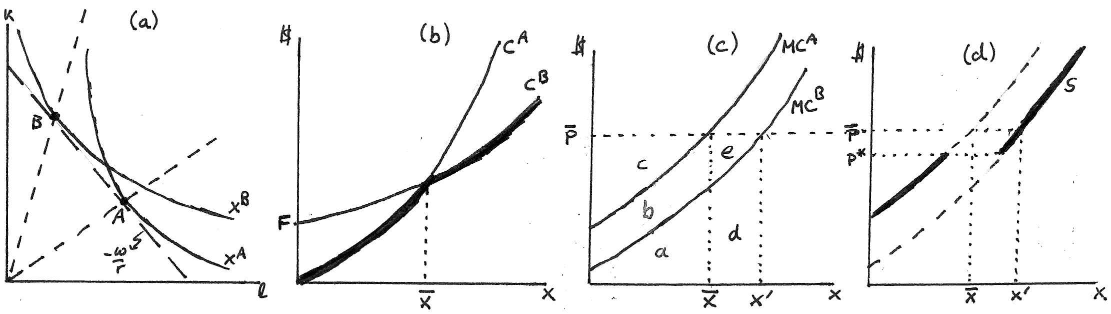
Business Application: *Switching Technologies: Suppose that a firm has two different homothetic, decreasing returns to scale technologies it could use, but one of these is patented and requires recurring license payments F to the owner of the patent. In this exercise, assume that all inputs — including the choice of which technology is used — are viewed from a long run perspective.*

A: *Suppose further that both technologies take capital k and labor ℓ as inputs but that the patented technology is more capital intensive.*

(a) *Draw two isoquants, one from the technology representing the less capital intensive and one representing the more capital intensive technology. Then illustrate the slice of each map that a firm will choose to operate on assuming the wage w and rental rate r are the same in each case.*

Answer: This is illustrated in panel (a) of Exercise Graph 13.7 where x^A labels an isoquant from the non-patented (labor intensive) technology and x^B labels an isoquant from the patented (capital intensive) technology. The isocost with slope $-w/r$ is tangent to the labor intensive technology at A and to the capital intensive technology at B. Since the production technologies are homothetic, the ray passing from the origin through A

represents the slice of the labor intensive production technology along which cost minimizing input bundles lie, and the ray passing from the origin through B represents the slice of the capital intensive technology along which cost minimizing bundles lie. Thus, the ratio of capital to labor used in production is greater in the patented (capital intensive) technology.



Exercise Graph 13.7 : Switching Technologies

(b) *Suppose that the patented technology is sufficiently advanced such that, for any set of input prices, there always exists an output level $\overline{x}$ at which it is (long run) cost effective to switch to this technology. On a graph with output x on the horizontal and dollars on the vertical, illustrate two cost curves corresponding to the two technologies and then locate $\overline{x}$. Then illustrate the cost curve that takes into account that a firm will switch to the patented technology at $\overline{x}$.*

<u>Answer</u>: This is illustrated in panel (b) of Exercise Graph 13.7. The non-patented technology gives rise to the cost curve labeled C^A and the patented technology gives rise to the cost curve C^B. Since there is an $\overline{x}$ such that it is cost effective to switch to the patented technology at $\overline{x}$, the two cost curves must intersect. The C^B curve must furthermore have positive intercept of F because use of the patented technology requires a fixed payment of F before production begins. The bold curve that connects C^A below $\overline{x}$ to C^B above $\overline{x}$ then represents the real cost curve for a firm in this industry — because the firm will, for any given output level, use the technology that minimizes costs.

(c) *What happens to $\overline{x}$ if the license cost F for using the patented technology increases? Is it possible to tell what happens if the capital rental rate r increases?*

<u>Answer</u>: If F increases, then C^B shifts up while C^A remains unchanged. Thus, it must be that the two cost curves intersect at a higher level of output — i.e. an increase in F causes an increase in the production level $\overline{x}$ at which a firm would switch from the non-patented to the patented technology. If the capital rental rate r increases, however, we cannot tell what

will happen to $\overline{x}$. This is because now both cost curves are affected, with the upward shift in C^A by itself causing $\overline{x}$ to decrease while the upward shift of C^B by itself would cause $\overline{x}$ to increase. Which of these effects dominates when both curves shift will depend on the precise nature of the underlying technology as well as the ratio of w to r.

(d) *At $\overline{x}$, which technology must have a higher marginal cost of production? On a separate graph, illustrate the marginal cost curves for the two technologies.*

Answer: In panel (b) of Exercise Graph 13.7, it is clear that the slope of C^A is steeper at $\overline{x}$ than the slope of C^B at that output level. Thus, the marginal cost of production at $\overline{x}$ is higher under the non-patented technology than under the patented technology. This is likely to be true for all output levels — leading to marginal cost curves such as those depicted in panel (c) of Exercise Graph 13.7 where MC^A indicates the marginal cost curve under the non-patented technology and MC^B indicates the marginal cost curve under the patented technology. (It is in principle possible that these marginal cost curves cross in some places — but that would require unusually shaped cost curves in panel (b) where C^B must have an intercept of F, C^A has no such intercept and the two curves cross at $\overline{x}$.)

(e) *At $\overline{x}$, the firm is cost-indifferent between using the two technologies. Recognizing that the marginal cost curves capture all costs that are not fixed — and that total costs excluding fixed costs can be represented as areas under marginal cost curves, can you identify an area in your graph that represents the recurring fixed license fee F?*

Answer: The total cost of producing $\overline{x}$ under the non-patented technology is simply the area under the MC^A curve in panel (c) of Exercise Graph 13.7 — i.e. area $a+b$. (This is because the marginal cost of each unit of output is the additional cost incurred — and when we sum all these "additional costs" we get the total cost if there is no fixed cost of production). Similarly, the total cost minus the fixed cost F under the patented technology is the area under MC^B — i.e. area a. These areas differ by b — i.e. not counting the fixed cost F under the patented technology, the cost of producing $\overline{x}$ is smaller under the patented technology by area b. At $\overline{x}$, however, the total cost (including fixed costs) is equal for the two technologies (as seen in panel (b) of the graph); i.e. $a+b=a+F$. Thus, $F=b$.

(f) *Suppose output price p is such that it is profit maximizing* under the non-patented technology *to produce $\overline{x}$. Denote this as $\overline{p}$. Can you use marginal cost curves to illustrate whether you would produce more or less if you switched to the patented technology?*

Answer: In order for it to be profit maximizing to produce $\overline{x}$ under the non-patented technology, $\overline{p}$ must fall as depicted in panel (c) of Exercise Graph 13.7 — i.e. $\overline{p}$ must intersect MC^A at $\overline{x}$. But $\overline{p}$ intersects MC^B at x' — which implies that, were the firm to switch to the patented technology, it would produce more.

(g) *Would profit be higher if you used the patented or non-patented technology when output price is $\overline{p}$. (*Hint*: Identify the total revenues if the firm produces at $\overline{p}$ under each of the technologies. Then identify the total cost of using the non-patented technology as an area under the appropriate marginal cost curve and compare it to the total costs of using the patented technology as an area under the other marginal cost curve and add to it the fixed fee F.)*

Answer: When selling $\overline{x}$ at $\overline{p}$, total revenue is equal to $\overline{p}$ times $\overline{x}$ — which is equal to the area $a+b+c$ in panel (c) of Exercise Graph 13.7. If the firm uses the non-patented technology to produce $\overline{x}$, its total costs are equal to the area under MC^A — which is equal to area $a+b$. Thus, the profit for a firm using the non-patented technology is $(a+b+c)-(a+b)=c$. If the firm uses the patented technology at price $\overline{p}$, it produces x' and thus earns revenues of $\overline{p}$ times x' — which is equal to area $a+b+c+d+e$. Its costs (not including the fixed F) are equal to the area under MC^B — which is $a+d$. We also concluded that the fixed F is equal to area b. Thus, total costs (including F) are $a+b+d$. Subtracting this from total revenue, we get profit of $a+b+c+d+e-(a+b+d)=c+e$. When price is $\overline{p}$, the firm would therefore earn profit of c by producing $\overline{x}$ under the non-patented technology and profit $c+e$ producing x' using the patented technology. Profit is therefore higher if the firm uses the patented technology when price is $\overline{p}$.

(h) True or False*: Although the total cost of production is the same under both technologies at output level $\overline{x}$, a profit maximizing firm will choose the patented technology if price is such that $\overline{x}$ is profit maximizing under the non-patented technology.*

Answer: This is true. We had identified $\overline{x}$ as the output level at which the two cost curves cross in panel (b) of Exercise Graph 13.7 — and thus total costs are the same under both technologies when firms produce $\overline{x}$. But we also just concluded that, at price $\overline{p}$ at which it is profit maximizing under the non-patented technology to produce $\overline{x}$, the firm can earn more profit by using the patented technology and producing x' (in panel (c) of the graph.)

(i) *Illustrate the firm's supply curve. (*Hint*: The supply curve is not continuous, and the discontinuity occurs at a price below $\overline{p}$.)*

Answer: This is illustrated in panel (d) of Exercise Graph 13.7. Up to some price level below $\overline{p}$, the profit maximizing firm will choose the non-patented technology. Over that range of prices, MC^A therefore forms the supply curve. But at some price — indicated as p^* in the graph — the firm will earn the same profit under both technologies but will produce more under the patented technology. We know that $p^* < \overline{p}$ because of our conclusion that profit using the patented technology is higher at $\overline{p}$ than profit under the non-patented technology. At prices higher than p^*, the firm will then have switched to the patented technology — causing the supply curve from then on to lie on MC^B.

B: *Suppose that the two technologies available to you can be represented by the production functions $f(\ell,k) = 19.125\ell^{0.4}k^{0.4}$ and $g(\ell,k) = 30\ell^{0.2}k^{0.6}$, but technology g carries with it a recurring fee of F.*

(a) *In exercise 13.2 you derived the general form for the 2-input Cobb-Douglas conditional input demands and cost function. Use this to determine the ratio of capital to labor (as a function of w and r) used under these two technologies. Which technology is more capital intensive?*

Answer: Plugging $\alpha = \beta = 0.4$ and $A = 19.125$ into the previously derived formula for input demands, we get that the f technology gives rise to conditional input demands

$$\ell_f(w,r,x) = 0.025\left(\frac{r}{w}\right)^{1/2} x^{5/4} \text{ and } k_f(w,r,x) = 0.025\left(\frac{w}{r}\right)^{1/2} x^{5/4}, \tag{13.7.i}$$

and plugging in $\alpha = 0.2$, $\beta = 0.6$ and $A = 30$, we get that the g technology gives rise to conditional input demands

$$\ell_g(w,r,x) = 0.00625\left(\frac{r}{w}\right)^{3/4} x^{5/4} \text{ and } k_g(w,r,x) = 0.01875\left(\frac{w}{r}\right)^{1/4} x^{5/4}. \tag{13.7.ii}$$

The ratio of capital to labor under the technologies f an g are then (respectively)

$$\frac{k_f(w,r,x)}{\ell_f(w,r,x)} = \frac{w}{r} \text{ and } \frac{k_g(w,r,x)}{\ell_g(w,r,x)} = 3\frac{w}{r}; \tag{13.7.iii}$$

i.e. the capital to labor ratio under technology g is three times as high as under f. (These ratios correspond to the slopes of the rays in panel (a) of Exercise Graph 13.7.) The g technology is therefore more capital intensive.

(b) *Determine the cost functions for the two technologies (and be sure to include F where appropriate).*

Answer: For the f technology, we plug in $\alpha = \beta = 0.4$ and $A = 19.125$ into the previously derived cost function for Cobb-Douglas production; and for the g technology we plug in $\alpha = 0.2$, $\beta = 0.6$ and $A = 30$ — and then we add the fixed technology fee F which has to be paid if g is used. This gives us

$$C_f(w,r,x) = 0.05w^{1/2}r^{1/2}x^{5/4} \text{ and } C_g(w,r,x) = 0.025w^{1/4}r^{3/4}x^{5/4} + F. \tag{13.7.iv}$$

(c) *Determine the output level $\overline{x}$ (as a function of w, r and F) at which it becomes cost effective to switch from the technology f to the technology g. If F increases, is it possible to tell whether $\overline{x}$ increases or decreases? What if r increases?*

Answer: To calculate $\overline{x}$, we need to find where the cost curves intersect (as in panel (b) of Exercise Graph 13.7). We therefore set the two cost functions in equation (13.7.iv)equal to each other and solve for x to get

$$\overline{x} = \left(\frac{40F}{2w^{1/2}r^{1/2} - w^{1/4}r^{3/4}}\right)^{4/5}. \qquad (13.7.v)$$

If F increases, $\overline{x}$ unambiguously increases as well — which makes sense since the fixed cost of using g has increased, it will not be cost effective to switch until a higher level of output. But if r increases, we cannot tell whether $\overline{x}$ will increase or decrease. (We gave some intuition for this in the answer to (c) of part A of this exercise.)

(d) *Suppose $w = 20$ and $r = 10$. Determine the price $\overline{p}$ (as a function of F) at which a firm using technology f would produce $\overline{x}$.*

Answer: From the first cost function in equation (13.7.iv), we can derive the marginal cost under technology f as $MC_f(w,r,x) = 0.0625w^{0.5}r^{0.5}x^{0.25}$. Plugging in $w = 20$ and $r = 10$, we then get $MC_f(20,10,x) = 0.8838835x^{0.25}$. Plugging these same input prices into equation (13.7.v), we get $\overline{x} = 2.0414474F^{0.8}$. Thus, the marginal cost of producing $\overline{x}$ under technology f is

$$MC_f(20,10,\overline{x}) = 0.8838835\left(2.0414474F^{0.8}\right)^{0.25} = 1.0565245F^{1/5} = \overline{p}, \qquad (13.7.vi)$$

where the last equality simply emerges from the fact that $p = MC$ for profit maximizing firms.

(e) *How much would the firm produce with technology g if it faces $\overline{p}$? Can you tell whether, regardless of the size of F, this is larger or smaller than $\overline{x}$ (which is the profit maximizing quantity when the firm used technology f and faces $\overline{p}$)?*

Answer: From the second cost function in equation (13.7.iv) we can derive the marginal cost function under technology g as $MC_g(w,r,x) = 0.03125w^{0.25}r^{0.75}x^{0.25}$ which becomes $MC_g(20,10,x) = 0.3716272x^{0.25}$ when $w = 20$ and $r = 10$. Setting this equal to $\overline{p} = 1.0565245F^{1/5}$ from equation (13.7.vi) and solving for x, we get $x \approx 65.33F^{0.8}$. We can then conclude that

$$x \approx 65.33F^{0.8} > 2.0414474F^{0.8} = \overline{x}; \qquad (13.7.vii)$$

i.e. regardless of what value F takes (as long as $F > 0$), the profit maximizing production level will be higher using technology g than when using technology f when the price is such that $\overline{x}$ is profit maximizing under technology f. We illustrated this intuitively in panel (c) of Exercise Graph 13.7.

(f) *The (long run) profit function for a Cobb-Douglas production function $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ is*

$$\pi(w,r,p) = (1-\alpha-\beta)\left(\frac{Ap\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \qquad (13.7.viii)$$

Can you use this to determine (as a function of p, w and r) the highest level of F at which a profit maximizing *firm will switch from f to g? Call this* $\overline{F}(w,r,p)$.

Answer: Plugging $\alpha = \beta = 0.4$ and $A = 19.125$ into the profit function, we get the profit function for the technology f; and plugging in $\alpha = 0.2$, $\beta = 0.6$ and $A = 30$, we get the profit function for technology g. This gives us

$$\pi_f(w,r,p) = 13,100\left(\frac{p^5}{w^2r^2}\right) \text{ and } \pi_g(w,r,p) = 209,952\left(\frac{p^5}{wr^3}\right). \quad (13.7.\text{ix})$$

For the production function g, however, we also need to take into account the fixed cost F — thus subtract F from $\pi_g(w,r,p)$. The fixed cost $\overline{F}$ at which the firm will switch to the technology g is the fixed cost at which profit is equal for both technologies. Thus, we need to solve

$$13,100\left(\frac{p^5}{w^2r^2}\right) = 209,952\left(\frac{p^5}{wr^3}\right) - F \quad (13.7.\text{x})$$

for F. This gives us

$$\overline{F} = 209,952\left(\frac{p^5}{wr^3}\right) - 13,100\left(\frac{p^5}{w^2r^2}\right). \quad (13.7.\text{xi})$$

(g) *From your answer to (f), determine (as a function of w, r and F) the price p^* at which a profit maximizing firm will switch from technology f to technology g.*

Answer: We simply need to solve equation (13.7.x) for p which gives us

$$p^* = \left(\frac{Fw^3r^3}{209,952w^2 - 13,100wr}\right)^{1/5}. \quad (13.7.\text{xii})$$

(h) *Suppose again that $w = 20$, $r = 10$. What is p^* (as a function of F)? Compare this to $\overline{p}$ you calculated in part (d) and interpret your answer in light of what you did in A(i).*

Answer: Plugging $w = 20$ and $r = 10$ into equation (13.7.xii), we get $p^* = 0.6288325F^{0.2}$. Comparing this to our answer in part (d), we conclude that

$$p^* = 0.6288325F^{1/5} < 1.0562545F^{1/5} = \overline{p}. \quad (13.7.\text{xiii})$$

Thus, the price p^* at which a profit maximizing firm switches from technology f to technology g lies below the price $\overline{p}$ at which a firm using production technology f would produce $\overline{x}$ at which the cost of production is equal for the two technologies. This implies that the supply curve switches from the MC_f curve to MC_g at p^* and below $\overline{p}$ — which we illustrated intuitively in panel (c) of Exercise Graph 13.7.

(i) *Suppose (in addition to the values for parameters specified so far) that $F =$ 1000. What is $\overline{p}$ and p^*? At the price at which the profit maximizing firm is indifferent between using technology f and technology g, how much does it produce when it uses f and how much does it produce when it uses g?*

Answer: Plugging $F = 1000$ into the equations for $\overline{p}$ and p^*, we get $\overline{p} = 1.0562545(1000)^{1/5} \approx \$4.21$ and $p^* = 0.6288325(1000)^{1/5} = 2.5034 \approx \$2.50$. Plugging $\alpha = \beta = 0.4$, $A = 19.125$, $w = 20$, $r = 10$ and $p = 2.50$ into the supply function $x(w,r,p)$, we get $x_f^* \approx 64.32$; and, plugging $\alpha = 0.2$. $\beta = 0.6$, $A = 19.125$, $w = 20$, $r = 10$ and $p = 2.50$ in the supply function, we get $x_g^* \approx 2,062$.

(j) *Continuing with the values we have been using (including $F = 1000$), can you use your answer to (a) to determine how much labor and capital the firm hires at p^* under the two technologies? How else could you have calculated this?*

Answer: Plugging $r = 10$, $w = 20$ and $x_f^* = 64.32$ into the conditional labor and capital demands from equation (13.7.i), we get $\ell_f^* = 3.22$ and $k_f^* = 6.44$ — the cost minimizing labor and capital inputs if the firm uses the f technology to produce $x_f^* = 64.32$ (which in turn is the profit maximizing output level at $p^* = 2.50$.) Similarly, if we plug $r = 10$, $w = 20$ and $x_g^* = 2,062$ into the conditional labor and capital demands from equation (13.7.ii), we get $\ell_g^* = 51.6$ and $k_g^* = 310$ — the cost minimizing labor and capital inputs if the firm uses the g technology to produce $x_g^* = 2,062$ (which in turn is the profit maximizing output level at $p^* = 2.50$.) You could also of course have derived the unconditional labor demand and capital demand functions — either by doing the profit maximization problems or using Hotelling's lemma.

(k) *Use what you have calculated in (i) and (j) to verify that profit is indeed the same for a firm whether it uses the f or the g technology when price is p^* (when the rest of the parameters of the problem are as we have specified them in (i) and (j).) (Note: If you rounded some of your previous numbers, you will not get exactly the same profit in both cases — but if the difference is small, it is almost certainly just a rounding error.)*

Answer: Profit is simply revenue minus costs. We can then calculate the profit under each technology as

$$\pi_f = p^* x_f^* - w\ell_f^* - rk_f^* = 2.5034(64.32) - 20(3.22) - 10(6.44) \approx \$32 \qquad (13.7.\text{xiv})$$

and

$$\pi_g = p^* x_g^* - w\ell_g^* - rk_g^* = 2.5034(2062) - 20(51.6) - 10(310) - 1000 \approx \$32. \qquad (13.7.\text{xv})$$

Exercise 13.9

Business and Policy Application: *Fixed amount of Land for Oil Drilling: Suppose that your oil company is part of a competitive industry and is using three rather than two inputs — labor ℓ, capital k and land L — to produce barrels of crude oil denoted by x. Suppose that the government, due to environmental concerns, has limited the amount of land available for oil drilling — and suppose that it has assigned each oil company $\overline{L}$ acres of such land. Assume throughout that oil sells at a market price p, labor at a market wage of w and capital at a rental rate r — and these prices do not change as government policy changes.*

A: *Assume throughout that the production technology is homothetic and has constant returns to scale.*

(a) *Suppose that, once assigned to an oil company, the company is not required to pay for using the land to drill for oil (but it cannot do anything else with it if it chooses not to drill.) How much land will your oil company use?*

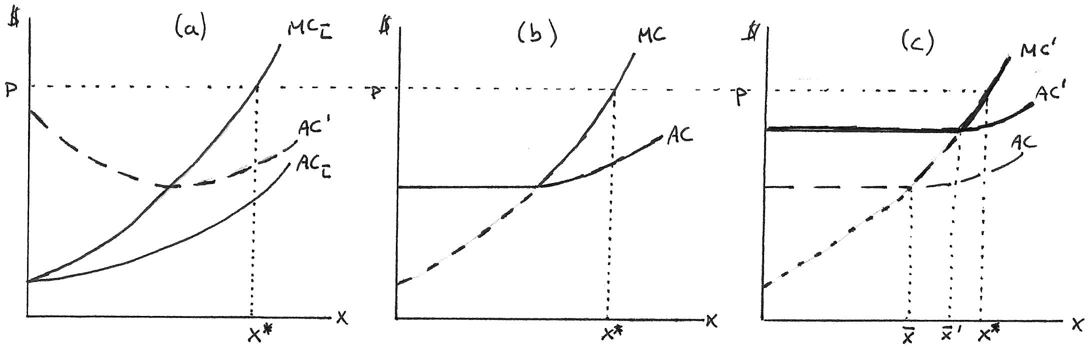
Answer: If land (up to $\overline{L}$ acres) is free to the company, it will use all of it so long as the marginal product of land is positive. This is because profit maximization implies that a firm hires inputs until the marginal revenue product equals the price of the input — and the price of land in this case is zero up to $\overline{L}$ (and effectively infinite thereafter).

(b) *While the 3-input production frontier has constant returns to scale, can you determine the effective returns to scale of production once you take into account that available land is fixed?*

Answer: Land essentially becomes a fixed input in both the short and long run. Constant returns to scale of the 3-input production frontier implies that we can double output by doubling all 3 of our inputs. But if land cannot be increased, doubling labor and capital will be insufficient to double output. The 2-input production frontier that holds land fixed at $\overline{L}$ therefore has decreasing returns to scale. (One way to think of this 2-imput production frontier is as a 3-dimensional "slice" (that holds land fixed at $\overline{L}$) of the original 4-dimensional production frontier that plotted production plans (x, ℓ, k, L)).

(c) *What do average and marginal cost curves look like for your company over the time frame when both labor and capital can be varied?*

Answer: Since the effective 2-input production frontier (that holds $\overline{L}$ fixed) has decreasing returns to scale, the average and marginal cost curves are upward sloping with marginal cost above average cost throughout. These curves are pictured as the solid curves $AC_{\overline{L}}$ and $MC_{\overline{L}}$ in panel (a) of Exercise Graph 13.9.



Exercise Graph 13.9 : Land Rented for Oil Drilling

(d) *Now suppose that the government begins to charge a per-acre rental price q for use of land that is assigned to your company, but an oil company that is assigned $\overline{L}$ acres of land only has the option of renting all $\overline{L}$ acres or none at all. Given that it takes time to relocate oil drilling equipment, you cannot adjust to this change in the short run. Will you change how much oil you produce?*

Answer: No. In the short run, the charge for renting land is simply an expense until the company can actually vacate the land.

(e) *In the long run (when you can move equipment off land), what happens to average and marginal costs for you company? Will you change your output level?*

Answer: If your company does not exit the industry, it has to pay $q\overline{L}$ to rent all the land assigned to it — which makes it exactly like a fixed recurring license fee. Once paid, the marginal cost of drilling for oil is no different than it was when the land was free. Thus, the $MC_{\overline{L}}$ curve in panel (a) of Exercise Graph 13.9 does not change and continues to be the marginal cost curve for this company. The long run average cost curve, however, incorporates the fixed payment for the land — and thus takes on the U-shape depicted by the dashed AC' curve in the graph. The difference between $AC_{\overline{L}}$ and AC' in panel (a) is the average fixed cost of the land — which is high when oil production is low but, as an average, falls as oil production increases. Thus, $AC_{\overline{L}}$ and AC' converge as x gets large. If you continue to produce in the long run, you will produce where the price of oil p intersects the marginal cost curve that has not changed as a result of the government's policy of charging for use of the land. Thus, *if* you continue to stay in the oil business, you will not change your output level in the long run. This is depicted for the price p in panel (a) of the graph — with production x^* unchanged when the government charges a land rental fee sufficiently low to keep p above AC'. However, if p lies below the dashed AC' curve, you will exit the industry in the long run (once you can vacate the land).

(f) *Suppose the government had employed a different policy that charges a per-acre rent of q but allowed companies to rent any number of acres between 0 and $\overline{L}$. What do long run average and marginal cost curves look like in that case? Would it ever be the case that a firm will rent fewer than $\overline{L}$ acres?(*Hint*: These curves should have a flat as well as an upward sloping portion.)*

Answer: In this case, your company would initially not be limited in how much land it can use for oil drilling — not until it drills enough to run into the $\overline{L}$ constraint imposed by the government. Thus, for low levels of x, your company is operating with full (long run) discretion in terms of how much land, labor and capital to use — and thus faces a constant returns to scale production process. This implies that, for low levels of x, the average cost function takes the form it does for constant returns to scale production functions — i.e. it is constant. However, once the output level reaches the point where your company would ordinarily want to rent more than $\overline{L}$ acres of land in order to produce output level x at minimum cost, the rent on $\overline{L}$ becomes a fixed cost, land becomes a fixed input and production from here on out has decreasing returns to scale. This implies that, at some output level, the constant AC curve begins to slope up. This is pictured in panel (b) of Exercise Graph 13.9 as the initially flat and eventually upward sloping AC curve. Since constant returns to scale implies a constant marginal cost and decreasing returns to scale implies an increasing marginal cost, the MC curve in the graph is similarly flat initially but upward sloping after some output level. It therefore overlaps with AC until the land constraint binds — at which point it slopes up and lies above AC causing AC to slope up as well.

(g) *How much will you produce now compared to the case analyzed in (d)?*

Answer: You will produce exactly the same as in part (d) — because the portion of the upward sloping (solid) portion of the MC curve in panel (b) is exactly the same as the $MC_{\overline{L}}$ curve in panel (a). (In both cases, this portion of the marginal cost curve is derived from the 2-input production frontier that holds land fixed at $\overline{L}$.) Thus, as shown in Exercise Graph 13.9, price will again intersect marginal cost at x^*.

(h) *Suppose that under this alternative policy the government raises the rental price to q'. Will your company change its output level in the short run?*

Answer: No — in the short run, your company is unable to move its oil drilling equipment — and thus forced to rent the land. Since there is nothing you can do about it — i.e. you cannot affect the expense by anything that you do — it is not an economic cost and therefore affects nothing.

(i) *How do long run average and marginal cost curves change? If you continue to produce oil under the higher land rental price, will you increase or decrease your output level, or will you leave it unchanged*

Answer: Panel (c) of Exercise Graph 13.9 illustrates the new marginal and average cost curves as the bold curves labeled MC' and AC' (with the two curves overlapping along the flat hold portion). Underneath these bold curves, the AC and MC curves from panel (b) are replicated for comparison. Under the higher land rental price q', the constant returns to scale portion extends to higher level of output — because when the government charges more for land, firms will substitute away from land and toward more capital and/or labor. (For instance, you might think of a firm that produces relatively little choosing to drill horizontally from one spot rather than drilling vertically from two spots that require more land.) As a result, the land constraint $\overline{L}$ does not bind until a higher output level is reached — $\overline{x}'$ as opposed to $\overline{x}$ when the land rental rate was lower. But, although the firm conserves on land when q increases, the average cost per barrel of oil still increases — which is why the flat portion of the AC' curve lies above AC. *Once the constraint of* $\overline{L}$ *is reached*, however, the firm operates on the 2-dimensional production frontier that holds land fixed and thus experiences decreasing returns to scale. This implies the upward sloping MC' curve — which lies on the previously derived $MC_{\overline{L}}$ curve (in panel (a)) and the MC curve in panel (b). You will continue to produce only if p lies above the lowest point of AC' as drawn in panel (c) of the graph — but this means you will produce at the intersection of price and the same marginal cost as in the previous panels. Thus, *if you continue to produce*, you will not change your output level as a result of the increase in the rental fee of land. The only possible change in your production plan would arise if p fell between the intercepts of the dashed AC and the bold AC' curves — in which case you would have produced before but would exit after the increase in q.

(j) True or False*: The land rental rate q set by the government has no impact on oil production levels so long as oil companies do not exit the industry. Explain. (*Hint*: This is true.)*

Answer: This is true as already explained. In fact, none of the policy changes we investigated has an impact on oil production unless it causes a firm to exit the industry. You can see this in Exercise Graph 13.9 by simply observing that p always intersects the same marginal cost curve to give us output level x^*.

B: *Suppose that your production technology for oil drilling is characterized by the production function* $x = f(\ell, k, L) = A\ell^{\alpha}k^{\beta}L^{\gamma}$ *where* $\alpha+\beta+\gamma = 1$ *(and all exponents are positive).*

(a) *Demonstrate that this production function has constant returns to scale.*

Answer: To demonstrate this, we simply need to show that multiplying all inputs by t results in a t-fold increase in output; i.e.

$$f(t\ell, tk, tL) = A(t\ell)^{\alpha}(tk)^{\beta}(tL)^{\gamma} = t^{(\alpha+\beta+\gamma)}A\ell^{\alpha}k^{\beta}L^{\gamma} = tA\ell^{\alpha}k^{\beta}L^{\gamma} = tf(\ell, k, L). \tag{13.9.i}$$

(b) *Suppose again that the government assigns $\overline{L}$ acres of land to your company for oil drilling, that there is no rental fee for the land but you cannot use the land for any other purpose. Given the fixed level of land available, what is your production function now? Demonstrate that it has decreasing returns to scale.*

Answer: The production function now is

$$x = f_{\overline{L}}(\ell, k) = \left[A\overline{L}^{\gamma}\right]\ell^{\alpha}k^{\beta}, \tag{13.9.ii}$$

where the term in brackets simply enters as a constant. This is simply a 2-input Cobb-Douglas production function with $\alpha + \beta < 1$ — which makes it a decreasing returns to scale production function. We can demonstrate this simply by showing

$$f_{\overline{L}}(t\ell, tk) = \left[A\overline{L}^{\gamma}\right](t\ell)^{\alpha}(tk)^{\beta} = t^{(\alpha+\beta)}\left[A\overline{L}^{\gamma}\right]\ell^{\alpha}k^{\beta} < t\left[A\overline{L}^{\gamma}\right]\ell^{\alpha}k^{\beta} = tf_{\overline{L}}(\ell, k). \tag{13.9.iii}$$

(c) *In exercise 13.2, you were asked to derive the (long run) cost function for a 2-input Cobb-Douglas production function. Can you use your result — which is also given in equation (13.5) of exercise 13.5 — to derive the cost function for your oil company? What is the marginal cost function associated with this?*

Answer: Since the production function $f_{\overline{L}}(\ell, k)$ is simply a 2-input Cobb-Douglas function with constant $\left[A\overline{L}^{\gamma}\right]$ (rather than just A) in the front, we can simply use the cost function previously derived and replace A with $\left[A\overline{L}^{\gamma}\right]$ to get

$$C_{\overline{L}}(w, r, x) = (\alpha + \beta)\left(\frac{w^{\alpha}r^{\beta}x}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.9.iv}$$

The marginal cost function is then

$$MC_{\overline{L}}(w, r, x) = \frac{\partial C_{\overline{L}}(w, r, x)}{\partial x} = \left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} = \left(\frac{w^{\alpha}r^{\beta}x^{\gamma}}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.9.v}$$

(d) *Next, consider the scenario under which the government charges a per-acre rental fee of q but only gives you the option of renting all $\overline{L}$ acres or none at* *all. Write down your new (long run) cost function and derive the marginal and average cost function. Can you infer the shape of the marginal and average cost curves?*

Answer: In order to drill for oil, you now need to pay a fixed cost of $q\overline{L}$ to rent the land. Thus, the cost function now is

$$C_{\overline{L}}(w,r,q,x) = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x}{\left[A\overline{L}^\gamma\right]\alpha^\alpha\beta^\beta}\right)^{1/(\alpha+\beta)} + q\overline{L}. \tag{13.9.vi}$$

This gives us a marginal cost curve

$$MC_{\overline{L}}(w,r,x) = \frac{\partial C_{\overline{L}}(w,r,q,x)}{\partial x} = \left(\frac{w^\alpha r^\beta x^\gamma}{\left[A\overline{L}^\gamma\right]\alpha^\alpha\beta^\beta}\right)^{1/(\alpha+\beta)} \tag{13.9.vii}$$

where the $q\overline{L}$ term drops out and the MC therefore is not a function of q. Finally, we get the average cost function

$$AC_{\overline{L}}(w,r,q,x) = \frac{C_{\overline{L}}(w,r,q,x)}{x} = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x^\gamma}{\left[A\overline{L}^\gamma\right]\alpha^\alpha\beta^\beta}\right)^{1/(\alpha+\beta)} + \frac{q\overline{L}}{x}. \tag{13.9.viii}$$

Since the marginal cost function is unchanged, the marginal cost curve with $q>0$ is the same upward sloping marginal cost curve as that with $q=0$; i.e. the land rental fee has no impact on marginal cost. (You can verify that the marginal cost curve is upward sloping by simply checking that the derivative of the marginal cost function with respect to x is positive.) The average cost function, however, does change with q — but only because of the last term $q\overline{L}/x$. For small x, this term is large — but as x increases, it converges to zero. This creates the U-shape of the AC' curve in panel (a) of Exercise Graph 13.9.

(e) *Does the (long run) marginal cost function change when the government begins to charge for use of the land in this way?*

Answer: As already demonstrated, it does not.

(f) *Now suppose that the government no longer requires your company to rent all $\overline{L}$ acres but instead agrees to rent you up to $\overline{L}$ acres at the land rental rate q. What would your conditional input demands and your (total) cost function be in the absence of the cap on how much land you can rent?*

Answer: To get the conditional input demands without a cap on how much land you can rent, you simply solve the problem

$$\min_{\ell,k,L} \; w\ell + rk + qL \text{ subject to } x = A\ell^\alpha k^\beta L^\gamma. \tag{13.9.ix}$$

Solving first order conditions in the usual way, we get

$$\ell(w,r,q,x) = \left(\frac{\alpha}{w}\right)^{(1-\alpha)} \left(\frac{r}{\beta}\right)^{\beta} \left(\frac{q}{\gamma}\right)^{\gamma} \frac{x}{A} \;;\; k(w,r,q,x) = \left(\frac{w}{\alpha}\right)^{\alpha} \left(\frac{\beta}{r}\right)^{(1-\beta)} \left(\frac{q}{\gamma}\right)^{\gamma} \frac{x}{A} \tag{13.9.x}$$

and

$$L(w,r,q,x) = \left(\frac{w}{\alpha}\right)^{\alpha} \left(\frac{r}{\beta}\right)^{\beta} \left(\frac{\gamma}{q}\right)^{(1-\gamma)} \frac{x}{A}. \tag{13.9.xi}$$

Multiplying these by their input prices, adding and simplifying, we then get the cost function

$$C(w,r,q,x) = w\ell(w,r,q,x) + rk(w,r,q,x) + qL(w,r,q,x) = \frac{w^{\alpha} r^{\beta} q^{\gamma} x}{A\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}}. \tag{13.9.xii}$$

Note that this is a constant returns to scale cost function that has the property that

$$MC = \frac{w^{\alpha} r^{\beta} q^{\gamma}}{A\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}} = AC, \tag{13.9.xiii}$$

where neither MC nor AC is dependent on x. In other words, marginal and average cost curves are overlapping and flat.

(g) *From now on, suppose that $A = 100$, $\alpha = \beta = 0.25$, $\gamma = 0.5$, $\overline{L} = 10{,}000$. Suppose further that the weekly wage rate is $w = 1000$, the weekly capital rental rate is $r = 1000$ and the weekly land rent rate is $q = 1000$. At what level of output $\overline{x}$ will your production process no longer exhibit constant returns to scale (given the land limit of $\overline{L}$)? What is the marginal and average cost of oil drilling prior to reaching $\overline{x}$ (as a function of x)?*

Answer: To determine at what level of output we reach the $\overline{L}$ constraint, we simply have to set equation (13.9.xi) to $\overline{L}$ and solve for x. This gives us

$$\overline{x} = A\overline{L}\left(\frac{\alpha}{w}\right)^{\alpha} \left(\frac{\beta}{r}\right)^{\beta} \left(\frac{q}{\gamma}\right)^{(1-\gamma)}. \tag{13.9.xiv}$$

Plugging in the values $A = 100$, $\alpha = \beta = 0.25$, $\gamma = 0.5$, $\overline{L} = 10{,}000$ and $w = r = q = 1000$, this gives us $\overline{x} \approx 707{,}107$. Plugging the same values into equation (13.9.xiii), we get that prior to reaching $\overline{x}$, $MC = AC \approx 28.28$.

(h) *After reaching this $\overline{x}$, what is the marginal and average long run cost of oil drilling (as a function of x)? Compare the marginal cost at $\overline{x}$ to your marginal cost answer in (g) and explain how this translates into a graph of the marginal cost curve for the firm in this scenario.*

Answer: After $\overline{x}$, we are employing the decreasing returns to scale production function $f_{\overline{L}=10000}$ for which we calculated marginal and average costs in equations (13.9.vii) and (13.9.viii). Substituting the various values into these equations, we get

$$MC_{\overline{L}=10000}(x) = 0.00004x \quad \text{and} \quad AC_{\overline{L}=10000}(x) = 0.00002x + \frac{10,000,000}{x}. \tag{13.9.xv}$$

Evaluating these at $\overline{x} = 707,107$, we get $AC = MC \approx 28.28$ — which is exactly what we arrived at in the previous part. This justifies the picture in panel (b) of Exercise Graph 13.9 where the constant returns to scale portion meets the upward sloping portion of the MC curve at $\overline{x}$.

(i) *What happens to $\overline{x}$ as q increases? How does that change the graph of marginal and average cost curves?*

<u>Answer</u>: From equation (13.9.xiv), we can easily see that $\overline{x}$ increases as q increases. This changes the graph for marginal and average cost curves as illustrated in panel (c) of Exercise Graph 13.9.

(j) *If the price per barrel of oil is $p = 100$, what is your profit maximizing oil production level?*

<u>Answer</u>: Setting p of 100 equal to $MC_{\overline{L}} = 0.00004x$ and solving for x, we get $x = 2,500,000$. You will therefore produce 2,500,000 barrels of oil per week.

(k) *Suppose the government now raises q from 1,000 to 10,000. What happens to your production of oil? What if the government raises q to 15,000?*

<u>Answer</u>: If you produce, you will still produce where p of 100 equals $MC_{\overline{L}} = 0.00004x$ — which implies you will still produce 2,500,000 barrels of oil per week. The question is whether the government has raised the land rental rate so high that it is more advantageous to exit and produce nothing. To determine at what q this happens, we have to determine at what q the lowest point of the average cost curve is equal to $p = 100$. The lowest point of average cost occurs along the flat, constant returns to scale portion where average cost is given by equation (13.9.xiii). Substituting the various values into that equation, we get $AC \approx 0.894427q^{0.5}$. Setting this equal to 100 and solving for q, we get $q = 12,500$. Thus, for any $q \leq 12,500$, output will remain unchanged at 2,500,000 barrels of oil per week, but for $q > 12,500$, output falls to zero as the firm exits the industry.

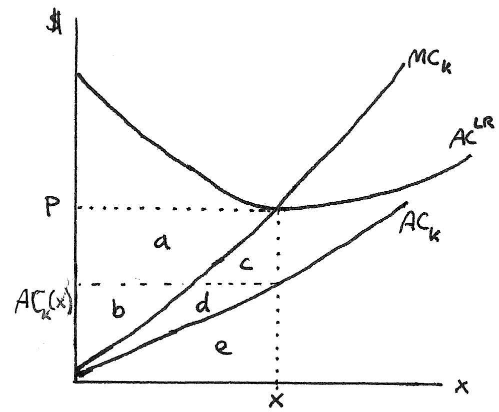
Exercise 13.11

<u>Policy and Business Application</u>: *Business Taxes*: *In this exercise, suppose that your hamburger business "McWendy's" has a homothetic decreasing returns to scale production function that uses labor ℓ and capital k to produce hamburgers x. You can hire labor at wage w and capital at rental rate r but also have to pay a fixed annual franchise fee F to the McWendy parent company in order to operate as a McWendy's restaurant. You can sell your McWendy's hamburgers at price p.*

A: *Suppose that your restaurant, by operating at its long run profit maximizing production plan (ℓ^*, k^*, x^*), is currently making zero long run profit. In each* *of the policy proposals in parts (b) through (h), suppose that prices w, r and p remain unchanged. In each part, beginning with (b), indicate what happens to your optimal production plan in the short and long run.*

(a) *Illustrate the short run AC and MC curves as well as the long run AC curve. Where in your graph can you locate your short run profit — and what is it composed of?*

Answer: These curves are illustrated in panel (a) of Exercise Graph 13.11 where the U-shape of the long run average cost curve derives from the fixed franchise fee.



Exercise Graph 13.11 : Short Run Profit

Since there are no fixed costs — only fixed expenses — in the short run, and since the decreasing returns to scale of the long run production process also implies decreasing returns to scale for the short run production process, the short run MC and AC curves have to be upward sloping (with the former above the latter). The short run profit is then composed of the expense on capital rk and the fixed franchise fee F — i.e. short run profit is $rk + F$ which disappears in the long run as these turn into costs. In the graph, there are two ways of seeing this short run profit. In both cases, we begin by identifying total revenue as the area $a + b + c + d + e$ — i.e. the price p times output x. (We know price has to be at the lowest point of the long run AC curve since we know the firm is making zero long run profit.) The short run costs can then be identified as the average (short run) cost of producing x — which is $AC_k(x)$ — times the output level x, or just area $b + d + e$. Alternatively, short run costs can also be seen as the area under the short run marginal cost curve — area $c + d + e$. Subtracting short run costs from revenues, we then get that short run profit is $a + c$ or, equivalently, $a + b$. (Logically this of course implies that $b = c$.)

(b) *Suppose the government determined that profits in your industry were unusually high last year — and imposes a one-time "windfall profits tax" of*

50% on your business's profits from last year.

Answer: There is nothing you can do in your business to avoid paying this windfall profits tax — it is a sunk "cost", an expense in the short run and irrelevant in the long run. Therefore you will not change your production plan.

(c) *The government imposes a 50% tax on short run profits from now on.*

Answer: If you are currently maximizing short run profits (which you are), you will not change anything in the short run if the government takes half of your short run profit. Half of the most you can make is still more than half of less than the most you can make in your business. In the long run, however, your profit will now no longer be positive — which means you will exit in the long run and stop producing (absent any changes in prices in the industry).

(d) *The government instead imposes a 50% tax on long run profits from now on.*

Answer: Your long run profit is zero — so the government will not collect any taxes from you. If what you were doing before was profit maximizing, you are still profit maximizing by doing exactly the same as you were doing before. This would be true even if your long run profits were positive. If you now only make half as much long run profit, you are still making a positive profit — which means you are still making more in this business than you could anywhere else.

(e) *The government instead taxes franchise fees causing the blood sucking McWendy's parent company to raise its fee to $G > F$.*

Answer: This will increase your long run costs — which means that, since you were making zero long run profit before, you will now be making negative long run profit. In the absence of any other changes (such as a change in price), you will therefore exit and stop producing hamburgers.

(f) *The government instead imposes a tax t on capital (which is fixed in the short run) used by your restaurant — causing you to have to pay not only r but also tr to use one unit of capital.*

Answer: Since capital is fixed in the short run, nothing changes for you in the short run (assuming you still are committed to the capital you have for now). Put differently, the tax payment tr is a short run expense, not a cost. In the long run, however, your average and marginal cost curves increase. If you were to continue to produce, this implies you will produce less (as p intersects MC at a lower quantity) — but you will in fact exit in the long run because your long run profit — which was zero before the increase in costs — must now be negative.

(g) *Instead of taxing capital, the government taxes labor in the same way as it taxed capital in part (f).*

Answer: Since labor is variable in the short run, this tax is an immediate cost since you affect you overall tax payment by changing how many workers you hire. Thus, the MC_k shifts up. If you continue to produce in the short run, you will then produce less because the new MC_k intersects price at a lower quantity. It is not clear, however, whether you will not fully shut down even in the short run. The crucial question is whether the tax on labor is sufficiently high for short run profit (which was positive at the outset) to fall below zero. If so, you will shut down. Depending on how large the tax rate t, you will therefore either produce less or not at all in the short run. In the long run, however, you will exit (unless something else changes) — because your previously zero long run profit is now negative.

(h) *Finally, instead of any of the above, the government imposes a "health tax" t on hamburgers — charging you \$$t$ for every hamburger you sell.*

Answer: The answer is similar to that given to part (g) — in the short run, you may stay open and produce fewer hamburgers or you may shut down depending on whether short run profits under the lower production level remain positive. In the long run, however, you will exit (unless something else changes).

B: *In previous exercises, we gave the input demand functions for a a firm facing prices (w, r, p) and technology $f(\ell, k) = A\ell^\alpha k^\beta$ (with $\alpha, \beta > 0$ and $\alpha + \beta < 1$) in equation (13.50) and the long run output supply function in equation (13.49) — both given in footnotes to earlier end-of-chapter exercises in this chapter.*

(a) *When you add a recurring fixed cost F, how are these functions affected? (*Hint*: You will have to restrict the set of prices for which the functions are valid — and you can use the profit function given in exercise 13.7) to do this strictly in terms of A, α, β and the prices (w, r, p).) What are the short run labor demand and output supply functions for a given $\bar{k}$?*

Answer: We now have to include the role of the recurring fixed cost F in the long run input demand and output supply functions. But we know from our graphical work that this simply causes these functions to become "shorter" — i.e. these functions remain valid but only for the set of input and output prices at which long run profit (which incorporates F) is not negative. In the absence of F, the profit function for Cobb-Douglas production functions was given in exercise 13.7 as

$$\pi(w, r, p) = (1-\alpha-\beta)\left(\frac{Ap\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)}. \tag{13.11.i}$$

To say that long run profit is positive at a given set of prices (p, w, r) is therefore to say that

$$(1-\alpha-\beta)\left(\frac{Ap\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)} \geq F \tag{13.11.ii}$$

or, rearranging terms, that

$$p \geq \left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)}. \tag{13.11.iii}$$

We can thus write the long run labor demand, capital demand and output supply functions as

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^\beta}{w^{(1-\beta)}r^\beta}\right)^{1/(1-\alpha-\beta)} \text{ if } p \geq \left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)} \tag{13.11.iv}$$

$$k(w,r,p) = \left(\frac{pA\alpha^\alpha\beta^{(1-\alpha)}}{w^\alpha r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \text{ if } p \geq \left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)} \tag{13.11.v}$$

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)} \text{ if } p \geq \left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)}, \tag{13.11.vi}$$

with all three equations equaling zero if the condition does not hold.

In the short run, however, F plays no role since it is not an economic cost. The short run production function given fixed $\overline{k}$ is $f_{\overline{k}}(\ell) = \left[A\overline{k}^\beta\right]\ell^\alpha$. Solving the short run profit maximization problem

$$\max_{\ell}\; p\left[A\overline{k}^\beta\right]\ell^\alpha - w\ell, \tag{13.11.vii}$$

we get the short run labor demand function

$$\ell_{\overline{k}}(w,p) = \left(\frac{\alpha p\left[A\overline{k}^\beta\right]}{w}\right)^{1/(1-\alpha)}. \tag{13.11.viii}$$

Substituting this back into the short run production function, we get the short run supply function

$$x_{\overline{k}}(w,p) = \left[A\overline{k}^\beta\right]^{1/(1-\alpha)}\left(\frac{\alpha p}{w}\right)^{\alpha/(1-\alpha)}. \tag{13.11.ix}$$

Each of the following correspond to the respective parts (b) through (h) in part A of the question:

(b) *For each of (b) through (h) in part A of the exercise, indicate whether (and how) the functions you derived in part (a) are affected.*

Answer: In (b), none of the functions are affected since last year's profits do not enter any of them.

In (c), the short run functions are not affected when a 50% tax on short run profits is imposed. To see more clearly why, you can write the short run profit maximization problem to include the 50% short run profits tax — and you would get

$$\max_{\ell} \; 0.5\left(pA\overline{k}^{\beta}\ell^{\alpha} - w\ell\right), \qquad (13.11.x)$$

which has first order conditions identical to those in the original problem in equation (13.11.vii).

In the long run, however, the 50% short run profit tax becomes a recurring fixed cost of doing business and would thus increase the F term in equations (13.11.iv), (13.11.v) and (13.11.vi). While this does not affect the functions themselves directly, it affects the range of prices under which the functions are not simply equal to zero (because the firm exits). If long run profit is originally zero, for instance, the 50% short run profit tax would then cause the input demand and output supply functions to go to zero because the inequality in each expression no longer holds.

In (d), none of the functions are affected. You can again see that the long run functions are unaffected by realizing that a tax on long run profits drops out as we solve the profit maximization problem

$$\max_{\ell,k} \; (1-t)\left(pf(\ell,k) - w\ell - rk\right) \qquad (13.11.xi)$$

where t stands for the tax rate applied to long run profit. Put differently, since the government taking *a fraction* of long run profit does not cause long run profit to become negative, this tax will never cause the inequality in equations (13.11.iv), (13.11.v) and (13.11.vi) to not hold.

In (e), since F does not appear in the short run equations, the new fixed cost G will also not appear — leaving the short run curves unaffected. F does, however, appear in equations (13.11.iv), (13.11.v) and (13.11.vi) — or, to be more precise, it appears in the inequality that restricts the prices for which the functions are applicable. As F increases, the inequality will no longer hold for some range of prices at the lower end — thus raising the price at which the firm exits. If, for instance, the firm was initially making zero long run profit, it would exit with an increase in F to G because the inequality in (13.11.iv), (13.11.v) and (13.11.vi) no longer holds.

In (f), since r appears in equations (13.11.iv), (13.11.v) and (13.11.vi), we know that the long run functions are affected. They are affected in two ways: First, the equations themselves are affected, with an increase in r causing a decrease in ℓ, k, and x; and second, the inequality is affected in the sense that the inequality now no longer holds for some range of prices at the lower end. This implies that, in the long run, the firm will reduce its output and its demand for labor and capital — and it will reduce these to zero if the inequality no longer holds. For instance, if long run profit is initially zero, the firm will exit (unless something else changes). In the short run, however, r does not appear in either the labor demand or output supply equations — and thus nothing changes in the short run.

In (g), the impact on the long run will mirror what we just described for a capital tax. In the short run, however, there was no impact of the tax on capital because r did not enter the short run labor demand or output supply functions — but w does appear in these, which implies that the labor tax has an immediate short run impact. In particular, an increase in w causes an immediate decrease in both labor demand and output supply.

In (h), the tax on hamburgers will also have short and long run impacts on our derived functions by changing the output price from p to $(p-t)$. This lower output price will shift short run supply and short run labor demand in the respective short run functions, reducing the quantity in each. In the long run, p appears in both the initial equation as well as the inequality of expressions (13.11.iv), (13.11.v) and (13.11.vi). In the equations to the left of each expression, p changes to $(p-t)$ — causing a drop in each. In the inequalities on the right, p changes to $(p-t)$ on the left-hand side of the inequality, or — alternatively, we can rewrite the inequality as

$$p \geq \left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)} + t. \tag{13.11.xii}$$

This implies that the price for which the input demand and output supply functions are valid increases — again "shortening" the input demand and output demand curves. If, for instance, the firm was making zero profit before the implementation of the tax, it will exit after the implementation (unless something else changes).

Conclusion: Potentially Helpful Reminders

1. Always be careful to call something a "cost" only if it is a real opportunity cost of doing business in the environment you are analyzing. Not everything that involves writing a check is a cost of doing business.
2. This is in fact what underlies the conclusion that the shut down price in the short run is (almost) always lower (and never higher) than the long run exit price. You don't have to cover the long run costs that are short run expenditures in order to justify staying open in the short run — but you do have to

cover all your long run costs in order to justify remaining in business in the long run.

3. When all is said and done, there are really only three types of costs we are analyzing: (1) Variable costs associated with inputs that can be changed in both the short and long run — and therefore enter MC and AC in both the short and long run; (2) Costs associated with inputs that are variable only in the long run and thus enter MC and AC only in the long run (while being "sunk" expenditures in the short run); and (3) long run recurring fixed costs that are not associated with inputs. (Later on in the text, we will also introduce one-time fixed entry costs associated with entering a market — but for now we assume that entry and exit are costless.)

4. Output supply curves are always more responsive to output price in the long run (than in the short run), and own-price input demand curves are similarly always more responsive to (own) input prices in the long run (than in the short run).

5. But cross-price input demand curves that map the relationship between an input and *a different input's* price can be more or less responsive in the long run (than in the short run) depending on the substitutability of inputs in production. Similarly, cross-input-price output supply curves — i.e. the response of output to input price changes — can be more or less responsive in the long run depending on the substitutability of inputs in production.

6. The most important part of this chapter in terms of building a foundation for future chapters is the first section. It is possible to make it through the remainder of the text quite easily without fully understanding the nuances of the rest of the chapter. If you do want to tackle the material in the latter sections, understanding Graph 13.9 is crucial.

CHAPTER

14

Competitive Market Equilibrium

Until now, we have only proceeded through the first two of three steps of the "economic way of thinking" — the crafting of a model and the process of optimizing within that model. We now proceed to the final step: to illustrate how an equilibrium emerges from the optimizing behavior of individuals — how the "economic environment" that individuals in a competitive setting take as given emerges from their actions. In the process, we begin to get a sense of how order can emerge "spontaneously" — an idea introduced briefly in the introduction as one of the big ideas that we should not loose as we dive into technical details of economic models.

Chapter Highlights

The main points of the chapter are:

1. An **equilibrium** arises in an economic model when no individual has an incentive to change behavior given what everyone else is doing. In a competitive model, it means that no individual has an incentive to change behavior given the economic environment that has emerged "spontaneously".

2. The **short run for an industry** is the time over which the number of firms in the industry is fixed because firms have not had an opportunity to enter or exit the industry. The **short run industry (or market) supply curve** is therefore the sum of the firm supply curves for the (short run) fixed number of firms in the industry, and the **short run equilibrium** is driven by the price at which the short run industry supply curve intersect the market demand curve (which is simply the sum of all individual demand curves).

3. The **long run for an industry** is the time it takes for sufficient numbers of firms to enter or exit the industry as conditions change. The **long run industry (or market) supply curve** therefore arises from the condition that the *marginal* firm in the industry must make zero profits so that no firm in the industry has an incentive to exit and no firm outside the industry has an incentive to enter. When all firms face identical costs, this implies a horizontal

long run industry supply curve at the price which falls at the lowest point of each firm's long run AC curve. The **long run equilibrium** then emerges at the intersection of market demand and long run industry supply.

4. To analyze what happens as conditions change in a competitive market, the most important curves to keep track of on the firm side are the (1) **long run** AC **curve** and (2) the **short run firm supply curve that crosses the long run** AC **curve at its lowest point** (but extends below it because shut down prices are lower than exit prices.) Any change that impacts short run firm supply curves will impact the short run industry supply curve, and any change that impacts the long run AC curve will impact the long run industry supply curve.

5. Changes that affect only a single firm in an industry do not affect the market equilibrium in either the short or the long run.

14A Solutions to Within-Chapter-Exercises for Part A

Exercise 14A.1

Can you explain why there is always a natural tendency for wage to move toward the equilibrium wage if all individuals try to do the best they can?

Answer: If wage were to drift below the equilibrium, firms would not be able to fill all their job vacancies because not enough workers are willing to work at a below-market wage. Thus, it is in each firm's interest to offer a slightly higher wage in order to fill its positions — and this continues to be true until all positions are filled at the equilibrium wage. If, on the other hand, the wage were to drift above the equilibrium, some workers who want to work will be unable to find a position. It would be in their interest to offer to work for slightly less so that they can get employed when there are fewer jobs than workers wishing to take them — and this continues until the wage falls at the equilibrium where the number of jobs is exactly equal to the number of workers willing to take them.

Exercise 14A.2

Suppose your firm only used labor inputs (and not capital) and that labor is always a variable input. If your firm had to renew an annual license fee, would the AC^{SR} and the long-run AC curves ever cross in this case?

Answer: No, in this case the only difference between AC and AC^{SR} is the cost of the license fee — which does not vary with output. Thus, AC lies above AC^{SR} but converges to it as x increases.

Exercise 14A.3

Why might the AC^{SR} and the long-run AC curves cross when the difference emerges because of an input (like capital) that is fixed in the short run? (*Hint*: Review Graphs 13.2 and 13.3.)

Answer: This is because the fixed *expense* associate with the input that is fixed in the short run becomes a *variable* cost in the long run. It is therefore different than a license fee that does not change with the level of output — the cost associated with the input increases as output increases. Suppose, for instance, that capital is the fixed input in the short run and therefore is not part of short run average cost. For high levels of output, the fixed capital level may be sufficiently low such that very high levels of labor are necessary to make up for it. This may cause the short run labor costs to exceed the long run costs of both labor and capital if the firm in the long run can substitute a lot of the labor for relatively little capital.

Exercise 14A.4

Explain why the MC curve in Graph 14.4 would be the same in the long and short run in the scenario of exercise 14A.2 but not in the scenario of exercise 14A.3.

Answer: This is because in the case of a fixed cost (like a license fee), the cost does not change with output — which implies the MC curve does not change even if the AC curve shifts up. But if the fixed expense in the short run becomes a variable cost in the long run (as happens with a fixed input that becomes variable), then the MC curve changes because the cost associated with the input changes with output as the input becomes variable in the long run.

Exercise 14A.5

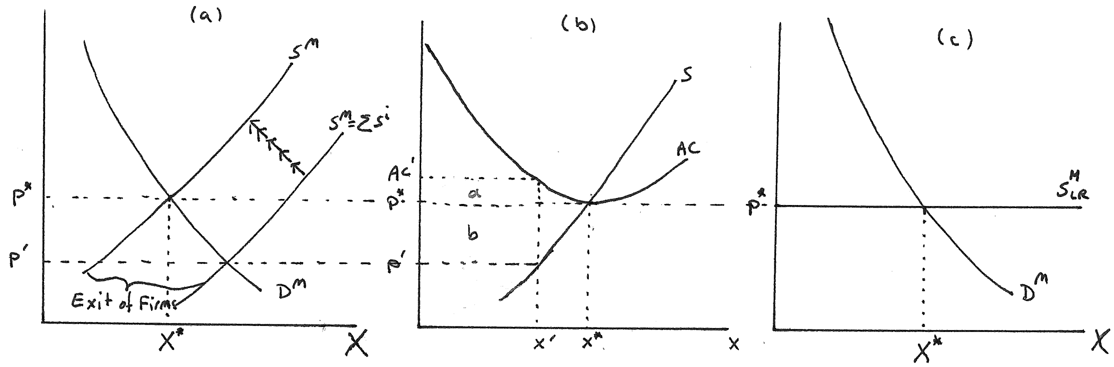
Can you draw the analogous sequence of graphs for the case when the short run equilibrium price falls below p^*?

Answer: This is illustrated in Exercise Graph 14A.5. The long run equilibrium price p^* falls at the lowest point of (long run) AC for the firms (where profit is zero). If the short run equilibrium price p' falls below p^* as drawn in panel (a), each firm produces x' along its short run supply curve as illustrated in panel (b). This implies that the average cost AC' is higher than p', leading to long run profit that is negative and equal to $(-a-b)$ in the graph. As a result, firms will exit — shifting the short run market supply curve in panel (a) to the left until price reaches p^*.

Exercise 14A.6

How does the full picture of equilibrium in Graph 14.2 look different in the long run?

Answer: The long run price would settle at the lowest point of the (long run) AC curve of firms. Thus, panel (f) would only need to show the long run AC curve, and the intersection of D^M and S^M in panel (e) would occur at the price equal to the lowest level of the AC curves of firms.



Exercise Graph 14A.5 : Movement to Long Run Equilibrium when price is below *AC*

Exercise 14A.7

How would you think the time-lag between short and long run changes in labor markets is related to the "barriers to entry" that workers face, where the barrier to entry into the PhD economist market, for instance, lies in the cost of obtaining a PhD.

Answer: The greater the barriers to entry, the longer it will take for the labor market to reach the long run equilibrium.

Exercise 14A.8

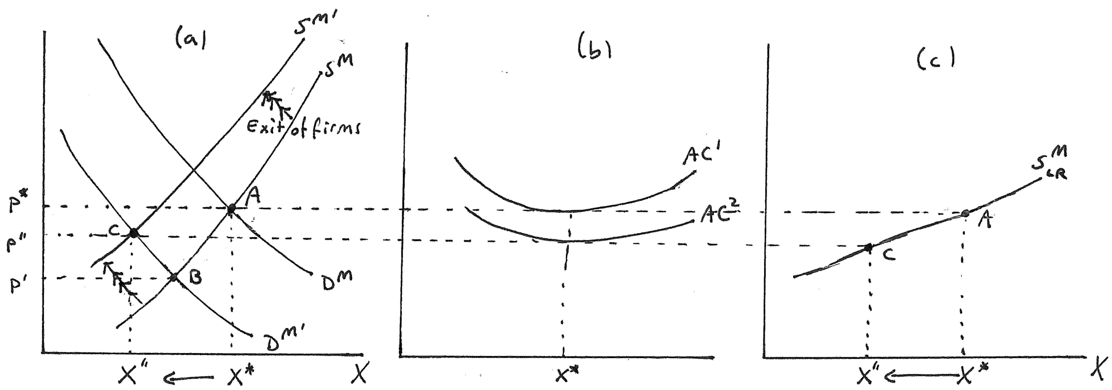
Can you explain why the previous sentence is true?

Answer: As long as the lowest points of the *AC* curves of firms are all at the same vertical height in the graph, all firms have the same exit (or entry) price — and thus exit and entry decisions by firms will drive long run price to that level. It does not matter whether these lowest points of *AC* curves occur at different output levels for different firms — i.e. whether these lowest points are horizontally different. If they are, it simply implies that different firms will produce different levels of output in long run equilibrium, but their exit/entry prices are all the same (so long as the *AC* curves do not differ vertically.)

Exercise 14A.9

Suppose market demand shifts inward instead of outward. Can you illustrate what would happen in graphs similar to those of Graph14.6?

Answer: Exercise Graph 14A.9 illustrates this. In panel (a), D^M and S^M intersect at the original equilibrium A — with price at p^* and industry output at X^*. At that price, any firm with AC at or below p^* is producing x^* — as, for instance, both the firms in panel (b). (This is assuming the lowest point of all AC curves occurs at the same output quantity). When demand shifts to $D^{M'}$ as illustrated in panel (a), the initial short run equilibrium shifts to B. Since the marginal firms were making zero profit before, they are now making negative long run profit — implying that they will begin to exit, which in turn causes the short run market supply curve in panel (a) to shift to the left. This continues to happen until the marginal firm left in the industry makes zero profit. This is illustrated as firm 2 with average cost curve AC^2 in panel (b). Since the highest cost firms exit, the new equilibrium price p'' will fall below p^* — leading to the upward sloping long run market supply curve in panel (c).



Exercise Graph 14A.9 : Inward shift in D^M

Exercise 14A.10

True or False: The entry and exit of firms in the long run insures that the long run market supply curve is always shallower than the short run market supply curve.

Answer: This is true. One way to see this is to think about changes in demand for an industry that is initially in long run equilibrium (before the change in demand). Suppose demand increases. This implies that industry output will rise as each firm produces more at the higher price (that results from the new intersection of (short run) market supply and demand. But, since we started in long run equilibrium, all firms that were initially making zero profit must now be making higher profit — which gives an incentive to firms outside the industry to *enter*. This will shift the short run market supply curves, driving down the price and increasing industry output until the marginal firm makes zero profit again. Thus, the entry of new firms causes the long run output increase to be larger than the short run increase. The reverse happens when demand falls. In that case, firms will produce less as price falls to the new intersection of demand and short run market supply. But since they were initially making zero profit, this implies they will not make negative (long run) profit — which implies some of the firms will exit the industry. This will shift the market supply curves inward, raising price back to the lowest point of the firms' (long run) AC curves. That shift then causes a further decrease in industry output. Thus, whether demand increases or decreases, the long run response is larger than the short run response — meaning that long run industry supply curves are shallower than short run industry supply curves because of entry and exit of firms in the long run.

Exercise 14A.11

True or False: While long run industry supply curves slope up (in increasing cost industries) because firms have different cost curves, long run industry supply curves in decreasing cost industries slope down even if firms have identical cost curves.

Answer: This is true. The text demonstrates how upward sloping long run industry supply curves arise from firms having different cost curves — with higher cost firms entering industries as industries expand — and thus price increasing to insure zero profit for marginal firms. In decreasing cost industries, however, the downward slope of industry supply curves is not due to different costs for firms — and in fact, if firms did have different cost curves, it would be much more likely that an industry could ever have a downward sloping long run supply curve. Rather, the downward sloping industry supply curve arises because *all* firms experience lower costs as the industry expands — i.e. firm costs are changing as the industry expands and input prices fall.

Exercise 14A.12

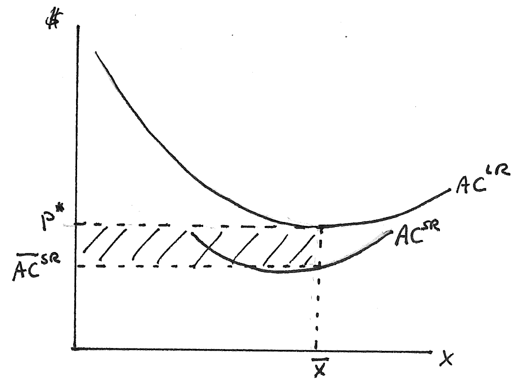
True or False: In the presence of fixed costs (or fixed expenditures), short-run profit is always greater than zero in long run equilibrium.

Answer: This is true. This is because short run profits do not include these fixed (long-run) costs while long run profits do.

Exercise 14A.13

Can you illustrate graphically the short and long run profits of the marginal firm in long run equilibrium? (*Hint*: You can do this by inserting into the graph the AC^{SR} curve as previously pictured in Graph 14.4.)

Answer: This is illustrated in Exercise Graph 14A.13 where the equilibrium price p^* causes the marginal firm to produce $\overline{x}$ at the lowest point of its long run average cost curve AC^{LR}. This implies short run average costs of AC^{SR}. From the short run perspective, the firm therefore incurs costs of $\overline{AC}^{SR} * \overline{x}$ but earns revenues of $p^* * \overline{x}$. The difference between these — indicated by the shaded area in the graph — is the short run profit earned by the marginal firm in long run equilibrium.



Exercise Graph 14A.13 : Short Run Profit in Long Run Equilibrium

Exercise 14A.14

Why does the increase in the fee result in a new (green) AC' curve that converges to the original (blue) AC curve?

Answer: This is because the increase in the fixed fee does not depend on the level of output — so, on average, the additional fee becomes less as output increases. Put differently, if the increased fixed fee is F, the average increased fixed fee is F/x — which is F when $x = 1$ but converges to zero as x gets large.

Exercise 14A.15

If you add the firm's long-run supply curve into panel (b) of the graph, where would it intersect the two average cost curves? Would the same be true for the firm's initial short-run supply curve? (*Hint*: For the second question, keep in mind that the firm will change its level of capital as its output increases.)

Answer: The firm's long run supply curve would intersect the two long run AC curves at their lowest points (because the long run supply curve for a firm is the portion of its long run MC curve that lies above its long run AC curve.) In fact, it would initially begin at the lowest point of the blue AC curve — and would get shorter as a result of the increase in the recurring fixed cost, starting at the lowest point of the green AC' curve after the increase in the recurring fixed cost.

The firm's initial short run supply curve would also cross the lowest point of the initial blue long run AC curve (since the industry is initially in long run equilibrium). But it would (almost certainly) not cross the lowest point of the green AC' curve. This is because the level of capital that the firm has in the initial long run equilibrium is unlikely to be the level of capital it will end up with in the new long run equilibrium — and its short run supply curve therefore takes a (long-run) sup-optimal level of capital as fixed.

Exercise 14A.16

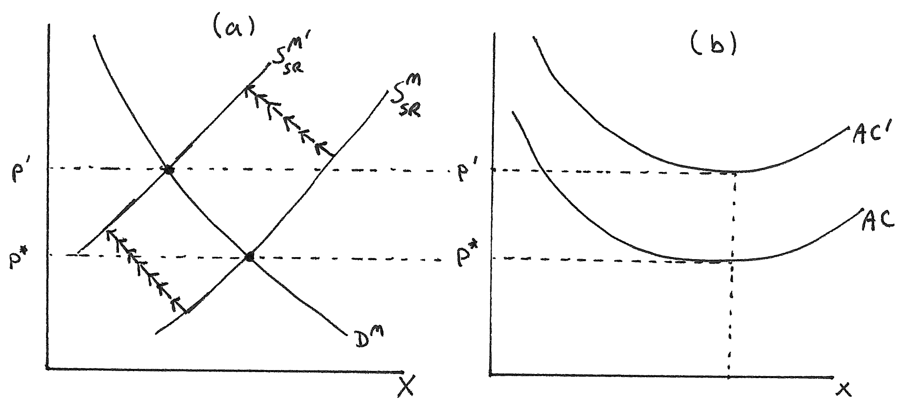
Could the *AC* curve shift similarly in the case where the increase in cost was that of a long run fixed cost?

Answer: No, it could not. In the case of a long run fixed cost, the (long run) *MC* curve remains unchanged because the fixed cost does not change the *additional* cost of producing any of the units of output. The *MC* curve also has to cross the *AC* curve both before and after the increase in the fixed cost — which means that the lowest point of the *AC* curve must shift to the right when the fixed cost increases.

Exercise 14A.17

How would you illustrate the transition from the short run to the long run using graphs similar to those in panels (a) and (b) in Graph 14.8?

Answer: When long run costs increase, firms exit the industry. This shifts the short run supply curve to the left — driving price up until it reaches the lowest point of the firms' *AC* curves. This is illustrated in Exercise Graph 14A.17.



Exercise Graph 14A.17 : Transitioning to Long Run Equilibrium

Exercise 14A.18

Consider two scenarios: In both scenarios, the cost of capital increases, causing the long run *AC* curve to shift up, with the lowest point of the *AC* curve shifting up by the same amount in each scenario. But in Scenario 1, the lowest point of the *AC* curve shifts to the right while in Scenario 2 it shifts to the left. Will the long run equilibrium price be different in the two scenarios? What about the long run equilibrium number of firms in the industry?

Answer: The only thing that matters for where the long run equilibrium price will settle is the vertical height of the lowest point of the long run *AC* curves of firms. Since this is the same in both scenarios, the long run equilibrium price will be the same for both cases. Thus, the long run market supply curve will intersect the market demand curve at the same point — which implies industry output will also be the same in both scenarios. But each firm will *increase* its production in the new equilibrium in Scenario 1 while each firm will *decrease* its production in the new equilibrium in Scenario 2. Thus, in order for the industry to produce the same in both scenarios, it must be the case that the equilibrium number of firms will be larger in Scenario 2 than in Scenario 1.

Exercise 14A.19

The previous side-quote makes the statement that a per-unit tax on output will be fully passed on to consumers in the long run. Can you explain what that means in the context of Graph 14.10? Does it also hold in the short run?

Answer: A per-unit tax will shift the long run *AC* curve up exactly as shown in panel (d) of Graph 14.10 – with the lowest point remaining at the same output level x^*. This is because a per-unit tax raises the marginal cost of each output unit by exactly the same amount – and thus causes every point on the average cost curve to shift up by that amount. Since the new long run zero-profit price therefore jumps from p^* to p'' – a difference exactly equal to the per unit tax, the tax is fully incorporated into the price that consumers pay in the long run. In the short run, however, the price rises by less – only to p' in panel (c) of Graph 14.10. Thus, the tax is not fully passed on to consumers in the short run, only in the long run as the exit of firms drives price higher.

Exercise 14A.20

If an increase in the wage causes an increase in the number of firms in an industry, does this give you enough information to know whether the lowest point of the long-run *AC* for firms shifted to the left or right? What if instead an increase in the wage causes a decrease in the number of firms in the industry?

Answer: An increase in the wage causes in increase in long run *AC* and thus an increase in the long run equilibrium price. As price increases, consumers will demand less – so total output in the industry falls. If the number of firms goes up, that implies more firms will be producing a smaller overall industry output – implying each firm is producing less than a firm produced before the wage increase. That tells us that the lowest point of long run *AC* must have shifted to the left.

If, on the other hand, the increase in the wage causes a decline in the number of firms, then a smaller number of firms is producing a smaller overall level of industry output. From this we cannot tell whether each individual firm is producing more or less than before the wage increase. It could be that industry output fell very little (due to a steep market demand curve) and that, since there are fewer firms, each firm actually produces more. Or it could be that industry output fell a lot (due to a shallow market demand curve), with each firm producing less than originally despite there being fewer firms. In the first case the lowest point of long run *AC* shifted to the right while in the second case it shifted to the left.

Exercise 14A.21

True or False: Regardless of what cost it is, if it increases for only one firm in a competitive industry, that firm will exit in the long run but it might not shut down in the short run.

Answer: This is true. Before the increase in costs, this firm was making zero long run profit. The increase in its costs does not change the equilibrium price since the firm is "small" — and thus long run profit is negative after the increase in the cost. This implies the firm will exit in the long run. In the short run, the firm continues to produce if the cost that increases is one associated with a short-run fixed input or a long run fixed cost — because these increases do not affect short run economic costs. If the increase in costs is an increase in a true short run cost (such as the cost of labor), then the firm's short run *MC* curve shifts to the left, causing it to either produce less (if short run profit does not become negative) or to shut down (if short run profit has become negative.)

Exercise 14A.22

What would a fifth row for an increase in per-unit taxes on output look like? Can you then also replicate Table 14.1 for the cases where the demand and the various cost examples decrease rather than increase.

Answer: An increase in per unit taxes would affect *AC* and *MC* in both the short and long run. Market price would rise in both the short and long run, but more in the long run than in the short run. Industry output would fall in both the short and long run, but more in the long run than in the short run. Firm output would fall in the short run but then return (for those firms that don't exit) to the original output level in the long run. The number of firms would decline.

The replication of the table follows.

Example	Affected Costs SR	Affected Costs LR	Market Price	Industry Output	Firm Output	LR # of Firms
↓ License Fee	None	*AC*	$-^{SR} \downarrow^{LR}$	$-^{SR} \uparrow^{LR}$	$-^{SR} \downarrow^{LR}$	↑
$\downarrow r$	None	*AC*, *MC*	$-^{SR} \downarrow^{LR}$	$-^{SR} \uparrow^{LR}$	$-^{SR} ?^{LR}$	?
$\downarrow w$	*AC*, *MC*	*AC*, *MC*	$\downarrow^{SR} \downarrow^{LR}$	$\uparrow^{SR} \Uparrow^{LR}$	$\uparrow^{SR} ?^{LR}$	?
↓ Demand	None	None	$\downarrow^{SR} -^{LR}$	$\downarrow^{SR} \Downarrow^{LR}$	$\downarrow^{SR} -^{LR}$	↓

14B Solutions to Within-Chapter-Exercises for Part B

Exercise 14B.1

Why is the demand function not a function of income?

Answer: This is because the utility function from which it was derived is quasi-linear — which removes income effects from demand.

Exercise 14B.2

Demonstrate that the average cost of production is U-shaped and reaches its lowest point at $x = 1280$ where AC=5. (*Hint*: You can illustrate the U-shape by showing that the derivative of AC is zero at 1280, negative for output less than 1280 and positive for output greater than 1280.)

Answer: Taking the derivative of $AC(x)$, we get

$$\frac{\partial AC(x)}{\partial x} = \frac{0.167185}{x^{3/4}} - \frac{1280}{x^2}. \qquad (14B.2.i)$$

Substituting $x = 1280$, we get $AC(1280) = 0$, and, for $x < 1280$, the function is indeed negative while for $x > 1280$ it is positive. We therefore have a U-shaped AC curve that reaches its lowest point at $x = 1280$. At that output level, the average cost is

$$AC(1280) = 0.66874(1280)^{1/4} + \frac{1280}{1280} = 5. \qquad (14B.2.ii)$$

Exercise 14B.3

Verify these individual production and consumption quantities.

Answer: Substituting $p = 5$ into the consumer demand equation, we get

$$x^d(5) = \frac{625}{5^2} = 25, \qquad (14B.3.i)$$

and substituting $p = 5$ into the firm supply equation $x^s(p)$, we get

$$x^s(5) = 437.754(5)^{2/3} = 1280. \qquad (14B.3.ii)$$

Exercise 14B.4

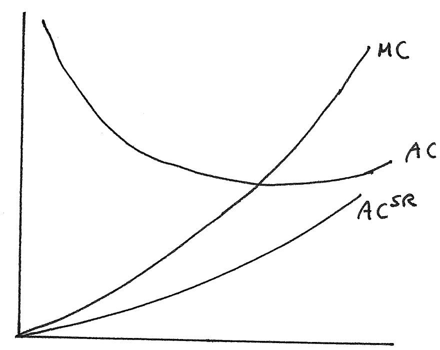
We have already indicated that $k = 256$ is in fact the optimal long run quantity of capital when (p, w, r)=(5,20,10). Can you then conclude that the industry is in long run equilibrium from the information in the previous paragraph? (*Hint*: This can only be true if no firm has an incentive to enter or exit the industry.)

Answer: Since we know $k = 256$ is the long run optimal quantity of capital, the short run expenditures of \$3,840 become economic costs in the long run. (If we did not know $k = 256$ was long run optimal, we would not be able to conclude this since the firm would further adjust capital and therefore the long run cost of capital would differ from the short run expense on capital.) Thus, from a long run perspective, the firm has \$3,840 more in costs than it has in the short run. Since we concluded that short run profit is \$3,840, this implies long run profit is \$0. The industry is therefore in long run equilibrium — with no firm in the industry wanting to exit and no firm outside the industry wanting to enter.

Exercise 14B.5

Can you graph the AC^{SR} into panel (c) of Graph 14.12?

Answer: This is illustrated in Exercise Graph 14B.5 where the short run average cost curve must lie below the short run MC curve.



Exercise Graph 14B.5 : AC^{SR} with Decreasing Returns to Scale Production

Exercise 14B.6

Why does the long-run profit become negative \$960 if nothing changes?

Answer: Since we started in long run equilibrium, we know that initially the firms were making zero long run profit. When the license fees are increased by \$960, this then implies that long run profit must fall to minus \$960 if nothing changes.

Exercise 14B.7

Verify these calculations.

Answer: The lowest point of the AC' curve occurs where its derivative is zero — i.e. where

$$\frac{\partial AC'(x)}{\partial x} = \frac{0.167185}{x^{3/4}} - \frac{2240}{x^2} = 0 \tag{14B.7.i}$$

which solves to $x = 2002.8$ or approximately $x = 2000$. At $x = 2000$, the average cost is

$$AC'(2000) = 0.66874(2000)^{1/4} + \frac{2240}{2000} \approx 5.59. \tag{14B.7.ii}$$

At that price, the market demand curve tells us that the consumers' demand is $D^M(5.59) = 40,000,000/(5.59^2) \approx 1,280,000$, with each individual consumer demanding $x^d(5.59) = 625/(5.59^2) \approx 20$. Each firm will supply about 2000 units of output — and with a total of about 1,280,000 produced by the industry, this implies that the new number of firms in the industry will be 1,280,000/2000≈ 640.

Exercise 14B.8

Compare the changes set off by an increase in the license fee to those predicted in Graph 14.8.

Answer: The graph suggests that we will see an increase in the quantity supplied by each firm with a decrease in quantity supplied by the industry at a higher price. Here we have calculated price increasing from $5 to $5.59, the industry quantity falling from 1,600,000 to 1,280,000 and and the amount produced by each firm *increasing* from 1,280 to 2,000. This is consistent with the directions of changes identified in the graph.

Exercise 14B.9

Verify these calculations.

Answer: The lowest point of the AC' curve occurs where its derivative is zero — i.e. where

$$\frac{\partial AC'(x)}{\partial x} = \frac{0.204759}{x^{3/4}} - \frac{1280}{x^2} = 0 \tag{14B.9.i}$$

which solves to $x = 1088.36$ or approximately $x = 1088$. At $x = 1088$, the average cost is

$$AC'(1088) = 0.819036(1088)^{1/4} + \frac{1280}{1088} \approx 5.88. \tag{14B.9.ii}$$

At that price, the market demand curve tells us that the consumers' demand is $D^M(5.88) = 40,000,000/(5.88^2) \approx 1,156,925$, with each individual consumer demanding $x^d(5.88) = 625/(5.88^2) \approx 18$. Each firm will supply 1088 units of output, which implies that the total number of firms will be 1,156,925/1088≈ 1063.

Exercise 14B.10

Are these results consistent with Graph 14.9?

Answer: The graph suggests that industry output will fall while firm output remains unchanged and price increases. We calculated that industry output falls from 1,600,000 to 1,156,925 and price rises from \$5 to \$5.88. These results are consistent with the graph. We also calculated that each firm's output will fall from 1,250 to 1,088 which is different from what is shown in the graph. However, in developing the graph, we noted that the lowest point of the long run *AC* curve might shift to the left or right as the rental rate of capital increases — and we just happened to draw it as shifting in neither direction. So, while the mathematical results in this example are not consistent with how we drew the graph, they are consistent with our discussion in part A of the chapter.

Exercise 14B.11

How much capital and labor are hired in the industry before and after the increase in r?

Answer: In Chapters 12 and 13, we calculated the input demand functions for this technolgoy to be

$$\ell(p,w,r) = 32768\frac{p^5}{r^2 w^3} \quad \text{and} \quad k(p,w,r) = 32768\frac{p^5}{w^2 r^3}. \qquad (14B.11.i)$$

When $p = 5$ and $w = 20$, these become

$$\ell(r) = \frac{12800}{r^2} \quad \text{and} \quad k(r) = \frac{256000}{r^3}. \qquad (14B.11.ii)$$

Evaluate at $r = 10$ and $r = 15$, this gives us $\ell = 128$ and $k = 256$ when $r = 10$ going to $\ell = 58.89$ and $k = 75.85$ when r increases to 15.

Exercise 14B.12

Verify these calculations.

Answer: Setting short run market supply equal to market demand implies

$$417{,}586p^{2/3} = \frac{40{,}000{,}000}{p^2}, \qquad (14B.12)$$

which implies $p^{8/3} = 95.789$ or $p = 5.533409 \approx \$5.53$. Substituting this into the firm's new short run supply function $x^{s'}(p) = 334.069p^{2/3}$, we get that each firm produces $xs'(5.53) \approx 1{,}045$. The industry output can be calculated by substituting the new price into either the market demand or supply curves — both of which tell us that overall industry output will rise to 1,306,395. (Because of rounding error, you will actually get slightly different answers depending on whether you plug the new price into the market demand or short run market supply functions — the output level 1,306,395 arises from using the price $p = 5.533409$ we calculated before rounding. Up to a rounding error, this is also the same as what we get if we multiply each firm's output of 1,045 by the total number of firms in the short run equilibrium (1,250).)Total revenue for each firm will then simply be the price times the output level 1,045 — or approximately \$5,782 if we use the un-rounded price or \$5,779 if we use $p = 5.53$ which is slightly rounded down.

Exercise 14B.13

How much does the industry production change in the short run?

Answer: Industry production falls from the original 1,600,000 calculated earlier to the 1,306,395 we calculated in the previous exercise — a short run drop of a little less than 300,000 output units.

Exercise 14B.14

Verify these calculations and compare the results to our graphical analysis of an increase in the wage rate in Graph 14.10.

Answer: First, to calculate the long run equilibrium price, we need to determine the lowest point of the long run average cost curve. The cost curve $C(w,r,x) = 2(wr)^{1/2}(x/20)^{5/4} + 1280$ (given at the beginning of part B of the text) becomes $C(30,10,x) = 0.819036x^{5/4} + 1280$ when the new wage (and original rental rate) are substituted for w and r. Thus, the average (long run) cost curve after the wage increase is

$$AC(x) = 0.819036x^{1/4} + \frac{1280}{x}. \tag{14B.14.i}$$

This reaches its lowest point when

$$\frac{\partial AC(x)}{\partial x} = \frac{0.204759}{x^{3/4}} - \frac{1280}{x^2} = 0. \tag{14B.14.ii}$$

Solving for x, we get that $x = 1088.36 \approx 1088$ — and the lowest average cost level reached at that output level is $AC(1088.36) \approx \$5.88$. Thus, the price rises from the original \$5.00 to \$5.53 in the short run to \$5.88 in the long run. Each firm, which originally produced 1280 units, reduces its output to 1045 in the short run, but that output level increases to 1088 in the long run for those firms that remain in the industry. At the new long run equilibrium price, the market demands $D^M = 40,000,000/(5.88^2) \approx 1,156,925$ — down from the original 1,600,000 and from the short run equilibrium industry output of 1,306,395. This implies that the number of firms that remain in the industry falls from the original 1,250 to $1,156,925/1088 \approx 1,063$ firms.

The graph in part A predicted that the industry would produce less in the short run and even less in the long run, and that the price will rise in the short run and even more in the long run. Both these predictions hold up in this example. The graph furthermore predicted that output by each firm will initially fall in the short run but will rise back to its original quantity in the long run for firms that stay in the industry. The latter does not hold in this example, but we had pointed out in part A that the long run output could in principle go up or down — and we simply graphed it as unchanged from the original quantity solely for convenience. Thus, the prediction in this example that firm output for those that remain in the industry will initially go down and then recover somewhat but not fully is not inconsistent with the discussion surrounding the graph in part A.

Exercise 14B.15

How much does individual consumption by consumers who were originally in the market change in the short run?

Answer: Individual demand was derived at the beginning of the chapter to be $x^d(p) = 625/p^2$. Substituting in $p = 5.91$, we then get that $x^d(5.91) = 625/(5.91^2) = 17.894 \approx 18$.

Exercise 14B.16

Verify these calculations and compare the results with Graph 4.11 where we graphically illustrated the impact of an increase in market demand.

Answer: Since the average cost curves for firms have not changed, the long run price falls to the previously calculated lowest point of the AC curve — which is \$5. Thus, firms will enter until price falls from the short run equilibrium of \$5.91 to the long run equilibrium of \$5. To meet market demand of 2,500,000 at that price, the new number of firms in the industry must be $2,500,000/1280 \approx 1,953$, up from the initial 1,250, with each firm producing at its original equilibrium quantity of 1,280 units of output. Thus, initially industry quantity rises in the short run because each firm produces more at a higher price, but in the long run each firm returns to its original quantity, more firms enter and price falls to its original level, with the industry increasing production beyond the short run increase. This is exactly what is demonstrated in Graph 14.11 of the text.

14C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 14.1

In Table 14.1, the last column indicates the predicted change in the number of firms within an industry when economic conditions change.

A: *In two cases, the table makes a definitive prediction, whereas in two other cases it does not.*

(a) *Explain first why we can say definitively that the number of firms falls as a recurring fixed cost (i.e. license fee) increases? Relate your answer to what we know about firm output and price in the long run.*

Answer: When a fixed cost increases, the long run MC curve does not change but the long run AC curve shifts up. Since the MC curve always crosses the lowest point of the AC curve, we know that this implies that the lowest point of the long run AC curve shifts to the right — i.e. to a higher level of output. This implies that the output level of firms that remain in the industry will increase in the new equilibrium — as will the price (since the AC curve has shifted up). But an increase in price means that, for any downward sloping demand curve, consumers will demand less of the good. The industry therefore produces less at a higher price — with each firm in the industry producing more. The only way this is possible is if some firms have exited — i.e. the number of firms in the industry has decreased.

(b) *Repeat (a) for the case of an increase in demand.*

Answer: When demand increases, none of the cost curves for firms change — with the long run AC curve in particular remaining unchanged. Thus, each firm in the new equilibrium will be producing at the same lowest point of its AC curve — and at the same price. The only thing that has changed is that demand has shifted — which implies that, at the same price, more output will be produced in the industry. With each firm in the industry producing the same output quantity, the only way more can be produced in the industry is for more firms to have entered — i.e. the number of firms in the industry has increased.

(c) *Now consider an increase in the wage rate and suppose first that this causes the long run AC curve to shift up without changing the output level at which the curve reaches its lowest point. In this case, can you predict whether the number of firms increases or decreases?*

Answer: In this case, the output level produced by each firm in the industry will remain the same but will be sold at a higher price. When price increases, however, less will be demanded (assuming a downward sloping market demand curve) — which implies the industry produces less. With each firm in the industry producing the same as before, the only way for the industry to produce less is for some firms to have exited. Thus, the number of firms in the industry falls in this case.

(d) *Repeat (c) but assume that the lowest point of the AC curve shifts up and to the right.*

Answer: If the lowest point of the AC curve shifts up and to the right, it means that firms that remain in the industry will produce *more* at a higher price — but the higher price implies that less will be demanded and thus the industry produces *less*. The only way each firm can produce more while the industry produces less is if some firms exited — and the number of firms in the industry therefore declines.

(e) *Repeat (c) again but this time assume that the lowest point of the AC curve shifts up and to the left.*

Answer: In this case, the lowest point of the AC curve occurs at a lower level of output and higher price — which means that firms in the industry will produce less and sell at that higher price. A higher price in turn means that consumers will demand less. Thus, each firm produces *less* as does the industry. Whether this implies more or fewer firms now depends on how much less each firm produces relative to how much the quantity demanded falls with the increase in price. Suppose each firm produces x% less and the industry as a whole produces y% less. Then if $x = y$, the number of firms stays exactly the same; if $x < y$, the number of firms falls and if $x > y$, the number of firms in the industry has to increase.

(f) *Is the analysis regarding the new equilibrium number of firms any different for a change in r?*

Answer: No, the analysis is no different for a change in r. Even if capital is fixed in the short run, it is variable in the long run — and treated just like the input labor.

(g) *Which way would the lowest point of the AC curve have to shift in order for us not to be sure whether the number of firms increases or decreases when w falls?*

Answer: When w falls, we know the long run AC curve will shift down — which implies that the long run equilibrium price will fall. At a lower price, the quantity demanded will increase — which implies that industry output will *increase*. Were each firm to continue to produce the same amount as before — or were each firm to produce less, then the only way for the industry to produce more would be for the number of firms to increase. Thus, in order for us not to be sure of whether the number of firms increases, it would have to be that each firm produces more (just as the industry produces more) — and this only happens if the lowest point of the AC curve shifts to the right as w falls.

B: *Consider the case of a firm that operates with a Cobb-Douglas production function $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ where $\alpha,\beta > 0$ and $\alpha+\beta < 1$.*

(a) *The cost function for such a production process — assuming no fixed costs — was given in equation (13.45) of exercise 13.5. Assuming an additional recurring fixed cost F, what is the average cost function for this firm?*
Answer: Including the fixed cost F, the total cost function becomes

$$C(w,r,x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} + F \tag{14.1.i}$$

which gives us an AC function

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} + \frac{F}{x}. \tag{14.1.ii}$$

(b) *Derive the equation for the output level x^* at which the long run AC curve reaches its lowest point.*
Answer: The AC curve reaches its lowest point where its derivative with respect to x is zero — i.e. where

$$\frac{\partial AC(w,r,x)}{\partial x} = \left[(1-\alpha-\beta)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}\right] - \frac{F}{x^2} = 0. \tag{14.1.iii}$$

Solving this for x, we then get the output level at the lowest point of the long run AC curve:

$$x^* = \left(\frac{A\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(\alpha+\beta)}. \tag{14.1.iv}$$

(c) *How does x^* change with F, w and r?*
Answer: Given the expression for x^*, it is easy to see that x^* increases with F and decreases with w and r. Put differently, the lowest point of the AC curve occurs at higher output levels as the fixed cost increases and at lower output levels when input prices increase.

(d) True or False*: For industries in which firms face Cobb-Douglas production processes with recurring fixed costs, we can predict that the number of firms in the industry increases with F but we cannot predict whether the number of firms will increase or decrease with w or r.*
Answer: This is true. As F increases, output price rises as does output by each firm. The higher output price, however, means that the quantity demanded — and thus the quantity supplied by the industry — decreases. The only way the industry output can decrease when firm output increases is if some firms have left the industry. When input prices increase, the equilibrium output price similarly rises (as the AC shifts up) — causing the industry to produce less. But, in the case analyzed here, each firm also produces less — so we cannot immediately tell whether the number of firms will increase or decrease.

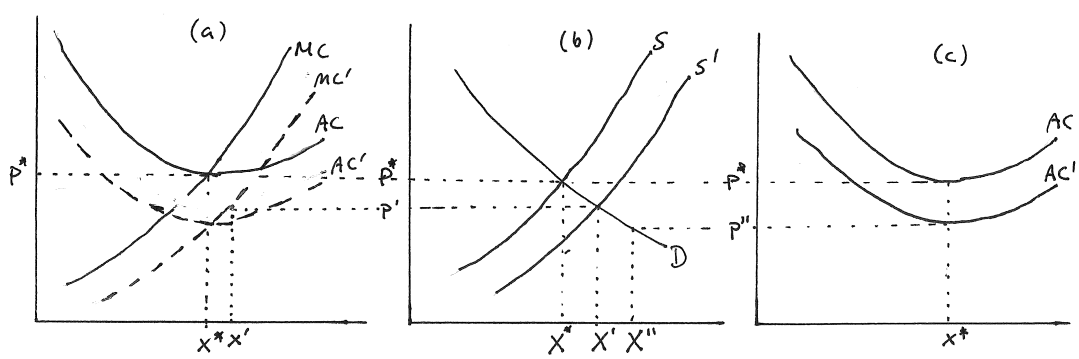
Exercise 14.3

Everyday and Business Application: *Fast Food Restaurants and Grease (cont'd): In exercise 12.8, you investigated the impact of hybrid vehicles that can run partially on grease from hamburger production on the number of hamburgers produced by a fast food restaurant. You did so, however, in the absence of considering the equilibrium impact on prices — assuming instead that prices for hamburgers are unaffected by the change in demand for grease.*

A: *Suppose again that you use a decreasing returns to scale production process for producing hamburgers using only labor and that you produce 1 ounce of grease for every hamburger. In addition, suppose that you are part of a competitive industry and that each firm also incurs a recurring fixed cost F every week.*

(a) *Suppose that the cost of hauling away grease is $q > 0$ per ounce. Illustrate the shape of your marginal and average cost curve (given that you also face a recurring fixed cost.)*

Answer: These are illustrated in panel (a) of Exercise Graph 14.3 as the solid curves labeled *MC* and *AC*. The marginal curve is upward sloping because it is unaffected by the recurring fixed cost. The average cost curve, however, is U-shaped as a result of *F*.



Exercise Graph 14.3 : Hamburgers and Hybrid Vehicles

(b) *Assuming all restaurants are identical, illustrate the number of hamburgers you produce in long run equilibrium.*

Answer: In long run equilibrium, you will be making zero profit — which means the long run price is p^* as illustrated in panel (a) of Exercise Graph 14.3. As a result, you produce x^*.

(c) *Now suppose that, as a result of the increased use of hybrid vehicles, the company you previously hired to haul away your grease is now willing to pay for the grease it hauls away. How do your cost curves change?*

Answer: The marginal and average cost curves will now shift down by the change in q. This is illustrated in panel (a) of Exercise Graph 14.3 with the dashed MC and AC curves.

(d) *Describe the impact this will have on the equilibrium price of hamburgers and the number of hamburgers you produce in the short run.*

Answer: In panel (b) of Exercise Graph 14.3, we illustrate the initial short run market supply curve S that intersects the demand curve D at the original price p^*. As a result of each firm's marginal cost curve shifting down, the short run market supply curve shifts down to S' — resulting in a decrease in the price to p'. The industry ends up producing more (X' rather than X^* in panel (b)) — which means each restaurant is producing more in the short run when the number of restaurants is fixed. This is illustrated as x' in panel (a).

(e) *How does your answer change in the long run?*

Answer: In the long run, the equilibrium price must settle at a point where all restaurants again make zero profit — i.e. to the lowest point of the new AC' curve. This is illustrated in panel (c) of Exercise Graph 14.3. Because the cost of each hamburger produced decreases by the same amount, the average cost curve shifts down in a parallel way — leaving the lowest point at that same output quantity x^* as before. Thus, in the long run, each restaurant will produce the same number of hamburgers as originally (x^*) and will sell them at price p''. The price has in essence fallen by the full decrease in the marginal cost. In the new equilibrium, there will be more restaurants than before.

(f) *Would your answers change if you instead assumed that restaurants used both labor and capital in the production of hamburgers?*

Answer: No — the cost curves would shift in exactly the same way.

(g) *In exercise 12.8, you concluded that the cholesterol level in hamburgers will increase as a result of these hybrid vehicles if restaurants can choose more or less fatty meat. Does your conclusion still hold?*

Answer: Yes, this conclusion still holds — if firms can profit from using fattier beef, they will all do so in equilibrium. If a firm did not do so, it would make negative profit.

B: *Suppose, as in exercise 12.8, that your production function is given by $f(\ell) = A\ell^{\alpha}$ (with $0 < \alpha < 1$)and that the cost of hauling away grease is q. In addition, suppose now that each restaurant incurs a recurring fixed cost of F.*

(a) *Derive the cost function for your restaurant.*

Answer: Solving the production function $x = A\ell^{\alpha}$ for ℓ, we get the conditional labor demand function

$$\ell(w,x) = \left(\frac{x}{A}\right)^{1/\alpha}. \tag{14.3.i}$$

Multiplying this by w and adding the cost of hauling grease as well as the fixed cost, we get

$$C(w,x,q) = w\left(\frac{x}{A}\right)^{1/\alpha} + qx + F. \tag{14.3.ii}$$

(b) *Derive the marginal and average cost functions.*

Answer: Taking the derivative of the cost function with respect to x, we get

$$MC(w,x,q) = \left(\frac{w}{\alpha A^{1/\alpha}}\right) x^{(1-\alpha)/\alpha} + q. \tag{14.3.iii}$$

Dividing the cost function by x, we get the average cost function

$$AC(w,x,q) = \left(\frac{w}{A^{1/\alpha}}\right) x^{(1-\alpha)/\alpha} + q + \frac{F}{x}. \tag{14.3.iv}$$

(c) *How many hamburgers will you produce in the long run?*

Answer: In the long run, you produce at the lowest point of your average cost curve. To determine the output quantity at which this occurs, we take the first derivative of our AC function, set it to zero and solve for x. This gives us

$$x^* = A\left(\frac{\alpha F}{(1-\alpha)w}\right)^{\alpha}. \tag{14.3.v}$$

(d) *What is the long run equilibrium price of hamburgers?*

Answer: To determine the long run equilibrium price, we plug x^* back into the average cost function and solve it; i.e.

$$p^* = AC(w,x^*,q) = \left(\frac{w}{A^{1/\alpha}}\right)\left[A\left(\frac{\alpha F}{(1-\alpha)w}\right)^{\alpha}\right]^{(1-\alpha)/\alpha} + q + \frac{F}{\left[A\left(\frac{\alpha F}{(1-\alpha)w}\right)^{\alpha}\right]} \tag{14.3.vi}$$

which, after some algebra, reduces to

$$p^* = \frac{w^{\alpha}F^{(1-\alpha)}}{A(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}} + q. \tag{14.3.vii}$$

(e) *From your results, determine how the long run equilibrium price and output level of each restaurant changes as q changes.*

Answer: The term q does not appear in our equation for x^* — which implies that it has no impact on the number of hamburgers produced by each restaurant in the long run. This is consistent with what we determined graphically. Our equation for p^*, on the other hand, has q simply entering as an additive term. Thus, in long run equilibrium, q is simply passed onto the consumer — as it falls (and even becomes negative), consumers therefore get the entire benefit.

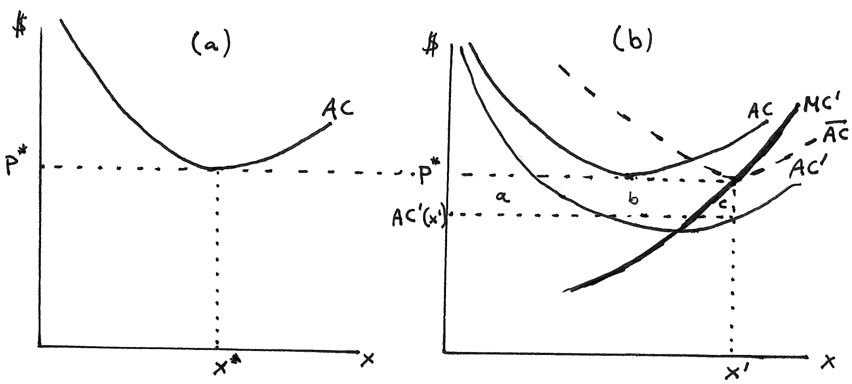
Exercise 14.5

Business Application: *"Economic Rent" and Profiting from Entrepreneural Skill: Suppose, as in exercise 14.4, that you are operating a hamburger restaurant that is part of a competitive industry. Now you are also the owner, and suppose throughout that the owner of a restaurant is also one of the workers in the restaurant and collects the same wage as other workers for the time he/she puts into the business each week. (In addition, of course, the owner keeps any weekly profits.)*

A: *Again, assume that all the restaurants are using the same homothetic decreasing returns to scale technology, but now the inputs include the level of entrepreneural capital c in addition to weekly labor ℓ and capital k. As in exercise 14.4, assume also that all restaurants are required to pay a recurring weekly fixed cost F.*

(a) *First assume that all restaurant owners possess the same level of entrepreneural skill c. Draw the long run AC curve (for weekly hamburger production) for a restaurant and indicate how many weekly hamburgers the restaurant will sell and at what price assuming that the industry is in long run equilibrium.*

Answer: This is illustrated in panel (a) of Exercise Graph 14.5 where the long run average cost curve of each firm is U-shaped because of the recurring fixed cost. Each restaurant will produce x^* and sell it at p^*.



Exercise Graph 14.5 : Rent on Entrepreneural Skill

(b) *Suppose next that you are special and possess more entrepreneural and management skill than all those other restaurant owners. As a result of your higher level of c, the marginal product of labor and capital is 20% greater for any bundle of ℓ and k than it is for any of your competitors. Will the long run equilibrium price be any different as a result?*

Answer: No, a single firm in a competitive industry is not large enough to affect market prices.

(c) *If your entrepreneural skill causes the marginal product of capital and labor to be 20% greater for any combination of ℓ and k than for your competitors, how does your isoquant map differ from theirs? For a given wage and rental rate, will you employ the same labor to capital ratio as your competitors?*

Answer: We know that the slopes of isoquants are $TRS = -MP\ell/MP_k$. If both marginal products increase by the same percentage, then the ratio is unchanged — which implies that your isoquant map looks exactly the same as your competitors' except that it is differently labeled because you can produce more with less capital and labor. Since the shapes of the isoquants are the same as those for your competitors', isocosts will be tangent along the same ray from the origin — which implies that you will employ the same labor to capital ratio as you minimize your costs. You will simply require less capital and labor for any given level of output.

(d) *Will you produce more or less than your competitors? Illustrate this on your graph by determining where the long run MC and AC curves for your restaurant will lie relative to the AC curve of your competitors.*

Answer: Since you need less labor and capital for any given level of output, your average costs are lower. Similarly, the lowest point of your AC curve will occur at a higher level of output than for your competitors. This is illustrated in panel (b) of Exercise Graph 14.5 where AC is your competitors' average cost curve and AC' is yours. Finally, we can put your long run MC' curve into the graph (making sure it crosses your average cost curve AC' at its lowest point. Since the market price is unchanged at p^*, we know you will profit maximize where p^* intersects MC' — at output level x'. You will therefore produce more than your competitors.

(e) *Illustrate in your graph how much weekly profit you will earn from your unusually high entrepreneural skill.*

Answer: In panel (b) of Exercise Graph 14.5, two rectangular boxes emerge from the dotted lines combined with the axes. The larger of these is equal to total revenues for your firm (price times output); the smaller one is your total cost (average cost times output); and the difference — area $a+b+c$ — is the difference between the two. Since those without your entrepreneural skill make zero profit, the profit you derive from your skill is therefore $a+b+c$.

(f) *Suppose the owner of MacroSoft, a new computer firm, is interested in hiring you as the manager of one of its branches. How high a weekly salary would it have to offer you in order for you to quit the restaurant business assuming you would work for 36 hours per week in either case and assuming the wage rate in the restaurant business is $15 per hour.*

Answer: It would have to offer you a salary equal to the level of compensation you currently get for spending your time in the restaurant business. Since you are one of the workers drawing an hourly wage $w = 15$, you are making $540 per week as one of the workers in your restaurant *plus* you earn a profit of $a+b+c$ as indicated in Exercise Graph 14.5.

MacroSoft would therefore have to offer you a minimum weekly salary of $a+b+c+540$.

(g) *The benefit that an entrepreneur receives from his skill is sometimes referred to as the* economic rent *of that skill — because the entrepreneur could be renting his skill out (to someone like MacroSoft) instead of using it in his own business. Suppose MacroSoft is willing to hire you at the rate you determined in part (f). If the economic rent of entrepreneural skill is included as a cost to the restaurant business you run, how much profit are you making in the restaurant business?*

Answer: You would then be making zero profit because $a+b+c$ in Exercise Graph 14.5 would become an additional periodic fixed cost in your restaurant business.

(h) *Would counting this economic rent on your skill as a cost in the restaurant business affect how many hamburgers you produce? How would it change the AC curve in your graph?*

Answer: Since the economic rent is a recurring long run fixed cost, it does not impact the long run MC curve in panel (b) of Exercise Graph 14.5. Thus, price p^* continues to intersect MC' at x' — implying you will produce exactly the same amount as if we did not count economic rent on your skill as a cost. The only thing that would change in panel (b) of the graph is that the average cost curve would be higher because $a+b+c$ is now included as a recurring average cost — and this average cost curve (denoted $\overline{AC}$ in the graph) — would reach its lowest point as it intersects the unchanged MC'. Thus, price equals MC' at the lowest point of $\overline{AC}$ — giving us zero profit for the firm if economic rents on entrepreneural skill are counted as a cost for the firm (and a payment to the owner).

B: *Suppose that all restaurants are employing the production function* $f(\ell,k,c)=30\ell^{0.4}k^{0.4}c$ *where* ℓ *stands for weekly labor hours,* k *stands for weekly hours of rented capital and* c *stands for the entrepreneural skill of the owner. Note that, with the exception of the* c *term, this is the same production technology used in exercise 14.4. The weekly demand for hamburgers in your city is, again as in exercise 14.4,* $x(p)=100,040-1,000p$.

(a) *First, suppose that* $c=1$ *for all restaurant owners, that* $w=15$ *and* $r=20$*, that there is a fixed weekly cost $4,320 of operating a restaurant, and the industry is in long run equilibrium. Determine the weekly number of hamburgers sold in each restaurant, the price at which hamburgers sell, and the number of restaurants that are operating. (If you have done exercise 14.4, you should be able to use your results from there.)*

Answer: Since $c=1$ for all restaurants, the production function becomes identical to that used in exercise 14.4. In parts (a) through (c) of exercise 14.4, you calculated that each restaurant will produce 4,320 hamburgers per week, that the long run equilibrium price will be $5 per hamburger and that there will be 22 restaurants in your city.

(b) *Next, suppose that you are the only restaurant owner that is different from all the others in that you are a better manager and entrepreneur and that this is reflected in* $c = 1.24573$ *for you. Determine your long run AC and MC functions. (Be careful not to use the cost function given in exercise 14.4 since c is no longer equal to 1. You can instead rely on the cost function derived for Cobb-Douglas technologies given in equation (13.45) in exercise 13.5 (and remember to add the fixed cost).)*

Answer: Your production function can then be written as

$$f(\ell, k) = [30(1.24573)]\,\ell^{0.4}k^{0.4} = 37.3719\ell^{0.4}k^{0.4}. \tag{14.5.i}$$

The cost function for a Cobb-Douglas production process $f(\ell, k) = A\ell^{\alpha}k^{\beta}$ is

$$C(w, r, x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{14.5.ii}$$

Substituting in $\alpha = \beta = 0.4$, $A = 37.3719$, $w = 15$ and $r = 20$, and adding the fixed cost of \$4,320, this gives us the cost function

$$C(x) = 0.374895x^{1.25} + 4320. \tag{14.5.iii}$$

The marginal and average cost functions are then

$$MC(x) = \frac{dC(x)}{dx} = 0.468619x^{0.25} \text{ and } AC(x) = \frac{C(x)}{x} = 0.374895x^{0.25} + \frac{4320}{x}. \tag{14.5.iv}$$

(c) *How many hamburgers will you produce in long run equilibrium?*

Answer: The long run equilibrium price must still be equal to \$5 per hamburger because the rest of the firms must be operating at zero long run profit. Your restaurant will, however, produce where price equals your MC curve which is different from those of the other firms. Thus, we set price equal to MC — i.e. $5 = 0.4686x^{0.25}$ and solve for x to get $x \approx 12{,}960$.

(d) *How many restaurants will there be in long run equilibrium given your higher level of c?*

Answer: Your production of 12,960 hamburgers per week is 3 times the production of the 4,320 hamburgers per week in the other restaurants. Before, there were 22 restaurants — which implies that now there will only be 20 including your restaurant. Thus, your entry into the restaurant market drives two of the other restaurants out of business.

(e) *How many workers (including yourself) and units of capital are you hiring in your business compared to those hired by your competitors? (Recall that the average worker is assumed to work 36 hours per week.)*

Answer: You can either solve the profit maximization problem to derive the labor and capital demand curves and use these to determine how many hours of labor and capital will be used. Alternatively, we could differentiate the cost function with respect to the input prices to get the conditional labor and capital demand functions — then plug in the input prices and output levels to get to the answer. Either way, we get that the other firms are hiring 576 labor hours and 432 capital hours per week, and your firm is hiring 1,728 labor hours and 1,296 hours of capital per week. At a work week of 36 hours, this implies that other firms hire 16 workers and your firm hires 36 workers.

(f) *How does your restaurant's weekly long run profit differ from that of the other restaurants?*

Answer: Other restaurants are selling 4,320 hamburgers at a price of \$5 to make total weekly revenues of \$21,600; and they pay a weekly fixed cost of \$4,320 and hire 576 worker hours at wage \$15 and 432 capital hours at rental rate \$20 for total cost of $4320 + 576(15) + 432(20) = \$21,600$. Thus, profits of other restaurants are zero (as we know has to be in long run equilibrium). Your firm, on the other hand, is selling 12,960 hamburgers at a price of \$5 for a total revenue of \$64,800. Your costs include the \$4,320 weekly fixed cost plus the cost of 1,728 hours of labor hired at a wage of \$15 and 1,296 hours of capital hired at a rental rate of \$20 for a total cost of $4320 + 1728(15) + 1296(20) = \$56,160$. This implies a profit for you of $64,800 - 56,160 = \$8,640$ per week.

(g) *Suppose Macrosoft is interested in hiring you as described in part A(f). How high a weekly salary would MacroSoft have to offer you in order for you to quit the restaurant business and accept the MacroSoft offer?*

Answer: Since you are also one of the workers who works 36 hours per week in your restaurant, your overall compensation is your labor income of $36(15) = \$540$ plus the profit of \$8,640 per week — for a total of \$9,180 per week. This is the least that MacroSoft would have to offer you in weekly compensation in order to attract you away from your restaurant business.

(h) *If you decide to accept the MacroSoft offer and you exit the restaurant business, will total employment in the restaurant business go up or down?*

Answer: With you in the restaurant business, we have 36 workers working in your business and 16 in each of the 19 others — for a total of 340 restaurant workers. With you out of the business, there are 22 restaurants employing 16 workers each — for a total of 352 workers. Thus, if you accept the MacroSoft offer, the number of workers (including owners) in restaurants increases by 12.

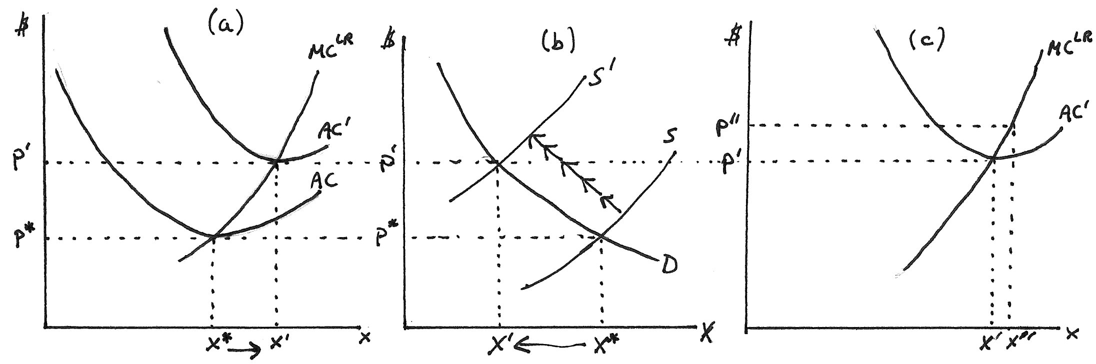
Exercise 14.7

Business and Policy Application: *Using License Fees to Make Positive Profit: Suppose you own one of many identical pharmaceudical companies producing a particular drug x.*

A: *Your production process has decreasing returns to scale but you incur an annually recurring fixed cost F for operating your business.*

(a) *Begin by illustrating your firm's average long run cost curve and identify your output level assuming that the output price is such that you make zero long run profit.*

Answer: This is illustrated in panel (a) of Exercise Graph 14.7 where the initial long run average cost curve — AC — is U-shaped because of the fixed cost F. Zero long run profit implies you are producing at the lowest point of that AC curve — quantity x^* sold at price p^*.



Exercise Graph 14.7 : A Large License Fee

(b) *Next to your graph, illustrate the market demand and short run market supply curves that justify the zero-profit price as an equilibrium price.*

Answer: This is done in panel (b) of the graph where the initial supply S intersects demand D at price p^*.

(c) *Next, suppose that the government introduces an annually recurring license fee G for any firm that produces this drug. Assume that your firm remains in the industry. What changes in your firm and in the market in both the short and long run as a result of the introduction of G and assuming that long run profits will again be zero in the new long run equilibrium?*

Answer: Since this is a fixed cost, it has no short run impact. In the long run, it does not shift the long run MC curve in panel (a) of Exercise Graph 14.7, only the long run AC curve which is illustrated as AC'. This implies that the lowest point of AC' lies to the right of the lowest point of the initial AC. Once the industry settles into a new long run equilibrium where all firms make zero profit, it must then be that the new equilibrium price p' falls at the lowest point of AC' causing each firm to produce x'. The license fee G therefore increases the output level in each firm that remains in the industry. However, overall production in the industry falls (in panel (b)) from X^* to X' as consumers demand less at the higher price p'. The fact that each firm is producing more but the industry is producing less implies that a number of firms must have exited on the way to the new long run equilibrium.

(d) *Now suppose that G is such that the number of firms required to sustain the zero-profit price in the new long run equilibrium is not an integer. In particular, suppose that we would require 6.5 firms to sustain this price as an equilibrium in the market. Given that fractions of firms cannot exist, how many firms will actually exist in the long run?*

Answer: If 7 firms produced, the price would be driven below p' and all firms would make negative long run profit. Thus, it must be that one more firm exits — and only 6 remain in the industry.

(e) *How does this affect the long run equilibrium price, the long run production level in your firm (assuming yours is one of the firms that remains in the market), and the long run profits for your firm?*

Answer: This is illustrated in panel (c) of Exercise Graph 14.7. We begin by replicating the new AC' as well as the (unchanged) long run MC curves from panel (a) — identifying again the zero profit price p' after the new fee G has been introduced. But if only 6 firms exist and it would have taken 6.5 to produce X' (in panel (b)) when each firm produces x', supply shifts further to the left as the 7th firm exits, driving price somewhat above p'. This reduces the quantity demanded and increases the quantity supplied by the remaining firms. In panel (c), the price p'' therefore lies above the zero profit price p' — with each of the remaining firms now producing $x'' > x'$. And, since price is now above the lowest point of the long run AC curve, each firm will make some positive economic profit.

(f) *True or False: Sufficiently large fixed costs may in fact allow identical firms in a competitive industry to make positive long run profits.*

Answer: This is, as we have just demonstrated, true.

(g) True or False*: Sufficiently large license fees can cause a competitive industry to become more* concentrated *— where by "concentrated" we mean fewer firms competing for each customer.*

Answer: This is true. As we have shown, an increase in fixed costs such as license fees will cause an upward and rightward shift of the long run average cost curves of firms — causing each firm to produce a larger quantity at a higher price in the new long run equilibrium. Because of the higher price, the industry produces less. Thus, we'll have fewer firms, with each firm attracting *more customers* — i.e. fewer firms compete for each given customer.

B: *Suppose that each firm in the industry uses the production function $f(\ell,k) = 10\ell^{0.4}k^{0.4}$ and each incurs a recurring annual fixed cost of \$175,646.*

(a) *Determine how much each firm produces in the long run equilibrium if $w = r = 20$. (You can use the cost function derived for Cobb-Douglas technologies given in equation (13.45) in exercise 13.5 (and remember to add the fixed cost).)*

Answer: For a Cobb-Douglas function of $f(\ell,k)=A\ell^{\alpha}k^{\beta}$, the cost function (in the absence of fixed costs) is

$$C(w,r,x)=(\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{14.7.i}$$

Dividing this by x and adding the average fixed cost FC/x, we then get the long run average cost function

$$AC(w,r,x)=\frac{C(w,r,x)}{x}+\frac{FC}{x}=(\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}+\frac{FC}{x}. \tag{14.7.ii}$$

Evaluating this at $\alpha=\beta=0.4$, $A=10$, $w=r=20$ and $FC=175,646$, this reduces to

$$AC(x)\approx 2.249x^{1/4}+\frac{175,646}{x}. \tag{14.7.iii}$$

The lowest point of this AC curve occurs where its derivative is equal to zero; i.e. where

$$\frac{dAC(x)}{dx}=\frac{0.5623}{x^{3/4}}-\frac{175646}{x^2}=0. \tag{14.7.iv}$$

Solving this for x, we get $x\approx 24,873$. This is how much each firm is producing in long run equilibrium.

(b) *What price are consumers paying for the drugs produced in this industry?*

Answer: Substituting $x=24,873$ back into the average cost function $AC(x)$ from equation (14.7.iii), we get that the long run equilibrium price must be approximately \$35.31.

(c) *Suppose consumer demand is given by the aggregate demand function $x(p)=1,000,000-10,000p$. How many firms are in this industry?*

Answer: Plugging in the long run equilibrium price $p=35.31$, we get consumer demand of $x=1,000,000-10,000(35.31)=646,900$. With each firm producing 24,873 units, this implies that there are $646,900/24873=26$ firms in the industry.

(d) *Suppose the government introduces a requirement that each company has to purchase an annual operating license costing \$824,354. How do your answers to (a), (b) and (c) change in the short and long run?*

Answer: Since this is an added fixed cost that is only a fixed expense in the short run, it does not affect anything in the short run. When added to the fixed cost of \$175,646, this annual fee increases the total fixed costs to \$1,000,000. This changes the long run AC curve to

$$AC(x)=2.249x^{1/4}+\frac{1,000,000}{x} \tag{14.7.v}$$

and its derivative to

$$\frac{dAC(x)}{dx} = \frac{0.5623}{x^{3/4}} - \frac{1,000,000}{x^2}. \tag{14.7.vi}$$

Setting the derivative to 0 and solving for x, we now get $x = 100,000$, up from the previous 24,873. Plugging this back into $AC(x)$, we get a long run equilibrium price of $p = 50$, up from the previous 35.31. At this price, consumers demand $x = 1,000,000 - 10,000(50) = 500,000$ units. With each firm producing 100,000 units, this leaves room for only 5 firms, down from the previous 26.

(e) *Are any of the firms that remain active in the industry better or worse off in the new long run equilibrium?*

Answer: No — they make zero profit before and again after the change to the new long run equilibrium.

(f) *Suppose instead that the government's annual fee were set at \$558,258. Calculate the price at which long run profits are equal to zero.*

Answer: When added to the fixed cost of \$175,646, this annual fee increases the total fixed costs to \$773,904. This changes the long run AC curve to

$$AC(x) = 2.249x^{1/4} + \frac{773,904}{x} \tag{14.7.vii}$$

and its derivative to

$$\frac{dAC(x)}{dx} = \frac{0.5623}{x^{3/4}} - \frac{773,904}{x^2}. \tag{14.7.viii}$$

Setting the derivative to 0 and solving for x, we now get approximately $x = 78,075$, up from the initial 24,873 but down from the 100,000 under the higher license fee. Plugging this back into $AC(x)$, we get a zero long run profit price of $p = 47$, up from the previous 35.31 but below the 50 under the higher license fee.

(g) *How many firms would this imply will survive in the long run assuming fractions of firms can operate?*

Answer: At a price of $p = 47$, consumers demand $x = 1,000,000 - 10,000(47) = 530,000$ units. With each firm producing 78,075 units, this would imply approximately $530,000/78,075 = 6.79$ firms in the market.

(h) *Since fractions of firms cannot operate, how many firms will actually exist in the long run? Verify that this should imply an equilibrium price of approximately \$48.2.(*Hint*: Use the supply function given for a Cobb-Douglas production process in equation (13.49) found in the footnote to exercise 13.7.)*

Answer: Since 7 firms cannot exist in the market, only 6 can survive. But if 6 firm produced 78,075 units at the zero long run profit price $p = 47$, only

468,450 units would be produced — which is 61,550 units less than the 530,000 units demanded by consumers at that price. Thus, in order for the market to clear in the long run, price has to increase. In order to determine by how much, we have to first derive the long run supply curve for each firm. The supply function for a production process $f(\ell,k) = A\ell^\alpha k^\beta$ is

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)} \tag{14.7.ix}$$

which, when evaluated at $\alpha = \beta = 0.4$, $A = 10$, and $w = r = 20$, becomes

$$x(p) = 0.016p^4. \tag{14.7.x}$$

Multiplying this by 6 — which is the number of firms remaining in the industry, we get

$$X^{LR}(p) = 0.096p^4. \tag{14.7.xi}$$

In an equilibrium with 6 firms, the equilibrium price then occurs where this supply function equals the demand function $x(p) = 1,000,000-10,000p$. The equation $0.096p^4 = 1,000,000-10,000p$ holds at $p = 48.197082195$ or approximately $p = 48.2$.

(i) *What does this imply for how much profit each of the remaining firms can actually make?*

Answer: At a price of \$48.2, equation (14.7.x) implies that the firm will produce output of approximately $x = 86,338$ which implies total revenues of $48.2(86,338) \approx 4,161,492$. Using equation (14.7.vii), we can determine its long run average cost at output 86,338 to be

$$AC = 2.249(86,338^{1/4}) + \frac{773,904}{86,338} \approx 47.058. \tag{14.7.xii}$$

This implies total costs of $47.058(86,338) \approx 4,062,894$. Subtracting this from total revenues of 4,161,492, we get long run profit of approximately \$98,598.

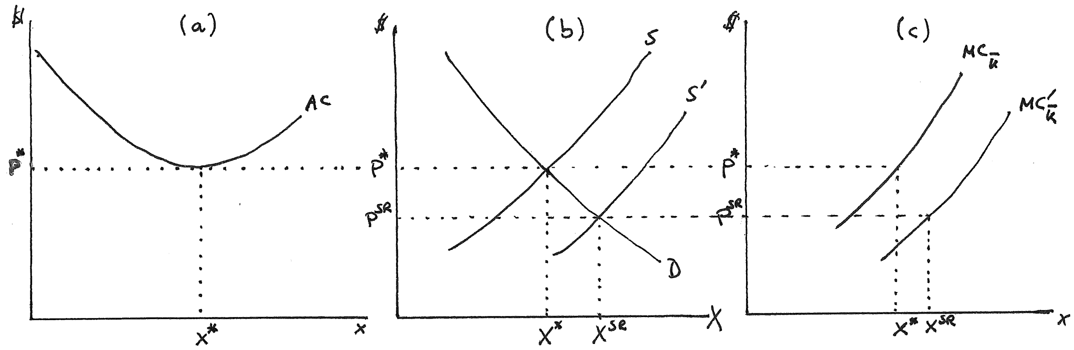
Exercise 14.9

Policy and Business Application: *Minimum Wage Labor Subsidy (cont'd)*: *In exercise 13.10, we investigated the firm's decisions in the presence of a government subsidy for hiring minimum wage workers. Implicitly, we assumed that the policy has no impact on the prices faced by the firm in question.*

A: *Suppose again that you operate a business that uses minimum wage workers ℓ and capital k. The minimum wage is w, the rental rate for capital is r and you are one of many identical businesses in the industry, each using a homothetic, decreasing returns to scale production process and each facing a recurring fixed cost F.*

(a) *Begin by drawing the average cost curve of one firm and relating it to the (short run) supply and demand in the industry assuming we are in long run equilibrium.*

Answer: This is done in panels (a) and (b) of Exercise Graph 14.9 where the long run equilibrium price p^* occurs at the intersection of the original market demand D and supply S curves (in panel (b)) and falls at the lowest point of each firm's average cost curve (in panel (a)).



Exercise Graph 14.9 : Minimum Wage Subsidy

(b) *Now the government introduces a wage subsidy s that lowers the effective cost of hiring minimum wage workers from w to* $(1-s)w$. *What happens in the firm and in the industry in the short run?*

Answer: This is illustrated in panels (b) and (c) of Exercise Graph 14.9. Each individual firm's short run marginal cost curve (given the fixed level of capital $\overline{k}$) shifts down — from $MC_{\overline{k}}$ to $MC'_{\overline{k}}$ in panel (c). Since this is happening to all firms, this implies that the short run market supply curve (in panel (b)) shifts down — from S to S'. As a result, the new short run equilibrium price falls to p^{SR} (in panel (b)) — and each firm produces more (in panel (c)) at that price. (In principle it is possible to draw these graphs such that the price falls and each firm produces less. However, that is a logical impossibility — because the number of firms is fixed in the short run. If the market overall produces more, it must be that each firm produces more in the short run).

(c) *What happens to price and output (in the firm and the market) in the long run compared to the original quantities?*

Answer: In the long run, the AC curve in panel (a) of Exercise Graph 14.9 shifts down — but we cannot be sure whether the lowest point shifts right or left. (The more likely case is that it shifts to the right). Thus, in the new long run equilibrium, we know the price has to settle below p^* — with each firm producing more or less than originally depending on whether the lowest point of the AC curve falls to the right or left of x^*. Since price is below p^*, we can also be sure that the overall quantity produced in the market will increase.

(d) *Is it possible to tell whether there will be more or fewer firms in the new long run equilibrium?*

Answer: No, it is not. It could be that each firm produces sufficiently more in the new long run equilibrium and the overall quantity demanded increases relatively less such that fewer firms can be sustained in the new equilibrium. But the reverse is also possible.

(e) *Is it possible to tell whether the long run price will be higher or lower than the short run price? How does this relate to your answer to part (d)?*

Answer: No, it is not possible to tell for sure. This relates to (d) in the sense that it relates to whether additional firms will enter or existing firms will exit in the transition from the short run to the long run equilibrium. If conditions are such that the number of firms falls, this implies that the exit of firms from the industry will put upward pressure on price relative to its short run value. If, on the other hand, conditions are such that the number of firms increases, then this implies that the entry of new firms puts additional downward pressure on price — causing the long run price to fall below the short run price.

B: *Suppose that the firms in the industry use the production technology $x = f(\ell, k) = 100\ell^{0.25}k^{0.25}$ and pay a recurring fixed cost of $F = 2,210$. Suppose further that the minimum wage is \$10 and the rental rate of capital is $r = 20$.*

(a) *What is the initial long run equilibrium price and firm output level?*

Answer: Plugging the production function values and input prices into the cost function for Cobb-Douglas production, we get a cost function of $C(x) = 0.00282843x^2$, and adding the fixed cost F, we get $C(x) = 0.00282843x^2 + 2210$. This gives us the average cost function $AC(x) = 0.00282843x + 2210/x$. Taking the first derivative, setting it to zero and solving for x then gives us the output level at the lowest point of the AC curve — which is $x \approx 884$. Plugging this back into the AC function, we then get that the long run equilibrium price is \$5. Each firm therefore sells 884 output units at a price of \$5 per unit.

(b) *Suppose that $s = 0.5$ — implying that the cost of hiring minimum wage labor falls to \$5. How does your answer to (a) change?*

Answer: The new long run equilibrium is then derived exactly as it was in (a) except that $w = 5$ is substituted in the first step when we derive the cost function from the general Cobb-Douglas form of the cost function. This gives us $C(x) = 0.002x^2$ and, once we go through the remaining steps, $x \approx 1,051$ as the output quantity at the lowest point of the AC curve. The long run (zero profit) price is approximately \$4.20.

(c) *How much more or less of each input does the firm buy in the new long run equilibrium compared to the original one? (The input demand functions for a Cobb-Douglas production process were previously derived and given in equation (13.50) of exercise 13.8.)*

Answer: Substituting $A = 100$, $\alpha = 0.25 = \beta$ and $r = 20$ into the labor and capital demand equations, we get

$$\ell(w,p) \approx 139.75\left(\frac{p^2}{w^{3/2}}\right) \text{ and } k(w,p) \approx 6.9877\left(\frac{p^2}{w^{1/2}}\right). \quad (14.9.i)$$

In the initial equilibrium, $(w,p) = (10,5)$ while in the new equilibrium, $(w,p) = (5,4.2)$. Substituting these into the equations, we then get that an initial input bundle $(\ell,k) = (110.5,55.25)$ and a new input bundle $(\ell,k) = (221,55.25)$. (Answers may differ slightly due to rounding errors.) Labor input therefore doubled but capital input remained unchanged.

(d) *If price does not affect the quantity of x demanded very much, will the number of firms increase or decrease in the long run?*

Answer: If the quantity demanded (and thus the quantity produced by the industry) remains roughly the same, the number of firms in the industry must decline since each firm is now producing 1,051 units of output rather than the initial 884.

(e) *Suppose that demand is given by $x(d) = 200,048 - 2,000p$. How many firms are there in the initial long run equilibrium?*

Answer: In the initial long run equilibrium, $p = 5$. This implies that the total quantity demanded is $200,048 - 2,000(5) = 190,048$. Each firm produces 884 units initially, which implies that we have 215 firms operating. (Your answer may be slightly below 215 because of rounding error when we use 884 units per firm rather than 883.84 which is the more exact number returned by the math.)

(f) *Derive the short run market supply function and illustrate that it results in the initial long run equilibrium price.*

Answer: To derive the short run market supply function, we need to first determine the short run supply function of each of the 215 firms in the initial equilibrium. We concluded in (c) that each firm is using 55.25 units of capital in that initial equilibrium. In the short run, when capital is fixed, each firm is therefore operating on the slice

$$f_{k=55.25}(\ell) = \left[100(55.25)^{0.25}\right]\ell^{0.25} = 272.636\ell^{0.25}. \quad (14.9.ii)$$

Solving the short run profit maximization problem

$$\max_{\ell}\ p\left(272.636\ell^{0.25}\right) - w\ell, \quad (14.9.iii)$$

we get

$$\ell_{k=55.25}(w,p) = 278.42\left(\frac{p}{w}\right)^{3/4} \text{ and } x_{k=55.25}(w,p) = 1113.67\left(\frac{p}{w}\right)^{1/3}, \quad (14.9.iv)$$

the latter of which is the firm's short run supply function. Multiplying this by the number of firms in the industry (which we derived as 215), we get a short run market supply function

$$X^{SR}(w,p) = 239,440\left(\frac{p}{w}\right)^{1/3} \tag{14.9.v}$$

which becomes

$$X^{SR}(p) = 111,138p^{1/3} \text{ when } w = 10. \tag{14.9.vi}$$

At the original equilibrium, it must be that demand is equal to this short run market supply — i.e.

$$200,048 - 2,000p = 111,138p^{1/3}, \tag{14.9.vii}$$

which holds (approximately, due to some rounding) for our initial long run equilibrium price $p = 5$.

(g) *Verify that the short run equilibrium price falls to approximately \$2.69 when the wage is subsidized.*

Answer: When wage falls to $w = 5$, the short run market supply curve in equation (14.9.v) becomes

$$X^{SR}(p) = 140,025p^{1/3}. \tag{14.9.viii}$$

The short run equilibrium then occurs where supply equals demand; i.e. where

$$140,025p^{1/3} = 200,048 - 2,000p \tag{14.9.ix}$$

which (approximately) holds when $p = 2.69$.

(h) *How much does each firm's output change in the short run?*

Answer: Plugging the price $p = 2.69$ and subsidized wage $w = 5$ into the short run supply function for each firm, we get $x_{k=55.25}(5, 2.69) \approx 906$. Note that this sums approximately to the overall quantity transacted in the market when there are (the initial) 215 firms in the market.

(i) *Determine the change in the long run equilibrium number of firms when the wage is subsidized and make sense of this in light of the short run equilibrium results.*

Answer: We previously determined that the long run equilibrium price will be roughly \$4.20 — which is lower than the initial price of \$5 but higher than the short run price of \$2.69. At \$4.20, we can determine the total level of output in the industry by substituting this price into the demand function to get $x = 191,648$ — and with each firm producing 1,051 in the new long run equilibrium, this implies approximately 182 firms — down from the initial 215. We can see the dynamics of what makes firms choose to exit by observing that the short run equilibrium price of \$2.69 lies below the long run zero-profit price of \$4.20 — thus firms are making negative long run profits in the short run equilibrium (while still making positive short run profits since the expense on capital and the fixed costs are not real costs in the short run).

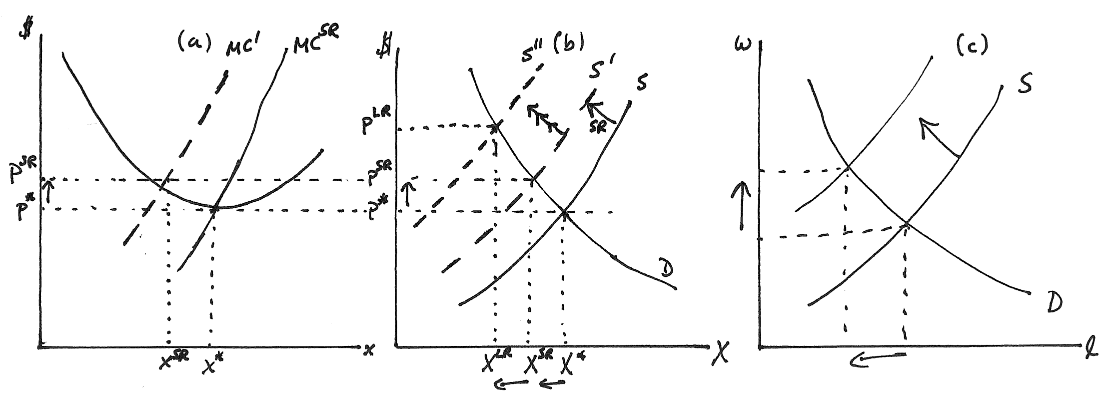
Exercise 14.11

Policy Application: *Public School Teacher Salaries, Class Size and Private School Markets: In exercise 14.10, we noted that private schools that charge tuition operate alongside public schools in U.S. cities. There is much discussion in policy circles regarding the appropriate level of public school teacher salaries (which are set by the local or state government) as well as the appropriate number of public school teachers (that determines class size in public schools).*

A: *Suppose again that private schools face U-shaped long-run AC curves for providing seats to children and that the private school market is currently in long run equilibrium.*

(a) *Begin by drawing two graphs — one with the long run AC curve for a representative private school and a second with the demand and (short run) aggregate supply curves (for private school seats) that are consistent with the private school market being in long run equilibrium (with private school tuition p on the vertical axis).*

Answer: This is done in panels (a) and (b) of Exercise Graph 14.11. In order for the private school market to be in long run equilibrium, each school makes zero profit and thus operates on the lowest point of its long run *AC* curve. Thus, the equilibrium price is p^* as reflected by the intersection of demand *D* and supply *S* in panel (b).



Exercise Graph 14.11 : Public School Teacher Salaries and Private School Markets

(b) *Now suppose the government initiates a major investment in public education by raising public school teacher salaries. In the market for private* *school teachers (with private school teacher salaries on the vertical and private school teachers on the horizontal), what would you expect to happen as a result of this public school investment?*

Answer: This is illustrated in panel (c) of Exercise Graph 14.11. I would expect the supply of private school teachers (of a given quality) to decrease as they are more attracted to the public schools. Thus, the equilibrium teacher wage in the private school market should increase.

(c) *How will this impact private school tuition levels, the number of seats in private schools and the overall number of children attending private schools in the short run?*

Answer: This increase in teacher salaries causes the short run MC curve of each private school to immediately shift to the left — indicated by MC' in panel (a) of Exercise Graph 14.11. Since this happens for all existing private schools, the market supply curve in panel (b) shifts immediately from S to S'. As a result, tuition price increases to p^{SR}, with the number of children attending private schools decreasing from the initial X^* to X^{SR} in panel (b). Each individual school will also admit fewer children (as shown in pane (a)). We know this because the same number of schools is admitting fewer students — and all schools are assumed to be roughly identical.

(d) *How does your answer change in the long run as private schools can enter and exit the industry?*

Answer: In principle, it could be that private schools either enter or exit as a result of these changes. Exercise Graph 14.11 illustrates the (more likely) case of schools exiting. If some private schools exit the market (because the short run increase in price is insufficient to cover long run costs), the supply curve in panel (b) of Exercise Graph 14.11 will shift further to S'', causing tuition price to increase further to p^{LR}. Thus, the market will serve fewer children in the long run — down from the initial drop to X^{SR} to X^{LR} in panel (b) of the graph. How many students are admitted by individual schools relative to the short run change is ambiguous — it depends on where the lowest point of the new long run AC falls, but it would typically be at a size smaller than the initial x^* (though it is in principle possible for it to be larger). If the lowest point of long run AC shifts sufficiently far to the left, private schools become very small — and if the demand curve is not too flat, this implies that the number of schools would actually *increase* in the long run. (This seems less likely and is not pictured in the graph.)

(e) *Suppose that instead of this teacher salary initiative, the city government decides to channel its public school investment initiative into hiring more public school teachers (as the city government is simply recruiting additional teachers from other states) and thus reducing class size. Assuming that this has no impact on the equilibrium salaries for teachers but does cause parents to feel more positively about public schools, how will the private school market be impacted in the short and long run?*

Answer: This would cause a decrease in demand for private schools — i.e. a leftward shift. The result would be the mirror image of what we concluded for vouchers in exercise 14.10: The lower demand would cause an initial drop in tuition levels in the short run, with each private school serving fewer children. In the long run, some private schools would exit, shifting the market supply curve to the left and raising tuition prices back to their original zero-profit level. In the long run, each remaining private school would therefore admit as many children as it did initially and would charge the same tuition it initially charged, but the private school market as a whole would serve fewer children.

(f) *How will your long run answer to (e) be affected if the government push for more public school teachers also causes equilibrium teacher salaries to increase?*

Answer: The shift in demand we just analyzed would shrink the private school sector but not change what the remaining private schools do (in terms of how many children they serve and what tuition they charge). If, however, there is an additional upward pressure on private school teacher salaries, then the smaller private school sector would change along the same lines as illustrated in Exercise Graph 14.11 — it would, in the long run, experience a further decrease in size, tuition levels would increase and existing private schools might be somewhat smaller or larger depending on how the increase in teacher salaries affects the lowest point of the AC curve.

B: *As in exercise 14.10, assume a total city-wide demand function* $x(p) = 24,710-2500p$ *for private school seats and let each private school's average long run cost function be given by* $AC(x) = 0.655x^{1/3} + (900/x)$*. Again, interpret all dollar values in thousands of dollars.*

(a) *If you have not already done so, calculate the initial long run equilibrium size of each school, what tuition price they charge and how many private schools there are in the market.*

Answer: As demonstrated in exercise 14.10 (b) and (d), each school has 515 students and charges $7,000 in tuition. There are 14 private schools in the city.

(b) *If you did B(a) in exercise 14.10 you have already shown that this* $AC(x)$ *curve arises from the Cobb-Douglas production function* $x = f(\ell, k) = 35\ell^{0.5}k^{0.25}$ *when* $w = 50$ *and* $r = 25$ *and when private schools face a fixed cost of 900. If you have not already done so, use this information to determine how many teachers and how much capital each school hires.*

Answer: With the labor and capital demand functions from (c) and (f) in exercise 14.10, we calculated that the initial long run profit maximizing production plan includes approximately 36 teachers per school as well as 36 units of capital per school.

(c) *Suppose that the increased pay for public school teachers drives up the equilibrium wage for private school teachers from 50 to 60 (i.e. from $50,000* *to \$60,000 per year). What happens to the equilibrium tuition price in the short run?*

Answer: The short run production function for fixed capital $\overline{k} = 36$ is

$$x = f_{\overline{k}}(\ell) = \left[35\left(36^{0.25}\right)\right]\ell^{0.5} \approx 85.73\ell^{0.5}. \qquad (14.11.i)$$

The short run profit maximization problem is then

$$\max_{\ell}\; p\left(85.73\ell^{0.5}\right) - w\ell. \qquad (14.11.ii)$$

Solving this, we get the short run labor demand function, and substituting it back into equation (14.11.i), we get the short run supply function:

$$\ell_{\overline{k}}(w,p) = 1{,}837.4\left(\frac{p}{w}\right)^2 \text{ and } x_{\overline{k}}(w,p) = 3674.8\left(\frac{p}{w}\right). \qquad (14.11.iii)$$

Setting w equal to the new level of 60 in the supply function $x_{\overline{k}}(w,p)$, we get each school's short run supply curve $x_{\overline{k}}(p) \approx 61.25p$, and multiplying it by 14 (i.e. the number of schools in the market), we get the market short run supply curve

$$X^{SR}(p) = 857.5p. \qquad (14.11.iv)$$

Setting this equal to the market demand curve $x(p) = 24710 - 2500p$ and solving for p, we get $p = 7.36$ for a tuition level of \$7,360, up from the initial \$7,000.

(d) *What happens to school size and class size?*

Answer: Plugging $w = 60$ and $p = 7.36$ into the expressions in equation (14.11.iii), we get

$$\ell_{\overline{k}} = 1{,}837.4\left(\frac{7.36}{60}\right)^2 \approx 27.65 \text{ and } x_{\overline{k}} = 3674.8\left(\frac{7.36}{60}\right) \approx 450.8. \qquad (14.11.v)$$

School size (i.e. the number of children in each school) therefore shrinks from the initial 515 to about 451, and the number of teachers per school shrinks from the initial 36 to around 27.65. This implies that class size increases from 14.3 to 16.3.

(e) *How will your answers on school size, tuition level and class size change in the long run? (*Hint*: You can use the cost function given in equation (13.45) of exercise 13.5 to derive the AC function — just make sure you keep track of the fixed cost of 900!)*

Answer: In the long run, schools again have to end up on the lowest point of their average cost curves. However, since w has increased, their average cost curves have shifted up. Dividing the cost function from exercise 13.5 by x and adding the average fixed cost $900/x$, we get

$$AC(w,r,x) = \frac{C(w,r,x)}{x} + \frac{FC}{x} = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} + \frac{FC}{x}. \tag{14.11.vi}$$

Substituting $\alpha = 0.5$, $\beta = 0.25$, $A = 35$, $w = 60$, $r = 25$ and $FC = 900$, we get

$$AC(x) = 0.74x^{1/3} + \frac{900}{x}. \tag{14.11.vii}$$

The lowest point on this U-shaped average cost curve arises when the derivative of AC is zero; i.e. when

$$\frac{dAC(x)}{dx} = \frac{0.247}{x^{2/3}} - \frac{900}{x^2} = 0. \tag{14.11.viii}$$

Solving this for x, we get $x \approx 469$. Thus, the school size, which started at 515 students and fell to 451 students in the short run, goes to 469 students in the long run. Plugging 469 back into the $AC(x)$ function in equation (14.11.vii), we can then determine that the bottom of the U-shaped AC curve occurs at an average cost of approximately 7.668 — which has to be the tuition price in the new long run equilibrium. Thus, tuition, which started at \$7,000 per student and went to \$7,360 in the short run, rises to to \$7,668 in the long run. Evaluating the long run labor demand equation at $p = 7.668$ and $w = 60$, we can determine that the number of teachers — which began at 36 and fell to 27.65 per school in the short run — goes to approximately 30 teachers per school. This causes class size — which began at 14.3 and rose to 16.3 in the short run — to go to 15.65.

(f) *How many private schools will remain in the market in the long run?*

Answer: We first have to determine the total quantity of school seats demanded — which we can do by evaluating the demand function at the new long run equilibrium price $p = 7.668$. This gives us

$$x(7.668) = 24710 - 2500(7.668) = 5,540. \tag{14.11.ix}$$

With each school serving about 469 children, this implies 11.8 or approximately 12 schools will remain. (Actually, given that 11.8 falls between 11 and 12, 11 schools would remain, with each producing at a slightly higher tuition price serving somewhat more students and making a small profit.)

Conclusion: Potentially Helpful Reminders

1. Students often find it irritating that we have defined the "short run" differently for firms than for industries — with firms operating in the "short run" as long as their capital level is fixed, and industries operating in the "short

run" so long as entry and exit of firms is not possible. You might be less irritated if you recognize that the two definitions actually boil down to the same definition: A firm needs to be able to invest in capital in order to enter (or to get rid of its capital in order to exit) the industry. Thus, a firm can enter or exit only if it can vary capital — which links our two ways of thinking about the "short run".

2. Of all the firm cost and expenditure curves we derived in the previous three chapters, only two are crucial in this chapter: (1) The long run *AC* curve (of the marginal firm) — or, to be even more specific, the lowest point of that long run *AC* curve. This determines long run price. (2) The short run *MC* curve for the fixed level of capital each firm has — which gives rise to the short run supply curve.

3. We often start out our analysis by assuming that an industry is initially in long run equilibrium. That implies that we start with a picture in which the marginal firm's short run supply curve crosses its long run *AC* curve at the lowest point of the *AC* curve. It extends below the *AC* curve because the short run shut-down price lies below the long run exit price. *Avoid drawing unnecessary curves that don't matter for your analysis.*

4. When you analyze a change that occurs within an industry that is initially in long run equilibrium, you therefore start with the initial long run picture of the marginal firm (with its long run *AC* curve and the short run supply curve) — and then ask whether any of the changes altered either of these curves. Then you can trace out what happens in the short run and the long run.

5. You can conclude what happens to the overall number of firms in the industry if you know what happens to each firm's output when we go from the initial to the new long run equilibrium and you know whether the overall quantity demanded at the new long run equilibrium price is higher or lower than it was originally. (The answer to the question of whether the number of firms increases or decreased will often be ambiguous if the change affecting the industry is a change in input prices). In the short run, the number of firms is fixed.

CHAPTER

15

The "Invisible Hand" and the First Welfare Theorem

Chapter 14 introduced the *positive* idea of equilibrium in the context of a competitive environment — and Chapter 15 now moves onto the more *normative* assessment of a competitive equilibrium within the context of the first welfare theorem. Put differently, Chapter 14 focuses on *predicting* changes in economic environments in competitive settings while Chapter 15 now focuses on *welfare* as defined by consumer surplus and profit (or producer surplus). In Chapter 14 the consumer side of the market did not play a prominent role — we simply said that the market demand curve arises from the sum of individual demands. This is all we need for prediction. In the Chapter 15, on the other hand, we return to some themes from consumer theory — particularly the insight that welfare is measured on marginal willingness to pay (or compensated demand) curves and that these are the same as regular (or uncompensated) demand curves (that we use for prediction) only in the case of quasilinear tastes.

Chapter Highlights

The main points of the chapter are:

1. It is generally not possible to interpret curves that emerge from aggregating individual consumer demand (or labor supply) curves as if they emerged from an individual's optimization problem. Interpreting aggregate economic relationships that emerge from utility maximization in such a way is possible only if redistributing resources within the aggregated group leads to individually offsetting changes in behavior — i.e. **offsetting income effects**.

2. **It is possible to treat aggregate (or market) demand curves as if they emerged from an individual optimization problem** if there are no income effects — i.e. **if the good of interest is quasilinear**. In that special case, (uncompensated) demand curves are also equal to marginal willingness to pay (or com-

pensated demand) curves, enabling us to **measure consumer surplus on the market demand curve**.

3. Since economic relationships emerging from profit maximization by firms do not involve income effects, there are **no analogous issues with interpreting aggregate or market supply curves** (or labor demand curves) as if they emerged from a single optimization problem. As a result, we can measure **producer surplus (or profit) on the market supply curve** without making any particular assumptions.

4. Under a certain set of conditions, market equilibrium leads to output levels that mirror what would be chosen by omniscient social planners that aim to maximize overall social surplus. This is known as the **first welfare theorem** of economics which **specifies the conditions under which markets allocate resources efficiently**.

5. The advantage of market allocations of resources is that they rely on the **self-interested behavior by individuals who know only their own circumstances and observe the market price** signal that coordinates actions of producers and consumers. The disadvantages of market allocation of resources arise first in real world **violations of the assumptions underlying the first welfare theorem** that lead to violations of efficiency and second on normative judgments about **equity versus efficiency** that may lead us to conclude that some market outcomes, while being efficient, are in some sense “unfair” or “unjust”.

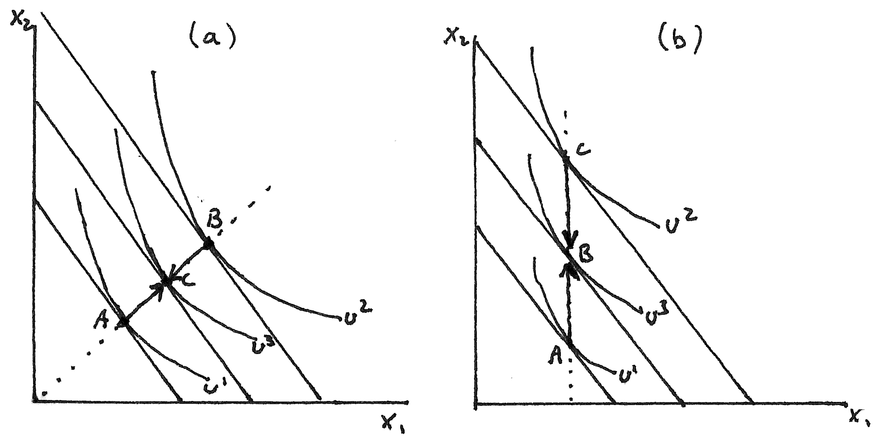
15A Solutions to Within-Chapter-Exercises for Part A

Exercise 15A.1

Suppose that my tastes and my wife’s tastes are exactly identical. If our tastes are also homothetic, does our household behave like a single representative agent? What if our tastes are quasilinear and neither individual is at a corner solution?

Answer: The answer is that, in both cases, our household will behave like an individual agent. This is illustrated in Exercise Graph 15A.1. In panel (a), we assume that our tastes are homothetic and identical. This implies that both my wife and I will optimize along a ray from the origin, with the precise ray depending on the output prices (and thus the slope of the budget constraints). Suppose that I initially have the lowest of these budget constraints and my wife initially has the highest. Then I will optimize at A and she will optimize at B. If you then redistribute income so we both face the same budget constraint, we will both face the middle one — and we will both optimize at C. Thus, my wife’s optimal bundle will move inward along the ray and mine will move outward along the ray — exactly offsetting each other.

Our overall bundle will thus remain the same as you redistribute income. The same is true in panel (b) where our tastes are quasilinear and neither of us is at a corner solution.



Exercise Graph 15A.1 : Individual Agents when Tastes are Identical

Exercise 15A.2

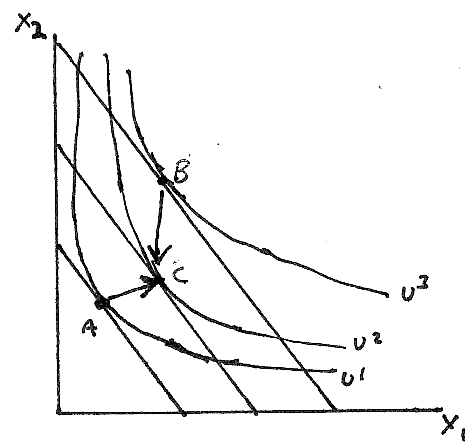
Can you illustrate a case where our tastes are identical but we do not behave as a representative agent?

Answer: One such case is illustrated in Exercise Graph 15A.2. Let's assume that the three indifference curves are drawn from the same map of indifference curves — i.e. the same tastes. Initially I have the low income and my wife has the high income — which means I choose A and she chooses B. Then you redistribute income so we both face the middle income — and we both choose C. The change in my wife's consumption bundle is then clearly not offset by the change in mine — because the arrows are not parallel to one another.

Exercise 15A.3

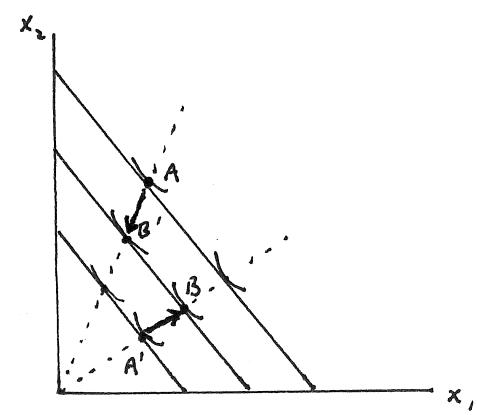
Suppose both my wife and I have homothetic tastes but they are not identical. Does this still imply that we behave like a single representative agent?

Answer: No, we will not behave as a representative agent (unless the tastes are over perfect substitutes). This is illustrated in Exercise Graph 15A.3. Suppose my wife has tastes that cause her to optimize on the steeper ray from the origin and I have tastes that cause me to optimize on the shallower one. If initially I have the low income and she has the high income, she will choose A and I will choose A'. After you redistribute income and we both face the middle budget constraint, she will



Exercise Graph 15A.2 : Individual Agents when Tastes are Identical: Part 2

choose B' and I will choose B. Because we are moving along rays that have different slopes, the changes in our consumption bundles will not offset one another.



Exercise Graph 15A.3 : Homothetic Tastes

Exercise 15A.4

True or False: As long as everyone has quasilinear tastes, the group will behave like a representative agent even if all the individuals do not share the same tastes (assuming no one is at a corner solution). The same is also true if everyone has homothetic tastes.

Answer: The first part of the statement is true but the second part is false. For quasilinear tastes, the graph in the text in fact has me and my wife having *different* tastes that are quasilinear. So we have demonstrated in the text that the first part is true — as long as tastes are quasilinear, the group will act as an individual agent. In exercise 15A.3, however, we have already demonstrated that the group will not act as an individual if tastes are homothetic but different.

Exercise 15A.5

Suppose that my wife and I share identical homothetic tastes (that are not over perfect substitutes). Will our household demand curve be identical to our marginal willingness to pay curve?

Answer: No. When tastes are homothetic, they give rise to income effects — which implies that individual demand curves and marginal willingness to pay curves will differ, and this difference continues to hold when we consider the aggregate demand curve. Thus, even though we behave as a representative agent, our demand and marginal willingness to pay curves will not be the same.

Exercise 15A.6

Does this measure of long run profit apply also when the firm encounters long run fixed costs?

Answer: Yes. This is because the fixed cost is included in the long run *AC* and is therefore counted, and the presence of fixed costs does not change the *additional* cost incurred by producing more than the quantity at the lowest point of the *AC* curve.

Exercise 15A.7

How would the picture be different if we were depicting an industry in long run equilibrium with all firms facing the same costs? What would long run producer surplus be in that case?

Answer: The long run supply curve would then be flat — which would eliminate the producer surplus area entirely from the graph. This should make sense: In a competitive industry where all firms face the same costs, entry and exit drive long run profit to zero. Thus, while each firm will earn short run profits (because certain long run costs are not costs in the short run), the industry will earn zero profit in the long run. The entire surplus in the market would then be earned by consumers.

Exercise 15A.8

Suppose we were not concerned about identifying producer and worker surplus but instead wanted to only predict the equilibrium wage and the number of workers employed. Would we then also have to assume that leisure is quasilinear for workers?

Answer: No — in order to predict the market equilibrium, we simply need to know the aggregate demand and supply curves in the market. We can aggregate these even if consumers (or workers) do not behave as one single representative agent. Put differently, we need the regular consumer demand or worker supply curves to predict the equilibrium, not the compensated consumer demand and worker supply curves.

Exercise 15A.9

Imagine that you are Barney and that you would like consumers to get a bigger share of the total "pie" than they would get in a decentralized market. How might you accomplish this? (*Hint*: Given your omnipotence, you are not restricted to charging the same price to everyone.)

Answer: All you would have to do is charge a lower price to some of the consumers. You could still give enough to producers so that their surplus is positive — but you could then redistribute some of the surplus from producers to consumers. In the extreme, you would simply cover the costs of producers and hand all the goods to the consumers who value them most, charging them only a price sufficient to raise enough money for you to pay off the producers.

Exercise 15A.10

Suppose the social marginal cost curve is perfectly flat — as it would be in the case of identical producers in the long run. Would you, as Barney, be able to give producers a share of the surplus?

Answer: Sure. All you would have to do is charge the consumers who really value the goods a lot more than the long run equilibrium price that would emerge in the market. That long run price is sufficient to cover all the long run costs for producers — but lots of consumers are willing to pay more. Thus, if you raise this additional revenue from consumers, you can redistribute some of the consumer surplus to producers who would, in the competitive long run market, make zero surplus.

Exercise 15A.11

How would Graph 15.8 look if good x were an inferior good for all consumers?

Answer: In this case the aggregate $MWTP$ curve would be shallower than the market demand curve, causing the actual consumer surplus to be smaller than what we would infer from just looking at the market demand curve.

Exercise 15A.12

True or False: If goods are normal, we will underestimate the consumer surplus if we measure it along the market demand curve, and if goods are inferior we will overestimate it.

Answer: This is true. You can see it for normal goods in the graph in the text — where the $MWTP$ curves are steeper than demand curves. Similarly, $MWTP$ curves are shallower than demand curves in the case of inferior goods — which implies the actual consumer surplus is smaller than what we would measure along the market demand curve.

15B Solutions to Within-Chapter-Exercises for Part B

Exercise 15B.1

Demonstrate that the conditions in equation (15.1) are satisfied for the demand functions in (15.2).

Answer: Taking the first derivatives with respect to I^m, we get

$$\frac{\partial x_i^m}{\partial I^m} = b_i(p_1, p_2) = \frac{\partial x_i^n}{\partial I^n}. \tag{15B.1.i}$$

Then, taking second derivatives, we get

$$\frac{\partial^2 x_i^m}{\partial (I^m)^2} = 0 = \frac{\partial^2 x_i^n}{\partial (I^n)^2}. \tag{15B.1.ii}$$

Exercise 15B.2

Can you see why equation (15.2) represents the most general way of writing demands that satisfy the conditions in equation (15.1)?

Answer: First, the only way the second derivatives can be zero is if income enters linearly and thus drops out when the first derivative is taken. Thus, we know that income can only enter as I multiplied by something that is not also a function of income — i.e. if income enters in the form $Ib(p_1, p_2)$ where the function b is at most a function of the prices (and not income). Second, the only way the first derivatives with respect to income can be the same across individuals is if the term following income is the same for both individuals — because that is the term that remains when we take the first derivative. Thus, the function b cannot be a individual specific — i.e. it cannot have an n or m superscript, but it can vary for goods — i.e. it can have an i subscript. Finally, other terms can enter the demand equations so long as they are not dependent on income — and thus do not affect the first derivative. Thus, we can have an a function that is not dependent on income but depends on prices — and that can vary across goods and individuals (since it drops out when we take the derivative with respect to income).

Exercise 15B.3

What are my household demand functions (for x_1 and x_2) if my wife's and my individual demands are those in equation (15.3)? Do the household demand functions also satisfy the Gorman Form?

Answer: Our household demand functions would simply be the sum of our individual demand functions. Since neither of the individual demand functions for x_1 is a function of income, our household demand function for x_1 will not be a function of income. Thus, our household demands arise from household preferences that are quasilinear in x_1, with household demand functions satisfying the Gorman form.

Exercise 15B.4

Given that the firms encounter a recurring fixed cost of $1,280, which of the above functions should actually be qualified to take account of this fixed cost?

Answer: The short run functions are not impacted, but the long run functions are. For instance, if $w = 20$ and $r = 10$, the lowest point of the AC function gives us a long run exit price of $p = 5$ — a price below which long run production falls to zero.

Exercise 15B.5

Draw the production possibility frontier described above. How would it look differently if the long run market supply curve slopes up? (*Hint*: With an upward-sloping supply curve, society is facing an increasing cost of producing x, implying that the trade-off in the society-wide production possibility frontier must reflect that increasing cost.)

Answer: In panel (a) of Exercise Graph 15B.5, the production possibility frontier given by $I = 5x + y$ is given — with the frontier having slope -5 throughout because the (social) opportunity cost of increasing x by one unit is always that 5 units of y must be sacrificed. When the cost of producing x increases with the level of x (as it does when the supply curve is increasing), then we would get a production possibility frontier with the shape illustrated in panel (b) — where the slope starts shallow (indicating a low opportunity cost for producing x) but increases (in absolute value) as x increases (indicating the increasing opportunity cost.)

Exercise 15B.6

Verify that this is indeed the case.

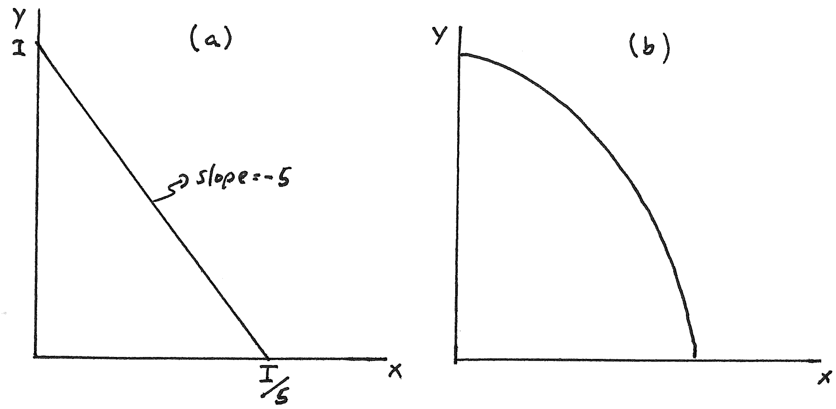
Answer: The Lagrange function is

$$\mathscr{L} = 12,649.11x^{1/2} + y + \lambda(I - 5x - y) \tag{15B.6.i}$$

which gives rise to the first order conditions

$$\frac{12,649.11}{2x^{1/2}} - 5\lambda = 0 \text{ and } 1 - \lambda = 0. \tag{15B.6.ii}$$

Plugging the latter into the former and solving for x, we get $x = 1,600,000$.



Exercise Graph 15B.5 : Production Possibility Frontiers

Exercise 15B.7

One way to verify that the representative consumer’s utility function is truly “representative” is to calculate the implied demand curve and see whether it is equal to the aggregate demand curve $D^M(p) = 40,000,000/p^2$ that we are trying to represent. Illustrate that this is the case for the utility function $U(x, y) = 12,649.11x^{1/2} + y$.

Answer: To derive the implied demand curve for the representative consumer, we solve the problem

$$\max_{x,y}\ 12,649.11x^{1/2} + y \text{ subject to } I = px + y. \tag{15B.7.i}$$

Setting up the lagrange function

$$\mathscr{L} = 12,649.11x^{1/2} + y + \lambda(I - px - y), \tag{15B.7.ii}$$

we can derive the first order conditions

$$\frac{12,649.11}{2x^{1/2}} - \lambda p = 0 \text{ and } 1 - \lambda = 0. \tag{15B.7.iii}$$

Substituting the latter into the former and solving for x, we get

$$x(p) = \frac{40,000,000}{p^2}, \tag{15B.7.iv}$$

precisely the aggregate demand function we are trying to represent with the representative consumer.

15C Solutions to Odd Numbered End-of-Chapter Exercises

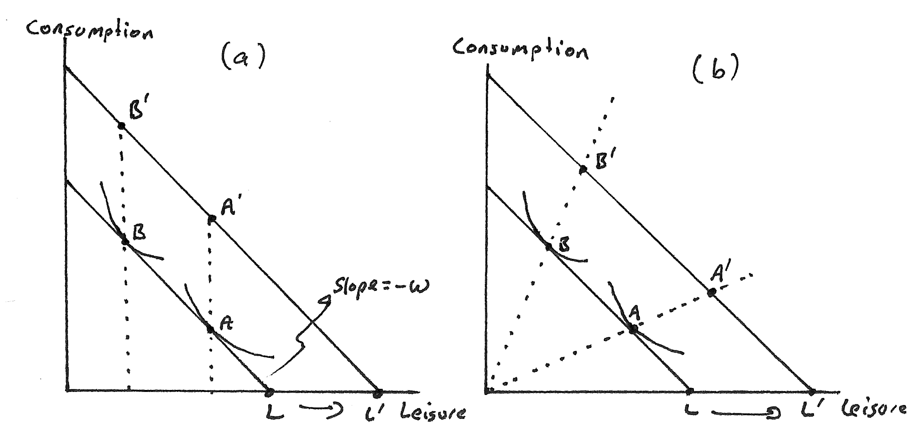
Exercise 15.1

Everyday Application: *Labor Saving Technologies*: *Consider inventions such as washing machines or self-propelled vacuum cleaners. Such inventions reduce the amount of time individuals have to spend on basic household chores — and thus in essence increase their leisure endowments.*

A: *Suppose that we wanted to determine the aggregate impact such labor saving technologies will have on a particular labor market in which the wage is w.*

(a) *Draw a graph with leisure on the horizontal axis and consumption on the vertical and assume an initially low level of leisure endowment for worker A. For the prevailing wage w, indicate this worker's budget constraint and his optimal choice.*

Answer: This is illustrated in panel (a) of Exercise Graph 15.1 as the lower of the two parallel budgets where worker A optimizes at bundle A.



Exercise Graph 15.1 : Labor Saving Household Technologies

(b) *On the same graph, illustrate the optimal choice for a second worker B who has the same leisure endowment and the same wage w but chooses to work more.*

Answer: This is also illustrated in panel (a) where worker B optimizes at bundle B — consuming less leisure and thus working more.

(c) *Now suppose that a household-labor saving technology (such as an automatic vacuum cleaner) is invented and both workers experience the same increase in their leisure endowment. If leisure is quasilinear for both workers, will there be any impact on the labor market?*

Answer: The increase in leisure endowment is indicated as an increase from L to L' in panel (a) of the graph. Since wage remains the same, this results in a parallel shift out of the budget constraints. If tastes are quasilinear in leisure, then worker A will optimize at A' and worker B will optimize at B'. Since their leisure consumption remains unchanged, this implies that workers will increase their labor supply by exactly the increase in leisure $(L' - L)$.

(d) *Suppose instead that tastes for both workers are homothetic. Can you tell whether one of the workers will increase his labor supply by more than the other?*

Answer: This is illustrated in panel (b) of Exercise Graph 15.1 where worker A will choose bundle A' and worker B will choose bundle B'. Worker A will therefore increase his leisure consumption by more than worker B — with neither worker committing the entire increase in leisure $(L' - L)$ to increased work hours. However, because worker B increases his leisure consumption by less than worker A, we know that worker B will increase his labor supply by more than worker A.

(e) *How does your answer suggest that workers in an economy cannot generally be modeled as a single "representative worker" even if they all face the same wage?*

Answer: In order for us to be able to use a "representative worker", it would have to be the case that, when leisure endowments are redistributed between workers, the overall amount of labor supplied remains unchanged. We can see in panel (a) of Exercise Graph 15.1 that, when leisure is quasilinear, leisure demand remains unchanged as leisure endowments are changed. Thus, were we to redistribute leisure endowments between individuals, the one who gets more leisure endowment would supply all of it as labor while the one who loses it would reduce his labor hours by the same amount. Thus, the actions of the two workers would exactly offset each other. The same is not, however, true in panel (b) where tastes are homothetic. Thus, a redistribution of leisure among workers would cause an increase in labor hours for the worker who receives more leisure endowment and reduce the labor hours of the worker who receives less — but the two would not offset each other unless the tastes were also identical.

B: *Consider the problem of aggregating agents in an economy where we assume individuals have an exogenous income.*

(a) *In a footnote in this chapter, we stated that, when the indirect utility for individual m can be written as $V^m(p_1, p_2, I^m) = \alpha^m(p_1, p_2) + \beta(p_1, p_2)I^m$, then demands can be written as in equation (15.2). Can you demonstrate that this is correct by using Roy's Identity?*

Answer: Applying Roy's identity, we get

$$x_i^m(p_1,p_2,I) = -\frac{\partial V/\partial p_i}{\partial V/\partial I} = -\frac{\left(\partial\alpha^m(p_1,p_2)/\partial p_i\right) + I^m\left(\partial\beta(p_1,p_2)/\partial p_i\right)}{\beta(p_1,p_2)}. \tag{15.1.i}$$

If we now define

$$a_i^m(p_1,p_2) = -\frac{\partial\alpha^m(p_1,p_1)/\partial p_i}{\beta(p_1,p_2)} \text{ and } b_i(p_1,p_2) = -\frac{\partial\beta(p_1,p_2)/\partial p_i}{\beta(p_1,p_2)}, \tag{15.1.ii}$$

we can write the demand function for good i by consumer m as

$$x_i^m(p_1,p_2,I) = a_i^m(p_1,p_2) + I^m b_i(p_1,p_2). \tag{15.1.iii}$$

Note that we can do this because the first term on the right hand side of equation (15.1.i) contains both an m superscript and an i subscript — thus causing the a function to contain both. But the second term contains (aside from I^m) only an i subscript (and no m superscript) — thus allowing us to write the b function without the m superscript.

(b) *Now consider the case of workers who choose between consumption (priced at 1) and leisure. Suppose they face the same wage w but different workers have different leisure endowments. Letting the two workers be superscripted by n and m, can you derive the form that the leisure demand equations $l^m(w,L^m)$ and $l^n(w,L^n)$ would have to take in order for redistributions of leisure endowments to not impact the overall amount of labor supplied by these workers (together) in the labor market?*

Answer: In order for redistributions in leisure endowments to have offsetting effects, it must be the case that the first derivative of $l^m(w,L^m)$ with respect to L^m is equal to the first derivative of $l^n(w,L^n)$ with respect to L^n and that the second derivative of each is zero. (This gives us the parallel linear (and offsetting) changes in consumption bundles as endowments are redistributed.) In order for this to be the case, the functions have to take the form

$$l^m(w,L^m) = a^m(w) + b(w)L^m \text{ and } l^n(w,L^n) = a^n(w) + b(w)L^n. \tag{15.1.iv}$$

The first derivatives with respect to the leisure endowments are then equal to $b(w)$, and the second derivatives are zero. Were the b functions superscripted by m and n, this would not be the case, nor would it be the case if leisure entered the b or a functions directly.

(c) *Can you re-write these in terms of labor supply equations $\ell^m(w,L^m)$ and $\ell^n(w,L^n)$?*

Answer: Since labor supply is just the leisure endowment minus leisure demand, we get

$$\ell^m(w, L^m) = L^m - \left(a^m(w) + b(w)L^m\right) = (1-b(w))\,L^m - a^m(w) \quad (15.1.v)$$

and

$$\ell^n(w, L^n) = L^n - \left(a^n(w) + b(w)L^n\right) = (1-b(w))\,L^n - a^n(w). \quad (15.1.vi)$$

(d) *Can you verify that these labor supply equations have the property that redistributions of leisure between the two workers do not affect overall labor supply?*

Answer: The first derivative of the labor supply functions with respect to the leisure endowments are now equal to $(1-b(w))$ and thus equal to each other — and the second derivatives are zero. Thus, a redistribution of endowments indeed causes an increase in labor supply by the worker who receives more endowment which is exactly offset by the decrease in labor supply by the worker who receives less endowment.

Exercise 15.3

Business and Policy Application: *License Fees and Surplus without Income Effects*: *In previous chapters, we explored the impact of recurring license fees on an industry's output and price. We now consider their impact on consumer and producer surplus.*

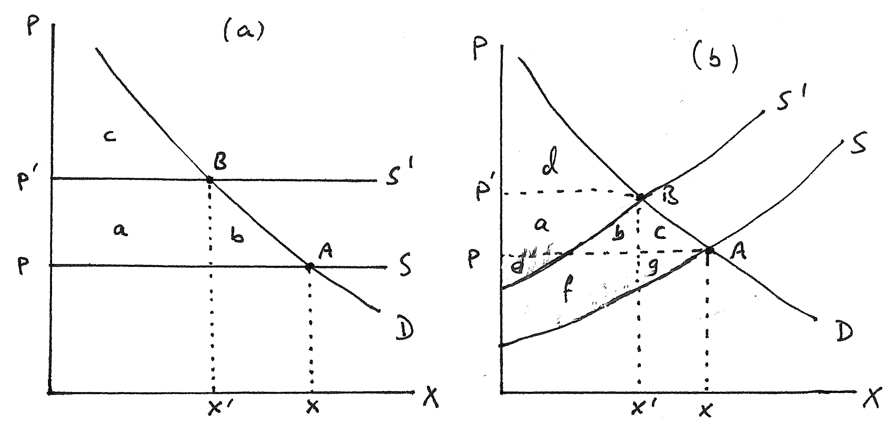
A: *Suppose that all firms in the fast food restaurant business face U-shaped average cost curves prior to the introduction of a recurring license fee. The only output they produce is hamburgers. Suppose throughout that hamburgers are a quasilinear good for all consumers.*

(a) *First, assume that all firms are identical. Illustrate the long run market equilibrium and indicate how large consumer and long run producer surplus (i.e. profit) are in this industry.*

Answer: This is illustrated in panel (a) of Exercise Graph 15.3. The long run supply curve S is flat because of entry and exit decisions by identical firms — leading to equilibrium price p and equilibrium output level x at point A. Producer surplus (or long run profit) is simply zero. Consumer surplus can be measured on the uncompensated demand curve because of the quasilinearity of x (that causes compensated demand curves to lie on top of the uncompensated demand curve). Thus, long run consumer surplus is $(a+b+c)$.

(b) *Illustrate the change in the long run market equilibrium that results from the introduction of a license fee.*

Answer: The long run supply curve shifts up (to S') because the long run average cost curves of each firm shift up. At the new equilibrium, price is p' and output is x' at point B. (Each firm ends up producing more, but the industry produces less as firms exit.)



Exercise Graph 15.3 : License Fees and Quasilinear Tastes

(c) *Suppose that the license fee has not yet been introduced. In considering whether to impose the license fee, the government attempts to ascertain the cost to consumers by asking a consumer advocacy group how much consumers would have to be compensated (in cash) in order to be made no worse off. Illustrate this amount as an area in your graph.*

Answer: Ordinarily, this would be measured along the compensated demand curve that goes through A. Since x is quasilinear, however, the compensated demand curves lie on the uncompensated demand curve that goes through A and B. Thus, the compensation is given by area $(a+b)$.

(d) *Suppose instead that the government asked the consumer group how much consumers would be willing to pay to avoid the license free. Would the answer change?*

Answer: Ordinarily this would be measured on the compensated demand curve that goes through B. However, all compensated demand curves lie on the uncompensated demand curve because of the quasilinearity of x. Thus, the amount consumers would be willing to pay is $(a+b)$.

(e) *Finally, suppose the government simply calculated consumer surplus before and after the license fee is imposed and subtracted the latter from the former. Would the government's conclusion of how much the license fee costs consumers change?*

Answer: No. The consumer surplus at A is $(a+b+c)$ and the consumer surplus at B is c — making the difference area $(a+b)$. (Again, this is only true because consumer surplus can, because of the quasilinearity of x, be measured on the uncompensated demand curve.)

(f) *What in your answers changes if, instead of all firms being identical, some firms had higher costs than others (but all have U-shaped average cost* *curves)?*

Answer: Not very much would be different — except for the fact that producer surplus would now be positive given that firms who are more cost effective can earn positive profit. This is illustrated in panel (b) of Exercise Graph 15.3 where supply shifts from S to S'. The change in consumer surplus is $(a+b+c)$ (which is equivalent to $(a+b)$ in panel (a)) — and the same measure would give the compensation required to consumers or the amount consumers would be willing to pay to prevent the fee from going into effect. Producer surplus, or long run profit, would be $(e+f+g)$ before and $(e+a)$ after.

B: *Suppose that each firm's cost function is given by $C(w,r,x) = 0.047287w^{0.5}r^{0.5}x^{1.25} + F$ where F is a recurring fixed cost.*[1]

(a) *What is the long run equilibrium price for hamburgers x (as a function of F) assuming wage $w = 20$ and rental rate $r = 10$?*

Answer: Each firm's cost function would then be

$$C(x,F) = 0.047287(20)^{0.5}(10)^{0.5}x^{1.25} + F = 0.66873917x^{1.25} + F. \quad (15.3.i)$$

From this, we can derive the long run average cost function as

$$AC(x,F) = 0.66873917x^{0.25} + \frac{F}{x}. \quad (15.3.ii)$$

To find the lowest point of this average cost function, we take the derivative with respect to x, set it to zero and solve for x to get $x = 4.18256389F^{0.8}$. Plugging this back into the average cost function, we get the long run equilibrium price (as a function of F):

$$p(F) = 0.66873917\left(4.18256389F^{0.8}\right)^{0.25} + \frac{F}{4.18256389F^{0.8}} = 1.195439F^{0.2}. \quad (15.3.iii)$$

(b) *Suppose that, prior to the imposition of a license fee, the firm's recurring fixed cost F was \$1,280. What is the pre-license fee equilibrium price?*

Answer: Using the equation $p(F)$, we can determine the initial equilibrium price

$$p(1280) = 1.195439\left(1280^{0.2}\right) = 5. \quad (15.3.iv)$$

(c) *What happens to the long run equilibrium price for hamburgers when a \$1,340 recurring license fee is introduced?*

Answer: Again, using the equation $p(F)$ and substituting the new fixed cost $F = 1280 + 1340 = 2620$, we get

$$p(2620) = 1.195439\left(2620^{0.2}\right) = 5.77. \quad (15.3.v)$$

[1] You can check for yourself that this is the cost function that arises from the production function $f(\ell,k) = 20\ell^{0.4}k^{0.4}$.

(d) *Suppose that tastes for hamburgers x and a composite good y can be characterized by the utility function $u(x,y) = 20x^{0.5} + y$ for all 100,000 consumers in the market, and assume that all consumers have budgeted \$100 for x and other goods y. How many hamburgers are sold before and after the imposition of the license fee?*

Answer: The demand function derived from this utility function is $x(p) = 100/p^2$. Summing over 100,000 consumers, we get a market demand function of

$$X(p) = \frac{10,000,000}{p^2}. \tag{15.3.vi}$$

Substituting the before and after prices of \$5 and \$5.77, this implies that 2,000,000 hamburgers were sold before the license fee and about 1,733,100 hamburgers are sold afterwards.

(e) *Derive the expenditure function for a consumer with these tastes.*

Answer: We need to solve the expenditure minimization problem

$$\min_{x,y} \; px + y \; \text{ subject to } \; u = 20x^{0.5} + y. \tag{15.3.vii}$$

This gives us the compensated demand functions

$$x(p) = \frac{100}{p^2} \; \text{ and } \; y(p,u) = u - \frac{200}{p}. \tag{15.3.viii}$$

Substituting this into the expenditure equation $px + y$, we get the expenditure function

$$E(p,u) = p\left(\frac{100}{p^2}\right) + u - \frac{200}{p} = u - \frac{100}{p}. \tag{15.3.ix}$$

(f) *Use this expenditure function to answer the question in A(c).*

Answer: First, we have to figure out how much utility consumers get in the absence of the license fee when $p = 5$. In that case, they consume 4 of x and 80 of y (given that they have budgeted \$100 for both goods) — which gives utility $u = 20(4^{0.5}) + 80 = 120$. In order to reach this utility level at the higher price $p = 5.77$, we have to evaluate the expenditure function $E(p,u)$ at $p = 5.77$ and $u = 120$; i.e.

$$E(5.77, 120) = 120 - \frac{100}{5.77} \approx 102.67. \tag{15.3.x}$$

Since each consumer has \$100 budgeted to start with, this implies that the government would have to compensate each consumer by \$2.67 — or a total of \$267,000 for the 100,000 consumers.

(g) *Use the expenditure function to answer the question in A(d).*

Answer: If consumers are asked how much they are willing to pay to not have the license fee implemented, they would first need to know how much utility they will get if the license fee in fact does get implemented. At $p = 5.77$, each consumer demands approximately 3 hamburgers (x) — down from 4 — and consumes \$82.67 of other goods (y) — up from 80 before. This implies that each consumer gets utility $u(3, 82.67) = 20(3^{0.5}) + 82.67 = 117.31$ if the license fee is implemented. (Had we not rounded a bit, this would actually be 117.33.) If the fee is not implemented, price falls to $p = 5$ — thus, in order to determine how much of a budget each consumer will need to be as well off without the fee as they are with it, we need to evaluate the expenditure function $E(p, u)$ at $p = 5.77$ and $u = 117.31$. This gives us

$$E(5.77, 117.31) = 117.31 - \frac{100}{5} = 97.31. \qquad (15.3.xi)$$

Thus, a consumer with a current budget of \$100 would be willing to pay \$2.69 — or, had we not rounded the utility figure and used 117.33, we would get that they are willing to pay \$2.67 each. Thus, the answer is the same as what we derived in the previous part — and consumers overall would be willing to pay approximately \$267,000 to avoid the license fee being implemented.

(h) *Take the integral of the demand function that gives you the consumer surplus before the license fee and repeat this to get the integral of the consumer surplus after the license fee is imposed.*

Answer: The consumer surplus before the license fee is

$$\int_5^{\infty} \frac{100}{p^2} dp = -\frac{100}{p}\Big|_5^{\infty} = 0 - \left(-\frac{100}{5}\right) = 20, \qquad (15.3.xii)$$

and the consumer surplus after the license fee is

$$\int_{5.77}^{\infty} \frac{100}{p^2} dp = -\frac{100}{p}\Big|_{5.77}^{\infty} = 0 - \left(-\frac{100}{5.77}\right) = 17.33. \qquad (15.3.xiii)$$

You could of course also have used the aggregate demand curve — and you would then have gotten the same answers (multiplied by 100,000).

(i) *How large is the change in consumer surplus from the price increase? Compare your answer to what you calculated in parts (f) and (g).*

Answer: The change in consumer surplus is therefore $20 - 17.33 = 2.66$ or (up to rounding errors) identical to what we calculated in parts (f) and (g). This is because, under quasilinear tastes, the (uncompensated) demand curve lies on top of the compensated demand curves — and we can thus use the (uncompensated) demand curve to measure changes in consumer surplus. (It furthermore implies that the two measures of changes in consumer surplus derived in (f) and (g) are identical because, even though they are measured on different compensated demand curves, they are identical because the compensated demand curves for different utility levels lie on top of one another.)

Exercise 15.5

Policy Application: *Redistribution of Income without Income Effects*: *Consider the problem a society faces if it wants to both maximize efficiency while also insuring that the overall distribution of "happiness" in the society satisfies some notion of "equity".*

A: *Suppose that everyone in the economy has tastes over x and a composite good y, with all tastes quasilinear in x.*

(a) *Does the market demand curve (for x) in such an economy depend on how income is distributed among individuals (assuming no one ends up at a corner solution)?*

Answer: If everyone's tastes are quasilinear in x, this means that each person's demand for x is independent of income (unless someone is at a corner solution). Thus, the aggregate demand curve in the market for x does not depend on the distribution of income in the population. Since the supply curve also does not depend on the distribution of income, the market equilibrium in the x market is independent of the income distribution.

(b) *Suppose you are asked for advice by a government that has the dual objective of maximizing efficiency as well as insuring some notion of "equity". In particular, the government considers two possible proposals: Under proposal A, the government redistributes income from wealthier individuals to poorer individuals before allowing the market for x to operate. Under proposal B, on the other hand, the government allows the market for x to operate immediately and then redistributes money from wealthy to poorer individuals after equilibrium has been reached in the market. Which would you recommend?*

Answer: Since the market outcome in the x market is independent of the distribution of income, it does not matter whether income is redistributed before or after the market equilibrium has been reached. The end result will be exactly the same. Thus, you should tell the government it does not matter which policy is put in place.

(c) *Suppose next that the government has been replaced by an omniscient social planner who does not rely on market processes but who shares the previous government's dual objective. Would this planner choose a different output level for x than is chosen under proposal A or proposal B in part (b)?*

Answer: No, the social planner would do exactly what the government would do under either of the two policies. This is because the social planner is not restricting his ability to achieve different notions of equity by allowing surplus in the x market to be maximized — which happens when the competitive equilibrium quantity of x is produced.

(d) True or False*: As long as money can be easily transferred between individuals, there is no tension in this economy between achieving many different notions of "equity" and achieving efficiency in the market for x.*

Answer: This is true (as already explained in the previous part).

(e) *To add some additional realism to the exercise, suppose that the government has to use distortionary taxes in order to redistribute income between individuals. Is it still the case that there is no tradeoff between efficiency and different notions of equity?*

Answer: In this case, a tradeoff does emerge — because redistribution through distortionary taxes implies the creation of deadweight losses as income is transferred between individuals. Thus, more redistribution implies a loss of social surplus — thus the tension between "equity" and efficiency.

B: *Suppose there are two types of consumers: Consumer type 1 has utility function $u^1(x,y) = 50x^{1/2} + y$, and consumer type 2 has utility function $u^2(x,y) = 10x^{3/4} + y$. Suppose further that consumer type 1 has income of 800 and consumer type 2 has income of 1,200.*

(a) *Calculate the demand functions for x for each consumer type assuming the price of x is p and the price of y is 1.*

Answer: Using the utility function $u(x,y) = Ax^{\alpha} + y$, we can solve for the demand function for x as

$$2x(p) = \left(\frac{\alpha A}{p}\right)^{1/(1-\alpha)}. \tag{15.5.i}$$

Substituting for the terms in the two utility functions for the two types, this implies demand functions

$$x^1(p) = \left(\frac{0.5(50)}{p}\right)^{1/(1-0.5)} = \frac{625}{p^2} \text{ and } x^2(p) = \left(\frac{0.75(10)}{p}\right)^{1/(1-0.75)} = \frac{3,164.0625}{p^4} \tag{15.5.ii}$$

for type 1 and 2 respectively.

(b) *Calculate the aggregate demand function when there are 32,000 of each consumer type.*

Answer: Multiplying each demand function by 32,000 and adding, we get

$$X(p) = \frac{32,000(625)}{p^2} + \frac{32,000(3,164.0625)}{p^4} = \frac{20,000,000p^2 + 101,250,000}{p^4}. \tag{15.5.iii}$$

(c) *Suppose that the market for x is a perfectly competitive market with identical firms that attain zero long run profit when $p = 2.5$. Determine the long run equilibrium output level in this industry.*

Answer: Substituting $p = 2.5$ into the equation $X(p)$, we get $X(2.5) = 5,792,000$.

(d) *How much x does each consumer type consume?*

Answer: Type 1 consumers consume $625/(2.5^2) = 100$ units of x and type 2 consumers consume $3,164.0625/(2.5^4) = 81$ units of x.

(e) *Suppose the government decides to redistribute income in such a way that, after the redistribution, all consumers have equal income — i.e. all consumers now have income of 1,000. Will the equilibrium in the x market change? Will the consumption of x by any consumer change?*

Answer: Income does not enter any demand function (because the good x is quasilinear) — which implies that the income distribution does not enter the aggregate demand function $X(p)$. Thus, redistributing income in this way does not change either the equilibrium level of output in the market or the level of x consumption of any individual.

(f) *Suppose instead of a competitive market, a social planner determined how much x and how much y every consumer consumes. Assume that the social planner is concerned about both the absolute welfare of each consumer as well as the distribution of welfare across consumers — with more equal distribution more desirable. Will the planner produce the same amount of x as the competitive market?*

Answer: Yes — social surplus is still maximized at the same output level regardless of how the planner decides to redistribute income (so long as no one ends up at a corner solution). Thus, the planner would want to maximize the surplus in the x market by picking the same output level as the market — and he can then worry about redistributing income to the desired level.

(g) True or False*: The social planner can achieve his desired outcome by allowing a competitive market in x to operate and then simply transferring y across individuals to achieve the desired distribution of happiness in society.*

Answer: This is true. In other words, in an economy where all tastes are quasilinear in x, the planner does not actually have to calculate the optimal quantity of x but can rather allow the market to determine that quantity since it is unaffected by how income is distributed. By shifting y from some people to others, the planner can then achieve whatever desired level of "equity" he desires.

(h) *Would anything in your analysis change if the market supply function were upward sloping?*

Answer: Since the market demand curve is unaffected by redistribution of income, the market demand would continue to intersect market supply at the same point regardless of whether or not the supply curve slopes up. Thus, nothing changes fundamentally in the problem if we assume an upward sloping supply curve.

(i) *Economists sometimes refer to economies in which all individuals have quasilinear tastes as "transferable utility economies" — which means that in economies like this, the government can transfer happiness from one* *person to another. Can you see why this is the case if we were using the utility functions as accurate measurements of happiness?*

Answer: If we use the two utility functions in this problem as accurate measurements of happiness, then the planner will increase utility by 1 unit for a person of type 1 and lower it by 1 unit for a person of type 2 if he transfers one unit of y from person 1 to person 2. Thus, he is in essence able to transfer utility between individuals.

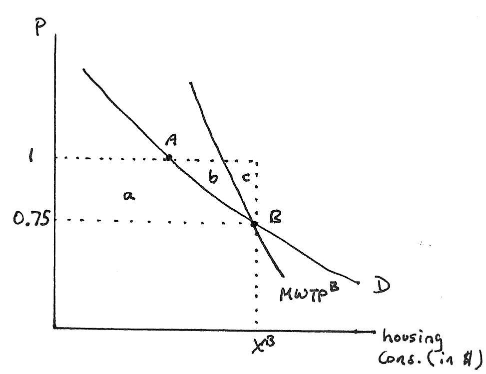
Exercise 15.7

Policy Application: *Dead Weight Loss from Subsidy of Mortgage Interest: The U.S. tax code subsidizes housing through a deduction of mortgage interest. For new homeowners, mortgage interest makes up the bulk of their housing payments which tend to make up about 25% of a household's income. Assume throughout that housing is a normal good.*

A: *For purposes of this problem, we will assume that all housing payments made by a household represent mortgage interest payments. If a household is in a 25% tax bracket, allowing the household to deduct mortgage interest on their taxes then is equivalent to reducing the price of \$1 worth of housing consumption to \$0.75.*

(a) *Illustrate a demand curve for a consumer, indicating both the with- and without-deductibility housing price.*

Answer: This is illustrated in Exercise Graph 15.7 where the demand curve is given by D. Without tax deductibility of housing costs, the consumer would locate at A where the price of a dollar of housing consumption is \$1. Under deductibility, however, the consumer faces a price of \$0.75 for every dollar in housing consumption — which implies she will locate at B.



Exercise Graph 15.7 : Tax Deductibility of Housing Costs

(b) *On the same graph, illustrate the compensated (or $MWTP$) curve for this consumer assuming that housing costs are deductible.*

Answer: If housing costs are deductible, the consumer locates at B. Thus, we would need to draw the $MWTP$ curve that runs through B — indicated as $MWTP^B$ in the graph. This is steeper than the uncompensated demand curve because housing is assumed to be a normal good.

(c) *On your graph, indicate where you would locate the amount that a consumer would be willing to accept in cash instead of having the subsidy of housing through the tax code.*

Answer: The consumer is equally happy all along $MWTP^B$ — which implies that we would need to give her the area $(a+b)$ in cash in order for her to be indifferent between the cash and the price subsidy.

(d) *On your graph, indicate the area of the deadweight loss.*

Answer: Under the subsidy implicit in the tax deductibility provision of the tax code, the government in essence pays \$0.25 for every \$1 in housing the consumer chooses. Under the subsidy the consumer chooses x^B — which implies that the total cost of the subsidy to the government is $0.25x^B$ — which is equal to the area $(a+b+c)$. Thus, the tax deductibility costs the government c more than the cash subsidy that would make the consumer just as well off — which implies c is the deadweight loss.

(e) *If you used the regular demand curve to estimate the deadweight loss, by how much would you over- or under-estimate it?*

Answer: If we used the regular demand curve to estimate the cash amount necessary to make the consumer just as happy, we would implicitly assume that housing is quasilinear (which it is not). As a result, we would conclude that the area a is how much cash the consumer would accept instead of tax deductibility of housing — which would lead us to conclude that the deadweight loss from tax deductibility is $(b+c)$ when it is actually just c. Thus, we would over-estimate the deadweight loss by area b.

B: *Suppose that a household earning \$60,000 (after taxes) has utility function $u(x,y) = x^{0.25}y^{0.75}$, where x represents dollars worth of housing and y represents dollars worth of other consumption. (Thus, we are implicitly setting the price of x and y to \$1.)*

(a) *How much housing does the household consume in the absence of tax deductibility?*

Answer: Letting p equal the price of housing, the demand function for this consumer is $x(p) = 0.25(60,000)/p$. When $p = 1$, this implies that $x = 15,000$.

(b) *If the household's marginal tax rate is 25% (and if all housing payments are deductible), how much housing will the household consume?*

Answer: The housing price for this household now falls to \$0.75 — which implies the household will choose $x = 0.25(60,000)/0.75 = 20,000$ in housing.

(c) *How much does the implicit housing subsidy cost the government for this consumer?*

Answer: Since the government effectively pays a quarter of the housing bill, it costs the government \$5,000.

(d) *Derive the expenditure function for this household (holding the price of other consumption at \$1 but representing the price of housing as p.)*

Answer: For a Cobb-Douglas utility function of the form $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$, we showed in Chapter 10 that the expenditure function takes the form

$$E(p_1, p_2, u) = \frac{u p_1^{\alpha} p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{15.7.i}$$

Setting $p_1 = p$, $p_2 = 1$ and $\alpha = 0.25$, this gives us

$$E(p, u) = 1.75476535 p^{0.25} u. \tag{15.7.ii}$$

(e) *Suppose the government contemplates eliminating the tax deductibility of housing expenditures. How much would it have to compensate this household for the household to agree to this?*

Answer: At the subsidized housing price, the household consumes \$20,000 in housing and \$45,000 in other goods. This gives utility of $u(20000, 45000) \approx 36,742$. To reach this level of utility at a non-subsidized price, the household's budget would have to be

$$E(1, 36742) = 1.75476535(36742) \approx 64,474. \tag{15.7.iii}$$

Since the consumer begins with \$60,000, this means the government would have to pay the household \$4,474.

(f) *Can you derive the same amount as an integral on a compensated demand function?*

Answer: For a Cobb-Douglas utility function of the form $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, we calculated in Chapter 10 that the compensated demand functions for x_1 is

$$h_1(p_1, p_2, u) = \left(\frac{\alpha p_2}{(1-\alpha) p_1}\right)^{(1-\alpha)} u. \tag{15.7.iv}$$

Plugging in $\alpha = 0.25$, $p_2 = 1$ and $p_1 = p$, we get the compensated demand function

$$h_x(p, u) = \left(\frac{1}{3p}\right)^{0.75} u. \tag{15.7.v}$$

Evaluating the integral of this between the prices 0.75 and 1 when utility is 36,742, we get

$$\begin{aligned} \int_{0.75}^{1} \left(\frac{1}{3p}\right)^{0.75} (36742)\, dp &= 4(36742)\left(\frac{1}{3}\right)^{0.75} p^{0.25}\Big|_{0.75}^{1} \\ &= 64474\left(1 - 0.75^{0.25}\right) \approx 4,474. \end{aligned} \tag{15.7.vi}$$

(g) *Suppose you only knew this household's (uncompensated) demand curve and used it to estimate the change in consumer surplus from eliminating the tax deductibility of housing expenditures. How much would you estimate this to be?*

Answer: You would take the integral of the uncompensated demand curve between prices 0.75 and 1 to get

$$\int_{0.75}^{1} \frac{15000}{p}\,dp = 15000\ln(p)|_{0.75}^{1} = 15000(\ln(1) - \ln(0.75)) \approx 4,315. \tag{15.7.vii}$$

(h) *Are you over- or under-estimating the deadweight loss from the subsidy if you use the (uncompensated) demand curve?*

Answer: If we use the uncompensated demand curve, we estimate the dead-weight loss from the subsidy as $5,000 - 4,315 = \$685$. If we use the compensated demand curve, we get $5,000 - 4,474 = \$526$. Thus, we are over-estimating the deadweight loss if we use the uncompensated demand function.

(i) *Suppose that all 50,000,000 home-owners in the U.S. are identical to the one you have just analyzed. What is the annual deadweight loss from the deductibility of housing expenses? By how much would you over- or under-estimate this amount if you used the aggregate demand curve for housing in this case?*

Answer: The annual deadweight loss is $526(50,000,000) = \$26,300,000,000$ or $26.3 billion dollars. If we used the uncompensated demand curve to estimate the deadweight loss, we would get $34.25 billion instead. Thus, by using the wrong demand curve, we would overestimate the deadweight loss by $7.95 billion.

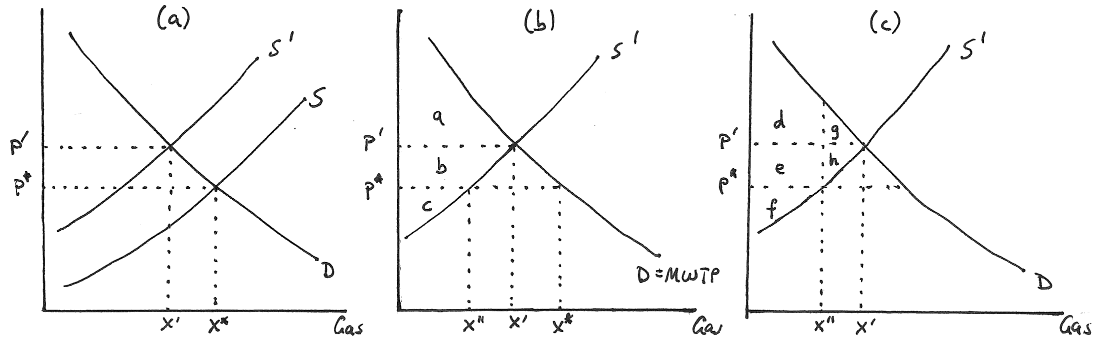
Exercise 15.9

Policy Application: *Anti-Price Gauging Laws*: *As we will discuss in more detail in Chapter 18, governments often interfere in markets by placing restrictions on the price that firms can charge. One common example of this is so-called "anti-price gauging laws" that restrict profits for firms when sudden supply shocks hit particular markets.*

A: *A recent hurricane disrupted the supply of gasoline to gas stations on the East Coast of the U.S. Some states in this region enforce laws that prosecute gasoline stations for raising prices as a result of natural disaster-induced drops in the supply of gasoline.*

(a) *On a graph with weekly gallons of gasoline on the horizontal and price per gallon on the vertical, illustrate the result of a sudden leftward shift in the supply curve (in the absence of any laws governing prices.)*

Answer: This is illustrated in panel (a) of Exercise Graph 15.9 where S is the original supply curve and S' is the new supply curve. The equilibrium



Exercise Graph 15.9 : Anti-Price Gauging Laws

shifts from one where price was p^* and gasoline consumption x^* to one where the price is p' and gasoline consumption is x'.

(b) *Suppose that gasoline is a quasilinear good for consumers. Draw a graph similar to the one in part (a) but include only the post-hurricane supply curve (as well as the unchanged demand curve). Illustrate consumer surplus and producer profit if price is allowed to settle to its equilibrium level.*

Answer: This is illustrated in panel (b) of Exercise Graph 15.9. Consumer surplus would be equal to area a and producer profit would be equal to area $(b+c)$.

(c) *Now consider a state that prohibits price adjustments as a result of natural disaster-induced supply shocks. How much gasoline will be supplied in this state? How much will be demanded?*

Answer: This is also illustrated in panel (b). At the pre-crisis price of p^*, firms would supply x'' — but consumers would want to buy x^*.

(d) *Suppose that the limited amount of gasoline is allocated at the pre-crisis price to those who are willing to pay the most for it. Illustrate the consumer surplus and producer profit.*

Answer: This is illustrated in panel (c) of Exercise Graph 15.9. If the limited amount of gasoline x'' is bought at p^* by those who value it the most, then consumer surplus is $(d+e)$. Producer profit is area f.

(e) *On a separate graph, illustrate the total surplus achieved by a social planner who insures that gasoline is given to those who value it the most and sets the quantity of gasoline at the same level as that traded in part (c). Is the social surplus different than what arises under the scenario in (d)?*

Answer: The social surplus would then be the same as in part (d) — equal to area $(d+e+f)$.

(f) *Suppose that instead the social planner allocates the socially optimal amount of gasoline. How much greater is social surplus?*

Answer: The socially optimal quantity is x'. If that much is produced, the total surplus is $(d+e+f+g+h)$ — which is greater than the surplus under the restricted quantity x'' by area $(g+h)$.

(g) *How does the total social surplus in (f) compare to what you concluded in (b) that the market would attain in the absence of anti-price gauging laws?*

Answer: It is identical.

(h) True or False*: By interfering with the price signal that communicates information about where gasoline is most needed, anti-price gauging laws have the effect of restricting the inflow of gasoline to areas that most need gasoline during times of supply disruptions.*

Answer: This is true, as demonstrated in the problem. The areas where gasoline would be most needed are those where the price would rise most in the absence of anti-price gauging laws. Thus, it is in these areas that the greatest shortages would emerge.

B: *Suppose again that the aggregate demand function $X^D(p) = 250,000/p^2$ arises from 10,000 local consumers of gasoline with quasilinear tastes (as in exercise 15.8).*

(a) *Suppose that the industry is in long run equilibrium — and that the short run industry supply function in this long run equilibrium is $X^S(p) = 3,906.25p$. Calculate the equilibrium level of (weekly) local gasoline consumption and the price per dollar.*

Answer: Setting $X^D(p) = X^S(p)$, we get $p = 4$. Substituting this back into either the demand or supply equation, we get $x = 15,625$.

(b) *What is the size of the consumer surplus and (short run) profit?*

Answer: The consumer surplus is

$$\int_4^\infty \frac{250,000}{p^2} dp = -\frac{250,000}{p}|_4^\infty = 0-(-62,500) = \$62,500. \quad (15.9.i)$$

The firm (short run) profits are

$$\int_0^4 3,906.25p\,dp = 1,953.125p^2|_0^4 = 31,250-0 = \$31,250. \quad (15.9.ii)$$

(c) *Next suppose that the hurricane-induced shift in supply moves the short run supply function to $\overline{X}^S = 2,000p$. Calculate the new (short run) equilibrium price and output level.*

Answer: We solve for the new equilibrium price by setting $X^D(p) = \overline{X}^S(p)$ and solving for $p = 5$. Plugging this back into either the demand or supply functions, we get $x = 10,000$.

(d) *What is the sum of consumer surplus and (short run) profit if the market is allowed to adjust to the new short run equilibrium?*

Answer: Consumer surplus is now

$$\int_5^{\infty} \frac{250,000}{p^2}dp = -\frac{250,000}{p}|_5^{\infty} = 0-(-50,000) = \$50,000. \quad (15.9.iii)$$

Profits for firms are

$$\int_0^5 2,000p\,dp = 1,000p^2|_0^5 = 25,000-0 = \$25,000. \quad (15.9.iv)$$

Thus, the sum of consumer surplus and (short run) firm profits is $75,000.

(e) *Now suppose the state government does not permit the price of gasoline to rise above what you calculated in part (a). How much gasoline will be supplied?*

<u>Answer</u>: At a price of $p = 4$, the gallons of gasoline supplied will be

$$\overline{X}^S(4) = 2,000(4) = 8,000. \quad (15.9.v)$$

(f) *Assuming that the limited supply of gasoline is bought by those who value it the most, calculate overall surplus (i.e. consumer surplus and (short run) profit) under this policy.*

<u>Answer</u>: The easiest way to calculate this is to find the area under the demand *curve* that lies above the supply curve up to $x = 8,000$. The area under the demand curve is

$$\int_0^{8000} \frac{500}{x^{0.5}}dx = 1,000x^{0.5}|_0^{8000} \approx \$89,442.72. \quad (15.9.vi)$$

The supply *curve* is the supply function solved for p — i.e. $p = 0.0005x$. The area under the supply curve up to $x = 8000$ is

$$\int_0^{8000} 0.0005x\,dx = 0.00025x^2|_0^{8000} = \$16,000. \quad (15.9.vii)$$

Thus, the overall surplus is $89442.72 - 16000 = \$73,442.72$.

(g) *How much surplus is lost as a result of the government policy to not permit price increases in times of disaster-induced supply shocks?*

<u>Answer</u>: In the absence of the policy, total surplus was $75,000 — which is $1,557.28 greater than the total surplus under the policy.

Conclusion: Potentially Helpful Reminders

1. The idea of representing different sides of the market as if they emerged from the behavior of a "representative agent" is a powerful one because it allows us to treat certain market curves using our insights from the development of consumer and producer theory.

2. It is only when a market relationship emerges from consumer theory — as it does in the case of consumer demand and labor supply — that we have to be careful as we are tempted to think about these market relationships as if they emerged from a single optimization problem. This is because of the presence of income effects that, when assumed away, remove the difficulty. It is for this reason that the analysis of welfare becomes significantly more straightforward when we assume quasilinear tastes.

3. Remember that you can always use market relationships to predict market outcomes — regardless of whether the condition of quasilinearity holds. It is only when we then try to determine exact welfare measures that we have to be careful if quasilinearity does not hold — which implies that it is only when quasilinearity does not hold that we have to worry about separately thinking about compensated rather than uncompensated relationships.

4. The complications of introducing income effects are explored in a particularly revealing way in end-of-chapter exercises 15.5 and 15.6 where we show how redistribution of income does not alter the equilibrium in a market under some conditions (quasilinearity) but does do so under other conditions (i.e. when there are income effects). Nevertheless, we show that the market equilibrium will retain its efficiency property under the assumptions of the first welfare theorem even in the presence of income effects — even though the nature of the equilibrium will depend on the initial distribution of income in that case.

5. It is important from the outset to be aware of the limitations of the first welfare theorem — limitations that arise from the underlying assumptions listed in the chapter and developed throughout the remainder of the text. End-of-chapter exercises 15.2, 15.7, 15.8 and 15.9 begin to explore these.

CHAPTER

16

General Equilibrium

Chapter 14 introduced the *positive* idea of equilibrium in the context of a competitive environment — and Chapter 15 moved onto the more *normative* assessment of a competitive equilibrium within the context of the first welfare theorem. We now extend the basic insights from these chapters to economies that are more "general" in the sense that we don't look just at one isolated market but rather treat the economy as an interconnected web of markets. It admittedly may at times not seem like we are extending the analysis to something more "general" — because the economies we discuss in this chapter seem very "simple" in that they only involve a few individuals and often abstract away from issues like production. But these "simple" economies can be extended to much more complex models with many goods, many individuals, many production processes, etc. And the basic insights that emerge from the "simple" models continue to hold in these more complicated models that indeed are more "general" than the partial equilibrium models of Chapters 14 and 15.

Chapter Highlights

The main points of the chapter are:

1. An **exchange economy** is an economy in which individuals trade what they own but no production takes place. Typically, individuals can improve their welfare in such economies by engaging in mutually beneficial trades. The set of efficient allocations of goods in such an economy is known as the **contract curve**.

2. A **competitive equilibrium** in an exchange economy consists of a set of *prices* and an *allocation* of the goods in the economy such that all individuals would agree to trade to that allocation at these prices. (The equilibrium is "competitive" because all individuals are assumed to be price takers.)

3. The *equilibrium allocation* of goods always lies on the contract curve (and is thus efficient). This is our generalization of the **first welfare theorem**. It

furthermore lies in the **core** of the economy — where the core is defined as the set of allocations under which no coalition of individuals could leave the economy with their endowment and do better on their own. (In the 2-person exchange economy, the core allocations are equal to the contract curve allocations that lie in the region of mutually beneficial trades.)

4. If governments can use *lump sum transfers*, then any allocation on the contract curve can become a competitive equilibrium allocation after appropriate lump sum transfers have been made. This is what is known as the **second welfare theorem**. If, however, the government can only use distortionary taxes, then a tradeoff emerges between notions of "equity" and "efficiency" because it is no longer possible for the government to redistribute and expect an efficient outcome.

5. Production can be introduced into general equilibrium economies, and the same basic welfare results hold. The simplest example of such an economy is the **Robinson Crusoe economy** where a single individual acts as producer, worker and consumer.

16A Solutions to Within-Chapter-Exercises for Part A

Exercise 16A.1

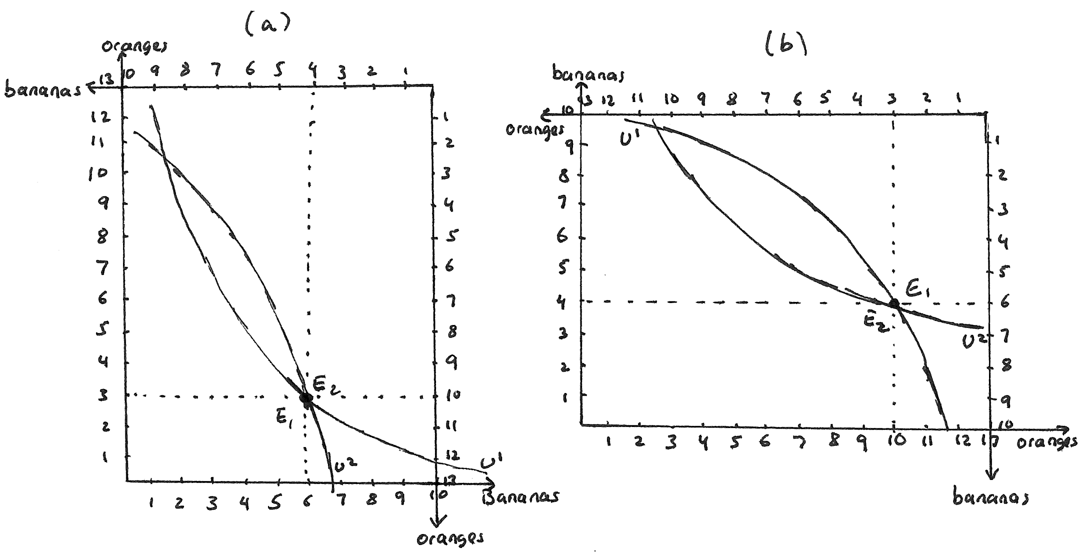
What would the Edgeworth Box for this example look like if oranges appeared on the vertical and bananas on the horizontal axis?

Answer: The Edgeworth Box would then be 10 units (bananas) long and 13 units (oranges) high, with the endowment point as pictured in panel (a) of Exercise Graph 16A.1.

Exercise 16A.2

What would the Edgeworth Box for this example look like if my wife's axes had the origin in the lower left corner and my axes had the origin in the upper right corner?

Answer: The box would have the same dimensions, but the endowment point would be located as pictured in panel (b) of Exercise Graph 16A.1 in the previous within-chapter exercise solution.



Exercise Graph 16A.1 : Edgeworth Boxes

Exercise 16A.3

True or False: Starting at point *A*, any mutually beneficial trade will involve me trading bananas for oranges, and any trade of bananas for oranges will be mutually beneficial. (*Hint*: Part of the statement is true and part is false.)

Answer: The first part of the statement is true — any mutually beneficial trade from *A* will involve me giving up bananas and getting oranges. This is because all the allocations of bananas and oranges that lie within the lens shape (and thus above both of our indifference curves through *A*) involve more oranges and fewer bananas for me. But it is not true that *any* trade of bananas for oranges will be mutually beneficial — only those lying above both our indifference curves. Put differently, the allocations that involve mutually beneficial trades from *A* do lie to the southeast of *A* (with fewer bananas and more oranges for me), but there are allocations that lie to the southeast of *A* that do not lie within the lens that represents allocations which are better for both me and my wife.

Exercise 16A.4

In Chapter 6, we argued that consumers leave Wal-Mart with the same tastes "at the margin" — i.e. with the same marginal rates of substitution between goods that they have purchased, and that this fact implies that all gains from trade have been exhausted. How is this similar to the condition for an efficient distribution of an economy's endowment in the exchange economy?

Answer: Once gains from trade have been exhausted in the Edgeworth Box, it is similarly true that the marginal rates of substitution of the two individuals will be equal to one another — just as when they come out of Wal-Mart after maximizing subject to facing the same prices. Thus, in both cases, the tastes are the same at the margin once all gains from trade have been exhausted. (This presumes that both individuals will optimize at an interior solution — but the efficiency logic extends to the case where individuals end up at corner solutions).

Exercise 16A.5

Starting at the point where my wife gets the entire endowment of the economy, are there points in the Edgeworth Box that make my wife worse off without making me better off (assuming that bananas and oranges are both essential goods for me)?

Answer: If both goods are essential, this means that the only way I can become better off than I am at the bundle (0,0) is for me to get at least some of each of the two goods. Thus, I do not become better off if you just take bananas away from my wife and give them to me, nor am I better off if you take oranges away from her and give them to me. Thus, any movement from (0,0) along either of the axes that refer to me would make my wife worse off without making me better off.

Exercise 16A.6

Is the point on the upper right hand corner of the Edgeworth Box Pareto efficient?

Answer: Yes — it is the point at which I get all bananas and oranges and my wife gets none. If that is the allocation, then any move away from this point within the box will make me worse off — and there is therefore no way to make someone better off without making anyone else worse off.

Exercise 16A.7

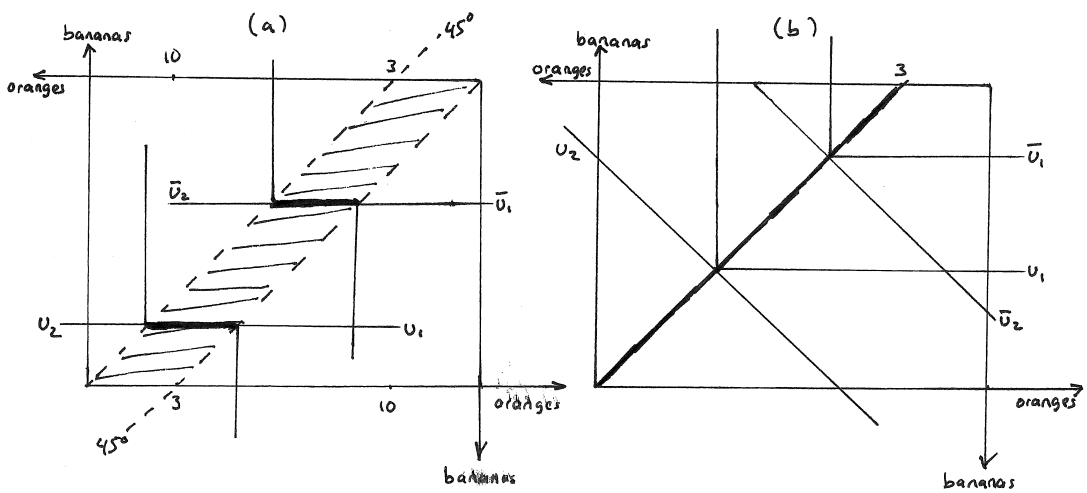
If bananas and oranges are essential goods for both me and my wife, can any points on the axes (other than those at the upper right and lower left corners of the Edgeworth Box) be Pareto efficient?

Answer: No. Consider a bundle that lies on one of the axes. If both goods are essential, it means that one of us is no better off than he/she would be if he/she had nothing. That means that we could simply give everything to the other person — make him/her better off without making the other worse off.

Exercise 16A.8

What would the contract curve look like if bananas and oranges were perfect complements for both me and my wife? (*Hint*: It is an area rather than a "curve".) What if they were perfect complements for me and perfect substitutes for my wife?

Answer: The case of perfect complements for both of us is depicted in panel (a) of Exercise Graph 16A.8 where u_1 and $\overline{u}_1$ are two indifference curves for me and u_2 and $\overline{u}_2$ are two indifference curves for my wife. These are "tangent" along the darkened parts which lie in the shaded region. And the shaded region is the region in which all such "tangencies" between two of our indifference curves lie — i.e. it is the contract curve. (Note that the region is bounded by the two 45 degree lines that emanate from our two origins of the two sets of axes — because it is along these 45 degree lines that the corners of our indifference curves lie.) Panel (b) shows the case where bananas and oranges are perfect complements for me and perfect substitutes for my wife. Again, u_1 and $\overline{u}_1$ are two indifference curves for me, and u_2 and $\overline{u}_2$ are two indifference curves for my wife. These are now "tangent" just at the corner of my indifference curves — making the 45 degree line emanating from *my* origin the contract curve.



Exercise Graph 16A.8 : Contract Curves

Exercise 16A.9

What does the contract curve look like if bananas and oranges are perfects substitutes (one for one) for both me and my wife? (*Hint*: You should get a large area within the Edgeworth Box as a result.)

Answer: The entire Edgeworth Box is then the contract curve. Because our indifference curves have the same slopes, there will be — for any arbitrary indifference curve for me — an indifference curve for my wife that lies right on top of mine and is thus "tangent" everywhere. Every allocation in the Edgeworth Box is therefore efficient. This should make intuitive sense: Suppose the two goods are Coke and Pepsi and neither of us can tell Pepsi apart from Coke. In that case, Coke and

Pepsi are perfect substitutes — and the only thing we care about is how much Coke and Pepsi we have *in total*. For any allocation in the Edgeworth Box, we will then not be able to make one of us better off without making the other worse off — because we either keep the overall quantity of soft drinks the same by changing only the mix of Coke and Pepsi in each of our bundles (thus making neither of us better or worse off), or we will increase the total quantity of soft drinks for one of us (thus making the other worse off). There is no way to make one of us better off without making the other worse off.

Exercise 16A.10

What are the intercepts of this budget on my wife's axes for oranges and bananas?

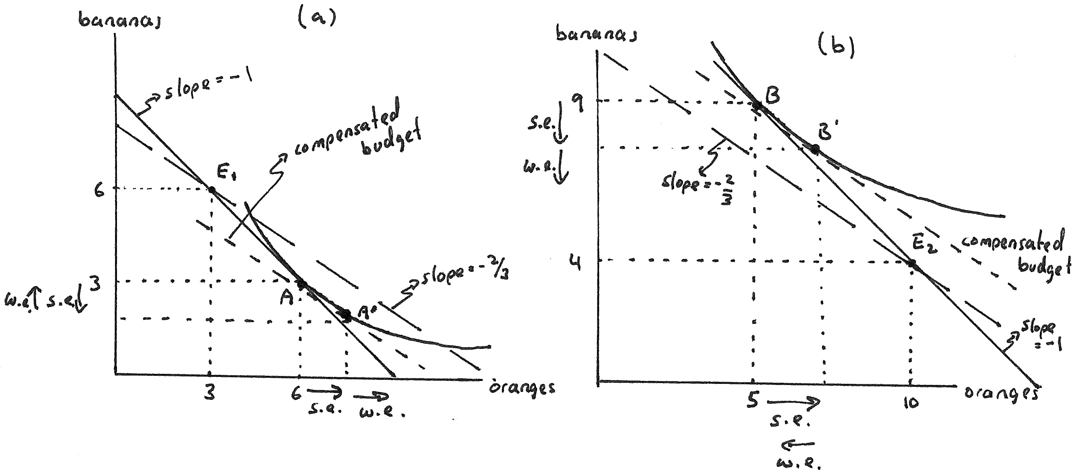
Answer: She begins at her endowment point $E_2 = (10,4)$, with 10 oranges and 4 bananas. If she sells the 4 bananas at a price of 1.5, she collects \$6 — for which she can buy 6 oranges at a price of 1. Thus, the intercept of her budget on the oranges axis is 16. Alternatively, she could sell her 10 oranges for \$10 at a price of 1, allowing her to buy 6.67 bananas at a price of 1.5. Her bananas intercept is therefore 10.67.

Exercise 16A.11

Suppose both oranges and bananas are normal goods for both me and my wife. Draw separate graphs for me and my wife — with the initial budget constraint when the prices were both equal to 1 and the new budget constraint when the price of bananas is raised to 1.5. Illustrate — using substitution and wealth effects — why my demand for oranges will unambiguously increase and my wife's demand for bananas will unambiguously decrease. Can you say unambiguously what will happen to my demand for bananas and my wife's demand for oranges?

Answer: This is illustrated in Exercise Graph 16A.11.

In panel (a), the graph for "me" is drawn — where the endowment point E_1 is the initial 3 oranges and 6 bananas and the solid original budget has slope -1 and passes through E_1. The new slope when bananas are priced at 1.5 (relative to oranges at 1) becomes $-2/3$ and must still pass through E_1. The bundle A is optimal on the original budget, and the move from A to A' is a substitution effect that implies more consumption of oranges and less of bananas. Since both goods are normal, the movement from the compensated budget to the new budget leads to an increase in consumption of both goods relative to A' — i.e. a wealth effect in the same direction as the substitution effect for oranges but not for bananas. We can therefore unambiguously say that I will consume more oranges when the price of bananas increases, but we cannot say whether I will consume more bananas (since we have offsetting substitution and wealth effects on the bananas axis). In panel (b), the graph for "my wife" is drawn, with the solid budget through E_2 again representing the initial budget with slope -1. The new budget again passes through E_2 but has shallower slope $-2/3$ — giving us a substitution effect that implies more oranges and fewer bananas (just as in panel (a)). For my wife, however, the move-



Exercise Graph 16A.11 : Price Changes in the Edgeworth Box

ment from the compensated budget to the new budget is a decrease in income — so the wealth effect points in the opposite direction (relative to the substitution effect) for oranges and in the same direction for bananas. Thus, for my wife we can unambiguously say that her consumption of bananas will fall as the price of bananas increases, but we cannot be sure whether her consumption of oranges will increase or decrease.

Exercise 16A.12

Suppose you had decided to leave the price of bananas at 1 and to rather change the price of oranges. What price (for oranges) would you have to set in order to achieve the same equilibrium outcome?

Answer: The equilibrium outcome will be achieved for *any* set of prices that result in a slope of the budget equal to $-2/3$. With oranges denoting good 1 and bananas denoting good 2, the slope of the budget constraint is $-p_1/p_2$. If we keep the price of bananas at 1, we therefore have to set the price of oranges (p_1) to 2/3 in order to get the right slope for the budget constraint.

Exercise 16A.13

Suppose you set the price of oranges equal to 2 instead of 1. What price for bananas will result in the same equilibrium outcome?

Answer: The equilibrium outcome will be achieved for *any* set of prices that result in a slope of the budget equal to $-2/3$. With oranges denoting good 1 and bananas denoting good 2, the slope of the budget constraint is $-p_1/p_2$. If we keep the price of oranges at $p_1 = 2$, we therefore have to set the price of bananas (p_2) to 3 in order to get the right slope for the budget constraint.

Exercise 16A.14

True or False: When the First Welfare Theorem holds, competitive equilibria in an exchange economy result in allocations that lie on the contract curve but not necessarily in the core.

Answer: This is false — the competitive equilibria in an exchange economy result in allocations that lie in the core but not necessarily on the contract curve. This is because no one would agree to trade at equilibrium prices if she became worse off — and lots of allocations on the contract curve have one person worse off than she is at the endowment point. Competitive equilibrium allocations *are*, however, efficient in addition to making no one worse of than she is at the endowment point — which implies that such allocations are on the portion of the contract curve that lies in the lens between the indifference curves that pass through the endowment point — i.e. in the core.

Exercise 16A.15

Can you think of other redistributions of oranges and bananas that would be "appropriate" for insuring that D is the competitive equilibrium outcome?

Answer: Any redistribution that results in a new endowment point that lies on the green budget (that is tangent to both indifference curves at D) would work. For instance, transferring 4 bananas and 3 oranges to me would leave my wife with 7 oranges and no bananas — which is the intercept of the green budget with her horizontal axis (oranges). Alternatively, you could give me 10 oranges and take 3 bananas away — leaving my wife with 7 bananas and no oranges. This would correspond to the allocation on the bananas axis. Of course there are many other possibilities in between these that would also work.

Exercise 16A.16

Does this production frontier exhibit increasing, decreasing or constant returns to scale? Is the marginal product of labor increasing, constant or decreasing?

Answer: This production frontier has decreasing returns to scale and diminishing marginal product of labor throughout.

Exercise 16A.17

As drawn, which of our usual assumptions about tastes — rationality, convexity, monotonicity, continuity — are violated?

Answer: The monotonicity assumption is violated — more is not better since I prefer less labor over more, all else being equal.

Exercise 16A.18

In votes on referenda on school vouchers, researchers have found that renters vote differently than homeowners. Consider a renter and homeowner in a bad public school district. Who do you think will be more likely to favor the introduction of school vouchers and who do you think will be more likely to be opposed?

Answer: The homeowner should be more likely to favor vouchers as demand for housing in bad public school districts will increase and thus drive up housing prices. Put differently, the homeowner in the bad public school district may favor private school vouchers because he expects it to increase the value of his home. The renter, on the other hand, would experience higher rents without the capital gain that comes from homeownership. Thus, the renter in the bad public school district is less likely to favor private school vouchers.

Exercise 16A.19

How do you think the elderly (who do not have children in school but who typically do own a home) will vote differently on school vouchers depending on whether they currently live in a good or bad public school district?

Answer: Demand for housing in bad public school districts will increase while demand for housing in good school districts will decrease — implying that housing prices will rise in bad school districts and fall in good school districts. The elderly would therefore be more likely to favor vouchers if they own a house in bad public school districts than if they owned a house in a good public school district.

Exercise 16A.20

If you were considering opening up a private school following the introduction of private school vouchers, would you be more likely to open your school in poor or in rich districts?

Answer: You would be more likely to open it in a poor district where public schools are likely to be relatively poor. This is because you will be able to attract no only parents who currently send their children to the local public school, but you will also be able to attract parents who currently live in better school districts but are willing to move to get better housing deals if they can send their children to private schools. (Some research suggests that the latter demand for private schools may be twice as large as the former.)

Exercise 16A.21

Suppose two different voucher proposals were on the table: The first proposal limits eligibility for vouchers only to families below the poverty line, while the second limits eligibility to those families who live in bad public school districts. Which policy is more likely to lead to general equilibrium effects in housing markets?

Answer: The second policy that targets geographically to bad (typically poor) public school districts will cause general equilibrium effects in housing markets because families — regardless of income — can qualify for the vouchers by moving to the targeted areas. The first policy targets primarily people who already tend to live in bad public school districts — and is therefore less likely to induce families to move (since middle income families could not qualify for the vouchers even if they moved.)

16B Solutions to Within-Chapter-Exercises for Part B

Exercise 16B.1

Can you see how the Edgeworth Box we drew in Section A contains all the allocations in this set?

Answer: The Edgeworth Box has length equal to the sum of the endowments of good 1 and height equal to the sum of the endowments of good 2 — i.e. length equal to $x_1^1 + x_1^2 = 13$ and height equal to $x_2^1 + x_2^2 = 10$ — just as is stated in the definition for FA. The set FA is then simply the set of all the possible ways of dividing 13 units of x_1 and 10 units of x_2 between two people — which is exactly what the Edgeworth Box is.

Exercise 16B.2

True or False: The Edgeworth Box represents a graphical technique that allows us to graph in a 2-dimensional picture points that lie in four dimensions.

Answer: This is true as is easily seen in the definition of the set FA which is in fact the definition of the Edgeworth Box. This definition has points that have four components — i.e. that lie in four dimensions. The clever trick used in graphing the Edgeworth Box is that we use two different two-dimensional axes to assign four values to each point — thus graphing a four dimensional space in 2 dimensions.

Exercise 16B.3

What are the reservation utilities for me and my wife in our example (given the utility functions specified above)?

Answer: The utility functions are $u^1(x_1, x_2) = x_1^{3/4} x_2^{1/4}$ and $u^2(x_1, x_2) = x_1^{1/4} x_2^{3/4}$, and our endowments are $(e_1^1, e_2^1) = (3,6)$ and $(e_1^2, e_2^2) = (10,4)$. Thus,

$$U^1 = 3^{3/4} 6^{1/4} \approx 3.5676 \text{ and } U^2 = 10^{1/4} 4^{3/4} = 5.0297. \tag{16B.3}$$

Exercise 16B.4

For the example of me and my wife, write the set of mutually beneficial allocations in the form of equation (16.6). Can you see that the lens-shaped area identified in the Edgeworth Box in Graph 16.2 is equivalent to this set?

Answer: The set would be

$$\begin{aligned} MB = \{(x_1^1, x_2^1, x_1^2, x_2^2) \in FA \mid \left(x_1^1\right)^{3/4} \left(x_2^1\right)^{1/4} \geq 3.5676 \text{ and} \\ \left(x_1^2\right)^{1/4} \left(x_2^2\right)^{3/4} \geq 5.0297\}, \end{aligned} \tag{16B.4.i}$$

where

$$FA = \left\{(x_1^1, x_2^1, x_1^2, x_2^2) \in \mathbb{R}_+^4 \mid x_1^1 + x_1^2 = 13 \text{ and } x_2^1 + x_2^2 = 10\right\}. \qquad (16B.4.ii)$$

Exercise 16B.5

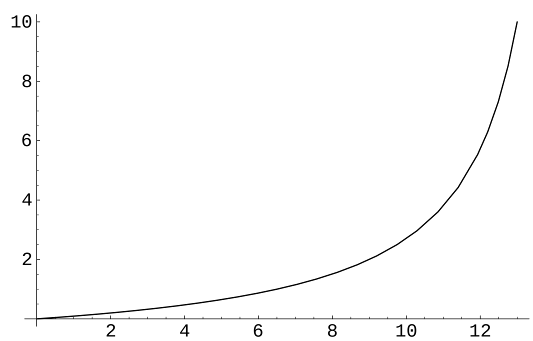
Can you see that no allocation in the set PE of an Edgeworth Box could have indifference curves pass through it in a way that creates a lens-shaped area between them?

Answer: Such a lens shape between indifference curves would imply a region that is feasible and contains bundles that lie above both indifference curve — i.e. there exist ways of making both people better off. If an allocation is such that we can make both people better off, it cannot lie in the PE set.

Exercise 16B.6

Verify that the contract curve we derived goes from one corner of the Edgeworth Box to the other.

Answer: Substituting $x_1^1 = 0$ into the equation for the contract curve, we get $x_2^1 = 0$ — i.e. the contract curve passes through the lower left hand corner of the Edgeworth Box at (0,0). Substituting $x_1^1 = 13$ into the equation for the contract curve, we get $x_2^1 = 10$ — i.e. the contract curve passes through (13,10), the upper left corner of the Edgeworth Box. The rest of the function is pictured in Exercise Graph 16B.6.



Exercise Graph 16B.6 : Contract Curve

Exercise 16B.7

A different way to find the contract curve would be to maximize my utility subject to the constraint that my wife's utility is held constant at utility level u^* and that her consumption bundle is whatever is left over after I have been given my consumption bundle. Put mathematically, this problem can be written as $\max_{x_1,x_2} x_1^{3/4} x_2^{1/4}$

s.t. $u^* = (13 - x_1)^{1/4}(10 - x_2)^{3/4}$ (where we drop the superscripts given that all variables refer to my consumption.) Demonstrate that this leads to the same solution as that derived in equation (16.9).

Answer: Setting up the Lagrange function for this problem, we get

$$\mathscr{L} = x_1^{3/4}x_2^{1/4} + \lambda\left(u^* - (13 - x_1)^{1/4}(10 - x_2)^{3/4}\right). \tag{16B.7.i}$$

Taking first order conditions, we get

$$\frac{\partial \mathscr{L}}{\partial x_1} = \frac{3x_2^{1/4}}{4x_1^{1/4}} + \lambda \frac{(10 - x_2)^{3/4}}{4(13 - x_1)^{3/4}} = 0 \text{ and } \frac{\partial \mathscr{L}}{\partial x_1} = \frac{x_1^{3/4}}{4x_2^{3/4}} + \lambda \frac{3(13 - x_1)^{1/4}}{4(10 - x_2)^{1/4}} = 0. \tag{16B.7.ii}$$

Collecting the λ terms on one side and dividing the two first order conditions by one another, we then get

$$\frac{3x_2}{x_1} = \frac{(10 - x_2)}{3(13 - x_1)} \tag{16B.7.iii}$$

which is identical to the condition in the text from which we derived the contract curve

$$x_2 = \frac{10x_1}{(117 - 8x_1)}. \tag{16B.7.iv}$$

Exercise 16B.8

Verify that these are the correct demands for this problem.

Answer: Consider the utility function $u(x_1, x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$ for an individual with endowments e_1 and e_2. At prices p_1 and p_2, the value of the endowment for this individual is $p_1e_1 + p_2e_2$, which allows us to write the utility maximization problem as

$$\max_{x_1, x_2} x_1^{\alpha}x_2^{(1-\alpha)} \text{ subject to } p_1x_1 + p_2x_2 = p_1e_1 + p_2e_2. \tag{16B.8.i}$$

We can then write the Lagrange function as

$$\mathscr{L} = x_1^{\alpha}x_2^{(1-\alpha)} + \lambda\left(p_1e_1 + p_2e_2 - p_1x_1 - p_2x_2\right). \tag{16B.8.ii}$$

The first two first order conditions are then

$$\frac{\partial \mathscr{L}}{\partial x_1} = \alpha x_1^{(\alpha-1)}x_2^{(1-\alpha)} - \lambda p_1 = 0 \text{ and } \frac{\partial \mathscr{L}}{\partial x_2} = (1-\alpha)x_1^{\alpha}x_2^{-\alpha} - \lambda p_2 = 0, \tag{16B.8.iii}$$

which solve to give us

$$x_2 = \frac{(1-\alpha)p_1x_1}{\alpha p_2}. \tag{16B.8.iv}$$

Substituting this into the budget constraint and solving for x_1, we then get

$$x_1 = \frac{\alpha(p_1e_1 + p_2e_2)}{p_1}, \qquad (16B.8.v)$$

and substituting this back into (16B.8.iv), we get

$$x_2 = \frac{(1-\alpha)(p_1e_1 + p_2e_2)}{p_2}. \qquad (16B.8.vi)$$

For me, $\alpha = 3/4$ and $(e_1, e_2) = (3,6)$ whereas for my wife $\alpha = 1/4$ and $(e_1, e_2) = (10,4)$. Substituting these into equations (16B.8.v) and (16B.8.vi), we can verify that these become equivalent to the equations in the text.

Exercise 16B.9

Write down the equilibrium condition in the x_2 market (from the second equation in expression (16.14)) using the appropriate expressions from (16.15) and solve for the equilibrium price ratio. You should get the same answer.

Answer: Equilibrium in the x_2 market is reached when

$$\frac{(3p_1 + 6p_2)}{4p_2} + \frac{3(10p_1 + 4p_2)}{4p_2} = 6 + 4. \qquad (16B.9)$$

Solving this for p_2, we again get $p_2 = 3p_1/2$.

Exercise 16B.10

Demonstrate that the same equilibrium allocation of goods will arise if $p_1 = 2$ and p_2 is one and a half times p_1 — i.e. $p_2 = 3$.

Answer: Plugging the prices into the demand equations for the two individuals, we get

$$x_1^1(2,3)) = \frac{3(3(2) + 6(3))}{4(2)} = 9 \text{ and } x_2^1(2,3) = \frac{(3(2) + 6(3))}{4(3)} = 2$$
$$x_1^2(2,3) = \frac{(10(2) + 4(3))}{4(2)} = 4 \text{ and } x_2^2(2,3) = \frac{3(10(2) + 4(3))}{4(3)} = 8. \qquad (16B.10)$$

Exercise 16B.11

Can you demonstrate that the equilibrium allocation we derived for me and my wife lies in the core that we defined in equation (16.12)?

Answer: To do this, we have to demonstrate that the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (9,2,4,8)$ lies in the PE set as well as the MB set. To demonstrate that it lies in the PE set, we have to show that

$$x_2^1 = \frac{10x_1^1}{(117 - 8x_1^1)}, x_1^2 = 13 - x_1^1 \text{ and } x_2^2 = 10 - x_2^1. \qquad (16B.11.i)$$

Plugging $x_1^1 = 9$ into the first two of these equations, we get

$$x_2^1 = \frac{10(9)}{(117-8(9))} = 2 \text{ and } x_1^2 = 13-9 = 4, \tag{16B.11.ii}$$

and plugging $x_2^1 = 2$ into the last of the three equations, we get $x_2^2 = 10-2 = 8$. Thus, the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (9,2,4,8)$ indeed lies in the PE set. To verify that it also lies in the MB set, we just need to verify that the utility for each person lies above their reservation utilities of $U^1 = 3.57$ and $U^2 = 5.03$. Evaluating utility for each of the individuals at the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (9,2,4,8)$, we get

$$u^1(9,2) = 9^{3/4}2^{1/4} \approx 6.18 \text{ and } u^2(4,8) = 4^{1/4}8^{3/4} \approx 6.73. \tag{16B.11.iii}$$

Exercise 16B.12

In our proof, we began by assuming that there exists a $(y_1^1, y_2^1, y_1^2, y_2^2)$ that is strictly preferred by everyone to $(x_1^1, x_2^1, x_1^2, x_2^2)$ and showed that there cannot be such an allocation within this economy. Can you see how the same logic also goes through if we assume that there exists an allocation $(z_1^1, z_2^1, z_1^2, z_2^2)$ that is strictly preferred by one of the individuals while leaving the other indifferent to $(x_1^1, x_2^1, x_1^2, x_2^2)$?

Answer: Suppose the allocation $(z_1^1, z_2^1, z_1^2, z_2^2)$ is strictly preferred by individual 1 to the allocation $(x_1^1, x_2^1, x_1^2, x_2^2)$ and that individual 2 is indifferent between the two allocations. Then (z_1^1, z_2^1) could not have been affordable for individual 1 under the equilibrium prices (p_1, p_2) (or else he would have chosen it) — i.e.

$$p_1 z_1^1 + p_2 z_2^1 > p_1 x_1^1 + p_2 x_2^1. \tag{16B.12.i}$$

If individual 2 is indifferent between $(z_1^1, z_2^1, z_1^2, z_2^2)$ and $(x_1^1, x_2^1, x_1^2, x_2^2)$, then we can conclude that the bundle (z_1^2, z_2^2) could not have been cheaper at the equilibrium prices than (x_1^2, x_2^2). This is because if (z_1^2, z_2^2) actually were to lie strictly inside the budget set (and not on the budget line), the indifference curve that contains (z_1^2, z_2^2) would cut the budget line — leaving some bundles on higher indifference curves that still lie on the budget constraint — which in turn implies that the optimal choice (x_1^2, x_2^2) could not have been one that yields the same utility as (z_1^2, z_2^2). Concluding that (z_1^2, z_2^2) could not have been cheaper at the equilibrium prices than (x_1^2, x_2^2) is then the same as concluding that

$$p_1 z_1^2 + p_2 z_2^2 \geq p_1 x_1^2 + p_2 x_2^2 \tag{16B.12.ii}$$

Adding equations (16B.12.i) and (16B.12.ii), we get

$$p_1(z_1^1 + z_1^2) + p_2(z_2^1 + z_2^2) > p_1(x_1^1 + x_1^2) + p_2(x_2^1 + x_2^2) \tag{16B.12.iii}$$

just as in the proof in the text (where we assumed that there exists a bundle which is strictly preferred by *both* individuals). The rest of the proof is then identical to the one in the text — with the ultimate contradiction that the allocation $(z_1^1, z_2^1, z_1^2, z_2^2)$ is not feasible.

Exercise 16B.13

Can you demonstrate that this is in fact the case for an N-person, M-good economy?

Answer: The proof is virtually identical to what we have done for the 2-person case. We begin with the equilibrium defined by prices $(p_1, p_2, ..., p_M)$ and the allocation

$$(x_1^1, x_2^1, ..., x_M^1, x_1^2, x_2^2, ..., x_M^2, ..., x_1^N, x_2^N, ...x_M^N) \in \mathbb{R}_+^{NM}. \quad (16B.13.i)$$

To say that this allocation is an equilibrium allocation under prices $(p_1, p_2, ..., p_M)$ is the same as saying that each individual n, taken his/her endowment and prices as given, in fact maximizes utility by choosing his/her portion of the allocation — i.e. $(x_1^n, x_2^n, ..., x_M^n)$. We are trying to prove that this equilibrium allocation must be efficient — i.e. there does not exist an alternative allocation

$$(y_1^1, y_2^1, ..., y_M^1, y_1^2, y_2^2, ..., y_M^2, ..., y_1^N, y_2^N, ...y_M^N) \in \mathbb{R}_+^{NM} \quad (16B.13.ii)$$

that is strictly preferred by at least one person and viewed as at least as good by all others. So suppose that such an allocation *did* exist, and suppose we identify the person for whom this allocation is strictly better as person 1. We then know that $(y_1^1, y_2^1, ..., y_M^1)$ could not have been affordable for person 1 at prices (p_1, p_2) (or else it would have been chosen) — i.e. we know

$$p_1 y_1^1 + p_2 y_2^1 + ... + p_M y_M^1 > p_1 x_1^1 + p_2 x_2^1 + ... + p_M x_M^1. \quad (16B.13.iii)$$

Similarly — and for reasons analogous to those explained in the previous exercise — it must be the case that

$$p_1 y_1^n + p_2 y_2^n + ... + p_M y_M^n \geq p_1 x_1^n + p_2 x_2^n + ... + p_M x_M^n \text{ for all } n = 2, 3, ..., N. \quad (16B.13.iv)$$

Adding equation (16B.13.iii) to the $(N-1)$ equations in expression (16B.13.iv), we get

$$p_1 \sum_{n=1}^N y_1^n + p_2 \sum_{n=1}^N y_2^n + ... + p_M \sum_{n=1}^N y_M^n > p_1 \sum_{n=1}^N x_1^n + p_2 \sum_{n=1}^N x_2^n + ... + p_M \sum_{n=1}^N x_M^n. \quad (16B.13.v)$$

Walras' Law tells us that the right hand side is equal to the value of the endowments — which implies we can re-write this as

$$p_1 \sum_{n=1}^N y_1^n + p_2 \sum_{n=1}^N y_2^n + ... + p_M \sum_{n=1}^N y_M^n > p_1 \sum_{n=1}^N e_1^n + p_2 \sum_{n=1}^N e_2^n + ... + p_M \sum_{n=1}^N e_M^n \quad (16B.13.vi)$$

or as

$$p_1\left(\sum_{n=1}^{N} y_1^n - \sum_{n=1}^{N} e_1^n\right) + p_2\left(\sum_{n=1}^{N} y_2^n - \sum_{n=1}^{N} e_2^n\right) + ... + p_M\left(\sum_{n=1}^{N} y_M^n - \sum_{n=1}^{N} e_M^n\right) > 0. \tag{16B.13.vii}$$

But, since all prices are positive, this implies that at least one of the bracketed terms in equation (16B.13.vii) must be greater than zero — i.e.

$$\sum_{n=1}^{N} y_m^n - \sum_{n=1}^{N} e_m^n > 0 \text{ for some good } m. \tag{16B.13.viii}$$

But that implies that the y allocation is not feasible — i.e. the assumption of inefficiency of the equilibrium allocation has led to a contradiction. Thus, the equilibrium allocation is efficient.

Exercise 16B.14

Verify the results in equation (16.27).

Answer: We can solve this either by setting up the Lagrange function or simply by substituting the constraint into the objective. Doing the former, we get the Lagrange function

$$\mathcal{L} = x^{\alpha}(L-\ell)^{(1-\alpha)} + \lambda\left(x - A\ell^{\beta}\right). \tag{16B.14.i}$$

The first two first order conditions are then

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \alpha x^{(\alpha-1)}(L-\ell)^{(1-\alpha)} + \lambda = 0 \text{ and}\\ \frac{\partial \mathcal{L}}{\partial \ell} &= -(1-\alpha)x^{\alpha}(L-\ell)^{-\alpha} - \lambda\beta A\ell^{(\beta-1)} = 0.\end{aligned} \tag{16B.14.ii}$$

which can also be written as

$$\alpha x^{(\alpha-1)}(L-\ell)^{(1-\alpha)} = -\lambda \text{ and } (1-\alpha)x^{\alpha}(L-\ell)^{-\alpha} = -\lambda\beta A\ell^{(\beta-1)}. \tag{16B.14.iii}$$

Dividing the two equations in expression (16B.14.iii) by one another and solving for x, we get

$$x = \frac{\alpha\beta A(L-\ell)\ell^{(\beta-1)}}{(1-\alpha)}, \tag{16B.14.iv}$$

and substituting this into the constraint from the optimization problem, we get

$$A\ell^{\beta} = \frac{\alpha\beta A(L-\ell)\ell^{(\beta-1)}}{(1-\alpha)} \tag{16B.14.v}$$

which we can in turn solve for ℓ to verify

$$\ell = \frac{\alpha\beta L}{1-\alpha(1-\beta)}. \tag{16B.14.vi}$$

Plugging this into (16B.14.iv) and solving for x, we furthermore verify the text's expression for x. This involves a bit of algebra — the quickest way to get there is as follows: Begin by substituting (16B.14.vi) into $(L-\ell)$ to get

$$(L-\ell) = L - \frac{\alpha\beta L}{1-\alpha(1-\beta)} = \frac{[1-\alpha(1-\beta)]L - \alpha\beta L}{1-\alpha(1-\beta)} = \frac{(1-\alpha)L}{1-\alpha(1-\beta)}. \tag{16B.14.vii}$$

Substituting this for $(L-\ell)$ into equation (16B.14.iv), we get

$$x = \frac{\alpha\beta A\ell^{(\beta-1)}}{(1-\alpha)}\left(\frac{(1-\alpha)L}{1-\alpha(1-\beta)}\right) = \frac{\alpha\beta AL\ell^{(\beta-1)}}{1-\alpha(1-\beta)}. \tag{16B.14.viii}$$

Substituting (16B.14.vi) into this for ℓ, we then get

$$x = \frac{\alpha\beta AL}{1-\alpha(1-\beta)}\left(\frac{\alpha\beta L}{1-\alpha(1-\beta)}\right)^{(\beta-1)} = A\left(\frac{\alpha\beta L}{1-\alpha(1-\beta)}\right)^{\beta}. \tag{16B.14.ix}$$

Exercise 16B.15

Verify that this is the correct solution.

Answer: Taking the first derivative of $(pA\ell^{\beta} - w\ell)$ (with respect to ℓ) and setting it to zero, we get

$$\beta pA\ell^{(\beta-1)} - w = 0, \tag{16B.15.i}$$

and solving for ℓ, we get

$$\ell = \left(\frac{w}{\beta pA}\right)^{1/(\beta-1)} = \left(\frac{\beta pA}{w}\right)^{1/(1-\beta)}. \tag{16B.15.ii}$$

Substituting this back into the production function, we get

$$x = A\left(\left(\frac{\beta pA}{w}\right)^{1/(1-\beta)}\right)^{\beta} = A\left(\frac{\beta pA}{w}\right)^{\beta/(1-\beta)} \tag{16B.15.iii}$$

Exercise 16B.16

Verify that these solutions for labor supply and banana demand are correct.

Answer: The Lagrange function for the maximization problem is

$$\mathscr{L} = x^{\alpha}(L-\ell)^{(1-\alpha)} + \lambda(w\ell + \pi(w,p) - px) \tag{16B.16.i}$$

giving rise to first order conditions

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{(\alpha-1)}(L-\ell)^{(1-\alpha)} - \lambda p = 0 \text{ and}$$
$$\frac{\partial \mathcal{L}}{\partial \ell} = -(1-\alpha)x^{\alpha}(L-\ell)^{-\alpha} + \lambda w = 0. \tag{16B.16.ii}$$

Rearranging these with the λ terms on one side, dividing the equations by each other and solving for x, we get

$$x = \frac{\alpha(L-\ell)w}{(1-\alpha)p}. \tag{16B.16.iii}$$

Substituting this into the constraint $w\ell + \pi(w,p) = px$ and solving for ℓ, we then get

$$\ell = \alpha L - \frac{(1-\alpha)\pi(w,p)}{w}, \tag{16B.16.iv}$$

and substituting this back into equation (16B.16.iii), we get

$$x = \frac{\alpha}{p}\left(wL + \pi(w,p)\right). \tag{16B.16.v}$$

These labor supply and banana demand equations can then be expanded by simply replacing $\pi(w,p)$ with the profit function

$$\pi(w,p) = (1-\beta)(Ap)^{1/(1-\beta)}\left(\frac{\beta}{w}\right)^{\beta/(1-\beta)} \tag{16B.16.vi}$$

to give us

$$\ell = \alpha L - \frac{(1-\alpha)(1-\beta)}{\beta}\left(\frac{\beta p A}{w}\right)^{1/(1-\beta)}$$
$$x = \frac{\alpha w}{p}\left(L + \frac{(1-\beta)}{\beta}\left(\frac{\beta p A}{w}\right)^{1/(1-\beta)}\right). \tag{16B.16.vii}$$

Exercise 16B.17

Verify that the same equilibrium relationship between prices and wages arises by solving $x^S = x^D$.

<u>Answer</u>: We need to solve

$$x^S(w,p) = A\left(\frac{\beta p A}{w}\right)^{\beta/(1-\beta)} = \frac{\alpha w}{p}\left(L + \frac{(1-\beta)}{\beta}\left(\frac{\beta p A}{w}\right)^{1/(1-\beta)}\right) = x^D(w,p). \tag{16B.17.i}$$

The problem is simplified if we re-write the right hand side as

$$x^D(w,p) = \frac{\alpha w}{p}\left(L + \frac{(1-\beta)}{\beta}\left(\frac{\beta p A}{w}\right)^{1/(1-\beta)}\right) = \\ = \frac{\alpha w}{p}L + \alpha(1-\beta)A\left(\frac{\beta p A}{w}\right)^{\beta/(1-\beta)}. \tag{16B.17.ii}$$

Equation (16B.17.i) then becomes

$$A\left(\frac{\beta p A}{w}\right)^{\beta/(1-\beta)} = \frac{\alpha w}{p}L + \alpha(1-\beta)A\left(\frac{\beta p A}{w}\right)^{\beta/(1-\beta)} \tag{16B.17.iii}$$

which can be re-written as

$$w^{1/(1-\beta)} = \frac{A(\beta p A)^{\beta/(1-\beta)}(1-\alpha(1-\beta))p}{\alpha L} = \\ = \frac{(A\beta)^{1/(1-\beta)}(1-\alpha(1-\beta))p^{1/(1-\beta)}}{\alpha\beta L}. \tag{16B.17.iv}$$

Taking both the left and right hand sides to the power $(1-\beta)$ and re-arranging terms slightly, we then arrive at

$$w = A\beta\left(\frac{1-\alpha(1-\beta)}{\alpha\beta L}\right)^{(1-\beta)} p. \tag{16B.17.v}$$

Exercise 16B.18

Can you tell from the graph of an equilibrium in Graph 16.10 that any combination of w and p that satisfies a particular ratio will generate the same equilibrium in the labor and banana markets?

Answer: In panel (b) of the graph we see that the equilibrium point C occurs at the tangency of the indifference curve u^* and the production frontier. The input and output prices that support this as an equilibrium are those that result in the isoprofit/budget line with slope w^*/p^* such that the line is tangent to both the indifference curve and the production frontier at C — and what matters for this is the *ratio* of the prices. (Economic theorists will often call the grey line in the graph a "separating hyperplane" — i.e. the "hyperplane" that separates the production frontier and indifference curve at precisely one point.)

Exercise 16B.19

Can you tell from the graph of an equilibrium in Graph 16.10 whether profit will be affected by different choices of w and p that satisfy the equilibrium ratio? Verify whether your intuition holds mathematically.

Answer: Suppose w^* and p^* satisfy the equilibrium ratio — resulting in profit π^*. The intercept of the grey isoprofit/budget line in panel (b) of the graph is π^*/p^*.

Now suppose that we increase both the wage and the output price by a factor t, resulting in profit π'. The intercept of the grey line that separates the indifference curve and production frontier in the graph does not change, which implies $\pi'/(tp^*) = \pi^*/p^*$ — which can only hold if $\pi' = t\pi^*$. Thus, as the input and output prices are scaled up and down, profit is scaled with it at the same rate. We can see this from the profit function by illustrating that $\pi(tw, tp) = t\pi(w, p)$ — i.e. by showing that the profit function is homogeneous of degree 1:

$$\begin{aligned}\pi(tw,tp) &= (1-\beta)(Atp)^{1/(1-\beta)}\left(\frac{\beta}{tw}\right)^{\beta/(1-\beta)} = \\ &= t(1-\beta)(Ap)^{1/(1-\beta)}\left(\frac{\beta}{w}\right)^{\beta/(1-\beta)} = t\pi(w,p).\end{aligned} \tag{16B.19}$$

Exercise 16B.20

Demonstrate that the equilibrium banana consumption (and production) is equal to the optimal level of banana consumption.

Answer: When first considering the optimal decision for Robinson Crusoe in the text, we derived the pareto optimal consumption level for bananas as

$$x = A\left(\frac{\alpha\beta L}{1-\alpha(1-\beta)}\right)^{\beta}. \tag{16B.20.i}$$

The *equilibrium* level can be determined by plugging our expression for the equilibrium wage w^* (in terms of p) into either the output supply or demand equation in the text. Using the output supply equation, we get

$$\begin{aligned}x^S &= A\left(\frac{\beta pA}{w^*}\right)^{\beta/(1-\beta)} = A\left(\beta pA\right)^{\beta/(1-\beta)}\left(\frac{1}{\beta Ap}\right)^{\beta/(1-\beta)}\left(\frac{\alpha\beta L}{1-\alpha(1-\beta)}\right)^{\beta} = \\ &= A\left(\frac{\alpha\beta L}{1-\alpha(1-\beta)}\right)^{\beta}.\end{aligned} \tag{16B.20.ii}$$

Substituting our expression for w^* into x^D gives the same answer.

Exercise 16B.21

Why is there no coalition to block A in the 2-person version of this economy?

Answer: In the 2-person economy, there are only three coalitions: (1) a coalition of both (and thus *all*) the individuals in the economy; (2) a coalition composed of a single individual of type 1; and (3) a coalition of a single individual of type 2. At A, "coalition" (2) is just as well off as at E while "coalition" (3) is strictly better off. Thus, neither "coalition" (2) nor "coalition" (3) would block the allocation A because neither can do better by splitting away from the economy. That leaves only coalition (1) — composed of both type 1 and type 2. At A, their indifference curves are tangent to one another — which implies there is no way we can make either of them better off without making the other worse off. Thus, coalition (1) has no reason to block A. As a result, none of the coalitions in the economy can do better by splitting off than the individuals in the coalitions do at A. The difference in the 4-person economy is that there are now more coalitions to consider — including 3-person coalitions composed of two of one type and one of the other.

Exercise 16B.22

Can you demonstrate that a coalition of two of the "type 2" individuals with on "type 1" individual will block the allocation B?

Answer: Such a coalition will block B for exactly the same reasons that the coalition of two type 1 individuals with 1 type 2 individual blocks A. Suppose B is proposed as the allocation for the economy — with each type getting what the axes for that type indicate at B. Now suppose the coalition of two type 2 and one type 1 individual splits off with their endowments. In the Edgeworth Box, draw a line that goes through E and B. With both types starting at E, a trade in which the two type 2 consumers give up 1 unit of x_2 results in the one type 1 individual getting *two* units of x_2. Thus, if the individuals in the coalition trade along the line that passes through E and B, the one type 1 individual moves down the line twice as quickly as the two type 2 individuals do. Thus, the coalition can trade among itself and end up with an allocation for the type 1 consumer that lies to the southeast of B and an allocation for the type 2 consumers that lies to the northwest of B — with both consumer types ending up on an indifference curve that is higher than the one that passes through B. Thus, the coalition can do better by splitting off the economy with its endowment than their members do under the allocation B — which implies that the coalition will "block" B from being implemented.

Exercise 16B.23

Why must the distance between E and D' be twice the distance from E to C'?

Answer: This is because there are two type 1 individuals and one type 2 individual — which means that everything a type 1 individual gives up is received twice by the single type 2 — and anything that the type 2 individual gives up in exchange is split between the two type 1 people.

Exercise 16B.24

Demonstrate that the competitive equilibrium allocation must lie in the core of the replicated exchange economy.

Answer: An equilibrium allocation X has the feature that the indifference curves for type 1 and type 2 consumers are tangent at X in the Edgeworth Box — with the line that passes through E and X "separating" the two indifference curves at X. If any coalition splits off, it will trade along a line that passes through E — and this line is either shallower or steeper than the one that passes through E and X. Suppose it is shallower. In that case all points on that line lie "below" the equilibrium indifference curve for type 2 consumers — and thus there is no coalition that includes a type 2 individual which could in fact do better for type 2. Suppose instead that the line is steeper. Then it lies entirely "below" the equilibrium indifference curve for type 1 consumers — implying that no type 1 consumer could be part of a blocking coalition. Thus, neither consumer could be part of a blocking coalition — which implies that the equilibrium allocation X lies in the core.

16C Solutions to Odd Numbered End-of-Chapter Exercises

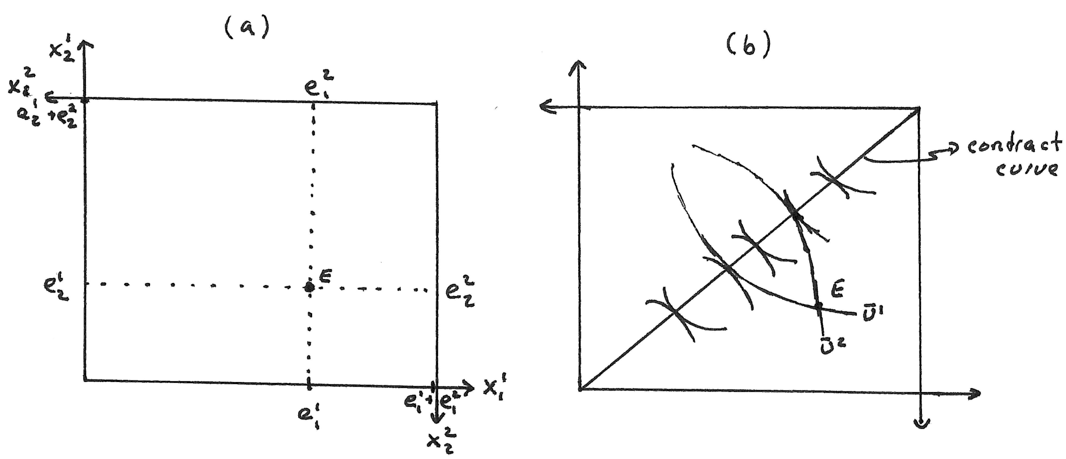
Exercise 16.1

Consider a 2-person/2-good exchange economy in which person 1 is endowed with (e_1^1, e_2^1) and person 2 is endowed with (e_1^2, e_2^2) of the goods x_1 and x_2.

A: *Suppose that tastes are homothetic for both individuals.*

(a) *Draw the Edgeworth Box for this economy, indicating on each axis the dimensions of the box.*

Answer: This is illustrated in panel (a) of Exercise Graph 16.1(1) where the width of the box is $(e_1^1 + e_1^2)$ and the height is $(e_2^1 + e_2^2)$.



Exercise Graph 16.1(1) : Contract Curve with Homothetic Tastes

(b) *Suppose that the two individuals have identical tastes. Illustrate the contract curve — i.e. the set of all efficient allocations of the two goods.*

Answer: Homothetic tastes have the characteristic that the MRS is the same along any ray from the origin. Consider the ray that passes from the lower left to the upper right corners of the box — i.e. from the origin for person 1 to the origin for person 2. If tastes are homothetic for each of the two individuals, this means that, for each individual, it is the case that the MRS is constant along this ray. And if their tastes are identical, then their MRS's are the same along that ray — i.e. on each point of the ray, the indifference curves that pass through that point are tangent to one another. Since the contract curve is the set of allocations where the indifference curves are tangent, this ray is then the contract curve. It is depicted in panel (b) of Exercise Graph 16.1(1).

(c) True or False*: Identical tastes in the Edgeworth Box imply that there are no mutually beneficial trades.*

Answer: This is false. In panel (b) of Exercise Graph 16.1(1), for instance, the indifference curves $\overline{u}^1$ and $\overline{u}^2$ contain the endowment bundle E — with allocations inside the lens created by these indifference curves representing mutually beneficial trades. The only way that there are no mutually beneficial trades when both individuals have identical homothetic tastes is if the endowment bundle falls on the contract curve — i.e. on the line connecting the origins for the two individuals.

(d) *Now suppose that the two individuals have different (but still homothetic) tastes.* True or False*: The contract curve will lie to one side of the line that connects the lower left and upper right corners of the Edgeworth Box — i.e. it will never cross this line inside the Edgeworth Box.*

Answer: This is true (almost). If the two individuals' tastes are not identical, then their indifference curves are not likely to be tangent on the line connecting the lower left and upper right corners of the box. Take one point on that line — it is likely the case that the indifference curve for person 1 is steeper or shallower than that for individual 2 at this point. Suppose first that individual 1's indifference curve is shallower. Then the two indifference curves form a lens shape — with the entire area of the lens lying *above* the line connecting the corners of the box. Since the slopes of the indifference curves are constant along this line, this same lens shape will appear above the line for *any* allocation on the line. This implies that the tangencies of indifference curves (which form the contract curve) must also lie *above* the line (because these tangencies will be found within the lens shapes formed from indifference curves that cross on the line.) The reverse will be true if individual 1's indifference curve is steeper along the line than indifference curves for individual 2 — with the entire contract curve now lying *below* the line. The reason the answer is true (*almost*) is that it is still possible that the marginal rates of substitution for the two individuals are in fact equal along the line connecting the lower left and upper right corners of the box. For instance, it might be that the tastes have different degrees of substitutability (and are therefore different) but still have the same marginal rates of substitution on that line. In that case, the contract curve lies on the line connecting the lower left and upper right corners. Thus, homothetic tastes imply that the contract curve lies either on the line connecting the corners or all to one side of that line.

B: *Suppose that the tastes for individuals 1 and 2 can be described by the utility functions $u^1 = x_1^{\alpha} x_2^{(1-\alpha)}$ and $u^2 = x_1^{\beta} x_2^{(1-\beta)}$ (where α and β both lie between 0 and 1). Some of the questions below are notationally a little easier to keep track off if you also denote $E_1 = e_1^1 + e_1^2$ as the economy's endowment of x_1 and $E_2 = e_2^1 + e_2^2$ as the economy's endowment of x_2.*

(a) *Let $\overline{x}_1$ denote the allocation of x_1 to individual 1, and let $\overline{x}_2$ denote the allocation of x_2 to individual 1. Then use the fact that the remainder of the economy's endowment is allocated to individual 2 to denote individual 2's allocation as $(E_1 - \overline{x}_1)$ and $(E_2 - \overline{x}_2)$ for x_1 and x_2 respectively. Derive the contract curve in the form $\overline{x}_2 = x_2(\overline{x}_1)$ — i.e. with the allocation of x_2 to person 1 as a function of the allocation of x_1 to person 1.*

Answer: You can derive this either by setting the MRS for individual 1 equal to the MRS for individual 2 — or you can solve the problem

$$\max_{\overline{x}_1, \overline{x}_2} \overline{x}_1^{\alpha} \overline{x}_2^{(1-\alpha)} \text{ subject to } u^2 = (E_1 - \overline{x}_1)^{\beta} (E_2 - \overline{x}_2)^{(1-\beta)} \tag{16.1.i}$$

where person 1's utility is maximized subject to getting person 2 to a particular indifference curve u^2. Either way, you will get to the point where you have an expression that sets the marginal rates of substitution equal to one another — i.e.

$$\begin{aligned} \frac{\partial u^1(\overline{x}_1, \overline{x}_2)/\partial x_1}{\partial u^1(\overline{x}_1, \overline{x}_2)/\partial x_2} = \frac{\alpha \overline{x}_2}{(1-\alpha)\overline{x}_1} = \frac{\beta(E_2 - \overline{x}_2)}{(1-\beta)(E_1 - \overline{x}_1)} = \\ = \frac{\partial u^2((E_1 - \overline{x}_1), (E_2 - \overline{x}_2))/\partial x_1}{\partial u^2((E_1 - \overline{x}_1), (E_2 - \overline{x}_2))/\partial x_2}. \end{aligned} \tag{16.1.ii}$$

Solving the middle of this expression for $\overline{x}_2$, we then get the contract curve

$$x_2(\overline{x}_1) = \frac{(1-\alpha)\beta E_2 \overline{x}_1}{\alpha(1-\beta)E_1 + (\beta - \alpha)\overline{x}_1}. \tag{16.1.iii}$$

(b) *Simplify your expression under the assumption that tastes are identical — i.e. $\alpha = \beta$. What shape and location of the contract curve in the Edgeworth Box does this imply?*

Answer: Replacing β with α, the expression then simplifies to

$$x_2(\overline{x}_1) = \frac{(1-\alpha)\alpha E_2 \overline{x}_1}{\alpha(1-\alpha)E_1 + (\alpha - \alpha)\overline{x}_1} = \frac{E_2}{E_1}\overline{x}_1. \tag{16.1.iv}$$

This is simply the equation of a line with zero vertical intercept and slope E_2/E_1 — which is the slope of the ray that passes from the lower left to the upper right corner of the Edgeworth Box. Thus, when tastes are identical, we get that the contract curve is the line that connects the origins for the two individuals in the Edgeworth Box — exactly as we did for homothetic tastes in part A of the question (and as depicted in panel (b) of Exercise Graph 16.1(1).)

(c) *Next, suppose that $\alpha \neq \beta$. Verify that the contract curve extends from the lower left to the upper right corner of the Edgeworth Box.*

Answer: Evaluating the contract curve equation (16.1.iii) at $\overline{x}_1 = 0$, we get $x_2(0) = 0$ — i.e. the contract curve passes through the lower left hand corner of the Edgeworth Box. Evaluating the contract curve at $\overline{x}_1 = E_1$, we get

$$x_2(E_1) = \frac{(1-\alpha)\beta E_2 E_1}{\alpha(1-\beta)E_1 + (\beta-\alpha)E_1} = \frac{(1-\alpha)\beta E_2 E_1}{(1-\alpha)\beta E_1} = E_2; \qquad (16.1.v)$$

i.e. the contract curve passes through the upper right corner of the box where individual 1 gets all the goods in the economy.

(d) *Consider the slopes of the contract curve when $\overline{x}_1 = 0$ and when $\overline{x}_1 = E_1$. How do they compare to the slope of the line connecting the lower left and upper right corners of the Edgeworth Box if $\alpha > \beta$? What if $\alpha < \beta$?*

Answer: The slope of the contract curve is the derivative of equation (16.1.iii) with respect to x_1 —

$$\begin{aligned}\frac{\partial x_2(\overline{x}_1)}{\partial x_1} &= \frac{(1-\alpha)\beta E_2}{\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1} - \frac{(\beta-\alpha)(1-\alpha)\beta E_2 \overline{x}_1}{\left[\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1\right]^2} = \\ &= \frac{\alpha\beta(1-\alpha)(1-\beta)E_1 E_2}{\left[\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1\right]^2}.\end{aligned} \qquad (16.1.vi)$$

Evaluated at $\overline{x}_1 = 0$ and at $\overline{x}_1 = E_1$, we get

$$\frac{\partial x_2(0)}{\partial x_1} = \frac{\beta(1-\alpha)E_2}{\alpha(1-\beta)E_1} \text{ and } \frac{\partial x_2(E_1)}{\partial x_1} = \frac{\alpha(1-\beta)E_2}{\beta(1-\alpha)E_1}. \qquad (16.1.vii)$$

The slope of the line connecting the two corners of the Edgeworth box is E_2/E_1. If $\alpha = \beta$, both derivatives in expression (16.1.vii) are equal to E_2/E_1 — i.e. the slope of the contract curve is exactly the slope of the line connecting the corners (as we already concluded previously). If $\alpha > \beta$, then

$$\frac{\beta(1-\alpha)}{\alpha(1-\beta)} < 1 \text{ and } \frac{\alpha(1-\beta)}{\beta(1-\alpha)} > 1 \qquad (16.1.viii)$$

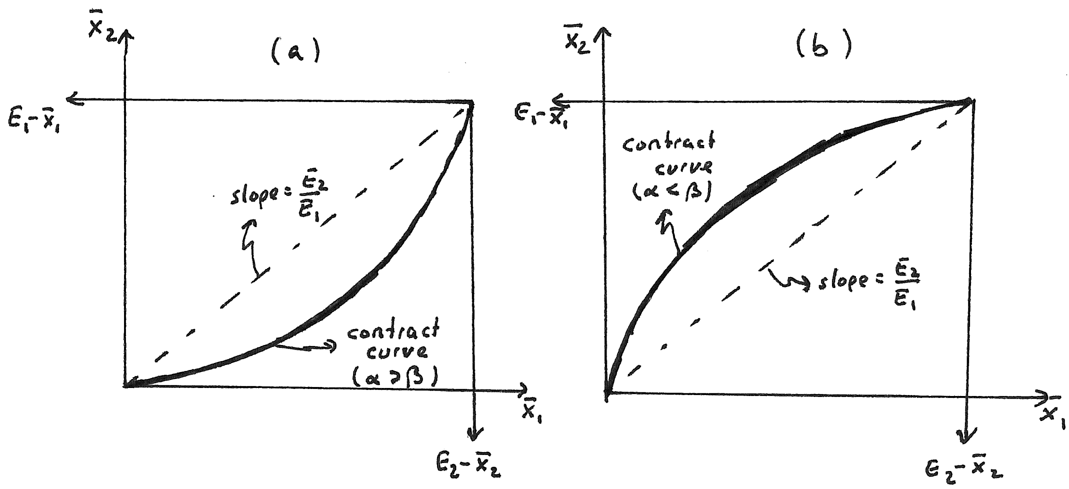
implying that

$$\frac{\partial x_2(0)}{\partial x_1} < \frac{E_2}{E_1} \text{ and } \frac{\partial x_2(E_1)}{\partial x_1} > \frac{E_2}{E_1}. \tag{16.1.ix}$$

The reverse relationship holds when $\alpha < \beta$.

(e) *Using what you have concluded, graph the shape of the contract curve for the case $\alpha > \beta$ and for the case when $\alpha < \beta$?*

The contract curves consistent with these relationships are graphed in Exercise Graph 16.1(2).



Exercise Graph 16.1(2) : Contract Curves when (a) $\alpha > \beta$ and when (b) $\alpha < \beta$

(f) *Suppose that the utility function for the two individuals instead took the more general constant elasticity of substitution form $u = (\alpha x_1 + (1-\alpha)x_2)^{-1/\rho}$. If the tastes for the two individuals are identical, does your answer to part (b) change?*

Answer: No, the answer does not change. The MRS for this utility function (derived in Chapter 5) is

$$MRS = -\left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{x_2}{x_1}\right)^{\rho+1}. \tag{16.1.x}$$

Using our notation and setting the MRS's equal to each other for the two individuals, we then get

$$\left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{\overline{x}_2}{\overline{x}_1}\right)^{\rho+1} = \left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{(E_2-\overline{x}_2)}{(E_1-\overline{x}_1)}\right)^{\rho+1} \tag{16.1.xi}$$

which we can solve for $\overline{x}_2$ to get the contract curve

$$x_2(\overline{x}_1) = \left(\frac{E_2}{E_1}\right)\overline{x}_1; \qquad (16.1.xii)$$

i.e. the contract curve again has zero vertical intercept and slope E_2/E_1, the slope of the ray that connects the two corners of the Edgeworth Box.

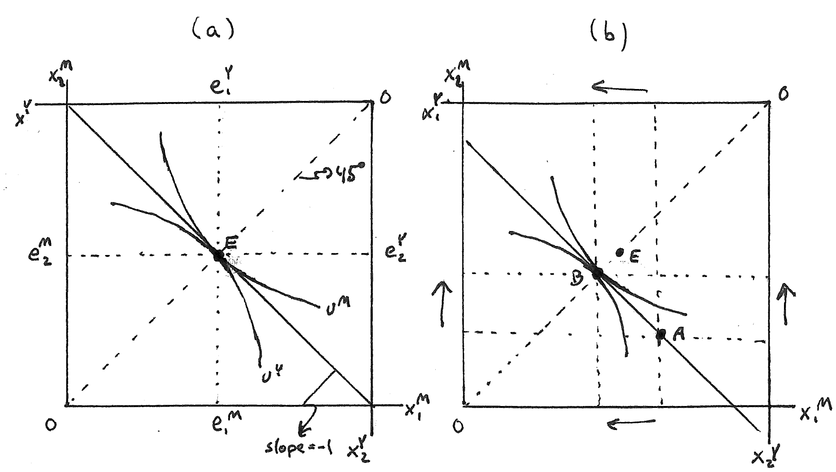
Exercise 16.3

Suppose you and I have the same homothetic tastes over x_1 and x_2, and our endowments of the two goods are $E^M = (e_1^M, e_2^M)$ for me and $E^Y = (e_1^Y, e_2^Y)$ for you.

A: *Suppose throughout that, when $x_1 = x_2$, our MRS is equal to -1.*

(a) *Assume that $e_1^M = e_2^M = e_1^Y = e_2^Y$. Draw the Edgeworth box for this case and indicate where the endowment point $E = (E^M, E^Y)$ lies.*

Answer: This is done in panel (a) of Exercise Graph 16.3 where the Edgeworth Box is drawn as a square (because the overall endowments of x_1 are equal to those of x_2), with the endowment bundle E located in the center (since we are endowed with equal amounts of everything.)



Exercise Graph 16.3 : Equal Endowments and Same Tastes

(b) *Draw the indifference curves for both of us through E. Is the endowment allocation efficient?*

Answer: This is also done in panel (a). Since our endowment lies on the 45-degree line and our MRS along the 45-degree line is always -1, the indifference curves through E are tangent to one another. This implies that the endowment allocation is efficient — because there is no lens shape between our indifference curves that would give us room to make both of us better off (or at least one better off without making the other worse off).

(c) *Normalize the price of x_2 to 1 and let p be the price of x_1. What is the equilibrium price p^*?*

Answer: The equilibrium price must pass through E and induce budget constraints for me and you such that both of us optimize at the same point within the Edgeworth Box. In this case, the only way to do this is to let $p^* = 1$ — resulting in a budget with slope -1 through E. Since the MRS at E is -1 for both of us, we will both choose to remain at our endowment bundle at this price. This is also illustrated in panel (a) of Exercise Graph 16.3.

(d) *Where in the Edgeworth Box is the set of all efficient allocations?*

Answer: The set of all efficient allocations lies on the 45-degree line — because along the 45 degree line, our MRS's are equal to 1 and thus equal to one another, implying indifference curves that are tangent to one another. This is the contract curve.

(e) *Pick another efficient allocation and demonstrate a possible way to re-allocate the endowment among us such that the new efficient allocation becomes an equilibrium allocation supported by an equilibrium price. Is this equilibrium price the same as p^* calculated in (c)?*

Answer: This is illustrated in panel (b) of Exercise Graph 16.3 where we consider the equilibrium if the endowment is redistributed so that it moves from E to A. The only place where the indifference curves are tangent to one another is on the 45-degree line where their slope is -1. Thus, the new equilibrium price must again be 1 — and the new budget must pass through the new endowment A as drawn. This will cause us to trade from A to B along the budget line with slope $p = 1$ — with me giving up x_1 to get more x_2 and you giving up an equal amount of x_2 to get more x_1 (as indicated by the arrows on the axes.)

B: *Suppose our tastes can be represented by the CES utility function $u(x_1, x_2) = \left(0.5x_1^{-\rho} + 0.5x_2^{-\rho}\right)^{-1/\rho}$.*

(a) *Let p be defined as in A(c). Write down my and your budget constraint (assuming again endowments $E^M = (e_1^M, e_2^M)$ for me and $E^Y = (e_1^Y, e_2^Y)$.)*

Answer: The value of or endowments has to be equal to the value of what we consume. For me, this implies

$$pe_1^M + e_2^M = px_1^M + x_2^M, \tag{16.3.i}$$

and for you it means

$$pe_1^Y + e_2^Y = px_1^Y + x_2^Y. \tag{16.3.ii}$$

(b) *Write down my optimization problem and derive my demand for x_1 and x_2.*

Answer: My optimization problem is then

$$\max_{x_1,x_2} \left(0.5x_1^{-\rho} + 0.5x_2^{-\rho}\right)^{-1/\rho} \text{ subject to } pe_1^M + e_2^M = px_1 + x_2 \qquad (16.3.iii)$$

where, for now, we suppress the M superscripts on the x variables. Setting up the Lagrangian and solving in the usual way, we get

$$x_1^M = \frac{pe_1^M + e_2^M}{p + p^{1/(\rho+1)}} \text{ and } x_2^M = \frac{p^{1/(\rho+1)}e_1^M + e_2^M}{p + p^{1/(\rho+1)}}. \qquad (16.3.iv)$$

(c) *Similarly, derive your demand for x_1 and x_2.*

Answer: Repeating the steps in the previous part for you, we get

$$x_1^Y = \frac{pe_1^Y + e_2^Y}{p + p^{1/(\rho+1)}} \text{ and } x_2^Y = \frac{p^{1/(\rho+1)}e_1^Y + e_2^Y}{p + p^{1/(\rho+1)}}. \qquad (16.3.v)$$

(d) *Derive the equilibrium price. What is that price if, as in part A, $e_1^M = e_2^M = e_1^Y = e_2^Y$?*

Answer: In equilibrium, the price has to be such that demand is equal to supply in both markets. Because of Walras' Law, we only have to solve for p in one of the markets though — and either one will work. Choosing the market for x_1, it must therefore be the case that $x_1^M + x_1^Y = e_1^M + e_1^Y$ or, plugging in our demands from the previous parts,

$$\frac{pe_1^M + e_2^M}{p + p^{1/(\rho+1)}} + \frac{pe_1^Y + e_2^Y}{p + p^{1/(\rho+1)}} = e_1^M + e_1^Y. \qquad (16.3.vi)$$

Multiplying both sides by the denominators on the left hand side, we get

$$pe_1^M + e_2^M + pe_1^Y + e_2^Y = \left(e_1^M + e_1^Y\right)\left(p + p^{1/(\rho+1)}\right) \qquad (16.3.vii)$$

and, rearranging terms,

$$p\left(e_1^M + e_1^Y\right) + \left(e_2^M + e_2^Y\right) = p\left(e_1^M + e_1^Y\right) + p^{1/(\rho+1)}\left(e_1^M + e_1^Y\right). \qquad (16.3.viii)$$

Subtracting out the first term on each side and then solving for p, we get

$$p^* = \left(\frac{e_2^M + e_2^Y}{e_1^M + e_1^Y}\right)^{(\rho+1)}. \qquad (16.3.ix)$$

When $e_1^M = e_2^M = e_1^Y = e_2^Y$, this simplifies to $p^* = 1$ — consistent with what we did in part A.

(e) *Derive the set of pareto efficient allocations assuming $e_1^M = e_2^M = e_1^Y = e_2^Y$. Can you see why, regardless of how we might redistribute endowments, the equilibrium price will always be $p = 1$?*

Answer: Let $e = e_1^M = e_2^M = e_1^Y = e_2^Y$. Then the economy is endowed with $2e$ of each good, which implies that, for any allocation (x_1^M, x_2^M) that I get, what's left over for you is $(2e - x_1^M), (2e - x_2^M)$. The pareto efficient set of (x_1^M, x_2^M) (with its implied consumption levels for you) is then defined as the set where our MRS's are equal to one another. The MRS for me at a bundle (x_1^M, x_2^M) is

$$MRS^M = -\frac{\partial u(x_1^M, x_2^M)/\partial x_1}{\partial u(x_1^M, x_2^M)/\partial x_2} = -\left(\frac{x_2^M}{x_1^M}\right)^{(\rho+1)} \quad (16.3.x)$$

and the MRS for you at the left-over bundle $((2e - x_1^M), (2e - x_2^M))$ is

$$MRS^Y = -\frac{\partial u((2e - x_1^M), (2e - x_2^M))/\partial x_1}{\partial u((2e - x_1^M), (2e - x_2^M))/\partial x_2} = -\left(\frac{2e - x_2^M}{2e - x_1^M}\right)^{(\rho+1)}. \quad (16.3.xi)$$

Setting MRS^M equal to MRS^Y and solving for x_2^M, we get

$$x_2^M = x_1^M; \quad (16.3.xii)$$

i.e. the contract curve is a straight line with slope 1 and intercept 0 — the 45-degree line in the Edgeworth Box. Since all efficient allocations happen on this line, and since equilibria are efficient, we know that any competitive equilibrium lies on the 45-degree line. This further implies that, when we plug $x_1^M = x_2^M$ and $2e - x_1^M = 2e - x_2^M$ into the equations for marginal rates of substitution, we get $MRS^M = -1 = MRS^Y$ in any equilibrium, which can only hold if the slope of the budget is -1. And that can only be true if $p = 1$.

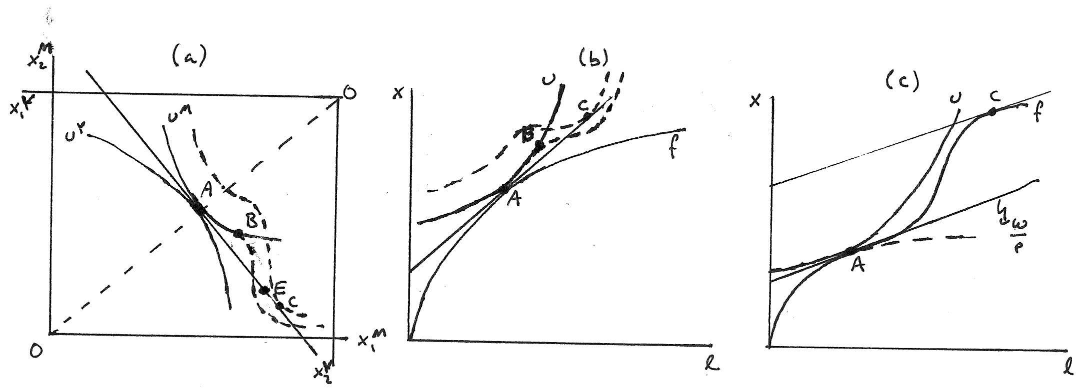
Exercise 16.5

In this exercise we explore some technical aspects of general equilibrium theory in exchange economies and Robinson Crusoe economies. Unlike in other problems, parts A and B are applicable to both those focused on A-Section material and those focused on B-Section material. Although the insights are developed in simple examples, they apply more generally in much more complex models.

A: The role of convexity in Exchange Economies: *In each part below, suppose you and I are the only individuals in the economy, and pick some arbitrary allocation E in the Edgeworth Box as our initial endowment. Assume throughout that your tastes are convex and that the contract curve is equal to the line connecting the lower left and upper right corners of the box.*

(a) *Begin with a depiction of an equilibrium. Can you introduce a non-convexity into my tastes such that the equilibrium disappears (despite the fact that the contract curve remains unchanged?)*

Answer: This is done in panel (a) of Exercise Graph 16.5 where the equilibrium budget passes through E and is tangent to both solid (and convex) indifference curves at A. Thus, A is an equilibrium allocation. However, if I permit my indifference curves to have non-convexities, I can maintain the tangency at A but lose the equilibrium at A by having my indifference curve continue along the dashed curve beginning at B and moving right. Notice that A is still efficient — but, when faced with the budget line that previously supported A as an equilibrium, I now no longer optimize at A but rather at C which lies on a higher dashed (and non-convex) indifference curve.



Exercise Graph 16.5 : Convexity Assumptions in General Equilibrium

(b) True or False*: Existence of a competitive equilibrium in an exchange economy cannot be guaranteed if tastes are allowed to be non-convex.*

Answer: This is true, as we have just shown.

(c) *Suppose an equilibrium does exist even though my tastes exhibit some non-convexity.* True or False*: The first welfare theorem holds even when tastes have non-convexities.*

Answer: The allocation A in panel (a) of Exercise Graph 16.5 would continue to be an equilibrium so long as the non-convexity that is introduced is not sufficiently pronounced so as to cause the indifference curve that is tangent at A to cross the budget line. Thus, had we drawn the non-convexity in a less pronounced manner, the budget line through A and E would still have been such that I optimize at A — and thus A would have continued to be an equilibrium. We can conclude that, *if an equilibrium exists in the presence of non-convex tastes,* then it will indeed still be efficient. The first welfare theorem therefore holds in the presence of non-convexities.

(d) True or False*: The second welfare theorem holds even when tastes have non-convexities.*

Answer: The second welfare theorem says that any efficient allocation can be an equilibrium allocation so long as endowments can be appropriately redistributed. We have just shown in panel (a) of Exercise Graph 16.5 an example of an efficient allocation A that cannot be supported as an equilibrium no matter where we move the endowment. This is because, in order to support A as an equilibrium, the budget line *has to be* the line that is draw in the graph — because that is the only budget that will cause *you* to optimize at A. But that line crosses the dashed extension of my indifference curve that is tangent at A — implying that I will not optimize at A if my tastes are the non-convex kind in the graph. Thus, we have identified a case where an efficient allocation cannot become an equilibrium allocation regardless of where we put the endowment. The statement is therefore false — the second welfare theorem may not hold when tastes have non-convexities.

B: The role of convexity in Robinson Crusoe Economies: *Consider a Robinson Crusoe economy. Suppose throughout that there is a tangency between the worker's indifference curve and the production technology at some bundle A.*

(a) *Suppose first that the production technology gives rise to a convex production choice set. Illustrate an equilibrium when tastes are convex. Then show that A may no longer be an equilibrium if you allow tastes to have non-convexities even if the indifference curve is still tangent to the production choice set at A.*

Answer: This is illustrated in panel (b) of Exercise Graph 16.5. The solid indifference curve is tangent to the convex production choice set at A, with both tangent to the isoprofit/budget line (that has slope w/p). When viewed as a budget line, the worker is doing the best he can by choosing A, and when viewed as an isoprofit line, the firm is doing the best it can at A, with the wage/price ration w/p supporting A as an equilibrium. But we can take the same indifference curve, keep it tangent to the budget at A, but then change its shape from B on to take the shape of the dashed curve. When we do this, we introduce a non-convexity — and, as a result, the worker is no longer doing the best he can by choosing A when confronting the budget formed by the former equilibrium wage/price ratio. In particular, the worker would now be better off optimizing at C — but that lies outside the production frontier and is therefore not an equilibrium. Thus, by introducing the non-convexity, A ceases to be a competitive equilibrium in this economy.

(b) *Next, suppose again that tastes are convex but now let the production choice set have non-convexities. Show again that A might no longer be an equilibrium (even though the indifference curve and production choice set are tangent at A).*

Answer: This is shown in panel (c) of Exercise Graph 16.5 where the production frontier f is tangent to the indifference curve u — thus making

A an efficient production plan. The budget that is tangent to both the production frontier and the indifference curve at A — with slope w/p — causes the worker to optimize at A where his indifference curve is tangent. However, the firm would not be optimizing at A — because it can reach a higher isoprofit curve and would maximize profit at C instead. The production plan A would be optimal for the firm (and would thus be an equilibrium) if the production frontier took on the dashed shape following A — i.e. if the production choice set were convex. But A is lost as an equilibrium because of the non-convexity of the solid production choice set.

(c) True or False*: A competitive equilibrium may not exist in a Robinson Crusoe economy that has non-convexities in either tastes or production.*

Answer: This is true as shown in the previous two parts.

(d) True or False*: The first welfare theorem holds even if there are non-convexities in tastes and/or production technologies.*

Answer: This is true. The non-convexities may cause there to be no equilibrium, but *if there is an equilibrium*, it will again have the feature that the indifference curve is tangent to the production frontier at that point — which will make it efficient. You can see this in panels (b) and (c) if you imagine the non-convexity that was introduced as being less pronounced. In panel (b), A would remain an equilibrium so long as the dashed portion of the indifference curve does not cross the budget line to the right of B — which is certainly possible even if there were a less pronounced non-convexity. And that equilibrium would be efficient. Similarly, in panel (c) you can imagine a non-convexity in the production choice set either to the left of A or some distance to the right of A — and you can imagine such a non-convexity to not be sufficiently pronounced so as to cross the isoprofit line that is tangent at A. In that case, A would remain as an equilibrium — and it would be efficient. Thus, the first welfare theorem holds — every equilibrium (that exists) is indeed efficient.

(e) True or False*: The second welfare theorem holds regardless of whether there are non-convexities in tastes or production.*

Answer: This is false. In panel (b) of the graph, we have shown an efficient point A that cannot be an equilibrium because the budget line that must support it crosses the indifference curve that is tangent at A. In panel (c) we have shown another efficient point A that cannot be supported as an equilibrium because the isoprofit line that is needed to support it as an equilibrium crosses the production frontier because of a non-convexity. We have therefore shown that, when there are non-convexities, there may be efficient outcomes that cannot be supported as equilibria.

(f) *Based on what you have done in parts A and B, evaluate the following: "Non-convexities may cause a non-existence of competitive equilibria in general equilibrium economies, but if an equilibrium exists, it results in an efficient allocation of resources. However, only in the absence of non-convexities can we conclude that there always exists some lump-sum re-*

distribution such that any efficient allocation can also be an equilibrium allocation." (Note: Your conclusion on this holds well beyond the examples in this problem — for reasons that are quite similar to the intuition developed here.)

Answer: The statement is fully consistent with everything we have done in this exercise. We have shown — in both exchange and Robinson Crusoe economies — that non-convexities may lead to a non-existence of equilibria; that *if* equilibria exist, they will be efficient (i.e. the first welfare theorem holds); but not all efficient outcomes can be supported as equilibria (i.e. the second welfare theorem fails in the presence of non-convexities).

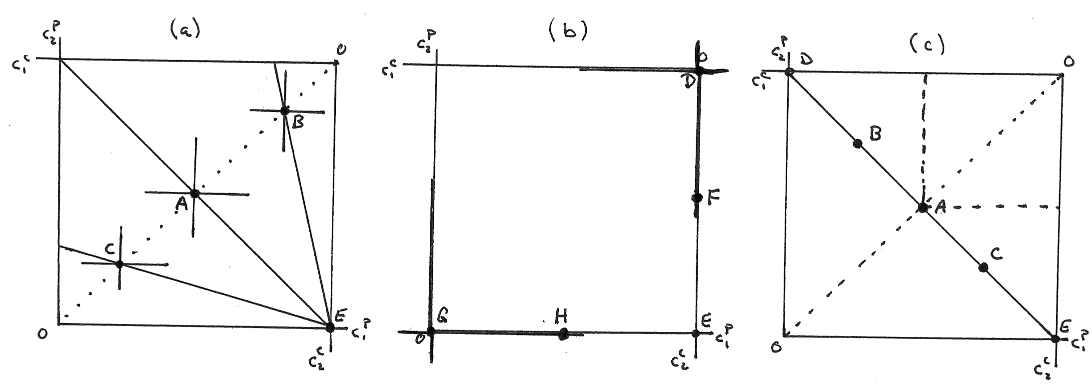
Exercise 16.7

Everyday Application: *Parents, Children and the Degree of Substitutability across Time: Consider again exactly the same scenario as in exercise 16.6.*

A: *This time, however, suppose that parent and child tastes treat consumption now and consumption in the future as perfect complements.*

(a) *Illustrate in an Edgeworth Box an equilibrium with a single parent and a single child.*

Answer: Perhaps the most obvious equilibrium is the one with price equal to 1 and thus a budget line that runs from E in the lower right corner to the upper left corner of the box — with the equilibrium allocation at A, pictured in panel (a) of Exercise Graph 16.7.



Exercise Graph 16.7 : Parents and Children: Part 1

(b) *Is the equilibrium you pictured in (a) the only equilibrium? If not, can you identify the set of all equilibrium allocations?*

Answer: It is not the only equilibrium — in fact, panel (a) of Exercise Graph 16.7 picture two others, with allocations at B and at C. Because of the sharp corners on indifference curves for perfect complements, any budget line with negative slope can be fit to any "tangency" of the two indifference curves on the 45 degree line. Thus, all allocations on the 45 degree line have some budget line that passes through the endowment allocation *E and* is "tangent" to both indifference curves on that point of the 45 degree line. The entire 45-degree line in the box is therefore the set of possible equilibrium allocations.

(c) *Now suppose that there were two children and one parent. Keep the Edgeworth Box with the same dimensions but model this by recognizing that, on any equilibrium budget line, it must now be the case that the parent moves twice as far from the endowment E as the child (since there are two children and thus any equilibrium action by a child must be half the equilibrium action by the parent). Are any of the equilibrium allocations for parent and child that you identified in (b) still equilibrium allocations? (*Hint*: Consider the corners of the box.)*

Answer: For any budget line that intersects the 45-degree line inside the box, both parent and child will optimize on the 45 degree line. But with two children and one parent, that cannot be an equilibrium — because the parent's action must be twice the children's in the opposite direction in order for demand to equal supply. Thus, none of the efficient allocations on the 45 degree line inside the box can be an equilibrium allocation. However, suppose that $p = \infty$. Then the budget line becomes vertical and passes through E. The parent will optimize at the top corner (point D in panel (b) of Exercise Graph 16.7), and the children don't care where on the budget they optimize because all the bundles on that budget lie on the same indifference curve. Thus, it is not inconsistent with optimization to assume that the children will choose F — halfway up the budget and halfway to D where the parent optimizes. Thus, children consume nothing now and give half of what they earn in the future to the parent, and parents consume everything now and half of everything (i.e. half of what each of the two children earns) in the future.

(d) *Suppose instead that there are two parents and one child. How does your answer change?*

Answer: No equilibrium allocation can lie on the 45 degree line for the same reason as in the previous case — and now we end up with the child optimizing at G in panel (b) of Exercise Graph 16.7 and the two parents optimizing at H, with $p = 0$. Thus, parents consume half their income now and nothing in the future, while children consume half of each parents' income now and everything in the future.

(e) *Repeat (a) through (d) for the case where consumption now and consumption in the future are perfect substitutes for both parent and child.*

Answer: When consumption across time is perfectly substitutable, the indifference curves have slope -1 at every allocation in the Edgeworth Box. Thus any equilibrium allocation inside the box must lie on the line connecting the upper left to the lower right corners of the box — the line pictured in panel (c) of Exercise Graph 16.7. Neither parent nor child cares where on that line they consume — and thus any split of the economy's endowment that falls on this line will be an equilibrium allocation with $p = 1$. For instance, when there is one child and one parent, A is a possible equilibrium allocation, as is C and B. When there are two children and 1 parent, any allocation that has the parent's bundle twice as far from E as the children's works — for instance A for the parent and C for the children. When there are two parents and one child, then any allocation that has the child twice as far as the parents from E works. In all cases, the equilibrium price continues to be $p = 1$ — because it makes no sense for individuals to trade on other terms when consumption now is the same as consumption in the future.

(f) *Repeat for the case where consumption now and consumption in the future are perfect complements for parents and perfect substitutes for children.*

Answer: Consider first the case of one parent and one child. For any budget with positive slope (not equal to infinity), the parent will optimize on the 45-degree line. For any price not equal to 1, the child will choose a corner solution (since consumption now and in the future are the same for her). Thus, the only way the child will trade to permit the parent to get to the 45 degree line is if $p = 1$ and the budget line takes the shape graphed in panel (c) of Exercise Graph 16.7. The equilibrium allocation is then A — where the parent's indifference curve is drawn as a dotted L-shape. Next, suppose there are two children. Nothing has changed in terms of the children's willingness to trade to an interior solution only at $p = 1$ and in terms of the parent's optimal bundle falling on the 45 degree line for any positive price. Thus, p will remain 1, the parent will optimize at A and the children will each optimize at C — halfway between A and E. Finally, suppose there are two parents and one child. Again, for the same reasons as before, price has to remain 1, and the parents' optimization has to lead to A. Thus, parents end up at A and the child ends up at the top left corner D — twice as far from E as the two parents.

(g) True or False*: The more consumption is complementary for the parent relative to the child, and the more children there are per parent, the more gains from trade will accrue to the parent.*

Answer: This is roughly true, as illustrated in the previous parts of the question. For instance, when parent viewed consumption as perfectly complementary across time while children viewed it as substitutable (in panel (c) of Exercise Graph 16.7), the children gain no utility from trading while the parent(s) get all gains from trade. Similarly, we saw in this and the previous exercise that more gains typically accrue to the party that is in control of the goods that are scarcer. Parents are in control of consumption now — which is relatively more scarce the more children there are per parent.

B: *Suppose that parent and child tastes can be represented by the CES utility function* $u(c_1, c_2) = \left(0.5c_1^{-\rho} + 0.5c_2^{-\rho}\right)^{-1/\rho}$. *Assume that the income earned by parents in period 1 and by children in period 2 is 100.*

(a) *Letting p denote the price of consumption now with price of future consumption normalized to 1, derive parent and child demands for current and future consumption as a function of ρ and p.*

Answer: We want to maximize utility (which is the same for parents and children) subject to the budget constraint — which is $100p = pc_1 + c_2$ for parents (who are endowed with 100 now) and $100 = pc_1 + c_2$ for children (who are endowed with 100 in the future). Solving this in the usual way, we get

$$c_1^P = \frac{100p^{1/(\rho+1)}}{p^{\rho/(\rho+1)}+1} \text{ and } c_2^P = \frac{100p}{p^{\rho/(\rho+1)}+1} \text{ for parents, and} \tag{16.7.i}$$

$$c_1^C = \frac{100}{p+p^{1/(\rho+1)}} \text{ and } c_2^C = \frac{100}{p^{\rho/(\rho+1)}+1} \text{ for children.} \tag{16.7.ii}$$

(b) *What is the equilibrium price — and what does this imply for equilibrium allocations of consumption between parent and child across time. Does any of your answer depend on the elasticity of substitution?*

Answer: This solves slightly more easily if we set demand and supply in the c_2 market equal to one another (rather than setting it equal to one another in the c_1 market. Of course the latter would give the same answer even if it is slightly more burdensome to get there.) Thus, we need to solve

$$\frac{100p}{p^{\rho/(\rho+1)}+1} + \frac{100}{p^{\rho/(\rho+1)}+1} = 100. \tag{16.7.iii}$$

Dividing by 100, multiplying by the denominator on the left hand side, and simplifying, we get

$$p = p^{\rho/(\rho+1)} \text{ or } 1 = \rho^{-1/(\rho+1)} \tag{16.7.iv}$$

which solves to $p = 1$. The answer therefore does not depend on ρ and thus is independent of the elasticity of substitution. (This is because the indifference curves for the utility function always have $MRS = -1$ along the 45 degree line no matter what elasticity of substitution is assumed.)

(c) *Next, suppose there are 2 children and only 1 parent. How does your answer change?*

Answer: We now have to sum twice the child demands with the parent demand for c_2 and set it equal to overall consumption in the future — which is 200 when there are two children. This implies we need to solve

$$\frac{100p}{p^{\rho/(\rho+1)}+1} + 2\left(\frac{100}{p^{\rho/(\rho+1)}+1}\right) = 200 \tag{16.7.v}$$

which solves to

$$p = 2^{\rho+1}. \tag{16.7.vi}$$

The equilibrium price now depends on ρ and thus on the elasticity of substitution. As ρ increases — which implies the elasticity of substitution falls — price increases. In the limit, as ρ approaches infinity — and consumption becomes perfectly complementary across time — price rises to infinity. This is exactly what we concluded in part A for perfect complements. As ρ falls to -1 — and consumption becomes perfectly substitutable across time, on the other hand, price becomes 1 — again exactly as we concluded in part A.

(d) *Next, suppose there are 2 parents and only 1 child. How does your answer change?*

Answer: We now have to sum twice the parent demands with the child demand for c_2 and set it equal to overall consumption in the future — which is 100 when there is only one child. This implies we need to solve

$$2\left(\frac{100p}{p^{\rho/(\rho+1)}+1}\right)+\frac{100}{p^{\rho/(\rho+1)}+1}=100 \tag{16.7.vii}$$

which solves to

$$p=\left(\frac{1}{2}\right)^{\rho+1}. \tag{16.7.viii}$$

The equilibrium price again depends on ρ and thus on the elasticity of substitution. As ρ increases — which implies the elasticity of substitution falls — price falls. In the limit, as ρ approaches infinity — and consumption becomes perfectly complementary across time — price falls to zero. This is exactly what we concluded in part A for perfect complements. As ρ falls to -1 — and consumption becomes perfectly substitutable across time, on the other hand, price becomes 1 — again exactly as we concluded in part A.

(e) *Explain how your answers relate to the graphs you drew for the extreme cases of both parent and child preferences treating consumption as perfect complements over time.*

Answer: We already did this. We showed that, as tastes become perfectly complementary, then p approaches infinity if there are two children and one parent and to zero if there are two parents and one child. We illustrated precisely this extreme case in panel (b) of Exercise Graph 16.7.

(f) *Explain how your answers relate to your graphs for the case where consumption was perfectly substitutable across time for both parents and children.*

Answer: Again, we already did this. We showed that, when consumption is perfectly substitutable across time, then price will be 1 regardless of the number of children relative to parents.

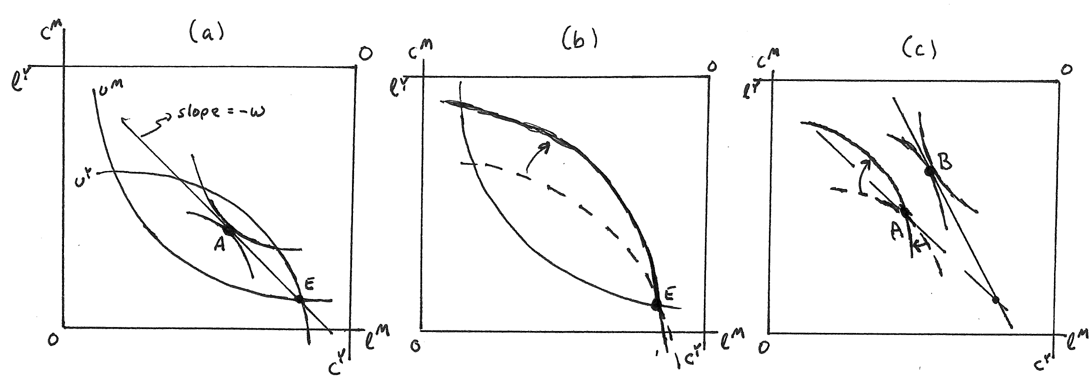
Exercise 16.9

Business Application: *Hiring an Assistant: Suppose you are a busy CEO — with lots of consumption but relatively little leisure. I, on the other hand, have only a part-time job and therefore lots of leisure with relatively little consumption.*

A: *You decide that the time has come to hire a personal assistant — someone who can do some of the basics in your life so that you can have a bit more leisure time.*

(a) *Illustrate our current situation in an Edgeworth Box with leisure on the horizontal and consumption on the vertical axis. Indicate an endowment bundle that fits the description of the problem and use indifference curves to illustrate a region in the graph where both of us would benefit from me working for you as an assistant.*

Answer: This is illustrated in panel (a) of Exercise Graph 16.9 where the endowment allocation E has me (on the lower axes) with lots of leisure but little consumption and you (on the upper axes) with the reverse. The mutually beneficial region is formed by the lens made from our indifference curves that pass through E. Both of us would prefer any allocation in that lens shape to the endowment bundle E.



Exercise Graph 16.9 : Cheerfulness in Office Assistants

(b) *Next, illustrate what an equilibrium would look like. Where in the graph would you see the wage that I am being paid?*

Answer: This is also illustrated in panel (a) of Exercise Graph 16.9 where the budget line that passes through A and E has slope $-w$ (where w is the wage when the price of consumption is normalized to 1).

(c) *Suppose that anyone can do the tasks you are asking of your assistant — but some will do it cheerfully and others will do it with attitude. You hate attitude — and therefore would prefer someone who is cheerful. Assuming* *you can read the level of cheerfulness in me, what changes in the Edgeworth box as your impression of me changes?*

Answer: As you think I am more cheerful, you will be willing to trade more of your consumption for an increase in your leisure. Thus, your indifference curves become steeper.

(d) *How do your impressions of me — i.e. how cheerful I am — affect the region of mutually beneficial trades?*

Answer: This is illustrated in panel (b) of Exercise Graph 16.9 where your original indifference curve through E is illustrated as a dashed indifference curve and your new indifference curve (that contains E) as my cheerfulness increases is illustrated as a bold curve. This increases the lens formed by our indifference curves through E — and thus the mutually beneficial region.

(e) *How does increased cheerfulness on my part change the equilibrium wage?*

Answer: This is illustrated in panel (c) of Exercise Graph 16.9 where A is the original equilibrium at low levels of cheerfulness and B is the new equilibrium at higher levels of cheerfulness. As my cheerfulness increases, your indifference curve through A becomes steeper — rotating from the dashed curve to the solid one. Thus, A can't be an equilibrium anymore because you now want more of me but I am not willing to offer any more at the original wage. Thus, the wage must increase in order to get me to offer more of myself and you to reduce your demand for me. This leads us to the steeper budget through B — with a higher wage. Cheerfulness is rewarded in the competitive market.

(f) *Your graph probably has the new equilibrium (with increased cheerfulness) occurring at an indifference curve for you that lies below (relative to your axes) the previous equilibrium (where I was less cheerful). Does this mean that you are worse off as a result of me becoming more cheerful?*

Answer: No, it does not. It is indeed true that your indifference curve through B in panel (c) of Exercise Graph 16.9 lies below A (relative to your axes). But this does not mean you are less happy — because my cheerfulness is what made your indifference curves get steeper. In terms of some of the earlier problems in our development of consumer theory, cheerfulness is a third good you care about — and as it changes in the problem, you switch to a different "slice" of your 3-dimensional indifference surfaces. Increased cheerfulness switches you to a slice where you are happier for any level of consumption and leisure than you were before — and so an indifference curve with more cheerfulness can lie below one with less cheerfulness and still be preferred.

B: *Suppose that my tastes can be represented by $u(c,\ell) = 200\ln\ell + c$ while yours can be represented by $u(c,\ell,x) = 100x\ln\ell + c$ where ℓ stands for leisure, c stands for consumption and x stands for cheerfulness of your assistant. Suppose that, in the absence of working for you, I have 50 leisure hours and 10 units of consumption while you have 10 leisure hours and 100 units of consumption.*

(a) *Normalize the price of c as 1. Derive our leisure demands as a function of the wage w.*

Answer: My budget constraint is $w\ell + c = 50w + 10$ while yours is $w\ell + c = 10w + 100$. Maximizing our utilities subject to these constraints, we get (by solving this in the usual way)

$$\ell^M = \frac{200}{w} \text{ for me and } \ell^Y = \frac{100x}{w} \text{ for you.} \tag{16.9.i}$$

(b) *Calculate the equilibrium wage as a function of x.*

Answer: The sum of our leisure demands has to be equal to the leisure supply of 60 in equilibrium — i.e.

$$\frac{200}{w} + \frac{100x}{w} = 60 \tag{16.9.ii}$$

which implies that the equilibrium wage is

$$w^* = \frac{10+5x}{3}. \tag{16.9.iii}$$

(c) *Suppose $x = 1$. What is the equilibrium wage, and how much will I be working for you?*

Answer: Substituting $x = 1$ into our equation for w^*, we get an equilibrium wage of 5. Plugging this wage into our leisure demand equations, we get that you will have 20 hours of leisure and I will have 40 — which is 10 less for me and 10 more for you than what we were endowed with. Thus, I'll be working for you for 10 hours.

(d) *How does your MRS change as my cheerfulness x increases?*

Answer: Your MRS is

$$MRS^Y = -\frac{\partial u(c,\ell,x)/\partial \ell}{\partial u(c,\ell,x)/\partial x} = -\frac{100x}{\ell}. \tag{16.9.iv}$$

Thus, for any bundle (ℓ, c), the MRS gets larger in absolute value as x increases — i.e your indifference curves become steeper as my cheerfulness increases.

(e) *What happens to the equilibrium wage as x increases to 1.2? What happens to the equilibrium number of hours I work for you? What if I get grumpy and x falls to 0.4?*

Answer: When x goes to 1.2, the equilibrium wage rises to 5.33 and the number of hours I work for you increases to 12.5. When x falls to 0.4, the equilibrium wage falls to 4 but you no longer hire me and we simply consume at our endowment bundles.

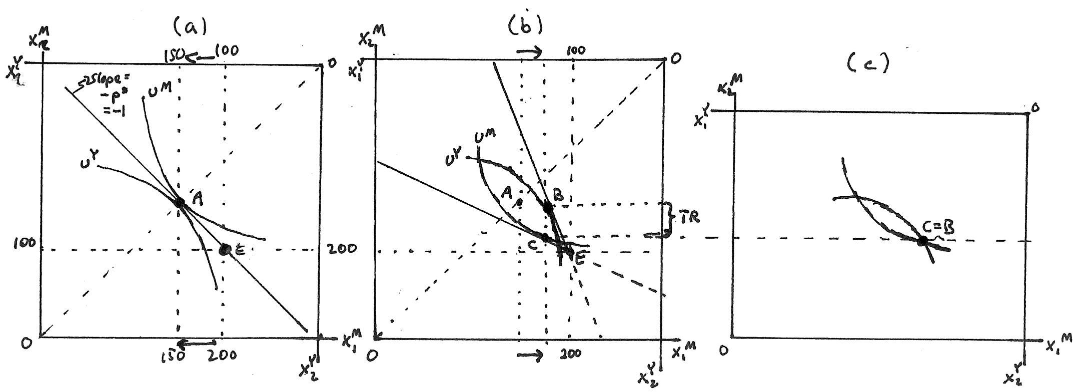
Exercise 16.11

Policy Application: *Distortionary Taxes in General Equilibrium: Consider, as in exercise 16.10, a 2-person exchange economy in which I own 200 units of x_1 and 100 units of x_2 while you own 100 units of x_1 and 200 units of x_2.*

A: *Suppose you and I have identical homothetic tastes.*

(a) *Draw the Edgeworth Box for this economy and indicate the endowment allocation E.*

Answer: This is illustrated in panel (a) of Exercise Graph 16.11 where the box takes on the shape of a square since the economy's endowment of both goods is 300.



Exercise Graph 16.11 : Distortionary Taxes

(b) *Normalize the price of good x_2 to 1. Illustrate the equilibrium price p^* for x_1 and the equilibrium allocation of goods in the absence of any taxes. Who buys and who sells x_1?*

Answer: This is also done in panel (a) of Exercise Graph 16.11 where A is the equilibrium allocation (which appears in the center of the box on the 45 degree line because of our identical homothetic tastes and endowments.) Thus, I sell 50 units of x_1 to you for the price of $p^* = 1$.

(c) *Suppose the government introduces a tax t levied on all transactions of x_1 (and paid in terms of x_2). For instance, if one unit of x_1 is sold from me to you at price p, I will only get to keep $(p-t)$. Explain how this creates a kink in our budget constraints.*

Answer: This implies that the price p paid by the buyer is greater than the price $(p-t)$ received by the seller. On my budget constraint, I am a seller to the left of E and a buyer to the right of E — implying that my budget has shallower slope $-(p-t)$ to the left of E and steeper slope $-p$ to the right of E, with a kink at E. The same is true for you — except that "right" and "left" are reversed when we flip your axes to create the Edgeworth Box. The portions along which I am a seller and you are a buyer of x_2 are illustrated as the solid lines in panel (b) of Exercise Graph 16.11, with the remaining portion of the constraints dashed to the right of E.

(d) *Suppose a post-tax equilibrium exists and that price increases for buyers and falls for sellers. In such an equilibrium, I will still be selling some quantity of x_1 to you. (Can you explain why?) How do the relevant portions of the budget constraints you and I face look in this new equilibrium, and where will we optimize?*

Answer: This is illustrated in panel (b) of Exercise Graph 16.11 where the steeper (solid) constraint is yours (with the higher post-tax price) and the shallower one is mine (with the lower pre-tax price). In equilibrium, it will still have to be the case that the amount of x_1 I sell to you is equal to the amount of x_1 you buy. Thus, in equilibrium, our two budgets have to be such that your optimum B lies right above my optimum C in the Edgeworth Box. We know that this will be to the right of the original equilibrium A — because your budget is steeper than before and mine is shallower than before. The fact that it is shallower for me means that I will be optimizing on a shallower ray from the origin (given that my tastes are homothetic), and the fact that it is steeper for you implies you will be optimizing on a steeper ray from your origin. Thus, the amount we trade will fall by the amount of the arrows in the graph. (The reason we know that I will still be selling (or at least not buying) x_1 under the tax is as follows: My budget under the tax has a kink at E — and becomes steeper to the right of E. Given that my tastes are homothetic, it cannot be that I optimize on that steeper portion — because the steeper parts of my indifference curves lie to the left of E).

(e) *When we discussed price changes with homothetic tastes in our development of consumer theory, we noted that there are often competing income (or wealth) and substitution effects. Are there such competing effects here relative to our consumption of x_1? If so, can we be sure that the quantity we trade* in equilibrium *will be less when t is introduced?*

Answer: Both of us experience a negative wealth effect — me because what I am selling has fallen in price, you because what you are buying has increased in price. Thus, the wealth effect says "consume less of x_1" for both of us. But the substitution effects operate in opposite directions for the two of us. For me, the price of x_1 falls as a result of the tax — which means the substitution effect will tell me to consume *more* of x_1. For you, on the other hand, the price of x_1 has increased — with the substitution effect therefore telling you to consume *less* of x_1. The wealth and substitution effects therefore point in opposite directions for me but not for you. This implies you will consume less x_1 under the tax, which means *in equilibrium* the prices have to adjust such that I will sell you less (and therefore consume more) even though the wealth effect tells me to consume less. (This implies that the equilibrium that we assume exists (with price increasing for buyers and falling for sellers) requires that the goods are sufficiently substitutable to create the necessary substitution effect.)

(f) *You should see that, in the new equilibrium, a portion of x_2 remains not allocated to anyone. This is the amount that is paid in taxes to the government. Draw a new Edgeworth box that is adjusted on the x_2 axes to reflect the fact that some portion of x_2 is no longer allocated between the two of us. Then locate the equilibrium allocation point that you derived in your previous graph. Why is this point not efficient?*

Answer: The portion of x_2 that remains not allocated in our tax-equilibrium in panel (b) of the graph is the vertical difference between B and C — labeled TR in the graph. Thus, the amount that gets allocated is TR less of x_2 than what is available — because the difference is collected by the government. If we shrink the Edgeworth Box by that vertical amount, we get the box illustrated in panel (c) of Exercise Graph 16.11. By shrinking the height of the box, we move B on top of C and now see even more clearly than in panel (b) that this allocation is not efficient. The reason it is inefficient is that both you and I would prefer to divide everything that was not taken by the government differently — with all the allocations in the lens shape between our indifference curves through $B = C$ all preferred by both of us. We could thus make everyone better off by moving the allocation into that lens shape without taking any of the tax revenue the government has raised back.

(g) True or False*: The deadweight loss from the distortionary tax on trades in x_1 results from the fact that our marginal rates of substitution are no longer equal to one another after the tax is imposed and* not *because the government raised revenues and thus lowered the amounts of x_2 consumed by us.*

Answer: This is true. The inefficiency we show in panel (c) arises from the fact that there is a lens shape between our indifference curves — and that lens shape arises from the fact that our marginal rates of substitution are not equal to one another (which is due to the fact that the prices we face as buyers and sellers is different when the government uses price-distorting taxes). The fact that the box has shrunk is not evidence of an inefficiency — because the government now has the difference and may well be doing some very useful things with the money. The problem is that what remains is not allocated efficiently due to distorted prices.

(h) True or False*: While the post-tax equilibrium is not efficient, it does lie in the region of mutually beneficial trades.*

Answer: This is true. In panel (b), the indifference curves through B and C still lie above E for both of us — i.e. trade is still making us better off than we would be without trade, just worse off than we would be if we could trade without price distortions. (Even if it is not obvious from the graph that our indifference curves through B and C lie above E, it should intuitively make sense that this has to be the case: After all, even in the presence of the distortionary tax, no one is forcing us to trade with one another — and we would not do so if trade made us worse off than we would be if we simply consumed our endowments.)

(i) *How would taxes that redistribute endowments (as envisioned by the Second Welfare Theorem) be different than the price distorting tax analyzed in this problem?*

Answer: Redistributions of endowments would involve lump sum taxes and subsidies that do not distort prices — because they would simply shift E around in the box. From the new E, markets could act as before — finding the competitive equilibrium price and causing the individuals to optimize where their indifference curves are tangent to one another and the resulting allocation is therefore efficient.

B: *Suppose our tastes can be represented by the utility function* $u(x_1, x_2) = x_1 x_2$. *Let our endowments be specified as at the beginning of the problem.*

(a) *Derive our demand functions for x_1 and x_2 (as functions of p — the price of x_1 when the price of x_2 is normalized to 1).*

Answer: My budget constraint is $px_1+x_2 = 200p+100$ while yours is $px_1+x_2 = 100p+200$. Solving our utility maximization problems subject to these constraints in the usual way, we get

$$x_1^M = \frac{100p+50}{p} \text{ and } x_2^M = 100p+50 \text{ for me, and} \tag{16.11.i}$$

$$x_1^Y = \frac{50p+100}{p} \text{ and } x_2^Y = 50p+100 \text{ for you.} \tag{16.11.ii}$$

(b) *Derive the equilibrium price p^* and the equilibrium allocation of goods.*

Answer: To derive the equilibrium price, we can sum the demands for x_1 and set them equal to 300 — the amount of x_1 that the economy is endowed with. Solving for p, we get $p^* = 1$. Substituting back into the demand equations, we get $x_1^M = x_2^M = x_1^Y = x_2^Y = 150$.

(c) *Now suppose the government introduces a tax t as specified in A(c). Given that I am the one that sells and you are the one that buys x_1, how can you now re-write our demand functions to account for t? (*Hint*: There are two ways of doing this — either define p as the pre-tax price and let the relevant price for the buyer be $(p+t)$ or let p be defined as the post-tax price and let the relevant price for the seller be $(p-t)$.)*

Answer: Letting p indicate the price paid by you and $(p-t)$ be equal to the price received by me (as the seller), we can substitute $(p-t)$ into my demand equations to get

$$x_1^M(t) = \frac{100(p-t)+50}{(p-t)} \text{ and } x_2^M(t) = 100(p-t)+50 \tag{16.11.iii}$$

Your demand functions would remain the same as before.

(d) *Derive the new equilibrium pre- and post-tax prices in terms of t. (*Hint*: You should get to a point where you need to solve a quadratic equation using the quadratic formula that gives two answers. Of these two, the larger one is the correct answer for this problem.)*

Answer: We again set demand for x_1 equal to supply to get the equation

$$x_1^M(t)+x_1^Y=\frac{100(p-t)+50}{(p-t)}+\frac{50p+100}{p}=300. \tag{16.11.iv}$$

Multiplying both sides by $(p-t)p$, taking all terms to one side, summing like terms and dividing by 50, we get

$$3p^2-3(t+1)p+2t=0. \tag{16.11.v}$$

Applying the quadratic formula (and accepting the higher of the two solutions), we get

$$p=\frac{3(t+1)+\sqrt{9(t+1)^2-4(3)(2t)}}{6}=\frac{(t+1)+\sqrt{t^2-\frac{2}{3}t+1}}{2} \tag{16.11.vi}$$

which is the post-tax equilibrium price that buyers pay. The pre-tax price that sellers receive is then simply t less; i.e.

$$(p-t)=\frac{(1-t)+\sqrt{t^2-\frac{2}{3}t+1}}{2}. \tag{16.11.vii}$$

(e) *How much of each good do you and I consume if $t=1$?*

Answer: Plugging $t=1$ into our equations for p and $(p-t)$, we get $p\approx 1.5774$ and $(p-t)\approx 0.5774$. Plugging these into our demand equations, we get

$$x_1^M\approx 186.60, x_2^M\approx 107.74, x_1^Y\approx 113.40 \text{ and } x_2^Y\approx 178.87. \tag{16.11.viii}$$

(f) *How much revenue does the government raise if $t=1$?*

Answer: The tax revenue must be the difference between the 300 units of x_2 that were available in the economy and the sum of our consumption levels of x_2; i.e. tax revenue must be $300-(107.74+178.87)=13.39$. We can verify that this is the case by multiplying $t=1$ times the quantity of x_1 that is sold by me to you in equilibrium — i.e. $(1)(200-186.60)=13.40$. (The difference between the two values for tax revenue is rounding error.)

(g) *Show that the equilibrium allocation under the tax is inefficient.*

Answer: To show that the equilibrium allocation is inefficient, all we have to show is that our marginal rates of substitution at the equilibrium consumption bundles are not the same. For the utility function we are using, the MRS is given by

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{x_2}{x_1}. \qquad (16.11.ix)$$

Plugging in our consumption levels from equation (16.11.viii), we get $MRS^M \approx 0.5774$ and $MRS^Y \approx 1.5774$ for you — which are, of course, equal to the negative $(p-t)$ and p values we calculated earlier and that form the slopes of our two equilibrium budget constraints.

Conclusion: Potentially Helpful Reminders

1. Keep in mind that the dimensions of the Edgeworth Box are determined by the *overall* endowment of each of the goods in the economy. A point in the Edgeworth Box has four components — two measured on each of the axes that correspond to the two individuals in the economy.

2. The set of mutually beneficial trades can easily be found by drawing the indifference curves (for the two individuals) that pass through the endowment point in the Edgeworth Box. (This usually gives us a lens-shaped set of mutually beneficial trades.) Within this set, only some of the allocations are efficient — because only some of them have the characteristic that the marginal rates of substitution for the two individuals are equal to one another.

3. A competitive equilibrium in the Edgeworth Box always has the following features: It consists of prices that form a budget line passing through the endowment, with indifference curves for both individuals tangent to this budget at the equilibrium allocation. The allocation is efficient because this tangency implies that the marginal rates of substitution for the two individuals are the same at that allocation — with no further gains from trade possible.

4. A competitive equilibrium in the Robinson Crusoe economy has similar features: It consists of an isoprofit line that also doubles as a worker budget constraint, with this line tangent to both the production frontier and the worker's indifference curve.

5. Keep in mind that we are still assuming that individuals are all price takers — and so we do not have to think about relative bargaining power when we investigate competitive equilibria. This is sometimes hard to keep in mind because the simple economies we are dealing with in this chapter only have two individuals in them — and it is therefore artificial for us to treat them as if they were price takers. (It seems even sillier in the Robinson Crusoe economy where we treat a single individual as if he had a split personality!) But the point here is to illustrate the basic intuitions that continue to hold when the economies get much larger and the assumption becomes natural.

6. The mathematical steps in calculating an equilibrium in a general equilibrium economy follow straightforwardly from the Edgeworth Box (or Robinson Crusoe) pictures — so keep going back to the underlying pictures if you get lost in the math steps.

CHAPTER

17

Choice and Markets in the Presence of Risk

In this chapter, we expand the consumer model to include the presence of risk. In the process, we are able to think about certain types of markets that insure against risk, markets that play important roles in modern life. And we can use the general equilibrium approach developed in Chapter 16 to investigate the market forces that arise when individuals attempt to insure against risk. In many classes, the primary emphasis will be on the material in the first section of the chapter, a section that deals with basic models of risk aversion when gambles are primarily about money. In the second section, we then see how this model is actually a special case of a model in which there are other things about the different "states of the world" that matter, and in the third section we introduce risk into the general equilibrium model.

Chapter Highlights

The main points of the chapter are:

1. When money is all that matters, we can model attitudes over risk in a straightforward way using a consumption/utility relationship whose shape determines the degree of **risk aversion**. Within this context, we can develop concepts like **certainty equivalence** and **risk premium**, both of which are related to the degree of risk aversion.

2. **Actuarily fair** insurance contracts have the feature that the expected value of consumption remains unchanged when the individual buys insurance — implying that the insurance company makes zero profits on average. In the simplest model of risk aversion, any risk averse individual will **fully insure** against risk when faced with a full menu of actuarily fair insurance contracts.

3. When tastes are **state-dependent**, the model becomes richer in that it allows for risk averse individuals to rationally choose to over- or under-insure.

The **state-independent** model is a special case of the more general state-dependent model.

4. In terms of the underlying math, the concept of a **von-Neumann Morgenstern expected utility function** can typically be used to model consumer tastes. Such a function has the feature that the utility over a risky gamble can be expressed as the probability-weighted average — or expected — utility. This expected utility is less than the utility of the expected value of the gamble whenever individuals are risk averse. (Expected utility theory, however, only holds as long as the *independence axiom* (developed in an appendix of the chapter) holds — and there exist well-known anomalies in consumer behavior that are not consistent with this axiom.)

5. In a general equilibrium setting, actuarily fair insurance contracts often cannot arise. The presence of **aggregate risk** in an economy, for instance, implies that there are not enough individuals willing to sell "recession insurance" on terms that would be actuarily fair. As a result, risk averse individuals in such an economy will not fully insure because of the equilibrium pricing of such insurance.

17A Solutions to Within-Chapter-Exercises for Part A

Exercise 17A.1

If the relationship depicted in Graph 17.1a were a single input production function, would it have increasing, decreasing or constant returns to scale?

Answer: It would have decreasing returns to scale — doubling inputs results in less than doubling of the "output".

Exercise 17A.2

Verify that my wife's expected household consumption is \$190,000.

Answer: The expected value of household consumption is

$$0.75(250,000) + 0.25(10,000) = \$190,000. \tag{17A.2}$$

Exercise 17A.3

What is the relationship between increasing, constant and decreasing marginal utility of consumption to risk loving, risk neutral and risk averse tastes?

Answer: Decreasing marginal utility of consumption implies risk averse tastes; constant marginal utility of consumption implies risk neutral tastes; and increasing marginal utility of consumption implies risk loving tastes.

Exercise 17A.4

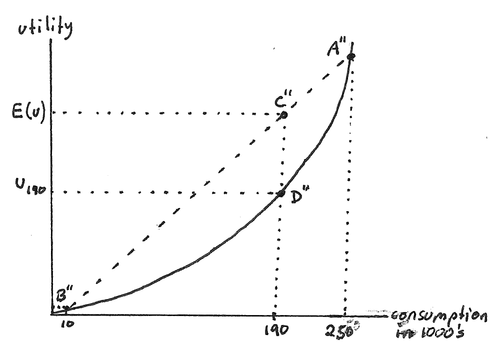
We said above that “a risk averse person’s utility of the expected value of a gamble is always higher than the expected utility of the gamble.” How does this statement change for risk neutral and risk loving tastes?

Answer: A risk neutral person’s utility of the expected value of a gamble is always the same as the expected utility of the gamble; and a risk loving person’s utility of the expected value of a gamble is always less than the expected utility of the gamble.

Exercise 17A.5

Illustrate that, if tastes are as described in panel (c), my wife prefers the “risky gamble” (of getting \$250,000 with probability 0.75 and \$10,000 with probability 0.25) over the “sure thing” (\$190,000 with certainty) that has the same expected value.

Answer: This is illustrated in Exercise Graph 17A.5 where the expected utility of the gamble is read off from point C'' as $E(u)$ and the utility of 190 for sure is read off from point D'' as u_{190}. Since $E(u) > u_{190}$, the expected utility from taking the gamble exceeds the utility from getting the expected value of the gamble for sure — i.e. the individual prefers more risk for the same expected outcome.



Exercise Graph 17A.5 : Risk Loving Tastes

Exercise 17A.6

What is the certainty equivalent and the risk premium for my wife if she had tastes that can be summarized as in panel (b) of Graph 17.2?

Answer: The certainty equivalent would be \$190,000 — because, in this case, she does not care about risk and simply cares about the expected value of the gamble. This implies that the risk premium — the difference between the expected value of the gamble and the certainty equivalent — is zero.

Exercise 17A.7

In panel (c) of Graph 17.2, is the risk premium positive or negative? Can you reconcile this with the fact that the tastes in this graph represent those of a risk lover?

Answer: In this case, the certainty equivalent is greater than the expected value of the gamble — which means that the risk premium is negative. Put differently, in this case the person is willing to pay to play the risky game rather than receive the expected value with certainty.

Exercise 17A.8

True or False: As an individual becomes more risk averse, the certainty equivalent for a risky gamble will fall and the risk premium will rise.

Answer: This is true. More risk averse tastes imply the person is willing to accept less for sure in order to step away from the gamble because she dislikes risk more — which implies the certainty equivalent falls with increasing risk aversion. Since the risk premium is the difference between the expected value of the gamble and the certainty equivalent, this implies that the risk premium increases with the degree of risk aversion.

Exercise 17A.9

Verify that the zero-profit relationship between b and p is as described in the previous sentence.

Answer: If the probability of the "bad" outcome is δ, the insurance company will incur costs that average δb per person. Since it collects premiums from everyone regardless of which outcome he/she faces, the average revenue per person is p. When profits are zero, revenues are equal to costs — i.e. $\delta b = p$ or, dividing both sides by δ, $b = p/\delta$.

Exercise 17A.10

Verify that my wife's expected income is still \$190,000 under this insurance policy.

Answer: Her expected income is

$$0.75(230,000) + 0.25(70,000) = \$190,000. \tag{17A.10}$$

Exercise 17A.11

What are some examples of other actuarily fair insurance contracts that do not provide full insurance? Would each of these also earn zero profit for insurance companies? Can you see why none of them would ever be preferred to full insurance by my wife?

Answer: With $\delta = 0.25$, any insurance contract that satisfies $b = 4p$ is actuarily fair; for instance $(b, p) = (10, 40)$, $(b, p) = (25, 100)$, etc. Each of these would indeed result in zero profit for the insurance company (assuming the insurance companies incur no other costs). But full insurance would be preferred (by someone with risk averse tastes) to any insurance policy that has a benefit level less than the one that insures fully (as well as any policy that has an insurance benefit greater than the full insurance amount.) This is because any actuarily fair insurance policy that does not fully insure has more risk than full insurance but the same expected consumption level.

Exercise 17A.12

Referring back to what you learned in Graph 17.3, what is my wife's consumer surplus if she fully insures in actuarily fair insurance markets?

Answer: Consumer surplus is the amount a consumer is willing to pay to participate in a market as opposed to not participating. We found earlier that my wife's certainty equivalent was equal to \$115,000 — i.e. she is indifferent between getting \$115,000 for sure as opposed to taking her chances without insurance. Getting rid of the risk is therefore worth \$75,000 — her risk premium. In other words, she would have been willing to pay up to \$75,000 more for the full insurance contract that has premium \$60,000 and benefit \$240,000. If she were to pay this maximum premium she is willing to pay, she would be buying a policy with benefit of \$240,000 and premium of \$135,000. This would result in a "good" outcome of $\$250,000 - \$135,000 = \$115,000$ and a "bad" outcome of $\$10,000 + \$240,000 - \$135,000 = \$115,000$.

Exercise 17A.13

What actuarily fair insurance policy would a risk loving consumer purchase? Can you illustrate your answer within the context of a graph that begins as in Graph 17.2c? (*Hint*: The benefit and premium levels will be negative.)

Answer: A risk loving consumer will want to transfer consumption from the "bad" state to the "good" state (rather than the other way around, as was the case for a risk averse consumer). In our case, my wife has \$10,000 available to transfer into the good state. This will leave her with zero consumption in the bad state as a result of receiving benefit b and paying premium p that, in order to be actuarily fair, have to satisfy the equation $b = 4p$. Thus,

$$0 = 10,000 + b - p = 10,000 + 4p - p = 10,000 + 3p. \qquad (17A.13)$$

Solving this for p, we get $p = -\$3,333.33$, and substituting this back into $b = 4p$, we get $b = -\$13,333.33$. Thus, the risk lover would want to pay $(b-p) = 13,333.33 - 3,333.33 = \$10,000$ in the bad state in order to increase consumption in the good state to $250,000 - p = 250,000 - (-3,333,33) = \$253,333.33$. All policy-holders for this insurance contract would therefore *receive* \$3,333.33 regardless of what state they end up in, the 25% that end up in the bad state would pay \$13,333.33. On average, the insurance company therefore has revenues of \$3,333.33 per customer and costs of 0.25(13,333.33)=\$3,333.33 — thus making zero profit.

Exercise 17A.14

True or False: A risk neutral consumer will be indifferent between all actuarily fair insurance contracts.

Answer: This is true. Actuarily fair insurance contracts reduce risk while keeping expected consumption levels the same. Since risk neutral consumers only care about the expected consumption level and are indifferent to different levels of risk, they are indifferent between insurance policies which keep the expected consumption levels the same while changing risk.

Exercise 17A.15

Verify the numbers on the horizontal axis of Graph 17.5.

Answer: The first policy is $(b_1, p_1) = (65,20)$; the second is $(b_2, p_2) = (100,40)$; and the third is $(b_3, p_3) = (122,60)$. Letting x denote the "bad" outcome and y the "good" outcome, we get outcomes $(x_1, y_1) = (55,230)$, $(x_2, y_2) = (70,210)$ and $(x_3, y_3) = (72,190)$ under the three policies. The expected consumption levels c_1, c_2 and c_3 are then

$$c_1 = 0.25(55) + 0.75(230) = 186.25, \quad c_2 = 0.25(70) + 0.75(210) = 175 \text{ and}$$
$$c_3 = 0.25(72) + 0.75(190) = 160.5. \tag{17A.15}$$

Exercise 17A.16

True or False: If firms in a perfectly competitive insurance industry face recurring fixed costs and marginal administration costs that are increasing, risk averse individuals will not fully insure in equilibrium.

Answer: This is true. The combination of recurring fixed costs and upward sloping marginal administrative costs creates U-shaped average cost curves for firms. In equilibrium, firms will have to cover these costs (that are in addition to the cost of honoring insurance claims) — and thus cannot price insurance at "actuarily fair" rates. Risk averse individuals will fully insure if insurance is actuarily fair but not when it is actuarily unfair (as it has to be in order for firms that face these additional costs to make zero profit).

Exercise 17A.17

Suppose only full insurance contracts were offered by the insurance industry — i.e. only contracts that insure that my wife will be equally well off financially regardless of what happens to me. What is the most actuarially unfair insurance contract that my wife would agree to buy? (*Hint*: Refer back to Graph 17.3.)

Answer: This relates to what we already did in answering within-chapter exercise 17A.12 where we calculated that my wife would be indifferent between not insuring and fully insuring with an actuarially unfair policy that has a benefit of $240,000 and a premium of $135,000. This is "full insurance" because the outcome in the "good" and "bad" states is the same for my wife if she carries this insurance: In the "bad" state, she pays $135,000 but receives $240,000 in addition to the $10,000 she starts with — leaving her with $115,000. In the "good" state, on the other hand, she just pays $135,000 from her initial $250,000 — again leaving her with $115,000. And since $115,000 is the certainty equivalent for her, we know that she is indifferent between getting $115,000 for sure or taking the uninsured gamble.

Exercise 17A.18

Why does consumption in the bad state rise only by 3 times the premium amount when actuarially fair insurance benefits are 4 times as high as the premium?

Answer: This is because even in the bad state, my wife has to pay the premium. Thus, when $b = 4p$ (with b the benefit in the bad state and p the premium of the policy), my wife gets $b - p = 4p - p = 3p$ once we take into account the fact that she still pays the premium.

Exercise 17A.19

Why is the slope of the budget constraint $-(1-\delta)/\delta$?

Answer: We concluded before that actuarially fair insurance implies that $b = p/\delta$ (where b is the benefit paid by the insurance company in the bad state, p is the insurance premium and δ is the probability of the bad state occurring.) Thus, if the consumer gives up $1 in the good state, she receives $b - 1 = (1/\delta) - 1 = (1-\delta)/\delta$ in the bad state (where she gets the benefit b but still has to pay the premium p).

Exercise 17A.20

What would indifference curves look like for a risk-neutral consumer? What insurance policy would she purchase?

Answer: The risk neutral consumer would have linear indifference curves with slope $-(1-\delta)/\delta$. Thus, there is an indifference curve that lies on the actuarially fair insurance contract budget line — which implies that the consumer is indifferent between all actuarially fair insurance policies. This makes sense — the expected value is the same all along the actuarially fair budget line, and all that risk neutral consumers care about is the expected value, not the risk.

Exercise 17A.21

What would indifference curves look like for a risk-loving consumer? What insurance policy would she purchase?

Answer: Risk loving consumers would have non-convex indifference curves that bend outward — implying that they will choose a corner solution with 0 consumption in the bad state.

Exercise 17A.22

We concluded previously that, when the two states are the same (aside from the income level associated with each state), $MRS = -(1-\delta)/\delta$ along the 45-degree line. In the case we just discussed, can you tell whether the MRS is greater or less than this along the 45-degree line?

Answer: The indifference curves along the 45 degree line are now steeper — which implies the MRS is larger in absolute value — implying we are willing to trade more consumption in the bad state for additional consumption in the good state when we are along the 45 degree line (compared to the case of state-independent tastes).

Exercise 17A.23

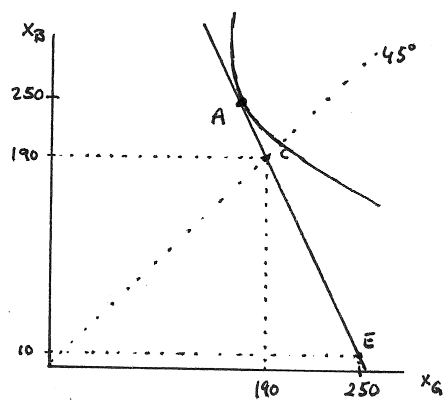
Suppose my wife was actually depressed by my presence and tolerates me solely for the paycheck I bring. Due to this depression, consumption is not very meaningful in the "good" state when I am around, but if I were not around, she would be able to travel the world and truly enjoy life. Might this cause her to purchase more than "full" life insurance on me? How would you illustrate this in a graph?

Answer: Yes, she would now "over-insure" in the sense that consumption in the bad state will be larger than consumption in the good state once she buys the optimal insurance policy. This is illustrated in Exercise Graph 17A.23 where indifference curves along the 45-degree line are shallower than would be the case for the optimal outcome bundle to lie on the 45 degree line (where full insurance would happen at C). Thus, my wife would choose A — an outcome bundle where she ends up consuming more in the bad state than in the good state.

Exercise 17A.24

Can you think of a different scenario in which it makes sense for the sports fan to bet against her own team?

Answer: Some sports fans might enter a near-full state of bliss when their team wins — requiring little additional consumption to gain further utility — i.e. their marginal utility from additional consumption is low when their team wins; but when their team loses, they need to consume various expensive libations to drown out their sorrows — i.e. the marginal utility of consumption is high for the same level of overall consumption as in the good state. In that case, the sports fan would



Exercise Graph 17A.23 : Over-Insurance

want to bet against her own team if she is risk averse — either her team wins and she is happy, or her team loses and she gets to consume a lot to forget about the loss.

Exercise 17A.25

Which assumption in our example results in the square shape of this Edgeworth Box?

Answer: The assumption of "no aggregate risk" — because that assumption implies that the *total* number of bananas on the island will be the same regardless of whether we are in the rainy or the dry state.

Exercise 17A.26

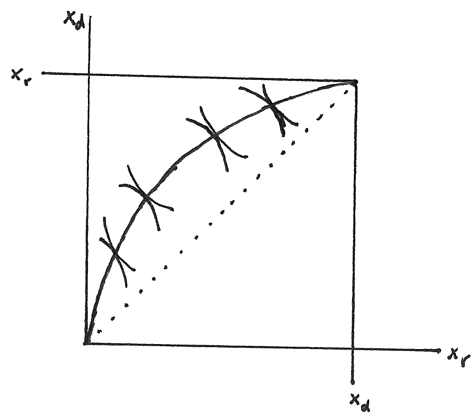
True or False: The 45-degree line is, in this case, the Contract Curve.

Answer: This is true. We know that, if tastes are not state-dependent, both individuals will have $MRS = -(1-\delta)/\delta$ along the 45-degree line — i.e. their indifference curves have the same slopes along the 45 degree line and are therefore tangent to one another. That is the defining characteristic of the contract curve, or the set of Pareto efficient outcomes.

Exercise 17A.27

What would the contract curve look like in this case?

Answer: This is depicted Exercise Graph 17A.27.



Exercise Graph 17A.27 : Contract Curve when you like Bananas more in the Rain

Exercise 17A.28

Suppose you liked bananas more when it rains than when it shines. Where would the equilibrium be?

Answer: The equilibrium would now lie below the 45-degree line, with $p_r^*/p_d^* < (1-\delta)/\delta$ — i.e. terms that are less favorable to me. This implies that the budget constraint is shallower than the actuarily fair budget constraint.

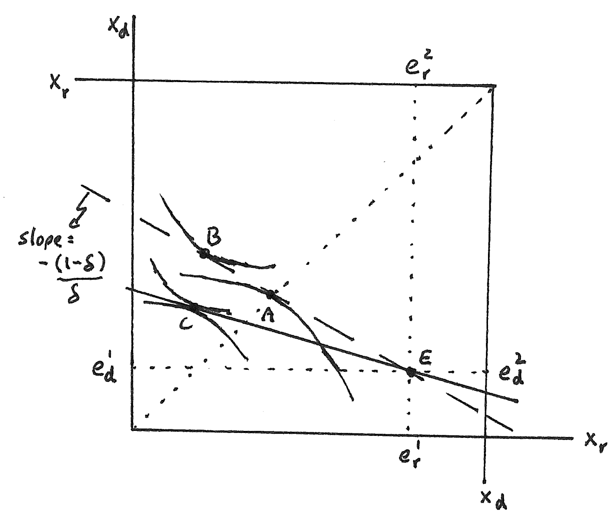
Exercise 17A.29

Suppose you were the one who had state-independent tastes and I was the one who values consuming bananas more when it shines than when it rains. Where would the equilibrium be?

Answer: This is depicted in Exercise Graph 17A.29. Since your tastes are state independent, we know that your indifference curves will have slope $-(1-\delta)/\delta$ along the 45 degree line. If the terms of trade were "actuarily fair", we would both be facing the dashed budget line, with you optimizing at A and me optimizing at B (since I value bananas more in the drought state). Since I want more bananas in the drought state than are supplied, the price for bananas in the drought state should rise relative to the price of bananas in the rainy state — leading to the solid budget line for both of us. In equilibrium, we will both optimize at some point like C.

Exercise 17A.30

Given that there are more bananas in the aggregate in the rainy state of the world, consider an endowment that has relatively more bananas in the dry state and another that has relatively more bananas in the rainy state. If you could choose your endowment, which endowment would you be more likely to want (assuming



Exercise Graph 17A.29 : Equilibrium when I like Bananas more in the Sunshine

we both have state-independent tastes and the overall endowments are not too different)?

Answer: We can see in the textbook graph that the equilibrium budget (in panel (b)) is shallower than the "actuarily fair" budget (in panel (a)). That means I will end up on a lower indifference curve in panel (b) than in panel (a), and you will end up on a higher indifference curve. You would therefore prefer to stick with your own endowment rather than switch with me. This makes intuitive sense: Since bananas are more scarce in the dry state, they are more valuable — which means, all else equal, you would want to have an endowment that has more bananas in the dry state and fewer in the wet state. (Of course you might still want the endowment that has relatively more bananas in the rainy state if that endowment is overall sufficiently larger than the other endowment option.)

Exercise 17A.31

In modeling equilibrium terms of trade that might emerge in financial markets, would you likely assume state-dependent or state-independent tastes?

Answer: In financial markets, investors care about the financial return to their investments — and there is no particular reason to expect the way they evaluate consumption to differ across different "states of the world." Thus, state-independent tastes might be a reasonable assumption for the model.

Exercise 17A.32

Suppose the two “states” of the world are recessions and economic booms. If you put consumption in economic booms on the horizontal axis, will the height of the Edgeworth box be larger or smaller than its width.

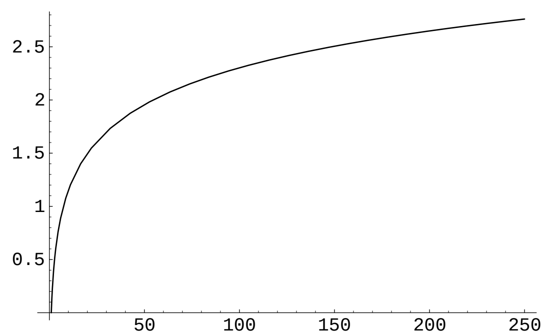
Answer: The overall level of consumption is larger during economic booms than during recessions — so you would expect the width of the box to be larger than its height.

17B Solutions to Within-Chapter-Exercises for Part B

Exercise 17B.1

Letting x denote consumption measured in thousands of dollars, illustrate the approximate shape of my wife's consumption/utility relationship in the range from 1 to 250 (interpreted as the range from \$1,000 to \$250,000.)

Answer: This is graphed in Exercise Graph 17B.1, with the function attaining $u(x) = 0$ when $x = 1$ (because $\ln(1) = 0$).



Exercise Graph 17B.1 : Graph of $u(x) = 0.5\ln(x)$ from x=1 through x=250

Exercise 17B.2

What does the graph of the utility function look like in the range of consumption between 0 and 1 (corresponding to 0 to \$1,000)?

Answer: Between 0 and 1, the function gives negative values that approach negative infinity as x approaches 0.

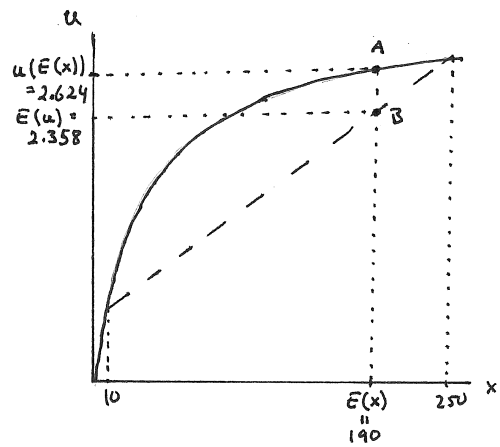
Exercise 17B.3

What is the utility of receiving the expected income, denoted $u(E(x))$? Illustrate $E(x)$, $E(u)$ and $u(E(x))$ on a graph of equation (17.2).

Answer: The utility of receiving the expected income $E(x)$ is

$$u(E(x)) = u(190) = 0.5\ln(190) \approx 0.5(5.247) = 2.6235. \qquad (17B.3)$$

This is illustrated on the vertical axis in Exercise Graph 17B.3, together with $E(x) = 190$ on the horizontal axis and $E(u) = 2.358$ on the vertical.



Exercise Graph 17B.3 : $E(x)$, $E(u)$ and $E(u(x))$

Exercise 17B.4

True or False: If u is a concave function, then $u(E(x))$ is larger than $E(u)$, and if u is a convex function, then $u(E(x))$ is smaller than $E(u)$.

Answer: This is true. The first part is illustrated in Exercise Graph 17B.3, and the second part is easily seen by drawing the same graph with a convex function.

Exercise 17B.5

What would $E(u)$ and $u(E(x))$ be for my wife if her utility of consumption were given instead by the convex function $u(x) = x^2$? Illustrate your answer in a graph.

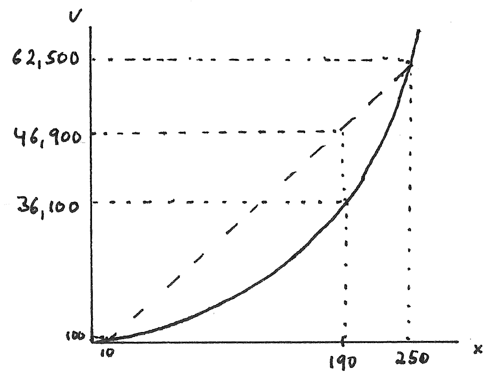
Answer: $E(x)$ would still be 190 as before. However, the utility of the outcomes in the good and bad states would now be

$$u(x_B) = u(10) = 10^2 = 100 \text{ and } u(x_G) = u(250) = 250^2 = 62{,}500. \tag{17B.5.i}$$

We thus get

$$\begin{aligned} &u(E(x)) = u(190) = 36{,}100 \text{ and} \\ &E(u) = 0.25u(x_B) + 0.75u(x_G) = 0.25(100) + 0.75(62500) = 46{,}900. \end{aligned} \tag{17B.5.ii}$$

which is illustrated in Exercise Graph 17B.5.



Exercise Graph 17B.5 : Risk Loving Tastes

Exercise 17B.6

The convexity of a function f is defined analogously to concavity, with the inequality in equation (17.7) reversed. Can you show that tastes which exhibit risk loving (as opposed to risk aversion) necessarily imply that any $u(x)$ used to define an expected utility function must be convex?

Answer: A function f is convex if and only if

$$\delta f(x_1) + (1-\delta)x_2 > f(\delta x_1 + (1-\delta)x_2). \qquad (17B.6.i)$$

For an individual to be risk loving, it must be the case that the expected utility of the gamble is greater than the utility of the expected value of the gamble; i.e. $U(x_G, x_B) > u(E(x))$. This can be expanded to read

$$\delta u(x_B) + (1-\delta)u(x_G) = U(x_G, x_B) > u(E(x)) = u(\delta x_B + (1-\delta)x_G). \qquad (17B.6.ii)$$

Thus, the definition of risk loving in equation (17B.6.ii) implies that $u(x)$ must be convex as defined in equation (17B.6.i).

Exercise 17B.7

Can you show in analogous steps that convexity of $u(x)$ must imply non-convexity of the indifference curves over outcome pairs (x_G, x_B)?

Answer: Consider again an average bundle (x_G^3, x_B^3) of two more extreme bundles such that

$$x_G^3 = \alpha x_G^2 + (1-\alpha)x_G^1 \text{ and } x_B^3 = \alpha x_B^2 + (1-\alpha)x_B^1, \qquad (17B.7.i)$$

with the more extreme bundles chosen to lie on the same indifference curve $\overline{U}$; i.e.

$$U(x_G^1, x_B^1) = U(x_G^2, x_B^2) = \overline{U}. \tag{17B.7.ii}$$

We can then again use the definition of the expected utility function and the convexity of $u(x)$ to show that

$$\begin{aligned}
U(x_G^3, x_B^3) &= \delta u(x_B^3) + (1-\delta) u(x_G^3) \\
&= \delta u\left(\alpha x_B^2 + (1-\alpha) x_B^1\right) + (1-\delta) u\left(\alpha x_G^2 + (1-\alpha) x_G^1\right) \\
&< \delta \left[\alpha u(x_B^2) + (1-\alpha) u(x_B^1)\right] + (1-\delta) \left[\alpha u(x_G^2) + (1-\alpha) u(x_G^1)\right] \\
&= \alpha \left[\delta u(x_B^2) + (1-\delta) u(x_G^2)\right] + (1-\alpha) \left[\delta u(x_B^1) + (1-\delta) u(x_G^1)\right] \\
&= \alpha U(x_G^2, x_B^2) + (1-\alpha) U(x_G^1, x_B^1) \\
&= \alpha \overline{U} + (1-\alpha)\overline{U} = \overline{U}.
\end{aligned} \tag{17B.7.iii}$$

Thus, we can conclude that

$$U(x_G^3, x_B^3) < \overline{U} = U(x_G^2, x_B^2) = U(x_G^1, x_B^1); \tag{17B.7.iv}$$

i.e. "averages are worse than extremes", implying non-convex indifference curves (that bend away from the origin). Put differently, we have now shown that risk loving tastes imply that any $u(x)$ used to construct an expected utility function that represents such tastes must be convex, and the convexity of $u(x)$ in turn implies that the indifference map is non-convex.

Exercise 17B.8

Illustrate x_{ce} and the risk premium on a graph with my wife's utility function $u(x) = 0.5 \ln x$.

Answer: This is illustrated in Exercise Graph 17B.8.

Exercise 17B.9

Verify the expressions for p^* and b^*.

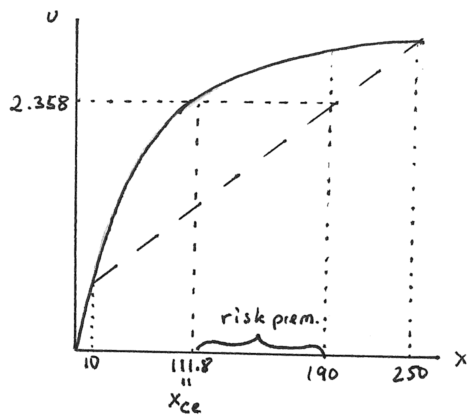
Answer: The problem

$$\max_p \, \delta \alpha \ln\left(x_B + \frac{(1-\delta)p}{\delta}\right) + (1-\delta)\alpha \ln(x_G - p), \tag{17B.9.i}$$

is an unconstrained optimization problem. So all we have to do is set the first derivative of the objective function with respect to the choice variable p to zero; i.e.

$$\alpha\delta\left(\frac{(1-\delta)}{\delta}\right)\left(\frac{\delta}{\delta x_B + (1-\delta)p}\right) - (1-\delta)\alpha\left(\frac{1}{x_G - p}\right) = 0. \tag{17B.9.ii}$$

Adding the second term to both sides and simplifying, we get



Exercise Graph 17B.8 : Certainty Equivalent and Risk Premium

$$\left(\frac{\delta}{\delta x_B + (1-\delta)p}\right) = \left(\frac{1}{x_G - p}\right) \tag{17B.9.iii}$$

or, cross-multiplying,

$$\delta x_B + (1-\delta)p = \delta(x_G - p). \tag{17B.9.iv}$$

Adding δp to both sides, and subtracting δx_B from both sides, we end up with

$$p^* = \delta(x_G - x_B). \tag{17B.9.v}$$

Substituting into the actuarily fair relationship between b and p — i.e. $b = p/\delta$ — we then also get $b^* = x_G - x_B$.

Exercise 17B.10

Even though we did not use the same underlying utility function as the one used to plot graphs in Section A, we have gotten the same result for the optimal actuarily fair insurance policy. Why is this?

Answer: This is because it is optimal for *all* risk averse individuals to fully insure — i.e. for all individuals with concave utility functions regardless of what precise shape the concavity takes. Full insurance just means a combination of premiums and benefits that results in the same consumption regardless of what state occurs — which does not depend on what utility functions are but just on what the good and bad outcomes as well as the associated probabilities are.

Exercise 17B.11

Derive the expression for the marginal rate of substitution for equation (17.20). Now suppose $\alpha{=}\beta$. What is the MRS along the 45 degree line on which $x_1 = x_2$? Compare this to the result we derived graphically in Graph 17.7.

Answer: The MRS is

$$MRS = -\frac{\partial U/\partial x_G}{\partial U/\partial x_B} = -\left(\frac{(1-\delta)\beta/x_G}{\delta\alpha/x_B}\right) = -\frac{(1-\delta)\beta x_B}{\delta\alpha x_G}. \qquad (17B.11)$$

On the 45-degree line, $x_B = x_G$ and thus cancel in the equation — and, if $\alpha = \beta$, these terms cancel as well — leaving us with $MRS = -(1-\delta)/\delta$. This implies a slope of $(-(1-\delta)/\delta)$ along the 45 degree line — precisely the result we derived graphically in Section A.

Exercise 17B.12

Can you see that the indifference curves generated by $U(x_1, x_2)$ in equation (17.20) are Cobb-Douglas? Write the function as a Cobb-Douglas function and derive the MRS. Does the property that must hold along the 45 degree line when tastes are not state-dependent hold?

Answer: The Cobb-Douglas utility function that gives the same indifference curves is

$$\overline{U}(x_G, x_B) = x_B^{\delta\alpha} x_G^{(1-\delta)\beta}. \qquad (17B.12.i)$$

(You can verify this by simply taking the natural log of this equation — which gives you back the original utility function with the expected utility form.) The MRS is

$$MRS = -\frac{\partial U/\partial x_G}{\partial U/\partial x_B} = -\left(\frac{(1-\delta)\beta x_B^{\delta\alpha} x_G^{-\delta\beta}}{\delta\alpha x_B^{(\delta\alpha-1)} x_G^{(1-\delta)\beta}}\right) = -\frac{(1-\delta)\beta x_B}{\delta\alpha x_G}. \qquad (17B.12.ii)$$

This is the same MRS as the one calculated in the previous exercise for the original utility function (showing once again that the two give rise to the same indifference curves.) On the 45 degree line, $x_B = x_G$ and, when $\alpha = \beta$ (as it must if tastes are not state-dependent), this reduces to $MRS = -(1-\delta)/\delta$. This is the condition that in fact must hold along the 45 degree line. (In this case, the utility function can in fact be further simplified as $U(x_1, x_2) = x_1^{\delta} x_2^{(1-\delta)}$.)

Exercise 17B.13

True or False: The expected utility function $U(x_G, x_B)$ can be transformed in all the ways that utility functions in consumer theory can usually be transformed without changing the underlying indifference curves, but such transformations will imply a loss of the expected utility form.

Answer: This is true. In transforming our original expected utility function $U(x_B, x_G) = \delta\alpha \ln x_B + (1-\delta)\beta \ln x_G$ to one that takes the usual Cobb-Douglas form in exercise (17B.12), for instance, we have preserved the shape of indifference curves but the new utility function is no longer the probability weighted sum of the utilities associated with each outcome as measured by a function $u(x)$.

Exercise 17B.14

On a graph with x_G on the horizontal and x_B on the vertical axis, illustrate this budget constraint using values derived from the example of my wife's choices over insurance contracts. Compare it to Graph 17.6b.

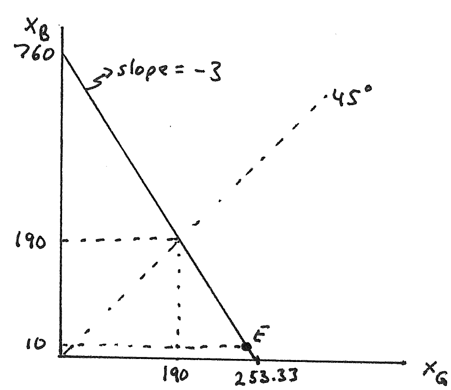
Answer: The budget line equation

$$x_B = \frac{\delta e_B + (1-\delta)e_G}{\delta} - \frac{(1-\delta)}{\delta} x_G \quad (17B.14.i)$$

becomes

$$x_B = \frac{0.25(10) + (1-0.25)(250)}{0.25} - \frac{(1-0.25)}{0.25} x_G = 760 - 3x_G. \quad (17B.14.ii)$$

The budget line therefore has vertical intercept of 760 and slope of −3 (giving us horizontal intercept of 253.33.) This is depicted in Exercise Graph 17B.14.



Exercise Graph 17B.14 : Actuarily Fair Budget Constraint

Exercise 17B.15

Verify the result in equation (17.25).

Answer: The Lagrange function for this problem is

$$\mathscr{L} = \delta\alpha \ln x_B + (1-\delta)\beta \ln x_G + \lambda\left(\delta e_B + (1-\delta)e_G - \delta x_B - (1-\delta)x_G\right) \qquad (17B.15.i)$$

giving us first order conditions

$$\frac{\partial \mathscr{L}}{\partial x_B} = \frac{\delta\alpha}{x_B} - \lambda\delta = 0 \text{ and } \frac{\partial \mathscr{L}}{\partial x_G} = \frac{(1-\delta)\beta}{x_G} - \lambda(1-\delta) = 0. \qquad (17B.15.ii)$$

These can be solved for $x_G = \beta x_B/\alpha$. When plugged into the budget constraint $\delta e_B + (1-\delta)e_G = \delta x_B + (1-\delta)x_G$, we can then solve for

$$x_B^* = \frac{\alpha\left(\delta e_B + (1-\delta)e_G\right)}{\delta\alpha + (1-\delta)\beta}. \qquad (17B.15.iii)$$

Plugging this back into $x_G = \beta x_B/\alpha$, we also get

$$x_G^* = \frac{\beta\left(\delta e_B + (1-\delta)e_G\right)}{\delta\alpha + (1-\delta)\beta}. \qquad (17B.15.iv)$$

Exercise 17B.16

Using the values of \$10 and \$250 as the consumption level my wife gets in state B and state G in the absence of insurance, what level of consumption does she get in each state when she chooses her optimal actuarily fair insurance policy (assuming, as before, that state B occurs with probability 0.25 and state G occurs with probability 0.75)?

Answer: She gets

$$x = \delta e_B + (1-\delta)e_G = 0.25(10) + (1-0.25)(250) = 190. \qquad (17B.16)$$

Exercise 17B.17

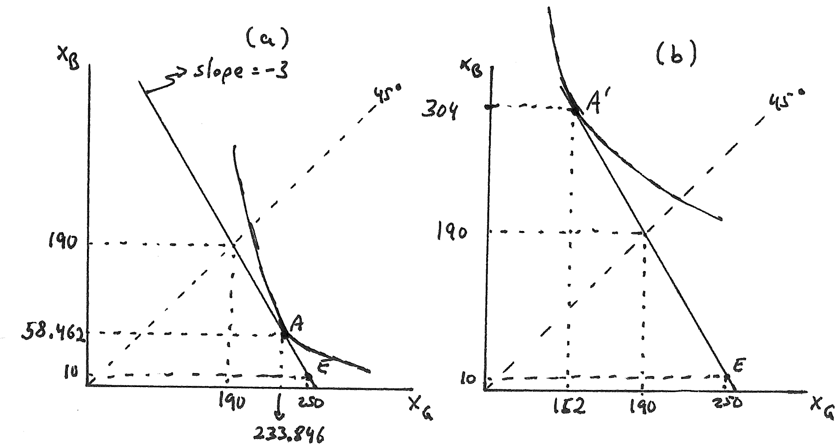
Using an Excel spreadsheet, can you verify the numbers in Table 17.1?

Answer: You can do this by simply specifying fields for e_B, e_G, δ and α while setting $\beta = 1$. Then program the formula for x_B^* and x_G^* derived in the text to calculate x_B and x_G in the table. The premium can then be derived simply as $p = e_G - x^G$, and the benefit is $b = p/\delta$.

Exercise 17B.18

Using a graph similar to Graph 17.8, illustrate the case of $\alpha/\beta = 1/4$ (row 2 in the table).

Answer: This is done in panel (a) of Exercise Graph 17B.18.



Exercise Graph 17B.18 : Two state-dependent tastes and actuarially fair insurance

Exercise 17B.19

Using a graph similar to Graph 17.8, illustrate the case of $\alpha/\beta = 2/1$ (row 7 in the table).

Answer: This is done in panel (b) of Exercise Graph 17B.18 in the answer to within-chapter exercise 17B.18.

Exercise 17B.20

For what values of α and β is utility state-independent for each of these consumers?

Answer: For $\alpha = \beta$.

Exercise 17B.21

Are we imposing any real restrictions by assuming that the utility weights placed on log consumption in the two states sum to 1 for each of the two consumers?

Answer: No. To see this, we can first derive the *MRS* for the expected utility functions as written, which are

$$\begin{aligned} MRS^1 &= -\frac{\partial U^1/\partial x_1}{\partial U^1/\partial x_2} = \frac{\delta \alpha x_2}{(1-\delta)(1-\alpha)x_1} \quad \text{and} \\ MRS^2 &= -\frac{\partial U^2/\partial x_1}{\partial U^2/\partial x_2} = \frac{\gamma \beta x_2}{(1-\gamma)(1-\beta)x_1}. \end{aligned} \tag{17B.21.i}$$

Now suppose we multiply α and $(1-\alpha)$ by k, and we multiply β and $(1-\beta)$ by t. We would then get the expected utility functions

$$V^1(x_1, x_2) = \delta k \alpha \ln x_1 + (1-\delta) k (1-\alpha) \ln x_2 \quad (17B.21.ii)$$

and

$$V^2(x_1, x_2) = \gamma t \beta \ln x_1 + (1-\gamma) t (1-\beta) \ln x_2. \quad (17B.21.iii)$$

The marginal rates of substitution for these utility functions are then identical to the ones above because the k and the t appears in both numerator and denominator and therefore cancels. Thus, while the labeling of the indifference curves changes, the shapes of the indifference curves do not. This is generally true for any *linear* transformation of the u functions — i.e. if we have identified u functions for each state that allow us to represent the indifference curves with an expected utility function, then any linear transformation of the u's will also allow us to represent the same indifference curves with an expected utility function using these transformed u's.

Exercise 17B.22

How would you write the analogous optimization problem for individual 2?
Answer: You would write it as

$$\max_{x_1^2, x_2^2} \gamma \beta \ln x_1^2 + (1-\gamma)(1-\beta) \ln x_2^2 \text{ subject to } p_1 e_1^2 + p_2 e_2^2 = p_1 x_1^2 + p_2 x_2^2. \quad (17B.22)$$

Exercise 17B.23

Suppose that the overall endowment in the economy is the same in each of the two states — i.e. $e_1^1 + e_1^2 = e_2^1 + e_2^2$; suppose that each consumer has state-independent utility (i.e. $\alpha = (1-\alpha)$ and $\beta = (1-\beta)$), and suppose that both consumers evaluate risk in the same way (i.e. $\delta = \gamma$). Can you then demonstrate that equilibrium terms of trade will be actuarily fair — i.e. $p_2^*/p_1^* = (1-\delta)/\delta$?

Answer: Using the equation derived in the text for p_2^* (when p_1^* is normalized to 1) and substituting α for $(1-\alpha)$, β for $(1-\beta)$ and δ for γ, we get

$$\begin{aligned}
p_2^* &= \frac{\alpha(1-\delta)\left(\beta\delta + \beta(1-\delta)\right) e_1^1 + \beta(1-\delta)\left(\alpha\delta + \alpha(1-\delta)\right) e_1^2}{\alpha\delta\left(\beta\delta + \beta(1-\delta)\right) e_2^1 + \beta\delta\left(\alpha\delta + \alpha(1-\delta)\right) e_2^2} \\
&= \frac{\alpha(1-\delta)\beta e_1^1 + \beta(1-\delta)\alpha e_1^2}{\alpha\delta\beta e_2^1 + \beta\delta\alpha e_2^2} \\
&= \frac{\alpha\beta(1-\delta)\left(e_1^1 + e_1^2\right)}{\alpha\beta\delta\left(e_2^1 + e_2^2\right)} \\
&= \frac{(1-\delta)}{\delta}.
\end{aligned} \quad (17B.23)$$

Exercise 17B.24

For the scenario described in the previous exercise, can you use individual demand functions to illustrate that each consumer will choose to equalize consumption across the two states? Where in the Edgeworth Box does this imply the equilibrium falls?

Answer: The demand functions derived in the text are

$$x_1^1(p_1,p_2) = \frac{\alpha\delta(p_1e_1^1 + p_2e_2^1)}{(\alpha\delta + (1-\alpha)(1-\delta))\,p_1} \text{ and}$$
$$x_1^2(p_1,p_2) = \frac{\beta\gamma(p_1e_1^2 + p_2e_2^2)}{\left(\beta\gamma + (1-\beta)(1-\gamma)\right)p_1}. \qquad (17B.24.i)$$

Substituting $p_1 = 1$, α for $(1-\alpha)$, β for $(1-\beta)$ and δ for γ, these become

$$x_1^1(p_2) = \frac{\alpha\delta(e_1^1 + p_2e_2^1)}{(\alpha\delta + \alpha(1-\delta))} = \frac{\alpha\delta(e_1^1 + p_2e_2^1)}{\alpha} = \delta(e_1^1 + p_2e_2^1) \qquad (17B.24.ii)$$
$$x_1^2(p_2) = \frac{\beta\delta(e_1^2 + p_2e_2^2)}{\left(\beta\delta + \beta(1-\delta)\right)} = \frac{\beta\delta(e_1^2 + p_2e_2^2)}{\beta} = \delta(e_1^2 + p_2e_2^2). \qquad (17B.24.iii)$$

Plugging in the equilibrium price $p_2^* = (1-\delta)/\delta$, we then get

$$x_1^1 = \delta\left(e_1^1 + \frac{(1-\delta)}{\delta}e_2^1\right) = \delta e_1^1 + (1-\delta)e_2^1 \text{ and} \qquad (17B.24.iv)$$
$$x_1^2 = \delta\left(e_1^2 + \frac{(1-\delta)}{\delta}e_2^2\right) = \delta e_1^2 + (1-\delta)e_2^2. \qquad (17B.24.v)$$

For each consumer, the budget constraint has to bind. Consider, for instance, consumer 1. His budget constraint is $p_1e_1^1 + p_2e_2^1 = p_1x_1^1 + p_2x_2^1$. Setting p_1 equal to its normalized value of 1, plugging in our equilibrium price for p_2 (i.e. $p_2 = (1-\delta)/\delta$), and substituting our consumption level $x_1^1 = \delta e_1^1 + (1-\delta)e_2^1$, this becomes

$$e_1^1 + \frac{(1-\delta)}{\delta}e_2^1 = \left(\delta e_1^1 + (1-\delta)e_2^1\right) + \frac{(1-\delta)}{\delta}x_2^1 \qquad (17B.24.vi)$$

which solves to

$$x_2^1 = \delta e_1^1 + (1-\delta)e_2^1. \qquad (17B.24.vii)$$

Using the same method, we can also show that $x_2^2 = \delta e_1^2 + (1-\delta)e_2^2$. Thus,

$$x_1^1 = x_2^1 \text{ and } x_1^2 = x_2^2; \qquad (17B.24.viii)$$

i.e. consumer 1 consumes the same quantity in each state, as does consumer 2. This equilibrium allocation therefore appears on the 45-degree line of the Edgeworth box — which, because we are assuming no aggregate risk, goes through the origins of both consumers' axes.

Exercise 17B.25

What is the shape of the Edgeworth Box representing an economy in which $e_1^1 + e_1^2 = e_2^1 + e_2^2$?

Answer: It is a square.

Exercise 17B.26

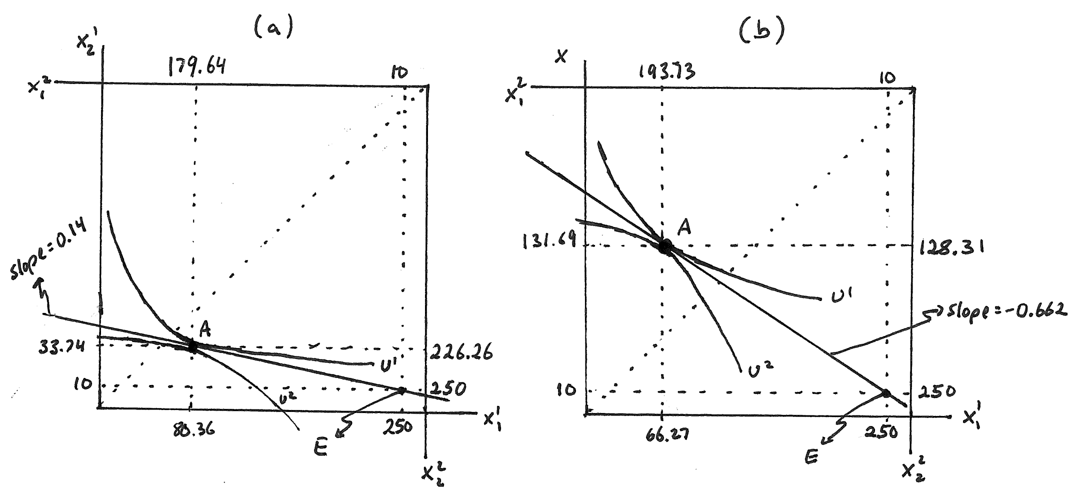
Why do you think individual 1 ends up with less consumption than individual 2 once they fully insure?

Answer: Individual 1 begins with an endowment of 250 in state 1 and 10 in state 2, but state 1 occurs only with probability 0.25 while state 2 occurs with probability 0.75. Individual 1's "expected endowment" is therefore only $0.25(250)+0.75(10) = 70$. Individual 2, on the other hand, begins with an endowment of 10 in state 1 and 250 in state 2 — giving him an "expected endowment" of $0.25(10)+0.75(250) = 190$. Thus, individual 1 is substantially richer. Even though the endowments appear to be symmetric, they are not because the different states do not arise with equal probability.

Exercise 17B.27

Can you draw out the equilibrium in rows 1 and 3 in Edgeworth Boxes?

Answer: These are illustrated in panels (a) and (b) where E is the initial endowment outcome bundle and A is the equilibrium. The slope of the budgets are equal to $-1/p_2^*$ since p_1 was normalized to 1.



Exercise Graph 17B.27 : Two Equilibria

Exercise 17B.28

Can you offer a similar intuitive explanation for the third set of results in Table 17.2?

Answer: In this set of simulations, individual 1's tastes are held constant, placing relatively little weight on state 1 consumption. In the first row, individual 2 places similarly little weight on state 1 consumption. As a result, the equilibrium price for buying state 2 consumption is high, which implies individual 2 who has most of the state 2 endowment is quite rich compared to individual 1 who has relatively little state 2 endowment. Both individuals end up fully insuring, but, because of the effective wealth disparity, individual 1 ends up with much less consumption than individual 2. As individual 2's β increases over the next three rows, he places increasingly more weight on consumption in state 1. As a result, demand for state 2 consumption falls, causing p_2^* to fall. Individual 1's tastes remain fixed throughout these simulations — so the only impact on him comes from the falling price of state 2 consumption — leading him to consume more in state 2 and less in state 1. Individual 2 ends up lowering state 2 consumption (despite the fact that it is becoming cheaper) because he is placing increasingly more weight on consumption in state 1 as β increases.

Exercise 17B.29

Suppose $\alpha = \beta = 0.5$. For what values of δ and γ will the equilibrium be the same as the one in the first row of Table 17.2? (*Hint*: This is harder than it appears. In row one of the table, $\beta\gamma = 1/16$ and $(1-\beta)(1-\gamma) = 9/16$. Thus, the overall weight placed on state 2 is 9 times the weight placed on state 1. When you now change β from 0.25 to 0.5, you need to make sure when you change γ that the overall weight placed on state 2 is again 9 times the weight placed on state 1.)

Answer: When β is raised to 0.5, we want γ to satisfy the condition that 0.5γ is 9 times as high as $0.5(1-\gamma)$ — i.e. $9(0.5)\gamma = 0.5(1-\gamma)$. Solving for γ, we get $\gamma = 0.10$. Thus, when $\beta = 0.5$ and person 2's belief about γ is equal to 0.10, the expected utility function for individual 2 will give rise to the same indifference curves as the expected utility function when $\beta = 0.25$ and $\gamma = 0.25$.

Exercise 17B.30

Can you see from the demand equations why consumption in the rainy season remains unchanged?

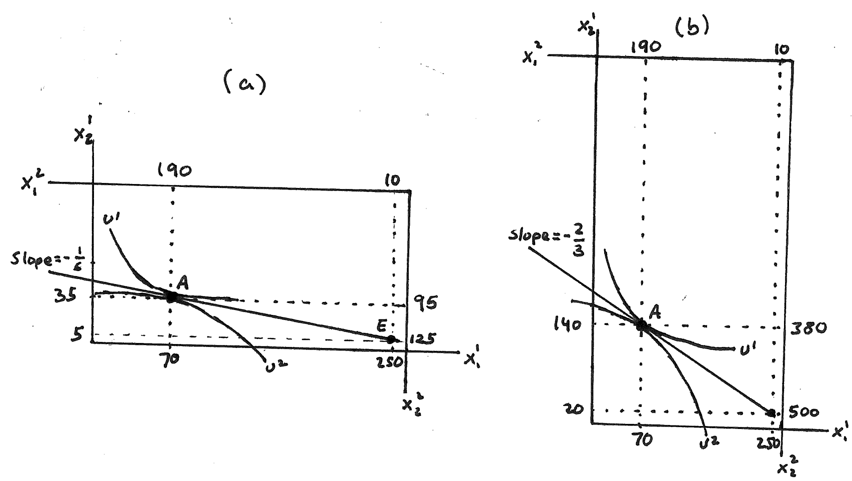
Answer: The demand equations for consumption in state 1 are derived in the text as

$$x_1^1(p_1,p_2) = \frac{\alpha\delta(p_1e_1^1 + p_2e_2^1)}{(\alpha\delta + (1-\alpha)(1-\delta))\,p_1} \text{ and}$$
$$x_1^2(p_1,p_2) = \frac{\beta\gamma(p_1e_1^2 + p_2e_2^2)}{\big(\beta\gamma + (1-\beta)(1-\gamma)\big)\,p_1}. \qquad (17B.30)$$

The only variables in the equation for $x_1^1(p_1,p_2)$ that are affected are e_2^1 which is cut in half and p_2 which has doubled. Thus, the two changes exactly offset one another. Similarly, the only variables in the equation for $x_1^2(p_1,p_2)$ that are affected are e_2^2 which is cut in half and p_2 which doubles — thus again offsetting one another.

Exercise 17B.31

Can you depict the equilibria in rows 1 and 3 in two Edgeworth Boxes?
Answer: These are depicted in panels (a) and (b) of Exercise Graph 17B.31



Exercise Graph 17B.31 : Different types of aggregate risk

Exercise 17B.32

In the third set of results of Table 17.4, we hold your land productivity constant while varyng mine. Can you make sense of the results?

Answer: In the first row, bananas in the rainy season (state 1) are relatively scarce — giving us a relatively low price for buying state contingent consumption in the drought season (state 2). This change in price is the only factor that is changing for individual 2 — and, as a result, she consumes somewhat more bananas in state 2 and less in state 1. In fact, from her perspective, the big factor is the increase in the relative price of consumption in state 1 which she is attempting to purchase given that she has very little endowment in state 1. Individual 1, on the other hand, suffers a large cut in his production (relative to row 2 of the table) — leading him to consume less in both states. In row 3, on the other hand, bananas in the rainy season (state 1) are relatively abundant — causing the price of state-contingent consumption in the drought season to rise (and the relative price of state contingent consumption in state 1 to fall). For individual 2, this change in relative prices is again the only factor that changes (relative to row 2) — making her endowment more valuable and lowering the price of the consumption she needs to buy (in state 1). As a result, her consumption in state 1 increases. Individual 1 is richer in that his endowment has doubled (relative to row 2), but the price of his state 1 endowment has fallen while the price of state 2 consumption (which he needs given his low endowment in state 2) has increased. We therefore see a modest increase in his consumption in state 2 because of his increased wealth and in spite of the increased price, and a much larger increase in his consumption in state 1. Once again, we see the terms of trade more favorable for whoever is buying in the state where bananas are more abundant: In row 1, bananas are more abundant in state 2 — and thus the terms of trade are more favorable for individual 1 who has little endowment in that state. In row 3, bananas are more abundant in state 1, and we thus see more favorable terms of trade for individual 2 who needs to purchase in state 1 given her low endowment there.

Exercise 17B.33

What is the probability of reaching outcome 2 if we play Gamble 1 half the time and Gamble 2 half the time?

Answer: The probability of reaching outcome 2 is

$$0.5(0.4)+0.5(0.8)=0.60. \tag{17B.33}$$

Exercise 17B.34

What weights would I have to put on Gambles 1 and 2 in order for the mixed gamble to result in a 0.50 probability of reaching outcome 1 and a 0.50 probability of reaching outcome 2?

Answer: Denoting the weight we would place on Gamble 1 as α, we would like to choose α such that

$$\alpha(0.60)+(1-\alpha)(0.2)=0.5 \text{ and } \alpha(0.40)+(1-\alpha)(0.80)=0.5. \tag{17B.34}$$

Solving either one of these, we get $\alpha=0.75$.

Exercise 17B.35

Does the paradox still hold if people's tastes are state-dependent? (*Hint*: The answer is yes.)

Answer: Suppose that tastes are state-dependent but can still be represented by an expected utility function. All this means is that there now exist three different

u functions — $u_A(x)$, $u_B(x)$ and $u_C(x)$, corresponding the states of A="winning $5 million", B="winning $1 million" and C="winning nothing", that allow us to write the expected utility of each gamble as a probability weighted average of u_5, u_1 and u_0 — i.e. of $u_5 = u_A(5,000,000)$, $u_1 = u_B(1,000,000)$ and $u_0 = u_c(0)$. The rest of the paradox then unfolds exactly the same way.

17C Solutions to Odd Numbered End-of-Chapter Exercises

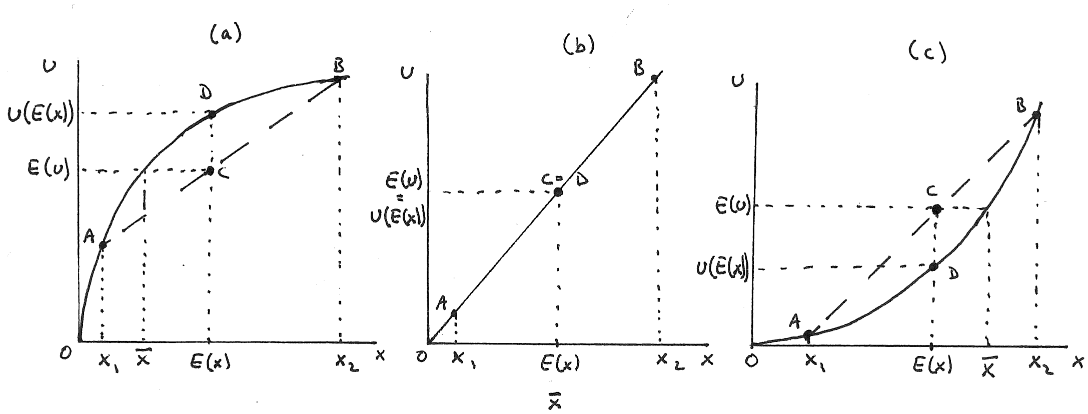
Exercise 17.1

In this exercise we review some basics of attitudes toward risk when tastes are state-independent and, in part B, we also verify some of the numbers that appear in the graphs of part A of the chapter.

A: *Suppose that there are two possible outcomes of a gamble: Under outcome A, you get \$$x_1$ and under outcome B you get \$$x_2$ where $x_2 > x_1$. Outcome A happens with probability $\delta = 0.5$ and outcome B happens with probability $(1-\delta) = 0.5$.*

(a) *Illustrate three different consumption/utility relationships — one that can be used to model risk averse tastes over gambles, one for risk neutral tastes and one for risk loving tastes.*

Answer: This is done in panels (a) through (c) of Exercise Graph 17.1.



Exercise Graph 17.1 : Different Attitudes about Risk

(b) *On each graph illustrate your expected consumption on the horizontal axis and your expected utility of facing the gamble on the vertical. Which of these — expected consumption or expected utility — does not depend on whether your degree of risk aversion*

Answer: The expected consumption level is simply $E(x) = 0.5x_1 + 0.5x_2$ and is illustrated in each panel as lying halfway between x_1 and x_2 on the horizontal axis. It does not depend on attitudes toward risk because it is simply a probability-weighted average of the two consumption levels that might happen. The expected utility $E(u)$ is the probability-weighted average of the utilities associated with each of the two possible outcomes — and is read off the line connecting A and B in each panel.

(c) *How does the expected utility of the gamble differ from the utility of the expected consumption level of the gamble in each graph?*

Answer: The expected consumption level of the gamble is $E(x)$. The utility associated with that level of consumption is read off the consumption/utility relationship itself — and is indicated as $u(E(x))$ in each panel of the graph. It is the utility the person gets from getting the expected consumption level without risk. It differs from the utility of the gamble because, although the gamble has the same expected consumption value, it involves risk. If you don't like risk, then the utility of the gamble will be less than the utility of the expected value of the gamble (as in panel (a)). But if you like risk, the utility of the gamble will be greater than the utility of the expected value of the gamble (as in panel (c)). The two will be the same in the case of risk neutrality (panel (b)) where the individual does not care one way or another about risk.

(d) *Suppose I offer you $\$\overline{x}$ to not face this gamble. Illustrate in each of your graphs where $\overline{x}$ would lie if it makes you just indifferent between taking $\overline{x}$ and staying to face the gamble.*

Answer: This is illustrated in each panel as the quantity that, if obtained without risk, will provide the same utility as the expected utility $E(u)$ of the gamble. It is what we called in the text the certainty equivalent.

(e) *Suppose I come to offer you some insurance — for every dollar you agree to give me if outcome B happens, I will agree to give you y dollars if outcome A happens. What's y if the deal I am offering you does not change the expected value of consumption for you?*

Answer: If the expected value of consumption is to remain unchanged, it must mean the expected value of what you are getting is the same as the expected value of what you are paying. When you agree to pay me \$1 if B happens, you agree to give me \$1 with probability 0.5 (since B happens with probability 0.5). Thus, the expected value of what you are giving me is 0.5. In return I give you \$y if A happens — which means the expected value of what I am giving you is 0.5y because A happens with probability 0.5. For the expected value of consumption to remain the same, it must therefore be the case that 0.5y=0.5 — i.e. y=\$1.

(f) *What changes in your 3 graphs if you buy insurance of this kind — and how does it impact your expected consumption level on the horizontal axis and the expected utility of the remaining gamble on the vertical?*

Answer: In each graph, x_1 increases by the same amount that x_2 decreases as I buy such insurance — thus reducing the risk of the gamble. However, the expected value $E(x)$ remains the same. In panel (a), however, the line on which expected utility is measured shifts up as a result of insurance — implying that $E(u)$ increases with insurance (as the expected value of the gamble remains unchanged but risk falls). But in panel (c), the line on which $E(u)$ is measured falls with insurance — implying the expected utility of the gamble falls as risk is decreased by insurance (while the expected value of consumption remains unchanged). This should make sense: In panel (a), you dislike risk — while in panel (c) you like it. Insurance that keeps the expected value of the gamble unchanged will therefore makes you better off in panel (a) and worse off in panel (c) — because such insurance reduces risk. In panel (b), on the other hand, we don't care about risk one way or another — which implies insurance that lowers risk without changing the expected consumption value of the gamble leaves you indifferent.

B: *Suppose we can use the function $u(x) = x^\alpha$ for the consumption/utility relationship that allows us to represent your indifference curves over risky outcomes using an expected utility function. Assume the rest of the set-up as described in A.*

(a) *What value can α take if you are risk averse? What if you are risk neutral? What if you are risk loving?*

Answer: When $0 < \alpha < 1$, we get the concave shape required for risk aversion; when $\alpha = 1$, we simply get the equation of a line $u(x) = x$ and thus get the shape required for risk neutrality; and if $\alpha > 1$, we get the convex shape required for risk loving. These correspond to the cases graphed in panels (a) through (c) of Exercise Graph 17.1.

(b) *Write down the equations for the expected consumption level as well as the expected utility from the gamble. Which one depends on α and why?*

Answer: The expected consumption value of the gamble is given by

$$E(x) = \delta x_1 + (1-\delta)x_2 = 0.5x_1 + 0.5x_2 \tag{17.1.i}$$

which does not depend on α because it has nothing to do with tastes. The expected utility is given by

$$U = E(u) = \delta u(x_1) + (1-\delta)u(x_2) = 0.5x_1^\alpha + 0.5x_2^\alpha. \tag{17.1.ii}$$

(c) *What's the equation for the utility of the expected consumption level?*

Answer: This is

$$u(0.5x_1 + 0.5x_2) = (0.5x_1 + 0.5x_2)^\alpha. \tag{17.1.iii}$$

(d) *Consider $\overline{x}$ as defined in A(d). What equation would you have to solve to find $\overline{x}$?*

Answer: It has to be the case that $u(\overline{x}) = E(u)$; i.e.

$$\overline{x}^\alpha = 0.5x_1^\alpha + 0.5x_2^\alpha. \tag{17.1.iv}$$

(e) *Suppose $\alpha = 1$. Solve for $\overline{x}$ and explain your result intuitively.*

Answer: In this case, equation (17.1.iv) simply becomes

$$\overline{x} = 0.5x_1 + 0.5x_2 \tag{17.1.v}$$

where the right hand side is simply $E(x)$. This is reflected in panel (b) of Exercise Graph 17.1 where tastes are risk neutral and the certainty equivalent of a gamble is simply equal to the expected consumption value of the gamble (since risk neutral individuals don't care one way or another about the risk of the gamble).

(f) *Suppose that, instead of 2 outcomes, there are actually 3 possible outcomes: A, B and C, with associated consumption levels x_1, x_2 and x_3 occurring with probabilities δ_1, δ_2 and $(1-\delta_1-\delta_2)$. How would you write the expected utility of this gamble?*

Answer: You would then simply write it as

$$U = E(u) = \delta_1 u(x_1) + \delta_2 u(x_2) + (1-\delta_1-\delta_2)u(x_3) = \\ = \delta_1 x_1^{\alpha} + \delta_2 x_2^{\alpha} + (1-\delta_1-\delta_2)x_3^{\alpha}. \tag{17.1.vi}$$

(g) *Suppose that u took the form*

$$u(x) = 0.1x^{0.5} - \left(\frac{x}{100{,}000}\right)^{2.5} \tag{17.1.vii}$$

This is the equation that was used to arrive most of the graphs in part A of the chapter, where x is expressed in thousands but plugged into the equation as its full value; i.e. consumption of 200 in a graph represents $x = 200{,}000$. *Verify the numbers in Graphs 17.1 and 17.3. (Note that the numbers in the graphs are rounded.)*

Answer: For Graph 17.1, the utility levels associated with points A, B, C and D are

$$\text{For } A\text{: } u(250{,}000) = 0.1(250{,}000)^{0.5} - \left(\frac{250{,}000}{100{,}000}\right) = 40.1179 \approx 40 \tag{17.1.viii}$$

$$\text{For } B\text{: } u(10{,}000) = 0.1(10{,}000)^{0.5} - \left(\frac{10{,}000}{100{,}000}\right) = 9.9968 \approx 10 \tag{17.1.ix}$$

$$\text{For } C\text{: } E(u) = 0.25(10) + 0.75(40) = 32.5 \tag{17.1.x}$$

$$\text{For } D\text{: } u(190{,}000) = 0.1(190{,}000)^{0.5} - \left(\frac{190{,}000}{100{,}000}\right) = 38.613 \approx 38.5 \tag{17.1.xi}$$

In Graph 17.3 of the text, we also calculated the certainty equivalent. Since the expected utility of the gamble is 32.5, the certainty equivalent $\overline{x}$ must satisfy $u(\overline{x}) = 32.5$. Plugging 115,000 (which appears in the graph as 115 on the horizontal axis) into the consumption/utility relationship $u(x)$, we get

$$u(115{,}000) = 0.1(115{,}000)^{0.5} - \left(\frac{115{,}000}{100{,}000}\right) = 32.4934 \approx 32.5 \tag{17.1.xii}$$

which verifies that indeed this consumer is indifferent between the gamble and getting \$115,000 for sure.

Exercise 17.3

We have illustrated in several settings the role of actuarily fair insurance contracts (b, p) *(where b is the insurance benefit in the "bad state" and p is the insurance premium that has to be paid in either state). In this problem we will discuss it in a slightly different way that we will later use in Chapter 22.*

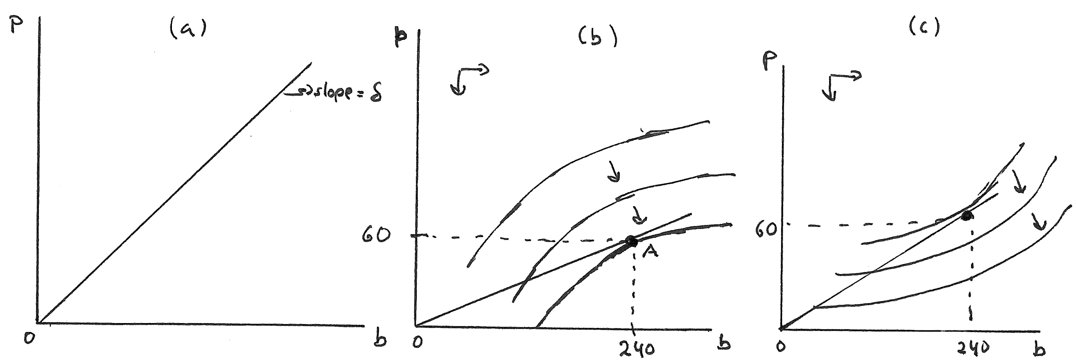
A: *Consider again the example, covered extensively in the chapter, of my wife and life insurance on me. The probability of me not making it is* δ, *and my wife's consumption if I don't make it will be 10 and her consumption if I do make it will be 250 in the absence of any life insurance.*

(a) *Now suppose that my wife is offered a full set of actuarily fair insurance contracts. What does this imply for how p is related to* δ *and b?*

Answer: Actuarial fairness implies that what my wife pays is equal to what she receives *in expectation.* She will receive $(b-p)$ with probability δ, and she will pay p with probability $(1-\delta)$. Thus, actuarial fairness implies that $\delta(b-p) = (1-\delta)p$ or simply $p = \delta b$.

(b) *On a graph with b on the horizontal axis and p on the vertical, illustrate the set of all actuarily fair insurance contracts.*

Answer: This is illustrated in panel (a) of Exercise Graph 17.3.



Exercise Graph 17.3 : Tastes over premiums p and benefits b

(c) *Now think of what indifference curves in this picture must look like. First, which way must they slope (given that my wife does not like to pay premiums but she does like benefits)?*

Answer: Indifference curves must slope up. Consider any initial bundle (b, p). We know that an increase in b to b' will make my wife unambiguously better off — which means that the bundle containing b' that is indifferent to (b, p) must have an offsetting increase in p which, by itself, would make my wife unambiguously worse off. You can thus think of this as indifference curves over two goods where one of the goods, namely the premium p, is really a "bad".

(d) *In which direction within the graph does my wife have to move in order to become unambiguously better off?*

Answer: She becomes unambiguously better off as p falls and b increases — thus, she becomes better off moving to the southeast in the graph.

(e) *We know my wife will fully insure if she is risk averse (and her tastes are state-independent). What policy does that imply she will buy if* $\delta = 0.25$*?*

Answer: As was shown in the text, this would imply buying a policy $(b, p) = (240, 60)$ which satisfies the actuarily fair relationship derived in (a). Under this policy, she would have consumption of only 190 in the "good" state (where she has income of 250 but needs to pay the premium of 60) but she also has consumption of 190 in the "bad" state (where she has income of 10, has to pay the premium of 60 but also gets a benefit of 240).

(f) *Putting indifference curves into your graph from (b), what must they look like in order for my wife to choose the policy that you derived in (e).*

Answer: This is illustrated in panel (b) of Exercise Graph 17.3.

(g) *What would her indifference map look like if she were risk neutral? What if she were risk-loving?*

Answer: If her tastes were risk neutral, she should be indifferent between all the actuarily fair insurance policies along the budget line $p = \delta b$. Thus, her indifference curves must be straight lines with slope δ. If she were risk loving, then she would still become better off moving to the southeast in the graph, but her indifference curves would bow in the opposite direction from those involving risk aversion. This is pictured in panel (c) of Exercise Graph 17.3.

B: *Suppose* $u(x) = \ln(x)$ *allows us to write my wife's tastes over gambles using the expected utility function. Suppose again that my wife's income is 10 if I am not around and 250 if I am — and that the probability of me not being around is* δ.

(a) *Given her incomes in the good and bad state in the absence of insurance, can you use the expected utility function to arrive at her utility function over insurance policies* (b, p)*?*

Answer: Her expected utility is

$$U(x_B, x_G) - \delta u(x_B) + (1-\delta)u(x_G) = \delta \ln x_B + (1-\delta)\ln x_G \qquad (17.3.i)$$

where x_B is her consumption in the event that I am not around and x_G is her consumption in the event that I am around. For any insurance policy (b, p), $x_B = (10 + b - p)$ and $x_G = (250 - p)$. We can therefore write her expected utility of the policy (b, p) as

$$U(b,p) = \delta \ln(10+b-p) + (1-\delta)\ln(250-p). \tag{17.3.ii}$$

(b) *Derive the expression for the slope of an indifference curve in a graph with b on the horizontal and p on the vertical axis.*

Answer: This is just the MRS which is

$$MRS = -\frac{\partial U(b,p)/\partial b}{\partial U(b,p)/\partial p} = -\frac{\delta/(10+b-p)}{\big(-\delta/(10+b-p)\big) - \big((1-\delta)/(250-p)\big)}$$
$$= \frac{\delta(250-p)}{\delta(250-p)+(1-\delta)(10+b-p)}. \tag{17.3.iii}$$

(c) *Suppose* $\delta = 0.25$ *and my wife has fully insured under policy* $(b,p) = (240,60)$. *What is her MRS now?*

Answer: Plugging $\delta = 0.25$, $b = 240$ and $p = 60$ into equation (17.3.iii) gives us

$$MRS = \frac{0.25(190)}{0.25(190)+0.75(190)} = 0.25. \tag{17.3.iv}$$

(d) *How does your answer to (c) compare to the slope of the budget formed by mapping out all actuarily fair insurance policies (as in A(b))? Explain in terms of a graph.*

Answer: We concluded in A(b) that the slope of the budget line is δ which is equal to 0.25 in our case. Now we concluded that, at the actuarily fair full insurance policy, the MRS of our indifference curve is also 0.25. Thus, the indifference curve is tangent to the budget line at the full insurance policy — implying that my wife is optimizing by fully insuring in the acuarily fair insurance market. We depicted this already in panel (b) of Exercise Graph 17.3 where tastes were assumed to be risk averse (as they are when we can use the concave function $u(x) = \ln x$ to represent tastes over gambles using the expected utility function.)

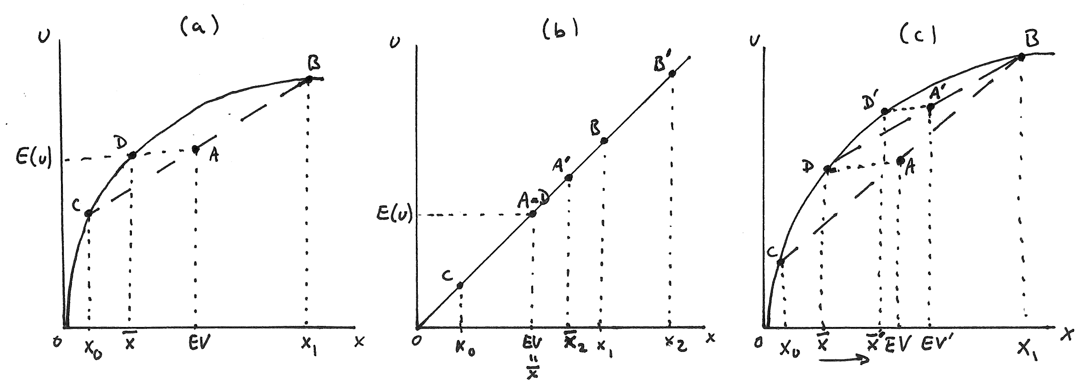
Exercise 17.5

Everyday Application: *Teenage Sex and Birth Control: Consider a teenager who evaluates whether she should engage in sexual activity with her partner of the opposite sex. She thinks ahead and expects to have a present discounted level of life-time consumption of x_1 in the absence of a pregnancy interrupting her educational progress. If she gets pregnant, however, she will have to interrupt her education and expects the present discounted value of her life-time consumption to decline to x_0 — considerably below x_1.*

A: *Suppose that the probability of a pregnancy in the absence of birth control is 0.5 and assume that our teenager does not expect to evaluate consumption any differently in the presence of a child.*

(a) *Putting the present discounted value of lifetime consumption x on the horizontal axis and utility on the vertical, illustrate the consumption/utility relationship assuming that she is risk averse. Indicate the expected utility of consumption if she chooses to have sex.*

Answer: This is done in panel (a) of Exercise Graph 17.5 where the concave shape of the relationship incorporates risk aversion. The expected utility of consumption if the teenager has sex is indicated as $E(u)$.



Exercise Graph 17.5 : Sex, Education and Birth Control

(b) *How much must the immediate satisfaction of having sex be worth in terms of lifetime consumption in order for her to choose to have sex?*

Answer: The expected utility of consumption if she has sex is $E(u)$ — which has a certainty equivalent of $\overline{x}$. Thus, she is giving up $(x_1 - \overline{x})$ by choosing to have sex — which is the least that the experience must be worth to rationalize the action.

(c) *Now consider the role of birth control which reduces the probability of a pregnancy. How does this alter your answers?*

Answer: Birth control reduces the probability of a pregnancy — and thus shifts EV in panel (a) up along the dotted line that connects B and C. Perfectly reliable birth control would imply that A shifts on top of B. As the probability of a pregnancy declines, $\overline{x}$ increases — implying that $(x_1 - \overline{x})$ falls. Thus, sex does not have to be as valued in order for the teenager to choose to engage in it.

(d) *Suppose her partner believes his future consumption paths will develop similarly to hers depending on whether or not there is a pregnancy — but he is risk neutral. For any particular birth control method (and associated probability of a pregnancy), who is more likely to want to have sex assuming no other differences in tastes?*

Answer: This is illustrated in panel (b) of Exercise Graph 17.5 where the consumption/utility relationship is graphed as linear to incorporate risk neutrality. The points A, B and C correspond to those in panel (a) — with $E(u)$ again representing the expected utility from consumption if sexual activity ensues. Note, however, that $\overline{x}$ — the certainty equivalent — is now equal to EV — which implies that $(x_1-\overline{x})$ is lower in panel (b) than in panel (a). The risk neutral partner therefore requires less immediate satisfaction from sexual activity to rationalize it than the risk averse partner.

(e) *As the payoff to education increases in the sense that x_1 increases, what does the model predict about the degree of teenage sexual activity assuming that the effectiveness and availability of birth control remains unchanged and assuming risk neutrality?*

Answer: Consider the case where the probability of a pregnancy is 0.5. We have already shown in panel (b) of Exercise Graph 17.5 that a risk neutral partner would need to place value of at least $(x_1-\overline{x})$ on sex in order to engage in it under these assumptions. Now suppose x_1 increases to x_2. This implies the expected value as well as the certainty equivalent increase to $\overline{x}_2$ — and the increase from $\overline{x}$ to $\overline{x}_2$ is half as much as the increase from x_1 to x_2. The new minimum value that this person must place on sex in order to justify it rationally is $(x_2-\overline{x}_2)$ — as compared to the previous $(x_1-\overline{x}_1)$. But, since the distance from x_1 to x_2 is twice the distance from $\overline{x}$ to $\overline{x}_2$, $(x_2-\overline{x}_2)>(x_1-\overline{x}_1)$ — meaning the value one must place on sex to engage in it has increased. Thus, fewer people will do so.

(f) *Do you think your answer to (e) also holds under risk aversion?*

Answer: Yes. Under risk aversion, the certainty equivalent changes more slowly as x_1 increases — which implies that the value that one must place on sex in order to engage in it (holding birth control constant) would increase more than in the case of risk neutrality.

(g) *Suppose that a government program makes daycare more affordable — thus raising x_0. What happens to the number of risk averse teenagers having sex according to this model?*

Answer: This is illustrated in panel (c) of Exercise Graph 17.5 where the original certainty equivalent is $\overline{x}$ and the original minimum value one must place on sex in order to engage in it is $(x_1-\overline{x})$. As x_0 increases, the certainty equivalent increases (to $\overline{x}'$) but x_1 remains unchanged — which implies that $(x_1-\overline{x}')$, the new minimum value one must place on sex, is less than the original $(x_1-\overline{x})$. Thus, more teenagers will have sex according to this model (assuming teenagers vary in the value they place on having sex).

B: *Now suppose that the function $u(x)=\ln(x)$ allows us to represent a teenager's tastes over gambles involving lifetime consumption using an expected utility function. Let δ represent the probability of a pregnancy occurring if the teenagers engage in sexual activity, and let x_0 and x_1 again represent the two lifetime consumption levels.*

(a) *Write down the expected utility function.*

Answer: The expected utility function is

$$U(x_0, x_1) = \delta \ln x_0 + (1-\delta) \ln x_1. \tag{17.5.i}$$

(b) *What equation defines the certainty equivalent? Using the mathematical fact that $\alpha \ln x + (1-\alpha)\ln y = \ln(x^\alpha y^{(1-\alpha)})$, can you express the certainty equivalent as a function x_0, x_1 and δ?*

Answer: The certainty equivalent $\overline{x}$ is the level of consumption whose utility is equal to the expected utility of the gamble; i.e. $\overline{x}$ is such that

$$\ln \overline{x} = \delta \ln x_0 + (1-\delta) \ln x_1. \tag{17.5.ii}$$

Using the mathematical fact pointed out in the question, this implies

$$\overline{x} = x_0^\delta x_1^{(1-\delta)}. \tag{17.5.iii}$$

(c) *Now derive an equation $y(x_0, x_1, \delta)$ that tells us the least value (in terms of consumption) that this teenager must place on sex in order to engage in it.*

Answer: This is

$$y(x_0, x_1, \delta) = x_1 - \overline{x} = x_1 - x_0^\delta x_1^{(1-\delta)}. \tag{17.5.iv}$$

(d) *What happens to y as the effectiveness of birth control increases? What does this imply about the fraction of teenagers having sex (as the effectiveness of birth control increases) assuming that all teenagers are identical except for the value they place on sex?*

Answer: To see this, we can take the partial derivative of y with respect to δ. This gives us

$$\frac{\partial y(x_0, x_1, \delta)}{\partial \delta} = -(\ln x_0) x_0^\delta x_1^{(1-\delta)} + (\ln x_1) x_0^\delta x_1^{(1-\delta)} = (\ln x_1 - \ln x_0) x_0^\delta x_1^{(1-\delta)} > 0. \tag{17.5.v}$$

Thus, as δ increases, y rises; and as δ decreases, y falls. Birth control becoming more effective implies δ decreases — which therefore implies that the consumption value placed on sex in order for a teenager to engage in it decreases. Put differently, as birth control becomes more effective, some teenagers for whom sex was not sufficiently valuable before will now find it worth it — and thus the fraction of teenagers having sex increases.

(e) *What happens to y as the payoff from education increases in the sense that x_1 increases? What does this imply for the fraction of teenagers having sex (all else equal)?*

Answer: Again, we take a partial derivative to find

$$\frac{\partial y(x_0, x_1, \delta)}{\partial x_1} = 1 - (1-\delta) x_0^\delta x_1^{-\delta} = 1 - (1-\delta)\left(\frac{x_0}{x_1}\right)^{0.5} > 0. \tag{17.5.vi}$$

The reason this expression is greater than zero is because $(x_0/x_1) < 1$ (since $x_0 < x_1$) and $(1-\delta) < 1$ — which implies the term that is being subtracted from 1 in the equation is the product of three numbers that are all below 1 (which must itself then be below 1). This then implies that an increase in x_1 results in an increase in y — i.e. the greater payoffs to education imply that the payoff from sex must increase in order for teenagers to be willing to engage in it. As a result, all else being equal, fewer teenagers will have sex.

(f) *What happens to y as the government makes it easier to continue going to school — i.e. as it raises x_0? What does this imply for the fraction of teenagers having sex?*

Answer: Again, taking the right partial derivative, we get

$$\frac{\partial y(x_0,x_1,\delta)}{\partial x_0} = -\delta x_0^{(\delta-1)} x_1^{(1-\delta)} = -\delta\left(\frac{x_1}{x_0}\right)^{(1-\delta)} < 0. \tag{17.5.vii}$$

Thus, as it gets easier to continue going to school despite a pregnancy, y falls — i.e. the value a teenager must place on sex in order to engage in it falls. This implies that more teenagers will have sex.

(g) *How do your answers change for a teenager with risk neutral tastes over gambles involving lifetime consumption that can be expressed using an expected utility function involving the function $u(x) = x$?*

Answer. The expected utility function would then be $U = \delta x_0 + (1-\delta)x_1$, and the certainty equivalent would be $\overline{x} = \delta x_0 + (1-\delta)x_1$. This implies that y is

$$y(x_0,x_1,\delta) = x_1 - (\delta x_0 + (1-\delta)x_1) = \delta(x_1 - x_0). \tag{17.5.viii}$$

Taking the three partial derivatives, we then get

$$\frac{\partial y}{\partial \delta} = (x_1 - x_0) > 0\,,\ \frac{\partial y}{\partial x_1} = \delta > 0 \text{ and } \frac{\partial y}{\partial x_0} = -\delta < 0. \tag{17.5.ix}$$

The signs of these derivatives are the same as before — implying changes in the same direction as δ, x_1 and x_0 change.

(h) *How would your answers change if $u(x) = x^2$?*

Answer: The expected utility function would be $U = \delta x_0^2 + (1-\delta)x_1^2$, and the certainty equivalent $\overline{x}$ is defined by the equation $u(\overline{x}) = \overline{x}^2 = \delta x_0^2 + (1-\delta)x_1^2$ which solves to

$$\overline{x} = \left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5} \tag{17.5.x}$$

which implies

$$y(x_0,x_1,\delta) = x_1 - \left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}. \tag{17.5.xi}$$

Taking the three partial derivatives, we then get

$$\frac{\partial y}{\partial \delta} = -\frac{1}{2}(x_0^2 - x_1^2)\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{-0.5} = \frac{x_1^2 - x_0^2}{2\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}} > 0 \quad (17.5.\text{xii})$$

$$\frac{\partial y}{\partial x_1} = 1 - \frac{1}{2}(2(1-\delta)x_1)\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{-0.5} =$$

$$= \frac{\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5} - (1-\delta)x_1}{\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}} > 0 \quad (17.5.\text{xiii})$$

$$\frac{\partial y}{\partial x_0} = -2\delta x_0\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{-0.5} = \frac{-2\delta x_0}{\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}} < 0. \quad (17.5.\text{xiv})$$

Thus we again get the same inequalities — implying all the effects still operate in the same direction. The first inequality is straightforward to see since $x_1 > x_0$. The second inequality holds so long as

$$\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5} - (1-\delta)x_1 > 0. \quad (17.5.\text{xv})$$

Adding the second term to both sides and squaring, we get

$$\delta x_0^2 + (1-\delta)x_1^2 > (1-\delta)^2 x_1^2 \quad (17.5.\text{xvi})$$

which can be re-written as

$$\delta x_0^2 > x_1^2\left[(1-\delta)^2 - (1-\delta)\right] = -\delta(1-\delta)x_1^2. \quad (17.5.\text{xvii})$$

This always holds given that $\delta x_0^2 > 0$ and the right hand side is less than zero.

Exercise 17.7

Everyday Application: *Venice and Regret: Suppose that you can choose to participate in one of two gambles: In Gamble 1 you have a 99% chance of winning a trip to Venice and a 1% chance of winning tickets to a movie about Venice; and in Gamble 2, you have a 99% of winning the same trip to Venice and a 1% chance of not winning anything.*

A: *Suppose you very much like Venice, and, were you to be asked to rank the three possible outcomes, you would rank the trip to Venice first, the tickets to the movie about Venice second, and having nothing third.*

(a) *Assume that you can create a consumption index such that getting nothing is denoted as 0 consumption, getting the tickets to the movie is $x_1 > 0$ and getting the trip is $x_2 > x_1$. Denote the expected value of Gamble 1 by $E(G_1)$ and the expected value of Gamble 2 by $E(G_2)$. Which is higher?*

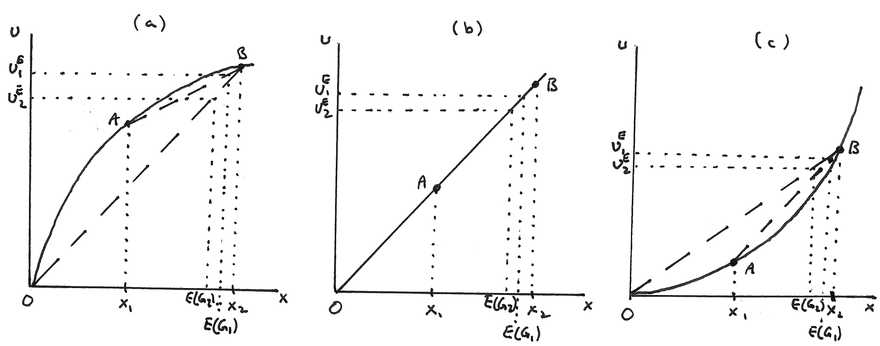
Answer: The expected value of Gamble 1, $E(G_1)$, is higher than the expected value of Gamble 2, $E(G_2)$ — because we take the same weighted average between x_2 and x_1 to get to $E(G_1)$ as we are between x_2 and 0 to get to $E(G_2)$. Thus

$$E(G_1) = 0.99x_2 + 0.01x_1 \text{ and } E(G_2) = 0.99x_2 + 0.01(0) = 0.99x_2. \quad (17.7.i)$$

Since $x_1 > 0$, it must therefore be the case that $E(G_1) > E(G_2)$.

(b) *On a graph with x on the horizontal axis and utility on the vertical, illustrate a consumption/utility relationship that exhibits risk aversion.*

Answer: This is illustrated in panel (a) of Exercise Graph 17.7 where the relationship takes on the concave shape necessary for risk aversion.



Exercise Graph 17.7 : Trips to and Movies about Venice

(c) *In your graph, illustrate the expected utility you receive from Gamble 1 and from Gamble 2. Which gamble will you choose to participate in?*

Answer: This is also illustrated in panel (a) of Exercise Graph 17.7. The expected utility of Gamble 1 is read off on the line connecting points A and B, and the expected utility of Gamble 2 is read off the line connecting (the origin) 0 to B. Thus, u_1^E is the expected utility of Gamble 1 and u_2^E is the expected utility of Gamble 2. We can see immediately that $u_1^E > u_2^E$ — thus you would choose to participate in Gamble 1 over Gamble 2.

(d) *Next, suppose tastes are risk neutral instead. Re-draw your graph and illustrate again which gamble you would choose. (*Hint*: Be careful to accurately differentiate between the expected values of the two gambles.)*

Answer: This is illustrated in panel (b) of Exercise Graph 17.7 where the shape of the consumption/utility relationship is now linear (as is required for risk neutrality). The expected utility of the gambles is again read off the lines that connect A and B (for Gamble 1) and 0 and B (for Gamble 2) — but these now lie on the consumption/utility relationship. Since $E(G_1) > E(G_2)$, we see that the expected utility of Gamble 1, u_1^E, is greater than the expected utility of Gamble 2, u_2^E. Again, you will choose Gamble 1 over Gamble 2.

(e) *It turns out (for reasons that become clearer in part B) that, risk aversion (or neutrality) is irrelevant for how individuals whose behavior is explained by expected utility theory will choose among these gambles. In a separate graph, illustrate the consumption/utility relationship again, but this time assume risk loving. Illustrate in the graph how your choice over the two gambles might still be the same as in parts (c) and (d). Can you think of why it in fact* has to be *the same?*

Answer: This is illustrated in panel (c) of Exercise Graph 17.7. Although the line connecting A and B now lies above the line connecting 0 and B, it is still the case that $E(G_1) > E(G_2)$. Thus, the graph can easily be drawn with the expected utility of Gamble 1 (u_1^E) greater than the expected utility of Gamble 2 (u_2^E). To see why this in fact *has to* be the case, denote the consumption/utility relationship $u(x)$. Thus, the utility of x_2 is given by $u(x_2)$, the utility of x_1 is given by $u(x_1)$ and the utility of 0 is simply $u(0) = 0$. The expected utility levels u_1^E and u_2^E (that lie on the lines connecting the outcomes) associated with Gambles 1 and 2 are then given by

$$u_1^E = 0.99u(x_2) + 0.01u(x_1) \text{ and } u_2^E = 0.99u(x_2) + 0.01u(0) = 0.99u(x_2). \tag{17.7.ii}$$

Since $u(x_1) > 0$, it must then be that $u_1^E > u_2^E$, and it is irrelevant whether the u function is concave or convex — so long as it slopes up and thus $u(x_1) > 0$. More intuitively, Gambles 1 and 2 place the same probability on winning the trip, but Gamble 1 places the remaining probability on winning a movie ticket while Gamble 2 does not. Thus, because Gamble 1 contains something "extra", it must be preferred to Gamble 2 under expected utility theory.

(f) *It turns out that many people, when faced with a choice of these two gambles, end up choosing Gamble 2. Assuming that such people would indeed rank the three outcomes the way we have, is there any way that such a choice can be explained using expected utility theory (taking as given that the choice implied by expected utility theory does not depend on risk aversion?)*

Answer: No, it cannot given the answer to (e). Put simply, the person gets the trip with probability 0.99 in both Gambles, but he gets something additional in Gamble 1 but not in Gamble 2. If that something additional — the movie ticket — is valuable, then Gamble 1 has to be better than Gamble 2 according to expected utility theory.

(g) *This example is known as* Machina's Paradox. *One explanation for it (i.e. for the fact that many people choose Gamble 2 over Gamble 1) is that expected utility theory does not take into account* regret. *Can you think of how this might explain people's paradoxical choice of Gamble 2 over Gamble 1?*

Answer: Having had such a high chance of actually winning the trip, not getting it might cause regret — and then watching a movie about Venice might make it worse. Thus, it is not that the person does not, all else equal, prefer the movie ticket to nothing. But the movie ticket — after coming so close to being able to get to Venice in person — might actually be worse than nothing because of the fact that the person is reminded of what he has lost. None of this fits into expected utility theory.

B: *Assume again, as in part A, that individuals prefer a trip to Venice to the movie ticket, and they prefer the movie ticket to getting nothing. Furthermore, suppose there exists a function u that assigns u_2 as the utility of getting the trip, u_1 as the utility of getting the movie ticket and u_0 as the utility of getting nothing, and suppose that this function u allows us to represent tastes over risky pairs of outcomes using an expected utility function.*

(a) *What inequality defines the relationship between u_1 and u_0?*

Answer: It must be that $u_1 > u_0$.

(b) *Now multiply both sides of your inequality from (a) by 0.01, and then add $0.99u_2$ to both sides. What inequality do you now have?*

Answer: Multiplying the two inequalities as instructed, we get $0.01u_1 > 0.01u_0$, and adding $0.99u_2$ to both sides, we get

$$0.99u_2 + 0.01u_1 > 0.99u_2 + 0.01u_0. \tag{17.7.iii}$$

(c) *Relate the inequality you derived in (b) to the expected utility of the two gambles in this example. What gamble does expected utility theory predict a person will choose (assuming the outcomes are ranked as we have ranked them)?*

Answer: The left hand side is the expected utility of Gamble 1 and the right hand side is the expected utility of Gamble 2. Since the left hand side is greater than the right hand side, expected utility theory implies that this person will choose Gamble 1 over Gamble 2.

(d) *When we typically think of a "gamble", we are thinking of different outcomes that will happen with different probabilities. But we can also think of "degenerate" gambles — i.e. gambles where one outcome happens with certainty. Define the following three such "gambles": Gamble A results in the trip to Venice with probability of 100%; Gamble B results in the movie ticket with probability of 100%; and Gamble C results in nothing with probability of 100%. How are these degenerate "gambles" ranked by someone who prefers the trip to the ticket to nothing?*

Answer: It must then be the case that $G_A \succ G_B \succ G_C$.

(e) *Using the notion of* mixed gambles *introduced in Appendix 1, define Gambles 1 and 2 as mixed gambles over the degenerate "gambles" we have just defined in (d). Explain how the* Independence Axiom *from Appendix 1 implies that Gamble 1 must be preferred to Gamble 2.*

Answer: Gamble 1 is simply Gamble A (which is equivalent to getting the trip) and Gamble B (which is equivalent to getting the movie ticket) mixed with weight 0.99 on G_A and 0.01 weight on G_B. Similarly, Gamble 2 is equivalent to mixing Gamble A with weight 0.99 and Gamble C with weight 0.01. We can thus write that

$$G_1 = 0.99G_A + 0.01G_B \text{ and } G_2 = 0.99G_A + 0.01G_C. \tag{17.7.iv}$$

(f) True or False*: When individuals who rank the outcomes the way we have assumed choose Gamble 2 over Gamble 1, expected utility theory fails because the independence axiom is violated.*

Answer: This is true. The independence axiom says that, if a Gamble B is preferred to a Gamble C, then the mixture of Gamble B with a third Gamble A must be preferred to the mixture of Gamble C with Gamble A so long as they are mixed with equal weights; i.e.

$$G_B \succ G_C \text{ implies } (\delta G_B + (1-\delta)G_A) \succ (\delta G_C + (1-\delta)G_A) \text{ for all } 0 < \delta < 1. \tag{17.7.v}$$

When $\delta = 0.01$, the left hand side of this implication becomes G_1 and the right hand side becomes G_2. Thus, the independence axiom implies that, if the movie ticket is worth more than nothing to the individual, then $G_1 \succ G_2$. Expected utility theory cannot predict that someone like this will choose Gamble 2 over Gamble 1 because such a prediction would imply a violation of the independence axiom on which expected utility theory is built.

(g) *Would the paradox disappear if we assumed state-dependent tastes? (*Hint*: As with the Allais paradox in Appendix 2, the answer is no.)*

Answer: The reason that assuming state-dependent tastes does not resolve the Machina Paradox is because it does not matter whether we choose one function u to assign utility to each outcome (as we would if tastes are state-independent) or we choose three separate functions u_2, u_1 and u_0 to assign utility to the outcomes. The question is whether we can assign utility values at all such that the expected utility of each gamble is a probability weighted average of the utilities associated with each outcome in the gamble. If we can find a way to assign such utility values, then expected utility theory can be applied, and it implies (as we have shown) that Gamble 1 is preferred to Gamble 2. Choosing Gamble 2 over Gamble 1 is then inconsistent with expected utility theory regardless of whether tastes are state dependent.

Exercise 17.9

Business Application: *Diversifying Risk along the Business Cycle*: *Suppose you own a business that does well during economic expansions but not so well during recessions which happen with probability δ. Let x_E denote your consumption level* *during expansions and let x_R denote your consumption level during recessions. Unless you do something to diversify risk, these consumption levels are $E=(e_E,e_R)$ where e_E is your income during expansions and e_R your income during recessions (with $e_E>e_R$). Your tastes over consumption are the same during recessions as during expansions and you are risk averse. For any asset purchases described below, assume that you pay for these assets from whatever income you have depending on whether the economy is in recessions or expansion.*

A: *Suppose I own a financial firm that manages asset portfolios. All I care about as I manage my business is expected returns, and any asset I sell is characterized by (p,b_R,b_E) where p is how much I charge for 1 unit of the asset, b_R is how much the asset will pay you (as, say, dividends) during recessions and b_E is how much the asset will pay you during expansions.*

(a) *Is someone like me — who only cares about expected returns — risk averse, risk loving or risk neutral?*

Answer: Someone who only cares about expected returns (but not risk) is risk neutral.

(b) *Suppose that all the assets I offer have the feature that those who buy these assets experience no change in their expected consumption levels as a result of buying my assets. Derive an equation that expresses the price p of my assets in terms of δ, b_R and b_E.*

Answer: In order for your expected consumption to remain unchanged, it must be that the expected change in consumption during recessions is exactly offset by the expected change in consumption during expansions — i.e.

$$\delta(-p+b_R)=-(1-\delta)(-p+b_E) \tag{17.9.i}$$

which solves to give us

$$p=\delta b_R+(1-\delta)b_E. \tag{17.9.ii}$$

(c) *What happens to my expected returns when I sell more or fewer of such assets?*

Answer: Just as your expected consumption is unchanged when you buy these assets, my expected returns are unchanged.

(d) *Suppose you buy 1 asset (p,b_R,b_E) that satisfies our equation from (b). How does your consumption during expansions and recessions change as a result?*

Answer: Your consumption during recessions will be

$$x_R=e_R-p+b_R=e_R-(\delta b_R+(1-\delta)b_E)+b_R=e_R+(1-\delta)(b_R-b_E), \tag{17.9.iii}$$

and your consumption during expansion would be

$$x_E = e_E - p + b_E = e_E - (\delta b_R + (1-\delta)b_E) + b_E = e_E + \delta(b_E - b_R). \quad (17.9.iv)$$

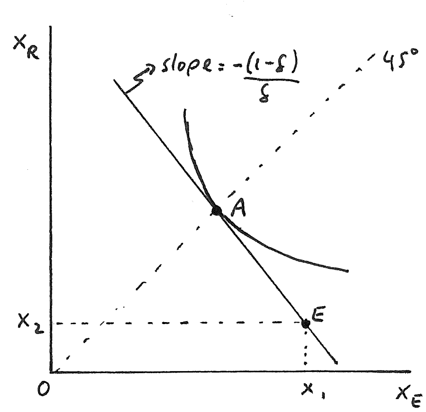
(e) *At what rate do assets of the kind I am offering allow you to transfer consumption opportunities from expansions to recessions? On a graph with x_E on the horizontal and x_R on the vertical axis, illustrate the "budget line" that the availability of such assets creates for you.*

Answer: In order to transfer consumption from expansions to recessions, you need to pick assets with $b_E < b_R$. Suppose you pick an asset with $b_R - b_E = 1$. Then, when you buy one unit of such an asset, your consumption in the two states becomes

$$x_R = e_R + (1-\delta) \text{ and } x_E = e_E - \delta. \quad (17.9.v)$$

Thus, you are trading δ in consumption during expansions for $(1-\delta)$ in consumption during recessions — or, put differently, for every \$1 in consumption you give up during expansions, you get $\$(1-\delta)/\delta$ during recessions.

This is then illustrated in a graph in Exercise Graph 17.9.



Exercise Graph 17.9 : Reducing Risk across Business Cycles

(f) *Illustrate in your graph your optimal choice of assets.*

Answer: This is also illustrated in Exercise Graph 17.9 where risk aversion and state-independence of tastes implies that you will "fully insure" because the terms of trade are "actuarily fair" in the sense that your expected consumption level does not change. As a result, you optimize at point A in the graph.

(g) *Overall output during recessions is smaller than during expansions. Suppose everyone is risk averse. Is it possible for us to all end up doing what you concluded you would do in (f)? (We will explore this further in exercise 17.10.)*

Answer: No — if the economy shrinks, it is not possible for everyone to fully insure in the sense of maintaining the same level of consumption regardless of the state of the economy.

B: *Suppose that the function $u(x) = x^\alpha$ is such that we can express your tastes over gambles using expected utility functions.*

(a) *If you have not already done so in part A, derive the expression $p(\delta, b_R, b_E)$ that relates the price of an asset to the probability of a recession δ, the dividend payment b_R during recessions and the dividend payment b_E during economic expansions assuming that purchase of such assets keeps expected consumption levels unchanged.*

Answer: Repeating our derivation from before: In order for your expected consumption to remain unchanged, it must be that the expected change in consumption during recessions is exactly offset by the expected change in consumption during expansions — i.e.

$$\delta(-p + b_R) = -(1-\delta)(-p + b_E) \tag{17.9.vi}$$

which solves to give us

$$p = \delta b_R + (1-\delta) b_E. \tag{17.9.vii}$$

(b) *Suppose you purchase k units of the same asset (b_E, b_R) which is priced as you derived in part (a) and for which $(b_R - b_E) = y > 0$. Derive an expression for x_R defined as your consumption level during recessions (given your recession income level of e_R) assuming you purchase these assets. Derive similarly an expression for your consumption level x_E during economic expansions.*

Answer: Your consumption during recessions will be equal to your recession income plus the dividends from your assets minus the price of the assets:

$$\begin{aligned} x_R &= e_R + kb_R - kp = e_R + kb_R - k(\delta b_R + (1-\delta) b_E) = \\ &= e_R + (1-\delta)k(b_R - b_E) = e_R + (1-\delta)ky. \end{aligned} \tag{17.9.viii}$$

Similarly,

$$\begin{aligned} x_E &= e_E + kb_E - kp = e_E + kb_E - k(\delta b_R + (1-\delta) b_E) = \\ &= e_E + \delta k(b_E - b_R) = e_E - \delta ky. \end{aligned} \tag{17.9.ix}$$

(c) *Set up an expected utility maximization problem where you choose k — the number of such assets that you purchase. Then solve for k.*

Answer: We have already determined the consumption levels x_R and x_B conditional on how many assets you buy subject to the pricing constraints. Thus, all we have to solve is the unconstrained optimization problem

$$\begin{aligned}\max_k \delta u(x_R) + (1-\delta)u(x_E) &= \delta x_R^\alpha + (1-\delta)x_E^\alpha = \\ &= \delta[e_R + (1-\delta)ky]^\alpha + (1-\delta)[e_E - \delta ky]^\alpha.\end{aligned} \tag{17.9.x}$$

Taking the first derivative of the right-hand side and setting it to zero, we get

$$\alpha\delta(1-\delta)y[e_R + (1-\delta)ky]^{(\alpha-1)} = \alpha(1-\delta)\delta y[e_E - \delta ky]^{(\alpha-1)} \tag{17.9.xi}$$

which simplifies to

$$e_R + (1-\delta)ky = e_E - \delta ky \tag{17.9.xii}$$

which we can solve for

$$k = \frac{(e_E - e_R)}{y} = \frac{(e_E - e_R)}{(b_R - b_E)}. \tag{17.9.xiii}$$

(For the last equality, we simply substituted back in for $y = (b_R - b_E)$.)

(d) *How much will you consume during recessions and expansions?*

Answer: Substituting (17.9.xiii) into (17.9.viii) and (17.9.ix), we get

$$x_R = e_R + (1-\delta)\left(\frac{(e_E - e_R)}{y}\right)y = \delta e_R + (1-\delta)e_E \tag{17.9.xiv}$$

$$x_E = e_E + \delta\left(\frac{(e_E - e_R)}{y}\right)y = \delta e_R + (1-\delta)e_E. \tag{17.9.xv}$$

Thus, you will buy sufficient numbers of assets such that consumption in recessions and expansions is equalized.

(e) *For what values of α is your answer correct?*

Answer: The answer is correct for $\alpha < 1$ when tastes over gambles are risk averse. It is not correct for $\alpha > 1$ when tastes are risk-loving. The calculus still produces the same answer, but the indifference curves now bow out and, while they are tangent to the budget at the "full insurance" bundle, they are tangent from below and therefore a local minimum rather than a maximum. When tastes are risk loving, the true solution is a corner solution. And when $\alpha = 1$, all outcome bundles with the same expected consumption value are optimal — including the one derived.

(f) True or False*: So long as assets that pay more dividends during recessions than expansions are available at "actuarily fair" prices, you will be able to fully insure against consumption shocks from business cycles.*

Answer: This is true, as we have just shown. We assumed $(b_R > b_E)$ for the assets that we are buying — and equation (17.9.xiii) shows that the smaller the difference between the recession and expansion dividends, the more assets we will buy — always with the ultimate goal of equalizing consumption across the business cycle.

(g) *Could you accomplish the same outcome by instead creating and selling assets with* $(b_E > b_R)$*?*

Answer: Yes — you can do exactly the same thing if you price such assets according to our pricing formula (equation (17.9.vii)) that keeps expected consumption constant. In this case, you would be paying someone else $(p - b_E)$ during expansions, but you would receive $(p - b_R) > 0$ during recessions.

Exercise 17.11

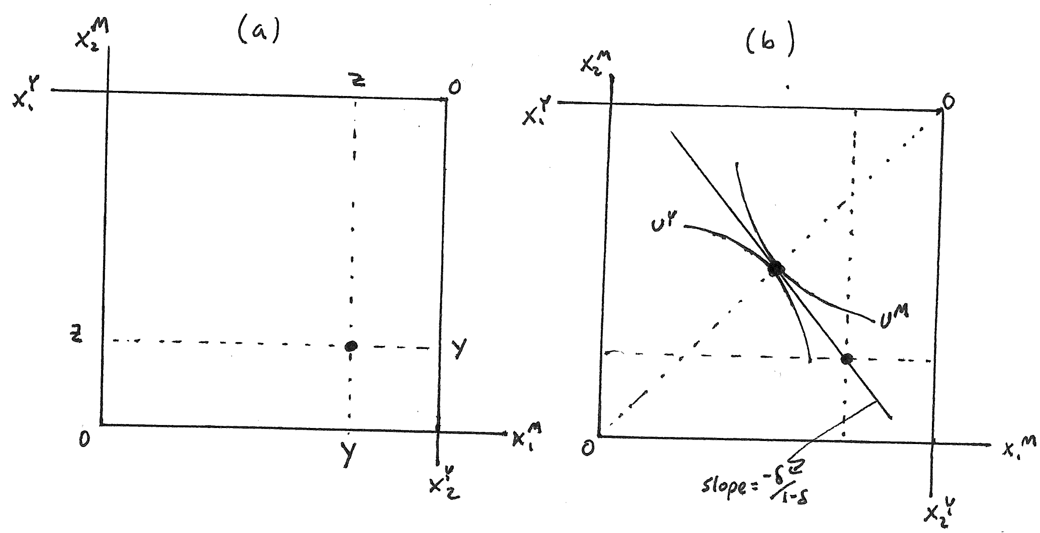
Business Application: *Local versus National Insurance*: *Natural disasters are local phenomena — impacting a city or a part of a state but rarely impacting the whole country, at least if the country is geographically large. To simplify the analysis, suppose there are two distinct regions that might experience local disasters.*

A: *Define "state 1" as region 1 experiencing a natural disaster, and define "state 2" as region 2 having a natural disaster. I live in region 2 while you live in region 1. Both of us have the same risk averse and state-independent tastes, and our consumption level falls from* y *to* z *when a natural disaster strikes. The probability of state 1 is* δ *and the probability of state 2 is* $(1-\delta)$.

(a) *Putting consumption* x_1 *in state 1 on the horizontal axis and consumption* x_2 *in state 2 on the vertical, illustrate an Edgeworth box assuming you and I are the only ones living in our respective regions. Illustrate our "endowment" bundle in this box.*

Answer: This is illustrated in panel (a) where my consumption is superscripted by M and yours by Y. Since I live in region 2, my consumption is high in state 1 when the disaster strikes in region 1; and since you live in region 1, your consumption is high in state 2 when the disaster strikes in region 2. The box is a square because, no matter where the disaster strikes, the overall level of consumption in the economy is $(y+z)$ — i.e. there is no aggregate risk.

(b) *Suppose an insurance company wanted to insure us against the risks of natural disasters. Under actuarily fair insurance, what is the opportunity cost of state 2 consumption in terms of state 1 consumption? What is the opportunity cost of state 1 consumption in terms of state 2 consumption? Which of these is the slope of the acuarily fair budget in your Edgeworth Box?*



Exercise Graph 17.11(1) : Disaster Insurance

Answer: Actuarily fair insurance means that our expected consumption remains unchanged from being insured. Thus, if I want insurance that gives me \$1 in state 2 that happens with probability $(1-\delta)$, I am asking for an expected net benefit of $(1-\delta)$. This needs to be offset by an expected net payment of δp that will be made to the insurance company in state 1 — and actuarial fairness implies $\delta p = (1-\delta)$ or $p = (1-\delta)/\delta$. Thus, the opportunity cost of \$1 in state 2 is $(1-\delta)/\delta$ in state 1. Alternatively, the opportunity cost of \$1 in state 1 is $\delta/(1-\delta)$ in state 2. The latter is the slope of the actuarily fair budget constraint when state 1 is on the horizontal axis.

(c) *Illustrate the budget line that arises from the set of all actuarily fair insurance contracts within the Edgeworth Box. Where would you and I choose to consume assuming we are risk averse?*

Answer: This is done in panel (b) of Exercise Graph 17.11(1). Under actuarily fair insurance terms, we would both choose to fully insure along the 45 degree line that connects the lower left to the upper right corners of the box.

(d) *How does this outcome compare to the equilibrium outcome if you and I were simply to trade state-contingent consumption across the two states?*

Answer: It is identical as can quickly be seen in panel (b) of Exercise Graph 17.11(1) where the budget line formed by the actuarily fair insurance terms causes us to optimize at the same point in the box.

(e) *Suppose there were two of me and two of you in this world. Would anything change?*

Answer: No, nothing would change as the same prices would still get us to optimize at the same point and, because there is no aggregate risk, there are enough resources in both states for the relevant trades to take place.

(f) *Now suppose that the two of me living in region 2 go to a local insurance company that operates only in region 2. Why might this company not offer us actuarily fair insurance policies?*

Answer: This insurance company may find it difficult to insure us because of the aggregate risk that the local economy faces. The insurance company needs to be able to write enough policies with risks that are offsetting so that it can in expectation meet the costs of all the benefits it has to pay with the premiums it is collecting in all those places where disaster does not strike. But if a local insurance company only sells local policies, it does not have people in other places where disaster won't strike to write offsetting policies.

(g) *Instead of insurance against the consequences of natural disasters, suppose we instead considered insurance against non-communicable illness. Would a local insurance company face the same kind of problem offering actuarily fair insurance in this case?*

Answer: No, the same problem would not arise for a local insurance company — because the "disasters" are not striking randomly without being clustered in geographic areas.

(h) *How is the case of local insurance companies insuring against local natural disasters similar to the case of national insurance companies insuring against business cycle impacts on consumption? How might international credit markets that allow insurance companies to borrow and lend help resolve this?*

Answer: In both cases, the problem is aggregate risk that does not make sufficient resources available in one state to make it possible to pay the necessary obligations. If insurance companies have access to full international credit markets, though, they can resolve this problem. They would do so by borrowing in such markets during times when bad times hit all at once (due to aggregate risk) and lend in good times.

B: *Suppose that, as in exercise 17.10, the function $u(x) = \ln x$ allows us to represent our tastes over gambles as expected utilities. Assume the same set-up as the one described in A.*

(a) *Let p_1 be defined as the price of \$1 of consumption if state 1 occurs and let p_2 be the price of \$1 of consumption in the event that state 2 occurs. Set $p_2 = 1$ and then denote the price of \$1 of consumption in the event of state 1 occurring as $p_1 = p$ and write down your budget constraint.*

Answer. Your budget constraint is then $pz + y = px_1 + x_2$, where the left hand side is the value of your endowment and the right hand side is the value of your consumption opportunity bundle to which you trade.

(b) *Solve the expected utility maximization problem given this budget constraint to get your demand x_1 for state 1 consumption as well as your demand x_2 for state 2 consumption.*

Answer: Your expected utility function is $U(x_1, x_2) = \delta \ln x_1 + (1-\delta)\ln x_2$, and your expected utility maximization problem is

$$\max_{x_1,x_2} \delta \ln x_1 + (1-\delta)\ln x_2 \text{ subject to } pz + y = px_1 + x_2. \tag{17.11.i}$$

Solving this in the usual way, we get your demands

$$x_1 = \frac{\delta(pz+y)}{p} \text{ and } x_2 = (1-\delta)(pz+y). \tag{17.11.ii}$$

(c) *Repeat (a) and (b) for me.*

Answer: For me, the budget constraint is $py + z = px_1 + x_2$ (because my endowment is the symmetric opposite of yours.) Otherwise everything is the same — giving us the following demands for me:

$$x_1 = \frac{\delta(py+z)}{p} \text{ and } x_2 = (1-\delta)(py+z). \tag{17.11.iii}$$

(d) *Derive the equilibrium price. Is this acutarily fair?*

Answer: In equilibrium, the demand for x_1 must be equal to the economy's endowment $(y+z)$ (as must the demand for x_2 since the economy's endowment is the same in both states.) Equilibrium in state 1 therefore implies

$$x_1^M + x_1^Y = y + z, \tag{17.11.iv}$$

where M superscripts my demand and Y superscripts yours. Plugging in the demands we calculated before, this can be written as

$$\frac{\delta(pz+y)}{p} + \frac{\delta(py+z)}{p} = y + z. \tag{17.11.v}$$

Solving this, we get

$$p = \frac{\delta}{(1-\delta)}. \tag{17.11.vi}$$

(Note that this is the inverse of what we have often derived under similar conditions because the probability of the state 1 rather than the probability of state 2 is δ here.)

(e) *How much do we consume in each state?*

Answer: Plugging the equilibrium price back into our demands from before, we get

$$x_1^Y = \delta y + (1-\delta)z = x_2^Y \text{ and } x_1^M = \delta z + (1-\delta)y = x_2^M \tag{17.11.vii}$$

where Y again superscripts you and M superscripts me. Thus, we both fully insure.

(f) *Does the equlibrium price change if there are 2 of you and 2 of me?*

Answer: We would still need that the demand is equal to the available endowment in each state. For state 1, this implies

$$2x_1^M + 2x_1^Y = 2(y+z), \tag{17.11.viii}$$

which, once we cancel the 2's, is identical to the previous equilibrium equation (17.11.iv). Thus, the equilibrium does not change as we increase the number of parties on each side of the market.

(g) *Finally, suppose that the two of me attempt to trade state-contingent consumption just between us. What will be the equilibrium price?*

Answer: The equilibrium in state 1 now requires twice my demand to sum to twice my endowment — i.e.

$$2\left(\frac{\delta(py+z)}{p}\right) = 2y \tag{17.11.ix}$$

which we can solve for

$$p = \frac{\delta z}{(1-\delta)y}. \tag{17.11.x}$$

(h) *Will we manage to trade at all?*

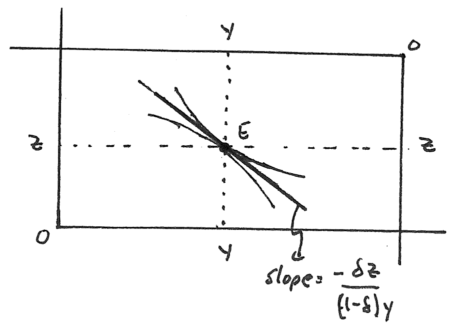
Answer: Plugging this back into my demand for x_1, we get

$$x_1 = \frac{\delta\left(\left(\frac{\delta z}{(1-\delta)y}y+z\right)\right)}{\frac{\delta z}{(1-\delta)y}} = y. \tag{17.11.xi}$$

Thus, at the equilibrium price, each of us simply consumes our endowment and no trade occurs.

(i) *Can you illustrate this in an Edgeworth Box? Is the equilibrium efficient?*

Answer: This is illustrated in Exercise Graph 17.11(2) where the Edgeworth Box is no longer a square since the economy's endowment in state 1 is now $2y$ and the endowment in state 1 is $2z$ (where $y > z$). Our individual endowment bundle is now in the center of the box, and the equilibrium price keeps both of us optimizing at that bundle. Since our indifference curves are tangent to one another, the equilibrium is efficient even though we both continue to face risk. The risk cannot be reduced because of the presence of aggregate risk.



Exercise Graph 17.11(2) : No Trade Equilibrium

Exercise 17.13

Policy Application: *More Police or More Teachers? Enforcement versus Education: Suppose again (as in exercise 17.12) that the payoff from engaging in a life of crime is x_1 if you don't get caught and x_0 (significantly below x_1) if you end up in jail, with δ representing the probability of getting caught. Suppose everyone has identical tastes but we differ in terms of the amount of income we can earn in the (legal) labor market — with (legal) incomes distributed uniformly (i.e. evenly) between x_0 and x_1.*

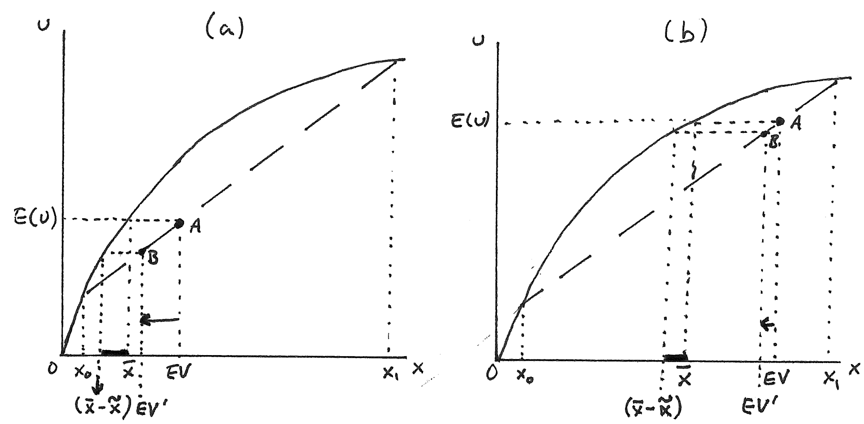
A: *Suppose there are two ways to lower crime rates: Spend more money on police officers so that we can make it more likely that those who commit crimes get caught, or spend more money on teachers so that we increase the honest income that potential criminals could make. The first policy raises δ; the second raises individual incomes through better education.*

(a) *Begin by drawing a risk averse individual's consumption/utility relationship and assume a high δ. Indicate the corresponding $\overline{x}$ that represents the (honest) income level at which a person is indifferent between an honest life and a life of crime.*

Answer: This is done in panel (a) of Exercise Graph 17.13 where the consumption/utility relationship is concave due to the assumption of risk aversion.

If δ is high, it means the probability of getting caught is high — which means the expected consumption value EV of a life of crime is relatively low. Thus, EV is drawn at a relatively low level — with the expected utility of a life of crime read off point A. The certainty equivalent is the level of consumption $\overline{x}$ that yields the same level of utility as the expected utility of the life of crime — and it is equal to the honest income level at which a person is indifferent between an honest life and a life of crime.

(b) *Consider a policy that invests in education and results in a uniform increase in all incomes by an amount $\tilde{x}$. On the horizontal axis of your graph,*



Exercise Graph 17.13 : More Police or More Teachers?

indicate which types of individuals (identified by their pre-policy income levels) will now switch from a life of crime to an honest life.

Answer: An individual who previously could make an honest living of $(\overline{x}-\tilde{x})$ will now end up earning $\overline{x}$ in the (legal) labor market. Thus, individuals whose pre-policy (honest) income falls in the darkened interval from $(\overline{x}-\tilde{x})$ to $\overline{x}$ will switch from a life of crime to an honest life as a result of the investments in education.

(c) *Next, consider the alternative policy of investing in more enforcement — thus increasing the probability of getting caught δ. Indicate in your graph how much the expected consumption level of a life of crime must be shifted in order for the policy to achieve the same reduction in crime as the policy in part (b).*

Answer: This is also illustrated in panel (a) of Exercise Graph 17.13. In order for a policy focused on raising δ to reduce crime by the same amount (without legal incomes rising), it must be that the expected consumption value of a life of crime falls sufficiently to make the expected *utility* of crime equal to the utility of an individual with (legal) income $(\overline{x}-\tilde{x})$. This would imply an increase in δ sufficiently high to move us to B — which requires a shift of the expected consumption value of crime from EV to EV'. Because the consumption/utility relationship is steep, $(EV-EV')$ is greater than $\overline{x}$.

(d) *If it costs the same to achieve a \$1 increase in everyone's income through education investments as it costs to achieve a \$1 reduction in the expected consumption level of a life of crime, which policy is more cost effective at reducing crime given we started with an already high δ.*

Answer: Since $(EV - EV')$ is greater than $\overline{x}$, it is more cost effective to reduce crime through investments in education in this case.

(e) *How does your answer change if δ is very low to begin with?*

Answer: This is pictured in panel (b) of Exercise Graph 17.13. Following the same steps as before, we now find that $(EV - EV')$ is less than $\overline{x}$ — implying it is more cost effective to reduce crime through increases in policing rather than investments in education.

(f) True or False*: Assuming people are risk averse, the following is an accurate policy conclusion from our model of expected utility: The higher current levels of law enforcement, the more likely it is that investments in education will cause greater reductions in crime than equivalent investments in additional law enforcement.*

Answer: This is true based on our analysis thus far. When δ is high, law enforcement levels are already high — in which case we found it is more likely to be cost effective to invest in education rather than additional law enforcement than when δ is low.

B: *Now suppose that, as in exercise 17.10, $x_0 = 20$ and $x_1 = 80$ (where we can think of these values as being expressed in terms of thousands of dollars).*

(a) *Suppose, again as in exercise 17.10, that expressing utility over consumption by $u(x) = \ln x$ allows us to express tastes over gambles using the expected utility function. If $\delta = 0.75$, what is the income level $\overline{x}$ at which an individual is indifferent between a life of crime and an honest life?*

Answer: When $\delta = 0.75$, the expected utility from a life of crime is given by

$$0.75\ln(20) + 0.25\ln(80) \approx 3.342. \qquad (17.13.i)$$

The certainty equivalent $\overline{x}$ is the obtained by setting $\ln(\overline{x}) = 3.342$ which solves to $\overline{x} = e^{3.342} \approx 28.28$ — the value of an honest income that makes individuals indifferent between an honest life and a life of crime.

(b) *If an investment in eduction results in a uniform increase of income of 5, what are the pre-policy incomes of people who will now switch from a life of crime to an honest life?*

Answer: Since we concluded before that 28.28 is the cut-off (honest) income level above which the expected utility from a life of crime is below the utility of an honest life, it is those with pre-policy incomes between 23.28 and 28.28 that will switch from lives of crime to honest lives.

(c) *How much would δ have to increase in order to achieve an equivalent reduction in crime? How much would this change the expected consumption level under a life of crime?*

Answer: In order for an increase in δ to accomplish the same thing, it must be that δ is sufficiently high for the expected utility of crime to lie below the utility from consumption of 23.28; i.e.

$$\delta \ln(20) + (1-\delta)\ln(80) = \ln(23.28). \qquad (17.13.ii)$$

Solving for δ, we get $\delta \approx 0.89$ — i.e. we have to increase enforcement from 0.75 to 0.89 in order to achieve the same reduction in crime as the education policy analyzed before.

(d) *If it is equally costly to raise incomes by \$1 through education investments as it is to reduce the expected value of consumption in a life of crime through an increase in δ, which policy is the more cost effective way to reduce crime?*

Answer: The education investment policy raises incomes for everyone by 5. The increased enforcement policy that raises δ to 0.89 results in a reduction in the expected consumption value of crime to

$$EV' = 0.89(20) + 0.11(80) = 26.60, \qquad (17.13.iii)$$

down 8.4 from the initial expected consumption value of crime $EV = 35$. If it is equally costly to raise everyone's income as it is to lower EV, the increased enforcement policy is therefore a significantly more costly way of reducing crime than the education investment policy.

(e) *How do your answers change if $\delta = 0.25$ to begin with?*

Answer: When $\delta = 0.25$, we have an initial expected consumption level of crime equal to $EV = 65$ and an expected utility of crime of $E(u) \approx 4.035$ — with certainty equivalent of $\overline{x} \approx 56.57$. Thus, initially everyone whose honest income falls below 56.57 lives a life of crime. Under the education investment policy, all incomes rise by 5 — which implies that those who previously could earn between 51.57 and 56.57 in the (legal) labor market would switch from a life of crime to an honest life under this policy. In order to achieve an equivalent reduction in crime through lowering δ (using an increased enforcement policy), we need to find the δ for which the expected utility from a life of crime is equal to the utility of consuming 51.57. This gives us $\delta \approx 0.317$ — up from the initial 0.25. And, $\delta = 0.317$ implies an expected consumption value of crime equal to approximately 61 — down by only 4 from the initial 65. If it is equally costly to fund education investments that raise everyone's income by a dollar as it is to reduce the expected consumption value of crime by one dollar through increased enforcement, it now costs less to reduce crime through increased enforcement rather than investments in education.

Conclusion: Potentially Helpful Reminders

1. The basic model of risk aversion (and risk loving) from the first section of the Chapter requires that you get comfortable with the difference between the *expected utility* of a gamble and the *utility of the expected value* of the gamble. Once you know how to read these two concepts on a graph of a

consumption/utility graph, you will have come a long way toward mastering this model. The concepts of certainty equivalence and risk premiums emerge straightforwardly from this.

2. The expected utility of a gamble involving two outcomes is always read off the line connecting the utility of the individual outcomes (which in turn are read off the consumption/utility relationship). The utility of the expected value of the gamble, on the other hand, is read directly off the consumption/utility relationship. The former is lower than the latter for risk averse individuals and higher for the risk loving individuals.

3. When calculating the set of actuarily fair insurance contracts, keep in mind that the expected benefit from holding the insurance contract must be equal to the expected cost. This equivalence then gives rise to the relationship between the price of an actuarily fair insurance policy and the benefit level of that policy.

4. When working with indifference curves in models of state-dependent utility, keep in mind that the interpretation of these indifference curves is different from what we developed in the consumer model without risk even though the indifference map looks the same. In previous chapters, the bundles contained on indifference curves were actual bundles of goods that the consumer consumed; in our model here, the bundles are "outcome pairs" — with only one of the outcomes actually coming about. When consumers make choices, they do not know which outcome will happen — they only know the probability with which each of the outcomes is likely to happen.

5. Because the indifference curves in the state-dependent model arise from an expected utility relationship (or function), their shape is determined in part by the probabilities with which the outcomes are thought to occur. As the probabilities change, so does the indifference map (and the underlying expected utility function).

6. There is no reason to expect that insurance pricing in a general equilibrium setting will turn out to be actuarily fair, particularly in the presence of aggregate risk. This point is developed further in a number of end-of-chapter exercises.

CHAPTER

18

Elasticities, Price Distorting Policies and Non-Price Rationing

This chapter is the first of five to treat violations of the first welfare theorem *in competitive settings* and the first of three that investigates the role of policy-induced price distortions. It begins, however, with a full treatment of the concept of elasticity, a topic we have saved until now so that we can discuss elasticity with respect to all the economic relationships we have derived. Our primary focus is on price elasticities as they are most relevant to the topic of policy-induced price distortions. We then introduce the most obvious way in which policy can distort prices — through explicit restrictions on prices in the form of price ceilings and price floors. Our treatment of these topics is an equilibrium treatment in the sense that we take the view that the emergence of shortages or surpluses under such policies is a *disequilibrium* phenomenon — and that, in the absence of rationing through prices, some non-price rationing mechanism will emerge and define the new equilibrium under any of these policies.

Chapter Highlights

The main points of the chapter are:

1. **Elasticity** refers to the responsiveness of behavior to some aspect of the economic environment and is defined as the percentage change in the observed behavior resulting from a one percent change in the economic variable. Elasticities of consumer demand, for instance, can be defined relative to changes in any price or changes in income; elasticities of output supply or input demand can be defined relative to changes in input or output prices; etc. The price elasticity of demand varies from zero to minus infinity along any **linear demand curve**, with consumer spending increasing as a result of a price increase when price elasticity lies between 0 and -1 and decreasing when the elasticity lies between -1 and $-\infty$.

2. The **division of surplus** in the market depends in part on the elasticity of demand and supply — and the **size of the impact of price distorting policies** on the market similarly **depends on price elasticities**.

3. The imposition of **price floors above the equilibrium price** results in a **disequilibrium surplus**. Since it cannot be an equilibrium for producers to produce goods they cannot sell, some **non-price rationing** mechanism then leads to a new equilibrium. This may arise from producers expending additional effort to attract consumers and thus incurring additional costs, or it may arise from explicit policies that accompany the price floor policy. The **size of the deadweight loss** depends on the type of non-price rationing mechanism that defines the new equilibrium

4. The imposition of **price ceilings below the equilibrium price** results in a **disequilibrium shortage**. In the absence of full price rationing, some **non-price rationing** mechanism will lead to a new equilibrium, with the size of deadweight loss again related to the mechanism that emerges.

5. Price ceilings and price floors may be implemented because of **ethical considerations** that are deemed by policy makers to outweigh the emergence of deadweight loss. Alternatively, they may be implemented in political processes that weight **concentrated benefits** more heavily than **diffuse costs**.

18A Solutions to Within-Chapter-Exercises for Part A

Exercise 18A.1

Knowing what you do from previous chapters, how would the social benefits from market interactions be distributed between producers and consumers in a long run competitive equilibrium in which all producers face the same costs?

Answer: In such an equilibrium, the long run market supply curve would be perfectly horizontal — and each producer would produce on the lowest point of the long run *AC* curve — thus making zero profit. Producer surplus (as measured by long run profit) would therefore be zero in the industry — and the entire social surplus would accrue to consumers.

Exercise 18A.2

True or False: If an individual consumer's demand curve is perfectly inelastic, the good is borderline between regular inferior and Giffen.

Answer: This is true — as regular inferior goods have demand curves that slope downward and Giffen goods have demand curves that slope up.

Exercise 18A.3

The price in Graph 18.3 is measured in dollars. What would the demand curve look like if instead we measured price in terms of pennies? Can you re-calculate price elasticity at 200, 400 and 600 units of output and demonstrate that you get the same answers we just derived?

Answer: The demand curve would still be linear, but the units on the horizontal axis would be 100 times greater than in the graph of the text. We could say that the demand curve would be 100 times steeper — because its slope would now be -50 instead of $-1/2$. Thus, for any increase in price of 100 cents, we would get a drop of $2x$ in the quantity demanded. Point A would then occur at quantity $x = 400$ and price $p = 20,000$. A 1 percent increase in price is equal to an increase of p by 200 — which would lead to a drop of 4 in the quantity demanded (to 396) — i.e. a 1% drop in quantity. Thus, the price elasticity is -1 at A. At B, $x = 200$ and $p = 30,000$. A 1 percent increase in p is then equal to 300 — which leads to a drop of 6 in the quantity demanded (to 194) — i.e. a 3 percent drop. Thus, the price elasticity at B is -3. Finally, at C we have $x = 600$ and $p = 10,000$. A 1 percent increase in p is equal to 100 — which leads to a drop in x of 2 (to 598) — i.e. a drop of 1/3 of a percent. Thus, the price elasticity at C is $-1/3$.

Exercise 18A.4

True or False: Unless a good is a Giffen good, price elasticity of demand is negative.

Answer: This is true because all demand curves slope down unless the good is Giffen. For a Giffen good, the price elasticity of demand would be positive.

Exercise 18A.5

Calculate the total spending this consumer undertakes at each of the two prices in panels (a) through (c) of Graph 18.4 and identify the magnitude and direction of the change in overall spending on good x.

Answer: In panel (a), the consumer spends \$50(700) = \$35,000 at the lower price and \$100(600) = \$60,000 at the higher price — i.e. her spending *increases* by \$25,000 when price increases. In panel (b), she spends \$175(450)=\$78,750 at the initial price and \$225(350) = \$78,750 at the higher price — i.e. her spending does not change when the price increases. Finally, in panel (c) she spends \$300(200) = \$60,000 at the initial price and \$350(100) = \$35,000 at the higher price — i.e. her spending *decreases* by \$25,000 as a result of the price increase. Thus, the consumer's spending increases when price elasticity is greater than -1 (in panel (a)), stays the same when price elasticity is equal to -1 (in panel (b)) and decreases when price elasticity is less than -1 (in panel (c)).

Exercise 18A.6

Suppose I notice that when long distance telephone rates came down, our monthly long-distance phone bill went up. What can you conclude about our price elasticity of demand for long distance telephone calls?

Answer: You should conclude that we must have been quite responsive to the drop in price — because as price dropped, we increased the number of phone calls sufficiently to cause overall spending to *increase*. Thus, we must be on the relatively elastic portion of demand — i.e. where the price elasticity of demand is less than -1.

Exercise 18A.7

The diamond industry's marketing efforts have convinced many of the convention that an engagement ring should always cost the lucky groom exactly 3 months salary. What does this imply about the price elasticity of demand for diamond size that the diamond industry is attempting to persuade us we should have?

Answer: The convention suggests that, for any given income, we should always spend exactly the same for an engagement ring *regardless of price*. A demand curve that has the feature that spending does not change as price changes is a demand curve with unitary price elasticity — i.e. with price elasticity of -1 everywhere.

Exercise 18A.8

Is the income elasticity of demand positive or negative? (*Hint*: Does your answer depend on whether the good is inferior or normal?)

Answer: The income elasticity of demand is the percentage change in the quantity demanded with a 1 percent change in income. If a good is normal, consumption of the good increases as income increases — implying a positive income elasticity. If a good is inferior, consumption of the good decreases as income increases — implying a negative income elasticity.

Exercise 18A.9

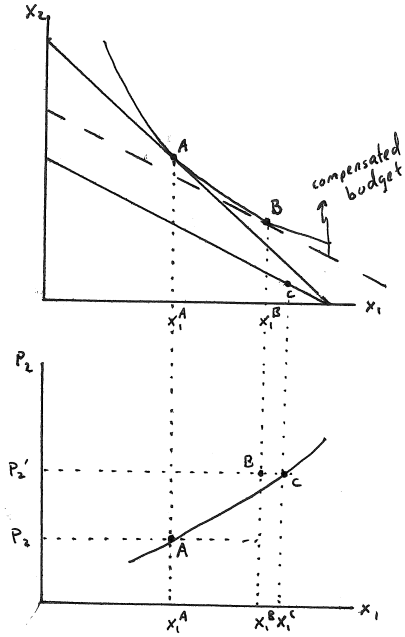
What kind of good does x have to be in order for the demand for x to be perfectly income inelastic?

Answer: Perfect income inelasticity implies that consumption of x does not change as income changes. Put differently, there would be no income effect on demand — which is true for goods that are quasilinear.

Exercise 18A.10

In a 2-good model, is the cross-price elasticity of demand for good x_1 positive or negative if x_1 is a regular inferior good? (*Hint*: Is the cross-price demand curve for good x_1 upward or downward sloping?)

Answer: In Exercise Graph 18A.10, the top panel illustrates an initially low price for x_2 (on the steeper budget line) and a higher price for x_2 (on the shallower budget line).



Exercise Graph 18A.10 : Cross-Price Elasticity

The consumer's original optimal bundle is A, and the substitution effect moves the consumer to B. If x_1 is regular inferior, the additional income effect implies that consumption of x_1 will increase as we move from the compensated budget to the final budget — implying that the final optimal bundle C lies to the right of B. In the lower panel, this is translated to a cross-price demand curve where we plot x_1 on the horizontal and p_2 on the vertical. The initial optimal consumption level of x_1 occurs at the lower price p_2, and the new optimal quantity of x_1 occurs at the higher price p_2'. Thus, the cross price demand curve is upward sloping — which implies a positive cross-price elasticity of demand for x_1. (If x_1 were a normal good, then C would fall to the left of B — which leaves open the possibility that C lies to the right or left of A. Thus, the cross-price demand curve for x_1 could slope up or down — implying that the cross-price elasticity of demand for x_1 could be positive or negative.)

Exercise 18A.11

Given what you learned in Chapter 13, is the price elasticity of supply for a competitive firm larger or smaller in the long run (than in the short run).

Answer: We concluded in Chapter 13 that output supply responses are always greater in the long run than in the short run when output price changes — which implies that the price elasticity of supply is larger in the long run than in the short run for any competitive firm.

Exercise 18A.12

Given what you learned in Chapter 14, what is the price elasticity of industry supply in the long run when all firms have identical costs?

Answer: The long run industry supply in the case of identical competitive firms is horizontal at the lowest point of the firms' average long run cost curves. Thus, the long run industry supply curve is perfectly elastic — i.e. it has price elasticity of supply of infinity.

Exercise 18A.13

Suppose a supply curve is linear and starts at the origin. What is its price elasticity of supply?

Answer: Such a supply curve would be characterized by the equation $p = \alpha x$ where α is just some positive constant. Consider some arbitrary output level $\overline{x}$ at price $\overline{p} = \alpha\overline{x}$. Now we can ask what percentage of a price increase would be necessary for the producer to increase output by 1 percent to $1.01\overline{x}$? The answer is that price (read off the supply curve) would have to be $\alpha(1.01\overline{x})$ which is equal to $1.01\overline{p}$. Thus, a one percent increase in price leads to a 1 percent increase in output — giving us price elasticity of supply of 1. Since we did this for an arbitrary initial point on the supply curve, this holds for all points on the supply curve — implying a price elasticity of supply of 1 everywhere.

Exercise 18A.14

If labor supply curves are "backward bending" (in the sense that they are upward sloping for low wages and downward sloping for high wages), how does the wage elasticity of labor supply change as wage increases?

Answer: The wage elasticity of labor supply would be positive for low wages, decrease as wage goes up and eventually become negative as the labor supply curve bends backwards.

Exercise 18A.15

True or False: The wage elasticity of labor demand is always negative.

Answer: True. Labor demand curves are always downward sloping — both in the short and long run — which implies that the wage elasticity of labor demand is always negative.

Exercise 18A.16

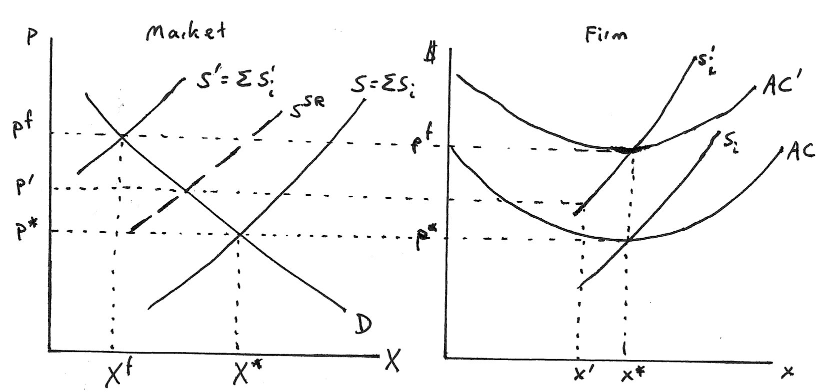
How does the size of the disequilibrium surplus change with the price elasticity of supply and demand?

Answer: The more inelastic the curves, the smaller the disequilibrium surplus.

Exercise 18A.17

Using the combination of industry and firm curves we employed in Chapter 14, illustrate what happens to each firm's cost curves as a result of the imposition of a price floor.

Answer: This is depicted in Exercise Graph 18A.17. Prior to the price floor, each firm operates in long run equilibrium at the lowest point of its AC curve, producing x^* at p^* with market demand and supply intersecting at the market quantity X^*. As a result of the imposition of the price floor, the marginal cost of producing each output unit increases — shifting up the short run firm supply curve from S_i to S_i' and shifting long run AC up parallel. The fact that each firm's supply curve has shifted up means that the short run industry supply curve shifts to S^{SR} — causing an increase in price to p' and a decrease in firm output to x' (as well as an industry decrease of output to X'). In the new long run equilibrium, however, firms will need to once again produce at the lowest point of their long run AC curves at p^f. Thus, firms exit, leading to the final equilibrium intersection of D and S' in the market — with each firm producing x^* once again, but the industry output falling to X^f.

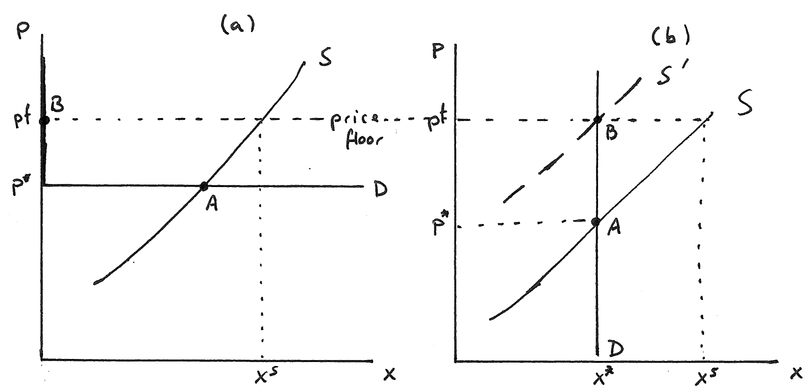


Exercise Graph 18A.17 : Price Floors and Firms

Exercise 18A.18

Depict the impact of a price floor on the quantity produced by the market when demand is perfectly price elastic. Repeat for the case when demand is perfectly price inelastic.

Answer: This is depicted in Exercise Graph 18A.18. In panel (a), demand is perfectly elastic — which implies that any price increase causes the quantity demanded to fall to zero. Thus, a price floor will lead to zero quantity demanded, with firms willing to supply x^s. Given that consumers demand zero, there is no amount of effort firms can exert and persuade consumers to consume at the price floor — thus, the equilibrium shifts from A to B, with output falling to zero and all firms exiting. In panel (b), demand is perfectly inelastic. Since consumers are not sensitive to price, they continue to demand x^* at the price floor, but producers are willing to supply x^s. Thus, they will compete for the limited number of customers, incurring additional marginal costs until supply intersects demand at B. Output quantity does not change from the original equilibrium because of the inelasticity of demand.

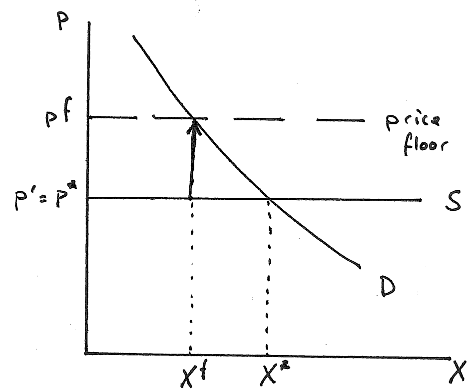


Exercise Graph 18A.18 : Price Floors and Price Elasticity

Exercise 18A.19

What is p' in long run equilibrium when all firms face the same costs?

Answer: When all firms face the same costs, long run industry supply is perfectly elastic at the price that falls at the lowest point of AC curves. In the initial equilibrium, this is equal to the long run equilibrium price p^*. The imposition of the price floor is then depicted in Exercise Graph 18A.19 where the effort exerted by producers shifts the long run supply curve up to the price floor, with the equilibrium output in the industry falling to X^f as firms exit. As a result, $p' = p^*$.



Exercise Graph 18A.19 : Price Floors and Identical Costs

Exercise 18A.20

Would you expect any entry or exit of producers as a result of the imposition of a price floor when it is complemented by a government program that guarantees surpluses will be purchased by the government at the price floor?

Answer: Yes, you would expect new firms to enter. This is because there is no change in any firm's costs when firms do not have to compete for a more limited pool of customers — but the price has increased. Thus, if firms were making zero profits before, they are now making positive profits — leading to new firms entering the industry.

Exercise 18A.21

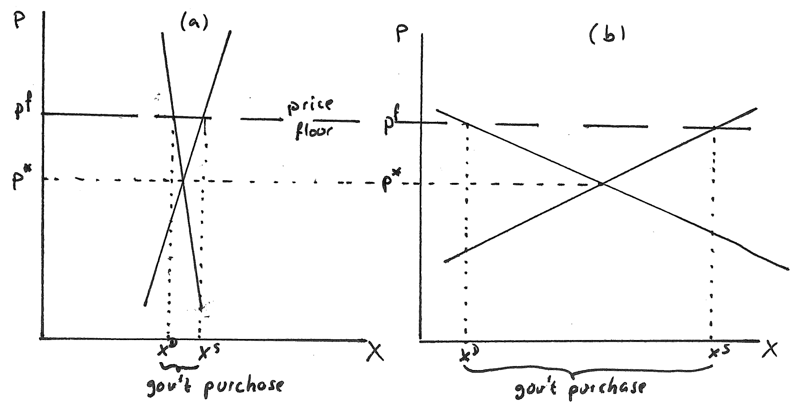
How will the amount that the government has to purchase change with price elasticities of demand and supply?

Answer: The more elastic the demand and supply curves, the more the government will have to purchase. This is illustrated in the two panels of Exercise Graph 18A.21 where panel (a) illustrates relatively inelastic demand and supply curves while panel (b) illustrates relatively elastic curves.

Exercise 18A.22

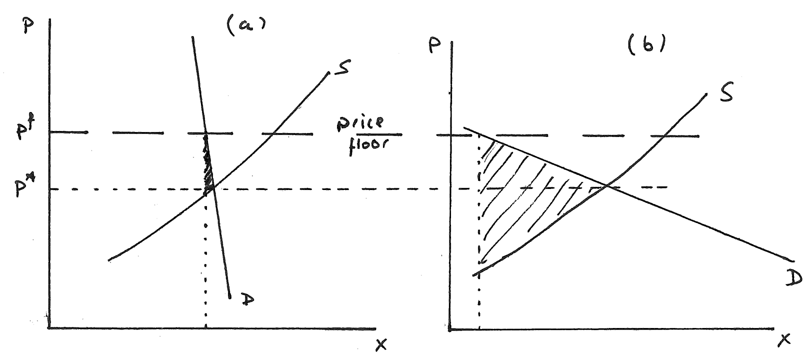
How does the dead weight loss change as the price elasticity of demand changes?

Answer: The answer depends in part on whether a portion (or all of) area $(b+d)$ is part of deadweight loss or not. If it is not — and only the triangle $(c+e)$ represents deadweight loss, then the deadweight loss will be larger the more elastic demand is. This is illustrated in Exercise Graph 18A.22 where demand is relatively price inelastic in panel (a) and relatively price elastic in panel (b). The shaded dead-weight



Exercise Graph 18A.21 : Price Floors, Government Purchases and Price Elasticity

loss triangle is clearly bigger in (b) than in (a). However, if the bulk of what is labeled $(b+d)$ in the text graph is also part of deadweight loss, then our conclusion could reverse — with the deadweight loss larger for inelastic demand. This should be intuitive: If the primary determinant of deadweight loss is the size of the quantity drop in the market (as in Exercise Graph 18A.22), then more elastic demand will create bigger deadweight loss as it causes a greater reduction in output as the price floor is imposed. If, on the other hand, the primary determinant of deadweight loss is the effort expended by firms to attract a smaller pool of customers, then the deadweight loss might be smaller if few firms actually remain in the market to expend such effort — i.e. if demand is very elastic and thus few firms are needed to meet demand at the price floor.

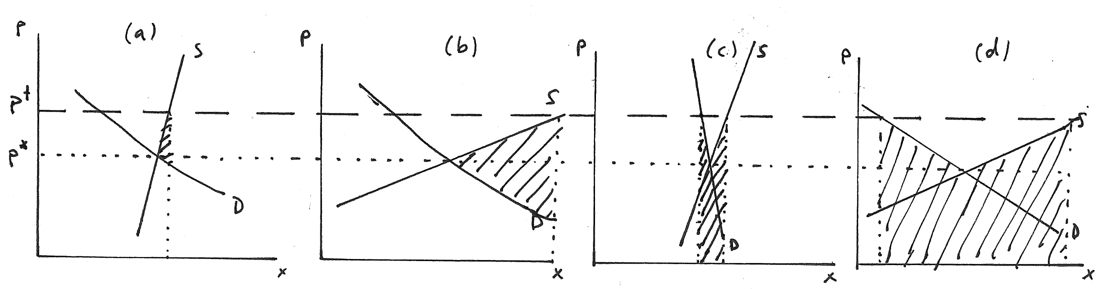


Exercise Graph 18A.22 : Price Floors, Dead Weight Loss and Price Elasticity

Exercise 18A.23

How does the deadweight loss change in size as the price elasticity of demand and supply changes?

Answer: Suppose first the government finds a way to get the goods it purchases to the consumers who most desire them — which implies the deadweight loss is just the triangle h. In that case, what matters for the size of the deadweight loss is the price elasticity of supply — with a relatively inelastic price elasticity leading to a small deadweight loss and a relatively large price elasticity of supply leading to a large deadweight loss. This is depicted in panels (a) and (b) of Exercise Graph 18A.23. Next, suppose that the deadweight loss emerges also because the government throws away the goods it purchases. In this case, both the price elasticity of demand and supply matter — with more elastic curves causing greater deadweight loss. This is depicted in panels (c) and (d) of Exercise Graph 18A.23.(Technically, the elasticity of demand also matters when just the triangle h is the deadweight loss — with a more elastic curve actually causing the triangle to become smaller. This, however, is a second order effect — the primary determinant of the deadweight loss in this case is the elasticity of supply as depicted in panels (a) and (b).)



Exercise Graph 18A.23 : Price Support Programs, Dead Weight Loss and Price Elasticity

Exercise 18A.24

How does the shortage that emerges in disequilibrium change as price elasticities of demand and supply change?

Answer: The shortage will be larger the more price elastic the demand and supply curves are.

Exercise 18A.25

How does the size of dead weight losses from price ceilings vary with the price elasticities of demand and supply?

Answer: If the deadweight loss is simply the triangle $(c+e)$, then we can see easily that (just as in the case of price floors), the triangle shrinks as demand and supply become more price inelastic. If, however, the deadweight loss includes the areas $(b+d)$ — as in the case where consumers stand in line, then the deadweight loss might actually increase with more price inelastic demand curves as consumers spend more time in line.

Exercise 18A.26

Consider our Robinson Crusoe Economy from Chapter 16 and suppose that the economy is currently in equilibrium with wage w^* and price p^*. Now suppose that a government requires that no wage lower than kw^* (with $k>1$) be paid in this economy. What will happen in order for this economy to return to equilibrium?

Answer: In order for this economy to return to equilibrium, prices will have to adjust such that the slope of the budget/isoquant returns to w^*/p^* — i.e. p has to increase to kp^* (thus returning the real wage back to what it was before.)

18B Solutions to Within-Chapter-Exercises for Part B

Exercise 18B.1

Could you also express the price elasticity as a function of only quantity? (*Hint*: Think of replacing the numerator rather than the denominator.)

Answer: Yes — we could write it as

$$-2\left(\frac{p(x)}{x}\right) = -2\left(\frac{400-\frac{1}{2}x}{x}\right) = \frac{x-800}{x}. \tag{18B.1}$$

Exercise 18B.2

Using the formula for price elasticity you derived in exercise 18.7, verify that you get the same price elasticity for x equal to 200, 400 and 600 (corresponding to points B, A and C in Graph 18.3.

Answer: At $x = 200$, we get an elasticity of $(200-800)/200 = -3$; at $x = 400$ we get $(400-800)/400 = -1$; and at $x = 600$ we get $(600-800)/600 = -1/3$ — exactly the same elasticities as those indicated in the graph.

Exercise 18B.3

Demonstrate that $\varepsilon_d < -1$ implies that consumer spending will fall with an increase in price and rise with a decrease in price.

Answer: With

$$\frac{d(TS)}{dp} = x(p) + p\frac{dx}{dp}, \tag{18B.3.i.i}$$

we can conclude that

$$\frac{d(TS)}{dp} < 0 \text{ if and only if } p\frac{dx}{dp} < -x(p) \tag{18B.3.i.ii}$$

which implies

$$\varepsilon_d = \frac{p}{x(p)}\frac{dx}{dp} < -1 \tag{18B.3.i.iii}$$

Exercise 18B.4

What is the price elasticity for x_1 and x_2 when tastes are Cobb-Douglas; i.e. when tastes can be represented by the utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$? (*Hint*: Recall that the demand functions in this case are $x_1(p_1, I) = \alpha I/p_1$ and $x_2(p_2, I) = (1-\alpha)I/p_2$.)

Answer:

$$\varepsilon_d = \frac{dx_1}{dp_1}\left(\frac{p_1}{x_1}\right) = \left(\frac{-\alpha I}{p_1^2}\right)\left(\frac{p_1}{\alpha I/p_1}\right) = -1. \quad (18B.4)$$

The price elasticity of demand for x_2 is similarly equal to -1.

Exercise 18B.5

Can you see from the expression for income elasticities that the sign of the elasticity will depend on whether the good x is normal or inferior?

Answer: The sign of the income elasticity of demand depends on the first term dx/dI which is positive for normal goods and negative for inferior goods.

Exercise 18B.6

Can you see that the sign of the cross price elasticity depends on the slope of the cross-price demand curve?

Answer: The sign of the cross price elasticity depends on the term dx_i/dp_j which is positive or negative depending on the slope of the cross price demand curve.

Exercise 18B.7

Can you express x^* in Graph 18.11 in terms of the demand and supply parameters A, α, B, β?

Answer: Plugging p^* into the demand equation $x_d(p)$, we get

$$x^* = x_d(p^*) = \frac{A - \frac{\beta A + \alpha B}{\alpha + \beta}}{\alpha} = \frac{A - B}{\alpha + \beta}. \quad (18B.7)$$

You could similarly plug p^* into the supply equation $x_s(p)$ to get the answer.

Exercise 18B.8

What is the surplus of x that exists in the initial disequilibrium?

Answer: The initial surplus is $(x_s^f - x_d^f)$ — which can be written as

$$\frac{p^f - B}{\beta} - \frac{A - p^f}{\alpha} = \frac{(\alpha+\beta)p^f - \alpha B - \beta A}{\alpha\beta}. \quad (18B.8)$$

You can check that this is right by plugging in p^* for p^f — and you will get that the surplus is zero (which it should be if the price floor is set at the market price).

Exercise 18B.9

By how much does the supply curve shift up? Express your answer purely in terms of demand and supply parameters and p^f.

Answer: The supply curve shifts up by $(p^f - p')$ which can be expressed as

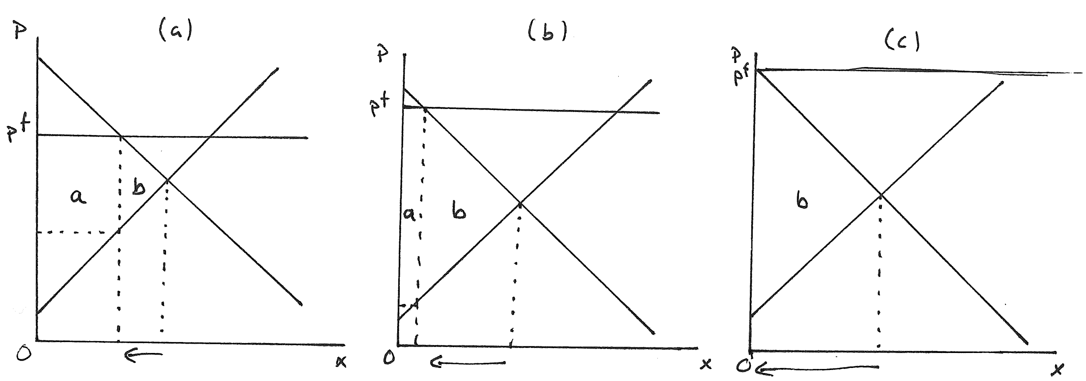
$$p^f - p' = p^f - \frac{\beta(A - p^f) + \alpha B}{\alpha} = \frac{(\alpha+\beta)p^f - \alpha B - \beta A}{\alpha} \tag{18B.9}$$

which again is zero when $p^* = p^f$.

Exercise 18B.10

Can you graphically illustrate why the lower and upper bounds of DWL ultimately converge as the price floor increases?

Answer: This is done in the three panels of Exercise Graph 18B.10 where the price floor p^f is increased in each panel. The lower bound on DWL is the area b in each graph, and the upper bound is $(a+b)$. As the price floor increases, b increases in size and a shrinks — eventually converging to zero when production stops.



Exercise Graph 18B.10 : Two Measures of DWL

Exercise 18B.11

Can you express the total effort cost incurred by producers as a function of demand and supply parameters and p^f?

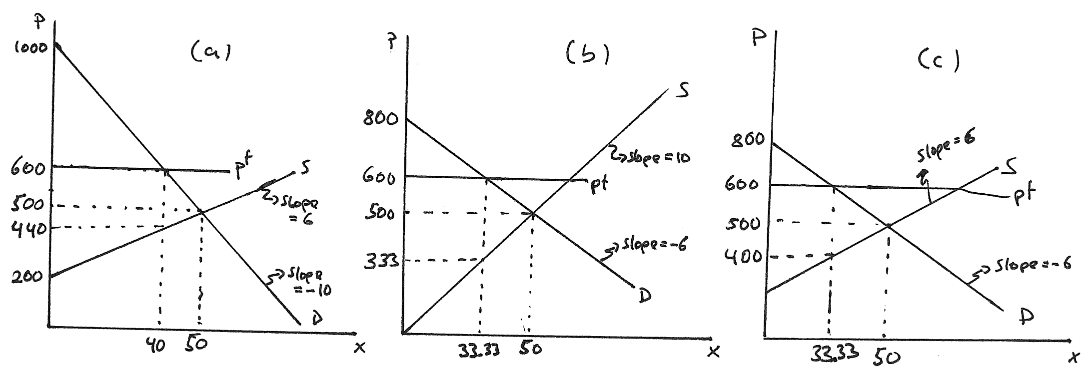
Answer: This is simply the rectangle formed by $(p^f - p')$ times x_d^f — which, given our equation (18B.9), is

$$\text{Effort Cost} = \left(\frac{A - p^f}{\alpha}\right)\left(\frac{(\alpha+\beta)p^f - \alpha B - \beta A}{\alpha}\right). \qquad (18B.11)$$

Exercise 18B.12

For each of the three sections of Table 18.2, graphically illustrate the third row using the information in the table to label everything on the axes that you can label.

Answer: This is illustrated in panels (a) through (c) of Exercise Graph 18B.12.

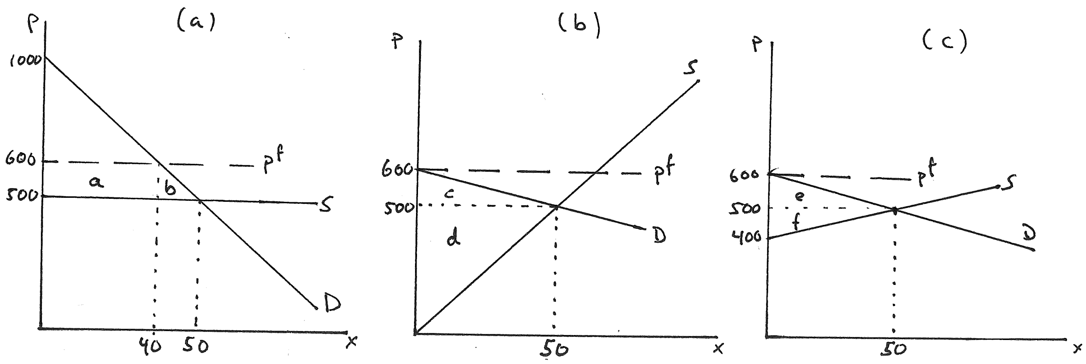


Exercise Graph 18B.12 : Changing slopes for D and S

Exercise 18B.13

Why do the lower and upper bounds for DWL converge in the lower two sections of Table 18.2 but not in the top portion?

Answer: In Exercise Graph 18B.13 we graph the last row of each of the three sections of the Table. In panel (a) (corresponding to the top section of the Table), the supply curve is perfectly elastic at $p = 500$. The imposition of the price floor then simply raises the supply curve by \$100 (i.e. the effort cost required to reach the equilibrium quantity 40). We thus have the lower bound of he deadweight loss in area b — and the upper bound in $(a+b)$, incorporating the effort cost a. This effort cost remains because a positive quantity is produced under the price floor. In panels (b) and (c), on the other hand, the same price floor results in output dropping to zero. Thus, there is no more effort cost — because no one produces. In panel (b) (corresponding to the lowest row in the middle portion of the Table), this results in deadweight loss of $(c+d)$ — with the upper and lower bounds on DWL the same since there are no effort costs in the new equilibrium. The same happens in panel (c) (corresponding to the lowest row in the last section of the Table) — where the upper and lower bound on DWL is $(e+f)$.



Exercise Graph 18B.13 : DWL with different D and S curves

Exercise 18B.14

What is the price elasticity of demand? What is the price elasticity of supply?
Answer: The price elasticity of demand is

$$\frac{dx_d}{dp}\frac{p}{x_d(p)} = -2\left(\frac{40,000,000}{p^3}\right)\left(\frac{p}{\frac{40,000,000}{p^2}}\right) = -2 \qquad (18B.14.i)$$

and the price elasticity of supply is

$$\frac{dx_s}{dp}\frac{p}{x_s(p)} = \frac{2}{3}\left(\frac{547,192}{p^{1/3}}\right)\left(\frac{p}{547,192p^{2/3}}\right) = \frac{2}{3}. \qquad (18B.14.ii)$$

Exercise 18B.15

Can you derive the lower and upper bound of deadweight loss as a function of p^f?

Answer: The upper bound on DWL is the area between PS and CS underneath the two curves (labeled as DWL plus C in the graph of the text). This area is composes of two parts. The portion underneath the demand function is

$$\int_{p^*=5}^{p^f} x_d(p)dp = \int_5^{p^f} \frac{40,000,000}{p^2}dp = 8,000,000 - \frac{40,000,000}{p^f} \qquad (18B.15.i)$$

and the portion underneath the supply function is

$$\int_{p'}^{p^*=5} x_s(p)dp = \int_{p'}^{5} 547,192p^{2/3}dp = 4,800,000 - 328,315(p')^{5/3}$$
$$\approx 4,800,000 - \frac{15,000,000,000}{(p^f)^5} \qquad (18B.15.ii)$$

where we substituted $p' = (625/(p^f)^3)$ (derived in the text) in the last step. Thus, the upper bound of DWL is the sum of these two equations. The lower bound is just the darkened triangle labeled DWL in the graph. Thus, we need to simply subtract the area C which was calculated in the text. We then get an upper bound of

$$\overline{DWL} = \int_5^{p^f} \frac{40,000,000}{p^2} dp + \int_{p'}^5 547,192p^{2/3} dp$$
$$= 12,800,000 - \frac{40,000,000(p^f)^4 + 15,000,000,000}{(p^f)^5} \tag{18B.15.iii}$$

and the lower bound of

$$\underline{DWL} = \overline{DWL} - C =$$
$$= 12,800,000 - \frac{40,000,000(p^f)^4 + 15,000,000,000}{(p^f)^5} - \frac{40,000,000((p^f)^4 - 625)}{(p^f)^5} =$$
$$= 12,800,000 - \frac{80,000,000(p^f)^4 - 10,000,000,000}{(p^f)^5}. \tag{18B.15.iv}$$

18C Solutions to Odd Numbered End-of-Chapter Exercises

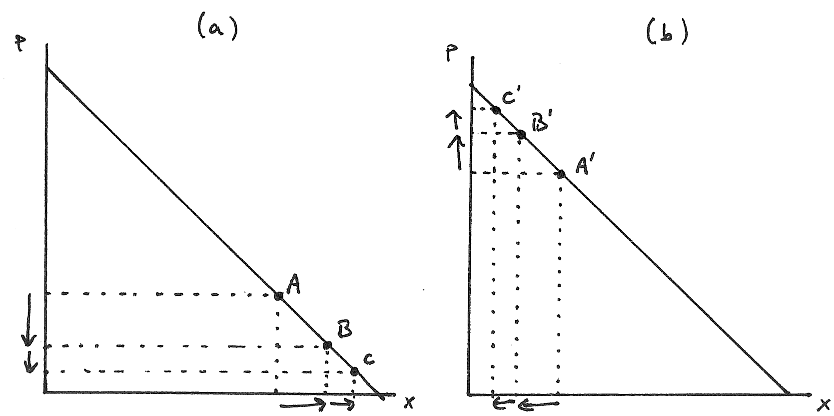
Exercise 18.1

Consider, as we did in much of the chapter, a downward sloping linear demand curve.

A: *In what follows, we will consider what happens to the price elasticity of demand as we approach the horizontal and vertical axes along the demand curve.*

(a) *Begin by drawing such a demand curve with constant (negative) slope. Then pick the point A on the demand curve that lies roughly three quarters of the way down the demand curve. Illustrate the price and quantity demanded at that point.*

Answer: This is depicted in panel (a) of Exercise Graph 18.1



Exercise Graph 18.1 : Price Elasticity as we Approach the Axes

(b) *Next, suppose the price drops by half and illustrate the point B on the demand curve for that lower price level. Is the percentage change in quantity from A to B greater or smaller than the absolute value of the percentage change in price?*

Answer: Since the price drops by half, we know that the absolute value of the percentage change in price is 50%. It should be obvious from the graph that the *percentage* increase in quantity from A to B is smaller than 50% — because the starting quantity at A is large relative to the change in quantity from A to B.

(c) *Next, drop the price by half again and illustrate the point C on the demand curve for that new (lower) price. The percentage change in price from B to*

C is the same as it was from A to B. Is the same true for the percentage change in quantity?

Answer: Since we again halved the price, the percentage change in price is 50%. The percentage change in quantity from B to C, however, is now smaller than it was from A to B — because the starting quantity at B is higher than it was at A while the absolute change from B to C is smaller. For the same percentage change in price, we therefore get smaller percentage changes in quantity as we move down the demand curve.

(d) *What do your answers imply about what is happening to the price elasticity of demand as we move down the demand curve?*

Answer: Price elasticity of demand is the percentage change in quantity over the percentage change in price. We have held the percentage change in price constant as we moved from A to B and then from B to C — implying the denominator of the price elasticity formula was kept the same. But we concluded that the percentage change in quantity (for the same percentage change in price) is less when we move from B to C than when we moved from A to B — implying that the absolute value of price elasticity gets smaller as we move down the demand curve.

(e) *Can you see what will happen to the price elasticity of demand as we get closer and closer to the horizontal axis?*

Answer: If you imagine repeating what we have done again — i.e. cutting price by half and checking what happens to the percentage change in quantity, we will keep getting the same result: The percentage change in quantity for the same percentage change in price will get smaller and smaller. Thus, the numerator of the price elasticity formula gets smaller and smaller while the denominator stays the same — implying that the fraction that represents price elasticity gets smaller and smaller — and closer and closer to zero — as we move toward the endpoint of the linear demand curve.

(f) *Next, start at a point A' on the demand curve that lies only a quarter of the way down the demand curve. Illustrate the price and quantity demanded at that point. Then choose a point B' that has only half the consumption level as at A'. Is the percentage change in price from A' to B' greater or less than the absolute value of the percentage change in quantity?*

Answer: This is illustrated in panel (b) of Exercise Graph 18.1. In moving from A' to B', we know that quantity drops by half — i.e. changes by 50%. But it should be obvious from the graph that price increases by less than 50%. This is because the beginning price at A' is already relatively high — and the incremental change in price from A' to B' is relatively low compared to its starting point.

(g) *Now pick the point C' (on the demand curve) where the quantity demanded is half what it was at B'. The percentage change in quantity from A' to B' is then the same as the percentage change from B' to C'. Is the same true of the percentage change in price?*

Answer: The percentage change is the same because we again dropped quantity by 50%. But the percentage change in price is now less as we go from B' to C' than it was when we went from A' to B'. This is because the starting price is higher at B' than it was at A' while the incremental increase in price from B' to C' is less than it was from A' to B'. Thus, as we move up the demand curve, the percentage change in price gets smaller and smaller (for the same percentage decrease in quantity).

(h) *What do your answers imply about the price elasticity of demand as we move up the demand curve? What happens to the price elasticity as we keep repeating what we have done and get closer and closer to the vertical intercept?*

Answer: We have shown that, for the same percentage decrease in quantity, the percentage increase in price gets smaller as we move up the demand curve. Price elasticity of demand is the percentage change in quantity divided by the percentage change in price. We have held the percentage change in quantity fixed — i.e. we have held the numerator of the price elasticity formula fixed; but we have concluded that the denominator becomes smaller as we move up the demand curve. This implies that we are dividing the same number by smaller and smaller numbers — which means the overall fraction is increasing in absolute value. The price elasticity of demand therefore gets larger and larger in absolute value as we move up the demand curve — and gets closer and closer to (negative) infinity as we get closer to the vertical intercept.

B: *Consider the linear demand curve described by the equation $p = A - \alpha x$.*

(a) *Derive the price elasticity of demand for this demand curve.*

Answer: First, we write the demand curve as a demand function by solving for x — i.e.

$$x(p) = \frac{A-p}{\alpha}. \tag{18.1.i}$$

The price elasticity of demand is then

$$\varepsilon_d = \frac{dx}{dp}\frac{p}{x(p)} = -\frac{1}{\alpha}\left(\frac{p}{(A-p)/p}\right) = \frac{-p}{A-p}. \tag{18.1.ii}$$

(b) *Take the limit of the price elasticity of demand as price approaches zero.*

Answer: As p approaches zero, the numerator goes to zero while the denominator goes to A — implying a limit of zero. Thus, the price elasticity of demand converges to zero as price converges to zero — i.e. as we approach the horizontal axis.

(c) *Take the limit of the price elasticity as price approaches A.*

Answer: As p approaches A, the numerator converges to $-A$ while the denominator approaches zero — implying that the price elasticity approaches minus infinity. Thus, as we approach the vertical axis along the demand curve, price elasticity approaches negative infinity.

Exercise 18.3

In the labor market, we can also talk about responsiveness — or elasticity — with respect to wages (and other prices) on both the demand and supply sides.

A: *For each of the following statements, indicate whether you think the statement is true or false (and why):*

(a) *The wage elasticity of labor supply must be positive if leisure and consumption are normal goods.*

Answer: This is false. As wages change, there are off-setting substitution and wealth effects relative to leisure when leisure is a normal good. This implies that the impact of a change in wage on labor supply is ambiguous when leisure is normal — and the relationship between wage and the quantity of labor supplied may be positive or negative (which further implies that this elasticity may be positive or negative.)

(b) *In end-of-chapter exercise 9.5, we indicated that labor supply curves are often "backward-bending". In such cases, the wage elasticity of labor supply is positive at low wages and negative at high wages.*

Answer: This is true. The backward bending labor supply that labor economists have identified as empirically important is upward sloping for low wages and downward sloping for high wages — implying a positive relationship (and thus a positive wage elasticity) between wage and quantity of labor supplied at low wages and a negative relationship at high wages.

(c) *The wage elasticity of labor demand is always negative.*

Answer: This is true — as we illustrated in our treatment of firms, labor demand is always downward sloping — which implies a negative relationship between wage and the quantity of labor demanded (and thus a negative wage elasticity of labor demand).

(d) *In absolute value, the wage elasticity of labor demand is at least as large in the long run as it is in the short run.*

Answer: This is also true. In Chapter 13, we showed that labor demand curves are shallower in the long run than in the short run — except for one special case where they have the same slope. Thus, labor demand is more responsive — or more elastic — in the long run than in the short run, implying a higher wage elasticity in absolute value.

(e) *(The compensated labor supply curve, which we will cover more explicitly in Chapter 19, is the labor supply curve that would emerge if we always insured you reached the same indifference curve regardless of the wage rate.) The wage elasticity of compensated labor supply must always be negative.*

Answer: This is false. Compensated labor supply curves only incorporate substitution effects (and not wealth effects). The substitution effect is unambiguous — if wage increases, leisure becomes relatively more expensive and is thus consumed in smaller amounts (absent wealth effects). This implies that, as wage increases, the compensated labor supply also increases — i.e. the relationship between wage and compensated labor supply is positive. Thus, the wage elasticity of compensated labor supply is positive.

(f) *The (long run) rental rate (of capital) elasticity of labor demand (which is a cross-price elasticity) is always positive.*

Answer: This is false. We showed in Chapter 13 that the relationship between the rental rate of capital and the quantity of labor demanded may be positive or negative depending on whether labor and capital are relatively substitutable or relatively complementary in production. Thus, the elasticity may be positive or negative.

(g) *The output price elasticity of labor demand is positive and increases from the short to the long run.*

Answer: This is partly true and partly false. As output price increases, firms want to produce more and thus hire more of all inputs — including labor. Thus, the relationship between output price and the quantity of labor demanded is positive, both in the short and long run. But we showed in Chapter 13 that such responses may be larger or smaller in in the long run than in the short run depending again on the degree of substitutability of labor and capital. (It may be that the firm will hire more labor in the short run but in the long run lets some labor go and substitutes into capital instead.)

B: *Suppose first that tastes over consumption and leisure are Cobb-Douglas.*

(a) *Derive the functional form of the labor supply function.*

Answer: Define c as dollars of consumption and ℓ as hours of labor. With L denoting leisure endowment, leisure consumption is $(L-\ell)$ and the utility maximization problem is

$$\max_{c,\ell} \; c^{\alpha}(L-\ell)^{(1-\alpha)} \text{ subject to } w\ell = c. \tag{18.3.i}$$

Solving this in the usual way, we get the labor supply function $\ell = \alpha L$.

(b) *What is the wage elasticity of labor supply in this case? Explain how this relates to the implicit elasticity of substitution in Cobb-Douglas tastes.*

Answer: Since w does not appear in the labor supply function $\ell = \alpha L$ that we derived in (a), the wage elasticity of labor supply is zero (since $d\ell/dw = 0$) — i.e. the labor supply curve is perfectly inelastic. This is a direct result of the fact that the elasticity of substitution between leisure and consumption is 1. We know that substitution and wealth effects point in opposite directions with respect to leisure (and thus labor supplied) — and with this elasticity of substitution, the wealth effect (which suggests the worker will work less as wage increases) is exactly offset by the substitution effect (which suggests the worker will work more as wage increases). If the elasticity of substitution goes above 1, the substitution effect will dominate the wealth effect — implying a positive wage elasticity of labor supply; and if the elasticity of substitution is less than 1, the wealth effect outweighs the substitution effect — implying a negative wage elasticity of labor supply.

(c) *Next, suppose that the decreasing returns to scale production process takes labor and capital as inputs and is also Cobb-Douglas. Derive the long run wage elasticity of labor demand.*

<u>Answer</u>: The wage elasticity of labor demand is

$$\begin{aligned}\frac{d\ell}{dw}\frac{w}{\ell} &= \frac{-(1-\beta)}{1-\alpha-\beta}\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{r^{\beta}}\right)^{1/(1-\alpha-\beta)} w^{\left(\frac{-(1-\beta)}{1-\alpha-\beta}-1\right)}\left[\frac{w}{\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}}\right]\\ &= \frac{-(1-\beta)}{1-\alpha-\beta}<0. \end{aligned} \tag{18.3.ii}$$

Since the production process has decreasing returns to scale, we know that $(1-\alpha-\beta)>0$ and $(1-\beta)>0$. Thus, the above expression is negative — i.e. the wage elasticity of labor demand is negative.

(d) *Derive the rental rate elasticity of labor demand. Is it positive or negative?*

<u>Answer</u>: The rental rate elasticity of labor demand is

$$\begin{aligned}\frac{d\ell}{dr}\frac{r}{\ell} &= \frac{-\beta}{1-\alpha-\beta}\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}}\right)^{1/(1-\alpha-\beta)} r^{\left(\frac{-\beta}{1-\alpha-\beta}-1\right)}\left[\frac{r}{\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}}\right]\\ &= \frac{-\beta}{1-\alpha-\beta}<0. \end{aligned} \tag{18.3.iii}$$

Since $\beta>0$ and $(1-\alpha-\beta)>0$, the above expression is negative — i.e. the rental rate elasticity of labor demand is negative.

(e) *Derive the long run output price elasticity of labor demand. Is it positive or negative?*

<u>Answer</u>: The output price elasticity of labor demand is

$$\begin{aligned}\frac{d\ell}{dp}\frac{p}{\ell} &= \frac{1}{1-\alpha-\beta}\left(\frac{A\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} p^{\left(\frac{1}{1-\alpha-\beta}-1\right)}\left[\frac{p}{\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}}\right]\\ &= \frac{1}{1-\alpha-\beta}>0. \end{aligned} \tag{18.3.iv}$$

Since $(1-\alpha-\beta)>0$, this expression is positive — implying a positive output price elasticity of labor demand.

(f) *In the short run, capital is fixed. Can you derive the short run wage elasticity of labor demand and relate it to the to long run elasticity you calculated in part (c)?*

Answer: If we start with the long run production function $f(\ell,k) = A\ell^{\alpha}k^{\beta}$, we can write the short run production function as $f(\ell) = \left[Ak^{\beta}\right]\ell^{\alpha}$ or simply $f(\ell) = B\ell^{\alpha}$ where B is defined as $B = Ak^{\beta}$. The short run profit maximization problem is then

$$\max_{\ell} \; pB\ell^{\alpha} - w\ell \qquad (18.3.v)$$

which solves in the usual way to give us the short run labor demand function

$$\ell(p,w) = \left(\frac{\alpha p B}{w}\right)^{1/(1-\alpha)}. \qquad (18.3.vi)$$

The short run wage elasticity of labor demand is then

$$\frac{d\ell}{dw}\frac{w}{\ell} = \frac{-1}{1-\alpha}(\alpha pB)^{1/(1-\alpha)}w^{\left(\frac{-1}{1-\alpha}-1\right)}\left[\frac{w}{\left(\frac{\alpha pB}{w}\right)^{1/(1-\alpha)}}\right] = \frac{-1}{1-\alpha}. \qquad (18.3.vii)$$

In (c), we calculated the long run elasticity as $-(1-\beta)/(1-\alpha-\beta)$ — which is greater in absolute value than $-1/(1-\alpha)$. (To prove this, suppose $(1-\beta)/(1-\alpha-\beta) \leq 1/(1-\alpha)$. Cross-multiplying, this implies $(1-\beta)(1-\alpha) \leq 1-\alpha-\beta$, and multiplying through on the left hand side, we get $1-\alpha-\beta+\alpha\beta \leq 1-\alpha-\beta$ — which has to be false since $\alpha\beta > 0$. Thus, we have a contradiction which proves that $(1-\beta)/(1-\alpha-\beta) > 1/(1-\alpha)$.)

(g) *Can you derive the short run output price elasticity of labor demand and compare it to the long run elasticity you calculated in part (e)?*

Answer: Using the short run labor demand function in (18.3.vi), we can calculate the short run output price elasticity of labor demand as

$$\frac{d\ell}{dp}\frac{p}{\ell} = \frac{1}{1-\alpha}\left(\frac{\alpha B}{w}\right)^{1/(1-\alpha)} p^{\left(\frac{1}{1-\alpha}-1\right)}\left[\frac{p}{\left(\frac{\alpha pB}{w}\right)^{1/(1-\alpha)}}\right] = \frac{1}{1-\alpha}. \qquad (18.3.viii)$$

In (e) we calculated the long run output price elasticity of labor demand as $1/(1-\alpha-\beta)$ which is larger than the short run elasticity; i.e. labor demand responds more to output price in the long run than in the short run.

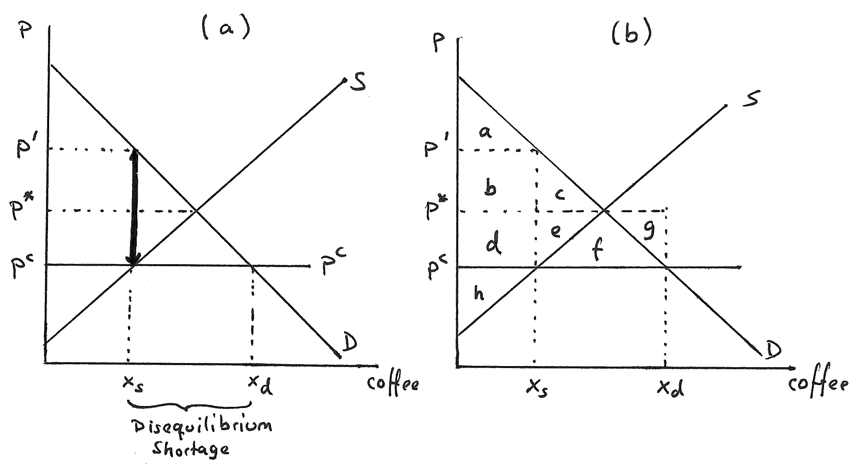
Exercise 18.5

In our treatment of price floors, we illustrated the case of a government program that purchases any surplus produced in the market. Now consider a price ceiling — and the analogous case of the government addressing disequilibrium shortages through purchases on international markets.

A: *Suppose, for instance, that the U.S. demand and supply curves for coffee intersect at p^* which is also the world price of coffee.*

(a) *Suppose that the government imposes a price ceiling p^c below p^* for domestic coffee sales. Illustrate the disequilibrium shortage that would emerge in the domestic coffee market.*

Answer: This is illustrated in panel (a) of Exercise Graph 18.5.



Exercise Graph 18.5 : Coffee Price Ceiling

(b) *In the absence of any further interference in the market, what would you expect to happen?*

Answer: Consumers would have to compete for the quantity x_s — expending effort that, in equilibrium, will equal the bold vertical distance in panel (a) of Exercise Graph 18.5. Thus, the effective price consumers will end up paying (including their effort cost) is p'.

(c) *Next, suppose that, as part of the price ceiling policy, the government purchases coffee in the world market (at the world market price p^*) and then sells this coffee at p^c domestically to any consumer that is unable to purchase coffee from a domestic produce. What changes in your analysis?*

Answer: Now consumers no longer have to compete with each other for the limited amount of coffee supplied domestically at the price ceiling — because the government supplies anything that domestic producers do not supply. Thus, consumers will actually only have to pay the price ceiling p^c (rather than p').

(d) *Illustrate — in a graph with the domestic demand and supply curves for coffee — the deadweight loss from this government program (assuming* *that your demand curve is a good approximation of marginal willingness to pay).*

Answer: This is illustrated in panel (b) of Exercise Graph 18.5. Consumers now get surplus $(a+b+c+d+e+f)$ while domestic producers get surplus (h). The government has to buy the difference between x_d and x_s at the world market price p^* — and then sells this quantity at p^c. Thus, the government loses $(p^* - p^c)$ on the quantity $(x_d - x_s)$. This is equal to area $(e+f+g)$. Adding up producer and consumer surplus — and subtracting the net cost of the government program, we then get overall surplus of $(a+b+c+d+h-g)$. Were these programs not in existence, overall surplus would be $(a+b+c+d+e+h)$. Thus, we lose $(e+g)$ as a result of this government program — i.e. the deadweight loss is $(e+g)$.

B: *Suppose demand and supply are given by $x_d = (A-p)/\alpha$ and $x_s = (p-B)/\beta$ (and assume that demand is equal to marginal willingness to pay).*

(a) *Derive the equilibrium price p^* that would emerge in the absence of any interference.*

Answer: Setting demand equal to supply and solving for p, we get (as in the text),

$$p^* = \frac{\beta A + \alpha B}{\alpha + \beta}. \tag{18.5.i}$$

(b) *Suppose the government imposes a price ceiling p^c that lies below p^*. Derive an expression for the disequilibrium shortage.*

Answer: Substituting p^c into the demand and supply equations and then subtracting x_s from x_d, we get

$$\text{Disequilibrium Shortage} = x_d(p^c) - x_s(p^c) = \frac{\beta A + \alpha B - (\alpha+\beta)p^c}{\alpha\beta}. \tag{18.5.ii}$$

(c) *Suppose, as in part A, that the government can purchase any quantity of x on the world market for p^* and it implements the program described in A(c). How much will this program cost the government?*

Answer: The program will cost the government $(p^* - p^c)$ times the disequilibrium shortage we calculated. By replacing p^* with our expression from part (a), we can derive

$$\begin{aligned}\text{Cost to Government} &= (p^* - p^c)\left[\frac{\beta A + \alpha B - (\alpha+\beta)p^c}{\alpha\beta}\right] = \\ &= \left[\frac{\beta A + \alpha B}{\alpha+\beta} - p^c\right]\left[\frac{\beta A + \alpha B - (\alpha+\beta)p^c}{\alpha\beta}\right] = \\ &= \frac{\left[\beta A + \alpha B - (\alpha+\beta)p^c\right]^2}{\alpha\beta(\alpha+\beta)}.\end{aligned} \tag{18.5.iii}$$

(d) *What is the deadweight loss from the combination of the price ceiling and the government program to buy coffee from abroad and sell it domestically at p^c?*

Answer: From panel (b) of Exercise Graph 18.5 and our answer to A(d) we can see that the deadweight loss area $(e+g)$ is equal to half the cost $(e+f+g)$ of the government program. Thus, deadweight loss is

$$\text{DWL} = \frac{\left[\beta A + \alpha B - (\alpha+\beta)p^c\right]^2}{2\alpha\beta(\alpha+\beta)}. \tag{18.5.iv}$$

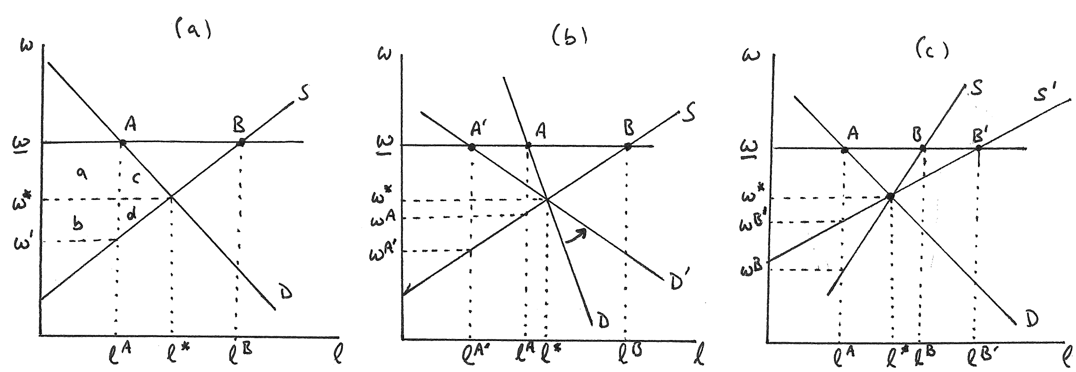
Exercise 18.7

Business and Policy Application: *Minimum Wage Laws: Most developed countries prohibit employers from paying wages below some minimum level $\underline{w}$. This is an example of a price floor in the labor market — and the policy has an impact in a labor market so long as $\underline{w} > w^*$ (where w^* is the equilibrium wage in the absence of policy-induced wage distortions.)*

A: *Suppose $\underline{w}$ is indeed set above w^*, and suppose that labor supply slopes up.*

(a) *Illustrate this labor market — and the impact of the minimum wage law on employment.*

Answer: This is illustrated in panel (a) of Exercise Graph 18.7 where ℓ^* is the pre-minimum wage level of employment and ℓ^A is the post-minimum wage level of employment. Thus, employment falls by $(\ell^* - \ell^A)$.



Exercise Graph 18.7 : Minimum Wage Laws

(b) *Suppose that the disequilibrium unemployment caused by the minimum wage gives rise to more intense effort on the part of workers to find employment. Can you illustrate in your graph the equilibrium cost of the additional effort workers expend in securing employment?*

Answer: The disequilibrium unemployment is the difference between ℓ^B and ℓ^A in panel (a) of Exercise Graph 18.7. If this disequilibrium is resolved by workers competing more intensely for jobs, then workers will require a higher wage in order to cover these additional costs. Thus, the labor supply curve shifts up as workers incur additional costs and point A does not become an equilibrium until the labor supply curve has shifted up by $(\underline{w} - w')$ which is the equilibrium increase in effort costs on the part of workers.

(c) *If leisure were quasilinear (and you could therefore measure worker surplus on the labor supply curve), what's the largest that deadweight loss from the minimum wage might become?*

Answer: The largest possible deadweight loss is composed of two parts (both represented in panel (a) of Exercise Graph 18.7): First, there is a deadweight loss from the decrease in employment (from ℓ^* to ℓ^A). This deadweight loss is equal to area $(c+d)$. The second part of deadweight loss emerges from the increased effort costs of workers to secure employment. These costs sum to $(a+b)$ — and how much of this is deadweight loss depends on how much of it is recouped by someone else in the economy. If none of it is recouped by anyone else (as when workers simply run from place to place applying frantically for jobs), the entire area is deadweight loss. Thus the largest possible area of deadweight loss is $(a+b+c+d)$.

(d) *How is the decrease in employment caused by the minimum wage (relative to the non-minimum wage employment level) related to the wage elasticity of labor demand? How is it related to the wage elasticity of labor supply?*

Answer: The more inelastic the demand curve, the smaller the distance between ℓ^* and ℓ^A — i.e. the smaller the impact of the minimum wage law on employment. The elasticity of the supply curve plays no role in determining ℓ^A — the post-minimum wage employment level — and therefore is not relevant for determining the employment impact of the minimum wage law. This can be seen in panels (b) and (c) of Exercise Graph 18.7. In panel (b), the demand curve D is more inelastic than the demand curve D' — with the former leading to a reduction in employment from ℓ^* to ℓ^A and the latter leading to a larger reduction from ℓ^* to $\ell^{A'}$. In panel (c), the supply curve S' is more elastic than the curve S — but for both, the new equilibrium moves to A which is unaffected by the elasticity of the supply curve.

(e) *Define unemployment as the difference between the number of people willing to work at a given wage and the number of people who can find work at that wage. How is the size of unemployment at the minimum wage affected by the wage elasticities of labor supply and demand?*

Answer: This definition if unemployment is the difference between ℓ^B and ℓ^A in panel (a) of the graph. In panels (b) and (c), it is obvious that greater elasticity of demand or supply will widen the gap between ℓ^A — the number of workers that can work under the minimum wage, and ℓ^B

— the number of workers who would like to work at the minimum wage. When labor demand curves get more wage elastic, the increase unemployment therefore comes from a larger reduction in actual employment; and when the labor supply curve becomes more elastic, the increase in unemployment comes from an increase in the number of workers that would like to work at the higher wage.

(f) *How is the equilibrium cost of effort exerted by workers to secure employment affected by the wage elasticities of labor demand and supply?*

Answer: Again, we can read the answers off the graphs in panels (b) and (c) of Exercise Graph 18.7. In panel (b), demand becomes more elastic from D to D' — leading to an increase in the equilibrium effort cost from $(\underline{w} - w^A)$ to $(\underline{w} - w^{A'})$. In panel (c), on the other hand, the supply curve becomes more elastic from S to S' — this time leading to a *decrease* in the effort cost from $(\underline{w} - w^B)$ to $(\underline{w} - w^{B'})$. An increase in the wage elasticity of labor therefore leads to an increase in the effort cost — while an increase in the elasticity of supply leads to a decrease in that cost.

B: *Suppose that labor demand is given by $\ell_D = (A/w)^\alpha$ and labor supply is given by $\ell_S = (Bw)^\beta$.*

(a) *What is the wage elasticity of labor demand and labor supply?*

Answer: The wage elasticity of labor demand is equal to

$$\frac{d\ell_D}{dw}\frac{w}{\ell_D} = -\alpha\left(\frac{A^\alpha}{w^{(\alpha+1)}}\right)\left(\frac{w}{\frac{A^\alpha}{w^\alpha}}\right) = -\alpha. \tag{18.7.i}$$

The wage elasticity of labor supply is

$$\frac{d\ell_S}{dw}\frac{w}{\ell_S} = \beta B^\beta w^{(\beta-1)}\left(\frac{w}{(Bw)^\beta}\right) = \beta. \tag{18.7.ii}$$

(b) *What is the equilibrium wage in the absence of any distortions?*

Answer: Setting ℓ_D equal to ℓ_S and solving for w, we get

$$w^* = \left(\frac{A^\alpha}{B^\beta}\right)^{1/(\alpha+\beta)}. \tag{18.7.iii}$$

(c) *What is the equilibrium labor employment in the absence of any distortions?*

Answer: Plugging our answer for w^* back into either the labor demand or supply function, we get

$$\ell^* = (AB)^{\alpha\beta/(\alpha+\beta)}. \tag{18.7.iv}$$

(d) *Suppose $A = 24{,}500$, $B = 500$ and $\alpha = \beta = 1$. Determine the equilibrium wage w^* and labor employment ℓ^*.*

Answer: Plugging these into our expression for w^* and ℓ^*, we get

$$\ell^* = 3{,}500 \text{ and } w^* = 7.00. \tag{18.7.v}$$

(e) *Suppose that a minimum wage of \$10 is imposed. What is the new employment level ℓ^A — and the size of the drop in employment ($\ell^* - \ell^A$)?*

Answer: To find the new employment level ℓ^A, we simply plug the minimum wage into the labor demand function to get $\ell^A = 2,450$, a drop of $(\ell^* - \ell^A) = 1,050$.

(f) *How large is unemployment under this minimum wage — with unemployment U defined as the difference between the labor that seeks employment and the labor that is actually employed at the minimum wage?*

Answer: We have already found the actual employment level $\ell^A = 2,450$ under the minimum wage. To find the level of employment desired by workers when the wage is 10, we simply plug this minimum wage into the labor supply function to get $\ell^B = 5,000$. Thus, unemployment, defined as $U = (\ell^B - \ell^A)$, is 2,550.

(g) *If the new equilibrium is reached through workers expending increased effort in securing employment, what is the equilibrium effort cost c^*?*

Answer: The equilibrium effort cost is the equal to the difference between the minimum wage and w' as depicted in panel (a) of Exercise Graph 18.7. We can find w' by first solving the labor supply function for the labor supply curve — i.e. solving for p to get $w = \ell^{1/\beta}/B$ — and then plugging $\ell^A = 2450$ in for ℓ. This gives us $w' = 4.9$ — which implies $c^* = 10 - 4.9 = 5.1$.

(h) *Create a table with w^*, ℓ^*, ℓ^A, $(\ell^* - \ell^A)$, U and c^* along the top. Then fill in the first row for the case you have just calculated — i.e. the case where $A = 24,500$, $B = 500$ and $\alpha = \beta = 1$.*

Answer: This is done in the following table:

Wage Elasticities and Minimum Wages						
Demand and Supply Parameters	w^*	ℓ^*	ℓ^A	$(\ell^* - \ell^A)$	U	c^*
$A = 24,500,\ B = 500,\ \alpha = \beta = 1$	\$7.00	3,500	2,450	1,050	2,550	\$5.10
$A = 11,668,\ B = 500,\ \alpha = 1.1,\ \beta = 1$	\$7.00	3,500	2,364	1,136	2,636	\$5.27
$A = 24,500,\ B = 238.1,\ \alpha = 1,\ \beta = 1.1$	\$7.00	3,500	2,450	1,050	2,731	\$4.92

(i) *Next consider the case where $A = 11,668$, $B = 500$, $\alpha = 1.1$ and $\beta = 1$. Fill in the second row of the table for this case — and explain what is happening in terms of the change in wage elasticities.*

Answer: This is also done in the table in part (h). The table shows that we have changed the labor demand parameters in such a way as to not change the equilibrium in the absence of wage distortions. Since we know from part (a) that the wage elasticity of labor demand is $-\alpha$, we know that what we have done is to make labor demand more elastic (by raising it in absolute value from 1 to 1.1). In panel (b) of Exercise Graph 18.7, we illustrated that this should lead to a larger drop in employment, an increase in unemployment and an increase in the effort cost for securing work. This is indeed what the figures in the second row of the table show.

(j) *Finally, consider the case where* $A = 24,500$, $B = 238.1$, $\alpha = 1$ *and* $\beta = 1.1$. *Fill in the third row of the table for this case — and again explain what is happening in terms of the change in wage elasticities.*

Answer: This is also done in the table in part (h). This time the labor supply parameters are changed in such a way as to keep the no-distortion equilibrium unchanged. By increasing β, however, we have now increased the wage elasticity of labor supply — just as we did in panel (c) of Exercise Graph 18.7. In that graph, we concluded that this should lead to no change in the drop of employment from the minimum wage (because firm demand is unchanged), an increase in unemployment (because more workers want to find work), and a decrease in the effort cost for securing work. This is again consistent with what the table indicates.

Exercise 18.9

Business and Policy Application: *Subsidizing Corn through Price Floors: Suppose the domestic demand and supply for corn intersects at* p^* *— and suppose further that* p^* *also happens to be the world price for corn. (Since the domestic price is equal to the world price, there is no need for this country to either import or export corn.) Assume throughout that income effects do not play a significant role in the analysis of the corn market.*

A: *Suppose the domestic government imposes a price floor* $\overline{p}$ *that is greater than* p^* *and it is able to keep imports of corn from coming into the country.*

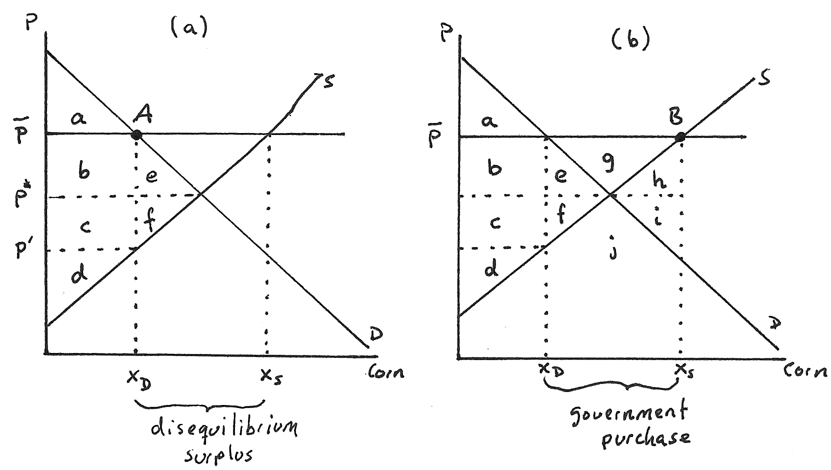
(a) *Illustrate the disequilibrium shortage or surplus that results from the imposition of this price floor.*

Answer: This is illustrated in panel (a) of Exercise Graph 18.9 where domestic supply and demand intersect at p^* and the price floor $\overline{p}$ is imposed above p^*. This results in a disequilibrium surplus, with x_S supplied but only x_D demanded.

(b) *In the absence of anything else happening, how will an equilibrium be re-established and what will happen to producer and consumer surplus?*

Answer: Consumer surplus will fall from $(a+b+e)$ to a while producer surplus will fall from $(c+d+f)$ to d. This is because, in equilibrium, producers will have to exert additional effort — i.e. incur additional costs — to compete for the limited number of consumers — which will cause the effective price they receive to fall to p'. (The additional marginal cost of effort on the part of producers must be $(\overline{p}-p')$ in order to make point A in panel (a) of Exercise Graph 18.9 the new equilibrium in which the disequilibrium shortage has been eliminated.)

(c) *Next, suppose the government agrees to purchase any corn that domestic producers cannot sell at the price floor. The government then plans to turn around and sell the corn it purchases on the world market (where its sales are sufficiently small to not affect the world price of corn). Illustrate how an equilibrium will now be re-established — and determine the change in domestic consumer and producer surplus from this government program.*



Exercise Graph 18.9 : Price Floor in Corn Market

Answer: This is illustrated in panel (b) of Exercise Graph 18.9 where the difference between x_S and x_D — previously labeled a "disequilibrium surplus" in panel (a) — now becomes the quantity of corn purchased by the government. In essence, the government purchasing program causes the equilibrium to settle at B rather than A (as in panel (a) of the graph) — because producers no longer have an incentive to expend additional effort to attract consumers since the government is guaranteeing it will purchase what cannot be sold at the price floor. Consumer surplus is then again a (since consumers purchase x_D at $\overline{p}$ as before; producer surplus, however, now increases to $(b+c+d+e+f+g)$ as producers supply x_S at the price $\overline{p}$.

(d) *What is the deadweight loss from the price floor with and without the government purchasing program?*

Answer: The greatest possible surplus achievable in this market is $(a+b+c+d+e+f)$. With just the price floor (and no government purchasing program), we concluded before that total surplus will be at most $(a+b+c+d)$ — implying a deadweight loss of $(e+f)$. When the price floor is supplemented with the government purchasing program, the sum of consumer and producer surplus becomes $(a+b+c+d+e+f+g)$. However, we now need to take into account that the government is also having to spend resources in order to buy the surplus at the price floor $\overline{p}$ and then sell it at a loss at p^*. It will therefore cost $(e+f+g+h+i+j)$ to buy the surplus corn and, when sold at p^*, it will raise revenues of $(f+i+j)$ — leaving a government loss of $(e+g+h)$. The total surplus is then the sum of producer and consumer surplus minus the government loss — which comes to $(a+b+c+d+e+f+g)-(e+g+h)=(a+b+c+d+f-h)$. Compared to the most possible surplus of $(a+b+c+d+e+f)$, this implies a deadweight loss of $(e+h)$.

(e) *In implementing the purchasing program, the government notices that it is not very good at getting corn to the world market — and all of it spoils before it can be sold. How does the deadweight loss from the program change depending on how successful the government is at selling the corn on the world market?*

Answer: The government loss now becomes $(e+f+g+h+i+j)$ — which gives us total surplus of $(a+b+c+d+e+f+g)-(e+f+g+h+i+j)=(a+b+c+d-h-i-j)$. Compared to the maximum possible surplus of $(a+b+c+d+e+f)$, this gives us a deadweight loss of $(e+f+h+i+j)$.

(f) *Would either consumers or producers favor the price floor on corn without any additional government programs?*

Answer: As illustrated in part (b) of the question, both producers and consumers lose surplus under the price floor policy without additional government programs. Thus, neither would favor such a program.

(g) *Who would favor the price floor combined with the government purchasing program? Does their support depend on whether the government succeeds in selling the surplus corn? Why might they succeed in the political process?*

Answer: As illustrated in part (c) of the question, producers gain substantial amounts of surplus when the government program is added to the price floor — and the amount of surplus they gain does not depend on what the government does with the surplus corn that was purchased. Thus, producers would favor the price floor when combined with the government purchasing program — and they might succeed in the political process because they are a relatively small group (compared to consumers and tax payers) experiencing concentrated benefits. This gives them an incentive to expend resources to lobby for such a program — and the diffuse nature of the costs (spread over many consumers and taxpayers) makes it unlikely that those who lose from the program will politically organize against it.

(h) *How does the deadweight loss from the price floor change with the price elasticity of demand?*

Answer: It decreases as demand becomes more inelastic.

B: *Suppose the domestic demand curve for bushels of corn is given by $p = 24 - 0.00000000225x$ while the domestic supply curve is given by $p = 1+0.00000000025x$. Suppose there are no income effects to worry about.*

(a) *Calculate the equilibrium price p^* (in the absence of any government interference). Assume henceforth that this is also the world price for a bushel of corn.*

Answer: Re-writing the demand and supply curves as demand and supply functions (i.e. solving for x to be on one side), we get

$$x_D = \frac{24-p}{0.00000000225} \text{ and } x_S = \frac{p-1}{0.00000000025}. \tag{18.9.i}$$

Setting these equal to one another and solving for p, we get the equilibrium price $p^* = 3.3$ per bushel.

(b) *What is the quantity of corn produced and consumed domestically? (Note: The price per bushel and the quantity produced is roughly equal to what is produced and consumed in the U.S. in an average year.)*

Answer: Plugging the equilibrium price of 3.3 into either the demand or supply function in equation (18.9.i), we get $x^* = 9,200,000,000$ or 9.2 billion bushels.

(c) *How much is the total social (consumer and producer) surplus in the domestic corn market?*

Answer: Calculating these as the relevant triangles above the supply and below the demand curves, we get

$$CS = \frac{(24-3.3)(9,200,000,000)}{2} = 95,220,000,000 \text{ and}$$
$$PS = \frac{(3.3-1)(9,200,000,000)}{2} = 10,580,000,000 \tag{18.9.ii}$$

for a total social surplus of \$105,800,000,000 or \$105.8 billion.

(d) *Next suppose the government imposes a price floor of* $\overline{p} = 3.5$ *per bushel of corn. What is the disequilibrium shortage or surplus of corn?*

Answer: Plugging this price floor into the demand and supply functions of equation (18.9.i), we get

$$x_D = 9,111,111,111 \text{ and } x_S = 10,000,000,000 \tag{18.9.iii}$$

bushels of corn — giving us a disequilibrium surplus of $(x_S - x_D) = 888,888,889$ bushels of corn.

(e) *In the absence of any other government program, what is the highest possible surplus after the price floor is imposed — and what does this imply about the smallest possible size of the deadweight loss?*

Answer: The consumer surplus under the price floor is easy to calculate as just the area under the demand curve down to the price floor $\overline{p} = 3.5$ — i.e.

$$CS = \frac{(24-3.5)(9,111,111,111)}{2} = 93,388,888,889. \tag{18.9.iv}$$

To calculate the producer surplus given that producers will have to incur additional costs in order to compete for the lower quantity demanded by consumers requires the additional step of calculating p' in panel (a) of Exercise Graph 18.9 — which we get by plugging in the quantity demanded into the supply curve equation; i.e.

$$p' = 1 + 0.00000000025(9,111,111,111) = 3.27777... \approx 3.278. \qquad (18.9.v)$$

The producer surplus triangle (equivalent to the triangle d in panel (a) of Exercise Graph 18.9) is then

$$PS = \frac{(3.27777778 - 1)(9,111,111,111)}{2} = 10,376,543,210. \qquad (18.9.vi)$$

Consumer and producer surplus together then sum to \$103,765,432,099 or approximately \$103.765 billion. If we want to arrive at the highest possible figure for social surplus, we need to assume that the costs paid by producers to compete for consumers were not socially wasteful — and these costs (equivalent to area $(b+c)$ in panel (a) of Exercise Graph 18.9) is $(3.5-3.278)(9,111,111,111) = 2,024,691,358$. Added to the sum of producer and consumer surplus, we therefore get the highest possible social surplus as approximately \$105,790,123,457. Compared the original surplus of \$105,800,000,000, we therefore get a deadweight loss of \$9,876,543.

(f) *Suppose next that the government purchases any amount that corn producers are willing to sell at the price floor $\overline{p}$ but cannot sell to domestic consumers. How much does the government have to buy?*

Answer: To determine the amount the government has to buy, we need to subtract the amount that consumers demand — i.e. 9,111,111,111 bushels — from the amount that producers will supply at the price floor. To determine the latter, we simply plug the price floor of 3.5 into the supply function to get 10,000,000,000. Thus, the difference is 888,888,889 bushels of corn — which is the disequilibrium surplus previously calculated in (d).

(g) *What happens to consumer surplus? What about producer surplus?*

Answer: Consumer surplus stays the same as before — because consumers continue to buy the same amount at the same price floor. Producer surplus, however, is now equal to the triangle $(b+c+d+e+f+g)$ in panel (b) of Exercise Graph 18.9 — which is

$$PS = \frac{(3.5-1)(10,000,000,000)}{2} = 12,500,000,000. \qquad (18.9.vii)$$

(h) *What happens to total surplus assuming the government sells the corn it buys on the world market at the price p^*?*

Answer: The total surplus is now the sum of consumer and producer surplus minus the loss the government takes by buying corn at the price floor of 3.5 and selling it at the world price of 3.3. This gives us

$$\text{Social Surplus} = 93,388,888,889 + 12,500,000,000 - (3.5 - 3.3)(888,888,889) = \\ = 105,711,111,111. \tag{18.9.viii}$$

In the absence of any program, the total surplus was $150,800,000,000.

This fell to $105,790,123,457 with the imposition of just the price floor, and we have now shown it falls further to $105,711,111,111 if the government purchasing program is added to the price floor. This implies that the deadweight loss jumps from $9,876,543 under just the price floor to $88,888,889 when the government purchasing program is added — an increase of $79,012,346.

(i) *How much does deadweight loss jump under just the price floor as well as when the government purchasing program is added if* $\overline{p} = 4$ *instead of 3.5? What if it is 5?*

Answer: Going through steps similar to those we just went through, the overall surplus falls from the original $105,800,000,000 to $105,679,012,346 under the price floor of $\overline{p} = 4$.

It furthermore falls to $104,711,111,111 if the government purchasing program is added. This implies a deadweight loss under just the price floor of $120,987,654 which increases to $1,088,888,889 when the purchasing program is added. If the price floor is raised to $\overline{p} = 5$, the overall social surplus falls from $105,800,000,000 to $105,086,419,753 under just the floor and $99,377,777,778 if the purchasing program is added. This implies a deadweight loss of $713,580,247 under just the price floor and $6,422,222,222 when the government purchasing program is added.

Exercise 18.11

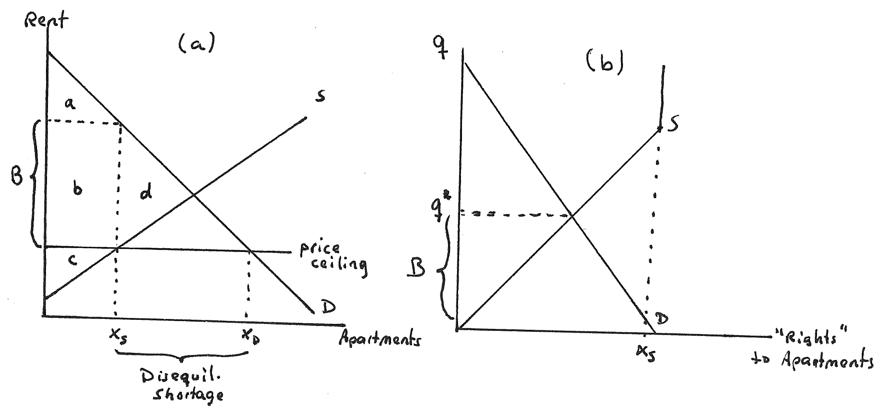
Policy Application: *Rent Control*: *A portion of the housing market in New York City (and many other cities in the world) is regulated through a policy known as* rent control. *In essence, this policy puts a price ceiling (below the equilibrium price) on the amount of rent that landlords can charge in the apartment buildings affected by the policy.*

A: *Assume for simplicity that tastes are quasilinear in housing.*

(a) *Draw a supply and demand graph with apartments on the horizontal axis and rents (i.e. the monthly price of apartments) on the vertical. Illustrate the "disequilibrium shortage" that would emerge when renters believe they can actually rent an apartment at the rent-controlled price.*

Answer: This is illustrated in panel (a) of Exercise Graph 18.11.

(b) *Suppose that the NYC government can easily identify those who get the most surplus from getting an apartment. In the event of excess demand for apartments, the city then awards the right to live (at the rent-controlled*



Exercise Graph 18.11 : Rent Controlled Apartments

price) in these apartments to those who get the most consumer surplus. Illustrate the resulting consumer and producer surplus as well as the deadweight loss from the policy.

<u>Answer</u>: This is illustrated in panel (a) of Exercise Graph 18.11. The area $(a+b)$ would now be consumer surplus — because these "high demanders" are the ones who get the apartments and only have to pay the price ceiling. Producer surplus is just area c — and deadweight loss is area d that no one receives because these apartments are not put on the market under rent control.

(c) *Next, suppose NYC cannot easily identify how much consumer surplus any individual gets — and therefore cannot match people to apartments as in (b). So instead, the mayor develops a "pay-to-play" system under which only those who pay monthly bribes to the city will get to "play" in a rent-controlled apartment. Assuming the mayor sets the required bribe at just the right level to get all apartments rented out, illustrate the size of the monthly bribe.*

<u>Answer</u>: This is illustrated as the distance B in panel (a) of Exercise Graph 18.11. By charging a bribe of this size, the mayor is able to collect a bribe from everyone who values these apartments at least as much as the price ceiling plus B — which is exactly the number of people we can fit into the rent-controlled apartments that are available.

(d) *Will the identity of those who live in rent-controlled apartments be different in (c) than in (b)? Will consumer or producer surplus be different? What about deadweight loss?*

<u>Answer</u>: In both cases, only the high demanders get into the apartments — so the identity of those living in the apartments is the same under both policies. The producer surplus and deadweight loss remains similarly the same. But now part of what is consumer surplus in (b) becomes revenue

from bribes in (c); i.e. consumer surplus is $(a+b)$ in (b) but only a in (c) — because now each consumer has to pay the bribe B on top of the price ceiling. The area b then becomes the bribe revenue for the mayor. Since this is a pure transfer from consumers to the mayor, it is not deadweight loss.

(e) *Next, suppose that the way rent-controlled apartments are allocated is through a lottery. Whoever wants to rent a rent-controlled apartment can enter his/her name in the lottery, and the mayor picks randomly as many names as there are apartments. Suppose the winners can sell their right to live in a rent-controlled apartment to anyone who agrees to buy that right at whatever price they can agree on. Who do you think will end up living in the rent-controlled apartments (compared to who lived there under the previous policies)?*

Answer: When all is said and done, the same people should once again end up in the apartments — whether they won in the lottery or not. This is because they value the apartments the most. If they win a ticket, they won't find someone that will buy it for an amount that exceeds how much they value the apartment. If they don't win a ticket, they will find someone that does not value the apartment as much as they do — and will thus buy the right to the apartment.

(f) *The winners in the lottery in part (e) in essence become the suppliers of "rights" to rent-controlled apartments while those that did not win in the lottery become the demanders. Imagine that selling your right to an apartment means agreeing to give up your right to occupy the apartment in exchange for a monthly check q. Can you draw a supply and demand graph in this market for "apartment rights" and relate the equilibrium point to your previous graph of the apartment market?*

Answer: The lottery will have made winners of some that really value the apartments highly and some that really don't value it very much at all. Thus, some of the winners will be willing to sell their rights at relatively low prices while others will demand higher prices. This results in a supply curve of the form in panel (b) of Exercise Graph 18.11 where the supply curve effectively ends — or becomes vertical — at x_s, the number of apartments that were raffled off in the lottery. The demand curve is made up of those who did not win in the lottery. We can't tell precisely what each of these curves will look like because of the randomness of the lottery — but they will intersect at an equilibrium price that allocates apartments to those who value them most. That price *has to be* equal to the size of the bribe B we identified in panel (a) as "clearing the market" — i.e. in equilibrium, those who get an apartment will again pay the price ceiling plus $B = q^*$.

(g) *What will be the equilibrium monthly price q^* of a "right" to live in one of these apartments compared to the bribe charged in (c)? What will be the deadweight loss in your original graph of the apartment market? How does your answer change if lottery winners are not allowed to sell their rights?*

Answer: As already explained in the answer to (f), $q^* = B$. The end result of the lottery combined with the market in "rights" will therefore again be the same as before — output is still limited to x_s, with high demanders living in the apartments. Those high demanders that won the lottery get a surplus equal to the difference between their marginal willingness to pay and the price ceiling; those who had to buy a right to an apartment because they did not win in the lottery only get a surplus of the difference between their $MWTP$ and the rental price inclusive of q^*. And those who won but sold their rights get a surplus q^*. Overall, consumers therefore get surplus a in panel (a) of Exercise Graph 18.11, and the sum of all the q^* surpluses — whether made by those who won and chose to live in an apartment or by those who won and sold their rights — is equal to area b. Landlords still get c — but no one gets d. Thus, d continues to be the deadweight loss. If, however, the lottery winners are not allowed to sell their rights, the deadweight loss will be larger because of the effective price ceiling of zero in the "rights" market where surplus is lost. Put differently, the "wrong" people will live in the apartments — in the sense that some who value the apartments more would be willing to pay these people an amount at which they would prefer not to live there and let others move in. The only way that the deadweight loss would not increase if we prohibit trade in the rights to apartments is if the lottery magically chose only the high demanders as winners — which would make everything identical to the case where the mayor magically knew who the high demanders were and allocated the apartments to them.

(h) *Finally, suppose that instead the apartments are allocated by having people wait in line. Who will get the apartments and what will deadweight loss be now? (Assume that everyone has the same value of time.)*

Answer: The same people will again live in the apartments — but they will now pay the cost B in the form of waiting in line. Since no one benefits from this, that implies that area b in Exercise Graph 18.11 now becomes part of deadweight loss.

B: *Suppose that the aggregate monthly demand curve is* $p = 10000 - 0.01x$ *while the supply curve is* $p = 1000 + 0.002x$. *Suppose further that there are no income effects.*

(a) *Calculate the equilibrium number of apartments* x^* *and the equilibrium monthly rent* p^* *in the absence of any price distortions.*

Answer: When written in terms of x rather than p, the demand and supply functions become $x = 1{,}000{,}000 - 100p$ and $x = 500p - 500{,}000$. Setting these equal to each other and solving for p, we get $p^* = 2{,}500$. Plugging this back into either the demand or supply function, we get $x^* = 750{,}000$.

(b) *Suppose the government imposes a price ceiling of \$1,500. What's the new equilibrium number of apartments?*

Answer: Plugging this into the supply function $x = 500p - 500{,}000$, we get $x = 250{,}000$.

(c) *If only those who are willing to pay the most for these apartments are allowed to occupy them, what is the monthly willingness to pay for an apartment by the person who is willing to pay the least but still is assigned an apartment?*

Answer: We can get this by plugging in the number of apartments under rent control — i.e. 250,000 — into the demand curve (or inverse demand function — which is equal to the marginal willingness to pay function when there are no income effects). We then get $p = 10{,}000 - 0.01(250{,}000) = \$7{,}500$.

(d) *How high is the monthly bribe per apartment as described in A(c)?*

Answer: This would simply be the difference between the value placed on the apartments by the marginal occupant — i.e. \$7,500 — and the amount this person has to pay in rent under rent control — i.e. \$1,500. Thus, the monthly bribe is \$6,000.

(e) *Suppose the lottery described in A(e) allocates the apartments under rent control, and suppose that the "residual" aggregate demand function by those who did not win in the lottery is given by $x = 750{,}000 - 75p$. What is the demand function for y — the "rights to apartments" (described in A(f))? What is the supply function in this market? (*Hint*: You will have to determine the marginal willingness to pay (or inverse demand) curves for those who did not win to get the demand for y and for those who did win to get the supply for y. And remember to take into account the fact that occupying an apartment is more valuable than having the right to occupy an apartment at the rent controlled price.)*

Answer: The residual demand curve (or inverse residual demand function) is the function $x = 750{,}000 - 75p$ solved for p — i.e. $p = 10{,}000 - (x/75)$. This is (in the absence of income effects) the marginal willingness to pay for these apartments by those who did not win the lottery. If someone who values an apartment at \$5,000 were asked his highest monthly price he would be willing to pay for a ticket that allows him to rent the apartment for the rent-controlled price of \$1,500, he would have to say he is willing to pay \$3,500 for such a ticket. Thus, the marginal willingness to pay for "rights" y to rent at the rent controlled price is \$1,500 lower than the marginal willingness to pay for the apartments. Letting q denote the marginal willingness to pay for y, we thus get $q = 8{,}500 - (y/75)$ — which is also the inverse demand function for y. Solving for y, we get the demand function

$$y_d = 637{,}500 - 75q. \tag{18.11.i}$$

The supply function comes from those who did win the lottery. The original aggregate demand function was $x = 1{,}000{,}000 - 100p$ while the residual demand function by those who did not win was $x = 750{,}000 - 75p$. Subtracting the latter from the former gives us the aggregate demand for apartments from those who won the lottery — i.e. $x = 250{,}000 - 25p$

which gives us a demand (and marginal willingness to pay) curve $p = 10,000-(1/25)x$. This is how much those who won tickets value the apartments. Someone who values an apartment at \$5,000 and owns a "right" to the apartment would then be willing to sell that right y for \$3,500 — because he gets that much consumer surplus from exercising his "right". The supply curve for these rights is then $q = -1500+(y/25)$. (To be slightly more accurate, we should specify this curve as flat at zero for those who value apartments at less than \$1,500, but, since the equilibrium will lie at a price q^* above zero, this does not matter for the math). Solving this for y, we get the supply function

$$y_s = 37,500 + 25q. \tag{18.11.ii}$$

(f) *What is the equilibrium monthly price of a right y to occupy a rent-controlled apartment? Compare it to your answer to (c).*

Answer: Now, all we have to do is set demand in the market for y equal to supply; i.e. $y_d = y_s$. Using functions we just derived, we therefore have

$$637,500 - 75q = 37,500 + 25q \tag{18.11.iii}$$

which solves to an equilibrium price $q^* = \$6,000$. Plugging this back into the demand (or supply) function for y, we get that 187,500 rights get traded after the lottery — with only 72,500 of the original lottery winners choosing to live in the rent-controlled apartment they won.

(g) *Calculate the deadweight loss from the rent control for each of the scenarios you analyzed above.*

Answer: In each of the scenarios, the deadweight loss is simply equal to the lower bound of the deadweight loss from a price ceiling — because in each case, those who live in the apartments are those who value them the most. (The producer price in each case is the price ceiling \$1,500 while the actual price necessary to clear the market for 250,000 apartments available at that price is \$7,500 — leaving a difference of \$6,000 that someone gets per apartment in each of the scenarios.) You can calculate this in a number of different ways by simply calculating the areas of the relevant triangles — and you should get that the deadweight loss is \$1,500,000,000 or \$1.5 billion.

(h) *How much would the deadweight loss increase if the rationing mechanism for rent-controlled apartments were governed exclusively by having people wait in line? (Assume that everyone has the same value of time.)*

Answer: This would imply that no one is now getting the \$6,000 times the 250,000 apartments — which implies the deadweight loss will increase by $6000(250,000) = \$1,500,000,000$ — i.e. the deadweight loss would double.

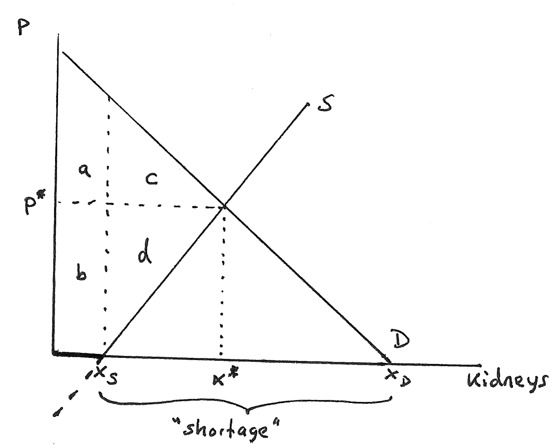
Exercise 18.13

Policy Application: *Kidney Markets: A large number of patients who suffer from degenerative kidney disease ultimately require a new kidney in order to survive. Healthy individuals have two kidneys but usually can live a normal life with just a single kidney. Thus, kidneys lend themselves to "live donations" — i.e. unlike an organ like the heart, the donor can donate the organ while alive (and live a healthy life with a high degree of likelihood). It is generally not permitted for healthy individuals to sell a kidney — kidney's can only be donated for free (with only the medical cost of the kidney transplant covered by the recipient or his insurance). In effect, this amounts to a price ceiling of zero for kidneys in the market for kidneys.*

A: *Consider, then, the supply and demand for kidneys.*

(a) *Illustrate the demand and supply curves in a graph with kidneys on the horizontal axis and the price of kidneys on the vertical. Given that there are some that in fact donate a kidney for free, make sure your graph reflects this.*

Answer: This is done in Exercise Graph 18.13 where the supply curve has to contain a flat spot on the horizontal axis in order to account for the fact that some give one of their kidneys away for free.



Exercise Graph 18.13 : Kidney Market

(b) *Illustrate how the prohibition of kidney sales results in a "shortage" of kidneys.*

Answer: In the graph, the quantity of kidneys supplied at a price of zero is x_S while the quantity demanded at that price is x_D. The difference is the shortage.

(c) *In what sense would permitting the sale of kidneys eliminate this shortage? Does this imply that no one would die from degenerative kidney disease?*

Answer: If the equilibrium price p^* were to be permitted to ration kidneys in this market, the quantity demanded *at that price* would equal the quantity supplied *at that price*. In this sense, there is no shortage. However, this does not imply that no one would die from kidney failure — as those not willing (or able) to pay the equilibrium price would still not get kidneys.

(d) *Suppose everyone has the same tastes but people differ in terms of their ability to generate income. What would this imply about how individuals of different income levels line up along the kidney supply curve in your graph? What does it imply in terms of who will sell kidneys?*

Answer: If everyone has the same tastes, then the income of kidney "suppliers" increases along the supply curve as we move to the left; i.e. poorer individuals would be willing to accept lower prices for one of their kidneys than richer people.

(e) *How would patients who need a kidney line up along the demand curve relative to their income? Who would not get kidneys in equilibrium?*

Answer: If tastes were again the same, higher income patients would be willing to pay more than lower income patients — thus lower income patients would be more likely not to get a kidney than higher income patients.

(f) *Illustrate in your graph the lowest that deadweight loss from prohibiting kidney sales might be assuming that demand curves can be used to approximate marginal willingness to pay. (*Hint*: The lowest possible deadweight loss occurs if those who receive donated kidneys under the price ceiling are also those that are willing to pay the most.)*

Answer: This is illustrated as the area $(c+d)$. This is because, under the price ceiling, only x_S kidneys are donated. If they are donated to those willing to pay the most, total surplus is $(a+b)$. If the equilibrium price p^* rationed kidneys in the market, the number of kidneys transplanted would rise to x^* — giving total surplus of $(a+b+c+d)$. We lose $(c+d)$ by prohibiting the selling of kidneys.

(g) *Does the fact that kidneys might be primarily sold by the poor (and disproportionately bought by well-off patients) change anything about our conclusion that imposing a price ceiling of zero in the kidney market is inefficient?*

Answer: No, it does not. By prohibiting sales that make both seller and buyer better off, we are prohibiting mutually beneficial trades from occurring — which is the source of the inefficiency. The fact that sellers might be relatively poor does not take away from the fact that they value their kidney less than they value the compensation the receive. And the fact that the kidneys might be disproportionately bought by the rich does not take away from the fact that they value the kidneys they buy more than the amount they pay for it — and more than sellers value the kidneys they are selling.

(h) *In the absence of ethical considerations that we are not modeling, should anyone object to a change in policy that permits kidney sales? Why do you think that opposition to kidney sales is so wide-spread?*

Answer: In principle, it is difficult to rationalize objections to permitting kidney sales in the absence of ethical considerations. To the extent that live kidney donations occur in the absence of kidney markets, these donations are typically among relatives who would likely still donate to their loved ones if kidney markets operated. (To the extent that kidney donations come from donors who have died, this is not the case — and some who would receive kidneys in this way may not be able to afford to buy a kidney if such kidneys could be sold). In principle, there may be some who lose in the transition to a market for kidneys — but the increase in the number of kidneys would likely lead to a large increase in the number of lives saved. Still, there are many reasons why one might object to permitting kidney sales — even though this would save many lives. Some might be concerned that some low income patients who might have received a kidney under the current system might not have the same chance in a kidney market. Others might be concerned that individuals who sell their kidneys might not always make such decisions in a rational state of mind. Yet others might point to potential abuses — with those in hierarchical power relationships able to coerce participation in kidney markets.

(i) *Some people might be willing to sell organs — like their heart — that they cannot live without in order to provide financially for loved ones even if it means that the seller will die as a result. Assuming that everyone is purely rational, would our analysis of deadweight loss from prohibiting such sales be any different? I think opposition to permitting such trade of vital organs is essentially universal. Might the reason for this also, in a less extreme way, be part of the reason we generally prohibit trade in kidneys?*

Answer: In principle, the analysis would be no different at all. To the extent to which there are potential sellers of their heart who would value the compensation their heirs receive more than they value their life, we are prohibiting trades that are mutually beneficial — and are therefore eliminating social surplus that could be generated. The natural opposition to the operation of such markets is founded in large part on the fact that those who might participate in such markets are likely to not be fully "rational" in the sense that their motives might emerge from psychological difficulties. Those contemplating suicide, for instance, would be natural candidates for sellers in this market, and we generally do not consider suicide as a rational act. In a less extreme way, concerns over the psychological issues that might lead one to participate in kidney markets may also play a role in the general opposition to permitting such markets.

B: *Suppose the supply curve in the kidney market is $p = B + \beta x$.*

(a) *What would have to be true in order for the phenomenon of kidney donations (at zero price) to emerge?*

Answer: It would have to be the case that $B < 0$ — implying a negative vertical intercept for the supply curve (as illustrated through the dashed portion in Exercise Graph 18.13).

(b) *Would those who donate kidneys get positive surplus? How would you measure this — and how can you make intuitive sense of it?*

Answer: Yes, those who donate at a zero price would still receive surplus — which would be measured as the triangle that emerges under the horizontal axis. In essence, those who donate kidneys would in fact be willing to pay to give their kidney away. This is not difficult to imagine in kidney donations where loved ones are involved.

1. When thinking about elasticities, always think of "the responsiveness of behavior to a change in an economic variable." The most common elasticity we think of is the price elasticity of demand — or the "responsiveness of demand to the change in price." But we can similarly think of any change in behavior as it relates to a change in an economic variable like price or income.

2. The formal definition of an elasticity is given in percentage terms: the percentage change in behavior from a 1 percent change in the economic variable. If the economic relationship that is involved is "downward sloping", it means that the change in behavior is in the direction opposite to the change in the economic variable — which means the elasticity is negative. Similarly, if the economic relationship that is involved is "upward sloping", the elasticity must be positive.

3. When asked questions about how surplus or deadweight loss changes as price elasticities change, it is often useful to sketch two quick graphs, one with a more elastic and one with a more inelastic curve. The answer then often jumps right out at you and you can then make intuitive sense of it.

4. The important thing to realize about price controls — whether they be price floors or price ceilings — is that the price that is set by the government cannot possibly be the "true" price that is paid by all market participants. The reason for that is that, if the undisturbed equilibrium price cannot emerge, the controlled price cannot equilibrate supply and demand. Thus, we are not in a real equilibrium — in a situation where no one can do better by changing behavior — unless something else happens. This "something else" is referred to in the text as non-price rationing.

5. As we will see throughout the coming chapters, government policies always create winners and losers. The important thing from an efficiency standpoint is to determine whether the winners win more than the losers lose, or whether the losers lose more than the winners win. In the first case, the policy is efficiency-enhancing, and in the second case, the policy is inefficient. Put differently, in the first case we could in principle (if we could transfer money between people) make everyone better off, whereas in the latter case someone will have to be made worse off no matter how much money is transferred from winners to losers.

6. This implies that if an inefficient policy is in place, removing the policy will create more gains for the winners than losses for the losers — so it is at least in principle possible to make everyone better off by eliminating the policy and transferring some of the gains by the winners to the losers. Put differently, an inefficient policy gives rise to a deadweight loss, and removing the policy will make that deadweight loss available to society and thus creates the possibility of making everyone better off. (All this assumes, of course, that efficiency is all we care about, and we have said many times — and will say many more times — that this is probably not always the case.)

CHAPTER

19

Distortionary Taxes and Subsidies

This chapter, the second in a series of three that focuses on implications of price distortions on the first welfare theorem, revisits the topics of distortionary taxes and subsidies, topics that had previously been treated solely within the consumer model and, to some extent (in end-of-chapter exercises) in the producer model. In these previous treatments, we simply assumed that a tax raised the price on consumers or the marginal costs of producers, or a subsidy lowered these. In the utility maximization framework, we were able to demonstrate that the deadweight losses from such price distortions could be measured on marginal willingness to pay (or compensated) curves — or, in the special case of quasilinear tastes, they could me measured on (uncompensated) demand curves. We return to these issues now because we now have the tools to illustrate how such taxes and subsidies affect consumer and producer prices *in equilibrium* — and how the work we previously did fits into this equilibrium framework. Building on our work with price elasticities in Chapter 18, we can furthermore differentiate these impacts based on relative price elasticities of demand and supply. And we can show how the pictures that you probably saw in previous economics courses apply for the case of quasilinear tastes but also how such pictures can be deeply misleading when the quasilinearity assumption is not appropriate.

Chapter Highlights

The main points of the chapter are:

1. The **statutory incidence** of taxes and subsidies bears no relation to the **economic incidence** of taxes and subsidies, with the latter arising from the price elasticities of demand and supply. The more price inelastic part of the market will end up bearing the greater burden of a tax because the tax can more easily be passed onto that side of the market regardless of how the tax law is written.

2. The size of **tax revenues** one can expect from a given tax rate is similarly dependent on the price elasticities of supply and demand, with revenues higher

the more price inelastic at least one side of the market is. For most taxes, we can derive a **Laffer curve** that illustrates tax revenues as initially increasing with the tax rate but eventually decreasing. As a rough rule of thumb, deadweight losses increase by t^2 for any t-fold increase in tax rates — leading to the conventional wisdom that **it is more efficient to tax a large base at a low rate rather than a small base at a high rate**.

3. When underlying tastes are **quasilinear**, deadweight losses can be measured as areas beneath the demand and above the supply curves — with deadweight losses on the consumer side arising from substitution effects as previously discussed in earlier chapters.

4. When underlying tastes are **not quasilinear**, measuring deadweight losses may be masked by offsetting income and substitution effects on the curve that arises from utility maximization. This is particularly relevant for **wage and capital taxes** as underlying tastes by workers and savers almost certainly give rise to such competing effects. In such cases, one cannot use the demand and supply picture as the sole guide for estimating deadweight losses but must instead isolate the deadweight loss from substitution effects using compensated curves.

5. **Land taxes** represent an unusual real world tax in that such taxes, when appropriately designed, give rise to no deadweight losses and no substitution effects.

19A Solutions to Within-Chapter-Exercises for Part A

Exercise 19A.1

Will the increase in price from the tax be larger or smaller in the long run? (*Hint*: How is the price elasticity of supply in the long run usually related to the price elasticity of supply in the short run?)

Answer: The increase in price will be larger in the long run since supply curves are more price-elastic in the long run.

Exercise 19A.2

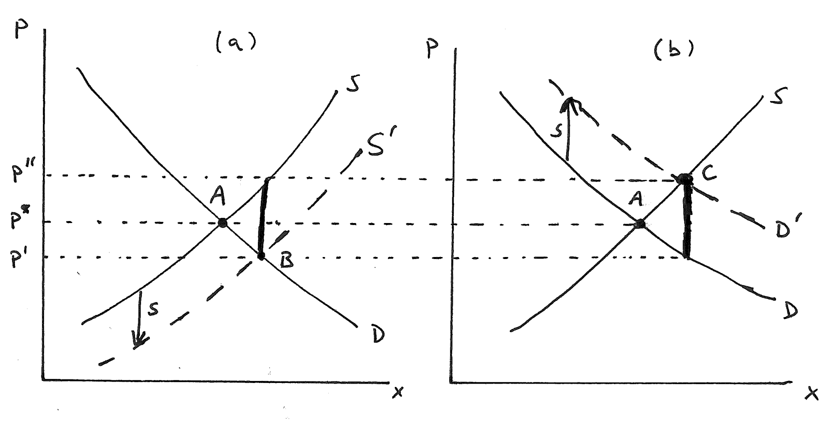
Using a pencil, redraw the graphs in panel (a) and (b) but this time label clearly which price buyers end up paying and sellers end up receiving, taking into account that sellers have to pay the tax in panel (a) and buyers have to pay the tax in panel (b). Then erase the shifted curves in your two graphs. Do the two graphs now look identical to each other and to the graph in panel (c)? (The answer should be yes.)

Answer: In panel (a), $p' = p_d$ and $p' - t = p_s$ (where p_d is the price paid by consumers and p_s is the price received by firms). In panel (b), $p'' + t = p_d$ and $p'' = p_s$. (Note that the graph is not drawn here.)

Exercise 19A.3

Illustrate how the equilibrium changes when the subsidy is paid to sellers (thus reducing their MC). Compare this to how the equilibrium changes when the subsidy is paid to buyers (thus shifting the demand curve). Can you see how both of these types of subsidies will result in an economic outcome summarized in Graph 19.2?

Answer: This is done in panels (a) and (b) of Exercise Graph 19A.3. In panel (a), the subsidy to producers shifts the supply curve down by the amount of the per-unit subsidy s — taking us from the initial equilibrium A to the new equilibrium B with equilibrium price p'. In panel (b), the subsidy to consumers shifts up the demand curve by s, taking us from the initial equilibrium A to the new equilibrium C with equilibrium price p''. Although the equilibrium prices are different in the two pictures, the end result is equivalent: In panel (a), p' represents the amount that consumers pay to producers — but producers still get the per-unit subsidy s on top of it. Thus, the consumer price is $p_d = p'$ and the supplier price is $p_s = p' + s = p''$. In panel (b), the equilibrium price p'' is the price that consumers pay to producers — but they then receive a per-unit subsidy s, which implies that the true price consumers pay is $p_d = p'' - s = p'$ while the price suppliers get is $p_s = p''$.



Exercise Graph 19A.3 : Price Subsidies for Producers and Consumers

Exercise 19A.4

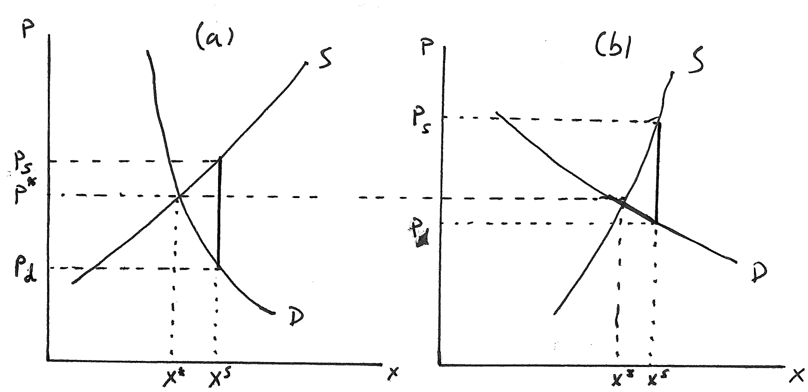
During the 2008 presidential campaign in the United States, oil prices increased sharply. Some candidates advocated a "tax holiday" on gasoline taxes to help consumers. Others argued that this would have little effect on gasoline prices in the short run. Assuming each side was honest, how must they have disagreed on their estimates of underlying price elasticities?

Answer: If you think the tax holiday will help consumers, you must think that the demand curve is inelastic relative to the supply curve — and thus the lower tax is primarily passed onto consumers. If you think the tax holiday will not help consumers very much, then you must think that the supply curve is inelastic relative to the demand curve — with the benefits of lower tax rates captured primarily by suppliers.

Exercise 19A.5

In graphs with demand and supply curves similar to those in Graph 19.3, illustrate the economic impact on buyers and sellers of subsidies. How does the benefit of a subsidy relate to relative price elasticities?

Answer: This is illustrated in Exercise Graph 19A.5 where the benefit of the subsidy disproportionately accrues to consumers in panel (a) where demand is relatively inelastic and to producers in panel (b) where supply is relatively inelastic. Thus, just as tax burdens are passed onto the side of the market that behaves relatively inelastically, so subsidy benefits are passed onto the side of the market that behaves relatively inelastically.



Exercise Graph 19A.5 : Price Subsidies and Elasticities

Exercise 19A.6

Does the impact of subsidies on market output also rise with the price-responsiveness of buyers and sellers?

Answer: Yes. This is easily seen by simply drawing the green price differential in Graph 19.4 of the text to the other side of the equilibrium price where the subsidy price differential is determined.

Exercise 19A.7

Suppose the government has already imposed the taxes graphed in Graph 19.4 and is now considering raising this tax. Can you see in these graphs under what circumstances this would result in a decrease in overall tax revenues?

Answer: It would result in a decline in tax revenues in panel (a) where the quantity response is large for even a small increase in the green tax differential.

Exercise 19A.8

Suppose the tax on fuel-efficient cars is low and the tax on gas-guzzling cars is high. Is it likely that our partial equilibrium estimate of a tax on gasoline will cause us to over- or underestimate the full impact on government revenues?

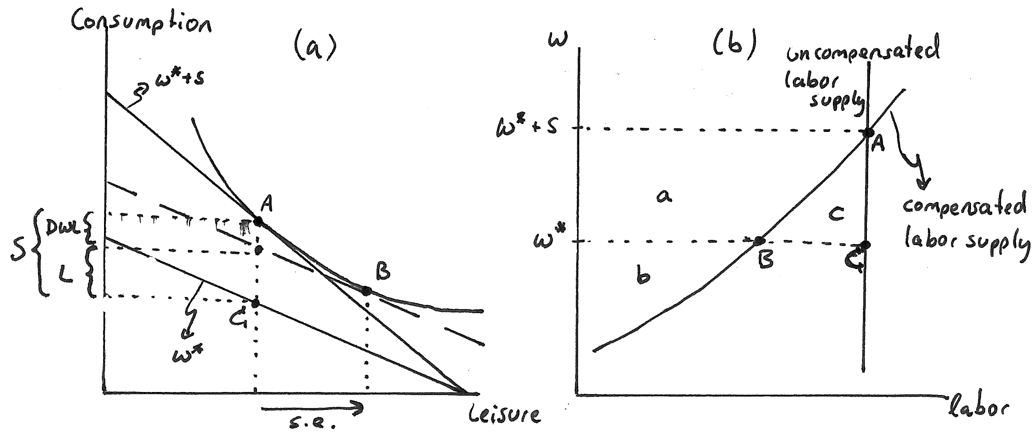
Answer: The high tax on gas-guzzling cars will cause a decrease in demand in that market and an increase in demand in the low-taxed fuel-efficient car market. Thus, we are likely to over-estimate the tax revenues in the high-tax market and under-estimate the tax revenues in the low tax market, with the former likely to outweigh the latter as consumers avoid paying taxes by changing behavior.

Exercise 19A.9

Illustrate, using an analogous set of steps we just used as we worked our way through Graph 19.6, how wage subsidies are inefficient even when workers are completely unresponsive to changes in wages. (*Hint*: If you get stuck, read the next section and come back.)

Answer: This is done in Exercise Graph 19A.9. In panel (a), we illustrate the shallower (pre-subsidy) wage w^* and the steeper (post-subsidy) wage $(x^* + s)$. After the subsidy, the worker chooses A. This implies that the government pays this worker the vertical distance S. The government could instead have given the worker a lump sum subsidy L and made him just as well off. The difference between L and S is the deadweight loss — and it arises solely because of the substitution effect from A to B. In panel (b) of the graph, the analogous points to A, B and C are plotted on the labor supply and compensated labor supply curves. The worker's surplus at the subsidized wage $(w^* + s)$ is area $(a + b)$. The total payment S made by the government is the difference between the two wages times the amount the worker works at A — giving us $S = a + c$. In the absence of the wage subsidy (and with just the lump sum subsidy), the worker's surplus is b. But the worker is indifferent between

A and B — which means that the lump sum subsidy must have been equal to $L = a$. Since we concluded that the price subsidy costs $(a + c)$, this leaves us with deadweight loss $DWL = c$.



Exercise Graph 19A.9 : Wage Subsidies and Deadweight Loss

Exercise 19A.10

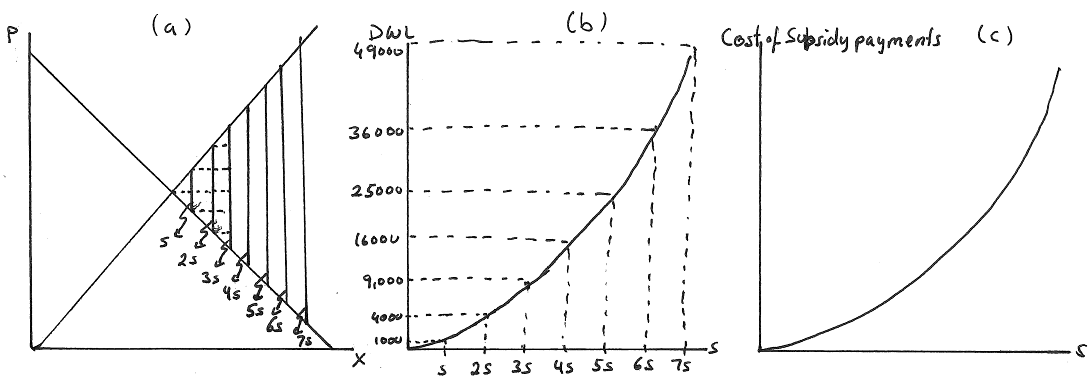
How large does deadweight loss get if the tax rate rises to $3t$? What if it rises to $4t$?

Answer: \$9,000 and \$16,000. (You can see this by adding up the relevant squares in Graph 19.8a, and of course this follows exactly the rule that a k-fold increase in tax rates causes a k^2-fold increase in the deadweight loss.

Exercise 19A.11

Illustrate the relationship between subsidy rates, the deadweight loss from a subsidy and the cost of the subsidy using the same initial graph of supply and demand as in Graph 19.8a in graphs analogous to panels (b) and (c) of Graph 19.8.

Answer: This is done in Exercise Graph 19A.11. Panel (a) is the same as the panel in the text, except that the subsidy shows up to the right of the equilibrium price. The deadweight loss graph in panel (b) is exactly identical to that in the text — because the deadweight loss triangles that accumulate are the same as those in the text except that they appear on the other side of the initial equilibrium. For instance, if s leads to a deadweight loss of 1,000, then the deadweight loss triangle to the right of the equilibrium in panel (a) for the small subsidy s has area 1,000. When the subsidy rate rises to $2s$, we are adding the equivalent of 3 further such deadweight loss triangles (with each of the two squares in panel (a) equal to 1,000 and the two triangles that remain jointly equal to 1,000). Thus, the new deadweight loss is the initial 1,000 plus 3,000 — or a total of 4,000. The major difference in the graphs comes in panel (c) which differs from panel (c) in the text. The reason for this is that tax revenues will eventually fall to zero as tax rates are increased, but subsidy costs for the government never fall as the government subsidizes more — demand for the subsidy just keeps on going up. Thus, the subsidy cost increases at an increasing rate as the subsidy rate *s* rises.



Exercise Graph 19A.11 : Subsidies, DWL and Costs

Exercise 19A.12

What would be the economic impact of a 100% tax on land rents (levied on owners)?

Answer: A 100% tax on land rents would reduce land prices to zero — thus in effect transferring (in a lump sum way) the wealth of land owners to the government. The reason that the transfer happens in a lump sum way is that there is nothing land owners can do to avoid paying any of this tax. They either continue to rent the land out and hand the rental revenues to the government, or they sell the land. But if they sell it, no one is willing to pay more than zero for the land. Either way, the landowners will pay the land rent tax.

19B Solutions to Within-Chapter-Exercises for Part B

Exercise 19B.1

Demonstrate that, whenever ε_d is less in absolute value than ε_s, consumers will bear more than half the incidence of the tax, and whenever the reverse is true, they will bear less than half of the incidence of the tax.

Answer: If $|\varepsilon_d| < \varepsilon_s$, this implies

$$|\varepsilon_d - \varepsilon_s| > |2\varepsilon_d| \tag{19B.1.i}$$

(because ε_d is negative while ε_s is positive). We can then write

$$\frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} < \frac{\varepsilon_d}{2\varepsilon_d} = \frac{1}{2}. \tag{19B.1.ii}$$

Given the expression for dp_s/dt we derived in the text, we then simply have to multiply this by negative 1 (which reverses the inequality) to get

$$\frac{dp_s}{dt} = -\frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} > -\frac{1}{2} \quad \text{or} \quad dp_s > -0.5dt. \tag{19B.1.iii}$$

We know that the only difference between p_d and p_s is t — which implies $dp_d = dp_s + dt$. Substituting our result from (19B.1.iii), we can then write this as

$$dp_d = dp_s + dt > -0.5dt + dt = 0.5dt \tag{19B.1.iv}$$

or simply $dp_d > 0.5dt$; i.e. the increase in the consumer's price is more than half the per unit tax and consumers therefore bear more than half the burden of the tax. To prove the reverse, the same steps can be used, with each inequality pointing in the opposite directions.

Exercise 19B.2

Can you show that $dp_d/dt = \varepsilon_s/(\varepsilon_s - \varepsilon_d)$? (*Hint*: Note that equation (19.2) implies $dp_d/dt = (dp_s/dt) + 1$.)

Answer: Dividing the expression $dp_d = dp_s + dt$ by dt, we get the expression given in the hint. Substituting $dp_s/dt = -(\varepsilon_d/(\varepsilon_d - \varepsilon_s))$, we then get

$$\frac{dp_d}{dt} = \frac{dp_s}{dt} + 1 = -\frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} + 1 \tag{19B.2.i}$$

$$= \frac{\varepsilon_d}{\varepsilon_s - \varepsilon_d} + \frac{\varepsilon_s - \varepsilon_d}{\varepsilon_s - \varepsilon_d} = \frac{\varepsilon_s}{\varepsilon_s - \varepsilon_d}. \tag{19B.2.ii}$$

Note that this again has an intuitive interpretation. It says that if price elasticity of supply is zero (i.e. supply is perfectly inelastic), consumers experience no price change as a result of a tax (because the entire burden of the tax rests with producers). If, on the other hand, price elasticity of demand is zero, $dp_d/dt = 1$ or $dp_d = dt$ and the entire burden of the tax is on consumers.

Exercise 19B.3

What is the price elasticity of demand for x? What is the cross-price elasticity of demand for y?

Answer: The price elasticity of demand is

$$\frac{dx}{dp}\frac{p}{x} = \frac{-\alpha}{p^2}\frac{p}{\frac{\alpha}{p}} = -1. \tag{19B.3.i}$$

The cross-price elasticity of demand for y is

$$\frac{dy}{dp}\frac{p}{y} = 0. \tag{19B.3.ii}$$

Exercise 19B.4

What is the price elasticity of supply?

Answer: The price elasticity of supply is

$$\frac{dx_s}{dp}\frac{p}{x_s} = \beta\frac{p}{\beta p} = 1. \tag{19B.4}$$

Exercise 19B.5

Can you verify that our answer for ΔPS is correct by simply calculating the area of the rectangle d and the triangle (e) in Graph 19.10?

Answer: The area of the rectangle d is $(10-6.18)(61.8) \approx 236$. The area of the triangle e is $(10-6.18)(100-61.8)/2 \approx 73$. Adding these together, we get a change in producer surplus equal to 309.

Exercise 19B.6

Can you derive this expenditure function more directly through an expenditure minimization problem?

Answer: The expenditure minimization problem is

$$\min_{x,y}\ px + y \ \text{ subject to } \ u = \alpha \ln x + y. \tag{19B.6.i}$$

Setting up the Lagrangian and solving in the usual way, we get compensated demand functions

$$x = \frac{\alpha}{p} \ \text{ and } \ y = u - \alpha \ln \frac{\alpha}{p}. \tag{19B.6.ii}$$

Plugging these in to the objective function $px+y$, we then get the expenditure function

$$E(p,u) = p\frac{\alpha}{p} + u - \alpha \ln\frac{\alpha}{p} = u + \alpha - \alpha \ln\frac{\alpha}{p}. \tag{19B.6.iii}$$

Exercise 19B.7

Can you verify that the expenditure necessary to reach the after tax utility at the pre-tax price is always less than (or equal to) I?

Answer: From what we derived in the text, we know that

$$E(p^*, u_t) = \alpha \ln\left(\frac{p^*}{p_d}\right) + I. \tag{19B.7}$$

Whether this is greater or less than I then depends on whether $\ln(p^*/p_d)$ is greater or less than 0. For any positive tax t, we know that $p_d \geq p^*$ — i.e. taxes will never lower the price consumers have to pay. Thus, $(p^*/p_d) \leq 1$ — which implies $\ln(p^*/p_d) \leq 0$ (since the natural log of 1 is zero and the natural log of anything between zero and 1 is negative). Thus, $E(p^*, u_t) \leq I$.

Exercise 19B.8

What has to be true for $E(p^*, u_t) = I$ to hold?

Answer: The only way $E(p^*, u_t) = I$ is if $\ln(p^*/p_d) = 0$ — which is the case if and only if $p^* = p_d$ (since the natural log of 1 is zero). Thus, the only way that the income necessary to reach the after-tax utility level at pre-tax prices is equal to I is if consumer price is unchanged as a result of the tax — which in turn can only happen if supply is perfectly inelastic. This should make intuitive sense: If consumer price is unaffected by that tax, then consumers are unaffected by the tax and their after-tax utility is the same as their before-tax utility. And the only way you can reach the before tax utility at pre-tax prices is for you to have I.

Exercise 19B.9

Can you show that in general, before substituting in specific pre- and post-tax prices, equation (19.13) (which we derived using integral calculus) and equation (19.18) (which we derived using the expenditure function) yield identical results?

Answer: The first equation tells us that $\Delta CS = \alpha(\ln p_d - \ln p^*)$. This can be re-written as $\Delta CS = -\alpha(\ln p^* - \ln p_d)$. Using the property of logarithm that $(\ln a - \ln b) = \ln(a/b)$, this further implies that

$$\Delta CS = -\alpha \ln\frac{p^*}{p_d} \tag{19B.9}$$

which is what we derived using the expenditure function.

Exercise 19B.10

Does the Laffer curve in this example have a peak? Why or why not?

Answer: No, in this case the Laffer curve does not have a peak but rather approaches 1000 as t gets large. This is because we have chosen a demand curve that has elasticity of -1 — which implies that total consumer spending on x remains the same as the price of x increases. For instance, with $\alpha = 1000$, the consumer buys $x = 1000/1 = 1000$ when $p_d = 1$ — for total spending of \$1000. As t goes up, the consumer reduces the quantity purchased such that $p_d x$ remains constant at \$1000. This further implies that, as t increases, a larger and larger fraction of the consumer's spending on x goes toward paying the tax. As t gets large, tax revenue then converges to the amount the consumer spends on x — which is \$1,000.

Exercise 19B.11

Verify that this labor supply function has zero wage elasticity of supply.

Answer: Since $dl_s/dw = 0$, the wage elasticity of supply $(dl_s/dw)(w/l_s) = 0$.

Exercise 19B.12

What is the wage elasticity of labor demand?

Answer: With labor demand function $l_d(w) = 25{,}000{,}000/w^2$, the wage elasticity of demand for labor is

$$\frac{dl_d}{dw}\frac{w}{l_d} = \frac{-50{,}000{,}000}{w^3}\frac{w}{\frac{25{,}000{,}000}{w^2}} = -2. \tag{19B.12}$$

Exercise 19B.13

Verify this.

Answer: The Lagrangian for the minimization problem is

$$\mathcal{L} = w\ell + c + \lambda\left(u_t - c^{\alpha}\ell^{(1-\alpha)}\right). \tag{19B.13.i}$$

The two first order conditions then imply that $\ell = (1-\alpha)c/(\alpha w)$. Plugging this into the constraint $u_t = c^{\alpha}\ell^{(1-\alpha)}$ and solving for c, we get the compensated consumption demand and plugging that back into $\ell = (1-\alpha)c/(\alpha w)$, we get the compensated leisure demand as given in the text. Plugging these into the objective function $w\ell + c$, we then get the expenditure function

$$\begin{aligned}
E(w,u_t) &= w\left(\frac{1-\alpha}{\alpha w}\right)^{\alpha} u_t + \left(\frac{\alpha w}{1-\alpha}\right)^{(1-\alpha)} u_t \\
&= w^{(1-\alpha)} u_t \left[\left(\frac{1-\alpha}{\alpha}\right)^{\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{(1-\alpha)}\right] \\
&= w^{(1-\alpha)} u_t \left[\frac{(1-\alpha)^{\alpha}(1-\alpha)^{(1-\alpha)} + \alpha^{(1-\alpha)}\alpha^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right] \\
&= w^{(1-\alpha)} u_t \left[\frac{1-\alpha+\alpha}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right] \\
&= \frac{w^{(1-\alpha)} u_t}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \qquad (19B.13.ii)
\end{aligned}$$

Exercise 19B.14

Can you find in a graph such as panel (b) of Graph 19.6 the various numbers calculated above?

Answer: Referring to Graph 19.6 (in the text), we calculated that $T = 400$, $L = 423.93$ and $DWL = 23.93$. The utility associated with A and B in panel (b) is $u^A = 193$.

Exercise 19B.15

Verify these.

Answer: For any per-unit tax t, we know that $p_d = p_s + t$. We can then write the demand function as $x_d = (A-(p_s+t))/\alpha$ and the supply function as $x_s = (p_s-B)/\beta$. Setting these equal to each other and solving for p_s, we get

$$p_s = \frac{\beta A + \alpha B - \beta t}{\alpha + \beta}. \qquad (19B.15.i)$$

From this, we can get the consumer after-tax price

$$p_d = p_s + t = \frac{\beta A + \alpha B - \beta t}{\alpha + \beta} + t = \frac{\beta A + \alpha B + \alpha t}{\alpha + \beta}. \qquad (19B.15.ii)$$

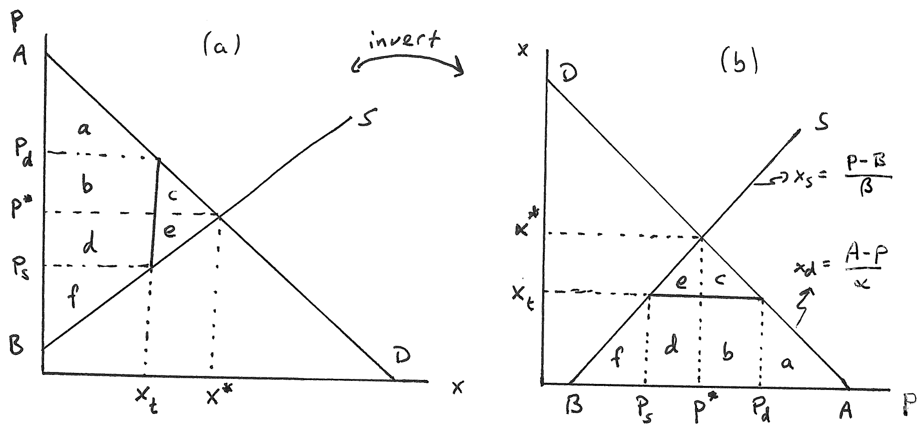
Plugging p_s into either $x_d = (A-(p_s+t))/\alpha$ or $x_s = (p_s - B)/\beta$, we get

$$x_t = \frac{A - B - t}{\alpha + \beta}. \qquad (19B.15.iii)$$

Exercise 19B.16

Verify the expression for deadweight loss. (*Hint*: There are two ways of doing this: You can either take the appropriate integrals of the supply and demand functions evaluated over the appropriate ranges of prices, or you can add rectangles and triangles in a graph.)

Answer: In this answer, we refer to Exercise Graph 19B.16, with the initial part based on panel (a). Note that the vertical intercepts of the demand and supply curves in panel (a) are A and B respectively. When $t=0$, output is $x^*=(A-B)/(\alpha+\beta)$ and price is $p^*=(\beta A+\alpha B)/(\alpha+\beta)$.



Exercise Graph 19B.16 : Taxes and Deadweight Loss

Thus, initial consumer surplus is area $(a+b+c)$ which is

$$CS^* = \frac{(A-p^*)x^*}{2} = \frac{1}{2}\left[A-\frac{\beta A+\alpha B}{\alpha+\beta}\right]\left(\frac{A-B}{\alpha+\beta}\right) = \frac{\alpha(A-B)^2}{2(\alpha+\beta)^2} \qquad (19B.16.i)$$

and, similarly, initial producer surplus is $(d+e+f)$ which is

$$PS^* = \frac{(p^*-B)x^*}{2} = \frac{\beta(A-B)^2}{2(\alpha+\beta)^2}. \qquad (19B.16.ii)$$

This gives us total initial surplus of $(a+b+c+d+e=F)$ or simply

$$TS^* = CS^*+PS^* = \frac{(\alpha+\beta)(A-B)^2}{2(\alpha+\beta)^2} = \frac{(A-B)^2}{2(\alpha+\beta)}. \qquad (19B.16.iii)$$

Consumer surplus under a tax t, on the other hand, is just area (a) or

$$CS^t = \frac{(A-p_d)x_t}{2} = \frac{1}{2}\left[A-\frac{\beta A+\alpha B+\alpha t}{\alpha+\beta}\right]\left(\frac{A-B-t}{\alpha+\beta}\right) = \frac{\alpha(A-B-t)^2}{2(\alpha+\beta)^2}. \qquad (19B.16.iv)$$

and producer surplus is area (f) or

$$PS^t = \frac{(p_s-B)x_t}{2} = \frac{1}{2}\left[\frac{\beta A+\alpha B-\beta t}{\alpha+\beta}-B\right]\left(\frac{A-B-t}{\alpha+\beta}\right) = \frac{\beta(A-B-t)^2}{2(\alpha+\beta)^2}. \qquad (19B.16.v)$$

Finally, tax revenue is area $(b+d)$ which is

$$TR^t = tx_t = \frac{t(A-B-t)}{\alpha+\beta}. \tag{19B.16.vi}$$

Total surplus under the per-unit tax t is then $(a+b+d+f)$ — or

$$TS^t = CS^t + PS^t + TR^t = \frac{(A-B-t)(A-B+t)}{2(\alpha+\beta)}. \tag{19B.16.vii}$$

The deadweight loss from the tax is then $TS^* - TS^t$ — which is equal to $(c+e)$ — or

$$DWL(t) = \frac{(A-B)^2}{2(\alpha+\beta)} - \frac{(A-B-t)(A-B+t)}{2(\alpha+\beta)} = \frac{(A-B)^2}{2(\alpha+\beta)} - \frac{(A-B)^2-t^2}{2(\alpha+\beta)} = \frac{t^2}{2(\alpha+\beta)}. \tag{19B.16.viii}$$

We could instead use integrals to calculate areas under demand and supply *functions* which are the inverses of demand and supply *curves*. These are graphed in panel (b) of Exercise Graph 19B.16. The deadweight loss — area $(e+c)$ — is then equal to the change in consumer surplus $(b+c)$ plus the change in producer surplus $(d+e)$ minus tax revenue $(d+b)$ where

$$\Delta CS = \int_{p^*}^{p_d} x_d(p)\,dp = \int_{p^*}^{p_d} \frac{A-p}{\alpha}\,dp = \left(\frac{(2A-p_d)p_d}{2\alpha}\right) - \left(\frac{(2A-p^*)p^*}{2\alpha}\right) \tag{19B.16.ix}$$

$$\Delta PS = \int_{p_s}^{p^*} x_s(p)\,dp = \int_{p_s}^{p^*} \frac{p-B}{\beta}\,dp = \left(\frac{(p^*-2B)p^*}{2\beta}\right) - \left(\frac{(p_s-2B)p_s}{2\beta}\right) \tag{19B.16.x}$$

$$TR = tx_t = \frac{t(A-B-t)}{\alpha+\beta}. \tag{19B.16.xi}$$

Adding the first two and subtracting the last, substituting our expressions for p^*, p_s and p_d and simplifying, we then get the same expression for deadweight loss.

Exercise 19B.17

For $t = 0.5$, verify that the marginal product columns of the table report the correct results.

Answer: When $t = 0.5$, $k_1 = 200$ and $k_2 = 800$. The marginal product of capital in housing is

$$MPK_1(k_1) = \frac{\alpha}{2k_1^{1/2}} = \frac{100}{2k_1^{1/2}} = \frac{50}{k_1^{1/2}}. \tag{19B.17.i}$$

When $k_1 = 200$ is plugged in, we therefore get $MPK_1 \approx 3.54$ — which means that the after tax return on capital in housing is $(1-t)MPK_1 = 0.5(3.54) = 1.77$. In the non-housing sector, marginal product is

$$MPK_2(k_2) = \frac{\beta}{2k_2^{1/2}} = \frac{100}{2k_2^{1/2}} = \frac{50}{k_2^{1/2}}. \qquad (19B.17.ii)$$

When $k_2 = 800$, we then get $MPK_2 \approx 1.77$. Thus, in equilibrium the after tax rate of return in housing is equal to the rate of return in the untaxed sector.

Exercise 19B.18

If capital is "sector-specific" and cannot move from one use to another, would you still expect the housing tax to be shifted? Explain.

Answer: No. If capital cannot be shifted, then the mechanism through which the general equilibrium effect arises is no longer operative.

Exercise 19B.19

Why is the relative size of the housing sector relevant for determining how much owners of capital in other sectors are affected by a tax on housing capital?

Answer: If the housing sector is small, then even a large tax on housing capital will result in a shift of capital that is small relative to the non-housing sector — and thus will have relatively little impact on the non-housing sector. Of course such a tax would raise relatively little in revenue if the taxed sector is small.

19C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 19.1

In our discussion of economic versus statutory incidence, the text has focused primarily on the incidence of taxes. This exercise explores analogous issues related to the incidence of benefits from subsidies.

A: *Consider a price subsidy for x in a partial equilibrium model of demand and supply in the market for x.*

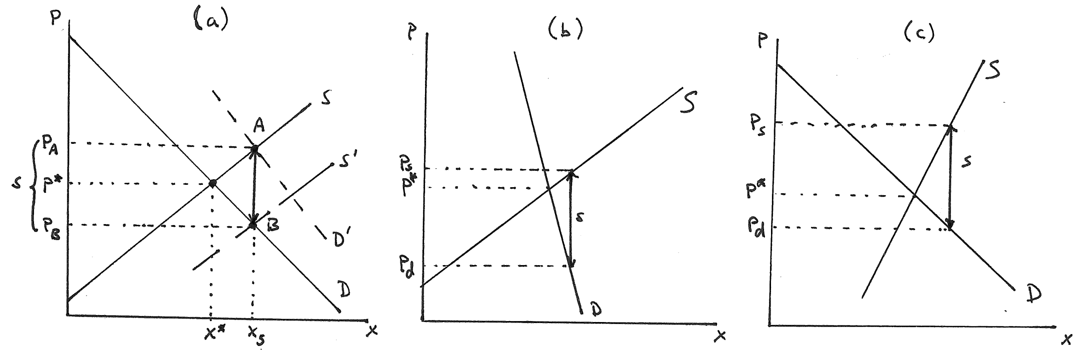
(a) *Explain why it does not matter whether the government gives the per-unit subsidy s to consumers or producers.*

Answer: This is illustrated in panel (a) of Exercise Graph 19.1 where the pre-subsidy demand curve D intersects the pre-subsidy supply curve S at price p^* and output level x^*. If the subsidy is given to consumers, demand curves shift up by s to D' — causing the equilibrium to shift to A with price p_A and output level x_s. If the subsidy is given to producers, supply shifts down by s to S' — causing the equilibrium to shift to B with price p_B and output level x_s. The output level in equilibrium A and B is therefore the same — and independent of whether the subsidy was given to consumers or producers. The equilibrium price differs — but the equilibrium impact on consumers and producers is identical. In equilibrium A, consumers pay p_A to producers but then receive s per unit back from the government — implying that the suppliers' price is $p_s = p_A$ and consumer prices are $p_d = p_A - s = p_B$. In equilibrium B, consumers pay p_B to producers — and producers now receive a per unit subsidy s for each good that is sold. Thus, the prices received by producers are $p_s = p_B + s = p_A$ and prices paid by consumers are $p_d = p_B$. Whether the subsidy is given to producers or consumers, it is therefore the case that $p_s = p_A$, $p_d = p_B$ and output is x_s.

(b) *Consider the case where the slopes of demand and supply curves are roughly equal in absolute value at the no-subsidy equilibrium. What does this imply for the way in which the benefits of the subsidy are divided between consumers and producers?*

Answer: This is what we just illustrated in panel (a) of Exercise Graph 19.1 where the price received by producers rises from p^* to p_A and the price paid by consumers falls from p^* to p_B. When the slopes of the demand and supply curves are roughly equal in absolute value at the initial equilibrium, the distance between p_A and p^* will be roughly equal to the distance between p^* and p_B — i.e. consumers will benefit by about as much as producers will from the subsidy.

(c) *How does your answer change if the demand curve is steeper than the supply curve at the no-subsidy equilibrium?*



Exercise Graph 19.1 : The Economic Incidence of Subsidies

Answer: This is illustrated in panel (b) of Exercise Graph 19.1 where the consumer price p_d drops substantially more than the producer price p_s rises from the initial p^*. Thus, when consumers are relatively less responsive to price changes, they will obtain the bulk of the benefit from the subsidy.

(d) *How does your answer change if the demand curve is shallower than the supply curve at the no-subsidy equilibrium?*

Answer: This is illustrated in panel (c) of Exercise Graph 19.1. Now the producer price p_s rises more than the consumer price p_d falls from the initial p^*. Thus, when consumers are relatively more responsive to price changes than producers, the bulk of the benefit from the subsidy accrues to producers.

(e) *Can you state your general conclusion — using the language of price elasticities — on how much consumers will benefit relative to producers when price subsidies are introduced. How is this similar to our conclusions on tax incidence?*

Answer: The general conclusion is that the benefits of subsidies are shifted disproportionately to the side of the market that is less sensitive to price — i.e. the side of the market that is more price inelastic. This is similar to what we concluded about the burden of a tax — which is also shifted to the side of the market that is more price inelastic.

(f) *Do any of your answers depend on whether the tastes for x are quasilinear?*

Answer: No — our prediction of market outcomes in terms of prices and quantities are based on uncompensated curves that include income and substitution effects — because if we want to know what actually happens, we have to look at actual or uncompensated curves. It is only when we try to assess changes in surplus — i.e. welfare changes — that we need to use compensated (or marginal willingness to pay) curves. Had we drawn into our graphs consumer surplus areas along the demand curves, we would therefore have had to assume that the uncompensated demand curves in the graphs are also equal to marginal willingness to pay curves — which would only be true if tastes are quasilinear in x.

B: *In Section 19B.1, we derived the impact of a marginal per-unit tax on the price received by producers — i.e.* dp_s/dt.

(a) *Repeat the analysis for the case of a per-unit subsidy and derive* dp_s/ds *where s is the per-unit subsidy.*

Answer: Consider the general case where demand is given by $x_d(p)$, supply is given by $x_s(p)$ and the no-tax equilibrium has price p^* and quantity x^*. Now suppose a small subsidy s (to be received by consumers for each unit of x that is purchased) is introduced. This implies that the price p_d effectively paid by buyers is s lower than the price p_s at which the good is purchased from suppliers; i.e. $p_d = p_s - s$. Taking the differential of this, we get

$$dp_d = dp_s - ds; \tag{19.1.i}$$

i.e. the change in the consumer price p_d is equal to the change in the producer price p_s minus the change in s. In the new equilibrium, demand has to equal supply, with each evaluated at the relevant price; i.e.

$$x_d(p_d) = x_s(p_s). \tag{19.1.ii}$$

Taking the differential of this, we can write

$$\frac{dx_d}{dp_d}dp_d = \frac{dx_s}{dp_s}dp_s \tag{19.1.iii}$$

and substituting equation (19.1.i) into equation (19.1.iii), this becomes

$$\frac{dx_d}{dp_d}(dp_s - ds) = \frac{dx_s}{dp_s}dp_s. \tag{19.1.iv}$$

Rearranging terms in this equation, we can write it as

$$\left(\frac{dx_d}{dp_d} - \frac{dx_s}{dp_s}\right)dp_s = \frac{dx_d}{dp_d}ds. \tag{19.1.v}$$

Before the subsidy is introduced, the equilibrium was at the intersection of supply and demand at p^* and x^* — a point on both the supply and demand curve. Multiplying equation (19.1.v) by p^*/x^*, it becomes

$$\left(\frac{dx_d}{dp_d}\frac{p^*}{x^*} - \frac{dx_s}{dp_s}\frac{p^*}{x^*}\right)dp_s = \frac{dx_d}{dp_d}\frac{p^*}{x^*}ds, \tag{19.1.vi}$$

which you should notice contains several price elasticity terms (evaluated at the no-tax equilibrium). Rewriting the equation in terms of these price elasticities, it becomes

$$(\varepsilon_d - \varepsilon_s)dp_s = \varepsilon_d ds \tag{19.1.vii}$$

where ε_d is the price elasticity of demand and ε_s is the price elasticity of supply. Re-arranging terms, we can also then write this as

$$\frac{dp_s}{ds} = \frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s}. \tag{19.1.viii}$$

(b) *What is dp_d/ds?*

Answer: Equation (19.1.i) implies that

$$\frac{dp_d}{ds} = \frac{dp_s}{ds} - 1. \tag{19.1.ix}$$

Substituting equation (19.1.viii) into this equation, we then get

$$\frac{dp_d}{ds} = \frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} - 1 = \frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} - \left(\frac{\varepsilon_d - \varepsilon_s}{\varepsilon_d - \varepsilon_s}\right) = \frac{\varepsilon_s}{\varepsilon_d - \varepsilon_s} \tag{19.1.x}$$

(c) *What do your results in (a) and (b) tell you about the economic incidence of a per-unit subsidy when the price elasticity of demand is zero? What about when the price elasticity of supply is zero?*

Answer: When $\varepsilon_d = 0$, we get

$$\frac{dp_d}{ds} = -1 \text{ and } \frac{dp_s}{ds} = 0; \tag{19.1.xi}$$

i.e. $dp_d = -ds$ and $dp_s = 0$. Thus, the entire benefit of the subsidy goes to consumers whose price drops by the amount of the subsidy. When $\varepsilon_s = 0$, we get

$$\frac{dp_d}{ds} = 0 \text{ and } \frac{dp_s}{ds} = 1; \tag{19.1.xii}$$

i.e. $dp_d = 0$ and $dp_s = ds$. Thus, the entire benefit of the subsidy goes to producers whose price increases by the amount of the subsidy.

(d) *What does your analysis suggest about the economic incidence of the subsidy when the price elasticities of demand and supply are equal (in absolute value) at the no-subsidy equilibrium?*

Answer: In the case where $|\varepsilon_d| = \varepsilon_s$, we get

$$\frac{dp_d}{ds} = \frac{\varepsilon_s}{\varepsilon_d - \varepsilon_s} = -\frac{1}{2} \text{ and } \frac{dp_s}{ds} = \frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} = \frac{1}{2}. \tag{19.1.xiii}$$

Thus, consumer prices fall by the same amount as producer prices increase — and the benefits of the subsidy are shared equally.

(e) *More generally, can you show which side of the market gets the greater benefit when the absolute value of the price elasticity of demand is less than the price elasticity of supply?*

Answer: When $|\varepsilon_d| < \varepsilon_s$, $|\varepsilon_d - \varepsilon_s| > 2|\varepsilon_d|$. This implies

$$\frac{|\varepsilon_d|}{|\varepsilon_d - \varepsilon_s|} < \frac{|\varepsilon_d|}{2|\varepsilon_d|} = \frac{1}{2} \tag{19.1.xiv}$$

which implies

$$\frac{dp_s}{ds} = \frac{\varepsilon_d}{\varepsilon_d - \varepsilon_s} = \frac{|\varepsilon_d|}{|\varepsilon_d - \varepsilon_s|} < \frac{1}{2}. \tag{19.1.xv}$$

Thus, producers will receive less than half the benefit of the subsidy when demand is more price inelastic than supply — which is exactly our conclusion in part A(e). Put differently, consumers receive a disproportionately larger share of the benefit of the subsidy when they are more price inelastic than producers. (We could similarly derive $dp_d/ds < -0.5$, which says the same thing.)

Exercise 19.3

In the text, we discussed deadweight losses that arise from wage taxes *even when labor supply is perfectly inelastic. We now consider wage* subsidies.

A: *Suppose that the current market wage is w^* and that labor supply for all workers is perfectly inelastic. Then the government agrees to pay employers a per-hour wage subsidy of \$$s$ for every worker hour they employ.*

(a) *Will employers get any benefit from this subsidy? Will employees?*

Answer: Employers will get no benefit from the subsidy because the entire economic incidence will fall on employees whose labor supply is perfectly inelastic. Thus, wages paid by employers will remain at w^* while wages received by employees will be $(w^* + s)$.

(b) *In a consumer diagram with leisure ℓ on the horizontal and consumption c on the vertical axes, illustrate the impact of the subsidy on worker budget constraints.*

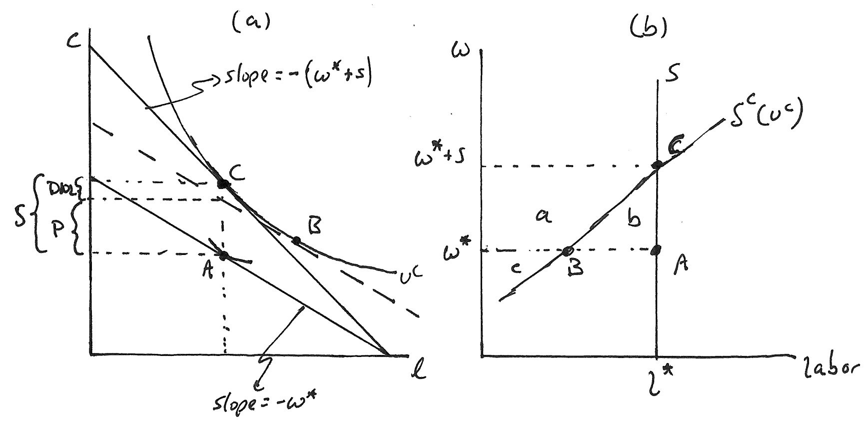
Answer: This is illustrated in panel (a) of Exercise Graph 19.3.

(c) *Choose a bundle A that is optimal before the subsidy goes into effect. Locate the bundle that is optimal after the subsidy.*

Answer: This is also illustrated in panel (a) of the Graph — where C lies vertically above A because the (uncompensated) labor response to the wage subsidy is perfectly inelastic.

(d) *Illustrate the size of the subsidy payment S as a vertical distance in the graph.*

Answer: This is again illustrated in panel (a) of Exercise Graph 19.3 where S is the vertical distance between A and C.



Exercise Graph 19.3 : Dead Weight Losses from Wage Subsidies with Inelastic Labor Supply

(e) *Illustrate how much P we could have paid the worker in a lump sum way (without distorting wages) to make him just as well off as he is under the wage subsidy. Then locate the deadweight loss of the wage subsidy as a vertical distance in your graph.*

<u>Answer</u>: This is also illustrated in panel (a) of Exercise Graph 19.3 where P is the difference between the parallel budgets. The higher of these is the budget necessary to get the worker to his post-subsidy utility level u^C at the pre-subsidy wage w^*. The deadweight loss is then simply the difference between S and P.

(f) *On a separate graph, illustrate the inelastic labor supply curve as well as the before and after-subsidy points on that curve. Then illustrate the appropriate compensated labor supply curve on which to measure the deadweight loss. Explain where this deadweight loss lies in your graph.*

<u>Answer</u>: This is done in panel (b) of Exercise Graph 19.3. The subsidy causes workers to move from A to C on their inelastic supply curve. The compensated labor supply curve that corresponds to the after-subsidy utility level u^C must then pass through C — and as long as there is at least some substitutability between leisure and consumption, it must slope up (because it only incorporates substitution effects). When measuring worker surplus on this compensated labor supply curve, we find surplus of $(a+c)$ under the wage subsidy but only surplus of (c) under the lump sum subsidy that eliminates the wage subsidy. Since the worker is equally happy at B and C, the lump sum subsidy must therefore be equal to (a); i.e. $P = a$. The actual wage subsidy paid, however, is s times l^* — which is equal to area $(a+b)$; i.e. $S = a + b$. Thus, the deadweight loss is $DWL = b$.

(g) True or False*: As long as leisure and consumption are at least somewhat substitutable, compensated labor supply curves always slope up and wage* *subsidies that increase worker wages create deadweight losses.*

Answer: This is true. Substitution effects tell us that consumption of leisure decreases when leisure becomes more expensive — which is equivalent to saying that labor supplied increases as w increases. Compensated labor supply curves only incorporate substitution effects — and thus, so long as there is any substitutability between leisure and consumption, more labor will be supplied as w goes up. Put differently, compensated labor supply curves must slope up — and it is that upward slope that gives rise to the deadweight loss from wage subsidies.

B: *Suppose that, as in our treatment of wage taxes, tastes over consumption c and leisure ℓ can be represented by the utility function $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$ and that all workers have leisure endowment of L (and no other source of income). Suppose further that, again as in the text, the equilibrium wage in the absence of distortions is $w^* = 25$.*

(a) *If the government offers a \$11 per hour wage subsidy for employers, how does this affect the wage costs for employers and the wages received by employees?*

Answer: Since labor supply is perfectly inelastic for these tastes (as shown in the text), the entire benefit of the subsidy accrues to workers. Thus, wages for workers increase to \$36 per hour while wages paid by employers remain unchanged at \$25 per hour.

(b) *Assume henceforth that $\alpha = 0.5$. What is the utility level u_s attained by workers under the subsidy (as a function of leisure endowment L)?*

Answer: From the utility maximization problem, we get that consumption demand is $c = 36(0.5)L = 18L$ while leisure demand is $0.5L$. (This is derived in the text — we simply plugged in the after-subsidy wage of \$36 and $\alpha = 0.5$.) Plugging these back into the utility function, we get

$$u_s = (18L)^{0.5}(0.5L)^{0.5} = 3L. \tag{19.3.i}$$

(c) *What's the least (in terms of leisure endowment L) we would need to give each worker in a lump sum way to get them to agree to give up the wage subsidy program?*

Answer: In the text, we derived the compensated leisure demand and consumption demand equations and, from these, the expenditure function

$$E(w,u) = \frac{w^{(1-\alpha)}u}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{19.3.ii}$$

The expenditure necessary to get the worker to the utility level u_s at the pre-subsidy wage $w = 25$ is $E(25, 3L)$. Plugging in $u = 3L$, $w = 25$ and $\alpha = 0.5$, we get $E = 30L$. Thus, the worker would have to have $30L$ in order to be just as happy without the subsidy as he is with the subsidy when the value of his leisure endowment is $25L$. We would therefore have to give the worker $5L$ in a lump sum way to make him as well off at a wage of $25 per hour as he is under the subsidized wage of $36 per hour.

(d) *What is the per worker deadweight loss (in terms of leisure endowment L) of the subsidy?*

Answer: The deadweight loss is then the difference between what we actually have to pay in a wage subsidy and what we could have paid in a lump sum way without making workers worse off. We just concluded that a lump sum payment of $5L$ would be just as good for the worker as the wage subsidy. Under the wage subsidy, the government has to pay $11 per hour worked — and workers always work $0.5L$ hours (taking the rest of their endowment as leisure). Thus, the wage subsidy costs $11(0.5)L = 5.5L$ — implying a deadweight loss of $DWL = 0.5L$ per worker.

(e) *Use the compensated labor supply curve to verify your answer.*

Answer: In the text, we derived the compensated labor demand curve as

$$l_s^c(w,u) = L - \left(\frac{1-\alpha}{\alpha w}\right)^{\alpha} u. \qquad (19.3.iii)$$

Plugging in $\alpha = 0.5$ and the after-subsidy utility level $u - 3L$, we get

$$l_s^c(w) = L - \left(\frac{3L}{w^{0.5}}\right). \qquad (19.3.iv)$$

The equivalent lump sum payment is the area under this function between the before and after-subsidy wage; i.e.

$$\text{Lump Sum Payment} = \int_{25}^{36} \left[L - \left(\frac{3L}{w^{0.5}}\right)\right] dw = 5L. \qquad (19.3.v)$$

Subtracting this from the actual subsidy cost of $11(0.5L) = 5.5L$, we get a deadweight loss of $DWL = 0.5L$ as before.

Exercise 19.5

(This exercise builds on exercise 19.4 which you should do before proceeding.) Through the income tax code, governments typically tax most interest income — but, through a variety of retirement programs, they often subsidize at least some types of interest income.

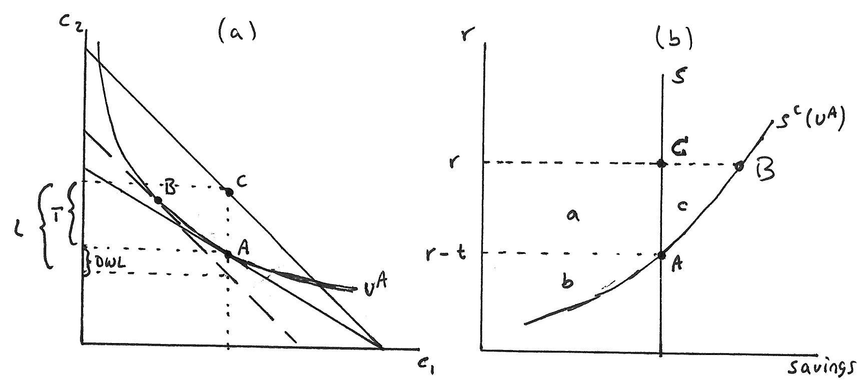
A: *Suppose all capital is supplied by individuals that earn income now but don't expect to earn income in some future period — and therefore save some of their current income. Suppose further that these individuals do not change their current consumption (and thus the amount they put into savings) as interest rates change.*

(a) *What is the economic incidence of a government subsidy of interest income? What is the economic incidence of a tax on interest income?*

Answer: The economic incidence of taxes and subsidies always falls more heavily on the side of the market that behaves more inelastically. In this case, savers are perfectly inelastic — which implies they will enjoy the full benefit of interest subsidies and pay the full cost of interest taxes. (If you have trouble seeing this, draw a graph with the savings supply curve *almost* perfectly inelastic — you should see that the incidence of taxes and subsidies then falls *almost* entirely on savers.)

(b) *In the text, we illustrated the deadweight loss from a subsidy on interest income when savings behavior is unaffected by changes in the interest rate. Now consider a tax on interest income. In a consumer diagram with current consumption c_1 on the horizontal and future consumption c_2 on the vertical axis, illustrate the deadweight loss from such a tax for a saver whose (uncompensated) savings supply is perfectly inelastic.*

Answer: This is done in panel (a) of Exercise Graph 19.5(1).



Exercise Graph 19.5(1) : Dead Weight Loss from Taxing Interest Income

The after tax optimal bundle (on the shallower budget) is A — giving utility u^A. At that bundle, the saver pays the distance T in taxes. But he would have been willing to pay up to L in order to keep the distortionary tax from being implemented — with the difference between L and T constituting the deadweight loss from the tax on interest income.

(c) *What does the size of the deadweight loss depend on? Under what special tastes does it disappear?*

Answer: The size of the deadweight loss depends on the distance between A and B — which in turn depends on the degree of substitutability between consumption now and in the future. The more complementary consumption is across time, the shorter this distance — and the less the deadweight loss. If consumption across the two periods is perfectly complementary, B and A would be the same point — and the deadweight loss would disappear because the substitution effect that gives rise to the deadweight loss would be gone. However, the uncompensated labor supply curve would then not be perfectly inelastic. Instead, for the deadweight loss to disappear *and* the labor supply curve to be perfectly inelastic, there would have to be a kink point at A that is sufficiently large to keep a substitution effect from appearing while still allowing point C to lie directly above A in panel (a) of the graph.

(d) *On a separate graph, illustrate the inelastic savings (or capital) supply curve. Then illustrate the compensated savings supply curve that allows you to measure the deadweight loss from the tax on interest income. Explain where in the graph this deadweight loss lies.*

Answer: This is illustrated in panel (b) of Exercise Graph 19.5(1). The deadweight loss will be measured on the compensated supply curve that corresponds to the after-tax utility level u^A (and therefore passes through A.) Saver surplus under the interest rate $(r - t)$ is just area b — while saver surplus under the interest rate r is $(a + b + c)$. But the saver is equally happy at A and B because at B he had to pay a lump sum tax that pushes him to the indifference curve u^A at the pre-tax interest rate. This implies that the lump sum tax he is willing to pay is $(a + c)$. But the tax revenue that is collected under the distortionary tax is just area a. This leaves us with deadweight loss c.

(e) *What happens to the compensated savings supply curve as consumption becomes more complementary across time — and what happens to the deadweight loss as a result?*

Answer: As consumption becomes more complementary across time, the distance between A and B in panel (a) of the graph decreases — which also implies the distance between C and B decreases in panel (b) of the graph Thus, the compensated savings supply curve becomes more inelastic as consumption becomes more complementary across time — because the substitution effect becomes smaller. This causes the deadweight loss area c in the graph to shrink. If consumption is perfectly complementary across time, C and B will lie on top of one another in panel (b) of the graph — with the compensated savings supply curve perfectly inelastic. In that case, the deadweight loss area disappears.

(f) *Is the special case when there is no deadweight loss from taxing interest income compatible with a perfectly inelastic* uncompensated *savings supply curve?*

Answer: Yes and no. We would need consumption across time to be perfectly complementary — but that would imply that there is only a wealth effect and no substitution effect. And this would imply that savings falls with an increase in the interest rate — i.e. C would lie to the right of A in panel (a) of the graph. This further implies a downward sloping savings supply curve, not a perfectly inelastic curve. But you could still have a kink at A that is not a right angle and that still allows C to lie directly above A (as discussed in the answer to (c)).

B: *Suppose everyone's tastes and economic circumstances are the same as those described in part B of exercise 19.4 — with $\alpha = 0.5$ and $I = 100,000$.*[1]

(a) *Suppose further that there are 10,000,000 consumers like this — and they are the only source of capital in the economy. How much capital is supplied regardless of the interest rate?*

Answer: The (inelastic) supply of capital is given by

$$K_s = 10,000,000k_s = 10,000,000(1-\alpha)I = 10,000,000(0.5)(100,000) = 500,000,000,000,000 \tag{19.5.i}$$

— i.e. $500 trillion.

(b) *Suppose next that demand for capital is given by $K_d = 25,000,000,000/r$. What is the equilibrium real interest rate r^* in the absence of any price distortions?*

Answer: Setting K_d equal to the inelastic capital supply of $500,000,000,000,000 and solving for r, we get $r^* = 0.05$.

(c) *Suppose that, for any dollar of interest earned, the government provides the person who earned the interest a 50 cent subsidy. What will be the new (subsidy-inclusive) interest rate earned by savers, and what will be the interest rate paid by borrowers? What if the government instead taxed 50% of interest income?*

Answer: Since the supply of savings is perfectly inelastic, savers will receive the full benefit of the subsidy and bear the full burden of the tax. Thus, under the subsidy, the interest rate earned by savers would be $1.5(r^*) = 1.5 * 0.05 = 0.075$. Under the tax, the interest rate earned by savers would fall to $0.5(r^*) = 0.5(0.05) = 0.025$. In both cases, borrowers would still pay interest $r^* = 0.05$.

(d) *Consider the subsidy introduced in (c). How much utility V will each saver attain under this subsidy?*

Answer: In exercise 19.4 we derived the indirect utility function as

$$V(r, I) = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(1+r)^{(1-\alpha)}I. \tag{19.5.ii}$$

Substituting in $\alpha = 0.5$, $I = 100,000$ and the post-subsidy interest rate for savers ($r = 0.075$), we get $V \approx 51,841.1$. (You can of course also simply use the demand functions for current and future consumption to get $c_1 = 50,000$ and $c_2 = 53,750$ and plug these into the utility function to get the same answer.)

(e) *How much current income would each saver have to have in order to obtain the same utility V at the pre-subsidy interest rate r^*? In terms of future dollars, how much would it therefore cost the government to make*

[1] Among other functions, you should have derived uncompensated and compensated savings function as

$$k_s(r, I) = (1-\alpha)I \text{ and } k_s^c = \left[1 - \alpha\left(\frac{1+\overline{r}}{1+r}\right)^{(1-\alpha)}\right]I. \tag{19.5}$$

each saver as well off in a lump sum way as it does using the interest rate subsidy?

Answer: In part (e) of exercise 19.4, we derived the expenditure function as

$$E(r,u) = \frac{u}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(1+r)^{(1-\alpha)}}. \tag{19.5.iii}$$

We now simply need to plug $\alpha = 0.5$, $r = 0.5$ and $u = 51,841.1$ to get $E \approx 101,183.47$. Thus, in current dollars, each saver's income would have to increase by \$1,183.47 — which is equal to $(1+0.05)1,183.47 \approx \$1,242.65$.

(f) *How much interest will the government have to pay to each saver (in the future) under the subsidy? Use this and your previous answer to conclude the amount of deadweight loss per saver in terms of future dollars. Given the number of savers in the economy, what is the overall deadweight loss?*

Answer: Since savers will always save \$50,000, they will earn $(1+0.075)50,000 = \$3,750$ in interest income under the subsidy — which is \$1,250 more than they would have earned in the absence of the subsidy. Thus, the subsidy costs the government \$1,250 per saver (in terms of future dollars). Subtracting the lump sum payment that would have made savers equally well off, we get a deadweight loss of

$$DWL = 1,250 - 1,242.65 = \$7.35 \tag{19.5.iv}$$

per saver. Given that there are 10 million savers, the overall deadweight loss is therefore about \$73.5 million in terms next period.

(g) *Derive the compensated savings function (as a function of r) given the post-subsidy utility level V.*

Answer: In exercise 19.4 we derived the compensated savings function as

$$k_s^c(r,\overline{r}) = \left[1 - \alpha\left(\frac{1+\overline{r}}{1+r}\right)^{(1-\alpha)}\right] I \tag{19.5.v}$$

where $\overline{r}$ is the interest rate that determines the compensated utility level. If we want the compensated savings function at the post-subsidy utility, we therefore set $\overline{r}$ to 0.075. Substituting in $\alpha = 0.5$ and $I = 100,000$, we then get

$$k_s^c(r) \approx 100,000 - \frac{51841.1}{(1+r)^{0.5}}. \tag{19.5.vi}$$

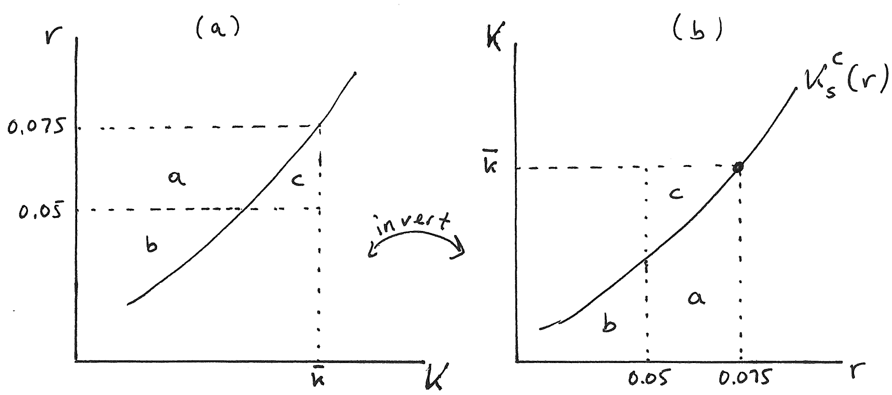
You can of course also derive this by substituting the compensated demand for current consumption evaluated at the post-subsidy utility level from I to get the same answer.

(h) *Use your answer to (g) to derive the aggregate compensated capital supply function — and then find the area that corresponds to the deadweight loss. Compare this to your answer in part (f).*

Answer: The aggregate compensated capital supply function is simply k_s^c multiplied by the number of savers — i.e.

$$K_s^c(r) = 10,000,000\left(100,000 - \frac{51841.1}{(1+r)^{0.5}}\right) \quad (19.5.vii)$$

In panel (a) of Exercise Graph 19.5(2), the inverse of this — which we have called the compensated capital supply *curve* — is depicted.



Exercise Graph 19.5(2) : Compensated Capital Supply Curve and Function

This is analogous to Graph 19.7 in the text where we concluded that area c in the graph is the deadweight loss. Panel (b) of the graph inverts this back to plot the compensated capital supply *function* $K_s^c(r)$ — with areas from panel (a) again labeled. We can now see that the deadweight loss area c is simply the rectangle $(a+c)$ minus the integral between $r = 0.05$ and $r = 0.075$. The rectangle $(a+c)$ is simply the total amount of capital $\overline{K}$ supplied under the subsidy multiplied by 0.025 — where $\overline{K}$ is simply 50,000 times the 10 million savers. Thus, $(a+c)$=500,000,000,000(0.025)=12,500,000,000. The deadweight loss area c is therefore

$$\begin{aligned} DWL &= 12,500,000,000 - 10,000,000\int_{0.05}^{0.075}\left(100,000 - \frac{51841.1}{(1+r)^{0.5}}\right)dr \\ &= 12,500,000,000 - 10,000,000\left[\left(100,000r - 2(51841.1)(1+r)^{0.5}\right)\Big|_{0.05}^{0.075}\right] \\ &= 73,531,136 \approx \$73.5 \text{ million.} \end{aligned} \quad (19.5.viii)$$

This is precisely what we calculated in (f).

(i) *Repeat parts (d) through (h) for the case of the tax on interest income described in part (c).*

Answer: To get the after-tax utility level, we would substitute $r = 0.025$ rather than $r = 0.075$ into our expression for V to get $V \approx 50,621.14$. To get the same utility level under the pre-tax interest rate $r^* = 0.05$, we would not substitute $r = 0.025$ (rather than $r = 0.075$) into our expression for the expenditure function — as well as our new value for V. This gives us $E \approx \$98,802.35$ — i.e. each saver would be willing to give up $100,000 - 98,802.35 = \$1,197.65$ in current dollars or $1,197.65(1+0.05) \approx 1,257.53$ in future dollars. The government would actually receive only \$1,250 in tax revenue (in terms of future dollars) — implying that individuals would be willing to give up \$7.53 to avoid the tax. For 10 million individuals, that amounts to a deadweight loss of approximately \$75.3 million. To calculate the same deadweight loss using the compensated capital supply function, we can again use $k_s^c(r,\overline{r})$ defined in part (g) — but now $\overline{r} = 0.025$ giving us the equation

$$k_s^c(r) = 100,000 - \frac{50621.14}{(1+r)^{0.5}}. \tag{19.5.ix}$$

Integrating this between $r = 0.025$ and $r = 0.05$, we get the lump sum payment an individual would be willing to make to not incur the tax — which is again \$1,257.53. Or, aggregating the function across 10,000,000 individuals, integrating in the same way and then subtracting the total tax payments received, we get approximately \$75.3 million in deadweight loss.

(j) *You have calculated deadweight losses for interest rates that are reasonable for 1-year time horizons. If we consider distortions in people's decisions over longer time horizons (such as when they plan for retirement), a more reasonable time frame might be 25 years. With annual market interest rates of 0.05 in the absence of distortions, can you use your compensated savings function (given in the footnote to the problem) to estimate again what the deadweight losses from a subsidy that raises the effective rate of return by 50% and from a tax that lowers it by 50% would be?*

Answer: If the undistorted interest rate is 0.05 annually, then a dollar invested now will result in $(1+0.05)^2 5 \approx \$3.39$ — giving us an effective interest rate of 2.39 (or 239 percent) over the 25 years. If a subsidy raised that by 50%, it would raise it to approximately 3.585. Using our compensated savings function, we would then get

$$\begin{aligned} k_s^c(r) &= \left[1 - \alpha\left(\frac{1+\overline{r}}{1+r}\right)^{(1-\alpha)}\right] I \\ &= \left[100,000 - 50,000\left(\frac{1+3.585}{1+r}\right)^{0.5}\right] \\ &= 100,000 - \frac{107,063}{(1+r)^{0.5}}. \end{aligned} \tag{19.5.x}$$

Using the same steps as in (h), we then get the deadweight loss

$$DWL = 50,000(3.585-2.39)(10,000,000) - 10,000,000\int_{2.39}^{3.585}\left(100,000 - \frac{107,063}{(1+r)^{0.5}}\right)$$
$$= 45,019,422,909 \approx \$45 \text{ billion} \qquad (19.5.xi)$$

or about $4,500 per individual. You can also verify this by using the expenditure function as we did in the earlier parts of this exercise. For a tax that cuts the effective return from 2.39 to 1.195, on the other hand, we would use the compensated savings function

$$\begin{aligned} k_s^c(r) &= \left[1-\alpha\left(\frac{1+\overline{r}}{1+r}\right)^{(1-\alpha)}\right]I \\ &= \left[100,000 - 50,000\left(\frac{1+1.195}{1+r}\right)^{0.5}\right] \\ &= 100,000 - \frac{74,077.66}{(1+r)^{0.5}}. \end{aligned} \qquad (19.5.xii)$$

Subtracting the actual tax revenue from the appropriate integral (that gives us the lump sum payment individuals would be willing to make to avoid the tax), we get

$$DWL = \left[10,000,000\int_{1.195}^{2.39}\left(100,000 - \frac{74,077.66}{(1+r)^{0.5}}\right)\right] - 1.195(50,000)(10,000,000)$$
$$= 64,671,207,082 \approx \$62.7 \text{ billion.} \qquad (19.5.xiii)$$

Exercise 19.7

Business and Policy Application: *Land Use Policies: In most Western democracies, it is settled law that governments cannot simply confiscate land for public purposes. Such confiscation is labeled a "taking" — and, even when the government has compelling reasons to "take" someone's property for public use, it must compensate the landowner. But, while it is clear that a "taking" has occurred when the government confiscates private land without compensation, constitutional lawyers disagree on how close the government has to come to literally confiscating private land before the action constitutes an unconstitutional "taking".*

A: *Any restriction that alters the way land would otherwise be used reduces the annual rental value of that land and, from the owner's perspective, can therefore be treated as a tax on rental value.*

(a) *Explain why the above statement is correct.*

Answer: There are two parts to the statement: First, restrictions on land use cause rental values to decline, and second, that this is equivalent to a tax on land rents from the owner's perspective. Prior to any restrictions on the land, the user of the land (whether this is a renter or the owner who can be though of as renting the land from himself) employs it in the optimal way. If the restrictions placed on the land still permit the land to be used in this way, the rental value of the land is unaffected. For instance, if the optimal use of the land is to have an office building on it and a restriction is passed that prohibits users to plant corn, nothing has really changed. But if the restriction impacts the use of the land, then by definition we are no longer using it the way we would have in the absence of the restriction — i.e. we are no longer employing it in the optimal way from the individual's perspective. Thus, the user gets less out of the land which lessens his demand for the land — i.e. the rental value declines. From the owner's perspective, it does not really matter what causes the rental value to decline — whatever it is, this reduces the value of the land. Since taxing land rents lowers land value, we can then set the tax just "right" so as to achieve a reduction in land value — including the reduction that occurs as a result of a land use regulation.

(b) *Suppose a land use regulation is equivalent (from the owner's perspective) to a tax of t% on land rents to be statutorily paid by landowners (where $0 < t < 1$). How does it affect the market value of the land?*

Answer: Under such a tax, the owners would still collect the same amount of land rent as before but would then have to give up t% of it. This lowers the land rents that owners can keep to $(1-t)$% of what they kept originally — implying that land value (which is just the present discounted value of all future land rents) falls by t%.

(c) *I am about to buy an acre of land from you in order to build on it. Right before we agree on a price, the government imposes a new zoning regulation that limits what I can do on the land. Who is definitively made worse off by this?*

Answer: You are definitively made worse off — because your land value drops immediately by the net present value of the decrease in land rents caused by the regulation.

(d) *Suppose you own 1000 acres of land that is currently zoned for residential development. Then suppose the government determines that your land is home to a rare species of salamander — and that it is in the public interest for no economic activity to take place on this land in order to protect this endangered species. From your perspective, what approximate tax rate on land rents that you collect is this regulation equivalent to? Do you think this is a "taking"?*

Answer: Since you are effectively prohibited from using (or renting) the land, this is equivalent to you paying a 100% tax on land rents. Were the government to actually impose such a tax, it would have essentially confiscated your land because all rents from it would go to the government instead of to you. But from your perspective, the regulation to protect the salamanders is no different — you again lose all the value from your land. The only difference is that the government has actually not taken possession of the value of the land the way it would under the 100% land rent tax. But from the owner's perspective, it sure looks like a taking.

(e) *Suppose that, instead of prohibiting all economic activity on your 1000 acres, the government reduces your ability to build residential housing on it to a single house. How does your answer change? What if it restricts housing development to 500 acres? Do you think this would be a "taking"?*

Answer: If you are now only allowed to build a single house, some small fraction of the original value of the land is retained — and the regulation therefore is not a complete "taking" from your perspective. But it's pretty close to a complete taking. If the government restricts housing development to only 500 of the 1000 acres, this is approximately equivalent to the government taxing your land rents at 50% — or confiscating half your land. Courts would almost certainly not call this a "taking" — but from the owner's perspective, it's just like the government just took half the land.

B: *Suppose that people gain utility from housing services h and other consumption x, with tastes described by the utility function $u(x,h) = \ln x + \ln h$. Consumption is denominated in dollars (with price therefore normalized to 1). Housing services, on the other hand, are derived from the production process $h = k^{0.5}L^{\alpha}$ where k stands for units of capital and L for acres of land. Suppose $0 < \alpha < 1$. Let the rental rate of capital be denoted by r, and assume each person has income of 1000.*

(a) *Write down the utility maximization problem and solve for the demand function for land assuming a rental rate R for land.*

Answer: Substituting the housing production function into the utility function, we can write the utility function as

$$u(x,k,L) = \ln x + \ln\left(k^{0.5}L^{\alpha}\right) = \ln x + 0.5\ln k + \alpha \ln L. \quad (19.7.i)$$

The utility maximization problem can then be written as

$$\max_{x,k,L} \ln x + 0.5\ln k + \alpha\ln L \text{ subject to } 1000 = x + rk + RL. \quad (19.7.ii)$$

We can then write the Lagrangian function as

$$\mathscr{L} = \ln x + 0.5\ln k + \alpha\ln L + \lambda\left(1000 - x - rk - RL\right). \quad (19.7.iii)$$

The first order conditions are then

$$\frac{1}{x} = \lambda; \ \frac{0.5}{k} = \lambda r \text{ and } \frac{\alpha}{L} = \lambda R. \quad (19.7.iv)$$

We can then use the first and third of these to write x in terms of L(i.e. $x = RL/\alpha$) and we can use the second and third of these to write k in terms of L (i.e. $k = 0.5RL/(\alpha r)$). Substituting these into the budget constraint, we can then solve for the demand function for land

$$L = \frac{1000\alpha}{(1.5+\alpha)R}. \tag{19.7.v}$$

(b) *Suppose your city consists of 100,000 individuals like this — and there are 25,000 acres of land available. What is the equilibrium rental rate per acre of land (as a function of α)?*

Answer: Setting demand equal to supply, we get the equation

$$100,000\left(\frac{1000\alpha}{(1.5+\alpha)R}\right) = 25,000 \tag{19.7.vi}$$

that solves for

$$R^* = \frac{4000\alpha}{(1.5+\alpha)}. \tag{19.7.vii}$$

(c) *Using your answers above, derive the amount of land each person will consume.*

Answer: Substituting equation (19.7.vii) into (19.7.v), we get

$$L = \frac{1000\alpha}{(1.5+\alpha)\left(\frac{4000\alpha}{(1.5+\alpha)}\right)} = \frac{1000\alpha(1.5+\alpha)}{(1.5+\alpha)(4000\alpha)} = \frac{1}{4} \tag{19.7.viii}$$

— i.e. everyone consumes a quarter of an acre in equilibrium.

(d) *Suppose the government imposes zoning regulations that reduce the coefficient α in the production function from 0.5 to 0.25. What happens to the equilibrium rental value of land?*

Answer: This implies that the equilibrium rental rate for an acre of land falls from

$$\frac{4000(0.5)}{(1.5+0.5)} = 1000 \tag{19.7.ix}$$

to

$$\frac{4000(0.25)}{(1.5+0.25)} \approx 571.43. \tag{19.7.x}$$

(e) *Suppose that what you have calculated so far is the monthly rental value of land. What happens to the total value of an acre of land as a result of these zoning regulations assuming that people use a monthly interest rate of 0.5% to discount the future?*

Answer: This implies that land values fall from $1000/0.005 = \$200,000$ to $571.43/0.005 \approx \$114,286$.

(f) *Suppose that, instead of lowering α from 0.5 to 0.25 through regulation, the government imposes a tax t on the market rental value of land and statutorily requires renters to pay. Thus, if the market land rental rate is R per acre, those using the land must pay tR on top of the rent R for every acre they use. Set up the renters' utility maximization problem, derive the demand for land and aggregate it over all 100,000 individuals. Then derive the equilibrium land rent per acre as a function of t (assuming $\alpha = 0.5$).*

Answer: The only thing that changes in the utility maximization problem is the rental price in the budget constraint — which must now include the tax. Thus, the utility maximization problem becomes

$$\max_{x,k,L} \ln x + 0.5 \ln k + 0.5 \ln L \text{ subject to } 1000 = x + rk + (1+t)RL. \quad (19.7.\text{xi})$$

The first order conditions are then

$$\frac{1}{x} = \lambda; \quad \frac{0.5}{k} = \lambda r \text{ and } \frac{0.5}{L} = \lambda(1+t)R. \quad (19.7.\text{xii})$$

Solving these as before, we get the demand function for land

$$L = \frac{250}{(1+t)R}. \quad (19.7.\text{xiii})$$

Aggregating across 100,000 consumers and setting equal to supply, we get the equation

$$100,000\left(\frac{250}{(1+t)R}\right) = 25,000. \quad (19.7.\text{xiv})$$

Solving for R, we get the equilibrium rental rate

$$R^* = \frac{1000}{(1+t)}. \quad (19.7.\text{xv})$$

(g) *Does the amount of land consumed by each household change?*

Answer: Plugging the equilibrium land rental rate into equation (19.7.xiii), we get $L = 1/4$ as before. So — no, the amount of land consumed by each household does not change — because the tax inclusive rental rate for land remains unchanged. (Before, the land rental rate was 1000 prior to the imposition of zoning; the tax inclusive rental rate now is $(1+t)R^* = 1000$.)

(h) *Suppose you own land that you rent out. What level of t makes you indifferent between the zoning regulation that drove α from 0.5 to 0.25 and the land rent tax that does not change α?*

Answer: We calculated before that the rental rate on an acre of land falls to 571.43 under the zoning regulation. You would therefore be indifferent between this and a land rent tax if the rental rate under the land rent tax also fell to 571.43; i.e. if

$$\frac{1000}{(1+t)} = 571.43. \tag{19.7.xvi}$$

This solves to $t = 0.75$.

(i) *Suppose the government statutorily collected the land rent tax from the owner instead of from the renter. What would the tax rate then have to be set at to make the land owner indifferent between the zoning regulation and the tax?*

Answer: In this case, the consumer does not statutorily face a tax — and so the utility maximization problem is the same as in (a) leading to the same equilibrium rental rate $R^* = 1000$ as calculated in (b). Landowners would therefore collect $1,000 per acre but would then have to pay $t(1000)$ to the government. In order for this to leave them with an after-tax rent of $571.43 (as under the zoning regulation), the tax rate would have to be set at $t \approx 0.4286$.

Exercise 19.9

Policy Application: *Rent Control: Is it a Tax or a Subsidy?: In exercise 18.11 we analyzed the impact of rent control policies that impose a price ceiling in the housing rental market. The stated intent of such policies is often to make housing more affordable. Before answering this question, you may wish to review your answers to exercise 18.11.*

A: *Begin by illustrating the impact of the rent control price ceiling on the price received by landlords and the eventual equilibrium price paid by renters.*

(a) *Why is it not an equilibrium for the price ceiling to be the rent actually paid by renters?*

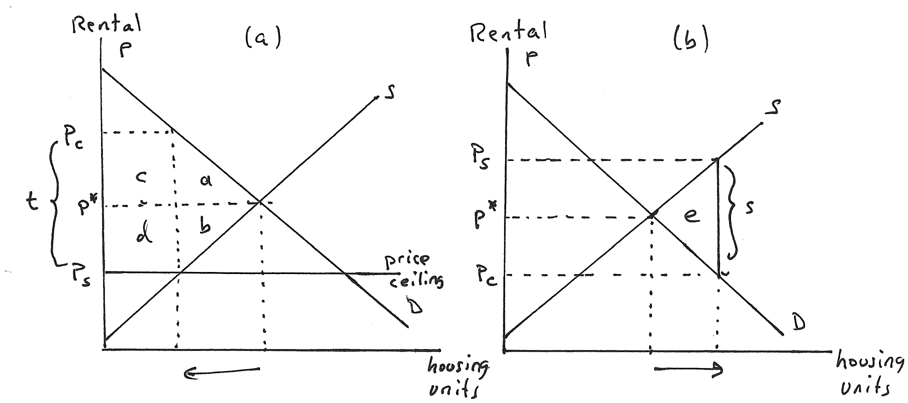
Answer: It is not an equilibrium because, at the rent controlled price, more people demand apartments than are supplied — which implies some non-price rationing mechanism must allocate the scarce apartments. This mechanism will raise the real cost of apartments to consumers until demand is once again equal to supply.

(b) *If you wanted to implement a tax or subsidy policy that achieves the same outcome as the rent control policy, what policy would you propose?*

Answer: This is illustrated in panel (a) of Exercise Graph 19.9(1).

The price ceiling results in a price of p_s received by suppliers and a price p_c paid by consumers (once all costs of the non-price rationing mechanism have been taken into account). We can achieve the same prices for consumers and producers (as well as the same reduction in housing units provided) by simply imposing a per-unit tax in the amount $t = (p_c - p_s)$.

(c) *Could you credibly argue that the alternative policy you proposed in (b) was designed to make housing more affordable?*



Exercise Graph 19.9(1) : Rent Control, Taxes and Subsidies

Answer: Since prices for consumers are raised from p^* to p_c, it would not be possible to argue this (since at least some of the tax will be passed to consumers).

(d) *If you did actually want to make housing more affordable (rather than trying to replicate the impact of rent control policies), would you choose a subsidy or a tax?*

Answer: You would want to use a subsidy — which would result in $p_c < p^* < p_s$ as well as an increase in housing units.

(e) *Illustrate your proposal from (d) — and show what would happen to the rental price received by landlords and the rents paid by renters. What happens to the number of housing units available for rent under your new policy?*

Answer: This is illustrated in panel (b) of Exercise Graph 19.9(1) where a subsidy of s raises producer prices to p_s and lowers consumer prices to p_c — while resulting in an increase in output. The subsidy s can in fact be set so as to insure that housing will be exactly as affordable as advocates of rent control wish when they set a price ceiling. (Under rent control, of course, this would not be successful.)

(f) True or False*: Policies that make housing more affordable must invariably increase the equilibrium quantity of housing — and rent control policies fail because they reduce the equilibrium quantity of housing while subsidies succeed for the opposite reason.*

Answer: This is true as illustrated in the previous parts and in the Graph.

(g) True or False*: Although rental subsidies succeed at the goal of making housing more affordable (while rent control policies fail to do so), we cannot in general say that deadweight loss is greater or less under one policy rather than the other.*

Answer: This is also true. In panel (a) of Exercise Graph 19.9(1), the deadweight loss from rent control is at least $(a+b)$ and may be as much as $(a+b+c+d)$ depending on the rationing mechanism used. In panel (b) of the Graph, the deadweight loss from the subsidy is equal to area (e). Depending on the relative elasticities of supply and demand, (e) may or may not be smaller than $(a+b)$. Of course the more the rationing mechanism under rent control causes deadweight loss to be larger than $(a+b)$, the more likely it is that the subsidy will definitively be more efficient. Still, the example illustrates that we may care more about the policy goal of creating more affordable housing than the precise size of deadweight loss — and if that is the case, then the subsidy is clearly the better policy as it actually achieves the policy goal rather than doing the opposite.

B: *Suppose, again as in exercise 18.11, that the aggregate monthly demand curve is $p = 10000 - 0.01x$ while the supply curve is $p = 1000 + 0.002x$. For simplicity, suppose again that there are no income effects.*

(a) *Calculate the equilibrium number of apartments x^* and the equilibrium monthly rent p^* in the absence of any price distortions.*

Answer: Re-writing the demand and supply curves as demand and supply functions (by writing them as functions of p), setting them equal to one another and solving, we get $p^* = 2,500$. Plugging this back into either the demand or supply function, we get $x^* = 750,000$.

(b) *In exercise 18.11, you were asked to consider the impact of a \$1,500 price ceiling. What housing tax or subsidy would result in the same economic impact?*

Answer: Imposing a price ceiling of 1,500 will result in a reduction of x supplied. We can solve for the quantity by substituting 1,500 for p into the supply curve and solving for x to get $x = 250,000$. Thus, suppliers will supply 250,000 housing units at the price ceiling of 1500. Consumers will then have to compete for the limited number of apartments — exerting effort that adds to their cost of renting these units. In equilibrium, the effort cost needs to be sufficient so that demand is equal to supply — which means the overall price (including effort cost) must be

$$p_c = 10000 - 0.01(250,000) = 7,500. \tag{19.9.i}$$

The rent control policy therefore results in a producer price of $p_s = 1,500$ and a consumer price of $p_c = 7,500$ — a result we could equally well get by simply imposing a per unit tax of \$6,000.

(c) *Suppose that you wanted to use tax/subsidy policies to actually reduce rents to \$1,500 — the stated goal of the rent control policy. What policy would you implement?*

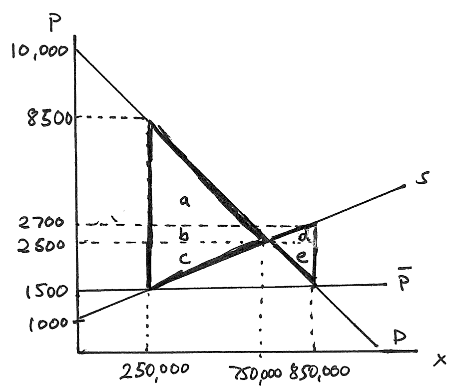
Answer: You would want to implement a subsidy program. In order to get the consumer price to be 1500, we will need sufficient numbers of apartments so that every consumer who wants to buy one at that price can get one. Thus, plugging 1,500 into the demand curve and solving for x, we get that we need a total of 850,000 apartments. In order for producers to supply that many apartments, they have to be able to charge

$$p_s = 1000 + 0.002(850,000) = 2,700. \tag{19.9.ii}$$

Thus, we need to provide a per unit subsidy of \$1,200 in order to create sufficient incentives for the market to provide housing at a consumer price of \$1,500 (and a seller's price of \$2,700).

(d) *Consider the policies you derived in (b) and (c). Under which policy is the deadweight loss greater?*

Answer: Exercise Graph 19.9(2) illustrates the supply and demand curves for this problem as well as the various numbers we have calculated. The deadweight loss from the tax is the triangle $(a + b + c)$ to the left of the equilibrium while the deadweight loss from the subsidy is the triangle $(d + e)$ to the right of the equilibrium. (The deadweight loss under rent control is at least the size of the deadweight loss from the tax but possibly more depending on the rationing mechanism). It is easily seen in the picture that the deadweight loss from the tax (and from rent control) is larger than the deadweight loss from the subsidy. (You can easily verify that the tax deadweight loss triangle is \$1,750,000,000 while the subsidy deadweight loss triangle is \$60,000,000.)



Exercise Graph 19.9(2) : Rent Control Taxes versus Housing Subsidies

Exercise 19.11

Policy Application: *Mortgage Interest Deductibility, Land Values and the Equilibrium Rate of Return on Capital: In the text, we suggested that the property tax can be thought of in part as a tax on land and in part as a tax on capital invested in housing. In the U.S., property taxes are typically levied by local governments — while the major piece of federal housing policy is contained in the federal income* *tax code which allows individuals to deduct (from income) the interest they pay on home mortgages prior to calculating the amount of taxes owed.*

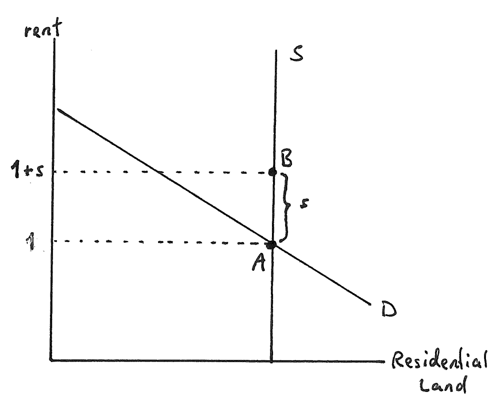
A: *Whereas we can think of the property tax as a tax on both land and housing structures, we can think of the homeownership subsidy in the federal tax code as a subsidy on land and housing structures.*

(a) *If your marginal federal income tax rate is 25% and you are financing 100% of your home value, how much of your housing consumption is being subsidized through the tax code? What if you are only financing 50% of the value of your home?*

Answer: In the first scenario, about 25% of your housing consumption is subsidized by the government through the tax code (because almost all of your mortgage payments are tax deductible since most of your payment is made up of interest that is deductible). If you are only financing 50%, only about 12.5% of your housing consumption is subsidized.

(b) *Suppose homeowners are similar to one another in terms of their marginal tax rate and how much of their home they are financing, and suppose that this implies a subsidy of s for every dollar of housing/land consumption. How would you predict the value of suburban residential land (assumed to be in fixed supply) is different as a result of this than it would have been in the absence of this policy?*

Answer: Exercise Graph 19.11 illustrates the market for residential land.



Exercise Graph 19.11 : Tax Code Subsidy for Residential Land

In the graph, land is initially denominated in units such that the original equilibrium price (at A) is \$1. The effective subsidy s per dollar then raises the rental price of residential land to $(1+s)$ — with the entire incidence of the subsidy going to land owners — i.e. land owners now collect $(1+s)$ per unit of land that they originally collected just \$1 on. We would therefore predict that residential land value increases by the present discounted value of all future streams of s for each unit of land.

(c) *When s was first introduced, who benefitted from the implicit land subsidy: current homeowners or future homeowners?*

Answer: Current homeowners (i.e. those who owned homes when the subsidy was introduced) benefitted — because all increases in rents from land (due to the subsidy) are immediately incorporated into the land price. If you owned land, your asset therefore became more valuable. If you did not own land when the subsidy was introduced, then you would not benefit from the subsidy because you will pay for the benefit of the subsidy streams when you buy land at the value that already incorporates all those future streams of subsidies.

(d) *Now consider s as a subsidy on housing capital. Do you think houses are larger or smaller as a result of the federal income tax code?*

Answer: This subsidy on housing capital reduces the price of building on land — and therefore will cause an increase in housing consumption (so long as housing is not a Giffen good). We would therefore tend to think that houses are larger as a result of the federal income tax code.

(e) *Suppose that the overall amount of capital in the economy is fixed and that capital is mobile across sectors. Thus, any given unit of capital can be invested in housing or alternatively in some other non-housing sector where it earns some rate of return. If the overall amount of capital in the economy is fixed, what happens to the fraction of capital invested in the housing sector?*

Answer: If houses are larger as a result of the tax subsidy (as we just concluded), this means that more capital is invested in housing than would otherwise be the case. Thus, the fraction of capital invested in the housing sector increases.

(f) *What would you predict will happen to the rate of return on capital in the non-housing sector? Explain.*

Answer: As capital shifts from the non-housing sector to the housing sector, the marginal product of capital must increase in the non-housing sector — which implies the rate of return on capital increases in the non-housing sector. Thus, the benefit of the subsidy on housing capital (as opposed to the subsidy on land) accrues to all forms of capital, not just housing capital.

(g) True or False*: Even though only housing capital is statutorily subsidized, the economic incidence of this subsidy falls equally on all forms of capital (so long as capital is mobile between sectors).*

Answer: This is true as just explained in the previous part.

B: *Suppose we model owners of capital as a "representative investor" who chooses to allocate K units of capital between the housing sector and other sectors of the economy. With k_1 representing capital invested in housing and k_2 representing capital invested in other sectors, suppose $f_1(k_1) = \alpha k_1^{0.5}$ and $f_2(k_2) = \beta k_2^{0.5}$ are the production functions of the two sectors.*

(a) *In the absence of any policy distortions, calculate the fraction of total capital (K) that is invested in the housing sector.*

Answer: Our representative investor then wants to maximize her total return by optimally choosing the allocation of her capital K across the two sectors. Put differently, she wants to solve the maximization problem

$$\max_{k_1,k_2} f_1(k_1)+f_2(k_2) \text{ subject to } k_1+k_2=K. \tag{19.11.i}$$

The solution to this problem is

$$k_1^* = \frac{\alpha^2 K}{\alpha^2+\beta^2} \text{ and } k_2^* = \frac{\beta^2 K}{\alpha^2+\beta^2}. \tag{19.11.ii}$$

(Note that at this solution, the marginal product of capital is the same in the two sectors). The fraction of total capital invested in housing is then $(\alpha^2/(\alpha^2+\beta^2)$.

(b) *What changes as a result of the federal income tax code's implicit housing subsidy s.*

Answer: The problem then becomes

$$\max_{k_1,k_2} (1+s)f_1(k_1)+f_2(k_2) \text{ subject to } k_1+k_2=K. \tag{19.11.iii}$$

The solution to this problem is

$$k_1^* = \frac{(1+s)^2\alpha^2 K}{(1+s)^2\alpha^2+\beta^2} \text{ and } k_2^* = \frac{\beta^2 K}{(1+s)^2\alpha^2+\beta^2}. \tag{19.11.iv}$$

Thus, the fraction of capital invested in the non-housing sector falls while the fraction invested in the housing sector increases.

(c) *What happens to the marginal product of capital in the non-housing sector?*

Answer: The marginal product of capital in the non-housing sector is given by

$$MPK_2 = \frac{\partial f_2(k_2)}{\partial k_2} = 0.5\beta k^{-0.5}. \tag{19.11.v}$$

Plugging k_2 from equation (19.11.iv) into this, we get

$$MPK_2 = 0.5\beta\left(\frac{\beta^2 K}{(1+s)^2\alpha^2+\beta^2}\right)^{-0.5} = 0.5\left(\frac{(1+s)^2\alpha^2+\beta^2}{K}\right). \tag{19.11.vi}$$

Thus, as s increases, the marginal product of non-housing capital also increases.

(d) *What happens to the equilibrium rate of return on capital?*

Answer: In equilibrium, the marginal product of capital in the non-housing sector increases (as just demonstrated) until it is equal to the subsidy-inclusive marginal product of capital in the housing sector — thus, the equilibrium marginal return on capital increases across all sectors.

(e) True or False*: The general equilibrium subsidy incidence of the implicit subsidy of housing capital falls equally on all forms of capital.*

Answer: This is true — capital moves between the sectors until the rate of return is equalized.

Conclusion: Potentially Helpful Reminders

1. Remember that the actual impact of a tax or subsidy — how much price changes for buyers and sellers, and how much the quantity transacted in the market changes — is determined solely by the uncompensated demand and supply curves. Thus, when thinking about tax incidence and output changes, we do not need to worry about substitution and income (or wealth) effects because the combined effect is what matters. The price elasticities of the curves then become the key fact to focus on.

2. It is when we start to evaluate the welfare changes from taxes and subsidies — the changes in surplus and deadweight loss — that isolating substitution effects may become important. Being told that the underlying good is quasi-linear, however, is your "get-out-of-jail-free" card: In that case, there are no income (or wealth) effects to worry about, and the regular (uncompensated) demand and supply curves are all we need.

3. The assumption of quasilinearity is in fact the key (and usually unspoken) assumption in Principles of Economics courses where supply and demand graphs are typically drawn with surplus and deadweight loss read directly off those curves.

4. It is only if there is a good reason to believe that the underlying tastes of consumers, workers or savers are not quasilinear that we have to deviate from the uncompensated demand and supply graphs to estimate welfare changes — i.e. changes in surplus and deadweight loss. To be more precise, in consumer goods markets, we have to worry about compensated demand curves; and in labor and capital markets, we have to worry about compensated supply curves.

5. This becomes particularly important when there is reason to believe that substitution and income (or wealth) effects are offsetting and thus mask each other in the uncompensated graph. In such cases, it becomes possible that the uncompensated graph shows little or no deadweight loss when in reality the deadweight loss is masked by these offsetting effects.

6. Finally, a quick graphing hint (that will carry through the next chapters): Even though demand and supply curves are shifting as a result of taxes and subsidies, it is much easier to find ways of identifying incidence and welfare changes without shifting the curves. In the case of taxes, this simply involves drawing the "tax wedge" to the left of the no-tax equilibrium; and in the case of subsidies, it involves drawing the "subsidy wedge" to the right of the no-tax equilibrium.

CHAPTER

20

Prices and Distortions across Markets

This chapter is the third and final chapter to investigate the way in which price distortions affect efficiency in competitive markets. While the previous chapters focused on a single market, this chapter deals with trade across markets — where the two different markets might be geographically or temporally separated. It therefore zeroes in on the role of exporters/importers as well as speculators — both of whom specialize in opportunities to make money by buying low and selling high. Under perfect competition, these economic agents end up making zero profit (just like other competitive firms) because their actions equilibrate prices across markets. Policy makers often interfere in this equilibrating process by preventing or limiting trade through taxes on imports (i.e. tariffs) or setting trade quotas. The chapter illustrates the deadweight losses that arise from the resulting price distortions, although it also points out that tariffs can be welfare improving for the importing market if supply is sufficiently inelastic in the exporting market to permit a sufficiently large burden of the tariff to be "exported". We furthermore trace the implication of free mobility of capital and labor for labor markets — illustrating the economic equivalence (in the absence of other frictions) of "outsourcing" (i.e. mobility of capital) and labor migration. Finally, we briefly explore the role of speculators that exploit price fluctuations over time, in the process introducing you to the idea of taking short and long positions in the market as well as the intuition of call and put options.

Chapter Highlights

The main points of the chapter are:

1. **Trade across markets** equilibrates prices across markets, with those that specialize in "buying low" and "selling high" earning zero profits so long as the export/import "industry" is competitive. Consumers in low price markets "lose" and producers in such markets "win" under trade, with the reverse being true in high price markets. Overall, however, **both markets experience a gain in consumer and producer surplus as a result of trade.** Policies that

prohibit trade therefore result in deadweight losses in both markets — although some in each market will benefit from such restrictions.

2. **Tariffs on imports** introduce a "wedge" between the price in export markets and the price in import markets just as taxes in a single market introduce a "wedge" between consumer and producer prices. This restricts the flow of goods from one market to the other and results in global deadweight losses. However, just as taxes in a single market are passed on to the side of the market that is less price elastic, the **tax burden of tariffs can be shifted out of the importing and into the exporting market, with such *tax exporting* more effective the more inelastic the supply curve in the importing market is.** This results in a decrease of deadweight loss in the importing market that is exactly offset by an increase in deadweight loss in the exporting market.

3. **Import quotas** have effects similar to those of tariffs in that they create a "wedge" between prices in the exporting and importing markets which in turn results in global deadweight losses.

4. **Outsourcing** arises when countries can trade goods but labor costs are lower in one place than another — thus permitting firms to move capital and locate in low cost markets. **Migration of labor**, on the other hand, attracts workers from low wage countries to high wage countries. While these forces move different curves in the labor market, they both result in downward pressure on wages in high wage countries and upward pressure on wages in low wage countries.

5. **Trading across Time** is conceptually similar to trading across geographically separated markets and has similar efficiency implications. Such trading is facilitated by financial instruments that can dampen price fluctuations across time but, unlike trade across space, such "trade across time" involves risks associated with guessing correctly where prices will head in the future.

20A Solutions to Within-Chapter-Exercises for Part A

Exercise 20A.1

During the transition from the initial to the new equilibrium, which producers make positive profits and which might make negative (long run) profits?

Answer: Since producers in both Florida and New York initially make zero profits, producers in Florida will make positive profits while some producers in New York make negative profits until they exit.

Exercise 20A.2

In our treatment of taxes within a single market in Chapter 19, we concluded that a doubling of a tax results in approximately a quadrupling of the deadweight loss. Is the same true for tariffs?

Answer: Yes, the same is approximately true for tariffs as well. When the tariff is doubled, the vertical side of each of the deadweight loss triangles doubles — which implies that the area of the triangle roughly quadruples.

Exercise 20A.3

How would the analysis change if supply were perfectly elastic in both regions (with the supply curve lying at a higher price in New York than in Florida)?

Answer: In that case, the equilibrium price would still adjust to the lower price in Florida — but production in New York would cease entirely as all firms in New York exit the market (with all goods now provided by Floridian firms that enter to meet the additional demand.)

Exercise 20A.4

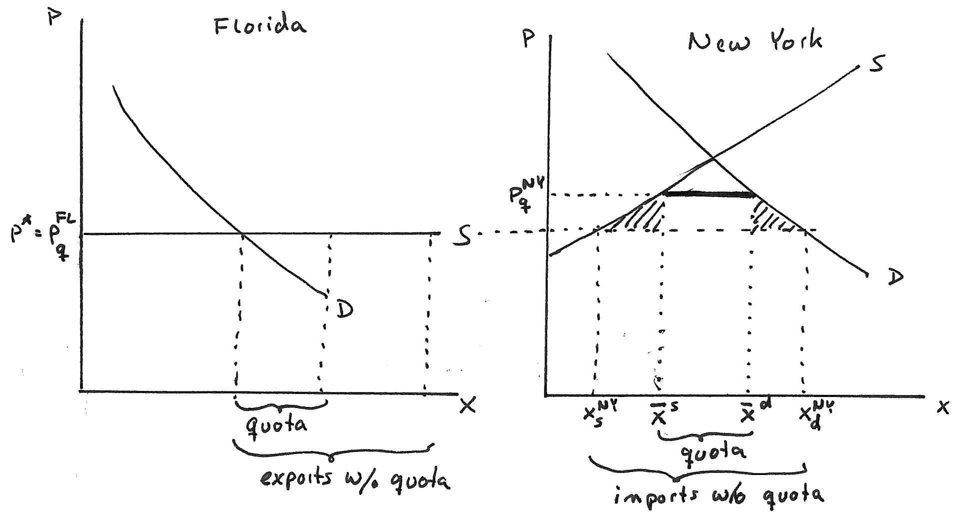
Identify separately consumer and producer surplus in both regions both before and after the import quota and check that the previous sentence is correct.

Answer: Before the imposition of the quota, consumer and producer surplus in Florida are a and $(b+c+d+e+f+g+h)$ respectively while consumer and producer surplus in New York are $(i+j+m+n+o+p)$ and $(k+l)$ respectively. The quota increases consumer surplus in Florida to $(a+b)$ and reduces producer surplus to $(c+g+f)$ — a gain of b (to consumers) and a loss of $(b+d+e+h)$ for producers, giving us an overall deadweight loss in Florida of $(d+e+h)$. In New York, consumer surplus falls to $(i+m)$ while producer surplus increases to $(j+k+l)$ — a loss of $(j+n+o+p)$ for consumers and a gain of j for producers, for an overall deadweight loss in New York of $(n+o+p)$.

Exercise 20A.5

What is the economic effect of an import quota in New York when the supply curve for hero cards in Florida is perfectly elastic?

Answer: This is illustrated in Exercise Graph 20A.5. In the absence of the quota, the price in both Florida and New York would be equal to p^* — the price at which supply in Florida is perfectly elastic. With the imposition of the quota, New York firms produce more ($\overline{x}^s$) and New York consumers demand less ($\overline{x}^d$) at the higher price p_q^{NY}. For Florida consumers, nothing changes as price remains unchanged, but some Florida firms will exit as demand from exporters falls. The deadweight loss from the import quota falls entirely onto New York — as indicated by the two shaded triangles.



Exercise Graph 20A.5 : Import Quotas

Exercise 20A.6

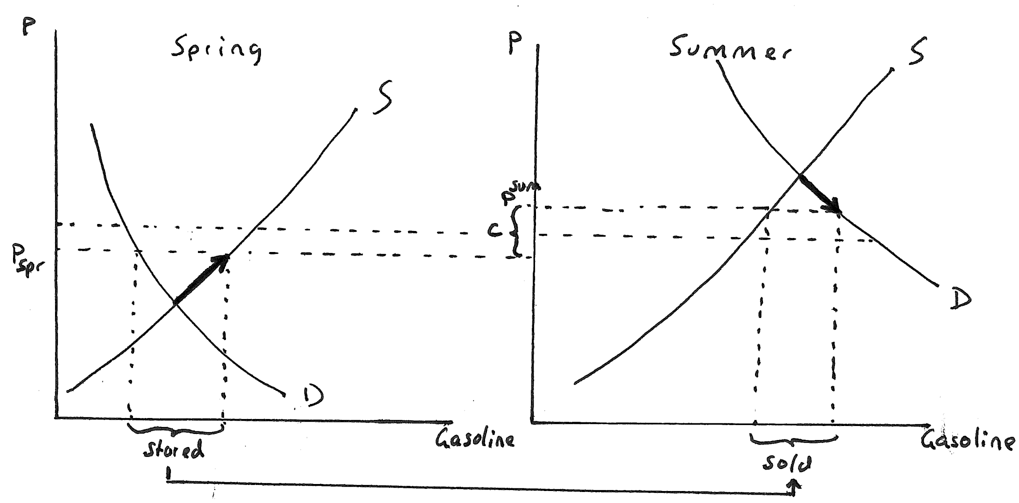
Suppose that the U.S. government attempts to alleviate suffering abroad by requiring that outsourcing firms apply some fraction of U.S. labor standards (i.e. good working conditions, health benefits, etc.) in any production facility abroad. Illustrate the impact this will have in Graph 20.6. Does the logic of the model suggest that this will improve the fortunes of workers abroad? Will it benefit domestic U.S. workers?

Answer: This would increase costs for outsourcing firms. As a result, demand for workers in India by outsourcing firms would not rise as much, and demand in the U.S. would not fall by as much. This suggests that wages in India would not rise as much, and wages in the U.S. would not fall as much — benefitting U.S. workers at the cost of Indian workers. If Indian workers value the higher labor standards that are imposing costs on outsourcing firms, these effects are mitigated — but they do not disappear unless Indian workers view each dollar spent on complying with labor regulations as equivalent to a dollar in wages.

Exercise 20A.7

Illustrate how Graph 20.8 changes as the cost to storing gasoline is introduced. Can you see such price fluctuations will worsen as the cost of storing gasoline increases?

Answer: This is illustrated in Exercise Graph 20A.7. The cost of storing gasoline now prevents a full equalization of prices across spring and summer — as speculators who buy gasoline in spring have to cover the per-gallon storage costs c for each gallon that they sell in the summer. For speculators to make zero profit, this implies that the price in spring p_{spr} is c less than the price in summer p^{sum}. The higher the storage cost c, the greater will be the difference between these prices.



Exercise Graph 20A.7 : Storage Costs

Exercise 20A.8

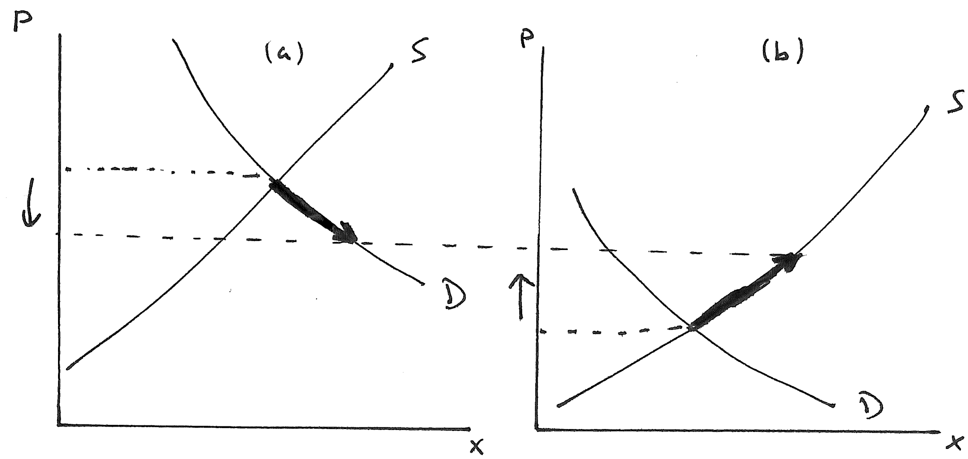
Illustrate in a graph similar to Graph 20.8 how this can contribute to stabilization of gasoline prices across time.

Answer: This is illustrated in Exercise Graph 20A.8 where the price in panel (a) is high and the price in panel (b) is low in the absence of speculators. The short selling speculator would borrow gasoline from someone who is holding it in reserve in panel (a) and sell it — thus shifting supply out (as indicated by the darkened arrow) and lowering price. Then, in the future, the speculator would purchase gasoline in order to pay back the amount that he borrowed — thus increasing demand and pushing up price as illustrated in panel (b).

Exercise 20A.9

Can you see why an investor would want to hold a call option if she believes the price of the asset is likely to go up, and why she would want to hold a put option if she believes prices are likely to fall?

Answer: A call option gives the owner of the contract the right to buy an asset in the future at a pre-determined price. If the actual price increases (beyond what



Exercise Graph 20A.8 : Selling Gasoline Short

is expected), this implies that the owner of the contract could (in the future) buy at the pre-determined low price and sell at the higher actual price, thus making a profit. (Alternatively, the value of the call option increases as actual prices rise — so the owner could simply sell the call option contract to someone else.) A put option, on the other hand, gives the owner the right to sell in the future at a pre-determined price. If prices fall (beyond what is expected), this would imply that the owner of the put option can (in the future) buy the commodity at the lower actual price and then sell it at the pre-determined higher price — thus making a profit. Alternatively, falling prices would increase the price of put options — and the owner could therefore simply sell the put option contract and make a profit that way.

20B Solutions to Within-Chapter-Exercises for Part B

Exercise 20B.1

Can you verify that, when exports are X^*, prices in the two regions are equal?

Answer: This gets a little tedious — so, to simplify notation, let's define $E = (\beta A - \alpha B)$, $F = (\delta C - \gamma D)$ and $G = (\gamma+\delta)\alpha\beta + (\alpha+\beta)\gamma\delta$. We can then write

$$X^* = \frac{(\alpha+\beta)F - (\gamma+\delta)E}{G} \qquad (20B.1.i)$$

and the two equilibrium prices as

$$\tilde{p}^1 = \frac{E + \alpha\beta X^*}{\alpha+\beta} = \frac{EG + \alpha\beta(\alpha+\beta)F - \alpha\beta(\gamma+\delta)E}{(\alpha+\beta)G} \qquad (20B.1.ii)$$

$$\tilde{p}^2 = \frac{F - \gamma\delta X^*}{\gamma+\delta} = \frac{FG - \gamma\delta(\alpha+\beta)F + \gamma\delta(\gamma+\delta)E}{(\gamma+\delta)G}. \qquad (20B.1.iii)$$

Multiplying the numerator and denominator of the $\tilde{p}^1$ equation by $(\gamma+\delta)$, and multiplying the numerator and denominator of the $\tilde{p}^2$ equation by $(\alpha+\beta)$, we get

$$\tilde{p}^1 = \frac{(\gamma+\delta)EG + \alpha\beta(\alpha+\beta)(\gamma+\delta)F - \alpha\beta(\gamma+\delta)^2 E}{(\alpha+\beta)(\gamma+\delta)G} \qquad (20B.1.iv)$$

$$\tilde{p}^2 = \frac{(\alpha+\beta)FG - \gamma\delta(\alpha+\beta)^2 F + \gamma\delta(\gamma+\delta)(\alpha+\beta)E}{(\alpha+\beta)(\gamma+\delta)G}. \qquad (20B.1.v)$$

Finally, expanding G in the numerators of both these equations (i.e. substituting $G = (\gamma+\delta)\alpha\beta + (\alpha+\beta)\gamma\delta$ in the numerators) and simplifying, we get

$$\tilde{p}^1 = \frac{(\alpha+\beta)(\gamma+\delta)\left[\gamma\delta E + \alpha\beta F\right]}{(\alpha+\beta)(\gamma+\delta)G} \qquad (20B.1.vi)$$

$$\tilde{p}^2 = \frac{(\alpha+\beta)(\gamma+\delta)\left[\gamma\delta E + \alpha\beta F\right]}{(\alpha+\beta)(\gamma+\delta)G}; \qquad (20B.1.vii)$$

i.e. $\tilde{p}^1 = \tilde{p}^2$ when exports are at the equilibrium level X^*.

Exercise 20B.2

Can you demonstrate that a tariff $t = p^2(\overline{X}) - p^1(\overline{X})$ will result in the same level of exports from region 1 as the import quota $\overline{X}$ — as well as the same equilibrium prices (in the two regions)?

Answer: In the text, we derived the equilibrium export level under a tariff t as

$$X^*(t) = \frac{(\alpha+\beta)(\delta C - \gamma D) - (\gamma+\delta)(\beta A - \alpha B) - (\alpha+\beta)(\gamma+\delta)t}{(\gamma+\delta)\alpha\beta + (\alpha+\beta)\gamma\delta}. \quad (20B.2.i)$$

Substituting the tariff

$$t = p^2(\overline{X}) - p^1(\overline{X}) = \frac{\delta C - \gamma D}{\gamma+\delta} - \frac{\beta A - \alpha B}{\alpha+\beta} + \frac{\left((\gamma+\delta)\alpha\beta - (\alpha+\beta)\gamma\delta\right)\overline{X}}{(\alpha+\beta)(\gamma+\delta)} \quad (20B.2.ii)$$

into this equation and simplifying, we get $X^*(t) = \overline{X}$. In the text we also derived the prices in the two regions for a given export level X as

$$\tilde{p}^1 = \frac{\beta A - \alpha B + \alpha\beta X}{\alpha+\beta} \text{ and } \tilde{p}^2 = \frac{\delta C - \gamma D - \gamma\delta X}{\gamma+\delta}. \quad (20B.2.iii)$$

At export level $\overline{X}$, these become what we calculated (in the text) as the prices $p^1(\overline{X})$ and $p^2(\overline{X})$ that arise under the quota.

Exercise 20B.3

Illustrate demand and supply curves in the two regions (with price on the vertical and quantity on the horizontal axes). Carefully label each intercept as well as the no-trade equilibrium prices and quantities. Then illustrate the equilibrium under free trade.

Answer: This is illustrated in Exercise Graph 20B.3. (Note that we have to solve the demand and supply *functions* in the text for p to give us the demand and supply *curves* we graph.)

Exercise 20B.4

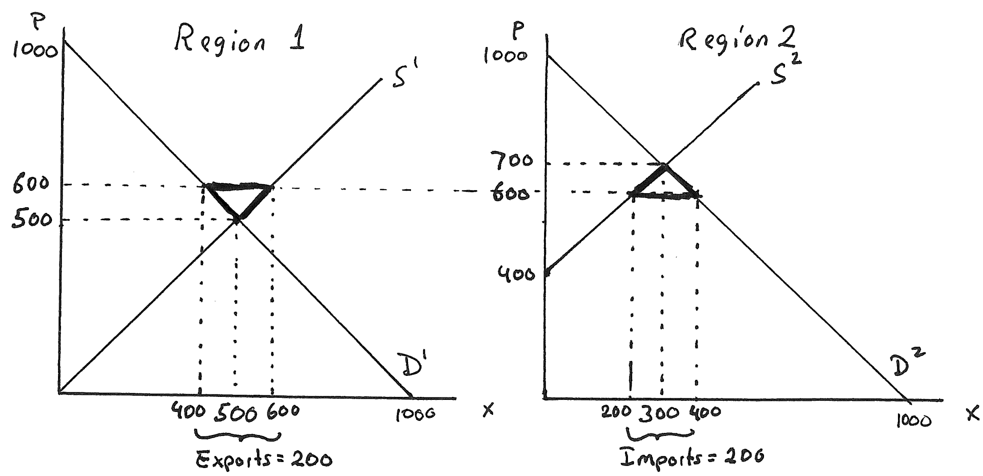
Assuming that demand curves are also marginal willingness to pay curves, what is the deadweight loss from prohibiting trade?

Answer: The deadweight loss is the sum of the two triangles in Exercise Graph 20B.3. The area of the triangle in each region is $200(100)/2 = 10,000$ — for a total deadweight loss across the two regions of 20,000.

Exercise 20B.5

Illustrate the impact of a \$100 per unit tariff on the equilibrium you have graphed in exercise 20B.3.

Answer: This is illustrated in Exercise Graph 20B.5 where exports and imports fall by 100.

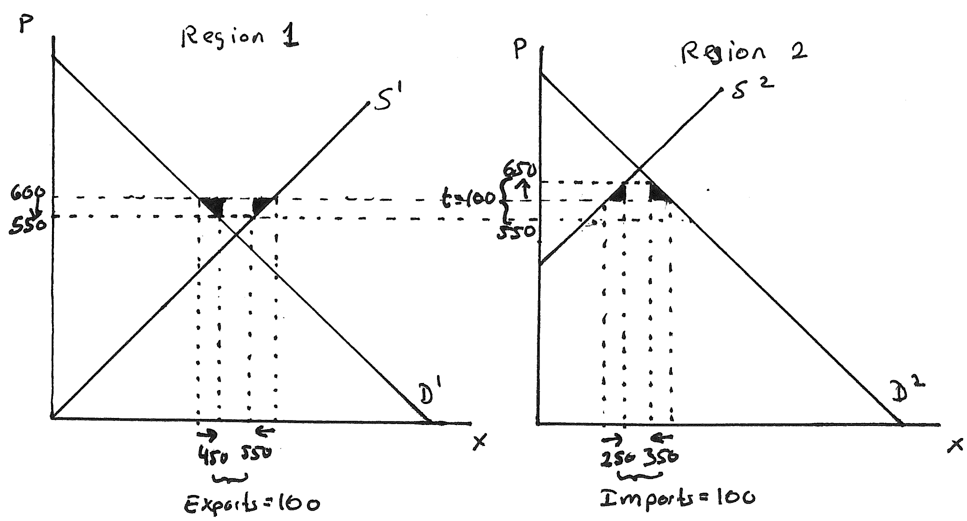


Exercise Graph 20B.3 : $A = 1,000 = C$, $\alpha = \beta = 1 = \gamma = \delta$, $B = 0$ and $D = -400$

Exercise 20B.6

Assuming again that demand curves are marginal willingness to pay curves, what happens to surplus in regions 1 and 2 when considering each in isolation? What happens to overall deadweight loss when considering both regions jointly?

Answer: The overall deadweight loss (across both regions) from is equal to the sum of the shaded triangles in Exercise Graph 20B.5. Each triangle has area $50(50)/2 = 1,250$ — for an overall deadweight loss equal to $1,250(4) = 5,000$. Considering region 1 in isolation, however, we would have to also consider the fact that this region is paying a portion of the tariff equal to the box in between the two triangles — i.e. they are paying $100(50) = 5000$. Thus, the deadweight loss for region 1 is $2,500 + 5,000 = 7,500$. In region 2, on the other hand, we have to consider the fact that the government collects 100 for every one of the 100 import units — only half of which is paid for by its own citizens. Thus, of the overall tax revenue of 10,000, half is a gain in social surplus. Subtracting the two shaded triangles, we get a net social *gain* of $5,000 - 2,500 = 2,500$. Region 1 therefore loses 7,500 while region 2 gains 2,500 (for, again, an overall deadweight loss of 5,000 across the two regions.)



Exercise Graph 20B.5 : Tariff of $100 per unit

20C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 20.1

In the text, we argued that the burden of tariffs is shifted across markets in ways that are analogous to how tax burdens are shifted between consumers and producers.

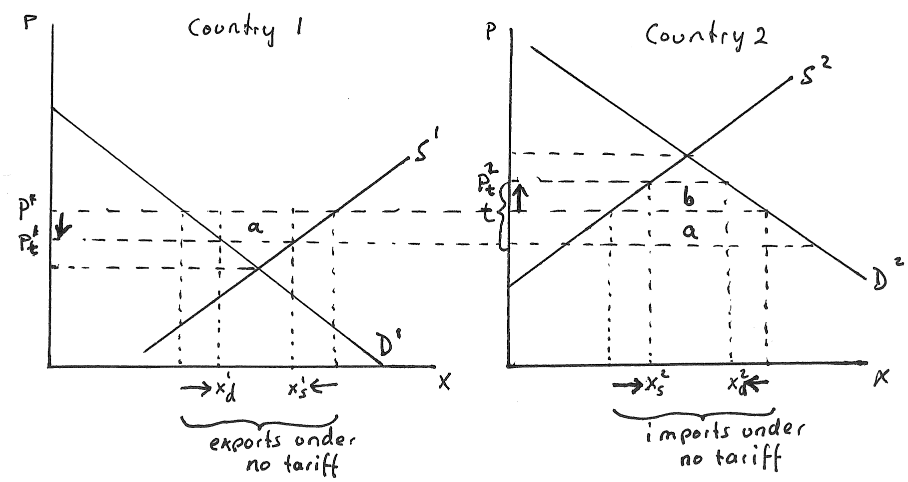
A: *Consider two countries — country 1 in which product x would sell at p^1 and country 2 in which it would sell at p^2 in the absence of any trade between the countries. Suppose throughout that $p^2 > p^1$.*

(a) *Begin by illustrating the free trade equilibrium assuming negligible transportation costs.*

Answer: This is illustrated in Exercise Graph 20.1(1) where p^* is the world price under free trade (with no transportation costs).

(b) *Illustrate how the imposition of an import tax (or tariff) of t per unit of x by country 2 changes the equilibrium.*

Answer: This is also illustrated in Exercise Graph 20.1(1) where p_t^1 becomes the price in country 1 under the tariff and p_t^2 becomes the price in country 2 under the tariff. Note that $(p_t^2 - p_t^1) = t$. Price therefore falls in country 1 and rises in country 2 — causing the quantity demanded by



Exercise Graph 20.1(1) : Economic Incidence of Tariffs

consumers in country 1 to increase (to x_d^1), the quantity supplied by producers in country 1 to decrease (to x_s^1), the quantity demanded by consumers in country 2 to decrease (to x_d^2) and the quantity supplied by producers in country 2 to increase (to x_s^2). As a consequence, exports and imports fall.

(c) *What in your answer to (b) would change if, instead of country 2 imposing a per unit import tax of t, country 1 had imposed a per unit* export *tax of the same amount t?*

Answer: Nothing changes. Exporters will still reduce their demand in country 1 and importers will reduce their supply in country 2 — because the per unit export tax is simply a cost of moving units of x from country 1 to country 2 just as the import tax is.

(d) *In your graph, illustrate the economic incidence of the tax t on trade; i.e. illustrate how much of the overall tax revenue is raised from country 1 and how much is raised from country 2.*

Answer: In Exercise Graph 20.1(1), the area (a) in the first panel is the amount of the tariff that is paid in country 1 while the area (b) in the second panel is the amount paid in country 2. Thus, (a) of the tax collections in country 2 are shifted to country 1.

(e) *How would your answer change if you made the supply curve in country 1 more elastic while keeping p^1 unchanged? What if you made the demand curve more elastic?*

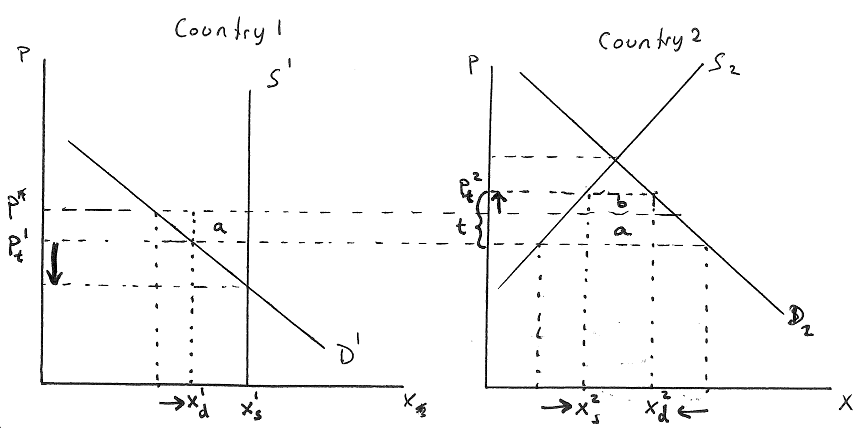
Answer: The more elastic the supply curve in country 1, the less price will fall from p^* to p_t^1 — and the more it will rise from p^* to p_t^2 in country 2. Thus, the more elastic the supply curve, the less able country 2 is to pass the burden of paying the tariff to country 1. The same would be true if you made the demand curve in country 1 more elastic.

(f) *In Chapter 19, we argued that it does not matter whether a per-unit tax is imposed on producers or on consumers within a market — the economic impact will be the same. How is what you have found in this exercise analogous to this result?*

Answer: Here it does not matter whether the tax on trade is imposed in the low price country through an export tax or in the high priced country as an import tax — either way, the tax becomes a cost of transporting goods from one country to the other. The impact is therefore the same (for an equally sized export and import tax) — with the economic incidence of the tax falling disproportionately on market that is more inelastic.

(g) *If the supply curve in country 1 were perfectly inelastic, would any of the tariff be paid by country 2?*

Answer: This is illustrated in Exercise Graph 20.1(2) which replicates what we did in Exercise Graph 20.1(1) except that it graphs S^1 as perfectly inelastic. Note that everything happens exactly as in Exercise Graph 20.1(1) — except that the quantity supplied in country 1 does not change with the tariff. More of the tariff will be paid by country 1 because it is easier to pass the tariff to that market (given the inelastic supply curve) — but some area (b) remains in country 2 — i.e. some of the tax burden remains in the taxing country even when the supply in country 1 is perfectly inelastic.



Exercise Graph 20.1(2) : Economic Incidence of Tariffs with perfectly inelastic Supply in Country 1

B: *Now consider demand and supply functions $x_d^1(p) = (A-p)/\alpha$ and $x_s^1(p) = (B+p)/\beta$ for country 1 and $x_d^2(p) = (C-p)/\gamma$ and $x_s^2(p) = (D+p)/\delta$ for country 2 (as in part B of the text.)*

(a) *Set up an Excel spreadsheet that calculates production and consumption levels in each country as a function of the demand and supply parameters A, B, α, β, C, D, γ and δ as well as the per-unit tariff t imposed by country 2. Would any of your spreadsheet differ if instead we analyzed a per-unit export tax in country 1?*

Answer: You can of course simply use the various formulas derived in part B of the text to create your spreadsheet. And no, none of the formulas would differ if we analyzed an export tax in country 1 instead — the steps for deriving the formulas would be exactly as they are in part B of the text.

(b) *Let $A = 1000 = C$, $\alpha = \beta = 1 = \gamma = \delta$, $B = 0$ and $D = -400$. Verify that you get the same result as what is reported in part B of the text for the same parameters when $t = 0$ and when $t = 100$.*

Answer: You should get a free trade equilibrium in which country 1 exports 200 units of x to country 2, with the world price of x equal to 600. When $t = 100$ is imposed, exports drop from 200 to 100, price in country 1 falls from 600 to 550 and price in country 2 rises from 600 to 650.

(c) *Set up a table in which the rows correspond to scenarios where we change the parameters B and β from $(49500, 100)$ in the first row to $(12000, 25)$, $(2000, 5)$, $(500, 2)$, $(0, 1)$, $(-250, 0.5)$, $(-375, 0.25)$, $(-450, 0.1)$, and $(-495, 0.01)$ in the next 8 rows. Then report in each row p^1 and x^1 — the price and quantity in country 1 in the absence of trade; $p^* = \tilde{p}^1 = \tilde{p}^2$ — the world price under free trade; X^* — the level of exports under free trade; $X^*(t)$ — the level of exports when $t = 100$ is imposed; $\tilde{p}^1(t)$ and $\tilde{p}^2(t)$ — the prices when a per unit tariff of $t = 100$ is imposed; and the fraction k of the tariff that is shifted to country 1.*

Answer: This is given in the following table:

$A = 1000 = C$, $\alpha = 1 = \gamma = \delta$, and $D = -400$								
(B, β)	p^1	x^1	p^*	X^*	$X^*(t)$	$\tilde{p}^1(t)$	$\tilde{p}^2(t)$	k
$(49500, 100)$	500	500	632.9	134.2	67.1	566.4	666.4	0.664
$(12000, 25)$	500	500	631.6	136.8	68.4	565.8	665.8	0.658
$(2000, 5)$	500	500	625.0	150.0	75.0	562.5	662.5	0.625
$(500, 2)$	500	500	614.3	171.4	85.7	557.1	657.1	0.571
$(0, 1)$	500	500	600.0	200.0	100.0	550.0	650.0	0.500
$(-250, 0.5)$	500	500	580.0	240.0	120.0	540.0	640.0	0.400
$(-375, 0.25)$	500	500	557.1	285.7	142.9	528.6	628.6	0.286
$(-450, 0.1)$	500	500	530.8	338.5	169.2	515.4	615.4	0.154
$(-495, 0.01)$	500	500	503.9	392.2	196.1	501.9	601.9	0.019

(d) *Explain what is happening as we move down the rows in your table.*

Answer: As one goes down the table, the supply curve in country 1 goes from almost completely inelastic to almost completely elastic — with the no-trade equilibrium in country 1 (i.e. the intersection of country 1 supply and demand) remaining constant at price and output of 500. As the supply curve in country 1 becomes more elastic, the world equilibrium price converges to the no-trade price in country 1 and export levels increase. When the tariff is introduced, the price in country 1 becomes less and less affected as the supply becomes more elastic — because less and less of the tariff can be passed to country 1. This is reflected in the final column where nearly two thirds of the tariff is being passed to country 1 in the top row (where supply is almost perfectly inelastic) while only about 2 percent is passed to country 1 in the last row (where supply is almost perfectly elastic). Note that even in the case where supply is near perfectly inelastic, not all the tariff is passed to country 1 — which is consistent with Exercise Graph 20.1(2) from part A.

(e) *Next, set up a table in which the rows correspond to scenarios where we change the parameters A and α from* $(50500, 100)$ *in the first row to* $(13000, 25)$, $(3000, 5)$, $(1500, 2)$, $(1000, 1)$, $(750, 0.5)$, $(625, 0.25)$, $(550, 0.1)$, *and* $(505, 0.01)$ *in the next 8 rows. (Keep the remaining parameters as originally specified in (b).) Then report the same columns as you did in the table you constructed for part (c).*

Answer: This is reported in the following table:

$C = 1000$, $B = 0$, $\beta = 1 = \gamma = \delta$ and $D = -400$								
(A, α)	p^1	x^1	p^*	X^*	$X^*(t)$	$\tilde{p}^1(t)$	$\tilde{p}^2(t)$	k
$(50500, 100)$	500	500	632.9	134.2	67.1	566.4	666.4	0.664
$(13000, 25)$	500	500	631.6	136.8	68.4	565.8	665.8	0.658
$(3000, 5)$	500	500	625.0	150.0	75.0	562.5	662.5	0.625
$(1500, 2)$	500	500	614.3	171.4	85.7	557.1	657.1	0.571
$(1000, 1)$	500	500	600.0	200.0	100.0	550.0	650.0	0.500
$(750, 0.5)$	500	500	580.0	240.0	120.0	540.0	640.0	0.400
$(625, 0.25)$	500	500	557.1	285.7	142.9	528.6	628.6	0.286
$(550, 0.1)$	500	500	530.8	338.5	169.2	515.4	615.4	0.154
$(505, 0.01)$	500	500	503.9	392.2	196.1	501.9	601.9	0.019

(f) *Are there any differences between your two tables? Explain.*

Answer: No, the two tables are identical. This is because we are changing the demand curve in the table in part (d) symmetrically to how we changed the supply curve in the table from part (c) — and thus altering the responsiveness of demand the way we altered the responsiveness of supply before. As demand becomes more elastic (going down the table), less of the tariff can be passed to country 1.

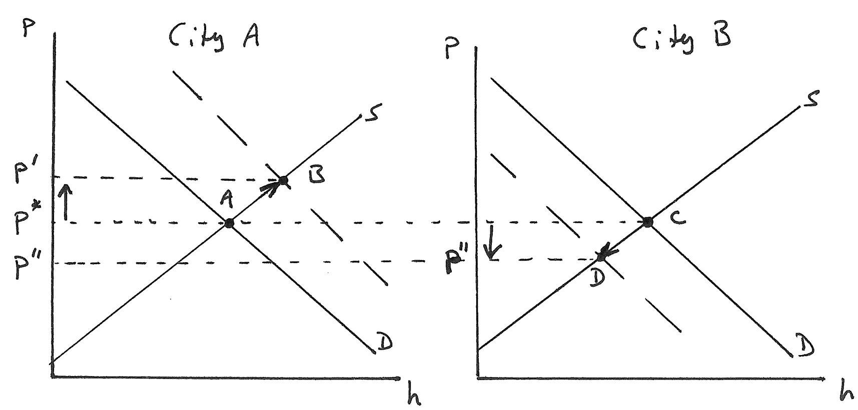
Exercise 20.3

Everyday Application: *Quality of Life Indexes*: *Every year, various magazines publish lists of "The 10 Best Cities for Living" or "The 10 Worst Cities". These lists are constructed by magazines weighting various factors such as climate, public amenities (like school quality and crime rates), local taxes and housing prices. Economist often sneer at these lists. Here is why.*

A: *Consider two cities that are identical in every way — same climate, same public amenities, same housing prices. Suppose for simplicity, unless otherwise stated, that everyone rents housing and everyone has the same tastes and income.*

(a) *Begin by drawing two side-by-side graphs of the housing markets in city A and city B.*

Answer: This is done by the solid demand and supply curves in Exercise Graph 20.3 — with equilibrium at A in city A and at C in city B — and with housing prices equal to p^* in both cities.



Exercise Graph 20.3 : Housing Prices and Quality of Life

(b) *Suppose city A elects a new mayor who is superb at what he does. He finds ways of improving the schools, lowering crime and building better public parks — all while lowering local tax rates. What will happen to the demand for housing in city A? What about in city B?*

Answer: Demand for living in city A will increase while demand for living in city B will decrease.

(c) *Depict the new equilibrium. Will housing prices still be the same in the two cities? Why or why not?*

Answer: This is depicted in Exercise Graph 20.3 with the shifted dashed curves — leading to the increase in housing prices to p' in city A and the decrease in housing prices in city B to p''.

(d) *Last year, two magazines independently ranked the quality of life in city A and city B as equal. This year, one magazine ranks the quality of life in city A higher than in city B and the other does the reverse. When pressed for an explanation, the first magazine highlights all the wonderful improvement in city A while the second one highlights the "excessively high" housing prices in city A and the "housing bargains" in city B. Which magazine is right?*

Answer: It seems that neither is right. The price difference between housing in city A and housing in city B emerges from the fact that city A has increased its amenity levels — and people move until prices have adjusted so that they are equally happy in the two cities. The price difference is a "compensating differential" — i.e. the lower housing prices in city B compensate for the fact that life is just not as nice, all else equal, there. But housing is cheaper — so you can make up for the fact that there are no public parks with having a bigger back yard!

(e) *What happens to the population size in cities A and B? What happens to the average house and lot size in cities A and B?*

Answer: People migrate from city B to city A — so population increases in city A and decreases in city B. However, housing prices increase in city A and decrease in city B — so we would expect that city B will have larger houses and lot sizes than city A. (Alternatively, people might just engage in more private consumption in city B given that housing is cheaper.)

(f) True or False*: If city A is large relative to the national housing market, the mayor's actions make everyone in the country better off — i.e. not all of the benefits of the mayor's ingenuity stay in city A.*

Answer: This is true, as we showed in the graph. In city B, nothing changes except that housing prices fall — which makes people better off since the cost of living in city B has fallen. (We are abstracting away from the fact that landlords will be worse off in city B because their wealth has declined).

(g) *If you like public amenities more than the average person, will you be better off? What if you like them less than the average person?*

Answer: If you like public amenities more than the average person, you will end up living in city A (whether you did so before or not) — because the public amenities are worth more to you than the price you pay through increased housing rents. If you like public amenities less than the average person, you will also be better off — and you will live in city B where you benefit from the drop in housing prices.

(h) True or False*: If city A is small relative to the national housing market, the primary beneficiaries of the mayor's actions are landlords in city A (i.e. those who owned land and housing in city A prior to the mayor's actions).*

Answer: This is also true. If city A is small, it has no impact on any other cities — so everyone elsewhere remains equally well off. Within city A, renters face housing prices that are higher precisely by an amount that incorporates the value of the mayor's actions — so they are no better off. But landlords — i.e. those who previously owned land and housing in city A — now see an increase in their wealth.

B: *Suppose that individuals have tastes over housing h, consumption x and public amenities y and these tastes can be represented by the utility function $u(h,x,y) = h^{0.25}x^{0.75}y$. Suppose everyone rents rather than owns housing.*

(a) *In city A, the average resident earns \$50,000 in income, faces a rental price for housing equal to \$5 per square foot, and enjoys amenity level $y = 10$. Assuming everyone maximizes utility, what utility level does the average resident attain? (*Hint*: Note that y is not a choice variable.)*

Answer: The price of consumption is by definition equal to 1. Letting the price of housing be equal to p and setting up the utility maximization problem in the usual way, we get that private consumption is equal to $x^* = 0.75(50,000) = 37,500$ while housing consumption is $h^* = 0.25(50,000)/5 = 2,500$ square feet. Thus, the average utility level would be

$$u^* = (2500)^{0.25}(37,500)^{0.75}(10) \approx 190,550. \tag{20.3.i}$$

(b) *Suppose the housing market across the nation is in equilibrium. If households can move across cities to maximize utility, can you tell what this implies about the utility level households attain in city B?*

Answer: Free mobility of households should imply equal utility everywhere — i.e. households in other cities also attain utility level of $u^* = 190,550$.

(c) *Now suppose the new mayor in city A is able to increase the public amenity level y from 10 to 11.25. If utility for residents remains unchanged because of an increase in housing prices, how much will housing consumption have to fall for each household?*

Answer: Given the Cobb-Douglas tastes we specified, private consumption is unaffected by changes in y or p — and thus x^* remains at 37,500. Thus, to get how much housing consumption will leave utility unchanged at $u^* = 190,550$ when y goes to 11.25, we simply solve

$$190,550 = h^{0.25}(37,500)^{0.75}(11.25) \tag{20.3.ii}$$

to get $h^* \approx 1,560.74$.

(d) *Suppose that city A is small relative to the nation — and thus does not affect housing price elsewhere. Can you tell how much the rental price of housing must have increased from the initial price of 5 as a result of the mayor's innovation?*

Answer: If city A is small, utility elsewhere remains at $u^* = 190,550$ — which means that housing prices in city A must rise to get the optimal housing consumption level to be such that utility in city A is also 190,550. We just calculated in the previous part that this implies $h^* = 1,560.74$. Given the Cobb-Douglas tastes, we know that housing demand is given by $0.25(50,000)/p$ — so we just need to solve

$$1,560.74 = \frac{0.25(50,000)}{p} \quad (20.3.iii)$$

to get $p^* \approx 8$.

(e) *Are renters in city A better off as a result of the mayor's innovations? What about landlords who own land and housing?*

Answer: Renters are just as well off as before — they pay more for housing (and thus consume less of it) but they get more public amenities to exactly offset this. Landlords, on the other hand, are definitely better off — they get an increase in their wealth from the increased price of housing.

Exercise 20.5

Everyday and Business Application: *Compensating Wage Differentials and Increased Worker Safety: Why would any worker choose to work in a profession (like coal mining) that is risky for the worker's health and safety? The answer is that such jobs tend to pay more than other jobs which require similar skill levels. The difference in wages between such "safe" jobs and risky jobs is what labor economists call a* compensating wage differential. *In the following exercise, suppose that it takes similar skills to work in coal mines as it does to work on oil rigs — and that workers in industries other than these two cannot easily switch to these industries.*

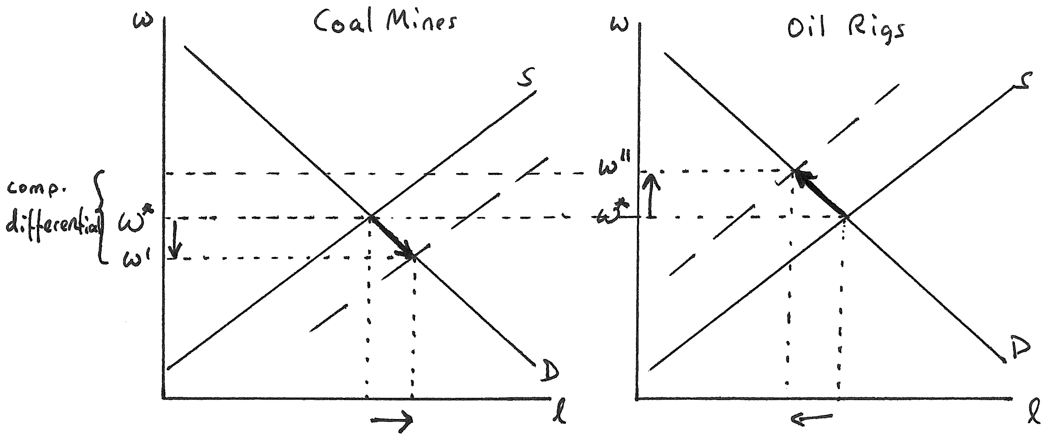
A: *Suppose that initially the wages in coal mines and oil rigs are the same.*

(a) *Illustrate demand and supply in the labor markets for oil workers and coal miners in two separate graphs. What does the fact that wages are identical in the two sectors tell you about the level of risk a worker takes on by working in coal mining relative to the level of risk he takes on by working on oil rigs?*

Answer: This is illustrated by the solid demand and supply curves in Exercise Graph 20.5 that intersect at w^* in both the coal and the oil industry.The fact that the wages are the same allows us to infer that the risk of the jobs must be roughly the same in both industries (assuming that one job isn't inherently more enjoyable than the other).

(b) *Suppose a new mining technology has just been invented — a technology that makes working in coal mines considerably safer than it was before. (For simplicity, suppose it is essentially costless to coal mining firms to put this technology in place.) What will happen to the supply of workers in the oil industry — and what will happen to the supply of workers in the coal industry?*

Answer: At the wage w^*, it has now become more attractive to work in coal mining and relatively less attractive to work on oil rigs. Thus, supply



Exercise Graph 20.5 : Compensating Differentials and Worker Safety

of workers in the coal mining industry will increase while supply of workers in the oil industry will fall — leading to downward pressure on wages in the coal industry and upward pressure on wages in the oil industry.

(c) *What happens to wages in the two industries? How does this relate to the idea of compensating wage differentials?*

Answer: This is illustrated in Exercise Graph 20.5 where supply increases in coal mining and decreases in oil rigging. The process continues until workers are again indifferent between working in one industry over the other — which is illustrated as happening when wages in coal mining fall to w' and wages in oil rigging rise to w''. The difference between these two wages is the compensating differential that compensates workers on oil rigs for the additional risk they take on (over the risk faced by coal miners).

(d) *Are workers in either industry better off?*

Answer: Workers in oil rigging are clearly better off — their working conditions are unchanged, but their wages have increased. This allows us to infer that workers in coal mining are also better off — because they were indifferent between the two industries before and are once again indifferent. Thus, even though their wages fall, coal miners are better off because they value the additional worker safety more than the loss in wages.

(e) *Suppose next that the oil industry if very large compared to the coal industry — so large that the change in wages in the oil industry is imperceptibly small. Are any workers better off as a result of the safety innovation in coal mines?*

Answer: In this case, the entire compensating differential will show up as a drop in wages in the coal industry — leaving workers in both industries equally well off as before. Put differently, the decline in wages is viewed by coal miners as exactly offset by the value they place on the increased safety. Workers therefore neither gain nor lose from the safety innovation.

(f) *In the case of the very large oil industry (relative to the coal industry), are any producers better off?*

Answer: Oil producers are unaffected since wages in the industry remain unchanged, but coal mining companies are clearly better off because they can payer lower wages.

(g) True or False*: The more competitive the labor market is across industries, the greater is the incentive for a producer in a competitive industry to find ways of improving employment safety conditions.*

Answer: This is true. We just showed the case of a perfectly competitive labor market where the coal industry is small overall in terms of its demand for labor. In this case, the entire benefit of worker safety innovations is captured by coal mining companies — because they can pay lower wages and attract more workers in the process as they innovate and create safer working conditions. If workers cannot easily move across all industries but only across some, then the benefit of safety innovations in one industry are more easily passed onto workers in other industries (as we showed in the earlier part of our exercise) — leaving less of an incentive for producers to invest in worker safety innovations.

B: *Suppose all workers' annual utility can be given by the function* $u(s,w) = (\alpha s^{-\rho} + (1-\alpha)w^{-\rho})^{-1/\rho}$ *where s is a work safety index that ranges from 0 to 10 (with 0 the least safe and 10 the most safe) and w is the annual wage denominated in tens of thousands of dollars.*

(a) *Suppose that workers of the skill type of coal miners are currently getting utility* u^* *in all sectors of the economy in which they are employed. Determine the relationship of the current wages offered to such workers in the economy as it relates to safety conditions — i.e. find* $w(s)$ *(which will itself be a function of* u^*, α *and* ρ*).*

Answer: We simply solve $u^* = (\alpha s^{-\rho} + (1-\alpha)w^{-\rho})^{-1/\rho}$ for w to get

$$w(s) = \left(\frac{(u^*)^{-\rho} - \alpha s^{-\rho}}{1-\alpha}\right)^{-1/\rho}. \qquad (20.5.i)$$

Note that this function is decreasing in s — i.e. as the job gets safer, the wage declines.

(b) *Suppose that* $\alpha = 0.5$ *and* $\rho = 0.5$*, and suppose that workers in the coal mining and in the oil rigging industries currently face safety conditions 5 and earn an annual wage (in tens of thousands) of 8. What level of utility* u^* *do workers like coal miners achieve in the economy?*

Answer: Plugging these values into the utility function, we get approximately $u^* = 6.238$.

(c) *Suppose that school teachers — who face safety of 10 — could equally well have chosen to become coal miners. What is their wage? How much of the coal miners' salary is therefore equilibrium compensation for the risk they face?*

Answer: Using equation (20.5.i) and plugging in $u^* = 6.238$, $s = 10$ and $\alpha = \rho = 0.5$, we get $w(10) \approx 4.2593$. This implies a compensating wage differential for the risk faced by oil riggers and coal miners equal to $8 - 4.2593 = 3.7407$; i.e. \$37,407 of the coal miners' \$80,000 salary is due to the risk they take on in their jobs.

(d) *Suppose that safety conditions in coal mines improve — to a safety index level of 6. Assuming the coal industry employs a small fraction of workers of this skill type, what will be the new equilibrium wage for coal miners? Are they better or worse off?*

Answer: Since the coal industry employs a small fraction of employees of this kind, the improvement in safety conditions will have no impact on wages elsewhere — and thus no impact on u^*. We can then calculate the new wage for coal miners in the same way we calculated the wage of teachers — except that we substitute $s = 6$ rather than $s = 10$. This gives us a new coal mining wage of approximately 6.4905 — a decrease of 1.5095 (or \$15,095 per year). Still, workers are neither better nor worse off — since their utility remains unchanged.

(e) *Next, construct a table that shows how compensating wage differentials vary with the elasticity of substitution of safety for wage. Let the first column of your table give ρ and let the next 4 columns give u^*, the wage of workers on oil rigs, the wage of workers in coal mines (after the safety improvements have been made) and the wage of teachers — all in tens of thousands of dollars. (Continue to assume $\alpha = 0.5$ and an initial annual wage of 8 in the coal and oil rigging industries (before the safety improvements in coal mining).) Fill in the table for the following values of ρ: -0.99, 0.01, 0.5, 5, 10.*

Answer: This is done in the following table:

Compensating Wage Differentials with $\alpha = 0.5$				
ρ	u^*	oil wage	coal wage	teacher w.
-0.99	6.4983	8.0000	6.9969	3.0163
0.01	6.3228	8.0000	6.6632	4.0062
0.50	6.2380	8.0000	6.4905	4.2593
5.00	5.6398	8.0000	5.3797	4.9382
10.00	5.3540	8.0000	5.0834	4.9960

(f) *Interpret the results in your table.*

Answer: When ρ is close to -1, safety and wage are nearly perfect substitutes. Thus, the substantially safer environment of the teacher results in a substantially lower wage, while the marginally safer environment for the coal miner (relative to the oil rigger) results in less of a lower wage (relative to the oil rigger). In fact, in the first row of the table, you can see that a 1 unit increase in safety pretty much is offset by a 1 unit reduction in salary, and a 5 unit increase in safety (for the teacher relative to the oil rigger) results in almost a 5 unit reduction in salary. As the elasticity of substitution between safety and wage increases (as we go down the table), the wage of the coal miner falls while the wage of the teacher increases. This is because workers are now willing to pay relatively more for some initial increase in safety, but as the job gets safer and safer, each additional unit of safety is worth less and less.

Compensating Wage Differentials with $\rho = 0.5$				
α	u^*	oil wage	coal wage	teacher w.
0.10	7.5924	8.0000	7.8076	7.3799
0.25	7.0370	8.0000	7.4431	6.3379
0.50	6.2380	8.0000	6.4905	4.2593
0.75	5.5678	8.0000	4.5183	1.7944
0.90	5.2162	8.0000	2.0162	0.4258

(g) *How do you think each row of the table would change if α is lowered or increased? Check you intuition in a table identical to the previous table except that you now fix ρ at 0.5 and let α take on the following values: 0.1, 0.25, 0.5, 0.75, 0.9.*

Answer: As α increases, workers place more weight on safety and less weight on the wage for any level of ρ. Thus, as α increases, we would expect workers to give up more wage to get an increase in safety — leading to larger reductions in wage for coal workers and teachers relative to oil riggers. This shows up clearly in the table.

Exercise 20.7

Business Application: *The Risks of Short Selling: In the text, we mentioned that short-selling can entail a lot more risk if the investor's guesses are wildly incorrect than taking the more conventional long position of buying and holding and asset.*

A: *Suppose oil currently sells for \$50 a barrel. Consider two different investors: Larry thinks that oil prices will rise, and Darryl thinks they will fall. As a result, Larry will take a long position in the oil market while Darryl will take a short position. Both of them have enough credit to borrow \$10,000 in cash or an equivalent amount (at current prices) in oil. (For purposes of this exercise, do not worry about any opportunity costs associated with the interest rate — i.e. simply assume an interest rate of 0 — and suppose oil can be stored without cost.)*

(a) *Consider Larry first. How much will he have one year from now if he carries through with his strategy of investing all his money in oil and oil one year from now stands at \$75 a barrel.*

Answer: Larry "goes long" — which means he invests the $10,000 he borrows in oil at $50 a barrel — and thus buys 200 barrels. One year from now when oil sells for $75 a barrel, he can then sell his 200 barrels for $15,000, pay back the $10,000 loan and be left with a cool $5,000.

(b) *Now consider the worst case scenario: A new energy source is found and oil is no longer worth anything 1 year from now. Larry's guess about the future was wildly incorrect. How much has he lost?*

Answer: Larry has lost all of the $10,000 he borrowed — and thus owes $10,000.

(c) *Next, consider Darryl. How much will he have 1 year from now (if he carries through with his strategy to sell oil short) if the price of oil one year from now stands at $25 a barrel.*

Answer: Selling oil short means that he will borrow oil, sell it now at the current price of $50 a barrel and then pay back the oil 1 year from now when it sells for $25 a barrel. He can borrow $10,000 worth of oil — which is 200 barrels at the current price. He turns around and sells the 200 barrels immediately — taking in $10,000. Then, when the price of oil falls to $25 a barrel, he buys 200 barrels of oil for $5,000 and pays back the 200 barrels. He thus has a cool $5,000 left.

(d) *Suppose instead that Darryl's prediction about the future was wildly incorrect and the price of oil stands at $100 a barrel next year. How much will he have lost if he leaves the oil market at that point?*

Answer: He will have borrowed 200 barrels and sold them for $10,000 immediately. When he has to pay back the oil 1 year from now, he has to spend $20,000 to buy 200 barrels at the price of $100 a barrel. Thus, he has lost $10,000.

(e) *Was the scenario in (d) the worst-case scenario for Darryl? Is there a limit to how much Darryl might lose by "going short"? Is there a limit to the losses that Larry might incur?*

Answer: No, that was not the worst-case scenario for Darryl — things would be even worse if the price of oil went even higher than $100. Suppose, for instance, it went to $200 a barrel. Then it will cost Darryl $40,000 to replace the 200 barrels he borrowed and sold for $10,000 — leaving him with a loss of $30,000. But of course things could still be worse if the price of oil jumps even more. In fact, there is no worst case scenario here — not until we get to an infinite price for oil and an infinite loss for Darryl. For Larry, however, we did identify the worst-case scenario in (b) — oil can't possibly sink lower in price than $0 a barrel. Thus, the most Larry can lose by "going long" is to lose his investment of $10,000.

(f) *Can you explain intuitively — without referring to this example — why short-selling entails inherently more risk for investors who are very wrong in their predictions than going long in the market does?*

Answer: Short selling entails more risk because short-sellers bet on the price of the commodity going *down*. If they are wrong and the price goes *up*, there is no limit to how high in principle it could go — and thus no limit on how big the losses might be if the investor's guess is wrong. Taking a long position means buying something now and planning to sell it later — and the worst that can happen is that what we bought now won't be worth anything in the future. We are betting that the price will go up, and the worst that can happen is that it goes to zero. So, there is a definite worst-case scenario — the scenario under which what I bought is not worth anything and thus I lose all my investment (but no more than that).

B: *Suppose more generally that a barrel of oil sells at price p_0 on the current "spot market" — which is defined as the market for oil that is currently being sold. Suppose further that you expect the price of a barrel of oil on the spot market n years from now to be p_n. Suppose the annual interest rate is r.*

(a) *Can you write down an equation $\pi_n^L(p_0, p_n, r, q)$ that gives the profit (expressed in current dollars) from going long in the oil market for n years by buying q barrels of oil today?*

Answer: This strategy would involve spending $p_0 q$ now in exchange for getting $p_n q$ in n years. The present discounted value of the latter is $p_n q/(1+r)^n$ — so we can write the profit as

$$\pi_n^L(p_0, p_n, r, q) = \frac{p_n q}{(1+r)^n} - p_0 q = \frac{p_n q - (1+r)^n p_0 q}{(1+r)^n}. \tag{20.7.i}$$

(b) *How high does the ratio p_n/p_0 have to be in order to justify going long in the oil market in this way? Can you make intuitive sense of this?*

Answer: Setting profit equal to zero in equation (20.7.i), we get

$$0 = \frac{p_n q - (1+r)^n p_0 q}{(1+r)^n}. \tag{20.7.ii}$$

Solving for p_n/p_0, we get $p_n/p_0 = (1+r)^n$ — i.e. as long as oil prices n years from now are $(1+r)^n$ higher than they are now, the strategy makes a positive profit because it justifies the opportunity cost of tying up the funds in oil (rather than in some other investment that returns r per year).

(c) *Next, can you write down the equation for $\pi_n^S(p_0, p_n, r, q)$ — the profit from selling q barrels of oil short by borrowing them now and repaying them in n years? (Assume that the person you are borrowing the oil from expects you to return $(1+r)^n$ times as much oil — i.e. he is charging the interest to be paid in terms of barrels of oil.)*

Answer: Under this strategy, you collect $p_0 q$ in revenue immediately as you sell the oil that you borrowed on the spot market. Then, in n years, you have to buy $(1+r)^n q$ barrels of oil at the spot price that year — i.e. at price p_n. Thus, in n years, you incur a cost of $(1+r)^n p_n q$ — but the present discounted value of that cost is just $p_n q$. The profit from short selling can then be stated as

$$\pi_n^S(p_0, p_n, r, q) = p_0 q - p_n q. \tag{20.7.iii}$$

(d) *How high can p_n/p_0 be to still warrant a short selling strategy of this type? Can you make intuitive sense of this?*

Answer: Setting $\pi_n^S(p_0, p_n, r, q) = p_0 q - p_n q$ to zero and solving for p_n/p_0, we get $p_n/p_0 = 1$. Thus, as long as the spot price n years from now is no greater than the current spot price, this short selling strategy can yield a profit. This makes sense for the following reason: We are paying back the oil with interest (paid in oil) — but we also get our revenues immediately and don't incur our costs for n years. Thus, the revenues we make can be invested and earn interest over n years — just as our obligation in terms of paying back oil accrues interest over that time. If the spot price now and n years from now is the same, we therefore come out even — but if the spot price falls from now to year n, we get to "buy low" in n years and "sell high" now.

Exercise 20.9

Business and Policy Application: *General Equilibrium Effects of a Property Tax: In Chapter 19, we introduced the idea that the property tax is really composed of two taxes: a tax on land, and a tax on improvements of land which we can think of as capital invested in housing.*

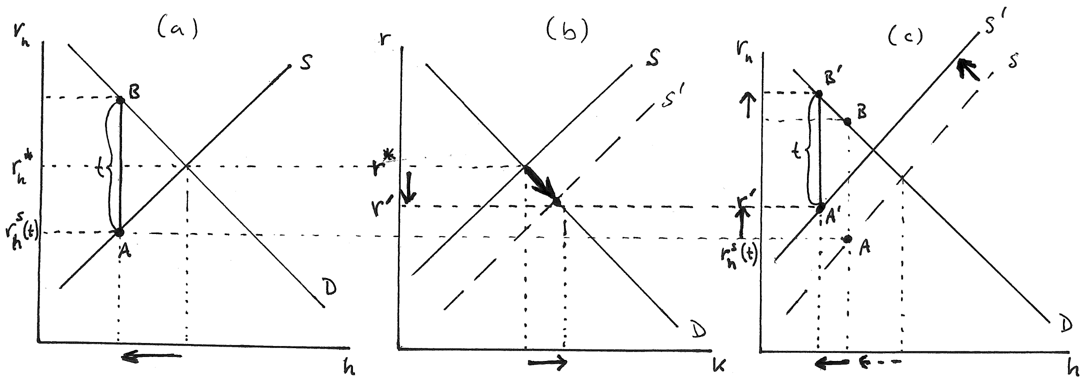
A: *For purposes of this problem, we focus only on the part of the property tax that is effectively a tax on housing capital. Assume, unless otherwise stated, that capital can move freely between housing and other uses.*

(a) *Begin by drawing a graph with housing capital h on the horizontal axis and the rental rate of housing capital r_h on the vertical. Draw demand and supply curves that intersect at r_h^* and illustrate the impact of the property tax t on the rental rate $r_h^s(t)$ earned by suppliers of capital when considering this market in isolation.*

Answer: This is illustrated in panel (a) of Exercise Graph 20.9 where, as we discovered in Chapter 19, it does not matter whether we view this as a tax imposed on demanders or suppliers. The tax ends up lowering the rental rate received by suppliers to $r_h^s(t)$ read off A in the graph — with the rental rate paid by demanders of housing capital t higher (read off point B).

(b) *Next to your graph from part (a), illustrate the demand and supply curves for non-housing capital prior to the imposition of the property tax on the housing market. Where must the equilibrium rental rate r^* be in relation to the pre-tax equilibrium housing capital rental rate r_h^*? Given that capital is mobile between the two sectors, can the after-tax "partial" equilibrium you identified for the housing market in (a) be the "general" equilibrium for the housing market once we take into account the mobility of capital across sectors?*

Answer: This is illustrated in panel (b) of Exercise Graph 20.9 where demand and supply intersect at the rental rate r^* that must be equal to r_h^*. This is because we are assuming that the capital market was in equilibrium prior to the imposition of the property tax — and the mobility



Exercise Graph 20.9 : Partial and General Equilibrium Incidence of the Property Tax

of capital across sector implies that such an equilibrium must equalize rental rates across the two sectors. When the property tax is then imposed, the rental rate received by suppliers of capital in the housing market falls in our partial equilibrium analysis of panel (a) — which implies we are no longer in general equilibrium across the two sectors because investors are now earning a lower return on their capital in the housing sector than they are in the non-housing sector.

(c) *What does your answer to (b) imply for what will happen to the supply curve for capital in the housing and non-housing sectors?*

Answer: Given that the rental rate for suppliers of capital is now lower in the housing sector than it is in the non-housing sector, investors will reduce supply in the housing sector and increase it in the non-housing sector; i.e. capital will flow from the housing sector to the non-housing sector until the rates of return for investors are equalized.

(d) *Illustrate the new general equilibrium that takes into account the movement of capital across sectors in response to the property tax. What happens to the rental rate of capital in the non-housing sector?*

Answer: This is illustrated in panels (b) and (c) of Exercise Graph 20.9. In panel (c) we begin by replicating panel (a) except for the fact that we show the original supply curve as a dashed line. We then illustrate the inward shift in supply in the housing sector in panel (c) — and the outward shift of supply in the non-husing sector of panel (b). These shifts will occur until the rates of return to investors are equalized across the sectors at the rental rate r' that falls in between the original equilibrium rental rate r^* and the partial equilibrium rental rate $r_h^s(t)$ that results in the housing sector from the imposition of the property tax. The rental rate in the non-housing sector therefore falls.

(e) *In what sense is a portion of the property tax burden shifted to non-housing capital?*

Answer: It is shifted in the sense that investors in the non-housing sector will now make lower rates of return on their capital than they did prior to the introduction of the property tax in the housing sector.

(f) *Are renters of housing capital better or worse off as a result of the general equilibrium shifting of some portion of the tax burden across sectors? Will they consume more or less housing compared to the initial partial equilibrium prediction?*

Answer: Renters — or demanders — of housing capital experience an increase in their rental rate beyond the partial equilibrium increase identified in panel (a) of the graph. This is illustrated in panel (c), with consumers ending up moving from B to B' as a result of the additional general equilibrium effect. The intuition for this is that, since investors will get a general equilibrium boost in their rate of return from the movement of capital, consumers must also receive a general equilibrium boost in the rental rate they pay — because in the end, the difference between the demander and supplier rental rates must be equal to t in general equilibrium as well.

(g) True or False*: The property tax will result in smaller houses and more investment in business machinery — but, if we do not take the general equilibrium effect of the tax into account, we will underestimate both how much smaller the houses will be and how many more business machines there will be.*

Answer: This is true. We see that the housing rental rate increases for consumers of housing — both in partial and general equilibrium — as a result of the property tax. But the rental rate for consumers of housing rises more once we take the general equilibrium effect of capital mobility into account — which means we will underestimate how much consumers of housing will conserve on their housing consumption. With respect to business machines, these represent one use of capital in the non-housing sector. If we do not take into account general equilibrium effects, the rate of return for capital in the non-housing sector does not change — which would lead us to predict no change in capital usage in the non-housing sector — and thus no change in the number of business machines. But, once we consider the general equilibrium effect that causes a decrease in the rental rate of capital, we see an increase in business investment outside the housing sector — thus more business machines. (Put differently, the cost of renting business machines falls as the rental rate of capital falls — which means firms will proceed down their marginal product of capital curves to increase the number of machines.)

B: *Suppose that demand and supply for capital are identical in the housing and non-housing sector — taking the form* $k_d(r) = (A - r)/\alpha$ *and* $k_s(r) = (B + r)/\beta$ *(as in the example of part B of the text). In this example, let* $A = 1$, $B = 0$, $\alpha = 0.00000015$ *and* $\beta = 0.00000001$.

(a) *Begin by determining the equilibrium rental rates* r^* *and* r_h^* *for non-housing capital and housing capital — and think of these as interest rates. How much capital is being transacted in each sector?*

Answer: Setting demand equal to supply and solving for r, we get $r^* = r_h^* = 0.0625$ — i.e. an interest rate of 6.25%.

(b) *Next, suppose that a tax of* $t = 0.04$ *is imposed through the property tax in the housing sector. If you assumed that there was no connection of the housing sector to any other sector of the economy, what would happen to the interest rate* r_h^s *received by suppliers of housing capital and the interest rate* r_h^d *paid by demanders of housing capital.*

Answer: A tax of t will cause a difference of t to emerge between the rental rate r_h^d paid by demanders and the rental rate r_h^s made by suppliers of housing capital. Setting demand equal to supply — while substituting r_h^d into the demand equation and $r_h^s = (r_h^d - t)$ into the supply equation, we get

$$\frac{A - r_h^d}{\alpha} = \frac{B + \left(r_h^d - t\right)}{\beta}. \tag{20.9.i}$$

Solving this for r_h^d and then plugging the solution into $r_h^s = (r_h^d - t)$, we get

$$r_h^d = \frac{\beta A - \alpha B + \alpha t}{(\alpha + \beta)} \text{ and } r_h^s = (r_h^d - t) = \frac{\beta A - \alpha B - \beta t}{(\alpha + \beta)}. \tag{20.9.ii}$$

Plugging in the values for A, B, α and β, we get

$$r_h^d = 0.10 \text{ and } r_h^s = 0.06. \tag{20.9.iii}$$

Suppliers of capital to the housing sector therefore see a drop in their rental rate from 0.0625 to 0.06 — i.e. a drop in the effective interest rate from 6.25% to 6%.

(c) *Next, suppose that capital is freely mobile across the two sectors. How much capital will flow out of the housing sector? (*Hint*: You can treat this just like any other problem involving trade between two sectors where the starting prices are not equal to one another. The flow of capital is then just defined exactly like* X^* *derived in the text. To apply this formula, you need to re-define the demand (or supply) curve in the housing sector to include the tax* $t = 0.04$ *— which simply shifts* A *down (or* B *up) by 0.04.)*

Answer: After the partial equilibrium effect of the property tax on the housing market is taken into account, we now have two markets for capital — with the rental rate in one lower than in the other. We would therefore expect capital to be "exported" from the market where the rental rate is low to the market where the rental rate is high — i.e. from the housing market to the non-housing capital market. In the text, we derived the equation for exports X^* as

$$X^* = \frac{(\alpha+\beta)(\delta C-\gamma D)-(\gamma+\delta)(\beta A-\alpha B)}{(\gamma+\delta)\alpha\beta+(\alpha+\beta)\gamma\delta} \tag{20.9.iv}$$

when the exporting region has demand and supply functions

$$x_d^1(p) = \frac{A-p}{\alpha} \text{ and } x_s^1(p) = \frac{B+p}{\beta} \tag{20.9.v}$$

and the importing region has demand and supply functions

$$x_d^2(p) = \frac{C-p}{\gamma} \text{ and } x_s^2(p) = \frac{D+p}{\delta}. \tag{20.9.vi}$$

The importing "region" in our example is the non-housing capital market where demand and supply curves are unchanged — i.e. $C = 1$, $D = 0$, $\gamma = 0.00000015$ and $\delta = 0.00000001$. Our exporting "region" is the housing market where the property tax has lowered the rate of return for capital investors. This can be captured by shifting the demand in the housing sector down by 0.04 — i.e. letting $A = 0.96$ while leaving $B = 0$, $\alpha = 0.00000015$ and $\beta = 0.00000001$. Substituting all these into our equation (20.9.iv), we get the level of exports to be

$$X^* = 133,333.33; \tag{20.9.vii}$$

i.e. 133,333.33 units of capital shift from the housing to the non-housing capital market.

(d) *What happens to the new equilibrium interest rate that suppliers of capital can get in the economy? In what sense has a portion of the property tax been shifted to all forms of capital?*

Answer: In the text, we derived the supplier "price" in the exporting "region" as

$$\tilde{p}^1 = \frac{\beta A-\alpha B+\alpha\beta X}{\alpha+\beta} \tag{20.9.viii}$$

when exports are X. Substituting $A = 0.96$, $B = 0$, $\alpha = 0.00000015$, $\beta = 0.00000001$ and $X = 133,333.33$, this gives us a new rental rate

$$r' = 0.06125 \tag{20.9.ix}$$

for suppliers of capital in the housing sector. The text also derives the "price" in the importing "region" as

$$\tilde{p}^2 = \frac{\delta C-\gamma D-\gamma\delta X}{\gamma+\delta} \tag{20.9.x}$$

when imports are X. Substituting $C = 1$, $D = 0$, $\gamma = 0.00000015$, $\delta = 0.00000001$ and $X = 133,333.33$, this again gives us $r' = 0.06125$ — i.e.

the movement of 133,333.33 units of capital from the housing to the non-housing sector equalizes the rental rates received by suppliers of capital in the two sectors. This implies that the interest rate will fall from 6.25% to 6.125% in the non-housing sector — and the interest rate received by suppliers of housing capital will rise (relative to the partial equilibrium level) from 6% to 6.125% as capital moves out of the housing sector and into the non-housing sector.

(e) *What happens to the rental rate of capital paid by consumers in the housing sector?*

Answer: In the housing sector, the rental rate paid by consumers must be $t = 0.04$ higher than the rental rate received by suppliers. We just calculated that the latter will receive a rental rate of 0.06125 — which means consumers will pay a rental rate of 0.10125 — up from 0.10 predicted from the partial equilibrium analysis.

(f) *Describe the general equilibrium economic incidence of the tax.*

Answer: In general equilibrium, all owners of capital incur the same incidence — regardless of which sector their capital is employed in, with their rate of return on capital falling from 6.25% to 6.125%. Consumers of housing incur (in this model) a higher incidence — their rental rate goes from 6.25% to 10.0125%.

Exercise 20.11

Policy Application: *U.S. Immigration Policy*: *U.S. immigration law is based on a quota system — i.e. a system under which there is a maximum number of immigrants allowed for each country, with different quotas set for different countries. In this exercise, we consider an alternative way of achieving the same level of immigration from each country. To make the exercise tractable, assume that all workers around the world are identical.*

A: *Assume throughout that the primary motive for migration is a search for higher wages.*

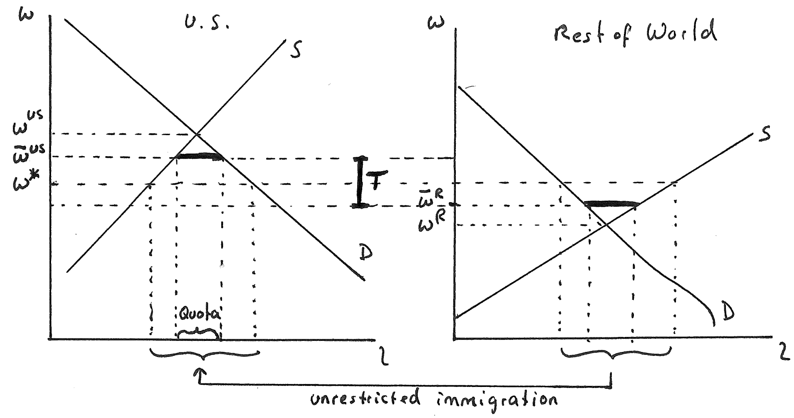
(a) *Begin by drawing the U.S. supply and demand curves for workers and, next to it, the supply and demand curves in the rest of the world. Assume that the equilibrium wage (in the absence of trade or migration) is higher in the U.S.*

Answer: This is illustrated in Exercise Graph 20.11 where w^{US} is the U.S. wage in the absence of immigration and w^R is the wage in the rest of the world in the absence of immigration.

(b) *Illustrate the equilibrium in which there are no restrictions to migration — assuming migration is relatively costless.*

Answer: The equalized wage w^* is the wage everywhere in the case of unrestricted migration of labor.

(c) *Now suppose the U.S. introduces an immigration quota that allows less migration than would naturally occur in the absence of restrictions. Illustrate the impact of such a quota on the labor markets in the U.S. and in the rest of the world.*



Exercise Graph 20.11 : Immigration Quotas and Taxes

Answer: The quota is illustrated as the horizontal bold line in both panels of the graph. The result is that the U.S. wage goes to $\overline{w}^{US}$ and the wage in the rest of the world goes to $\overline{w}^{R}$.

(d) *Suppose that the U.S. had not imposed the immigration quota but instead rations access to the U.S. from rest of the world by charging an immigration tax of T per worker. Illustrate how large T would have to be to result in the same level if immigration from the rest of the world.*

Answer: This is illustrated as the vertical bold distance between the two graphs in Exercise Graph 20.11. This has the same effect as the quota because both create a difference of the same amount between the wage in the US and the wage in the rest of the world.

(e) True or False*: Within the context of this example, country-specific immigration quotas are equivalent to country-specific immigration taxes.*

Answer: This is true. We showed this assuming the rest of the world can be lumped into a single labor market (which would be true if there were no migration restrictions between countries in the rest of the world). If there are many countries with their own different wage levels partially protected through immigration laws, then the immigration tax would have to differ for each country depending on what quota the U.S. immigration system assigns to that country. (Furthermore, if the quota system has different quotas for skill levels, then the taxes would vary by both country and skill level of the worker.)

B: *Now consider labor demand and supply functions $l_d^1(w) = (A - w)/\alpha$ and $l_s^1(w) = (B + w)/\beta$ for the rest of the world and $l_d^2(w) = (C - w)/\gamma$ and $l_s^2(w) = (D + w)/\delta$ for the U.S.*

(a) *Let $A = C = 100,000$, $B = -1,000$, $D = 0$, $\alpha = 0.002$, and $\beta = \gamma = \delta = 0.001$. What would be the equilibrium wage in the U.S. and in the rest of the world if they were isolated from one another?*

Answer: Setting demand equal to supply in each country and solving for w, we get

$$w^1 = 34,000 \text{ and } w^2 = 50,000. \tag{20.11.i}$$

(b) *What would be the equilibrium wage if labor was fully and costlessly mobile? How high would immigration to the U.S. be?*

Answer: In the text we derived X^* — the level of "exports" from the low priced country to the high priced country for an example with identical functional forms for demand and supply curves (except that now we have labor as the "good" that is being exported through migration). Using this formula, we get that the level of immigration to country 2 from country 1 is 13,714,286. In the text we also derived the equilibrium "prices" in the two countries for any export level X. Plugging in this "export" level of labor, we get a worldwide equilibrium wage of

$$w^* = 43,143. \tag{20.11.ii}$$

(c) *Suppose the U.S. governments sets a 1,000,000 quota for immigration from the rest of the world. How will the equilibrium wage in the U.S. and the rest of the world be affected by this?*

Answer: Using 1,000,000 rather than 13,714,286 as our level of "exports" in the equations for equilibrium "prices" in the two countries, we get

$$\overline{w}^1 = 34,667 \text{ and } \overline{w}^2 = 49,500. \tag{20.11.iii}$$

(d) *How high would the U.S. have to set an immigration tax in order to achieve the same outcome?*

Answer: The (annual) immigration tax would have to be set at the difference between the two wage levels under the quota — i.e. $T = 49,500 - 34,667 = 14,833$. You can check that this is correct by plugging this into our equation for $X^*(t)$ in the text — the equation for how much in "exports" there would be under tariff t. Setting $t = 14,833$ results in "export" (or immigration) level of 1,000,000.

Conclusion: Potentially Helpful Reminders

1. In this chapter, we continue the practice of not showing shifts in curves but rather showing the impact of the underlying shift within pictures that simply show the original demand and supply curves. The reason we can do that here — and use the original curves to show consumer and producer surpluses — is

that the underlying shifts in curves occur solely because of the activities of exporters and importers (or speculators): The consumer and producer curves in fact remain unchanged.

2. In exporting regions, this implies that we can simply slide along the supply curve (as exporters are shifting demand even as consumer demand in the region remains unchanged.) In importing regions, this implies that we can slide along demand curves (as importers shift supply while the supply curves of local producers remain unchanged.)

3. Remember that we usually do not have to keep track of profits or surplus for exporters, importers or speculators as long as they operate in a competitive market where their profits are driven to zero.

4. Taxes are always shifted disproportionately to parties whose behavior is less price elastic. We showed that in Chapter 19 for taxes within a given market, and the same intuition applies to tariffs that tax transfers of goods across markets. It is for this reason that a portion of the burden of the tariff can be "exported".

5. Once you understand the pictures that describe trade in goods markets, it is easy to make the jump to "trade in labor markets". Remember that such "trade" can happen because goods are manufactured abroad where labor is cheaper, with labor itself not moving. (We call this "outsourcing".) It can also happen by labor moving from low wage regions to high wage regions — with the impact on labor markets similar to those of outsourcing except that labor now literally moves.

6. Similarly, the same types of graphs allow us to describe "trade across time", with speculators buying low and selling high. To the extent that speculators are accurately predicting future prices, this becomes identical to buying low and selling high across regions. It is the fact that speculators might make mistakes that makes the situation different — something that is explored in a number of the end-of-chapter exercises.

CHAPTER

21

Externalities in Competitive Markets

This chapter moves beyond the consideration of policy-induced distortions of market prices to externality-induced distortions of such prices. It illustrates that markets will over- or under-produce (relative to the efficient level of production) in the presence of negative or positive externalities on either the consumption or the production side. Along the way, we illustrate the potential role for Pigouvian taxes or subsidies, for markets in pollution vouchers (or "cap-and-trade" policies), and for civil society organizations. We furthermore explore how how, at a fundamental level, the "market failure" in the presence of externality is a "failure of markets to exist" — and how thinking of externalities through this lens can motivate policies that introduce property rights-based approaches to solving inefficiencies. Finally, the chapter concludes with a discussion of the Coase Theorem — in part building on the insight that property rights play an important role in the emergence of externality problems, and in part adding to that insight by illustrating how "small" externalities differ from "large" externalities in the role that transaction costs play in allowing individuals to bargain their way out of inefficiency.

Chapter Highlights

The main points of the chapter are:

1. Relative to efficient output levels, competitive markets will over-produce when subject to **negative exernalities** and under-produce when subject to **positive externalities**. One possible form of non-market intervention to correct these "market failures" lies in **Pigovian taxes and subsidies** set at rates that cause markets to produce at the efficient output levels.

2. An alternative policy in the realm of pollution externalities lies in the creation of **cap-and-trade** or **pollution voucher** systems. While these do not guarantee an efficient level of pollution, they put incentives in place to insure that pollution reductions are undertaken in the least costly (and thus most efficient) way.

3. Civil society — i.e. non-market and non-government — organizations often play a role in abating the effect of externalities. The **free rider problem** faced by such organizations is sometimes partially addressed through government subsidies — either directly to particular organizations or in the form of tax deductibility (of, for instance, **charitable giving**).

4. At its core, the market failure that arises in the presence of externalities arises because of **the failure of some market(s), and property rights, to exist**. The non-existence of property rights often gives rise to a **Tragedy of the Commons**.

5. Not all externalities require government intervention. In particular, the **Coase Theorem** suggests that, in the presence of well-established property rights and low transactions costs, individuals will solve small externality problems on their own.

21A Solutions to Within-Chapter-Exercises for Part A

Exercise 21A.1

Suppose that the "pollution" emitted in the production of hero cards is of a kind that has no harmful effects for humans but does have the benefit of killing the local mosquito population — i.e. suppose the pollution is good rather than bad. Would the market produce more or less than Barney?

Answer: In this case, Barney would produce more than the market — i.e. the market would under-produce. This is because — in addition the marginal benefit captured in the marginal willingness to pay curves of consumers, there is now a benefit that consumers do not take into account. Thus, the *social marginal benefit* curve is higher than the demand curve (which we have assumed is equal to the marginal willingness to pay curve for consumers). As a result, the social marginal benefit curve intersects the supply curve at a higher quantity than the demand curve does.

Exercise 21A.2

Would anything fundamental change in our analysis if we let go of our implicit assumption that the aggregate demand curve is also equal to the marginal willingness to pay curve? (Your answer should be no. Can you explain why?)

Answer: Nothing fundamental changes in the sense that the (uncompensated) market curves still give rise to the same prediction of what the equilibrium will be, and this will involve over-production. The aggregate marginal willingness to pay curve would cross the supply curve at the equilibrium price, giving rise once again to a deadweight loss triangle. This triangle will be larger or smaller than the one depicted on the uncompensated curves, but it is there nonetheless. And, the optimal Pigouvian tax will again involve the marginal social damage at the desired optimum production level.

Exercise 21A.3

What if the government only knows the marginal social damage of pollution at the equilibrium output level x^M and sets the tax rate equal to this quantity? Will this result in the optimal quantity being produced? If not, how do the SMC and the supply curve have to be related to one another in order for this method of setting the tax to work?

Answer: The graph in the text is drawn with SMC diverging from S as quantity increases. If the government uses the social marginal damage observed at the competitive equilibrium, and if it then sets the tax rate equal to that marginal externality cost, it will choose a tax rate higher than the one illustrated — and will thus tax too much, causing output to fall below its socially optimal level. In order for the government to be able to use the marginal externality cost of pollution at the competitive equilibrium to efficiently implement a Pigovian tax, it must be the case that the marginal externality cost does not increase or decrease with output — which would imply that the SMC curve is parallel to the S curve.

Exercise 21A.4

In Chapter 18, we discussed the efficiency losses from government mandated price ceilings or price floors. Could either of these policies be efficiency enhancing in the presence of pollution externalities (assuming the government has sufficient information to implement these policies)?

Answer: Yes, both a price ceiling and a price floor would reduce output — and, if appropriately set, could get output to converge to the socially optimal level.

Exercise 21A.5

Explain how firms face a cost for pollution regardless of whether the government gives them tradable pollution vouchers or whether firms have to purchase these.

Answer: If the government gives away the tradable pollution voucher to a firm, the firm's opportunity cost of using the voucher is equal to the price it could have gotten for the voucher in the pollution market. If the firm has to purchase a pollution voucher, the opportunity cost of using the voucher is once again the price that has to be paid for it.

Exercise 21A.6

If the government, after creating the pollution voucher market, decides to tax the sale of pollution vouchers, will there be any further reduction in pollution? (*Hint*: The answer is no.)

Answer: If the government taxes the sale of pollution vouchers, it will simply lower the price of pollution vouchers by the amount of the tax (since the number of pollution vouchers is fixed — implying a perfectly inelastic supply curve for vouchers). The overall number of vouchers will, however, remain the same — meaning the amount of pollution remains the same as pollution vouchers continue to be traded at the same price.

Exercise 21A.7

In one of the 2008 Presidential Primary debates, one candidate advocated the cap-and-trade system over a carbon tax on the grounds that the carbon tax would be partially passed onto consumers in the form of higher prices. Another candidate who also supported the cap-and-trade system corrected this assertion — suggesting that, to whatever extent a carbon tax would be passed onto consumers, the same is true costs (of tradable permits) under the cap-and trade system. Who was right?

Answer: The second candidate was right. Both systems impose a cost on polluters — one by taxing carbon emissions directly, the other by forcing polluters to rent tradable permits for each unit of pollution emitted. The incidence of these costs — i.e. the extent to which they are passed to consumers — then depends entirely on the relative magnitudes of price elasticities of demand and supply in different industries.

Exercise 21A.8

Suppose that advocates of pollution taxes proposed a reduction in such taxes for key industries that would otherwise be opposed to the policy. How is this different than giving pollution vouchers away for free to such key industries in a cap-and-trade system?

Answer: It is no different. This is, in fact, one of the political arguments in favor of cap-and-trade systems as opposed to pollution tax systems. In the cap-and-trade system, it is politically easier to simply give vouchers to some industries in order to get their political support — but it seems politically more difficult to exempt industries from broad-based pollution taxes.

Exercise 21A.9

Less-developed countries often point out that countries like the United States did not have to confront the fact that they caused a great deal of pollution during their periods of development, and thus suggest that developed countries should disproportionately incur the cost of reducing worldwide pollution now. Can you suggest a way for this to be incorporated into a global cap-and-trade system?

Answer: Such a system could simply allocate disproportionately more pollution rights to less-developed countries, with richer countries having the option of "putting their money where their mouth is" and purchasing some of the extra pollution right allocated to those countries.

Exercise 21A.10

Suppose that, instead of generating positive consumption externalities, hero cards actually divert the attention of children from studying and thus impose negative consumption externalities. Can you see how such externalities can be modeled exactly like negative production externalities?

Answer: In essence, hero cards would cause a particular kind of pollution — thus imposing an externality cost in addition to the costs that producers incur in production. Thus, the social marginal cost would exceed the supply curve — implying that supply and demand intersect at a quantity that lies above the socially optimal quantity.

Exercise 21A.11

In what sense does the tax-deducibility of charitable contributions represent another way of subsidizing charities?

Answer: This provision of the income tax code lowers the price for giving money to charities. For instance, if someone is paying a marginal income tax rate of 30 percent, she can give $1 to charity but only lose 70 cents in consumption — because she does not have to pay the 30 percent tax on the $1 she would have had to pay had she not given the dollar to the charity. This represents a subsidy to voluntary giving.

Exercise 21A.12

In a progressive income tax system (with marginal tax rates increasing as income rises), are charities valued by high income people implicitly favored over charities valued by low income people? Would the same be true if everyone could take a tax *credit* equal to some fraction k of their charitable contributions?

Answer: Yes — because the implicit subsidy to giving is based on your marginal tax rate. For someone in a 50 percent tax bracket, it costs 50 cents to give one dollar to charity, while for someone in the 20 percent tax bracket, it costs 80 cents to give that same dollar to the same charity. If lower income people favor different charities than higher income people, this implies the government policy is favoring the charities that are favored by high income people. The same would not be true of the government subsidy of charities through the tax code came in the form of a tax credit that allowed everyone to take a credit for a fraction of their charitable contributions. This is because tax credits are directly deducted from the taxpayer's tax obligation — and thus not dependent on the taxpayer's marginal tax rate. (For this conclusion to be fully correct, the tax credit would have to be *refundable* — i.e. when someone has a tax obligation of $\$x$ before the tax credit is applied, the person would receive a tax refund if his charitable contributions y were sufficiently large to exceed $\$x$; i.e. if $\$ky > x$ the taxpayer would get a refund equal to $\$(ky - x)$.)

Exercise 21A.13

We did not explicitly discuss a role for civil society institutions in correcting market failures due to negative externalities. Can you think of an example of such efforts in the real world?

Answer: One could think of many such examples. Lots of environmental groups, for instance, spend substantial resources to foster a culture that includes individuals paying attention to the environmental impact of their actions; some such organizations buy environmentally sensitive land and set it aside; others offer ways for individuals to pay for their "carbon footprint" with the intent of then using the payments for counteracting the impact of that footprint.

Exercise 21A.14

Large portions of the world's forests are publicly owned — and not protected from exploitation. Identify the tragedy of the commons — and the externalities associated with it — that this creates.

Answer: Since no one owns these forests, the cost to producers of extracting resources is simply equal to the cost of getting the resources out. Thus, producers are not forced to confront any social costs associated with reducing the size of these forests, nor do they consider whether this is the optimal time to harvest particular trees (or whether it would be more optimal to allow them to grow taller before they are harvested). As a result, the forests are over-harvested from a social — or efficiency — perspective.

Exercise 21A.15

Why do you think there is a problem of over-fishing in the world's oceans?

Answer: Again, there is a tragedy of the commons that emerges from no one owning the oceans — and the fish in them. The problem is no different than the over-harvesting of forests in the previous exercise.

Exercise 21A.16

Can you think of an other costs that we do not think about as we decide to get onto public roads?

Answer: The other obvious cost is that we do not think about the wear and tear on the roads that are caused by us.

Exercise 21A.17

Are there other externality-based reasons to tax gasoline?

Answer: The other obvious reason is pollution. This makes for a more compelling efficiency reason to tax gasoline — because no matter when we drive, we will pollute roughly in proportion to the amount of gasoline we burn in our cars. Such a tax is of course a classic Pigouvian tax.

Exercise 21A.18

Some have argued against using tolls to address the congestion externality on the grounds that wealthier individuals will have no problem paying such tolls while the poor will. Is this a valid argument against the efficiency of using tolls?

Answer: No, it is not a valid efficiency argument — which is not to say it is not a valid argument. Efficiency, however, only considers the overall social surplus — with a situation efficient when social surplus is maximized. The poor may well end up being the ones to drive less under congestion tolls — because their willingness to pay for driving is lower. The differential impact on the poor my bother us on equity grounds, but it does not take away from the efficiency argument for congestion tolls. One possible way to deal with the equity concern is to use a portion of the congestion toll revenue to write checks to poor people — thus making them better off without substantially altering their incentive to conserve on the time spend on roads.

Exercise 21A.19

True or False: While it might not matter for efficiency which way the judge rules, you and I nevertheless care about the outcome of his ruling.

Answer: This is true. If the judge rules in my favor, I either get to build my addition or you end up paying me an amount that makes me at least as well off as the addition would. If the judge rules in your favor, I either do not get to build the addition or I get to build it but have to pay you compensation. I clearly would prefer having the judge rule in my favor, and you would clearly prefer to have the judge rule in your favor. The efficiency argument of the Coase theorem, however, is that it does not matter which way the judge rules *in terms of whether the surplus-maximizing action will be taken.* If it is socially efficient for the addition to be built, it will be built no matter how the judge rules — and if it is socially inefficient for the addition to be built, it will not be built regardless of how the judge rules. But the ruling does have an impact on how the social surplus is divided between us.

Exercise 21A.20

Use the Coase Theorem to explain why the government probably does not need to get involved in the externality that arises when I play my radio sufficiently loud that my neighbors are adversely affected, but it probably does need to get involved in addressing pollution that causes global warming.

Answer: The Coase Theorem suggests that, *so long as property rights are well defined and transaction costs are low*, individuals will arrive at arrangements that implement the socially optimal outcome. In the case of me playing my radio too loud, property rights are either well defined or can quickly be well-defined by a judge — and transaction costs that keep us from getting together to solve our problems are low. In the case of global warming, we have two problems: First, property rights are not well defined — no one owns the air that is being damaged. Even if it became well understood who had the right to pollute and be left alone from pollution, the number of individuals involved is in the billions — and thus transaction costs are too high for us to think that individuals will come to an efficient resolution just because property rights have been established.

Exercise 21A.21

In what sense do you think the relevant property rights in this case are in fact well established?

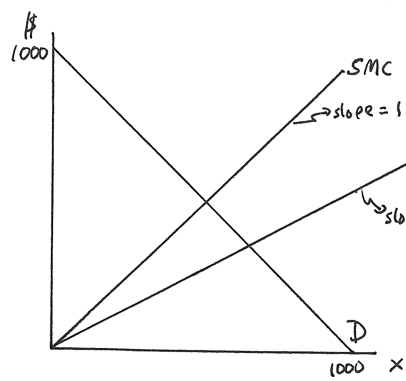
Answer: The relevant property rights here involve the right of bee keepers to not release their bees on orchards — and the right of orchard owners not to have bees released without their consent. These are pretty basic rights — rights that one can reasonably expect courts to enforce.

21B Solutions to Within-Chapter-Exercises for Part B

Exercise 21B.1

Suppose $A = 1{,}000$, $\alpha = 1$, $\beta = 0.5$, $\delta = 0.5$ and $B = 0$. Illustrate the market demand and supply as well as the SMC curves in a graph with x on the horizontal axis.

Answer: The demand *curve* is then simply the demand function solved for p — i.e. $p = 1000 - x$, while the supply *curve* is the supply function solved for p — i.e. $p = 0.5x$. Finally, the SMC function is $SMC = x$. These are illustrated in Exercise Graph 21B.1.



Exercise Graph 21B.1 : Demand, Supply and SMC Curves

Exercise 21B.2

Suppose the "pollution" emitted is actually not harmful and simply kills the mosquito population in the area. The SMC of the pollution might then be negative — i.e. this kind of pollution might actually produce social benefits. Will the efficient quantity now be greater or less than the market quantity? Show this within the context of the example.

Answer: In this case, the marginal "cost" from pollution would result in SMC being less than private marginal cost. Using the example from the text, we get this by letting $C_E(x) = -(\delta x)^2$ (i.e. we make the non-private cost negative) — which would result in a social marginal cost of $SMC = -B + (\beta - 2\delta^2)x$. Setting this equal to the inverse demand function gives us

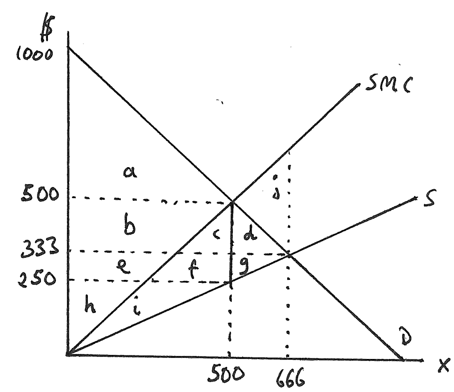
$$x^{opt} = \frac{A+B}{\alpha+\beta-2\delta^2} \qquad (21B.2)$$

which is now less than the market quantity $x^M = (A+B)/(\alpha+\beta)$ (given that the denominator in the former is smaller than the latter.)

Exercise 21B.3

Complete exercise 21B.1 by illustrating and labeling the Pigouvian tax for this example.

Answer: This is done in Exercise Graph 21B.3.



Exercise Graph 21B.3 : Demand, Supply and SMC Curves: Part 2

Exercise 21B.4

Using the graph from the previous exercise, calculate consumer surplus, producer surplus, the externality cost and overall surplus in the absence of the Pigouvian tax. Then calculate these again under the Pigouvian tax, taking into account the tax revenue raised. What is the deadweight loss from not having the Pigouvian tax?

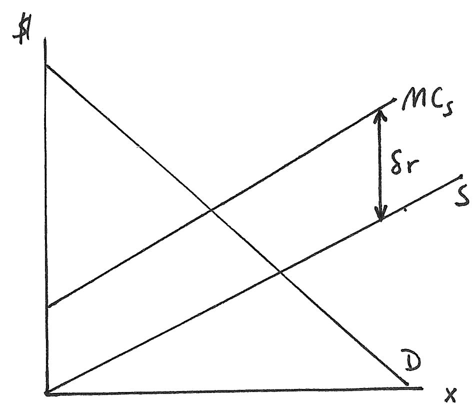
Answer: Calculating the various triangles, we get the results as outlined in the following table. (When calculating the externality cost, it is easiest to simply subtract the area under the supply curve from the area under the SMC curve.) Total surplus is equal to 222,222 without the Pigouvian tax and 250,000 with it, implying a deadweight loss of about 27,778.

Exercise 21B.5

$A = 1000 = C$, $\alpha = 1$, $\beta = \delta = 0.5$, and $B = 0$					
	Cons Surplus	Producer Surplus	Externality	Tax Revenue	Total Surplus
w/o t	$a+b+c+d$	$e+f+g+h+i$	$-(c+d+f+g+i-j)$		$a+b+e+h-j$
	222,222	111,111	-111,111	0	222,222
with t	a	$h+i$	$-(c+f+i)$	$b+c+e+f$	$a+b+e+h$
	125,000	62,500	-62,500	125,000	250,000

Illustrate how this shifts the supply curve in your graph (where you assume $A = 1,000$, $\alpha = 1$, $\beta = 0.5$, $\delta = 0.5$ and $B = 0$.).

Answer: This is illustrated in Exercise Graph 21B.5.



Exercise Graph 21B.5 : The Impact of the Cost of a Pollution Voucher

Exercise 21B.6

Verify that a voucher price of zero results in the market output according to this demand function.

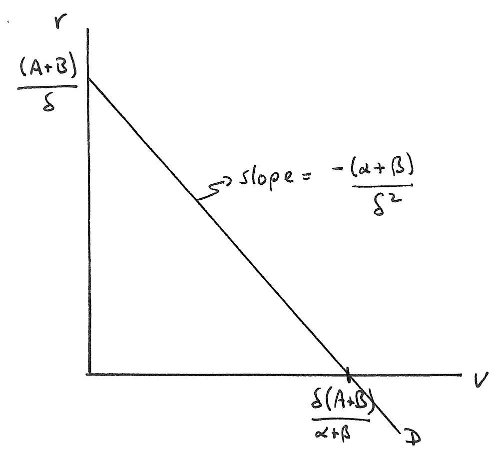
Answer: Setting $r = 0$, we get $v(0) = \delta(A+B)/(\alpha+\beta)$. It takes δ vouchers to produce 1 unit of output — so output will be $(1/\delta)$ times the number of vouchers. We therefore get that, when $r = 0$,

$$x = \frac{1}{\delta}\left(\frac{\delta(A+B)}{\alpha+\beta}\right) = \frac{A+B}{\alpha+\beta} = x^M. \tag{21B.6}$$

Exercise 21B.7

Illustrate the demand curve for pollution vouchers and label its slope and intercept.

Answer: This is done in Exercise Graph 21B.7.



Exercise Graph 21B.7 : Demand Curve for Pollution Vouchers

Exercise 21B.8

What is the relationship between the length of the blue and magenta lines in panels (a) and (b)?

Answer: The lines in panel (a) are δ times as long as the lines in panel (b).

Exercise 21B.9

Implicitly we are assuming $\delta = 2$ in panel (b) of 21.6. How would this graph change if $\delta < 1$ — i.e. if each unit of output produces less than one unit of pollution?

Answer: Then the lines in panel (b) would be longer than those in panel (a).

Exercise 21B.10

If $V = \delta x_2$, which distance in panels (a) or (b) of Graph 21.6 is equal to r^*?

Answer: The magenta distance in panel (b).

Exercise 21B.11

For the case when $A = 1{,}000$, $\alpha = 1$, $\beta = 0.5$, $\delta = 0.5$ and $B = 0$ (as you have assumed in previous exercises), what is the rental rate of the pollution voucher when $V = 250$? What is the price of a pollution voucher if the interest rate is 0.05?

Answer: Plugging the relevant values into our equation for the rental rate, we get $r = 500$. The price of a voucher is the present discounted value of all future rental rates — and at interest rate 0.05, this is $500/(0.05) = \$10,000$.

Exercise 21B.12

Suppose the government simply gives away the pollution vouchers. Why is the deadweight loss the same under tax and cap-and-trade policies that satisfy $t = \delta r^*(V)$ (even though one makes revenue for the government while the other does not)?

Answer: In the case where vouchers are given away, the owners of the voucher are earning the rental rate per voucher. It takes δ vouchers per unit of output — and for the policy to be equivalent to a per unit tax t, it must be that the rental rate on the voucher is $r(t) = t/\delta$. For any output unit produced, the government makes t in revenue under the tax — and the owner of δ vouchers makes $\delta r(t) = \delta(t/\delta) = t$. The surplus that accrues to the government under a tax therefore accrues to those who were given the vouchers under cap-and-trade; in either case, it is surplus, and the surpluses are the same if the two policies are equivalent. (If the government sold the vouchers instead, it would immediately make the present discounted value of all future rents (or all future tax payments).

Exercise 21B.13

Illustrate on a graph where the deadweight loss falls when $t = 400$ in Table 21.1. What about when it falls at $t = 100$?

Answer: This is illustrated in Exercise Graph 21B.13 where area (a) denotes the deadweight loss when the tax is set to $t = 400$ and area (b) denotes the deadweight loss when the tax is set to $t = 100$. The former is set inefficiently high — causing output to drop below the optimum of 500, while the latter is set inefficiently low.

Exercise 21B.14

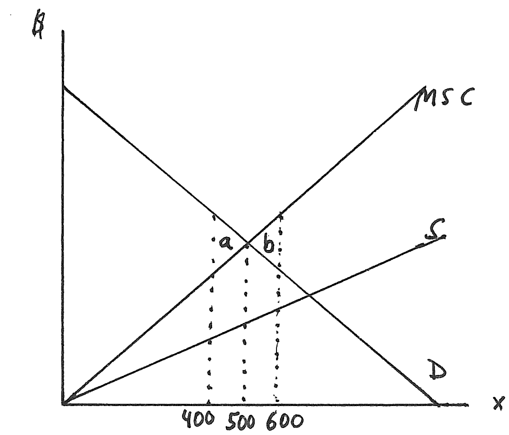
Which two of these four goods represent the consumption levels x_1^1 and x_2^1 that exist for person 1 in an exchange economy without externalities?

Answer: These goods would be $x_1^1 = x_{11}^1$ and $x_2^1 = x_{21}^1$ — i.e. person 1's "impression" of her own consumption of goods 1 and 2.

Exercise 21B.15

If there are two consumers and two goods, how many missing markets are there potentially? More generally, how many missing markets could there be when there are M goods and N consumers?

Answer: In the case of two consumers and two goods, there are potentially 4 missing markets: your "impression" of my two consumption levels and my "impression" of yours. If there are M goods and N consumers, then every consumer



Exercise Graph 21B.13 : Deadweight Loss from Inefficiently High or Low Tax

consumes M goods that enter $(N-1)$ other consumers' utility. Thus, consumer 1 "produces" $M(N-1)$ goods, as does every other consumer — resulting in a potential of $M^2(N-1)$ missing markets.

Exercise 21B.16

Verify that individual 2's demand functions for x_1 and x_2 are unchanged as a result of the inclusion of x_3 in her utility function.

Answer: Recall from the general equilibrium chapter that we can set one of the prices to 1 and simply solve for the other (as a price relative to the other price). So suppose you set the price of good 1 to 1 and denote the price of good 2 by p. Person 2's utility maximization problem is now

$$\max_{x_1,x_2} \ x_1^{1/4}x_2^{3/4}x_3^{\gamma} \ \text{ subject to } \ x_1+px_2=10+4p. \tag{21B.16.i}$$

Note that x_3 is not a choice variable for individual 2 in the absence of a market for x_3 — i.e. individual 2 will simply have to accept the level of x_3 that is produced by individual 1. The Lagrange function for this problem is

$$\mathscr{L}=x_1^{1/4}x_2^{3/4}x_3^{\gamma}+\lambda(10+4p-x_1-px_2). \tag{21B.16.ii}$$

The first two first order conditions are then

$$\frac{\partial\mathscr{L}}{\partial x_1}=\frac{x_2^{3/4}x_3^{\gamma}}{4x_1^{3/4}}-\lambda=0 \ \text{ and } \ \frac{\partial\mathscr{L}}{\partial x_2}=\frac{3x_1^{1/4}x_3^{\gamma}}{4x_2^{1/4}}-\lambda p=0. \tag{21B.16.iii}$$

Taking the λ terms to the right hand side of each equation and then dividing the equations by one another, the x_3^{γ} term cancels — and the rest of the problem proceeds the same as if x_3 had not been in the utility function. Thus, the demands will be the same as before.

Exercise 21B.17

Do you think the conclusion (in exercise 21B.16) that demands for x_1 and x_2 do not change will hold regardless of what form the utility function takes?

Answer: No — this is a result of the Cobb-Douglas specification. In general, x_3 might enter utility in a way where it is either more or less complementary with the choice variables x_1 and/or x_2 — which would imply that the level of x_3 that the consumer takes as given affects her demands for x_1 and/or x_2.

Exercise 21B.18

Verify these demand functions. (*Hint*: It becomes significantly easier algebraically to first take logs of the utility function.)

Answer: The Lagrange function for this problem is

$$\mathscr{L} = \beta \ln(x_1) + (1-\beta)\ln(x_2) + \gamma \ln(x_3) + \lambda(p_1 e_2^1 + p_2 e_2^2 + p_3 x_3 - p_1 x_1 - p_2 x_2) \quad (21B.18.i)$$

giving first order conditions

$$\frac{\beta}{x_1} - \lambda p_1 = 0\,, \quad \frac{1-\beta}{x_2} - \lambda p_2 = 0 \text{ and } \frac{\gamma}{x_3} + \lambda p_3 = 0. \quad (21B.18.ii)$$

Solving the first 2 for x_2 and the first and third for x_3, we get

$$x_2 = \frac{(1-\beta)p_1 x_1}{\beta p_2} \text{ and } x_3 = \frac{-\gamma p_1 x_1}{\beta p_3}. \quad (21B.18.iii)$$

Substituting these into the constraint and solving for x_3, we get

$$x_1^2 = \frac{\beta(p_1 e_1^2 + p_2 e_2^2)}{(1+\gamma)p_1}. \quad (21B.18.iv)$$

Finally, substituting this into the equations in (21B.18.iii), we get

$$x_2^2 = \frac{(1-\beta)(p_1 e_1^2 + p_2 e_2^2)}{(1+\gamma)p_2} \text{ and } x_3 = \frac{-\gamma(p_1 e_1^2 + p_2 e_2^2)}{(1+\gamma)p_3} \quad (21B.18.v)$$

Exercise 21B.19

Do the demand functions converge to those we derived in the absence of an externality as the externality approaches zero (i.e. as γ approaches zero?

Answer: Yes, they do. In the absence of the externality, the Cobb-Douglas demands take the form

$$x_1^2 = \frac{\beta(p_1 e_1^2 + p_2 e_2^2)}{p_1} \text{ and } x_2^2 = \frac{(1-\beta)(p_1 e_1^2 + p_2 e_2^2)}{p_2} \quad (21B.19)$$

which is what we get as γ converges to zero.

Exercise 21B.20

Suppose the judge rules in my favor instead. What optimization problem do you solve as you come over to have coffee in order to offer me a payment for not building the addition? Can it again be the case that the efficient outcome does not happen for certain beliefs δ you might have about my true benefit from the addition?

Answer: The benefit to you of offering me p in order not to built the addition (and having me accept) is $(c-p)$ — because you would incur cost c if the addition is built but you now incur only a cost of p. The probability that I will accept an offer p is equal to the probability of my true benefit being less than p — i.e. $\delta(p)$. You would therefore choose p to maximize $\delta(p)(c-p)$. Again it will be possible for $c > b$ but the offer that you make to be less than b — i.e. it is possible for your offer to be rejected by me even though it would be efficient for me not to built the addition.

Exercise 21B.21

Why did our mathematical methods of solving for consumer 2's demand for x_3 not uncover this problem?

Answer: Our calculus methods do not reveal corner solutions.

21C Solutions to Odd Numbered End-of-Chapter Exercises

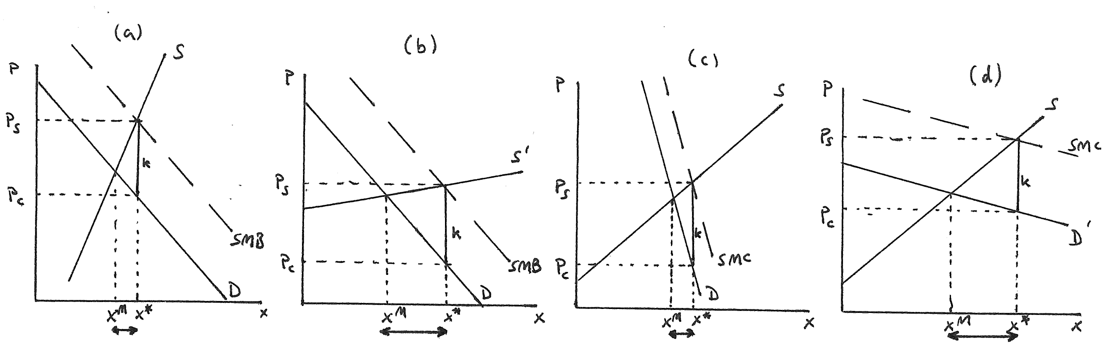
Exercise 21.1

Consider the case of a positive consumption externality.

A: *Suppose throughout this exercise that demand and supply curves are linear, that demand curves are equal to marginal willingness to pay curves and that the additional social benefit from each consumption unit is k and is constant as consumption increases.*

(a) *Draw two graphs with the same demand curve but one that has a fairly inelastic and one that has a fairly elastic supply curve. In which case is the market output closer to the optimal output?*

Answer: This is illustrated in panels (a) and (b) of Exercise Graph 21.1 where x^M is the market output and x^* is the optimal output. The difference between the two increases as the supply curve becomes more elastic.



Exercise Graph 21.1 : Positive Externalities and Elasticities

(b) *Does the Pigouvian subsidy that would achieve the optimal output level differ across your two graphs in part (a)?*

Answer: No, it does not. In each case, the marginal social benefit associated with the externality is k for all output units — which implies the optimal Pigouvian subsidy is $s = k$ in both cases.

(c) *Draw two graphs with the same supply curve but one that has a fairly inelastic demand curve and one that has a fairly elastic demand curve. In which case is the market output closer to the optimal output?*

Answer: This is illustrated in panels (c) and (d) of Exercise Graph 21.1. The greater the elasticity of demand, the greater the difference between the market and the socially optimal output.

(d) *Does the Pigouvian subsidy that would achieve the optimal output level differ across your two graphs in part (c)?*

Answer: No, it does not — for the same reason as in (b).

(e) True or False*: While the size of the Pigouvian subsidy does not vary as the slopes of demand and supply curves change, the level of under-production increases as these curves become more elastic.*

Answer: This is true, as already illustrated.

(f) *In each of your graphs, indicate who benefits more from the Pigouvian subsidy — producers or consumers.*

Answer: In panels (b) and (c), the bulk of the benefit goes to consumers as p_c falls more than p_s rises from the market price. This is because in both these cases, the demand curve is relatively more inelastic than the supply curve. In panels (a) and (d), producers benefit more as p_s rises more than p_c falls relative to the market price. This is because in these cases the supply curve is relatively more inelastic than the demand curve. The result is as it always is for subsidies: The side of the market that behaves relatively more inelastically will get the bulk of the benefit of a subsidy.

B: *Suppose demand is given by $x_d = (A-p)/\alpha$ and supply is given by $x_s = (B+p)/\beta$.*

(a) *Derive the competitive equilibrium price and output level.*

Answer: Setting demand and supply equal to one another and solving for p, we get the competitive market price p^M — and substituting this back into either demand or supply function, we get the market output x^M —

$$p^M = \frac{\beta A - \alpha B}{\alpha + \beta} \text{ and } x^M = \frac{A+B}{\alpha+\beta}. \tag{21.1.i}$$

(b) *Suppose that the marginal positive externality benefit is k per unit of output. What is the function for the social marginal benefit SMB curve?*

Answer: The equation for the demand *curve* is $p = A - \alpha x$. If the marginal externality benefit is k, this implies the social marginal benefit curve is

$$SMB = (A+k) - \alpha x. \tag{21.1.ii}$$

(c) *What is the optimal output level?*

Answer: To get the optimal output level, we have to set the supply curve equal to the SMB curve. The supply *curve* is $p = -B + \beta x$. Setting this equal to SMB and solving for x, we get

$$x^* = \frac{A+B+k}{\alpha+\beta} \tag{21.1.iii}$$

which is greater than x^M for any $k > 0$.

(d) *What is the Pigouvian subsidy? Show the impact it has on prices paid by consumers and prices received by producers — and illustrate that it achieves the optimal outcome.*

Answer: Since the marginal benefit from the externality is constant, we know the Pigouvian subsidy is $s = k$. Thus, $p_c = p_s - k$. We can thus write the demand and supply functions as

$$x_d = \frac{A-(p_s-k)}{\alpha} \quad \text{and} \quad x_s = \frac{B+p_s}{\beta}. \tag{21.1.iv}$$

Setting these equal to each other and solving for p_s (and then substituting this into $p_c = p_s - k$), we get

$$p_s = \frac{\beta A - \alpha B + \beta k}{\alpha+\beta} \quad \text{and} \quad p_c = \frac{\beta A - \alpha B - \alpha k}{\alpha+\beta}. \tag{21.1.v}$$

Substituting p_c into the demand function and p_s into the supply function, we then get $x_d = x_s = x^*$ — implying that the subsidy implements the optimal output level.

(e) *Next, suppose that the total externality social benefit is given by $SB = (\delta x)^2$. Does the market outcome change? What about the optimal outcome?*

Answer:The market outcome will still occur where supply intersects demand —and is thus unchanged. A social benefit $SB = (\delta x)^2$ implies that the *marginal* benefit from the externality is $2\delta^2 x$ — implying a social marginal benefit curve (that includes the marginal consumer benefit) of

$$SMB = A - \alpha x + 2\delta^2 x = A + (2\delta^2 - \alpha). \tag{21.1.vi}$$

Setting this equal to the supply curve $p = -B + \beta x$ and solving for x, we get

$$x^* = \frac{A+B}{\alpha+\beta-2\delta^2} \tag{21.1.vii}$$

which is greater than x^M for any $\delta > 0$.

(f) *Derive the Pigouvian subsidy now — and illustrate again that it achieves the social optimum.*

Answer: The optimal subsidy will result in a price p_c for consumers and p_s for producers such that both choose x^* at those prices. Thus, we want to calculate p_c and p_s solving the equations

$$\frac{A-p_c}{\alpha} = \frac{A+B}{\alpha+\beta-2\delta^2} \quad \text{and} \quad \frac{B+p_s}{\beta} = \frac{A+B}{\alpha+\beta-2\delta^2}. \tag{21.1.viii}$$

Solving these, we get

$$p_c = \frac{\beta A - \alpha B - 2A\delta^2}{\alpha+\beta-2\delta^2} \quad \text{and} \quad p_s = \frac{\beta A - \alpha B + 2B\delta^2}{\alpha+\beta-2\delta^2}. \tag{21.1.ix}$$

The optimal Pigouvian subsidy rate is then

$$s = p_s - p_c = \frac{2\delta^2(A+B)}{\alpha+\beta-2\delta^2}. \tag{21.1.x}$$

Plugging p_s into the supply function and p_c into the demand function, we furthermore get that $x_s = x_d = x^*$ under the prices that emerge with this subsidy.

Exercise 21.3

We discussed in the text that the "market failure" that emerges in the presence of externalities can equally well be viewed as a "failure of markets to exist", and we discussed the related idea that establishing property rights may allow individuals to resolve externality issues even when markets are not competitive.

A: *We will explore this idea a bit further by asking whether there is a "right way" to establish property rights in the case of pure consumption externalities.*

(a) *Suppose we consider the case where your consumption of music in your dorm room disturbs me next door. Let x denote the number of minutes you choose to play music each day, and let e be the number of minutes you are allowed to play music. If e is set at 0, who is given the "property rights" over the air on which the soundwaves travel from your room to mine?*

Answer:When $e = 0$, I am given complete property rights — because you are not allowed to play any music unless I agree to it.

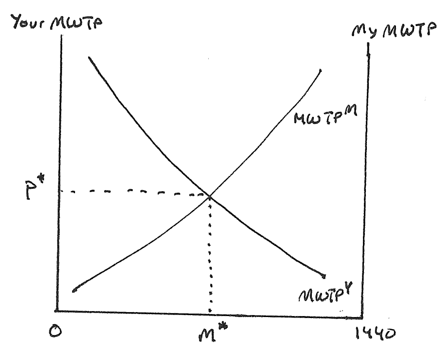
(b) *What if e is set to 1,440 (which is equal to the number of minutes in a day)?*

Answer: In this case, you are given the complete property rights — you can play music as much as you like without getting my permission.

(c) *Draw a graph with minutes of music per day on the horizontal axis — ranging from 0 to 1,440. Draw a vertical axis at 0 minutes and another vertical axis at 1,440 minutes. Then illustrate your marginal willingness to pay for minutes of music (measured on the left vertical axis) and my marginal willingness to pay for reductions in the number of minutes of music (measured on the right axis) — and assume that these are invariant to how e is set. What is the efficient number of minutes m^*?*

Answer: This is illustrated in Exercise Graph 21.3.

$MWTP^Y$ is your marginal willingness to pay for additional minutes and $MWTPY^M$ is my marginal willingness to pay for fewer minutes of music. (When lots of minutes are being played at the right, I am willing to pay a lot to reduce those minutes — but I am willing to pay less if few minutes are played — thus the upward slope). The efficient number of minutes is at the intersection — m^*. For any fewer minutes than that, additional minutes of music are worth more to you than they cost me — and for more minutes of music, additional minutes are worth less to you than they cost me.



Exercise Graph 21.3 : Minutes of Music in Your Dorm Room

(d) *The assignment of e in part (a) represents the extreme case where you have no right to play your music while the assignment in (b) represents the polar opposite extreme where I have no right to peace and quiet. Review, within the context of this example, the logic behind the Coase Theorem that suggests the efficient outcome will be reached regardless of whether* $e = 0$ *or* $e = 1,440$ *so long as transaction costs are low.*

Answer: If $e = 0$, you are not allowed to play music unless I agree to it. But for minutes up to m^*, you would be willing to pay me more than I need to receive in order to play music. Thus, there are gains from you coming to talk to me and negotiating the purchase of rights to play music — with those gains maximized when you offer me p^* per minute for accepting m^* minutes of music. If $e = 1,440$, on the other hand, you can play as much music as you would like. It is then worth my time to come chat with you — and pay you to reduce the number of minutes of music you play. Whenever we are above m^*, I am willing to pay you more to reduce your music than you are willing to pay to play it — so we can both gain by me paying you to play less. Our gains from trade will be maximized when I pay you p^* per minute to reduce your music from 1,440 to m^*.

(e) *Since* $e = 0$ *and* $e = 1,440$ *are two extreme assignments of property rights, we can now easily think of many cases in between. Does the Coase Theorem apply also to these in between cases? Why or why not?*

Answer: The exact same logic applies to any assignment of e. As long as e is not set to m^*, there are similar gains from trade — and we will again agree to a price p^* to get to m^*. Only if $e = m^*$ are we at a point where there are no gains from trade — but that is because we are already at the efficient level of music. We therefore will trade toward the efficient level of music so long as property rights e are assigned in *some* way and transaction costs are sufficiently low.

(f) *From a pure efficiency standpoint, if the Coase Theorem is right, is there any case for any particular assignment of e?*

Answer: No — the only thing that matters is that e is set at some level and transaction costs are low.

B: *Suppose that your tastes can be described by the utility function $u(x,y) = \alpha \ln x + y$, where x is the number of minutes per day of music and y is a composite consumption good. My tastes, on the other hand, can be described by $u(x,y) = \beta \ln(1440 - x) + y$, with $(1440 - x)$ representing the number of minutes per day without your music. Both of us have some daily income level I, and the price of y is 1 given that y is a composite good denominated in dollars.*

(a) *Let e be the allocation of rights as defined in part A — i.e. e is the number of minutes that you are permitted to play music without my permission. When $x < e$, I am paying you $p(e-x)$ to play less than you are allowed to — and when $x > e$, you are paying me $p(x-e)$ for the minutes above your "rights". What is your budget constraint?*

Answer: Another way of phrasing this is as follows: You will receive $p(e-x)$ from me — with this amount being negative (i.e. you will actually pay me) when $x > e$. Thus, your budget constraint is $y = I + p(e-x)$ which can also be written as

$$px + y = I + pe. \tag{21.3.i}$$

(b) *What is my budget constraint?*

Answer: I, on the other hand, will have to pay $p(e-x)$ to you — with this amount being negative (i.e. you will actually be paying me) when $x > e$. Thus, my budget is $y = I - p(e-x)$ which can also be written as

$$y - px = I - pe. \tag{21.3.ii}$$

(c) *Set up your utility maximization problem using the budget constraint you derived in (a) — then solve for your demand for x.*

Answer: We need to solve

$$\max_{x,y} \alpha \ln x + y \text{ subject to } px + y = I + pe. \tag{21.3.iii}$$

Solving this in the usual way, we get your demand for x to be

$$x^Y = \frac{\alpha}{p}. \tag{21.3.iv}$$

(d) *Set up my utility maximization problem and derive my demand for x.*

Answer: We need to solve

$$\max_{x,y} \beta \ln(1440 - x) + y \text{ subject to } y - px = I - pe. \tag{21.3.v}$$

Solving this in the usual way, we get my demand for x as

$$x^M = \frac{1440p - \beta}{p}. \tag{21.3.vi}$$

(e) *Derive the p^* we will agree to if transaction costs are zero — and derive the number of minutes of music you will play. Does your answer depend on the level at which e was set?*

Answer: Setting x^Y equal to x^M and solving for p, we get

$$p^* = \frac{\alpha + \beta}{1440}. \tag{21.3.vii}$$

Plugging this back into either x^Y or x^M, we get the total number of minutes of music as

$$x^* = \frac{1440\alpha}{\alpha + \beta}. \tag{21.3.viii}$$

This is independent of e — so it does not matter how property rights were assigned.

(f) *According to your results, how much music is played if I don't care about peace and quiet (i.e. if $\beta = 0$? How much is played if you don't care about music — i.e. $\alpha = 0$?*

Answer: Substituting $\beta = 0$ into the equation for x^*, we get $x^* = 1440$ — i.e. if I don't care about my peace and quiet, you will play music all the time. Substituting $\alpha = 0$ into x^*, we get $x^* = 0$ — i.e. if you don't care about the music, no music will be played.

(g) True or False*: The total number of minutes of music played does not depend on e — but you and I still care how e is assigned.*

Answer: This is true. We showed already that x^* does not depend on e — so the amount of music played does not depend on e. But if e is set below x^*, you have to pay me — and if e is set above x^*, I have to pay you. Thus, while x^* is unaffected by e, the consumption level of y we can enjoy *is* affected by e — and thus we care about whether you or I get the bulk of the property rights.

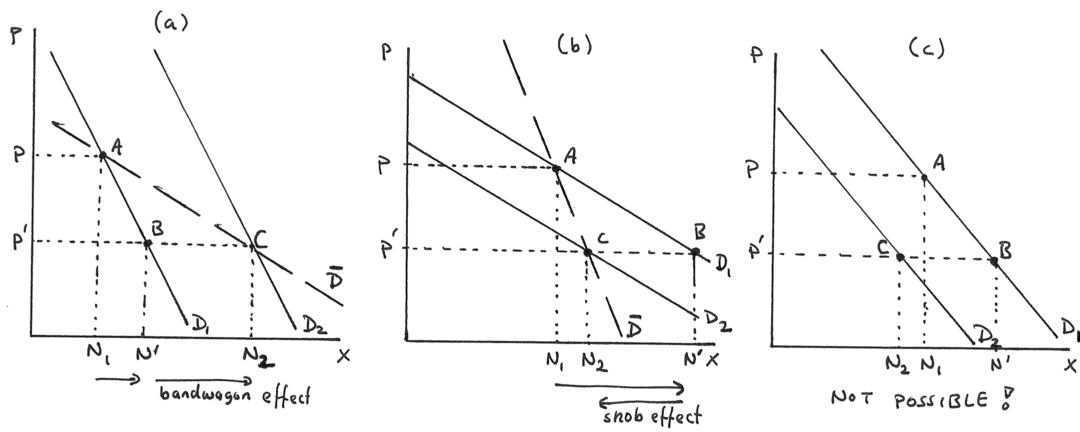
Exercise 21.5

Everyday Exercise: *Children's Toys and Gucci Products*: *In most of our development of consumer theory, we have assumed that tastes are independent of what other people do. This is not true for some goods. For instance, children are notorious for valuing toys more if their friends also have them — which implies their marginal willingness to pay is higher the more prevalent the toys are in their peer group. Some of my snooty acquaintances, on the other hand, like to be the center of attention and would like to consume goods that few others have. Their marginal willingness to pay for these goods thus falls as more people in their peer group consume the same goods.*

A: *The two examples we have cited are examples of positive and negative* network externalities..

(a) *Consider children's toys first. Suppose that, for a given number N of peers, demand for some toy x is linear and downward sloping — but that an increase in the "network" of children (i.e. an increase in N) causes an upward parallel shift of the demand curve. Illustrate two demand curves corresponding to network size levels $N_1 < N_2$.*

Answer: This is done in panel (a) of Exercise Graph 21.5(1) where demand D_1 corresponds to the demand curve when the network is of size N_1 and D_2 is the demand curve when the network size is N_2.



Exercise Graph 21.5(1) : Bandwagon and Snob Effects

(b) *Suppose every child at most buys one of these toys which are produced at constant marginal cost. For a combination of p and x to be an equilibrium, what must be true about x if the equilibrium lies on the demand curve for network size N_1?*

Answer: It must be that $x = N_1$ — because only then is the demand curve D_1 the correct demand curve given the size of the network.

(c) *Suppose you start in such an equilibrium and the marginal cost (and thus the price) drops. Economists distinguish between two types of effects: A* direct *effect that occurs along the demand curve for network size N_1, and a* bandwagon effect *that results from increased demand due to increased network size. Label your original equilibrium A, the "temporary" equilibrium before network externalities are taken into account as B and your new equilibrium (that incorporates both effects) as C. Assume that this new equilibrium lies on the demand curve that corresponds to network size N_2.*

Answer: This is also done in panel (a) of Exercise Graph 21.5(1). The initial equilibrium A occurs on the demand curve D_1 at price p — with output equal to N_1. When price drops to p', consumers will respond along the D_1 demand curve so long as they do not realize that the drop in price is expanding the network N. Thus, as long as their perception of network size remains at N_1, consumption will increase only from N_1 to N' — which is the direct price effect and puts us at the "temporary" equilibrium B. But this can't really be an equilibrium — because the network is of size N' but children behave as if it was only of size N_1. Thus, as children notice the increase in the network size, demand shifts up — causing increases in consumption — which in turn causes an increase in the network — which pushes up the demand curve — etc. The process does not end until the consumption level is equal to the network size. If the new equilibrium lies on D_2, it must be that demand along D_2 is equal to N_2 when price is p'. This is illustrated as the equilibrium C.

(d) *How many toys are sold in equilibrium C? Connect A and C with a line labeled $\overline{D}$. Is $\overline{D}$ the true demand curve for this children's toy? Explain.*

Answer: Since C lies on D_2 — the demand curve that applies when the network consists of N_2 children — it must be that N_2 toys are sold in equilibrium C.

(e) *If you were a marketing manager with a limited budget for a children's toy company, would you spend your budget on aggressive advertising early as the product is rolled out or wait and spread it out? Explain.*

Answer: You would want to spend aggressively initially in order to get network effects to start working in your favor by raising the value of the toys for the children that have not yet bought.

(f) *Now consider my snooty acquaintances who like Guuci products more if few of their friends have them. For any given number of friends N that also have Gucci products, their demand curve is linear and downward sloping — but the intercept of their demand curve falls as N increases. Illustrate three demand curves for $N_1 < N_2$.*

Answer: This is done in panel (b) of Exercise Graph 21.5(1) where D_1 corresponds to the demand curve when the network consists of relatively few N_1 people and thus lies above D_2 that represents the demand curve when more people (N_2) have the Gucci products.

(g) *Assume for convenience that everyone buys at most 1 Gucci product. Identify an initial equilibrium A under which N_1 Gucci products are sold at some initial price p — and then a second equilibrium C at which N_2 Gucci products are sold at price $p' < p$. Can you again identify two effects — a* direct *effect analogous to the one you identified in (c) and a* snob effect *analogous to the bandwagon effect you identified for children's toys? How does the snob effect differ from the bandwagon effect?*

Answer: This is also done in panel (b) of Exercise Graph 21.5(1) where the direct price effect occurs prior to people adjusting their awareness of how many others have Gucci products while the snob effect results in a reduction of Gucci products because they are no longer worth as much to my snooty acquaintances when more of their friends also have them. The actual demand curve that takes into account both effects is $\overline{D}$. Note that the snob effect points in the opposite direction from the direct price effect while the bandwagon effect points in the same direction as the direct price effect.

(h) True or False*: Bandwagon effects make demand more price elastic while snob effects make demand less price elastic.*

Answer: This appears to be true — as demonstrated in panels (a) and (b) of Exercise Graph 21.5(1).

(i) *In exercise 7.9 we gave an example of an upward sloping demand curve for Gucci products, with the upward slope emerging from the fact that utility was increasing in the price of Gucci products. Might the demand that takes both the direct and snob effects into account also be upward sloping in the presence of the kinds of network externalities modeled here?*

Answer: No, this is not possible in the case of negative network externalities of the type we describe in this exercise. In order for it to be the case, the snob effect would have to outweigh the direct price effect — giving us a graph like panel (c) of Exercise Graph 21.5(1) where C lies to the left of A. But this implies that $N_2 < N_1$ — which contradicts the fact that $N_1 < N_2$ (which we used to draw D_1 as the higher demand curve).

B: *Consider again the positive and negative network externalities described above.*

(a) *Consider first the case of a positive network externality such as the toy example. Suppose that, for a given network size N, the demand curve is given by $p = 25N^{1/2} - x$. Does this give rise to parallel linear demand curves for different levels of N, with higher N implying higher demand?*

Answer: Yes — the set of demand curves all have slope of minus 1, with the intercept $25N^{1/2}$ increasing as N increases.

(b) *Assume that children buy at most one of this toy. Suppose we are currently in an equilibrium where $N = 400$. What must the price of x be?*

Answer: In an equilibrium, $N = x$ — which implies that $x = 400$ in this equilibrium. Thus,

$$p = 25\left(400^{1/2}\right) - 400 = 100. \tag{21.5.i}$$

(c) *Suppose the price drops to \$24. Isolate the direct effect of the price change — i.e. if child perception of N remained unchanged, what would happen to the consumption level of x?*

Answer: For the direct price effect, we hold N equal to 400 and write the demand curve as $p = 25(400^{1/2}) - x = 500 - x$. Setting $p = 24$ and solving for x, we get $x = 476$ — and increase of 76 from the previous 400.

(d) *Can you verify that the real equilibrium (that includes the bandwagon effect) will result in $x = N = 576$ when price falls to \$24? How big is the direct effect relative to the bandwagon effect in this case?*

Answer: In any equilibrium, $N = x$ — which implies the real demand curve is given by

$$p(x) = 25x^{1/2} - x. \tag{21.5.ii}$$

Substituting $x = 576$ into this equation, we get $p = 24$. Thus, it is indeed an equilibrium for $N = x = 576$ when $p = 24$. Thus, the direct price effect took us from consumption of 400 (at the original price) to 476 (at the new price), and the bandwagon effect took us from 476 to 576 — an additional jump of 100 toys. (Note: In exercise 21.8, we will actually see that there are several equilibria in this model — but, given that we started in the equilibrium that was initially assumed, the analysis above is correct. The reason for the multiple equilibria lies in the fact that the demand function (that incorporates the bandwagon effect) has an inverse U-shape.)

(e) *Consider next the negative network externality of the Gucci example. Suppose that, given a network of size N, the market demand curve for Gucci products is $p = (1000/N^{1/2}) - x$. Does this give rise to parallel linear demand curves for different levels of N, with higher N implying lower demand?*

Answer: Yes — the slope again remains equal to minus 1 in all these demand curves regardless of N, while the intercept is $1000/N^{1/2}$ which falls as N increases.

(f) *Assume again that no one buys more than one Gucci item. Suppose we are currently in equilibrium with $N = 25$. What must the price be?*

Answer: In equilibrium, it must then again be the case that $N = x$ — which implies that $x = 25$ and the price is

$$p = \frac{1000}{25^{1/2}} - 25 = 175. \tag{21.5.iii}$$

(g) *Suppose the price drops to \$65. Isolate the direct effect of the price change — i.e. if people's perception of N remained unchanged, what would happen to the consumption level of x?*

Answer: Keeping N fixed at 25, the demand curve is

$$p = \frac{1000}{25^{1/2}} - x = 200 - x. \tag{21.5.iv}$$

Substituting $p = 65$ and solving for x, we get $x = 135$ — an increase of 110 over the original 25.

(h) *Can you verify that the real equilibrium (that includes the snob effect) will result in $x = N = 62$? How big is the direct effect relative to the snob effect in this case?*

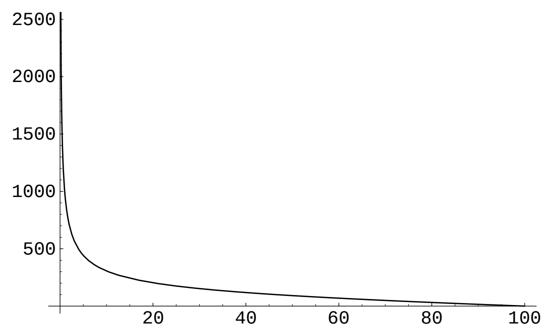
Answer: In equilibrium, $N = x$ — which implies we can write the demand curve that includes all effects as

$$p = \frac{1000}{x^{1/2}} - x. \tag{21.5.v}$$

Plugging $x = 62$ into this equation, we get $p = 65$. Thus, the snob effect *reduces* consumption from 135 (which was implied by the direct effect) to 62. In our example, the direct effect therefore raises consumption by 110 while the snob effect lowers it by 73 — for a net increase of 37 — when price falls from the initial $175 to the new $65.

(i) *Although the demand curves for a fixed level of N are linear, can you sketch the demand curve that includes both direct and snob effects?*

Answer: The demand curve that includes both effects was given in equation (21.5.v) — and it is pictured in Exercise Graph 21.5(2) (with x — or N — on the horizontal axis and p on the vertical).



Exercise Graph 21.5(2) : Demand for Gucci (including Snob Effect)

Exercise 21.7

Business and Policy Application: *The Externality when Fishing in the Commons: In exercise 21.6, we showed that free access to a fishing lake causes overfishing because fishermen will continue to fish until the cost of inputs (i.e. fishing nets, in our example) equals* average *rather than* marginal *revenue product.*

A: *Suppose that the lake in exercise 21.6 is publicly owned.*

(a) *What is the externality that fishermen impose on one another in this lake?*

Answer: The externality is that each individual fisherman does not take into account the reduction in the average catch of fish that he causes by bringing one more net onto the lake. This is a negative externality — and it is in the presence of negative externalities that resources are overused as the lake is overfished in our example.

(b) *Seeing the problem as one involving this externality, how would you go about setting a Pigouvian tax on fishing nets to remedy the problem? What information would you have to have to calculate this?*

Answer: You would want to set the Pivouvian tax such that each fisherman faced the full cost he imposes on all the other fishermen when he brings one more net onto the lake — and the efficient number of nets are brought to the lake as a result. In order to do this, you have to know what the efficient number N^* of nets is. You would then need to calculate the reduction in average revenue product from one more net — and multiply that by N^* because everyone who is already on the lake will suffer this reduction in average revenue product.

(c) *Suppose instead that the lake is auctioned off to someone who then charges per-net fees to fishermen who would like to fish on the lake (as in A(e) of exercise 21.6). How do you think the fees charged by a profit maximizing lake-owner compare to the optimal Pigouvian tax?*

Answer: The profit maximizing lake owner would want to maximize his revenue from fees. For each net, he can charge up to the average revenue product (pAP) — because the average product is the number of fish that are caught in each net and the price is what those fish sell for on the market. Or, put differently, he can charge the total profit that fishermen would make in the absence of the fees divided by the number of nets (since each net gets its proportional share of the overall profit). If the overall profit is the highest it can be, then the overall fees collected will be the highest they can be — so the lake owner would want to charge a fee equal to the maximum profit from the lake divided by the number of nets N^* necessary to get to that profit. Just like the Pivouvian taxer, he would need to know N^* and use it to arrive at the fee that will cause only N^* nets to be used on the lake. But that is exactly the same thing as what is achieved through the Pivouvian per-net tax — which must mean that the per-net fee charged by the lake-owner is the same as the Pivouvian tax.

(d) *Do you think it is easier for the government to collect the information necessary to impose a Pigouvian tax in part (b) or for a lake-owner to collect the information necessary to impose the per-net fees in part (c). Who has the stronger incentive to get the correct information?*

Answer: It seems that it should be easier for the lake-owner who can actually get to know his lake to determine N^* and the profit maximizing fee — and he certainly has the incentive given that he wants to maximize profit. The government may have a more difficult time gathering the information — and it is not clear that it has the incentive to do so.

(e) *How would the price of the lake that the government collects in (c) compare to the tax revenues it raises in (b)?*

Answer: The price for the lake should be equal to the present discounted value of all future rents from fees. Since the per-net weekly fees are the same as the Pivouvian weekly tax per net, this implies that the price of the lake is equal to the discounted present value of all Pigouvian tax payments that the government would otherwise be able to collect.

(f) *Suppose instead that the government tries to solve the externality problem by simply setting a limit on per-net fishing licenses that fishermen are now required to use when fishing on the public lake. If the government sets the* *optimal cap on licenses and auctions these off, what will be the price per license?*

Answer: The optimal cap is N^* — and if the government limits the number of nets on the lake to N^* through fishing licenses, the value of each license is equal to the profit that fishermen can make per net (in the absence of a fee) — which is exactly equal to the fee that a profit maximizing lake owner would set and thus exactly equal to the Pigouvian tax per net.

(g) *What do each of the above solutions to the Tragedy of the Commons share in common?*

Answer: In each case, the fishermen are forced to confront the externality cost they impose on all the other fishermen — and thus "internalize" the costs they impose on others.

(h) *Legislators who represent political districts (such as Congressmen in the U.S. House of Representatives) can be modeled as competing for pork barrel projects to be paid for by the government budget. Could you draw an analogy between this and the problem faced by fishermen competing for fish in a public lake? (This is explored in more detail in end-of-chapter exercise 28.2 in Chapter 28.)*

Answer: The direct analogy is as follows: The national government's budget is analogous to the lake — capable of funding many pork barrel projects in political districts much as the lake is capable of providing fish for many fishermen. The problem is that the national budget, like the lake, is a "commons" — with neither fishermen nor politicians forced to confront the costs imposed on others by their actions. The fishermen don't take into account the cost imposed by their nets on other fishermen just as the politicians don't take into account the cost of their pork barrel projects on taxpayers in other districts. This would suggest that we have too many pork barrel projects just as we have too many fish being taken out of the publicly owned lake.

B: *Let N denote the total number of fishing nets used by everyone and $X = f(N) = AN^{\alpha}$ the total catch per week. As in exercise 21.6, let r be the weekly rental cost per net, let p be the market price for fish and let $A > 0$ and $0 < \alpha < 1$.*

(a) *The lake is freely accessible to anyone who wants to fish. How much revenue does each individual fisherman make when he uses one net?*

Answer: Each net catches the average product $AP = AN^{\alpha}/N = AN^{(\alpha-1)}$ — which implies that each fisherman makes $pAP = pAN^{(\alpha-1)}$ in revenue per net.

(b) *What is the loss in revenue for everyone else who is fishing the lake when one fisherman uses one more net?*

Answer: The additional net has some impact on the average product of each net — i.e. the average yield for each net will decline by $\partial AP/\partial N$. This decline will affect all the N nets that are already on the lake — which implies the loss in revenue for everyone on the lake is

$$pN\frac{\partial AP}{\partial N} = pN(\alpha-1)AN^{(\alpha-2)} = -(1-\alpha)pAN^{(\alpha-1)}. \tag{21.7.i}$$

(c) *Suppose that each fisherman took the loss of revenue to others into account in his own profit maximization problem when choosing how many nets n to bring. Write down this optimization problem. Would this solve the externality problem?*

Answer: The problem would be

$$\max_n \; npAN^{(\alpha-1)} - (1-\alpha)pAN^{(\alpha-1)} - nr = n\alpha pAN^{(\alpha-1)} - rn. \tag{21.7.ii}$$

Taking the partial derivative with respect to n and solving for N, we then get

$$N^* = \left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)} \tag{21.7.iii}$$

which is exactly equal to the efficient number of nets we calculated in exercise 21.6, part B(a). Thus, if every fisherman took into account the loss in revenue he causes to everyone else by bringing one more net, the externality problem would be solved.

(d) *A Pigouvian tax is optimally set to be equal to the marginal social damage an action causes when evaluated at the optimal market level of that action. Evaluate your answer to (b) at the optimal level of N to derive the optimal Pigouvian tax on nets.*

Answer: This would give us an optimal per-net tax t of

$$t = (1-\alpha)pA(N^*)^{(\alpha-1)} = (1-\alpha)pA\left(\left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)}\right)^{(\alpha-1)} = \frac{(1-\alpha)r}{\alpha}. \tag{21.7.iv}$$

(e) *Suppose that all fishermen just consider their own profit but that the government has imposed the Pigouvian per-net tax you derived in (d). Write down the fisherman's optimization problem and illustrate its implications for the overall level of N. Does the Pigouvian tax achieve the efficient outcome?*

Answer: The problem would then be

$$\max_n \; npAN^{(\alpha-1)} - \left(r + \frac{(1-\alpha)r}{\alpha}\right)n = npAN^{(\alpha-1)} - \frac{rn}{\alpha}. \tag{21.7.v}$$

Taking the derivative with respect to n and solving for N, we get

$$N^* = \left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)} \tag{21.7.vi}$$

— the efficient number of nets. The Pigouvian tax therefore achieves the efficient outcome.

(f) *Suppose the government privatized the lake and allowed the owners to charge per-net fees. The owner might do the following: First, calculate the maximum profit (not counting the rental value of the lake) he would be able to make by simply fishing the lake himself with the optimal number of nets — then set the fee per net at this profit divided by the number of nets he himself would have used. What per-net fee does this imply?*

Answer: Profit for the lake-owner if he fished the lake himself (not counting the rental value of the land) is $\pi = px* - rn^*$ where x^* and n^* are the optimal number of fish to take out of the lake and the optimal number of nets to use. We calculated x^* and n^* in exercise 21.6 as

$$x^* = A^{1/(1-\alpha)}\left(\frac{\alpha p}{r}\right)^{\alpha/(1-\alpha)} \text{ and } n^* = \left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)}. \tag{21.7.vii}$$

Plugging this into our equation for π (and doing some tedious algebra), we get

$$\begin{aligned}\pi^* &= pA^{1/(1-\alpha)}\left(\frac{\alpha p}{r}\right)^{\alpha/(1-\alpha)} - r\left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)} = \\ &= (pA)^{1/(1-\alpha)}\left[\left(\frac{\alpha}{r}\right)^{\alpha/(1-\alpha)} - \left(\frac{\alpha}{r}\right)^{1/(1-\alpha)}\right] = \\ &= \left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)}\left[\frac{r}{\alpha} - r\right] = \\ &= \left(\frac{\alpha pA}{r}\right)^{1/(1-\alpha)}\left[\frac{(1-\alpha)r}{\alpha}\right].\end{aligned} \tag{21.7.viii}$$

Dividing this by the optimal number of nets n^*, we get the per-net-fee F

$$F = \frac{\pi^*}{n^*} = \frac{(1-\alpha)r}{\alpha}. \tag{21.7.ix}$$

(g) *Compare your answer to (f) to your answer to (d). Can you explain why the two are the same?*

Answer: They are the same because both the Pigouvian tax t and the per-net fee F were derived with the implicit aim of getting fishermen to limit the number of nets to N^*.

(h) *Suppose $A = 100$, $\alpha = 0.5$, $p = 10$ and $r = 20$. What is the optimal Pigouvian (per-net) tax and the profit maximizing per-net fee that an owner of the lake would charge?*

Answer: Plugging these into the relevant expression, we get $t = F = 20$.

Exercise 21.9

Business Application: *Pollution that increases Firm Costs — The Market Outcome: In the text, we assumed for convenience that the ill effects of pollution are felt by people other than producers and consumers. Consider instead the following case: An entire competitive industry is located around a single lake that contains some vital property needed for the production of x. Each unit of output x that is produced results in pollution that goes into the lake. The only effect of the pollution is that it introduces a chemical into the lake — a chemical that requires firms to reinforce their pipes to keep them from corroding. The chemical is otherwise harmless to the population as well as to all wildlife in the area.*

A: *We have now constructed an example in which the only impact of pollution is on the firms that are creating the pollution. Suppose that each unit of x that is produced raises every firm's fixed cost by δ.*

(a) *Suppose all firms have identical decreasing returns to scale production processes, with the only fixed cost created by the pollution. For a given amount of industry production, what is the shape of an individual firm's average cost curve?*

Answer: Decreasing returns to scale imply an upward sloping marginal cost (MC) curve. Combined with the presence of a fixed cost introduced by pollution, this results in a U-shaped average cost (AC) curve. As always, the MC curve crosses the AC curve at the lowest point of the AC curve.

(b) *In our discussion of long run competitive equilibria, we concluded in Chapter 14 that the long run industry supply curve is horizontal when all firms have identical cost curves. Can you recall the reason for this?*

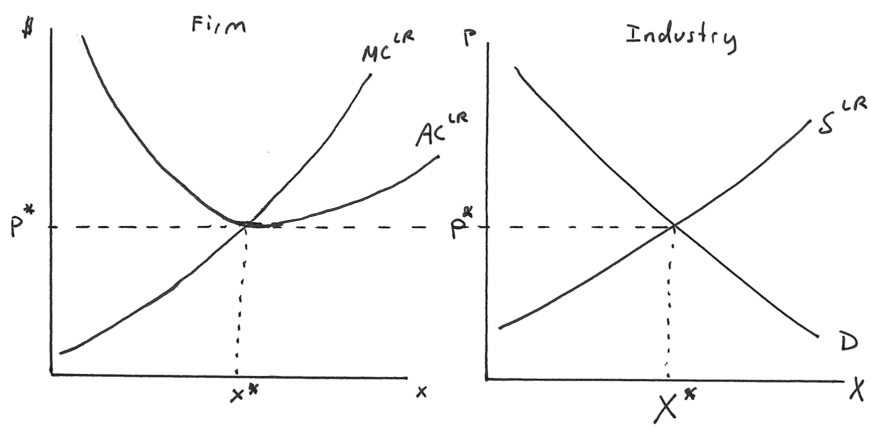
Answer: The reason for this was entry and exit of firms. If firms in the industry made positive profit, then identical firms from outside the industry would have an incentive to enter and thereby raise supply — and lower price. This would continue until price reaches the point where all firms in the industry make zero profit — a price that settles at the lowest point of the U-shaped AC curve. If, on the other hand, firms in the industry made negative profit, then some firms will exit the industry — thus lowering supply and driving up price. This continues until price settles where firms make zero profit — i.e. the lowest point of the AC curves. From this, we can conclude that the industry will supply (in the long run) any quantity that is demanded at the zero-profit price.

(c) *Now consider this example here. Why is the long run industry supply curve now upward sloping despite the fact that all firms are identical?*

Answer: The more firms enter this industry and produce, the higher the level of pollution in the lake will be. As the pollution goes up, the fixed costs of all the firms go up — which causes the AC curves of all firms to rise as the industry around the lake gets bigger. And as the AC curves rise, the zero-profit price that must hold in long-run equilibrium (due to entry and exit) must rise. We can therefore conclude that the greater the demand for x, the higher will be the price in the long run — because the increased pollution is raising the cost of all the identical producers.

(d) *In side-by-side graphs of a firm's cost curves and the industry (long run) supply and demand curves, illustrate the firm and industry in long run equilibrium.*

Answer: This is done in Exercise Graph 21.9 where the firm makes zero profit by producing x^* at p^* and the industry produces X^* with a sufficient number N^* of firms such that $N^*x^* = X^*$.



Exercise Graph 21.9 : Increasing Cost Industry due to Pollution Externality

(e) *Usually we can identify producer surplus — or firm profit — as an area in the demand and supply picture. What is producer surplus here? Why is your answer different from the usual?*

Answer: The producer surplus — or long run profit — is zero because all firms are producing at the lowest point of the AC curves (as illustrated in the graph). Usually we could denote the area below p^* and above the supply curve as producer surplus (or profit) — because usually an upward slope of the long run supply curve arises from firms that have different cost structures that permit more cost efficient firms to earn profit in equilibrium (and only the marginal firm to make zero profit). Here, however, the upward slope arises because of an externality that raises everyone's costs as the industry expands — and this is why everyone makes zero profit at the equilibrium.

(f) *In chapter 14, we briefly mentioned the term* decreasing cost industries — *industries in which the long run industry supply curve is downward sloping despite the fact that all firms might have identical production technologies. Suppose that in our example the pollution causes a decrease rather than an increase in fixed costs for firms. Would such a positive externality be another way of giving rise to a decreasing cost industry?*

Answer: Yes. In this case, the AC curve for each firm falls as the industry expands — much like it would if labor costs fell as the industry expanded. Since all firms have to make zero profit in long run equilibrium, that implies the zero-profit price — i.e. the price at the lowest point of the AC curve — must fall as the industry gets larger. As demand expands, the industry therefore provides additional output at lower prices, with each firm in the industry still making zero profit.

B: *Suppose that each firm's (long run) cost curve is given by $c(x) = \beta x^2 + \delta X$ where x is the firm's output level and X is the output level of the whole industry. Note that x is contained in X — and thus we could write the cost function as $c(x) = \beta x^2 + \delta x + \delta \overline{X}$ where $\overline{X}$ is the output produced by all other firms. When each firm is small relative to the industry, however, the impact of a single firm's pollution output on its own production cost is negligible — and it is a good approximation (that makes the problem a lot easier to solve) to simply write a single firm's cost curve as $c(x) = \beta x^2 + \delta X$. Furthermore, if all firms are identical, it is reasonable to assume that all firms produce the same output level $\overline{x}$. Letting N denote the number of firms in the industry, we can therefore write $X = N\overline{x}$ and re-write the cost function for an individual firm as $c(x) = \beta x^2 + \delta N\overline{x}$.*

(a) *How is our treatment of a producer's contribution to her own costs similar to our "price-taking" assumption for competitive firms?*

Answer: We are assuming that the producer is sufficiently small relative to the market that she can take the amount of pollution in the lake as given — just as she takes the price set in the market as given as a price taker. Both assumptions are derived from the "smallness" of each firm relative to the market — and in this sense, we are simply assuming that each firm is a "pollution-taker" and not sufficiently large to strategically think about her own contribution to pollution.

(b) *Derive the marginal and average cost functions for a single firm (using the final version of our approximate cost function). (Be careful to realize that the second part of the cost function is, from the firm's perspective, simply a fixed cost.)*

Answer: Treating the fixed cost $\delta N\overline{x}$ as fixed, we get

$$MC(x) = \frac{\partial c(x)}{\partial x} = 2\beta x \quad \text{and} \quad AC(x) = \frac{c(x)}{x} = \beta x + \frac{\delta N\overline{x}}{x}. \tag{21.9.i}$$

(c) *Assuming the firm is in long run equilibrium, all firms will make zero profit. Use your answer to (b) to derive the output level produced by each firm as a function of δ, β, N and $\overline{x}$.*

Answer: Zero profit implies that each firm produces at the lowest point of its AC curve — which we can derive by either setting the derivative of AC to zero or by setting MC equal to AC. Either way, we get

$$x = \left(\frac{\delta N\overline{x}}{\beta}\right)^{1/2}. \tag{21.9.ii}$$

(d) *Since all firms are identical, in equilibrium the single firm we are analyzing will produce the same as each of the other firms — i.e. $x = \overline{x}$. Use this to derive a single firm's output level $x(N)$ as a function of δ, N, and β. What does this imply about the equilibrium price $p(N)$ (as a function of δ and N) given that firms make zero profit in equilibrium?*

Answer: Replacing $\overline{x}$ in equation (21.9.ii) with x and solving for x, we get

$$x(N) = \frac{\delta N}{\beta}. \tag{21.9.iii}$$

Since firms make zero profit in equilibrium, we know that price will settle at the lowest point of the AC curve. Plugging $x(N)$ into our expression for AC, we get $AC = 2\delta N$ — which is also what we get if we plug $x(N)$ into our expression for MC. Thus,

$$p(N) = 2\delta N. \tag{21.9.iv}$$

(e) *Since each firm produces $x(N)$, multiply this by N to get the aggregate output level $X(N)$ — then invert it to get the number of firms $N(X)$ as a function of β, δ and X.*

Answer: Multiplying $x(N)$ by N, we get $X(N) = \delta N^2/\beta$. Inverting, we get the number of firms as a function of X — i.e.

$$N(X) = \left(\frac{\beta X}{\delta}\right)^{1/2}. \tag{21.9.v}$$

(f) *Substitute $N(X)$ into $p(N)$ to get a function $p(X)$. Can you explain why this is the long run industry supply curve with free entry and exit?*

Answer: This gives us

$$p(X) = 2\left(\beta\delta X\right)^{1/2}. \tag{21.9.vi}$$

We derived this function by assuming that all firms make zero profit and that there are sufficient firms in the market (all of whom make zero profit) to produce X. Put differently, we have derived the price that will emerge in the market when entry and exit of firms have driven profits to zero — taking into account the pollution cost. Thus, $p(X)$ is the long run industry supply curve — and it is upward sloping even though the firms are all identical because, as the industry grows, the fixed cost faced by each firm increases from the increase in pollution.

(g) *Suppose the aggregate demand for X is given by the demand curve $p_D(X) = A/(X^{0.5})$. Set the industry supply curve equal to the demand curve to get the equilibrium market output X^* (as a function of A, δ and β).*

Answer: Setting the demand curve equal to the supply curve gives us the equation

$$\frac{A}{X^{1/2}} = 2\left(\beta\delta X\right)^{1/2}. \tag{21.9.vii}$$

Solving this for X, we get the long run equilibrium output level

$$X^* = \frac{A}{2(\beta\delta)^{1/2}}. \tag{21.9.viii}$$

(h) *Use your answer to (g) to determine the equilibrium price level p^* (as a function of A, δ and β).*

Answer: Plugging this into either the supply curve of equation (21.9.vi) or the demand curve, we get

$$p^* = (2A)^{1/2}(\beta\delta)^{1/4}. \tag{21.9.ix}$$

(i) *Use your answer to (g) to determine the equilibrium number of firms N^* (as a function of A, δ and β).*

Answer: Plugging X^* into equation (21.9.v), we get

$$N^* = \left(\frac{A}{2}\right)^{1/2}\left(\frac{\beta}{\delta^3}\right)^{1/4}. \tag{21.9.x}$$

(j) *Suppose that $\beta = 1$, $\delta = 0.01$ and $A = 10,580$. What are X^*, p^* and N^*? How much does each individual firm produce? (Do exercise 21.10 to compare these to what is optimal.)*

Answer: Plugging these values into our equations, we get

$$p^* = 46,\ X^* = 52,900 \text{ and } N^* = 2,300. \tag{21.9.xi}$$

Thus, 2,300 firms produce 52,900 output units and sell them at a price of \$46 each. This implies that each firm produces 52900/2300=23 units. This is the same answer we get when we plug N^* into equation (21.9.iii).

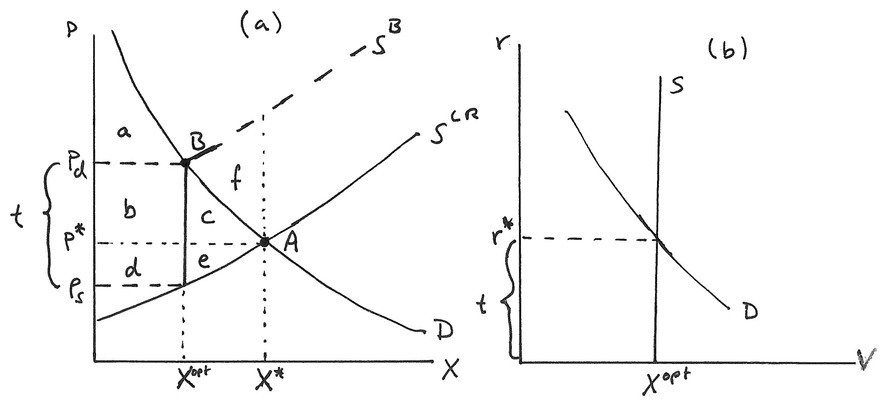
Exercise 21.11

Policy Application: *Pollution that Increases Firm Costs — Policy Solutions: This exercise continues to build on exercises 21.9 and 21.10. Assume the same basic setup of firms located around a lake producing pollution that causes the fixed costs of all firms to increase.*

A: *Continue to assume that each output unit that is produced results in an increase of fixed costs of δ for all firms in the industry.*

(a) *Begin by illustrating the market demand and long run industry supply curves, labeling the market equilibrium as A.*

Answer: This is illustrated in panel (a) of Exercise Graph 21.11 where S^{LR} is the long run industry supply curve and X^* is the market outcome.



Exercise Graph 21.11 : Policy on the Lake

(b) *Next, without drawing any additional curves, indicate the point B in your graph where the market would be producing if firms were taking the full cost of the pollution they emit into account.*

Answer: This is also done in panel (a) of Exercise Graph 21.11. (The point would arise from the intersection of demand with S^B — the long run industry supply curve that would emerge if all firms accounted for the full cost of the pollution they produce (as Barney does in the previous exercise).)

(c) *Illustrate the Pigouvian tax that would be necessary to get the market to move to equilibrium B.*

Answer: This, too, is illustrated in panel (a) of Exercise Graph 21.11 where the tax is t — resulting in output level X^{opt}, price p_d for consumers and price p_s for producers.

(d) *Suppose N^* is the number of firms in the industry in the market outcome, N^{opt} is the optimal number of firms and δ continues to be as defined throughout. What does the government have to know in order to implement this Pigouvian tax? Is what the government needs to know easily observable prior to the tax?*

Answer: Each unit of output results in an increase in fixed costs by δ for *each* of the firms in the industry. For the tax to be equal to the marginal social cost of pollution, it must therefore be equal to δN for the N that will exist in the optimal outcome. The optimal Pigouvian tax is then $t = \delta N^{opt}$ — implying the government must know how many firms *should* exist. Simply observing how many firms N^* *do* exist is not enough — implying that it is not sufficient for the government to know what is easily observable in the absence of the tax.

(e) *Where in your graph does consumer surplus before and after the tax lie?*

Answer: The consumer surplus before the tax is illustrated as area $(a+b+c)$ in panel (a) of Exercise Graph 21.11, while the consumer surplus after the tax is illustrated by area (a).

(f) *Keeping in mind what you concluded in exercise 21.9, has (long run) producer surplus — or long run industry profit — changed as a result of the tax?*

Answer: No — long run industry profit is zero before and after the tax because of entry and exit. Our usual practice of measuring producer surplus on the supply curve below the price does not apply here because the reason for the upward sloping industry supply curve is not that some firms are better at producing outcome than others — all firms here are equal, and the upward slope comes from the externality. Thus, all firms make zero profit before and after because of entry and exit.

(g) True or False*: The pollution cost under the Pigouvian tax is, in this example, equal to the tax revenue that is raised under the tax.*

Answer: This is true. The total pollution cost when the industry settles at B is equal to $\delta N^{opt}X^{opt}$ because the pollution damage is δN^{opt} for *each* unit of output produced in the industry. The tax revenue is tX^{opt} — and $t=\delta N^{opt}$.

(h) *Is there* additional pollution damage *under the market outcome (in the absence of the tax)?*

Answer: Yes, there is additional pollution damage. In the market outcome, the total pollution damage is $\delta N^* X^*$ — and both the number of firms and the overall output level are higher at A than they are at B. One way to approximate this additional damage is through area $(c+e+f)$ in panel (a) of Exercise Graph 21.11 — but this is actually an underestimate of the additional damage because the marginal social cost with N^* firms is δN^* while the marginal cost with N^{opt} firms is δN^{opt}; i.e. the marginal social damage for the first X^{opt} units of output is larger when there are N^* firms — and thus larger than $(b+d)$, and the additional damage for the units between X^{opt} is also larger than indicated by the areas for the same reason. The externality cost under the market outcome (in the absence of taxes) is therefore larger than the area $(b+c+d+e+f)$.

(i) *Is there a deadweight loss from not using the tax?*

Answer: Yes, there is. We concluded already that consumer surplus shrinks from $(a+b+c)$ to (a) under the tax, the pollution cost decreases from something larger than $(b+c+d+e+f)$ to $(b+d)$ and the tax revenue under the tax is $(b+d)$. Given that industry profits are zero before and after, we get total surplus increasing from something less than $(a-d-e-f)$ to (a) — implying a deadweight loss larger than $(d+e+f)$ in the absence of the tax.

(j) *Suppose the government instead wanted to impose a cap-and-trade system on this lake — with pollution permits that allow a producer to produce the amount of pollution necessary to produce one unit of output. What* *is the "cap" on pollution permits the government would want to impose to achieve the efficient outcome? What would be the rental rate of such a permit when it is traded?*

Answer: The government would then want to set the overall pollution permits (or vouchers) to X^{opt} — creating a market in pollution voucher that would settle at an equilibrium rental rate of $r^* = t$ (where t is the optimal Pigouvian tax). This voucher market is illustrated in panel (b) of Exercise Graph 21.11.

(k) *What would the government have to know to set the optimal cap on the number of pollution permits?*

Answer: The government would have to know X^{opt} — the optimal output level in the industry. This appears different than the information required for the creation of an optimal Pigouvian tax — which was δ and N^{opt}. In the end, however, the government would have to know δ and N^* to really estimate the optimal level of industry output — so there is not that much difference in how much the government needs to know to set optimal tax or voucher policies in this example — and in the real world it probably never has enough information to really conclude what is "optimal". The advantage of tradable pollution permits does not actually show up in this example: Conditional on deciding how much pollution is acceptable generally, tradable vouchers have the advantage that firms that might differ in their ability to reduce pollution will self select — with those for whom it is difficult buying permits and those for whom it is easy simply reducing pollution instead.

B: *Continue with the functional forms for costs and demand as given in exercises 21.9 and 21.10. Suppose, as you did in parts of the previous exercises, that* $\beta = 1$*,* $\delta = 0.1$ *and* $A = 10,580$ *throughout this exercise.*

(a) *If you have not already done so in part (f) of exercise 21.9, determine the Pigouvian tax that would cause producers to behave the way the social planner would wish for them to behave. What price will consumers end up paying and what price will firms end up keeping under this tax?*

Answer: As we calculated in the previous exercise, the Pigouvian tax is $t = 18.78$ — with consumer prices rising from \$46 to \$56.34 and producer price falling from \$46 to \$37.56.

(b) *Calculate (for our numerical example) consumer surplus with and without the Pigouvian tax. (Skip this if you are not comfortable with integral calculus.) Why is (long run) producer surplus — or long run profit in the industry — unchanged by the tax?*

Answer: Answer: Since we know from the text note in the previous exercise that the demand curve is one that can arise from a single representative consumer for whom x is quasilinear, we know we can treat the demand curve as a marginal willingness to pay curve — and the demand *function* $x_d = (10580/p)^2$ as a compensated demand function. This implies that consumer surplus is

$$\int_p^\infty \left(\frac{10580}{p}\right)^2 dp = \frac{10580^2}{p}. \quad (21.11.i)$$

Evaluating this at the original price of 46 and the after tax price of 56.34, we get

$$CS_{before} = 2,433,400 \text{ and } CS_{after} = 1,986,802. \quad (21.11.ii)$$

(Without rounding error, $CS_{after} = 1,986,863$.)

Producer surplus — or long run profit — remains zero in both cases as discussed in part A(f) of this exercise.

(c) *Determine the total cost of pollution before and after the tax is imposed.*

Answer: In the previous exercise, we calculated the number of firms to be $N^* = 2,300$ and the industry output to be $X^* = 52,900$ in the absence of taxes. The total cost of pollution before the tax is imposed is

$$\delta N^* X^* = 0.01(2300)(52900) = 662,288. \quad (21.11.iii)$$

We also calculated the optimal number of firms to be $N^{opt} = 1,878$ and the optimal industry output to be $X^{opt} = 52,267$. The total cost after the optimal tax is imposed is therefore

$$\delta N^{opt} X^{opt} = 0.01(1878)(35267) = 662,314. \quad (21.11.iv)$$

(Without rounding error, the latter would be 662,288 instead.)

(d) *Determine tax revenue from the Pigouvian tax.*

Answer: The tax rate we calculated is $t = 18.78$ per unit of output, and the after-tax industry output level was calculated as $X^{opt} = 52,267$, giving us a tax revenue of

$$TR = 18.78(53,267) = 662,314. \quad (21.11.v)$$

(In the absence of rounding error, this would be 662,288.)

(e) *What is the total surplus before and after the tax — and how much deadweight loss does this imply in the absence of the tax?*

Answer: Subtracting the pollution cost from the consumer surplus prior to the tax, we get total surplus before the tax equal to $1,216,700. After the tax, the tax revenue is exactly offset by the pollution cost — leaving us with total surplus equal to consumer surplus. This is $1,986,802 (or, without rounding error, $1,986,863). Subtracting the surplus before from the surplus after, we get the deadweight loss (from not imposing the Pigouvian tax) equal to $770,102 (or $770,163 in the absence of rounding error).

(f) *Suppose next that the government instead creates a tradable pollution permits — or voucher — system in which one voucher allows a firm to produce the amount of pollution that gets emitted from the production of 1 unit of output. Derive the demand curve for such vouchers.*

Answer: The pollution level allowed by each voucher is set such that 1 voucher v allows production of 1 unit of x. We can thus replace x with v — and the demand will be the difference between the market demand curve for x ($p = A/x^{0.5}$) and the long run industry supply curve for x ($p = 2(\beta\delta x)^{0.5}$). The rental rate r for a voucher is therefore

$$r(v) = \frac{A}{v^{1/2}} - 2(\beta\delta v)^{1/2} = \frac{10,580}{v^{1/2}} - 0.2x^{1/2}. \qquad \text{(21.11.vi)}$$

(g) *What is the optimal level of vouchers for the government to sell — and what will be the rental rate of the vouchers if the government does this?*

Answer: The optimal level of vouchers is the level that results in the optimal level of industry output (which we calculated to be 35,267). Thus, $v^{opt} = 35,267$. Plugging this into the demand curve for vouchers, we get a rental rate of

$$r(35,267) = \frac{10,580}{(35,267)^{1/2}} - 0.2(35,267)^{1/2} = 18.78. \qquad \text{(21.11.vii)}$$

Note that this is exactly equal to the optimal per-unit Pigouvian tax because we defined the amount of pollution permitted by one voucher to be sufficient to produce one unit of output.

Conclusion: Potentially Helpful Reminders

1. In the case of polluting firms, a Pigouvian tax is a tax on the output produced by such firms. It differs from a pollution tax which is a tax that is directly levied on the pollution that is emitted, not on the level of output that is produced. Firms can alter the amount of Pigouvian tax that they owe by altering their production level; they can alter how much in a pollution tax to pay by either altering their output level or finding less polluting ways to produce output.

2. A pollution tax therefore prices pollution directly. It is for this reason that pollution taxes are quite similar to cap-and-trade systems (or pollution voucher systems). These systems also price pollution directly.

3. The difference between pollution taxes and cap-and-trade systems is that the former sets the price of pollution and then allows the level of pollution to be determined in the markets — while the latter sets the amount of pollution and allows the price for pollution to be determined in the market.

4. Yet another way in which externalities like pollution can be priced is through the establishment of property rights. It is for this reason that we say that the "market failure" from externalities is actually a failure of markets to exist — because in the presence of markets that price externalities, there would be no market failure. This relates closely to the "Tragedy of the Commons" — the "tragedy" of market failure that arises when property rights have not been assigned and individuals therefore have an incentive to over-exploit resources.

5. The Coase Theorem is closely linked to the idea of property rights — because it again tells us that, when property rights are assigned, there are ways for individuals to overcome externality problems even if no explicit market is established. But the theorem also focuses on the importance of transactions costs which often keep individuals from solving externality problems even when property rights are assigned.

CHAPTER

22

Asymmetric Information in Competitive Markets

This is our final chapter dealing with competitive markets — and our final investigation of violations of the first welfare theorem within competitive settings. The issue we are concerned about in this chapter is hidden information; i.e. information that is relevant to a transaction but which only one party to the transaction has. Such hidden information opens the possibility of one side of the market "exploiting" the other — such as the owner of a used car misinforming the potential buyer of its quality. It also opens the possibility of high quality producers not being able to sell their goods because low quality producers are successfully imitating them, with consumers unable to tell the difference when purchasing the product. One market in which we know the issue of hidden, or asymmetric, information to be important is the insurance market where those seeking insurance might have information about their circumstances that insurance companies cannot see. And it is within the context of insurance markets that many of the ideas related to adverse selection are treated in much of this chapter.

Chapter Highlights

The main points of the chapter are:

1. The **adverse selection problem** (in output markets) arises when a full market cannot be sustained in equilibrium because of the negative externality imposed by high cost consumers on low cost consumers. Put differently, when firms cannot tell high cost from low cost consumers (in areas like health insurance), they may be unable to price their product so that low cost consumers are willing to buy it — which may also imply that the price would rise to a point where even high cost consumers will not buy it.

2. If less informed parties (like insurance companies) know that certain identifiable groups are more likely to contain high cost consumers, they will use

group characteristics to **statistically discriminate** against particular groups (even if they have no prejudice against that group.)

3. **Moral hazard** refers to the change in behavior an individual will undertake once a contract is entered into — such as the tendency to engage in riskier behavior once health insurance is obtained. If firms cannot tell who is more and who is less likely to engage in moral hazard, this can aggravate the adverse selection problem.

4. **Signals** and **screens** are often used by firms, consumers and workers in order to overcome at least part of the adverse selection problem. Signals are used by the more informed party to "signal" information to the less informed party; screens are used by the less informed party to "screen" for information that the other party has.

5. **Information** is costly — and so, even if it can be used to re-establish markets through signals and screens, the efficiency problem raised through adverse selections is not necessarily fully addressed. At the same time, policy solutions are problematic to the extent to which they rely on gathering information that is costly to obtain.

6. There are different types of equilibria that might emerge under asymmetric information: **pooling equilibria** (in which different types of consumers (or producers or workers) "pool" and behave identically despite being different; **separating equilibria** in which different types behave differently and information about them is therefore revealed; or some combination of the two.

7. Understanding asymmetric information allows us to understand better issues such as **discrimination** which may happen because of inherent prejudice or because of statistical discrimination due to asymmetric information.

22A Solutions to Within-Chapter-Exercises for Part A

Exercise 22A.1

What would be the equilibrium insurance premium if, in a system that forced all students to buy insurance, the only insurance policy offered were one that guarantees a B? What if the only policy that were offered was one that guaranteed a C?

Answer: If only B-insurance were sold, I would have to pay $3c$ for the 10% of the class that would otherwise get an F, $2c$ for everyone that would otherwise get a D, c for everyone that would otherwise get a C and nothing for those who earn a B or A. Thus, the insurance premium would be

$$p_B = 0.1(3c) + 0.25(2c) + 0.3(c) = 1.1c. \qquad (22A.1.i)$$

Similarly, the mandatory C-insurance would cost

$$p_C = 0.1(2c) + 0.25(c) = 0.45c. \tag{22A.1.ii}$$

Exercise 22A.2

In an efficient allocation of grade insurance (when only A-insurance is offered), who would have A-insurance? (*Hint*: Compare the total cost of raising each student type's grade to the total benefit that this would yield for each student type.)

Answer: The cost of raising the grade for a B student to an A is c; for a C student it is $2c$; for a D student it is $3c$ and for an F student it is $4c$. The benefit to the B student is $2c$; for the C student it is $2.5c$, for the D student it is $3c$ and for the F student it is $3.5c$. The cost therefore exceeds the benefit for F students — which implies they should not hold A insurance in an efficient allocation. B and C students definitely should. For D students, it costs the same as it benefits them — so it would be efficient for them to hold or not hold A insurance.

If you are tempted to say that the costs translate to benefits for the professor who is paid off, that is not so — we are paying him to overcome his scruples, not simply transferring money. And c is the least we have to pay him for each grade level change.

Exercise 22A.3

If all types of insurance policies were available — i.e. A-insurance, B-insurance, etc. — who would have what type of insurance under efficiency? (*Hint*: Compare the marginal cost of raising each student type's grade by each level to the marginal benefit of doing so.)

Answer: For all students, the benefit of the first level change in the grade is $2c$ — and the cost for that is c. So it is certainly efficient to raise every student's grade by one level. But the benefit of each additional level is only $0.5c$, with the marginal cost remaining c. Thus, it is not efficient to raise any student's grade more than one level. B students should therefore hold A insurance; C students should hold B insurance; D students should hold C insurance and F students should hold D insurance.

Exercise 22A.4

Verify that my break-even insurance premium for A-insurance would have to be approximately $2.69c$ if only the 65 C, D and F students bought the insurance.

Answer: I would have to pay $4c$ for the 10 F-students, $3c$ for the 25 D-students and $2c$ for the 30 C-students — for a total of $175c$ for all 65 students who bought insurance. The average cost per student is then $175/65 \approx 2.69$.

Exercise 22A.5

Would I be able to sell A insurance if students were always willing to pay $2c$ for every increase in their letter grade? Would the resulting equilibrium be efficient?

Answer: We know that, if all students buy A-insurance, the zero-profit premium for the insurance is $2c$. But A-students (assuming they know they are A-students) will not buy the insurance at any price. We also know that, if all students other than A students buy the insurance, the zero-profit premium is approximately $2.22c$. But at that price, B-students are not willing to buy the insurance (since they are only willing to pay $2c$.) If all students other than A and B-students buy the insurance, we have concluded that the zero-profit price is approximately $2.69c$. A C-student is now willing to pay $4c$ for the A-insurance — $2c$ to raise the grade to a B and $4c$ to raise it to an A. Thus, C-students are willing to pay $2.69c$, as are D and F-students. Thus, I would indeed be able to sell A-insurance if students are willing to pay $2c$ for every increase in their letter grade by one level — with 65% of the class purchasing such insurance and the remaining 35% choosing not to purchase it at the price $2.69c$. But this equilibrium is not efficient — because efficiency would require B students to also hold insurance (since it costs only c to raise their grade to an A but they would benefit by $2c$.)

Exercise 22A.6

What would be the equilibrium price p_A^F for an F student if that student will earn an F with 75% probability and a D with 25% probability?

Answer: Such a student would cost me $3c$ 25% of the time and $4c$ 75% of the time — implying a zero-profit price of $p_A^F = 3.75c$.

Exercise 22A.7

Conditional on only B insurance being allowed, is this equilibrium efficient?

Answer: Yes, this equilibrium is efficient. Every student that buys insurance obtains a positive surplus, and no one who could benefit from insurance is excluded from the market.

Exercise 22A.8

Conditional on only B insurance being allowed, is this equilibrium efficient?

Answer: No, this equilibrium is not efficient because we could increase overall surplus by forcing all C, D and F students to buy the insurance. In that case, the price would fall from $2.29c$ to $1.69c$, increasing the surplus for D and F-students by $0.60c$. In a class of 100 students, there are 35 such students — causing the increase in surplus for them to be $35(0.6c)$=$21c$. The C-students would be forced to buy insurance they value at $1.5c$ at a price of $1.69c$ — causing each of them to incur negative surplus of $0.19c$. Since there are 30 of them in every class of 100, this implies an overall loss of $30(0.19c)$=$5.7c$. Subtracting this from the gain of $21c$ for the

D and F students, we get an overall increase in surplus of $15.3c$. Thus, the D and F-students would be willing to pay the C students to enter the insurance market — and in principle everyone could be better off.

Exercise 22A.9

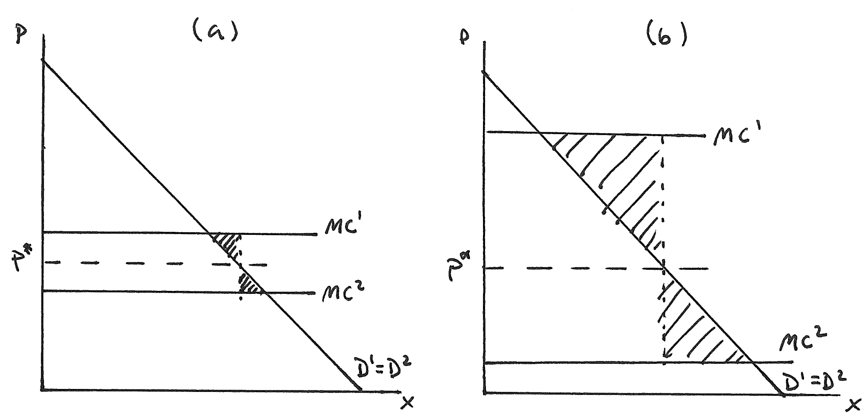
Suppose the current market price for car insurance were less than p^*. What would happen under perfect competition with free entry and exit? What if instead the market price for car insurance were greater than p^*?

Answer: If price were less than p^*, some insurance firms would exit the market given that all firms are making negative long run profit — and firms would keep exiting until price is at p^*. The reverse would happen if price were above p^* — firms would be making positive long run profit, which implies new firms will enter and drive down the price until it reaches p^*.

Exercise 22A.10

True or False: The greater the difference between MC^1 and MC^2, the greater the deadweight loss from the introduction of asymmetric information.

Answer: This is true, as indicated by the shaded deadweight loss areas in panels (a) and (b) of Exercise Graph 22A.10.

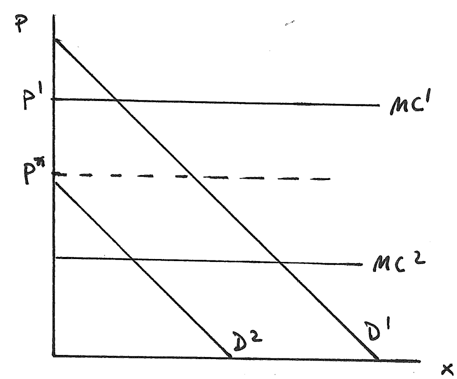


Exercise Graph 22A.10 : Inefficiency grows as MC differences grow

Exercise 22A.11

Suppose that type 1 consumers valued car insurance more highly — implying D^1 lies above D^2. Can you illustrate a case where the introduction of asymmetric information causes type 2 consumers to no longer purchase any car insurance? What price would type 1 consumers then pay?

Answer: This is illustrated in Exercise Graph 22A.11 where the price p^* falls above the intercept for the demand curve for type 2 consumers. Since only type 1 consumers will buy insurance in equilibrium, $p^1 = MC^1$ would be their price.



Exercise Graph 22A.11 : Asymmetric Information and the Disappearance of a Market

Exercise 22A.12

How much do type 1 consumers lose? How much do type 2 consumers gain? What is the net effect on overall consumer surplus?

Answer: Consumers of type 1 lose $(b+c)$ while consumers of type 2 gain $(d+e+f)$. Area (b) is equal to area (d) — so those cancel out. Area (e) is equal in size to area $(c+g)$ — which then implies that type 2 consumers gain the equivalent of $(f+g)$ more than type 1 consumers lose. This is exactly the size of the deadweight loss from asymmetric information — i.e. the full deadweight loss is recovered in the separating equilibrium in which screening is costless.

Exercise 22A.13

Why is the screening cost equal to area $(d+e)$?

Answer: The reason that safe drivers are charged p^* instead of MC^2 in the separating equilibrium is the screening costs that firms have to pass onto type 2 consumers. Summing this difference between p^* and MC^2 across all the type 2 consumers gives us area $(d+e)$.

Exercise 22A.14

Why do firms in this case pay a screening cost that does not allow them to lower any premiums? (*Hint*: Think about whether — given that everyone else pays for the screening costs and discovers who are the safe and unsafe drivers — an individual firm can do better by not discovering which of its potential customers are type 1 and which are type 2.)

Answer: Suppose every firm pays the screening costs and charges type 1 consumers p^1 while charging type 2 consumers p^*. Now imagine owning a firm and deciding not to screen. As a result, you don't have to pay the screening costs — but you also don't know who is type 1 and who is type 2. You therefore have to charge everyone the same price. If you set your price above p^1, you will get no customers. If you set it between p^* and p^1, all the type 1 drivers will buy from you but all the type 2 drivers will continue to buy at p^* from the other firms. Thus, you would make negative profits. And if you set price below p^*, both type 1 and 2 drivers will buy from you — implying again a negative profit. Thus, the best you could do is set price to p^* — which is the zero profit price. You therefore cannot do better than zero profit by deviating from the equilibrium in which everyone pays the screening cost and a separating equilibrium results.

Exercise 22A.15

Could there be a screening-induced separating equilibrium in which p^2 is higher than p^*?

Answer: No. Suppose there were such an equilibrium. We know that any separating equilibrium must have $p^1 = MC^1$. (This is because otherwise new firms could emerge and make profit by charging less and getting all the type 1's.) The screening cost must, as noted in the text, be passed onto the type 2 consumers. If this leads to $p^2 > p^*$, however, the following would happen: New firms would emerge, avoid all the screening costs and just charge a price above p^* and below p^2. All consumers would want to buy from such firms (since we have assumed that p^1 and p^2 lie above p^* in the separating "equilibrium") — which would imply that firms charging this price between p^* and p^2 could make positive profit. Since firms have an incentive to not do what we proposed in the initial separating equilibrium, that initial equilibrium was in fact not an equilibrium.

Exercise 22A.16

Would your analysis be any different if the insurance companies did the screening themselves rather than hiring firms in a separate industry to do it for them?

Answer: No, all of our analysis would be exactly the same whether the costs are paid to a screening firm or whether they are paid internally within the firm.

Exercise 22A.17

True or False: When it is costless to tell the truth and very costly to lie, consumer signaling will unambiguously eliminate the inefficiency from adverse selection.

Answer: This is true — the safe driver for whom it is costless to reveal their type will do so, and the unsafe drivers for whom it is too difficult to lie will stay silent. Thus, firms will know who the safe drivers are (because they reveal it) and who the unsafe drivers are (because they stay silent) — and thus all information is revealed. Without any further asymmetric information, we then get $p^1 = MC^1$ and $p^2 = MC^2$ as efficiency demands.

Exercise 22A.18

Suppose $\delta = (p^* - MC^2)$ and $\gamma > (MC^1 - MC^2)$. What is the *increase* in dead weight loss in going from the initial pooling equilibrium to the separating equilibrium?

Answer: In this case, type 1 consumers will not signal but type 2 consumers will be indifferent between signaling and not signaling (since the cost of signaling to them is exactly equal to the benefit they get from signaling). We can therefore have a separating equilibrium in which type 2 consumers signal and type 1 consumers do not, with $p_1 = MC^1$ and $p^2 = MC^2$. Type 1 consumers therefore lose $(b + c)$ in surplus. Type 2 consumers will only buy x^* because their effective price continues to be p^* (once we include the signaling cost), and thus type 2 consumers gain no surplus. They do, however, spend $(d + e)$ resources to signal their type, and no one benefits from this. Thus, the overall increase in deadweight loss is $(b + c + d + e)$.

Exercise 22A.19

True or False: If δ and γ are such that a separating equilibrium emerges from consumer signaling, the question of whether the resulting resolution of asymmetric information enhances efficiency rests only on the size of δ, not the size of γ.

Answer: This is true. In the separating equilibrium, only type 2 consumers signal — which means only they incur a cost of signaling. The only sense in which γ — the cost of falsely signaling your type if you are type 1 — matters is in terms of whether a separating equilibrium *can* emerge. If it *does emerge*, it is because γ is too high for it to be worth it to type 1 consumers to signal falsely — so they will not engage in any signaling and simply accept $p^1 = MC^1$.

Exercise 22A.20

Is it possible under these conditions for there to also be a pooling equilibrium in which no one sends any signals? (*Hint*: What would insurance companies have to believe in such an equilibrium if they did see someone holding up the "I am safe" sign?)

Answer: Yes, it is possible if insurance companies believe that anyone holding up an "I am safe" sign is a lying type 1. In that case, no one would claim to be safe — because doing so would be counterproductive. When no one sends signals, everyone just pays p^* — and the pooling equilibrium is simply one without signaling. In part B of our game theory chapter, this is discussed as a "Bayesian" equilibrium in which beliefs off the equilibrium can take on any form — including one where insurance companies believe everyone who does send a signal (which no one does in equilibrium) is a liar.

Exercise 22A.21

Suppose $(p^* - MC^2) < \delta = \gamma < (MC^1 - MC^2)$. Will there be a separating equilibrium? (*Hint*: The answer is no.)

Answer: The best that the safe drivers can hope to do by credibly signaling their type is to get the insurance premium $p^2 = MC^2$. Thus, the most they have to gain is $(p^* - MC^2)$ — and if their signaling cost is above this, it is not worth signaling. Type 1 can gain more from falsely signaling if the signals are believed — but if type 2 consumers don't signal and type 1 consumers falsely signal, insurance companies will be able to tell type 1 from type 2 because only type 1 signals. Thus, if they observe a signal, they will actually offer $p^1 = MC^1$ as the premium — but type 1 consumers get p^* as a premium if they just don't signal.

Exercise 22A.22

Why is it possible for a signaling equilibrium to result in a pooling equilibrium in which no information is revealed — but it is not possible to have such a pooling equilibrium emerge when firms screen?

Answer: In a signaling equilibrium, individual consumers decide whether to signal — and if all of them signal and insurance companies believe that anyone not signaling is unsafe, it is an equilibrium for all of them to do so (assuming the costs of signaling fall within the right range). No information is revealed, but not sending a signal would reveal "bad" information about yourself. In the screening equilibrium, the firms decided whether to pay for information — and there is no point for an insurance company to pay for useless information that does not permit them to identify who the safe and who the unsafe drivers are. Since the only way firms will pay for information is if it is informative, screening can only result in a separating equilibrium.

Exercise 22A.23

Another factor that lessens the adverse selection problem in life insurance markets is that the bulk of demand for life insurance comes from people who are young to middle aged and not from the elderly. How does this matter?

Answer: The young to middle aged are much less likely to have particular knowledge of health conditions that might cause premature death within the horizon of a life insurance policy. Their demand for life insurance is usually motivated by the fact that children in the household are not yet grown up and might therefore need resources in the event of a premature death. Were there a high demand for life insurance from the elderly, one might worry a lot more about such demand arising from specific information about their health that individuals know but the insurance company does not know.

Exercise 22A.24

In our car insurance example, asymmetric information caused the market to create a pooling equilibrium in which some over-consumed and others under-consumed. Why might this not be the case in the unemployment insurance market where those with high demand are much more likely to be those with high probability of being laid off? (*Hint*: Can you imagine an unraveling of the market for reasons similar to what we explored in the grade insurance case?)

Answer: In the car insurance market, we assumed that both types have the same demand curve for insurance. If the demand for unemployment insurance is significantly higher the more likely someone is to get laid off, we would get something more like what we explored in exercise 22A.11 — where the pooling equilibrium price does not attract any low demand types and thus only the high demand individuals buy insurance. If we make the example richer — with more types — we can get an unraveling of a pooling equilibrium from the bottom up — with first the tenured professors exiting the market which then drives up premiums which then causes some others to exit the market which further drives up premiums, etc. This is exactly analogous to the grade insurance example where demand for A insurance increased the worse a student you were (just as we might assume the demand for unemployment insurance increases the more likely you are to be unemployed)— and this caused the unraveling of an equilibrium in which the A insurance ended up not being sold at all.

Exercise 22A.25

Is mandatory participation in government unemployment insurance efficient — or do you think it might just be more efficient than market provision?

Answer: Forcing everyone into the same insurance pool is almost certainly not efficient, but — if the market for unemployment insurance would unravel due to adverse selection in the absence of government interference, it might well be significantly more efficient than no government involvement.

Exercise 22A.26

Consider used car dealerships in small towns. How might *reputation* play a role similar to brand names in addressing the asymmetric information problem?

Answer: It is often said that in small towns, everyone knows everyone. That's not quite true — but it is more likely that everyone knows someone that knows how someone else generally behaves. Used car dealers might know more about the cars they sell than customers — and can therefore take advantage. But they can't do so for long if they develop a reputation for over-selling the quality of their cars. Just like brand names in some markets are signals that convey information through establishing reputations for quality, small towns might have informal mechanisms for such reputations to develop more directly and thus discipline the behavior of those who have more information in markets like the used car market.

Exercise 22A.27

What is *Consumer Reports* analogous to in our discussion of car insurance?

Answer: It is analogous to the "screening industry" that identifies high and low cost types for car insurance companies — and then sells that information to them in order for the car insurance industry to be able to price policies differentially depending on how safe a driver you are. In the Consumer Reports case, the information asymmetry lies on the other side of the market — firms know more about their product's quality than consumers. Thus, the screening industry that Consumer Reports is part of provides a service to consumers (rather than firms) by revealing information that is otherwise too costly to access.

Exercise 22A.28

One of the most popular features of "Obamacare" is the requirement that insurance companies can no longer discriminate based on preexisting health conditions, and one of the least popular features of "Obamacare" is the mandate that everyone has to buy insurance. Explain why many analysts argued during the 2010 health care debate that if you like the first feature, you will have to accept the second?

Answer: If you prohibit insurance companies from discriminating based on preexisting health conditions, the adverse selection problem is unleashed in the following sense: Healthy individuals, knowing they can get insurance if they get sick, may choose to opt out of purchasing health insurance – thus driving up premiums as predominantly sicker individuals select into the health insurance market, making insurance unaffordable for those most in need of health care. The mandate that everyone has to buy insurance eliminates this adverse selection problem by forcing the healthy to be part of the insurance pool, thus keeping premiums low.

Exercise 22A.29

Explain the moral hazard problem that is referred to in the above paragraph. In what way would the proposal for individual health savings accounts address this problem?

Answer: Moral hazard occurs when individuals change behavior after entering a contract. In this case, the behavior we are concerned about is the overconsumption of routine health care once individuals are fully insured and don't face the marginal cost of health care procedures. Health savings accounts would take care of this in the sense that individuals would now confront the marginal cost of each health care procedure since they would have to pay for it through their health savings account.

Exercise 22A.30

Which of the following possibilities makes it more likely that widespread college attendance is efficient: (1) Colleges primarily provide skills that raise marginal product, or (2) colleges primarily certify who has high marginal product.

Answer: The first of these unambigously creates an efficiency role for colleges so long as students pay the bulk of the expense of their education. (If they do not, then students may attend college despite the fact that the cost is higher than the present discounted value of their increased marginal product — which would not be efficient.) In the second scenario, colleges add little value but simply reveal information. Given the high cost of college eduction, this seems less promising as a potential efficiency enhancing role if it became the main role for colleges. (Put differently, there are surely cheaper ways to uncover the information as to who has high and who has low marginal product.)

Exercise 22A.31

What are we implicitly assuming about the costs of screening applicants in these markets?

Answer: We are assuming that it is too costly to screen for individual characteristics — and cheaper to just statistically discriminate based on easily observed characteristics that correlate with group probabilities.

Exercise 22A.32

True or False: Statistical Discrimination leads to equilibria that have both "separating" and "pooling"' features.

Answer: This is true. The whole point of statistically discriminating is to divide people into groups based on easily observable characteristics — and then to offer *different* contracts to members of different groups. In this sense, statistical discrimination leads to a separating equilibrium. At the same time, firms are unable to get to all the information — and thus unable to fully separate individuals into different contracts based on their true underlying characteristics. By using characteristics like gender as a proxy, they are in effect *pooling* individuals with different underlying characteristics and offering those individuals (that all belong to the same group) the same contracts (despite the fact that individuals within a group would be offered different contracts if all the information were fully known.)

Exercise 22A.33

Suppose public schools invested more resources into gender sensitivity training in hopes of lessening gender discrimination in the future. Would you recommend this if you knew that gender discrimination was purely a form of statistical discrimination?

Answer: This would not be a well-targeted policy if the problem is statistical discrimination. Under statistical discrimination against women, the discrimination does not arise from any inherent bias in favor or against women. Rather, the discrimination arises from underlying group differences. So long as these group differences exist and so long as it is too expensive for firms to uncover individual differences within groups, statistical discrimination remains economically rational (i.e. profit maximizing).

Exercise 22A.34

In the past, gender discrimination was often enshrined in statutory laws — making it illegal for firms to hire women into certain roles or schools to admit women as students. If you are one of the corporate board members in company A, why might you favor such laws even if all you care about is not having women in your own company? If you are one of the corporate board members in company C, would you similarly favor such a law?

Answer: Such a law will protect company A from the market penalty its stock might incur if other companies hire from a larger pool of talent and thus become more dollar profitable because they don't discriminate based on prejudice. If you are one of the company C board members, on the other hand, you discriminate only to the extent that asymmetric information makes statistical discrimination profitable. You would in fact prefer others to not discriminate — because company C is more profitable than company B. Thus, you should not favor a law mandating discrimination against women.

Exercise 22A.35

True or False: In the above example, the asymmetric information that leads to statistical discrimination against African Americans is still rooted in discrimination based on prejudice — but it may be rooted primarily in prejudice-based discrimination from the past.

Answer: This is true — the reason there are group differences that lead to statistical discrimination may be found in past discrimination based on prejudice. For instance, African American children attend worse public schools because public schools in general are worse in poorer areas and African American children are more likely to live in poor areas. They are more likely to live in poor areas because their parents are more likely to be poor — and their parents may be more likely to be poor because of past discrimination.

22B Solutions to Within-Chapter-Exercises for Part B

Exercise 22B.1

Explain why such contracts are actuarily fair.

Answer: Consider an individual of type δ. If this individual buys an insurance contract (b,p), it means that she will receive the benefit b with probability δ while paying the premium p with probability 1 (since the premium is due regardless of whether the good or bad state of the world hits.) The insurance contract is actuarily fair if the expected benefit is equal to the expected cost of the contract to the consumer — i.e. if $\delta b = p$. An analogous argument implies that an actuarily fair insurance contract for a θ type has $\theta b = p$.

Exercise 22B.2

Why is $(b,p) = (240,60)$ an insurance contract that provides full insurance to a δ type consumer?

Answer: Under this contract, consumption in state 1 becomes $x_1 = 10+240-60 = 190$ and consumption in state 2 becomes $x_2 = 250-60 = 190$. Thus, $x_1 = x_2$ — i.e. the consumer enjoys the same consumption in the good and bad states under this insurance contract.

Exercise 22B.3

What would indifference curves look like for risk neutral consumers? What about risk loving consumers?

Answer: For risk neutral individuals, indifference curves would be straight lines with slope equal to the slope of the actuarily fair contract "budget". For risk averse individuals, the indifference curves would point in the opposite direction.

Exercise 22B.4

Demonstrate that full insurance for type θ implies the same benefit level as for type δ.

Answer: Under the insurance contract (b,p), consumption in the bad state is $x_1 - p + b$ while it is $x_2 - p$ in the good state. Actuarial fairness implies $p = \theta b$ — which implies that the consumption level in the bad state can be written as $x_1 + (1-\theta)b$ and the consumption level in the good state can be written as $x_2 - \theta b$. Setting x_1 equal to x_2 and solving for b, we then get

$$b = x_2 - x_1 = 250 - 10 = 240; \quad (22B.4)$$

i.e. b is independent of the probability θ.

Exercise 22B.5

Are these consumer types risk averse?

Answer: Yes. There are two ways to tell: One is by looking at the utility function $u(x) = \alpha x$ that is concave. (You can check that the second derivative of the function with respect to x is negative.) The second way is to look at the expected utility functions U^{δ} and U^{θ}. Note that these have the Cobb-Douglas form — i.e. they take the form of a typical Cobb-Douglas utility function where we have simply transformed the function by taking logs. Thus, indifference curves in the space with x_1 on the horizontal and x_2 on the vertical axis take the usual convex shape — and we showed in Chapter 17 that this implies risk aversion.

Exercise 22B.6

Set up the expected utility maximization problem for θ types and derive the optimal choice assuming they face an actuarily fair insurance menu?

Answer: The optimization problem is

$$\max_{b,p} \ \theta\alpha\ln(x_1+b-p)+(1-\theta)\alpha\ln(x_2-p) \ \text{ subject to } \ p=\theta b. \tag{22B.6.i}$$

You can either set up the corresponding Lagrange function or simply substitute the constraint into the objective and set the derivative with respect to b to zero. Either way, you will get the answer that the optimal contract has

$$b=x_2-x_1 \ \text{ and } \ p=\theta(x_2-x_1). \tag{22B.6.ii}$$

Exercise 22B.7

How do these results relate to the values in Graph 22.3?

Answer: For $\delta=0.25$, $\theta=0.5$, $x_1=10$ and $x_2=250$, this implies $(b,p)=(240,60)$ for δ types and $(b,p)=(240,120)$ for θ types — exactly as drawn in the graph.

Exercise 22B.8

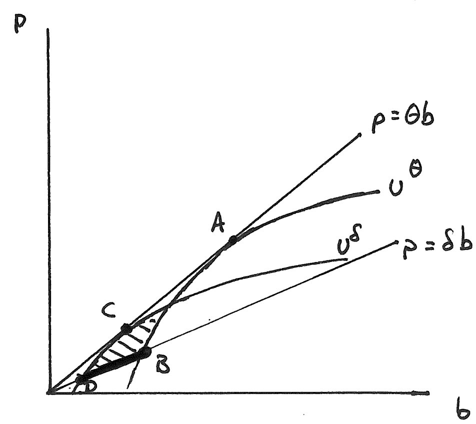
Suppose insurance companies offer all actuarily fair insurance contracts to type θ. Can you identify in panel (b) of Graph 22.4 the area representing all insurance contracts that consumers of type δ would purchase rather than choosing from the menu of contracts aimed at type θ?

Answer: Of the contracts on the $p=\theta b$ lines, C is the most preferred contract for consumers of type δ. Thus, all contracts in the region that lies to the southeast of the indifference curve U^{δ} are preferred by δ types.

Exercise 22B.9

From the area of contracts you identified in exercise 22B.8, can you identify the subset which insurance companies would be interested to offer assuming they are aware that high risk types might try to get low cost insurance?

Answer: This is illustrated as the shaded region in Exercise Graph 22B.9. The region satisfies three conditions: First, none of the contracts in the region would be chosen by high risk θ types who prefer A; second, all of these contracts are preferred by low risk δ types to any of the actuarily fair insurance contracts aimed at high risk types (of which C is the most preferred for δ types); and all of the shaded contracts lie on or above the actuarily fair insurance line $p = \delta b$ — and thus all of them would make zero or greater profit for the company that sold them to low risk consumers.



Exercise Graph 22B.9 : Insurance contracts

Exercise 22B.10

From the contracts identified in exercise 22B.9, can you identify which of these contracts could not be offered in equilibrium when the insurance industry is perfectly competitive?

Answer: Perfect competition implies that profits must be zero (in the long run) — and thus none of the shaded area in Exercise Graph 22B.9 that results in a positive profit can occur in long run equilibrium. The only contracts from the shaded region that yield zero profit (given that only low risk consumers would buy them) lie on the bold portion (from D to B) of the $p = \delta b$ line.

Exercise 22B.11

Can you verify that full insurance implies consumption of $((1-\theta)x_2 + \theta x_1)$?

Answer: In within-chapter exercise 22B.6, we showed that the full insurance benefit is $b = (x_2 - x_1)$ and the actuarily fair premium for that benefit level is $p = \theta(x_2 - x_1)$. Thus, consumption under the "bad" state would be

$$x_1 + b - p = x_1 + (x_2 - x_1) - \theta(x_2 - x_1) = (1-\theta)x_2 + \theta x_1. \quad (22B.11.i)$$

Similarly, consumption in the "good state" is

$$x_2 - p = x_2 - \theta(x_2 - x_1) = (1-\theta)x_2 + \theta x_1. \quad (22B.11.ii)$$

Exercise 22B.12

Can you show mathematically (by evaluating utilities) that this equilibrium is inefficient relative to the equilibrium identified in Graph 22.3c?

Answer: The high risk types buy the same policy in the separating equilibrium as they do in the full information equilibrium — and are therefore equally well off. The low risk types buy $(b^\delta, p^\delta) = (85.2, 21.3)$ in the separating equilibrium and $(b, p) = (240, 60)$ in the full insurance equilibrium. The expected utility for low risk δ types of an insurance contract (b, p) is

$$\begin{aligned} U^\delta(b,p) &= \alpha\delta \ln(x_1 + b - p) + \alpha(1-\delta)\ln(x_2 - p) = \\ &= \alpha\left[0.25\ln(10 + b - p) + 0.75\ln(250 - p)\right] \end{aligned} \quad (22B.12.i)$$

which gives us

$$U^\delta(240, 60) \approx 5.25\alpha \text{ and } U^\delta(85.2, 21.3) \approx 5.15\alpha. \quad (22B.12.ii)$$

Thus, utility for low risk types falls in the separating equilibrium. This implies that low risk types become worse off in the separating equilibrium while high risk types remain equally well off — which constitutes a move toward less efficiency.

Exercise 22B.13

True or False: Under perfect competition (and assuming that insurance companies incur no costs other than the benefits they pay out), risk averse individuals with state-independent tastes will fully insure in the absence of asymmetric information but may insure less than fully in its presence.

Answer: In the absence of asymmetric information, insurance companies can compete for each type's business in a separate market — with entry and exit driving profit to zero and thus causing insurance contracts to be actuarily fair. When insurance contracts are actuarily fair, risk averse individuals with state-independent tastes will fully insure (see Chapter 17). But we have just shown that the introduction of asymmetric information causes competitive firms to no longer offer full insurance to low risk types — causing some risk types to insure less than fully. The statement is therefore true.

Exercise 22B.14

Can you verify the intercepts for point C in Graph 22.4b?

Answer: Point C is the utility maximizing insurance contract for δ types when they choose from contracts that fall on the line $p = \theta b = 0.5b$. To solve for C, we therefore solve

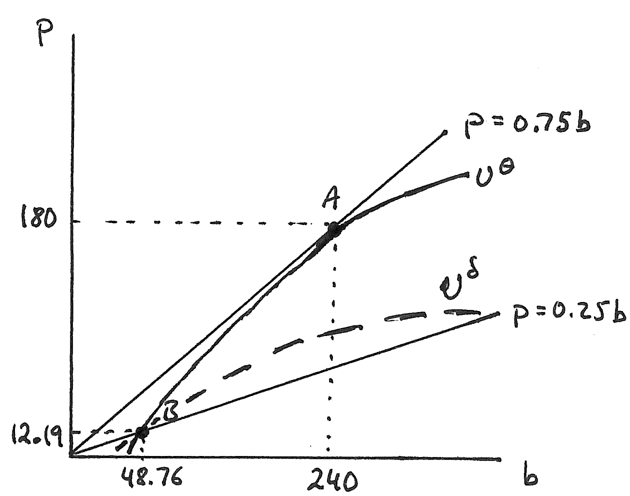
$$\max_{b,p}\ 0.25\alpha \ln(10+b-p) + 0.75\alpha \ln(250-p) \text{ subject to } p = 0.5b. \tag{22B.14}$$

Substituting the constraint into the objective, taking the derivative with respect to b and setting it to zero, we can solve for $b = 110$, and substituting this back into the constraint, we get $p = 55$. Both are as shown in the text graph.

Exercise 22B.15

Draw a graph with b on the horizontal and p on the vertical axis illustrating the separating equilibrium in row 4 of Table 22.2.

Answer: This is done in Exercise Graph 22B.15.



Exercise Graph 22B.15 : Equilibrium when $\theta = 0.75$

Exercise 22B.16

Can you show, using equation (22.12), that the last sentence is correct?

Answer: The slope of this line is $\left[\gamma\delta + (1-\gamma)\theta\right]$. Since $\delta < \theta$, and increase in γ causes this slope to decrease.

Exercise 22B.17

What is the expected value of consumption for θ types at point D? Is it higher or lower than under full insurance? Explain.

Answer: At D, the contract is $(b,p) = (176.25, 58.75)$. This implies that consumption in the bad state is $x_1 = 10 + 176.25 - 58.75 = 127.5$ while consumption in the bad state is $x_2 = 250 - 58.75 = 191.25$. Since both states are reached with probability $\theta = 0.5$, this implies an expected value for consumption for θ types of $EV^{\theta} = 0.5(127.5) + 0.5(191.5) = 159.375$. Under the actuarily fair full insurance contract $(b,p) = (240,120)$, the expected consumption is only 130. Thus, the contract D raises the expected consumption value for θ types.

Exercise 22B.18

What is the expected value of consumption for δ types at point D? Is it higher or lower than the expected value of consumption without insurance? Explain.

Answer: At D, the contract is $(b,p) = (176.25, 58.75)$. This implies that consumption in the bad state is $x_1 = 10+176.25-58.75 = 127.5$ while consumption in the bad state is $x_2 = 250 - 58.75 = 191.25$. Since the bad state is reached with probability $\delta = 0.25$ and the good state is reached with probability $(1-\delta) = 0.75$, this implies an expected value for consumption for δ types of $EV^{\delta} = 0.25(127.5) + 0.75(191.5) = 175.3125$. The expected consumption without insurance is $0.25(10) + 0.75(250) = 190$. Thus, the contract D lowers the expected consumption value for δ types — but it also lowers the risk.

Exercise 22B.19

Why must any potential pooling equilibrium contract D lie on the zero-profit pooling line?

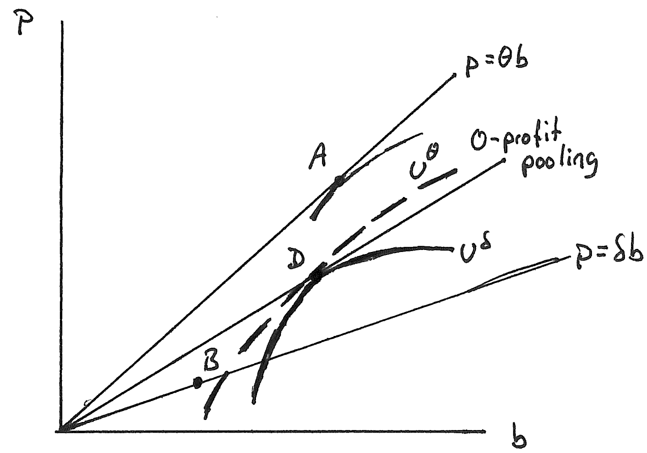
Answer: Because entry and exit of firms implies that profits for insurance companies must be zero in the long run.

Exercise 22B.20

Can you think of what would have to be true about how the blue and magenta indifference curves relate to one another at D in order for the problematic area (j) to disappear? Explain why this would then imply that D is a competitive equilibrium pooling contract.

Answer: This can happen only if the magenta indifference curve is tangent to the blue indifference curve at D. We illustrate this in Exercise Graph 22B.20.

The solid indifference curve U^{δ} is tangent to the zero profit pooling line at D and lies to the right of B, the separating equilibrium contract for low risk types. The



Exercise Graph 22B.20 : Existence of Pooling Equilibrium

dashed indifference curve U^θ is tangent to U^δ at D. To see why D is now a competitive pooling equilibrium contract, we have to check whether any other contract exists that a single insurance firm could offer and make a positive profit with. First, note that *all* contracts that are preferred by δ types to D lie to the southeast of U^δ — and *all* of them are also preferred to D by θ types. Thus, if any contract to the southeast of U^δ were offered by an insurance company, both risk types would demand it — but since all of these contracts lie below the zero-profit pooling line, this implies negative profits for the firm. Next, consider the contracts that lie between the dashed U^θ indifference curve and the solid U^δ indifference curve. Such contracts would be taken up only by high risk θ types (since they lie to the northwest of U^δ) — but all these contracts also lie below $p = \theta b$ and will thus earn negative profit if only high risk types choose them over D. Finally, consider all contracts to the northwest of U^θ. Such contracts are all less preferred than D by both risk types — and thus there is no demand for them. This exhausts all possible contracts — which implies that any firm which offers a contract other than D (given that all other firms offer D) will either make a negative profit or attract no customers. The contract D is therefore an equilibrium contract.

Exercise 22B.21

For the case where $\gamma = 1/2$ and where a pooling equilibrium therefore does not exist (as shown in Graph 22.5b), can you divide the set of possible insurance contracts into different regions and illustrate that no firm would have an incentive to offer any contracts other than those that are provided in the separating equilibrium?

Answer: Using the text graph for when $\gamma = 0.5$: Any contract below the blue indifference curve will attract both risk types — but all such contracts lie below the zero-profit pooling line and therefore result in negative profit. The same is true for any contract below the $p = \delta B$ line. Contracts that lie above the $p = \delta b$ line and between the blue and magenta indifference curves would attract only high risk types — but all those contracts lie below the $p = \theta b$ line and would thus result in negative profit. All other contracts would not he demanded by anyone.

Exercise 22B.22

Can you demonstrate mathematically that θ types also prefer D to their separating contract A (which has (b, p)=(240,120)?

Answer: Plugging these contracts into the utility function for θ types, we get utility of 5.05α for contract D and utility of 4.87α for contract A.

Exercise 22B.23

When $\gamma = 0.5$ (as in Graph 22.5), equations (22.14) and (22.15) give the contract (b, p)=(154.67,58). Can you demonstrate that the indifference curve containing this point lies "below" the indifference curve that δ types can attain by purchasing the contract B that allows them to separate?

Answer: Plugging (b, p)=(154.67,58) into the utility function for δ types, we get utility of 5.11α — as opposed to utility of 5.15α in contract B.

Exercise 22B.24

Can you explain intuitively the change in pooling contracts as you move down Table 22.3? What happens to the problematic (j) region from our graph as we go down the table?

Answer: As there are fewer and fewer high risk types, the adverse selection problem lessens since there are fewer high cost consumers that can adversely select into the low risk insurance pool. For this reason, the pooling contract we have been calculating comes closer and closer to the actuarily fair full insurance contract for low risk types. As we go down the table, the (j) region that lies to the southeast of the δ indifference curve containing the pooling contract and *above* the $p = \delta b$ line necessarily shrinks (and goes to zero) since D converges to the full insurance point on $p = \delta b$ at which the δ indifference curve is tangent.

22C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 22.1

Consider again the example of grade insurance. Suppose students know whether they are typically A, B, C, D or F students, with A students having a 75% chance of getting an A and a 25% chance of getting a B; with B, C and D students having a 25% chance of getting a grade above their usual, a 50% chance of getting their usual grade and a 25% chance of getting a grade below their usual; and with F students having a 25% chance of getting a D and a 75% chance of getting an F. Assume the same bell-shaped grade distribution as in the text — i.e. in the absence of grade insurance, 10% of grades are A's, 25% are B's, 30% are C's, 25% are D's and 10% are F's.

A: *Suppose, as in the text, that grade insurance companies operate in a competitive market and incur a cost c for every level of grade that is changed for those holding an insurance policy. And suppose that A through D students are willing to pay 1.5c to ensure they get their usual grade and 0.5c for each grade level above the usual; F students are willing to pay 2c to get a D and* $0.5c$ *for each grade level above that.*

(a) *Suppose first that your instructor allows me only to sell A insurance in your classroom. Will I be able to sell any?*

Answer: If everyone were to buy the insurance in a class of 100, I would incur an average cost of $0.25c$ for the 10 "A students", an average cost of c, $2c$ and $3c$ for the 25 "B", the 30 "C" and the 25 "D" students respectively, and an average cost of $3.75c$ for the 10 "F students". My average cost would therefore be $2c$. However, "A students" are only willing to pay $1.5c$ for A-insurance and would therefore not buy at a premium of $2c$. If only non-"A students" bought my A-insurance, you can verify that my premiums would have to rise to approximately $2.19c$, which implies "B students" are no longer willing to participate; if only the remaining students bought the A-insurance, premiums would have to rise to approximately $2.65c$, which implies that "C students" would no longer participate which would necessitate a premium of approximately $3.21c$ that would cause "D students" to no longer participate. This would lead to premiums of $3.75c$ which not even the "F students" are willing to pay. Thus, the A-insurance market won't exist due to adverse selection.

(b) *Suppose next that your professor only allowed me to sell B insurance. Would I be able to sell any?*

Answer: The expected cost for B-insurance is 0, $0.25c$, c, $2c$ and $2.75c$ for A, B, C, D and F students respectively. We know that A students will not buy the insurance because all A students make at least a B anyhow. The zero-profit price for B-insurance assuming that all 90 (out of 100) non-A students purchase the insurance is then

$$p_B = \frac{25(0.25c) + 30(c) + 25(2c) + 10(2.75c)}{90} \approx 1.26c. \tag{22.1.i}$$

B-students are willing to pay up to $1.5c$ to insure a B — so they would buy the B-insurance at that price. C, D and F students are willing to pay more than that — so they would all buy. Thus, the insurance market exists and covers all students that can benefit at the zero-profit price $1.26c$ — and the adverse selection problem is not sufficient to eliminate the B-insurance market.

(c) *What if I were only allowed to sell C or only D insurance?*

Answer: Only C, D and F students are potentially interested in C insurance. C insurance costs $0.25c$, c and $1.75c$ for C, D and F students respectively. Thus, the insurance premium if all three student types bought the insurance would be

$$p_C = \frac{30(0.25c) + 25(c) + 10(1.75c)}{65} \approx 0.77c \tag{22.1.ii}$$

The insurance is worth $1.5c$ to C students and more to all the others. Thus, I could sell C insurance. With respect to D insurance, only D and F students would be interested — and they would cost $0.25c$ and $0.75c$ respectively. The insurance premium for D insurance assuming both types bought it would then be

$$p_D = \frac{25(0.25c) + 10(0.75c)}{35} \approx 0.39 \tag{22.1.iii}$$

which is less than what both types value it at. Thus, both would buy it if it were the only insurance offered.

(d) *If they were the only policies offered, could policies A and D attract customers in a competitive equilibrium at the same time? In equilibrium, who would buy which policy? (*Hint*: Only C, D and F students buy insurance in equilibrium.)*

Answer: Yes — the C students would buy the A policy at price $p_A = 2$ and the D and F students would buy the D policy at price $p_D = 0.39c$. This is an equilibrium because no student type can do better by doing anything else and insurance companies make zero profit. Given $p_A = 2c$, A students (who value A insurance at $1.5c$) do not think it is worth it and so won't want to buy; B students (who value A insurance at $2c$) will be indifferent between being insured and not being insured — so is not inconsistent with optimization for them to not buy. C students value A insurance at $2.5c$ but only have to pay $2c$ — so they are doing better than if they were not insured. D students value A insurance at $3c$ and would thus get c in surplus by buying it; but they value D insurance at $1.5c$ and only have to pay $0.39c$ — giving surplus of $1.11c$. Thus, D students are better off buying D insurance than A insurance given the prices. F students similarly get more surplus from D insurance than from A insurance. Thus, given the prices, everyone is doing the best they can. Furthermore, $p_A = 2c$ is a zero-profit insurance rate for A insurance when only C types buy — because the expected cost of insuring C students with A insurance is $2c$.

Similarly, $p_D = 0.39c$ is a zero profit price for D insurance given that D and F types buy it (as we showed in part(c)).

How can you arrive at this as the equilibrium? You can simply start by assuming only the A types buy A insurance and note that, at the zero profit price, this will imply that B types will want to buy it as well; and, at the zero-profit price for A insurance when both A and B types buy it, the C types will want to buy it as well. If A insurance is then priced at the zero profit price (of $1.35c$) for A, B and C students buying it, D students will get surplus of $1.65c$ from buying at that price — more than they could get from buying D insurance even if it were given away for free. So D types would buy A insurance — pushing the price to $1.81c$ at which point A students will no longer buy it. If only B, C and D students buy it, the zero profit price rises to $2c$ at which point B students become indifferent between buying and not buying. But at a price of $2c$ for A insurance, D students only get surplus of c from buying it, whereas they get surplus of $1.11c$ from buying D insurance at the zero-profit price of $0.39c$ when D and F students buy D insurance. We can keep the price of A insurance at $2c$ if we then suppose that neither B nor D students buy it — which gets us the equilibrium that only C students buy A insurance while D and F students buy D insurance at $p_A = 2c$ and $p_D = 0.39c$.

(e) *If they were the only policies offered, could policies A and C attract customers in a competitive equilibrium at the same time? (*Hint*: The answer is no.)*

Answer: We know from what we have done already, that it can't be an equilibrium for only A types to buy A insurance. If A and B students buy A insurance, the zero-profit price is $0.79c$ — sufficiently low such that C, D and F students would all buy A insurance even if C insurance were given away for free. (This is because each student values the move from a C to an A at c — and the price of the A policy if only A and B students buy it is $0.79c$.) So now suppose A, B and C students buy A insurance — which would put the zero profit price at $1.35c$. If this leaves D and F students buying C insurance, the zero profit price for C insurance is $1.21c$ — barely lower than the A insurance — which implies that D and F students would want the A insurance. It can't be that A, B, C, and D students buy A insurance — because that would require a zero-profit price of $1.81c$ which is more than the A students are willing to pay. So we know A students won't be buying insurance in any potential equilibrium. If only B students buy A insurance, the price would be c — with the price of C insurance at $0.77c$ assuming that C, D and F students buy it, which they would not since they value a move from C to A at c but the difference in price is only $0.23c$. If B and C students buy A insurance, then the price of A insurance is $1.55c$, with price of C insurance at $1.21c$ when D and F students buy it. The prices are still too close to keep D and F students from wanting to buy the A insurance over the C insurance. So we can try B, C and D buying A insurance at the zero-profit price of $2c$ — leaving only F

students to buy C insurance at a price of $1.75c$. Again the difference in price is not enough to keep F students from buying A insurance — and if they do, the equilibrium fully unravels as we illustrated in part (a). So we know B students can't be buying insurance in any potential equilibrium. Suppose then that only C students buy A insurance — giving us a zero profit price of $2c$ — which we already concluded is not enough to keep the D and F students from buying A insurance. Suppose next that C and D students buy A insurance — leaving only F students to buy C insurance. Then the zero profit prices are $2.45c$ for A insurance and $1.75c$ for C insurance — but the jump from a C to an A is worth c to F students while costing only $0.70c$ at these prices. So this, too, cannot be an equilibrium and C students therefore cannot buy insurance in equilibrium. Finally, suppose D students buy A insurance and F students buy C insurance. Then the zero profit prices are $3c$ for A insurance and $1.75c$ for C insurance. F students do now prefer the C policy to the A policy — but D students get no surplus under the A policy while getting $0.25c$ surplus under the C policy. So D students would want to switch to the C policy — leaving us no customers for the A policy.

(f) *If they were the only policies offered, could policies B and D attract customers in a competitive equilibrium at the same time? (*Hint*: The answer is again no.)*

Answer: We know from the outset that no A student will buy either of these insurances. We can then ask if B students might in equilibrium. If they are the only ones to buy B insurance, the price would be $0.25c$ — so low that C, D and F students would all buy it even if D insurance is given away for free. If B and C students both buy B insurance, then the price would be $0.66c$ — still low enough for D and F students (who value a move from a D to a B at c) to jump in. If B, C, and D students buy B insurance, the price goes to $1.08c$ — with the price for D insurance equal to $0.75c$ when sold only to F types who will then want to also buy B insurance. So B students cannot be buying insurance in any equilibrium. Suppose then that only C students buy B insurance — which would imply a zero profit price of c. But that implies that D and F students will also want B insurance if the price of D insurance is even a penny above 0. If C and D students buy B insurance, the zero profit price is $1.45c$ — with a zero profit price of $0.75c$ for D insurance bought only by F students who will therefore want to buy B insurance instead (since the difference in price is less than 1). Thus, C students cannot be buying B insurance if we also want someone to still be buying D insurance. This leaves us with the possibility of D students buying B insurance and F students buying C insurance. The zero profit prices would then be $2c$ for B insurance and $0.75c$ for D insurance. The F students now do prefer D insurance over B insurance (since the price difference is greater than 1), but the D students now get surplus of $0.5c$ from B insurance and $0.75c$ from D insurance. They therefore prefer D insurance and no one will buy B insurance.

(g) *Without doing any further analysis, do you think it is possible to have an equilibrium in which more than 2 insurance policies could attract customers?*

Answer: No. We had enough difficulty finding a single case where two policies could attract customers — in part (d). That is, in fact the only case in which two policies are bought in equilibrium. The problem with attempting to offer more policies is that there is too much adverse selection — causing too many students to pretend to be of a type different than they are.

(h) *Are any of the equilibria you identified efficient? (*Hint*: Consider the marginal cost and marginal benefit of each level of insurance above insuring that each student gets his/her typical grade.)*

Answer: The benefit to each student of insuring their usual grade is $1.5c$. The cost of insuring their usual grade is $0.25c$ (since only a quarter of students would need to have their grade raised). Thus, it is efficient to insure students to the point of guaranteeing their usual grade. But the cost of each additional grade level of insurance is c, and the additional benefit to the student is only $0.5c$ per level. Thus, it is not efficient to insure students at a level above their usual grade. In none of the equilibria we identified are we insuring every student gets their at least their usual grade — and in all of them do we have students who are insured above their usual grade. Thus, all the equilibria we have identified are inefficient.

B: *In A(c), you identified a particular equilibrium in which A and D insurance are sold — when it was not possible to sell just A insurance.*

(a) *How is this conceptually similar to the self-selecting separating equilibrium we introduced in Section B of the text?*

Answer: It is conceptually similar in the sense that the insurance industry is structuring its insurance offerings in such a way as to get consumer to self-select into different policies. In both the example here and in Section B, the insurance company knows nothing about your type before you buy a policy. In the text, the insurance company knew exactly what type you were after you bought insurance. Here, this is not the case — since C and D consumers buy A insurance. The insurance companies therefore know for sure that you are an F type if you buy the D policy, but they don't know if you are a C or a D type if you buy the A policy — nor do they know whether you are an A or B student if you do not buy.

(b) *How is it different?*

Answer: It is different in the sense that students are not truthfully revealing their type — with C and D students "pooling" and buying A insurance while F students separate.

Exercise 22.3

In exercise 22.2, we showed how an efficient equilibrium with a complete set of insurance markets can be re-established with truthful signaling of information by consumers. We now illustrate that signaling might not always accomplish this.

A: *Begin by once again assuming the same set-up as in exercise 22.1. Suppose that it costs* c *to truthfully reveal who you are and* $0.25c$ *more for each level of exaggeration; i.e. for a C student, it costs c to reveal that he is a C student,* $1.25c$ *to falsely signal that he is a B student and* $1.5c$ *to falsely signal that he is an A student.*

(a) *Begin by assuming that insurance companies are pricing A, B, C and D insurance competitively under the assumption that the signals they receive are truthful. Would any student wish to send false signals in this case?*

Answer: Yes, everyone would want to pretend to be an *A* student. The table gives the surplus that each student would get from buying insurance of different types — each of which is priced at $0.25c$ when insurance companies believe they are only getting truthful signals. Thus, a *C* student will pay the premium $0.25c$ for *A* insurance plus the signaling cost of $1.5c$ — and he values the insurance at $2.5c$. His surplus from getting *A* insurance is then $2.5c - 1.5c - 0.25c = 0.75c$. As is evident in the table, each student maximizes his surplus by signaling that he is an *A* student when insurance is priced as if signals were truthful. Thus, the equilibrium from the previous exercise is not an equilibrium here.

	Type of Insurance			
	A-Insurance	*B*-Insurance	*C*-Insurance	*D*-Insurance
A Student	$0.25c$	—	—	—
B Student	$0.50c$	$0.25c$	—	—
C Student	$0.75c$	$0.50c$	$0.25c$	—
D Student	$1.00c$	$0.75c$	$0.50c$	$0.25c$
F Student	$1.25c$	$1.00c$	$0.75c$	$0.50c$

(b) *Could A insurance be sold in equilibrium (where premiums have to end up at zero-profit rates given who is buying insurance)? (*Hint*: Illustrate what happens to surplus for students as premiums adjust to reach the zero profit level.)*

Answer: We have shown already in the table in part (a) that everyone will pretend to be an *A* student when insurance policies are all priced as if signals were truthful. This would then imply that insurance rates for *A* insurance would have to rise to $2c$ in order for profit to be zero — but at that price, surplus would be negative for all students is they bought *A* insurance. If only *A* and *B* students bought *A* insurance, the price would have to be $0.79c$ — which would again cause both types to have negative surplus. Since premiums would go up if other students joined in, *A* and *B* cannot be buying *A* insurance in equilibrium. If only *C* students bought *A* insurance (and sent the necessary signal to do so), premiums would again have to be $2c$ — leaving only negative surplus — and premiums will only have to go up as *D* and *F* students join in — giving even more negative surplus. *A* insurance therefore cannot be sustained in equilibrium.

(c) *Could B insurance be sold in equilibrium? What about C and D insurance?*

Answer: In the absence of A insurance being offered, the table in part (a) shows that all non-A students would want to buy B insurance if insurance were priced as if everyone told the truth. But then the zero-profit premium would have to be $2.19c$ — which would cause everyone's surplus from owning B insurance to become negative. Even if only B and C students were buying B-insurance (and sending the necessary signals), the premiums would be $0.66c$ — too high for B students to make positive surplus. Thus B students cannot buy B insurance in equilibrium. If only C students bought B-insurance, their premiums would have to be c — which makes their surplus negative — and this gets worse if D and F students join in. So C students cannot be buying B insurance in equilibrium. The same will happen if premiums go to $2c$ when only D students buy B insurance — unraveling the B-insurance market.

C insurance suffers a similar fate. The table in part (a) shows that, in the absence of A and B insurance, everyone who is not an A or B student would buy C insurance if such insurance were priced as if everyone told the truth. But this would require a zero profit premium of $1.65c$ — which would result in everyone getting negative surplus. If only C and D students bought, the premium would be $1.45c$ — still giving negative surplus for C and D students. If only D and F students bought, surplus for both is negative under the zero profit premium of $2.21c$ — which gets worse for F types if they were the only ones buying C insurance.

Finally, we get to D insurance — which the table in part (a) again shows both D and F students would buy if it were priced as if everyone told the truth. But if both these types buy D insurance, the zero profit premium has to go to $0.39c$. This will give surplus of $0.11c$ to D students (who pay c to signal their type truthfully — thus paying a total of 1.39 for insurance they value at $1.5c$) and $0.36c$ to F students (who pay $1.25c$ to falsely signal that they are D students — thus paying $1.64c$ for insurance they value at $2c$). So they would both still buy. Thus, D insurance is the only insurance that can be sold in equilibrium.

(d) *Based on your answers to (b) and (c), can you explain why the equilibrium in this case is to have only D-insurance sold — and bought by both D and F students? Is it efficient?*

Answer: We have shown that A, B and C insurance cannot be sold in equilibrium because the zero-profit price is either too low to keep everyone from wanting it or too high to attract any customers. Put differently, either too many students lie or no one wants to buy. We also showed that D insurance can survive — with both D and F students getting positive surplus under the zero profit premium of $0.39c$. In this case, firms are making zero profit — and no student can do better by doing something else. We therefore have an equilibrium where consumer signals do not result in the efficient allocation of insurance.

(e) *Now suppose that the value students attach to grades is different: They would be willing to pay as much as* $4c$ *to guarantee their usual grade and* $0.9c$ *more for each level of grade above that. Suppose further that the cost of telling the truth about yourself is still* c *but the cost of exaggerating is* $0.1c$ *for each level of exaggeration about the truth. How much surplus does each student type get from signaling that he is an* A *student if* A*-insurance is priced at* $2c$*?*

Answer: A students would get surplus of c because they would pay a signaling cost c plus the premium of $2c$ to get a benefit of $4c$. B students would get surplus of $1.8c$ because they would pay a signaling cost of $1.1c$ plus the premium of $2c$ to get a benefit of $4.9c$. Similarly, C students would get a surplus of $2.6c$; D students would get a surplus of $3.4c$ and F students would get a surplus of $4.2c$.

(f) *Suppose that insurance companies believe that any applicant for* B *insurance is a random student from the population of* B*,* C*,* D *and* F *students; that any applicant for* C *insurance is a random student from the population of* C*,* D *and* F *students; and any applicant for* D *insurance is a random student from the population of* D *and* F *students. How would they competitively price* B*,* C *and* D *insurance?*

Answer: They would price B insurance at $1.26c$, C insurance at $0.77c$ and D insurance at $0.39c$.

(g) *Suppose that, in addition, insurance companies do not sell insurance to students who did not send a signal as to what type they are. Under these assumptions, is it an equilibrium for everyone to signal that they are* A *students?*

Answer: The only way for students to buy insurance is to signal. The consumer surplus — assuming students signal — for each student under the different insurance policies is then depicted in the table below — where every student gains the maximum surplus buy signaling he is an A student. Thus, it is an equilibrium for only A insurance to be sold under the beliefs of insurance companies regarding any potential applicant for B, C and D insurance.

	Type of Insurance			
	A-Insurance	B-Insurance	C-Insurance	D-Insurance
A Student	$1.00c$	—	—	—
B Student	$1.80c$	$1.74c$	—	—
C Student	$2.60c$	$2.54c$	$2.23c$	—
D Student	$3.40c$	$3.34c$	$3.03c$	$2.61c$
F Student	$4.20c$	$4.14c$	$3.83c$	$3.41c$

(h) *There are two sources of inefficiency in this equilibrium. Can you distinguish between them?*

Answer: The first source is that insurance is not allocated efficiently because it costs c to raise a grade for a student above his usual grade — but it is only valued at $0.9c$. This implies the efficient allocation of insurance is to have everyone insured for his usual grade only. The second source of inefficiency comes from the signaling: All students pay signaling costs, but no information is revealed in the process.

B: *In exercise 22.2B, we introduced a new "signaling technology" that restored the efficient allocation of insurance from an initially inefficient allocation in a self-selecting separating equilibrium. Suppose that insurance companies believe anyone who does not send a signal that he is a δ type must be a θ type.*

(a) *Suppose that c_f is below the range you calculated in B(d) of exercise 22.2. Can you describe a pooling equilibrium in which both types fully insure and both types send a signal that they are δ types?*

Answer: In this case, the θ types will want to pretend to be δ types if the δ types can actually buy actuarily fair full insurance — which means that insurance companies can't let the δ types buy actuarily fair full insurance. Instead they might offer full insurance at the pooled price based on how the fraction of the population is δ and the fraction that is θ. If companies believe a non-signal is conclusive evidence of the customer being a θ type, then they would only offer the higher θ rates for insurance to anyone that does not signal. Thus, it may be in the interest of both types to signal that they are δ types in order to avoid being treated like a θ type — and then to buy the pooled contract that in actuarily unfair for δ types and too generous for θ types.

(b) *In order for this to be an equilibrium, why are the beliefs about what a non-signal would mean important? What would happen if companies believed that both types are equally likely not to signal?*

Answer: The beliefs about what a non-signal means are important because they determine how someone will be treated by the insurance company if they don't signal — and this determines whether it is optimal for δ types to stay in the pooled contract. If companies believed everyone was equally likely not to signal, it would treat a non-signal as containing the same information as a signal — and thus would offer the same pooling contract. But if, prior to the invention of the signaling device, a separating equilibrium existed, that meant that there are sufficient numbers of θ types for the δ types to prefer to have the restricted actuarily fair contract they get in the separating equilibrium. Thus, if companies believed everyone was equally likely to signal as not signal, we would revert back to the self-selecting separating equilibrium.

(c) *True or False*: For an equilibrium like the one you described in part (a) to be an equilibrium, it matters what firms believe about events that never happen in equilibrium.

Answer: This is true — as we just saw in the previous answer. (Note: If you do part B of the game theory chapter, you will see a similar role for beliefs in what we will call Bayesian games.)

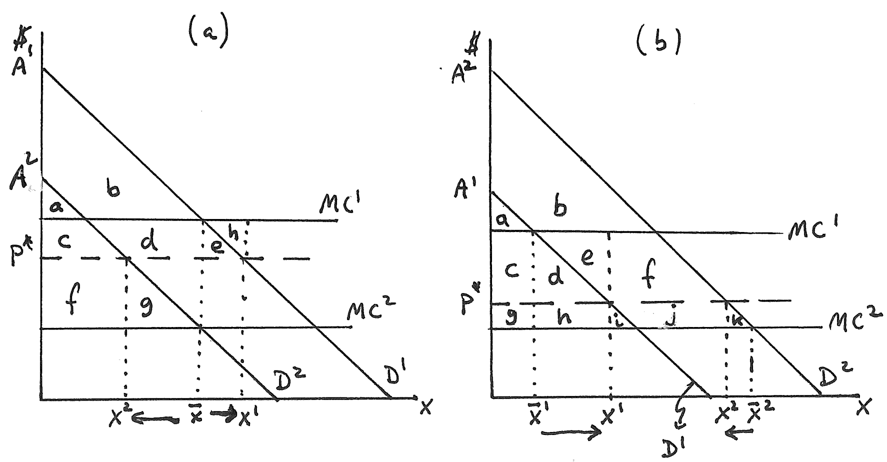
Exercise 22.5

We developed our first graphical model of adverse selection in the context of car insurance in section 22A.2 where we assumed that the marginal cost MC^1 of providing car insurance to unsafe drivers of type 1 is greater than the MC^2 of providing insurance to safe drivers of type 2.

A: *Continue with the assumption that $MC^1 > MC^2$. In this exercise, we will investigate how our conclusions in the text are affected by altering our assumption that $D^1 = D^2$ — i.e. our assumption that the demand (and marginal willingness to pay) curves for our two driver types are the same.*

(a) *Suppose demand curves continue to be linear with slope α — but the vertical intercept for type 1 drivers is A^1 while the intercept for type 2 drivers is A^2. Suppose first that $A^1 > A^2 > MC^1 > MC^2$. Illustrate the equilibrium. Would p^* — the price that emerges in the asymmetric equilibrium — still be halfway between MC^1 and MC^2 as was the case in the text?*

Answer: This is treated in panel (a) of Exercise Graph 22.5(1) where D^1 lies above D^2.



Exercise Graph 22.5(1) : Different Demands for Different Consumers

We have drawn it here in such a way that, where insurance priced at the MC specific to each consumer type, both consumer types would buy the same amount of insurance $\overline{x}$ — but that does not necessarily have to be the case depending on exactly how far apart the demand curves are relative to the marginal cost curves. If producers can tell consumers apart, they would thus charge each type a price equal to that type's marginal cost — causing consumption levels of $\overline{x}$ for the two types (where MC^1 intersects D^1 and MC^2 intersects D^2.) When producers cannot tell types apart, a single price p^* emerges, with losses from type 1 consumers exactly offset by gains from type 2 consumers. At p^*, the two consumer types consume x^1 and x^2 — i.e. type 1 increases consumption (because for him price drops from MC^1) and type 2 decreases consumption (because for him price increased from MC^2). Losses from type 1 consumers are equal to $(c+d+e+h)$ — and are exactly offset by gains from type 2 consumers equal to (f). Thus, $f=(c+d+e+h)$ — and in order for this to hold, it must be that p^* lies closer to MC^1 than to MC^2. (Put differently, since more high cost type 1 consumers will buy insurance at p^* than low cost type 2 consumers, the price cannot drop by as much for the type 1 consumers as it must rise for the type 2 consumers in order for firms to make zero profit.)

(b) *Identify the deadweight loss from asymmetric information in your graph.*

Answer: Consumer surplus for type 1 increases from $(a+b)$ to $(a+b+c+d+e)$ while consumer surplus for type 2 decreases from $(a+c+f+g)$ to $(a+c)$. Thus, total consumer surplus goes from $(2a+b+c+f+g)$ to $(2a+b+2c+d+e)$. Since we know that $f=(c+d+e+h)$, we can equivalently say that total consumer surplus goes from $(2a+b+2c+d+e+g+h)$ to $(2a+b+2c+d+e)$ (by replacing f in the initial total consumer amount). Thus, the asymmetric information problem that gives rise to a single price p^* results in a deadweight loss of $(g+h)$. These two deadweight loss triangles have an interpretation that is intuitively equivalent to the two deadweight loss triangles in the text where the demands were the same; i.e. we lose (g) from type 2 consumers because they lower their consumption to x_2 from $\overline{x}$, an amount that previously got them consumer surplus of (g). We also lose (h) because we are providing insurance all the way to x^1 for type 1 consumers, but for the amount from $\overline{x}$ to x^1, it costs (h) more than these consumers benefit.

(c) *What is the equilibrium if instead $A^2 > A^1 > MC^1 > MC^2$? How does p^* compare to what you depicted in (a)?*

Answer: This is pictured in panel (b) of Exercise Graph 22.5(1) where D^1 lies *below* D^2. For reasons similar to those covered in the previous parts, the consumption levels for the two types in the absence of asymmetric information (when producers know each consumer's marginal cost and can therefore charge a price equal to each consumer's marginal cost) are $\overline{x}^1$ and $\overline{x}^2$. In the presence of asymmetric information, the single price p^* emerges — giving rise to consumption levels x^1 and x^2 — i.e. type 1 consumers increase consumption (because price falls from MC^1) and type 2 consumers decrease consumption (because price rises from MC^2). The price p^* is such that the losses from type 1 consumers (equal to $(c+d+e)$) are exactly offset by the profits from type 2 consumers (equal to $(g+h+i+j)$). Thus, we know that $(c+d+e)=(g+h+i+j)$. In order for this to be the case, p^* now has to be closer to MC^2 than to MC^1.

(d) *Identify again the deadweight loss from asymmetric information.*

Answer: Consumer surplus for type 1 consumers goes from an initial (a) to $(a+c+d)$, and consumer surplus for type 2 consumers goes from $(a+b+c+d+e+f+g+h+i+j+k)$ to $(a+b+c+d+e+f)$. Adding across consumers, we therefore get that total consumer surplus goes from $(2a+b+c+d+e+f+g+h+i+j+k)$ to $(2a+b+2c+2d+e+f)$. Since we know that $(c+d+e)=(g+h+i+j)$, we can equivalently say that consumer surplus goes from $(2a+b+2c+2d+2e+f+k)$ to $(2a+b+2c+2d+e+f)$ when asymmetric information is introduced — i.e. $(e+k)$ are lost. The intuition is again the same as in the previous cases.

(e) *What would have to be true about the relationship of A^1, A^2, MC^1 and MC^2 for safe drivers not to buy insurance in equilibrium?*

Answer: We would have to have $A^1 > MC^1 > A^2 > MC^2$. In that case, we can get an equilibrium where insurance is only offered at the price $p^* = MC^1$, with no safe drivers buying since $MC^1 > A^2$. This could not happen if $MC^1 < A^2$ (as in panel (a) of our graph.)

(f) *What would have to be true about the relationship of A^1, A^2, MC^1 and MC^2 for unsafe drivers not to buy insurance in equilibrium?*

Answer: It would have to be the case that $A^1 < MC^2 < MC^1 < A^2$ — with unsafe drivers caring so little about insurance that they would not even buy it at the safe driver rates. In that case, $p^* = MC^2$, with the entire demand curve for type 1 drivers lying below p^*. This cannot happen if $A^1 > MC^2$.

B: *In our model of Section B, we assumed that the same consumption/utiltiy relationship $u(x)$ can be used for high cost θ and low cost δ types to represent their tastes over risky gambles with an expected utility function.*

(a) *Did this assumption imply that tastes over risky gambles were the same for the two types?*

Answer: The expected utility function for θ types is then

$$U^{\theta}(x_1,x_2) = \theta u(x_1) + (1-\theta)u(x_2) \quad (22.5.i)$$

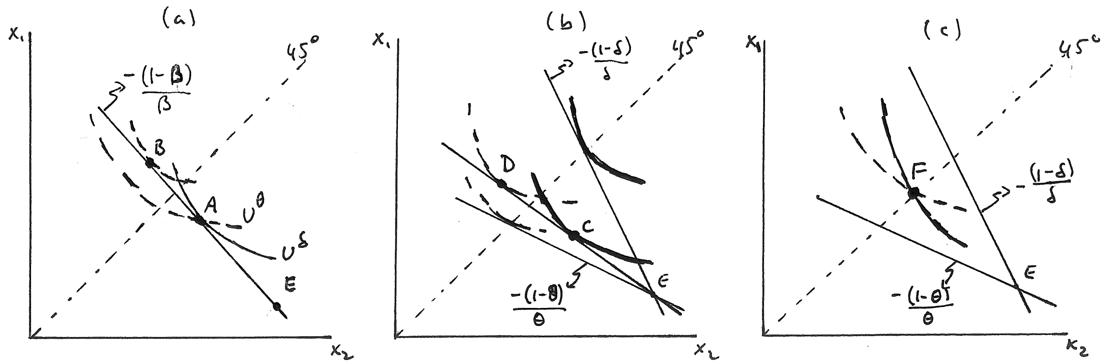
while the expected utility function for δ types is

$$U^{\delta}(x_1,x_2) = \delta u(x_1) + (1-\delta)u(x_2). \quad (22.5.ii)$$

Since $\theta \neq \delta$, these are different functions that give rise to different indifference curves.

(b) *Illustrate the actuarily fair insurance contracts in a graph with x_2 — the consumption in the good state — on the horizontal and x_1 — the consumption in the bad state — on the vertical. Then illustrate the choice set created by a set of insurance contracts that all satisfy the same terms — i.e. insurance contracts of the form $p = \beta b$ (where b is the benefit level and p is the premium).*

Answer: This is done in panel (a) of Exercise Graph 22.5(2) where E represents the consumption levels in the two states without insurance. The set of insurance contracts $p = \beta b$ then results in a contract line with slope $-(1-\beta)/\beta$. This is because, under this contract, a consumer gives up the



Exercise Graph 22.5(2) : Different Ways of Pooling

premium βp in good times in order to get $(b - \beta b)$ in bad times (where she gets the benefit b but also pays the premium βb).

(c) *Can you tell whether θ or δ types will demand more insurance along this choice set?*

Answer: Yes. Using the expected utility functions from (a), we can derive the marginal rates of substitution as

$$MRS^\theta = -\frac{(1-\theta)}{\theta}\frac{\partial u(x_2)/\partial x}{\partial u(x_1)/\partial x} \text{ and } MRS^\delta = -\frac{(1-\delta)}{\delta}\frac{\partial u(x_2)/\partial x}{\partial u(x_1)/\partial x}. \quad (22.5.iii)$$

(Note that, since we are putting x_2 on the horizontal axis, the MRS derivation has the partial derivative of u with respect to x_2 in the numerator.) From this, we see the, at any (x_2, x_1), the slope of the indifference curves for δ types is steeper than the slope of the indifference curves for θ types (because $\theta > \delta$). This implies that θ types will optimize at greater insurance levels than δ types for any contract line.

This is illustrated in panel(a) of Exercise Graph 22.5(2) where δ types optimize at A and θ types optimize at B. At A, you can see the shallower slope for the dashed indifference curve U^θ than for the solid indifference curve U^δ — which implies that B must lie to the left of A. (Note: It does not necessarily have to be that B lies to one side and A to the other side of the 45 degree line — this has to be true only for contract lines that fall in between the actuarily fair contract lines for the two types.)

(d) True or False*: Our θ types would be analogous to the car insurance consumers of type 1 in part A of the exercise while our δ types would be analogous to consumers of type 2.*

Answer: This is true for A(a) but not for A(b). In A(a), we assumed that the high cost type 1 consumers had greater demand for insurance for any given price than low cost type 2 consumers — which is what we are finding emerges in our model here: The high cost θ types buy more insurance than the low cost δ types for any given relationship between p and b. In A(b) we assumed that the high cost types have lower demand for insurance than the low cost types - which seems fundamentally less reasonable (since those who are likely to get into trouble are in greater need of insurance.)

(e) *Suppose there are an equal number of δ and θ types and suppose that the insurance industry for some reason offered a single full set of insurance contracts $p = \beta b$ and that this allowed them to earn zero profits. Would the $p = \beta b$ line lie halfway between the actuarily fair contract lines for the two risk types?*

Answer: No, it would not. This is because (as we have just shown), when the two types face the same contract line, the θ types will buy more insurance than the δ types. Thus, the cost to the insurance company will not be the average between the two — because the low cost types buy less insurance. Rather, it will be more than the average — giving us a contract line closer to the actuarily fair contract line for θ types than that for δ types. This is illustrated in panel (b) of Exercise Graph 22.5(2) where the δ types pick C and the θ types pick D..

(f) *Suppose instead that the insurance industry offered a single insurance policy that provides full insurance — and that firms again make zero profits. Would the contract line that contains this policy lie halfway between the two actuarily fair contract lines in your graph? What is different from the previous part?*

Answer: Now the contract line that contains this policy indeed lies halfway between the two actuarily fair lines — because now the two types are forced to consume the same policy — which means that the cost of providing the insurance is the average of what it costs for the two types (given that there is an equal number of them). This is illustrated in panel (c) of Exercise Graph 22.5(2) where the policy is F. What is different from the previous part is that we are now forcing the two types to consume the same policy whereas before they were allowed to pick different policies — with the high cost types buying more insurance than the low cost types.

Exercise 22.7

Business Application: *Competitive Provision of Health Insurance: Consider the challenge of providing health insurance to a population with different probabilities of getting sick.*

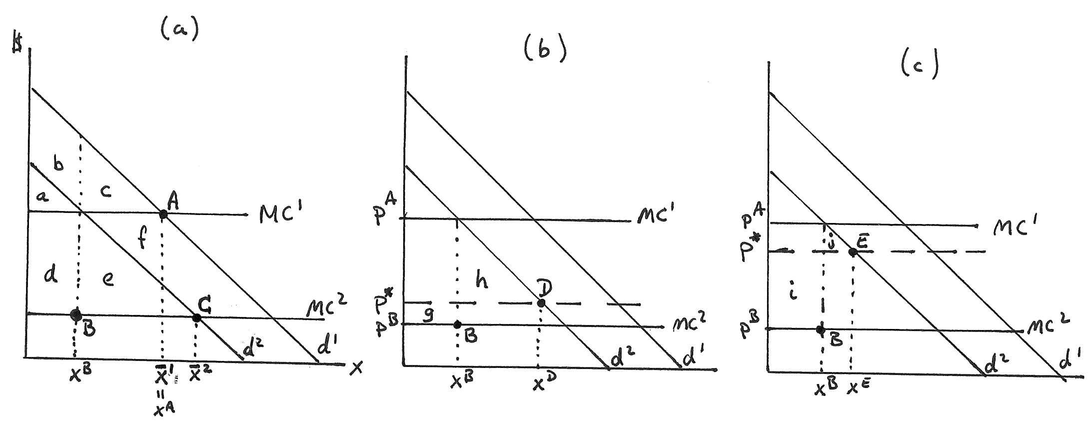
A: *Suppose that, as in our car insurance example, there are two consumer types — consumers of type 1 that are likely to get sick, and consumers of type 2 that are relatively healthy. Let x represent the level of health insurance, with $x = 0$ implying no insurance and higher levels of x indicating increasingly generous health insurance benefits. Assume that each consumer type has linear demand curves (equal to marginal willingness to pay), with d^1 representing the demand curve for a single consumer of type 1 and d^2 representing the demand curve for* *a single consumer of type 2. Suppose further that the marginal cost of providing additional health coverage to an individual is constant, with $MC^1 > MC^2$.*

(a) *For simplicity, suppose throughout that d^1 and d^2 have the same slope. Suppose further, unless otherwise stated, that d^1 has higher intercept than d^2. Do you think it is reasonable to assume that type 1 has higher demand for insurance?*

Answer: It seems reasonable to assume that those who are more likely to get sick have higher demand for health insurance — which is what we are assuming when we assume that the intercept of d^1 is higher than the intercept of d^2.

(b) *Begin by drawing a graph with d^1, d^2, MC^1 and MC^2 assuming that the vertical intercepts of both demand curves lie above MC^1. Indicate the efficient level of insurance $\overline{x}^1$ and $\overline{x}^2$ for the two types.*

Answer: This is done in panel (a) of Exercise Graph 22.7(1).



Exercise Graph 22.7(1) : Equilibrium in the Insurance Market

Here the efficient level of insurance for type 1 consumers occurs where MC^1 intersects D^1 at A and the efficient level of insurance for type 2 consumers occurs where MC^2 intersects D^2 at C. (Note: There is no particular reason for $\overline{x}^1$ to lie to the left of $\overline{x}^2$ — had we drawn the difference between d^1 and d^2 larger — or the difference between MC^1 and MC^2 smaller, the reverse would hold. Nothing fundamental changes in the analysis regardless of how the graph is drawn.)

(c) *Suppose the industry offers any level of x at price $p = MC^1$. Illustrate on your graph the consumer surplus that type 1 individuals will get if this were the only way to buy insurance and they buy there optimal policy A. How much consumer surplus will type 2 individuals get?*

Answer: Type 1 consumers will buy $x^A = \overline{x}^1$ and thus get consumer surplus $(a + b + c)$. Consumers of type 2 will buy only up to the point where MC^1 crosses d^2 — thus getting consumer surplus (a).

(d) *Next, suppose you want to offer an additional insurance contract B that earns zero profit if bought only by type 2 consumers, that is preferred by type 2 individuals to A and that makes type 1 consumers just as well off as they are under the options from part (c). Identify B in your graph.*

Answer: This is also done in panel (a) of Exercise Graph 22.7(1). Note that there is no particular reason that B lies vertically underneath the intersection of MC^1 and d^2 — it could lie to the right or left. It must be, however, that B lies on the MC^2 curve — otherwise firms offering it would not make zero profits. In order for type 1 individuals to be indifferent between A and B, it must be that their consumer surplus is the same under both contracts. Since their consumer surplus at A is $(a+b+c)$ and their consumer surplus as B is $(a+b+d)$, this implies that (c) has to be equal to (d). Notice that (c) gets larger and (d) gets smaller as we move B to the left, with (c) small and (d) large when B is horizontally close to A. Thus, starting B vertically underneath A and moving it to the left, there will come some x^B at which (d) is exactly equal to (c). Finally, it has to be the case that type 2 consumers are better off at B than would be otherwise — which has to be the case. (It is trivial to see when B lies right below the intersection of d^2 and MC^1 because then consumer surplus simply increases from (a) to $(a+d)$ — but it is also true if B lies to the right or left of that intersection point.)

(e) *Suppose for a moment that it is an equilibrium for the industry to offer only contracts A and B (and suppose that the actual B is just slightly to the left of the B you identified in part (d)).* True or False*: While insurance companies do not know what type consumers are when they walk into the insurance office to buy a policy, the companies will know what type of consumer they made a contract with after the consumer leaves.*

Answer: This is true — consumers of type 1 would be just slightly better off buying A while consumers of type 2 are better off buying B. Thus, you will know that the consumer is a type 1 if he bought A and a type 2 if he bought B.

(f) *In order for this to be an equilibrium, it must be the case that it is not possible for an insurance company to offer a "pooling price" that makes at least zero profit while attracting both type 1 and 2 consumers. (Such a policy has a single price p^* that lies between MC^1 and MC^2.) Note that the demand curves graphed thus far were for only one individual of each type. What additional information would you have to know in order to know whether the zero-profit price p^* would attract both types?*

Answer: You would need to know no additional information to know that type 1 individuals would prefer the pooling contract price p^* — because it would be below the price at which they are otherwise buying A. But we don't know if such a price would attract consumers of type 2. It is a higher price, but if it allowed type 2 consumers to buy a larger quantity, that might make up for the loss in consumer surplus from the higher price. This is illustrated in panels (b) and (c) of Exercise Graph 22.7(1) where a low p^* is graphed in panel (b) and a high p^* is graphed in panel (c). At p^*, type 2 consumers will buy where p^* intersects d^2 — i.e. at point D in panel (b) and at point E in panel (c). In panel (b), this implies that consumers will lose area (g) in consumer surplus because of the increase in price but will gain area (h) from being able to purchase more insurance. Since (h) > (g), the consumer is better off and thus will choose the pooling contract. But in panel (c), type 2 consumers lose (i) and gain (j) — with the former larger than the latter. Thus, the higher p^*, the less likely it is that a pooling price p^* could attract both consumers.

(g) True or False*: The greater the fraction of consumers that are of type 1, the less likely it is that such a "pooling price" exists.*

<u>Answer</u>: This is true — because the greater the fraction of type 1 consumers, the higher the price p^* will have to be in order for firms offering that price to make zero profit.

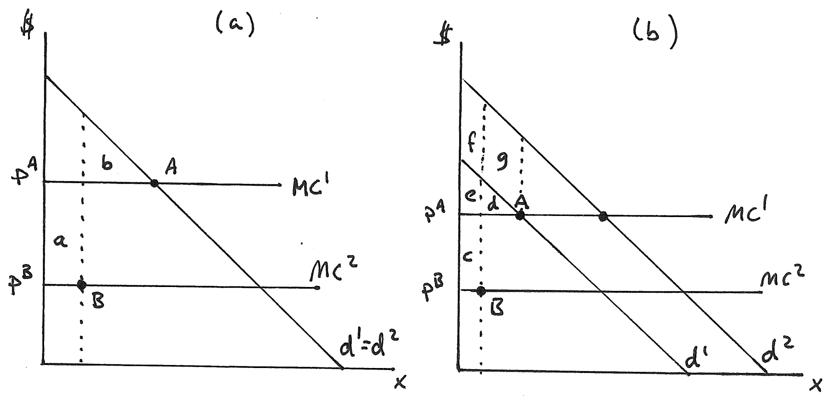
(h) *Suppose that no such pooling price exists. Assuming that health insurance firms cannot observe the health conditions of their customers, would it be a competitive equilibrium for the industry to offer contracts A and B? Would this be a pooling or a separating equilibrium?*

<u>Answer</u>: Yes, this would be a separating equilibrium because the two types end up revealing who they are by choosing different contracts. In fact, the equilibrium could simply offer any insurance amount at price $p^A = MC^1$ and any insurance amount up to x^B at price $p^B = MC^2$. But no insurance above x^B can be offered at p^B — otherwise type 2 consumers will buy at p^B — which means p^B would no longer be a zero profit price.

(i) *Would you still be able to identify a contract B that satisfies the conditions in (d) if $d^1 = d^2$? What if $d^1 < d^2$?*

<u>Answer</u>: Panel (a) of Exercise Graph 22.7(2) illustrates the case where $d^1 = d^2$. The contract B has to be such that area (a) is equal to area (b) so that type 1 individuals would lose as much (i.e. (b)) as they would gain (i.e. (a)) from switching from A to B. But because $d^1 = d^2$, this implies type 2 individuals will be similarly indifferent — and not strictly prefer B to A. In this borderline case, it is therefore barely possible to find B that satisfies the necessary conditions for a separating equilibrium to emerge. The case where $d^2 > d^1$ is graphed in panel (b) of Exercise Graph 22.7(2). In order for type 1 consumers to be indifferent between A and B, it has to now be that area (c) is equal to area (d). The B that satisfies this is graphed. But type 2 consumers would now prefer A to B — because their surplus under B is $(c+e+f)$ while their surplus under A is $(d+e+f+g)$. Since (c) is equal to (d), the surplus under A can also be written as $(c+e+f+g)$ — implying that consumers of type 2 are better off by area (g) if they pick A. The separating equilibrium can therefore not emerge.

B: *Part A of this exercise attempts to formalize a key intuition we covered in section B of the text with a different type of model for insurance.*



Exercise Graph 22.7(2) : Almost no — and no possible separating equilibrium

(a) *Rather than starting our analysis by distinguishing between marginal costs of different types, our model from section B starts by specifying the probabilities θ and δ that type 1 and type 2 individuals will find themselves in the "bad state" that they are insuring against. Mapping this to our model from part A of this exercise, with type 1 and 2 defined as in part A, what is the relationship between δ and θ?*

Answer: With θ as the probability of the "bad state" for type 1 and δ the probability of the "bad state" for type 2, it must be that $\theta > \delta$.

(b) *To fit the story with the model from section B, we can assume that what matters about bad health shocks is only the impact they have on consumption — and that tastes are state independent. (We will relax this assumption in exercise 22.8). Suppose we can, for both types, write tastes over risky gambles as von-Neumann Morgenstern expected utility functions that employ the same function $u(y)$ as "utility of consumption" (with consumption denoted y). Write out the expected utility functions for the two types.*

Answer: Let y_1 be consumption when sick and y_2 consumption when healthy (with, presumably, $y_1 < y_2$). We would get

$$U^1(y_1, y_2) = \theta u(y_1) + (1-\theta)u(y_2) \tag{22.7.i}$$

for type 1 consumers and

$$U^2(y_1, y_2) = \delta u(y_1) + (1-\delta)u(y_2). \tag{22.7.ii}$$

(c) *Does the fact that we can use the same $u(y)$ to express expected utilities for both types imply that the two types have the same tastes over risky gambles — and thus the same demand for insurance?*

Answer: No, they do not. The expected utility functions U^1 and U^2 differ because the probabilities θ and δ differ. The expected utility functions in fact take the Cobb-Douglas form — with U^1 placing heavier emphasis on y_1 than U^2.

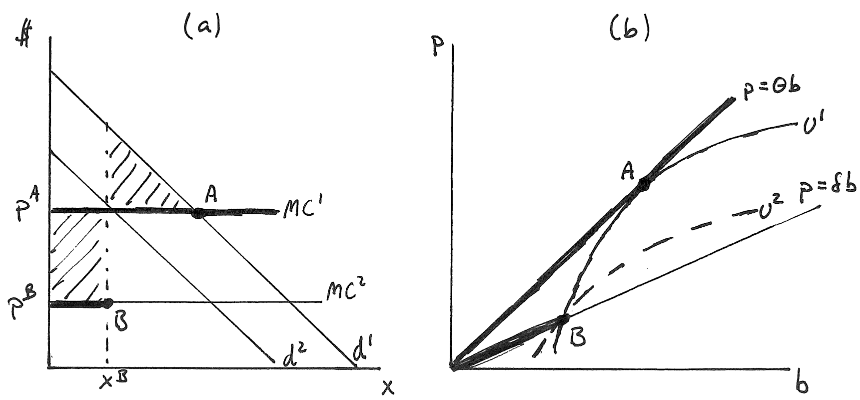
(d) *If insurance companies could tell who is what type, they would (in a competitive equilibrium) simply charge a price equal to each type's marginal cost. How is this captured in the model developed in section B of the text?*

Answer: This is captured by the zero-profit (or actuarily fair) contract lines — which differ for the two types. In particular, for type 1, the zero-profit contracts are $p = \theta b$ (where p is the insurance premium and b is the insurance benefit), and for type 2 they are $p = \delta b$.

(e) *In the separating equilibrium we identified in part A, we had insurance companies providing the contract A that is efficient for type 1 individuals — but providing an inefficient contract B to type 2. Draw the model from section B of the text and illustrate the same A and B contracts. How are they exactly analogous to what we derived in part A?*

Answer: Exercise Graph 22.7(3) illustrates our model from part A in panel (a) and our analogous model from Section B of the text in panel (b). In panel (a), high cost types have higher demand for insurance levels x — and B is structured so that high cost types are indifferent between their efficient insurance choice A and the option intended for low cost types — B. This is done by insuring that the shaded areas in the panel are equal to one another — because that insures that the loss in surplus from switching between A and B is equal to the gain for type 1 consumers. That's exactly what we do in panel (b) for the new model. There, $p = \theta b$ represents the zero profit contract line for type 1 consumers and $p = \delta b$ represents the zero profit line for type 2 consumers who cost less and thus have more generous benefits for any insurance premium. The efficient insurance choice for type 1 consumers is A — the point at which they fully insure (given their risk aversion) at the actuarily fair rate. Type 1 consumers are then indifferent between all insurance contracts that fall on the indifference curve U^1 — which goes through A and B. Thus, B is the actuarily fair insurance contract aimed at type 2 consumers that makes type 1 consumers indifferent to A — just as it is in panel (a). In both cases, type 2 consumers are better off at B than at A. The analogy extends even further: In panel (a), all the darkened insurance packages can be offered — any insurance level at price p^A and all insurance levels up to x^B at price p^B. The analogous darkened lines in panel (b) say the same thing: Any actuarily fair (or zero-profit) insurance contract aimed at high cost types can be offered, but only actuarily fair insurance packages aimed at type 2 to the left of B can be offered. If the restriction on what can be offered at the zero-profit rates for type 2 were not included, then type 1 individuals would buy at the type 2 price in both cases.

(f) *In part A we also investigated the possibility of a potential pooling price — or pooling contract — breaking the separating equilibrium in which A and*



Exercise Graph 22.7(3) : Two Models — Same Idea

B are offered. Illustrate in the different model here how the same factors are at play in determining whether such a pooling price or contract exists.

Answer: In the model of panel (a) of Exercise Graph 22.7(3), the crucial factor is whether the pooling price p^* is such that it would in fact attract type 2 consumers away from B. The closer p^* is to p^B, the more likely this is the case — and p^* gets closer to p^B the fewer high cost type 1's there are. In panel (b), the zero-profit pooling line falls between $p = \theta b$ and $p = \delta b$ — getting closer to the former as the fraction of type 1 consumers increases and getting closer to the latter as the fraction of type 1 consumers falls. The separating equilibrium cannot be broken in panel (b) unless the zero profit pooling line crosses the dashed indifference curve U^2 — which is more likely to happen the fewer type 1 consumers there are. Once again, the conclusion and intuition is exactly the same.

(g) *Evaluate again the* True/False *statement in part A(g).*

Answer: This is true as already discussed in the previous part. Again, the two models give exactly the same punch line.

Exercise 22.9

Policy Application: *Moral Hazard versus Adverse Selection in Health Care Reform: We mentioned moral hazard only briefly — and primarily in the context of how this might aggravate the adverse selection problem. In this exercise, we explore moral hazard a bit more in the context of health insurance. (Both part A and part B of this exercise can be done without having done section B in the chapter.)*

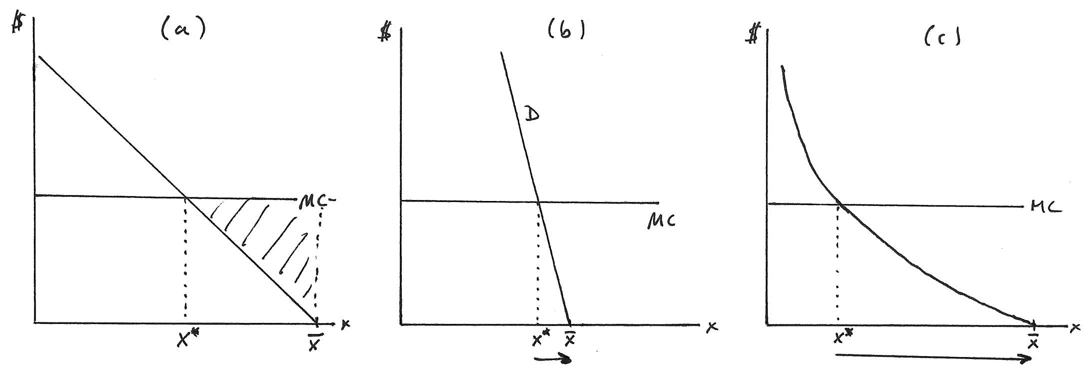
A: *Suppose throughout that individuals do not engage in riskier life-styles as a result of obtaining health insurance.*

(a) *How does this assumption eliminate one form of moral hazard that we might worry about?*

Answer: If some individuals would alter their behavior — go ski-diving or start smoking as a result of knowing that they have health insurance, this would be a classic source of moral hazard. If insurance companies could not identify how severe the moral hazard would be with different individuals, this would furthermore strengthen the adverse selection problem in health insurance markets — because individuals that tend toward risky behavior may adversely select into the insurance pool.

(b) *Suppose that a unit of health care x is such that it can be provided at constant marginal cost that is the same for all patients. Illustrate a patient's demand curve for x as well as the MC curve for providing x.*

Answer: This is done in panel (a) of Exercise Graph 22.9.



Exercise Graph 22.9 : Moral Hazard in Health Insurance Markets

(c) *Suppose demand for health care services is equal to marginal willingness to pay. If the patient pays out-of-pocket for health care, how much would she consume assuming that health care services are competitively priced (with health care providers facing negligible recurring fixed costs)?*

Answer: Under competition, price would be equal to marginal cost — and consumers would demand x^* in panel (a) of Exercise Graph 22.9.

(d) *Suppose next that the patient has insurance coverage that pays for all health related expenses. How much x does she consume now?*

Answer: Since she does not have to pay for the service, she would consume until the marginal benefit is zero — which gets her to $\bar{x}$ in the graph.

(e) *Moral hazard refers to the change in behavior that arises once a person enters a contract. Have you just uncovered a source of moral hazard in the health insurance market? Explain how this results in inefficiency.*

Answer: Yes. The patient would consume x^* in the absence of insurance and $\bar{x}$ with insurance —i.e. getting insurance changes the amount of health care services she consumes. Note that, for the amount between x^* and $\bar{x}$, the consumer is valuing the services less than what they cost.

(f) *Now replicate your picture two times: Once for a patient where the moral hazard problem is small, and once for a patient where it is large. If insurance companies cannot tell the difference between these two individuals, how does this asymmetric information potentially give rise to adverse selection?*

Answer: This is done in panels (b) and (c) of Exercise Graph 22.9. If insurance companies cannot tell the difference between these types, they do not know whether they are insuring a high cost (panel (c)) or low cost (panel (b)) type — even if they look observably similar. It may then be that the high cost type adversely selects into the insurance pool.

B: *Consider two alternative proposals for health care reform: Under proposal A, the government mandates that everyone buy health insurance, restricts insurance companies to provide a single type of policy with generous benefits — and then lets the companies compete for customers to sell that policy. Under proposal B, the government sets up "health care savings accounts" for everyone and allows insurance companies to offer only policies with high "deductibles". Under this latter policy, consumers would then pay for most health related expenditures using funds in their health care savings accounts and could convert any balance to retirement accounts when they reach the age of 65 (and thus become eligible for government health care for the elderly — called Medicare in the U.S.) Insurance under policy B is therefore aimed only at "catastrophic" events that cost more than the deductible of the policy.*

(a) *Suppose you were concerned about excessive health care costs. How would the two different proposals aim at addressing this?*

Answer: The first proposal attempts to deal with the typical adverse selection problem. As we have seen throughout this chapter, this problem can cause high cost types to adversely select into insurance — leaving lower cost types uninsured (or under-insured). By forcing everyone into a single type of policy, the adverse selection problem is taken care of in the sense that everyone is "selecting" into the policy — with competition driving price to marginal cost. Proposal *B*, on the other hand, is concerned with moral hazard. As we saw in part A of the question, moral hazard can cause people who are insured to over-consume health care — thus pushing up the cost of health care (if the supply curve eventually slopes up) — and it can create its own adverse selection problem. By setting up health savings accounts that can eventually be turned into consumption (through retirement accounts), individuals would confront the cost of the health care services they consume. The accounts are set up to insure the cash is there for minor medical expenses — but it's real cash and thus health care consumption entails a real cost for consumers. The catastrophic health insurance policies are less likely to be subject to adverse selection and moral hazard problems — and would be there for large medical expenses. Thus, proposal *B* is targeting moral hazard as the big cost problem.

(b) *If you thought the primary problem arose from the moral hazard analyzed in part A of this exercise, which policy would you favor?*

Answer: You would probably favor policy B which largely eliminates the moral hazard problem discussed in part A of the exercise (as explained in the answer to B(a).) Your view would essentially be that the biggest problem to fix is the fact that people hold so much insurance that they severely over-consume health care.

(c) *Suppose instead that you thought the primary problem arose from the rising cost of health insurance linked to increasingly severe adverse selection (unrelated to the moral hazard problem analyzed in part A) and a growing pool of uninsured people. Which policy might you more likely favor?*

Answer: Policy A aims to address this — by forcing everyone into the same insurance pool and then allowing competition to drive down price.

Conclusion: Potentially Helpful Reminders

1. While we have developed the underlying concepts related to asymmetric information primarily for insurance markets, it is important to step back and realize how pervasive issues of asymmetric information are in other parts of life. Some of the end-of-chapter exercises try to get at that.

2. We have developed different models that allow us to get at different ideas related to asymmetric information. Keep in mind throughout, however, that they all deal with the basic underlying externality problem that some can impose on others so long as information remains hidden.

3. Part A of the chapter deals with asymmetric information largely in the absence of thinking about risk. Part B of the chapter can really only make sense if you have covered some of the material on risk in Chapter 17 — and it is there that we are able to deal more fully with insurance markets that, in the end, are all about dealing with risk.

4. The key to seeing whether a particular equilibrium can be sustained is to ask whether the different types of individuals are in fact willing to participate in the market under the proposed equilibrium. If the answer is no for some (of the low cost consumers, for instance), then the equilibrium unravels. Still, it may be that it does not fully unravel — that some (higher cost) consumers are still able to purchase, albeit at a higher price.

5. When signals or screens are possible, we have to think carefully about whether it is in anyone's interest to use these, and what that implies about the equilibrium that is possible. Many of the end-of-chapter exercises directly confront this.

CHAPTER

23

Monopoly

With this chapter, we leave the world of perfectly competitive markets and begin our analysis of violations of the final of the first welfare theorem conditions we first outlined in Chapter 15. To be more precise, we have so far always assumed that everyone is "small" relative to the economy — and that everyone therefore acts as a "price taker". An equilibrium could therefore be described as a situation in which everyone is doing the best he/she can *given the circumstances he/she faces*. When individuals become "large" relative to the economic environment, however, this definition of an equilibrium no longer holds. This is because now the individual has some control over the circumstances he/she is facing — and "doing the best he/she can" now involves using his/her market power to optimally arrange his/her circumstances. We begin in this chapter with the simplest and most extreme case of such market power: the case of a monopoly.

Chapter Highlights

The main points of the chapter are:

1. A firm enjoys a **monopoly** if it produces a product that has no close substitutes in an environment where potential competitors face high **barriers to entry**.

2. When charging a single per-unit price to all customers, a monopolist will choose to sell on the point on the demand curve where **marginal revenue is equal to marginal cost** — which logically implies that it will not produce some goods that are valued more by consumers than they cost to produce. This creates a deadweight loss.

3. If a monopolist can identify different consumer types with different demand curves, *and* if the monopolist can prevent consumers from re-selling the product they buy, the firm will engage in **market segmentation** through either **first-degree** or **third-degree price discrimination.** Under first-degree price discrimination, the monopolist captures the entire surplus while producing

the efficient quantity; under third-degree price discrimination, the monopolist charges different per-unit prices to different consumer types.

4. If a monopolist cannot identify different consumer types but knows the proportion of each type in the population, the firm will engage in **second-degree price discrimination** under which different output/price packages are designed in such a way so as to get consumers to reveal which type they are.

5. In the presence of downward sloping average cost curves (that arise from either increasing returns to scale technologies or high fixed entry costs), it is often "natural" for a single firm to dominate the market. Such cases are known as **natural monopolies**. For such monopolies, the barriers to entry arise from the nature of the underlying production technology. Alternatively, barriers to entry can arise from legal protection offered to firms through the political process.

23A Solutions to Within-Chapter-Exercises for Part A

Exercise 23A.1

What is the marginal revenue of producing an additional good if the producer is at point C on the demand curve in Graph 23.1?

Answer: At C, an increase of output from 600 to 601 implies that price will fall from \$100 to \$99.50. At output of 600, total revenue for the firm is 600(100)=\$60,000, and at output of 601, total revenue is 601(99.50)=\$59,799.50. The marginal revenue of producing the 601st good is therefore -\$200.50.

Exercise 23A.2

Where does MR lie when price elasticity falls between -1 and 0?

Answer: It lies below the horizontal axis — implying that total revenue will fall if additional output is produced.

Exercise 23A.3

Where does a monopolist maximize revenue if she faces a unitary elastic demand curve such as the one in Graph 18.5?

Answer: If price elasticity of demand is -1 everywhere, then revenue is the same for all output quantities.

Exercise 23A.4

True or False: If recurring fixed costs are \$40,000, then the monopolist will earn \$80,000 in short run economic profit and \$40,000 in long run economic profit.

Answer: This is true. With zero marginal cost, the MC curve would intersect with MR at output of 400 and price would be \$200 per unit. Thus, the firm would make 400(200)=\$80,000 in revenue with no short run costs — implying a short run profit of \$80,000. In the long run, the recurring fixed cost would reduce this to \$40,000.

Exercise 23A.5

Suppose MC is equal to \$200 for all quantities for a monopolist who faces a market demand curve of the type in Graph 23.1. At what point on the demand curve will she choose to produce?

Answer: The MC curve would be horizontal at \$200 — and would intersect MR at output of 200 units giving a price (from the demand curve) of \$300 per unit. Thus, the monopolist would produce at point B on the demand curve.

Exercise 23A.6

Suppose a deep freeze causes the Florida orange crop to be reduced by 50% causing the price for oranges to increase. As a result, we observe that the total revenues of Florida orange growers increases. Could the Florida orange industry be a monopoly?(*Hint*: The answer is no.)

Answer: We observe an increase in price resulting in an increase in overall revenues — which implies that the Florida orange growers were operating on the *inelastic* portion of the demand curve (i.e. the portion where price elasticity of demand lies between -1 and 0). No monopolist would ever choose to produce on the inelastic portion of demand — which implies that the Florida orange industry could not be a monopoly. Put differently, if the Florida orange growers were indeed a monopoly, they would not have waited for a freeze to reduce output to get more revenue — they would have already done that prior to the freeze.

Exercise 23A.7

Suppose that demand is as depicted in Graph 23.1 and MC=0. What is the monopolist's profit maximizing output level and what is the efficient output level? What if MC=300?

Answer: If $MC = 0$, the monopolist's output level is 400 (where MC intersects MR) but the efficient output level is 800 (where MC intersects demand). Similarly, if $MC = 300$, the monopolist's output level is 100 but the efficient output level is 200.

Exercise 23A.8

True or False: Depending on the shape of the MC curve, the efficient output level might lie on the elastic or the inelastic portion of the demand curve.

Answer: This is true. The efficient output level lies where MC intersects demand — which can fall on the inelastic portion if MC is sufficiently low. The monopolist's quantity, on the other hand, lies where MC intersects MR — which always happens on the *elastic* portion of demand no matter how low the MC goes (except for the case where $MC = 0$ when the monopolist's quantity falls on the border between elastic and inelastic demand — i.e. where price elasticity of demand is exactly -1.)

Exercise 23A.9

True or False: In the presence of negative production externalities, a monopolist may produce the efficient quantity of output.

Answer: This is also true. Negative production externalities (like pollution) cause the social MC to lie above the firm's MC — and the efficient output level lies where social MC intersects demand. If the externality is sufficiently big (and not too big), the efficient quantity might lie exactly where the monopolist chooses to produce by setting her firm's MC equal to MR.

Exercise 23A.10

True or False: If demand were not equal to marginal willingness to pay (due to the presence of income effects on the consumer side), the deadweight loss area may be larger or smaller but would nevertheless arise.

Answer: True — if $MWTP$ were steeper than demand, the deadweight loss would be larger, and if it were shallower, it would be smaller.

Exercise 23A.11

We simplified the analysis by assuming that each person will buy only 1 piece of art. How would you extend the idea of perfect price discrimination (resulting in demand being equal to marginal revenue) to the case where consumers bought multiple pieces?

Answer: We would simply assume that the monopolist knows everyone's demand (or marginal willingness to pay) curves — and can sell each individual unit to each person for exactly the price that the person is willing to pay.

Exercise 23A.12

The practice of charging a fixed fee plus a per unit price is called a "two-part tariff." It consists of a fixed payment that is independent of the quantity a consumer buys and a per-unit price for each unit the consumer chooses to purchase. Can you identify in the Graph 23.4 which portion would be the fixed payment and what would be the per-unit price for each of the two consumers if the two-part tariff is implemented by a perfectly price-discriminating monopolist?

Answer: The perfectly price discriminating monopolist who uses a two-part tariff would charge each consumer a fixed fee equal to the triangle above the MC curve and below the consumer's demand curve. He would then also charge a per-unit price equal to $MC = 10$. The fixed fee would therefore differ for the two consumer types but the per-unit price would be the same.

Exercise 23A.13

In our example of me running my art studio and selling to consumers who place value only on the first piece of art they purchase, is there a difference between first and third degree price discrimination? Explain. (*Hint*: The answer is no.)

Answer: In this special case, there would be no distinction between first and third degree price discrimination. Under third degree price discrimination, the producer would simply charge each consumer her marginal willingness to pay for the first output she buys (because her marginal willingness to pay for the second output unit is zero). This is exactly identical to what the producer would do under first degree price discrimination.

Exercise 23A.14

Why do we not run into similar problems of ambiguity in thinking about the welfare effects of first degree price discrimination?

Answer: Under first degree price discrimination, the monopolist captures the entire efficient level of surplus. Making such first degree price discrimination illegal implies that monopolist will charge a single price and reduce output — and the reduction in output implies a decrease in overall surplus. While consumers would get surplus in this case and would (as a group) benefit from making first degree price discrimination illegal, the increase in consumer surplus is offset by a larger decrease in monopoly profit — with overall surplus unambiguously shrinking.

Exercise 23A.15

Explain how this represents separate "two-part tariffs" for the two consumer types (as defined in exercise 23A.12).

Answer: The two packages can in fact be phrased this way: Package 1 has 200 units sold at a fixed fee of $(a + b + c)$ and a per-unit price of \$10; and the second package has 100 units sold at a fixed fee of (a) and a per-unit price of \$10.

Exercise 23A.16

Why would the monopolist not be able to offer two per-unit prices as in Graph 23.4?

Answer: He would — but he would not be able to force type 1 consumers to buy at p^1 and type 2 consumers to buy at p^2 because we are now assuming that he does not know who is which type. As a result, if he did offer these two prices, everyone would just buy at the lower price.

Exercise 23A.17

In exercise 23A.12, we introduced the notion of a "two-part tariff". Can you express the pricing suggested above in terms of two-part tariffs?

Answer: Formulated as a two-part tariff, the monopolist would offer the following two options: Any consumer can purchase 100 units for a per-unit price of $10 plus a fixed fee of a, or she can buy 200 units for a per unit price of $10 plus a fixed fee of $(a+c)$.

Exercise 23A.18

What price will the profit maximizing monopolist charge for x^* and for 200 units in panel (c) of Graph 23.5?

Answer: For quantity x^*, the monopolist charges a price of $(10x^* + i)$, and for quantity 200 she charges a price of $(2,000 + i + l + k)$. Alternatively, put into the language of a two-part tariff, the monopolist offers x^* at a per-unit price of $10 plus a fixed fee of (i) and 200 output units at a per-unit price of $10 plus a fixed fee of $(i + l + k)$.

Exercise 23A.19

We have assumed in our example that there is an equal number of type 1 and type 2 consumers in the economy. How would our analysis change if the monopolist knew that there were twice as many type 1 consumers as type 2 consumers?

Answer: This would imply that, whenever the monopolist lowers the quantity for type 2 consumers, she loses the revenue she could have gotten from that consumer but gains the additional charge she can charge type 1 consumers *two times.* In terms of the text graph, it means that the monopolist loses (g) in panel (b) but gains $(2f)$ (rather than just (f)). It therefore no longer makes sense for the monopolist to set x^* in panel (c) such that the red vertical distance is equal to the blue vertical distance — rather, the monopolist would want to keep cutting the quantity in the type 1 package until the red distance is *twice* the blue distance. Second degree price discrimination then means that x^* is set so that the loss from a type 1 consumer when x^* is reduced by one more unit is just equal to half the gain from type 2 consumers — because we get the gain from type 2 consumers twice for every one loss from the type 1 consumers.

Exercise 23A.20

In Chapter 22, we analyzed situations in which there is asymmetric information between consumers and producers (as in the insurance market). Can you see how the problems faced by an insurance company that does not know the risk-types of its consumers are similar to the problem faced by the monopolist who is trying to second-degree price discriminate?

Answer: It is similar in the sense that consumers in both types of markets have information that producers would like to have in order to price their product. In the insurance case, (competitive) firms would like to know the risk type of those looking for insurance in order to be able to target insurance packages to different types of producers (rather than not being able to offer insurance to some because of the adverse selection of high cost consumers). We showed that, through signaling and screening, it might be possible for insurance companies to get this information and sustain separate insurance markets for different types of consumers. In the case of a second-degree price discriminating monopolist, the monopolist similarly does not know the type of consumer she is facing — and is attempting to structure incentives so that consumers will signal their type by the choices they make.

Exercise 23A.21

Review the logic of how a production process can have diminishing marginal product of all inputs while still exhibiting increasing returns to scale.

Answer: Recall from Chapter 12 that diminishing marginal product implies that a producer cannot increase *a single input* by t and expect output to increase by a factor of t. If there is diminishing marginal product of all inputs, this implies increasing a single input at a time will result in less and less of an increase in output. But this does not preclude the possibility that we can increase *all inputs* at the same time by a factor of t and get an increase in output that is larger than a factor of t — which would imply increasing returns to scale.

Exercise 23A.22

Can you see in Graph 23.6a that a price taking firm facing a downward sloping AC curve would produce either no output or an infinite amount of the output depending on what the price is?

Answer: If price is at or below MC, the price taking firm would not produce, but if it is above MC, it would produce an infinite amount as AC converges to MC when output gets large — and thus profit from producing large amounts is positive.

Exercise 23A.23

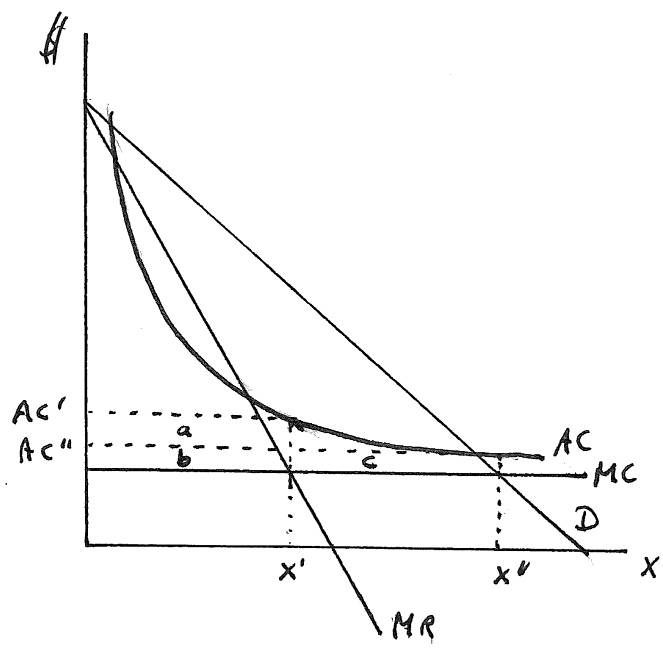
Suppose the technology is such that AC is U-shaped but the upward sloping part of the U-shape happens at an output level that is high relative to market demand. Can the same "natural monopoly" situation arise?

Answer: Yes, because even if AC eventually slopes up, the relevant portion (given market demand) may be downward sloping throughout.

Exercise 23A.24

In a graph similar to Graph 23.6b, illustrate the negative profit that arises when the monopolist is forced to price at MC.

Answer: This is illustrated two different ways in Exercise Graph 23A.24. If the monopolist produces x' (where $MR = MC$), her average costs would be AC' and her *negative* profit would be $(a+b)$. If, on the other hand, she produced where $MC = D$ at x'', her average cost would be AC'' and her negative profit would be $(b+c)$. Since the negative profit here is equal to the fixed cost, it must be that $a = c$.



Exercise Graph 23A.24 : Negative Profit when $p = MC$

Exercise 23A.25

Suppose the fixed cost is a one-time fixed entry cost that is sufficiently large to result in a picture like panel (c). *True or False*: If the government pays the fixed cost for the firm, it will not have to regulate the firm in order to make sure the firm makes a profit — but he monopoly outcome will be inefficient.

Answer: This is true — once the fixed costs are paid, they are sunk and all that matters is the MC. Thus, the monopoly would produce where MC intersects MR and price in the usual way — leading to positive profit for the monopolist but inefficiently low output because the monopolist strategically reduces quantity in order to raise price.

Exercise 23A.26

Is this an example of a two-part tariff? Does it result in efficiency?

Answer: Yes, it is a two part tariff that combines a fixed fee with a per-unit charge. And it does achieve efficiency if price is regulated to be equal to MC, with the fixed fee transferring some of what would otherwise be consumer surplus to the monopolist in order to make sure she covers her fixed costs.

Exercise 23A.27

Suppose that instead a private company is charged with laying all the infrastructure and then charges competing electricity firms to use the electrical grid. How might this raise a different set of efficiency issues related to monopoly pricing? Would these issues still arise if the government auctioned off the right to build an electricity grid to a single private company?

Answer: The company that owns the infrastructure would own a monopoly on this infrastructure — giving rise to the usual efficiency concerns for a monopoly. It the government auctioned off the right to build the grid, the winner of the auction would still be a single monopoly — so the competitive auction does not resolve this problem.

Exercise 23A.28

In the 1970s when OPEC countries raised world prices for oil substantially by exercising their market power, the Saudi oil minister is said to have warned them: "Remember, the Stone Age did not end because we ran out of stones." Explain what he meant and how his words relate to constraints that monopolies face.

Answer: The Saudi oil minister was warning about the threat of alternative energy sources — and the fact that, as oil prices increase, the incentives for entrepreneurs to develop such alternatives increases as well. Thus, if OPEC exerts its market power, it implicitly will speed up the development of alternative fuels — which places an implicit constraint on how much OPEC may wish to exert its market power with an eye toward postponing competition from such alternative fuels.

23B Solutions to Within-Chapter-Exercises for Part B

Exercise 23B.1

Explain why the cost minimization problem in the firm's duality picture is identical for firms regardless of whether they are monopolies or perfect competitors.

Answer: The cost minimization problem simply asks what the least cost way of producing different output levels is. The answer to that question depends on the technology the firm is dealing with and the input prices it faces. So long as input markets are competitive, the least cost way of producing a particular output level is then the same for competitive firms and monopolists. Put differently, cost minimization is only concerned with the cost side — not the revenue side, and it is on the revenue side that the monopolist faces different circumstances than the competitor.

Exercise 23B.2

Use equation (23.10) to verify the vertical intercept of the marginal revenue curve in Graph 23.1b.

Answer: At the vertical intercept of a linear demand curve, $\epsilon_D = -\infty$. As the elasticity of demand approaches minus infinity, the term in parentheses in the equation goes to 1 — implying $MR = p(x)$.

Exercise 23B.3

Set up a revenue maximization problem for the firm. Then verify that this is indeed the revenue maximizing output level and that, at that output, $\epsilon_D = -1$.

Answer: The revenue maximization problem is

$$\max_{p,x} \; px \text{ subject to } p = p(x) = \left(\frac{A}{\alpha} - \frac{1}{\alpha}x\right) \tag{23B.3.i}$$

which can also be written as

$$\max_{x} \left(\frac{A}{\alpha} - \frac{1}{\alpha}x\right)x. \tag{23B.3.ii}$$

Setting the first derivative equal to zero and solving for x, we get $x = A/2$. The price elasticity of demand when $x(p) = A - \alpha p$ (and inverse demand $p(x) = (A - x)/\alpha$) is

$$\epsilon_D = \frac{dx}{dp}\frac{p(x)}{x} = -\alpha\left(\frac{(A-x)/\alpha}{x}\right) = -\left(\frac{A-x}{x}\right). \tag{23B.3.iii}$$

Plugging in $x = A/2$, we get $\epsilon_D = -1$.

Exercise 23B.4

Can you use equation (23.10) to now prove that, so long as $MC > 0$, the monopolist will produce where $\epsilon_D < -1$?

Answer: Substituting the elasticity expression for MR into the first order condition, we get

$$p(x)\left(1+\frac{1}{\epsilon_D}\right) = MC. \tag{23B.4.i}$$

Solving this for ϵ_D, we get

$$\epsilon_D = -\left(\frac{p(x)}{p(x)-MC}\right). \tag{23B.4.ii}$$

As long as $MC > 0$, the denominator in the term in parenthesis is smaller than the numerator — which means that the term in parenthesis is larger than 1. Thus, ϵ_D is greater than 1 in absolute value, or — given the negative sign, $\epsilon_D < -1$.

Exercise 23B.5

Illustrate that profit maximization approaches revenue maximization as $MC = c$ approaches zero.

Answer: As c approaches 0, x^M approaches $A/2$ — which we concluded before is the revenue maximizing output quantity.

Exercise 23B.6

Verify for the example of our linear demand curve and constant marginal cost c that it does not matter whether the firm maximizes profit by choosing x or p (as in the problems defined in equations (23.2) and (23.3) above).

Answer: If the firm chooses x, it solves the problem

$$\max_x \; p(x)x - cx = \left(\frac{A}{2}-\frac{1}{\alpha}x\right)x - cx. \tag{23B.6.i}$$

Taking the derivative with respect to x, setting it to zero and solving for x, we get $x = (A-\alpha c)/2$. Plugging this into the inverse demand function $p(x) = (A-x)/\alpha$, we also get $p = (A+\alpha c)/2\alpha$.

If the firm chooses p, it solves the problem

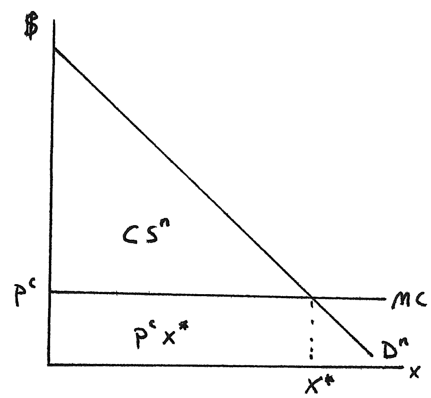
$$\max_p \; px(p) - cx(p) = p(A-\alpha p) - c(A-\alpha p). \tag{23B.6.ii}$$

Setting the first derivative with respect to p to zero and solving for p, we get $p = (A+\alpha c)/2\alpha$. Plugging this back into $x(p) = A-\alpha p$, we get $x = (A-\alpha c)/2$.

Exercise 23B.7

Illustrate graphically the two different parts of the two-part tariff in equation (23.17).

Answer: This is illustrated in Exercise Graph 23B.7



Exercise Graph 23B.7 : First Degree Price Discrimination with Two-Part Tariff

Exercise 23B.8

Verify that equation (23.22) holds for this example. (Be sure to evaluate elasticities at the optimal output levels.)

Answer: The price elasticities of demand are

$$\epsilon_{D^1} = \frac{\partial dx^1}{\partial p}\frac{p}{x^1(p)} = \frac{-\alpha p}{A - \alpha p} \text{ and } \epsilon_{D^2} = \frac{-\beta p}{B - \beta p}. \tag{23B.8.i}$$

Substituting in our values for p^1 and p^2 and simplifying, we get

$$\epsilon_{D^1} = \frac{-(A + \alpha c)}{A - \alpha c} \text{ and } \epsilon_{D^2} = \frac{-(B + \beta c)}{B - \beta c}. \tag{23B.8.ii}$$

Using these, the equation we are trying to verify implies

$$\frac{p^1}{p^2} = \frac{(\epsilon_{D^2} + 1)\epsilon_{D^1}}{(\epsilon_{D^1} + 1)\epsilon_{D^2}} = \frac{\beta(A + \alpha c)}{\alpha(B + \beta c)}. \tag{23B.8.iii}$$

This is the same thing we get when we simply divide our expression for p^1 and p^2 by one another.

Exercise 23B.9

True or False: The higher priced market under (third degree) price discrimination is more price inelastic.

Answer: Begin with the equation (derived in the text):

$$p^1\left(1+\frac{1}{\epsilon_{D^1}}\right)=MC=p^2\left(1+\frac{1}{\epsilon_{D^2}}\right). \tag{23B.9.i}$$

Suppose $p^1 > p^2$. Then the term in parenthesis on the left hand side has to be smaller than the term in parentheses on the right hand side for the equation to hold; i.e.

$$1+\frac{1}{\epsilon_{D^1}}<1+\frac{1}{\epsilon_{D^2}} \tag{23B.9.ii}$$

which implies

$$\frac{1}{\epsilon_{D^1}}<\frac{1}{\epsilon_{D^2}}. \tag{23B.9.iii}$$

Multiplying both sides by ϵ_{D^1} and ϵ_{D^2} (which are both negative numbers — thus switching the sign of the inequality each time), we conclude that $\epsilon_{D^1} > \epsilon_{D^2}$; i.e. when $p^1 > p^2$, it must be that the price elasticity of demand by consumer type 1 is greater than price elasticity for consumer type 2. Put differently, consumer type 1 is more price inelastic than consumer type 2.

Exercise 23B.10

Intuitively, why does the fixed charge F from the two-part tariff not show up in the demand function?

Answer: The fixed charge F is a sunk cost as soon as it is paid — and thus does not affect the quantity choice.

Exercise 23B.11

Derive the price charged to consumer n by a third-degree price discriminating monopolist with constant marginal cost c.

Answer: The inverse demand function is $p(x)=\theta_n-\theta x$. The marginal revenue function once again has the same intercept but twice the slope — i.e.

$$MR=\theta-2\theta x. \tag{23B.11.i}$$

Setting this equal to marginal cost c and solving for p, we get

$$p^n=\frac{\theta_n-c}{2\theta_n}. \tag{23B.11.ii}$$

Exercise 23B.12

Verify that this equation is correct.

Answer: The steps for doing so are given in the two lines immediately preceding the equation in the text.

Exercise 23B.13

Verify that this equation is correct.

Answer: We are solving the problem

$$\max_p \frac{(\theta_1 - p)^2}{2\theta_1} + (p-c)\left[1 - \left(\frac{\gamma}{\theta_1} + \frac{(1-\gamma)}{\theta_2}\right)p\right]. \tag{23B.13.i}$$

Setting the first derivative of the objective function equal to zero, we get the equation

$$\frac{-(\theta_1 - p)}{\theta_1} + \left[1 - \left(\frac{\gamma}{\theta_1} + \frac{(1-\gamma)}{\theta_2}\right)p\right] - (p-c)\left(\frac{\gamma}{\theta_1} + \frac{(1-\gamma)}{\theta_2}\right) = 0. \tag{23B.13.ii}$$

We can re-write this as

$$\frac{-\theta_2(\theta_1 - p) + \theta_1\theta_2 - (\gamma\theta_2 + (1-\gamma)\theta_1)p - (p-c)(\gamma\theta_2 + (1-\gamma)\theta_1)}{\theta_1\theta_2} = 0. \tag{23B.13.iii}$$

Multiplying both sides by $\theta_1\theta_2$ and then simplifying, this becomes

$$\theta_2 p - 2p(\gamma\theta_2 + (1-\gamma)\theta_1) + c(\gamma\theta_2 + (1-\gamma)\theta_1) = 0 \tag{23B.13.iv}$$

which we can solve for p to get

$$p^* = \frac{c(\gamma\theta_2 + (1-\gamma)\theta_1)}{2(\gamma\theta_2 + (1-\gamma)\theta_1) - \theta_2}. \tag{23B.13.v}$$

Exercise 23B.14

Are these preferences convex?

Answer: Yes — any convex combination of two bundles on an indifference curve lies to the southeast of the indifference curve — which is in the preferred region. Thus, averages are better than extremes.

Exercise 23B.15

Note that each set of blue and magenta indifference curves cross once, with the magenta indifference curve having a steeper slope at that point than the blue indifference curve. Can you give an intuitive explanation for this?

Answer: With the tastes that we have assumed, θ is the parameter that indicate whether someone is a "high" or "low" demander. In our case, $\theta_1 < \theta_2$ — which implies type 1 consumers are low demanders relative to type 2 demanders. At any (x,P) combination, type 2 consumers should therefore be willing to pay more for one more unit of x than type 1 consumers — which is the same as saying that the slope of the indifference curve should be steeper for type 2 than for type 1. We can also see this by simply taking the marginal rate of substitution. We have assumed a function $u(x) = (1-(1-x)^2)/2$ and represent tastes over x and P with the function

$$U(x,P) = \theta u(x) - P = \theta\left(\frac{1-(1-x)^2}{2}\right) - P. \qquad (23B.15.i)$$

The MRS for the indifference curves in our graph is then

$$MRS = -\frac{\partial U/\partial x}{\partial U/\partial P} = -\left(\frac{\theta(1-x)}{-1}\right) = \theta(1-x). \qquad (23B.15.ii)$$

As θ increases, the MRS increases — giving us a steeper slope to the indifference curve at any (x,P).

Exercise 23B.16

Given what you know of how the firm constructed the two-part tariff, can you give an intuitive explanation for this?

Answer: The monopolist set the fixed fee equal to the consumer surplus that type 1 consumers would have otherwise gotten — thus, no consumer surplus is left for type 1 consumers when the two-part tariff is implemented.

Exercise 23B.17

Explain why, for the preferences we have been working with, the two demand curves have the same horizontal intercept.

Answer: We derived the demand functions as $x(p) = (\theta - p)/\theta$ — giving us inverse demand functions (or demand curves) of $p(x) = \theta - \theta x$. As θ increases (from type 1 to type 2), the vertical intercept therefore shifts up — but the slope increases as well (in absolute value)! Since the slope is the same parameter as the intercept, the horizontal intercept remains unchanged (at $x = 1$).

Exercise 23B.18

Why is $F^2 = (a+b+c+d)$ the highest possible fixed fee the firm can charge to type 2 consumers given that it sets per unit prices at MC and charges type 1 $F^1 = (a+b+c)$?

Answer: If consumer type 2 were to buy the type 1 package, she would get surplus $(f+e)$. If she buys the type 2 package with fixed charge $F^2 = (a+b+c+d)$, she would get the same surplus. Thus, were the firm to set F^2 any higher, type 2 consumers would simply by the type 1 package.

Exercise 23B.19

Why is the expected profit from the single two-part tariff $(a+b+(1-\gamma)(c+e))$?

Answer: Under the single two-part tariff, the fixed fee is (a) and the per-unit price is p^*. Thus, from consumer type 1 we would get profit of $(a+b)$, and from consumer type 2 we would get $(a+b+c+e)$. There are γ type 1 consumers and $(1-\gamma)$ type 2 consumers — thus we have an expected profit of

$$\gamma(a+b)+(1-\gamma)(a+b+c+e) = a+b+(1-\gamma)(c+e). \tag{23B.19}$$

Exercise 23B.20

Can you think of alternative scenarios under which the single two-part tariff yields more profit?

Answer: Suppose $\gamma = 0$. Then the profit from the single two-part tariff is $(a+b+c+e)$ and the profit from the new pricing policy is $(a+b+c+d)$. Area (d) is smaller than (e) — so the single two part tariff yields more profit. This must also be true for small γ.

Exercise 23B.21

If the monopolist is restricted to offering a single two-part tariff (rather than two separate tariffs intended for the two consumer types), is she more or less likely to forego second degree price discrimination in favor of first degree price discrimination with respect to the high demand type?

Answer: If the type of second degree price discrimination is limited to only a single two-part tariff, then second degree price discrimination is less attractive from the monopolist's perspective than it would be if it were unrestricted. Thus, the monopolist would be more likely to just first degree price discriminate relative to the high demand type.

Exercise 23B.22

From looking at Table 23.1, it seems that the firm is unambiguously less restricted in its pricing under second degree price discrimination than under third degree price discrimination. So how could it theoretically be the case that profit is higher under third degree price discrimination?

Answer: The restriction that is not shown in the table is that firms do not know who is which type under second degree price discrimination — but it does know under third degree price discrimination. Because it knows less under second degree price discrimination, the firm has to leave some consumer types with some surplus — as it does under third degree price discrimination under which it faces the restriction of no fixed fees.

Exercise 23B.23

Can you think of a scenario under which all the inequalities turn to equalities in equations (23.44) and (23.45)? (*Hint:* Think of goods for which consumers demand only 1 unit.)

Answer: If everyone only demands one good — and if everyone is identical, then the inequalities all turn to equalities.

Exercise 23B.24

Can you give an intuitive explanation for why this has to hold?

Answer: Under first degree price discrimination, monopolists get all the surplus — thus all consumer types get zero surplus. Under the two part tartiff — and under second degree price discrimination, the monopolist is able to take all of the low demanders' consumer surplus — and only needs to leave the high demanders with some surplus to keep them from acting like low demanders. In the absence of any price discrimination, consumers of type 1 get lumped together with the high demanding type 2 consumers — which means that the price charged to both of them together will be higher than the price for type 1 consumers under third degree price discrimination (which allows the monopolist to treat the low demanders as a separate market).

Exercise 23B.25

Can you think of any definitive policy implications if the goal of policy is to maximize consumer welfare (with no regard to firm profit)?

Answer: If only consumer welfare matters, the first degree price discrimination is unambiguously the worst case scenario as consumer surplus is zero. While it does not matter for consumers of type 1, consumers of type 2 get higher surplus under 2nd degree price discrimination (with multiple two-part tariffs) and even higher consumer surplus under a single 2-part tariff (when monopolists do not know who is what type). Thus, we can say that the two-part tariff (with no type-knowledge by monopolists) is preferred to second degree price discrimination which is preferred to first degree price discrimination if consumer surplus is used to rank these. The relationship between these and no discrimination as well as 3rd degree price discrimination is ambiguous — and type 1 consumers rank 3rd degree price discrimination over no price discrimination while the reverse holds for type 2 consumers. Thus, the only other definitive statement we can make is that both no discrimination and third degree price discrimination are preferred on consumer surplus grounds to first degree price discrimination, but we cannot rank which is preferred more.

Exercise 23B.26

Explain all the zeros in Table 23.2.

Answer: The zeros for the fixed fees in the first two rows simply arise from the fact that fixed fees are not permitted under no price discrimination and under 3rd degree price discrimination. The zeroes for consumer surplus under first degree price discrimination come from the fact that monopolists expropriate all surplus in this case. And the zeros for type 1 consumer surplus under the 2-part tariff and 2nd degree discrimination arises from the fact that monopolies can expropriate all the surplus from low demanders and only have to watch how much they take from high demanders in order to make sure they don't find it more attractive to take the low-demand price/quantity package.

Exercise 23B.27

In Table 23.1 we note that there are no restrictions on per-unit prices for the two consumer types under either first or second degree price discrimination, with firms being able to tell consumer types apart in the former case but not the latter. Yet in Table 23.2, the firm appears to be charging exactly the same per unit prices to the two consumers under first degree price discrimination when it can tell the consumers apart and *different* per-unit prices under second-degree price discrimination when the firm cannot tell the consumer types apart. Explain this intuitively.

Answer: Under first degree price discrimination, firms can charge a per-unit price equal to marginal cost (which is $25 in the example) — and then charge different fixed fees to the two consumer types, with each fee equal to the consumer surplus that type would get if it only paid marginal cost. Under second degree price discrimination, the firm makes sure to set the per-unit price for high demanders equal to marginal cost (causing them to purchase the efficient quantity) — and then sets the fee for those consumers as high as possible without making the low-demand package more attractive than the high demand package. To maximize profit, we saw that monopolists will reduce the quantity in the low demand package so as to make it less attractive to high demand consumers — and thus low demand consumers will not buy at the intersection of marginal cost and demand. As a result, the per-unit price is set higher for low demand types. While it is true that the monopolist does not know who is what type to start out with, she designs the two packages in such a way as to get the types to reveal themselves.

23C Solutions to Odd Numbered End-of-Chapter Exercises

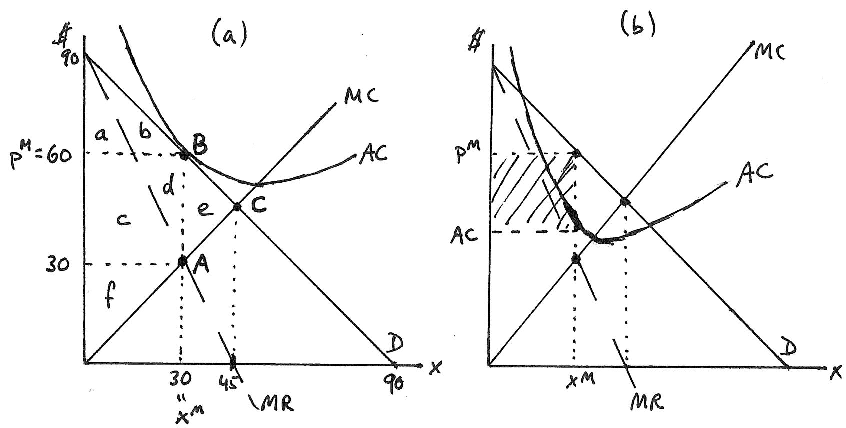
Exercise 23.1

Suppose that the demand curve for a product x provided by a monopolist is given by $p = 90 - x$ and suppose further that the monopolist's marginal cost curve is given by $MC = x$.

A: *In this part, we will focus on a graphical analysis — which we ask you to revisit with some simple math in part B. (It is not essential that you have done Section B of the chapter in order to do (a) through (d) of part B of this question.)*

(a) *Draw a graph with the demand and marginal cost curves.*

Answer: This is done in panel (a) of Exercise Graph 23.1.



Exercise Graph 23.1 : Profit Maximizing Monopolist

(b) *Assuming that the monopolist can only charge a single per-unit price for x, where does the marginal revenue curve lie in your graph?*

Answer: The marginal revenue curve has the same vertical intercept as the demand curve but twice the slope — and thus intersects the horizontal axis at half the distance.

(c) *Illustrate the monopolist's profit maximizing "supply point".*

Answer: The profit maximizing quantity is determined by the intersection of MR and MC — i.e. point A in panel (a) of Exercise Graph 23.1. The profit maximizing price is then the highest possible price at which this quantity can be sold — and is read off point B.

(d) *In the absence of any recurring fixed costs, what area in your graph represents the monopolist's profit. (There are actually two areas that can be used to represent profit — can you find both?)*

Answer: There are two ways you can represent profit in this case: First, we can simply use the usual way of indicating producer surplus as the area below the price down to the MC curve to get profit of $(c+d+f)$. Second, we can also illustrate profit as the difference between MR and MC up to point A — i.e. area $(a+c+f)$. This implies that $(a) = (d)$ in the graph.

(e) *Assuming that the demand curve is also the marginal wilingness to pay curve, illustrate consumer surplus and deadweight loss.*

Answer: Consumer surplus is simply the area above the price paid by consumers up to the demand curve — i.e. $(a+b)$. Deadweight loss arises from the fact that the efficient level of output would have occurred at C where the marginal benefit is equal to the marginal cost — and by producing only x^M, no one gets the potential surplus (e). Thus, deadweight loss is equal to area (e).

(f) *Suppose that the monopolist has recurring fixed costs of an amount that causes her actual profit to be zero. Where in your graph would the average cost curve lie? In particular, how does this average cost curve relate to the demand curve?*

Answer: The average cost curve would be U-shaped and would be tangent to the demand curve at point B as illustrated in panel (a) of Exercise Graph 23.1. This is because, when there are recurring fixed costs, we can illustrate profit as the difference between p^M and AC multiplied by x^M; i.e. profit=$(p^M - AC)x^M$. In order for this to be zero, it must be that $p^M = AC$ when x^M is produced — which must mean that the AC curve touches point B. And it cannot be that AC crosses the demand curve — because then there would be ways to make positive profit by choosing a quantity where AC lies below D — but the profit maximizing output level occurs where $MR = MC$.

(g) *In a new graph, illustrate again the demand, MR and MC curves. Then illustrate the monopolist's average cost curve assuming the recurring fixed costs are half of what they were in part (f).*

Answer: This is illustrated in panel (b) of Exercise Graph 23.1.

(h) *In your graph, illustrate where profit lies.* True or False*: Recurring fixed costs only determine* whether *a monopolist produces — not how much she produces.*

Answer: Profit is illustrated as $(p^M - AC)x^M$ — the shaded area in panel (b) of Exercise Graph 23.1. The statement is true: If the monopolist produces, she will produce where $MR = MC$ and charge the maximum price at which this output level can be sold. Recurring fixed costs have nothing to do with where MR, MC or D curves are. But if recurring fixed costs get too high — i.e. higher than area $(c+d+f)$ in panel (a) of Exercise Graph 23.1 — then the monopolist will simply not produce since her profit would be negative.

B: *Consider again the demand curve and MC curve as specified at the beginning of this exercise.*

(a) *Derive the equation for the marginal revenue curve.*

Answer: This equation is $MR = 90 - 2x$ — same vertical intercept as the demand curve, but twice the slope. You can also derive it by recognizing that total revenue is $TR = px = (90 - x)x = 90x - x^2$ and then taking the derivative to get

$$MR = \frac{dTR}{dx} = 90 = 2x. \tag{23.1.i}$$

(b) *What is the profit maximizing output level x^M? What is the profit maximizing price p^M (assuming that the monopolist can only charge a single per-unit price to all consumers)?*

Answer: To get the profit maximizing output level, we set MR equal to MC — i.e. $90 - 2x = x$ — and solve for x to get $x^M = 30$. Plugging this into the demand curve, we then get $p^M = 90 - x^M = 90 - 30 = 60$.

(c) *In the absence of recurring fixed costs, what is the monopolist's profit?*

Answer: There are various ways we could calculate this. The total cost (in the absence of recurring fixed costs) is the area under the MC curve up to $x_M = 30$ — which is $30(30)/2 = 450$ while the total revenue is $60(30) = 1,800$. Subtracting the former from the latter, we get profit of $1800 - 450 = \$1,350$. We could also have calculated the producer surplus area $(c+d+f)$ in panel (a) of Exercise Graph 23.1. This area is composed of the rectangle $30(30) = 900$ and the triangle $30(30)/2 = 450$. Adding these together, we again get profit of \$1,350. Finally, we could calculate the area between MR and MC up to x^M — area $(a+c+f)$ in the graph. This area consists of two triangles — $(a+c) = 60(30)/2 = 900$ and $(f) = 30(30)/2 = 450$ — which sum again to \$1,350.

(d) *What is consumer surplus and deadweight loss (assuming that demand is equal to marginal willingness to pay).*

Answer: Consumer surplus is $(a+b) = 30(30)/2 = 450$. The deadweight loss area (e) is equal to $15(30)/2 = 225$.

(e) *What is the cost function if recurring fixed costs are sufficiently high to cause the monopolist's profit to be zero?*

Answer: Since the monopolist profit in the absence of recurring fixed costs is \$1,350, we know that recurring fixed costs have to be \$1,350 in order for profit to be zero. Thus, the cost function is

$$c(x) = 1350 + \frac{x^2}{2}. \tag{23.1.ii}$$

The second term in this cost function is such that the derivative is x — which is equal to the marginal cost. (In other words, the second term is the integral of x.)

(f) *Use this cost function to set up the monopolist's optimization problem and verify your answers to (b).*

Answer: The profit maximization problem is

$$\max_{x} \; px - c(x) = (90 - x)x - \left(1350 + \frac{x^2}{2}\right). \tag{23.1.iii}$$

Setting the derivative of the objective function with respect to x equal to zero, we get the equation

$$90 - 2x = x \tag{23.1.iv}$$

which is exactly $MR = MC$. This solves to give us $x^M = 30$ — which, plugged back into the demand curve, gives us $p^M = 60$.

(g) *Does the average cost curve relate to the demand curve as you concluded in part A(f)?*

Answer: We concluded that the average cost curve (when recurring fixed costs result in zero profit) must be tangent to the demand curve at B in panel (a) of Exercise Graph 23.1. The average cost curve is given by

$$AC(x) = \frac{c(x)}{x} = \frac{1350}{x} + \frac{x}{2}. \tag{23.1.v}$$

At B, $x = 30$ — which implies $AC = (1350/30) + (30/2) = 60$. Thus, the AC curve certainly passes through B. To check that it is tangent at B, we need to derive the derivative of AC —

$$\frac{dAC}{dx} = -\frac{1350}{x^2} + \frac{1}{2}. \tag{23.1.vi}$$

Setting $x = 30$ in this equation, we get $dAC/dx = (-1350/900) + 0.5 = -1$ — which is equal to the slope of the demand curve. Thus, we can conclude that the AC curve passes through B and is tangent at B to the demand curve.

(h) *How does the profit maximization problem change if the recurring fixed costs are half of what we assumed in part (e)? Does the solution to the problem change?*

Answer: The profit maximization problem becomes

$$\max_{x} \; px - c(x) = (90 - x)x - \left(675 + \frac{x^2}{2}\right) \tag{23.1.vii}$$

because the recurring fixed cost is now $1350/2 = 675$. When we take the derivative with respect to x, however, the recurring fixed cost drops out — so having the lower recurring fixed cost does not change the solution to the problem. Put differently, it is still the case that profit is maximized where $MR = MC$.

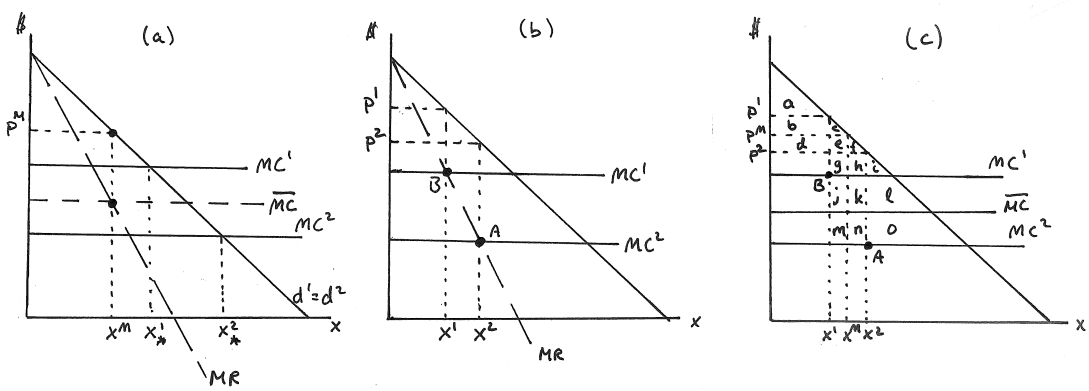
Exercise 23.3

Business and Policy Application: *Monopoly Pricing in Health Insurance Markets: In Chapter 22, we worked with models in which high and low cost customers compete for insurance. Consider the level x of health insurance that consumers might choose to buy, with higher levels of x indicating more comprehensive insurance coverage.*

A: *Suppose that there are relatively unhealthy type 1 consumers and relatively healthy type 2 consumers. The marginal cost of providing additional insurance coverage is then MC^1 and MC^2, with $MC^1 > MC^2$. Unless otherwise stated, assume that $d^1 = d^2$ — i.e. the individual demand curves for x are the same for the two types. Also, suppose that the number of type 1 and type 2 consumers is the same, and some portion of each demand curve lies above MC^1.*

(a) *Begin by drawing a graph with the individual demands for the two types, d^1 and d^2, as well as the marginal costs. Indicate the efficient levels of health insurance x^1_* and x^2_* for the two types.*

Answer: This is done in panel (a) of Exercise Graph 23.3 where the efficient consumption level occurs where each type's marginal cost curve crosses the demand curve.



Exercise Graph 23.3 : Monopolies in Insurance Markets

(b) *Suppose the monopolist cannot tell consumers apart and can only charge a single price to both types. What price will it be and what level of insurance will each type purchase?*

Answer: This is also illustrated in panel (a) of Exercise Graph 23.3. Since the monopolist cannot tell the consumer types apart but knows that half of them are type 1 and half are type 2, he knows only that the average MC that he will face when he insures someone is $\overline{MC}$ halfway between MC^1 and MC^2. Since the two types have the same demand curve, the MR curve is the same for both. The monopolist therefore maximizes profit where $\overline{MC} = MR$ — selling x^M to each type and charging a price p^M per unit of insurance.

(c) *How does your answer change if the monopolist can first-degree price discriminate?*

Answer: Under first degree price discrimination, the monopolist can identify the types and can charge both a per unit price as well as a fixed fee. Thus, the monopolist would charge a per unit price equal to the marginal cost for each type — and then a fixed fee equal to each type's consumer surplus. Put differently, the monopolist will sell x_*^1 to type 1 and x_*^2 to type 2 — and charge both the entire area underneath their demand curve up to the quantity they are sold. The outcome is efficient — but the monopolist gets all the surplus.

(d) *What if he can third-degree price discriminate?*

Answer: This is illustrated in panel (b) of Exercise Graph 23.3. Under third degree price discrimination, the monopolist can again tell the two types apart — but now he can only charge separate per-unit prices and no fixed fees. He will therefore treat the two types as if they were entirely different markets — selecting the quantity that arises from the intersection of MR with MC and then pricing accordingly on the demand curve. That intersection occurs at A for type 2 and at B for type 1 — leading to x^2 and p^2 for type 2 and x^1 and p^1 for type 1.

(e) *Suppose you worked for the Justice Department's anti-trust division and you only cared about efficiency. Would you prosecute a first-degree price discriminating monopolist in the health insurance market? What if you cared only about consumer welfare?*

Answer: If you cared only about efficiency, you would not prosecute (and would rather cheer the monopolist on as she first-degree price discriminates — because the efficient level of insurance is provided to both types under such price discrimination.) If you cared rather about consumer welfare, you would prosecute — and would prefer the monopolist just act as a regular monopolist charging a single price. This would result in inefficient insurance levels — but both types would get at least some consumer surplus (while being "under-insured").

(f) *In the text we suggested that it is generally not possible without knowing the specifics of a case whether third degree price discrimination is more or less efficient than no price discrimination by a monopolist. For the specifics in this case, can you tell whether type 1 consumers are better off without this price-discrimination? What about consumer type 2?*

Answer: This is dealt with in panel (c) of Exercise Graph 23.3 where points A and B are transferred from panel (b) and the prices under price discrimination and in the absence of price discrimination are indicated (from panel (b) and (a)). Type 1 consumers are clearly better off because their price falls from p^1 under price discrimination to p^M — while type 2 consumers are clearly worse off since their price rises from p^2 to p^M. To quantify this, type 1 consumers go from consumer surplus (a) to $(a+b+c)$ while type 2 consumers go from surplus $(a+b+c+d+e+f)$ to $(a+b+c)$ when we abandon third degree price discrimination in favor of a single monopoly price.

(g) *Would it improve average consumer surplus to prohibit the monopolist from third-degree price discriminating? Would it be more efficient?*

Answer: Using the areas in Exercise Graph 23.3 and the results from the previous part, we can see that type 1 consumers gain surplus of $(b+c)$ while consumers of type 2 lose surplus $(d+e+f)$. Since area (b) is equal to area (d) and area (c) is equal to area (f). we can see that, for any pair of one type 1 and one type 2 consumer, surplus of (e) is lost when we abandon third degree price discrimination — or an average of half of (e). Would it be more efficient? For this we have to compare deadweight loss: When type 2 consumers purchase at p^2, there is a deadweight loss of $(i+l+o)$, and when they buy at p^M, there is a deadweight loss of $(f+h+k+n+i+l+o)$ — an increase in deadweight loss of $(f+h+k+n)$. If a type 1 consumer buys at p^1, there is a deadweight loss of $(c+e+f+g+h+i)$ which falls to $(f+h+i)$ when she buys at p^M — a decline in deadweight loss of $(c+e+g)$. For any pair of a type 1 and a type 2 consumer, we therefore increase deadweight loss by $(f+h+k+n)$ and decrease it by $(c+e+g)$ when we abandon third degree price discrimination in favor of a single monopoly price. Since the increase is larger than the decrease, deadweight loss increases and abandoning third degree price discrimination would be inefficient.

B: *Suppose next that we normalize the units of health insurance coverage such that the demand function is $x^n(p)=(\theta_n-p)/\theta_n$ for type n. You can interpret $x=0$ as no insurance and $x=1$ as full insurance. Let $\theta_1=20$ and $\theta_2=10$ for the two types of consumers, and let $MC^1=8$ and $MC^2=6$.*

(a) *Determine the efficient level of insurance for each consumer type.*

Answer: The efficient level of insurance for each type occurs where the demand curve intersects the relevant marginal cost curve. From the demand *function* $x^n(p)$, we can determine the demand *curve* by simply solving for p to get $p^n(x)=\theta_n-\theta_n x$. To solve for the efficient insurance levels, we then need to solve the equations

$$p^1=20-20x=8=MC^1 \text{ and } p^2=10-10x=6=MC^2 \tag{23.3.i}$$

to get $x^1_*=0.6$ and $x^2_*=0.4$.

(b) *If a monopolist cannot tell who is what type and can only charge a single per-unit price for insurance, what will she do assuming there are γ type 1 consumers and $(1-\gamma)$ type 2 consumers, with $\gamma<0.5$? (*Hint*: Define the monopolist's expected profit and maximize it.)*

Answer: The expected profit is

$$E(\pi) = \gamma\pi^1 + (1-\gamma)\pi^2 = \\ = \gamma\left[p\left(\frac{20-p}{20}\right) - 8\left(\frac{20-p}{20}\right)\right] + (1-\gamma)\left[p\left(\frac{10-p}{10}\right) - 6\left(\frac{10-p}{10}\right)\right] \tag{23.3.ii}$$

Setting the derivative of this with respect to p to zero and solving for p, we get

$$p^M = \frac{16-2\gamma}{2-\gamma} \tag{23.3.iii}$$

(c) *What would the monopoly price be if $\gamma = 0$? What if $\gamma = 2/7$? What is the highest that γ can be and still result in type 2 consumers buying insurance?*

Answer: When $\gamma = 0$ — i.e. when there are no high cost/high demand types, $p^M = 8$. This makes sense — in this case, the monopolist simply sets MR for type 2 equal to $MC = 6$. The marginal revenue for type 2 is $MR = 10-20x$. When set equal to 6, we can solve for $x = 0.2$ and, plugging this back into the demand curve for type 2, we get $p = 8$. When $\gamma = 2/7$, we get $p^M = 9$. The highest price that a type 2 consumer would ever pay (given her demand curve intersects the vertical axis at 10) is 10. Setting our expression for p^M equal to 10 and solving for γ, we get $\gamma = 0.5$ — i.e. if the fraction of high cost/high demand types goes above 0.5, the monopoly price will rise above the level at which type 2 consumers would buy insurance.

(d) *Suppose that the monopolist first-degree price discriminates. How much insurance will each consumer type purchase? How much will each type pay for her coverage?*

Answer: The monopolist will price insurance at marginal cost for each type — plus charge a fixed fee equal to consumer surplus. Consumers will therefore purchase the efficient level of insurance — 0.4 for type 2 and 0.6 for type 1. The payments for this level of coverage will be

$$P^1 = \frac{0.6(20-8)}{2} + 8(0.6) = 8.40 \text{ and } P^2 = \frac{0.4(10-6)}{2} + 6(0.4) = 3.20. \tag{23.3.iv}$$

(e) *How do your answers to (d) change if the monopolist third-degree price discriminates?*

Answer: In this case, the monopolist simply treats the two consumer types as separate markets and sets $MR = MC$ in both markets. The marginal revenues in the two markets are

$$MR^1 = 20-40x \text{ and } MR^2 = 10-20x. \tag{23.3.v}$$

We thus solve $20-40x=8$ and $10-20x=6$ to get $x^1=0.3$ and $x^2=0.2$. Plugging these back into the demand curves for the two types, we get $p^1=20-20(0.3)=14$ and $p^2=10-10(0.2)=8$. Consumers of type 1 will therefore pay a total of $P^1=4.20$ to get coverage of 0.3, and consumers of type 2 will pay a total of $P^2=1.6$ to get coverage of 0.2. (This gives us consumers surplus of $CS^1=0.3(20-14)/2=0.9$ and $CS^2=0.2(10-8)/2=0.2$.)

(f) *Let the payment that individual n makes to the monopolist be given by $P^n=F^n+p^n x^n$. Express your answers to (c), (d) and (e) in terms of F^1, F^2, p^1 and p^2.*

Answer: In the case of first degree price discrimination,

$$F^1=3.6,\ F^2=0.8,\ p^1=8 \text{ and } p^2=6. \tag{23.3.vi}$$

For third degree price discrimination, we get

$$F^1=0,\ F^2=0,\ p^1=14 \text{ and } p^2=8. \tag{23.3.vii}$$

(g) *Suppose $\gamma=0.5$ — i.e. half of the population is type 1 and half is type 2. Can you rank the three scenarios in (c), (d) and (e) from most efficient to least efficient?*

Answer: First degree price discrimination is most efficient — because the efficient insurance coverage levels $x_*^1=0.6$ and $x_*^2=0.4$ are sold to both types. So the question is really how to rank third degree price discrimination versus no price discrimination. Under third degree price discrimination, we have concluded that $p^1=14$, $p^2=8$, $x^1=0.3$ and $x^2=0.2$. When $\gamma=0.5$, we have concluded that the non-discriminating monopolist will sell only to type 1 consumers. Thus, she will treat type 1 consumers just like she would under third degree price discrimination — but type 2 consumers would buy 0 instead of 0.2. Thus, third degree price discrimination is more efficient than no price discrimination — type 1's buy the same inefficient amount under both scenarios, but type 2 individuals buy nothing under no discrimination.

(h) *Can you rank them in terms of their impact on consumer welfare for each type? What about in terms of population weighted average consumer welfare?*

Answer: Given what we have just said in the previous part, type 1 consumers are indifferent between no discrimination and third degree price discrimination, but type 2 consumers prefer third degree price discrimination to no price discrimination. On average, consumers therefore prefer third degree price discrimination to no discrimination. Neither type gets any consumer surplus under first degree price discrimination (even though they would both insure efficiently) — so type 1 consumers would prefer both third degree discrimination and no discrimination to first degree discrimination and type 2 consumers would prefer third degree price discrimination to no discrimination — but would be indifferent between no discrimination and first degree price discrimination (even though they would insure efficiently under the latter and not insure at all under the former). On average, we can conclude that third degree price discrimination is preferred to no price discrimination which is preferred to first degree price discrimination on consumer welfare grounds.

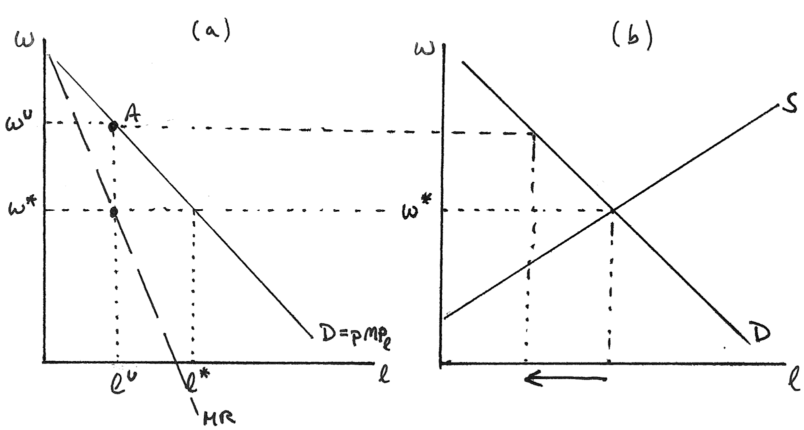
Exercise 23.5

Business and Policy Application: *Labor Unions Exercising Market Power. Federal anti-trust laws prohibit many forms of collusion in price setting between firms. Labor unions, however, are exempt from anti-trust laws and are allowed to use market power to raise wages for their members.*

A: *Consider a competitive industry in which workers have organized into a union that is now renegotiating the wages of its members with all the firms in the industry.*

(a) *To keep the exercise reasonably simple, suppose that each firm produces output by relying solely on labor input. How does each firm's labor demand curve emerge from its desire to maximize profit? Illustrate a single firm's labor demand curve (with the number of workers on the horizontal axis). (Note: Since these are competitive firms, this part has nothing to do with market power.)*

Answer: This is done in panel (a) of Exercise Graph 23.5. The firm's labor demand curve is simply the downward sloping part of its marginal revenue product curve, where $MRP_\ell = pMP_\ell$.



Exercise Graph 23.5 : Labor Unions as Monopolists

(b) *On a graph next to the one you just drew, illustrate the labor demand and supply curves for the industry as a whole prior to unionization.*

Answer: This is done in panel (b) of Exercise Graph 23.5.

(c) *Label the competitive wage w^* and use it to indicate in your first graph how many workers an individual firm hired before unionization.*

Answer: The competitive wage w^* is indicated in panel (b) — and firms will hire until $MRP_\ell = w^*$. Our firm in panel (a) therefore hires ℓ^* workers in panel (a).

(d) *Suppose that the union that is negotiating with the firm in your graph is exercising its market power with an aim toward maximizing the overall gain for its members. Suppose further that the union is sufficiently strong to be able to dictate an outcome. Explain how the union would go about choosing the wage in this firm and the size of its membership that will be employed by this firm. (*Hint*: The union here is assumed to have monopoly power — and the marginal cost of a member is that member's competitive wage w^*.)*

Answer: If the union wants to maximize its gains, it wants to behave like a profit maximizing monopolist. Like all profit maximizing monopolists, it will choose a supply point at a quantity where MR crosses MC. The MR is the same as it always is — starts at the vertical intercept of the demand curve and has twice the slope. (This is because, as the union admits more members, it has to lower the wage for all current members). The marginal cost of a member is the wage that member could make elsewhere. MC therefore intersects MR at ℓ^U, allowing the union to charge a wage of w^U and thus pick the supply point A.

(e) *If all firms in the industry are becoming unionized, what impact will this have on employment in this industry? Illustrate this in your market graph.*

Answer: Employment in the industry will fall — as indicated in panel (b) of the graph.

(f) *Suppose that those workers not chosen to be part of the union migrate to a non-unionized industry. What will be the impact on wages in the non-unionized sector?*

Answer: Assuming the unionized industry is sufficiently large relative to the labor market, the migration of workers into the non-unionized sector will cause an increase in supply — and a decrease in wages in that sector.

B: *Suppose that each firm in the industry has the same technology described by the production function $f(\ell) = A\ell^\alpha$ with $\alpha < 1$, and suppose that there is some fixed cost to operating in this industry.*

(a) *Derive the labor demand curve for each firm.*

Answer: To derive the labor demand curve, we solve the maximization problem (where FC stands for the recurring fixed costs)

$$\max_\ell \; pf(\ell) - w\ell - FC = pA\ell^\alpha - w\ell - FC \tag{23.5.i}$$

to get

$$\ell(w,p) = \left(\frac{\alpha p A}{w}\right)^{1/(1-\alpha)}. \tag{23.5.ii}$$

(b) *Suppose that the competitive wage for workers of the skill level in this industry is w^*. Define the optimization problem that the labor union must solve if it wants to arrive at its optimal membership size and the optimal wage according to the objective defined in A(d). (It may be more straightforward to set this up as a maximization problem with w rather than ℓ as the choice variable.)*

Answer: The union solves the problem

$$\max_w \; w\ell(w,p) - w^*\ell = w\left(\frac{\alpha pA}{w}\right)^{1/(1-\alpha)} - w^*\left(\frac{\alpha pA}{w}\right)^{1/(1-\alpha)}$$
$$= w^{-\alpha}\left(\alpha pA\right)^{1/(1-\alpha)} - w^*\left(\frac{\alpha pA}{w}\right)^{1/(1-\alpha)}. \quad (23.5.iii)$$

(c) *Solve for the union wage w^U that emerges if the union is able to use its market power to dictate the wage. What happens to employment in the firm?*

Answer: Solving the problem in the usual way, we get

$$w^U = \frac{w^*}{\alpha}. \quad (23.5.iv)$$

Substituting this into the labor demand function, we get

$$\ell^U = \left(\frac{\alpha pA}{w^U}\right)^{1/(1-\alpha)} = \left(\frac{\alpha^2 pA}{w^*}\right)^{1/(1-\alpha)}. \quad (23.5.v)$$

In a competitive industry, firms cannot have increasing returns to scale — which implies that $\alpha < 1$. Thus, $w^U > w^*$ — i.e. the union wage is higher than the previous competitive wage. This also implies that $\ell^U < \ell^*$ — i.e. fewer workers are hired by the firm under unionization.

(d) *Can you verify your answer by instead finding MR and MC from the perspective of the union — and then setting these equal to one another?*

Answer: You start by deriving the labor demand *curve* from the labor demand function in equation (23.5.ii) which gives us $w(\ell) = \alpha pA/(\ell^{(1-\alpha)})$. The total revenue in wages for union workers is then

$$TR = \ell w(\ell) = \ell\frac{\alpha pA}{\ell^{(1-\alpha)}} = \alpha pA\ell^{\alpha}. \quad (23.5.vi)$$

The derivative of this with respect to ℓ is marginal revenue; i.e. $MR = \alpha^2 pA\ell^{\alpha-1}$. Setting this equal to $MC = w^*$. we can solve for ℓ to get

$$\ell^U = \left(\frac{\alpha^2 pA}{w^*}\right)^{1/(1-\alpha)} \quad (23.5.vii)$$

and substituting this back into the demand curve, we get

$$w^U = \frac{\alpha p A}{(\ell^U)^{(1-\alpha)}} = \frac{\alpha p A}{\left(\frac{\alpha^2 p A}{w^*}\right)} = \frac{w^*}{\alpha}. \qquad (23.5.viii)$$

(e) *Given the fixed cost to operating in the industry, would you expect the number of firms in the industry to go up or down?*

Answer: When fixed costs stay the same but variable costs (i.e. labor costs) increase, each firm ends up having to produce more in the new equilibrium (where price has adjusted to the new lowest point of the AC curve). But, with price increasing, the quantity demanded falls — thus, the number of firms in the unionized industry will fall.

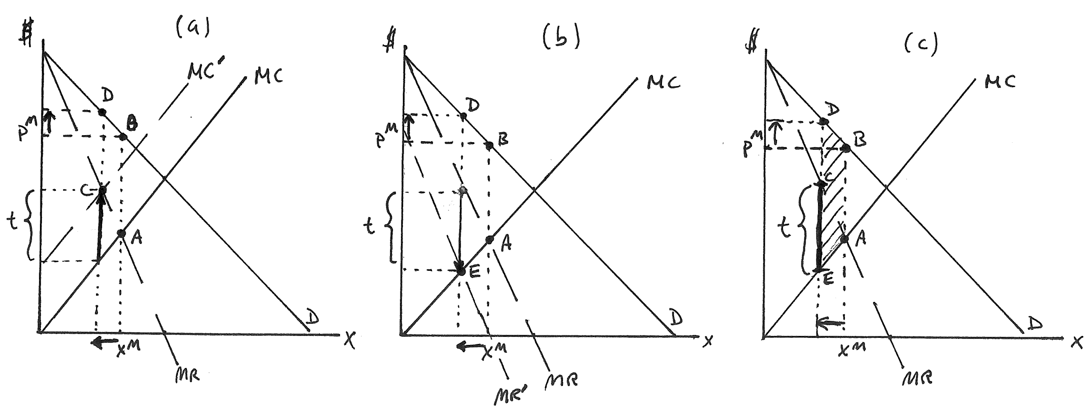
Exercise 23.7

Business and Policy Application: *Taxing Monopoly Output: Under perfect competition, we found that the* economic *incidence of a tax — i.e. who ends up paying a tax — had nothing to do with* statutory *incidence — i.e. who the law said should pay the tax.*

A: *Suppose the government wants to tax the good x which is exclusively produced by a monopoly with upward sloping marginal cost.*

(a) *Begin by drawing the demand, marginal revenue and marginal cost curves. On your graph, indicate the profit maximizing supply point (x^M, p^M) chosen by the monopolist in the absence of any taxes.*

Answer: This is done in panel (a) of Exercise Graph 23.7 where the initial MC crosses MR at point A — leading the firm to choose B on the demand curve, producing x^M and selling it at p^M.



Exercise Graph 23.7 : Taxing Monopoly Output

(b) *Suppose the government imposes a per-unit tax of t on the* production *of x — thus raising the marginal cost by t. Illustrate how this changes the profit maximizing supply point for the monopolist.*

Answer: This is also illustrated in panel (a) of Exercise Graph 23.7 where the new MC' is higher than the original MC by t. The new marginal cost then intersects MR at C — causing the monopolist to choose D as her supply point.

(c) *What happens to the price paid by consumers? What happens to the price that monopolists get to keep (given that they have to pay the tax)?*

Answer: The price paid by consumers goes up as indicated by the arrow in the graph. The price kept by monopolists is t lower than this — and lies somewhere below the original price p^M. Thus, monopolists pass part of the tax onto consumers.

(d) *Draw a new graph as in (a). Now suppose that the government instead imposes a per-unit tax t on* consumption. *Which curves in your graph are affected by this?*

Answer: This is done in panel (b) of Exercise Graph 23.7 where MC again intersects MR at A — leading to the profit maximizing supply point B. When the government taxes consumption of x, the demand curve shifts down by t — and with it the MR curve (that always has the same vertical intercept and twice the slope) also shifts down by t.

(e) *In your graph, illustrate the new marginal revenue curve — and the impact of the consumption tax for the monopolist's profit maximizing output level.*

Answer: This is done in panel (b) of Exercise Graph 23.7 where the new marginal revenue curve MR' lies t dollars below the original MR. This implies that MR' and MC now cross at point E — leading to the same drop in output for the same per unit tax rate. (This is seen more clearly in panel (c) of the graph where we simply include the tax wedge t that appears in both panels (a) and (b) between the original MR and MC curves.) For that quantity, the original demand curve suggests we can price at point D — which is the price that consumers will pay once we include the per unit tax they pay. Of course they pay t less than that to the monopolist — along the new demand curve that lies t below the original demand curve (but is not pictured).

(f) *What happens to the price paid by consumers? What happens to the price that monopolists get to keep (given that they have to pay the tax)?*

Answer: The price paid by consumers — including the per-unit tax they pay — can again be read off point D — i.e. the consumers' price increases. The monopolist's price falls — because it lies t below D along the new demand curve (that is not pictured).

(g) *In terms of who pays the tax, does it matter which way the government imposes the per-unit tax on x?*

Answer: No, it does not matter — the economic incidence of the tax is once again independent of the statutory incidence. This is most easily seen in panel (c) of Exercise Graph 23.7 where, rather than shifting any curves, we simply include the tax wedge t that falls between MC and

MR in both panels (a) and (b). This tax wedge determines by how much output will fall — and that drop in output is independent of whether we shifted MC or MR to get there. Once we know how much output falls, we know how much consumers are willing to pay for that output level at point D. When the tax is imposed on the firm, that is simply what consumers will pay to the firm, but the firm will then have to pay t for every unit of output in tax — thus receiving t per unit of output less than the price paid by consumers. When the tax is imposed on consumers, the same reduction in output still means that consumers will overall pay as suggested by point D on their original demand curve — except that part of the price is not the tax t — leaving monopolists again with t per unit less than what consumers pay.

(h) *By how much does deadweight loss increase as a result of the tax? (Assume that demand is equal to marginal willingness to pay.)*

Answer: Deadweight loss increases by the shaded area in panel (c) of Exercise Graph 23.7 — because the tax causes a drop in the already inefficiently low output level.

(i) *Why can't monopolists just use their market power to pass the entire tax onto the consumers?*

Answer: For the same reason that the monopolist cannot choose a supply point above the demand curve. To say that the monopolist has market power is not to imply that the monopolist can do anything he wants to — he is still constrained by the demand curve. If he were to try to pass on the entire tax, he would have to raise price by t — which would result in an output level (bought by consumers) that lies to the left of the intersection of the new MC and MR. In other words, if the monopolist passed along the entire per-unit tax, he would make less profit than he does by passing along only part of it.

B: *Suppose the monopoly has marginal costs $MC = x$ and faces the demand curve $p = 90 - x$ as in exercise 23.1.*

(a) *If you have not already done so, calculate the profit maximizing supply point (x^M, p^M) in the absence of a tax.*

Answer: Setting MR equal to MC, we get the equation $90 - 2x = x$ which solves to $x^M = 30$. Plugging this into the demand curve, we get $p^M = 90 - x^M = 90 - 30 = 60$.

(b) *Suppose the government introduces the tax described in A(b). What is the new profit maximizing output level? How much will monopolists charge?*

Answer: The new marginal cost function is $MC = x + t$. Setting this equal to MR, we get the equation $x + t = 90 - 2x$. Solving for x, we then get

$$x_t^M = \frac{90 - t}{3}. \tag{23.7.i}$$

Plugging this into the (inverse) demand function, we get

$$p_t^M = \frac{180+t}{3}. \quad (23.7.ii)$$

(c) *Suppose the government instead imposed the tax described in A(d). Set up the monopolist's profit maximization problem and solve it.*

Answer: With the per unit tax t on consumption, the demand curve is now $p = 90 - x - t$ — implying that total revenue is $(90 - x - t)x$. The cost of producing x (in the absence of recurring fixed costs) is $c(x) = x^2/2$ — because the derivative of this cost function gives us $MC = x$. Thus, the profit maximization problem is

$$\max_x \,(90 - x - t)x - \frac{x^2}{2}. \quad (23.7.iii)$$

Taking the derivative with respect to x and setting it to zero, we get $90 - 2x - t - x = 0$. Solving for x, this gives us

$$x_t^M = \frac{90-t}{3}. \quad (23.7.iv)$$

Plugging this into the new demand curve $p = 90 - x - t$, we get the price $p = (180 - 2t)/3$. This is the price charged by monopolists — consumers have to pay an additional amount t to the government — making their price equal to $(180 + t)/3$.

(d) *Compare your answers to (b) and (c). Is the economic incidence of the tax affected by the statutory incidence?*

Answer: Our answer for the monopolist output level is the same in (b) as in (c) — as is the price paid by consumers and the price kept by the firm. Thus, the economic incidence of the tax is independent of the statutory incidence.

(e) *What fraction of the tax do monopolists pass onto consumers when monopolists are statutorily taxed? What fraction of the tax do consumers pass onto monopolists when consumers are statutorily taxed?*

Answer: In the absence of the tax, the price is $p^M = 60$. With a tax t — regardless of how it is statutorily designed — the monopolist's price p_t^M and the consumers' price p_t^C are

$$p_t^M = \frac{180-2t}{3} = 60 - \frac{2}{3}t \text{ and } p_t^C = \frac{180+t}{3} = 60 + \frac{1}{3}t. \quad (23.7.v)$$

Thus, monopolists pass one third of the tax onto consumers when the government levies the tax on production, and consumers pass on two thirds of the tax to monopolists when the government levies the tax on consumption.

Exercise 23.9

Policy Application: *Some Possible "Remedies" to the Monopoly Problem: At least when our focus is on efficiency, the core problem with monopolies emanates from the monopolist's strategic under-production of output — not from the fact that monopolists make profits. But policy prescriptions to deal with monopolies are often based on the presumption that the problem is that monopolies make excessive profits.*

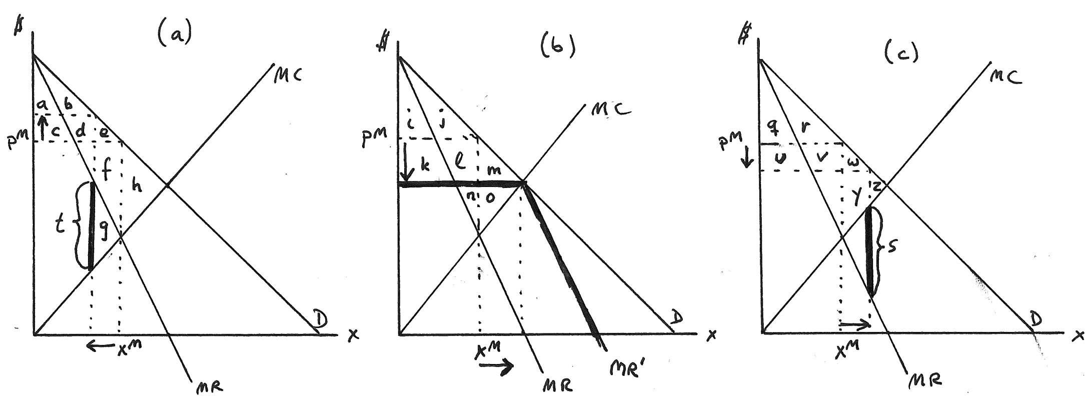
A: *Suppose the monopoly has marginal costs $MC = x$ and faces the demand curve $p = 90 - x$ as in exercise 23.1. Unless otherwise stated, assume there are no recurring fixed costs. In each of the policy proposals below, indicate the impact the policy would have on consumer welfare and deadweight loss.*

(a) *The government imposes a 50% tax on all economic profits.*

Answer: If the government taxes pure economic profits, it has no impact on the monopolist's behavior. If (x^M, p^M) was the profit maximizing supply point before the profits tax, then we know that economic profit is as large as it can be at (x^M, p^M). When the government takes half of it, only half is left over — but half of the largest possible profit is still greater than half of any alternative profit. Since the monopolist does not change behavior under a pure profits tax, the tax has no impact on consumer welfare or deadweight loss.

(b) *The government imposes a per-unit tax t on x. (In problem 23.7, you should have concluded that it does not matter whether the tax is levied on production or consumption.)*

Answer: This is pictured in panel (a) of Exercise Graph 23.9 where the original (x^M, p^M) is indicated.



Exercise Graph 23.9 : Different Policy "Remedies"

The vertical darkened line is the tax rate t — which we showed in exercise 23.7 creates a wedge between MC and MR, leading to a decrease in output and an increase in consumer price as indicated by the arrows on the two axes. Consumer surplus therefore shrinks from $(a+b+c+d+e)$ to $(a+b)$, and deadweight loss increases from (h) to $(e+f+g+h)$.

(c) *The government sets a price ceiling equal to the intersection of MC and demand. (*Hint*: How does this change the marginal revenue curve?)*

Answer: The new marginal revenue curve MR' is graphed in panel (b) of Exercise Graph 23.9 as the darkened line consisting of an initial flat portion and then a portion that has twice the slope of D. With the price ceiling set at the level where MC intersects demand, the additional revenue the monopolist gets for each output unit up to the efficient level is exactly equal to the price ceiling — because he now does not lower the price on all previous units when he produces one more unit. The new MR' line therefore intersects MC at the same point where MC intersects D — leading to the efficient output level being produced and sold at the price ceiling. Thus, consumer surplus increases from $(i+j)$ to $(i+j+k+l+m)$ and deadweight loss falls from $(m+o)$ to zero.

(d) *The government subsidizes production of the monopoly good by s per unit.*

Answer: This is pictured in panel (c) of Exercise Graph 23.9. Just as a tax creates a wedge to the left of the intersection of MR and MC, a subsidy creates an analogous wedge to the right of MR and MC — thus causing an increase in output and drop in consumer price as indicated on the axes. Consumer surplus therefore increases from $(q+r)$ to $(q+r+u+v+w)$, while deadweight loss falls from $(w+y+z)$ to (z). If the subsidy level s were set just right, the efficient output level might result — causing the deadweight loss to disappear entirely. If, however, s were set higher than that, deadweight loss would re-appear to the other side of the intersection of MC and D — and for very high s, this could get larger than the initial deadweight loss without a subsidy.

(e) *The government allows firms to engage in first-degree price discrimination.*

Answer: Under first degree price discrimination, the monopolist produces the efficient level of output but usurps all consumer surplus. Thus, consumer surplus shrinks to zero, as does deadweight loss.

(f) *Which of the analyses above might change if the firm also has recurring fixed costs.*

Answer: Recurring fixed costs have no influence other than to determine whether the monopolist produces at all. If the monopolist produces prior to the policy intervention, then the only way our answers might change is if the policy intervention reduces profit. This is the case for the 50% profits tax, the per-unit tax t and the price-ceiling. In each of these cases, profit declines as a result of the policy — and, in the presence of recurring fixed costs, it could decline sufficiently to become negative. This would mean that the monopolist ceases production — causing the disappearance of consumer surplus and the emergence of a very large deadweight loss. This is not a worry for the per-unit subsidy s or the legalization of first degree price discrimination — both of which increase rather than decrease profit for the monopolist.

(g) True or False*: In the presence of distortions from market power, price distorting policies* can *be efficient.*

Answer: This is true — as it has been for some of the other distortions we have discussed. (For instance, we saw that price-distorting taxes and subsidies can be efficiency enhancing in the presence of externalities.) It is generally the case that policies which create deadweight losses in a world where there are no deadweight losses to begin with *might* reduce deadweight losses in a world where we already have distortions that lead to deadweight losses. Economists refer to a world without distortions as "first-best" — and we often say that policies which are inefficient in a "first-best" world might be efficient in a "second-best" world.

B: *Suppose demand and marginal costs are as specified in part A. Unless otherwise stated, assume no recurring fixed costs.*

(a) *Determine the monopolist's optimal supply point (assuming no price discrimination). Does it change when the government imposes a 50% tax on economic profits?*

Answer: Setting $MR = 90 - 2x$ equal to $MC = x$ and solving for x, we get $x^M = 30$, and substituting this back into the demand curve, we get $p^M = 60$. This results in profit of \$1,350. If the government taxes profits at 50%, the behavior of the monopolist is unchanged — but her profit falls to \$675. You can see that her profit is unchanged by simply recognizing that the profit maximization problem in the presence of a profit tax of 50% is

$$\max_{p,x} 0.5(px - c(x)) \text{ rather than } \max_{p,x} (px - c(x)) \tag{23.9.i}$$

where px is revenue and $c(x)$ is cost. Note that, when the derivative of the objective functions in these problems is taken and set to zero, we can simply multiply both sides of the resulting equation in the first case by 2 to get the equation from the latter problem. Thus, multiplying profit by any number does not change the optimal choice for the monopolist.

(b) *Suppose the government imposes a \$6 per unit tax on x. Solve for the new profit maximizing supply point.*

Answer: We can model this simply as a \$6 increase in MC to $MC' = x + 6$. Setting this equal to MR, we get the equation $x + 6 = 90 - 2x$ — which solves to $x = 28$. Consumer price increases to $p = 90 - 28 = 62$ — with \$2 of the \$6 tax therefore passed onto consumers.

(c) *Is there a price ceiling at which the monopolist will produce the efficient output level?*

Answer: Yes — a price ceiling equal to the vertical intersection of MC and D. This intersection would give us a price ceiling of $\overline{p} = 45$.

(d) *For what range of recurring fixed costs would the monopolist produce prior to the introduction of the policies in (a), (b) and (c) but not after their introduction?*

Answer: In the absence of any of these policies and in the absence of recurring fixed costs, the monopolist makes profit of \$1,350. Thus, so long as recurring fixed costs are less than \$1,350, the monopolist will produce prior to any government intervention. This profit falls to \$675 under the 50% profits tax. Thus, if the recurring fixed costs lie in the range of \$675 to \$1,350, the monopolist will produce prior to the introduction of the profits tax but not afterwards. Under the per-unit tax $t = 6$, the monopolist raises price to \$62 but only gets to keep \$56 per unit of output. Thus, revenue falls to $56(28) = 1,568$. Variable production costs are $28(28/2) = 392$. This implies that profit under the tax is 1,568-392=\$1,176 prior to figuring in any recurring fixed costs. Thus, if recurring fixed costs are in the range of \$1,176 to \$1,350, the monopolist will produce before the tax but not after. Finally, under the price ceiling, revenues are 45(45)=\$2,025 while variable costs are 45(45)/2=\$1,012.50 — giving us profit (before recurring fixed costs) of \$1,012.50. Thus, if recurring fixed costs are in the range from \$1,012.50 to \$1,350, the monopolist produces before the price ceiling but not afterwards.

(e) *What is the profit maximizing output level if the monopolist can perfectly price discriminate?*

Answer: It is the efficient output level of $x^* = 45$.

(f) *How high a per-unit subsidy would the government have to introduce in order for the monopolist to produce the efficient output level?*

Answer: We can model a per-unit subside s as decreasing MC to $MC' = x - s$. Setting this equal to $MR = 90 - 2x$, we get the equation $x - s = 90 - 2x$. Solving for x, we get $x = (90 + s)/3$. The efficient output level is 45. Thus, setting $(90 + s)/3$ equal to 45 and solving for s, we get the efficient per unit subsidy $s^* = 45$.

(g) *For what range of recurring fixed costs does the monopolist not produce in the absence of a subsidy from part (f) but produces in the presence of the subsidy? If recurring fixed costs are in this range, will the monopolist produce the efficient quantity under the subsidy?*

Answer: We know that the monopolist will not produce in the absence of the subsidy if recurring fixed costs exceed \$1,350. Under $s = 45$, consumer price is \$45 — but the firm gets another \$45 in a subsidy for a total of \$90 per unit sold. When she sells 45 units, this implies that total revenues under the subsidy are 90(45)=\$4,050. The variable cost of producing 45 units of x is 45*45/2=\$1,012.50. Thus, profit before figuring in any recurring fixed costs is 4,050-1,012.50=\$3,037.50. Thus, if recurring fixed costs are in the range of \$1,350 to \$3,037.50, the monopolist would not produce in the absence of a subsidy but would produce if the efficient subsidy $s^* = 45$ were introduced. And if she does produce, she will produce the efficient output level $x^* = 45$.

Exercise 23.11

Policy Application: *Regulating Market Power in the Commons: In exercises 21.9 and 21.10, we investigated the case of many firms emitting pollution into a lake. We assumed the only impact of this pollution was to raise the marginal costs for all firms that produce on the lake.*

A: *Revisit part A(g) of exercise 21.10.*

(a) *How does a merging of all firms around the lake (into one single firm) solve the externality problem* regardless *of how large the pollution externality is?*

Answer: In this case, the externality problem is the following: Each firm around the lake considers only its own marginal costs and not the marginal costs that its own production imposes on all the other firms around the lake. If all these firms merge into a single firm operated by a single company, then that one company is affected by all the costs that were previously externalities; i.e. if output from one facility imposes costs on other facilities, those costs are now fully borne by the same company. Thus, the company will "internalize" all these costs, and there is no more externality.

(b) *Suppose you are an anti-trust regulator who cares about efficiency. You are asked to review the proposal that all the firms around this lake merge into a single firm. What would you decide if you found that, despite being the only firm that produces output x* on this lake*, there are still plenty of other producers of x such that the output market remains competitive.*

Answer: In this case, the merging of all the firms has no anti-comptetitive effects but it does have the effect of eliminating an externality problem. It is therefore efficiency enhancing to allow all the firms to merge into one.

(c) *Suppose instead that, by merging all the firms on the lake, the newly emerged firm will have obtained a monopoly in the output market for x. How would you now think about whether this merger is a good idea?*

Answer: In this case, we are trading one problem for another when we allow the firms to merge; on the one hand, the externality problem disappears, but on the other hand, we are creating a monopoly that will use its market power to restrict output and raise price. While before the market was producing too much because it did not take into account the externality, the monopolist now will produce too little — he will fully take into account what was previously an externality but will now exert market power in pricing. Whether an efficiency-motive would lead a regulator to permit these firms to merge would then depend on the size of the externality in the absence of the merger — and the loss of efficiency from monopoly power if the merger is put in place.

(d) *How would your answers to (b) and (c) change if the externality emitted by firms on the lake* lowered *rather than raised everyone's marginal costs?*

Answer: The answer to (c) would remain the same: By merging the firms, the new company that owns all the facilities will take into account all the benefits it imposes on other facilities as production is increased — and we will therefore have eliminated the externality. The answer to (d) would similarly remain the same — with the externality problem solved, the introduction of market power will lead to underproduction of the good x (relative to the efficient amount). Regulators therefore have to weight the gains from eliminating the externalities through the merger against the costs of the market power we are creating in the process.

B: *Suppose, as in exercise 21.9 and 21.10, that each of the many firms around the lake has a cost function $c(x) = \beta x^2 + \delta X$ where x is the firm's output level and X is the total output by all firms around the lake.*

(a) *In exercise 21.10B(a), we discussed how a social planner's cost function for each firm would differ from that of each individual firm. Review this logic. How does this apply when all the firms merge into a single company that owns all the production facilities around the lake?*

Answer: The same logic applies here: The social planner needs to take into account all costs from each firm's actions because they are all real costs even if they affect firms other than the one that is producing. When firms around the lake merge to create a single company with many production facilities, the single company has the same incentives as the social planner in terms of how it views costs. Thus, it will use the same cost function for each of the facilities as the social planner would.

(b) *Will the single company make decisions different from that of the social planner in exercise 21.10? What does your answer depend on?*

Answer: If the single company remains a price taker in the market for x, then it will make the same decisions as the social planner would — and all the answers to the problem in Chapter 21 remain the same here. If, however, the company acquires market power as it merges all the previously competitive firms, it's marginal revenue curve will differ from the one that the social planner would use — because the social planner considers the marginal social benefit (which, in the absence of other externalities, is given by the marginal willingness to pay of consumers) while the monopolist considers the steeper marginal revenue curve that only considers marginal benefits to the firm. Whether or not the new company will behave like the social planner therefore depends on whether the merger of firms around the lake results in an acquisition of market power.

Conclusion: Potentially Helpful Reminders

1. The rule that monopolists (in the absence of market segmentation) price where $MR = MC$ is exactly the rule that perfect competitors use to determine how much to produce. The difference is that, for perfect competitors, the demand curve for each firm (as opposed to the market demand curve) is

perfectly elastic — which implies that the MR curve for competitors is equal to the demand curve they face.

2. Make sure that you understand the intuition for why monopolists will *never* produce on the inelastic part of demand — and why the demand curve they face is different from the demand curve that competitors face (even if the market demand curve is the same for both).

3. When we first introduced the theory of the firm in Chapters 11 and 12, we argued that there are two ways to think of profit maximization: the "one-step" method where we think of the competitive firm choosing the profit maximizing production plan that lies at the tangency of isoprofits and production frontiers; and the "two-step method" where the firm first derives its cost curves and then sets price equal to marginal cost. In the graphs for monopolists, we are essentially using the "two-step method". The monopolist's marginal cost curve arises in exactly the same way as it does for competitive firms — and by drawing that curve, we are implicitly doing step 1. We then do step 2 by finding where MR intersects MC — which is, as we said above, exactly the same thing we do for competitive firms where the only difference is that, for competitive firms, $p = MR$.

4. In terms of the underlying math, we can again refer back to the initial development of the theory of the firm in Chapters 11 and 12. The *cost minimization* problem is exactly the same for monopolists because that problem only asks what the least cost way of producing different levels of output is — which in turn has nothing to do with the firm's market power in the output market. We can therefore use cost functions in exactly the same way as we have done throughout. Even the *profit maximization* problem looks very similar to what we have done before — with the only difference being that p (which was taken as given by the competitive firm) is now replaced by the demand function $p(x)$.

5. The crucial difference between first and third degree price discrimination on the one hand and second degree price discrimination on the other is the information that the monopolist has. In the case of first and third degree price discrimination, the monopolist can identify what consumer type each customer is; in the case of second degree price discrimination, all the consumer types look the same to the monopolist. Thus, the monopolist has to be a lot more "strategic" in second degree price discrimination — essentially designing output/price combinations that force consumers to reveal who they are.

CHAPTER

24

Strategic Thinking and Game Theory

This chapter is a fresh start, so to speak. It does not directly build on any of the previous material in the text but instead constructs a new set of tools that allow us to think more clearly about the idea of strategic thinking when it matters. We have had little need for these tools up until now — because so far, almost everything we have done relied on the assumption that individuals are "small" relative to their economic environment and thus in no need of strategic thinking. (There have been some exceptions to this — in our chapter on adverse selection as well as in our chapter on monopoly where we have muddled through without the formal tools we are now introducing.) The new tool kit is known as *game theory*. Simply put, models will now take the form of "games" that incorporate real world incentives and thus allow us to focus on strategy. In part A of the chapter, we introduce complete information games, whereas in part B we focus on incomplete information games.

Chapter Highlights

The main points of the chapter are:

1. A **game** is defined by the set of players, actions (including the sequence in which actions are taken) and payoffs. When all actions are taken at the same time, the game is called a **simultaneous move game**, and when some players move ahead of other players, the game is called a **sequential move game**.

2. It is common for us to use a **payoff matrix** to represent a simultaneous move game and a **game tree** to represent a sequential move game (but it is possible to represent either type of game in either form.)

3. A **strategy** is a complete plan of action for the game — not just the set of actions that is actually taken. In sequential move games, this implies that at least some players will have strategies that consist of actions which are not taken but which would be taken if the game evolved in a way that is different than the actual evolution of the game. Such plans for actions can play an

important role in determining how players interact in practice — because knowing what other players would do if I changed by actions is important for me to think strategically about what I should do.

4. A **Nash equilibrium** is a strategy for each player such that every player's strategy is a best response to the other players' strategies. A **subgame-perfect** Nash equilibrium is a Nash equilibrium (in a sequential move game) that involves no non-credible threats. In sequential move games, every subgame-perfect Nash equilibrium is a Nash equilibrium but not every Nash equilibrium is subgame perfect.

5. A **dominant strategy** is a strategy that is a best response to all strategies that other players might play. Most games do not have dominant strategies, but the **Prisoner's Dilemma** is an important game that does. It is important in part because it is the starkest example of a game in which strategic thinking leads to sub-optimal outcomes.

6. **Repeated games** are sequentially played simultaneous move games in which players meet repeatedly. Subgame perfect equilibria in such games might result in actions other than those we observe in single-shot simultaneous games — but not always. Finitely repeated Prisoner's Dilemma games, for instance, result in the same (subgame-perfect) equilibrium actions as those we predict in the single-shot Prisoner's Dilemma. The same is not true for **infinitely repeated** Prisoners' Dilemma games (which can also be thought of as repeated Prisoner's Dilemma games in which there is always a certain probability that players meet again.)

7. One of the important questions we will face repeatedly is how **cooperation** can be sustained in repeated games that have Prisoner's Dilemma incentives.

8. **Mixed strategies** are strategies that involve playing different actions with probabilities between 0 and 1 (while **pure strategies** are strategies that involve playing each action with either probability 0 or 1.) Some games only have mixed strategy equilibria — and such equilibria can be interpreted as pure strategy equilibria in related games that involve some uncertainty.

9. The concepts developed in part A of the chapter for games of complete information have direct analogs in the concepts (developed in part B) for games of **incomplete information** where at least some players are uncertain about they payoffs that other players face. Such games require the introduction of **beliefs** on the part of players — beliefs about what types of players are playing in the game. A common technique for introducing uncertainty into games is to model an initial stage in which the fictional player "Nature" assigns types to players, with all players knowing the probability with which different types are assigned but then only finding out their own type before the actual game begins.

10. A **Bayesian Nash equilibrium** is the incomplete information game analog to a Nash equilibrium for games of complete information. It consists not only

of strategies for all players — but also beliefs that are consistent with how the game is played.

11. A **perfect (Bayesian) Nash equilibrium** is the incomplete information game analog to a subgame-perfect Nash equilibrium in complete information games. Just as off-the-equilibrium-path plans play important roles in subgame-perfect equilibria, off-the-equilibrium-path beliefs can play important roles in perfect (Bayesian) Nash equilibria.

24A Solutions to Within-Chapter-Exercises for Part A

Exercise 24A.1

In the last example (of employer and worker), which set of actions might be more appropriately modeled as continuous?

Answer: The employer's set of possible wage offers is more appropriately modeled as continuous, while the worker's decision is either to accept or to reject the offer.

Exercise 24A.2

Suppose that for every player n in a game, the payoffs for player n depend on player n's action as well as the sum of all the other players' actions, but no single other player has, alone, a perceptible influence on player n's payoff. Would such a game characterize a setting in which strategic thinking was important?

Answer: This would essentially be a competitive setting where each player takes her economic circumstances (determined by everyone else) as given. Thus, strategic thinking is not important in this setting.

Exercise 24A.3

True or False: In simultaneous move games, the number of pure strategies available to a player is necessarily equal to the number of actions a player has available.

Answer: This is true. Since the players move simultaneously, there are not multiple "nodes" a player might need a plan for — and thus the player just needs to settle on the action that she wishes to play.

Exercise 24A.4

Verify that the payoffs listed in Table 24.3 are consistent with those given in the game tree of Graph 24.2.

Answer: The top cell (10,10) on the left of the table represents the payoff to player 1 going “Left” and player 2 always going “Left” — which puts us on the end node farthest to the left in the graph, with payoffs of 10 for each player. The other cells in the table similarly correspond to the payoffs at the end of the game tree given the actions taken in accordance with the strategies listed.

Exercise 24A.5

Are the two pure strategy Nash equilibria we have identified efficient?

Answer: Yes, both sides achieve the maximum level of payoff, with all other outcomes making at least one worse off without making anyone else better off.

Exercise 24A.6

True or False: If a simultaneous move game gives rise to a dominant strategy for a player, then that strategy is a best response for any strategy played by the other players.

Answer: This is true. We defined a dominant strategy as a strategy where a player has the incentive to play a single action regardless of what the opponent does — which means that this action is in fact the best response to anything the opponent might do.

Exercise 24A.7

Suppose that player 2 has payoffs as in Table 24.4 while player 1 has payoffs as in Table 24.5. Write out this payoff matrix. Is there a dominant strategy equilibrium? Is there a unique Nash equilibrium? If so, is it efficient?

Answer: The payoff matrix for this new game is given in the table here. In this game, “Down” is a dominant strategy for player 1 and “Up” is a dominant strategy for player 2. Thus, there is a unique Nash equilibrium — with player 1 playing “Down” and player 2 playing “Up” — resulting in payoffs of 15 for player 1 and 7 for player 2. The equilibrium is efficient — because there is no other outcome in which both parties are at least as well off. (There is also no other outcome for which the sum of the payoffs is greater — implying that, even if one player could compensate the other, there would be no other outcome that could be preferred by both players.)

		Player 2	
		Up	Down
Player 1	Up	10,10	0,7
	Down	15,7	5,5

Exercise 24A.8

Suppose both player's payoffs are as in Table 24.5 except that player 1's payoff when both players play "Up" is 20. Is there a dominant strategy equilibrium? Is there a unique Nash Equilibrium? If so, is it efficient?

Answer: The payoff matrix is given in Table here. Player 1 now no longer has a dominant strategy — if player 2 goes "Up", player 1's best response is to also go "Up" (and get 20 rather than 15), but if player 2 goes "Down", player 1's best response is to also go "Down" (and get 5 instead of 0). For player 2, on the other hand, playing "Down" is a dominant strategy — regardless what player 1 does, player 2 does better playing "Down" rather than "Up". So we know player 2 will play "Down", with player 1 best responding by also playing "Down". Both players playing "Down" is therefore the Nash equilibrium — which is unique though not a dominant strategy equilibrium (since one of the players does not have a dominant strategy). It is not efficient since both players would prefer to get the outcome (20,10) that they could get if they both played "Up" instead.

		Player 2	
		Up	Down
Player 1	Up	20,10	0,15
	Down	15,0	5,5

Exercise 24A.9

Suppose payoffs are as in exercise 24A.8 except that player 2's payoff from playing "Down" is 10 less than before (regardless of what player 1 does). Is there a dominant strategy equilibrium? Is there a unique Nash Equilibrium? If so, is it efficient?

Answer: The payoff matrix is depicted here. As in exercise 24A.8, player 1 does not have a dominant strategy — if player 2 goes "Up", player 1's best response is to also go "Up" (and get 20 rather than 15), but if player 2 goes "Down", player 1's best response is to also go "Down" (and get 5 instead of 0). Player 2 does have a dominant strategy — regardless of what player 1 does, it is best for player 2 to play "Up". Thus, there is a unique Nash equilibrium (that does not involve dominant strategies by both players) in which both players play "Up" — with player 1 getting payoff of 20 and player 2 getting payoff of 10. This equilibrium is efficient — both players prefer this outcome over any other outcome in the payoff matrix.

		Player 2	
		Up	Down
Player 1	Up	20,10	0,5
	Down	15,0	5,-5

Exercise 24A.10

Can you find which strategies in the game depicted in Table 24.3 constitute a Nash equilibrium? (*Hint*: You should be able to find four combinations of strategies that constitute Nash equilibria.)

Answer: There are four Nash equilibria: (1) Player 1 plays "Left" and Player 2 plays ("Left","Left"); (2) Player 1 plays "Left" and Player 2 plays ("Left","Right"); (3) Player 1 plays "Right" and Player 2 plays ("Right","Right"); and (4) Player 1 plays "Right" and Player 2 plays ("Left","Right").

Exercise 24A.11

Is it also a Nash equilibrium for player 1 to play $Right$ and player 2 to play $(Left,Right)$? If not, why was it a Nash equilibrium before when players were indifferent between coordinating on the left or the right side of the road?

Answer: No, that is not a Nash equilibrium because, if player 2 plays $(Left,Right)$, it is a best response for player 1 to play $Left$. When players were indifferent between which side of the road to choose, the two players were best responding to each other when player 1 to played $Right$ and player 2 to played $(Left,Right)$.

Exercise 24A.12

True or False: In sequential move games, all pure strategy subgame perfect equilibria are pure strategy Nash equilibria but not all pure strategy Nash equilibria are subgame perfect.

Answer: This is true. A subgame perfect equilibrium involves strategies that are best responses to one another (and thus constitute Nash equilibrium strategies) *and* involve no non-credible threats. Nash equilibria, on the other hand, may involve non-credible threats — which implies not all Nash equilibria are subgame perfect.

Exercise 24A.13

What are the Nash equilibria and the subgame perfect equilibria if player 2 rather than player 1 gets to move first in this version of the game?

Answer: First, consider subgame perfect equilibria with player 1 predicting what player 2 will do at each node: In particular, Player 2 knows that Player 1 will go "Left" at the left node and "Right" at the right node — which means Player 2 knows she can get her preferred outcome by going "Right" and then having Player 1 follow suit. The subgame perfect equilibrium is then Player 2 playing "Right" and Player 1 playing ("Left","Right"). This is the only subgame perfect equilibrium. But it would also be a Nash equilibrium for Player 2 to play "Right" and Player 1 to play ("Right","Right") — this would lead to the same outcome as the subgame perfect equilibrium. And it would be a Nash equilibrium for Player 2 to play "Left" and Player 1 to play ("Left","Left") — which would lead to the opposite outcome of both players driving on the left — but it would also involve the non-credible threat by Player 1 to play "Left" if she gets to the "Right' node.

Exercise 24A.14

Suppose the game had a third stage in which the existing firm gets a chance to re-evaluate its price in the event that a new firm has entered the market. This would imply that the game tree in Graph 24.4 continues as depicted in Graph 24.5. What is the subgame perfect equilibrium in this case?

Answer: We can solve the game backward by first looking at what the existing firm will do in the final stage. Regardless of which node the existing firm finds itself in the final stage, it is best to pick the "Low" price. This then implies that the potential firm will get a payoff of -10 if it enters — regardless of which node it finds itself on. Thus, the potential firm does not enter, and the existing firm chooses the "High" price in the first stage. The subgame perfect equilibrium is then given by the existing firm playing the strategy ("High", ("Low","Low")) and the potential firm playing the strategy ("Don't Enter","Don't Enter").

Exercise 24A.15

In our example in Graph 24.4, we say that the subgame perfect equilibrium is not efficient from the perspective of the two players. Could it be efficient from the perspective of "society"?

Answer: Yes, from the perspective of society, the threat of entry from the potential firm implies that the existing firm cannot use its market power — and thus behaves more competitively.

Exercise 24A.16

Why is this outcome inefficient from the perspective of the two players? Could it be efficient from the perspective of "society"?

Answer: It is inefficient from the perspective of the players because both players would prefer the outcome where each goes to prison for only 1 year (when both play "Deny"). But it may be efficient from the perspective of "society" in that society's larger aim is to deter crime — and to make potential criminals face a relatively high cost to committing crimes.

Exercise 24A.17

Why is this a Prisoners' Dilemma Game?

Answer: This game has the same incentives as the classic prisoners' dilemma game, with both players having a dominant strategy of "Don't Cooperate" which leads to the inefficient outcome of each getting payoffs of 10 when both could be getting payoffs of 100 by playing "Cooperate".

Exercise 24A.18

Does the same logic hold for any repeated simultaneous game in which the simultaneous game has a single pure strategy Nash equilibrium? Put differently, does subgame perfection require that players in such games always simply repeat the simultaneous game Nash equilibrium?

Answer: Yes, it does. If there is a single pure strategy Nash equilibrium to the game, then both players know what will be played in the last stage. This makes the last stage irrelevant to the second to last stage — and thus the second to last stage is played as if it were a single-shot game, with the pure strategy Nash equilibrium once again being played. But this makes the second to last stage irrelevant to what is played in the third to last stage, etc.

Exercise 24A.19

True or False: In an infinitely repeated Prisoners' Dilemma game, every subgame of the sequential game is identical to the original game.

Answer: This is true. If the game never stops, then it is an infinitely repeated Prisoners' Dilemma game even if we have already played once, or twice, or three times, etc.

Exercise 24A.20

True or False: If two players play "nice" strategies in the repeated Prisoner's Dilemma, they will always cooperate with one another every time they meed.

Answer: This is true. A "nice" strategy begins with cooperation and does not stop cooperating unless the other party did not cooperate at some point. If both players play "nice", they will cooperate the first time and each time thereafter.

Exercise 24A.21

Explain why this type of trigger strategy is "nice".

Answer: It is "nice" because non-cooperation is never initiated by the person who plays this strategy.

Exercise 24A.22

Would you playing "Cooperate Always" also potentially be a best response for you to my trigger strategy? Would my trigger strategy then be a best response to your "Cooperate Always" strategy?

Answer: Yes, you playing "Cooperate Always" is also a best response to my trigger strategy — resulting in the same outcome. But my trigger strategy would not be a best response to your "Cooperate Always" strategy — because if you play that strategy, it is a best response for me to play "Never Cooperate".

Exercise 24A.23

Why can't the same type of "trigger strategy" sustain cooperation in a repeated Prisoner's Dilemma that has a definitive end?

Answer: It is because of the fact that the finitely repeated game unravels from the bottom up. Both players know that neither will cooperate in the last stage, which implies neither will cooperate in the second to last stage, which implies neither will cooperate in the third to last stage, etc. In light of this, both players know in the first stage that there will be no cooperation after the first stage — which implies that not cooperating in the first stage becomes a dominant strategy the first time we meet. Put differently, there is no reason to be "nice" the first time we meet — because I already know we'll never cooperate. The whole point to being "nice" initially in the infinitely repeated game is that it, combined with the threat of not cooperating if cooperation is not reciprocated, can sustain cooperation down the tree — but in the finitely repeated game we know that this cooperation unravels from the bottom up.

Exercise 24A.24

If you model the decision about whether to be friendly to someone you run into as part of a Prisoners' Dilemma, why might you expect people in small towns to be friendlier than people in big cities?

Answer: People in small towns face a high probability of running into each other again and again (with each encounter involving the same relatively high probability that you'll run into each other again in the future). People in large towns face a low probability of running into each other again and again. Part of what is necessary for cooperation to be sustained in such repeated encounters (with no clear end) is that we know we are likely to meet again.

Exercise 24A.25

True or False: Whenever individuals find themselves in a Prisoner's Dilemma game, there is profit to be made if someone can determine a way to commit players to change behavior.

Answer: This is true. We know the non-cooperative Nash equilibrium outcome in the Prisoner's Dilemma is inefficient — which implies that there is a way to make everyone better off. If someone can figure out a way to commit the players to change behavior and reach the efficient outcome, there is enough to go around to get that person paid for his effort.

Exercise 24A.26

How might your answer to the previous exercise help explain why we see more cooperation in real-world Prisoner's Dilemma games than we expect from the incentives contained in the game?

Answer: There is an incentive for people — whether the ones that play the game or outsiders that observe the game — to find a way to get people to cooperate. In essence, there is money left on the table (so to speak) when people play the Nash equilibrium in the Prisoner's Dilemma, and when there is money on the table, there is an incentive for someone to find a way to get to it. Put differently, in the real world there are incentives for people to solve these incentive problems, and sometimes they do.

Exercise 24A.27

Note that it is always possible to write a pure strategy in the form of a mixed strategy with one probability set to 1 and the others set to zero. How would you write my pure strategy of "Heads" in the form of a mixed strategy?

Answer: This would be written as (1,0), with 1 indicating a probability of 1 that I play "Heads" and 0 indicating a probability of 0 that I will play "Tails".

Exercise 24A.28

What would be my expected payoff if I play "Heads" all the time when you play the mixed strategy that places probability 0.5 of "Heads"?

Answer: If I play "Heads" all the time and you play "Heads" half the time, then half the time I will get a payoff of 1 and half the time I will get a payoff of -1. Thus, my expected payoff is $0.5(1)+0.5(-1)=0$.

Exercise 24A.29

Is the mixed strategy equilibrium more or less efficient than the pure strategy equilibria in the "Left/Right" game?

Answer: Under each of the pure strategy equilibria, we get a payoff of 10 each. In the mixed strategy strategy equilibrium, on the other hand, we end up in each of the outcomes of the payoff matrix with equal probability — thus making payoff of 10 half the time and payoff of 0 half the time, for an expected payoff of 5 each. Thus, the mixed strategy equilibrium is less efficient than both the pure strategy equilibria.

Exercise 24A.30

Determine the mixed strategy Nash equilbrium for the game described in the previous sentence.

Answer: The payoff matrix for this game is illustrated in table here.

Consider player 1's best response to player 2's possible mixed strategies first. We know that if player 2 plays (1,0) — the pure strategy "Left" — it is a best response for player 1 to also play (1,0), i.e. the same pure strategy. Now consider player 2 playing the strategy $(\lambda,1\text{-}\lambda)$ (with $0 \leq \lambda \leq 0$). The expected payoff to player 1 from playing

		Player 2	
		Left	Right
Player 1	Left	10,5	0,0
	Right	0,0	5,10

(1,0) (i.e. "Left" always) is then $10\lambda+0(1-\lambda) = 10\lambda$. The expected payoff to player 1 of playing (0,1) (i.e. "Right" always), on the other hand, is $0(\lambda)+5(1-\lambda) = 5-5\lambda$. To find the λ for which player 1 is indifferent between playing (1,0) and playing (0,1), we set the two payoffs equal to each other; i.e.

$$10\lambda = 5-5\lambda. \tag{24A.30.i}$$

Solving for λ, we get $\lambda = 1/3$. Thus, player 1's best response to player 2's mixed strategy (λ,1-λ) is (1,0) if $\lambda > 1/3$, (0,1) if $\lambda < 1/3$ and (ρ,1-ρ) if $\lambda = 1/3$ where ρ can take any value from 0 to 1.

Next, consider player 2's best response to player 1 playing (ρ,1-ρ) (where $0 \le \rho \le 1$). Player 2's payoff from playing (1,0) (i.e. "Left" always) is then $5\rho+0(1-\rho) = 5\rho$, and her payoff from playing (0,1) (i.e. "Right" always) is $0(\rho)+10(1-\rho) = 10-10\rho$. Player 2 will then be indifferent between playing (1,0) and playing (0,1) when

$$5\rho = 10-10\rho. \tag{24A.30.ii}$$

Solving for ρ, we get $\rho = 2/3$. Thus, player 2's best response to player 1's mixed strategy (ρ,1-ρ) is (1,0) when $\rho > 2/3$, (0,1) when $\rho < 2/3$ and (λ,1-λ) when $\rho = 2/3$ where λ can take any value from 0 to 1. The mixed strategy equilibrium then has player 1 playing $(2/3, 1/3)$ and player 2 playing $(1/3, 2/3)$.

Exercise 24A.31

Plot the best response functions to mixed strategies for the Prisoners' Dilemma game and illustrate that there exists only a single, pure strategy equilibrium.

Answer: For a prisoners' dilemma game in which the first pure strategy is "Cooperate" and the second pure strategy is "Don't Cooperate", player 1 does not care what value λ takes in player 2's mixed strategy (λ,1-$lambda$) because playing (0,1) is a dominant strategy. Similarly, player 2 does not care what value ρ takes in player 1's mixed strategy (ρ, 1-ρ) because playing (0,1) is a dominant strategy. Putting λ on the horizontal axis and ρ on the vertical, player 1's best response function is therefor a horizontal line from 0 to 1 on the horizontal axis, and player 2's best response function is a vertical line from 0 to 1 on the vertical axis. The two best response functions meet only once — at the origin where $\lambda = 0$ and $\rho = 0$. This is the pure strategy equilibrium in which both players choose "Don't Cooperate" with probability 1.

24B Solutions to Within-Chapter-Exercises for Part B

Exercise 24B.1

If there are N players and T possible types, how many probabilities constitute my beliefs about the other players in the game?

Answer: This would involve $T(N-1)$ probabilities — T because that is how many types there are, and $(N-1)$ because I know my own type.

Exercise 24B.2

In what sense does the distinction between Nash and subgame perfect equilibrium illustrate how "off the equilibrium path" plans — i.e. plans that are never executed in equilibrium — can be important?

Answer: In repeated games, there may exist Nash equilibria that are not subgame perfect. These equilibria are sustained as Nash equilibria by non-credible threats that are never executed in equilibrium — i.e. off-the-equilibrium path behavior that is never executed sustains the Nash equilibrium. Subgame perfection requires that the off-the-equilibrium path behavior also satisfy rationality — i.e. that all threats that are not executed in equilibrium are credible.

Exercise 24B.3

Do you agree or disagree with the following statement: "Both complete and incomplete information simultaneous move games can be modeled as games in which Nature moves first, but Nature plays only pure strategies in complete information games while it plays mixed strategies in incomplete information games."

Answer: The statement is correct. In complete information games, there is no ambiguity about the type of player that everyone is. Another way of saying that is to say that "Nature" assigned types to players with probability 1 each, which is the same as saying that Nature played a pure strategy at the stage of the game that precedes the simultaneous game between players.

Exercise 24B.4

What is player 2's dominant strategy in each of the two games?

Answer: Player 2's dominant strategy in the first game is to play $Right$ — because no matter what Player 1 does, Player 2 is better off playing $Right$. In the second game, Player 2's dominant strategy is to play $Left$ for the same reason.

Exercise 24B.5

How can we be sure that player 1 will play D in equilibrium in the left-hand side game but U in the right-hand side game?

Answer: Although these strategies are not dominant strategies for Player 1, we can nevertheless be sure he will play them because we know Player 2's dominant strategy in each game. Thus, Player 1 knows that Player 2 will play $Right$ in the left-hand-side game — which implies that his best response should be $Down$. Similarly, Player 1 knows that player 2 will play $Left$ in the right-hand-side game, which means his best response is to play Up.

Exercise 24B.6

Since all players know the probabilities with which types are assigned, how would you characterize player 1's beliefs about which node he is playing from once the game reaches his information set?

Answer: Player 1's beliefs are $(\rho, (1-\rho))$, with ρ being the probability that he is facing a type I player and $(1-\rho)$ being the probability that he is facing a type II player.

Exercise 24B.7

True or False: If we depict a simultaneous move (complete information) game in a game tree, each player only has one information set.

Answer: This is true because, in such a game, each player can only be one possible type.

Exercise 24B.8

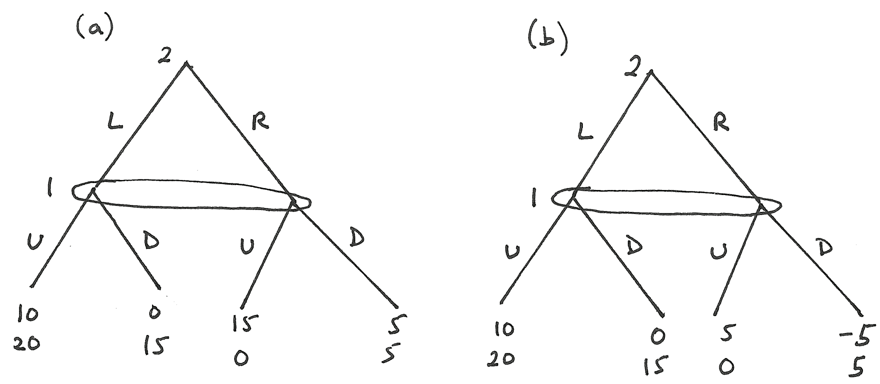
How would you depict the complete information game from either of the payoff matrices in the graph if you had player 2 rather than player 1 at the top of the game tree?

Answer: The left-hand-side game is depicted in Exercise Graph 24B.8.

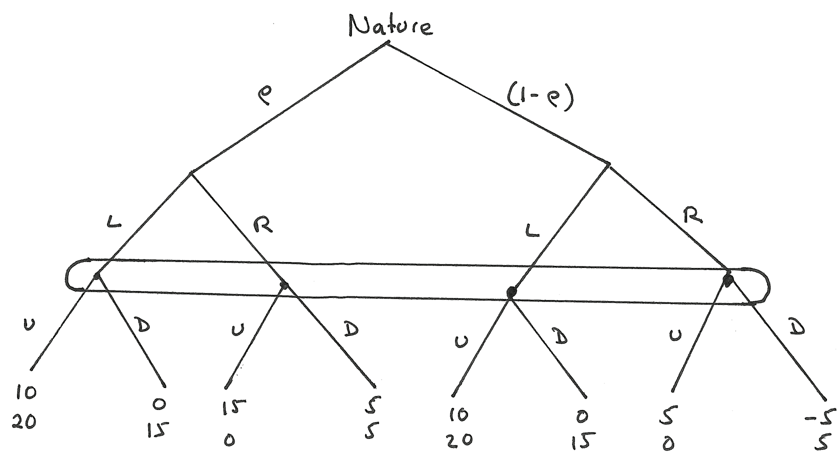
Exercise 24B.9

You could also draw the game tree in Graph 24.8 with Player 2 going first and player 1 going second. What do the information sets look like if you depict the game in this way?

Answer: This is depicted in Exercise Graph 24B.9.



Exercise Graph 24B.8 : Complete Information Game in Extensive Form



Exercise Graph 24B.9 : Text Graph 24.8 with Player positions reversed

Exercise 24B.10

True or False: Player 2 has 4 possible strategies while player 1 has 2 possible strategies.

Answer: This is true. Player 1 can play the strategy "Up" or the strategy "Down" — since he does not know which node he is at when he gets to play, he simply picks an action for the one information set from which he plays. Player 2, on the other hand, has two possible information sets: One where she knows she is type I and one where she knows she is type II. Within the information set, she does not know whether player 1 has played "Up" or "Down" (because the game is a simultaneous game) — so she cannot specify her strategy based on what player 1 has done. But she can specify her strategy based on what type she is — and thus has the following strategies available: (Left, Left) — i.e. play "Left" regardless of which type she is; (Left, Right) — i.e. play "Left" if she is type I and play "Right" if she is type II; (Right, Left) — i.e. play "Right" when she is type I and "Left" when she is type II; and (Right, Right) — i.e. play "Right" regardless of what type she is.

Exercise 24B.11

How would the outcome be different if the two games at the bottom of Graph 24.8 were the games in Table 24.5 and exercise 24A.7 (with player 2's actions labeled "left" and "right" instead of "up" and "down")?

Answer: This would imply that player 1 faces the game

		Player 2	
		Left	*Right*
Player 1	*Up*	10,10	0,15
	Down	15,0	5,5

with probability ρ and the game

		Player 2	
		Left	*Right*
Player 1	*Up*	10,10	0,7
	Down	15,7	5,5

with probability $(1-\rho)$. For player 2, it is a dominant strategy to play "Right" in the first game and to play "Left" in the second game — and player 1 should know this. Thus, player 1's expected payoff from playing "Up" is $0\rho + 10(1-\rho) = 10 - 10\rho$ while his expected payoff from p laying "Down" is $5\rho + 15(1-\rho) = 15 - 10\rho$. Notice that the latter is always larger than the former — regardless of what value ρ takes. Thus, player 1 will play "Down" in equilibrium, and player 2 will play (Right,Left) — i.e. player 2 will play "Right" if she is type I and "Left" if she is type II.

Exercise 24B.12

What is the probability that Nature assigns a type greater than $\overline{t}$ to a player?

Answer: That probability is $(1-\overline{t})$.

Exercise 24B.13

What is the set of possible actions A for this game?

Answer: The set of possible actions is $A = [0,1]$. In other words, players will choose a bid from the interval $[0,1]$. (Technically, it is actually the case that players could submit bids higher than 1, but since we know that no player has a value greater than 1 for the painting, we can ignore the actions above 1.)

Exercise 24B.14

Verify that this is correct.

Answer: Taking the derivative of the objective function with respect to a^i and setting it to zero, we get the result.

Exercise 24B.15

Suppose that both bidders know how much each of them value the painting; i.e. suppose the game was one of complete information. What would be the Nash equilibrium bidding behavior then? How does it differ from the incomplete information game?

Answer: Suppose my value is v and yours is $\overline{v}$, with $v > \overline{v}$. If you bid $\overline{v}$ and I bid anything above that, I will win the auction — so my best response is to bid slightly above $\overline{v}$. Given that I bid slightly above $\overline{v}$ bidding anything below my bid is a best response for you — since you will lose the auction and don't want to win given I am bidding above your valuation. Thus, it is a Nash equilibrium for me to bid slightly above $\overline{v}$ and you to bid $\overline{v}$. (It could not be part of a Nash equilibrium for you to bid under $\overline{v}$ — because then my best response would be to bid slightly above that — but your best response to that would be to bid slightly above that (which would still be below $\overline{v}$). If $\overline{v} < v$, the reverse holds — with you bidding slightly above my valuation v and me bidding v in equilibrium. If $v = \overline{v}$, the we are both best responding to each other if we both bid our exact valuation — and both of us win with probability 0.5. The outcome, then, is that the winning bid will be approximately equal to the lower of the two valuations — whereas in the incomplete information context, we derived in the text that the winning bid is half the higher valuation. The outcomes under complete and incomplete information therefore differ (unless the higher valuation is twice the lower valuation).

Exercise 24B.16

Check that the payoffs listed in Graph 24.9 correspond to the payoffs in Graph 24.8.

Answer: They do — keeping in mind that the person called "player 2" before is now the "sender", and the person we called "player 1" before is now the "receiver".

Exercise 24B.17

Determine for each of the three remaining sender pure strategies why the strategy cannot be part of a (subgame perfect) equilibrium.

Answer: The three remaining sender pure strategies are (L,R), (R,L) and (R,R). Consider first (L,R). This implies that the sender will go R if she is type II. Since the receiver is playing (U,D) (assuming subgame perfection), this implies that the receiver would go D — resulting in payoff of -5 for the sender of type II. She can do better by playing L and having the receiver play U — which will give her 10. Consider next (R,L). This implies that sender type I will go R — with the receiver playing D and getting payoff of 5 for the sender. But sender type I can do better by going L and having the receiver play U — thus getting 10. Finally, consider (R,R). In this case both sender types would be better off going L and then seeing the receiver go U.

Exercise 24B.18

Suppose the minus 5 payoff in the lower right corner of the game tree were 0 instead. Would we still get the same subgame perfect equilibrium? Could (R,R) be part of a Nash equilibrium that is not subgame perfect?

Answer: Nothing has changed for the receiver — for whom (U,D) is still the only strategy that is subgame perfect. Nothing has changed for sender type I — who would still best respond by going L. And sender type II will get 20 by going L and 0 by going R — and so will still want to go L. Thus, the previously identified subgame perfect equilibrium $\{(L,L),(U,D)\}$ is still a subgame perfect equilibrium. But now (R,R) can also be part of a Nash equilibrium where the receiver plays (D,D). If the receiver plays (D,D), she is always going D no matter what. Thus, a sender type I would get 0 by going L and 5 by going R — and would thus prefer to go R. And a sender type II would get 0 whether she goes R or L — and thus is best responding by going R. Conditional on the sender playing (R,R), it is furthermore a best response for the receiver to always play D. Thus, $\{(R,R),(D,D)\}$ is a Nash equilibrium. It is not subgame perfect, however, because it requires the receiver to say she'll play D if she observes L even though it is in her best interest to play U if she observed L — i.e. the Nash equilibrium we have identified relies on non-credible actions being planned off the equilibrium path.

Exercise 24B.19

Suppose that we changed the -5 payoff in Graph 24.9 to 20. Demonstrate that this would imply that only the separating strategy (L,R) can survive in equilibrium.

Answer: Nothing has changed for the receiver for whom it is still best to go U if he observes L and D if he observed R — regardless of what type of sender he faces. Thus, the receiver will play (U,D). The sender knows this. If she is assigned type I by Nature, she therefore knows that she will get payoff of 10 by going L and payoff of 5 if she goes R — which implies she'll go L. If, on the other hand, she is assigned type II by Nature, she gets 10 by going L and 20 if she goes R — and so she will go R. So, in equilibrium, the sender plays (L,R) — L if she is type I and R if she is type II — and the receiver plays (U,D) — i.e. U if she observes L and D if she observes R. The receiver will know the sender type because the sender is playing a separating strategy — one that results in a different action being played depending on what type she is.

Exercise 24B.20

In the previous section, we talked about subgame perfect strategies in ways that we cannot do here. What is different?

Answer: In the previous section, the receiver had a clear optimal action from each of her two information sets. Thus, if she observed L, it was best for her to go U and if she observed R it was best for her to go D — regardless of what her beliefs where about which node in her information set was relevant. The difference now is that the receiver would like to take different actions depending on which of the two nodes of the information set following R she is at.

Exercise 24B.21

Is there any way for (R,R) to be an equilibrium sender strategy? (Your answer should be no — can you explain why?)

Answer: Subgame perfection requires that the receiver has to go U when he observes L — because regardless of what type the sender is, the receiver gets a higher payoff from playing U when L has been played by the sender. The sender knows this. If she is type II, she therefore knows that she will get 10 by going L — which is higher than anything she could get by going R. Thus, given that the receiver plays (U,D), it cannot be a best response for the sender to play R when she is of type II. And this implies (R,R) cannot be part of an equilibrium in which the receiver plays U after observing L. We might then ask if (R,R) could be sustained as an equilibrium that does not respect subgame perfection — which would require the receiver to play (D,D). Suppose this were the case. Then a sender of type II is better off going L (and getting 0 rather than -5) — which means (R,R) is not a best response even if the receiver were to non-credible plan to always play D. So (R,R) cannot be part of a Nash equilibrium whether it is subgame perfect or not.

Exercise 24B.22

How much higher a payoff would a type II sender get by switching her signal in this way?

Answer: Rather than getting -5, she would get 10 — i.e. she would get 15 more in payoff.

Exercise 24B.23

For the game in Graph 24.10, we have therefore found both a separating and a pooling equilibrium, but for the pooling equilibrium we needed to place a restriction on out-of-equilibrium beliefs. Do you find these restrictions "reasonable" in this example?[1]

Answer: To me, the separating equilibrium seems more reasonable. Here is why: A type I sender does better in the separating equilibrium than in the pooling equilibrium (getting 15 instead of 10) while a type II player does equally well in either. A type I sender therefore has something to gain from trying to get to the separating equilibrium in which the receiver believes it likely that an R action is an indication that the sender is a type I individual. A type II sender has nothing to gain from trying this. Thus, if I am a receiver in this game and I observed an R, I would bet it's a type I sender that just took an action — i.e. my δ would be above 0.5 — which would cause me to go U. I therefore find it counterintuitive for a receiver to believe $\delta < 0.5$ when observing R — which is required for the pooling equilibrium. Thus, I don't find the pooling equilibrium in this case to be intuitive — while I do find the separating equilibrium to be intuitive. (This is the basic logic behind the intuitive criterion that is hinted at in the footnote — a criterion that favors the separating equilibrium as the one more likey to have predictive power.)

Exercise 24B.24

What has to be true about δ in order for (L,L) to be an equilibrium pooling strategy when $\rho < 0.5$?

Answer: We just concluded in the text that the (L,L) pooling equilibrium requires the receiver to play U when observing L. It also requires the receiver to play D when observing R — otherwise type I senders will deviate and play R (thus getting 15 rather than 10 from going L). And we have concluded that the only (subgame perfect) way for the receiver to play D when observing R is for δ to be less than 0.5.

Exercise 24B.25

Could the separating strategy (L,R) be part of an equilibrium in this case?

Answer: No. If the sender in fact played this strategy, the receiver would know with certainty that she is facing a type I sender after observing L and a type II sender after observing R. Thus, the receiver would play the strategy (D,D) as a best response. Now consider a type I sender who gets 0 under this proposed set of strategies. If she switched to playing R, she would get 5 instead — which means (L,R) is not a best response to (D,D).

Exercise 24B.26

[1] This is far from a trivial question and it has concerned game theorists a great deal. After all, what does it mean for beliefs related to events that do not happen in equilibrium to be reasonable? An approach to this, known as the "Intuitive Criterion" has been derived. You can read more about this in Section 4.4 of Gibbons.

Suppose that, in the game in Graph 24.10, we had changed the receiver's payoff from playing U when facing a sender of type II who plays left from 20 to 5 instead. Could there be a separating equilibrium in that game? Is there a pure strategy equilibrium for all values of ρ?

Answer: There could not be a separating equilibrium in this game. Consider first (L,R). The receiver would then know that he is facing a sender type I if he saw L and a sender type II if he saw R being played. That implies he would best respond by playing (U,D). A type II sender would then get -5 but could get 10 by deviating and going L (followed by U on the part of the receiver). Thus, (L,R) cannot be an equilibrium. Next, consider (R,L). This would imply that the receiver knows he is facing a sender type I if he observes R and a sender type II if he observes L — implying a best response of (U,D). This gives a type II sender a payoff of 0 — but she could get a payoff of 5 by deviating and going L (followed by U on the part of player 1). Thus, (R,L) cannot be part of an equilibrium.

Since neither of the two separating strategies could be part of an equilibrium, consider next the two pooling strategies. If (L,L) is played — i.e. if both sender types play L, then no information has been revealed to the receiver. For this to be an equilibrium from the sender perspective, it must be that neither sender type can do better by going R. That immediately rules out D as an equilibrium action from the left information set for the receiver — because were he to play D, sender type I could do better by going R (because anything that he would get by going R dominates 0). Thus, the pooling equilibrium with (L,L) must have the receiver playing U from his left information set. His payoff (given he is in his left information set) is $20\gamma+5(1-\gamma)=15\gamma+5$ for playing U and $15\gamma+15(1-\gamma)=15$ for playing D. This implies a higher payoff for playing U if $\gamma>2/3$ — implying that the receiver will play U if $\gamma>2/3$ and D if $\gamma<2/3$. Since no information has been revealed, there is no reason for the receiver to have updated his beliefs as a result of observing L — which implies we also know that he will play U if $\rho>2/3$ and D if $\rho<2/3$. Since it must be that U is played from this information set for (L,L) to be part of the equilibrium, we therefore know that this pooling equilibrium can only exist when $\rho>2/3$ — and the receiver therefore best-responds with U if he observes L in a pooling equilibrium. We now only have to check whether either of the sender types could do better by deviating from the pooling strategy. Type II definitely cannot do better since she is getting 10 which is higher than anything she could get by going R. Type I, however, could do better by going R if the receiver would play U upon observing R. Thus, in order for (L,L) to be a pooling equilibrium, it must be that the receiver plans to go D when observing R — which we concluded previously he will do so long as $\delta<0.5$. *We therefore have a pooling equilibrium* $\{(L,L),(U,D)\}$ *when* $\rho>2/3$ *and the out-of-equilibrium beliefs* $\delta<0.5$.

Finally, we can ask under what conditions we could get a pooling equilibrium with (R,R). In order for this to be part of an equilibrium, it cannot be that the receiver plays D following R — because then type II senders would be unambiguously better off playing L (where any possible payoff is higher than -5). Thus, the only way (R,R) can be part of an equilibrium is if U is the best response action when R is observed — which we previously concluded in the text is only the case when $\delta>0.5$. Since the pooling strategy reveals no information, this implies that $\rho>0.5$.

Suppose, then, that $\rho > 0.5$ and (R,R) is played with U played when the receiver reaches the right information set. We now only have to check that neither sender type could benefit from switching to L. Type I is getting 15 which is higher than anything she could get from switching, but type II would do better switching to L if L would be followed by U. Thus, to sustain this as an equilibrium, it must be that D is played following L — which we concluded above will happen only if $\gamma < 2/3$. *We therefore have a pooling equilibrium* $\{(R,R),(D,U)\}$ *when* $\rho > 1/2$ *and the out-of-equilibrium beliefs* $\gamma < 2/3$.

We have therefore identified pure strategy pooling equilibria only for the case where ρ is at least 0.5.

Exercise 24B.27

If the sender plays a pooling strategy (L,L), why is the receiver's belief about nodes in the information set I_R undefined according to Bayes' Rule?

Answer: Under this pooling strategy, the probability of reaching I_R is zero — which means that we divide by zero when applying Bayes' Rule to get updated probability believes conditional on I_R being reached. It is for this reason that we do not have an immediate way of restricting beliefs in information sets that are not reached in equilibrium.

Exercise 24B.28

Is every Bayesian Nash equilibrium also a perfect Bayesian Nash equilibrium? Is every perfect Bayesian Nash equilibrium also a Bayesian Nash equilibrium? Explain.

Answer: The relationship between the set of Bayesian Nash equilibria and the set of perfect Bayesian Nash equilibria is exactly the same as the relationship between Nash and subgame perfect equilibria in complete information games — every perfect Baysian Nash equilibrium is also a Bayesian Nash equilibrium but not every Bayesian Nash equilibrium is a perfect Bayesian Nash equilibrium. In both cases, the "non-perfect" equilibrium can employ non-credible threats off the equilibrium path.

Exercise 24B.29

True or False: When a game tree is such that all information sets are single nodes, then subgame perfect Nash equilibrium is the same as perfect Bayesian Nash equilibrium.

Answer: This is true — because then the game is one of complete information, with Bayes' rule implicitly telling you that the probability of reaching a node given that one reached the information set that contains the node is 1 — because all information sets only have one node.

Exercise 24B.30

Verify the last sentence.

Answer: If I am a tit-for-tat player and you play C first (getting a payoff of 10 since I will also play C) and D second (getting a payoff of 15 since I will still be playing C), you get a payoff of 25 across the two periods. If I am a "rational" player, and you play C followed by D, you will be caught getting 0 the first time and 5 the second time (since I will play D both times). The first scenario happens with probability ρ, and the second happens with probability $(1-\rho)$. Thus, your expected payoff from playing C followed by D is $25\rho+5(1-\rho)=20\rho-5\rho$. If you play D both periods and I am a tit-for-tat player, you will get 15 the first time and 5 the second time — for a total of 20, and if I am a "rational player" (who also plays D both times), you will get 5 each time — for a total of 10. Thus, your expected payoff from playing D both times is $20\rho+10(1-\rho)=10\rho+10$.

Exercise 24B.31

What are the beliefs that support this as a perfect Bayesian equilibrium?

Answer: You will know after the first period whether I am a tit-for-tat player — if you see me playing C, you know I am tit-for-tat; if you see me playing D, you will know I am "rational". So the second time we meet, you know who I am with certainty. The first time we meet, nothing has happened to cause you to update your beliefs from your initial belief that I am tit-for-tat with probability ρ and "rational" with probability $(1-\rho)$.

Exercise 24B.32

Verify that, if we play these strategies, your expected payoff will be $35\rho+15(1-\rho)=20\rho+15$, and my payoff as a T_2 type will be 30.

Answer: My payoffs as a non-tit-for-tat T_2 type are easy: On the equilibrium path, we both cooperate the first time — giving me 10; you cooperate and I deviate the second time — giving me 15; and we both don't cooperate the last time, giving me 5 — for a total of 30 (for me) across the three interactions. Your expected payoff, on the other hand, depends on the probability ρ that I am a tit-for-tat type. If I am, then we will cooperate the first and second times — giving you 10 each time; then you will defect while I still cooperate the third time — giving you 15. If I am a tit-for-tat type, you will therefore get 35. If I am not a tit-for-tat type, then we will cooperate the first time — giving you 10; you will cooperate while I will defect the second time — giving you 0; and we will both defect the last time, giving you 5 — for a total of 15 across all three interactions. Your expected payoff is therefore $35\rho+15(1-\rho)=20\rho+15$.

Exercise 24B.33

Propose a way that average payoffs could be 5 for player 1 and 12.5 for player 2.

Answer: Play (C,C) half the time and (C,D) the other half. That would give both players payoff (10,10) half the time and (0,15) the other half of the time. This would give player 1 an average payoff of 0.5(10)+0.5(0)=5 and player 2 a payoff of 0.5(10)+0.5(15)=12.5.

24C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 24.1

In the Hollywood movie "A Beautiful Mind", Russel Crowe plays John Nash who developed the Nash Equilibrium concept in his PhD thesis at Princeton University. In one of the early scenes of the movie, Nash finds himself in a bar with three of his fellow (male) mathematics PhD students when a group of five women enters the bar.[2] *The attention of the PhD students is focused on one of the five women, with each of the four PhD students expressing interest in asking her out. One of Nash's fellow students reminds the others of Adam Smith's insight that pursuit of self interest in competition with others results in the socially best outcome, but Nash — in what appears to be a flash if insight — claims "Adam Smith needs revision".*

A: *In the movie, John Nash then explains that none of them will end up with the woman they are all attracted to if they all compete for her because they will block each other as they compete — and that furthermore they will not be able to go out with the other women in the group thereafter (because none of them will agree to a date once they know they are at best everyone's second choice). Instead, he proposes, they should all ignore the woman they are initially attracted to and instead ask the others out — it's the only way they will get a date. He quickly rushes off to write his thesis — with the movie implying that he had just discovered the concept of Nash Equilibrium.*

(a) *If each of the PhD students were to play the strategy John Nash suggests — i.e. each one selects a woman other than the one they are all attracted to, could this in fact be a pure strategy Nash Equilibrium?*

<u>Answer</u>: No, it could not. In order for this to be a pure strategy Nash Equilibrium, it must be the case that everyone is playing a best response relative to everyone else. If everyone else is ignoring the woman that is deemed most attractive, it is not a best response for any single player to also ignore the woman. Thus, none of the PhD students is playing a best response strategy to the others if they are all ignoring the woman they find most attractive.

(b) *Is it possible that any pure strategy Nash equilibrium could result in no one pursuing the woman they are all attracted to?*

<u>Answer</u>: No, it is not possible — since this would imply everyone choosing to ignore the woman deemed most attractive — which would in turn imply that a single player's best response is to approach this woman.

(c) *Suppose we simplified the example to one in which it was only Nash and one other student encountering a group of two women. We then have two pure strategies to consider for each PhD student: Pursue woman A or pursue woman B. Suppose that each viewed a date with woman A as yielding a "payoff" of 10 and a date with woman B as yielding a payoff of 5.*

[2] Nash is actually with 4 others, but the rest of the scene unfolds as if there were 4 of them in total.

Each will in fact get a date with the woman that is approached if *they approach different women, but neither will get a date if they approach the same woman in which case they both get a payoff of 0. Write down the payoff matrix of this game.*

Answer: The payoff matrix to this game is

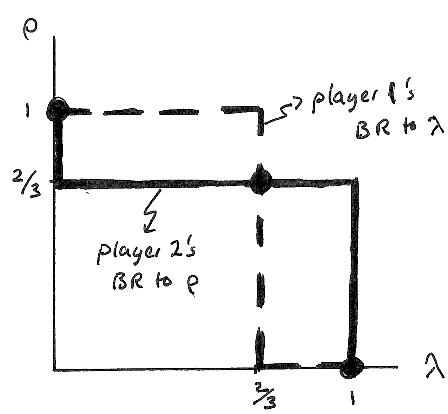
		Player 2	
		A	B
Player 1	A	0,0	10,5
	B	5,10	0,0

(d) *What are the pure strategy Nash Equilibria of this game?*

Answer: The pure strategy equilibria of this game are $\{A,B\}$ and $\{B,A\}$ — where the first action in each pairing is player 1's strategy and the second action is player 2's strategy.

(e) *Is there a mixed strategy Nash Equilibrium in this game?*

Answer: Yes, there is a mixed strategy equilibrium. Consider first the probability ρ placed by player 1 on A. We know that if $\rho = 1$, player 2's best response is to play B. Letting λ denote player 2's probability of playing A, we therefore know that player 2's best response to $\rho = 1$ is to play $\lambda = 0$ — which continues to be the case so long as $\rho > 2/3$. This is because player 2's payoff from playing A is $0(\rho) + 10(1-\rho) = 10(1-\rho)$ and player 2's payoff from playing B is $5\rho + 0(1-\rho) = 5\rho$ — and these are equal when $\rho = 2/3$. Thus, when $\rho = 2/3$, player 2 will best respond with any λ ranging from 0 and 1 (and when $\rho < 2/3$, player 2 best responds with $\lambda = 1$). This gives the solid best response function in Exercise Graph 24.1(1).



Exercise Graph 24.1(1) : Mixed Strategy Best Responses

Similarly, for player 1 the payoff from playing A is $0(\lambda)+10(1-\lambda) = 10(1-\lambda)$, and his playoff from playing B is $5\lambda + 0(1-\lambda) = 5\lambda$. He is therefore indifferent between playing A and B when $\lambda = 2/3$ — implying he would best respond by setting ρ anywhere from 0 to 1 when $\lambda = 2/3$ but best responds by playing $\rho = 0$ when $\lambda > 2/3$ and by playing $\rho = 1$ when $\lambda < 2/3$. This gives the dashed best response function for player 1 in Exercise Graph 24.1(1). The mixed strategy equilibrium occurs where the two best response functions intersect — i.e. at $\{\rho = 2/3, \lambda = 2/3\}$. Note that $\{\rho = 0, \lambda = 1\}$ and $\{\rho = 1, \lambda = 0\}$ are also at intersections of the two best response functions — and these are equivalent to the pure strategy Nash equilibria we found in part (d).

(f) *Now suppose there is also a woman C in the group of women — and a date with C is viewed as equivalent to a date with B. Again, each PhD student gets a date if he is the only one approaching a woman, but if both approach the same woman, neither gets a date (and thus both get a payoff of zero). Now, however, the PhD students have 3 pure strategies: A, B and C. Write down the payoff matrix for this game.*

Answer: This is illustrated as

		Player 2		
		A	B	C
	A	0,0	10,5	10,5
Player 1	B	5,10	0,0	5,5
	C	5,10	5,5	0,0

(g) *What are the pure strategy Nash Equilibria of this game? Does any of them involve woman A leaving without a date?*

Answer: We now have the following Nash equilibria: $\{A,B\}$, $\{A,C\}$, $\{B,A\}$, $\{C,A\}$ (where the first action is again the strategy of player 1 and the second is the strategy of player 2). In each of these, one of the students goes out with woman A.

(h) *In the movie, Nash then explains that "Adam Smith said the best result comes from everyone in the group doing what's best for themselves." He goes on to say "...incomplete ... incomplete ... because the best result will come from everyone in the group doing what's best for themselves and the group ... Adam Smith was wrong." Does the situation described in the movie illustrate any of this?*

Answer: Not really. It is true that each Nash equilibrium is efficient in the sense that the sum of the payoffs is the largest it can possibly be — but that would in fact illustrate that the "best result comes from everyone in the group doing what's best for themselves". No one in the game considers what's best for the group.

(i) *While these words have little to do with the concept of Nash Equilibrium, in what way does game theory — and in particular games like the Prison-*

ers' dilemma — challenge the inference one might draw from Adam Smith that self interest achieves the "best" outcome for the group?

Answer: It is true that game theory can illustrate how the "best" outcome for the group is not obtained by each individual doing what is best for himself. The prisoners' dilemma is one game where this is the case — the Nash equilibrium is one where both the individuals and the group do worse than they could if they considered the group. Put differently, the Nash equilibrium in the prisoners' dilemma is inefficient — and the individuals could in fact do better if they found a way to force each other to go against their self-interest. But that is not something that will happen as a result of Nash equilibrium behavior — the Nash equilibrium in that case simply illustrates that self-interest does not always serve the group (or the individual) when we leave the idealized setting of the first welfare theorem. The game depicted in the movie, however, does not illustrate anything like this when properly modeled and analyzed.

B: *Consider the 2-player game described in part A(c). (Note: Part (a) and (b) below can be done without having read Section B of the Chapter.)*

(a) *Suppose that the players move sequentially — with player 1 choosing A or B first — and player 2 making his choice after observing player 1's choice. What is the subgame perfect Nash equilibrium?*

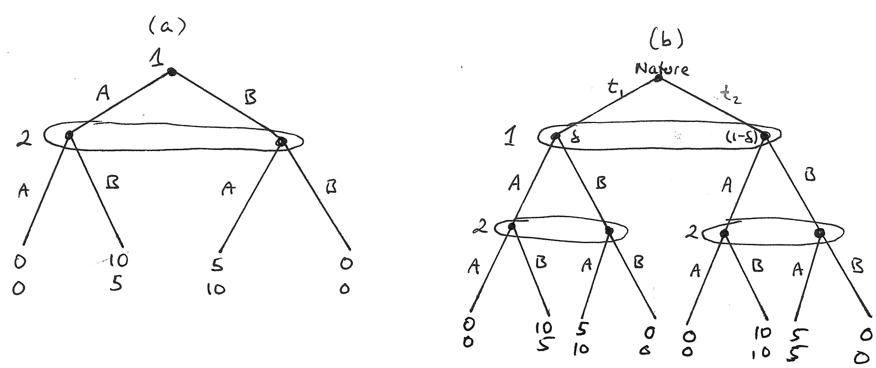
Answer: Solving from the end of the game, we first consider what player 2 will do if he observes player 1 having approached woman A or B. In both cases, it is optimal for player 2 to choose the other woman. Player 1 can predict this — and therefore will approach woman A. The subgame perfect equilibrium is therefore player 1 playing A and player 2 playing (A, B) — with the first action indicating what player 2 will do if he observes A and the second indicating what he will do if he observes B on the part of player 1.

(b) *Is there a Nash equilibrium in which player 2 goes out with woman A? If so, is there a non-credible threat that is needed to sustain this as an equilibrium?*

Answer: Player 1 playing B and player 2 playing (A, A) is a Nash equilibrium. (Given that player 1 plays B, it is a best response for player 2 to always choose A — and given that player 2 always chooses A, it is a best response by player 1 to choose B. But this Nash equilibrium requires that player 2 threatens to play A even if player 1 plays A in the first stage of the game — and that player 1 believes this threat.It is not, however, a credible threat since player 2 will optimally choose B if he observes A having been chosen in stage 1.

(c) *Next, consider again the simultaneous move game from A(c). Draw a game tree for this simultaneous move game — with player 1's decision on the top. (*Hint*: Use the appropriate information set for player 2 to keep this game a simultaneous move game). Can you state different beliefs for player 2 (when player 2 gets to his information set) such that the equilibria you derived in A(d) and A(e) arise?*

Answer: This is illustrated in panel (a) of Exercise Graph 24.1(2). Player 1 starts by choosing A or B — and player 2 goes next but is unable to tell whether player 1 chose A or B. Thus, the two nodes for player 2 are contained in one information set. This is equivalent to the two players moving simultaneously — because player 2 has no more information than he would if he were to move simultaneously. Let ρ be the probability with which player 2 thinks he is at the first node when it is his turn. His payoff from going A is then $0\rho + 10(1-\rho) = 10(1-\rho)$, and his payoff from going B is $5\rho + 0(1-\rho) = 5\rho$. Player 2 is therefore indifferent between moving A and moving B if $10(1-\rho) = 5\rho$ — i.e. if $\rho = 2/3$. Given such indifference, it would then be one (of many) best responses for player 2 to choose A with probability $\gamma = 2/3$ and B with probability $(1-\gamma) = 1/3$ — and if player 2 plays this way, it is player 1's best response to play A with probability $\rho = 2/3$. This is the mixed strategy equilibrium from part A(e) — and it requires player 2 to believe with probability 2/3 that player 1 went A — which in turn is consistent with individual 1's strategy. If player 2 believes player 1 went A with probability greater than 2/3, it is a best response for individual 2 to choose B with probability 1 — and if player 2 chooses B with probability 1, player 1's best response is to choose A with probability 1. Thus, the equilibrium in which player 1 chooses A and player 2 chooses B requires player 2 to believe individual 1 chooses A with probability 1. This is one of the two pure strategy equilibria found in A(d). The other is similar, with player 2 now believing that player 1 will play B with probability 1.



Exercise Graph 24.1(2) : Nash and Friend

(d) *Continue to assume that both players get payoff of 0 if they approach the same woman. As before, player 1 gets a payoff of 10 if he is the only one to approach woman A and a payoff of 5 if he is the only one to approach woman B. But player 2 might be one of two possible types: If he is type 1, he has the same tastes as player 1, but if he is of type 2, he gets a payoff of only 5 if he is the only one to approach woman A and a payoff of 10 if he* *is the only one to approach women B. Prior to the beginning of the game, Nature assigns type 1 to player 2 with probability δ (and thus assigns type 2 to player 2 with probability $(1-\delta)$.) Graph the game tree for this game — using information sets to connect nodes where appropriate.*

Answer: This is done in panel (b) of Exercise Graph 24.1(2).

(e) *What are the pure strategy equilibria in this game? Does it matter what value δ takes?*

Answer: In any pure strategy equilibrium, player 1 either plays A or he plays B. If he plays A, then player 2 will best respond by playing B regardless of what type he was assigned. Similarly, if player 1 plays B, player 1 will best respond by playing A regardless of what type he was assigned. Thus, we have two pure strategy Nash equilibria: (1) Player 1 plays A and player 2 plays (B, B) — i.e. B if type 1 and B if type 2; and (2), player 1 plays B and player 1 plays (A, A). The value of $delta$ is irrelevant.

Exercise 24.3

Consider a simultaneous game in which both players choose between the actions "Cooperate", denoted by C, and "Defect", denoted by D.

A: *Suppose that the payoffs in the game are as follows: If both players play C, each gets a payoff of 1; if both play D, both players get 0; and if one player plays C and the other plays D, the cooperating player gets α while the defecting player gets β.*

(a) *Illustrate the payoff matrix for this game.*

Answer: This is illustrated as

		Player 2	
		C	D
Player 1	C	1,1	α,β
	D	β,α	0,0

(b) *What restrictions on α and β would you have to impose in order for this game to be a Prisoners' dilemma? Assume from now on that these restrictions are in fact met.*

Answer: In order for this to be a Prisoners' Dilemma game, it must be that D is a dominant strategy for both players. Consider player 2 first: If player 1 plays C, it must be that it is optimal for player 2 to play D — which implies it must be that $\beta > 1$. Similarly, if player 1 plays D, it must be that D is optimal for player 2 — which implies it must be that $\alpha < 0$. Player 1's situation is symmetric — so the same restrictions will work to make D a dominant strategy for player 1. This is therefore a prisoners' dilemma game if $\alpha < 0$ and $\beta > 1$.

B: *Now consider a repeated version of this game in which players 1 and 2 meet 2 times. Suppose you were player 1 in this game, and suppose that you knew* *that player 2 was a "Tit-for-Tat" player — i.e. a player that does not behave strategically but rather is simply programed to play the Tit-for-Tat strategy.*

(a) *Assuming you do not discount the future, would you ever cooperate with this player?*

Answer: You know that player 2 will play C the first time and will then mimic what you do the first time when you meet the second time. You have four possible pure strategies to play: (C,C), (C,D), (D,C) and (D,D). Assuming you do not discount the future, your payoffs from these will be (1+1), $(1+\beta)$, $(\beta+\alpha)$ and $(\beta+0)$ respectively. Since $\beta>1$ and $\alpha<0$, we know that

$$(1+\beta)>(1+1)\ ,\ (\beta+0)>(\beta+\alpha)\ \text{ and }\ (1+\beta)>(\beta+0). \qquad (24.3.i)$$

The first inequality implies that (C,D) strictly dominates (C,C) and you will therefore not play (C,C). The second inequality implies that (D,D) strictly dominates (D,C) and you will therefore not play (D,C). And the third equality implies that (C,D) strictly dominates (D,D) and you will therefore not play (D,D). Thus, you will play (C,D) — and you will therefore cooperate in the first stage.

(b) *Suppose you discount a dollar in period 2 by* δ *where* $0<\delta<1$. *Under what condition will you cooperate in this game?*

Answer: Your payoffs from strategies (C,C), (C,D), (D,C) and (D,D) will now be $(1+\delta)$, $(1+\delta\beta)$, $(\beta+\delta\alpha)$ and $(\beta+0)$ respectively. We can then say unambiguously that

$$(1+\delta\beta)>(1+\delta)\ \text{ and }\ (\beta+0)>(\beta+\delta\alpha), \qquad (24.3.ii)$$

which implies that (C,D) strictly dominates (C,C) and (D,D) strictly dominates (D,C). You will therefore definitely never play (C,C) or (D,C). This leaves only (C,D) and (D,D), with payoffs of $(1+\delta\beta)$ and $(\beta+0)$ respectively. The former is larger than the latter so long as

$$\beta<\frac{1}{(1-\delta)}, \qquad (24.3.iii)$$

which implies you will play (C,D) when this condition holds and (D,D) otherwise. (Both are possible if the equation holds with equality.) Thus, if β is sufficiently large relative to the discount factor δ, you will still cooperate in the first stage.

(c) *Suppose instead that the game was repeated 3 rather than 2 times. Would you ever cooperate with this player (assuming again that you don't discount the future)? (*Hint*: Use the fact that you should know the best action in period 3 to cut down on the number of possibilities you have to investigate.)*

Answer: In the third encounter, it will always be best to play D since D is a dominant strategy in the single shot game. Thus, the only question is what you would do in the first and second encounter. We therefore have 4 possible strategies to consider: (C,C,D), (C,D,D), (D,C,D) and (D,D,D). The payoffs (given that the other player plays Tit-for-Tat) for these strategies are: $(1+1+\beta)$, $(1+\beta+0)$, $(\beta+\alpha+\beta)$ and $(\beta+0+0)$ respectively. Since

$$(1+1+\beta) > (1+\beta+0) > (\beta+0+0), \tag{24.3.iv}$$

we know that (C,C,D) dominates (C,D,D) which dominates (D,D,D) and thus neither (C,D,D) nor (D,D,D) get played. That leaves only (C,C,D) and (D,C,D) to consider, and (C,C,D) dominates (D,C,D) so long as $(1+1+\beta) > (\beta+\alpha+\beta)$. This simplifies to

$$\beta < 2-\alpha, \tag{24.3.v}$$

and as long as this condition holds, you will play (C,C,D). If, however, $\beta > 2-\alpha$, you will play (D,C,D) (and if $\beta = 2-\alpha$, either of the two strategies will maximize your payoff.) In either case, you will cooperate at some point, though only under (C,C,D) do both players ever cooperate at the same time.

(d) *In the repeated game with 3 encounters, what is the intuitive reason why you might play D in the first stage?*

Answer: If the reward from defecting when your opponent cooperates is sufficiently large relative to the loss one takes when cooperating in the face of the opponent defecting, it makes sense to take advantage of the Tit-for-Tat opponent right away, and then trick him into cooperating again in the last stage.

(e) *If player 2 is strategic, would he ever play the "Tit-for-Tat" strategy in either of the two repeated games?*

Answer: No, it would not make sense because player 2 should realize that player 1 will play D in the last encounter. This implies that the last encounter does not matter for what action will be played in the second to last encounter — which again implies player 1 will play D. Thus, the logic of subgame perfection should imply that player 2 will play D always.

(f) *Suppose that each time the two players meet, they know they will meet again with probability $\gamma > 0$. Explain intuitively why "Tit-for-Tat" can be an equilibrium strategy for both players if γ is relatively large (i.e. close to 1) but not if it is relatively small (i.e. close to 0).*

Answer: If γ is close to 1, the probability of meeting again is large. Thus, the short term payoff for player 1 from deviating from "Tit-for-Tat" and playing D without provocation is outweighed by the fact that the opponent will now play D until player 1 unilaterally starts playing C again. Put differently, rather than getting a payoff of 1 by playing C against the "Tit-for-Tat" strategy this period, player 1 can get $\beta > 1$, but it also implies that player 1 will face payoffs of 0 (rather than 1) from now on as both players switch to D, or player 1 will have to incur a payoff of $\alpha < 1$ (rather than 1) in a future period in order to get his opponent to cooperate again. If the chance of meeting again is sufficiently large, that's not worth it. If it is sufficiently small, however, it makes sense to graph β while you can. Thus, "Tit-for-Tat" can be a best response to "Tit-for-Tat" only if the chance of another encounter is large enough.

Exercise 24.5

Everyday Application: *Splitting the Pot: Suppose two players are asked to split \$100 in a way that is agreeable to both.*

A: *The structure for the game is as follows: Player 1 moves first — and he is asked to simply state some number between zero and 100. This number represents his "offer" to player 2 — the amount player 1 offers for player 2 to keep, with player 1 keeping the rest. For instance, if player 1 says "30", he is offering player 2 a split of the \$100 that gives \$70 to player 1 and \$30 to player 2. After an offer has been made by player 1, player 2 simply chooses from two possible actions: either "Accept" the offer or "Reject" it. If player 2 accepts, the \$100 is split in the way proposed by player 1; if player 2 rejects, neither player gets anything. (A game like this is often referred to as an* ultimatum game.*)*

(a) *What are the subgame perfect equlibria in this game assuming that player 1 is restricted to making his "offer" in integer terms — i.e. assuming that player 1 has to state a whole number.*

Answer: As always, we begin by thinking about player 2 who gets zero by playing "Reject" and whatever was offered by playing "Accept". For any positive amount x, the payoff from accepting therefore strictly dominated the payoff from rejecting — implying that, in any subgame perfect equilibrium, player 2 should always "Accept" any positive offer. If the offer is zero, player 2 is indifferent between accepting and rejecting — so either "Accept" or "Reject" could then be subgame perfect when zero is offered. Player 1 knows this — and thus knows he is playing someone whose strategy is either "Always Accept" or "Accept unless the offer is zero". The best response to the first ("Always Accept") strategy is to offer zero — giving us the subgame perfect equilibrium in which player 1 plays "Zero" and player 2 plays "Always Accept". The best response to the second strategy by player 2 (i.e. "Accept unless the offer is zero") is to offer \$1 (since only integer offers are allowed). Thus, we have two possible subgame perfect equilibria — one under which player 1 walks away with \$100, the other under which player 1 walks away with \$99.

(b) *Now suppose that offers can be made to the penny — i.e. offers like \$31.24 are acceptable. How does that change the subgame perfect equilibria? What if we assumed dollars could be divided into arbitrarily small quantities (i.e. fractions of pennies)?*

Answer: If offers can be made to the penny, the two subgame perfect equilibria described in (a) still exist — except that the latter involves an offer of a penny instead of an offer of a dollar by player 1. Thus, player 1 walks away with either \$100 or \$99.99. The more divisible we make a dollar, the smaller the quantity that is offered in the second subgame perfect equilibrium — with that quantity converging to zero. If we remove all assumptions about money being indivisible beyond a certain amount, the second subgame perfect equilibrium vanishes entirely — because for any positive amount that player 1 might offer, it is in fact better (for player 1) to offer half that amount. Thus, when we assume a dollar can be divided into arbitrarily small quantities, the only subgame perfect equilibrium is one in which player 2 plays "Accept any offer" and player 1 offers zero.

(c) *It turns out that there are at most two subgame perfect equilibria to this game (and only 1 if dollars are assumed to be fully divisible) — but there is a very large number of Nash equilibria regardless of exactly how player 1 can phrase his offer (and an infinite number when dollars are assumed fully divisible). Can you, for instance, derive Nash equilibrium strategies that result in player 2 walking away with \$80? Why is this not subgame perfect?*

Answer: The following would be a Nash equilibrium in which player 2 gets \$80: Player 1 offers \$80, and player 2 plays the strategy "Reject all offers below \$80, accept all others." Given this strategy by player 2, it is a best response for player 1 to offer \$80, and given the offer of \$80, player 2's strategy is a best response as well. Thus we have a Nash Equilibrium. However, this Nash equilibrium involves player 2 playing a strategy that involves a non-credible threat — the threat to reject all offers below \$80. For this reason, this Nash Equilibrium is not subgame perfect.

(d) *This game has been played in experimental settings in many cultures — and, while the average amount that is "offered" differs somewhat between cultures, it usually falls between \$25 and \$50, with players often rejecting offers below that. One possible explanation for this is that individuals across different cultures have somewhat different notions of "fairness" — and that they get utility from "standing up for what's fair". Suppose player 2 is willing to pay \$30 to stand up to "injustice" of any kind, and anything other than a 50-50 split is considered by player 2 to be unjust. What is now the subgame perfect equilibrium if dollars are viewed as infinitely divisible? What additional subgame perfect equilibrium arises if offers can only be made in integer amounts?*

Answer: This sense of fairness has changed the payoffs of the game from what we have assumed so far — with player 2 getting a payoff of 30 for rejecting any offer of less than 50. Thus, it is now credible for player 2 to play the strategy "Reject any offer below 30" — because we know player 2 gets the equivalent of \$30 in utility from rejecting unfair offers. It is not, however, credible for player 2 to play a strategy that rejects offers above \$30 — because accepting such offers yields more than rejecting them (which only yields \$30). It would therefore be a subgame perfect equilibrium for player 1 to offer \$30 and for player 2 to play the strategy "Reject any offer below \$30, otherwise accept". In the case where player 1 is limited to making offers in integer terms, we would also have a subgame perfect equilibrium in which player 1 offers \$31 and player 2 plays the strategy "Accept only offers *greater than* \$30".

(e) *Suppose instead that player 2 is outraged at "unfair" outcomes in direct proportion to how far the outcome is removed from the "fair" outcome, with the utility player 2 gets from rejecting an unfair offer equal to the difference between the amount offered and the "fair" amount. Suppose player 2 believes the "fair" outcome is splitting the \$100 equally. Thus, if the player faces an offer* $x < 50$*, the utility she gets from rejecting the offer is* $(50 - x)$*. What are the subgame perfect equilibria of this game now under the assumption of infinitely divisible dollars and under the assumption of offers having to be made in integer terms?*

Answer: For any offer of less than \$25, the payoff from rejecting is greater than the payoff from accepting. Thus, subgame perfection requires that player 2 reject any offer less than \$25. For any offer greater than \$25, the payoff from accepting is greater than the payoff from rejecting — thus subgame perfection implies player 2 will accept all offers greater than \$25. Offers of \$25 leave player 2 indifferent between accepting and rejecting the offer. In the case of infinitely divisible dollars, the only subgame perfect equilibrium therefore involves player 1 offering \$25 and player 2 playing the strategy "Reject all offers below \$25, accept all others" — which results in player 1 walking away with \$75 and player 2 walking away with \$25. In the case of offers restricted to integer amounts, the following additional subgame perfect equilibrium is possible: Player 1 offers \$26, and player 2 plays the strategy "Accept offers *greater than* \$25", which results in player 1 getting \$74 and player 2 getting \$26.

B: *Consider the same game as that outlined in A and suppose you are the one that splits the \$100 and I am the one who decides to accept or reject. You think there is a pretty good chance that I am the epitome of a rational human being who cares only about walking away with the most I can from the game. But you don't know me that well — you think there is some chance* ρ *that I am a self-righteous moralist who will reject any offer that is worse for me than a 50-50 split. (Assume throughout that dollars can be split into infinitesimal parts.)*

(a) *Structure this game as an incomplete information game.*

Answer: The game has 3 stages: In stage 1, Nature moves and assigns me type 1 with probability ρ and type 2 with probability $(1-\rho)$. In stage 2, you split the \$100 without knowing what type I am — i.e. both nodes in the game tree lie in the same information set for you. In stage 3, I (knowing my type) decide whether or not to accept the offer you have made.

(b) *There are two types of pure strategy equilibria to this game (depending on what value* ρ *takes). What are they?*

Answer: You will either decide to offer the minimum to make the moralist happy — i.e. \$50 — or you will take your chances and offer zero (or very close to zero). It makes no sense to offer something between 0 and 50

— because your offer will then be rejected by the moralist while leaving you with less than a zero offer if you face a non-moralist. It also makes no sense to offer more than \$50 — since we know even the moralist accepts the \$50 offer. If you offer 0, you expected payoff is then \$100 with probability $(1-\rho)$ and 0 with probability ρ — for a total expected payoff of $100(1-\rho)$. If, on the other hand, you offer \$50, you know you will walk away with \$50 because the offer will be accepted for sure. You will therefore offer 0 so long as $100(1-\rho) > 50$ — i.e. so long as $\rho < 0.5$. If $\rho > 0.5$, you will offer \$50. In all cases, the non-moralist will accept what is offered. (If $\rho = 0.5$, both offers are consistent with equilibrium behavior.)

(c) *How would your answer change if I, as a self-righteous moralist (which I am with probability ρ) reject all offers that leave me with less than \$10?*

Answer: In this case, you will either offer 0 or 10. Your expected payoff from offering 0 is $100(1-\rho)$ while your expected payoff from offering 10 is 90. Thus, you will offer 0 if $\rho < 0.1$ and 10 if $\rho > 0.1$ (with both offers possible when $\rho = 0.1$.

(d) *What if it's only less than \$1 that is rejected by self-righteous moralists?*

Answer: The cut-off for when you will offer \$1 rather than 0 is now $\rho = 0.01$.

(e) *What have we implicitly assumed about risk aversion?*

Answer: We have implicitly assumed that you are indifferent between gambles that leave you with the same expected value — i.e. we have assumed risk neutrality.

Exercise 24.7

Everyday Application: *Real World Mixed Strategies*: *In the text, we discussed the "Matching Pennies" game and illustrated that such a game only has a mixed strategy equilibrium.*

A: *Consider each of the following and explain (unless you are asked to do something different) how you might expect there to be no pure strategy equilibrium — and how a mixed strategy equilibrium might make sense.*

(a) *A popular children's game, often played on long road trips, is "Rock, Paper, Scissors". The game is simple: Two players simultaneously signal through a hand gesture one of three possible actions: $Rock$, $Paper$ or $Scissors$. If the two players signal the same, the game is a tie. Otherwise, $Rock$ beats $Scissors$, $Scissors$ beats $Paper$ and $Paper$ beats $Rock$.*

Answer: Suppose player 1 plays anything other than a mixed strategy with equal probabilities (of 1/3) on each of the three actions. Then we know that he places probability greater than 1/3 on at least one of his actions. Choose the action that he plays with highest probability — for illustration, suppose it is $Rock$. Then player 2's best response is to play the action that beats the most played action by player 1 — $Paper$, in this case. But if player 2 plays $Paper$, player 1's best response does not include any strategy that involves positive probabilities of playing $Rock$. So it cannot be that a strategy that places probabilities other than equal probabilities of 1/3 on each action is part of an equilibrium. This leaves us with mixed strategies that place equal probability on each action — which is the mixed strategy equilibrium. That seems reasonable — as we play this game, the best we can do is to pick randomly between the three options in a way that does not permit our opponent to predict that we might be favoring one option.

(b) *One of my students objects: "I understand that $Scissors$ can beat $Paper$, and I get how $Rock$ can beat $Scissors$, but there is no way $Paper$ should beat $Rock$. What ... $Paper$ is supposed to magically wrap around $Rock$ leaving it immobile? Why can't $Paper$ do this to $Scissors$? For that matter, why can't $Paper$ do this to people?... I'll tell you why: Because $Paper$ can't beat anybody!"*[3] *If $Rock$ really could beat $Paper$, is there still a mixed strategy Nash Equilibrium?*

Answer: In this case, $Paper$ does not beat anything — and so it should never be played in equilibrium. Effectively, this implies that we can treat the game is if there were in fact only two actions — $Scissors$ and $Rock$. $Rock$ beats $Scissors$ — so it is a dominant strategy to play $Rock$ — and all games are then ties as both players play $Rock$ in equilibrium.

(c) *In soccer, penalty kicks often resolve ties. The kicker has to choose which side of the goal to aim for, and, because the ball moves so fast, the goalie has to decide simultaneously which side of the goal to defend.*

Answer: This is exactly the matching pennies game: The goalie would like to match the side that the kicker is kicking to, and the kicker would like to kick to the side opposite the one being defended by the goalie. From our analysis of the matching pennies game, we know that we only have a mixed strategy equilibrium in which the goalie defends each side with probability 0.5 and the kicker kicks to each side with probability 0.5. This makes a reasonable amount of sense for the example. (Some economists have actually tried to see whether the goal kicking in professional soccer games — or the serve patterns by professional tennis players (see the next part of the exercise) conforms with mixed strategy equilibrium behavior. It turns out, that it does not quite — players are not sufficiently randomizing their actions to qualify them as true mixed strategies. But it may be that, while technically not playing mixed strategies, players are playing strategies sufficiently complex so that the opponent cannot predict the next move — which is similar to what happens under mixed strategies.)

(d) *How is the soccer example similar to a situation encountered by a professional tennis player whose turn it is to serve?*

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[3] My student continues (with some editing on my part to make it past the editorial censors): "When I play "Rock, Paper, Scissors", I always choose $Rock$. Then, when someone claims to have beaten me with $Paper$, I can punch them in the face with my already clenched fist and say — oh, sorry — I thought paper would protect you, moron."

Answer: The exact same issue comes up — do you serve to the right or to the left — with the defending player having to decide which side to defend in the instant that the serve happens. If the serve is sufficiently fast, the game is a simultaneous matching pennies game.

(e) *For reasons I cannot explain, teenagers in the 1950's sometimes played a game called "chicken". Two teenagers in separate cars drove at high speed in opposite directions on a collision course — and whoever swerved to avoid a crash lost the game. Sometimes, the cars crashed and both teenagers were severely injured (or worse). If we think behavior in these games arose within an equilibrium, could that equilibrium be in pure strategies?*

Answer: No, we could not have a pure strategy equilibrium that involved crashes. If player 1 knew that player 2 would not swerve, his best response would presumably be to swerve. There are therefore two pure strategy Nash equilibria — one in which player 1 swerves and one in which player 2 swerves. But there is also a mixed strategy equilibrium in which the two players randomize between swerving and not swerving — which can lead to equilibrium crashes.

B: *If you have done part B of exercise 24.4, appeal to incomplete information games with almost complete information to explain intuitively how the mixed strategy equilibrium in the chicken game of A(e) can be interpreted.*

Answer: The argument in exercise 24.4 is that mixed strategy behavior might emerge from uncertainty over the payoffs of the opponent. The chicken game is in fact quite similar to the Battle of the Sexes game in exercise 24.4 — with both cases being examples of coordination games. (In the Battle of the Sexes, the players try to coordinate on the same action — in the chicken game, they try to coordinate on opposing actions.) So the same argument can be made here: A player might not be certain of just how much the opponent values being the one that does not swerve. We would model such a game as a game of imperfect information in which Nature assigns a type to one (or both) players before the game begins — with each player only knowing his own type and the probability with which Nature assigned the different types to the opponent. Even if we make the range of possible payoff differences across types narrow, we showed that such uncertainty can lead to exactly the behavior that looks like mixed strategies in the complete information game.

Exercise 24.9

Everyday and Business Application: *Bargaining over a Fixed Amount: Consider a repeated version of the game in exercise 24.5. In this version, we do not give all the proposal power to one person but rather imagine that the players are* bargaining *by making different proposals to one another until they come to an agreement. In part A of the exercise we analyze a simplified version of such a bargaining game, and in part B we use the insights from part A to think about an infinitely repeated bargaining game. (Note: Part B of the exercise, while conceptually building on part A, does not require any material from Section B of the Chapter.)*

A: *We begin with a 3-period game in which \$100 gets split between the two players. It begins with player 1 stating an amount x_1 that proposes she should receive x_1 and player 2 should receive $(100 - x_1)$. Player 2 can then accept the offer — in which case the game ends with payoff x_1 for player 1 and $(100 - x_1)$ for player 2; or player 2 can reject the offer, with the game moving on to period 2. In period 2, player 2 now has a chance to make an offer x_2 which proposes player 1 gets x_2 and player 2 gets $(100 - x_2)$. Now player 1 gets a chance to accept the offer — and the proposed payoffs — or to reject it. If the offer is rejected, we move on to period 3 where player 1 simply receives x and player 2 receives $(100 - x)$. Suppose throughout that both players are somewhat impatient — and they value \$1 a period from now at \$$\delta (< 1)$. Also suppose throughout that each player accepts an offer whenever he/she is indifferent between accepting and rejecting the offer.*

(a) *Given that player 1 knows she will get x in period 3 if the game continues to period 3, what is the lowest offer she will accept in period 2 (taking into account that she discounts the future as described above)?*

Answer: The amount x that player 1 is guaranteed in period 3 is worth δx in period 2. Thus, player 1 will accept any offer that is at least equal to δx — i.e. any offer $x_2 \geq \delta x$.

(b) *What payoff will player 2 get in period 2 if he offers the amount you derived in (a)? What is the present discounted value (in period 2) of what he will get in this game if he offers less than that in period 2?*

Answer: If player 2 offers $x_2 = \delta x$, he will get a payoff of $(100 - \delta x)$ in period 2. If player 2 makes an offer $x_2 < \delta x$, the offer will be rejected and he will get $(100 - x)$ one period later. Such an amount is worth $\delta(100 - x)$ now — so the present discounted value of making an offer that gets rejected is $\delta(100 - x)$.

(c) *Based on your answer to (b), what can you conclude player 2 will offer in period 2?*

Answer: We determined that player 2 will get $(100 - \delta x)$ if he makes the lowest possible offer that is accepted in period 2 — while he makes $\delta(100 - x)$ if he makes an offer that gets rejected. Since $0 < \delta < 1$, the former is larger than the latter — which implies that player 2 will make an offer of $x_2 = \delta x$ in period 2, with player 1 accepting the offer.

(d) *When the game begins, player 2 can look ahead and know everything you have thus far concluded. Can you use this information to derive the lowest possible period 1 offer that will be accepted by player 2 in period 1?*

Answer: Player 2 knows that he will get $(100 - \delta x)$ in period 2 if the game continues to period 2 — and this amount is worth $\delta(100 - \delta x)$ in period 1. Thus, player 2 will accept any offer that allocates at least this amount to player 2 — i.e. any x_1 such that

$$(100 - x_1) \geq \delta(100 - \delta x) \tag{24.9.i}$$

which can also be written as

$$x_1 \leq 100 - \delta(100 - \delta x). \tag{24.9.ii}$$

(e) *What payoff will player 1 get in period 1 if she offers the amount you derived in (d)? What will she get (in present value terms) if she offers an amount higher for her (and lower for player 2)?*

Answer: If player 1 offers $x_1 = 100 - \delta(100 - \delta x)$, this is the amount she gets (because the offer will be accepted). If player 1 offers $x_1 > 100 - \delta(100 - \delta x)$ (which means player 2 gets less), then the offer is rejected and player 1 gets δx in period 2 — which is worth $\delta^2 x$ in period 1.

(f) *Based on your answer to (e), can you conclude how much player 1 offers in period 1 — and what this implies for how the game unfolds in subgame perfect equilibrium?*

Answer: Player 1 will therefore offer $x_1 = 100 - \delta(100 - \delta x)$ so long as it is greater than $\delta^2 x$ — which it is for any $\delta < 1$. Thus, player 1 offers $x_1 = 100 - \delta(100 - \delta x)$ in period 1 — and player 2 accepts. In the subgame perfect equilibrium, the game therefore ends in period 1.

(g) True or False*: The more player 1 is guaranteed to get in the third period of the game, the less will be offered to player 2 in the first period (with player 2 always accepting what is offered at the beginning of the game).*

Answer: This is true — because $x_1 = 100 - \delta(100 - \delta x) = 100(1 - \delta) + \delta^2 x$ — with x_1 increasing as x increases. Thus, the greater the guaranteed amount x for player 1 in period 3, the more player 1 will propose for herself in period 1 — with less going to player 2.

B: *Now consider an infinitely repeated version of this game; i.e. suppose that in odd-numbered periods — beginning with period 1 — player 1 gets to make an offer that player 2 can accept or reject, and in even-numbered periods the reverse is true.*

(a) True or False*: The game that begins in period 3 (assuming that period is reached) is identical to the game beginning in period 1.*

Answer: This is true — in period 3 we begin with player 1 making a proposal, just as we do in period 1, and in the game that begins in period 3, the proposals and counter-proposals continue indefinitely until someone accepts, just as is the case in the game that begins in period 1.

(b) *Suppose that, in the game beginning in period 3, it is part of an equilibrium for player 1 to offer x and player 2 to accept it at the beginning of that game. Given your answer to (a), is it also part of an equilibrium for player 1 to begin by offering x and for player 2 to accept it in the game that begins with period 1?*

Answer: Yes, this must be the case, since the two games — i.e. the one that begins in period 1 and the one that begins in period 3 are identical. Any equilibrium in one game must therefore also be an equilibrium in the other.

(c) *In part A of the exercise, you should have concluded that — when the game was set to artificially end in period 3 with payoffs x and* $(100 - x)$*, player 1 ends up offering* $x_1 = 100-\delta(100-\delta x)$ *in period 1, with player 2 accepting. How is our infinitely repeated game similar to what we analyzed in part A when we suppose, in the infinitely repeated game beginning in period 3, the equilibrium has player 1 offer x and player 2 accepting the offer?*

Answer: If the players know that, in the game that begins in period 3, player 1 will offer x (to herself) and player 2 will accept (leaving her with $(100 - x)$), then, when viewed from period 1, it is just as if period 3 simply gives x to player 1 and $(100 - x)$ to player 2 if we end up reaching that period.

(d) *Given your answers above, why must it be the case that* $x = 100 - \delta(100 - \delta x)$*?*

Answer: In part A, we concluded that it is subgame perfect for player 1 to offer $x_1 = 100 - \delta(100 - \delta x)$ in period 1 (and for player to to accept in period 1) when the guaranteed period 3 payoffs are x and $(100 - x)$. If x and $(100-x)$ is the allocation that happens in the game that begins in period 3, the same must therefore hold in period 1 for the infinitely repeated game. But we also concluded that, if x and $(100-x)$ is the equilibrium allocation in the game that begins in period 3, it must also be the equilibrium allocation in the game that begins in period 1 — i.e. it must be that $x_1 = x$. Combining this with our conclusion that $x_1 = 100 - \delta(100 - \delta x)$, we can conclude that $x = 100 - \delta(100 - \delta x)$.

(e) *Use this insight to derive how much player 1 offers in period 1 of the infinitely repeated game. Will player 2 accept?*

Answer: We then have to solve $x = 100 - \delta(100 - \delta x)$ for x. The equation can also be written as $x = 100(1-\delta) + \delta^2 x$ or, collecting the x terms on the left hand side, as

$$(1 - \delta^2)x = 100(1 - \delta). \quad (24.9.iii)$$

Since $(1 - \delta^2) = (1 + \delta)(1 - \delta)$, this reduces to $x = 100/(1 + \delta)$ — implying that the \$100 is split into

$$x = \frac{100}{1-\delta} \text{ for player 1, and } (100 - x) = \frac{100\delta}{1-\delta} \text{ for player 2.} \quad (24.9.iv)$$

Player 2 will accept this offer.

(f) *Does the first mover have an advantage in this infinitely repeated bargaining game? If so, why do you think this is the case?*

Answer: Yes, the first player has an advantage because she gets $100/(1-\delta)$ compared to $(100\delta/(1 - \delta))$ for the second player. For any $0 < \delta < 1$, the former is larger than the latter. Thus, the first mover advantage derives from the fact that our players are impatient — that they do not value a dollar in the future the same as a dollar now. Player 1 can exploit this and therefore offer player 2 less than half of the \$100 and still get him to accept — because a little less now is better than a little more in the future.

Exercise 24.11

Business Application: *Monopoly and Price Discrimination*: *In Chapter 23, we discussed first, second and third degree price discrimination by a monopolist. Such pricing decisions are strategic choices that can be modeled using game theory — which we proceed to do here. Assume throughout that the monopolist can keep consumers who buy at low prices from selling to those who are offered high prices.*

A: *Suppose a monopolist faces two types of consumers — a high demand consumer and a low demand consumer. Suppose further that the monopolist can tell which consumer has low demand and which has high demand; i.e. the consumer types are observable to the monopolist.*

(a) *Can you model the pricing decisions by the monopolist as a set of sequential games with different consumer types?*

Answer: In this case, the monopolist essentially plays a different game with each consumer type (since he can tell the consumer type before the interaction begins). Thus, for each consumer type, we have a sequential game in which the monopolist makes a price/quantity offer to the consumer and the consumer decides to accept or reject it.

(b) *Suppose the monopolist can construct any set of two-part tariffs — i.e. a per-unit price plus fixed fee for different packages. What is the subgame perfect equilibrium of your games?*

Answer: In each game with a consumer type, the monopolist will make an offer that maximizes his surplus — which results in a quantity where per-unit price is equal to marginal cost and the fixed fee is equal to the consumer surplus that the consumer type would get if we were not to charge him a fixed fee. Each consumer type will then accept an offer if it results in at least zero consumer surplus and reject offers that result in negative consumer surplus. The result is that the monopolist engages in first degree price discrimination that results in consumers getting zero consumer surplus.

(c) True or False*: First degree price discrimination emerges in the subgame perfect equilibrium but not in other Nash equilibria of the game.*

Answer: This is true. We already demonstrated the first part of the statement in the previous exercise. The second part arises from the possibility of consumers engaging in non-credible threats that are taken seriously by monopolists in the first stage of the game. Such a threat might, for instance, involve a strategy by a consumer that goes as follows: “I will accept any offer that gets me at least x consumer surplus, and I will reject all other offers.” Given that the consumer plays this strategy, the monopolist’s best response is to offer a package with consumer surplus x — and given that this is the monopolist’s strategy, the consumer’s strategy is a best response. But it involves the non-credible threat that packages with positive consumer surplus will be rejected.

(d) *How is this analysis similar to the game in exercise 24.5?*

Answer: It is similar in the sense that the monopolist gets to make an offer of how the overall surplus is to be split between the consumer and the firm — and subgame perfection suggests that this structure of the game allows the monopolist to capture the entire surplus (just as the person who splits the $100 in exercise 24.5 offers the second player zero in expectation that the offer will be accepted.)

(e) *Next, suppose that the monopolist cannot charge a fixed fee but only a per-unit price — but he can set different per-unit prices for different consumer types. What is the subgame perfect equilibrium of your games now?*

Answer: The subgame perfect equilibrium now is what we called third degree price discrimination in Chapter 23.

B: *Next, suppose that the monopolist is unable to observe the consumer type but knows that a fraction ρ in the population are low demand types and a fraction $(1-\rho)$ are high demand types. Assume that firms can offer any set of price/quantity combinations.*

(a) *Can you model the price setting decision by the monopolist as a game of incomplete information?*

Answer: Yes. In the first stage, Nature moves and assigns low demand type to the consumer with probability ρ and high demand type with probability $(1-\rho)$. The firm does not know what nature did — i.e. it begins at an information set that contains two nodes, one for each of the two possible moves by Nature. From this information set, the firm has to determine what price/quantity combinations to offer. Then, in the third stage, consumers (who know their type) choose one of the offers or no offer at all.

(b) *What is the (subgame) perfect Bayesian equilibrium of this game in the context of concepts discussed in Chapter 23? Explain.*

Answer: The equilibrium outcome is second-degree price discrimination in which the monopolist structures the two price/quantity offers in such a way as to get the consumer types to self-identify; and he will reduce the attractiveness of the low demand package sufficiently in order to maximize the overall profit he gets from the two consumer types.

Exercise 24.13

Policy Application: *Negotiating with Pirates, Terrorists (and Children)*: *While we often think of Pirates as a thing of the past, piracy in international waters has been on the rise. Typically, pirates seize a commercial vessel and then demand a monetary ransom to let go of the ship. This is similar to some forms of terrorism where, for instance, terrorists kidnap citizens of a country with which the terrorists have a grievance — and then demand some action by the country in exchange for the hostages.*

A: *Oftentimes, countries have an explicit policy that "we do not negotiate with terrorists" — but still we often discover after the fact that a country (or a company that owns a shipping vessel) paid a ransom or took some other action demanded by terrorists in order to resolve the crisis.*

(a) *Suppose the ships of many countries are targeted by pirates. In every instance of piracy, a country faces the decision of whether or not to negotiate, and the more likely it is that pirates find victims amenable to negotiating a settlement, the more likely it is that they will commit more acts of piracy. Can you use the logic of the Prisoners' Dilemma to explain why so many countries negotiate even though they say they don't? (Assume Pirates cannot tell who owns a ship before they board it.)*

Answer: If no country negotiated, pirates and terrorists would have the lowest possible incentives to continue. But each country faces Prisoners' Dilemma incentives in the following sense: If the country negotiates, it reaps the benefits but the costs get spread over all countries. Thus, regardless of what other countries do, it may be a dominant strategy for each country to itself choose negotiation when faced with a pirate or terrorist threat. (There are, of course, other things at play, like large countries attempting to exercise leadership that makes it easier for others to follow, politicians facing political consequences if negotiations become public, etc.).

(b) *Suppose that only a single country is targeted by terrorists. Does the Prisoner's Dilemma still apply?*

Answer: No, it does not apply. The single country will face all the costs and benefits of engaging (or not engaging) with terrorists.

(c) *If you had to guess, do you think small countries or large countries are more likely to negotiate with pirates and terrorists?*

Answer: Small countries would be more likely to negotiate with terrorists. The benefit of such negotiations is that the country will be able to get something in return for negotiating; the cost is that terrorism is more likely to pay off — which implies that the cost of the small country negotiating will be borne by all countries (with only a small fraction borne by the small country). A larger country faces more of this cost.

(d) *Children can be like terrorists — screaming insanely to get their way and implicitly suggesting that they will stop screaming if parents give in. In each instance, it is tempting to just give them what they want, but parents know that this will teach children that they can get their way by screaming, thus leading to an increased frequency of outbursts by the little terrors. If a child lives with a single parent, is there a Prisoners' dilemma?*

Answer: Just as there is no Prisoner's Dilemma in part (a), there is none here — the single parent faces all the costs and benefits of giving into the screaming child.

(e) *What if the child lives in a two-parent household? What if the child is raised in a commune where everyone takes care of everyone's children?*

Answer: The Prisoner's Dilemma problem is introduced when there are two parents — and gets worse as the number of adults in charge of the children increases (for the same reasons as in (b)).

(f) *All else being equal, where would you expect the most screaming per child: in a single-parent household, a two-parent household or in a commune?*

Answer: For reasons articulated already, we would expect the most screaming in the commune.

Exercise 24.15

Everyday, Business and Policy Application: *To Fight or not to Fight: In many situations, we are confronted with the decision of whether to challenge someone who is currently engaged in a particular activity. In personal relationships, for instance, we decide whether it is worthwhile to push our own agenda over that of a partner; in business, potential new firms have to decide whether to challenge an incumbent firm (as discussed in one of the examples in the text); and in elections, politicians have to decide whether to challenge incumbents in higher level electoral competitions.*

A: *Consider the following game that tries to model the decisions confronting both challenger and incumbent: The potential challenger moves first — choosing between staying out of the challenge, preparing for the challenge and engaging in it, or entering the challenge without much preparation. We will call these three actions O (for "out"), P (for "prepared entry") and U (for "unprepared entry"). The incumbent then has to decide whether to fight the challenge (F) or give into the challenge (G) if the challenge takes place; otherwise the game simply ends with the decision of the challenger to play O.*

(a) *Suppose that the payoffs are as follows for the five potential combinations of actions, with the first payoff indicating the payoff to the challenger and the second payoff indicating the payoff to the incumbent: (P,G) leads to (3,3); (P,F) leads to (1,1); (U,G) leads to (4,3); (U,F) leads to (0,2); and O leads to (2,4). Graph the full sequential game tree with actions and payoffs.*

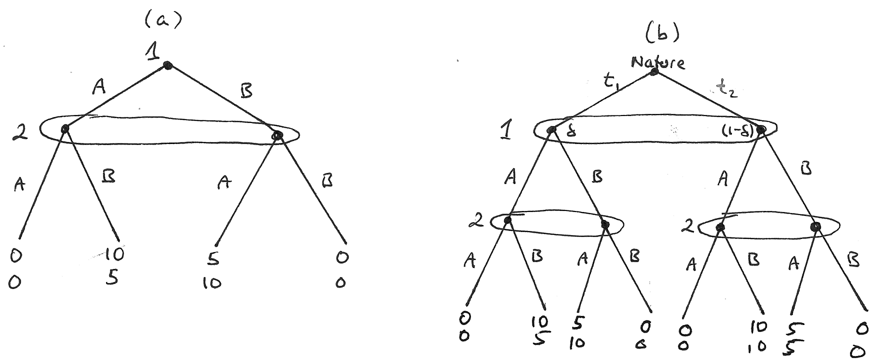
Answer: This is illustrated in Exercise Graph 24.15.

(b) *Illustrate the game using a payoff matrix (and be careful to account for all* strategies*).*

Answer: This is illustrated in the table here. The strategy pairs for the incumbent are such that the first action is the one planned if the challenger plays P and the second is the action planned if the challenger plays U. For instance, the pair (G,F) indicates a strategy that responds to a prepared challenge with G and an unprepared challenge with F.

(c) *Identify the pure strategy Nash equilibria of the game and indicate which of these is subgame perfect.*

Answer: If the challenger plays P, the incumbent best responds by playing either (G,G) or (G,F). If the incumbent plays (G,G), however, the



Exercise Graph 24.15 : The Challenge to an Incumbent

		Incumbent			
		(G,G)	(G,F)	(F,G)	(F,F)
	P	3,3	3,3	1,1	1,1
Challenger	U	4,3	0,2	4,3	0,2
	O	2,4	2,4	2,4	2,4

challenger would best respond by playing U but if the incumbent plays (G,F), the challenger best responds by playing P. Thus, P is a best responds to (G,F) and (G,F) is a best response to P — implying that $\{P, (G,F)\}$ is a Nash equilibrium. If the challenger plays U, the incumbent best responds by playing (G,G) or (F,G) — and U by the challenger is a best responds to both of these. Thus, both $\{U, (G,G)\}$ and $\{U, (F,G)\}$ are both Nash equilibria. Finally, if the challenger plays O, all four of the incumbent's strategies are best responses, but O is a best response by the challenger only to (F,F). Thus, $\{O, (F,F)\}$ is a Nash equilibrium. We therefore have identified four pure strategy Nash equilibria. But of these, only $\{U, (G,G)\}$ is subgame perfect. You can see this by solving the game tree in part (a) backwards: If the incumbent encounters a prepared challenger, his best response is to play G (thus giving him payoff of 3 rather than 1), and if he encounters an unprepared challenger, his best response is also to play G (thus getting 3 rather than 2). Knowing that the incumbent will respond by playing G to any challenge, the challenger knows he will get payoff of 3 if he plays P, 4 if he plays U and 2 if he plays O — and he will therefore choose U.

(d) *Next, suppose that the incumbent only observes whether or not the challenger is engaging in the challenge (or staying out) but does not observe whether the challenger is prepared or not. Can you use the logic of subgame perfection to predict what the equilibrium will be?*

Answer: Yes. The incumbent's best response to a challenge is the same — i.e. play G — regardless of whether or not the challenger prepared.

Thus, the challenger knows that the incumbent will play G if challenged, and thus he knows he can get his highest payoff of 4 by playing U. The strategies in this case would then be for the challenger to play U and the incumbent to play G when facing a challenge.

(e) *Next, suppose that the payoffs for (P,G) changed to (3,2), the payoffs for (U,G) changed to $(4,2)$ and the payoffs for (U,F) changed to (0,3) (with the other two payoff pairs remaining the same). Assuming again that the incumbent fully observes both whether he is being challenged and whether the challenger is prepared, what is the subgame perfect equilibrium?*

Answer: Solving the game tree backwards again, we can consider what the incumbent's best response to P and U are. If he encounters a prepared challenger, he will do best by playing G (thus getting a payoff of 2 rather than 1), but if he encounters an unprepared challenger, he will do best playing F (and getting a payoff of 3 rather than 2). Thus, the subgame perfect strategy by the incumbent — i.e. the strategy that employs to non-credible threats — is to play (G,F). The challenger knows this — and thus expects a payoff of 3 from P, a payoff of 0 from U and a payoff of 2 from O. He will therefore play P — leading to the subgame perfect equilibrium $\{P, (G,F)\}$ where the challenger issues a prepared challenge and the incumbent gives in.

(f) *Can you still use the logic of subgame perfection to arrive at a prediction of what the equilibrium will be if the incumbent cannot tell whether the challenger is prepared or not as you did in part (d)?*

Answer: No, it is no longer as clear. The reason it was clear in (d) derived from the fact that the challenger knew that the incumbent would always give in — regardless of whether he believed the challenger to be prepared or not. But now the best response by the incumbent depends on whether the incumbent believes the challenger to be prepared or not. Without knowing what beliefs the incumbent operates with as he interprets a challenge, we cannot apply the logic of subgame perfection. (For this reason, we need to move to the material of section B of the chapter where beliefs play a role in the formation of an equilibrium.)

B: *Consider the game you ended with in part A(f).*

(a) *Suppose that the incumbent believes that a challenger who issues a challenge is prepared with probability δ and not prepared with probability $(1-\delta)$. What is the incumbent's expected payoff from playing G? What is his expected payoff from playing F?*

Answer: If he plays G, he will get a payoff of 2 with probability δ when the challenger is prepared and a payoff of 2 with probability $(1-\delta)$ when the challenger is unprepared. Thus, his expected payoff from G is 2. If he plays F, he gets a payoff of 1 with probability δ and a payoff of 3 with probability $(1-\delta)$. Thus, the expected payoff from F is $\delta+3(1-\delta) = (3-2\delta)$.

(b) *For what range of δ is it a best response for the incumbent to play G? For what range is it a best response to play F?*

Answer: The incumbent best responds by playing G if $2 \geq (3-\delta)$ — i.e. if $\delta \geq 1/2$. He best responds by playing F if $\delta \leq 1/2$.

(c) *What combinations of strategies and (incumbent) beliefs constitute a pure strategy subgame perfect equilibrium? (Be careful: In equilibrium, it should not be the case that the incumbent's beliefs are inconsistent with the strategy played by the challenger!)*

Answer: If $\delta \leq 1/2$, the incumbent will play F when encountering a challenge — implying that the challenger will get payoff of 1 by playing P, 0 by playing U and 2 by playing O. Thus, the challenger plays O. Given that the challenger never challenges, any beliefs about whether the challenger prepared or not when he issues a challenge are consistent with the strategies played — including the beliefs $\delta \leq 1/2$. The following is then a set of pure strategy Bayesian Nash equilibria: players play the strategies $\{O, F\}$ and incumbents believe challengers who challenge are prepared with probability $\delta \leq 1/2$. The incumbent will fight when challenged but never is challenged in equilibrium.

If the incumbent has beliefs $\delta \geq 1/2$, on the other hand, he will play G in response to a challenge. The challenger then expects a payoff of 3 from playing P, 4 from laying U and 2 from playing O. He will therefore choose U. This might suggest equilibrium $\{U, G\}$ with beliefs $\delta \geq 1/2$. The only problem is that the beliefs are now not consistent with the strategies that are played — because the challenge plays U with probability 1 while the incumbent believes the challenger to have played U with only probability $(1-\delta) \leq 1/2$ when a challenge is observed. Thus, we do not have pure strategy equilibria with beliefs $\delta < 1/2$.

(d) *Next, suppose that the payoffs for (P,G) changed to (4,2) and the payoffs for (U,G) changed to (3,2) (with the remaining payoff pairs remaining as they were in A(f)). Do you get the same pure strategy subgame perfect equilibria?*

Answer: Yes, the same equilibria still hold. Nothing has changed in the payoffs for the incumbent — who will therefore still play F if $\delta \leq 1/2$. If the incumbent fights any challenge, the challenger is still better off playing O — and given he does not challenge, the beliefs of $\delta \leq 1/2$ are not inconsistent with the strategies that are being played.

(e) *In which equilibrium — the one in part (c) or the one in part (d) — do the equilibrium beliefs of the incumbent seem more plausible?*

Answer: Consider first the game in (c): If the challenger challenges here, he is better off preparing if he expects F and not preparing if he expects G. Next, consider the game in (d): In this version, if the challenger challenges, he is better off preparing *regardless* of whether the incumbent gives in or fights. In the latter case, it would seem reasonable that the incumbent then expects that any challenge is one by a prepared challenger

— but the equilibrium beliefs we have derived require the incumbent to believe that a challenger is prepared with only probability $\delta \leq 1/2$. That seems unreasonable in a way that the same beliefs are not unreasonable in the game in (c) where there is no clear dominant choice between P and U for the challenger.

Exercise 24.17

Policy Application *Some Prisoners' Dilemmas*: *We mentioned in this chapter that the incentives of the prisoners' dilemma appear frequently in real world situations.*

A: *In each of the following, explain how these are prisoners' dilemmas and suggest a potential solution that might address the incentive problems identified in such games.*

(a) *When I teach the topic of prisoners' dilemmas in large classes that also meet in smaller sections once a week, I sometimes offer the following extra credit exercise: Every student is given 10 points. Each student then has to decide how many of these points to donate to a "section account" and convey this to me privately. Each student's payoff is a number of extra credit points equal to the number of points they did* not *donate to their section* plus *twice the average contribution to the section account by students registered in their section. For instance, if a student donates 4 points to his section and the average student in the section donated 3 points, then this student's payoff would be 12 extra credit points — 6 because the student only donated 4 of his 10 points, and 6 because he gets twice the average donated in his section.*

Answer: This is a prisoners' dilemma because — although it is a dominant strategy for everyone to keep all their points (and not contribute to the section account), everyone would be better off if everyone contributed all their points. (In the case where everyone holds onto their points, each student in the section gets 10 extra credit points; in the case where everyone donates all points, the average donation is 10 and thus all students in the section would get 20 extra credit points; but no matter what everyone else in the section does, it is better for an individual student to not contribute his own points so long as there are more than 2 students in the class.) This is the classic prisoners' dilemma. And students find all sorts of creative ways out of it. One student once designed a program through which students would send their e-mails to me, with the program set so that it would submit all points from each student so long as all students agreed — but would send no points otherwise. This changed the incentives in the game — because now students knew that the would get 10 for sure if they submitted anything other than all their points — but they had a chance at 20 points if they submitted all their points. Other students organized ways of checking each other's e-mails before they were sent. When given the option of voting for a 100% tax that would transfer all points in the section to the section account, most students voted for this — essentially hiring the government to force them to do what they otherwise knew they would have trouble getting done.

(b) *People get in their cars without thinking about the impact they have on other drivers by getting on the road — and at certain predictable times, this results in congestion problems on roads.*

Answer: Everyone would be better off if all drivers would take into account the costs they impose on others when getting on the road — but it is in no one's interest to do so. Put differently, everyone would like better an outcome where we cooperated but it is in everyone's incentive not to cooperate no matter what others do — the classic prisoners' dilemma. This is the same for all externalities. One possible solution is to have congestion pricing on roads — i.e. fees that increase during times when there is otherwise much congestions. This changes the payoffs of the prisoners' dilemma, with drivers essentially hiring the government to change those payoffs. Alternatively, private road-owners might charge differential tolls that achieve a similar outcome.

(c) *Everyone in your neighborhood would love to see some really great neighborhood fireworks on the next national independence day — but somehow no fireworks ever happen in your neighborhood.*

Answer: This is another externality problem that creates a prisoners' dilemma — we all would be better off if we contributed to such fireworks, but it is in each of our individual interests not to contribute regardless of what everyone else does. We might solve this by hiring a local home owner's association to pay for the fireworks with mandatory dues paid by all homeowners in the neighborhood; or someone who is a great social entrepreneur might make everyone feel guilty enough to overcome the prisoners' dilemma incentives.

(d) *People like downloading pirated music for free but would like to have artists continue to produce lots of great music.*

Answer: In order for lots of great music to be produced, it might be necessary for artists to be able to benefit financially from their work. But pirating music circumvents this — and everyone has an incentive to do this regardless of whether others are also doing it. Put differently, we might all be better off by not having any of us pirate music, but we are individually better off by getting what we can for free regardless of how much everyone else is doing this. We might overcome this through strict enforcement of copyright laws, or there might be public campaigns to make people feel guilty for pirating music — thus changing the payoffs in the prisoners' dilemma.

(e) *Small business owners would like to keep their businesses open during "business hours" and not on evenings and weekends. In some countries, they have successfully lobbied the government to force them to close in the evenings and weekends. (Laws that restrict business activities on Sunday are sometimes called* blue laws.*)*

Answer: Each business owner has an incentive to remain open to attract customers — regardless of whether other businesses stay open. (If they don't, then I get all the customers by staying open; if they do, I lose customers because I don't offer hours that are as convenient.) But if all businesses close at the same time, no one loses any customers. So the incentives are precisely those of the prisoners' dilemma — and forcing everyone to close at the same time enforces cooperation that could not otherwise be sustained.

B: *In Chapter 21, we introduced the* Coase Theorem*, and we mentioned in Section 21A.4.4 the example of bee keeping on apple orchards. Apple trees, it turns out, don't produce much honey (when frequented by bees), but bees are essential for cross-polination.*

(a) *In an area with lots apple orchards, each owner of an orchard has to insure that there are sufficient numbers of bees to visit the trees and do the necessary cross-polination. But bees cannot easily be kept to just one orchard — which implies that an orchard owner who maintains a bee hive is also providing some cross-polination services to neighboring orchards. In what sense to orchard owners face a prisoners' dilemma?*

Answer: This is another externality problem resulting in prisoners' dilemma incentives. Each orchard owner can get away with providing fewer bee hives than is optimal because the bees from neighboring orchards will come by — and the fact that individual orchard owners do not take into account the benefits of each bee hive for other orchard owners, too few bee hives will be set up and everyone agrees that everyone would be better off if all were forced to have more bee hives.

(b) *How does the Coase Theorem suggest that orchard owners will deal with this problem?*

Answer: The Coase Theorem suggests that, when property rights are well specified, orchard owners will find a way to contract with each other to find the efficient solution.

(c) *We mentioned in Chapter 21 that some have documented a "custom of the orchards" — an implicit understanding among orchard owners that each will employ the same number of bee hives per acre as the other owners in the area. How might such a custom be an equilibrium outcome in a repeated game with indefinite end?*

Answer: Suppose each orchard owner believes that they will be in the same vicinity as other orchard owners in the area with some probability again and again — i.e. there is no definitive end to the relationship. We have shown that there is no dominant strategy in this case. So suppose all orachard owners adopt the "custom of the orchards" so long as all others did so the previous year but otherwise switch to the non-cooperative number of bee hives on their land. If the probability of future interactions is sufficiently large and orchard owners do not discount the future too much, then each orchard owner is best responding to every other orchard owner by playing this strategy — and the custom of the orchards is observed in equilibrium. This is similar to the tit-for-tat strategy leading to cooperation in repeated prisoners' dilemmas that have no clear end — so long as it is sufficiently likely that future interactions occur and so long as the future is not heavily discounted. The increased payoff from future cooperation is larger in expected value terms than the gains one could make by violating the custom of the orchards this year.

Conclusion: Potentially Helpful Reminders

1. As with all the tools we have introduced, there is no way to really learn them without applying them. For helpful hints on how to approach finding Nash and subgame-perfect equilibria, see Section 24A.2.6 in the text — and then practice!

2. Subgame perfection should become a very intuitive idea — and the reason for why we solve sequential games "from the bottom up" should make sense. Applying this idea to repeated simultaneous games is then relatively straightforward. (It should, for instance, become quite intuitive why the subgame perfect equilibrium in the finitely repeated Prisoner's Dilemma is for everyone to never cooperate — no matter how often the players meet (as long as the number of meetings is finite).

3. For mixed strategy equilibria, practice thinking about best response functions of the kind introduced in Section 24A.4.2. (I suspect that many instructors will focus primarily on pure strategy equilibria — so feel free to skip the section on mixed strategies if it is not emphasized in your class.)

4. Often, as you practice solving for game theory equilibria of all kinds (and particularly perfect Bayesian equilibria), you will develop your own shortcuts and your own way of approaching the problems. If you find approaches that work, use them — and check your answers to make sure you are capturing what you need to.

5. Remember that a Nash equilibrium (of any kind) is a list of strategies — one for each player — and *not* a list of actions that are played in equilibrium. For Bayesian equilibria, you also need to specify beliefs that support the equilibrium strategies.

6. The Prisoner's Dilemma is a particularly important game in the social sciences. Often students think that players will not cooperate because they think no one else will cooperate. This is not true. Players in the game do not cooperate because it is a *dominant strategy* not to cooperate. In other words, *regardless* of what other strategies are played by other players, it is payoff maximizing for any one player to not cooperate. Thus, even if you thought everyone else was cooperating, it would still be in your interest to not cooperate.

CHAPTER

25

Oligopoly

Prior to Chapter 23, we always dealt with competitive markets, and in Chapter 23, we introduced an extreme version of market power in the form of a monopolist who produces a good without close substitutes and is protected by high barriers to entry. We now merge these two ideas — the ideas of competition and market power — in our analysis of oligopolies. An oligopoly, like a monopoly, is protected by barriers to entry and by the fact that there are no close substitutes to the output that is being produced. The (small number of) firms within an oligopoly, however, compete with one another. The question we confront in this chapter is how this competition between firms in an oligopoly affects the market power of the oligopoly, and we will see that the answer depends on what form the competition takes. Regardless, however, there is always an incentive for firms in an oligopoly to collude with one another and behave as if the oligopoly was a monopoly — but Prisoner's Dilemma incentives for the individual firms often make such collusion difficult to sustain.

Chapter Highlights

The main points of the chapter are:

1. When firms in an oligopoly compete on price, we call this **Bertrand competition**; when they instead compete on quantity (and allow the price to form), we call it **Cournot competition**.

2. The **strategic variable** used in oligopoly competition turns out to matter a great deal, with Bertrand competition eliminating all market power while Cournot competition allows firms to benefit from the market power of the oligopoly. (In part B of the chapter, we show mathematically that Cournot competition converges to perfect competition as the number of firms in an oligopoly gets large. Bertrand competition, on the other hand, converges to perfect competition as soon as the number of firms jumps from 1 to 2.)

3. Another way to differentiate between different forms of oligopoly competition involves asking whether some firms announce their strategic variable

before others have to. If one firm moves before the other, the first is called the *Stackelberg leader* and the second is called the *Stackelberg follower*, with this type of competition referred to as **Stackelberg competition**.

4. In environments of simultaneous competition (like Bertrand or Cournot competition), we use the concept of Nash equilibrium to predict behavior; and in environments of sequential competition (like Stackelberg competition), we use the concept of subgame perfect (Nash) equilibrium.

5. Another feature of sequential settings involves **entry deterrence** where an *incumbent firm* that has a monopoly has to decide whether to strategically deviate from monopoly behavior in order to deter entry by a competitor. Whether this is necessary or effective depends on the size of the fixed entry costs for potential competitors.

6. **Cartels** are collusive agreements between firms in an oligopoly — agreements that typically involve setting production quotas for each firm such that the oligopoly can raise price closer to the monopoly level. Fortunately (for consumers) and unfortunately (for firms), each firm within an oligopoly has an incentive to cheat on cartel agreements — thus pushing price down.

25A Solutions to Within-Chapter-Exercises for Part A

Exercise 25A.1

Can you see how this is the only possible Nash equilibrium? Is it a dominant strategy Nash equilibrium?

Answer: We already argued in the text that, for any price I set above the MC, your best response is to charge below my price and my best response to that is to charge below your price until we get to $p = MC$. Similarly, no price below MC is ever a best response. Thus, the equilibrium is unique. But it is not an equilibrium with dominant strategies because setting $p = MC$ is *not* my best response to *any* price that you might set.

Exercise 25A.2

Is there a single Nash equilibrium if more than two firms engage in Bertrand competition within an oligopoly?

Answer: The same logic applies if there are more than two firms — which again implies that the only Nash equilibrium is one where we all announce $p = MC$.

Exercise 25A.3

How would you think about subgame perfect equilibria under sequential Bertrand competition with 3 firms (where firm 1 moves first, firm 2 moves second and firm 3 moves third)?

Answer: We would want to solve for the subgame perfect equilibrium "from the bottom up" — i.e. we would start by considering firm 3 which moves last. The best response for firm 3 to any set of prices announced by firms 1 and 2 is to set a price just below the lowest price that has already been announced — unless that lowest price is at or below MC (in which case it is a best response to simply announce $p = MC$.) Firm 2 knows this when setting its price — and so it knows that it will sell no output unless it announced $p = MC$. Firm 1 knows all of this, and thus also sets $p = MC$.

Exercise 25A.4

Suppose our two firms know that we will encounter each other n times and never again thereafter. Can $p > MC$ still be part of a subgame perfect equilibrium in this case assuming we engage in pure price competition?

Answer: If there is a definitive end to the game after n periods, we can apply the concept of subgame perfection by again solving "from the bottom up" — i.e. beginning in period n. In that period, the firms know they'll never see each other again and they are thus playing a one-shot Bertrand price competition game. We know that the only equilibrium to this game is for both firms to set $p = MC$. We can then think about period $n-1$ — the second to last period. Given that we know that we will both play $p = MC$ in the last period, there is nothing we can do in period $n-1$ that will change this. Thus, period n becomes irrelevant — and we once again will play as if this is the only time we see each other, implying that we will again set $p = MC$. The same holds true as we move up the game tree — implying that any potential for cooperation unravels and the only subgame perfect equilibrium is for us to both set $p = MC$ in every period. Thus, $p > MC$ cannot be sustained as in a subgame perfect equilibrium when there is a definitive end to our interactions.

Exercise 25A.5

Can you identify in panel (b) of Graph 25.2 the quantity that corresponds to the horizontal intercept of firm 2's best response function in panel (a)?

Answer: As long as the quantity $\overline{x}_1$ that I set leaves price above MC, there is room for you to produce some quantity and make profit. Thus, you will cease to produce only when I produce an amount sufficient to drive price down to MC — which occurs at quantity x^* in panel (b) of the graph.

Exercise 25A.6

What is the slope of the best response function in panel (a) of Graph 25.2? (*Hint*: Use your answer to exercise 25A.5 to arrive at your answer here.)

Answer: We concluded in exercise 25A.5 that the horizontal intercept of the best response function in panel (a) is x^*. Since x^M arises from the intersection of MR with MC while x^* arises from the intersection of D with MC in panel (b) — and since MR has twice the slope of D, it must then be the case that $x^M = (1/2)x^*$. And since x^M is the vertical intercept of the best response function in panel (a), this implies that the slope of the best response function is -1/2.

Exercise 25A.7

Which type of behavior under simultaneous decision making within an oligopoly results in greater social surplus: quantity or price competition?

Answer: Price (or Bertrand) competition results in the socially efficient output level that occurs where MC intersects demand. Quantity (or Cournot) competition results in an output level below the efficient quantity (with price above MC) — and thus yields less overall social surplus (but still more than would happen under a strict monopoly).

Exercise 25A.8

True or False: Under Bertrand competition, $x_1^B = x_2^B = x^M$.

Answer: This is true — under Bertrand competition (assuming the firms each get half the customers), the overall output is twice the monopoly output — which implies that each firm produces the monopoly output.

Exercise 25A.9

Determine the Stackelberg price in terms of p^M — the price a monopolist would charge — and MC.

Answer: In Graph 25.4b in the text, you can see that the Stackelberg price — which we can read off the blue residual demand curve D^r at output quantity x^M — lies halfway between the monopoly price p^M and the MC. Thus, the Stackelberg price p^S is

$$p^S = \frac{p^M - MC}{2} + MC = \frac{p^M + MC}{2}. \qquad (25A.9)$$

Exercise 25A.10

Where is the predicted Stackelberg outcome in Graph 25.3c?

Answer: Firm 1 produces x^M while firm 2 produces $x^M/2$. Thus, the Stackelberg outcome lies at the intersection of the dashed horizontal line that passes through point M and the dashed vertical line that passes through point B. Put differently, the Stackelberg outcome lies on the (shallow) Cournot best response function for firm 2 (which passes through point C) at the output quantity x^M for firm 1.

Exercise 25A.11

True or False: Once the entrant has paid the fixed entry cost, this cost becomes a sunk cost and is therefore irrelevant to the choice of how much to produce.

Answer: This is true. The fixed entry cost is a real economic cost only relative to the decision of whether or not to enter — once it is paid, the firm no longer considers it for output decisions because it is unaffected by any output decisions that are made.

Exercise 25A.12

Is the smallest fixed cost of entering that will prevent firm 2 from coming into the market greater in panel (a) or in panel (b)?

Answer: In panel (a), the potential entrant will in fact enter so long as $FC < \pi^C$ while in panel (b) the potential entrant will enter so long as $FC < \pi^{SF}$. So the answer boils down to whether π^C is greater or less than π^{SF}. A Stackeberg follower is at a competitive disadvantage relative to a simultaneous Cournot competitor — thus $\pi^{SF} < \pi^C$. Put differently, we know that price will be higher under Cournot competition (because overall quantity will be less) than under Stackelberg competition — and the Cournot competitor will produce *more* than the Stackelberg follower. Thus, the potential entrant gets a higher price and produces a larger quantity in panel (a) than in panel (b) if she enters. As a result, the smallest fixed cost that will keep the potential entrant from entering is greater under Cournot competition — i.e. in panel (a).

Exercise 25A.13

How might the cartel agreement have to differ if we were currently engaged in Stackelberg competition? (*Hint*: Think about how the cartel profit compares to the Stackelberg profits for both firms, and use the Stackelberg price you determined in exercise 25A.9 along the way.)

Answer: If we are currently engaged in Stackelberg competition, I (as the leader) am producing the monopoly quantity x^M while you are producing half that quantity. If we both agreed to produce only half the monopoly quantity, you would unambiguously be better off since you would not have to cut your output but you'd now be able to benefit from the higher monopoly price. As the Stackelberg leader, however, I am currently producing the monopoly quantity x^M and selling at the Stackelberg price that we derived in exercise 25A.9 as

$$p^S = \frac{p^M + MC}{2}. \tag{25A.13.i}$$

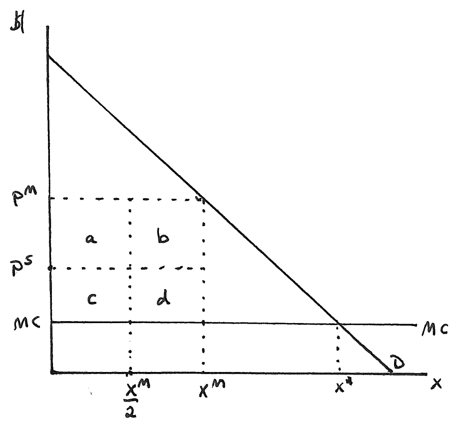
Thus, the Stackelberg profit for the leader is

$$\pi^{SL} = x^M\left(\frac{p^M + MC}{2}\right) - x^M(MC) = (p^M - MC)\left(\frac{x^M}{2}\right). \tag{25A.13.ii}$$

Under the cartel agreement, I would now produce $x^M/2$ and sell at p^M. This implies a profit of

$$\pi^{Cartel} = p^M\left(\frac{x^M}{2}\right) - MC\left(\frac{x^M}{2}\right) = (p^M - MC)\left(\frac{x^M}{2}\right), \qquad (25A.13.iii)$$

which is exactly the same profit I was already making as a Stackelberg leader.



Exercise Graph 25A.13 : Stackelberg Leader and Cartel Agreement

Thus, I don't gain anything from the cartel agreement unless we divide the monopoly quantity between in such a way that I get to produce more than $x^M/2$ and you produce less than $x^M/2$. You can see this graphically in Exercise Graph 25A.13.

In the absence of a cartel agreement, the Stackelberg leader sells x^M at p^S — which gives him profit of area $(c+d)$. Under a cartel agreement that gets both cartel members to produce $x^M/2$, the price will rise to p^M. Thus, each cartel member gets the profit area $(a+c)$. But since p^S lies halfway between p^M and MC, and since $x^M/2$ lies halfway between 0 and x^M, each of the labeled squares (i.e. a, b, c and d) has the same area. Thus, the Stackelberg leader profit $(c+d)$ is equal to the cartel member profit $(a+c)$ if the cartel agreement specifies that each firm will produce half of x^M.

Exercise 25A.14

Can you verify the last sentence by just looking at the best response functions we derived earlier in Graph 25.2?

Answer: Your best response function in panel (a) of Graph 25.2 has vertical intercept of x^M and slope of -1/2. We can therefore write it as $x_2 = x^M - 0.5x_1$. Thus, if I produce $x_1 = 0.5x^M$, this equation gives us $x_2 = x^M - 0.5(0.5x^M) = 0.75x^M$.

Exercise 25A.15

The Prisoners' Dilemma you and I face as we try to maintain a cartel agreement works toward making us worse off. How does it look from the perspective of society at large?

Answer: From the perspective of society at large, the prisoners' dilemma incentives we face as producers result in the positive tendency for the cartel agreement — which aims to restrict output — to break down and result in greater output. Since the monopoly quantity set by the cartel is inefficient, a breakdown in the agreement that results in greater output leads to greater efficiency.

Exercise 25A.16

Why would oligopolists who cannot voluntarily sustain cartel agreements want to have such agreements enforced?

Answer: They would want such an agreement enforced because the prisoners' dilemma incentives are such that each cartel member has an incentive to cheat — with the result that the agreement will break down and result in less profit for each firm unless it is enforced.

Exercise 25A.17

In circumstances where firms are not certain about demand conditions in any given period, why might a more forgiving trigger strategy (like Tit-for-Tat) that allows for the re-emergence of cooperation be better than the extreme trigger strategy that forever punishes perceived non-cooperation in one period?

Answer: The extreme punishment strategy offers the maximum deterrent to keep the other firm(s) from cheating on the cartel agreement — and thus is most likely to help sustain cooperation when it is easy to catch someone cheating. If, however, the economic environment is such that other influences in the market keep firms from knowing for sure whether anyone is cheating on the cartel agreement, this maximum deterrent runs up against the problem that mistakes might be made. In other words, firms that did not in fact cheat might be kept from cooperating forever, never permitting cooperation to re-emerge. The expected cost of such a mistake might be higher than the benefit from using a strategy with the maximum deterrent built in.

25B Solutions to Within-Chapter-Exercises for Part B

Exercise 25B.1

Verify x^M and p^M in equation (25.2).

Answer: Solving the demand function for p to get the demand curve, we get $p(x) = (A - x)/\alpha$. Total revenue is $p(x)x = (Ax - x^2)/\alpha$, which implies marginal revenue is $MR = (A - \alpha c)/2$. Setting MR equal to marginal cost c and solving for x, we get $x = (A - \alpha c)/2$, and substituting into $p(x)$, we get $p = (A + \alpha c)/2\alpha$.

Exercise 25B.2

Verify that p_1^r is in fact the correct inverse demand function.

Answer: Simply solve for x_1^r for p to get this inverse demand function (or demand curve).

Exercise 25B.3

Derive this MR function using calculus.

Answer: Total revenue is

$$TR = p_1^r x_1 = \left(\frac{A - \overline{x}_2}{\alpha}\right) x_1 - \left(\frac{1}{\alpha}\right) x_1^2. \qquad (25B.3.i)$$

Differentiating with respect to x_1 we get

$$MR_1^r = \left(\frac{A - \overline{x}_2}{\alpha}\right) - \left(\frac{2}{\alpha}\right) x_1. \qquad (25B.3.ii)$$

Exercise 25B.4

Verify that this is correct.

Answer: To calculate your equilibrium output level, we can substitute $x_1(x_2)$ into $x_2(x_1)$ and solve for x_2 — which gives us $x_2^C = (A - \alpha c)/3$.

Exercise 25B.5

Verify that these quantities are in fact the Nash equilibrium quantities; i.e. show that, given you produce this amount, it is best for me to do the same, and given that I produce this amount, it is best for you to do the same.

Answer: Suppose you produce $\overline{x}_2 = (A - \alpha c)/3$. Plugging this into my best response function $x_1(x_2)$, we get

$$x_1(\overline{x}_2) = \frac{A - [(A-\alpha c)/3] - \alpha c}{2} = \frac{A-\alpha c}{3}. \tag{25B.5}$$

Exercise 25B.6

How does the monopoly price p^M (derived in equation (25.2)) compare to the price that will emerge in the Cournot equilibrium? How does it compare to the Bertrand price?

Answer: To find the Cournot and Bertrand prices, we simply plug the overall output levels into the inverse demand function. We then get

$$p^C = p(2x_i^C) = \frac{A - 2x_i^C}{\alpha} = \frac{A - [2(A-\alpha c)/3]}{\alpha} = \frac{A + 2\alpha c}{3\alpha}, \tag{25B.6.i}$$

and

$$p^B = p(2x_i^B) = \frac{A - 2x_i^B}{\alpha} = \frac{A - [2(A-\alpha c)/2]}{\alpha} = c. \tag{25B.6.ii}$$

So long as $A > \alpha c$, we then get that $p^M > p^C > p^B$ as we would expect. (The condition that $A > \alpha c$ simply implies that demand is positive when output is priced at marginal cost.)

Exercise 25B.7

Compare this equation to equation (23.15) in our chapter on monopolies. How are they related?

Answer: The equation we derived for monopoly is

$$p\left(1 + \frac{1}{\epsilon_D}\right) = MC \tag{25B.7}$$

which is identical to the equation for oligopolies except for the additional term N. Put differently, our equation for oligopolies is equal to our equation for monopolies if we set N equal to 1 — and $N = 1$ simply means there is a single firm in the oligopoly, which makes it a monopoly.

Exercise 25B.8

Can you make a case for why the Cournot model gives intuitively more plausible predictions than the Bertrand model for oligopolies in which identical firms produce identical goods?

Answer: In what we have just shown, we illustrate how the Cournot model can fill in the gap between the extremes of monopoly and perfect competition. When $N = 1$, we have the condition we derived for monopolies (with the implicit monopoly markup related to price elasticity). As N gets large, the condition converges to $p = MC$ — the result we expect under perfect competition. This differs from the stark Bertrand prediction of an immediate jump from the monopoly solution to the perfectly competitive solution as N goes from 1 to 2.

Exercise 25B.9

Verify that this is the correct inverse residual demand function for me.

Answer: Solving for p, we get

$$p_1 = \frac{A - x_1}{\alpha} - \frac{A - x_1 - \alpha c}{2\alpha} = \frac{A + \alpha c}{2\alpha} - \frac{1}{2\alpha} x_1. \qquad (25B.9)$$

Exercise 25B.10

In Graph 25.4b, the residual demand curve has a kink at the level of MC. Verify that the function we derived above in fact meets the market demand curve at $p = MC$. How would you fully characterize the residual demand curve mathematically (taking into account the fact that it is kinked)?

Answer: Letting $p_1^r = c$ in equation (25B.9) and solving for x, we get $x = A - \alpha c$ — which is the level of x we get from the regular demand function when $p = c$. Thus, the residual demand function meets the market demand function at $p = MC$. To full specify the residual demand curve, we would then write

$$\begin{aligned} p_1^r &= \frac{A + \alpha c}{2\alpha} - \frac{1}{2\alpha} x_1 \;\; \text{for} \;\; x \leq A - \alpha c \\ &= \frac{A - x}{\alpha} \;\; \text{for} \;\; x > A - \alpha c. \end{aligned} \qquad (25B.10)$$

Exercise 25B.11

How does the overall level of Stackelberg output relate to the monopoly quantity and the Cournot quantity? What is more efficient in this setting (from society's vantage point): Cournot or Stackelberg competition?

Answer: The overall Stackelberg quantity is

$$x^S = x_1^{SL} + x_2^{SF} = \frac{A - \alpha c}{2} + \frac{A - \alpha c}{4} = \frac{3}{4}(A - \alpha c), \qquad (25B.11.i)$$

compared to the overall Cournot quantity

$$x^C = x_1^C + x_2^C = \frac{A - \alpha c}{3} + \frac{A - \alpha c}{3} = \frac{2}{3}(A - \alpha c). \qquad (25B.11.ii)$$

Thus, $x^S > x^C$ — which means Stackelberg competition leads to an output level closer to the efficient (Bertrand) level (of $x^B = (A - \alpha c)$) than Cournot competition. Stackelberg competition is therefore more efficient.

Exercise 25B.12

What will be the output price under Stackelberg competition, and how does this relate to the Cournot and monopoly prices?

Answer: The output price under Stackelberg competition is then

$$p^S = \frac{A - x^S}{\alpha} = \frac{A + 3\alpha c}{4\alpha} \tag{25B.12.i}$$

as opposed to the Cournot and monopoly prices that we previously calculated as

$$p^C = \frac{A + 2\alpha c}{3\alpha} \text{ and } p^M = \frac{A + \alpha c}{2\alpha}. \tag{25B.12.ii}$$

As long as $A > \alpha c$ (i.e. as long as demand is positive when output is priced at marginal cost), we then get

$$p^M > p^C > p^S > p^B. \tag{25B.12.iii}$$

Exercise 25B.13

Can you draw a graph analogous to Graph 25.3c, indicating the monopoly outcome (assuming the two firms would split the monopoly output level), the Cournot outcome, the Stackelberg outcome and the Bertrand outcome? Carefully label all the points.

Answer: This is done in Exercise Graph 25B.13.

Exercise 25B.14

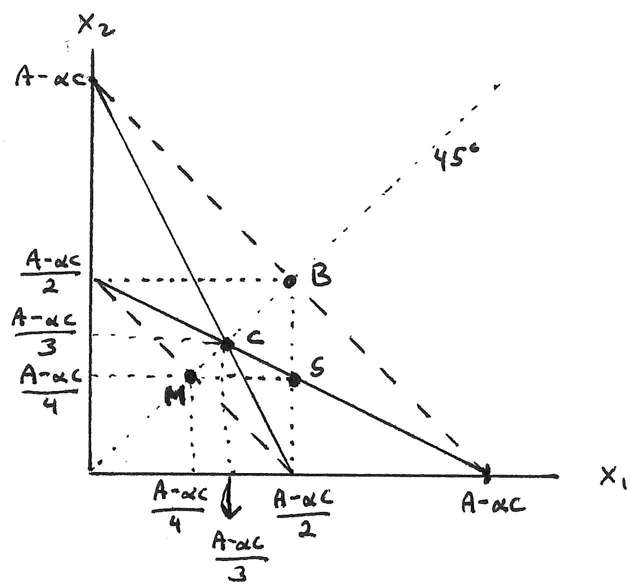
Show that the first order condition for firm 1 approaches an expression similar to the first order condition for each of the firm 2 types as firm 1's uncertainty diminishes; i.e. as ρ approaches zero or 1.

Answer: Note simply that x_1 approaches $(A - \alpha c - x_2^H)/2$ as ρ approaches 1 and $(A - \alpha c - x_2^L)/2$ as ρ approaches 0.

Exercise 25B.15

Verify the last equation.

Answer: Substituting x_1^* into $x_2^L = (A - x_1 - \alpha c^L)/2$, we get



Exercise Graph 25B.13 : Comparing Monopoly, Cournot, Stackelberg and Bertrand

$$x_2^L = \frac{A-\alpha c^L}{2} - \frac{A-2\alpha c+\alpha(\rho c^H+(1-\rho)c^L)}{6} =$$
$$= \frac{3A-3\alpha c^L - A + 2\alpha c - \alpha(\rho c^H+(1-\rho)c^L)}{6} =$$
$$= \frac{2A+2\alpha c-4\alpha c^L+\alpha\rho c^L-\alpha\rho c^H}{6}$$
$$= \frac{A+\alpha c-2\alpha c^L}{3} - \frac{\alpha\rho}{6}(c^H-c^L). \tag{25B.15}$$

Exercise 25B.16

Can you tell whether the Cournot price will be higher or lower under this type of asymmetric information than it would be under complete information? (*Hint*: For both the case of a high cost and a low cost type, can you see if overall production is higher or lower in the absence of asymmetric information?)

Answer: Suppose first that firm 2 has high costs. By comparing x_1^* with $\rho < 1$ to x_1^* with $\rho = 1$ (where there is no asymmetric information in that firm 1 knows firm 2 has high costs), we see that firm 1 produces $(1/3)\alpha(1-\rho)(c^H - c^L)$ more when $\rho = 1$. We know from our expression of x_2^{H*} that firm 2 will produce $(1/6)\alpha(1-\rho)(c^H - c^L)$ less when $\rho = 1$. Thus, overall output is $(1/6)\alpha(1-\rho)(c^H - c^L)$ *higher* under complete information if firm 2 has high costs — implying a lower price in the absence of asymmetric information.

Next, suppose that firm 2 has low costs. By comparing x_1^* with $\rho > 0$ to x_1^* with $\rho = 0$ (where there is no asymmetric information in that firm 1 knows firm 2 has low costs), we see that firm 1 produces $(1/3)\alpha\rho(c^H - c^L)$ less when $\rho = 0$. We know from our expression of x_2^{L*} that firm 2 will produce $(1/6)\alpha\rho(c^H - c^L)$ more when $\rho = 0$. Thus, overall output is $(1/6)\alpha\rho(c^H - c^L)$ *lower* under complete information if firm 2 has low costs — implying a higher price in the absence of asymmetric information.

We conclude that price will be higher with asymmetric information if firm 2 has high costs and lower if firm 2 has low costs.

Exercise 25B.17

Suppose the two firms engage in price (Bertrand) competition, and suppose $c > c^H$. What price do you expect will emerge?

Answer: We know that price will be equal for the two firms under Bertrand competition. If price were above c, both high and low cost firms could do better by charging slightly less than c and thus getting the entire market. Thus, price cannot be above c. It cannot be c either — because both cost types can still do better by slightly underbidding in this case. Thus, price will be just below c, with firm 1 producing nothing and firm 2 producing above its marginal cost.

Exercise 25B.18

Suppose again the two firms engage in price (Bertrand) rather than quantity competition, and suppose $c^L < c < c^H$. This case is easier to analyze if we assume sequential Bertrand competition — with firm 1 setting its price first and firm 2 setting it after it observes p_1 (and after it finds out its cost type). What equilibrium prices would you expect? Does your answer change with ρ?

Answer: First, note that firm 1 will never set p_1 below its marginal cost c. Firm 1 does not know firm 2's cost type — but firm 1 can predict what each cost type will do once p_1 is observed. In particular, a low cost firm 2 will set p_2 just below p_1 so long as $p_1 > c$, and a high cost firm 2 will set p_2 just below p_1 so long as $p_1 > c^H$. Thus, if firm 1 sets $p_1 > c^H$, it will never sell anything; if it sets $p_1 = c^H$ is will sell nothing if firm 2 has low costs and split the market if firm 2 has high costs; and if it sets $p_1 \geq c$ but below c^H, it will get the whole market if firm 2 is high cost and none of the market if firm 2 has low cost. Thus, if firm 1 sets its price just below c^H, it will get the whole market with probability ρ and there is no way it will ever get more of the market. Thus, firm 1 sets p_1 just below c^H, and firm 2 ends up setting p_2 at c^H if it is a high cost type and p_2 just below p_1 if its a low cost type. This is a subgame perfect equilibrium because firm 2's low cost strategy is to always set its price just below p_1 (for any $p_1 \geq c$ contemplated by firm 1) — and firm 1 therefore give up on getting any market share in the event that firm 2 is a low cost firm. This leaves it only able to maximize its market share and profit when firm 2 is high cost. The answer does not change with ρ.

Exercise 25B.19

Verify that this equation is correct.
Answer: We get

$$
\begin{aligned}
\pi_2 &= \left(\frac{A+3\alpha c}{4\alpha}\right)\left(\frac{A-\alpha c}{4}\right) - c\left(\frac{A-\alpha c}{4}\right) - FC \\
&= \left(\frac{A+3\alpha c}{4\alpha} - c\right)\left(\frac{A-\alpha c}{4}\right) - FC \\
&= \left(\frac{A+3\alpha c}{4\alpha} - \frac{4\alpha c}{4\alpha}\right)\left(\frac{A-\alpha c}{4}\right) - FC \\
&= \left(\frac{A-\alpha c}{4\alpha}\right)^2 - FC \\
&= \frac{(A-\alpha c)^2}{16\alpha^2} - FC
\end{aligned} \tag{25B.19}
$$

Exercise 25B.20

Verify that this derivation of $p(x_1)$ is correct.
Answer: The inverse demand function is $p(x) = (A-x)/\alpha$, and, if firm 1 produces x_1 and firm 2 enters, the total output will be $x = x_1 + x_2(x_1)$ where $x_2(x_1)$ is firm 2's best response function is

$$x_2(x_1) = \frac{A - x_1 - \alpha c}{2}. \tag{25B.20.i}$$

We then get

$$p(x_1) = \frac{A - x_1 - x_2(x_1)}{\alpha} = \frac{A - x_1 - \frac{A - x_1 - \alpha c}{2}}{\alpha} = \frac{A - x_1 + \alpha c}{2\alpha}. \tag{25B.20.ii}$$

Exercise 25B.21

Again, verify that this derivation is correct.
Answer: Substituting $p(x_1)$ and $x_2(x_1)$ into $(p(x_1)x_2(x_1) - cx_2(x_1)) > FC$, we get

$$
\begin{aligned}
&\frac{A - x_1 + \alpha c}{2\alpha}\left(\frac{A - x_1 - \alpha c}{2}\right) - c\left(\frac{A - x_1 - \alpha c}{2}\right) = \\
&= \left(\frac{A - x_1 - \alpha c}{2}\right)\left(\frac{A - x_1 + \alpha c}{2\alpha} - c\right) = \\
&= \frac{(A - x_1 - \alpha c)^2}{4\alpha} > FC.
\end{aligned} \tag{25B.21}
$$

Exercise 25B.22

Verify that this is correct. Does it make sense that profit for the Stackelberg leader is exactly twice the profit of the Stackelberg follower (which we calculated in equation (25.31) when $FC = 0$?

Answer: Profit for the Stackelberg leader is

$$\pi^{SL} = (p^S - c)x^{SL} = \left(\frac{A+3\alpha c}{4\alpha} - c\right)\left(\frac{A-\alpha c}{2}\right) = \frac{(A-\alpha c)^2}{8\alpha}. \tag{25B.22}$$

And it makes sense that this would be twice the profit (not counting fixed costs) of the Stackelberg follower — because the Stackelberg follower produces half as much as the leader.

Exercise 25B.23

As noted in the footnote, the quadratic formula also gives a second solution, namely $x = (2 - 2^{1/2})(A - \alpha c)/4$. Can you locate this solution in panel (a) of Graph 25.6?

Answer: It is the level of x that occurs at the lower intersection of the green curve and the vertical dashed line emanating from π^{SL}.

Exercise 25B.24

Verify π_i^D. Is it unambiguously larger than π_i^{Cartel}?

Answer: If one firm sticks with the cartel agreement and the other produces x_i^D, the total output level is

$$x = \frac{A-\alpha c}{4} + \frac{3(A-\alpha c)}{8} = \frac{5(A-\alpha c)}{8}. \tag{25B.24.i}$$

Plugging this into the inverse demand function $p(x) = (A - x)/\alpha$, we get

$$p(x) = \frac{3A+5\alpha c}{8\alpha}. \tag{25B.24.ii}$$

Profit for the deviating firm is then

$$\pi_i^D = (p(x) - c)x_i^D = \left(\frac{3A+5\alpha c}{8\alpha} - c\right)\frac{3(A-\alpha c)}{8} = \frac{9(A-\alpha c)^2}{64\alpha}. \tag{25B.24.iii}$$

Profit from not deviating is $\pi^{Cartel} = (A - \alpha c)^2/8\alpha$ which can also be written as

$$\pi^{Cartel} = \frac{8(A-\alpha c)^2}{64\alpha} \tag{25B.24.iv}$$

which is clearly less than π_i^D.

Exercise 25B.25

Verify that this is the correct per period profit in the Cournot equilibrium.
Answer: The Cournot profit is

$$\pi^C = (p^C - c)x^C = \left(\frac{A + 2\alpha c}{3\alpha} - c\right)\frac{A - \alpha c}{3} = \frac{(A - \alpha c)^2}{9\alpha}. \qquad (\text{25B.25})$$

25C Solutions to End-of-Chapter Exercises

Exercise 25.1

In the text, we demonstrated the equilibrium that emerges when two oligopolists compete on price when there are no fixed costs and marginal costs are constant. In this exercise, continue to assume that firms compete solely on price and can produce whatever quantity they want.

A: *We now explore what happens as we change some of these assumptions. Maintain the assumptions we made in the text and change only those referred to in each part of the exercise. Assume throughout that costs are never so high that no production will take place in equilibrium, and suppose throughout that price is the strategic variable.*

(a) *First, suppose both firms paid a fixed cost to get into the market. Does this change the prediction that firms will set $p = MC$?*

Answer: These would be sunk costs and would have no impact on the Bertrand equilibrium — thus $p = MC$ remains the equilibrium.

(b) *Suppose instead that there is a recurring fixed cost FC for each firm. Consider first the sequential case where firm 1 sets its price first and then firm 2 follows (assuming that one of the options for both firms is to not produce and not pay the recurring fixed cost). What is the subgame perfect equilibirum? (If you get stuck, there is a hint in part (f).)*

Answer: We start with firm 2 that observes p_1. If firm 2 sets price $p_2 < p_1$, it will get the entire market. This will be a best response if profit from doing so is non-negative; i.e. if $x_d(p_2)(p_2 - MC) \geq FC$ (where $x_d(p_2)$ is the demand for the good at p_2). And this in turn will hold for some p_2 close to p_1 so long as profit at p_1 (when one firm gets the whole market) is positive; i.e. so long as $x_d(p_1)(p_1 - MC) > FC$ which can also be written as

$$p_1 > \frac{FC + cx_d}{x_d} = AC(x_d) \tag{25.1.i}$$

where $AC(x_d)$ is the average cost of producing $x_d = x_d(p_1)$. Firm 2 will therefore set price just below p_1 whenever (25.1.i) holds. For all other p_1, it will not attempt to compete. Firm 1 knows all this and therefore sets

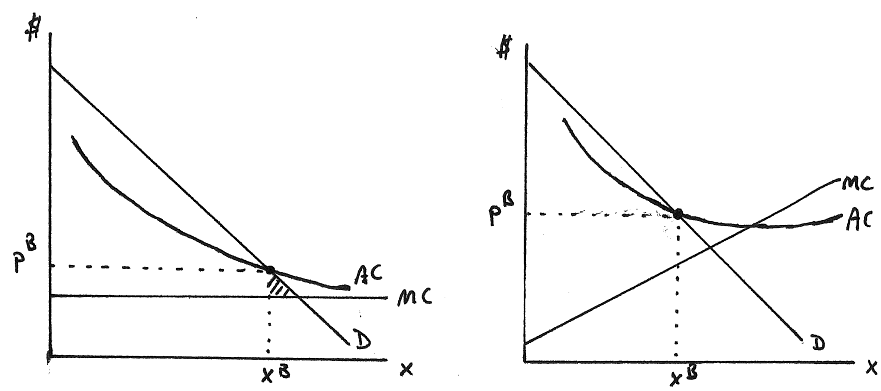
$$p_1 = \frac{FC + cx_d}{x_d} = AC(x_d); \tag{25.1.ii}$$

i.e. firm 1 sets price at average cost (thus getting zero profit) while firm 2 does not compete.

(c) *Consider the same costs as in (b). Can both firms produce in equilibrium when they move simultaneously?*

Answer: Suppose both firms do produce in equilibrium. Then it must be that both have set the same price $\overline{p}$ and both are making at least zero profit — which further implies that $\overline{p} > MC$ because the firms are covering their FC. But that means that either firm could do better by setting a price just below $\overline{p}$, getting the whole market rather than splitting it and thus essentially doubling revenues while lowering average costs (because MC is fixed and thus the fixed cost is spread over greater output). This implies that $\overline{p}$ cannot be an equilibrium price — implying that both firms cannot be producing in equilibrium.

(d) *What is the simultaneous move Nash Equilibrium? (There are actually 2.)*
Answer: Note: My original answer to this question read as follows: "The simultaneous move Nash Equilibrium outcome is the same as the sequential one in part (b). We already concluded in (c) that both firms cannot produce in equilibrium. This implies that one firm is producing while the other is not attempting to be in the market. If the producing firm is making positive profit, then the second firm is not best responding by staying out — it should instead price just below the first firm, get all the market and make positive profit. Thus, the one firm in the market cannot be making positive profit. Thus, $p = AC$ where AC crosses demand (as in part (b).) This is illustrated in panel (a) of Exercise Graph 25.1 where one of the firms ends up producing x^B at p^B. There are two pure strategy Nash equilibria — because either firm could be the one that sets p^B."



Exercise Graph 25.1 : Bertrand Model with recurring FC

However, it turns out that this answer is not correct — an error I discovered thanks to one of my students pointing it out. While it is true that the non-producing firm is best-responding by not entering the market given the $p = AC$ price set by the producing firm, the reverse is not true: Given that the non-producing firm is not producing, it is not a best response by the producing firm to set $p = AC$. Thus, there is no pure strategy Nash equilibrium in this game — and the statement in the text that "There are actually 2" is wrong.

(e) True or False*: The introduction of a recurring fixed cost into the Bertrand model results in $p = AC$ instead of $p = MC$.*

Answer: This is true, as demonstrated already.

(f) *You should have concluded above that the recurring fixed cost version of the Bertrand model leads to a single firm in the oligopoly producing. Given how this firm prices the output, is this outcome efficient — or would it be more efficient for both firms to produce?*

Answer: The outcome is not efficient — there is a shaded deadweight loss triangle in panel (a) of Exercise Graph 25.1. (This is because there are additional output units that could be produced at a marginal cost that lies below the value consumers place on them.) But if both firms were to produce, they would have to charge a higher price in order to maintain non-negative profit — implying even less output would be produced. Thus, it would be less efficient for two firms to split the market.

(g) *Suppose next that, in addition to a recurring fixed cost, the marginal cost curve for each firm is upward sloping. Assume that the recurring fixed cost is sufficiently high to cause AC to cross MC to the right of the demand curve. Using logic similar to what you have used thus far in this exercise, can you again identify the subgame perfect equilibrium of the sequential Bertrand game as well as the simultaneous move (pure strategy) Nash equilibria?*

Answer: This is illustrated in panel (b) of Exercise Graph 25.1. By the same logic as before, only a single firm will produce in equilibrium and will set p^B as illustrated in the graph with, once again, $p^B = AC$. The best response of the opposing firm is to not contest this. Were the price set any higher, the opposing firm could make positive profit by setting the price slightly lower. The producing firm once again makes zero profit. In the sequential case, firm 1 is the producing firm. In the simultaneous case, we have the same issue as the one raised in the answer to part (d).

B: *Suppose that demand is given by $x(p) = 100 - 0.1p$ and firm costs are given by $c(x) = FC + 5x^2$.*

(a) *Assume that $FC = 11,985$. Derive the equilibrium output x^B and price p^B in this industry under Bertrand competition.*

Answer: We know from our work in part A that a single firm will produce at $p = AC$ in equilibrium. Solving the demand function $x(p)$ for x, we get the demand curve $p(x) = 1000 - 10x$. Dividing $c(x)$ by x, we get $AC(x) = (11985/x) + 5x$. Setting these equal to each other, we get the equation

$$1000 - 10x = \frac{11,985}{x} + 5x \qquad (25.1.\text{iii})$$

which can be re-written (in a form required for use of the quadratic formula) as

$$15x^2 - 1000x + 11985 = 0. \qquad (25.1.\text{iv})$$

Applying the quadratic formula, we get

$$x^B = \frac{1,000 + \sqrt{1,000,000 - 4(15)(11985)}}{30} = 51. \qquad (25.1.v)$$

Plugging this back into the demand curve, we get

$$p^B = 1000 - 10(51) = 490. \qquad (25.1.vi)$$

(b) *What is the highest recurring fixed cost FC that would sustain at least one firm producing in this industry? (*Hint*: When you get to a point where you have to apply the quadratic formula, you can simply infer the answer from the term in the square root.)*

Answer: Setting $p(x)$ equal to AC once again (but this time leaving fixed cost unspecified), we get the equation

$$1000 - 10x = \frac{FC}{x} + 5x \qquad (25.1.vii)$$

which can be re-written (in a form required for use of the quadratic formula) as

$$15x^2 - 1000x + FC = 0. \qquad (25.1.viii)$$

The quadratic formula allows us to solve this for x — and the formula contains in it the term $\sqrt{1,000,000 - 60FC}$. This term becomes undefined if $(1,000,000 - 60FC)$ falls below zero — which is when the AC curve lies above the demand curve and therefore does not intersect it. Thus, the highest that FC can get and still preserve at least a tangency between demand and AC is when $(1,000,000 - 60FC) = 0$ or $FC = 16,667$. This is, as you see in upcoming exercises, also the highest fixed cost under which an unencumbered monopolist will produce. Thus, another way to arrive at this answer it to derive monopoly profit, set it to zero and see what FC it implies.

Exercise 25.3

In exercise 25.2, we considered quantity *competition in the simultaneous Cournot setting. We now turn the sequential Stackelberg version of the same problem.*

A: *Suppose that firm 1 decides its quantity first and firm 2 follows after observing x_1. Assume initially that there are no recurring fixed costs and that marginal cost is constant as in the text.*

(a) *Suppose that both firms have a recurring FC (that does not have to be paid if the firm chooses not to produce). Will the Stackelberg equilibrium derived in the text change for low levels of FC?*

Answer: For low levels of FC, the recurring fixed costs effectively become sunk costs for both firms — implying that firms will behave as if there were no FC. Thus, the Stackelberg equilibrium in the text would not change — with firm 1 producing the monopoly quantity and firm 2 producing half that.

(b) *Is there a range of FC under which firm 1 can strategically produce in a way that keeps firm 2 from producing?*

Answer: Yes, this is exactly like the entry deterrence game in the text. For a range of FC, it will be worthwhile for firm 1 to produce more than the monopoly quantity in order to drive profits for firm 2 to zero (or just below zero) were it to produce and pay FC.

(c) *At what FC does firm 1 not have to worry about firm 2?*

Answer: Firm 1 will not have to worry about firm 2 producing if firm 2's profit from producing half the monopoly quantity when firm 1 produces the monopoly quantity is less than zero. This will happen for sufficiently high FC.

(d) *Could FC be so high that no one produces?*

Answer: Yes, if FC is sufficiently high for AC to lie above the demand curve, not even a monopolist will be able to make profit (unless it can price discriminate). At that point, neither firm will produce.

(e) *Suppose instead (i.e. suppose again $FC = 0$) that the firms have linear upward sloping MC curves, with MC for the first output unit equal to what the constant MC was in the text. Can you guess how the Stackelberg equilibrium will change?*

Answer: Since marginal costs are now higher, best response functions will be farther in (as in the previous exercise) and output quantities will therefore be lower than they were in the text. As a result, the Stackelberg equilibrium price increases.

(f) *Will firm 1 be able to engage in entry deterrence to keep firm 2 from producing?*

Answer: The only way firm 1 can engage in strategic entry deterrence to keep firm 2 out of the market is if there are fixed costs. Since there are none, both firms will produce.

B: *Consider again the demand function $x(p) = 100 - 0.1p$ and the cost function $c(x) = FC + 5x^2$ (as you did in exercise 25.1 and implicitly in the latter portion of exercise 25.2).*

(a) *Suppose first that $FC = 0$. Derive firm 2's best response function to observing firm 1's output level x_1.*

Answer: Firm 2's residual demand function given x_1 is

$$x_2^r = 100 - x_1 - 0.1p \tag{25.3.i}$$

which, solved for p, gives us the residual demand curve

$$p_2^r = 1000 - 10x_1 - 10x_2 \tag{25.3.ii}$$

which in turn implies a residual marginal revenue function

$$MR_2^r = 1000 - 10x_1 - 20x_2. \tag{25.3.iii}$$

Setting this equal to $MC = 10x_2$ and solving for x_2, we then get firm 2's best response function

$$x_2(x_1) = \frac{100 - x_1}{3}. \tag{25.3.iv}$$

(b) *What output level will firm 1 choose?*

<u>Answer</u>: Firm 1 will take $x_2(x_1)$ as given and solve for its optimal output level based on this expectation. Thus, firm 1's residual demand function is

$$x_1^r = 100 - 0.1p - \frac{100 - x_1}{3} \tag{25.3.v}$$

which, when solved for p, gives us firm 1's residual demand curve

$$p_1^r = \frac{2000 - 20x_1}{3} \tag{25.3.vi}$$

and residual marginal revenue function

$$MR_1^r = \frac{2000 - 40x_1}{3}. \tag{25.3.vii}$$

Setting this equal to $MC = 10x_1$ and solving for x_1, we get

$$x_1 = \frac{200}{7} \approx 28.57. \tag{25.3.viii}$$

(c) *What output level does that imply firm 2 will choose?*

<u>Answer</u>: Plugging $x_1 = 200/7$ into firm 2's best response function $x_2(x_1)$, we then get

$$x_2 = \frac{100 - (200/7)}{3} = \frac{500}{21} \approx 23.81. \tag{25.3.ix}$$

(d) *What is the equilibrium Stackelberg price?*

<u>Answer</u>: The overall quantity produced is then equal to

$$x^S = x_1 + x_2 = \frac{200}{7} + \frac{500}{21} = \frac{1100}{21} \approx 52.38. \tag{25.3.x}$$

Plugging this into the equation for the demand curve, we get the Stackelberg price

$$p^S = 1000 - 10x^S = 1000 - 10\left(\frac{1100}{21}\right) \approx 476.19. \tag{25.3.xi}$$

(e) *Now suppose there is a recurring fixed cost $FC > 0$. Given that firm 1 has an incentive to keep firm 2 out of the market, what is the highest FC that will keep firm 2 producing a positive output level?*

Answer: If firm 1 sets output so that firm 2 produces, it means that they will again produce the Stackelberg quantities — i.e. $x_1 = 28.57$ and $x_2 = 23.81$ — with output price settling at $p^S = 476.19$. This will make firm 1's profit

$$\pi_1^S = 476.19(28.57) - FC - 5(28.72)^2 \approx 9{,}524 - FC. \tag{25.3.xii}$$

But firm 1 might be able to do better by raising its output level so as to push firm 2's profit if it produces to zero. How much does firm 1 have to produce to make sure firm 2 has zero (or slightly less than zero) profit? If firm 2 produces, firm 1 knows it will do so according to its best response function $x_2(x_1)$ which we calculated in (a) as $x_2(x_1) = (100 - x_1)/3$. When firm 1 produces x_1 and firm 2 produces $(100 - x_1)/3$, the industry output is $(2x_1 + 100)/3$ — implying a price

$$p(x_1) = 1000 - 10\left(\frac{2x_1 + 100}{3}\right) = \frac{2000 - 20x_1}{3}. \tag{25.3.xiii}$$

Firm 2's profit will then be

$$\begin{aligned}\pi_2(x_1) &= \left(\frac{2000 - 20x_1}{3}\right)\left(\frac{100 - x_1}{3}\right) - FC - 5\left(\frac{100 - x_1}{3}\right)^2 \\ &= (500 - 5x_1)\left(\frac{100 - x_1}{3}\right) - FC.\end{aligned} \tag{25.3.xiv}$$

If firm 1 succeeds in getting $x_2 = 0$, price will be $p(x_1) = 1000 - 10x_1$, with firm 1's profit

$$\pi_1(x_1) = (1000 - 10x_1)x_1 - FC - 5x_1^2 = 1000x_1 - 15x_1^2 - FC. \tag{25.3.xv}$$

Since firm 1 knows it can get the Stackelberg profit $\pi_1^S = 9{,}524 - FC$ by allowing firm 2 to produce, it will only produce strategically more if $1000x_1 - 15x_1^2 > 9{,}524$. Thus, the highest x_1 it would produce to keep firm 2 out is an amount that solves the equation

$$15x_1^2 - 1000x_1 + 9524 = 0. \tag{25.3.xvi}$$

Employing the quadratic formula, this gives us $x_1 \approx 55.15$. Plugging this into equation (25.3.xiv), we get firm 2 profit $\pi_2(55.15) = 3{,}352 - FC$. Firm 1 can therefore keep firm 2 out so long as $FC > 3{,}352$. The lowest recurring fixed cost that keeps firm 2 in the market is therefore $FC \approx 3{,}352$.

(f) *What is the lowest FC at which firm 1 does not have to engage in strategic entry deterrence in order to keep firm 2 out of the market?*

Answer: If firm 1 knows it is a monopoly, it will produce the monopoly quantity at the monopoly price. The monopoly quantity and price in this case is

$$x^M = \frac{1000}{30} = 33.33 \text{ and } p^M = 666.67. \tag{25.3.xvii}$$

If firm 1 produces this, firm 2's best response (not considering FC) is

$$x_2(x^M) = \frac{100 - 33.33}{3} \approx 22.22. \tag{25.3.xviii}$$

Together with x^M, this would imply total industry output of $x = 33.33 + 22.22 = 55.55$ — which implies output price of $p(x) = 1000 - 10(55.55) = 444.50$. This in turn would imply a profit (in the absence of FC for firm 2 equal to $\pi_2 = 444.50(22.22) - 5(22.22)^2 \approx 7{,}408$. If $FC > 7{,}408$, firm 2 thus best responds by not producing when firm 1 produces the monopoly quantity.

(g) *What is the lowest FC at which neither firm will produce?*

Answer: Neither firm will produce if not even an unencumbered monopolist can make a profit. The monopoly profit is $\pi^M = 666.67(33.33) - FC - 5(33.33)^2 = 16{,}666 - FC$. Thus, for $FC > 16{,}666$, no firm will produce.

(h) *Characterize the equilibrium in this case for the range of FC from 0 to 20,000.*

Answer: Given our results so far, we can now say the following. For $FC < 3{,}352$, firm 1 produces $x_1 = 28.57$, firm 2 produces $x_2 = 23.81$ and output is sold at $p = 476.19$. For $3{,}352 < FC < 7{,}408$, firm 2 produces $x_2 = 0$, firm 1 produces an amount between 33.33 and 55.15, with the amount declining as FC increases. Price rises (from 448.50 to 666.67) as FC increases and output falls (from 55.15 to 33.33). For $7408 < FC < 16{,}666$, firm 1 produces $x_1 = 33.33$, firm 2 produces $x_2 = 0$ and output price is 666.67. And for $FC > 16{,}666$, no one produces and nothing is sold.

Exercise 25.5

Business Application: *Quitting Time: When to Exit a Declining Industry*: *We illustrated in the text the strategic issues that arise for a monopolist who is threatened by a potential entrant into the market — and in Chapter 26, we will investigate firm entry into an industry where demand increases. In this exercise, suppose instead that an industry is in decline in the sense that demand for its output is decreasing over time. Suppose there are only two firms left — a large firm L and a small firm S.*

A: *Since our focus is on the decision of whether or not to exit, we will assume that each firm i has fixed capacity k^i at which it produces output in any period in which it is still in business; i.e. if a firm i produces, it produces $x = k_i$. Since*

L is larger than S, we assume $k^L > k^S$. The output that is produced is produced at constant marginal cost $MC = c$. (Assume throughout that, once a firm has exited the industry, it can never produce in this industry again.)

(a) *Since demand is falling over time, the price that can be charged when the two firms together produce some output quantity $\overline{x}$ declines with time — i.e. $p_1(\overline{x}) > p_2(\overline{x}) > p_3(\overline{x}) > ...$ where subscripts indicate the time periods $t = 1,2,3,....$ If firm i is the only firm remaining in period t, what is its profit π_t^i? What if both firms are still producing in period t?*

Answer: If firm i is the only remaining firm, profit is $\pi_t^i = (p_t(k^i) - c)k^i$. If both firms are producing in period t, profit for firm i will be $\pi_t^i = (p_t(k^S + k^L) - c)k^i$.

(b) *Let t^i denote the last period in which demand is sufficiently high for firm i to be profitable (i.e. to make profit greater than or equal to zero) if it were the only firm in the market. Assuming they are in fact different, which is greater: t^L or t^S?*

Answer: Since the large firm has larger capacity and thus produces more if it produces, the price it will get if it is the only one in the industry is always lower than the price the small firm would get if it were the only firm in the industry. Thus, if it were the only firm in the industry, the small firm could make positive profit longer than the large firm as demand declines — i.e. $t^L < t^S$.

(c) *What are the two firms' subgame perfect strategies beginning in period $(t^S + 1)$?*

Answer: If the two firms reach periods after t^S, neither one of them can make a profit even if it is the only firm in the industry. Thus, for all periods beginning with $t^S + 1$, the subgame perfect strategies for both firms must be to exit regardless of whether the other firm is still in the industry.

(d) *What are the two firms' subgame perfect strategies in periods $(t^L + 1)$ to t^S?*

Answer: The large firm cannot make a profit after t^L even if it is the only firm in the industry — thus, it's strategy for periods $(t^L + 1)$ to t^S must be to exit (if it were still in the industry — which, in equilibrium, it would not be). The small firm can still make a profit from t^L to t^S if it is the only one in the industry. In any subgame perfect equilibrium, it will therefore remain the only firm in the industry during this period. Put differently, any threat by the large firm that it will stay is non-credible — implying the small firm can confidently play the strategy "stay" during this period.

(e) *Suppose both firms are still in business at the beginning of period t^L before firms make their decision of whether to exit. Could both of them producing in this period be part of a subgame perfect equilibrium? If not, which of the two firms must exit?*

Answer: Since t^L is the last period in which the large firm by itself could make positive profit, it seems reasonable to assume that profit will be negative for both firms if both are still in the industry. Assuming this is the case, it cannot be an equilibrium for both firms to stay. The large firm knows it will exit next period if it reaches that period — and so it knows its profit will be zero starting next period. It can also get a profit of zero this period by exiting — and staying together with the small firm implies a negative profit. So the large firm is not best responding by staying. The small firm also knows the large firm will exit next period if it remains this period — and so it knows that it will make a positive profit from next period until t^S if it stays. Thus it is possible (and seems likely) that any loss the small firm would make in this period if the large firm stayed is offset by the profit from being the only firm in the coming periods (where price will rise a lot since the large firm will exit). Assuming this to be the case, the small firm is best responding by staying regardless of whether or not the large firm stays this period or not. Thus, the large firm exits and the small firm stays.

(f) *Suppose both firms are still in business at the beginning of period* ($t^L - 1$) *(before exit decisions are made). Under what condition will both firms stay? What has to be true for one of them to exit — and if one of them exits, which one?*

Answer: The argument from the previous part remains the same in period $t^L - 1$ so long as both firms would make negative profit if both remained — i.e. so long as $p_t(k^S + k^L) - c < 0$ when $t = t^L - 1$. As long as this is the case, the large firm therefore exits while the small firm stays if both reach period $t^L - 1$.

(g) *Let* $\overline{t}$ *denote the last period in which* $(p_t(k^S + k^L) - c) \geq 0$. *Describe what happens in a subgame perfect equilibrium, beginning in period* $t = 1$, *as time goes by — i.e. as* $\overline{t}$, t^L *and* t^S *pass. Is there ever a time when price rises as the industry declines?*

Answer: In subgame perfect equilibrium, both firms produce their capacity until and including $\overline{t}$. (The fact that they produce at capacity is assumed in the problem set-up.) At the beginning of $(\overline{t}+1)$, the large firm exits and the small firm continues to produce its capacity until and including t^S (which comes after t^L). Following t^S, the small firm exits and no further output is produced. This implies that price falls consistently through $\overline{t}$ and approaches c, then jumps up when the large firm exits at $\overline{t}$, and then continues to fall again toward c until the small firm exits (prior to price falling below c).

(h) *Suppose that the small firm has no access to credit markets — and therefore is unable to take on any debt. If the large firm knows this, how will* *this change the subgame perfect equilibrium?* True or False*: Although the small firm will not need to access credit markets in order to be the last firm in the industry, it will be forced out of the market before the large firm exits if it does not have access to credit markets.*

Answer: It continues to be true that subgame perfection requires that the small firm will exit if it ever reaches (t^S+1) or later and the large firms exits if it ever reaches (t^L+1) or later. It also has to continue to be true in equilibrium that both firms do not remain in the market past $\overline{t}$. The question is whether the new constraint on the small firm alters what happens in $(\overline{t}+1)$ — i.e. can we still say that the large firm will exit in $(\overline{t}+1)$? Before, we could say that the small firm stays and the large firm exits at $(\overline{t}+1)$. This is because, for any $\tilde{t}$ such that $\overline{t} < \tilde{t} \leq t^L$, the small firm was previously willing to sustain a one period loss in exchange for the future profits it will get when L exits the next period — whereas L was not willing to sustain a loss in $\tilde{t}$ knowing that it would exit in $(\tilde{t}+1)$. But now the small firm cannot in fact sustain a one-period loss because it is unable to access credit markets to borrow the amount necessary to survive a one-period loss. Thus, the small firm will now exit in $\tilde{t}$ if both firms reach $\tilde{t}$. Since this holds for all $\tilde{t}$ such that $\overline{t} < \tilde{t} \leq t^L$, it holds in $(\overline{t}+1)$. Thus, both firms produce at capacity until $\overline{t}$ after which the small firm exits and the large firm stays until t^L. Even though the small firm never needed to access credit markets in our previous subgame perfect equilibrium, this ceases to be an equilibrium when the small firm cannot credible threaten to access credit markets if it needs to. And therefore the small firm now exits before the large firm exits — and the market shuts down at t^L before t^S (when it shut down in our previous subgame perfect equilibrium.

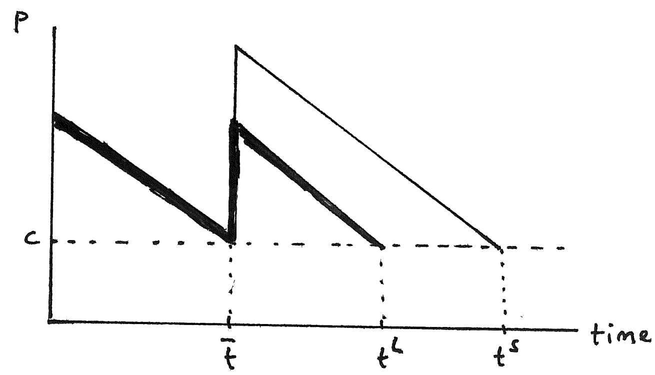
(i) *How does price now evolve differently in the declining industry (when the small firm cannot access credit markets)?*

Answer: Before, price fell consistently (approaching c) until (and including) $\overline{t}$ when both firms remained in the market. Then it increased in $(\overline{t}+1)$ to $p(k^S)$ and then falls again consistently (approaching c) until (and including) t^S after which the market shuts down. Now, the price evolves exactly the same way until (and including) $\overline{t}$. But at that point, the price now rises by less — to $p(k^)$ (which is less than $p(k^S)$ because $k^L > k^S$), and it then falls again consistently (approaching c) until (and including) t^L at which time the market shuts down.

The two price paths — with and without credit constraints for the small firm — are graphed in Exercise Graph 25.5.

B: *Suppose* $c = 10$, $k^L = 20$, $k^S = 10$ *and* $p_t(\overline{x}) = 50.5 - 2t - \overline{x}$ *until price is zero.*

(a) *How does this example represent a declining industry?*



Exercise Graph 25.5 : Price with and without credit constraints for the Small Firm

Answer: This represents a declining industry because the demand curve intercept $(50.5-2t)$ falls with time — i.e. demand is shifting down as time passes.

(b) *Calculate t^S, t^L and $\bar{t}$ as defined in part A of the exercise.*

Answer: The time period $\bar{t}$ is defined as the last period in which both firms can be in the market and make non-zero profit — i.e. the greatest t such that $p_t(k^S+k^L) \geq c = 10$. The demand curve is $p_t(k^S+k^L) = 50.5 - 2t - (20+10)$ which gives us $p_5 = 10.5$ and $p_6 = 8.5$. Thus, the last period in which price is greater than or equal to the marginal cost 10 when both firms produce is period $\bar{t} = 5$. The period t^L is defined as the last period in which the large firm can make non-negative profit if it is the only firm in the market — i.e. the highest t such that $p_t(k^L) \geq 10$ where $k^L = 20$. Given $p_t(k^L) = 50.5-2t-20$, we get $p_10(k^L) = 10.5$ and $p_11(k^L) = 8.5$ — implying $t^L = 10$. Through similar reasoning, we can conclude that $t^S = 15$.

(c) *Derive the evolution of output price as the industry declines.*

Answer: In the table depicted here, the first set of columns (labeled "No Credit Constraints") then gives the evolution of firm outputs over 16 periods — with the resulting evolution of price p. Note that price falls until (and including) $\bar{t} = 5$, then spikes up when the large firm exits, then declines until the small firm exits in $(t^S+1) = 16$.

(d) *How does your answer change when firm S has the credit constraint described in A(h) — i.e. when the small firm has no access to credit markets.*

Answer: In this case, we concluded that the large firm will stay and the small firm will exit in period $(\bar{t}+1)$. This results in the evolution of output and prices as given in the second set of columns (labeled "Firm 2 Credit Constraint") in the table. Note that price evolves as in the first case until (and including) $\bar{t} = 5$, then spikes more because the large firm that now

No Credit Constraints				Firm 2 Credit Constraint		
t	x^L	x^S	p	x^L	x^S	p
1	20	10	\$18.5	20	10	\$18.5
2	20	10	\$16.5	20	10	\$16.5
3	20	10	\$14.5	20	10	\$14.5
4	20	10	\$12.5	20	10	\$12.5
$\overline{t}$=5	20	10	\$10.5	20	10	\$10.5
6	0	10	\$28.5	20	0	\$18.5
7	0	10	\$26.5	20	0	\$16.5
8	0	10	\$24.5	20	0	\$14.5
9	0	10	\$22.5	20	0	\$12.5
t^L=10	0	10	\$20.5	20	0	\$10.5
11	0	10	\$18.5	0	0	—
12	0	10	\$16.5	0	0	—
13	0	10	\$14.5	0	0	—
14	0	10	\$12.5	0	0	—
t^S=15	0	10	\$10.5	0	0	—
16	0	0	—	0	0	—

remains produces more, then declines until (and including) $t^L = 10$ after which the market shuts down.

(e) *How would your answer change if the large rather than the small firm had this credit constraint?*

Answer: The initial subgame perfect equilibrium in which there are no credit constraints does not require a credible threat by the large firm to be able to take on debt. Thus, imposing a credit constraint on the large firm would not change the initial equilibrium, giving us the evolution of the declining market that is represented in the first set of columns of the table in part (d) (labeled "No Credit Constraints").

(f) *Suppose firm S can only go into debt for n time periods. Let $\overline{n}$ be the smallest n for which the subgame perfect equilibrium without credit constraints holds, with $n < \overline{n}$ implying the change in equilibrium you described in part A(h). What is $\overline{n}$? (Assume no discounting).*

Answer: We already explored why it was important for the small firm to have access to credit markets even though it does not actually need to use credit markets if we are to sustain a subgame perfect equilibrium in which the large firm exits in period $(t^L + 1)$. In essence, the large firm needs to be able to take seriously the threat of the small firm that it will stay in the market after it becomes unprofitable for both firms to stay in the market following period $\overline{t}$. In order for this threat to remain credible at $(\overline{t}+1)$, it must then be the case that the small firm can sustain the threat sufficiently long for the large firm to know that the losses it would incur were the small firm to stay for as long as it can outweigh the profit the large firm can make once the small firm is forced to exit.

Suppose, then, that the small firm's strategy at $(\bar{t}+1) = 6$ is to remain in the market until $(\bar{t}+n) = (5+n)$ after which it exits. Suppose further that $n < 5$ — which is the time period between $\bar{t}$ and $(t^L + 1)$ (i.e. periods 6 through 10) during which the large firm can still be profitable if it is the only one in the market. This implies that the large firms will make losses in periods 6 though 6 through $(5+n)$ and profits in periods $(6+n)$ through 10 before it exits in period 11. In order for the large firm to remain in the market in period 6, it must then be that the initial losses are offset by the later profits. The following table lists the price p that would emerge in the market if both firms remain from period 6 through 10 — with the resulting per period large firm loss (i.e. negative profit) π_S^L in the event that the smaller firm remains in the market and the cumulative loss $\sum \pi_S^L$ as time passes and the smaller firm continues to remain.

Partial Credit Constraints for Small Firm					
t	p	π_S^L	$\sum \pi_S^L$	π_E^L	$\sum \pi_E^L$
6	\$8.5	-\$30	-\$30	\$170	\$450
7	\$6.5	-\$70	-\$100	\$130	\$280
8	\$4.5	-\$110	-\$210	\$90	\$150
9	\$2.5	-\$150	-\$360	\$50	\$60
10	\$0.5	-\$190	-\$550	\$10	\$10

It also lists the large firm profit π_E^L in each period if the smaller firm has exited — as well as the cumulative profit $\sum \pi_E^L$ from each period on if the smaller firm has exited. For instance, if the smaller firm does not exit until the beginning of period 8, the large firm will have lost \$100 over period 6 and 7 and then earn a total of \$150 in profit over periods 8 through 10 when it is the only firm left. Thus, if the firm can only get access to credit markets for 2 periods, it is unable to credibly threaten to remain in period 6. But if it can access credit markets for 3 periods (through the end of period 8), it can credibly threaten to do so (because, as you can calculate, it would more than make up for that in profits in the long run) — it the large firm would not have enough profit left to make to force the small firm to borrow for 3 periods and then exit. Thus, $\bar{n} = 3.0$

(g) *If $n < \bar{n}$, how will output price evolve as the industry declines?*

Answer: If $n < \bar{n}$, the small firm exits at the beginning of period 6 and price evolves as in the last column of the table in part (d).

Exercise 25.7

Business Application: *Financing a Strategic Investment under Price Competition: In exercise 25.6, we investigated the incentives of firms to finance technologies that lower marginal costs. We did so in a sequential setting where firms compete by setting quantity, with the incumbent firm moving first. Can you repeat the exercise under the assumption that firms are sequentially competing on price (with firm 1 moving first)?*

A: The answers below correspond to parts (a) through (f) in exercise 25.6:

(a) Before the patent runs out, you are a monopolist producing the monopoly quantity. If you do not invest in the technology, you will then face a Bertrand competitor whose marginal cost is the same as yours. The equilibrium is then for both of you to set $p = MC$, with each of you producing roughly half the quantity demanded. It turns out (as shown in the text) that, for the linear demand curve assumed in the problem, half of the resulting overall quantity is equal to the previous monopoly quantity. Thus, you would not change your output, but your profit would drop to zero.

(b) If you invest in this technology, you'll be able to continue to produce (at just below) $p = MC$ — i.e. you only have to price slightly below the lowest your competitor can price — and you will serve the whole market. Thus, since price essentially remains the same, overall output will remain the same but you will produce all of it. Thus, you will produce roughly twice what you produced in part (a).

(c) As soon as $MC' < MC$, you will be able to force your competitor out by pricing just below MC. Thus, if you invest in a technology that reduces your marginal cost (and your opponent cannot do this, then you will force your opponent to not produce.

(d) We already concluded that you will force your opponent out even if MC' is slightly below MC — so you will surely do so if the difference is bigger.

(e) Unlike for the Stackelberg case in the previous exercise, the amount that is produced does not increase as a result of the investment. It does, however, get produced at lower cost — unless FC is exactly equal to the profit made by your firm after you invest in the technology. And this enhances efficiency.

(f) In the Stackelberg case there is a range of FC under which firm 1 invests in the technology but firm 2 does not — because the technology is worth less to the Stackelberg follower than to the leader. Now we have a slightly different situations: Firm 2 moves second and knows if firm 1 invested in the technology and also knows what price firm 1 set. If FC is relatively low, this means firm 1 can no longer set its price close to MC — because then firm 2 could invest in the technology and price below p_1 — thus getting the entire market. Thus, if firm 1 knows that firm 2 will adopt the technology in stage 2, it has to set its price such that firm 2's profit from investing and setting p_2 just below p_1 is negative. For any FC under which firm 1 would adopt the technology when firm 2 is not allowed to do so, there exists some price p_1 for which this is the case — but the lower FC is, the lower p_1 has to be set in order to keep firm 2 from adopting it. So in the sequential Bertrand case, it will never be the case that both firms adopt the technology.

B: Below are answers corresponding to parts (a) through (e) of the question:

(a) Your output level when you were a monopolist is $x^M = (A - \alpha c)/2$. If you engage in sequential Bertrand competition, you will increase your output to $x^B = (A - \alpha c)$ where $p = c$ — i.e. you will double output but cease making profits.

(b) It will no longer be the case here that your competitor will produce if you successfully lower your marginal cost. You will price at (just below) c — which means output will still be $x^B = (A - \alpha c)$. Your profit is then

$$\pi_1 = (c - c')x^B = (c - c')(A - \alpha c). \tag{25.7.i}$$

(c) In the absence of considering the fixed cost, your profit from having lowered your marginal cost is π_1 — which is the highest that FC can go and still cause you to invest in the technology. Thus,

$$\overline{FC}_1(c') = (c - c')(A - \alpha c) = (40 - c')(1000 - 10(40)) = 600(40 - c'). \tag{25.7.ii}$$

(d) As noted in the answer A(f), there is no $FC > 0$ at which firm 2 will invest in the technology. Thus $\overline{FC}_2 = 0$.

(e) This will be true for $0 < FC < 12,000$.

Exercise 25.9

Business and Policy Application: *Production Quotas under Cartel Agreements: In exercise 25.8, we investigated the acquisition price that an incumbent firm might pay to acquire a competitor under different bargaining and economic settings. Instead of one firm acquiring or merging with another, two firms in an oligopoly might choose to enter a cartel agreement in which they commit to each producing a quota of output (and no more).*

A: *Suppose again that both firms face a linear downward sloping demand curve, the same constant marginal cost, and no recurring fixed costs.*

(a) *Under the different bargaining settings and economic environments described in exercise 25.8,*[1] *what are the profits that the two firms in the cartel will make in terms of π^M, π^C, π^{SL} and π^{SF} (as these were defined in A(d) of exercise 25.8)?*

Answer: These are given in the following table for the three bargaining settings (first row) and the three economic settings (first column). In each cell of the table, the first expression is firm 1's profit and the 2nd is firm 2's profit.

(b) *It turns out that $\pi^C = (4/9)\pi^M$, $\pi^{SL} = (1/2)\pi^M$ and $\pi^{SF} = (1/4)\pi^M$ for examples like this. Using this information, can you determine the relative share of profit that each firm in the cartel will get for each of the bargaining and economic settings from (a)?*

Answer: These are given in table given here:

[1] There is a total of 9 such cases: 3 market settings (Bertrand, Cournot, Stackelberg) and three bargaining settings (ultimatum game with firm 1 proposing, ultimatum game with firm 2 proposing, and the alternating offer game).

Cartel Profits			
	Ultimatum 1	Ultimatum 2	Alternating Offers
Bertrand	$\pi^M, 0$	$0, \pi^M$	$0.5\pi^M, 0.5\pi^M$
Cournot	$(\pi^M - \pi^C), \pi^C$	$\pi^C, (\pi^M - \pi^C)$	$0.5\pi^M, 0.5\pi^M$
Stackelberg	$(\pi^M - \pi^{SF}), \pi^{SF}$	$\pi^{SL}, (\pi^M - \pi^{SL})$	$0.5(\pi^M + \pi^{SL} - \pi^{SF}), 0.5(\pi^M - \pi^{SL} + \pi^{SF})$

Profit and Output Quota Shares			
	Ultimatum 1	Ultimatum 2	Alternating Offers
Bertrand	1, 0	0,1	0.5, 0.5
Cournot	(5/9), (4/9)	(4/9), (5/9)	0.5, 0.5
Stackelberg	(3/4), (1/4)	0.5, 0.5	(5/8), (3/8)

They calculated from the relationships of the various terms to monopoly profit (as given in the question). In the table, the first fraction in each cell is equal to the fraction of the monopoly profit that accrues to firm 1 in the cartel, and the second fraction in each cell is equal to the fraction of the monopoly profit that accrues to firm 2 in the cartel. For instance, firm 1's fraction of the monopoly (or cartel) profit under alternating offer bargaining when the economic environment is Stackelberg (with firm 1 leading) is derived by starting with the profit level $0.5(\pi^M + \pi^{SL} - \pi^{SF})$ from the table in part (a) and then substituting for the terms that are not π^M:

$$0.5(\pi^M + \pi^{SL} - \pi^{SF}) = 0.5(\pi^M + (1/2)\pi^M - (1/4)\pi^M) = 0.5(3/4)\pi^M = \frac{5}{8}\pi^M. \tag{25.9.i}$$

(c) *Assuming the cartel agreement sets x^M — the monopoly output level — as the combined output quota across both firms, what fraction of x^M will be produced by firm 1 and what fraction by firm 2 under the different bargaining and economic settings we are analyzing?*

Answer: These fractions are logically the same as the fractions of the monopoly profits that firms get — and these are given in table in part (b).

(d) *Assume that any cartel agreement results in x^M being produced, with each firm producing a share depending on what was negotiated.* True or False*: For any such cartel agreement, the payoffs for firms could also have been achieved by one firm acquiring the other at some price.*

Answer: This is true. You can compare the acquisition prices for each of the bargaining and economic settings in the table you should have arrived at in A(d) of exercise 25.8 to the firm 2 payoffs in the table in part (a) — and they are the same. In the case where firm 2 is acquired by firm 1, the acquisition price is equal to firm 2's payoff — whereas in the case of cartels, firm 2's cartel profit is its payoff. They are the same for a given bargaining and economic setting because the merger of firms is the surest way for two firms to actually behave like a single monopolist — which is exactly what a cartel is attempting to do.

(e) *Explain why the firms might seek government regulation to force them to produce the prescribed quantities in the cartel agreement.*

Answer: In a cartel agreement, each firm has an incentive to produce more than its quota because the cheating firm gets the whole benefit of the increased production (in terms of the profit from the additional output that is produced) but only pays half the cost (in terms of the lower price that results). This is why we said in the text that cartels face a prisoners' dilemma — both firms like the cartel profits better than the profits that obtain when they both cheat, but it is in each firm's best interest to cheat on the agreement regardless of whether the other firm cheats; i.e. cheating is a dominant strategy. For this reason, it is economically rational for cartels to try to get the government to regulate them to their cartel quota production and thus enforce the cartel agreement.

(f) *In the early years of the Reagan administration, there was a strong push by the US auto industry to have Congress impose protective tariffs on Japanese car imports. Instead, the administration negotiated with Japanese car companies directly — and got them to agree to "voluntary export quotas" to the US, with the US government insuring that companies complied. How can you explain why Japanese car companies might have agreed to this?*

Answer: We can explain this from the insight that cartels face a prisoners' dilemma — they would like to reduce output in order to raise price, but each firm in the cartel has an incentive to cheat. For Japanese car manufacturers, entering "voluntary export quotas" that are enforced by the US government may well have represented a way for the Japanese car firms to enter a cartel agreement among each other — and hire the US government as an enforcer that keeps them from cheating. (Empirical evidence in fact suggests that the average price of a Japanese car sold in the US increased by about $2,500 as a result of these export quotas.)

(g) *Suppose the firms cannot get the government to enforce their cartel agreement. Explain how such cartel agreements might be sustained as a subgame perfect equilibrium if, each time the firms produce, they expect there is a high probability that they will again each produce as the only firms in the industry in the future?*

Answer: This is equivalent to turning the cartel agreement into an infinitely repeated prisoners' dilemma in which firms can use trigger strategies to enforce a cooperative equilibrium in which they stick by their cartel quotas. For instance, the strategy "I will produce the cartel quota the first time and then as long as everyone has always produced the cartel quota in the past; otherwise, I will cheat forever." We showed in the game theory chapter that this is a subgame perfect equilibrium so long as firms meet sufficiently often (which certainly holds for firms in an oligopoly) and don't discount the future too much.

(h) *If you are a a lawyer with the antitrust division of the Justice Department and were charged with detecting collusion among firms that have entered a cartel agreement — and if you thought that these agreements were typically sustained by trigger strategies, in which market setting (Bertrand, Cournot or Stackelberg) would you expect this to happen most frequently?*

Answer: I would expect it to happen most frequently in market settings where the trigger strategies contained the biggest punishment threat — i.e. in settings were the cost from deviating from the cartel agreement is the greatest for any given trigger strategy. This cost is largest in the Bertrand setting — because the trigger strategy that results in an end to the cartel agreement results in zero profit for each firm after the agreement falls apart.

B: *Suppose again that firms face the demand function $x(p) = A - \alpha p$, that they both face marginal cost c and neither faces a recurring fixed cost.*

(a) *For each of the bargaining and economic settings discussed in exercise 25.8, determine the output quotas x_1 and x_2 for the two firms.*

Answer: The easiest cases are those involving Bertrand settings — and these are displayed in the first row of the table below where the first entry in each cell is x_1 and the second is x_2.

	Output Quotas		
	Ultimatum 1	Ultimatum 2	Alternating Offers
Bertrand	$(A-\alpha c)/2$, 0	0, $(A-\alpha c)/2$	$(A-\alpha c)/4$, $(A-\alpha c)/4$
Cournot	$5(A-\alpha c)/18$, $2(A-\alpha c)/9$	$2(A-\alpha c)/9$, $5(A-\alpha c)/18$	$(A-\alpha c)/4$, $(A-\alpha c)/4$
Stackelberg	$3(A-\alpha c)/8$, $(A-\alpha c)/8$	$(A-\alpha c)/4$, $(A-\alpha c)/4$	$5(A-\alpha c)/16$, $3(A-\alpha c)/16$

Moving onto the Cournot case, we do the following: We know that Cournot competitors produce $(A-\alpha c)/3$ and sell it at $p^C = 2(A+\alpha c)/(3\alpha)$ — which implies a Cournot profit for each firm of $\pi^C = (A-\alpha c)^2/(9\alpha)$. Under the ultimatum bargaining setting, we further know that the non-proposer will end up getting this Cournot profit under the cartel agreement (since the proposer ends up with all the gains from the agreement.) We furthermore know that the cartel will produce as a monopolist — and thus the price of the output is the monopoly price $p^M = (A+\alpha c)/(2\alpha)$. In order for the non-proposer in the ultimatum bargaining to get the Cournot profit given that his goods will be sold at the price p^M, the quota $\overline{x}$ he is producing must then satisfy the equation

$$\pi^C = \frac{(A-\alpha c)^2}{9\alpha} = \left(\frac{A+\alpha c}{2\alpha} - c\right)\overline{x} = (p^M - c)\overline{x}. \quad (25.9.ii)$$

Solving this, we get

$$\overline{x} = \frac{2(A-\alpha c)}{9}. \quad (25.9.iii)$$

This is then the output quota for the non-proposer in an ultimatum bargaining setting where the default (if no agreement is reached) is Cournot competition. The proposer's quota is then simply

$$x^M - \overline{x} = \frac{A-\alpha c}{2} - \frac{2(A-\alpha c)}{9} = \frac{5(A-\alpha c)}{18}. \tag{25.9.iv}$$

The rest of the entries in the table are calculated similarly.

(b) *Verify that the fraction of the overall cartel production undertaken by each firm under the different scenarios is what you concluded in A(c).*

Answer: This is indeed the case. In each case the profits sum to $x^M = (A-\alpha c)/2$ — the monopoly output level — and the share of this output by each firm corresponds to the entries in the table in part (b).

(c) *Suppose* $A = 1000$, $c = 20$ *and* $\alpha = 40$. *What is the cartel quota for each of the two firms under each of the economic and bargaining settings you have analyzed?*

Answer: These are given in in the following table:

Output Quotas			
	Ultimatum 1	Ultimatum 2	Alternating Offers
Bertrand	300, 0	0, 300	150, 150
Cournot	166.7, 133.3	133.3, 166.7	150, 150
Stackelberg	225, 75	150, 150	187.5, 112.5

(d) *In terms of payoffs for the firms, is the outcome from the cartel agreement any different than the outcome resulting from the negotiated acquisition price in exercise 25.8?*

Answer: In the table from part B(c), we depict the profits for each firm under the different bargaining and economic settings. (These are calculated simply by taking the output quotas from the table in part B(c) and multiplying them by $(p^M - c) = 30$ (for the parameters we are using). Firm 2's payoffs are highlighted — and these are identical to the acquisition prices you might have calculated in the previous exercise. Thus, the payoffs for firm 2 are the same whether the firm is negotiating a cartel agreement or an acquisition price — it depends only on the bargaining and economic environment.

(e) *Suppose the two firms enter a cartel agreement with a view toward an infinite number of interactions. Suppose further that \$1 one period from now is worth \$$\delta$ now. What is the lowest level of* $\delta < \$1$ *for each of the bargaining settings such that the cartel agreement will be respected by both firms if they would otherwise be Cournot competitors?*

Answer: Let the one-period profit that firm i gets from being in the cartel be π_i^{Cartel}; let the one period profit from deviating from the cartel when the opponent still abides by the cartel agreement be π_i^D; and let the profit

Cartel Firm Profits			
	Ultimatum 1	Ultimatum 2	Alternating Offers
Bertrand	\$9,000, **\$0**	\$0, **\$9,000**	\$4,500, **\$4,500**
Cournot	\$5,000, **\$4,000**	\$4,000, **\$5,000**	\$4,500, **\$4,500**
Stackelberg	\$6,750, **\$2,250**	\$4,500, **\$4,500**	\$5,625, **\$3,375**

from both firms reverting to the Cournot outcome be given by π_i^C. The cartel profits for each firm are given in row 2 of the table below.

The profits that each firm gets under the Cournot outcome are $\pi_i^C = (A-\alpha c)^2/(9\alpha) = \$4,000$. This leaves us to calculate the profit from deviating. In the "Alternating Offers" bargaining setting, both firms produce 150 under the cartel — and we can simply use the best response function

$$x_1(x_2) = \frac{A - x_2 - \alpha c}{2} = \frac{1000 - x_2 - 10(40)}{2} = \frac{600 - x_2}{2} \tag{25.9.v}$$

to derive $x_i^D = 225$ when the other firm continues to produce the cartel quota of 150. With overall quantity $X = 225 + 150 = 375$, output price is then $p = (A - X)/\alpha = (1000 - 375)/10 = 62.5$ — giving the deviating firm a profit of $\pi_i^D = (62.5 - 40)225 = \$5,062.5$.

In the appendix to the chapter, we derive the condition that has to be satisfied in order for the harshest trigger strategy to sustain an equilibrium as

$$\frac{\pi_i^{Cartel}}{(1-\delta)} > \pi_i^D + \frac{\delta \pi_i^C}{(1-\delta)} \tag{25.9.vi}$$

which can also be written as

$$\delta > \frac{\pi_i^D - \pi_i^{Cartel}}{\pi_i^D - \pi_i^C}. \tag{25.9.vii}$$

Plugging in our values for the cartel quotas under "Alternating Offers", we get

$$\delta > \frac{5062.5 - 4500}{5062.5 - 4000} \approx 0.5294. \tag{25.9.viii}$$

In the "Ultimatum" bargaining setting, the non-proposer j then similarly maximizes his one-period payoff when the opponent produces 166.7 by producing $x_j^D = 216.7$. At the overall output level of $X = 166.7 + 216.7 = 383.4$, price is $p = (1000-383.4)/10 = 61.66$ resulting in $\pi_i^D = 4693.7$. Plugging this into equation (25.9.vii), we get

$$\delta > \frac{4693.7 - 4000}{4693.7 - 4000} \approx 1. \tag{25.9.ix}$$

The proposer k, on the other hand, best-responds to the non-proposers quantity of 133.3 with 233.4 — resulting in output price 63.34 and $\pi_k^D = 5,445.2$. Plugging this into equation (25.9.vii), we get

$$\delta > \frac{5,445.2 - 4500}{5,445.2 - 4000} \approx 0.654. \qquad (25.9.x)$$

Thus, the minimum value for δ required to sustain the cartel when the "Alternating offer" quotas are assigned is 0.5295, and the minimum δ when the "Ultimatum" quotas are assigned is greater than 1. This should make sense: In the ultimatum game the non-proposer is held to his non-cooperative payoff in the cartel — so a one time gain from deviating is followed by just reverting what he makes in the cartel. Thus, it's always worth it to cheat.

(f) *Repeat (e) for the case of Bertrand and Stackelberg competitors.*

Answer: Equation (25.9.vii) must still hold — except that now we need to replace π_i^C with π_i^B which represents the Bertrand profit of 0.

$$\delta > \frac{\pi_i^D - \pi_i^{Cartel}}{\pi_i^D - \pi_i^B} = \frac{\pi_i^D - \pi_i^{Cartel}}{\pi_i^D}. \qquad (25.9.xi)$$

In the "Alternating offers" case, $x_i^{Cartel} = 4500$. If the opponent continues to produce the cartel quantity 150, firm i would best respond for a 1-period payoff by producing to $x_i^D = 225$ which we calculated in (e) will give him $\pi_i^D = (62.5-40)225 = \$5,062.5$. Thus, the cartel can be sustained so long as

$$\delta > \frac{5062.5 - 4500}{5062.5} = 0.1111. \qquad (25.9.xii)$$

In the "Ultimatum" bargaining case, the result is the same as for the Cournot case — since the non-proposer gets no benefit from the cartel, he will cheat for all $\delta \leq 1$. The same holds in the Stackelberg case for the "Ultimatum" game.

Finally, we consider the Stackelberg case under the "Alternating offers" quotas of 187.5 and 112.5. The Stackelberg follower gets $\pi_i^{Cartel} = 3375$, and his Stackelberg follower profit is $\pi_i^{SF} = 2,250$. If he deviates for one period, he best-responds to the leader's quantity of 187.5 by producing 206.25. The total output level of $X = 187.5 + 206.25 = 393.75$ results in output price 60.625 — giving us $\pi_i^D = (60.625 - 40)206.25 = 4253.9$. Using the same formula as before, we then get

$$\delta > \frac{\pi_i^D - \pi_i^{Cartel}}{\pi_i^D - \pi_i^{SF}} = \frac{4253.9 - 3375}{4253.9 - 2250} = 4386. \qquad (25.9.xiii)$$

In this case, we do not even need to consider the Stackelberg leader — because any cheating on his part would immediately be detected given that firm 2 can observe firm 1's output before deciding on its own output. Since the leader knows this, the best he can do by cheating is to pick the Stackelberg leader quantity — which will be followed by the follower quantity; i.e. cheating would immediately get the players to the Stackelberg outcome. Thus, the leader has no incentive to cheat.

(g) *Assuming that cartel quotas are assigned using alternating offer bargaining, which cartels are most likely to hold: Those that revert to Bertrand, Cournot or Stackelberg? Can you explain this intuitively? Which is second most likely to hold?*

Answer: We calculated so far that the lowest δ required to sustain the cartel is 0.1111 under Bertrand, 0.5295 under Cournot, and 0.4386 under Stackelberg. Thus, the cartel is easiest to sustain under Bertrand and second easiest under Stackelberg. This makes intuitive sense: Under Bertrand, the punishments are severe — zero profit forever — and so one must really discount the future heavily in order to take a one-time advantage of the opponent.

Exercise 25.11

Policy Application: *Subsidizing an Oligopoly: It is common in many countries that governments subsidize the production of goods in certain large oligopolistic industries. Common examples include aircraft industries and car industries.*

A: *Suppose that a 2-firm oligopoly faces a linear, downward sloping demand curve, with each firm facing the same constant marginal cost and no recurring fixed cost.*

(a) *If the intent of the subsidy is to get the industry to produce the efficient output level, what should be the subsidy for Bertrand competitors?*

Answer: Since Bertrand competitors already produce the efficient quantity in this setting, the subsidy should be zero.

(b) *How would your answer to (a) change if each firm faced a recurring fixed cost?*

Answer: If each firm faced recurring fixed costs FC, then only one of the Bertrand firms will produce in equilibrium — and it will produce at price $p = AC$. If x^* is the efficient quantity, then the subsidy needs to be $s = FC/x^*$ so that the Bertrand firm will produce at $p = MC$ instead; i.e. suppose $MC = c$, then

$$AC(x^*) = \frac{cx^* + FC}{x^*} = c + \frac{FC}{x^*}. \tag{25.11.i}$$

But if the government sets $s = FC/x^*$, it lowers MC to $(c-s) = (c-FC/x^*)$ which implies

$$AC_s(x^*) = \frac{(c-s)x^* + FC}{x^*} = c - \frac{FC}{x^*} + \frac{FC}{x^*} = c. \tag{25.11.ii}$$

When the firm then sets price equal to AC, it is setting price equal to the marginal cost of production c — which is what efficiency requires.

(c) *What happens (as a result of the subsidy) to best response functions for firms who are setting quantity (rather than price)? How does this impact the Cournot equilibrium?*

Answer: For any quantity $\overline{x}$ that the opposing firm produces, my best response is now to produce more than before the subsidy. Thus, the best response functions shift out — and with both best response functions shifting out, the intersection will shift out (along the 45 degree line). Thus, both firms produce more and overall output increases.

(d) *How would you expect this to impact the Stackelberg equilibrium?*

Answer: With best response functions shifting out, equilibrium output in the Stackelberg equilibrium must rise as well.

(e) *Suppose policy-makers can either subsidize quantity-setting oligopoly firms in order to get them to produce the efficient quantity, or they can invest in lowering barriers to entry into the industry so that the industry becomes competitive. Discuss how you would approach the trade-offs involved in choosing one policy over the other.*

Answer: Both policies involve cost for the government, and both policies (if successfully implemented) lead to the efficient outcome. Thus, the question is which policy costs more to implement. Under the subsidy policy, the government has to pay the subsidies but it does not have to invest in lowering barriers to entry — and under the policy of removing barriers to entry, the government incurs costs involved in this activity but does not have to pay any subsidies.

(f) *How would your answer be affected if you knew that it was difficult for the government to gather information on firm costs?*

Answer: If the government cannot easily determine firm costs, it will be difficult to find the "right" subsidy rate to induce efficient production. You should then be more inclined to favor spending resources on removing barriers to entry so that competition can result in efficient outcomes.

(g) *Suppose there are recurring fixed costs that are sufficiently high for only one firm to produce under quantity competition. Might the subsidy result in the entry of a second firm?*

Answer: In exercise 25.2, we investigated the role of recurring fixed costs in Cournot models (and in the next exercise we did the same for Stackelberg competition). We showed that, if FC is sufficiently high, only one firm will produce in equilibrium. Since the per-unit subsidy shifts out best response functions, this will raise the level of FC at which only a single firm will produce in equilibrium — and thus it is possible for such subsidies to cause an increase in the number of firms from 1 to 2.

B: *Suppose demand is given by $x(p) = A - \alpha p$, that all firms face constant marginal cost c and there are no recurring fixed costs.*

(a) *If the government introduces a per-unit subsidy $s < c$, what happens to the marginal costs for each firm?*

Answer: The marginal costs for each firm fall to $(c - s)$.

(b) *How do the Monopoly, Bertrand, Cournot and Stackelberg equilibria change as a result of the subsidy?*

Answer: Since we know marginal costs fall to $(c - s)$, all we have to do is replace c with $(c - s)$ in our usual solutions. These are presented here.

Equilibrium Output and Prices with Subsidy s			
	x_1	x_2	p
Monopoly	$(A - \alpha(c-2))/2$	0	$(A + \alpha(c-s))/(2\alpha)$
Bertrand	$(A - \alpha(c-2))/2$	$(A - \alpha(c-2))/2$	$(c-s)$
Cournot	$(A - \alpha(c-2))/3$	$(A - \alpha(c-2))/3$	$(A + 2\alpha(c-s))/(3\alpha)$
Stackelberg	$(A - \alpha(c-2))/2$	$(A - \alpha(c-2))/4$	$(A + 3\alpha(c-s))/(4\alpha)$

(c) *Suppose $A = 1000$, $c = 40$ and $s = 15$. What is the economic incidence of the subsidy in each economic environment — i.e. what fraction of the subsidy is passed onto consumers and what fraction is retained by producers?*

Answer: In the following table, we begin in the first column with the pre-subsidy price. The second column has the price that actually emerges and is paid by consumers (as calculated using the results from the table in part (b)), and the third column gives the price that firms actually receive (including the subsidy). (The latter is just the former plus $s = 15$). The final two columns then give the tax incidence — with the first column giving the percentage of the subsidy that is passed to consumers and the second giving the percentage of the subsidy that is retained by firms.

Prices with Subsidy $s = 15$					
	$p_{s=0}$	p_s^c	p_s^f	% Cons.	% Firms
Monopoly	\$70.00	\$62.50	\$77.50	50%	50%
Bertrand	\$40.00	\$25.00	\$40.00	100%	0%
Cournot	\$60.00	\$50.00	\$65.00	66.67%	33.33%
Stackelberg	\$55.00	\$43.75	\$58.75	75%	25%

(d) *How would your answer to (c) change if the government instead imposed a per unit tax $t = 15$?*

Answer: All we have to do is change $(c - s)$ to $(c + t)$ in the table from part (b) to get the new equilibrium quantities. The following table then provides results analogous to those for subsidies in the table from part (c). Note that the relative incidence does not change — i.e. the percentage passed onto consumers (and the percentage falling on producers) remains the same as for subsidies.

(e) *How much of a tax or subsidy has to be set in order to get the efficient level of output under each of the four market conditions?*

Answer: The efficient overall quantity is $x^* = A - \alpha c = 1000 - 10(40) = 600$ units of output. This is already what is produced under Bertrand competition — which means no tax or subsidy is needed. In each of the other

Prices with Tax $t = 15$					
	$p_{t=0}$	p_s^c	p_s^f	% Cons.	% Firms
Monopoly	\$70.00	\$77.50	\$62.50	50%	50%
Bertrand	\$40.00	\$55.00	\$40.00	100%	0%
Cournot	\$60.00	\$70.00	\$55.00	66.67%	33.33%
Stackelberg	\$55.00	\$66.25	\$51.25	75%	25%

cases, less than 600 units is produced in the absence of intervention — which implies that subsidies would be needed to raise output. In each case, we can simply set the overall output level (using the formulas in the table from part (b)) equal to 600 and solve for s. This implies a subsidy rate of $s^M = 60$ for monopolists, a rate of $s^C = 30$ under Cournot competition and a rate $s^S = 20$ under Stackelberg competition.

(f) *Suppose you are advising the government on policy and you have two choices: Either you subsidize the firms in the oligopoly, or you lower the barriers to entry that keep the industry from being perfectly competitive. For each of the four market conditions, determine what cost you would be willing to have the government incur to make the industry competitive rather than subsidize it?*

Answer: A perfectly competitive industry would produce 600 output units at price $p^* = c = 40$. We just calculated in the previous part what subsidy rates are required to get each of the four oligopoly/monopoly markets to produce this amount — $s^M = 60$, $s^C = 30$, $s^S - 20$ and $s^B = 0$. Since these are per-unit subsidies, the total subsidy costs will then be $S^M = 60(600) =$ \$36,000, $S^C = 30(600) =$ \$18,000, $S^S =$ \$12,0000 and $S^B = 0$. Since this is how much it costs in each case to get the market to produce the efficient level of output, you would be willing to pay up to that much in each case to instead remove all barriers to entry.

(g) *Suppose that pollution was produced in this industry — emitting a constant level of pollution per unit of output, with a cost of b per unit of output imposed on individuals outside the market. How large would b have to be under each of the market conditions in order for the outcome to be efficient (without any government intervention)?*

Answer: The efficient level of output under this pollution externality is the level at which social marginal cost $(c+b) = (40+b)$ is equal to marginal benefit. The marginal benefit of x is given by the demand curve $100-0.1x$. Setting this equal to marginal social cost, we get $(40+b) = (100-0.1x)$ or just

$$x^* = 600 - 10b. \tag{25.11.iii}$$

The monopoly, Bertrand, Cournot and Stackelberg output levels for our example are $x^M = 300$, $x^B = 600$, $x^C = 400$ and $x^S = 450$. Substituting these in for x^* and solving for b, we get pollution levels $b^M = 30$, $b^B = 0$,

$b^C = 20$ and $b^S = 15$ that would make each of the outcomes efficient. (The ordering makes intuitive sense: Since $x^M < x^C < x^S < x^B$, the pollution levels have to be highest for monopolies and lowest for Bertrand competitors in order for behavior to be efficient. In fact, since Bertrand competitors produce the efficient output level in the absence of pollution, the Bertrand outcome is efficient if there is no pollution — i.e. $b^B = 0$).

Conclusion: Potentially Helpful Reminders

1. In this chapter, we work a lot with the idea of *best response functions* as a way of thinking about game theory equilibria. Remember that these follow the basic logic of best responses as first introduced in Chapter 24 — so they really are nothing new. All we are doing is recognizing that, as we think about firms setting price or quantity, their set of possible actions is continuous.

2. Be sure to understand why a Nash equilibrium in simultaneous oligopoly competition occurs where best response functions for firms cross. Each function tells us the best response of a firm to another firm's action, and when the firms' best response functions cross, they are therefore best responding to each other.

3. It is often difficult at first to see how the Stackelberg leader's best response to the follower's actions is different from the Cournot competitor's best response. The key lies in the sequential nature of the Stackelberg game (as opposed to the simultaneous nature of the Cournot game). In the sequential setting, subgame perfection implies the leader knows exactly what the follower will do once the follower observes the leader's action — and it is this advantage that allows the Stackelberg leader to take advantage in a way that a Cournot competitor cannot.

4. Entry deterrence has a similar sequential structure — allowing the incumbent to credibly commit to a quantity that will keep competitors out, something that is not possible in a simultaneous setting.

CHAPTER

26

Product Differentiation and Innovation in Markets

This chapter builds on the merging of competitive forces with market power as first introduced in our treatment of oligopolies in Chapter 25. The distinguishing feature of oligopolies is that, like monopolies, they produce goods without close substitutes and are insulated by high barriers to entry. Unlike monopolies, however, the firms in an oligopoly face competition from one another. We now see how firms can soften this competition — even when it takes the form of Bertrand price competition — by differentiating products within the oligopoly and thus carving up the consumer market. We also revisit the decision of firms to enter — and derive a model of strategic pricing and entry decisions in markets where products can be differentiated and tailored to consumer needs. As in Chapter 25, the fixed entry cost plays an important role, but we now see innovation as a vehicle for potential entrants to carve out sufficient market power after entering to overcome the fixed entry cost.

Chapter Highlights

The main points of the chapter are:

1. Firms in oligopolies can **soften price competition** through innovation and product differentiation.

2. The ability to innovate and differentiate products then opens the possibility for evolving markets characterized by increasing **product diversity**. Unlike our model of perfect competition, such **monopolistically competitive** markets produce differentiated products that enable firms to retain some market power even as they compete fiercely within the marketplace.

3. Because of the presence of **fixed entry costs**, a monopolistically competitive industry has firms that are making positive profits while identical firms outside the industry would make negative profit by entering.

4. Product differentiation can take many forms, including differentiation that happens through **advertising**. Advertising might play the socially beneficial role of providing consumers information, or it might simply serve the interests of firms by "artificially" differentiating products so as to soften price competition.

26A Solutions to Within-Chapter-Exercises for Part A

Exercise 26A.1

True or False: Suppose that Coke knows that it has positive consumer demand if it sets $p = MC$. Then it must be the case that Coke will price above MC.

Answer: This is true. If it sets $p = MC$, it will make zero profit. But if there is positive demand for Coke at $p > MC$, then Coke can make positive profit by pricing above MC.

Exercise 26A.2

We have said that under product differentiation we would expect the quantity of Coke that is demanded to be affected by both the price of Coke and the price of Pepsi. Can you see how the models of product differentiation result in firms facing precisely this kind of demand when they locate at different points in the product characteristics interval (or circle)?

Answer: When Coke and Pepsi locate at different points on the product characteristics circle, they have differentiated their product. As a result, as Pepsi raises its price, Coke will be able to attract more customers at its current price since it will be able to get additional customers that now find Coke at the current price more desirable given that Pepsi has gotten more expensive. Of course if Coke raises its price, demand will fall as some customers will now switch to Pepsi.

Exercise 26A.3

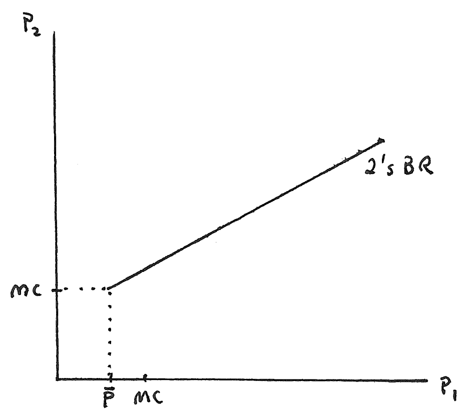
Would the equilibrium outcome be different if one firm announced its product characteristic prior to the other one having to do so?

Answer: No, the outcome would be the same. For any product characteristic other than 0.5 announced by firm 1, firm 2 would choose a characteristic that is slightly closer to 0.5 — thus capturing a larger market share. This leaves firm 1 to announce 0.5, with firm 2 best responding by also choosing 0.5.

Exercise 26A.4

Suppose the demand for firm 2's output is zero for any p_2 at or above MC when firm 1 sets price p_1 to zero. Furthermore, suppose that demand for firm 2's output becomes positive at $p_2 = MC$ when firm 1 sets a price $\overline{p}$ that lies between 0 and MC. What would firm 2's best response function look like?

Answer: If firm 1 sets its price to zero and firm 2 cannot sell in that environment, it can in principle set its price anywhere at or above MC since it will have no impact on its sales. This will continue to be true as firm 1 raises its price above zero up to the price $\overline{p}$ when firm 2 faces positive demand when it sets its price to MC. As firm 1 sets its price above $\overline{p}$, firm 2 can begin to raise its price above MC. Firm 2's best response function will then (for all practical purposes) begin where $p_1 = \overline{p}$ and look as depicted in Exercise Graph 26A.4.

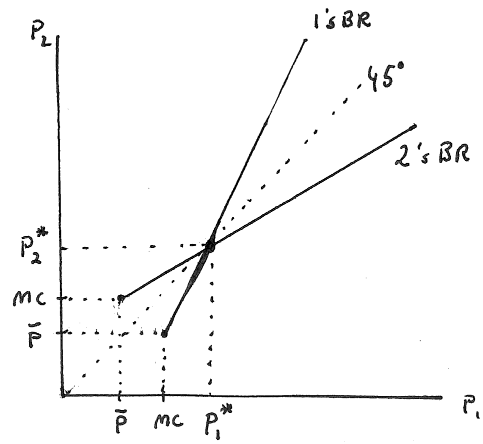


Exercise Graph 26A.4 : Firm 2's best response function

Exercise 26A.5

Consider the case described in exercise 26A.4 and assume the two firms are symmetric relative to one another. Will it still be the case that $p > MC$? Can you see how decreasing product differentiation in the minds of consumers will lead to a result that approaches $p = MC$? (*Hint*: As $\overline{p}$ gets closer to MC, product differentiation diminishes.)

Answer: If the two firms are symmetric, firm 1's best response function will be similar to firm 2's best response function (as derived in exercise 26A.4. This givens the picture in Exercise Graph 26A.5 where both best response functions are graphed on the same graph. The two best response functions intersect at p_1^* and p_2^*, where (because of the symmetry) $p_1^* = p_2^*$. The equilibrium price will therefore still be above MC. This is true so long as $\overline{p}$ lies below MC. But as $\overline{p}$ gets closer to MC, the equilibrium price p^* converges to MC. Thus, as product differentiation disappears, price competition leads back to the Bertrand result of both firms pricing at MC. (Product differentiation disappears when firm 2 cannot sell anything at $p = MC$ as long as firm 1's price is even slightly below MC — thus diminishing product differentiation is equivalent to $\overline{p}$ approaching MC.)



Exercise Graph 26A.5 : Equilibrium Prices

Exercise 26A.6

If there is no first stage entry decision and the number of firms is simply fixed as in an oligopoly with barriers to entry, can you see how this represents the full equilibrium of the game?

Answer: If the number of firms N is fixed, we simply have a Bertrand price competition game with N firms — and all that remains is for firms to choose price. Thus, there is nothing further to consider — the equilibrium we described is the Nash equilibrium of the game.

Exercise 26A.7

In the context of this model, why is the last sentence slightly more correct than the second to last sentence in the previous paragraph?

Answer: The second sentence permits for the possibility that profit (including fixed costs) is actually positive in equilibrium — so long as one additional firm entering would make profit negative for all firms. In equilibrium, it has to be the case that no potential firm that did not enter in stage 1 could do better by entering — which implies that all firms that don't enter in stage 1 must not be able to make a positive profit by entering. Thus, if profits (including fixed costs) are positive in stage 2 but any potential entering would tip the equilibrium in stage 2 to everyone making negative profit (including fixed costs), then no potential firm that did not enter in stage 1 would have an incentive to actually enter. It is therefore possible for entering firms to make positive profit (including fixed costs) so long as those profits are sufficiently low to keep anyone else from wanting to enter.

Exercise 26A.8

True or False: As long as the fixed entry cost $FC > 0$, firms in the industry will make positive profits while firms outside the industry would make negative profits by entering the industry.

Answer: This is true once we consider fixed costs as sunk costs in stage 2. Firms that entered in stage 1 paid the fixed cost — which is now sunk — and thus make unambiguously positive economic profit (even if profit including fixed cost is zero). But firms that did not enter would make negative profit by entering — because for them the fixed cost is not yet sunk.

Exercise 26A.9

True or False: While we needed a model of product differentiation to allow for Bertrand competition to be able to fully fill the gap between perfect competition and monopoly, we do not need anything in addition to what we introduced in Chapter 25 to do the same for Cournot competition.

Answer: This is true. Cournot competition has the feature that, as the number of firms increases, the Nash equilibrium converges to the competitive equilibrium. Put differently, the more firms there are in an oligopoly, the more intense the competition between Cournot competitors gets — and as it gets more intense, the firms begin to behave as if they are perfectly competitive (and the price converges to marginal cost).

Exercise 26A.10

Where in panel (a) of Graph 26.4 is the firm's total revenue given that it charges p^i? Where is its variable cost given that it produces x^i?

Answer: The firm's total revenue is p^i times x^i — which is simply the box created by the dashed blue lines and the axes. The variable cost is given by the area under the MC curve up to x^i. Profit that excluded fixed cost is therefore equal to the shaded blue are in the graph.

Exercise 26A.11

True or False: With economic profit appropriately defined for each firm, the profit of firms in the industry is positive while the profit of a firm outside the industry would be zero or negative if it entered the monopolistically competitive market in equilibrium.

Answer: This is true. For firms inside the industry, the fixed entry costs have already been paid and are therefore sunk costs (i.e. not real economic costs). Thus, when using only economic costs to calculate profit, the firms in the industry make positive profit. For firms outside the industry, on the other hand, the fixed entry costs are not yet sunk — they are real economic costs of starting production. Thus, firms outside the industry would make zero or negative profit from entering the industry.

Exercise 26A.12

The innovation discussed above was in terms of product characteristics and thus impacted demand. Can you tell the same kind of story where a firm instead innovates in a way that reduces costs?

Answer: Yes. If a firm in a monopolistically competitive equilibrium finds a way to reduce costs, it's picture will look exactly as panel (a) of Graph 26.5 – except the reason now is that its cost curves shifted down (rather than its demand shifting up). Over time, other firms will adopt the cost innovation and equilibrium will be restored.

Exercise 26A.13

Many of the advocates for lowering n in patent laws draw on the burst of innovation in open source software communities. Can you see why?

Answer: The open source software community has demonstrated that there are ways to spur innovation without patent protection. It is difficult to fully reconcile this with our economic models — much of the innovation in open source communities appears to happen without the innovators directly profiting in the usual way. Rather, there are other motivators that cause the innovation — including developing reputations and gaining expertise as open source innovators become known for their innovations.

Exercise 26A.14

Consider an oligopoly with consumers being only partially aware of each firm's products and prices, and suppose that firms in the oligopoly decide to engage in informational advertising. In what sense might they be facing a prisoners' dilemma?

Answer: If products are similar or identical across firms but consumers are not aware of all the firms and their products, then each firm has an incentive to make consumers aware of its products — but as all firms engage in such advertising, price competition increases. Thus, were no firms to advertise, the equilibrium price could be in substantial excess of MC — leading to economic profit for firms. As firms advertise and consumers become aware of their choices, price competition intensifies, driving price down to MC. While all firms have an incentive to advertise, the outcome is one where profits are lower than if no firm advertised. Thus, firms would actually make more profit if none of them advertise, but the dominant strategy equilibrium is one where they all advertise.

Exercise 26A.15

Suppose that you hear that an industry group is attempting to persuade the government to ban advertising in its industry. Given your answer to exercise 26A.14, might you be suspicious of the industry group's motives?

Answer: I would be suspicious because the firms might be caught up in the prisoners' dilemma described in exercise 26A.14 — and they might be looking to the government to force them out of the prisoners' dilemma. This, however, might simply be to lessen price competition — and such lessening of price competition would benefit the firms at the expense of consumers.

Exercise 26A.16

In my experience, car advertisements on television are different. Can you argue that they are more in the category of informational advertising than the Coke and Pepsi ads we just discussed?

Answer: At least in my experience, car advertisements on TV provide information — information about sales, about pricing policies, about particular cars that are being pushed for sale, etc. This is different from Coke and Pepsi ads that seem to primarily shape the image of the product rather than provide price or quality information.

26B Solutions to Within-Chapter-Exercises for Part B

Exercise 26B.1

Can you think of why it is reasonable to assume $\alpha > \beta$?

Answer: Suppose, for instance, that both p_i and p_j rise by the same amount. If $\alpha = \beta$, this implies that the quantity of x_i demanded remains unchanged. But we generally think demand curves are downward sloping — with demand for x_i falling as p_i increases. This would then no longer be the case if $\alpha = \beta$ when both prices rise by the same amount. Put differently, when both prices rise by the same amount (from, say, the same initial level), we would not expect people to switch from one good to another — but we would expect them to reduce their consumption. When $\alpha < \beta$, the demand equation would predict something even less intuitive: as price of both goods goes up by the same amount, demand for x_i would *fall*. The most reasonable assumption is that, if both prices rise by the same amount, demand for each good will still fall — which implies $\alpha > \beta$.

Exercise 26B.2

Suppose $p_j = 0$. Interpret the resulting best price response for firm i in light of what we derived as the optimal monopoly quantity and price when $x = A - \alpha p$.

Answer: The monopoly price for a firm that faces $x = A - \alpha p$ is $p^M = (A + \alpha c)/(2\alpha$ — exactly what our $p_i(p_j)$ equation gives us when $p_j = 0$. This is of course due to the fact that we have now assumed a demand equation for x_i that reduced to $x_i = A - \alpha p_i$ when $p_j = 0$.

Exercise 26B.3

Before going to our concrete example, we argued that Bertrand competition will lead to prices above marginal cost when $x_i(c, p_j) > 0$. In our example, we find that, in equilibrium, $p > c = MC$ so long as $c < A/(\alpha - \beta)$. Can you reconcile the general conclusion with the conclusion from the example?

Answer: In our example, $x_i(c, p_j) = 0$ implies

$$x_i(c, p_j) = A - \alpha c + \beta p_j = 0. \tag{26B.3.i}$$

But firm j would never set p_j below c in equilibrium — which implies that the lowest it will ever set p_j is equal to c. If we then substitute c for p_j in our equation, we have a sufficient condition for Bertrand competition to lead to price above marginal cost. Doing so and solving for c, we get

$$c < \frac{A}{\alpha - \beta} \tag{26B.3.ii}$$

which is the condition we derived as having to hold in our example for the equilibrium price to be above marginal cost.

Exercise 26B.4

Derive the right hand side of equation (26.9).
Answer: We can derive this in the following way:

$$\begin{aligned} D^2(p_1,p_2,y_1,y_2) &= 1-\overline{n} = 1-\left[\frac{y_2+y_1}{2}+\frac{(p_2-p_1)}{2\alpha(y_2-y_1)}\right] \\ &= 1-y_2+\frac{2y_2}{2}-\frac{y_2+y_1}{2}+\frac{(p_1-p_2)}{2\alpha(y_2-y_1)} = \\ &= (1-y_2)+\frac{y_2-y_1}{2}+\frac{(p_1-p_2)}{2\alpha(y_2-y_1)}. \end{aligned} \tag{26B.4}$$

Exercise 26B.5

Set up firm 2's optimization problem and verify the best response function $p_2(p_2)$.
Answer: The optimization problem is

$$\max_{p_2}\ (p_2-c)\left((1-y_2)+\frac{y_2-y_1}{2}+\frac{(p_1-p_2)}{2\alpha(y_2-y_1)}\right). \tag{26B.5.i}$$

The first derivative set to zero gives us

$$(1-y_2)+\frac{y_2-y_1}{2}+\frac{p_1}{2\alpha(y_2-y_1)}-\frac{c}{2\alpha(y_2-y_1)}-\frac{p_2}{\alpha(y_2-y_1)}=0 \tag{26B.5.ii}$$

which solves to

$$p_2(p_1)=\frac{p_1}{2}+\frac{c-\alpha(y_2^2-y_1^2)+2\alpha(y_2-y_1)}{2}. \tag{26B.5.iii}$$

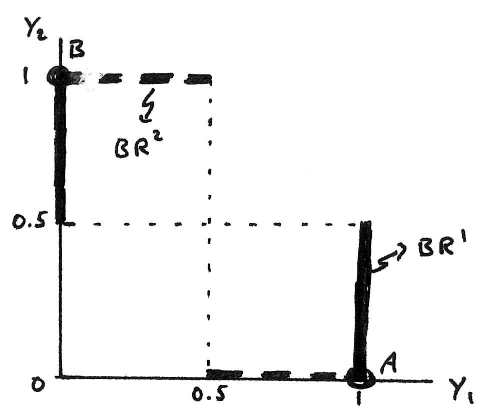
Exercise 26B.6

Explain the last sentence in parenthesis.
Answer: If $y_2<0.5$, then the market to the right of y_2 is bigger than the market to the left of y_2. Given this, firm 1 will settle its product in the larger market — and thus $y_1>y_2$. We have restricted ourselves to labeling the firms in such a way that the product characteristics of firm 1 are no larger than those of firm 2.

Exercise 26B.7

Suppose we do not restrict y_1 to be less than y_2. Given what we have done, can you plot the two firms' best response functions to the product characteristics chosen by the other firm and illustrate the stage 1 pure strategy equilibria? How many such equilibria are there? (*Hint*: Once the restriction that $y_1 \leq y_2$ is removed, there are two pure strategy equilibria.)

Answer: The table in the text indicates that firm 1's best response to any $y_2 \geq 0.5$ is $y_1 = 0$. This is illustrated in Exercise Graph 26B.7 by the bold vertical line segment on the vertical axis. The symmetry of the problem then tells us that firm 2's best response to any $y_1 \leq 0.5$ is $y_2 = 1$ — indicated by the bold dashed line segment emanating from the vertical axis at 1. If $y_2 \leq 0.5$, on the other hand, firm 1's best response is to pick $y_1 = 1$ — indicated by the vertical bold line segment emanating from the horizontal axis at 1; and if firm 1 chooses $y_1 \geq 0.5$, firm 2's best response is to set $y_2 = 0$ as indicated by the dashed line segment on the horizontal axis. (Note that for $y_i = 0.5$,, firm j in indifferent between choosing $y_j = 1$ or $y_j = 0$ — but not indifferent to settling between 0 and 1. Thus, the best response functions are not continuous at 0.5. They therefore intersect twice: Once at point A (where $y_1 = 0$ and $y_2 = 1$) and once at point B (where $y_1 = 1$ and $y_2 = 0$). Both equilibria are characterized by maximal product differentiation.



Exercise Graph 26B.7 : Hotelling First Stage Best Response Functions

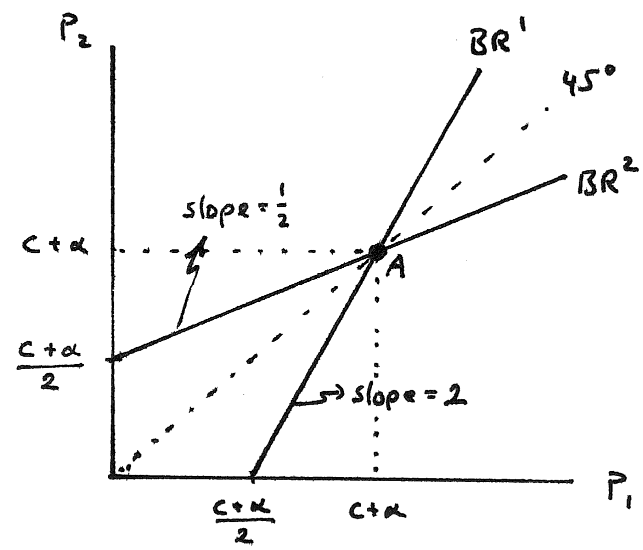
Exercise 26B.8

Can you plot the two firms' best response functions in stage 2 of the game given that $y_1 = 0$ and $y_2 = 1$ were chosen in the first stage? Carefully label slopes and intercepts. Are these prices the same for the two pure strategy equilibria in stage 1 that you identified in exercise 26B.7?

Answer: Plugging $y_1 = 0$ and $y_2 = 1$ into our equations for $p_1(p_2)$ and $p_2(p_1)$, we get

$$p_1(p_2) = \frac{c+\alpha}{2} + \frac{p_2}{2} \quad \text{and} \quad p_2(p_1) = \frac{c+\alpha}{2} + \frac{p_1}{2}. \tag{26B.8}$$

These are plotted in Exercise Graph 26B.8 — with intersection at A giving the equilibrium prices of $p_1 = p_2 = c + \alpha$.



Exercise Graph 26B.8 : Hotelling Second Stage Best Response Functions

Exercise 26B.9

Suppose that, instead of being quadratic as we have modeled them here, the cost that consumer n pays for consuming a product with characteristic $y \neq n$ is linear; i.e. suppose that this cost is $\alpha|n-y|$ where $|n-y|$ represents the distance between y and n. If the two oligopolists engage in maximal product differentiation (i.e. $y_1 = 0$ and $y_2 = 1$), is that product differentiation still socially excessive?

Answer: In this case, the marginal cost of deviating from the ideal point does not change as the distance from the ideal increases. It is therefore efficient for firms to settle at the extreme ends of the interval [0,1] — just as it is efficient for the firms to settle at 0.25 and 0.75. To see this, consider all the consumers whose ideal point falls in the interval [0.0.5]. For both cases, these consumers go to firm 1 — whether it is located at $y_1 = 0$ or at $y_1 = 0.25$. In case 1, consumer 0 does not pay a cost while consumer 0.5 pays the cost 0.5. In the second case, consumer 0 pays 0.25 and consumer 2 pays 0.25. In both cases, they pay 0.25 *on average*. The same is true for the next pair of consumers created by pairing the consumer next to consumer 0 with the consumer next (to the left) of consumer 0.5. Thus, when y_1 moves from 0 to 1, the increased cost to some is exactly offset by a decreased cost for others.

Exercise 26B.10

Why does the fixed entry cost FC not enter this problem? If you did include it in the definition of profit, would it make any difference?

Answer: By the time stage 2 happens, the firms have already chosen to enter and paid the fixed cost. Thus, the fixed cost is now a sunk cost and no longer an economic cost. If it were included in the expression for profit in the optimization problem, it would simply drop out as the first derivative is taken — thus leaving everything unchanged.

Exercise 26B.11

Verify $p_i(p)$.

Answer: Taking the first derivative of the objective function with respect to p_i and setting it to zero, we get

$$\left(\frac{p-p_i}{\alpha}+\frac{1}{N}\right)-\frac{(p_i-c)}{\alpha}=0 \qquad (26B.11.i)$$

which solves to

$$p_i(p)=\frac{p+c}{2}+\frac{\alpha}{2N}. \qquad (26B.11.ii)$$

Exercise 26B.12

Verify p^* and N^*.

Answer: N^* emerges straightforwardly from $(c+(\alpha/N)-c)(1/N)-FC=0$ when this is solved for N. Plugging $N^*=(\alpha/FC)^{1/2}$ into our previous equation for $p^*(N)=c+(\alpha/N)$. we get $p^*=c+(\alpha FC)^{1/2}$.

Exercise 26B.13

In both the Hotelling case and the "circle model", we have assumed for convenience that each consumer always just consumes one good from the firm that produces a product closest to her ideal. How does this assumption alleviate us from having to consider the price of output in our efficiency analysis (even though we know that firms end up pricing above MC)?

Answer: The assumption essential means we do not have to worry about changes in consumer surplus that result from changes in output price — because as price increases, each consumer still buys exactly 1 unit of output. This turns the increased expense to the consumer into a mere transfer to producers, with no efficiency loss in the process. It therefore allows us to focus solely on the issue of whether the number of firms is efficient.

Exercise 26B.14

What is the elasticity of substitution between x and the sub-utility over the y goods?

Answer: Since we have have combined the sub-utility over the y goods into a Cobb-Douglas function over x and the y goods, the elasticity of substitution between x and the sub-utility over the y goods is 1 (as it is for all Cobb-Douglas functions).

Exercise 26B.15

Demonstrate that the price elasticity of demand for y_j is $-1/(\rho+1)$.

Answer: Applying our formula for price elasticity of demand, we get

$$\begin{aligned}\frac{dy_j}{dp_j}\frac{p_j}{y_j(p_j)} &= \left(\frac{-\beta}{\rho+1}\right)p_j^{-(2+\rho)/(\rho+1)}\left(\frac{p_j}{\beta p_j^{-1/(\rho+1)}}\right)\\ &= \left(\frac{-\beta}{(\rho+1)p_j^{(2+\rho)/(\rho+1)}}\right)\left(\frac{p_j^{(2+\rho)(\rho+1)}}{\beta}\right)\\ &= \frac{-1}{\rho+1}. \end{aligned} \tag{26B.15}$$

Exercise 26B.16

Why do fixed entry costs not enter this problem?

Answer: Because they are sunk costs by the time the firm decides what price to charge.

Exercise 26B.17

Can you give an intuitive explanation for each of the 3 factors that causes firm output in the y market to increase?

Answer: As the goods become more substitutable, there is a need for fewer firms in the market, with each firm producing more; as fixed costs increase, fewer firms will enter — leaving more consumers for each particular firm; and as marginal costs decrease, price falls and consumers increase consumption (given their downward sloping demands).

Exercise 26B.18

Can you give an intuitive explanation for each of the 4 factors that increase product diversity in the y market?

Answer: The more value consumers place on the y goods, the greater the quantity demanded from each consumer and thus the greater the room for more firms in the market; the less substitutable the goods, the more room for specialized firms ; the greater the income of the consumers, the greater the overall demand for y goods and thus the more room for firms to enter the market; and the lower the fixed entry cost, the easier it is for firms to enter.

Exercise 26B.19

Verify the values for the column $\rho = -0.5$.

Answer: Plugging in the various parameters of the example, we get

$$\begin{aligned} p^* &= -\frac{c}{\rho} = -\frac{100}{-0.5} = 200 \\ y^* &= \frac{-\rho}{1+\rho}\left(\frac{FC}{c}\right) = \frac{0.5}{1-0.5}\left(\frac{100,000}{100}\right) - 1,000 \\ N^* &= \frac{(1-\alpha)(1+\rho)I}{FC} = \frac{(1-0.9)(1-0.5)(1,000,000,000)}{100,000} = 500. \end{aligned} \tag{26B.19}$$

Exercise 26B.20

What values in the table change if consumer income rises? What if consumers develop more of a taste for "eating out" — i.e. what if α falls? What if the fixed cost of setting up restaurants increases?

Answer: If income rises, only N^* changes. The same is true if α changes. If the FC changes, both y^* and N^* change — with firms producing more but fewer firms in the market as FC increases.

Exercise 26B.21

What is the equilibrium if $s < c$? What if $s = c$?

Answer: If $s < c$, no output would be produced; if $s = c$, price is set at c and consumers are indifferent between buying and not buying — implying the size of the market is not pinned down.

Exercise 26B.22

What is the equilibrium if $c \leq s < c + c_a$?

Answer: Since firms cannot price below $(c + c_a)$, and since consumers are not willing to pay more than $s > (c + c_a)$, the lack of information about available products keeps the market from operating.

Exercise 26B.23

Suppose the social planner decides to sell goods at $p = c$. Is consumer surplus the same in the market with advertising as under this social planner's solution? If not, how is overall surplus the same?

Answer: Since prices in the monopolistically competitive market are above marginal cost, consumer surplus will not be the same. However, since the social planner engages in exactly as much advertising as the market, the expected number of goods consumed will be the same in each case. Thus, the fact that consumers are paying a higher price in the market than under the social planner does not have an efficiency cost — the difference is simply a transfer from consumers to firms.

Exercise 26B.24

Verify that this best response function is correct.

Answer: Substituting our equation for $p(a_1, a_2)$ into the profit equation, we get firm 1's profit as

$$\pi^1 = 0.5\left(c + f(a_1, a_2) - c\right) - c_a a_1 = 0.5 f(a_1, a_2) - c_a a_1 = 0.5 a_1^{1/3} x_2^{1/3} - c_a a_1. \tag{26B.24.i}$$

The profit maximization problem for firm 1 is then

$$\max_{a_1} \pi^1 = 0.5 a_1^{1/3} x_2^{1/3} - c_a a_1. \tag{26B.24.ii}$$

Setting the derivative of π^1 with respect to a_1 equal to zero and solving for a_1, we get

$$a_1 = \frac{a_2^{1/2}}{6^{3/2} c_a^{3/2}} = \left(\frac{a_2}{216 c_a^3}\right)^{1/2}. \tag{26B.24.iii}$$

Exercise 26B.25

Can you determine whether firms are making positive profits in equilibrium? What happens as the cost of image advertising gets large? What happens as it approaches zero? Can you make sense of this within the context of the model?

Answer: The equilibrium price is

$$\begin{aligned} p^* = p(a_1^*, a_2^*) &= c + f(a_1^*, a_2^*) = c + (a_1^*)^{1/3}(a_2^*)^{1/3} \\ &= c + \left(\frac{1}{216 c_a^3}\right)^{1/3}\left(\frac{1}{216 c_a^3}\right)^{1/3} \\ &= c + \frac{1}{36 c_a^2}. \end{aligned} \tag{26B.25.i}$$

Firm profit is then

$$\pi = (p^* - c)\frac{1}{2} - c_a a^* = \frac{1}{2}\left(c + \frac{1}{36c_a^2} - c\right) - c_a\left(\frac{1}{216c_a^3}\right)$$
$$= \frac{1}{72c_a^2} - \frac{1}{216c_a^2} = \frac{1}{108c_a^2} > 0. \qquad (26B.25.ii)$$

As c_a gets large, profit therefore approaches zero since it is becoming harder to manipulate people's preferences. As c_a falls to zero, on the other hand, profit approaches infinity. This is because, it it costless to manipulate people's preferences, there is nothing to keep the firms from going all out — thus raising price toward infinity. Of course this latter prediction is not realistic — and it emerges in the model because we are assuming consumers will each consume 1 unit of the output *no matter what the price*.

Exercise 26B.26

Suppose boys tend to like "Fred Flinstone" and girls tend to like "Dora the Explorer". Interpret our model in terms of a cereal company placing "Dora the Explorer" on a cereal box with the intent of differentiating the cereal from otherwise identical cereal by a second firm that instead places "Fred Flinstone" on its cereal box. Do you think $\gamma > 0$ for the intended consumers (i.e. children)?

Answer: Children seem to view the cereal box as part of the product — with their favorite cartoon character appearing on the box increasing their utility from consuming the cereal. If this is so, $\gamma > 0$.

26C Solutions to End-of-Chapter Exercises

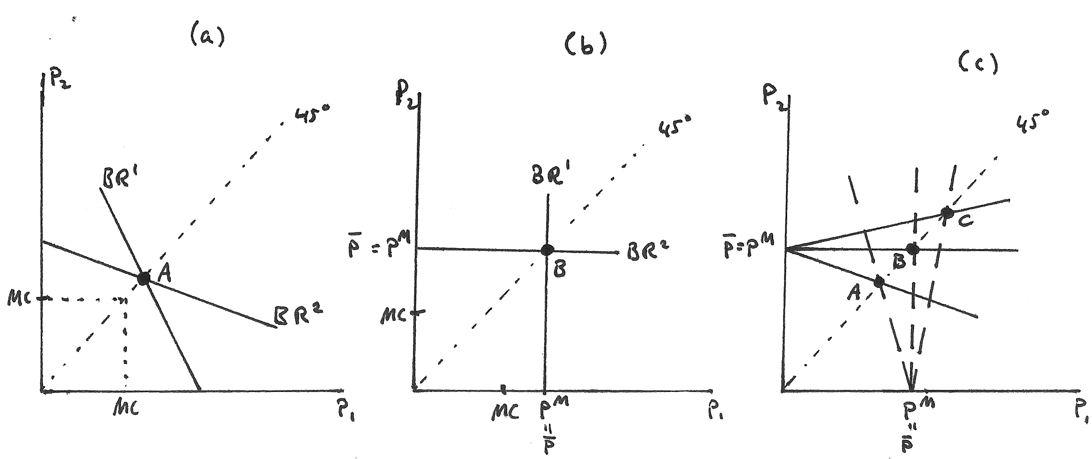
Exercise 26.1

We introduced the topic of differentiated products in a simple 2-firm Bertrand price setting model in which each firm's demand increases with the price of the other firm's output. The specific context we investigated was that of imperfect substitutes.

A: *Assume throughout that demand for each firm's good is positive at $p = MC$ even if the other firm sets its price to 0. Suppose further that firms face constant MC and no fixed costs.*

(a) *Suppose that instead of substitutes, the goods produced by the two firms are complements — i.e. suppose that an increase in firm j's price causes a decrease rather than an increase in the demand for firm i's good. How would Graph 26.3 change assuming both firms end up producing in equilibrium?*

Answer: This is illustrated in panel (a) of Exercise Graph 26.1. The complementarity of the goods now means that, as p_2 increases, the "residual" demand for firm 1's good *decreases* (or shifts *inward.*) This implies that firm 1 will set a lower price as a result of p_2 increasing. (You can convince yourself of this by drawing out two parallel demand curves, drawing the corresponding MR curves, and including a constant MC. Then see where the monopolist's solution for p^M falls in the two cases). As a result, best response functions slope down rather than up — leading to the Nash equilibrium indicated by A in the graph.



Exercise Graph 26.1 : Price competition under Different Assumptions

(b) *What would the in-between case look like in this graph — i.e. what would the best response functions look like if the price of firm j's product had no influence on the demand for firm i's product?*

Answer: In this case, the two firms are simply unrelated monopolies — and both will simply solve the monopoly problem — thus setting price at p^M regardless of what price the other firm sets. The best response functions are then as indicated in panel (b) of Exercise Graph 26.1 — with both firms ending up at B where both set their respective monopoly prices.

(c) *Suppose our three cases — the case of substitutes (covered in the text), of complements (covered in (a)) and of the in-between case (covered in (b)) — share the following feature in common: When $p_j = 0$, it is a best response for firm i to set $p = \overline{p} > MC$. How does $\overline{p}$ relate to what we would have called the monopoly price in Chapter 23?*

Answer: As already discussed in the answer to the previous part, $\overline{p}$ is equal to the monopoly price that each firm sets assuming the other operates in an entirely unrelated market.

(d) *Compare the equilibrium price (and output) levels in the three cases assuming both firms produce in each case.*

Answer: This is done in panel (c) of Exercise Graph 26.1 where A and B are the equilibria from panels (a) and (b) while C is the equilibrium for the case where the two firms produce imperfect substitutes. Clearly $p^A < p^B < p^C$ — implying that output is smallest when the firms produce imperfect substitutes and largest when they produce imperfect complements.

(e) *In which of the three cases might it be that there is no equilibrium in which both firms produce?*

Answer: Since we are assuming throughout that demand for each firm's good is positive at $p = MC$ even if the other firm sets its price to 0, we know that the best response functions start above MC. Thus, as long as the best response functions are not downward sloping, they will cross at $p > MC$ and both firms produce. But if the best response functions are downward sloping (as they are in the complements case in panel (a)), it may be that they cross on the 45 degree line below MC. In that case, the true best response function actually ends prior to reaching the 45-degree line. In that case, both firms cannot produce in equilibrium (since they would be making negative profit) and only one firm will be in the market.

B: *Consider identical firms 1 and 2, and suppose that the demand for firm i's output is give by $x_i(p_i, p_j) = A - \alpha p_i - \beta p_j$. Assume marginal cost is a constant c and there are no fixed costs.*

(a) *What range of values correspond to goods x_i and x_j being substitutes, complements and in-between goods as defined in part A of the exercise.*

Answer: If $\beta > 0$, then an increase in the price p_j causes causes a decrease in demand for good x_i — which implies the two goods are complements. If $\beta < 0$, an increase in the price p_j causes an increase in demand for x_i — thus the two goods are substitutes. When $\beta = 0$, a change in p_j has no effect on x_i — implying that we are in the in-between case.

(b) *Derive the best response functions. What are the intercepts and slopes?*
Answer: Following the steps in the text (where an almost identical demand equation was used — except that βp_j entered positively), we get

$$p_i(p_j) = \frac{A+\alpha c - \beta p_j}{2\alpha} \text{ and } p_j(p_i) = \frac{A+\alpha c - \beta p_i}{2\alpha}. \tag{26.1.i}$$

(c) *Are the slopes of the best response functions positive or negative? What does your answer depend on?*
Answer: The slope of $i's$ best response function to p_j is $-\beta/(2\alpha$ — which is positive when $\beta < 0$ (and the two goods are complements) and positive when $\beta > 0$ (when the two goods are substitutes.)

(d) *What is the equilibrium price in terms of A, α, β and c. Confirm your answer to A(d).*
Answer: Following the same steps as in the text — i.e. substituting $p_j(p_i)$ into $p_i(p_j)$ and solving for p_i, we get the equilibrium price

$$p^* = \frac{A+\alpha c}{2\alpha+\beta} \tag{26.1.ii}$$

at which the two firms are best responding to one another. Letting p_s^* denote the equilibrium price when the two goods are substitutes (i.e. $\beta <$ 0), p_n^* the price when they are neither substitutes nor complements (i.e. $\beta = 0$) and p_c^* the price when they are complements (i.e. $\beta > 0$), we then get

$$p_c^* < p_n^* < p_s^* \tag{26.1.iii}$$

as intuitively derives in part A. (Note that this holds so long as α is greater than the absolute value of β when $\beta < 0$ — which we explained in the text is a natural assumption to make for a model like this. Note also that p_n^* is equal to what we get for the monopoly price when we have a stand-alone monopoly — which we have two of when $\beta = 0$.)

(e) *Under what conditions will only one firm produce when the two goods are relatively complementary?*
Answer: Both firms will produce if $p^* \geq MC$; i.e. if

$$p^* = \frac{A+\alpha c}{2\alpha+\beta} \geq c. \tag{26.1.iv}$$

Solving this for c, we get $c = A/(\alpha+\beta)$. Thus, as long as marginal costs are not greater than $A/(\alpha+\beta)$, both firms produce in equilibrium.

Exercise 26.3

Business and Policy Application: *Mergers and Antitrust Policy in Related Product Markets: In exercise 26.1, we investigated different ways in which the markets for good x_i (produced by firm i) and good x_j (produced by firm j) may be related to each other under price competition. We now investigate the incentives for firms to merge into a single firm in such environments — and the level of concern that this might raise among antitrust regulators.*

A: *One way to think about firms that compete in related markets is to think of the externality they each impose on the other as they set price. For instance, if the two firms produce relatively substitutable goods (as described in (a) below), firm 1 provides a positive externality to firm 2 when it raises p_1 because it raises firm 2's demand when it raises its own price.*

(a) *Suppose that two firms produce goods that are relatively substitutable in the sense that, when the price of one firm's good goes up, this increases the demand for the other firm's goods. If these two firms merged, would you expect the resulting monopoly firm to charge higher or lower prices for the goods previously produced by the competing firms? (Think of the externality that is not being taken into account by the two firms as they compete.)*

Answer: As firm 1 raises its price, it only considers its own profit and not firm 2's profit. But as firm 1 raises price, it is raising demand — and thus profit — for firm 2. This is a positive externality that firm 1 is not taking into account — and, as a result, it will set "too low" a price relative to what the firms would do if they could make joint decisions (as they can if they merge into a single monopoly). Thus, if the two firms merge, the price of both goods will increase.

(b) *Next, suppose that the two firms produce goods that are relatively complementary in the sense that an increase in the price of one firm's good* decreases *the demand for the other firm's good. How is the externality now different?*

Answer: When firm 1 raises price, it now lowers demand (and profit) for firm 2 — thus emitting a negative (rather than a positive) externality that it is not taking into account when it sets its own price.

(c) *When the two firms in (b) merge, would you now expect price to increase or decrease?*

Answer: Since firm i is not taking into account the damage it does to firm j when it raises price, firm i will raise its price "too high" relative to what the two firms would decide jointly. Thus, as the firms merge, I would expect prices to fall.

(d) *If you were an antitrust regulator, which merger would you be worried about: The one in (a) or the one in (b)?*

Answer: You would worry about the merger in (a) — the merger of firms that produce relatively substitutable goods. In those cases, we determined that the merged firm will raise price — whereas in the case of complements we determined the merged firm will lower price. In both cases, the firms exercise market power when they merge into a monopoly — but in the case of complements, they are (by merging) eliminating an externality that resulted in too high a price. Anti-trust regulators worry about mergers resulting in higher prices as firms collude — and would probably not be concerned about mergers that result in a reduction in output price.

(e) *Suppose that instead the firms were producing goods in unrelated markets (with the price of one firm not affecting the demand for the goods produced by the other firm). What would you expect to happen to price if the two firms merge?*

Answer: If the two markets are unrelated and a change in price in one market has no impact on the demand for goods in the other, then the initial firms were already two separate monopolists. Without any relationship between the markets, a merger does not change this — the merged firm would continue to behave as a monopolist in the individual markets. Thus, in this case, we would not expect price to change.

(f) *Why are the positive externalities we encountered in this exercise good for society?*

Answer: The positive externality we encountered was for the case of competing oligopolistic firms in markets that produce relative substitutes. Here, neither firm took into account the "benefit" it creates for the other firm's profits as it raises price and thereby increases demand for the other firm. Because of this, each firm will set price lower than it would if it were colluding with the other firm — and this is good for consumers (which are the rest of society in this example).

B: *Suppose we have two firms — firm 1 and 2 — competing on price. The demand for firm i is given by $x_i(p_i, p_j) = 1000 - 10p_i + \beta p_j$, and each firm faces constant marginal cost $c = 20$ (and no fixed costs).*

(a) *Calculate the equilibrium price p^* as a function of β.*

Answer: Using either the results from the text or referring to results from exercise 26.1,

$$p^* = \frac{1,200}{20-\beta}. \tag{26.3.i}$$

(b) *Suppose that the two firms merged into one firm that now maximized overall profit. Derive the prices for the two goods (in terms of β) that the new monopolist will charge — keeping in mind that the monopolist now solves a single optimization problem to set the two prices. (Given the symmetry of the demands, you should of course get that the monopolist will charge the same price for both goods).*

Answer: The monopolist then solves the problem

$$\max_{p_1,p_2} \pi = (p_1 - 20)(1000 - 10p_1 + \beta p_2) + (p_2 - 20)(1000 - 10p_2 + \beta p_1). \tag{26.3.ii}$$

From the two first order conditions, we get

$$p_1 = \frac{600 - 10\beta + \beta p_2}{10} \text{ and } p_2 = \frac{600 - 10\beta + \beta p_1}{10}. \quad (26.3.iii)$$

Substituting the latter into the former and solving for p_1, we get

$$p_1 = \frac{6000 + 500\beta - 10\beta^2}{(100 - \beta^2)} = \frac{(600 - 10\beta)(10 + \beta)}{(10 - \beta)(10 + \beta)} = \frac{600 - 10\beta}{10 - \beta}. \quad (26.3.iv)$$

Substituting this back into our equation for p_2, we get the same for p_2. Thus,

$$p^M = p_1 = p_2 = \frac{600 - 10\beta}{10 - \beta}. \quad (26.3.v)$$

(c) *Create the following table: Let the first row set different values for β ranging from minus 7.5 to 7.5 in 2.5 increments. Then, derive the equilibrium price (for each β) when the two firms compete and report it in the second row. In a third row, calculate the price charged by the monopoly (that results from the merging of the two firms) for each value of β.*

Answer: This is done in the following table:

Competitive Oligopoly versus Monopoly							
β	-7.5	-5.0	-2.5	0	2.5	5.0	7.5
p^*	\$46.63	\$48.00	\$53.33	\$60.00	\$68.57	\$80.00	\$96.00
p^M	\$38.57	\$43.33	\$50.00	\$60.00	\$76.67	\$110.00	\$210.00
$2\pi_i^*$	\$11,174	\$15,680	\$22,222	\$32,000	\$47,184	\$72,000	\$115,520
π^M	\$12,071	\$16,333	\$22,500	\$32,000	\$48,167	\$81,000	\$180,500

(d) *Do your results confirm your intuition from part A of the exercise? If so, how?*

Answer: In part A, we discussed that fact that price setting oligopolists create externalities for each other as they raise price — a positive externality in the case of substitutes and a negative externality in the case of complements. As a result, they will price "too low" relative to the monopoly outcome in the case of substitutes and "too high" in the case of complements. This implies that price will rise as a result of a merger if the two goods are substitutes — and it will fall when they are complements. This is precisely what the table shows — with $\beta > 0$ representing cases where the two firms produce relative substitutes and with $\beta < 0$ implying they produce relative complements.

(e) *Why would firms merge if, as a result, they end up charging a lower price for both goods than they were able to charge individually?*

Answer: By merging, they can internalize the externality that keeps them from jointly attaining the maximum profit when they are in the oligopoly setting — and this is true whether the firms are creating positive or negative externalities for one another.

(f) *Add two rows to your table — calculating first the profit that the two firms together make in the competitive oligopoly equilibrium and then the profit that the firms make as a monopoly following a merger. Are the results consistent with your answer to (e)?*

Answer: This is done in the final two rows of the table from part (c). Note that profit increases as a result of the merger in all cases except for the case where $\beta = 0$ and the two markets are therefore entirely unrelated. In that special case, the two firms are independent monopolies to begin with — and merging simply makes them continue to do what they did before. But in all other cases, there is an externality that can be internalized when the firms merge — allowing the monopoly that owns both firms to make more profit than the firms can independently.

Exercise 26.5

Business Application: *Price Leadership in Differentiated Product Markets: We have considered how oligopolistic firms in a differentiated product market price output when the firms simultaneously choose price. Suppose now that two firms have maximally differentiated products on the Hotelling line [0,1] and that the choice of product characteristics is no longer a strategic variable. But let's suppose now that your firm gets to move first — announcing a price that your opponent then observes before setting her own price. This is similar to the Stackelberg quantity-leadership model we discussed in Chapter 25 except that firms now set price rather than quantity.*

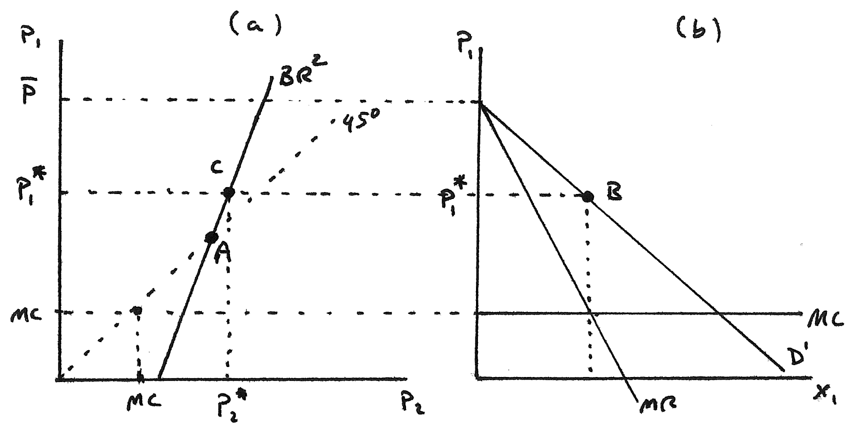
A: *Suppose you are firm 1 and your opponent is firm 2, with both firms facing constant marginal cost (and no fixed costs).*

(a) *Begin by reviewing the logic behind sequential pricing in the pure Bertrand setting where the two firms produce undifferentiated products. Why does the sequential (subgame perfect) equilibrium price not differ from the simultaneous price setting equilibrium?*

Answer: In the undifferentiated products case, your firm knows (when it announces price first) that your competitor will price just below you and get the whole market unless you price at marginal cost. Thus, you price at marginal cost, and your competitor does the same, with the two firms splitting the market. This is the same result that emerges in the case of simultaneous pricing decisions.

(b) *Now suppose that you are producing maximally differentiated products on the Hotelling line. When firm 2 sees your price p_1, illustrate its best response in a graph with p_2 on the horizontal and p_1 on the vertical axis.*

Answer: This is illustrated in panel (a) of Exercise Graph 26.5, with your competitor's best response denoted BR^2.



Exercise Graph 26.5 : Price Leadership with Differentiated Products

(c) *Include in your graph the 45-degree line and indicate where the price equilibrium falls if you and your competitor set prices simultaneously.*

Answer: The price equilibrium in the simultaneous move case falls at point A in panel (a) of Exercise Graph 26.5 — with both firms setting the same price above MC (because product differentiation softens the price competition that leads to price equal to marginal cost in the undifferentiated products case).

(d) *Let $\overline{p}$ be the price that results in zero demand for your goods assuming that your competitor observes $\overline{p}$ before setting her own price. Indicate $\overline{p}$ in a plausible place on your graph. Then, on a graph next to it, put p_1 on the vertical axis and x_1 — the good produced by your firm — on the horizontal. Where does your demand curve start on the vertical axis given that you take into account your competitor's response?*

Answer: This is illustrated in panels (a) and (b) of Exercise Graph 26.5 — with the demand curve for your good starting on the vertical axis at $\overline{p}$.

(e) *Draw a demand curve for x_1 and let this be the demand for x_1 given you anticipate your competitor's response to any price you set. Include MC and MR in your graph and indicate p_1^* — the price you will choose given that you anticipate your competitor's price response once she observes your price.*

Answer: This is done in panel (b) of Exercise Graph 26.5 where your firm sets the monopoly price given the demand curve it faces.

(f) *Finally, find your competitor's price p_2^* on your initial graph. Does it look like p_1^* is greater or less than p_2^*?*

Answer: Bringing p_1^* back to panel (a) of the graph, we now read off your competitor's best response from her best response function BR^2 — giving us her price response p_2^*. It looks like $p_1^* > p_2^*$ — which is exactly what we find mathematically in part B.

(g) *Who will have greater market share on the Hotelling line — you as the price leader, or your competitor?*

Answer: Given that you charge a higher price than your competitor, your competitor will have a larger market share than you.

B: *Suppose that the costs (other than price) that consumers incur is quadratic as in the text — i.e. a consumer n whose ideal point is $n \in [0,1]$ incurs a cost $\alpha(n-y)^2$ from consuming a product with characteristic $y \in [0,1]$. Continue to assume that firm 1 has located its product at 0 and firm 2 has located its product at 1 — i.e. $y_1 = 0$ and $y_2 = 1$. Firms incur constant marginal cost c (and no fixed costs).*

(a) *For what value of α is this the Bertrand model of Chapter 25? In this case, does the equilibrium price differ depending on whether one firm announces a price first or whether they announce price simultaneously? (Assume subgame perfection in the sequential case.)*

Answer: The product differentiation disappears when $\alpha = 0$ — in which case the two firms become pure Bertrand competitors that price at MC in equilibrium. As we point out in Chapter 25, this conclusion does not depend on whether the pricing game is sequential or simultaneous.

(b) *Now suppose $\alpha > 0$. If the firms set price simultaneously, what is the equilibrium price?*

Answer: Using the equations $p_1^*(y_1, y_2)$ and $p_2^*(y_1, y_2)$ from the text and substituting $y_1 = 0$ and $y_2 = 1$, we get

$$p_1^* = p_2^* = c + \alpha. \tag{26.5.i}$$

(c) *Next, suppose firm 1 announces its price first, with firm 2 then observing firm 1's price before setting its own price. Using the same logic we used in the Stackelberg model of quantity competition, derive the price firm 1 will charge (as a function of c and α.) (*Hint*: You can use the best response function for firm 2 derived in the text — substituting $y_1 = 0$ and $y_2 = 1$ — to set up firm 1's optimization problem.)*

Answer: Firm 1 knows firm 2's best price response function which is given in the text as $p_2(p_1)$. Substituting $y_1 = 0$ and $y_2 = 1$ into this equation, we get

$$p_2(p_1) = \frac{p_1 + c + \alpha}{2}. \tag{26.5.ii}$$

Since firm 1 moves first, subgame perfection requires that firm 1 then chooses its price by taking this best response function as given and solving the problem

$$\max_{p_1} (p_1 - c)\left(\frac{1}{2} + \frac{p_2(p_1) - p_1}{2\alpha}\right) \quad (26.5.\text{iii})$$

where the second term is demand for firm 1's output (derived from our results in the text by setting $y_1 = 0$ and $y_2 = 1$). Substituting equation (26.5.ii) into this maximization problem, we get that firm 1 will solve

$$\max_{p_1} (p_1 - c)\left(\frac{1}{2} + \frac{[(p_1 + c + \alpha)/2] - p_1}{2\alpha}\right) = (p_1 - c)\left(\frac{3\alpha + c - p_1}{4\alpha}\right). \quad (26.5.\text{iv})$$

Solving this in the usual way, we get

$$p_1 = c + \frac{3\alpha}{2}. \quad (26.5.\text{v})$$

(d) *What price does this imply firm 2 will set after it observes p_1? Which price is higher?*

Answer: Substituting $p_1 = c + (3/2)\alpha$ into equation (26.5.ii), we get

$$p_2 = c + \frac{5\alpha}{4}. \quad (26.5.\text{vi})$$

Thus, $p_1 > p_2$.

(e) *Derive the market shares for firms 1 and 2. In the Stackelberg quantity setting game, the firm that moved first had greater market share. Why is that not the case here?*

Answer: We can now determine the market share for firm 1 be finding the consumer type $\overline{n}$ that is indifferent between shopping from the two firms; i.e. we need to find n such that

$$p_1 + \alpha\overline{n}^2 = p_2 + \alpha(1 - \overline{n})^2 \quad (26.5.\text{vii})$$

which, given our solutions for p_1 and p_2, is

$$c + \frac{3\alpha}{2} + \alpha\overline{n}^2 = c + \frac{5\alpha}{4} + \alpha(1 - \overline{n})^2. \quad (26.5.\text{viii})$$

Solving this for $\overline{n}$, we get $\overline{n} = 0.375$. Thus, market share for firm 1 is 0.375 and market share for firm 2 is $(1 - 0.375) = 0.625$. Firm 1 now has lower market share because it sets a higher price in equilibrium — with firm 2 undercutting firm 1's price when it gets to observe it.

(f) *Derive profit for the two firms. Which firm does better — the leader or the follower?* True or False*: The quantity leader in the Stackelberg model has a first mover advantage while the price leader in the Hotelling model has a first mover* disadvantage.

Answer: Profit for firm 1 is

$$\pi_1 = (p_1 - c)\overline{n} = \left(c + \frac{3\alpha}{2} - c\right)(0.375) = 0.5625\alpha. \tag{26.5.ix}$$

Profit for firm 2 is

$$\pi_2 = (p_2 - c)(1 - \overline{n}) = \left(c + \frac{5\alpha}{4} - c\right)(0.625) = 0.78125\alpha. \tag{26.5.x}$$

The statement in the exercise is therefore true — firm 1 makes less profit than firm 2 because firm 2 has an advantage from knowing the price that firm 1 announces. Thus, firm 1 is at a disadvantage from having to move first.

(g) True or False*: Both firms prefer sequential pricing in the Hotelling model over simultaneous pricing (given maximal product differentiation).*

Answer: In the simultaneous pricing model, we derived the equilibrium price that both firms charge as $p^* = c + \alpha$ — with both firms then getting equal market shares of 0.5. Using this to calculate profit, we get

$$\pi = (p^* - c)(0.5) = (c + \alpha - c)(0.5) = 0.5\alpha. \tag{26.5.xi}$$

This is less than the profit either of the firms make when one of them is a price leader in the sequential setting. Thus, while firm 1 is at a disadvantage relative to firm 2 when firm 1 moves first, even firm 1 is better off in the sequential setting than when there is no price leadership.

Exercise 26.7

Business Application: *Deterring Entry of Another Car Company: Suppose that there are currently two car companies that form an oligopoly in which each faces constant marginal costs. Their strategic variables are price and product characteristics.*

A: *Use the Hotelling model to frame your approach to this exercise and suppose that the two firms have maximally differentiated their products, with company 1 selecting characteristic 0 and company 2 selecting characteristic 1 from the set of all possible product characteristics [0,1].*

(a) *Explain why such maximal product differentiation might in fact be the equilibrium outcome in this model.*

Answer: As we have shown repeatedly, two oligopolists can soften price competition by differentiating their product — thus both pricing above *MC* (which would not be possible if they did not differentiate their product).

(b) *Next, suppose a new car company plans to enter the market and chooses 0.5 as its product characteristic. If the new company enters in this way and existing companies can no longer vary their product characteristics, what happens to car prices? In what way can we view this as two distinct Hotelling models?*

Answer: We know that consumers whose ideal point lies to the left of 0.5 will buy either from firms 1 (whose product characteristic is 0) or from the new entrant — and consumers whose ideal point lies above 0.5 will either buy from firm 2 or from the new entrant. We can then consider the interval [0,0.5] as a Hotelling model with firms 1 and 3 located at the extremes of the interval, and we can consider the interval [0.5,1] as a second Hotelling model with firms 3 and 2 located at the extremes. Considering each model separately, we know that the equilibrium will involve each firm in each model to set the same price and thus split the market — with firms 1 and 2 getting one fourth of the overall [0,1] market and firm 3 getting half the market (in the interval [0.25,0.75]). Since products are less differentiated, price competition will imply that price will be lower if the third firm enters the market.

(c) *How much profit would the new company make relative to the original two?*

Answer: Since the new company has twice the market share of the other two, it will make twice the profit of the other two.

(d) *Suppose that the existing companies announce their prices prior to the new company making its decision on whether or not to enter. Suppose further that the existing companies agree to announce the same price. If the new company has to pay a fixed cost prior to starting production, do you think there is a range of fixed costs such that companies 1 and 2 can strategically deter entry?*

Answer: Firms 1 and 2 know that, if firm 3 enters, they will each see their prices fall and their market share reduced — thus experiencing a decrease in profit. They would therefore be willing to lower their prices somewhat if this has the effect of keeping firm 3 out of the market. Suppose the fixed entry cost is such that firm 3 would make a small profit (when fixed costs are included in the profit calculations) by entering. Then, if the firms can credibly commit to a price prior to firm 3 entering, they can lower prices a little bit and turn firm 3's profit from entering into a loss — thus causing firm 3 not to enter. As fixed costs fall, the amount that firms 1 and 2 would have to lower their price in order to deter entry increases — and eventually fixed entry costs will be sufficiently low such that the two firms would make less of a profit by lowering price and keeping half the market share that by just letting entry happen and restricting themselves to serving only a quarter of the market share. Thus, there is a range of fixed costs for which it would make sense for the two firms to announce a lower price that deters entry (assuming they can do so credibly.)

(e) *What determines the range of fixed costs under which the existing companies will successfully deter entry?*

Answer: We already reasoned through this in the previous part. The key for the incumbent firms is whether they will make more profit by lowering price and deterring entry or by just letting entry happen. In the former case, they keep half the market share each, in the latter they know they will be reduced to only a quarter of the market share.

(f) *If the existing companies had foreseen the potential of a new entrant who locates at 0.5, do you think they would have been as likely to engage in maximum product differentiation in order to soften price competition between each other?*

Answer: Since the appearance of this entry causes causes them to lower price — either because of strategic entry deterrence or because of entry, they might instead have chosen to accept stiffer price competition between themselves by not maximally differentiating their products in order to make it less attractive for this potential entrant to come into the market.

(g) *We have assumed throughout that the entrant would locate at 0.5. Why might this be the optimal location for the entrant?*

Answer: If the entrant locates anywhere else, he differentiating his product less from one of the two other firms — thereby engaging in fiercer price competition on one side of the market. By choosing 0.5, he is maximally differentiating his product from both other firms.

B: *Consider the version of Hotelling's model from Section 26B.2 and suppose that two oligopolistic car companies, protected by government regulations on how many firms can be in the car industry, have settled at the equilibrium product characteristics of 0 and 1 on the interval [0,1]. Suppose further that* $\alpha = 12,000$ *and* $c = 10,000$ *and assume throughout that car companies cannot change their product characteristics once they have chosen them.*

(a) *What prices are the two companies charging? How much profit are they making given that they do not incur any fixed costs (and given that we have normalized the population size to 1)?*

Answer: Using the equations $p_1^*(y_1, y_2)$ or $p_2^*(y_1, y_2)$ derived in the text and plugging in $c = 10,000$, $\alpha = 12,000$, $y_1 = 0$ and $y_2 = 1$, we get $p^* = \$22,000$. Profit for each firm is then

$$\pi^* = 0.5(p^* - c) = 0.5(22000 - 10000) = 6,000. \tag{26.7.i}$$

Note that this is normalized given we normalized the population to size 1. With population of 1 million, this would imply a profit of \$6 billion for each firm.

(b) *Now suppose that the government has granted permission to a third company to enter the car market at 0.5. But the company needs to pay a fixed cost FC to enter. If the third company enters, we can now consider the intervals* $[0, 0.5]$ *and* $[0.5, 1]$ *separately — and treat each of these as a separate Hotelling model. Derive* $D^1(p_1, p_3)$. *Then derive* $D^3(p_1, p_3)$ *(taking care to note that the relevant interval is now* $[0, 0.5]$ *rather than [0,1].)*

Answer: We can now consider a Hotelling model with two firms located at the extremes of the [0,0.5] interval — i.e. $y_1 = 0$ and $y_3 = 0.5$. The consumer $\overline{n}$ that is indifferent between shopping at these two companies would satisfy the equation

$$p_1 + \alpha(\overline{n} - y_1)^2 = p_3 + \alpha(\overline{n} - y_3)^2. \qquad (26.7.ii)$$

Substituting $y_1 = 0$ and $y_3 = 0.5$ and solving for $\overline{n}$, we get

$$\overline{n} = \frac{1}{4} + \frac{(p_3 - p_1)}{\alpha} = \frac{1}{4} + \frac{(p_3 - p_1)}{12{,}000}. \qquad (26.7.iii)$$

This is then demand for firm 1's cars — i.e.

$$D^1(p_1, p_3) = \frac{1}{4} + \frac{(p_3 - p_1)}{12{,}000}. \qquad (26.7.iv)$$

Demand (from the interval [0,0.5]) for firm 3's cars is then $(0.5 - \overline{n})$ or simply

$$D^3(p_1, p_3) = \frac{1}{4} + \frac{(p_1 - p_3)}{12{,}000}. \qquad (26.7.v)$$

(c) *Determine the best response functions $p_1(p_3)$ and $p_3(p_1)$. Then calculate the equilibrium price.*

Answer: To determine firm 1's best response function, we solve

$$\max_{p_1} (p_1 - c)D^1(p_1, p_3) = (p_1 - 10{,}000)\left(\frac{1}{4} + \frac{(p_3 - p_1)}{12{,}000}\right) \qquad (26.7.vi)$$

which gives us

$$p_1(p_3) = 6{,}500 + \frac{p_3}{2}. \qquad (26.7.vii)$$

For firm 3, we similarly solve

$$\max_{p_3} (p_3 - c)D^3(p_1, p_3) = (p_3 - 10{,}000)\left(\frac{1}{4} + \frac{(p_1 - p_3)}{12{,}000}\right) \qquad (26.7.viii)$$

which gives us

$$p_3(p_1) = 6{,}500 + \frac{p_1}{2}. \qquad (26.7.ix)$$

Substituting $p_3(p_1)$ into $p_1(p_3)$ and solving, we get $p_1 = 13{,}000$, and substituting that back into $p_3(p_1)$ we get the same for p_3. Thus, the equilibrium price is $p^* = \$13{,}000$. (The same analysis using firms 2 and 3 leads to the same result.)

(d) *How much profit will the 3 companies make (not counting the FC that any of them had to pay to get into the market)?*

Answer: Firms 1 and 2 will make profit $0.25(p^*-c) = 0.25(13,000-10,000) = 750$ since they each now have market share of 1/4 — while firm 3 will make twice that (since it has market share of 0.5). Thus,

$$\pi_1 = \pi_2 = 750 \text{ and } \pi_3 = 1,500. \tag{26.7.x}$$

(e) *If company 3 makes its decision of whether to enter and what price to set at the same time as companies 1 and 2 make their pricing decisions, what is the highest FC that will still be consistent with the new car company entering?*

Answer: Since firm 3 knows that it will make a profit of 1,500 if it enters, the highest fixed cost consistent with entry when everyone moves simultaneously is $FC = 1,500$.

(f) *Suppose instead that companies 1 and 2 can commit to a price before company 3 decides whether to enter. Suppose further that companies 1 and 2 collude to deter entry — and agree to announce the same price prior to company 3's decision. What is the most that companies 1 and 2 would be willing to lower price in order to prevent entry?*

Answer: The incumbent firms know that they will make a profit of 750 if they allow entry. The most they are willing to lower price is therefore an amount that will give them 750 in profit assuming firm 3 does not enter; i.e. the lowest $\overline{p}$ they would be willing to set to keep firm 3 out has to satisfy

$$0.5(\overline{p} - 10000) = 750. \tag{26.7.xi}$$

Solving this for $\overline{p}$, we get $\overline{p} = \$11,500$.

(g) *What is the lowest FC that would now be consistent with company 3 not entering? (Be careful to consider firm 3's best price response and the implications for market share.)*

Answer: Were the incumbent firms to set their price to \$11,500, firm 3's best price response is given by $p_3(p_1)$ as

$$p_3(11500) = 6,500 + \frac{11,500}{2} = \$12,250. \tag{26.7.xii}$$

But now firm 3 would not get half the market share because it is charging a price higher than the other two firms. To determine what market share it would get, we can determine what $\overline{n}$ is indifferent between firm 1 and firm 3; i.e. for what $\overline{n}$ does the following hold:

$$11,500 + 12,000\overline{n}^2 = 12,250 + 12,000(0.5 - \overline{n})^2. \tag{26.7.xiii}$$

Solving this, we get $\overline{n} = 0.3125$ — which is firm 1's market share. Firm 2's market share is the same — leaving firm 3 with market share of 0.375 and profit (absent fixed costs) of

$$\pi_3 = 0.375(12{,}250 - 10{,}000) = 843.75. \tag{26.7.xiv}$$

Thus, firms 1 and 2 cannot deter entry unless $FC > 843.75$. If incumbent firms can collude to set an entry-deterrent price prior to firm 3 deciding whether to enter, they will therefore be able to do so for fixed costs between 843.75 and 1,500.

Exercise 26.9

Business and Policy Application: *The Software Industry*: *When personal computers first came onto the scene, the task of writing software was considerably more difficult than it is today. Over the following decades, consumer demand for software has increased as personal computers became prevalent in more and more homes and businesses at the same time as it has become easier to write software. Thus, the industry has been one of expanding demand and decreasing fixed entry costs.*

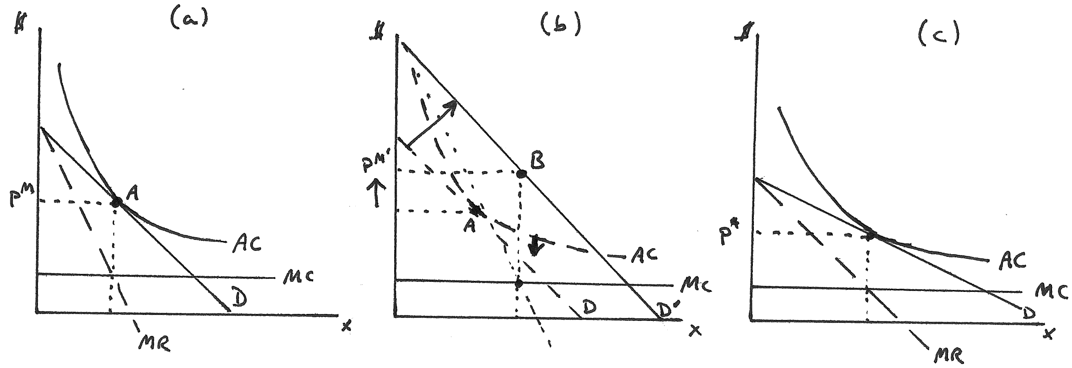
A: *In this part of the exercise, analyze the evolution of the software industry using both the monopolistic competition model from Section 26A.4 as well as insights from our earlier oligopoly models.*

(a) *Begin with the case where the first firm enters as a monopoly — i.e. the case where it has just become barely profitable to produce software. Illustrate this in a graph with a linear downward sloping demand curve, a constant MC curve and a fixed entry cost.*

Answer: This is done in panel (a) of Exercise Graph 26.9 where the usual monopoly picture (of price being set at the quantity where $MR = MC$) is supplemented by the inclusion of the AC curve that includes the fixed entry cost. If it has just become profitable for a single firm to enter, this AC curve must be tangent at A on the demand curve where profit maximizing production occurs. This then implies that $AC = p^M$ which in turn implies a profit from entering of zero (even as the firm makes positive profits once it has entered and fixed costs are sunk).

(b) *Suppose that marginal costs remain constant throughout the problem. In a separate graph, illustrate how an increase in demand impacts the profits of the monopoly and how a simultaneous decrease in fixed entry costs alters the potential profit from entering the industry.*

Answer: This is done in panel (b) of Exercise Graph 26.9 where demand increases from the (dashed) D to the (solid) D'. As a result, the monopoly price increases — thus raising monopoly profits for the firm that is in the market. This by itself already increases to profit for any potential firm that is considering entry — and this is further exacerbated by a drop in AC due to falling fixed entry costs.



Exercise Graph 26.9 : Evolution of software industry

(c) *Given the possibility of strategic entry deterrence, what might the monopolist do to forestall entry of new firms?*

<u>Answer</u>: The monopolist might decide to charge a price below p^M in order to make it less profitable for entrants to come into the market. In order to do this, the monopolist has to be able to credibly commit to a price even if entry of new firms happens.

(d) *Suppose the time comes when a strategic entry deterrence is no longer profitable and a second firm enters. Would you expect the entering firm to produce the same software as the existing firm? Would you expect both firms to make a profit at this point?*

<u>Answer</u>: The second firm has an incentive to differentiate its product so as not to set of the fiercest form of Bertrand competition. As a result, both firms will make a positive profit when the second firm enters — with entry costs still sufficiently high to forestall the entry of a third firm.

(e) *As the industry expands, would you expect strategic entry deterrence to play a larger or smaller role? In what sense is the industry never in equilibrium?*

<u>Answer</u>: Strategic entry deterrence with more than one incumbent firm would imply coordination of pricing by the incumbent firms — which becomes increasingly difficult as there are more incumbent firms. One would therefore expect a decline in strategic entry deterrence as the industry grows. The industry is never in equilibrium as long as demand continues to rise and fixed entry costs continue to fall — with new entry of firms becoming profitable as time passes.

(f) *What happens to profit for firms in the software market as the industry expands? What would the graph look like for each firm in the industry if the industry reaches equilibrium?*

<u>Answer</u>: As the market expands, competition becomes more intense — implying that firm profits will fall with time. If the industry were to reach

equilibrium, it would imply that all firms in the industry are making positive profit but entry costs are such that potential entrants make non-positive profits. This implies that profits including fixed entry costs are zero. This is illustrated for a monopolistically competitive firm in an industry that has reached equilibrium in panel (c) of Exercise Graph 26.9.

(g) *If you were an antitrust regulator charged with either looking out for consumers or maximizing efficiency, why might you not want to interfere in this industry despite the presence of market power? What dangers would you worry about if policy-makers suggested price regulation to mute market power?*

Answer: Your biggest fear might be that regulating this market will dampen innovation and will thus lead to a less diverse set of offerings of software for consumers. While it is true that, at any given moment, it is possible to raise overall surplus through price regulation (because of the presence of market power), it is also true that regulation to address this would dampen the incentives for new firms to emerge — and thus would cause future deadweight losses to be larger.

(h) *In what sense does the emergence of open-source software further weaken the case for regulation of the software industry? In what sense does this undermine the case for long-lasting copyrights on software?*

Answer: The emergence of open-source software provides another disciplining force on firms in the software industry. The threat of open source software supplanting a firm's software product provides incentives for software firms to continue to innovate and keep price relatively low — exactly what one would want in order to maximize both short run and long run efficiency and consumer welfare. The fact that the software industry continues to thrive despite open source software competition suggests that innovation in this market may not be primarily due to the existence of patents and copyrights.

B: *In this part of the exercise, use the model of monopolistic competition from Section 26A.4. Let disposable income I be \$100 billion, $\rho = -0.5$ and marginal cost $c = 10$.*

(a) *What is the assumed elasticity of substitution between software products?*

Answer: The elasticity of substitution σ is

$$\sigma = \frac{1}{1+\rho} = \frac{1}{1-0.5} = 2. \qquad (26.9.i)$$

(b) *Explain how increasing demand in the model can be viewed as either increasing I or decreasing α. Will either of these change the price that is charged in the market? Explain.*

Answer: The Cobb-Douglas functional form into which the y goods are incorporated implies that overall spending on the software goods is a fraction $(1-\alpha)$ of income. Thus, the overall share of income devoted to software increases in income and decreases in α. The equilibrium price in this model, however, is simply

$$p^* = -\frac{c}{\rho} = -\frac{10}{(-0.5)} = 20. \tag{26.9.ii}$$

which implies neither α nor I has an impact on price. (Price remains constant so long as c and ρ are unchanged because more firms will simply enter the industry to meet increased demand.)

(c) *We noted in part A of the exercise that fixed entry costs in the software industry have been declining. Can that explain falling software prices within this model?*

Answer: Since FC does not appear in the equilibrium equation for p^*, a decline in fixed entry costs cannot explain a change in price.

(d) True or False*: As long as the elasticity of substitution between software products remains unchanged, the only factor that could explain declining software prices in this model is declining marginal cost. (Can you think of real world changes in the software industry that might be consistent with this?)*

Answer: If the elasticity of substitution does not change, then ρ does not change — which only leaves c in our equation for p^* to alter the equilibrium price. Falling software prices can thus only be explained (as long as ρ stays constant) by decreasing marginal costs. As the software industry has developed, for instance, distribution has increasingly shifted to downloads from the internet rather than distribution through brick-and-mortar stores — which could be viewed as a decrease in marginal cost.

(e) *Now consider how increases in demand and decreases in costs translate to the equilibrium number of software firms. Suppose $\alpha = 0.998$ initially. What fraction of income does this imply is spent on software products? How many firms does this model predict will exist in equilibrium under the parameters of this model assuming fixed entry costs are \$100 million? What happens to the number of firms as FC falls to \$10 million, \$1 million and \$100,000?*

Answer: With $\alpha = 0.998$, the fraction of income spent on software is 0.002 or 0.2% of I. The equilibrium number of firms in this model is

$$N^* = \frac{(1-\alpha)(1+\rho)I}{FC}. \tag{26.9.iii}$$

With income of \$100 billion, $\alpha = 0.998$, and $\rho = -0.5$, we then get 1 firm when FC is equal to \$100 million, 10 software firms when FC is equal to \$10 million, 100 firms when FC is \$1 million and 1,000 firms when FC is \$100,000.

(f) *Suppose that FC is \$1,000,000. What happens as α falls from 0.998 to 0.99 in 0.002 increments as demand for software expands through changes in representative tastes when more consumers have computers?*

Answer: Using the same formula for N^*, the model predicts 100 firms when $\alpha = 0.998$ — increasing to 200 firms when $\alpha = 0.996$, 300 firms when $\alpha = 0.994$, 400 firms when $\alpha = 0.992$ and 500 firms when $\alpha = 0.99$.

(g) *Suppose FC is $1,000,000 and* $\alpha = 0.99$. *What happens if demand increases because income increases by 10%?*

Answer: The number of firms would increase from 500 to 550.

Exercise 26.11

Policy Application: *To Tax or Not to Tax Advertising*: *In the text, we discussed two different views of advertising — one of which we said arises primarily from an economist's perspective, the other primarily from a psychologist's. The nature of public policy toward the advertising industry will depend on which view of advertising one takes.*

A: *Consider the two views* — informational advertising *and* image marketing.

(a) *In what sense does information advertising potentially address a market condition that represents a violation of the first welfare theorem?*

Answer: Information advertising is aimed at providing information to consumers who otherwise lack sufficient information to make informed choices. As such, there is an asymmetric information problem in the market — firms know something that consumers do not. Thus, information advertising addresses an asymmetric information problem and can thus be efficiency enhancing — and may even be the efficient way for information to be conveyed. At the same time, the market may end up providing too little information — or it may provide too much. Still, it is an example of how a violation of the first welfare theorem may be addressed by the formation of a new market.

(b) *In what sense does image marketing result in potentially negative externalities? Might it result in positive externalities?*

Answer: If image marketing creates artificial preferences for products that are otherwise identical, then firms are simply using it to artificially create the appearance of product differentiation in order to soften price competition. In this case, image marketing creates a negative externality for consumers because it imposes costs on consumers who are not able to consume at their artificially created ideal points — and it creates market power that raises prices. Both parts of consumer costs — the prices they pay and the costs they incur from not consuming at their ideal points — are therefore raised without any meaningful underlying product differentiation. At the same time, it may be that image marketing actually changes products in a way that is not only meaningful to consumers (in the sense that they respond to it) but that also makes consumers better off (because they value both the product and the image of the product in an absolute sense). In that case — the case where, for instance, children get added utility from seeing cartoon characters on cereal boxes — it may be that image marketing creates positive externalities. If the welfare gain from increased utility on the part of consumers is sufficiently large, this may result in the benefits outweighing the costs from higher prices (that result from softened price competition).

(c) *If you wanted to make an efficiency case for taxing advertising, how would you do it? What if you wanted to make an efficiency case for subsidizing it?*

Answer: Within the model of information advertising, you would have to argue that the advertising market provides too much information — with a tax on advertising reducing the amount to a more efficient level. Alternatively, within the model of information advertising, you might find that firms provide inefficiently low levels of information — and a subsidy can get them closer to the efficient level. Within the model of image marketing, a tax can be efficiency enhancing if image marketing simply causes consumers to develop greater preferences for one good over another without changing their absolute utility from consuming the good. On the other hand, if image marketing also creates an increase in the absolute level of enjoyment of goods, one might even argue that advertising should be subsidized.

(d) *Suppose a public interest group lobbies for regulatory limits on the amount of advertising that can be conducted. Explain how this might serve the interests of firms?*

Answer: This is most easily seen in a model of information advertising where firms increase price competition by informing consumers of the presence of consumer goods. The Nash equilibrium may well be an equilibrium to a game that is a prisoners' dilemma for firms: All of them could make greater profit under less intense price competition in the absence of information advertising, but, regardless of what all other firms do, it is in the best interest of each firm to advertise its own products. The result is fiercer price competition which is good for consumers and bad for firms. If firms could then have the government enforce a collusive agreement to advertise less, they can essentially use the government to escape the prisoners' dilemma — thus leaving consumers worse off but themselves better off.

B: *Consider the three-stage image marketing model in Section 26B.6 but assume that $f(a_1, a_2) = a_1^{1/2} + a_2^{1/2}$. Suppose further that the cost for consumer n from consuming y is $\alpha(n-y)^2 - \gamma\alpha$, with $\gamma = 0$ unless otherwise stated.*

(a) *Solving the game backwards (in order to find subgame perfect equilibria), does anything change in stages 2 and 3 of the game?*

Answer: No, nothing changes. In stage 3, firms will set prices given the level of production differentiation y_1 and y_2 in stage 2 — and these best response price functions take α as given (just as derived earlier in the text). In stage 2, firms choose their product characteristics y_1 and y_2 — and we have shown that they will choose maximal differentiation $y_1 = 0$ and $y_2 = 1$ for $\alpha > 0$ given the other assumptions of the model.

(b) *What would be the advertising levels chosen by each firm.*

Answer: Each firm i in stage 1 then maximizes

$$\max_{a_i} (p(a_1,a_2)-c)\frac{1}{2}-c_a a_i = (a_1^{1/2}+a_2^{1/2}-c)\frac{1}{2}-c_a a_i \tag{26.11.i}$$

which results in equilibrium levels of advertising of

$$a_1^* = a_2^* = \frac{1}{4c_a^2}. \tag{26.11.ii}$$

(c) *Suppose the two firms can collude on the amount of advertising each undertakes (but the rest of the game remains the same). Would they choose different levels of a_1 and a_2?*

Answer: If the two firms maximized joint profit in the first stage (with the rest of the game remaining the same — i.e. no collusion in the rest of the game), then they maximize

$$\begin{aligned}\max_{a_1,a_2} \pi &= (p(a_1,a_2)-c)\frac{1}{2}-c_a a_1+(p(a_1,a_2)-c)\frac{1}{2}-c_a a_2 = \\ &= (p(a_1,a_2)-c)-c_a(a_1+a_2)\end{aligned} \tag{26.11.iii}$$

or, substituting $p(a_1,a_2)=a_1+a_2$,

$$\max_{a_1,a_2} \pi = (a_1^{1/2}+a_2^{1/2}-c)-c_a(a_1+a_2). \tag{26.11.iv}$$

The first order conditions to this optimization problem then imply

$$a_1^* = a_2^* = \frac{1}{4c_a^2} \tag{26.11.v}$$

just as in the case where they optimize separately. Thus, there are no gains from colluding in advertising.

(d) *For what level of $\gamma=\overline{\gamma}$ is there no efficiency case for either subsidizing or taxing advertising? What if $\gamma>\overline{\gamma}$? What if $\gamma<\overline{\gamma}$?*

Answer: For consumer costs of $\alpha(n-y)^2$, we concluded in Section B.2.3 that the cost incurred by consumers (when firms maximally differentiate their products) is $\alpha/12$. (The reasoning here is identical to the text's). To offset this cost given the decrease in consumer costs from $\gamma\alpha$, we therefore have to have $\overline{\gamma}=1/12$ in order for image advertising to have no efficiency consequences. (While firms will be able to charge higher prices as a result of the product differentiation — and thus consumers will be worse off from these higher prices, the model assumes consumers always buy 1 unit of the good, making the increased price a simple transfer from consumers to producers.)

If $\gamma>1/12$, image marketing is efficient, and if $\gamma<1/12$ is it inefficient.

(e) *Is there any way to come to a conclusion about the level of γ from observing consumer and firm behavior?*

Answer: No — the behavioral predictions are the same regardless of the value of γ. This is because γ does not enter the firms' optimization problems — and consumers are always assumed to buy one unit of the good.

Conclusion: Potentially Helpful Reminders

1. The idea that product differentiation can soften price competition should become quite intuitive. Starting with the stark Bertrand model prediction that only two firms are needed in an oligopoly in order for price to be set to marginal cost, the introduction of product differentiation gives us a way of softening this stark prediction and seeing how, even under Bertrand competition, market power can be preserved when the number of firms is small.

2. Our models of product differentiation allow us to finally fill in completely the gaps between the extremes of perfect competition and monopoly. We already were able to do that with the Cournot model of quantity competition in Chapter 25 — because the Cournot model predicts a convergence of price to marginal cost as the number of firms in an oligopoly increases.

3. More precisely, our models of firm entry into differentiated product markets with fixed entry costs allow us to see how we can "fill in the gaps" between perfect competition and monopoly even under Bertrand price competition. Barriers to entry can create an equilibrium in which firms inside a differentiated product market each have some market power and thus earn positive profit — while firms outside the market will not enter because of fixed entry costs.

4. If you got to this chapter, the hope is that it is among the most satisfying of chapters — because it really builds on all the insights from our perfectly competitive model as well as our models of market power. The end result is to get a more complete picture of the variety of market types that lie on the continuum between the extremes that we have investigated before. As such, the chapter should have a "real world flavor", with insights more easily translating to what we see in the world around us.

CHAPTER

27

Public Goods

For much of the text, we have so far neglected a large category of goods known as public goods. These are goods that can be consumed by more than one person at a time, and their existence is related closely to the topic of externalities that we first covered in Chapter 21. You can see this connection to externalities by simply thinking of producing a public good for yourself and then noticing that someone else is able to also consume it. You have, in effect, imposed a positive externality on someone else by producing the public good. Such goods can be *exclusionary* if it is possible to exclude people from the positive externality, or they can be *non-exclusionary* if it is not possible to exclude others. And it is when the good is non-exclusionary (such as national defense) that it becomes particularly difficult for markets to be effective providers of such goods. In the larger picture, one of the fundamental points that emerges is the problem faced by voluntary civil society efforts — the problem we identify in this Chapter as the "free-rider problem."

Chapter Highlights

The main points of the chapter are:

1. The reason that we would expect public goods to be under-produced (and under-consumed) can be found in the **free-rider problem** — the problem that everyone is tempted to "free-ride" on the public goods of others. This incentive to free-ride is in fact exactly the incentive modeled in the Prisoner's Dilemma (where it is a dominant strategy not to cooperate.)

2. The optimal level of public good output occurs where the **sum of the marginal benefits of consumption is equal to the marginal cost of producing the public good**. This is the same optimality condition that holds at the margin for private goods — but the difference is that, with private goods, the sum of the marginal benefits is the same as the marginal benefit to a single consumer.

3. There are essentially three ways to attack the free-rider problem inherent in public goods provision: (1) government provision through taxation or gov-

ernment subsidies of private contributions; (2) fostering markets — particularly for exclusionary public goods; and (3) fostering civil society engagement.

4. A fourth possible way to provide public goods efficiently lies in the area of **preference revelation mechanisms**, but these have not developed to a point where they are widely used.

27A Solutions to Within-Chapter-Exercises for Part A

Exercise 27A.1

True or False: The efficient level of public good production therefore occurs where marginal cost crosses the aggregate demand for public goods as drawn in Graph 27.1b.

Answer: This is true, particularly when the public good is quasilinear and demand curves are equal to marginal willingness to pay curves. If the public good is not quasilinear, the intersection of (vertically aggregated) demand still gives us an efficient level of public good — but now there are other efficient levels that correspond to different distributions of income in the population (as outlined more in the next exercise).

Exercise 27A.2

Can you explain how there is a single efficient level of the public good when tastes for public goods are quasilinear — but there are multiple levels of efficient public good provision when this is not the case? (*Hint*: Consider how redistributing income (in a lump sum way) affects demand in one case but not the other.)

Answer: When tastes are quasilinear in the public good, there are no income effects and thus the (vertically aggregated) demand curve does not change as income is redistributed (so long as it is not so dramatically redistributed that individuals end up at corners solutions). When tastes are not quasilinear, however, we can redistribute income among individuals (in an efficient lump-sum way) and thereby shift the aggregate demand curve for the public good. Thus, the intersection of marginal cost with demand will change as income is differently distributed — but the resulting provision level at the intersection will be efficient.

Exercise 27A.3

Does this production technology exhibit increasing or decreasing returns to scale?

Answer: It exhibits decreasing returns to scale since it becomes increasingly difficult to convert private goods into public goods as production of public goods increases.

Exercise 27A.4

What would the relationship in the graph look like if the technology had the opposite returns to scale as what you just concluded?

Answer: It would then bend inward rather than outward.

Exercise 27A.5

Why must the shaded areas in panels (b) and (c) of Graph 27.2 be equal to one another?

Answer: For any level of public good y, the amount of private good we have left to give to consumer 1 after making sure consumer 2 is on the indifference curve $\overline{u}_2$ is the vertical difference between the magenta indifference curve and the green production possibilities frontier in panel (b). It is precisely that distance which the forms the height of the green hill in panel (c) at that output level.

Exercise 27A.6

Is there any reason to think that y^* — the optimal level of the public good — will be the same regardless of what indifference curve for consumer 2 we choose to start with? How does your answer change when tastes are quasilinear in the public good? And how to does this relate to your answer to exercise 27A.2?

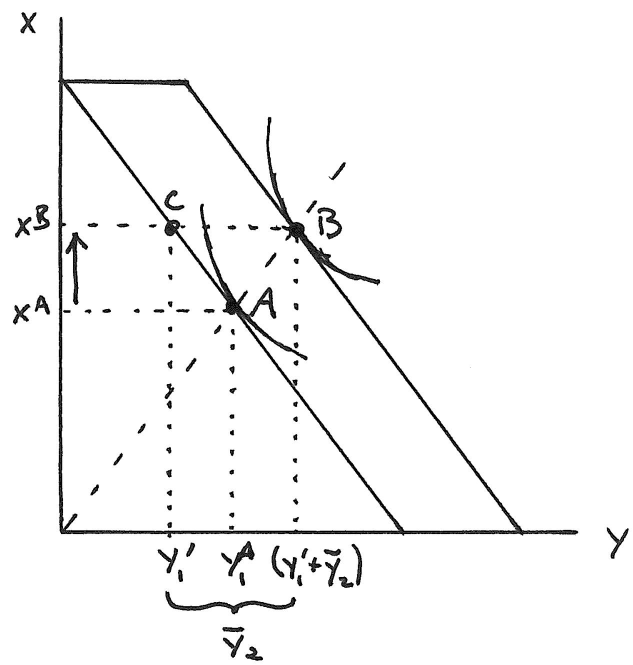
Answer: In general, there is no reason to think that y^* will always be the same regardless of what indifference level for consumer 2 we have chosen — the only thing we can conclude from our analysis is that, regardless of what y^* turns out to be, it will be the case that $MB_1 + MB_2 = MC_y$ at y^*. But if tastes are quasilinear, then the slopes of indifference curves do not change along any vertical line in our graph. Thus, as we move the magenta indifference curve for consumer 2 up and down in panel (b) of our graph, the MRS remains constant along any vertical line we might draw. When we then subtract the magenta curve from the green production possibilities frontier to get our "hill" in panel (b), the slopes of this hill at any given level of y will be unaffected by what indifference curve for consumer 2 we have chosen — because the slopes of the hill at any level of y is just the difference of the slopes of the production possibilities frontier and the magenta indifference curve (neither of which have changed at a given y if tastes are quasilinear). In essence, while the hill gets bigger (with $\underline{y}$ and $\overline{y}$ farther apart) as we go to lower magenta indifference curves in panel (b) of the graph, the slopes of the hill at any given level of y remain unchanged. And if consumer 1 then also has quasilinear tastes, this implies that his tangency will occur at the same y^*.

This relates to our answer to exercise 27A.2 where we concluded that the optimal public goods level is the same regardless of how we redistribute income between individuals if individual tastes are quasilinear — but it will not be the same if tastes are not quasilinear. In essence, we are doing the same thing now — as we move to higher or lower indifference curves in panel (b) of our graph, we are redistributing income between the two individuals — and if their tastes are quasilinear, this does not affect their marginal benefits at any public good level.

Exercise 27A.7

In a graph with y on the horizontal axis and a composite private good x on the vertical, illustrate my budget constraint assuming that $\overline{y}_2 = 0$. How does this budget constraint change when $\overline{y}_2 > 0$? Show that, if tastes are homothetic, I will end up consuming more y when $\overline{y}_2 > 0$ but will myself purchase less y. Does this hold whenever y and x are both normal goods? Does it hold if y is an inferior good?

Answer: This is illustrated in Exercise Graph 27A.7.



Exercise Graph 27A.7 : Contributions as $\overline{y}_2$ increase

The initial budget line when $\overline{y}_2$ is zero is the budget line that contains the optimal bundle A — with individual 1 contributing y_1^A to the public good (while individual 2 contributes nothing). The new budget constraint when $\overline{y}_2 > 0$ is the one containing B — it allows individual 1 to consume $\overline{y}_2$ even if he contributes nothing. If tastes are homothetic, B is the new optimal bundle for individual 1 — implying that his private consumption increases (from x^A to x^B), as does his public good consumption (from y_1^A to $(y_1' + \overline{y}_2)$. But individual 1's contribution to the public good can now be read off point C — which subtracts from his consumption level the amount $\overline{y}_2$ contributed by person 2. Thus, individual 1 now gives y_1' — which is less than y_1^A. Thus, he contributes less to the public good while consuming more of it. This holds anytime that both x and y are normal goods — because when both goods are normal, bundle B lies to the northeast of A. In fact, it holds anytime x — the composite good — is a normal good, because whenever x is normal, B will lie above A — implying that person 1's private consumption increases as individual 2 gives more to the public good. An increase in person 1's private consumption necessarily implies that he must be contributing less to the public good.

Exercise 27A.8

If you and I have identical tastes but I have more income than you, would the equilibrium fall above, on or below the 45 degree line (assuming all goods are normal goods)?

Answer: If we increase my income and decrease yours, then my (blue) best response function will shift up and your (magenta) best response function will shift down. This necessarily implies that — compared to the graph in the text — the intersection of our best response functions will now lie above the 45 degree line. Thus, I will buy more fireworks than you if my income is higher than yours (and our tastes are the same).

Exercise 27A.9

True or False: If everyone is currently giving to a public good — including the government — then this model would predict that the government's involvement has not done anything to alleviate the inefficiency of private provision of public goods.

Answer: This is true. If everyone is giving and the government taxes individuals to spend on the public good, then individuals would simply give that amount less to the public good. The government is in essence taking some income from each individual to spend on the public good — but each individual is still able to consume the same consumption bundle as before so long as she reduces her giving. Assuming no relative prices have changed — which is to assume that the government used lump sum taxes, the individual should therefore choose the same consumption bundle as before, replacing the taxes taken from her with less giving to the public good. In this way, the model predicts that individuals will undo what the government is doing by taxing and spending on the public good.

Exercise 27A.10

Could it be that an increase of government support for a public good causes someone who previously chose to give to that public good to cease giving? How would such a person's best response function look?

Answer: Yes, it could be that the government contribution is sufficiently large for a person to stop contributing to the public good. In that case, her best response function will have reached zero at the government contribution level.

Exercise 27A.11

Given what we have learned about the rate at which deadweight loss increases as tax rates rise, what would you expect to happen to the optimal level of government provision of a particular public good as the number of public goods financed by government increases?

Answer: The more public goods are finance by the government, the higher is the tax rate that the government has to use. Since deadweight loss increases roughly with the square of the tax rate, this implies that the optimal level of government public good provision declines the larger the government is — i.e. the more public goods it funds.

Exercise 27A.12

If a particular public good is subject to some partial "crowd-out" when governments contribute to its provision, might it be optimal for the government not to contribute to the public good in the presence of distortionary taxation?

Answer: Yes. If government contributions to a public good crowd out private contributions, it may be more efficient to have individuals contribute on their own (even though they will not contribute the efficient amount) — because the efficiency gains from increased public good provision through government contributions is outweighed by the loss in efficiency from distortionary taxation.

Exercise 27A.13

In Section B we show mathematically that the optimal subsidy will involve the government paying for half the cost of the fireworks if you and I have the same preferences. By thinking about the size of the externality — i.e. how much of the total benefit that is not taken into account by an individual consumer — does this make intuitive sense?

Answer: In contributing to the public good, I will take into account my marginal benefit MB_1 but not yours (MB_2). If our tastes are the same, $MB_1 = MB_2$ — which means I am taking into account exactly half the social benefit of contributing to the public good. Thus, if the government matches my contributions dollar for dollar, it is providing exactly the right Pigouvian subsidy — i.e. the subsidy that makes me take take into account twice the marginal benefit I took into account in the absence of the subsidy.

Exercise 27A.14

Could the government induce production of the efficient level of fireworks if it only subsidized the purchases of one of the consumers?

Answer: Yes. If it only subsidizes y_1, it will shift out only the blue best response function in our text graph — but if it shifts it out enough, we can get the blue best response function to intersect the magenta (unsubsidized) function at y^*.

Exercise 27A.15

True or False: Under an income tax that has increasing marginal tax rates as income goes up, the rich get a bigger per-dollar subsidy for charitable giving than the poor when charitable giving is tax deductible.

Answer: This is true. If your marginal tax rate is x%, it costs you \$$(1-x)$ to give \$1 to a charity. As x increases, the effective subsidy the government is giving you for contributing to charities therefore rises. In a progressive tax system, this implies the rich get a greater subsidy for charitable giving than the poor.

Exercise 27A.16

If the only way to finance the subsidy for private giving is through distortionary taxation, would you expect the optimal subsidy to be larger or smaller than if the subsidy can be finance through efficient lump sum taxes?

Answer: You would expect the optimal subsidy to then be smaller — because it would have to weight the efficiency gain from subsidizing charitable contributions against the efficiency loss of raising the revenues to fund such subsidies.

Exercise 27A.17

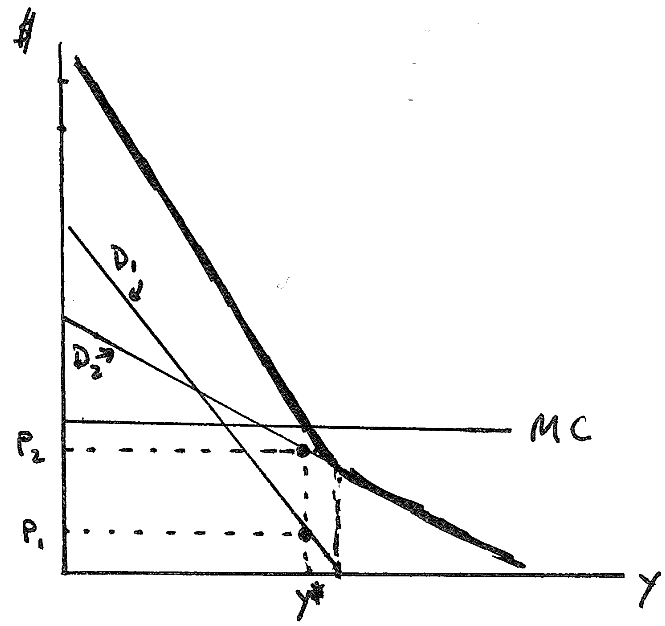
Can you think of other goods that are non-rivalrous (at least to some extent) but also excludable?

Answer: Movies in movie theaters; concerts; private schools; chess clubs; Disney World; golf courses; etc.

Exercise 27A.18

Illustrate, using a graph of two different demand curves for two different consumers, how a producer would calculate y^* — and what prices she would charge to each individual in order to get them to in fact choose y^* as their most preferred bundle.

Answer: This is illustrated in Exercise Graph 27A.18 where D_1 is individual 1's demand curve and D_2 is individual 2's. The bold curve is then the vertically summed total demand for the public good — with marginal cost intersecting at y^* which is the optimal quantity of the public good. If the producer produces y^*, she would then charge p_1 to consumer 1 and p_2 to consumer 2 — at which prices both consumers would choose to purchase y^* in their individual optimization problem. By construction, $p_1 + p_2 = MC$.



Exercise Graph 27A.18 : Lindahl Prices

Exercise 27A.19

Does the producer collect enough revenues under such individualized pricing to cover marginal costs?

Answer: The producer is essentially producing where MC intersects the (vertically aggregated) demand curve for the public good — with the sum of the individualized prices equaling the MC of the last good produced. Thus, the producer is able to finance his marginal costs (just as producers of private goods can finance their marginal costs when the produce until MC crosses demand.)

Exercise 27A.20

Consider the entrance fees to movie theaters on days when not every seat in the movie theater fills up. If it is generally true that older people and students have lower demand for watching new releases in movie theaters, can you explain entrance discounts for the elderly and for students as an attempt at Lindahl pricing?

Answer: When movie theaters charge different prices to different demographic groups, they are (statistically) price discriminating based on the average marginal willingness to pay by members of these groups. The correlation of marginal willingness to pay differences with price differences that are charged to members of different groups is then precisely what Lindahl had in mind.

Exercise 27A.21

Why do consumers not face the same incentive to lie about their tastes in such a "Tiebout" equilibrium as they do in a Lindahl equilibrium?

Answer: Consumers in the Tiebout model are just like consumers in private goods markets so long as there is a relatively competitive "market" of communities. As such, they become price takers who choose between alternatives, with communities having no more need to inquire about the consumers' tastes than Wal Mart has to inquire about my tastes for pants. In equilibrium, consumers with similar tastes then settle in the same community just as consumers with similar tastes for pants and shirts will buy similar pants and shirt. This differs from the Lindahl equilibrium where the producer tries to sell the same public good to many different types of consumers and thus has to ascertain the tastes of different types of consumers in order to set individualized prices.

Exercise 27A.22

In recent years, gated communities that provide local security services privately have emerged in many metropolitan areas that are growing quickly. Can you think of these from "club" perspective?

Answer: Such gated communities are very much like clubs in the sense that they have figured out a way to make their public good excludable. Only those who get into the gate experience the privately provided public security of the community. Such services are then typically paid for through homeowners association fees by those who live in the community.

Exercise 27A.23

Can you think of the provision of free access to swimming pools in condominium complexes in a way that is analogous to Coase's findings about lighthouses?

Answer: The public good of a lighthouse was bundled with the private good of docking rights in the harbor — just as the public good of swimming pools in condo complexes is tied to ownership of condos. Whether through homeowner fees (that are analogous to docking fees in harbors) or through higher sales values for condos when there is access to a swimming pool, the funds for providing the pool can therefore be raised because of the bundling of a private and a public good.

Exercise 27A.24

Explain how it is rational for me to give to both relieving poverty in the third world and to Alzheimer research in the presence of "warm glow" but not in its absence.

Answer: Both these public goods are large projects — and my contribution can only make a very tiny difference. In the absence of the warm glow effect, it would be rational for me to pick the one that I think is more important and give only to that effort. But if I also get a warm glow from giving to many different organizations — or particularly to these two, then I may well think that relieving poverty in the third world is more important than Alzheimer's research but give to both because I get some private utility from writing checks to the Alzheimer foundation and remembering my grandmother in the process.

Exercise 27A.25

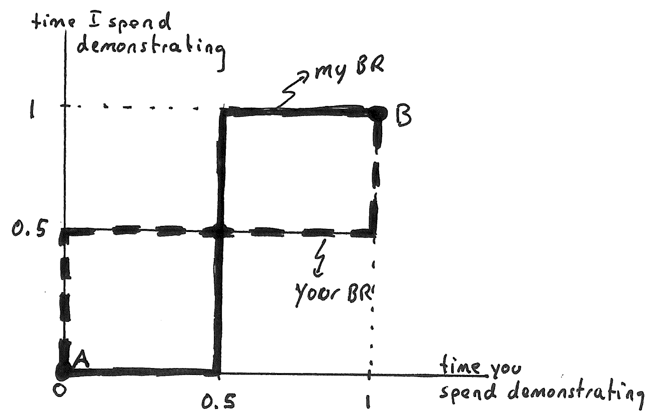
Can you use the "warm glow effect" to explain why government contributions to public goods (such as public radio) do not fully crowd out private contributions?

Answer: When the government taxes us and then gives our money to public radio, we may not get the same warm glow from being forced to pay taxes as we do from voluntarily writing a check to public radio. In such a case, I will still want to write a check to public radio in order to capture the private benefit from doing so — even if I realize that the government is already giving my money to public radio. After all, I still want the "I support public radio" sticker for my car!

Exercise 27A.26

Suppose my warm glow from demonstrating in the streets (for some worthy cause) depends on how much you demonstrate in the streets and vice versa. Letting the fraction of our time spent demonstrating go from 0 to 1, suppose that I do not get enough of a warm glow from demonstrating unless you spend at least half your time on the streets, and you feel similarly (about your warm glow and my participation). Illustrate our best response functions to each other's time on the streets. Where are the two stable pure strategy Nash equilibria — and where is the tipping point?

Answer: This is illustrated in Exercise Graph 27A.26. The two stable pure strategy equilibria are at points A and B — where neither of us demonstrates at A and both of us demonstrate all the time at B. The intersection of our best response functions at (0.5,0.5) is technically also an equilibrium — if you demonstrate half the day, I can best respond by also demonstrating half the day and vice versa. But it is not stable — slightly less protesting and we both slide to A, slightly more protesting and we both slide to B. Thus, we can think of this unstable in between "equilibrium" as the tipping point from one equilibrium to the other.



Exercise Graph 27A.26 : Demonstrating and Tipping Points

Exercise 27A.27

Suppose you have a piece of art that you would like to give to the person who values it the most but you do not know people's tastes. Explain how a second-price sealed bid auction (as described in exercise 27.10) represents a mechanism that accomplishes this while eliciting truthful messages from all interested parties.

Answer: A second-price sealed bid auction is an auction in which the highest bid wins and the winner pays an amount equal to the second highest bid. As demonstrated in the end-of-chapter exercise that is referenced, the equilibrium in such an auction is for everyone to bid their true value, with the person who values the item most highly therefore ending up receiving it. Thus, the incentives are such that everyone tells the truth about how much they value the item — and the person who values it most gets the item.

Exercise 27A.28

Suppose three people lived at the end of the culdesac and suppose the mayor proposes the same mechanism except that he now asks you for a \$333.33 check at the start (instead of \$500) and you are told (as player 1) that you will get a refund equal to $m_2 + m_3$ if $(m_1 + m_2 + m_3) \geq 0$ and the light is built. (Otherwise, you just get your \$333.33 back and no light is built.) Can you show that truth telling is again a dominant strategy for you?

Answer: The reasoning is virtually identical to that for 2 individuals. Suppose you are individual 1. At the time you decide what m_1 message to send to the mayor, you do not know what m_2 and m_3 messages sent by the others are. It may be that $-(m_2 + m_3) \leq v_1$ or it may be that $-(m_2 + m_3) > v_1$. If $-(m_2 + m_3) \leq v_1$, we can add m_2 and m_3 to both sides of the inequality and get $v_1 + m_2 + m_3 \geq 0$. Thus, if you send a truthful message of $m_1 = v_1$, $m_1 + m_2 + m_3 \geq 0$ and the street light will be built. Your resulting payoff is then $v_1 + m_2 + m_3 \geq 0$ which is at least as good as getting a payoff of 0 that would occur if you sent a false message that caused the light not to be built. Thus, if $-(m_2 + m_3) \leq v_1$, you should send a truthful message $m_1 = v_1$. Now suppose the other scenario is true — i.e. $-(m_2 + m_3) > v_1$. If, under that scenario, you again sent a truthful message $m_1 = v_1$, then $m_1 + m_2 + m_3 < 0$ and the street light does not get built and you get a payoff of 0. If you instead sent a false message that is high enough to get the street light built, your payoff will be $v_1 + m_2 + m_3 < 0$ — so again it's best to send the truthful message $m_1 = v_1$. Thus, *regardless of what messages m_2 and m_3 are sent by the other two people, it is your best strategy to send a truthful message about your own preferences.* Put differently, truth telling in this game is a dominant strategy.

Exercise 27A.29

Can you think of a case where our simple mechanism generates sufficient revenues to pay for the street light?

Answer: It generates a sufficient amount of revenue if $v_1 = v_2 = 0$ — i.e. if we both value the light at exactly \$500 and therefore send truthful messages that we value it 0 more than \$500. In that case, the government does not have to make any side-payments to us (because $m_1 = m_2 = 0$) and simply gets to keep the \$500 we contributed at the outset. The street light costs \$1,000 — so the government has just enough money.

Exercise 27A.30

Can you think of a case where the mechanism results in an outcome under which the city needs to come up with more money than the cost of the street light in order to implement the mechanism?

Answer: This is true for any $v_1 + v_2 > 0$ other than $v_1 = v_2 = 0$. Under truth telling, such values result in the street light being built — and the government therefore having to come up with \$1000. But unless $v_1 = v_2 = 0$, it will necessarily be the case that $m_1 + m_2 > 0$. Since the government is paying m_2 to person 1 and m_1 to person 2, the fact that $m_1 + m_2 > 0$ implies that it will have to return some of the \$1,000 it initially collected at the outset — leaving it short of the \$1,000 it needs to build the light.

27B Solutions to Within-Chapter-Exercises for Part B

Exercise 27B.1

Explain the constraint $y = f(X - \sum x_n)$.

Answer: The private good inputs left over for producing public goods after we have assigned the private good levels $(x_1, x_2, ..., x_N)$ is equal to the total X available minus the sum $\sum x_n$ of what has been allocated to individuals. Thus, the amount available as an input into public good production is $(X - \sum x_n)$ — and the amount of public good efficiently produced is therefore $f(X - \sum x_n)$.

Exercise 27B.2

Verify the outcome of this optimization problem. (*Hint*: Solve the first two first order conditions for λ and use your answer to derive the equation for $(x_1 + x_2)$.)

Answer: The Lagrange function is

$$\begin{aligned} \mathscr{L} = \alpha \ln x_1 + (1-\alpha)\ln(I_1 + I_2 - x_1 - x_2) + \\ + \lambda(\overline{u} - \alpha \ln x_2 - (1-\alpha)\ln(I_1 + I_2 - x_1 - x_2)). \end{aligned} \tag{27B.2.i}$$

The first order conditions imply that

$$\frac{\alpha}{x_1} = \frac{(1-\lambda)(1-\alpha)}{(I_1 + I_2 - x_1 - x_2)} \text{ and } -\frac{\lambda\alpha}{x_2} = \frac{(1-\lambda)(1-\alpha)}{(I_1 + I_2 - x_1 - x_2)} \tag{27B.2.ii}$$

which implies $\alpha/x_1 = -\lambda\alpha/x_2$ or $\lambda = -x_2/x_1$. Substituting this into the first of the first order conditions, we get

$$\frac{\alpha}{x_1} = \frac{(x_1 + x_2)(1-\alpha)}{x_1(I_1 + I_2 - x_1 - x_2)} \tag{27B.2.iii}$$

which can be solved to give us

$$x_1 + x_2 = \alpha(I_1 + I_2) \tag{27B.2.iv}$$

from which y^* follows as described in the text.

Exercise 27B.3

What is y^* if there are N rather than 2 consumers of the type described in our example (i.e. with the same Cobb-Douglas tastes but different incomes)? What if everyone's income is also the same?

Answer: The (vertically) aggregated demand for the public good is then

$$\frac{(1-\alpha)I_1}{y} + \frac{(1-\alpha)I_2}{y} + ... + \frac{(1-\alpha)I_N}{y} = \frac{(1-\alpha)\left(\sum_{n=1}^N I_n\right)}{y}. \tag{27B.3.i}$$

Setting demand equal to the marginal cost of producing y — which, given our assumption about the production process, is 1, the social planner solves

$$\frac{(1-\alpha)\left(\sum_{n=1}^{N} I_n\right)}{y} = 1 \tag{27B.3.ii}$$

to get

$$y^* = (1-\alpha)\left(\sum_{n=1}^{N} I_n\right). \tag{27B.3.iii}$$

If everyone has the same income, this reduces to

$$y^* = N(1-\alpha)I. \tag{27B.3.iv}$$

Exercise 27B.4

Explain why $p_1 = p_2 = 1$ for both consumers in the absence of subsidies for giving to the public good.

Answer: A contribution of \$1 costs the consumer exactly \$1 in the absence of a subsidy — thus it costs each individual n \$1 to buy one dollar's worth of contribution of z_n.

Exercise 27B.5

Draw the best response functions for the two individuals in a graph similar to Graph 27.3. Carefully label intercepts and slopes.

Answer: This is done in Exercise Graph 27B.5.

Exercise 27B.6

Why do private contributions to the public good result in the optimal level of the public good when $\alpha = 0$?

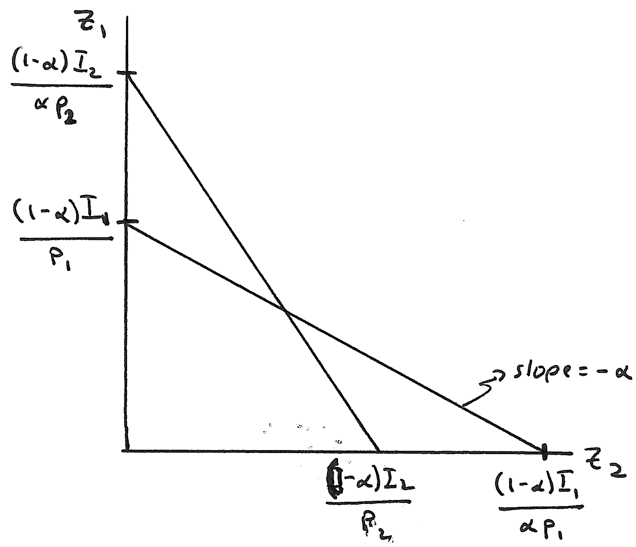
Answer: Because when $\alpha = 0$, the individuals do not care about anything other than the public good — which means they will spend all their income on the public good because there is nothing else of value to spend it on.

Exercise 27B.7

Consider the equilibrium public good level as a fraction of the optimal public good level. In our example, what is the lowest this fraction can become, and what is the critical variable?

Answer: We get that

$$\frac{y^{eq}}{y^*} = \frac{1}{1+\alpha}. \tag{27B.7}$$



Exercise Graph 27B.5 : Best Response Functions

The lowest value this fraction can take is 1/2 — when $\alpha = 1$. The greater α is — i.e. the more individuals value private over public good consumption, the more the public good is underprovided through voluntary contributions when measured as a fraction of the efficient level of contributions.

Exercise 27B.8

As N gets larger, what do y^* and y^{eq} converge to for the example in Table 27.2? What does the equilibrium level of public good as a fraction of the optimal level converge to?

Answer: The optimal level y^* is $500N$ while the equilibrium level converges to \$1,000. Thus, as N gets large, y^{eq}/y^* approaches zero.

Exercise 27B.9

Can you explain in a bit more detail why the tax in this case is efficient?

Answer: We have assumed in our setting that income for the two individuals is exogenous — that is, it does not depend on what the individuals do (as it would if the individuals needed to sell endowments to get income). When the government imposes a tax rate t on exogenous income, it then simply shifts the budget constraint in without changing opportunity costs — i.e. without changing the slopes of the budget constraints and thus without giving rise to substitution effects. Since substitution effects are what gives rise to deadweight losses from taxation, a tax such as the one in this example is efficient.

Exercise 27B.10

Demonstrate that these best response functions converge to those in equations (27.20) and (27.21) as g goes to zero.

Answer: When $g=0$, the best response for individual 1 becomes

$$z_1(z_2,0) = \frac{(1-\alpha)I_1(I_1+I_2)}{(I_1+I_2)p_1} - \alpha z_2 = \frac{(1-\alpha)I_1}{p_1} - \alpha z_2 \tag{27B.10}$$

which is equal to what we derived previously in the absence of government contributions. The same holds for $x_2(x_1,0)$.

Exercise 27B.11

Can you tell if there is any crowd-out for the last dollar spent by the government if the government provides the optimal level of the public good in this case?

Answer: The optimal level of public good in the case when $I_1 = I_2$ is $2(1-\alpha)I$. We just concluded in the text that individuals will stop contributing when $2(1-\alpha)I/(1+\alpha)$ — which is less than the optimal public good level. Thus, when the government provides the optimal level of the public good, individuals are at their corner solution where they already give nothing — which implies there is no crowd-out at the margin.

Exercise 27B.12

Can you offer an intuitive explanation for why $s^* = 1/2$? How would you expect this to change as the number of consumers increases?

Answer: Each individual is taking only half the social benefit into account when giving to the public good — so reducing the price to half insures that they are both internalizing this externality.

Exercise 27B.13

We previously concluded that the optimal level of the public good is $(1-\alpha)(I_1+I_2)$. Can you use our solutions for s^* and t^* to show that this level is achieved through the voluntary contributions of the 2 individuals when the policy (s^*, t^*) is implemented?

Answer: We concluded in the text that the total contributions by the two individuals under policy (s,t) is

$$y^v(t,s) = \frac{(1-\alpha)(1-t)(I_1+I_2)}{(1+\alpha)(1-s)}. \tag{27B.13.i}$$

Substituting $s^* = 1/2$ and $t^* = (1-\alpha)/2$, we get

$$\begin{aligned} y^v(t^*, s^*) &= \frac{(1-\alpha)(1-(1-\alpha)/2)(I_1+I_2)}{(1+\alpha)(1-(1/2))} = \\ &= \frac{(1-\alpha)((1+\alpha)/2)(I_1+I_2)}{(1+\alpha)/2} = \\ &= (1-\alpha)(I_1+I_2). \end{aligned} \tag{27B.13.ii}$$

Exercise 27B.14

What do you think p_n will be in the N-person case if everyone shares the same Cobb-Douglas tastes? What if they also all have the same income level?

Answer: We would then have

$$p_n = \frac{I_n}{\sum_1^N I_j} \tag{27B.14}$$

which simply becomes $p_n = I/NI = 1/N$ when all incomes are the same.

Exercise 27B.15

What is different for Bill Gates that might make him rationally contribute to multiple charities?

Answer: Bill Gates gives billions of dollars. Suppose $\partial F/\partial Y_a$ is greater than $\partial F/\partial Y_b$ and $\partial F/\partial Y_c$. This implies that Bill Gates would first give to charity a since he can make the most difference there. But, given how much he is giving, it may well be that, after he gives to a, $\partial F/\partial Y_a$ is now smaller than $\partial F/\partial Y_b$ and $\partial F/\partial Y_c$, — in which case he should move onto giving to other charities. It is therefore quite conceivable that he does not believe $\partial F/\partial Y_a = \partial F/\partial Y_b = \partial F/\partial Y_c$ before he gives — but that he still ends up giving to all three charities because he thinks (quite likely correctly) that his money is making a large difference in each.

Exercise 27B.16

Suppose I only give to small local charities. In what way might I then be like Bill Gates and give rationally to more than one?

Answer: In this case, even if I give substantially less than Bill Gates, it may be that my contribution to charity a can alter $\partial F/\partial Y_a$ sufficiently to cause me to rationally give to the next charity in line. This time it is not because my contribution is large in an absolute sense — it is simply that the charities are small and so my contributions become large in a relative sense.

Exercise 27B.17

Can you explain why it is rational to diversify a private investment portfolio in the presence of risk and uncertainty but the same argument does not hold for diversifying our charitable giving?

Answer: When we balance our financial portfolios, we are trying to trade off our own personal risk and return. We are not attempting to influence the welfare of corporations in which we invest — if we were, we would similarly not make a significant enough difference to invest in more than one corporation. But in our charitable giving, we are attempting to make a difference in the cause in which the charity is involved — and unless we give enough to make a sizable difference in this way, we would invest in only a single charity.

Exercise 27B.18

Verify our derivation of z^{eq} and z^*. Then demonstrate that z^{eq} converges to z^* as β goes to zero. Can you make intuitive sense of this?

Answer: The first order condition of our maximization problem to derive z^{eq} is

$$\frac{-\alpha}{I-z_1}+\frac{\beta}{z_1+(N-1)z}+\frac{\gamma}{z_1}=0. \qquad (27B.18.i)$$

Multiplying through by the three denominators, this becomes

$$-\alpha z_1(z_1+(N-1)z)+\beta z_1(I-z_1)+\gamma(I-z_1)(z_1+(N-1)z)=0. \qquad (27B.18.ii)$$

Collecting terms and adding the negative terms to both sides, we then get

$$(\alpha+\beta+\gamma)z_1^2+(\alpha+\gamma)(N-1)zz_1=(\beta+\gamma)Iz_1+\gamma(N-1)Iz. \qquad (27B.18.iii)$$

Replacing z_1 with z, this becomes

$$(\alpha+\beta+\gamma)z^2+(\alpha+\gamma)(N-1)z^2=(\beta+\gamma)Iz+\gamma(N-1)Iz. \qquad (27B.18.iv)$$

Dividing through by z, we get

$$(\alpha+\beta+\gamma)z+(\alpha+\gamma)(N-1)z=(\beta+\gamma)I+\gamma(N-1)I \qquad (27B.18.v)$$

which we can solve to get

$$z^{eq}=\frac{(\beta+\gamma N)I}{\beta+(\alpha+\gamma)N}. \qquad (27B.18.vi)$$

To derive z^*, we recall our previous result in the absence of a warm glow effect where, when $u(x_n,y)=x_n^{\alpha}y^{(1-\alpha)}$, the optimal public good level is $(1-\alpha)NI$ (assuming there are N identical individuals). Where we to simply write this utility function as $u(x_n,y)=x_n^{\alpha}y^{\beta}$ without restricting $\alpha+\beta$ to equal 1, the same optimality condition would be written as an optimal public good level of $\beta IN/(\alpha+\beta)$. The only difference now is that we have an additional z_n^γ term in the utility function. But z_n is just a private good for each individual — with the optimal solution in the Cobb-Douglas case implying that the consumer should devote a fraction $\gamma/(\alpha+\beta+\gamma)$ to this good and $\alpha/(\alpha+\beta+\gamma)$ to the x good. This leaves us with the optimal fraction $\beta/(\alpha+\beta+\gamma)$ to be contributed to the y good — and, since the z good also gets private utility, a total optimal contribution of

$$z^* = \frac{(\beta+\gamma)I}{\alpha+\beta+\gamma}. \qquad (27B.18.vii)$$

When $\beta = 0$, we can then substitute this into z^{eq} and z^* to get

$$z^{eq}_{\beta=0} = \frac{\gamma I}{(\alpha+\gamma)} = z^*_{\beta=0}. \qquad (27B.18.viii)$$

This makes intuitive sense in the following way: When $\beta = 0$, individuals no longer care about the sum of all the contributions to the public good — only about the level of their own individual contribution. And when this happens, we are essentially left with only private goods from the individual's perspective — consumption of x and contributions to the public good z. The free rider problem therefore no longer exists.

Exercise 27B.19

Suppose the above example applies to a pastor whose congregation has 1,000 members that get utility from overall donations y to the church as well as their own individual contribution z_n. Each member makes \$50,000 and tastes are defined as in equation (27.51) with $\alpha = 0.5$, $\beta = 0.495$ and $\gamma = 0.005$. The pastor needs to raise \$1 million for a new church. He can either put his effort into doubling the size of his congregation, or he can put his energy into fiery sermons to his current congregation — sermons that will change γ to 0.01 and β to 0.49. Can you show that these will have roughly the same impact on how much he collects?

Answer: We can calculate the various amounts of giving by using the equation

$$z^{eq} = \frac{(\beta+\gamma N)I}{\beta+(\alpha+\gamma)N} \qquad (27B.19)$$

that we derived in the text. Before the pastor does anything, $I = 50,000$, $\alpha = 0.5$, $\beta = 0.495$, $\gamma = 0.005$ and $N = 1,000$. Plugging these into the equation and multiplying by 1,000, we get the current level of giving that the pastor can expect as \$543,527. If we change N to 2,000 in the equation — and then multiply the individual contributions by 2,000, we get that overall donations will rise to \$1,038,600. If we leave N as 1,000 but alter β and γ to 0.49 and 0.01 (respectively), we get overall donations of \$1,027,444 — roughly the same as if we simply doubled the congregations.

Exercise 27B.20

Suppose $\delta_2 = 1$. Using $\delta_1 = -0.01$ and the values z^{eq}_{low} and z^{eq}_{high} in the table, derive the implied level of γ in the two equilibria. (Note that these will not match the ones discussed in the text because the table does not normalize all exponents in the utility function to sum to 1). Then, using the parameters for I, N, α and β provided in the table, employ equation (27.54) to verify z^{eq}_{low} as well as z^{eq}_{high}.

Answer: For the two equilibria, we get

$$\gamma_{low} = -0.01 + \frac{16.79}{1,000} = 0.00679 \text{ and } \gamma_{low} = -0.01 + \frac{593.17}{1,000} = 0.58317. \quad (27B.20.i)$$

Using these, together with the other parameter values, in our equation for z^{eq}, we get

$$z^{eq}_{low} = \frac{(0.4 + 0.00679(10,000))(1,000)}{0.4 + (0.4 + 0.00679)(10,000)} = 16.79 \quad (27B.20.ii)$$

and

$$z^{eq}_{high} = \frac{(0.4 + 0.58317(10,000))(1,000)}{0.4 + (0.4 + 0.58317)(10,000)} = 593.17. \quad (27B.20.iii)$$

Exercise 27B.21

Illustrate in a graph similar to Graph 27.6 what the payment $P_i(p_i)$ for this individual would be if p_i is sufficiently high such that $\overline{y}_i > y^*$.

Answer: This is illustrated in Exercise Graph 27B.21. Here, p_i falls above the intersection of the two curves. The optimal quantity (given reported demands) is still y^* — but now $\overline{y}_i > y^*$. The fixed part of the Groves-Clarke charge P_i is still $p_i\overline{y}_i$ — which is equal to area $(a+b+c+d+e)$. But the part of the charge that depends on i's reported demands is now subtracted rather than added — and this part is again the area under the upward sloping curve between y^* and $\overline{y}_i$. The net charge is then

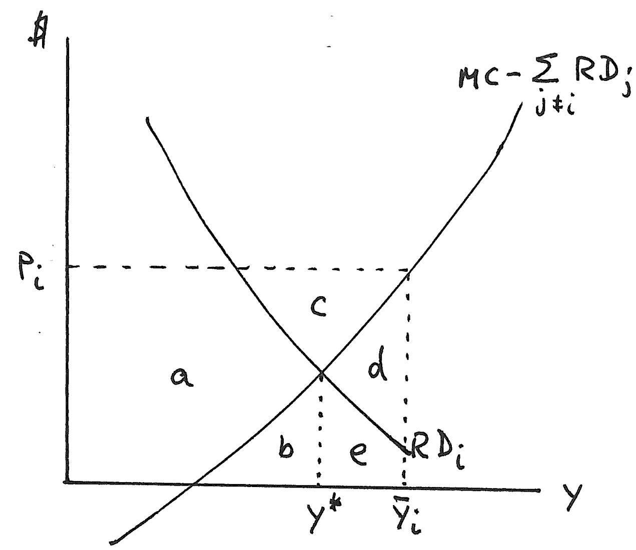
$$P_i = a + b + c. \quad (27B.21)$$

Exercise 27B.22

In Graph 27.6, we considered the case in which $\overline{y}_i < y^t$. Repeat the analysis above to show that over- and under-reporting is similarly counterproductive when p_i is sufficiently high to cause $\overline{y}_i > y^t$.

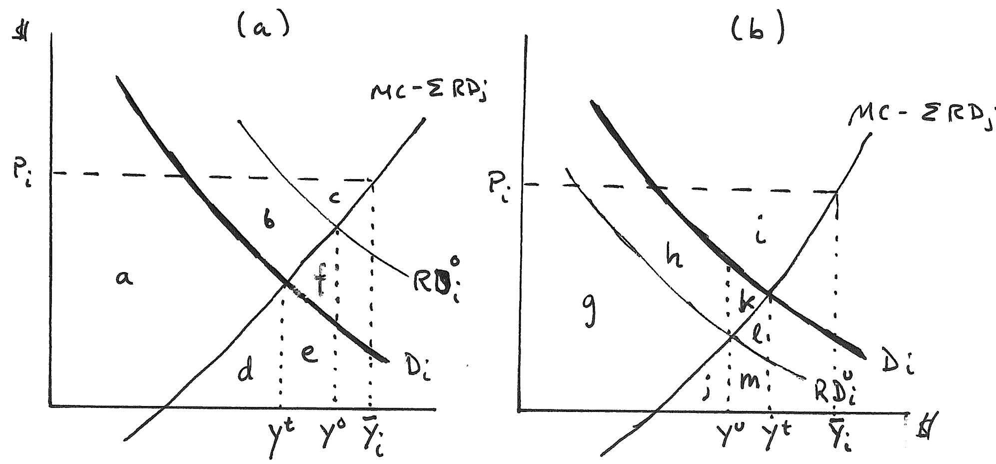
Answer: This is illustrated in the two panels of Exercise Graph 27B.22.

In both panels, the darkened curve D_i is person i's true demand curve — which, where it reported, would result in public good level y^t. In panel (a) we consider i over-reporting his demand by reporting RD^o_i and thus pushing the public good level to y^o. If the demand in panel (a) is reported truthfully, the Groves-Clarke



Exercise Graph 27B.21 : Groves-Clarke Charge when p_i is high

charge would be $(a+b+c+d)$. If the individual over-reports, the charge becomes $(a+b+c+d+e+f)$ — i.e. it increases by $(e+f)$. But the increased public good production from y^t to y^o is valued by i only by the area under his true demand curve between y^t and y^o — i.e. by area (e). Thus, over-reporting results in an additional charge of $(e+f)$ but only an additional benefit of (e). It is therefore not in individual i's interest to over-report his demand regardless of what everyone else is reporting. In panel (b), we then consider the case of under-reporting. By reporting RD_i^u rather than the true demand D_i, the individual would cause public good production to fall from y^t to y^u. His Groves-Clarke charge if he reports his true demand is area $(g+h+i+j+k+l+m)$ — and this charge falls to $(g+h+i+j+k)$ if he under-reports. Thus, under-reporting causes his Groves-Clarke charge to fall by $(l+m)$. The value he places on the public good level from y^u to y^t, however, is equal to the area under his true demand curve D_i between these two quantities — i.e. area $(k+l+m)$. Individual i would therefore give up the public good benefit of $(k+l+m)$ in exchange for a reduction in his Groves-Clarke charge of only $(l+m)$ — making him worse of by area (k). It is therefore also not in individual i's interest to under-report his demand for the public good regardless of what everyone else reports.



Exercise Graph 27B.22 : Incentives when p_i is high

27C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 27.1

We discussed in the text the basic externality problem that we face when we rely on private giving to public projects. In this exercise, we consider how this changes as the number of people involved increases.

A: *Suppose that there are N individuals who consume a public good.*

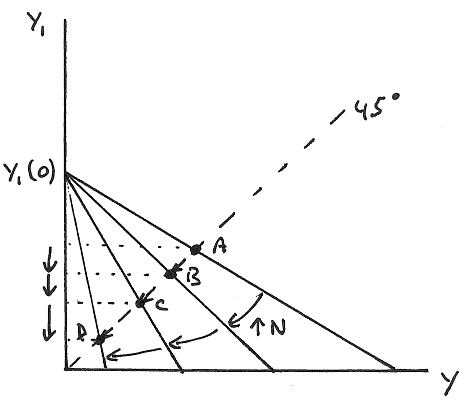
(a) *Begin with the best response function in panel (a) of Graph 27.3 — i.e. the best response of one person's giving to another person's giving when N = 2. Draw the 45 degree line into your graph of this best response function.*

Answer: This is illustrated in Exercise Graph 27.1, with the shallowest of the solid lines representing the best response function in the 2-person case.

(b) *Now suppose that all N individuals are the same — just as we assumed the 2 individuals in Graph 27.3 are the same. Given the symmetry of the problem (in terms of everyone being identical), how must the contributions of each person relate to one another in equilibrium?*

Answer: In equilibrium, everyone will contribute the same — thus best responding to everyone else.

(c) *In your graph, replace y_2 — the giving by person 2, with y — and let y be the giving that each person other than person 1 undertakes (assuming they all give the same amount). As N increases, what happens to the best*



Exercise Graph 27.1 : Private Giving as N Increases

response function for person 1? Explain, and relate your answer to the free rider problem.

Answer: The only point on the best response function that remains the same is $y(0)$ — if everyone else is giving zero, then my decision about how much to give is the same regardless of how many others there are. But as there are more "others" who give, my best response to each of them giving some amount y will be to give less than if there was only one other person giving y. This is for the same reason as our original best response function was downward sloping: If there are only two of us, the more you give, the less I will give even though my overall consumption of the public good will increase. If instead another person joins up and gives the same amount as you give, it is to me the same as if you had given twice as much — and I will therefore best respond by giving less. Thus, my best response function will rotate inward as illustrated in Exercise Graph 27.1 when N increases. I will, in essence, have more people to free ride on — and will therefore free ride more by giving less.

(d) *Given your answers to (b) and (c), what happens to person 1's equilibrium contribution as N increases? (*Hint*: Where on the best response function will the equilibrium contribution lie?)*

Answer: Since all of us will end up giving the same in equilibrium, we must end up on the 45 degree line in the graph. Thus, as person 1's best response function rotates inward when N increases, we move from A to B to C to D and beyond (as N increases further). This implies that person 1's contribution — and by implication everyone's equilibrium contribution — will fall as N increases. When N gets really large, this equilibrium private contribution converges to zero.

(e) *When $N = 2$, how much of the overall benefit from his contribution is in-*

dividual 1 taking into account as he determines his level of giving? How does this change when N increases to 3 and 4? How does it change as N gets very large?

Answer: When $N = 2$, there are two of us, and each one therefore takes into consideration half the overall benefit from his private contribution to the public good. When $N = 3$, the amount of the overall benefit each of us takes into account falls to $1/3$ — and when $N = 4$, it falls to $1/4$. For a population of N, each person therefore only takes into account $1/N$ of the overall benefit from his private contribution — leaving $(N-1)/N$ that he does not take into account. As N get large, the fraction of the overall benefit of a private contribution that each person takes into account approaches zero.

(f) *What does your answer imply for the level of subsidy s that is necessary to get people to contribute to the efficient level of the public good as N increases? (Define s as the level of subsidy that will cause a \$1 contribution to the public good to cost the individual only* $\$(1-s)$.*)*

Answer: If I don't take into account half the benefit that I create by giving to the public good, then a subsidy of $s = 0.5$ will cause me to internalize the externality by having me pay for only half of the actual contributions I am making. When I don't take into account $2/3$ of the benefits I create (as when $N = 3$), the subsidy must rise to $s = 2/3$ — and when $N = 4$, it must rise to $s = 3/4$ because I now only take into account $1/4$th of the benefits I create. For a population of N, the subsidy must therefore rise to $s = (N-1)/N$ because that is the fraction of the benefit I create with my contribution that I do not take into account. This implies that the optimal subsidy will approach $s = 1$ as N gets large.

(g) *Explain how, as N becomes large, the optimal subsidy policy becomes pretty much equivalent to the government simply providing the public good.*

Answer: Since the optimal subsidy approaches 1 as N becomes large, the government ends up paying essentially all of the "private" contributions. This is logically equivalent to the government simply funding the public good.

B: *In Section 27B.2.2, we considered how two individuals respond to having the government subsidize their voluntary giving to the production of a public good. Suppose again that individuals have preferences that are captured by the utility function* $u(x, y) = x^{\alpha} y^{(1-\alpha)}$ *where x is dollars worth of private consumption and y is dollars spent on the public good. All individuals have income* I*, and the public good is financed by private contributions denoted* z_n *for individual* n*. The government subsidizes private contributions at a rate of* $s \leq 1$ *and finances this with a tax t on income.*

(a) *Suppose there are N individuals. What is the efficient level of public good funding?*

Answer: The optimal level of the public good is $y^* = (1-\alpha)NI$.

(b) *Since individuals are identical, the Nash equilibrium response to any policy (t,s) will be symmetric — i.e. all individuals end up giving the same in equilibrium. Suppose all individuals other than n give z. Derive the best response function $z_n(t,s,z)$ for individual n. (As in the text, this is most easily done by defining n's optimization as an unconstrained optimization problem with only z_n as the choice variable and the Cobb-Douglas utility function written in log form.)*

Answer: The amount n will have left over for private consumption after giving z_n to the public good is $[(1-t)I-(1-s)z_n]$, and the public good amount will be $[z_n+(N-1)z]$ given everyone other than N contributes z and n contributes z_n. Thus, individual n will solve the problem

$$\max_{z_n} \alpha \ln[(1-t)I-(1-s)z_n]+(1-\alpha)\ln[z_n+(N-1)z]. \tag{27.1.i}$$

Setting the derivative with respect to z_n equal to zero (and taking the negative term to the other side), we get the first order condition

$$\frac{\alpha(1-s)}{(1-t)I-(1-s)z_n}=\frac{(1-\alpha)}{z_n+(N-1)z}. \tag{27.1.ii}$$

Solving this for z_n, we get

$$z_n(t,s,z)=\frac{(1-\alpha)(1-t)I-\alpha(1-s)(N-1)z}{(1-s)}. \tag{27.1.iii}$$

(c) *Use your answer to (b) to derive the equilibrium level of individual private giving $z^{eq}(t,s)$. How does it vary with N?*

Answer: Setting the left-hand side of equation (27.1.iii) equal to z and then solving for z, we get

$$z^{eq}=\frac{(1-\alpha)(1-t)I}{(1-s)[\alpha(N-1)+1]}. \tag{27.1.iv}$$

Since N only appears in the denominator, the private giving to the public good will fall as N increases.

(d) *What is the equilibrium quantity of the public good for policy (t,s)?*

Answer: The equilibrium level of the public good is then simply N times the equilibrium private giving level — i.e.

$$y^{eq}(t,s)=\frac{N(1-\alpha)(1-t)I}{(1-s)[\alpha(N-1)+1]}. \tag{27.1.v}$$

(e) *For the policy (t,s) to result in the optimal level of public good funding, what has to be the relationship between t and s if the government is to cover the cost of the subsidy with the tax revenues it raises?*

Answer: The efficient level of the public good is $y^*=(1-\alpha)NI$. If the government subsidizes private giving with a subsidy level of s, and if the efficient public good level is reached, then the government will therefore have to pay $s(1-\alpha)NI$ in subsidies. Since it is taxing income to do so, it therefore has to be the case that $tNI = s(1-\alpha)NI$ which solves to

$$t = s(1-\alpha). \tag{27.1.vi}$$

(f) *Substitute your expression for t from (e) into your answer to (d). Then determine what level of s is necessary in order for private giving to result in the efficient level of output you determined in (a).*

Answer: Substituting $t = s(1-\alpha)$ into the right hand side of equation (27.1.v) and setting it equal to the efficient level of $y^* = (1-\alpha)NI$, we get the equation

$$\frac{N(1-\alpha)(1-s(1-\alpha))I}{(1-s)[\alpha(N-1)+1]} = (1-\alpha)NI. \tag{27.1.vii}$$

Solving this for s, we then get the efficient subsidy

$$s^* = \frac{N-1}{N}. \tag{27.1.viii}$$

(g) *Derive the optimal policy (t^*, s^*) that results in efficient levels of public good provision through voluntary giving. What is the optimal policy when $N = 2$? (Your answer should be equal to what we calculated for the 2-person case in Section 27B.2.2.) What if $N = 3$ and $N = 4$?*

Answer: We already derived s^*, and we from (e) the relationship of s to t to give us

$$t^* = s^*(1-\alpha) = \frac{(1-\alpha)(N-1)}{N}. \tag{27.1.ix}$$

When $N = 2$, this implies $(t^*, s^*) = (0.5(1-\alpha), 0.5)$ — i.e. a subsidy rate of 0.5 and a tax rate of $0.5(1-\alpha)$ — which is exactly what we derived in the text. When N increases, we get

$$s^* = \frac{2}{3} \text{ and } t^* = \frac{2(1-\alpha)}{3} \text{ for } N = 3 \tag{27.1.x}$$

and

$$s^* = \frac{3}{4} \text{ and } t^* = \frac{3(1-\alpha)}{4} \text{ for } N = 4. \tag{27.1.xi}$$

(h) *Can you explain s^* when N is 2, 3, and 4 in terms of how the externality changes as N increases? Does s^* for $N = 1$ make intuitive sense?*

Answer: When $N = 2$, each individual is only taking into account half of the overall benefit of his giving — because there is one other person who also benefits. Thus $s^* = 0.5$. When $N = 3$, each person only takes into account one third of the overall benefit — because there are now 2 others who also benefit — thus giving us $s^* = 2/3$. And when $N = 4$, there are 3 others that benefit and thus each person only takes into account one fourth of the benefit of his actions — implying $s^* = 3/4$. But when $N = 1$, there is only one beneficiary of the public good who therefore takes the entire benefit of the public good into account — giving us $s = (N-1)N = 0$. No subsidy is necessary when there is no positive externality from giving.

(i) *What does this optimal policy converge to as N gets large? Interpret what this means.*

Answer: As N gets large, $s^* = (N-1)/N$ converges to 1. This is because the externality of private giving gets very large as N gets large — each person only takes into account his own private benefit from his contribution and not the $(N-1)$ benefits he bestows on others. As 1 becomes small relative to N, the government has to subsidize virtually all of private giving — which becomes equivalent to the government simply providing the public good.

Exercise 27.3

Everyday Application: *Sandwiches, Chess Clubs, Movie Theaters and Fireworks: In the introduction, we mentioned that, while we often treat public and private goods as distinct concepts, many goods actually lie in between the extremes because of "crowding".*

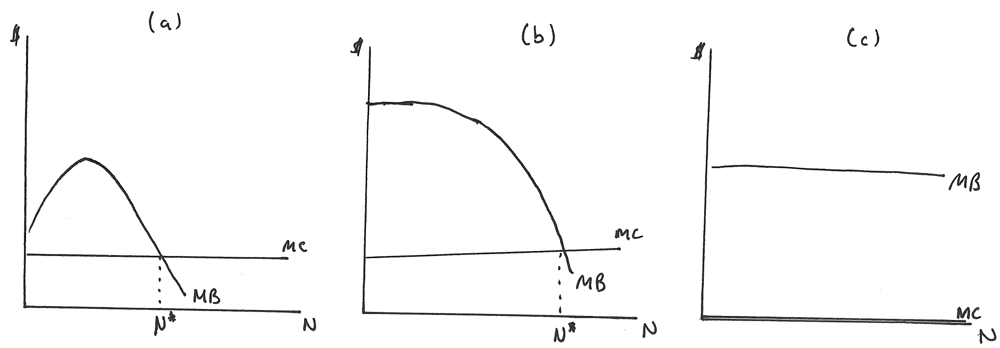
A: *We can think of the level of crowding as determining the optimal group size for consumption of the good — with optimal group size in turn locating the good on the continuum between purely private and purely public goods.*

(a) *One way to model different types of goods is in terms of the marginal cost and marginal benefit of admitting additional group members to enjoy the good. Begin by considering a bite of your lunch sandwich. What is the marginal benefit of admitting a second person to the consumption of this bite? What is therefore the optimal "group size" — and how does this relate to our conception of the sandwich bite as a private good?*

Answer: The marginal benefit of admitting a second person is less than zero — it's just gross to even contemplate what it would mean. As a result, the optimal group size — i.e. the size where marginal benefit crosses marginal cost — is 1. That is the definition of a purely private good — a good that can be enjoyed by only a single person.

(b) *Next, consider a chess club. Draw a graph with group size N on the horizontal axis and dollars on the vertical. With additional members, you'll have to get more chess-boards — with the marginal cost of additional members plausibly being flat. The marginal benefit of additional members might initially be increasing, but if the club gets too large, it becomes impersonal and not much fun. Draw the marginal benefit and marginal cost curves and indicate the optimal group size. In what way is the chess club not a pure public good?*

Answer: This is done in panel (a) of Exercise Graph 27.3 where the optimal



Exercise Graph 27.3 : Between Public and Private Goods

club membership is N^*. The chess club does not provide a purely private or a purely public good. It is not purely private because optimal group membership lies above 1 — but it is not purely public because the optimal membership is finite.

(c) *Consider the same exercise with respect to a movie theater that has N seats (but you could add additional people by having them sit or stand in the isles). Each customer adds to the mess and thus the cleanup cost. What might the marginal cost and benefit curves now look like?*

Answer: This is graphed in panel (b) of Exercise Graph 27.3. The marginal cost might be flat — or it might be somewhat increasing. The marginal benefit is probably flat initially — as we add additional people and they sit scattered around theater, no one's enjoyment is affected but additional people are enjoying the good. As theater gets more crowded, it might become somewhat less enjoyable as we have to endure people's chattering and sit right next to people and feel more cramped. While additional people get benefit from the movie, everyone's benefit is affected — and this will happen at an increasing rate as the movie theater fills up. The optimal group size N^* again appears where MC intersects MB — and this might occur before or after N — the number of seats — depending on how the crowding of theater affects those already in theater.

(d) *Repeat the exercise for fireworks.*

Answer: This is done in panel (c) of the Graph. Additional group members do not add to the cost of the fireworks — and they do not take away from anyone else's enjoyment of the fireworks — at least not until N gets very large. Thus, we would not expect marginal benefit to cross marginal cost until the group size gets very large — when some stand so far away that we might have to get more expensive fireworks that go higher into the sky (for instance). This is then an example of something much closer to a pure public good.

(e) *Which of these do you think the market and/or civil society can provide relatively efficiently — and which might require some government assistance?*

Answer: Sandwiches are private goods — the types of goods the first welfare theorem envisions. Markets should be able to provide these quite efficiently. Movie theaters have reasonable optimal group sizes — and can exclude people who do not pay. Thus, we see the market provide these quite effectively as well despite the fact that there is some "publicness" (i.e. some non-rivalry). Fireworks differ — both in that optimal group size is large and, perhaps more importantly, the fact that it is difficult to exclude non-payers. Thus, markets would have the greatest difficulty providing these effectively.

(f) *Why do you think fireworks on national holidays are usually provided by local governments — but Disney World is able to put on fireworks every night without government help?*

Answer: Fireworks on national holidays are primarily provided by local governments because of the issues mentioned in the previous part. Disney World is unusual — it covers a large land area, with fireworks not easily visible from outside the park (unless you are in an airplane). Thus, Disney World can in fact exclude non-payers — and displays fireworks every night as part of the package of goods that individuals buy access to when they pay for park entry. This suggests that excludability is the key reason that we rarely see private provision of large fireworks — although we do see it in some circumstances.

B: *Consider in this part of the exercise only crowding on the cost side — with the cost of providing some discrete public good given by the function $c(N) = FC + \alpha N^{\beta}$ with $\alpha > 0$ and $\beta \geq 0$. Assume throughout that there is no crowding in consumption of the public good.*

(a) *Derive the marginal cost of admitting additional customers. In order for there to be crowding in production, how large must β be?*

Answer: The marginal cost (relative to admitting additional group members) is

$$MC = \frac{\partial C}{\partial N} = \beta\alpha N^{(\beta-1)}. \tag{27.3.i}$$

In order for there to be crowding in production, it must be that β is greater than 1 (and the marginal cost function therefore slopes up.) If $\beta \leq 1$, the marginal cost curve is either flat or downward sloping.

(b) *Find the group membership at the lowest point of the average cost function. How does this relate to optimal group size when group size is sufficiently small for multiple providers to be in the market?*

Answer: The average cost function is

$$AC = \frac{C(N)}{N} = \frac{FC}{N} + \alpha N^{(\beta-1)}. \tag{27.3.ii}$$

We can find the lowest point of this function in one of two ways — either by checking where MC crosses AC or by taking the derivative of AC and setting it to zero. Doing the former, we solve $MC = AC$ to get

$$N^* = \left(\frac{FC}{\alpha(\beta-1)}\right)^{1/\beta} \tag{27.3.iii}$$

which is the same answer we get by solving for it the other way. If there are multiple providers of this public good, the least cost way of providing the public good is for clubs of size N^*.

(c) *What is the relationship between α, β and FC for purely private goods?*

Answer: For purely private goods, $N^* = 1$. Setting the left hand side of equation (27.3.iii) equal to 1, we therefore get

$$\alpha(\beta-1) = FC. \tag{27.3.iv}$$

(d) *Suppose that the good is a purely public good. What value of α could make this so? If $\alpha > 0$, what value of β might make this so?*

Answer: If the good is a purely public good, there is no marginal cost of admitting additional group members. If $\alpha = 0$, we have one case where this is true. It would similarly be true if $\beta = 0$. If $\beta < 1$, MC is downward sloping — and the optimal group size would still be such that the good is a pure public good in the sense that optimal group membership is the entire population.

(e) *How does α affect optimal group size? What about FC and β? Interpret your answer.*

Answer: As α goes up, the marginal cost of any additional group member is higher — and the optimal group size therefore falls. As β increases (above 1), the marginal cost curve slopes up and gets steeper — implying again that optimal group size increases. Finally, as the fixed cost of providing the public good increases, we need a larger group to support the public good — and thus the optimal group size rises.

Exercise 27.5

Everyday and Business Application: *Raising Money for a Streetlight through a "Subscription Campaign": Sometimes, a civil society institution's goal can be clearly articulated in terms of a dollar value that is needed. Consider, for instance, the problem you and I face when we want to fund a streetlight on our dark culdesac. We know the total cost of the light will be C — and so we know exactly how much money we need to raise. One way we can raise the money is through what is known as a* subscription campaign. *Here is how a subscription campaign would work: We put a money "pledge jar" in between our two houses, and you begin by pledging an amount x_1^Y. We then agree that we will alternate putting a pledge for a contribution into the jar on a daily basis — with me putting in a pledge x_2^M the second day, then you putting in a pledge x_3^Y the third day, me putting in x_4^M the fourth day, etc. When* *enough money is pledged to cover the cost C of the street light, we pay for the light — with you writing a check equal to the total that you have pledged and me writing a check for the total I have pledged.*

A: *Suppose you and I each value the light at \$1,000 but the light costs \$1,750. We are both incredibly impatient people — with \$1 tomorrow valued by us at only \$50 cents today. For simplicity, assume the light can be put up the day it is paid for.*

(a) *Suppose it ends up taking T days for us to raise enough pledges to fund the light. Let x_T^i be the last pledge that is made before we reach the goal. What does subgame perfection imply x_T^i is? (*Hint*: Would it be subgame perfect for person j who pledges the day before to leave an amount to be pledged that is less than the maximum person i is willing to pledge on day T?)*

Answer: Given that person j would leave money on the table if he pledged on day $(T-1)$ more than he needed to in order for i to finish out the pledges on day T, it must be that person i is indifferent between pledging x_T^i and waiting for person j to finish up the project pledged in period $(T+1)$. If i pledges x_T^i, the light gets put up on day T — giving a payoff of $(1000 - x_T^i)$ to individual i (given that previous pledges have already been made and are now "sunk" on day T). If i does not pledge today, j will finish out the pledges tomorrow but that means the light does not get put up until tomorrow. Given how impatient we are, the light tomorrow is worth only 0.5(1000) = 500. Thus, for i to be indifferent between pledging x_T^i today or waiting another day for the light to go up, it must be that

$$1000 - x_T^i = 500 \tag{27.5.i}$$

or simply $x_T^i = 500$.

(b) *Next, consider person j whose turn it is to pledge on day $(T-1)$. What is x_{T-1}^j? (*Hint*: Person j knows that, unless he gives the amount necessary for i to finish off the required pledges on day T, he will end up having to give again (an amount equal to what you calculated for x_T^i) on day $(T+1)$ and have the light delayed by one day.)*

Answer: When person j pledges enough on day $(T-1)$ to insure that person i will finish off the required pledges tomorrow, the streetlight gets put up tomorrow. Thus, person j's payoff is $(1000 - x_{T-1}^j)$ tomorrow — or, given our impatience, $0.5(1000 - x_{T-1}^j)$ today (on day $(T-1)$.) But person i who pledged on day $(T-2)$ will have made sure that the amount j has to pledge on day $(T-1)$ in order to insure the light will be put up on day T is the highest that j is willing to pledge; i.e. it has to be that the level of x_{T-1}^j necessary to have the light built on day T makes j indifferent between pledging and delaying the streetlight by one day to day $(T+1)$ by not pledging. If j skips a pledge, he will switch positions with person i — and will end up finishing up the pledges on day $(T+1)$ by pledging

500 — the subgame perfect equilibrium pledge in the last period as calculated in (a). Thus, by delaying the street light, person j knows he will get $1000-500=500$ on day $(T+1)$ — two days after today (where today is the day he has to decide on x^j_{T-1}). Given our impatience, this means that he gets a present discounted payoff of $0.5^2(500)=125$ by delaying the project. In order for him to be indifferent between delaying and pledging now, it must therefore be that

$$0.5(1000-x^j_{T-1})=125 \tag{27.5.ii}$$

which gives us $x^j_{T-1}=750$.

(c) *Continue working backwards. How many days will it take to collect enough pledges?*

<u>Answer</u>: We have already determined that the last period pledge will be \$500 and the second to last period pledge will be \$750. Thus, in the last two periods we collect pledges equal to \$1,250. Now consider day $T-2$ when it is once again i's turn to pledge. By the same reasoning as the one used in (b), it must be that i will have to give the most that is necessary in order for the next two days to unfold the way we have concluded — with the street light put on on day T. Thus, the equilibrium pledge amount x^i_{T-2} must make i indifferent between pledging and knowing that he will pledge \$500 on day T to get the light put up on day T — and pledging nothing now, thus reversing positions with j and pledging \$750 on day T with the light not going up until day $(T+1)$. The payoff in the first scenario is $0.5^2(1000-500-x^i_{T-2})$ and the payoff in the second scenario is $0.5^3(1000-750)$. In equilibrium, it must then be the case that

$$0.5^2(1000-500-x^i_{T-2})=0.5^3(1000-750). \tag{27.5.iii}$$

This solves to give us $x^i_{T-2}=375$.

So now we know that in the last 3 days of the campaign, we will collect \$1,625. We are still short of the \$1,750 we need. So now we have to consider period $(T-3)$ — and if we go through the very same reasoning again, we get that individual j would be willing to contribute up to \$187.50 on day $(T-3)$. But we only need \$125 more — so he will contribute $x^i_{T-2}=125$. It therefore takes us 4 days to raise the number of pledges we need — with j pledging \$125 today and \$750 the day after tomorrow, and person i pledging \$375 tomorrow and \$500 three days from now.

(d) *How much does each of us have to pay for the streetlight (assuming you go first)?*

<u>Answer</u>: If you go first, you are person j. Thus, based on our answer so far, you will end up paying \$875 and I will end up contributing \$875 as well.

(e) *How much would each of us be willing to pay the government to tax us an amount equal to what we end up contributing — but to do so today and thus put up the light today?*

Answer: We concluded that each of us will contribute \$875 — and each of us will get a benefit of \$1,000 when the light is put up. By engaging in the subscription campaign, we therefore each get a surplus of \$125 in three days. The present discounted value of \$125 three days from now (given our level of impatience) is $0.5^3(125) = 15.63$. Thus, getting \$125 in surplus *today* rather than getting it three days from now is worth \$109.37 — which is the most we'd be willing to pay to employ the government to tax us rather than engaging in the subscription campaign.

(f) *What is the remaining source of inefficiency in the subscription campaign?*

Answer: The efficient outcome is to build the street light *now*, but subscription campaigns take time. Thus, while the campaign results in the efficient outcome to the extent to which efficient street light gets built, it inefficiently delays the building of the streetlight. The reason for this is once again due to the free rider problem — each of us keeps delaying in order to get the other to contribute, but we'd both we better off if someone just made us put what would be our pledges at the end of the game into the pot immediately.

(g) *Why might a subscription campaign be a good way for a pastor of a church to raise money for a new building but not for the American Cancer Association to raise money for funding cancer research?*

Answer: The pastor can put a clear dollar figure on the goal — building a new church. Thus, we can implement a subscription campaign where members of the church are repeatedly approached as the campaign nears its target. In the case of the American Cancer Association, it is not as clear that a clean dollar figure that is meaningful to contributors can be announced. The goal is to cure cancer, but no one knows how much that will cost. It is therefore more difficult to design an effective subscription fundraising campaign in this case.

B: *Now consider the more general case where you and I both value the street light at* $\$V$, *it costs* $\$C$, *and \$1 tomorrow is worth* $\$\delta < 1$ *today. Assume throughout that the equilibrium is subgame perfect.*

(a) *Suppose, as in A(a), that we will have collected enough pledges on day T when individual i puts in the last pledge. What is* x_T^i *in terms of* δ *and* V*?*

Answer: Suppose you are the one to make that final pledge x_T on day T. Knowing that you will do this, I will have made sure with my own pledge on day $T-1$ that you are indifferent between pledging x_T or waiting one more day for me to make it. After all, if I had made a pledge on day $T-1$ that left you with less than x_T to reach overall pledges of C, I could have done better by pledging less and letting you take care of the rest. Let's assume that \$1 tomorrow is worth $\$\delta < 1$ today for each of us. You being indifferent between pledging x_T and waiting another day for me to finish out the pledges is the same as saying that you are indifferent between getting $(V - x_T)$ now or waiting another day to get V tomorrow (which is worth δV today). Thus $(V - x_T) = \delta V$ which implies the last period pledge is

$$x_T = x_T^i = (1-\delta)V. \tag{27.5.iv}$$

(b) *What is x_{T-1}^j? What about x_{T-2}^i?*
Answer: On day $(T-1)$, it is my turn to pledge — and I know you will pledge x_T tomorrow if I pledge an amount x_{T-1}^j that is just enough to get the total pledges to sum to $(C - x_T)$. If I do this, I will get a benefit of $\delta(V - x_{T-1}^j)$. If I don't pledge and wait for you to pledge tomorrow, I am in essence delaying us by a day, and I will be the one pledging x_T when it's my turn again. Thus, not pledging on day $(T-1)$ gives me a payoff of $\delta^2(V - x_T)$. You will have made sure the day before that I will be just indifferent between these two options — so $\delta(V - x_{T-1}^j) = \delta^2(V - x_T)$ where we previously calculated $x_T = (1-\delta)V$. Solving for x_{T-1}^j, we get the second to last period pledge of

$$x_{T-1} = x_{T-1}^j = (1-\delta^2)V. \tag{27.5.v}$$

On day $(T-2)$, it is your turn, and you know that I will pledge x_{T-1} tomorrow and you will pledge x_T the day after so long as you make sure to pledge and amount x_{T-2}^i that is enough such that the total pledges get to $C - x_{T-1} - x_T$ by the end of today. I will have made sure the day before $T-2$ that x_{T-2}^j makes you indifferent between making the pledge and not making a pledge (thereby delaying us for one day). The payoff from making the pledge is then $\delta^2(V - x_T - x_{T-2}^i)$ while the payoff from delaying by a day is $\delta^3(V - x_{T-1})$. Setting these equal and substituting the quantities we previously calculated for x_T and x_{T-1}, we can solve for the third to last period pledge as

$$x_{T-2} = x_{T-2}^i = \delta(1-\delta^2)V. \tag{27.5.vi}$$

(c) *From your answers to (b), can you infer the pledge amount x_{T-t} for t ranging from 1 to $(T-1)$?*
Answer: The pledge on day t (with $1 \le t \le (T-1)$) is

$$x_{T-t} = \delta^{(t-1)}(1-\delta^2)V. \tag{27.5.vii}$$

(d) *What is the amount pledged today — i.e. in period 0?*
Answer: In period 0, the amount that is pledged is the shortfall between all the following T pledges and the cost of the streetlight — i.e.

$$x_0 = C - \sum_{t=0}^{T-1} x_{T-t}. \tag{27.5.viii}$$

(e) *What is the highest that C can be in order for $(T+1)$ pledges — i.e. pledges starting on day 0 and ending on day T — to cover the full cost of the light.*

Answer: The highest that C can be is equal to all the equilibrium pledges from periods 1 through T plus x_0. Thus, the highest that C depends on the highest that x_0 can be — and that follows the same rule as what we derived from periods 1 through $(T-1)$ in equation 27.5.vii (for the same reasons); i.e. $x_0^{max} = \delta^{(T-1)}(1-\delta^2)V$. Summing all the equilibrium pledges from periods 1 through T and adding x_0^{max}, we get the highest possible cost that could be raised in T periods as

$$\begin{aligned}
C &= x_0^{max} + \sum_{t=0}^{T-1} x_{T-t} \\
&= x_0^{max} + x_T + \sum_{t=1}^{T-1} x_{T-t} \\
&= x_T + \sum_{t=1}^{T} x_{T-t} \\
&= (1-\delta)V + \sum_{t=1}^{T} \delta^{(t-1)}(1-\delta^2)V \\
&= (1-\delta)V + \sum_{t=0}^{T-1} \delta^{t}(1-\delta^2)V
\end{aligned} \tag{27.5.ix}$$

(f) *Recalling that $\sum_{t=0}^{\infty}\delta^t = 1/(1-\delta)$, what is the greatest amount that a subscription campaign can raise if it goes on sufficiently long such that we can approximate the period of the campaign as an infinite number of days?*

Answer: Letting T go to infinity, equation (27.5.ix) becomes

$$\begin{aligned}
C &= (1-\delta)V + \sum_{t=0}^{\infty} \delta^{t}(1-\delta^2)V \\
&= (1-\delta)V + \frac{(1-\delta^2)V}{1-\delta} = \frac{(1-\delta)^2 + 1 - \delta^2)V}{1-\delta} \\
&= \frac{(1-2\delta+\delta^2+1-\delta^2)V}{1-\delta} = 2V.
\end{aligned} \tag{27.5.x}$$

(g) True or False*: A subscription campaign will eventually succeed in raising the necessary funds so long as it is efficient for us to build the street light.*

Answer: We have just illustrated this by showing that the most that can be raised through the subscription campaign is an amount $2V$. Since there are only two of us, each valuing the streetlight at V, it is efficient for the street light to get built so long as $C \leq 2V$.

(h) True or False*: In subscription campaigns, we should expect initial pledges to be small — and the campaign to "show increasing momentum" as time passes, with pledges increasing as we near the goal.*

Answer: This is true — up to the second to last pledge. We have concluded that pledges in periods 1 through $(T-1)$ follow the pattern $x_{T-t} = \delta^{(t-1)}(1-\delta^2)V$ (and the pledge in period 0 is no greater than what is suggested by this pattern). Thus, $x_{T-1} = (1-\delta^2)V$, $x_{T-2} = \delta(1-\delta^2)V$, $x_{T-3} = \delta^2(1-\delta^2)V$, etc. — or, put differently,

$$x_t = \frac{x_{t-1}}{\delta}. \tag{27.5.xi}$$

With $\delta < 1$, this implies $x_{T-1} > x_{T-2} > x_{T-3} > ... > x_0$ — i.e. pledges increase with time. The only exception is the last period where we have concluded $x_T = (1-\delta)V$ which is less than $x_{T-1} = (1-\delta^2)V$.

Exercise 27.7

Policy Application: *Demand for Charities and Tax Deductibility: In end-of-chapter exercise 9.9 of Chapter 9, we investigated the impact of various U.S. income tax changes on the level of charitable giving. If you have not already done so, do so now and investigate the different ways that tax policy changes in the U.S. over the past few decades might have impacted the level of charitable giving.*

Answer: The answer is detailed in the solutions to Chapter 9.

Exercise 27.9

Policy Application: *Distortionary Taxes and National Security: In the real world, government provision of public goods usually entails the use of distortionary taxes to raise the required revenues. Consider the pure public good "national defense", a good provided exclusively by the government (with no private contributions).*

A: *Consider varying degrees of inefficiency in the nation's tax system.*

(a) *In our development of the concept of deadweight loss from taxation, we found that the deadweight loss from taxes tends to increase at a rate k^2 for a k-fold increase in the tax rate. Define the "social marginal cost of funds" $SMCF$ as the marginal cost society incurs from each additional dollar spent by the government. What is the shape of the $SMCF$ curve?*

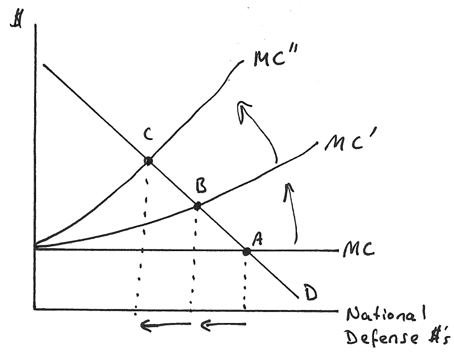
Answer: For an efficient tax system, the $SMCF$ curve is flat at 1 — i.e. the social cost of $1 of government spending is $1. As a tax system becomes less efficient, the social cost of government spending rises above $1 per dollar spent. And, it the tax system exhibits the feature that the deadweight loss increases by k^2 for a k-fold increase in the tax rate, the $SMCF$ curve not only lies above 1 but is also upward sloping (as it becomes increasingly costly to raise additional revenues).

(b) True or False*: If the public good is defined as "spending on national defense", then the marginal cost of providing $1 of increased funding for the public good is $1 under an efficient tax system.*

Answer: This is true — if the tax system is efficient, then it costs $1 to pay for $1 of the public good "national defense spending" — i.e. the marginal cost of providing the public good (as defined) is flat at 1.

(c) *How does the marginal cost of providing this public good change as the tax system becomes more inefficient?*

Answer: This is illustrated in Exercise Graph 27.9



Exercise Graph 27.9 : Distortionary Taxes and National Defense

– with the marginal cost increasing from the efficient case MC to MC' and MC'' as the tax system becomes less efficient.

(d) *Use your answer to (c) to explain the following statement: "As the inefficiency of the tax system increases, the optimal level of national defense spending by the government falls."*

Answer: Included in Exercise Graph 27.9 is the (vertically aggregated) demand curve for national defense. As the tax system becomes less efficient, the optimal national defense spending level can be read off A, B and C in the graph — with optimal spending declining as the tax system becomes less efficient.

(e) *What do you think of the following statement: "Nation's that have devised more efficient tax systems are more likely to win wars than nations with inefficient tax systems."*

Answer: Given our answers so far, this is true so long as governments produce public goods in some relation to efficiency. It is certainly true for the case where governments produce efficient levels conditional on the tax system they use (as demonstrated in the graph). But it is also true if, for instance, governments always produce some fraction $\delta < 1$ of the efficient level.

B: *Suppose we approximate the demand side for goods by assuming a representative consumer with utility function $u(x, y) = x^{1/2}y^{1/2}$ and income I, where x is private consumption (in dollars) and y is national defense spending (in dollars).*

(a) *If the government can use lump sum taxes to raise revenues, what is the efficient level of national defense spending?*

Answer: As calculated in the text for the Cobb-Douglas case with exponents α and $(1-\alpha)$, the efficient level of the public good under an efficient tax system is $(1-\alpha)I$. In our case, $\alpha = 1/2$ — implying that the efficient level of national defense spending is $I/2$.

(b) *Next, suppose that the government only has access to inefficient taxes that give rise to deadweight losses. Specifically, suppose that it employs a tax rate t on income I, with tax revenue equal to $TR = tI/(1+\beta t)^2$. How does this capture the idea of deadweight loss? What would β be if the tax were efficient?*

Answer: Given this formulation of tax revenue, we get that tax revenue is 0 at $t = 0$ and less than tI for any $t > 0$ so long as $\beta > 0$, with the shortfall between the efficient case (where $TR = tI$) and the inefficient case (where $TR = tI/(1+\beta t)^2$) increasing with t. For this tax system to be efficient, it would have to be that $\beta = 0$.

(c) *Given that it has to use this tax to fund national defense, derive the efficient tax rate and level of national defense. (It is easiest to do this by setting up an optimization problem in which t is the only choice variable, with the utility function converted to logs.) How does it compare to your answer to (a)?*

Answer: The tax revenue TR that is raises is spend on national defense — so the public good level is equal to TR. The private good level is then simply the after tax income $(1-t)I$. Thus, we can write the representative consumer's utility maximization problem as

$$\max_t \frac{1}{2}\ln(1-t)I + \frac{1}{2}\ln\left(\frac{tI}{(1+\beta t)^2}\right). \qquad (27.9.i)$$

Setting the first order condition to zero and taking the negative term to the other side, we get

$$\frac{1}{2(1-t)} = \frac{(1+\beta t)^2}{2tI}\left(\frac{(1+\beta t)I - 2tI\beta}{(1+\beta t)^3}\right). \qquad (27.9.ii)$$

This can be simplified to

$$\frac{1}{2(1-t)} = \frac{1-\beta t}{2t(1+\beta t)} \qquad (27.9.iii)$$

which solves to give us the optimal tax rate

$$t^* = \frac{1}{\beta+2}. \qquad (27.9.iv)$$

This results in tax revenues — and thus national defense spending — of

$$y^* = \frac{I}{\beta+2}. \qquad (27.9.v)$$

When $\beta = 0$ this implies an optimal defense spending level of $I/2$ — exactly what we conclude in (a) for a lump sum tax system (that is efficient). But as the tax system becomes less efficient (with β increasing above 0), y^* falls.

(d) *Suppose $I = 2,000$. What is national defense spending and the tax rate t when $\beta = 0$? How does it change when $\beta = 0.25$? What if $\beta = 1$? $\beta = 4$? $\beta = 9$?*

Answer: When $\beta = 0$, $y^* = 1,000$ — the efficient level of spending in the absence of tax distortions. When $\beta = 0.25$, the optimal defense spending level falls to 720 — with it further falling to 375, 120 and 55 as β goes to 1, 4 and 9 respectively.

(e) *Suppose next that the government provides two pure public goods — spending on national defense y_1 and spending on the alleviation of poverty y_2 (where the latter is a public good in the ways developed in exercise 27.8). Suppose that the representative consumer's tastes can be described by $u(x, y_1, y_2) = x^{0.5} y_1^{\gamma} y_2^{(0.5-\gamma)}$. Modify the optimization problem in (c) to one appropriate for this setting — with the government now choosing both t* and *the fraction k of tax revenues spent on national defense (versus the fraction $(1-k)$ spent on poverty alleviation.)*

Answer: This is done with the problem:

$$\max_{t,k} 0.5\ln(1-t)I + \gamma \ln\left(\frac{ktI}{(1+\beta t)^2}\right) + (0.5-\gamma)\ln\left(\frac{(1-k)tI}{(1+\beta t)^2}\right). \tag{27.9.vi}$$

(f) *Does the optimal tax rate differ from what you derived before? What fraction of tax revenues will be spent on national defense?*

Answer: When we take the derivative of the objective function in our optimization problem with respect to the choice variable t, we get exactly the same answer as we did in (c) — allowing us to solve this first order condition for

$$t^* = \frac{1}{\beta+2} \text{ implying tax revenue } TR = \frac{I}{\beta+2}. \tag{27.9.vii}$$

The derivative of the objective function with respect to k simplifies to

$$k = 2\gamma \text{ implying } (1-k) = 1-2\gamma. \tag{27.9.viii}$$

Thus, when the utility function puts equal weight on national defense and poverty alleviation (with $\gamma = 0.25$), the tax revenue is split equally between the two public goods — but if the utility function puts greater weight on national defense, the same overall tax revenues will be apportioned more heavily to national defense.

Exercise 27.11

Policy Application: *The Pork Barrel Commons*: *In representative democracies where legislators represent geographic districts in legislative bodies (such as the U.S. House of Representatives), we often hear of "pork barrel spending". Typically, this refers to special projects that legislators include in bills that pass the legislature — projects that have direct benefits for the legislator's district but not outside the district. In this exercise we will think of these as publicly funded private goods whose benefits are confined to some fraction of residents of the geographical boundaries of the district. (In exercise 27.12, we will consider the case of different types of local public goods.)*

A: *Suppose that there are N different legislative districts, each with an equal proportion of the population. Suppose for simplicity that all citizens are identical — and that tax laws affect all individuals equally. Suppose further that all projects cost C, and that the total benefits B of a project are entirely contained in the district in which the project is undertaken.*

(a) *How much of the cost of a project that is passed by the legislature do the citizens in district i pay?*

Answer: Since the cost of projects is spread across the nation's population, the residents of district i only pay C/N.

(b) *How much of a benefit do the citizens in district i receive if the project is located in district i? What if it is not?*

Answer: Given the local nature of the publicly funded goods, the citizens in district i collectively receive the entire benefit B if the project is located in district i — and none of it if it is not.

(c) *Suppose the possible projects that can be brought to district i range in benefits from $B = 0$ to $B = \overline{B}$ where $\overline{B} > C$. Which projects should be built in district i if the legislature cares only about efficiency?*

Answer: Only projects with $B \geq C$ should be built from an efficiency perspective.

(d) *Now consider a legislator who represents district i and whose payoff is proportional to the surplus his district gets from the projects he brings to the district. What projects will this legislator seek to include in bills that pass the legislature?*

Answer: From the district's perspective, the benefit of a project is B and the cost is only C/N. Thus, the legislator will request all projects with $B \geq C/N$.

(e) *If there is only a single district — i.e. if $N = 1$, is there a difference between your answer to (c) and (d)?*

Answer: No, if $N = 1$, then both answers are to recommend building a project only if $B \geq C$.

(f) *How does the set of inefficient projects that the legislator includes in bills change as N increases?*

Answer: The legislator will seek to build projects with benefits in the range of C/N to $\overline{B}$ while the efficient projects like in the subset of that range that goes from C to B. As N increases, the lower end of the legislator's range falls — thus increasing the number of inefficient projects (with benefit levels between C/N and C).

(g) *In what sense do legislator's have an incentive to propose inefficient projects even though all of their constituents would be better off if no inefficient projects were located in any district? Can you describe this as a prisoners' dilemma? Can you also relate it to the Tragedy of the Commons (where you treat taxpayer money as the common resource)?*

Answer: Regardless of what other legislators do, the legislator from district i has an incentive to keep getting projects for his district so long as the benefit is larger than the cost *to his district*. It is therefore a dominant strategy for each legislator to request projects with benefits greater than C/N rather than with benefits greater than C. Suppose that all districts built one such inefficient project with benefit B'. The cost for all these projects is CN — with each district bearing a share $1/N$ of that cost — i.e. each district will pay C for all the inefficient projects. But their benefit in each district is only $B' < C$ — so they would all be better off if none of the inefficient projects were built. This is a classic prisoners' dilemma where all legislators would have happier districts if the legislators could agree to no inefficient projects anywhere, but in the absence of such an enforceable agreement, they all will request inefficient projects for their districts. In essence, they view the taxpayer revenues as a common pool of resources — much as fishermen in commonly owned lakes view the lake as a common pool resource. And, just as the fishermen overfish the lake, the legislators overfish the taxpayer revenues.

B: *Consider the same set of issues modeled slightly differently. Instead of thinking about a number of different projects per district, suppose there is a single project per district but it can vary in size. Let y_i be the size of a government project in district i. Suppose that the cost of funding a project of size y is $c(y) = Ay^{\alpha}$ where $\alpha > 1$, and suppose that the total benefit to the district of such a project is $b(y) = By^{\beta}$ where $\beta \leq 1$.*

(a) *What do the conditions $\alpha > 1$ and $\beta \leq 1$ mean? Do they seem like reasonable assumptions?*

Answer: The condition $\alpha > 1$ means that the production function for public goods has decreasing returns to scale — giving rise to increasing marginal costs $MC = \alpha A y^{\alpha-1}$. The condition that $\beta \leq 1$ implies that marginal benefits of the project are not increasing with size. These are typical assumptions one would make — decreasing returns to scale and downward sloping marginal benefit curves.

(b) *Suppose all districts other than district i get projects of size $\overline{y}$ and district i gets a project of size y_i. Let district i's legislator get a payoff π^i that is some fraction k of the net benefit that citizens within his district get from* *all government projects. What is $\pi^i(y_i, N)$ assuming that the government is paying for all its projects through a tax system that splits the cost of all projects equally across all districts?*

Answer: The residents of his district will get the full benefit of the project y_i but bear only $1/N$th the cost — while also paying for $1/N$th the cost of all the other $(N-1)$ projects of size $\overline{y}$. Thus, if the legislator from district i gets a payoff that is a fraction k of his constituents' surplus, his payoff is

$$\pi^i(y_i, N, \overline{y}) = k\left[By_i^\beta - \frac{Ay_i^\alpha}{N} - \frac{A\overline{y}^\alpha(N-1)}{N}\right]. \tag{27.11.i}$$

(c) *What level of y_i will legislator i choose to include in the government budget? Does it matter what $\overline{y}$ is?*

Answer: Legislator i will solve the problem

$$\max_{y_i} \pi^i(y_i, N, \overline{y}) = k\left[By_i^\beta - \frac{Ay_i^\alpha}{N} - \frac{A\overline{y}^\alpha(N-1)}{N}\right]. \tag{27.11.ii}$$

Setting the derivative of π^i with respect to y_i equal to zero and solving for y_i, we then get

$$y_i = \left(\frac{\beta BN}{\alpha A}\right)^{1/(\alpha-\beta)}. \tag{27.11.iii}$$

It is irrelevant what $\overline{y}$ is — i.e. it is a dominant strategy for the legislator to choose this level of the project for his district.

(d) *What level of y^{eq} will all legislators request for their districts?*

Answer: Since all legislators face exactly the same incentives, they will all request

$$y^{eq} = \left(\frac{\beta BN}{\alpha A}\right)^{1/(\alpha-\beta)}. \tag{27.11.iv}$$

(e) *What is the efficient level of y^* per district? How does it differ from the equilibrium level?*

Answer: The efficient level for each district is the level that sets the marginal benefit equal to the *total* marginal cost, not just the marginal cost borne by residents of the district. Thus, at the efficient level of y, it must be true that

$$MB = \beta By^{\beta-1} = \alpha Ay^{\alpha-1} = MC. \tag{27.11.v}$$

Solving for y, we get

$$y^* = \left(\frac{\beta B}{\alpha A}\right)^{1/(\alpha-\beta)}. \tag{27.11.vi}$$

Comparing y^* to y^{eq}, we get

$$y^* = \frac{y^{eq}}{N}; \tag{27.11.vii}$$

i.e. the efficient level of the project in each district is $1/N$th the equilibrium level. (This has led some to refer to the result as the "Law of 1/N.")

Conclusion: Potentially Helpful Reminders

1. While we tend to think of public goods as a separate category from private goods, it is useful to remember that the extremes of "pure public goods" and "pure private goods" are just that — extremes. A whole lot of goods lie in between these extremes — with some but not complete non-rivalry. This is explored in some detail in end-of-chapter exercise 27.3.

2. It is also useful to continue to remember that the non-rivalry that characterizes public goods has nothing to do with whether or not it is possible to exclude non-payers from consuming the good. And it is the fact that some public goods are non-excludable that makes it difficult for markets to provide them. Excludable public goods — like swimming pools or movies in movie-theaters, can more easily be provided by competitive firms or clubs. Even geographically excludable goods — like public schools — can be thought of as arising in competitive environments where communities compete with one another for residents. (This is what we called the Tiebout model of local public good provision.)

3. In the text, we introduce the concept of a *Lindahl Equilibrium* — an equilibrium in which a single public good quantity is produced and each person is charged a different price depending on that person's marginal willingness to pay. While interesting in theory, the concept runs into the problem that it is in every person's interest to under-report their willingness to pay in order to not be charged much and thus free-ride on what others are providing. This problem does not arise in the private good competitive equilibrium where no one can alter the price that she is being charged.

4. You should understand intuitively how this free-rider problem — and the incentive to misrepresent one's tastes for the public good if payment is linked to willingness to pay — gets worse as the number of consumers increases.

5. It is for this reason that economists have explored preference revelation mechanisms — i.e. incentive mechanisms that achieve the dual purpose of providing public goods *and* getting accurate information about people's willingness to pay.

CHAPTER

28

Government and Politics

If we can view consumers, workers and firms through the economist's lens, then there seems to be no reason in principle not to do the same for politicians. This is what Chapter 28 is about. The topic is one that has occupied economists and political scientists for decades — and it arose from the insight that there is nothing magical about democratic political processes, no reason in particular to expect them to come up with an efficient or in any way ideal outcome. What matters in most cases is how the voting is set up, what the rules are, etc. One of the important lessons in this is that, just like markets face problems when the conditions of the first welfare theorem are violated, and just as civil society efforts confront the free-rider problem, so government approaches have to come to terms with the imperfect nature of political processes.

Chapter Highlights

The main points of the chapter are:

1. The **median voter theorem** gives precise conditions under which a majority rule voting process leads to a **Condorcet winner** — i.e. a policy that beats all other policies in pairwise voting.

2. When the conditions of the median voter theorem are violated, there is no Condorcet winner. This opens a door for the **agenda setter** to devise the sequence of pairwise votes in such a way as to get to his most preferred outcome.

3. The starkest depiction of how chaotic democratic processes can potentially get is given in what we call the **"anything can happen theorem."**

4. Imposing **structure on democratic voting** can alleviate the chaos of the "anything can happen theorem" and thus limit the power of agenda setters.

5. Not all democratic processes are based on simple voting. Often, either by design or simply because of informal processes that emerge, voters (particularly in legislatures) can express the intensity of their preferences through such practices as vote trading.

6. We see again in this chapter a concept we developed in earlier chapters — the idea of inefficient legislating that results from **concentrated benefits and diffuse costs.**

7. Part B of the chapter proves the influential **Arrow's Theorem** — a theorem that posits 5 basic conditions that democratic processes should satisfy and then demonstrates that no such processes exist.

28A Solutions to Within-Chapter-Exercises for Part A

Exercise 28A.1

Suppose y is defined as per pupil spending in public schools. If there are N different taxpayers and an equal number of school children, and if all taxpayers share the financing of pubic schools equally, what is the slope of this "budget line"?

Answer: A taxpayer then needs to give up $1 to get an increase in per pupil spending of $1 — so the slope is -1.

Exercise 28A.2

Suppose instead that y is defined as the overall spending in public schools. What is the slope of the "budget line" under the same conditions as described in exercise 28A.1?

Answer: To get a $1 increase in overall spending on public schools, each family would not have to give up $\$1/N$ — thus the slope now would be $-1/N$.

Exercise 28A.3

Suppose that tax rates were progressive — implying that the tax rate increases as more tax revenue is being raised. Would preferences over y still be single peaked?

Answer: Yes, they would still be single peaked. Each consumer might be in a different income bracket — and therefore face a different marginal tax rate, but at that rate, she would still face the same kind of linear budget constraint and tastes as those graphed in the text.

Exercise 28A.4

In the graph, we have depicted all the single peaked preferences as having the same shape and differing only in the placement of the ideal point. Would the same Condorcet winner arise under single peaked preferences that differ in their shapes but not the horizontal location of ideal points?

Answer: Yes, the only thing that matters for the result is where on the issue dimension y the ideal points are and that the preferences are single peaked. It does not matter how high the peaks are — or whether the slopes from the ideal point are steep or shallow.

Exercise 28A.5

Can you see how this equilibrium prediction conforms to the equilibrium in the Hotelling model when firms are restricted to charging the same output price (and where the ideological spectrum is replaced by product differentiation)?

Answer: In the absence of price competition, the firms' goal on the Hotelling line is to maximize market share — which is done by settling at the median point just as the politician settles at the median voter's position.

Exercise 28A.6

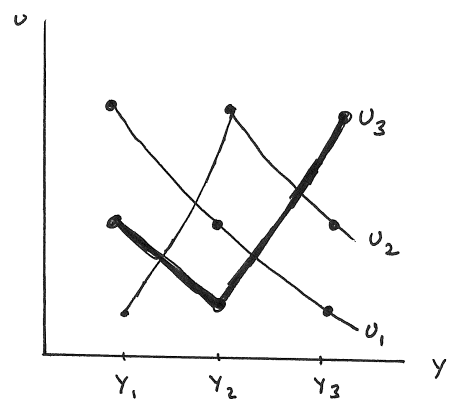
In the U.S., prior to running as a party nominee in a Presidential election, a potential candidate first has to win primary elections that are restricted to members of the potential candidate's party. For instance, multiple candidates for the Democratic Party's nomination must compete first among only Democrats to earn the party's nomination — before then competing against the Republican Party's nominee in the general election. Can you use our median voter model to argue that candidates for the Democratic Party's nomination will initially position themselves to the left of the median voter but will then succeed in the general election (against the Republican nominee) only to the extent to which they can "move to the center" in the general election campaign?

Answer: In the party nomination battle, the successful Democratic candidate has to appeal to the median voter within the Democratic part — who will lie to the left of the general election median voter. Similarly, in the Republican nomination battle, the successful Republican candidate has to appeal to the Republican median voter who lies to the right of the general election median voter. In the general election, the two candidates will then moderate their positions — attempting to get toward the median voter whose position they would have taken from the start had it not been for the need to fight for the party nomination first. This basic intuition emerges from the median voter theorem — and has been most famously articulated by Richard Nixon who advised future Republicans to run to the right in the primaries and then to the center in the general election.

Exercise 28A.7

Which of these voters have single peaked preferences over public school spending?

Answer: Voter 3's tastes are not single peaked. This is illustrated in Exercise Graph 28A.7 where utility levels consistent with the preference orderings given for the three voters are graphed — with u_i the utility of voter i. Voters 1 and 2 have single peaked preferences — with voter 1's peak (at least for these three alternatives) at y_1 and voter 2's peak at y_2. But voter 3 only has a "trough" at y_2 — with utility increasing as public school spending rises and falls from y_2.



Exercise Graph 28A.7 : Preferences for School Spending y

Exercise 28A.8

Legislators — like Senators — who run for executive office — like Governor or President — are often confronted by the media with votes they have taken in the legislature that seem to contradict their stated positions. Sometimes you will hear politicians respond that their vote against something was actually a vote for something given the sequence of votes that was scheduled. Are they being sly or might they be telling the truth?

Answer: We have just shown that sophisticated voters will sometimes do better by voting against their most preferred policy in order to get their second most preferred policy implemented in the next voting stage — knowing that if they instead voted for their most preferred policy, it would lose in the next stage to something that is at best third-preferred. Thus, legislators may well find themselves voting against their ideal when they look down the agenda tree and forecast what will happen next. This may be a reason why it has been difficult for sitting U.S. Senators who frequently run for President to get elected — there are too many seemingly contradictory votes to explain effectively. While many Senators run for both party nominations each election cycle — and some succeed in getting their party's nomination, only two have succeeded in winning the Presidency in the general election since the early 1900's, and both — John F. Kennedy and Barack Obama — had relatively short and undistinguished careers in the Senate but charismatic general appeal. (Other Presidents served as Senators at one point or another — but did not get elected as sitting Senators. These include Harry Truman and Lyndon Johnson who were named Vice-President and became President when their predecessor died, and Richard Nixon who had served in the Senate for 3 years prior to becoming Vice-President under Dwight Eisenhower — before ultimately getting elected President in 1968 following an unsuccessful run in 1960 (against John F. Kennedy).)

Exercise 28A.9

How would ideal points differ for voters who report that they are conservative, liberal, or libertarian?

Answer: Conservatives would tend to favor military spending over domestic spending — with their ideal point therefore lying relatively high in the graph but close to the z (military) axis. Liberals would tend to favor domestic spending over military spending — with their ideal point lying relatively low in the graph but far to the right. And libertarians would tend to like all spending to be relatively low — with their ideal point therefore lying close to the origin.

Exercise 28A.10

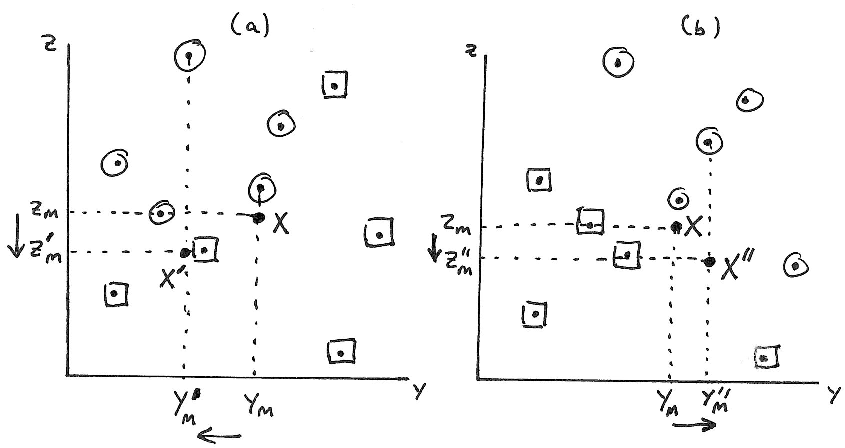
Can you explain how panels (d) through (f) complete the argument that G can be implemented through a sequence of pair-wise votes?

Answer: Panel (d) inherits policy D from panel (c) — and we now put up E against D. The policy E is constructed to be closer to 3 and 1 than D — thus getting support from individuals 1 and 3 and defeating D. In panel (e), we give up on voter 3 and focus on 1 and 2 by constructing the policy F to lie closer to 1 and 2 than E does. As a result, F beats E. Finally, in panel (f), we gan put G against F and have 2 and 3 support G — because their ideal point lies closer to G than to F.

Exercise 28A.11

How would you assign the members to committees if you wanted to get less spending on y and z than we did in panel (a) of Graph 28.9? What if you wanted more of y but less of z?

Answer: This is done in panels (a) and (b) of Exercise Graph 28A.11. In panel (a), we are able to change the outcome from the original X to X' — with less spending on both policies. In panel (b), we are able to change the policy from X to X'' — with more spending on y and less on z.



Exercise Graph 28A.11 : Committee Assignments

Exercise 28A.12

Relate the idea of concentrated benefits and diffuse costs to the free rider problem faced by interest groups that represent beneficiaries and victims of policies.

Answer: Any interest group that wishes to mobilize those it represents faces a free rider problem — i.e. it is in the interest of those it represents to free ride on the efforts put in by others. The smaller the number of individuals represented by the interest group, however, the smaller will be the free rider problem faced by the interest group in organizing its members. Thus, if benefits are concentrated among a few people, they will face less of a free rider problem in organizing and lobbying for a policy than those who experience costs diffused over large numbers of people that face a much bigger free rider problem.

28B Solutions to Within-Chapter-Exercises for Part B

Exercise 28B.1

How does Pareto Optimality as a concept differ from Pareto Unanimity?

Answer: Pareto optimality simply means efficiency — which is satisfied whenever we have a policy which cannot be altered in such a way as to make someone better off without making anyone else worse off. Pareto unanimity refers to a choice between two social states — neither of which must necessarily be efficient — and in particular refers to a property of the social choice function that determines which of the two choices should be implemented. If the social choice functions satisfies pareto unanimity, it simply means it will pick unanimously preferred social states in pair-wise competitions. Were the social choice function pareto optimal, we might think that it means that the social choice function will always choose efficient policies over inefficient policies — which is not the case under pareto unanimity (because there are many examples of inefficient policies that are preferred by some individuals to efficient policies that are preferred by others, in which case the property of pareto unanimity gives us no guidance on what the social choice function should pick).

Exercise 28B.2

In Section A, we developed the "median voter theorem" that says, when voters' preferences over a single dimensional issue are single peaked, the outcome under majority rule is the outcome preferred by the median voter. If we define that social choice function to be majority rule, does this make the median voter an Arrow Dictator under that rule? (*Hint*: The answer is no.)

Answer: No, it does not. An Arrow dictator essentially gets his way *no matter what the profile of preferences is for everyone else.* The median voter (in the median voter model) only gets his way because the preferences of others just happen to have fallen in a way where the median voter is pivotal. Any change in others' ideal points from one side of the median to the other, and the median voter is no longer pivotal and no longer gets his way — but the Arrow dictator would.

Exercise 28B.3

Is the agenda setter in Graph 28.5 an Arrow dictator? Is the agenda setter in our discussion of the "Anything-Can-Happen" Theorem an Arrow dictator?

Answer: The agenda setter in Graph 28.5 is not an Arrow dictator — because he is able to get his favorite outcome only in the case where someone has non-single-peaked preferences. Were all his voters to have single-peaked preferences, then there would be a Condorcet winner — the median voter — and the agenda setter cannot get anything he wants. In the "Anything-Can-Happen" Theorem, on the other hand, the agenda setter really can get his way always even if everyone disagrees — so long as he can structure the voting sequence and he knows everyone's tastes. In this unstructured setting, the agenda setter becomes the dictator who simply manipulates the process to give the appearance of democracy.

Exercise 28B.4

Explain why this conclusion follows from the definition of the decisiveness over (x, y) of the coalition D.

Answer: Everyone in D prefers x over y but everyone outside D prefers y over x. Since D is decisive, it must be that f chooses x over y — i.e. $x \succsim y$.

Exercise 28B.5

By changing the individual preference orderings in (28.5) and then proceeding through similar steps, can you show that it is similarly true that whenever members of D prefer z over x and everyone else disagrees, the social preference ordering that arises from f must pick z over x as well.

Answer: Consider the preference profile

$$\begin{aligned} z \succsim^i y \succsim^i x \;\; \forall \; i \in D \\ x \succsim^j z \succsim^j y \; \forall \; j \notin D. \end{aligned} \tag{28B.5.i}$$

The decisiveness of D with respect to (x, y) then implies that y us chosen over x by the social choice functions (because everyone in D prefers y to x while everyone outside D disagrees). We therefore have

$$y \succsim x. \tag{28B.5.ii}$$

Pareto Unanimity further implies that $z \succsim y$ since everyone prefers z to y. Rationality then implies transitive social preferences — and since $z \succsim y$ and $y \succsim x$, it must be that the social choice function picks z over x — i.e.

$$z \succsim x. \tag{28B.5.iii}$$

But the only people that prefer z to x are members of D — with everyone outside D preferring x to z. Regardless of how we now switch around y in the preference orderings, it must continue to hold by the IIA axiom that the social choice function picks z over x — because y is irrelevant to the choice over z and x. But that implies that, whenever everyone in D likes z better than x and everyone else disagrees, D will get its way.

Exercise 28B.6

The above reasoning implies that *every* coalition of everyone but one person must be decisive. How can it be that both $D = \{2, 3, ..., N\}$ and $D' = \{1, 2, 3, ..., N-1\}$ can be decisive?

Answer: Remember that the definition of decisiveness of D only has bite — i.e. it only applies — when the preferences are such that everyone in D feels one way and everyone else feels another way about a pair-wise choice. So, when we say that D is decisive, it means that D gets its way when everyone in D agrees and individual 1 disagrees. When we say that D' is also decisive, we mean that D' gets its way when everyone in D' agrees and only individual N disagrees. If everyone in D agrees and only individual 1 disagrees, then the decisiveness of D' does not tell us anything — because everyone in D' does not agree. Similarly, if everyone in D' agrees and only individual N disagrees, the definition of decisiveness does not apply to coalition D (which included N) in which not everyone agrees. To say that D is decisive and that D' is decisive is therefore not to make mutually exclusive statements.

28C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 28.1

In Chapter 4 we considered different ways of thinking about single-peaked preferences over two-dimensional issue spaces. We did so in particular in end-of-chapter exercise 4.11 which you can now re-visit.

Answer: See the solutions for Chapter 4 exercises.

Exercise 28.3

Everyday Exercise: *Why Vote?: Voting is costly. If you vote in person, you have to find your polling place and often stand in line until you get to the voting booth to vote. If you vote by absentee ballot, you have to figure out how to get one and then be sure to mail it in. In both cases, you probably have to do some work figuring out who the candidates and what the issues are.*

A: *Many people purposefully choose not to vote — and they often give the following reason: "I don't think it matters." As we will see in this exercise, they might mean one of two things by this — and they appear to be right in at least one sense.*

(a) *First, suppose we take the median voter model really seriously and believe it accurately predicts the position of the two candidates from which we choose. How might this justify the excuse given by voters who don't vote?*

Answer: If we really believed the median voter model 100%, we would conclude that, in 2-person elections, both people must be taking identical positions — and therefore "it does not matter" which one we elect.

(b) *In the real world, there are many frictions that keep the median voter model's prediction from fully coming to fruition. For instance, candidates might have to win party nominations first, and then run in the general election — which means we tend to end up with right-of-center and left-of-center candidates. In light of this, is it reasonable to think that the excuse given in (a) is justified in the real world?*

Answer: The "it does not matter because the two candidates are identical" excuse in part (a) then does not work in the real world where real world factors cause candidates to ultimately differ in their positions and in what they are likely to do if elected.

(c) *Next, consider a different way in which the "it does not matter" statement might be meant: Perhaps a voter recognizes that it matters which candidate wins (in the sense that the world will change differently depending on which one wins), but she believes the candidate who will win will almost certainly win whether any individual voter goes to the polls or not. Do you think this is true in the real world?*

Answer: It is very rare that an election ends up in a tie or is won by a 1-vote margin — but that is required if one person's vote is to actually make a difference in the outcome of the election. It therefore seems almost certainly true that a voter's vote will not change the outcome of an election — and in that sense "does not matter". (Some might argue that it would matter if everyone took that view and stays home. While that is certainly true, rational voters should know from observing past elections that not everyone will stay home — and thus the argument continues to hold. My wife, a political science major, still hates me for saying this.)

(d) *In light of your answer to (c), might it be rational for many people not to vote?*

Answer: Yes, if a voter rationally believes that her vote will almost certainly not change the outcome of an election, and if it is costly to vote — even just a little bit costly — the marginal benefit (of essentially zero) is not worth the marginal cost of voting.

(e) *In the 2008 U.S. Presidential election, Barack Obama won by close to 10 million votes. In what sense is the puzzle not so much why more people didn't vote but rather why so many — about 60% of eligible voters — did.*

Answer: This follows straight from our answers to the previous part. In elections where 130 million people vote, one vote will almost certainly not make a difference in the outcome. This suggests that the marginal benefit of voting (in terms of the probability of affecting the outcome) is zero. So if there is any marginal cost to voting, it is surprising that so many people still vote. In 2008, some people actually stood in line for 8 hours to vote — a large marginal cost indeed given that the outcome was going to be the same whether any one of the individuals in that 8 hour line had voted or not.

(f) *Suppose we believe that governments are more effective the more voters engage in elections. In what sense does this imply that voters have prisoners' dilemma incentives that give rise to free-riding?*

Answer: This would mean that voting has a positive externality — i.e. by voting, I make the government just a bit more effective even though I don't change the outcome of the election. Even if that positive externality is tiny, multiply it by 300 million (i.e. the population of the U.S.), and even a tiny externality adds up. For instance, if the positive externality is one hundredthousandth of a cent, it would add up to $30 per vote. We thus find ourselves in a situation where we want effective government — and thus would like governments to be elected by a sizable fraction of the electorate — but it is in our individual interest to not vote since the marginal private benefit is zero and the marginal cost is positive. Put differently, our incentive is to free ride on the effectiveness of government that others produce by voting. (In some countries, everyone is in fact required to vote or pay a fine — which is one way to address the externality problem.)

(g) *In Chapter 27, we suggested that one way charitable organizations overcome free-rider problems among potential donors is to find ways of eliciting within donors a "warm glow" from giving. Can you think of an anal-*

ogous explanation that can rationalize why so many people vote in large elections?

Answer: From a very early age, children in democratic societies are typically told how important it is to vote. Thus, from an early age, we seem to attempt to shape preferences in favor of voting so that we in essence produce a private benefit from voting (because not voting makes us feel guilty.) On the day of elections in the U.S., voters get “I voted” stickers that many wear all day. Much effort therefore goes into causing people to experience part of the voting process as a private good — presumably in order to compensate for the free-rider incentives that are otherwise present.

(h) *Suppose that the voters who do not vote are those who are “disillusioned”. What positions might two candidates take on the Hotelling interval [0,1] if the disillusioned voters (that do not vote) are those that cannot find a candidate whose position is within 3/16ths of their ideal point? Could we have an equilibrium where the extreme ends of the political spectrum do not vote? Could we have one where the center does not vote?*

Answer: In this case, any two positions that are at least 3/16ths away from the endpoints of the interval can be an equilibrium with two candidates. This is because, no matter which two points in the interval [3/16,3/16] the two candidate choose, they will split the votes of those who vote. The farther apart they are, the more people will vote — but they will split 50-50. (If one were to pick a position closer to the end-points, then the other could win the election outright by picking a position within the [3/16,3/16] interval sufficiently far from the other candidate.) It may therefore be that people on the extremes end up being disillusioned and not voting — or it may be that people in the center don’t vote — depending on where the two candidates stake out their position.

B: *In the 2000 U.S. Presidential election, George W. Bush defeated Al Gore by 537 votes in the State of Florida — and with those 537 votes won the election.*

(a) *The close margin in the 2000 election is often cited by politicians as evidence that you should “not believe your vote does not matter”. I would argue that it shows the opposite: Even in close elections, it is almost never the case that one vote counts. What do you think?*

Answer: This is as close to a tied election as one is ever likely to see in a country of this size. Yet even in this case, one vote more for the losing candidate would simply have meant that George W. Bush would have won by 536 rather than 537 votes. There appears to be no way to argue that any single voter’s behavior could have changed the outcome — and no reason to question the presumption that the probability of a single voter affecting the outcome of a large election is essentially zero.

(b) *Ralph Nader, the Green Party candidate, received nearly 100,000 votes in the State of Florida in the 2000 Presidential election. Many believe that, had Ralph Nader’s name not been on the ballot, Al Gore would have won* *the State of Florida — and with it the Presidency. If so, which one of Arrow's axioms does this suggest is violated by the way the U.S. elects Presidents? Explain.*

Answer: The Independence of Irrelevant Alternatives (IIA) axiom states essentially that the social ranking over a pair of alternatives should be independent of how those alternatives are ranked relative to other alternatives. Yet the argument is that the inclusion of Nader on the ballot caused a different social ranking of Bush and Gore than would have happened in the absence of this "irrelevant" alternative.

(c) *Some election systems require the winning candidate to win with at least 50% plus 1 votes — and, if no candidate achieves this, require a run-off election between the top two candidates. Since this seems difficult to implement in the 50 state-wide elections that result in electoral college votes that determine the winner of a U.S. Presidential election, some have proposed a system of* instant run-off voting. *In such a system, voters rank the candidates from most preferred to least preferred. In the first round of vote counting, each voter's top ranked candidate is considered as having receive a vote from that voter, and if one candidate gets 50% plus 1 votes, he is declared the winner. If no candidate receives that many votes, the election authorities find which candidate received the lowest first round votes and then re-assigns that candidate's votes to the candidates that were ranked second by these voters. If one candidate reaches 50% plus 1 votes, he wins — otherwise the election authorities repeat the exercise, this time re-assigning the votes of the candidate who initially received the second to last number of first place rankings. This continues until someone gets 50% plus 1 votes. Had the State of Florida used this system in 2000, do you think the Presidential election outcome might have been different?*

Answer: If it is true that a sizable majority of Nader voters would have been Gore voters had Nader not been on the ballot, these voters would presumable have been more likely to rank Gore rather than Bush second in an instant run-off election. As a result, in the "instant run-off" where Nader's first-round voters are reassigned based on their second choice, Gore would have ended up with more additional votes than Bush — quite likely more than 537 (which is what he needed to win). At the same time, it is almost impossible to know for sure whether this would have changed the election outcome — because the awareness that votes would be counted differently might well have affected how candidates allocated campaign resources and to what extent voters voted strategically. Nader might well have received more first round votes because many voters who preferred Nader to Gore might have voted for Gore given they knew Nader had no chance of winning — but they also would have ranked Gore second so that Gore would eventually have gotten those votes. But Gore might have spent less time in Florida if he thought Nader was not as big a threat to his chances of winning the state (because he would get most of the Nader voters in the second round) — in which case Bush might have won by enough of a margin in the first round to win outright.

(d) *Nader is often referred to as a "spoiler" — because of many people's belief that he "spoiled" the election outcome for Gore.* True or False*: It is much less likely that a third candidate plays the role of spoiler in an instant run-off election — but it is still possible if the third candidate is sufficiently strong.*

Answer: The statement is true — in both its parts. First, it is much less likely that a candidate like Nader can spoil the election outcome for someone like Gore — because Gore will ultimately get the Nader votes that would have otherwise come to Gore in the first round (had Nader not been on the ballot). But it is still possible for the spoiler candidate to keep a candidate that would have otherwise won in pair-wise voting from winning — if the spoiler candidate is sufficiently strong to come in second but not with enough second round votes to win the election. Consider, for instance, the example of the 1992 election where Bush Sr. ran against Bill Clinton and a strong independent candidate, Ross Perot. At times, Perot was polling upwards of 25-30% in national opinion polls (although he ultimately only won 19% because he dropped out and then came back into the race, citing some bizarre reasons.) Suppose Perot had won 30% of the vote with Bush winning 41% and Clinton winning 29%. This would have triggered Clinton being eliminated — with his second choice votes being apportioned to Perot and Bush. Suppose a third of Clinton voters ranked Bush second and two thirds ranked Perot second. Then Bush would have been assigned one third of 29% or 9.67% in additional overall votes — taking him to 50.67% and causing him to win the election. But suppose that 75% of Perot voters preferred Clinton over Bush. Then, in a head-to-head election between Bush and Clinton, Clinton would have gotten not only his 29% but also three quarters of Perot's 30% — for a total of 51.5% and thus the Presidency. Perot could thus have spoiled the election for Clinton — if he was sufficiently strong to get into the top two, not sufficiently strong to win and if his voters were sufficiently tilted toward Clinton.

Exercise 28.5

Everyday Application: *To Enter or Not to Enter the Race*: *Suppose there are three possible candidates that might run for office, and each has to decide whether or not to enter the race. Assume the electorate's ideal points can be defined by the Hotelling line from Chapter 26 — i.e. the ideal points are uniformly distributed on the interval [0,1].*

A: *Let π_i denote the probability that candidate i will win the election. Suppose that the payoff to a candidate jumping into the race is $(\pi_i - c)$ where c is the cost of running a campaign.*

(a) *How high must the probability of getting elected be in order for a candidate to get into the race?*

Answer: It must be that $\pi_i \geq c$.

(b) *Consider the following model: In stage 1, three potential candidates decide simultaneously whether or not to get into the race and pay the cost c. Then, in stage 2, they take positions on the Hotelling line — with voters then choosing in an election where the candidate who gets the most votes wins.* True or False*: If there is a Nash equilibrium in stage 2 of the game, it must be that the probability of winning is the same for each candidate that entered the race in stage 1.*

<u>Answer</u>: True. If only one candidate entered, then the statement is trivially true. If two entered, then we know the only equilibrium is one where they both take the median position 0.5 in the middle of the Hotelling line. If there are three candidates and the probabilities of winning are not the same for all, then the candidate closest in position to the one whose probability is highest can increase her probability of winning by moving slightly closer to her opponent's position.

(c) *Suppose there is a Nash equilibrium in stage 2 regardless of how many of the three candidates entered in stage 1. What determines whether there will be 1, 2 or 3 candidates running in the election?*

<u>Answer</u>: If there is a Nash equilibrium in stage 2, then three candidates will enter in stage 1 so long as $c < 1/3$ — because the probability of winning in stage 2 with three candidates running is 1/3. If $1/3 < c < 1/2$, only two candidates would enter in stage 1 — and if $c > 1/2$, only a single candidate enters.

(d) *Suppose that the probability of winning in stage 2 is a function of the number of candidates that are running as well as the amount spent in the campaign, with candidates able to choose different levels of c when they enter in stage 1 but facing an increasing marginal cost $p(c)$ for raising campaign cash. (The payoff for a candidate is therefore now $(\pi_i - p(c))$.) In particular, suppose the following: Campaign spending matters only in cases where an election run solely on issues would lead to a tie (in the sense that each candidate would win with equal probability). In that case, whoever spent the most wins the election. What might you expect the possible equilibria in stage 1 (where entry and campaign spending are determined simultaneously) to look like?*

<u>Answer</u>: It could not be an equilibrium for candidates to spend different amounts — since spending anything when your opponent spends more is a sure way to lose the election in stage 2 where the probability of winning the election on issues alone can only result in a tie. Thus, any candidate that enters must spend the same mount c^*. The fact that the marginal cost of raising campaign cash is increasing then still permits different possibilities for how many candidates might enter in the first stage — but the cost of the campaign is now determined endogenously, with the payoff from running in the election $\pi - p(c^*) = 0$.

(e) *Suppose the incumbent is one of the potential candidates — and he decides whether to enter the race and how much to spend first. Can you in this* *case see a role for strategic entry deterrence similar to what we developed for monopolists who are threatened by a potential entrant?*

Answer: Yes, there is now an analogous case for the incumbent if the incumbent gets to announce his campaign war chest before the others decide whether or not to enter the race (just as the monopolist has to be able to set quantity before the potential entrant decides whether to enter). The incumbent then has to determine how costly it is to amass the war chest necessary to limit the field of opponents — and can credibly commit to a level of campaign spending because he gets to move first.

(f) *With the marginal cost of raising additional funds to build up a campaign war chest increasing, might the incumbent still allow entry of another candidate?*

Answer: Yes, how much entry the incumbent allows depends on the ease with which he can raise the campaign war chest.

B: *Consider the existence of a Nash equilibrium in stage 2.*

(a) *What are two possible ways in which 3 candidates might take positions in the second stage of our game such that your conclusion in A(b) holds?*

Answer: If our conclusion from A(b) holds, then any Nash equilibrium with three candidates competing in stage 2 must have $\pi_1 = \pi_2 = \pi_3$. One way that this could happen is if all candidates take the median position 0.5 — and thus end up in a three-way tie (in expectation). Another way for it to happen is for the candidates to locate at points 1/6, 1/2 and 5/6 — which would give each of them 1/3 of the vote.

(b) *Can either of these be an equilibrium under the conditions specified in part A?*

Answer: No. For the case where all three candidates take the position 0.5, one candidate could move slightly to the left or right and increase his vote share to just under 1/2 — virtually assuring his election. For the case where the candidates take the positions 1/6, 1/2 and 5/6, the candidate at 1/6 could increase his vote share to close to one half by changing his position to lie just to the left of 1/2 — again virtually assuring his election. (Something similar applies to the candidate at 5/6).

(c) *Suppose that, instead of voter ideal points being uniformly distributed on the Hotelling line, one third of all voters hold the median voter position. How does your conclusion about the existence of a stage 2 Nash equilibrium with three candidates change? Does your conclusion from A(c) still hold?*

Answer: If a third of the population holds the median position, then all candidates picking the median position is an equilibrium in stage 2. This results in a probability of winning the election equal to 1/3 for each of the three candidates. If any of the candidates deviated from the median position, he would get less than 1/3 of a vote share. Our conclusion from A(c) still holds.

Exercise 28.7

Business and Policy Application: *Voting with Points*: *Jean-Charles de Borda (1733-99), a contemporary of Condorcet in France, argued for a democratic system that deviates from our usual conception of majority rule. The system works as follows: Suppose there are M proposals. Each voter is asked to rank these — with the proposal ranked first by a voter given M points, the one ranked second given $(M-1)$ points, and so forth.*[1] *The points given to each proposal are then summed across all voters, and the top N proposals are chosen — where N might be as low as 1. This voting method, known as the* Borda Count *is used in a variety of corporate and academic settings as well as some political elections in countries around the world.*

A: *Suppose there are 5 voters denoted 1 through 5, and there are 5 possible projects $\{A, B, C, D, E\}$ to be ranked. Voters 1 through 3 rank the projects in alphabetical sequence (with A ranked highest). Voter 4 ranks C highest, followed by D, E, B and finally A. Voter 5 ranks E highest, followed by C, D, B and finally A.*

(a) *How does the Borda Count rank these? If only one can be implemented, which one will it be?*

Answer: Table table here presents the number of points assigned to each of the 5 alternatives by the 5 voters and sums these in the last row. The Borda Count therefore ranks C highest, followed by A, B, D and E.

Borda Count					
	A	B	C	D	E
Voter 1	5	4	3	2	1
Voter 2	5	4	3	2	1
Voter 3	5	4	3	2	1
Voter 4	1	2	5	4	3
Voter 5	1	2	4	3	5
Total	17	16	18	13	11

[1] There exist other versions of Borda's method — such as assigning $(M-1)$ points to the top ranked choice and leaving zero for the last ranked. For purposes of this problem, define the method as it is defined in the problem.

(b) *Suppose option D was withdrawn from consideration before the vote in which voters rank the options. How does the Borda Count now rank the remaining projects? If only one can be implemented, which one will it be?*

Answer: The Borda Count is now re-done in the following table — with A (rather than C) winning, followed by B and C tied and E bringing up the rear.

Borda Count				
	A	B	C	E
Voter 1	4	3	2	1
Voter 2	4	3	2	1
Voter 3	4	3	2	1
Voter 4	1	2	4	3
Voter 5	1	2	3	4
Total	14	13	13	10

(c) *What if both D and E are withdrawn?*

Answer: The following table re-calculates the Borda Count with the initial winner C now coming in third, and with A again winning.

Borda Count			
	A	B	C
Voter 1	3	2	1
Voter 2	3	2	1
Voter 3	3	2	1
Voter 4	1	2	3
Voter 5	1	2	3
Total	11	10	9

(d) *Suppose I get to decide which projects will be considered by the group and the group allows me to use my discretion to eliminate projects that clearly do not have widespread support. Will I be able to manipulate the outcome of the Borda Count by strategically picking which projects to leave off?*

Answer: Yes, as we have shown already. By leaving all proposals on the table, I know C will get picked for sure, but by dropping D I can get A for sure instead (with C moving into second place) and by dropping E as well I can get C to move into third place (with A still winning). If I really want A implemented, I should therefore remove either D or both D and E from the agenda — and if I want to make sure that C is last in line, I want to do the latter.

B: *Arrow's Theorem tells us that any non-dictatorial social choice function must violate at least one of his remaining four axioms.*

(a) *Do you think the Borda Count violates Pareto Unanimity? What about Universal Domain or Rationality?*

Answer: Given the way the points are assigned, unanimously preferred choices will always be chosen by the Borda count. The method does not restrict preferences — and it results in a preference ranking that, by construction, does not give rise to intransitivities. Thus, none of these axioms are violated.

(b) *In what way do your results from part A of the exercise tell us something about whether the Borda Count violates the Independence of Irrelevant Alternatives (IIA) axiom?*

Answer: In part A, we showed that an agenda setter can eliminate alternatives that are ranked 5th and 4th in the initial Borda count — and this can have an impact on what is ranked first, second and third. Put differently, the inclusion or elimination of D and E affects how we rank A, B and C — but a social choice function that satisfies IIA would rank A, B and C independent of what other options that are irrelevant are also included.

(c) *Derive again the Borda Count ranking of the five projects in part A given the voter preferences as described.*

Answer: This is already done in A(a). It results in the Border Count ranking of

$$C \succ A \succ B \succ D \succ E. \tag{28.7}$$

(d) *Suppose voter 4 changed his mind and now ranks B second and D fourth (rather than the other way around). Suppose further that voter 5 similarly switches the position of B and D in his preference ordering — and now ranks B third and D fourth. If a social choice function satisfies IIA, which social rankings cannot be affected by this change in preferences?*

Answer: If the social choice function represented by the Borda Count satisfied IIA, no social ordering of a pair that does not involve B and/or D should change — because no individual preferences over pairs involving A, C or E have changed. Similarly, we have not changed how anyone feels over B and C — voter 4 still likes C better than B as does voter 5. Thus, the social preference ordering over B and C should not change. There are a number of others like this.

(e) *How does the social ordering of the projects change under the Borda Count? Does the Borda Count violate IIA?*

Answer: This is calculated in the next table where the numbers that have changed from the initial table in part (a) are in bold. The Borda count ranking is now

$$B \succ C \succ A \succ E \succ D. \tag{28.7.i}$$

Borda Count					
	A	*B*	*C*	*D*	*E*
Voter 1	5	4	3	2	1
Voter 2	5	4	3	2	1
Voter 3	5	4	3	2	1
Voter 4	1	**4**	5	**2**	3
Voter 5	1	**3**	4	**2**	5
Total	17	**19**	18	**10**	11

But we see that C was preferred by the Borda Count to A before the change and now A is preferred to C. If the alternatives B and D were irrelevant to how the social preference order ranks A and C, this should not happen. Note also that no voter changes their ranking of B against C — yet the Borda count had C winning before and now has B winning. We see again that the Borda Count violates IIA.

Exercise 28.9

Policy Application: *Political Coalitions and Public School Finance Policy: In this exercise, we consider some policy issues related to public support for schools — and the coalitions between income groups that might form to determine the political equilibrium.*

A: *Throughout, suppose that individuals vote on only the single dimension of the issue at hand — and consider a population that is modeled on the Hotelling line [0,1] with income increasing on the line. (Thus, individual 0 has the lowest income and individual 1 has the highest income, with individual 0.5 being the median income individual.)*

(a) *Consider first the case of public school funding in the absence of the existence of private school alternatives. Do you think the usual median voter theorem might hold in this case — with the public school funding level determined by the ideal point of the median income household?*

Answer: While it depends on the nature of the tax system used to fund the public schools, it is not unlikely that, given the tax system used to fund the schools, demand for public school spending increases with household income. In that case, the ideal points for public spending would be increasing along the Hotelling line — with the median voter's most preferred public spending level representing the Condorcet winning policy.

(b) *Next, suppose private schools compete with public schools, with private schools charging tuition and public schools funded by taxes paid by everyone. How does this change the politics of public school funding?*

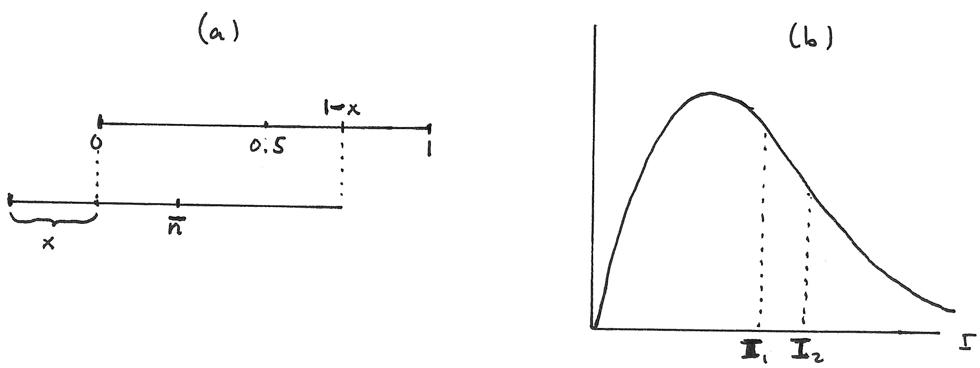
Answer: Conditional on sending one's children to private schools, one's preferred tax — and thus one's preferred public school spending level — would drop to zero. Thus, the politics of public school funding changes in the sense that those who choose private schools become low demanders of public school quality.

(c) *Some have argued that political debates on public school funding are driven by "the ends against the middle". In terms of our model, this means that the households on the ends of the income distribution on the Hotelling line will form a coalition with one another — with households in the middle forming the opposing coalition. What has to be true about who disproportionately demands private schooling in order for this "ends against the middle" scenario to unfold?*

Answer: For the "ends against the middle" scenario to unfold, it must be that demand for private schools comes disproportionately from high income families who would favor high public school spending in the absence of private schools but now favor low public school spending.

(d) *Assume that the set of private school students comes from high income households. What would this model predict about the income level of the new median voter?*

Answer: This is illustrated in panel (a) Exercise Graph 28.9. The first line from 0 to 1 represents the original Hotelling line with income and ideal levels of public school spending rising along the line. As a result, 0.5 is the original median voter whose preferred public school spending level is implemented in the absence of private schools. If then a private school opens and draws the highest x income households away from the public system, that x segment of the Hotelling line now moves to the other side of the original median voter in terms of its ideal point for public good spending. As a result, the new median voter is $\overline{n} = 0.5 - x$ — which implies the new median voter (in the presence of private schools) has less income than the original median voter (in the absence of private schools).



Exercise Graph 28.9 : Median Voters for Public School Spending

(e) *Consider two factors: First, the introduction of private schools causes a change in the income level of the median voter, and second, we now have private school attending households that pay taxes but do not use public* *schools. In light of this, can you tell whether per pupil public school spending increases or decreases as private school markets attract less than half the population? What if they attract more than half the population?*

Answer: As we have seen, the median voter's income will fall — which, all else being equal, would imply a decrease in public school spending. But fewer kids attend public school and thus every dollar in tax revenue results in a larger increase in per pupil spending. Thus, "all else is not equal" — and the two forces point in opposite direction. This makes it impossible to tell without further information whether per pupil public spending on education rises or falls as more high income individuals go to private schools — at least so long as fewer than 50% do so. If more than 50% attend private schools, the median voter will be someone who sends her child to private schools — which would cause us to predict a sharp drop in public school funding (to zero, if we take the model completely literally). But if private school attendance is less than 50%, you can think of public school attendees actually getting a subsidy from private school attendees — so, although they would vote for less spending if the price remained the same, they might vote for more given the implicit subsidy that allows them to free ride on the tax payments of private school attending households.

(f) *So far, we have treated public school financing without reference to the local nature of public schools. In the U.S., public schools have traditionally been funded locally — with low income households often constrained to live in public school districts that provide low quality. How might this explain an "ends against the middle" coalition in favor of private school vouchers (that provide public funds for households to pay private school tuition)?*

Answer: In a system with widely varying school quality based on local income levels, it may be the case that low income parents would be the first to switch to private schools if they received a voucher — and would find this preferable to their current public school. High income households already send their children to private schools — so vouchers would be a pure income transfer to them (as they would not have to pay as much of their children's tuition bills anymore). Thus, there are two natural constituencies for private school vouchers: the poor who are constrained in the public system to the worst schools, and the rich who already use private schools.

(g) *In the 1970's, California switched from local financing of public schools to state-wide (and equalized) financing of its public schools. State-wide school spending appears to have declined as a result. Some have explained this by appealing to the fact that the income distribution is skewed to the left, with the statewide median income below the statewide mean income. Suppose that local financing implies that each public school is funded by roughly identical households (who have self-seleted into different districts as our Chapter 27 Tiebout model would predict), while state financing im-*

plies that the public school spending level is determined by the state median voter. Can you explain how the skewedness of the state income distribution can then explain the decline in state-wide public school spending as the state switched from local to state financing?

Answer: Such a skewed income distribution is graphed in panel (b) of Exercise Graph 28.9 where I_1 is the median income level and I_2 is the mean income level. In a local system where income types separate into distinct communities, there is broad agreement within the community on how much to spend on the local school — because everyone has the same income. Thus, each community funds its school in relation to the common income level of its residents — with the average per pupil spending level in the state therefore approximately equal to the income level that the average person would have chosen. When per pupil spending is determined in state-wide elections, however, the median income level will determine the spending level — and since the median is below the mean, we would expect the skewedness of the income distribution to result in less overall public spending on education in the statewide system than in the local system.

B: *Suppose preferences over private consumption x, a public good y and leisure ℓ can be described by the utility function $u(x,y,\ell) = x^\alpha y^\beta \ell^\gamma$. Individuals are endowed with the same leisure amount L, share the same preferences but have different wages. Until part (e), taxes are exogenous.*

(a) *Suppose a proportional wage tax t is used to fund the public good y and a tax rate t results in public good level $y = \delta t$. Calculate the demand function for x and the labor supply function. (Note: Since t is not under the control of individuals, neither t nor y are choice variables at this point.)*

Answer: The consumer then solves the problem

$$\max_{x,\ell} x^\alpha y^\beta \ell^\gamma \text{ subject to } x = (1-t)w(L-\ell). \tag{28.9.i}$$

Solving this in the usual way, we get leisure demand of

$$\ell = \frac{\gamma L}{\alpha+\gamma}. \tag{28.9.ii}$$

Subtracting this from L gives the labor supply l^*, and plugging it into the budget constraint gives us x^* — which are

$$l^* = \frac{\alpha L}{\alpha+\gamma} \text{ and } x^* = (1-t)w\left(\frac{\alpha L}{\alpha+\gamma}\right). \tag{28.9.iii}$$

(b) *Suppose instead that a per-capita tax T is used to fund the public good; i.e. everyone has to pay an equal amount T. Suppose that a per-capita tax T results in public good level $y = T$. Calculate the demand function for x and the labor supply function.*

Answer: We now solve

$$\max_{x,\ell} x^{\alpha} y^{\beta} \ell^{\gamma} \text{ subject to } x = w(L-\ell) - T. \tag{28.9.iv}$$

Solving this in the usual way, we get leisure demand

$$\ell = \frac{\gamma(wL-T)}{\alpha+\gamma}. \tag{28.9.v}$$

Subtracting from L gives labor supply l^* and substituting into the budget constraint gives demand for x — i.e.

$$l^* = \frac{\alpha wL - \gamma T}{(\alpha+\gamma)w} \text{ and } x^* = \frac{\alpha wL - (2\gamma+\alpha)T}{\alpha+\beta}. \tag{28.9.vi}$$

(c) True or False*: Since the wage tax does not result in a distortion of the labor supply decision while the per-capita tax does, the former has no deadweight loss while the latter does.*

Answer: This is false. Efficiency losses from taxes happen from substitution effects — which occur when taxes change opportunity costs and not when they do not. The wage tax changes the price of leisure — and thus gives rise to substitution effects which happen to be masked by an offsetting wealth effect in our example. But the substitution effect creates deadweight loss. The lump sum tax T, on the other hand, does not give rise to any substitution effects — even though it's wealth effect causes a change in the optimal labor supply decision. But the wealth effect does not cause a deadweight loss.

(d) *Calculate the indirect utility function for part (a) (as a function of L, w and t).*

Answer: To get the indirect utility function, we plug ℓ^* and x^* from (a) into the utility function and replace y with δt to get

$$\begin{aligned} V &= \left[(1-t)w\left(\frac{\alpha L}{\alpha+\beta}\right)\right]^{\alpha} (\delta t)^{\beta} \left(\frac{\gamma L}{\alpha+\gamma}\right)^{\gamma} \\ &= (1-t)^{\alpha} t^{\beta} \left[\delta^{\beta} w^{\alpha} \left(\frac{\alpha L}{\alpha+\gamma}\right)^{(\alpha+\gamma)}\right]. \end{aligned} \tag{28.9.vii}$$

(e) *Now suppose that a vote is held to determine the wage tax t. What tax rate will be implemented under majority rule? (*Hint*: Use your result from (d) to determine the ideal point for a voter.)*

Answer: To determine a voter's ideal tax rate t^*, all we have to do is maximize the indirect utility function with respect to t. Taking the derivative of V with respect to t and setting it to zero, we can then solve for voter's optimal tax rate as

$$t^* = \frac{\beta}{(\alpha+\beta)}. \tag{28.9.viii}$$

Since this optimal tax rate for our voter is not a function of wage (which is the only dimension on which voters differ), all voters agree that this is the optimal tax rate — i.e. all voters have the same ideal point. This is because, although higher income voters demand more y all else equal, they also have to pay more of a tax share — with the latter effect offsetting the former.

(f) *Suppose that y is per pupil spending on public education. What does this imply that δ is (in terms of average population income $\overline{I}$, number of taxpayers K and number of kids N in school)?*

Answer: Tax revenue from a tax rate t is then $tK\overline{I}$ — and per pupil spending is revenue divided by N. Thus,

$$\delta = \frac{K\overline{I}}{N} \text{ when } y = \delta t. \tag{28.9.ix}$$

(g) *Now suppose there exists a private school market that offers spending levels demanded by those interested in opting out of public education (and assume that spending is all that matters in people's evaluation of school quality). People attending private school no longer attend public school but still have to pay taxes. Without doing any additional math, what are the possible public school per pupil spending levels y that you think could emerge in a voting equilibrium (assuming that public education is funded through a proportional wage tax)? Who will go to what type of school?*

Answer: As more children go to private school, N — the number of children going to public school — falls. This implies that δ increases as more children go to private school. Our expression of t^* is independent of δ — which implies that those who remain in public schools continue to favor the same tax rate as before, but those choosing private school will now favor a tax rate of zero. So long as the fraction going to private school is less than 0.5, however, t^* remains the majority rule equilibrium — which implies that per pupil public spending increases as private school markets expand. Those who attend private school will be the richest households.

(h) *Can you think of necessary and sufficient conditions for the introduction of a private school market to result in a Pareto improvement in this model?*

Answer: So long as less than half of the population goes to private school, the introduction of private schools represents a Pareto improvement in this model. This is because per pupil spending rises in public schools while tax rates remain the same — implying that public school attending households are better off than they would be in the absence of private schools. Private school attending households could choose to consume the higher levels of public school spending at the same tax rates as well but choose instead to opt for private school — which implies they are at least as well off as they would be in the public schools that now spend more. Thus, private school attending households must also be better off than they would be in the absence of a private school market. With everyone benefitting from the introduction of private schools, the model therefore predicts a pareto improvement.

(i) *In (e), you should have concluded that, under the proportional wage tax, everyone unanimously agrees on what the tax rate should be (when there are no private schools). Would the same be true if schools were funded by a per-capita tax T?*

Answer: No. Under a per capita tax T, everyone pays the same amount (rather than the same rate). This implies that ideal points will not all be the same in the absence of private school market — with ideal points for T increasing with household income and the median income household determining the level of T (and thus the level of per pupil spending) in a voting equilibrium. Thus, only the median income voter would get his preferred level of taxation.

Exercise 28.11

Policy Application: *The Pork Barrel Commons and the "Law of* $1/N$*"*: If you did not do these in Chapter 27, you can now do end-of-chapter exercises 27.11 and 27.12 to explore the problem of pork-barrel projects and the "Law of $1/N$."

Answer: See our answer in the solutions to Chapter 27.

Conclusion: Potentially Helpful Reminders

1. If voting only occurs over a single dimension, ask yourself if it is reasonable to assume that tastes are single-peaked over that dimension. If the answer is yes, you know that the median voter sits on the winning proposal if voting is governed by majority rule.

2. But if the issue space is two-dimensional — or if taste are not single-peaked in one dimension — you know that the problem will have a role for agenda setters.

3. The chapter certainly does not cover all the different approaches to voting and politics — some are further developed in end-of-chapter exercises, but all use similar logic to what is contained in the text examples.

4. Arrow's Impossibility Theorem is often misunderstood, and the underlying axioms sometimes misinterpreted. If you are covering Section B of the chapter, pay careful attention to the definitions of the axioms — in particularly the Independence of Irrelevant Alternatives Axiom. And remember what Arrow's theorem tries to show: It tries to find a coherent voting process that works for *any* (rational) configuration of underlying voter tastes.

CHAPTER

29

What is Good? Challenges from Psychology and Philosophy

In this final full chapter, we explore some of the challenges to the underlying economic framework we have outlined — challenge that emanate from the disciplines of psychology and philosophy. The influence of psychology has given rise to the sub-field of behavioral economics, a sub-field that explores potential explanations for systematic biases in decision making. Philosophy, on the other hand, attempts to explore some of the deeper aspects of the human condition, aspects that positive economists have no particular need to explore given their focus on predicting behavior. Normative economists, on the other hand, try to attach normative meaning to positive predictions — and thus need to connect to some of the deeper questions we find in the field of philosophy.

Chapter Highlights

The main points of the chapter are:

1. The idea of **present-bias** in decision making suggests that there is something special about the "present" — that our decisions about tradeoffs between now and the future are framed in our minds differently than our decisions about tradeoffs between the future and the more distant future. The **beta-delta** model is the simplest possible framework for modeling such present-bias — and it illustrates the potential for **self-control problems**.

2. **Self-control problems** are distinct from the concept of **impatience**. Individuals with self-control problems make future plans that they cannot keep when the future comes, while impatient individuals simply like to experience benefits soon and costs later.

3. **Reference-dependent** decision making is a form of decision making where a reference-point — typically the perceived status quo — takes on a special meaning, with losses from that reference point given more weight than gains

from that reference point. This leads to phenomena like **loss aversion** and **endowment effects** that do not arise in the typical economic model.

4. **Prospect Theory** is an alternative to the expected utility framework, an alternative that makes room for reference-dependence, diminishing sensitivity, and probability weighting.

5. Just because a positive economic model uses the ideas of "utility" or "happiness" to make predictions does not mean that those ideas have deeper normative value when thoroughly examined in a more inter-disciplinary way.

6. The concept of a **utility possibility frontier** can be combined with the concept of a **social welfare function** to explore fundamentally different philosophical approaches to judging whether particular outcomes are "good". One extreme philosophical position is captured by the **Rawlsian** social welfare function that places exclusive weight on the least well off member of society.

7. Economics can help construct realistic utility possibility frontiers by introducing the fact that distortions in behavior will tend to impact the distribution of utilities (or income or wealth) that are possible. This implies that even the extreme Rawlsian social welfare function will often choose unequal levels of utility (or income or wealth).

8. Social welfare functions are tools that can be used in **consequentialist** approaches to normative economics — i.e. approaches that place all the emphasis on outcomes. An alternative approach is to place more emphasis on *process* — asking not so much whether outcomes are just but rather whether processes used to get to outcomes are just.

29A Solutions to Within-Chapter-Exercises for Part A

Exercise 29A.1

Suppose $c = 100$, $b = 125$, $\delta = 0.95$ and $\beta = 0.8$. What is the expected value of undertaking the investment c in period 1 when viewed from $t = 0$? What is it when viewed from $t = 1$?

Answer: When viewed from $t = 0$, the expected value of undertaking the action in the next period is $[-0.8(0.95)(100) + 0.8(0.95)^2(125)] = 14.75$. When viewed from $t = 1$, it is $[-100 + 0.8(0.95)(125) = -5$. So the individual would plan to undertake the action but then change his mind.

Exercise 29A.2

When psychologists offer people the choice of \$50 today or \$100 next year, they tend to pick the \$50 immediately. But when the same people are offered the choice of \$50 five years from now or \$100 six years from now, they usually pick the \$100. Explain how this does not fit into the usual model of intertemporal choice — but it does fit into the modified model in the previous paragraph.

Answer: Under the usual model, the decision rule today would be to accept the \$50 now if $50 \geq 100\delta$. The decision of whether to plan to accept the \$50 five years from now (as opposed to waiting to get \$100 six years from now) would depend on whether $\delta^5(50) \geq \delta^6(100)$ which reduces to the decision rule to accept \$50 in 5 years if $50 \geq 100\delta$. Thus, at least as long as δ is constant over time, the same decision rule applies. But in the beta-delta model, the decision rule today is to accept \$50 now if $50 \geq 100\beta\delta$ and the decision rule of whether to accept the \$50 five years from now is to accept if $50 \geq 100\delta$. The rules differ, and if $\beta < 1$, the individual might in fact choose the immediate \$50 now and plan to choose the delayed \$100 in the future.

Exercise 29A.3

What does it mean for β to be greater than 1 in the beta-delta model?

Answer: If $\beta > 1$, the individual is anti-present biased — which is to say that he may plan not to undertake an investment c in the future but then change his mind when the future becomes the present. Put differently, such individuals actually suddenly become more patient in the present.

Exercise 29A.4

Consider again the example in within-chapter-exercise 29A.2. Suppose $\delta = 1/1.05 \approx 0.952$. What is the highest level of β that could lead to the choices in the example? What would δ have to be now if $\beta = 1$ to lead to the present choice — and why does this not help us explain the dual result described in the example?

Answer: The beta-delta model predicts the individual will choose the \$50 now if $50 \geq 100\beta\delta$. Substituting $\delta = 1/1.05 \approx 0.952$, this implies the individual will choose \$50 now if $50 \geq 95.24\beta$ which holds with equality when $\beta = 0.525$. Thus, the highest β for which we can then explain the individual's decision today is $\beta = 0.525$. The highest δ for which this choice could be explained when $\beta = 1$ is $\delta = 0.5$. But $\delta = 0.5$ by itself — while explaining the individual's decision today, does not explain the individual's decision to plan to accept \$100 six years from now over \$50 five years from now.

Exercise 29A.5

Many people buy health club memberships only never to use them. Yet they hold onto them and continue paying their monthly fees for long periods of time. How can the purchase of such memberships be explained, and what does the fact that individuals hold onto their memberships without using them tell us about their awareness of how they are making decisions?

Answer: The purchase of such memberships indicates that individuals are planning to use them. When they do not use them, we can use present-bias to explain this — they planned to use them in the future, but when the future becomes the present, it is too costly (in terms of time and effort) to actually use them. The fact that individuals continue to hold onto such memberships without using them suggests they are unaware of their present-bias — always planning to start using them soon, but then always finding it to burdensome to actually start using them when “soon” becomes “now”.

Exercise 29A.6

Some financial advisors recommend that people choose 15-year mortgages with higher monthly payments rather than 30-year mortgages with lower monthly payments even if the interest rates on both mortgages are the same and even if the 30-year mortgages allow people to pre-pay (and thus pay them off in 15 years) if they want to. How does this make sense from a behavioral economist’s perspective when it makes less sense when viewed through a traditional economic model?

Answer: It does not make sense when viewed through a traditional model because the 30-year mortgage (as described) allows you more options while not precluding the options that are available under the 15-year mortgage. Thus, under the 30-year mortgage, you can do everything you can do under the 15-year mortgage — and more. From a behavioral economics perspective, however, the 15-year mortgage can serve as a commitment device to save (in the form of creating equity in the home) — precisely because it cuts off the option of taking longer to pay off the debt.

Exercise 29A.7

In the period prior to the 2007 housing crisis, it was easy for people to re-finance their homes. If people choose 15-year (rather than 30-year) mortgages as a savings commitment device (as suggested in exercise 29A.6), might the ready option to re-finance have made self-aware but present-biased people worse off?

Answer: This would, in fact, undo the commitment device function of the 15-year mortgage — because re-finaning allows people to access the equity in their homes through borrowing and thus drawing down the savings that the 15-year mortgage was intended to create. Present-biased individuals would therefore have lost a commitment device that made them better off — and would be worse off because they cannot bind themselves to a savings plan as they had intended when choosing the 15-year mortgage.

Exercise 29A.8

True or False: If individual tastes are quasilinear in basketball tickets, the prices people were willing to accept should be identical to the prices they were willing to pay. (Hint: You may have done a detailed exercise that is identical to this in end-of-chapter exercise 10.7 of Chapter 10.)

Answer: This is true — for the conditions of this experiment where the randomness with which tickets are allocated allows us to assume that tastes are roughly identical for those who have tickets and those who do not. This is explained in considerable detail in the answer to end-of-chapter exercise 10.7 in Chapter 10. In essence, the two individuals are identical in every way except that one owns a ticket and the other does not — which makes the one that owns the ticket richer than the one that does not. Any difference in willingness to accept and willingness to pay is then due to this difference in "wealth" — i.e. it is due to an income effect. When tastes are quasilinear, there are no income effects. When tastes are normal, then the willingness to accept will be higher than the willingness to pay — but only by the size of the income effect which cannot plausibly be sufficiently large to explain the reported answers.

Exercise 29A.9

In end-of-chapter exercise 10.7, we considered a very similar situation in which two individuals are identical except that one has a pizza coupon. We concluded that the two individuals will be able to agree on a price at which to trade the coupon so long as pizza is not a normal good. If there is an endowment effect, will the two people be more or less likely to trade the coupon?

Answer: They will be less likely to trade — i.e. they may not be able to trade even if pizza is an inferior good.

Exercise 29A.10

On several occasions, I have observed one of my colleagues insist on taking a special trip to the movie rental place in order to return a movie that would otherwise be overdue when he could have just waited to the next day and returned the movie on his way to work. The late fee is $1. If I called this same colleague (on a night when he did not have a movie due) with "special information" that there was $1 hidden behind one of the movies in the movie place — and that he can be virtually assured of getting the dollar if he comes by now, he would never think it worth it to take that special trip for $1. Can you explain my colleague's behavior using reference-based preferences with loss aversion?

Answer: In the case of the movie rental return, the reference point is that he has the $1 — and incurring a late fee is interpreted as a "loss" from that reference point. In the second case, he does not have the $1 — and getting the $1 by taking the special trip would be interpreted as a "gain". If losses are psychologically more harmful than gains, then we can explain why my colleague takes the special trip "to avoid losing $1" but not in order "to gain $1".

Exercise 29A.11

In his book *Predictably Irrational*, my psychologist colleague Dan Ariely suggests (incorrectly, it turns out) that taxing gasoline may not have much impact on long run gasoline consumption because, he hypothesizes, people will adjust their reference-point and thus will respond primarily in the short run and not that much in the long run. This is exactly the opposite prediction that a neoclassical economist would make. Can you see how he arrives at his prediction?

Answer: The argument would go somewhat like this: If individuals make their gasoline consumption decisions with reference to "what gasoline should cost", they might initially respond a lot to a large increase in gasoline prices — because their reference point is that "gasoline should not cost this much" and they therefore conserve because they do not want to pay an "inflated price". But, as gasoline prices stay high, their reference point might adjust — i.e. "what gasoline should cost" adjusts. As a result, they begin to purchase more gasoline again.

Exercise 29A.12

Explain how the two sets of options are equivalent.

Answer: A gas guzzler that gets 8 MPG will use 1000/8=125 gallons for 1,000 miles — and a gas guzzler that gets 10 MPG will use 100 gallons to go 1,000 miles. This is equivalent to Option A in the alternative framing of the problem. Similarly, a car that gets 25 MPG will use 40 gallons to go 1,000 miles while a car that gets 40 MPG will use 25 gallons per 1,000 miles. This is exactly what Option B says. And a car that gets 50 MPG will use 20 gallons per 1,000 miles while a car that gets 100 MPG will use only 10 gallons per 1,000 miles. That's Option C.

Exercise 29A.13

Some years ago, Congress passed a law permitting stores to charge different amounts to cash customers as they do to credit card customers. When it became clear that the law would pass, the credit card lobby insisted on language that would permit "cash discounts" but not "credit card surcharges". In light of reference-based preferences with loss aversion, can you think of why credit card companies might have lobbied so hard for this?

Answer: Under reference-based preferences, "cash discounts" are "gains" relative to the reference point (of the initial price) — while "credit card surcharges" are "losses" (from the initial price). If losses are psychologically more painful, more consumers will be dissuaded from using their credit cards (and using cash instead) if they think they will "lose" something from using their credit cards than if they think they will "gain" something from paying cash instead. Thus, if consumers really do use such reference-based decision making, credit card companies can expect more use of credit cards if cash discounts rather than credit card surcharges are permitted.

Exercise 29A.14

How can reference-based preferences explain the empirical facts on enrollments in retirement programs?

Answer: If workers view the default option as their "endowment", they place particular value on the default option and are therefore more likely to stick with it. Thus, if workers are automatically enrolled in the retirement plan, they will be more likely to stick with it than they would be to enroll in it if the default was to not be enrolled.

Exercise 29A.15

Explain the last sentence.

Answer: The marginal utility of income (or consumption) is the additional utility an individual gets from one more dollar of income (or consumption). If our notion of utility is equal to (or correlated with) what people report on happiness studies, we would therefore expect the first dollars earned (or consumed) to be reported as resulting in greater additional happiness than later dollars. Thus, someone with twice as much income (or consumption) will have less than twice as much happiness.

Exercise 29A.16

In earlier end-of-chapter exercises, we introduced the notion of "compensating differentals" in labor markets — wage differences that emerge because some jobs are inherently less pleasurable or involve more risk, factors that in equilibrium will be reflected in wages. How might the existence of such compensating differentials bias researchers into finding the marginal utility of income to be diminishing when it actually is not?

Answer: Suppose you have two identical individuals choosing different jobs — one that pays little but involves deeply meaningful work and the other that pays a lot but involves work that is less meaningful and involves greater risk. Since the identical individuals are choosing these jobs, they must be indifferent between them — and they might therefore report the same level of happiness. The researcher might then interpret this as implying that additional income does not produce additional happiness — but the whole reason the two individuals are indifferent between the two jobs is that the additional income made up (in terms of utility) for the different levels of pleasure that the jobs themselves produced. Thus, income does cause more happiness in the example even though two identical individuals with different incomes are equally happy.

Exercise 29A.17

Explain how reference-based preferences can provide such an explanation for the two sets of findings.

Answer: Suppose individuals within a country evaluate their level of happiness with the average level of income as the reference point. This would result in additional income being correlated with additional happiness (just as the non-reference based model would predict). But if everyone's income increases over time, individuals would still evaluate their happiness relative to the average — and thus (average) happiness itself may not increase (unless the distribution of income has also changed) — with the non-reference based model instead predicting that average happiness should be going up.

Exercise 29A.18

If the reference-based preference explanation for the Easterlin paradox is correct, how would this imply that we are all caught up in a big prisoners' dilemma?

Answer: In this case, we could all remain just as happy as we are now by simply working less and not causing average income to increase. (In fact, we would be happier because we'd presumably get some happiness from increased leisure consumption). But because we are all trying to move up on the income distribution in order to get closer to or surpass the average income reference point, we keep working hard — causing average income to increase without happiness changing. Some have referred to this as the "hedonic treadmill" — we keep running on the treadmill to get ahead, but because everyone else is doing the same, we never really get ahead of our position relative to the reference point and thus none of us get happier as we try to get happier. We'd all therefore prefer to stop the treadmill and stay where we are, but it is in each of our individual incentives to stay on the treadmill.

Exercise 29A.19

Suppose we consider our brain as the outcome of an evolutionary process aimed at maximizing the survival of our species. How might this be consistent with the memory bias we have just discussed?

Answer: In order for the species to survive and thrive, it is important that everyone work hard to move forward. It may then be important from an evolutionary perspective for people to think that getting toward various milestones will make them really happy even if it does not. Evolution, in this sense, would not really "care" about whether we actually get happy when we reach the milestone — what matters is that we pursue it rather than slack off.

Exercise 29A.20

In what sense do you think this memory bias might work in the opposite direction as present-bias discussed earlier?

Answer: Memory bias, as discussed in the previous exercise, makes us work harder and invest more in anticipation of happiness when the work or the investment pays off. Present-bias, on the other hand, makes us work less and consume more now.

Exercise 29A.21

Suppose the marginal utility of income is constant — and we can costlessly redistribute income across individuals. What would that imply for the shape of the utility possibility frontier?

Answer: It would mean that the utility possibility frontier is a straight line because each dollar that gets redistributed results in the same increase in utility for the person that gets the dollar (and the same decrease in utility for the person from whom it is taken) regardless of how much we have redistributed.

Exercise 29A.22

True or False: Every allocation on the contract curve in panel (a) translates to a point on the utility possibility frontier in panel (b).

Answer: This is true. Take any allocation on the contract curve. The utility values associated with the two indifference curves that pass through this allocation then becomes one point on the utility possibility frontier — with any other point on the contract curve giving us different utility values and thus a different point on the utility possibility frontier.

Exercise 29A.23

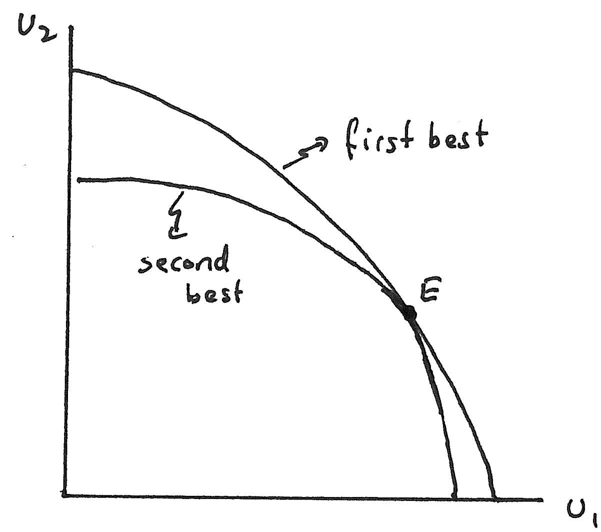
Suppose that initial wealth were more equally distributed. Illustrate how the first and second best utility possibilities would then be related to one another. What point do they share in common?

Answer: The one point that the first and second-best utility possibility frontiers must always share is the point that requires no redistribution — i.e. the utility allocation produced by the initial wealth distribution. This is indicated in Exercise Graph 29A.23 as point E. As money is then redistributed in either direction, we can no longer reach the first-best utility possibility frontier in the second-best world of distortionary taxation. This results in a second best utility possibility frontier as illustrated in Exercise Graph 29A.23.

Exercise 29A.24

True or False: Symmetric social preferences that view utilities as somewhat substitutable — i.e. social preferences that give rise to indifference curves between the extremes of perfect complements and perfect substitutes — would result in optimal allocations that lie between A and B in Graph 29.4a.

Answer: This is true. We can see this in panel (a) by noting that the blue social indifference curve (that is tangent at B) has slope of -1 — which implies that the slope of the second-best utility possibility frontier on the 45-degree line (at A) is less (in absolute value). Any symmetric social preference would have a marginal rate of substitution equal to -1 along the 45 degree line — getting steeper above the 45 degree line and shallower below it. This implies that it must be tangent to the green constraint between A and B.



Exercise Graph 29A.23 : Utility Possibility Frontier

Exercise 29A.25

Suppose that government income redistribution programs cause no change in behavior. Then the Rawlsian social indifference curves would imply full redistribution of income — i.e. full income equality after redistribution.

Answer: This is true. If there is no change in behavior, then income can be redistributed without causing overall income to decline. As a result, we provide the most income to the poorest person by simply equalizing the incomes between the two individuals — giving us the Rawlsian outcome of full income redistribution.

Exercise 29A.26

Now suppose that government redistribution programs cause changes in behavior (such as those predicted by the Laffer curve from Chapter 8. Can you argue that Rawlsian social indifference curves would now imply less than full redistribution — i.e. some income inequality would remain after the Rawlsian redistribution program has been implemented?

Answer: This can result in an income allocation possibility frontier that is initially upward sloping — similar to the shape of the utility possibility frontier in panel (b) of Graph 29.4 in the text. For reasons identical to those in that graph, the Rawlsian optimum then may involve leaving one individual with greater income than the other — because redistributing further will result in a loss of income for both. (The Rawlsian solution would, in essence, imply setting the redistributive rate at the top of the Laffer curve.)

Exercise 29A.27

Can the relationship in these graphs ever cross the 45-degree line?

Answer: No, this is logically not possible since we are lining the population on the horizontal axis up starting with the lowest income individual and ending with the highest income individual. The highest the Lorenz curve can go is if all individuals have the same income — in which case it would lie on the 45 degree line.

Exercise 29A.28

Using the points analogous to A and B from panel (a) in Graph 29.5, show how panels (b) and (c) represent an increasingly equal income distribution.

Answer: In panel (a), the lowest 40% of the income distribution earns 5% of all income — whereas in panels (b) and (c) the percentage of income earned by the lowest 40% of the distribution rises to 15% and then 30% respectively. Using the A points instead, we can similarly see that panel (a) implies that the bottom 80% of the income distribution earns half of all income while that same portion of the income distribution earns 60% of all income in panel (b) and 75% of all income in panel (c).

Exercise 29A.29

Under what conditions would Nozick's just society lead to efficient outcomes?

Answer: Under the conditions of the first welfare theorem where voluntary exchange leads to all gains from trade being exploited — or, alternatively, if the government and the civil society together efficiently corrected for any violations of the first welfare theorem.

29B Solutions to Within-Chapter-Exercises for Part B

Exercise 29B.1

Demonstrate that the last sentence is true.

Answer: For an investment c at t with payoff b at $(t+n)$, the beta-delta model would then predict that, when viewed from the present, we will plan to undertake the investment if

$$\beta\delta_1\delta_1...\delta_t c < \beta(\delta_1\delta_1...\delta_t)\delta_{t+1}\delta_{t+2}...\delta_{t+n}b \tag{29B.1.i}$$

which reduces to

$$c < \delta_{t+1}\delta_{t+2}...\delta_{t+n}b \tag{29B.1.ii}$$

just as in the usual model without β. But, when t becomes the present, we will choose to undertake the investment only if

$$c < \beta\delta_{t+1}\delta_{t+2}...\delta_{t+n}b \tag{29B.1.iii}$$

Exercise 29B.2

In within-chapter-exercise 29A.2, we implicitly assumed that δ is constant over time. Would allowing for the possible change in δ over time allow for the standard model to explain what we previously concluded only the beta-delta model can explain?

Answer: Yes — if we anticipate that we will become more patient over time in the sense that $\delta_1 < \delta_2 < \delta_3 < ...$, we may make an impatient decision (like choosing \$50 now over \$100 a year from now) and then also choose \$100 in six years over \$50 in five years. In our example, it would need to be that $\delta_1 \leq 0.5$ and $\delta_5 \geq 0.5$. The crucial difference is that in this case we accurately forecast that we will become more patient with time — and indeed we do become more patient with time.

Exercise 29B.3

Explain the last sentence.

Answer: When $\pi(\delta) = \delta$, $\pi(1-\delta) = (1-\delta)$ and $r = 0$, the prospect theory equation becomes

$$\delta u(x_1) + (1-\delta)u(x_2) \tag{29B.3}$$

which is simply the von-Neumann Morgenstern expected utility form.

Exercise 29B.4

In one set of experiments, individuals were asked how much they would be willing to pay to participate in a gamble in which they receive $8 when a coin comes up heads but owe $5 if it comes up tails. Close to two thirds were not willing to pay anything — which can be explained in the standard expected utility framework only if we are willing to assume a level of risk aversion that is roughly equivalent to such individuals never leaving their house for fear of all the risks they will encounter. Can you rationalize the results of the experiment for a risk averse individual using only the parts of prospect theory that incorporate reference bias and loss aversion?

Answer: The expected value of this gamble is 1.5. Under the ordinary framework, a risk neutral individual would therefore accept the gamble. If the individual takes zero as the reference point and evaluates gains different from losses, however, he may not accept the gamble. In particular if losses are experienced as at least 8/5ths as psychologically painful as gains, then the same risk neutral individual would not agree to the gamble.

Exercise 29B.5

Demonstrate that the same conclusion — i.e. that $u_{1A} = u_{2A}$ and $u_{1B} = u_{2B}$ — arises when tastes are risk loving. How are the options ranked differently by each group relative to risk aversion?

Answer: This is illustrated in Exercise Graph 29B.5 where the u function is convex instead of concave, indicating risk loving. The expected outcomes of the gambles are exactly what they were before — leaving us with the same points on the horizontal axis. But now $u_{1A} = u_{2A}$ lies *below* $u_{1B} = u_{2B}$ — implying that individuals would choose the risky B option over the safe A option in both cases when tastes are not reference based.

Exercise 29B.6

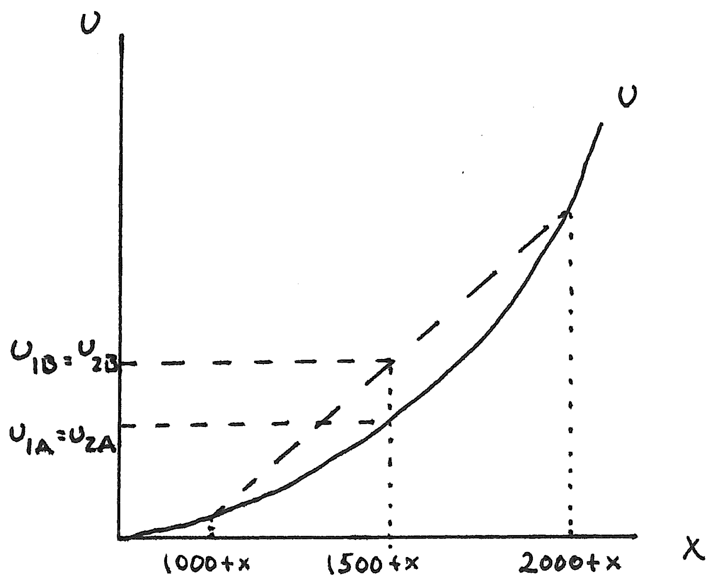
Can you explain how *diminishing sensitivity* gives rise to the switch between risk loving and risk aversion at the reference point?

Answer: Diminishing sensitivity implies that the u function is concave above the reverence point and convex below the reference point — with concave u functions implying risk aversion and convex u functions implying risk loving.

Exercise 29B.7

Explain how probability weighting can make sense of the fact that risk averse individuals play in state lotteries. How can it explain purchases of insurance against small, low-probability risks when insurance policies are priced far from actuarily fair?

Answer: If lottery players over-estimate small probabilities, the may be over-estimating the small probability of winning the lottery — and may therefore be-



Exercise Graph 29B.5 : Risk loving tastes

lieve the expected value to be positive when it is actually negative. The same reasoning explains how people might think that buying insurance against small, low-probability risks at actuarially unfair prices is actually actuarially fair.

Exercise 29B.8

Verify the derivation of this first-best utility possibility frontier.

Answer: First, we determine individual 1's leisure and consumption decision for a given lump sum tax T by solving

$$\max_{\ell, c_1} 2c_1^{1/2}\ell^{1/2} \text{ subject to } c_1 = 1 - T - \ell. \tag{29B.8.i}$$

Solving this in the usual way gives us $\ell^* = (1-T)/2$ and $c_1 = (1-T)/2$. Plugging these back into the utility functions, we get

$$u_1 = 2\left(\frac{(1-T)}{2}\right)^{1/2}\left(\frac{(1-T)}{2}\right)^{1/2} = (1-T). \tag{29B.8.ii}$$

Since $c_2 = T$ and $u_2 = c_2$, we know that $u_2 = T$. Plugging this into the equation, we get the first best utility possibility frontier

$$u_1 = 1 - u_2. \tag{29B.8.iii}$$

Exercise 29B.9

* In Section A, we suggested that the shape of the utility possibility frontier has something to do with our assumptions about the marginal utility of income. Can you apply this insight here to explain the linear utility possibility frontier in our example?

Answer: Suppose individual 1 gets one additional dollar in income transferred to him. We see from ℓ^* and c_1^* that this implies the individual will increase both leisure and consumption equally — with the two together yielding one additional unit of utility when plugged into the utility function. Thus, the marginal utility of income is 1 for individual 1. (You can also view this as thinking of the utility function for individual 1 having constant returns to scale.) Individual 2 similarly has a marginal utility of income equal to 1 because every dollar in additional consumption results in an increase of 1 unit of utility. The fact that both individuals have marginal utility of income equal to 1 implies we can redistribute 1 unit of utility from one individual to the other by redistributing \$1 from one to the other.

Exercise 29B.10

Verify the derivation of the second-best utility possibility frontier.

Answer: First we determine individual 1's leisure and consumption choice depending on t by solving

$$\max_{c_1,\ell} 2c_1^{1/2}\ell^{1/2} \text{ subject to } c_1 = (1-t)(1-\ell). \tag{29B.10.i}$$

Solving this in the usual way, we get

$$c^* = \frac{1-t}{2} \text{ and } \ell^* = \frac{1}{2}. \tag{29B.10.ii}$$

Plugging these back into individual 1's utility function, we get

$$u_1 = 2\left(\frac{(1-t)}{2}\right)^{1/2}\left(\frac{1}{2}\right)^{1/2} = (1-t)^{1/2}. \tag{29B.10.iii}$$

Individual 2's utility is $u_2 = t(1-\ell^*) = t/2$, which we can write as $t = 2u_2$. Plugging this into the equation for u_1, we get

$$u_1 = (1-2u_2)^{1/2} \tag{29B.10.iv}$$

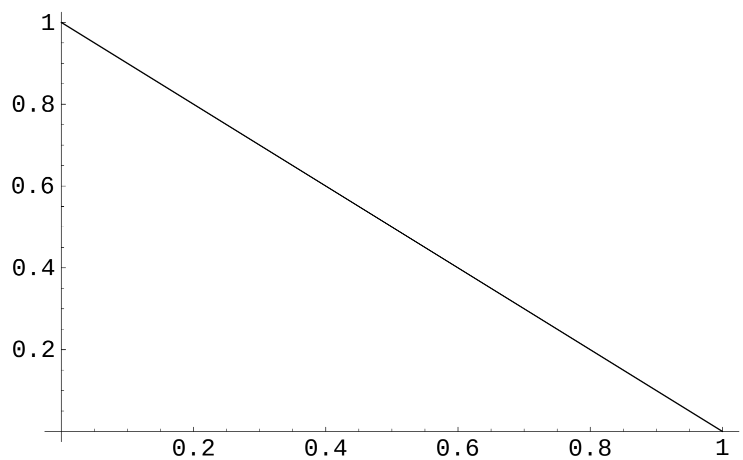
or, solving for u_2,

$$u_2 = \frac{1-u_1^2}{2}. \tag{29B.10.v}$$

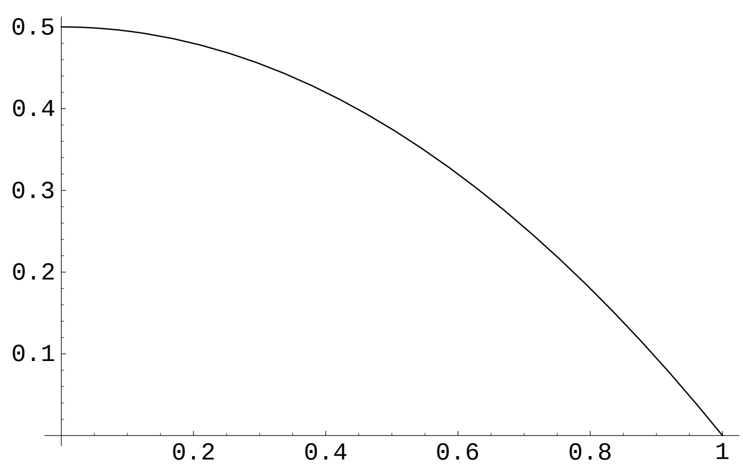
Exercise 29B.11

Can you graph these two utility possibility frontiers and explain their relationship intuitively?

Answer: The first-best utility possibility frontier is graphed in Exercise Graph 29B.11(1) and the second best frontier is graphed in Exercise Graph 29B.11(2) — with u_2 on the vertical axis and u_1 on the horizontal in both cases.



Exercise Graph 29B.11(1) : First best utility possibility frontier



Exercise Graph 29B.11(2) : Second best utility possibility frontier

Exercise 29B.12

Verify these solutions for the different social welfare functions.

Answer: We can write the maximization problems in a form that is slightly easier to solve by taking logs of W and substituting the constraint into the objective function. For the first problem, this gives us

$$\max_{u_1} W = u_1^{\alpha}(1-u_1)^{(1-\alpha)} \tag{29B.12.i}$$

which solves straightforwardly to give us $u_1^{FB} = \alpha$ and $u_2^{FB} = (1-\alpha)$. The second problem can be written as

$$\max_{u_1} W = u_1^{\alpha}\left(\frac{1-u_1^2}{2}\right)^{(1-\alpha)} \tag{29B.12.ii}$$

which, with a bit more algebra, solves to

$$u_1^{SB} = \left(\frac{\alpha}{2-\alpha}\right)^{1/2} \text{ and } u_2^{SB} = \frac{1-\alpha}{2-\alpha}. \tag{29B.12.iii}$$

Exercise 29B.13

We have implicitly assumed that we can measure individual utilities in order to construct first and second-best utility possibility frontiers. Suppose instead that we can only measure consumption. What would the first and second-best consumption possibilities frontiers look like for our example?

Answer: Since individual 1's leisure choice is independent of t, we can transfer consumption from individual 1 to individual 2 dollar for dollar. Put differently, individual 1's labor supply curve is perfectly inelastic — causing his before-tax labor income to be independent of t. Thus, the Laffer curve is simply upward sloping and linear — implying that we can raise t all the way to 1 and extract all of individual 1's consumption to give it to individual 2.

Exercise 29B.14

Might a government that derives the first and second-best consumption possibilities frontiers from exercise 29B.13 mistakenly think that there is no efficiency loss from redistribution? How does your conclusion illustrate our conclusion from earlier chapters that deadweight loss from labor taxes cannot be derived by simply looking at uncompensated labor supply curves?

Answer: Since the first and second-best consumption possibilities frontiers look the same in our example, one might conclude that there is no efficiency loss from redistribution because it appears that one can simply transfer consumption from individual 1 to individual 2 costlessly (because individual 1's labor supply curve is perfectly inelastic). But, as we saw in the utility possibility frontiers for the same example, there is indeed an efficiency loss from redistribution — with the second-best utility possibility frontier giving rise to only half the u_2 intercept as the first best. The reason for this is that efficiency losses cannot be gleaned from the elasticity of the uncompensated labor supply curve — because this curve masks offsetting income and substitution effects. The substitution effects give rise to deadweight losses — and these are captured in the smaller second-best utility possibility set (relative to the first best).

Exercise 29B.15

If a government used the second-best consumption possibility frontier as if it were the appropriate utility possibility frontier, would it redistribute too much or too little relative to what it would do if it could measure utilities?

Answer: It would redistribute too much because it is not considering the deadweight loss from redistributive taxation (that is hidden in substitution effects which do not appear on the second-best consumption possibilities frontier.)

Exercise 29B.16

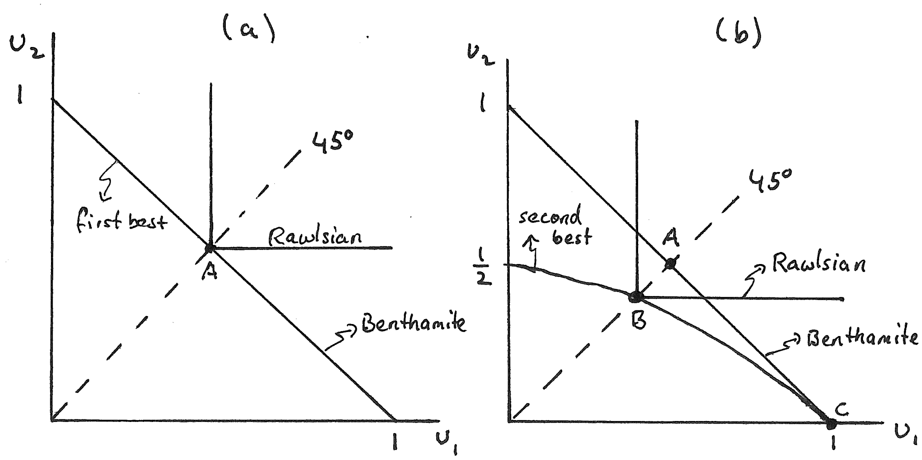
Can you draw the first and second best utility possibility frontiers and indicate how you would graphically arrive at the same results?

Answer: This is done in panels (a) and (b) of Exercise Graph 29B.16. In panel (a), we replicate the first best utility possibility frontier from Exercise Graph 29B.11(1) — with the Rawlsian optimum at A. Since the Benthamite indifference curves have slope -1 just as the utility possibility frontier, all utility allocations on the frontier (including A) are optimal from the Benthamite perspective. In panel (b) we replicate the second-best utility possibility frontier from Exercise Graph 29B.11(2). Since Benthamite indifference curves have slope -1, they now result in the corner solution C — while the Rawlsian solution remains on the 45 degree line but now results in lower (equal) utility allocations than we had at the first-best point A.

Exercise 29B.17

We concluded that the Benthamite, Rawlsian and Cobb-Douglas social welfare function with $\alpha = 0.5$ all agree that the first-best utility allocation $u_1 = u_2 = 0.5$ is optimal — but we also found that they quite dramatically disagree on what the second-best utility allocation is. Explain why.

Answer: In the first best case, the sum of utilities is always the same along the utility possibility frontier — implying that a social welfare function that maximizes the sum of utilities is indifferent between all allocations on the frontier. This is the Benthamite result. But in the second-best case, some utility is lost as redistribution occurs — which implies only the no-redistribution utility allocation maximizes the sum of utilities.



Exercise Graph 29B.16 : First and Second-Best Rawlsian and Benthamite Optima

Exercise 29B.18

Verify that this is in fact a positive monotone transformation.
Answer: We can write this transformation as

$$v^n(x) = -\left(\frac{1}{u^n(x)}\right)^{\rho}, \tag{29B.18}$$

with increases in u^n appearing in the denominator. If it were not for the negative sign up front, this would imply that v^n decreases as u^n increases — which would not be a positive monotone transformation. But, by including the negative sign, we reverse this, with v^n now increasing in u^n. We therefore have a positive monotone transformation when we take u^n to the power $(-\rho)$ while at the same time multiplying it by minus 1.

Exercise 29B.19

How is what we have just done a positive monotone transformation?
Answer: We can write this transformation as

$$W = \left(-\frac{1}{\overline{V}}\right)^{1/\rho}. \tag{29B.19}$$

Taking $\overline{V}$ to the power $1/\rho$ preserves the ordering (as long as $\rho > 0$). Taking the inverse of $\overline{V}$ reverses the ordering, but multiplying the inverse by negative 1 restores it.

29C Solutions to Odd Numbered End-of-Chapter Exercises

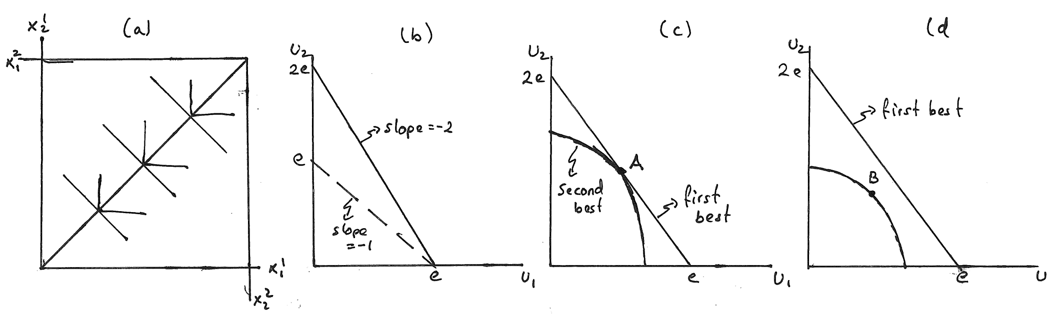
Exercise 29.1

For students who have read Chapter 16, we have indicated that, for exchange economies, the utility possibility frontier corresponds to utility levels on the contract curve that is contained in the Edgeworth Box.

A: *Consider the special 2-good case where consumer 1 views the goods x_1 and x_2 as perfect complements — with utility equal to the lower of the quantities of x_1 and x_2 in her basket. Consumer 2, on the other hand, views the goods as perfect substitutes, with utility equal to the sum of the quantities of x_1 and x_2 in his basket.*

(a) *Illustrate the contract curve for these two consumers in the Edgeworth box assuming the overall endowment of each of the two goods in the economy is e.*

Answer: This is illustrated in panel (a) of Exercise Graph 29.1(1), with the contract curve represented by the diagonal from one corner of the box to the other.



Exercise Graph 29.1(1) : Edgeworth Boxes and Utility Possibility Frontiers

(b) *What does the utility possibility frontier that derives from this contract curve look like? Carefully label intercepts and slopes.*

Answer: This is illustrated in panel (b) of Exercise Graph 29.1(1) as the solid utility possibility frontier. The most utility that individual 1 can get is e — because his utility is equal to the lower of the two consumption items in his basked. So if he gets everything in the economy, his basked has (e, e) and the lower of the two amounts is $e = u_1$. Individual 2 can get twice that utility when everything is given to him — because his utility is the sum of the goods in his basked. Thus, we get a utility possibility frontier with u_2 intercept of $2e$ and slope -2.

(c) *How would the utility possibility frontier be different if the utility of consumer 2 were given by half the sum of the quantities of x_1 and x_2 in his basket?*

Answer: Now the most utility individual 2 could get is e — implying that the new utility possibility frontier is given by the dashed line with slope -1 in panel (b) of Exercise Graph 29.1(1).

(d) *Consider the original utility possibility frontier from part (b). Suppose the two individuals are currently endowed with the midpoint of the Edgeworth box. Locate the point on the utility possibility frontier that corresponds to this allocation of goods.*

Answer: This is done in panel (c) of Exercise Graph 29.1(1), with the endowment utilities labeled A.

(e) *Suppose that the government does not have access to efficient taxes for the purpose of redistributing resources. Rather, the government uses distortionary taxes, with the marginal cost of redistributing $1 increasing with the level of redistribution. What do you think the second-best utility possibility frontier now looks like relative to the first best? Do the two share any points in common?*

Answer: Point A would still lie on the second best utility possibility frontier because it involves no redistribution and thus no inefficient taxation. But, the farther we get from A, the more redistribution is taking place — and the more the second-best utility possibility frontier must therefore deviate from the first best. This is also illustrated in panel (c) of the graph.

(f) *Suppose instead that the current endowment bundle lies off the contract curve on the diagonal that runs from the upper left to the lower right corner of the Edgeworth box. If competitive markets are allowed to operate, do your first and second best utility possibility frontiers differ from those you derived so far?*

Answer: No, they will not differ. The competitive equilibrium price in this economy must produce budget lines with slope -1 — otherwise individual 2 chooses a corner solution. The diagonal that is perpendicular to the contract curve intersects at the midpoint of the Edgeworth box which was the endowment we previously assumed. This perpendicular also has slope -1 — and thus is equal to the equilibrium budget line. The two individuals therefore trade to the midpoint in equilibrium — giving us the same utility levels as those labeled A in panel (c) of the graph. The utility possibility frontiers are then also the same.

(g) *If markets were not allowed to operate in the case described in (e), where would the second best utility possibility frontier now lie relative to the first best?*

Answer: Now the initial allocation of goods is not efficient — implying that the utility allocations associated with the initial allocation lie inside the first best utility possibility frontier. This is labeled as point B in panel (d) of Exercise Graph 29.1(1). The second best utility possibility frontier would then pass through B and retain its previous shape because of the inefficiency of redistributive taxes. Thus, the entire second best utility possibility frontier lies strictly inside the first best utility possibility set if competitive markets are not permitted to operate.

B: *Suppose we have an Edgeworth economy in which both individuals have the utility function $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$ and where the economy's endowment of each of the two goods is e.*

(a) *Set up a maximization problem in which the utility of consumer 1 is maximized subject to the economy-wide endowment constraints and subject to keeping individual 2's utility at $\overline{u}$. (By defining individual 2's consumption of each good as the residual left over from individual 1's consumption, you can write this problem with just a single constraint.)*

Answer: The resource constraints are that $e = x_1^1 + x_1^2$ and $e = x_2^1 + x_2^2$ where superscripts indicate individuals and subscripts indicate goods. These constraints can then be re-written as $x_1^2 = e - x_1$ and $x_2^2 = e - x_2$ where we drop individual 1's superscripts. The maximization problem is then

$$\max_{x_1,x_2} x_1^{\alpha}x_2^{(1-\alpha)} \text{ subject to } (e-x_1)^{\alpha}(e-x_2)^{(1-\alpha)} = \overline{u}. \tag{29.1.i}$$

(b) *Derive the contract curve for this economy.*

Answer: Solving the maximization problem in the usual way, the first two first order conditions reduce to $x_1 = x_2$. Plugging x_2 in for x_1 in the constraint, we then get

$$\overline{u} = (e-x_2)^{\alpha}(e-x_2)^{(1-\alpha)} = e - x_2. \tag{29.1.ii}$$

This solves to $x_2 = e - \overline{u}$ and, since the first order conditions gave us $x_1 = x_2$, we conclude that

$$x_1 = e - \overline{u} = x_2. \tag{29.1.iii}$$

The contract curve in the Edgeworth box is then simply the diagonal connecting the two corners of the box — where $x_1 = x_2$ along that diagonal (since the box in this example is a square).

(c) *Use this to derive the utility possibility frontier. What shape does it have?*

Answer: Plugging the pareto optimal $x_1 = e - \overline{u} = x_2$ into individual 1's utility function, we get

$$u_1 = (e-\overline{u})^{\alpha}(e-\overline{u})^{(1-\alpha)} = e - \overline{u}. \tag{29.1.iv}$$

Since $\overline{u}$ was an arbitrarily chosen level of utility for u_2, we can replace it by u_2 to get the utility possibility frontier

$$u_1 = e - u_2. \tag{29.1.v}$$

Thus, the utility possibility frontier is a straight line with slope -1 and intercept e.

(d) *How would your answers change if we had specified the utility function as* $u(x_1, x_2) = x_1^{\beta} x_2^{(0.5-\beta)}$ *with* $0 < \beta < 0.5$.

Answer: The first two first order conditions of the maximization problem would still give us $x_1 = x_2$. Plugging this into the constraint, we get

$$\overline{u} = (e - x_2)^{\beta}(e - x_2)^{(0.5-\beta)} = (e - x_2)^{1/2}. \tag{29.1.vi}$$

Solving for x_2, we get $x_2 = e - \overline{u}^2$, and, since $x_1 = x_2$,

$$x_1 = e - \overline{u}^2 = x_2. \tag{29.1.vii}$$

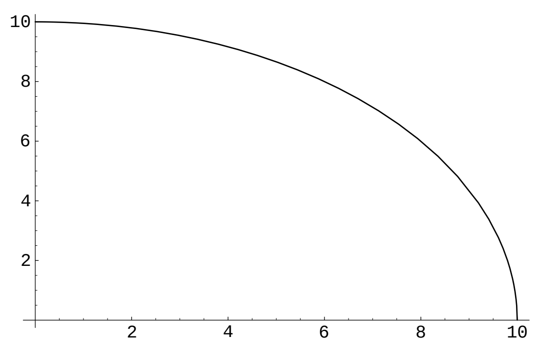
Plugging this into individual 1's utility function, we get

$$u_1 = (e - \overline{u}^2)^{\beta}(e - \overline{u}^2)^{(0.5-\beta)} = \left(e - \overline{u}^2\right)^{1/2}. \tag{29.1.viii}$$

Substituting individual 2's utility level u_2 for $\overline{u}$, we then get the utility possibility frontier

$$u_1 = (e - u_2^2)^{1/2}. \tag{29.1.ix}$$

This is graphed in Exercise Graph 29.1(2) for $e = 100$ — illustrating that the linearly shaped utility possibility we derived for the previous utility function now takes on a strictly convex shape.



Exercise Graph 29.1(2) : Utility Possibility Frontier when $u(x_1, x_2) = x_1^{\beta} x_2^{(0.5-\beta)}$ and $e = 100$

(e) *Do the two different utility functions represent the same underlying (ordinal) preferences? If so, explain the difference in the two utility possibility frontiers.*

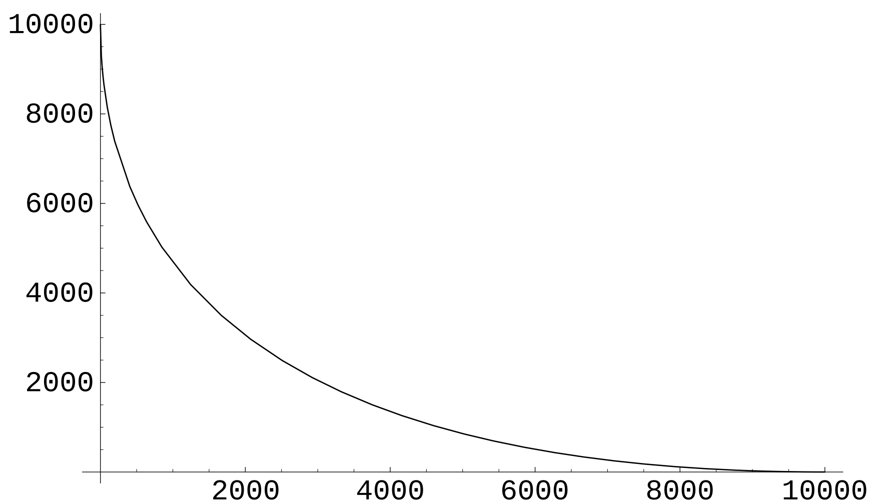
Answer: The two utility functions do represent the same underlying ordinal preferences when $\alpha = 2\beta$. (You can see this by simply squaring our second utility function.) The difference in the utility possibility frontiers arises from the fact that utility possibility frontiers rely on cardinal (not ordinal) measurements on utility. Thus, we can change the labeling on individual indifference curves without changing the shape of the curves (as we did by changing the utility function) and impact the shape of the utility possibility frontier.

(f) *How could you keep the same (ordinal) preferences but transform the utility function in such a way as to cause the utility possibility set to be non-convex? Explain.*

Answer: If, instead of taking the original utility function to the power 1/2 we had squared it, we would get the non-convex shape. Following the same steps as in the previous part, the utility possibility frontier would then in fact be

$$u_1 = \left(e - u_2^{1/2}\right)^2 \qquad (29.1.x)$$

which is graphed in Exercise Graph 29.1(3).



Exercise Graph 29.1(3) : Utility Possibility Frontier when $u(x_1, x_2) = x_1^{2\alpha} x_2^{(2-2\alpha)}$ and $e = 100$

Exercise 29.3

One topic investigated by behavioral economists but not covered in the text relates to how individuals assess probabilities of random events occurring repeatedly. The hot-hand fallacy *is the fallacy that a randomly generated event is* more *likely to occur again if it has just been observed to have occurred multiple times. For instance, a poker player that has had a streak of "hot hands" might believe that he is on a winning streak and will again be dealt a "hot hand" in the next game. The* gambler's fallacy, *on the other hand, occurs when people believe that, once a randomly generated event has occurred, it is* less *likely to occur again. For instance, a lottery player might observe that a particular number has just won in a lottery and conclude that it is less likely that this number will win in the next run of the same lottery. (Note that* *neither part A nor part B of this exercise requires any material presented in Section B.)*

A: *Both types of fallacies arise, for instance, for naive investors in stocks.*

(a) *When a stock falls in value, people often hold onto it based on the argument that "what goes down must come up". What fallacy is this an example of?*

Answer: This is an example of the gambler's fallacy. The stock value has presumably gone down because of some bad news about the company — but all the available information is now incorporated into the stock price. There is no way of knowing whether the next piece of news is any more likely to be good or bad than it is for another stock.

(b) *When a stock rises in value, people sometimes hold onto it because "the company must be doing well and will thus continue to rise in value." What fallacy is at play now?*

Answer: This is an example of the hot-hand fallacy. At any given time, it should be the case that the available information about a company's health is incorporated into the stock price — and there is no way of knowing whether the next piece of news about the company is any more likely to be positive than it is for any other company that one could invest in.

(c) *If you know that lots of other people believe that stocks which have risen in value will rise again in the near future, might this affect your investment choices even if you do not yourself operate under any particular illusion about probabilities of random events?*

Answer: If you believe others believe in the hot-hand fallacy for stocks, you might also want to invest in upward moving stocks knowing that those with gambler's fallacy will demand the stock and will cause its value to rise.

(d) *In the period leading up to the housing market crash in 2007, housing values were increasing at dramatic rates — by as much as 20 to 25% annually in some markets. Lots of people invested with the expectation that this would continue. Can you use the hot-hand fallacy to explain such financial "bubbles"?*

Answer: If lots of people believe in the hot-hand fallacy in a market like housing, their investment behavior can itself produce the "hot-hand" and drive up prices. Even those who do not believe in the hot-hand fallacy might invest knowing that others do and will drive prices up. Of course eventually prices will again reflect the real underlying economic realities — but on the way toward that point, a financial "bubble" can be driven by the hot-hand fallacy. When the bubble bursts, the same fallacy might lead to an initial over-correction.

(e) *The empirical evidence suggests that investors generally are less likely to dispose of losing stocks than they are of disposing winning stocks. Is there another aspect of behavioral economics, one that is explicitly covered in* *the text, that might explain this (rather than either of the fallacies we have mentioned in this exercise)?*

Answer: This might also be interpreted as evidence of reference-based preferences with loss aversion — with the reference point being the price that the investor originally paid for the stock.

(f) *In lotteries where people guess what number will be chosen, the total money pot gets split between the winners. In light of the fact that the gambler's fallacy appears to be strong among lottery players, why might it be best to choose last week's winning number when playing the lottery this week?*

Answer: If a lot of people believe in the gambler's fallacy with respect to winning lottery numbers, then you can predict that last week's winning number will not be chosen by many people. (This is in fact empirically true in real-world lotteries.) As a result, since the probability of last week's number winning this week is just as high as the probability of any other number winning this week, the expected value of choosing last year's number is higher because, if it wins, the payoff will be split between fewer people (since fewer people pick the number).

B: *One of my friends had four children, each a boy. She had really been hoping for a girl for some time and reasoned that she should try again — after all, having four boys in a row was an unlikely enough event — what were the chances of the even less likely event of 5 boys in a row?*

(a) *How many possible gender sequences are there for a woman who gives birth to 4 children? What does this imply for the probability that the sequence will be "all boy"?*

Answer: There are 16 possible gender sequences — which implies that the probability of the event "all boys" is 1/16=0.0625 or 6.25%. (If you know some probability theory, you may recall that you can also calculate this probability as 0.5^4.) Thus, in a pool of 100 women who have four children, we would expect to observe the outcome "all boys" about 6 times.

(b) *What is the probability that her first 5 births are all boys?*

Answer: In five births, there is a total of 32 possible gender sequences — implying that the probability that the first 5 births are all boys is 1/32 = 0.03125 or 3.125%. Out of a pool of 100 women who have 5 children, we would expect the "all boys" outcome about 3 times.

(c) *What is the probability that the sequence of a woman's first 4 children is boy-girl-boy-girl? What about any other gender sequence?*

Answer: This probability is exactly the same as the probability of a woman's first 4 children being all boys (0.0625)— because both the event "boy" and the event "girl" occur each time with probability 0.5 — and the sequence boy-girl-boy-girl is one of 16 possible sequences just as the "all boys" sequence is one of 16 possible sequences that each happen with equal probability. The same is true for any of the other 14 gender sequence.

(d) *What is the probability that my friend's next (and fifth) child will be a boy? How does it compare to the probability that a woman who has had the boy-girl-boy-girl sequence will have a girl as her fifth child?*

Answer: Each birth is an independent event — which implies the probability that the fifth child is a boy is 0.5 or 50%. The same is true for the probability of a girl — regardless of what the previous gender sequence for the first 4 children was.

(e) *My friend used the evidence that four boys in a row was an unlikely event — and five boys in a row would be an even more unlikely event, as her reason for why she thought she had a good chance of her fifth child being a girl. What part of her reasoning is correct, and what part is incorrect?*

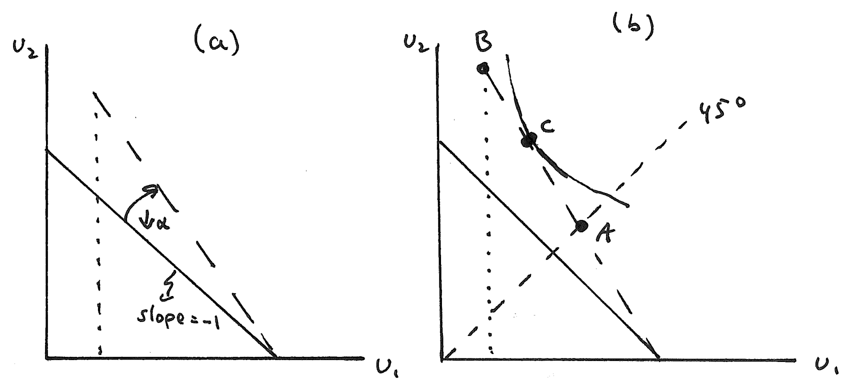
Answer: My friend is certainly right that having 4 boys in a row is an unlikely event — one that occurs with probability 0.0625 to be exact. And having 5 boys in a row is an even more unlikely event — one that occurs in only about 3 out of 100 cases. (The same is of course true for any other gender sequence — each of them will happen with probability 0.0625 in the 4-children case and 0.03125 in the 5-children case). But none of that has anything to do with the probability of the next child being a boy — which is 0.5 or 50% regardless of what the previous gender sequence was.

Exercise 29.5

Everyday Application: *Extreme Altruism and Normative Economics*: *In exercise 29.4 we considered the case of two individuals where one had the usual preferences that depend only on his own consumption level and the other had preferences partially characterized by envy. Consider now the opposite case where individual 2 still has the usual preferences and individual 1 has the same preferences as in exercise 29.4 except that now $\alpha < 0$. Put differently, individual 1 now derives positive utility from individual 2's consumption. How do your conclusions now change?*

A: The change in the utility possibility frontier as α falls below zero is now illustrated in panel (a) of Exercise Graph 29.5, with the only point the two utility possibilities frontier sharing is again the case where all resources are given to individual 1. Now, however, individual 1's utility is higher for any other division of resources because of the warm glow he gets from individual 2's consumption. Thus, the warm glow causes the utility possibility frontier to get steeper. In panel (b), we then plot the utility allocations for the case of Rawlsian social indifference curves (point A), Benthamite social indifference curves (point B) and the in between social indifference map that is symmetric around the 45 degree line (point C). While envy was rewarded before, altruism is not now rewarded — with each of the social indifference maps resulting in fewer resources for the altruistic individual 1.

B: The general equations derived in exercise 29.4 are the same as before, but the interpretation is different because $\alpha < 0$. For the case where $I = 1{,}000$,



Exercise Graph 29.5 : Social Welfare and Extreme Altruism

$\alpha = -0.5$ and $\beta = 0$, this gives the following results:

$$\begin{aligned}
\text{Rawlsian Social Welfare Function implies: } & u_1 = u_2 = 666.67 \text{ and } (x,y) = (333.33, 666.67) \\
\text{Cobb-Douglas Social Welfare Function implies: } & u_1 = 500, u_2 = 1{,}000 \text{ and } (x,y) = (0, 1000) \\
\text{Benthamite Social Welfare Function implies: } & u_1 = 500, u_2 = 1{,}000 \text{ and } (x,y) = (0, 1000)
\end{aligned} \tag{29.5.i}$$

When $\alpha = -0.5$, we therefore already reach the corner solution under the Cobb-Douglas social welfare function. For $\alpha = -1/3$, the results are less extreme:

$$\begin{aligned}
\text{Rawlsian Social Welfare Function implies: } & u_1 = u_2 = 600 \text{ and } (x,y) = (400, 600) \\
\text{Cobb-Douglas Social Welfare Function implies: } & u_1 = 500, u_2 = 750 \text{ and } (x,y) = (250, 750) \\
\text{Benthamite Social Welfare Function implies: } & u_1 = 333.33, u_2 = 1{,}000 \text{ and } (x,y) = (0, 1000)
\end{aligned} \tag{29.5.ii}$$

Exercise 29.7

Everyday and Business Application: *Teaser Rates on Credit Cards and Mortgages: Credit card companies often offer "teaser rates" to new customers — i.e. interest rates that are initially very low but then increase dramatically after a year. Mortgage companies did the same during the sub-prime mortgage period prior to the financial crisis of 2007.*

A: *Consider present-bias as modeled by the beta-delta framework and the explanation it might offer for how students and homeowners end up taking on "too much debt".*

(a) *Consider first a college student who receives credit card offers with teaser rates that charge low interest until the student graduates. As a high school* *senior, our student decides on how much consumption he will undertake once he gets to college (knowing that he will have access to such credit cards. Assuming that δ is the same for present-biased students as it is for students who do not have present-bias, will the plans such students make for consumption while in college differ?*

Answer: No, they will not differ because present-biased students in the beta-delta model evaluate future tradeoffs no differently than non-present biased students. The only distortion in their decision making that is introduced by β regards tradeoffs between the present and the future — but as a high school senior, all plans that are being contemplated lie in the future.

(b) *Next, consider the student in his freshman year. How will students deviate in their actual consumption from their previous plans if they are present-biased?*

Answer: Present-biased students will now consume more in their freshman year than they anticipated — because β now alters the way they actually trade freshman year consumption off against future consumption.

(c) *As our student consumes in his freshman year, he plans for consumption in his remaining three years in college. Will the present-biased student's plans for consuming over the coming years now differ from the non-present biased student (given how each may have deviated from their initial plans during the freshman year)?*

Answer: Yes, they will differ. Even though the present-biased student will trade future periods off against one another the same way that non-present-biased students do, he knows he is consuming more currently and will therefore have to adjust his plans for future consumption downward. (This is therefore a pure income effect.)

(d) *Now consider our student in his sophomore year. Will the present-biased student now take on more debt than he planned as he was contemplating his sophomore year during his time as a freshman?*

Answer: Yes, he will again take on more debt than he planned for the same reasons as he did in his freshman year — because, when he becomes a sophomore, the sophomore year is now the "present" and so his present-bias kicks in.

(e) *Explain how students might end up with considerably more debt than they had planned to — and how limits on credit card borrowing might improve the welfare of some students.*

Answer: Every year, the present-biased student ends up taking on more debt than he had planned — because every year he runs into the same self-control problem as the future becomes the present and therefore revises his consumption upward from his plan. Thus, he ends up taking on more debt each year than he would have optimally planned at the beginning. As a result, limits on credit card borrowing can benefit present-biased students by controlling their spending when they lack the self-control to do so.

(f) *Prior to 2007, mortgage companies offered low teaser interest rates to new home buyers. Home values increased dramatically from 2001 through 2005 — allowing homeowners to re-finance at new teaser rates throughout. How might behavioral economists explain the explosion of home foreclosures beginning in 2006 and 2007 when home prices began to level off and then fall?*

Answer: In some sense, present-biased homeowners behaved like our present-biased college student — each year consuming more than they had planned, and further enabled by rising home prices that allowed them to get more money on credit. Present-bias therefore exacerbated the level of borrowing, a behavioral economist would argue — leading to a much bigger problem when homeowners began to see the value of their homes drop in 2006 and 2007.

B: *Consider a three period model of a college student in his junior year. Suppose this student has no income in periods 0 and 1 while he is in school but then expects an income I in period 2 after he graduates. Suppose that utility of consumption in period i is given by $u(c_i) = \ln c_i$, where c_i is consumption in period i. Suppose further that the student discounts in accordance with the beta-delta model — with his utility of a consumption stream (c_0, c_1, c_2) given by $U = u(c_0) + \beta\delta u(c_1) + \beta\delta^2 u(c_2)$.*

(a) *The student is unable to consume in periods 0 and 1 unless he borrows on his income from period 2. A credit card company offers him a credit card that charges no interest while he is in school and an interest rate r thereafter. Thus, he pays no interest for consumption he undertakes in periods 0 and 1 until period 2 when he has to pay interest $(1+r)(c_1+c_2)$. Set up this student's optimization problem subject to a 3-period budget constraint.*

Answer: The maximization problem is

$$\max_{c_0,c_1,c_2} \ln c_0 + \beta\delta \ln c_1 + \beta\delta^2 \ln c_2 \text{ subject to } c_2 = I - (1+r)(c_0+c_1) \quad (29.7.\text{i})$$

which can also be written as an unconstrained maximization problem

$$\max_{c_0,c_1} \ln c_0 + \beta\delta \ln c_1 + \beta\delta^2 \ln[I - (1+r)(c_0+c_1)]. \quad (29.7.\text{ii})$$

(b) *Derive his optimal consumption plan c_0, c_1 and c_2 as a function of I, r, β and δ.*

Answer: Solving either of these problems (where the latter is probably less tedious to solve), we get

$$c_0 = \frac{I}{(1+r)(1+\beta\delta+\beta\delta^2)}, \quad c_1 = \frac{\beta\delta I}{(1+r)(1+\beta\delta+\beta\delta^2)}, \quad c_2 = \frac{\beta\delta^2 I}{(1+\beta\delta+\beta\delta^2)}. \quad (29.7.\text{iii})$$

(c) *Suppose $I = \$100,000$, $r = 0.2$ and $\delta = 0.95$. If the student does not have present-bias, what consumption levels will the student plan to have in each period — and how much credit card debt does the student plan to have when he graduates?*

Answer: If the student is not present-biased, $\beta = 1$. Substituting these parameters into the equations derived in (29.7.iii), we get

$$c_0 = \$29,214;\ c_1 = \$27,753;\ \text{and}\ c_2 = \$31,639. \tag{29.7.iv}$$

This implies planned credit card debt of $56,968.

(d) *Suppose that $\beta = 0.5$. How much credit card debt does the student plan to have when he graduates?*

Answer: The planned consumption levels are now

$$c_0 = \$43,262;\ c_1 = \$20,549;\ \text{and}\ c_2 = \$23,426 \tag{29.7.v}$$

implying a planned credit card debt of $63,811.

(e) *Calculate the ratio of his period 1 to period 2 consumption plans in the two scenarios. Why are they the same?*

Answer: In both cases, this ratio is 0.8772. It is the same because the beta-delta model does not alter the planned tradeoffs people make between future periods — only between the present and the future.

(f) *How much credit card debt will the student from part (c) actually have when he graduates? What about the student from part (d)?*

Answer: The student from part (c) will carry through on his plan — and will therefore have the credit card debt of $56,968 we calculated in (c). The present-biased student, however, will not follow his plan. When he gets to period 1, period 1 will now be the "present". He will therefore now solve the problem

$$\max_{c_1,c_2} \ln c_1 + \beta\delta \ln c_2 \ \text{subject to}\ c_3 = I - (1+r)c_0 - (1+r)c_1 \tag{29.7.vi}$$

where c_0 was already spent in the previous period and is therefore no longer a choice variable. Solving this in the usual way, we get

$$c_1 = \frac{I-(1+r)c_0}{(1+r)(1+\beta\delta)} \ \text{and}\ c_2 = \frac{\beta\delta[I-(1+r)c_0]}{(1+\beta\delta)}. \tag{29.7.vii}$$

Plugging in $\beta = 0.4$, $I = 100,000$, $r = 0.2$ and $c_0 = 43,262$ (as calculated before), we therefore get

$$c_1 = \$27,167 \ \text{and}\ c_2 = \$15,485. \tag{29.7.viii}$$

Adding c_0 and c_1, we then get an actual credit card debt of $70,429 when the student graduates.

(g) *Now consider the student with $\beta = 0.5$ as a sophomore looking ahead to being a junior. He is fully supported by his parents in his sophomore year, but he knows they will no longer support him in his junior year when he is able to get credit. (Assume that credit card companies do not offer cards to sophomores but only to juniors). As he thinks about how much he will end up borrowing, will his plan differ from the student who is not present-biased? How much more credit card debt will he end up with than he planned to as a sophomore?*

Answer: As a sophomore, the present-biased student will evaluate the future just as the non-present-biased student — because all the β terms in the future will end up canceling as he considers making tradeoffs between future periods. Thus, his plans would be the same as the ones we calculated for the student in part (c) — implying that, as a sophomore, the present-biased student will plan for a total credit card debt of $56,968 when he graduates. We just calculates that his actual credit card debt will be $70,429 — or $13,461 more than he plans as a sophomore.

(h) True or False*: Regulations that limit the amount of credit card debt that students can take on can improve the welfare of present-biased sophomores.*

Answer: This is true — present-biased sophomores will have self-control problems as juniors and seniors — and will therefore over-consume relative to what they optimally plan as sophomores. Limiting the amount they can borrow can then act as a self-control device.

Exercise 29.9

Everyday and Business Application: *Endowment Effects and Housing Markets: In end-of-chapter exercises 6.9 and 7.6, we derived the curious prediction that homeowners are made better off by housing price fluctuations — regardless of whether housing prices go up or down. This was due to the fact that, assuming some degree of substitutability between housing and other goods (and no transactions costs), homeowners will sell their homes whenever housing prices change — buying smaller homes and consuming more when price increases and buying larger homes and consuming less when housing prices fall.*

A: *Revisit the logic behind this conclusion before proceeding.*

(a) *One reason that homeowners do not constantly switch homes when housing prices fluctuate arises from the fact that there are moving costs that make switching homes not worthwhile for small price fluctuations. Now consider another explanation rooted in endowment effects uncovered by behavioral economists. Within the context of the model you used in exercises 6.9 and 7.6, how might you be able to model such endowment effects in terms of the shapes of indifference curves for homeowners?*

Answer: You would need to introduce a kink of the indifference curve — not necessarily as extreme as the kink for perfect complements — at the endowment point. With such a kink, the budget line can rotate around the endowment point (as it does when housing prices fluctuate) without causing the homeowner to re-optimize by getting a different house. Without a kink, there is a substitution effect and homeowners will move (in the absence of transactions costs.)

(b) *Next, consider the problem faced by a homeowner who needs to move during a "down" market. Suppose the homeowner originally purchased his home at price p_0 — and suppose that this price has become a "reference point" — with the homeowner interpreting a sales price above p_0 as a "gain" and a sales price below p_0 as a "loss". Explain how behavioral economists might predict that the level of p_0 will affect the asking price that the homeowner sets.*

Answer: Since this homeowner exhibits "loss aversion" — with a loss interpreted as any price below the original purchase price, the homeowner will be less willing to lower the price below p_0 than he is to lower it as long as the price lies above p_0. The higher p_0, the greater the loss aversion effect — and the more the asking price will deviate from what a homeowner would set in the absence of reference-based preferences.

(c) *Housing economists have uncovered the following empirical fact: During times when housing demand is falling (putting downward pressure on home prices), houses that are for sale typically take longer to sell — resulting in an increase in the number of houses on the market. Can you explain this using reference-based preferences with "loss aversion"?*

Answer: In "down markets", more homeowners that are looking to sell their homes will face the fact that they will incur a "loss" relative to the price at which they purchased the home. Thus, the loss aversion effect on asking prices — discussed in the previous part — becomes more intense the more severe the down market is. Behavioral economists would then expect homeowners to ask for artificially high prices (relative to market conditions) — with homes therefore selling more slowly.

B: *Consider the optimization problem faced by a homeowner who is moving and is determining an asking price for his home. Such a homeowner faces the following trade-off: A higher asking price p means a lower probability of selling the home, but it also means greater utility for the homeowner if the home sells. Suppose that the probability of a sale is given by $\delta(p) = 1 - 0.00001p$. Suppose further that $p_0 = \$100,000$ was the price at which the homeowner had originally bought the home, and his utility from not selling the home is $\overline{u} = (10,000 - \alpha p_0)$. His utility of selling the home depends on the price p and is given by $u(p, p_0) = (p - \alpha p_0)$ when $p \geq p_0$ and $v(p, p_0) = \beta(p - \alpha p_0)$ when $p < p_0$.*

(a) *What values do α and β take in a model without reference-based preferences?*

Answer: If preferences are not reference-based, then the purchase price p_0 should not matter — implying that $\alpha = 0$ and $\beta = 1$.

(b) *Set up the optimization problem for the homeowner under the assumptions in (a) and solve for the optimal asking price.*

Answer: The optimization problem is then

$$\begin{aligned}\max_p \delta(p)u(p) + (1-\delta(p))\overline{u} &= (1-0.00001p)p + 0.00001p(10,000) \\ &= p - 0.00001p^2 + 0.1p \qquad \text{(29.9.i)} \\ &= 1.1p - 0.00001p^2.\end{aligned}$$

The first order condition is $1.1 - 0.00002p = 0$ which solves to $p = \$55,000$.

(c) *Next, suppose (from here on out) that $\alpha = 1$ and $\beta = 2.25$. Repeat the optimization exercise assuming that the homeowner uses the function u (and not v). What would be the optimal asking price?*

Answer: We then have the problem

$$\begin{aligned}\max_p \delta(p)u(p,p_0) + (1-\delta(p))\overline{u} &= (1-0.00001p)(p-100,000) + 0.00001p(10,000-100,000) \\ &= p - 0.00001p^2 - 100,000 + p - 0.9p \qquad \text{(29.9.ii)} \\ &= 1.1p - 0.00001p^2 - 100,000.\end{aligned}$$

This gives us the same first order condition as in the previous part, with $p = \$55,000$.

(d) *If the homeowner has reference-based preferences as specified by u and v, is the price you calculated in (c) the true optimal asking price?*

Answer: The price we calculates, \$55,000, lies below the reference price \$100,000. We have therefore used the wrong utility function — i.e. the one intended for evaluating prices above the reference price, which implies that this is not the true optimal asking price.

(e) *Next, set up the optimization problem again — but this time use v instead of u. What is the optimal asking price you now get? Is this the true optimal asking price for this homeowner? Explain.*

Answer: Now we have the problem

$$\begin{aligned}\max_p \delta(p)v(p,p_0) + (1-\delta(p))\overline{u} &= (1-0.00001p)[2.25(p-100,000)] + 0.00001p(10,000-100,000) \\ &= 2.25p - 0.0000225p^2 - 225,000 + 2.25p - 0.9p \qquad \text{(29.9.iii)} \\ &= 3.6p - 0.0000225p^2 - 225,000.\end{aligned}$$

The first order condition is now $3.6 - 0.000045p = 0$ which solves to $p = 80,000$. This is now the true asking price — below the reference point \$100,000 and appropriately evaluated with the v function.

(f) *What is the probability that the home will sell for the price you calculated in (b) and (c) — and how does it compare to the probability that the home will sell at the price you calculated in (e)? Can you reconcile this with the empirical fact stated in A(c)?*

Answer: The probability that the home will sell at price $55,000 is

$$\delta(55,000) = 1 - 0.00001(55,000) = 0.45 \tag{29.9.iv}$$

and the probability that it will sell at price $80,000 is

$$\delta(80,000) = 1 - 0.00001(80,000) = 0.20. \tag{29.9.v}$$

The home is therefore less likely to sell at the price chosen under reference-based preferences with loss aversion — meaning that it will take longer to sell.

Exercise 29.11

Business and Policy Application: *Increased Liquidity, Procrastination and National Savings*: *Over the past few decades, increasingly sophisticated financial investment possibilities have enabled individuals to place their savings into assets that can be sold instantly if need be — as opposed to investments in more "illiquid" assets like land, assets that require time and effort to convert to cash.*

A: *Consider individual 1 whose intertemporal tastes can be characterized by the beta-delta model versus individual 2 whose intertemporal tastes are characterized in the more usual neoclassical "delta" model. Both individuals just inherited some money and intend to invest this for their retirement.*

(a) *Could an increase in the availability of liquid assets for investment purposes make individual 2 better off? Could it make individual 2 worse off?*

Answer: The increased availability of liquid investment opportunities expands the choice set for individual 2 — and any expansion of the individual's choice set cannot make him worse off because the previous options are still available. Thus, this might make him just as well off or, by making other alternatives available, might make him better off.

(b) *Now consider individual 1. Suppose this individual consults an investment planner who has observed this individual's past savings and consumption decisions and recommends an investment strategy. Why might he recommend a strategy that focuses on illiquid assets?*

Answer: The beta-delta individual has every intention of saving for retirement but will have difficulty keeping his hands off the money he saved for retirement prior to retiring as the future becomes the present and his present-bias tempts him to consume earlier. Having his money tied up in illiquid assets increases the cost of grabbing the money early — thus making it less likely that he will give into his self-control problems.

(c) *If individual 1 is aware of his time-inconsistency problem, will he accept the financial planner's advice? Would he have any reason not to accept it if he is unaware of his self-control problem?*

Answer: The individual who is aware of his problem will in fact actively seek to bind himself in the future to keep his "future self" from robbing his retirement savings. He will therefore accept the advice. If the individual is not aware of his self-control problem, he is blissfully planning to not grab his retirement money early for pre-retirement consumption — and so he has no reason not to follow his investment advisor's advice even though he'll be frustrated with his decision later on when he really wants to consume from his savings but cannot do so easily.

(d) *Suppose that, instead of just having inherited money, the two individuals have just accepted a job in which their company contributes to a 401K retirement plan. The individuals now must choose between two investments for their retirement account: investment A consists of a mix of stocks and bonds that can be sold easily, while investment B consists of 10-year savings "certificates of deposit" that cannot be cashed out without a substantial penalty. (In both cases, there would be a tax penalty for withdrawing funds from the 401K plan, but, since it is the same for any 401K withdrawal, ignore this feature of 401K plans here.) Assuming identical rates of return on the two investments, which will cause individual 1 to accumulate more savings for retirement? What about individual 2?*

Answer: For the same reason as in the previous parts, individual 1 will end up saving more under the illiquid strategy *B* because he will have greater difficulty giving into the temptation to consume out of his retirement funds when the future becomes the present. Individual 2, on the other hand, will save the same under both plans (so long as he accurately forecasts his future income).

(e) *Suppose individual 1 also has reference-based preferences subject to endowment or status quo effects. If the company gets to choose the initial investment strategy but allows individuals to opt into a different strategy if they want to, which investment strategy would the company choose for its workers (assuming it cares about the level of retirement savings that employees undertake)?*

Answer: Choosing the strategy *B* for its workers is a form of "libertarian paternalism". It has little effect on individual 2 who can switch to strategy *A* without much effort, but it puts the beta-delta individual into a strategy that helps him avoid his self-control problems and, given that the individual views the default strategy as his "endowment strategy", he will tend to stick with this strategy.

(f) *Over the past few decades, there has been a substantial decrease in national savings in the U.S. How might a behavioral economist use the idea of procrastination to explain this?*

Answer: This is a case where an increase in the available options for investment may have caused present-biased individuals to have more liquid savings — thus making it easier for them to consume out of their savings as the future becomes the present. If this is so, then savings will decline as a result of the increased availability of liquid assets.

Exercise 29.13

Policy Application: *Confirmation Bias, Politics, Research and Last-Minute Studying*: *Individuals have lots of assumptions about the way the world works, assumptions that help frame how they make decisions. These assumptions are often challenged or confirmed by empirical evidence. However, psychologists who have analyzed how people change their assumptions about the world suggest that we tend to seek out evidence that confirms our assumptions and ignore evidence that contradicts our assumptions. This phenomenon is known as* confirmation bias, *and one of the early experiments uncovering this bias is described in part B.*

A: *Over the past few decades, there has been a vast increase in the number of sources that individuals can use to inform themselves about what is going on in the world. For instance, most individuals used to rely on their local newspaper (which often drew its material primarily from a handful of national news outlets) and the evening newscast on one of three networks. Today, on the other hand, there are lots of cable news channels people can choose from throughout the day, and an increasing number of people rely on news from internet sources.*

(a) *Many observers of public discourse have suggested that the assumptions individuals bring to policy discussions are now often more diametrically opposed than in the past, with different camps often no longer able to hold civil dialogue because they so fundamentally disagree about the underlying "facts". If this is true, how can this be explained by the increased number of news and opinion outlets?*

Answer: If individuals actively seek information that confirms their current views, they can now choose a set of news outlets that share their assumptions about how the world works and report news and opinion accordingly. As a result, assumptions about how the world works are "confirmed" by the selective evidence that people seek out — thus hardening their positions, in some cases to the point where they are unable to see "the other side" to the story.

(b) *In the past, opinion polls often suggested that public disapproval of a U.S. President was in the single digits, but more recently, a President is considered as doing well if his disapproval ratings are in the 20 to 30 percent range. Can confirmation bias in the more recent news environment explain this?*

Answer: If those with slight pre-dispositions against the current President seek out news sources that tend to support their skepticism and report news accordingly, then slight pre-dispositions can be "confirmed" and thus harden to strong opposition. When individuals had to get their news from the same sources, they were less able to select news that would confirm their biases — and thus one would expect less polarizing view points to have emerged in the past.

(c) *Until the mid-1980's, the Federal Communication Commission in the U.S. enforced a rule known as the "Fairness Doctrine". This rule required news outlets — particularly on radio and TV — to present opposing viewpoints. It was argued at the time that some media markets only had one or two such news outlets, and thus the Fairness Doctrine was required to allow people to get alternative points of view so that they could then form informed opinions. Since the mid-1980's, the Fairness Doctrine is no longer applied — allowing news outlets to present news and opinions in any way they see fit. It was argued that increased competition has led to competing news outlets in virtually all markets — thus automatically allowing individuals to gather alternative viewpoints to form their own opinions. Now some are arguing for a re-instatement of the Fairness Doctrine but others view it as a violation of free speech and free competition of ideas in the product-differentiated marketplace. Can you argue both sides of this issue?*

Answer: The re-imposition of the Fairness doctrine would certainly limit free speech in the sense that those espousing a particular viewpoint on TV or radio would be forced to also present alternative views. It would furthermore restrict competition in a differentiated product market — and we have seen that product differentiation in the presence of competition can lead to increases in consumer welfare. And, if we consider a model without confirmation bias, the increased set of choices of news outlets would make consumers better off by increasing their choice sets, with nothing keeping them from gathering opposing viewpoints from different sources. Advocates of the Fairness Doctrine, on the other hand, may believe that confirmation bias is sufficiently severe as to keep individuals from actively considering alternative viewpoints and instead seeking out only those new sources that confirm their biases. They might then argue that this imposes larger externalities in the sense that society is made worse off by political polarization based on assumptions that individuals do not confront with empirical evidence.

(d) *Some have observed an increase in the number of people who believe in a variety of "conspiracy theories" — theories such as that the 9/11 attack was orchestrated by the government or that a politician secretly adheres to a religious view that differs from his stated view. How might this be explained in light of the fact that most individuals find evidence against such theories conclusive?*

Answer: Given that some news outlets, particularly on the internet, report only "facts" that support such theories and at the same time cast doubt on the reliability of other outlets that contradict such "facts", it is now easier for people pre-disposed toward believing in conspiracy theories to have their beliefs confirmed.

(e) *Empirical social scientists often do econometric regression analysis on real-world data to ascertain the direction and magnitude of people's responses to different policies. As computational analysis has become less costly, such* *researchers are now able to run literally tens of thousands of different regressions — using combinations of different variables and empirical specifications — whereas in the past they have had to limit themselves to a few regressions. Suppose that researchers have prior beliefs about what an empirical investigation might show. How might you view statistically significant empirical results reported in research papers more skeptically as a result of knowing about confirmation bias?*

Answer: Given the large number of regression specifications that can now be run on a data set, researchers with strong prior assumptions about what the empirical evidence will show might engage in confirmation bias in the sense that they discount regression results that contradict their bias as resulting from "mis-speficied modeling" while accepting regression results that confirm their bias. They would therefore, without any intent to mislead, focus on reporting results that confirm their assumptions. Statistical tests of significance report confidence levels of results under the assumption that no such selection of results driven by confirmation bias occurs. What could accurately be described as statistically significant may then not be statistically significant in light of confirmation-bias driven selection of reporting.

(f) *In the final hours before an exam, students often "study" intensely by scanning their notes and focusing on key terms that they have highlighted. Some students find that this dramatically increases their sense of being prepared for the exam — and then find that they do not do nearly as well on the exam as they had thought they would given their last-minute studying. Can you explain this using the idea of confirmation bias?*

Answers: In their last minute studying, students may simply be focusing on the terms and concepts they know and selectively disregard other aspects of their notes. This then confirms their "bias" that they are prepared for the exam without actually increasing their level of preparation. This is the same reason why it is often hard for students to find their mistakes on exams once they have made them — even if they are simple mistakes. Once an answer is written, students have a tendency to proof-read their answers with the bias of trying to confirm that their answers are in fact correct.

B: *The following experiment, first conducted in the early 1960's, is an illustration of confirmation bias. Suppose that you are given the following sequence of numbers: 2-4-6. You are told that this sequence conforms with a particular rule that was used to generate the sequence and are asked to figure out what the underlying rule is. To do so, you can generate your own 3-number sequences and ask the experimenter for feedback on whether your sequence also conforms with the underlying rule. You can do this as often as you need to until you are certain you know what the underlying rule is — at which time you tell the experimenter your conclusion.*

(a) *Suppose that, when you first see the 2-4-6 sequence, you recognize it as a sequence of even numbers and believe that the underlying rule probably* *requires the even numbers. What is an example of a sequence that you might use to test this assumption if you have confirmation bias?*

Answer: You might use any sequence of even numbers — a sequence like 4-6-8 or 12-14-16.

(b) *What sequence of numbers might you propose to test your assumption if you did not have confirmation bias and were open to your assumption being incorrect?*

Answer: You might then try a sequence of numbers that violates your assumption — a sequence like 5-6-8. If you were then told that this sequence conforms with the underlying rule, you would know your assumption that the rule requires even numbers to be incorrect.

(c) *The underlying rule was simple: in order to comply with the rule, it simply had to be an ascending sequence. Very few subjects correctly identified this rule — instead very confidently concluding that the rule was much more complex. The experimenters concluded that people consistently derived an incorrect rule because they gave examples that would confirm their assumptions rather than attempt to* falsify *them. (A sequence intended to "falsify" an assumption would be one that violates the assumption.) How is this consistent with your answers to (a) and (b)?*

Answer: The sequences in (a) are sequences intended to "confirm" the assumption that the sequence requires even numbers. Whenever the subject gives such a sequence, he would be told that it in fact conforms to the underlying rule so long as the sequence was an ascending sequence of even numbers. But that does not tell the subject whether even numbers are really required by the underlying rule — just giving sequences with even numbers simply does not allow the subject to test whether the underlying rule might allow odd numbers. The sequence in (b) does allow the subject to falsify his assumption — but unless one is willing to consider the possibility that one's assumption is wrong, one might never think to provide a sequence intended to falsify the assumption that is presumed to be correct.

Conclusion: Potentially Helpful Reminders

1. Keep in mind the difference between present-bias and impatience. Even many economists who are only casually familiar with behavioral economics confuse the two, but only the former is a violation of the neoclassical model of intertemporal decision-making in that only present-bias (and not impatience) leads to self-control problems.

2. The reference-dependence model often creates a special role for "ownership" where owning something creates a different reference point. This leads to different behavioral predictions across scenarios that are identical from the neoclassical economist's perspective.

3. While the insights from behavioral economics are compelling, be careful not to overreach with them. Just because people sometimes exhibit systematic biases does not mean they always do — nor does it mean that policy can always correct for them. One of the potentially appealing middle-grounds lies in what we have called "libertarian paternalism" — policies that gently nudge people in directions against their biases without forcing them to go there.

4. The idea of "first-best" and "second-best" is an important one — with normative decisions about "what is good" often being tempered by the fact that policy confronts "second-best" scenarios. Our discussion of this in terms of utility possibility frontiers illustrates how economics and philosophical ideas can interact fruitfully.

5. If you understand indifference curves (and utility functions), you understand social indifference curves (and social welfare functions). The latter are, in fact, the same as the former except that what appears on the axes has changed — with social indifference curves just being indifference curves over bundles of utilities (or incomes or wealth) rather than over bundles of consumption goods.