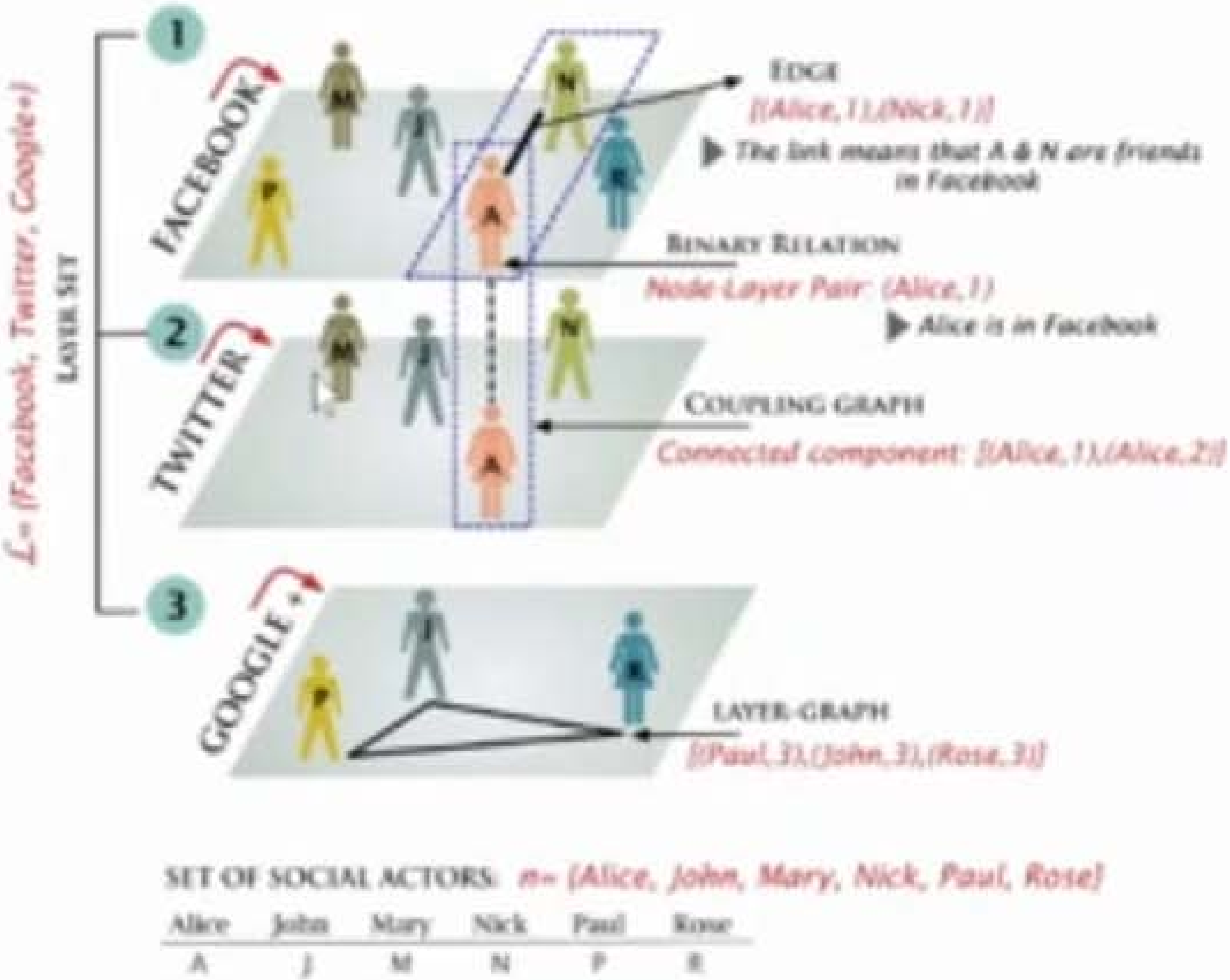
超扩散

多层网络的**超扩散现象**是指多层网络的驰豫时间比构成它的**各层网络**的**驰豫时间都短**。

$$t < \max\{t_1, \cdots, t_M\} \Leftrightarrow \mathrm{Re}(\lambda_2(\mathcal{L})) > \max\{\mathrm{Re}(\lambda_2(L_1)), \cdots, \mathrm{Re}(\lambda_2(L_M))\}$$



层内连边:
同一媒介**内**好友信息传播

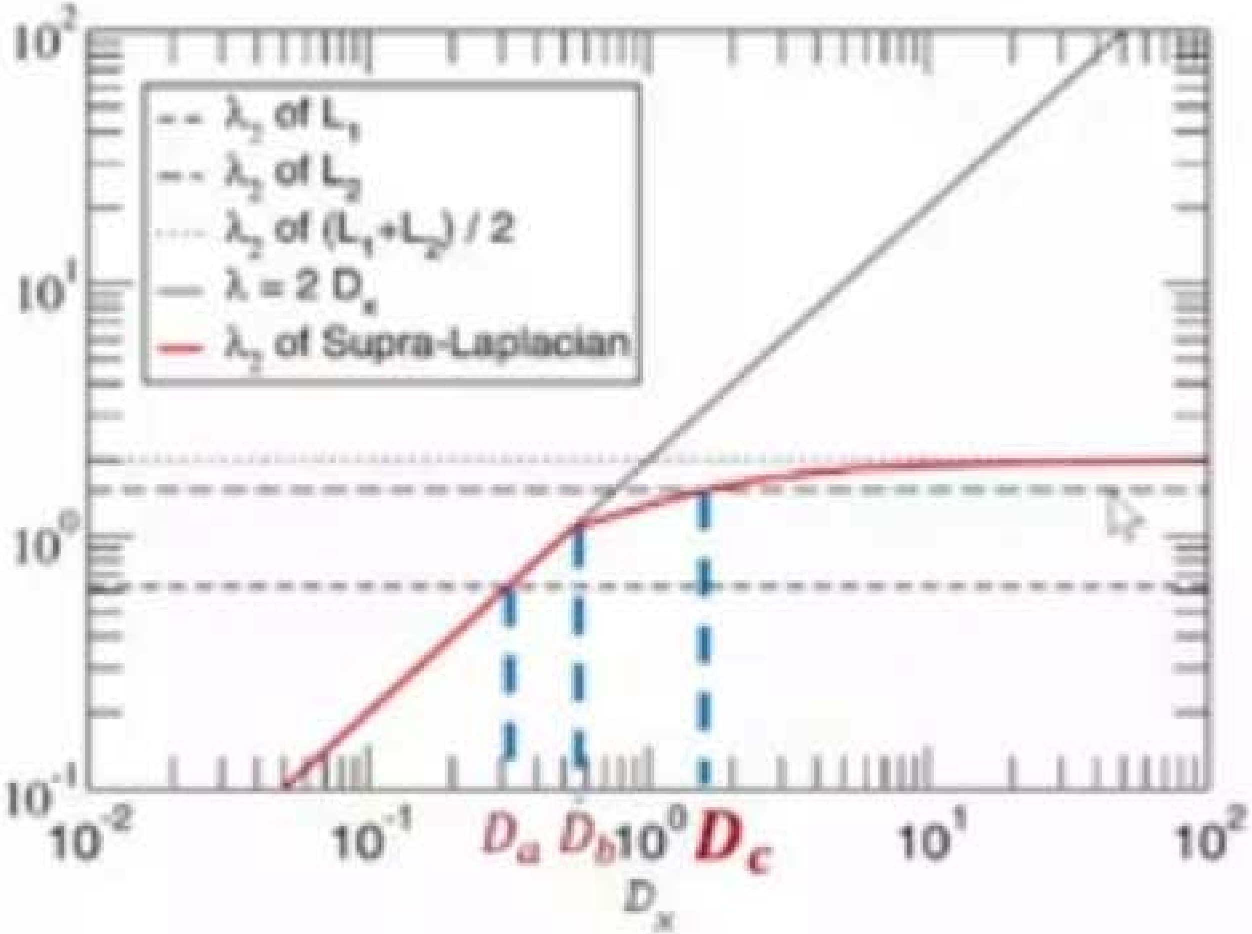
层间连边:
自身不同媒介**间**信息传播

多重社交网络

一定条件下

信息**超扩散**

超扩散分析



$$\mathcal{L} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} + D_x \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

- $0 < D_x \ll 1 = D_1 = D_2$ 时，$2D_x$ 接近于0，$\lambda_2(\mathcal{L}) = 2D_x$
- $D_x \gg 1$时，$\lambda_2(\mathcal{L}) \approx \lambda_s \triangleq \lambda_2\left(\frac{L_1+L_2}{2}\right)$

- 若$\lambda_s > max\{\lambda_2(L_1), \lambda_2(L_2)\}$，存在某个临界值 $\boldsymbol{D_c}$，当 $D_x > \boldsymbol{D_c}$ 时，$\lambda_2(\mathcal{L}) > max\{\lambda_2(L_1), \lambda_2(L_2)\}$，称**两层网络**发生**超扩散**。
- $\lambda_s > max\{\lambda_2(L_1), \lambda_2(L_2)\}$，称**两层网络**是**可超扩散**的。

超扩散分析

对于任意Laplacian矩阵 L，其次小特征值 $\lambda_2 = \min_{x\perp\mathbf{1}} x^{\mathrm{T}} L x$，$\mathbf{1}$ 为全1向量，是对应于 L 的最小特征值 0 的特征向量。

$$\lambda_s = \min_{x\perp\mathbf{1}} x^{\mathrm{T}}\left(\frac{L_1 + L_2}{2}\right)x \geq \min_{x\perp\mathbf{1}} x^{\mathrm{T}}\frac{L_1}{2}x + \min_{x\perp\mathbf{1}} x^{\mathrm{T}}\frac{L_2}{2}x$$

$$= \frac{\lambda_2(L_1) + \lambda_2(L_2)}{2} \geq \min\{\lambda_2(L_1), \lambda_2(L_2)\}$$

- 两层网络的**扩散**一定比扩散得最慢的子网络要快。
- $\mathrm{Re}(\lambda_2(\mathcal{L}(D_x))) > max\{\mathrm{Re}(\lambda_2(L_1)), \mathrm{Re}(\lambda_2(L_2))\}$ $\xLeftrightarrow{\text{定义}}$ **超扩散**

可超扩散 $\xLeftrightarrow{\text{定义}}$ $\mathrm{Re}(\lambda_s(L_1, L_2)) > max\{\mathrm{Re}(\lambda_2(L_1)), \mathrm{Re}(\lambda_2(L_2))\}$

超网络模型

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{u=1}^{M}\sum_{j=1, j\neq i}^{N} b_{ij}^{u}\Gamma^{u}(x_j(t) - x_i(t))$$

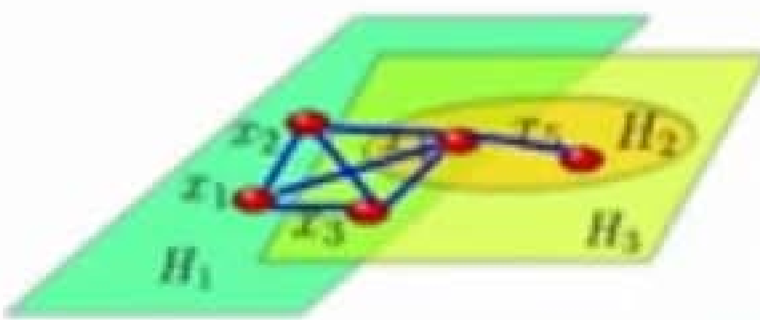
$$i = 1, 2, \cdots, N$$

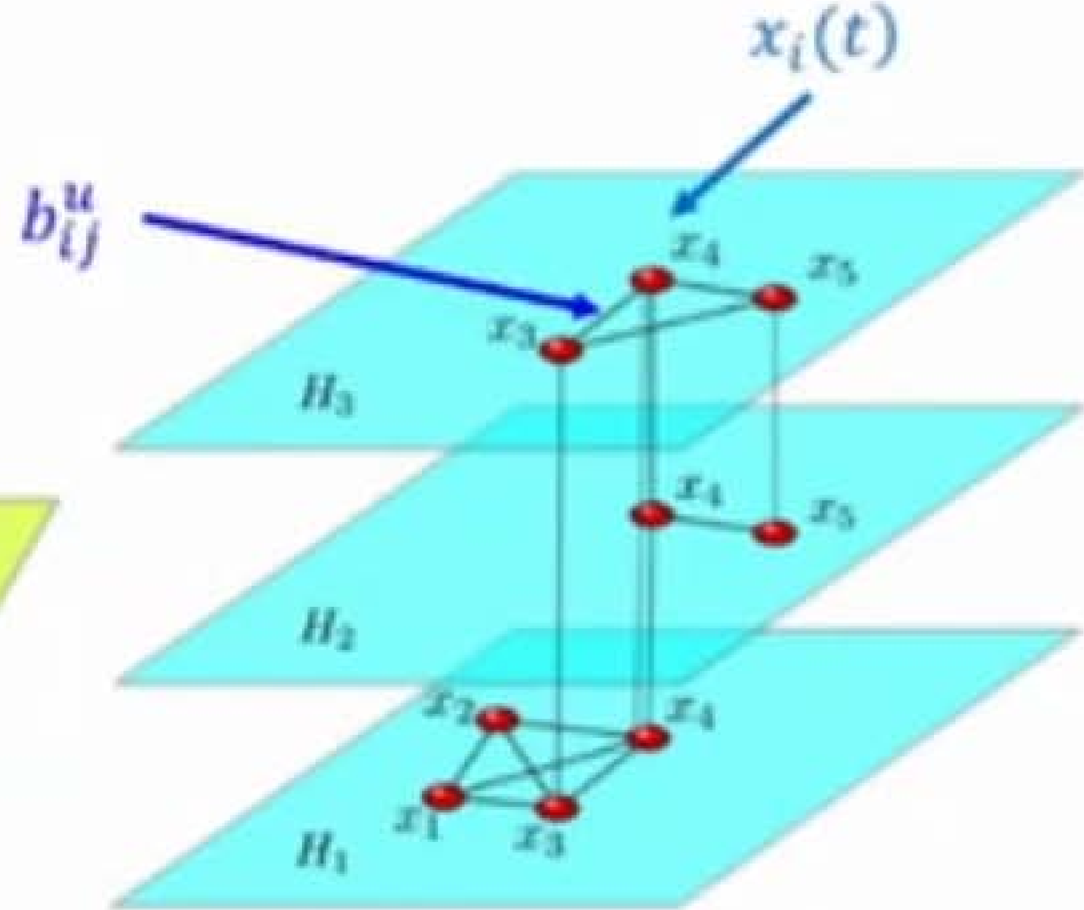
$$u = 1, 2, \cdots, M$$

$$\Rightarrow \quad \dot{x}_i(t) = f(x_i(t)) - \sum_{u=1}^{M}\sum_{j=1}^{N} \bar{b}_{ij}^{u}\Gamma^{u}(x_j(t))$$

$$\bar{b}_{ij}^{u} := \begin{cases} d_i^u, & i = j \\ -b_{ij}^u, & i \neq j \end{cases},$$

$$d_i^u = \sum_{j\neq i} b_{ij}^u,$$





网络模型（Recall）

复杂动力网络是刻画多个个体耦合的复杂系统模型，每个个体均是一个动力系统，并且个体之间存在特定的耦合关系. 因此，**模型有两个主要部分，一是节点动力学，二是网络结构**.

考虑由N个相同的节点构成的一般连续时间动力网络模型:

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} \bar{a}_{ij}(H(x_j(t)) - H(x_i(t))),\ i = 1, \cdots, N \tag{1}$$

其中，$x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in R^n$为节点$i$的**状态变量**，$f \in C[R^n, R^n]$，常数$c > 0$为网络的**耦合强度**，$\bar{A} = (\bar{a}_{ij})_{N\times N}$ 为反映网络拓扑结构的**邻接矩阵**，可以不对称，$H \in C[R^n, R^n]$为**内连函数**.

电力-计算机网络

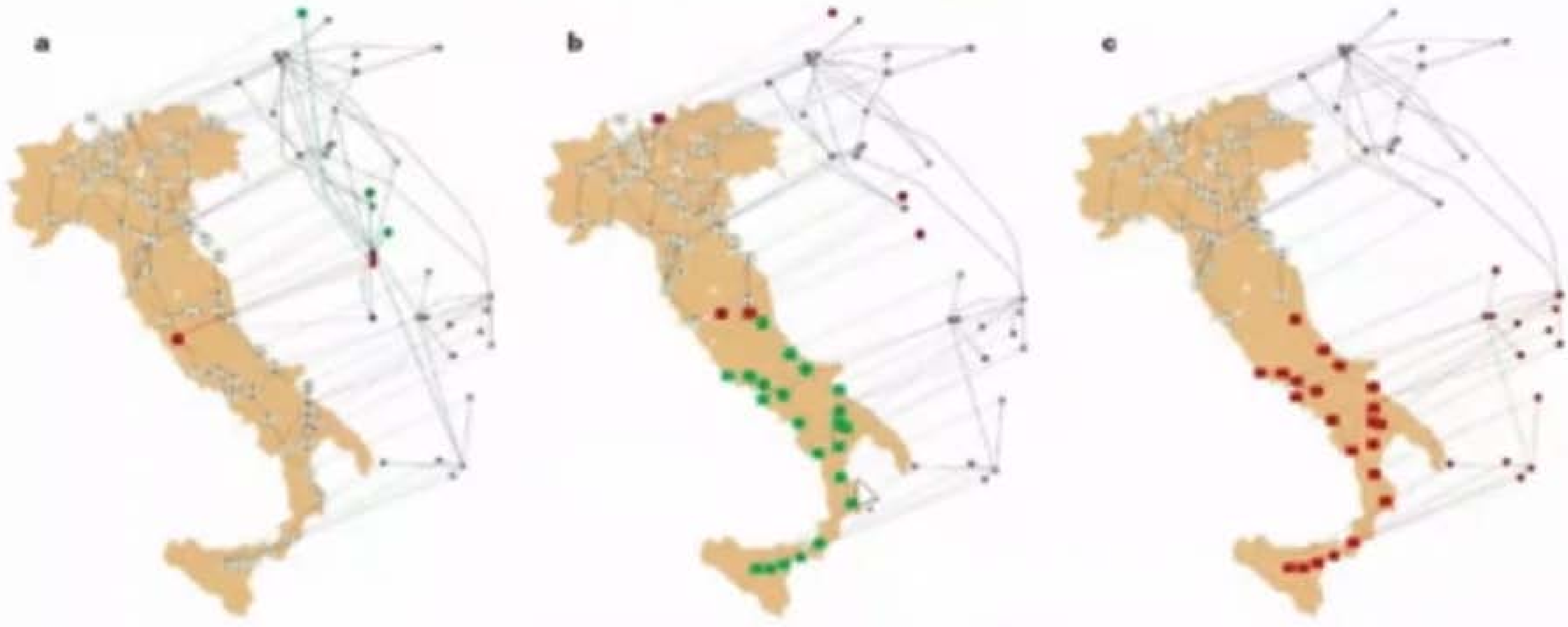
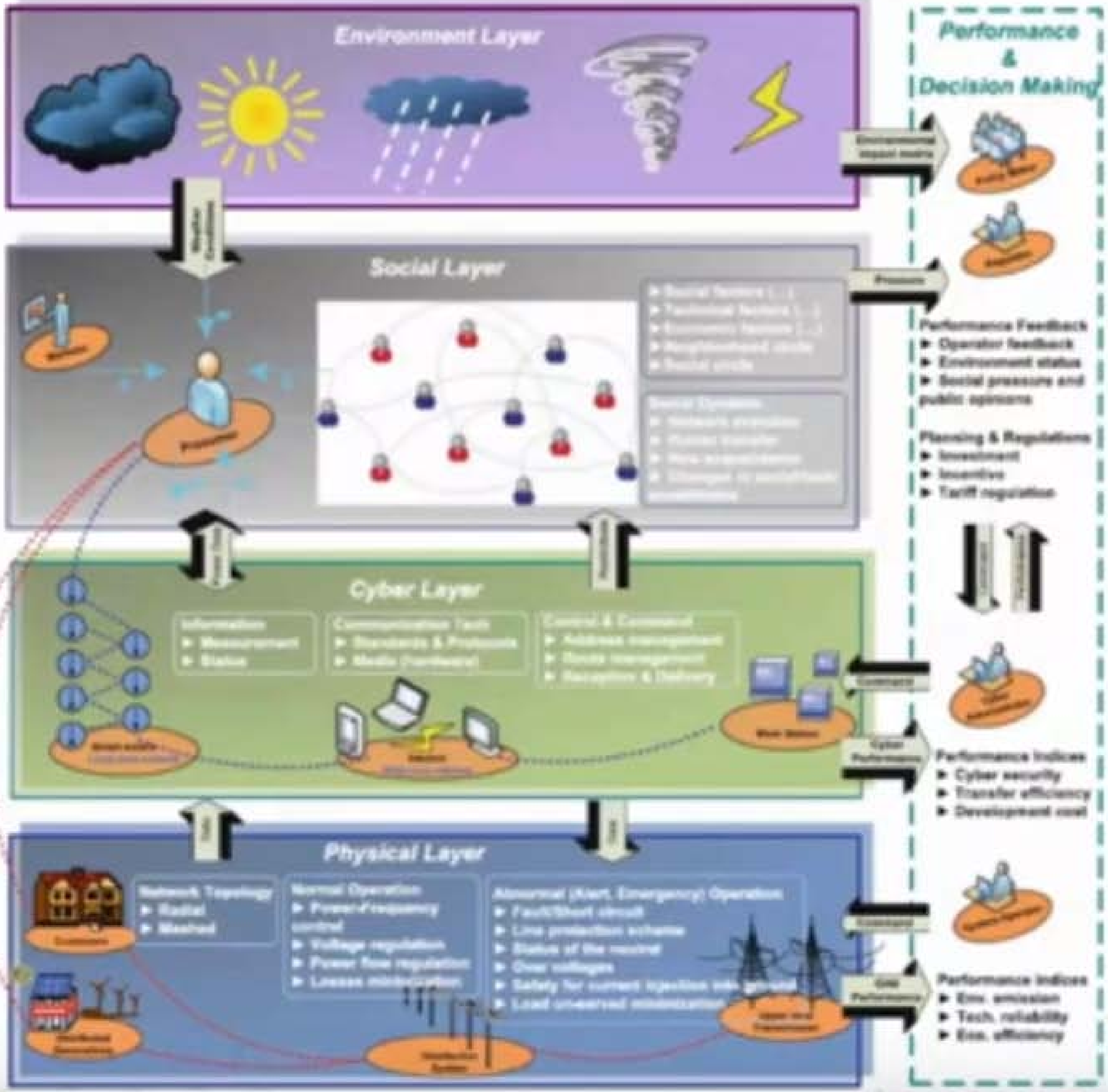


Figure 1 | Modelling a blackout in Italy. Illustration of an iterative process of a cascade of failures using real-world data from a power network (located on the map of Italy) and an Internet network (shifted above the map) that were implicated in an electrical blackout that occurred in Italy in September 2003[20]. The networks are drawn using the real geographical locations and every Internet server is connected to the geographically nearest power station. **a**, One power station is removed (red node on map) from the power network and as a result the Internet nodes depending on it are removed from the Internet network (red nodes above the map). The nodes that will be disconnected from the giant cluster (a cluster that spans the entire network) at the next step are marked in green. **b**, Additional nodes that were disconnected from the Internet communication network giant component are removed (red nodes above map). As a result the power stations depending on them are removed from the power network (red nodes on map). Again, the nodes that will be disconnected from the giant cluster at the next step are marked in green. **c**, Additional nodes that were disconnected from the giant component of the power network are removed (red nodes on map) as well as the nodes in the Internet network that depend on them (red nodes above map).

[1] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, S. Havlin, Catastrophic cascade of failures in interdependent networks." *Nature*, 2010, 464: 1025-8.

多层大型系统网络

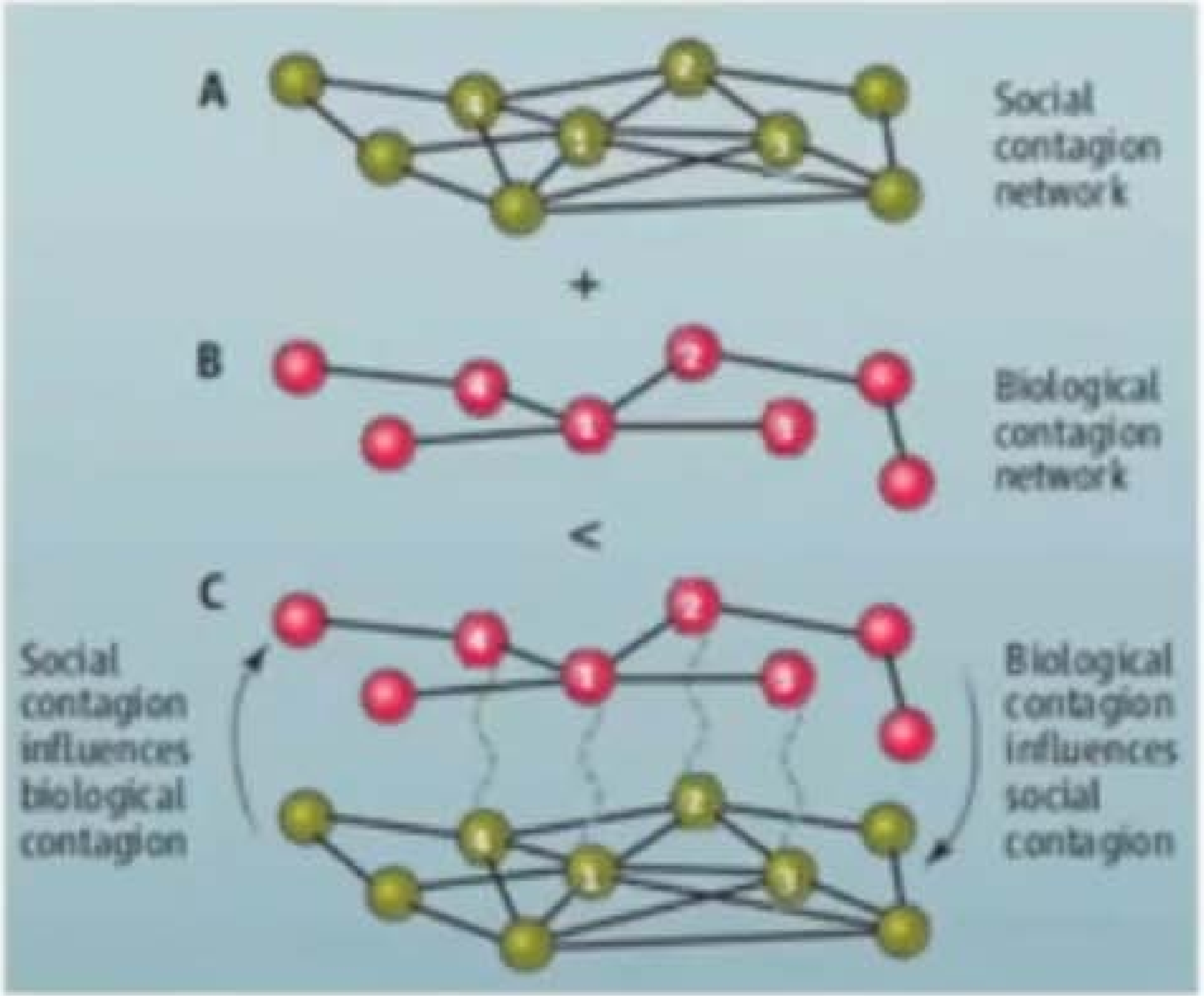


Sometimes life is complicate,

sometimes it is just complex

[1] G. D'Agostino, A. Scala, Networks of networks: The last frontier of complexity, *Springer*, 2014, Ch 9.

多层疾病传播网络



社交接触网络层:
疾病信息传播

疾病接触网络层:
疾病传播

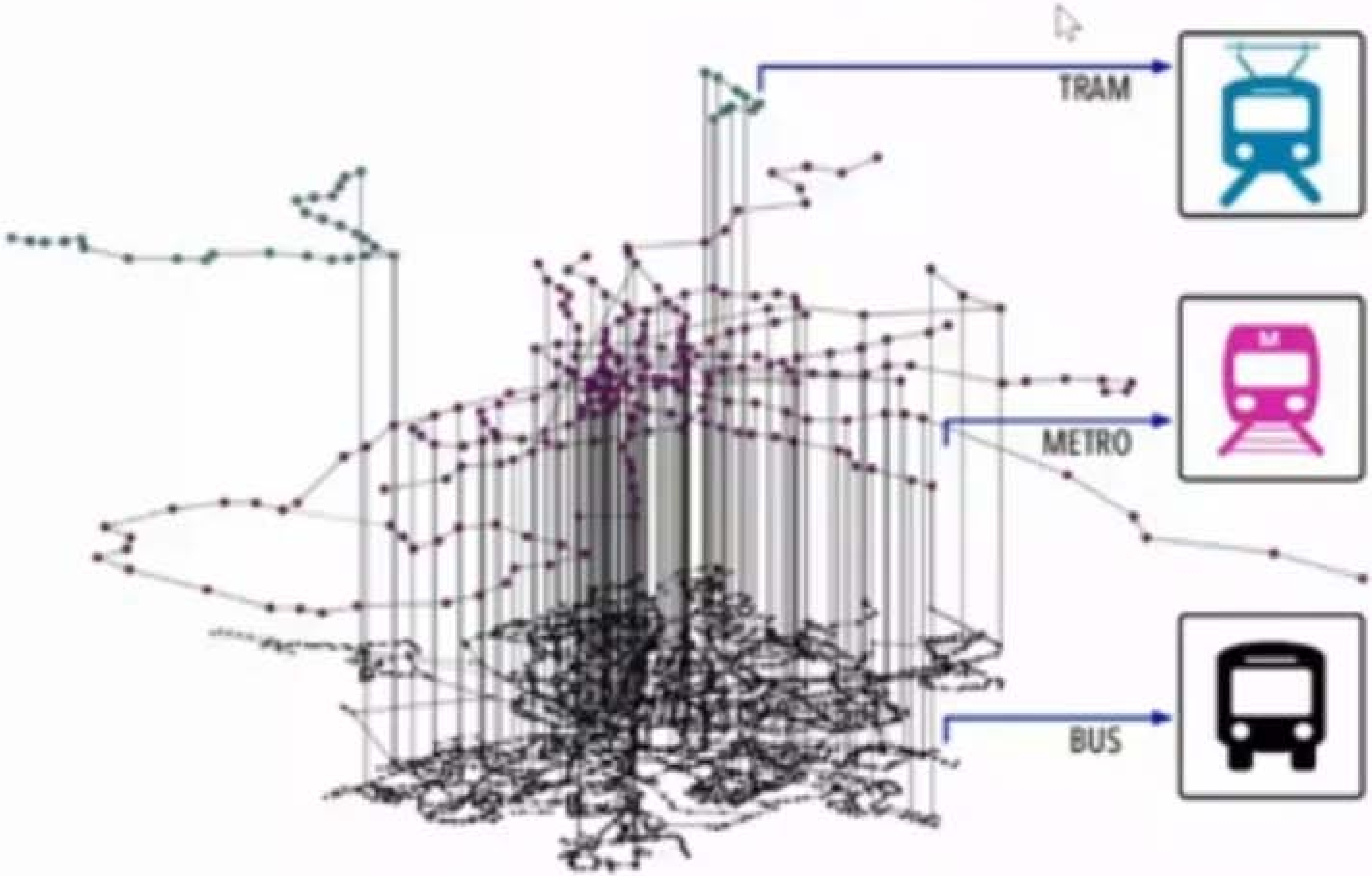
初期——疾病信息的传播会极大阻碍疾病传播

中期——动力学很复杂

爆发——疾病爆发加速信息息传播；信息传播阻碍疾病传播

[1] C. T. Bauch, A. P. Galvani, Social Factors in Epidemiology. *Science*, 2013, 342:47-49.

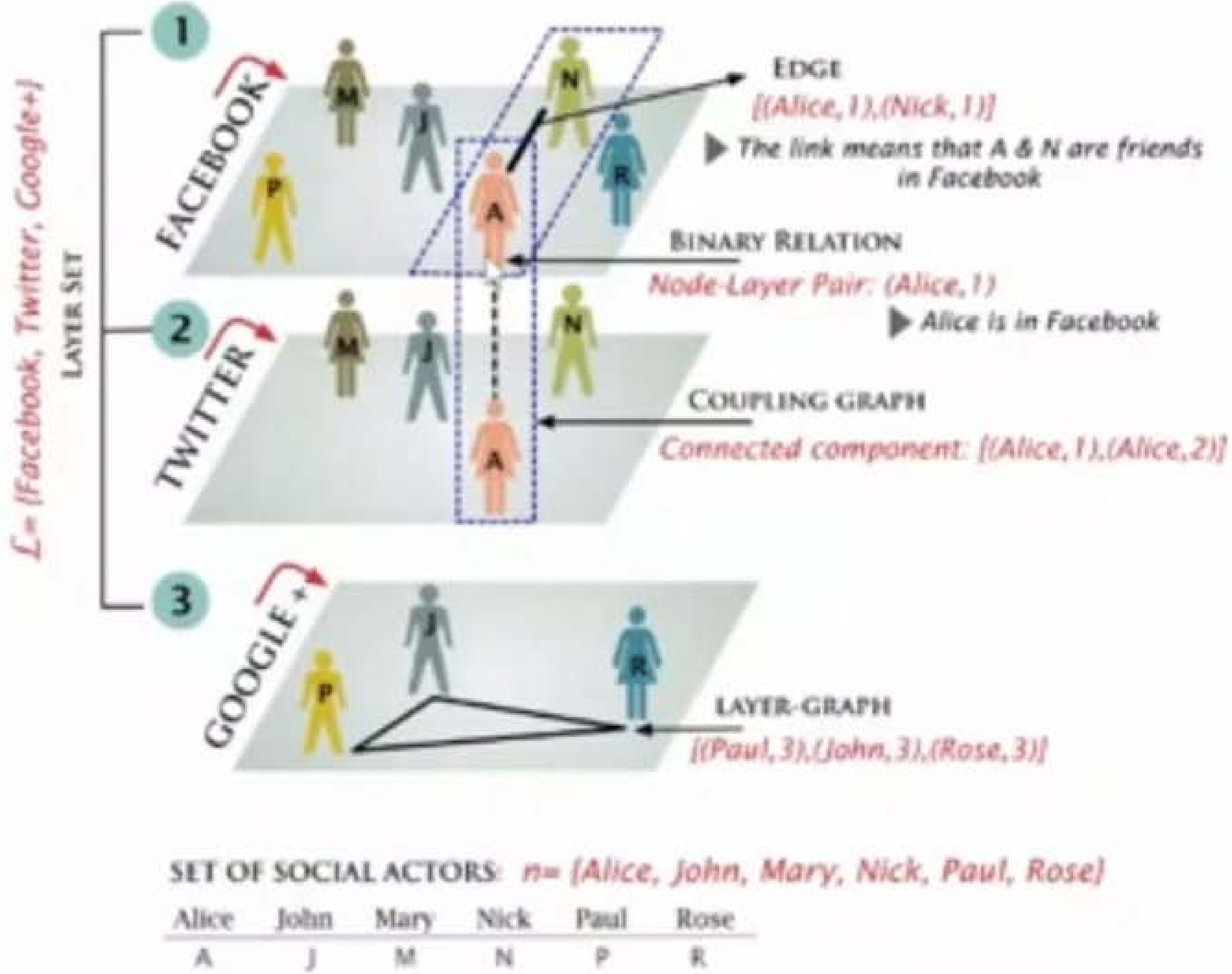
多层交通网络



[1] A. Aleta, S. Meloni, Y. Moreno. A multilayer perspective for the analysis of urban transportation systems. *Scientific Reports*, 2017, 7:44359.

[2] A. Aleta, Y. Moreno. Multilayer networks in a nutshell. *Annual Review of Condensed Matter Physics*, 2019, 10:45-62.

多层社交网络



层内连边:
同一媒介内
好友信息传播

层间连边:
自身不同媒介
间信息传播

多层网络

信息传播加快

[1] G. F. de Arruda, F. A. Rodrigues, Y. Moreno. Fundamentals of spreading processes in single and multilayer complex networks. *Physics Reports*, 2018, 756: 1-59.

多层网络结构分类

- 固定结构网络
  - 相互连接网络 (Interconnected network)
    - 多重网络 (多路复用网络, Multiplex network)
    - ……
    - 一般相互连接网络
  - 超网络 (Hypernetwork)
- 相互依存网络 (Interdependent network)
- 自适应网络 (Adaptive network)
- 时序网络 (Temporal network)

……

[1] 陆君安. 从单层网络到多层网络——结构、动力学和功能. *现代物理知识*. 27(4): 3-8, 2015.

[2] 吴宗柠. 狄增如. 樊瑛. 多层网络的结构与功能研究进展. *电子科技大学学报*. 50(1): 106-120, 2021.

多层网络结构分类

- 固定结构网络
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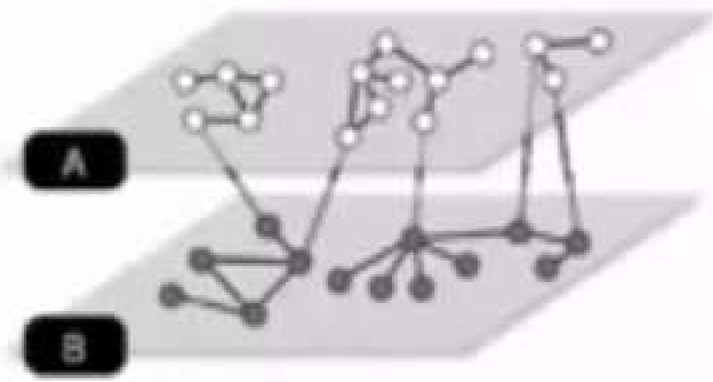
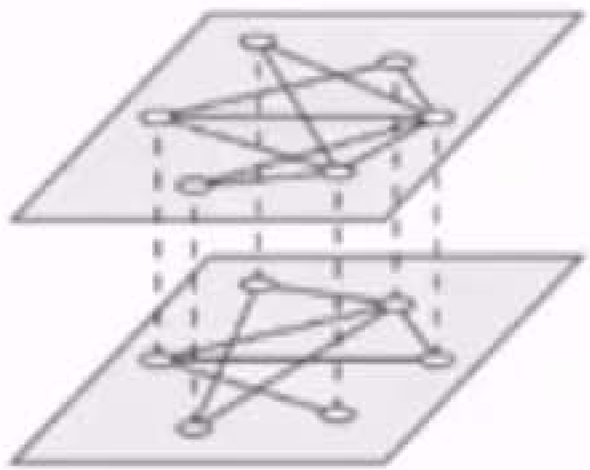
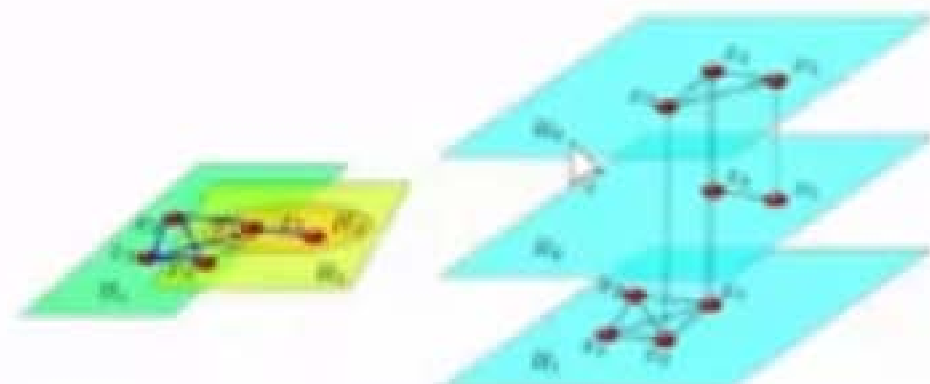
    - ……
    - 一般相互连接网络
  - 超网络 (Hypernetwork)



相互连接网络



多重网络



超网络

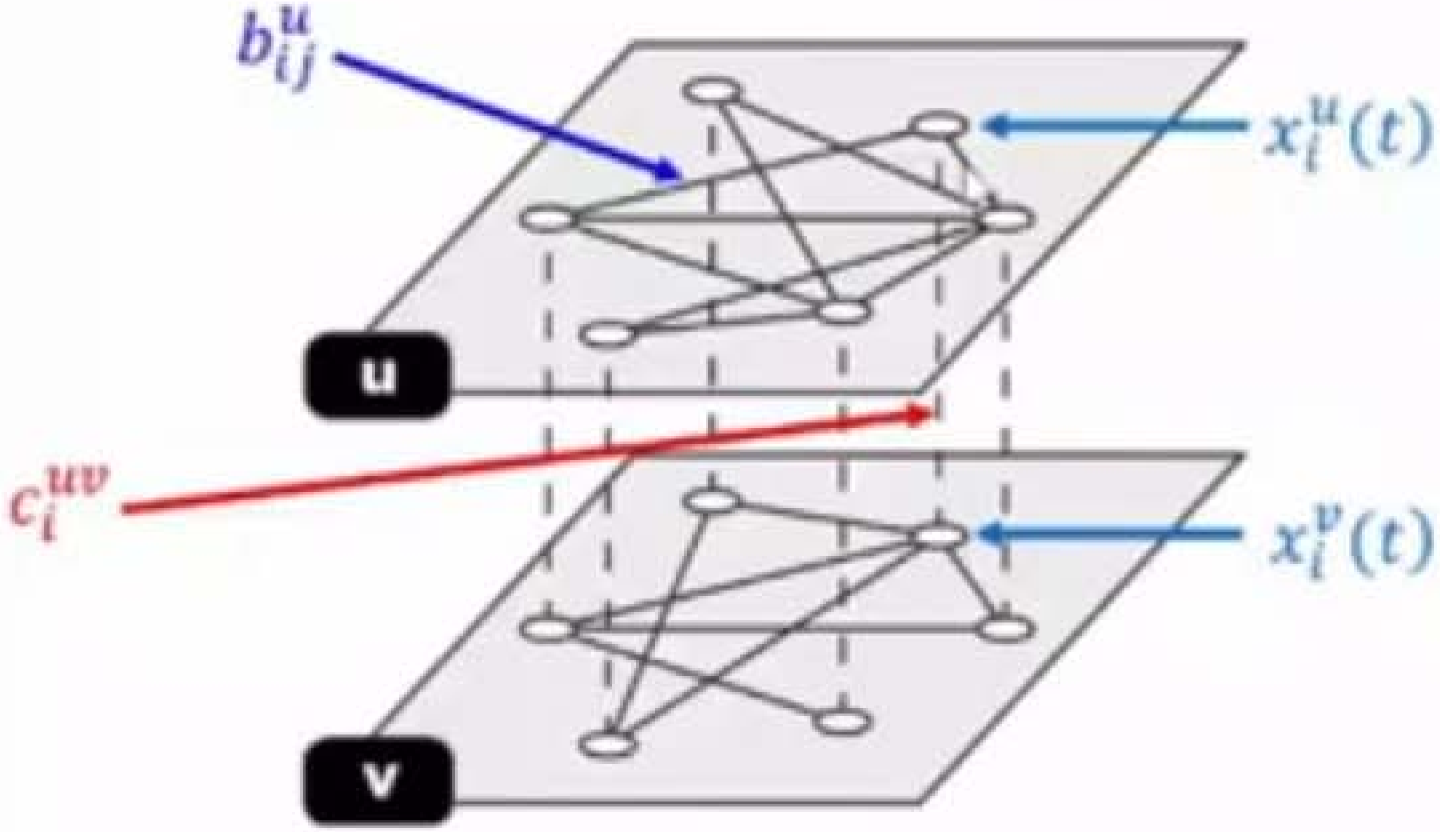
多重网络模型

$$\dot{x}_i^u(t) = f^u(x_i^u(t)) + \sum_{j=1, j\neq i}^{N} b_{ij}^u \Gamma^u (x_j^u(t) - x_i^u(t)) + \sum_{v=1, v\neq u}^{M} c_i^{uv} \Gamma^{uv} (x_i^v(t) - x_i^u(t))$$

$$\Rightarrow \quad \dot{x}_i^u(t) = f^u(x_i^u(t)) - \sum_{j=1}^{N} \bar{b}_{ij}^u \Gamma^u (x_j^u(t)) + \sum_{v=1, v\neq u}^{M} c_i^{uv} \Gamma^{uv} (x_i^v(t) - x_i^u(t))$$

$$i = 1, 2, \cdots, N$$

$$u = 1, 2, \cdots, M$$



$$\bar{b}_{ij}^u := \begin{cases} d_i^u, & i = j \\ -b_{ij}^u, & i \neq j \end{cases},$$

$$d_i^u = \sum_{j\neq i} b_{ij}^u,$$

网络模型

每层N个节点的M层多重网络的扩散方程：

$$\frac{dx_i^K}{dt} = D_K \sum_{j=1}^{N} w_{ij}^K (x_j^K - x_i^K) + \sum_{L=1}^{M} D_{KL} (x_i^L - x_i^K),$$

$$i = 1, \ldots, N, K = 1, \ldots, M.$$

假设：

每层网络结构**双向**、连通、可加权——层内Laplacian对称、不可约

w_{ij}^K——第K层中节点 i 和节点 j 间的权重

D_k ——第K层的层内耦合强度

D_{KL}——第K层和第L层的层间耦合强度