Cubes and Towers

Submit on Dec 12th and 14th, 2017
Course slides are online https://weidongshao.github.io/presos/cubes.html.
YouTube Video Link https://youtu.be/sQ3FSFOEpLo

1. How many 1-cm cubes are needed to build a cube with an edge of 8-cm? What is the sum of lengths of all edges?

   Answer: In the lecture, we noticed that the number of cubes can be calculated as

   $$\#of cubes = width * height * depth = 8 \times 8 \times 8 = 512$$

   There are 12 edges in a cube. Each edge has a length of 8cm. So the sum of length of all edges is

   $$12 \times 8cm = 96cm$$

2. A rectangular box has 4cm in width, 12cm in height, and 8 in depth. What is the sum of lengths of all edges? How many 1-cm cubes are needed to build this box?

   Answer: We use the same formula as above,

   $$\#of cubes = width * height * depth = 4 \times 12 \times 8 = 384$$

   There are 12 edges in a rectangular box. The edges, however, are not of equal length. Instead,

   - 4 edges have a length of 4cm.
   - 4 edges have a length of 12cm.
   - 4 edges have a length of 8cm.

   So the sum of length of all edges is

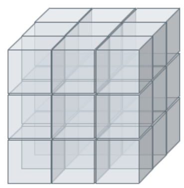
   $$4 \times 4cm + 4 \times 12cm + 4 \times 8cm = 96cm$$

   Please note that, total edge length is also 96cm, same as that of the cube in Problem 1. But the cube in Problem 1 has a larger volumes (or more number of unit cubes).

3. **Reference Problem:** A wooden cube that measures 3 cm along each edge is painted red. The painted cube is then cut into 1-cm cubes as shown in the diagram.

   (a) How many of 1-cm cubes do not have red paint on any face?

   (b) How many of 1-cm cubes that only have 1 face painted red?



Answer:

   (a) Only the 1-cm cube in the middle is not colored. All other 1-cm cubes have at least one face painted. So the answer is 1.

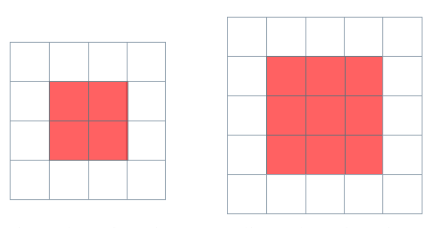
   (b) The middle 1-cm cube of each face has just one red face. There are 6 faces of the 3-cm cube. Hence 6 1-cm cubes have red paint on just 1 face.

   Repeat the above questions for 4-cm edge length

   Answer: In this type of problem, please note that only the cubes in the most outfacing layers has some faces painted. Depending on the position of the unit cubes in the layer, the unit cube might have either 1, 2, or 3 faces painted.

   (a) The unit-cubes in the inner (center) $2cm \times 2cm \times 2cm$ cube are not painted at all. So there are 8 unit(1-cm) cubes that do not have red paint on any face.

   (b) On each of the 6 faces of cube, there are 4x4=16 unit cubes. Only the ones in the center have one face painted in red. See part a) of the below figure. There are 4 such cubes (from one face of the big cube). In total, there are $6 \times 4 = 24$ cubes that has just one face painted in red.



Part b) of Problems 4 and 5

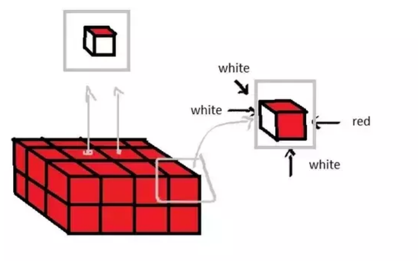
4. Repeat the above problem for 5-cm edge length

   Answer:

   (a) The unit-cubes in the inner (center) $3cm \times 3cm \times 3cm$ cube are not painted at all. So there are 27 unit(1-cm) cubes that do not have red paint on any face.

   (b) On each of the 6 faces of cube, there are $5 \times 5 = 25$ unit cubes. Only the ones in the center have one face painted in red. See part b) of the below figure. There are 9 such cubes (from one face of the big cube). In total, there are $6 \times 9 = 54$ cubes that has just one face painted in red.

5. A rectangular $4 \times 3 \times 2$ block has its surface painted red and is cut into cubes with each edge equaling 1 unit.



   (a) How many cubes have exactly one face painted red?

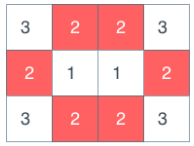
   (b) How many cubes have exactly two faces painted red?

   (c) How many cubes have exactly three faces painted red?

   Answer: Note that in this problem, there are total $4 \times 3 \times 2 = 24$ unit cubes. All of them have at least one face painted.

   (a) Only 2 cubes on the top and 2 cubes on the bottom layer have exactly one face painted red. Total 4 cubes have exactly one face painted red.

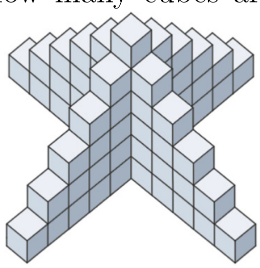
   (b) On the top layer, there are 6 cubes that has 2 faces painted. See illustration in the figure below. On the bottom layer, there are also 6 cubes that has 2 faces painted. Total 12 cubes.



16 cubes as seen from the top.
The colored ones have 2 faces painted red.
The ones marked with 3 have 3 faces painted red.
The ones marked with 1 has 1 face painted red.

(c) The cubes at the corner have exactly three faces painted red. There are total 8 cubes.

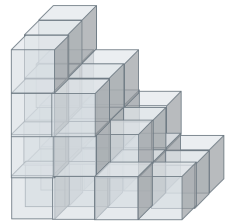
6. How many cubes are needed to build a tower like this, but 12 cubes high?



Answer: The key is to identify the pattern in the number of cubes in each layer of the tower. As discussed in the class, in this tower, the top layer has 1 cube, and each layer has 4 more cubes than the layer above it. Use the following table, we know that 276 cubes are needed to build a tower with 12 cubes high.

Layer	# of Cubes	Subtotal (from the top layer)
1 (top)	1	1
2	5	6
3	9	15
4	13	28
5	17	45
6	21	66
7	25	91
8	29	120
9	33	153
10	37	190
11	41	231
12	45	276

7. The builder wants to paint the staircase he has built. Each pint of paint can paint one cube on all faces. i.e. 6 faces. How many pints of paint does he



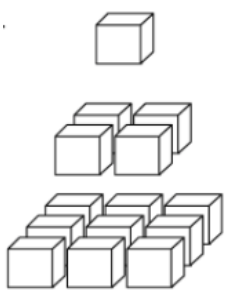
need? (note you dont have to paint the bottom face).

Answer: We can look at the tower and find the number of faces that need to be painted layer by layer

Layer	# of Faces to be Painted	Subtotal (from the top layer)
1 (top)	11	11
2	13	24
3	15	39
4	17	56

Total 56 faces need to be painted. Each pint of paint can paint 6 faces. So the builder needs 10 pints of paint to pain the stair.

8. Kelly is building a tower with cubes. He wants the top of the tower to have only one cube. The next layer down will have four cubes. The third layer from the top will have nine cubes. Kelly wants to make the tower 10 stories high. How many cubes does he need to use for the base of the tower? Hint: Each layer has an edge length increased by 1 from the layer above.



Answer: Let's find the pattern in the number of cubes in each layer of the tower. The top layer has 1 cube, and 2nd layer has $2 \times 2 = 4$ cubes. The 3rd layer from the top has $3 \times 3 = 9$ cubes. Use the following table, we know that 276 cubes are needed to build a tower with 12 cubes high.

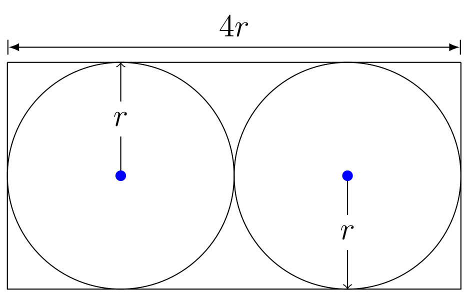
Layer	# of Cubes	Subtotal (from the top layer)
1 (top)	1	1
2	$2\times2 = 4$	5
3	$3\times3 = 9$	14
4	$4\times4 = 16$	30
5	$5\times5 = 25$	55
6	$6\times6 = 36$	91
7	$7\times7 = 49$	140
8	$8\times8 = 64$	204
9	$9\times9 = 81$	285
10	$10\times10 = 100$	385

9. A rectangular prism can fit two spherical balls, each having radius r. Each ball touches 5 sides of the rectangular prism. What is the volume of the prism, in terms of r?

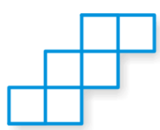
   Answer:

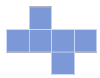
   We need to figure out the dimension of the prism and express it in terms of r. It is helpful to look at the geometry of the cross section of the prism. (Note: A cross section is the shape we get when cutting straight through an object.) One cross section is illustrated in the following diagram with the enclosing rectangular of dimension $4r$-by-$2r$.

   The prism has a dimension of $4r \times 2r \times 2r$. So its volume is $16r^3$



10. Each of the following shapes below can be folded into a cube (along the edges). Do you see it? In your mind, try to figure out how it happens.





Note: A good web resource is from this link:
https://illuminations.nctm.org/activity.aspx?id=3544