Д/З №6

787 $\dot{x} = x - y, \quad \dot{y} = y - 4x$

За методом Ейлера:

$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}, \quad A, B, \lambda$ — cons

$A\lambda = A - B, \quad B\lambda = B - 4A$

$$\det \begin{vmatrix} \lambda - 1 & 1 \\ 4 & \lambda - 1 \end{vmatrix} = 0$$

або $(\lambda - 1)^2 - 4 = 0$

$\lambda_1 = 3 \quad \lambda_2 = -1$ прості корені

$x_1 = A_1 e^{3t}, \quad x_2 = A_2 e^{-t}$

$y_1 = B_1 e^{3t} \quad y_2 = B_2 e^{-t}$

$A_1 \lambda_1 = A_1 - B_1, \quad A_2 \lambda_2 = A_2 - B_2$

Оскільки A_1 і A_2 — довільні, то, припустимо, що $A_1 = A_2 = 1$. Тоді

$B_1 = -2, \quad B_2 = 2$

$$X(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ -2e^{3t} & 2e^{-t} \end{pmatrix}$$

$$x(t) = \begin{pmatrix} x \\ y \end{pmatrix} = X(t)C = \begin{pmatrix} e^{3t} & e^{-t} \\ -2e^{3t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 e^{3t} + C_2 e^{-t} \\ -2C_1 e^{3t} + 2C_2 e^{-t} \end{pmatrix} \Longrightarrow$$

$\Rightarrow x = C_1 e^{3t} + C_2 e^{-t}, \quad y = -2C_1 e^{3t} + 2C_2 e^{-t}$

790 $\begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = 0$

$1 - \lambda = \pm 3i$

$\lambda = 1 \pm 3i$

Власний вектор:

$$\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3ix_1 = 3x_2, \quad x_1 = ix_2$

Отже, як власний вектор можна взяти $\begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\begin{cases} x = ie^{(1+3i)t} \\ y = e^{(1+3i)t} \end{cases} \Rightarrow \begin{cases} x = ie^{t}(\cos 3t + i\sin 3t) \\ y = e^{t}(\cos 3t + i\sin 3t) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = ie^{t}\cos 3t - e^{3t}\sin 3t \\ y = e^{t}\cos 3t + ie^{t}\sin 3t \end{cases}$$

Загальний розв'язок:

$$\begin{cases} x = e^{t}(C_1 \cos 3t - C_2 \sin 3t) \\ y = e^{t}(C_1 \sin 3t + C_2 \cos 3t) \end{cases}$$

(793) $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 4x - y \end{cases}$

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}; \quad \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$-(3-\lambda)(1+\lambda) + 4 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda_{1,2} = 1, \text{ то}$$

$$x = (\alpha_1 + \beta_1 t)e^t$$

$$y = (\alpha_2 + \beta_2 t)e^t$$

$$\alpha_1 + \beta_1 t + \beta_1 = 3\alpha_1 + 3\beta_1 t - \alpha_2 - \beta_2 t$$

$$\alpha_2 + \beta_2 t + \beta_2 = 4\alpha_1 + 4\beta_2 t - \alpha_2 - \beta_2 t$$

$$\begin{cases} \beta_1 = 3\beta_1 - \beta_2 \\ \alpha_1 + \beta_1 = 3\alpha_1 - \alpha_2 \\ \beta_2 = 4\beta_1 - \beta_2 \\ \alpha_2 + \beta_2 = 4\alpha_1 - \alpha_2 \end{cases} \qquad \begin{aligned} \beta_2 &= 2\beta_1 \\ \beta_1 &= 2\alpha_1 - \alpha_2 \\ \beta_2 &= 4\alpha_1 - 2\alpha_2 \end{aligned}$$

$$\alpha_1 = C_1, \quad \alpha_2 = C_2, \quad \beta_1 = 2C_1 - C_2, \quad \beta_2 = 2(2C_1 - C_2)$$

$$x = (C_1 + (2C_1 - C_2)t)e^t$$

$$y = (C_2 + 2(2C_1 - C_2)t)e^t$$

(797) $\begin{cases} \dot{x} = x - 2y - z \\ \dot{y} = -x + y + z \\ \dot{z} = x - z \end{cases}$ $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\lambda = 0$ $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

$\Rightarrow \begin{matrix} \alpha = \gamma = 1 \\ \beta = 0 \end{matrix}$ $\begin{cases} x_1 = C_1 \\ y_1 = 0 \\ z_1 = C_1 \end{cases}$

$\lambda = 2$ $\begin{pmatrix} -1 & -2 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha = 3 \\ \beta = -2 \\ \gamma = 2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x_2 = 3C_2 e^{2t} \\ y_2 = -2C_2 e^{2t} \\ z_2 = C_2 e^{2t} \end{cases}$

$\lambda = -1$ $\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = -1 \\ \gamma = 2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x_3 = 0 \\ y_3 = -C_1 e^{-t} \\ z_3 = 2C_3 e^{-t} \end{cases}$

$\begin{cases} x = C_1 + 3C_2 e^{2t} \\ y = -2C_2 e^{2t} - C_3 e^{-t} \\ z = C_1 + C_2 e^{2t} + 2C_3 e^{-t} \end{cases}$

801 $\begin{cases} x' = x - y - z \\ y' = x + y \\ z' = 3x + z \end{cases}$ $\begin{cases} x' - x + y + z = 0 \\ y' - x - y = 0 \\ z' - 3x - z = 0 \end{cases}$ $X = y' - y$ $\begin{cases} y'' - 2y' + 2y + z = 0 \\ z' - 3y' + 3y + z = 0 \end{cases}$

$\begin{cases} x = y' - y \\ y = -y'' + 2y' - 2y \\ y''' - 3y'' + 7y' - 5y = 0 \end{cases}$

$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$

	1	-3	7	-5
1	1	-2	5	0

$\lambda_1 = 1$, $\lambda^2 - 2\lambda + 5 = 0$

$\lambda_1 = -4$ $\lambda_{2,3} = 1 \pm 2i$

$y = (C_1 + C_2 \cos 2t + C_3 \sin 2t) e^t$

$y' = (C_1 + (C_2 + 2C_3) \cos 2t + (C_3 - 2t_2) \sin 2t) e^t$

$y'' = (C_1 + (4C_3 - 3C_2) \cos 2t - (4C_2 + 3C_3) \sin 2t) e^t$

$x = (2C_3 \cos 2t - 2C_2 \sin 2t) e^t$, $z = (-C_2 + 3\cos t +$

$+ 3C_2 \sin 2t) e^t$

$x = (2C_3 \cos 2t - 2C_2 \sin 2t) e^t$

$y = (C_1 + C_2 \cos 2t + C_3 \sin 2t) e^t$

$z = (-C_1 + 3C_2 \cos 2t + 3C_3 \sin 2t) e^t$

809 $\begin{cases} \dot{x} = y - 2x - x \\ \dot{y} = 4x + y \\ \dot{z} = 2x + y - z \end{cases}$ $\lambda_1 = 1$

$\lambda_{2,3} = -1$

$A = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$

$\underline{\lambda = 1}$:

$$\begin{pmatrix} -2 & 1 & -2 \\ 4 & 0 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2V_{11} + V_{12} = 2V_{13} \\ 4V_{11} = 0 \\ 2V_{11} + V_{12} = 2V_{13} \end{cases} \qquad \begin{cases} V_{11} = 0 \\ V_{12} = 2V_{13} \\ V_{13} = 2V_{13} \end{cases} \qquad \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda = -1$:

$$\cancel{\begin{pmatrix} -2 & 1 & -2 \\ 4 & 0 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$$\cancel{\begin{cases} -2V_{11} + V_{12} = 2V_{13} \\ 4V_{11} = 0 \\ 2V_{11} + V_{12} = 2V_{13} \end{cases} \quad V_{11} = 0}$$

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} V_{12} = 2V_{13} \\ 4V_{11} = -2V_{12} \\ 2V_{11} = -V_{12} \end{cases} \qquad \begin{aligned} V_{11} &= 1 \\ V_{12} &= -2 \\ V_{13} &= -1 \end{aligned} \qquad \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

$$\begin{cases} V_{12} - 2V_{13} = 1 \\ 4V_{11} + 2V_{12} = -2 \\ 2V_{11} + V_{12} = -1 \end{cases} \qquad \begin{aligned} V_{11} &= 0 \\ V_{12} &= -1 \\ V_{13} &= -1 \end{aligned} \qquad \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

В-дь: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + C_3 e^{-t} \cdot$

$\cdot \left(t \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right) = C_1 e^{t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} t \\ -2t- \\ -t-1 \end{pmatrix}$

(826) $\begin{cases} \dot{x} = y + 2e^{t} \\ \dot{y} = x + t^{2} \end{cases}$

одн. с-ма:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$$

$|A - \lambda E| = 0$, $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$,

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$.

$-\xi_1 + \xi_2 = 0$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $x_1 = e^{t}$

$\xi_1 = \xi_2$ $y_1 = e^{t}$.

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$

$\xi_2 = -\xi_1$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $x_2 = e^{-t}$

$y_2 = -e^{-t}$

$x_0 = C_1 e^{t} + C_2 e^{-t}$

$y_0 = C_1 e^{t} - C_2 e^{-t}$

$f_2(t) = t^2$

1 a) $x_{ч2} = At^2 + Bt + D$

$y_{ч2} = \widehat{A}t^2 + \widetilde{B}t + \widehat{D}$

$\begin{cases} a + at + b = ct + d + 2 \\ c + ct + d = at + b \end{cases}$

$a = c$

$a + b = d + 2$

$c = a$

$c + d = b$

$a = 1$, $b = d + 1$, $c = 1$, $b = 0$, тоді $d = -1$

$z_{21} = te^t$

$y_{21} = (t-1)e^t$

$$\begin{cases} 2At + B = \tilde{A}t^2 + \tilde{B}t + \tilde{D} \\ 2\tilde{A}t + \tilde{B} = At^2 + Bt + D + t^2 \end{cases}$$

$\tilde{A} = 0$ $\quad A = 0$

$2A = \tilde{B}$ $\quad B = 0$

$B = \tilde{D}$ $\quad \tilde{D} = 0$

$A + 1 = 0$ $\quad A = -1$

$2\tilde{A} = B$ $\quad \tilde{B} = -2$

$\tilde{B} = D$ $\quad D = -2$

$x_{22} = -t^2 - 2$

$y_{22} = -2t$

В-дь: $\begin{cases} x = C_1e^t + C_2e^{-t} + te^t - t^2 - 2 \\ y = C_1e^t - C_2e^{-t} + (t-1)e^t - 2t \end{cases}$

834 $\begin{cases} \dot{x} = x + 2y \\ \dot{y} = x - 5\sin t \end{cases}$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. $\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0$

$-\lambda + \lambda^2 - 2 = 0$

$\lambda_1 = -1$

$\lambda_2 = 2$

1) $\lambda_1 = -1$

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

$\xi_1 = -\xi_2 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, то $x_1 = e^{-t}$

$y_1 = -e^{-t}$

2) $\lambda_2 = 2$

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

$\xi_1 = 2\xi_2 \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, то $x_2 = 2e^{2t}$

$y_2 = e^{2t}$

$x = C_1 e^{-t} + 2C_2 e^{2t}$

$y = -C_1 e^{-t} + C_2 e^{2t}$

$x_{21} = A\cos t + B\sin t$

$y_{21} = C\cos t + D\sin t$, то

$$\begin{cases} -A\sin t + B\cos t = A\cos t + B\sin t + 2C\cos t + 2D\sin t \\ -C\sin t + D\cos t = A\cos t + B\sin t - 5\sin t \end{cases}$$

$$\begin{cases} -A = B + 2D \\ B = A + 2C \\ -C = B - 5 \\ D = A \end{cases} \qquad \begin{cases} D = A \\ B = -3A \\ B = A + 2C \\ B = 5 - C \end{cases}$$

$2C = -4A \qquad C = -2A$

$-3A = 5 - C \qquad -3A = 5 + 2A$, $\quad A = -1$

$C = 2$

$D = -1$

$B = 3$

$x = C_1 e^{-t} + 2C_2 e^{2t} - \cos t + 3\sin t$

$y = C_1 e^{-t} + C_2 e^{2t} + 2\cos t - \sin t$

835 $\begin{cases} \dot{x} = 2x - 4y \\ \dot{y} = x - 3y + 3e^t \end{cases}$

одн. с-ма

$A = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$, $\det |A - \lambda E| = \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = 0$

$-(3+\lambda)(2-\lambda) + 4 = 0$

$-6 - 2\lambda + 3\lambda + \lambda^2 + 4 = 0$

$\lambda^2 + \lambda - 2 = 0$

$\lambda_1 = 1 \qquad \lambda_2 = -2$

1) $\lambda = 1$

$\begin{pmatrix} 1 & -4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$

$\xi_1 = 4\xi_2 \qquad \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, то $\quad x_1 = 4e^t$

$y_1 = e^t$

2) $\lambda = -2$

$\begin{pmatrix} 4 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$

$\xi_1 = \xi_2 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, то $\quad x_2 = e^{-2t}$

$y_2 = e^{-2t}$

$x_{21} = (a+b)e^t$

$y_{21} = (c+d)e^t$

$$\begin{cases} a + at + b = 2at + 2b - 4ct - 4d \\ c + ct + d = at + b - 3ct - 3d + 3 \end{cases}$$

$$\begin{cases} a + b = 2b - 4d \\ a = 2a - 4c \\ c + d = b - 3d + 3 \\ c = a - 3c \end{cases}$$

$a = 4c$

$a = b - 4d$

$c = b - 4d + 3$

$b = 0$, то

$4c = -4d$

$c = -4d + 3$

$c = 4c + 3$

$c = -1$, $a = -4$, $d = 1$, то

$x = 4C_1 e^{t} + C_2 e^{-2t} - 4te^{-t}$

$y = C_1 e^{t} + C_2 e^{-2t} - (t-1)e^{t}$

851 $\dot{x} = Ax$, $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$(3-\lambda)^2 = 0$

$\lambda = 3$ кратності 2

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 \cdot h_1 + 0 \cdot h_2 = 0 \\ 0 \cdot h_1 + 0 \cdot h_2 = 0 \end{cases} \Rightarrow$$

$\Rightarrow \forall$ вектор буде власним, візьмемо, наприклад $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

866 $\dot{x} = Ax$, $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & -1 & -1-\lambda \end{vmatrix} = 0$$

$-\lambda^3 = 0$, $\lambda = 0$ кр. 3

$\begin{cases} 2h_1 - h_3 = 0 \\ h_1 - h_2 = 0 \\ 3h_1 - h_2 - h_3 = 0 \end{cases}$ $\begin{cases} 2h_1 = h_3 \\ h_1 = h_2 \\ 3h_1 - h_2 - h_3 = 0 \end{cases}$ $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

порядок матр. $(A - \lambda E) = 3$

ранг $= 2$

к-сть власних вект. $= 3 - 2 = 1$

$\begin{cases} 2h_1 - h_3 = 1 \\ h_1 - h_2 = 1 \\ 3h_1 - h_2 - h_3 = 2 \end{cases}$ нехай $h_1 = 0 \Rightarrow \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

$\{ \ldots$ нехай $h_1 = 0 \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

В-дь: $x = C_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} t \\ t-1 \\ 2t-1 \end{pmatrix} + C_3 \begin{pmatrix} t^2/2 \\ t^2/2 - t + 1 \\ t^2/2 \cdot 2 - t \end{pmatrix}$

або

$$x = C_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} t \\ t-1 \\ 2t-1 \end{pmatrix} + C_3 \begin{pmatrix} t^2 \\ t^2 - 2t + 2 \\ 2t^2 - 2t \end{pmatrix}$$