Chapter 9

Root Locus Techniques

Outline

1 Complex variables
2 Characteristic equation
3 Rule to draw root locus
4 example

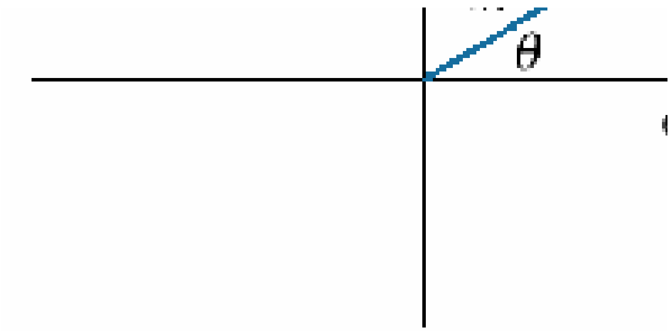
Complex variables

$$s = \sigma + j\omega \qquad s = M \measuredangle \theta \qquad s = M e^{j\theta}$$

When

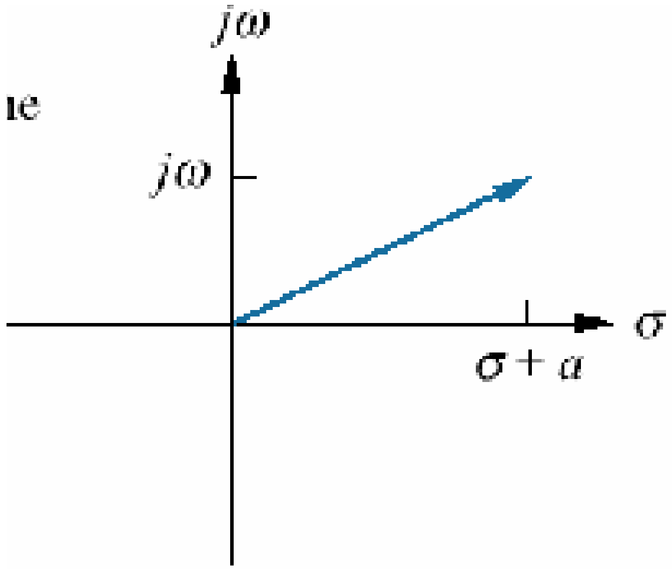
$$M = \sqrt{\sigma^2 + \omega^2} \qquad \theta = \tan^{-1}\frac{\omega}{\sigma}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$



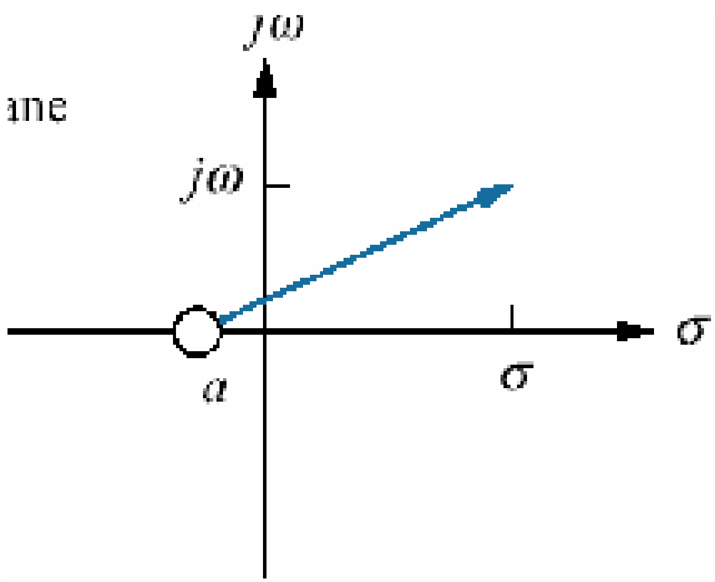
Vector representation of complex numbers :

$s = \sigma + j\omega$

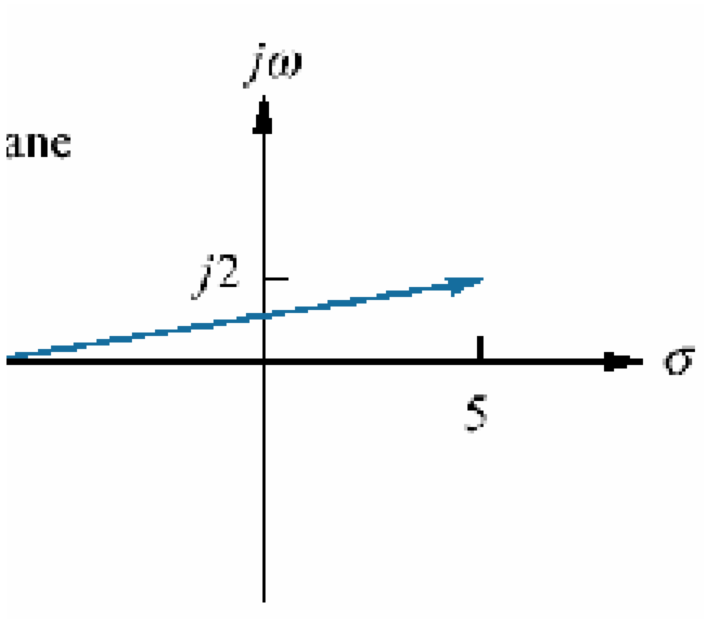


Vector representation of complex numbers :

(s + a)



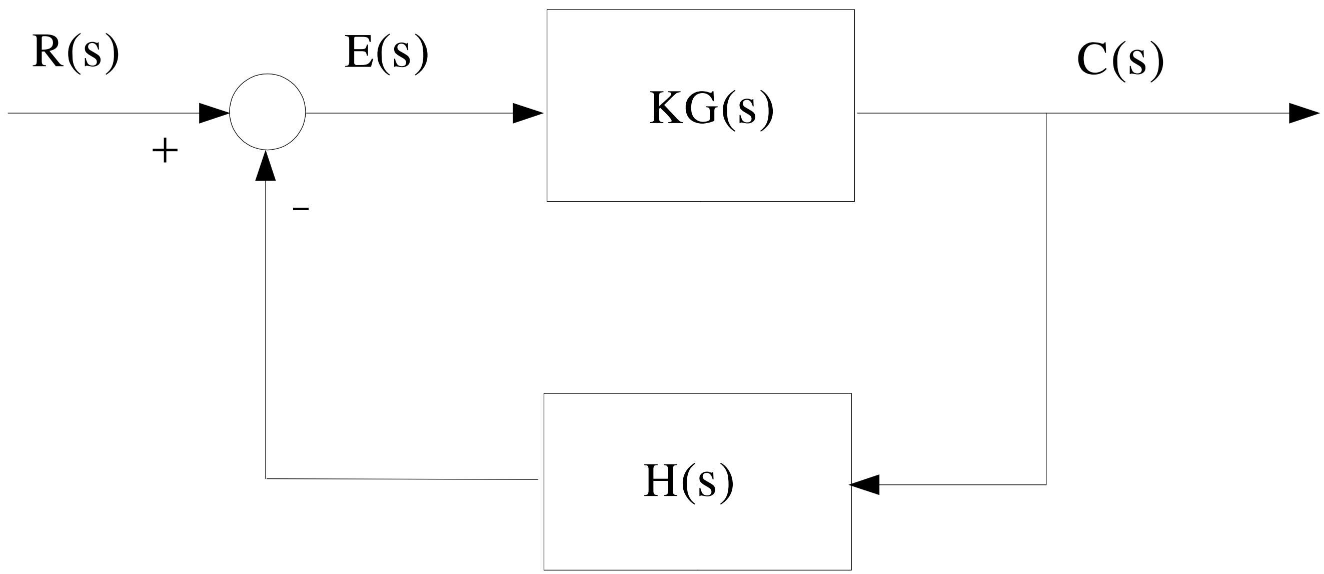
Alternate representation of : (s + a)



Vector representation of complex numbers :

$$(s + 7)|_{s5 + j2}$$

Characteristic equation



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

Char. Eq. $$1 + KG(s)H(s) = 0$$

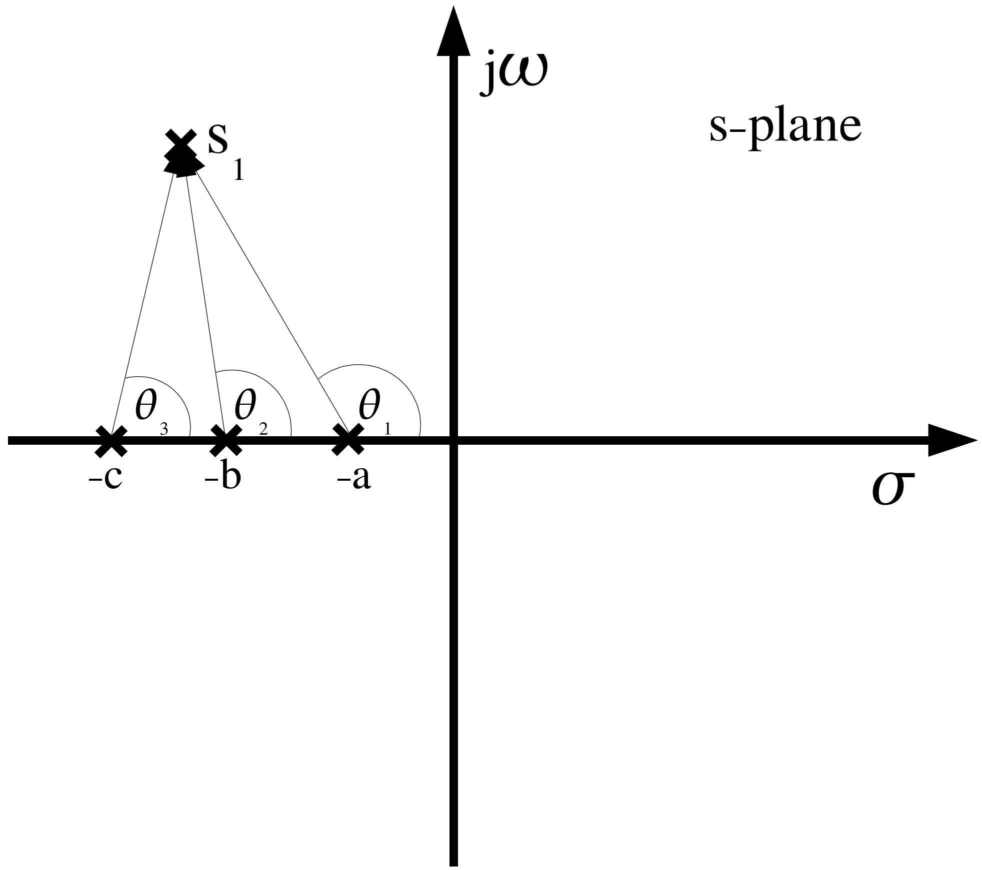
Characteristic Equation

$$1+KG(s)H(s) = 0$$

$$KG(s)H(s) = -1$$

G(s)H(s) is a complex function. We have :

$$\boxed{KG(s)H(s) = M \measuredangle \theta = 1 \left(2k+1\right)\pi}$$



$$\theta_1 + \theta_2 + \theta_3 = 180^\circ$$

$$|KG(s)H(s)| = 1$$

$$\measuredangle KG(s)H(s) = (2k+1)\pi$$

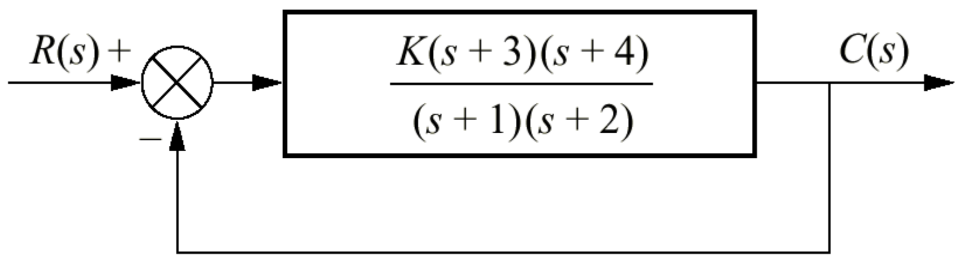
Value of K :

$$K = \frac{1}{|G(s)|\,|H(s)|}$$

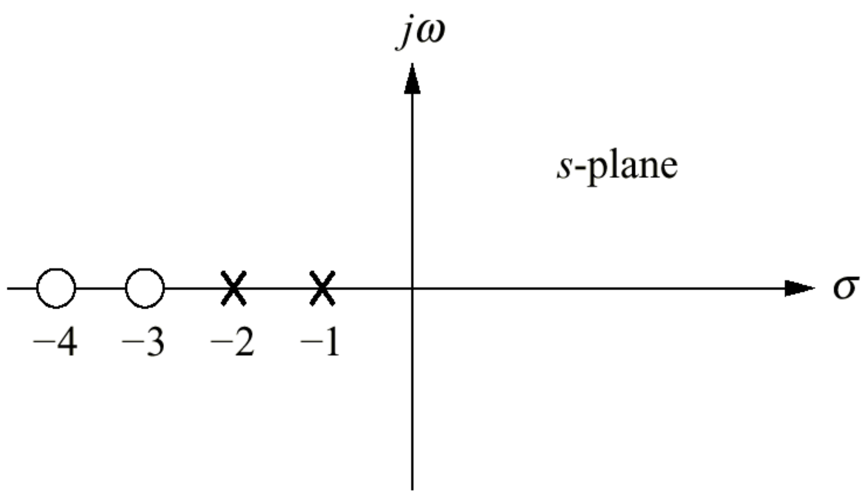
$$G(s) = K \frac{\prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_i)}$$

$$= \frac{K\,|s+z_1|\,|s+z_2|\ldots|s+z_m|}{|s+p_1|\,|s+p_2|\ldots|s+p_n|} \frac{e^{j(\theta_1+\ldots+\theta_m)}}{e^{j(\phi_1+\ldots+\phi_n)}}$$

Example (Nise)

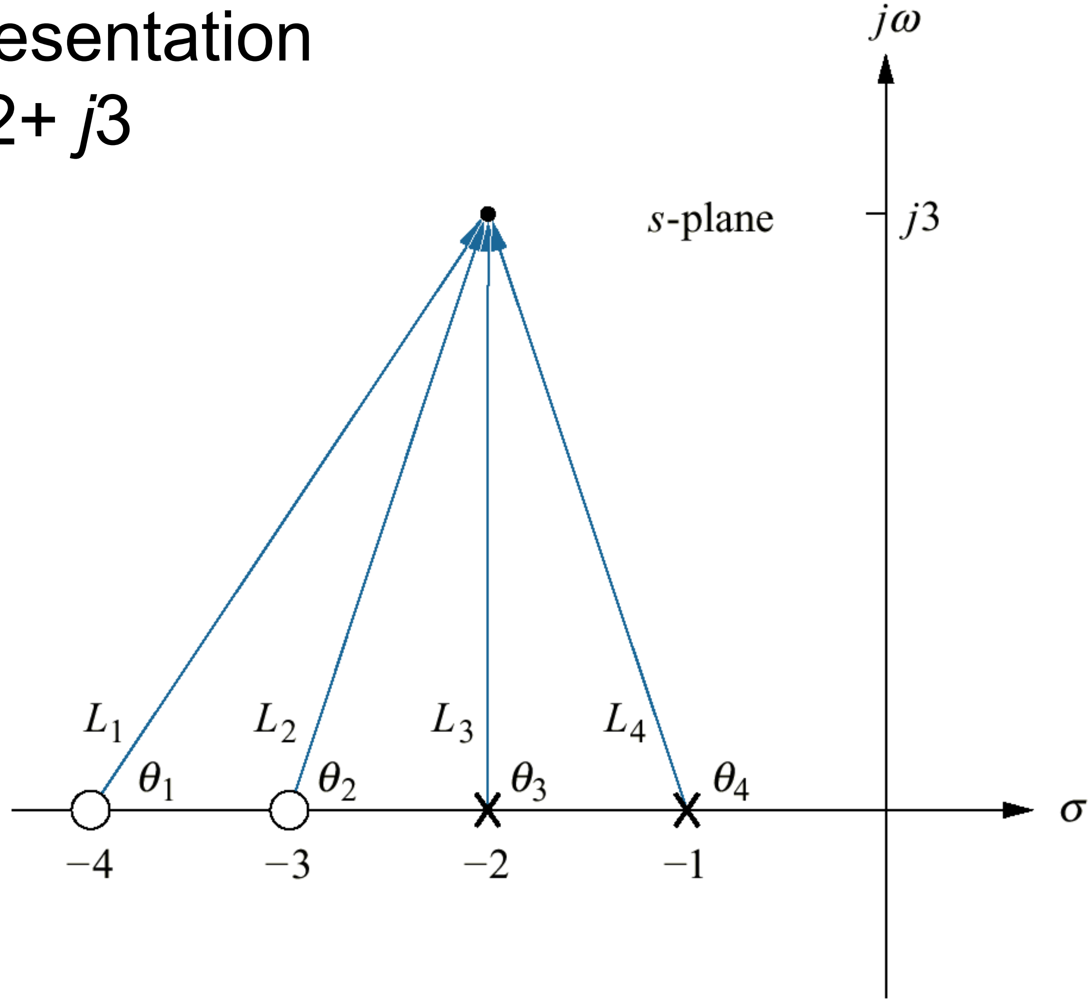


(*a*)



(*b*)

Vector representation of $G(s)$ at $-2+j3$



$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ$$

$$= -70.55^\circ$$

Point -2+j3 is not on root locus.

Point $-2+j\sqrt{2}/2$ is on the Root Locus. We have 180 degree of angle of Ch. Eq.

For the gain

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\Pi \text{ pole lengths}}{\Pi \text{ zero lengths}}$$

Sketching the root locus

1. Number of branch

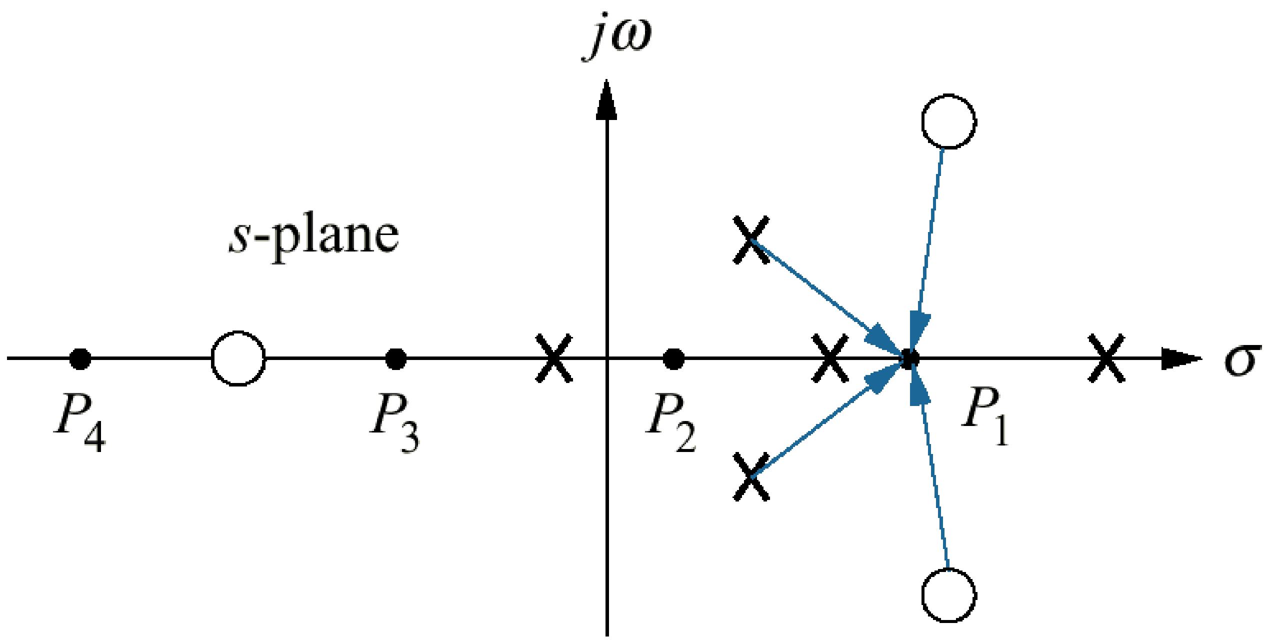
The number of branch of the root locus is equals to the number of the open-loop poles.

2. Symmetry

The root locus is symmetrical about the real axis.

3. Real-axis segments

On the real axis, for K>0, the root locus exists to the left of an odd number of real-axis finite open-loop poles and/or finite open-loop zeros.



Poles and zeros of a general open-loop system with test points, P_j, on the real axis

From the figure, we observe the following :

1 At each point p1, p2, p3, p4, the angular contribution of a pair of open-loop complex poles or zeros is zero.

2 The angular contribution of the open-loop poles and zeros to the left of the respective point is zero.

The angle at each point using only the open-loop, real-axis poles and zeros to the right of each point, we note :

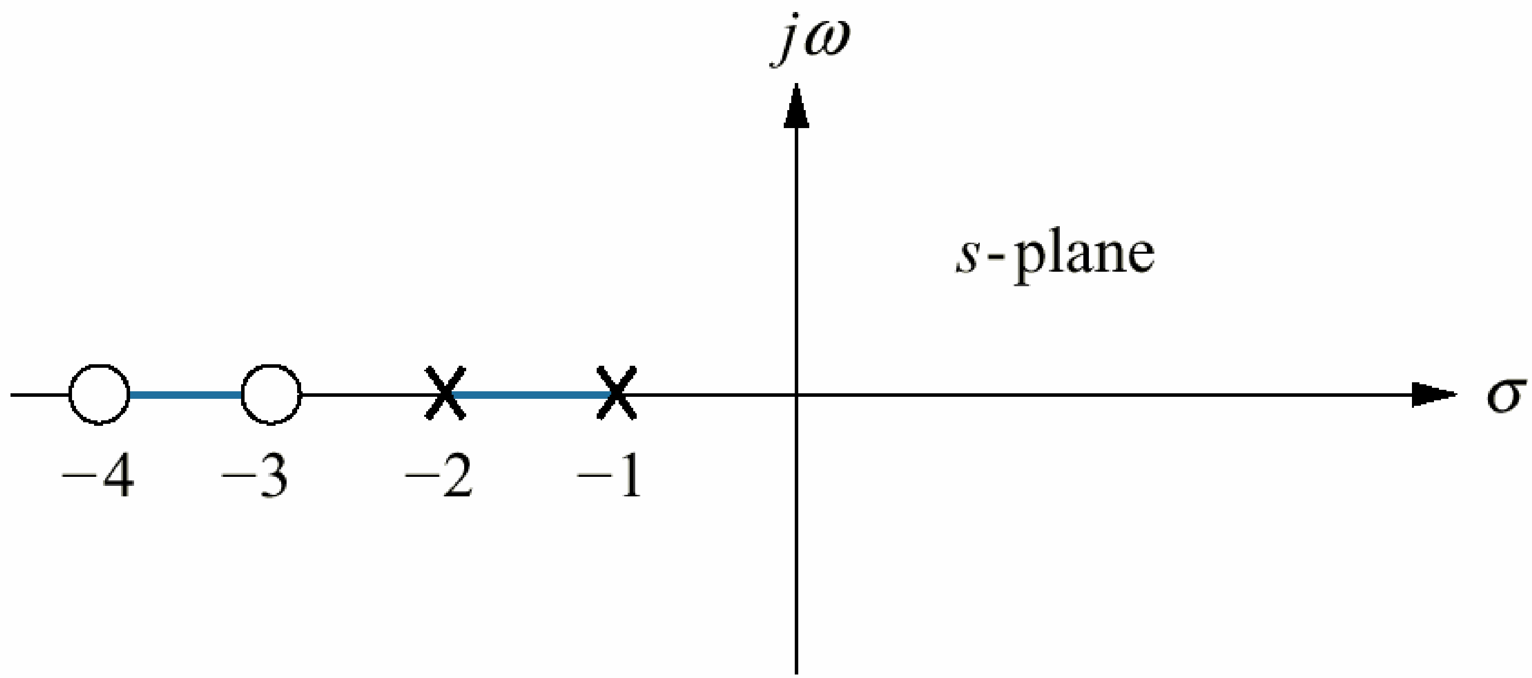
1 The angles on the real axis alternate between 0 and 180 degree.

2 the angles 180 degree on the real axis that exist to te left of an odd number of poles and/or zeros.

Example

$$G(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

Plot of Real-axis segments of the root locus



4. Starting and ending points

The root locus begins at the finite and infinite poles of G(s)H(s) and ends at the finite and infinite zeros of G(s)H(s).

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

Closed-loop Transfer function :

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$= \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

When K $\rightarrow$ 0 (Small gain)

$$T(s) \approx \frac{K N_G(s) D_H(s)}{D_G(s) D_H(s) + \epsilon}$$

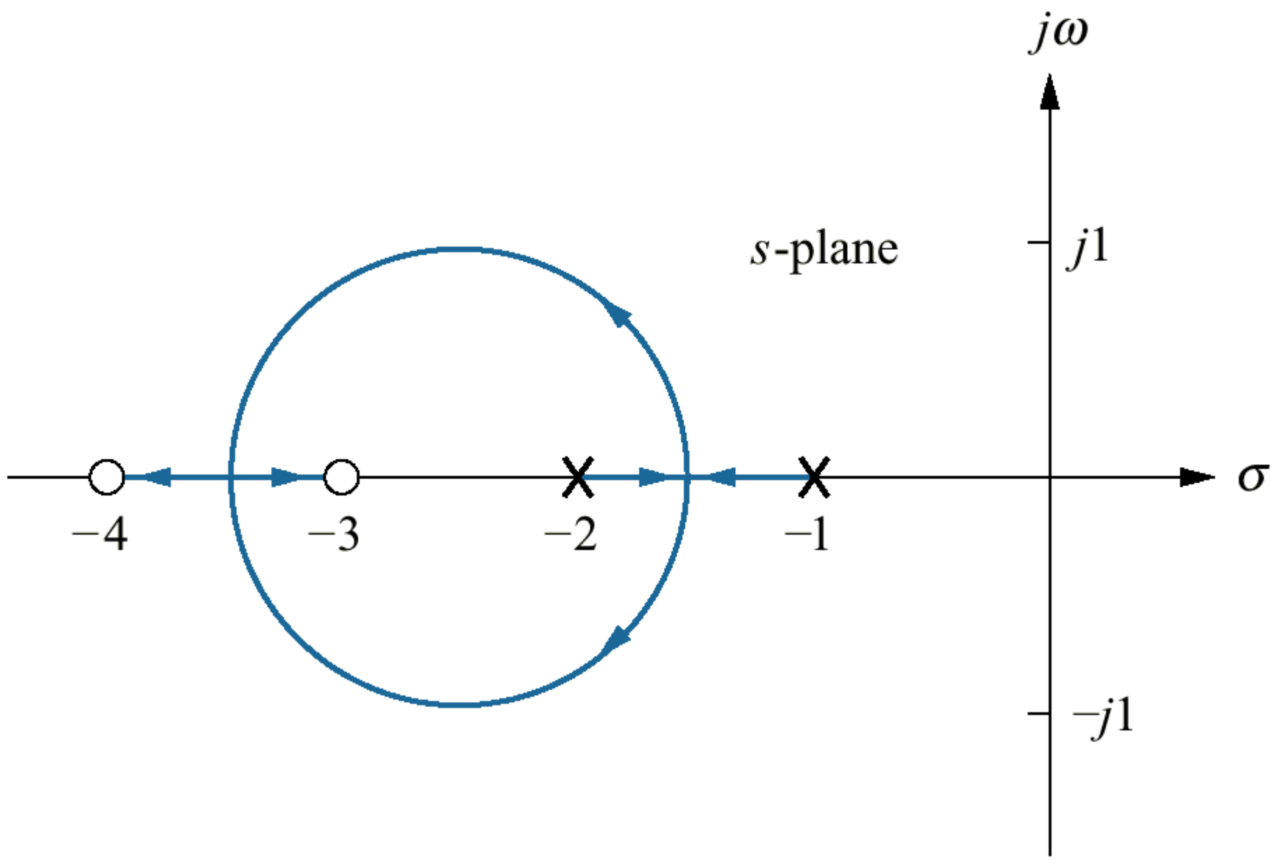
Closed-loop poles equal $D_G(s)D_H(s)$

When K $\rightarrow$ ∞ (High gain)

$$T(s) \approx \frac{KN_G(s)D_H(s)}{\epsilon + KN_G(s)N_H(s)}$$

Closed-loop poles equal $N_G(s)N_H(s)$

Complete root locus for the system example



5. Behavior at infinity

Every function of s has an equal number of poles and zeros.

Pole at infinity

If the function approches infinity as s approaches infinity, then the function has a pole at infinity.

Zero at infinity

If the function approaches zero as s approaches infinity, then the function has a zero at infinity.

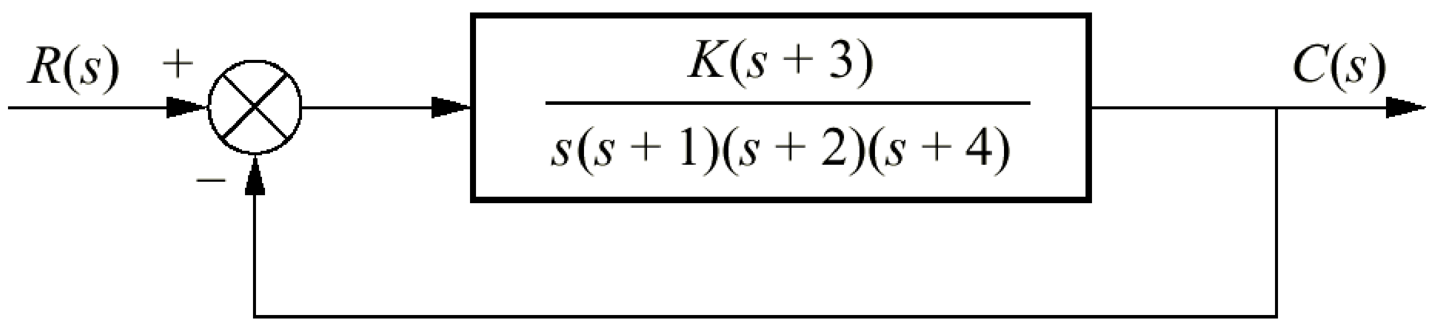
The root locus approaches staight lines as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a, and ,θ_a, as follows :

$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{Number\ of\ finite\ poles - Number\ of\ finite\ zeros}$$

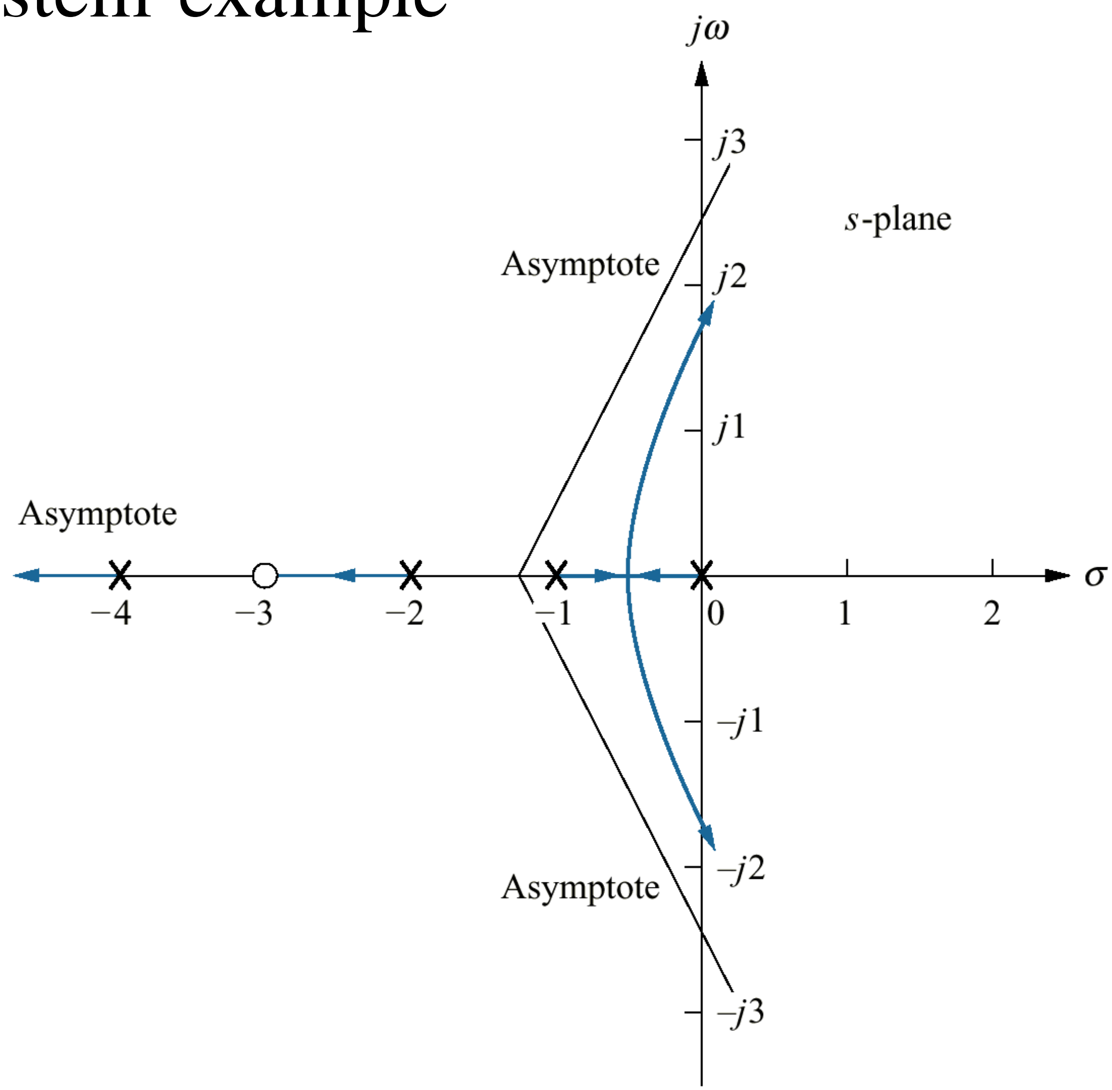
$$\theta_a = \frac{(2k+1)\pi}{Number\ of\ finite\ poles - Number\ of\ finite\ zeros}$$

$$k = 0, \pm 1, \pm 2, \ldots$$

Example (Nise)



Root locus and asymptotes for the system example



6. Real-axis Breakaway and Breakin points

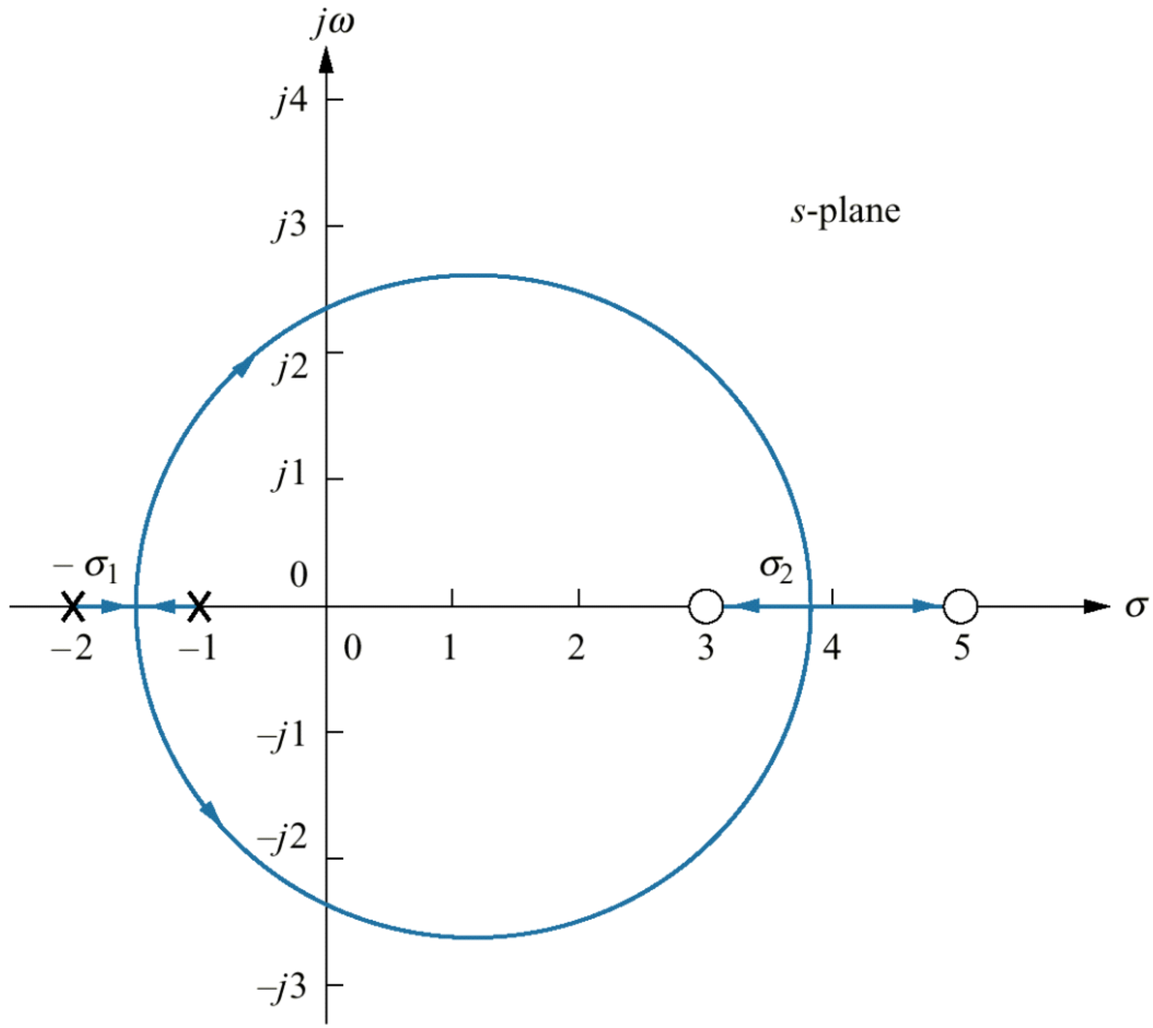
6.1 Breakaway and Breakin points satisfy the relationship (Transition Method):

$$\sum_{1}^{m} \frac{1}{\sigma + z_i} = \sum_{1}^{n} \frac{1}{\sigma + p_i}$$

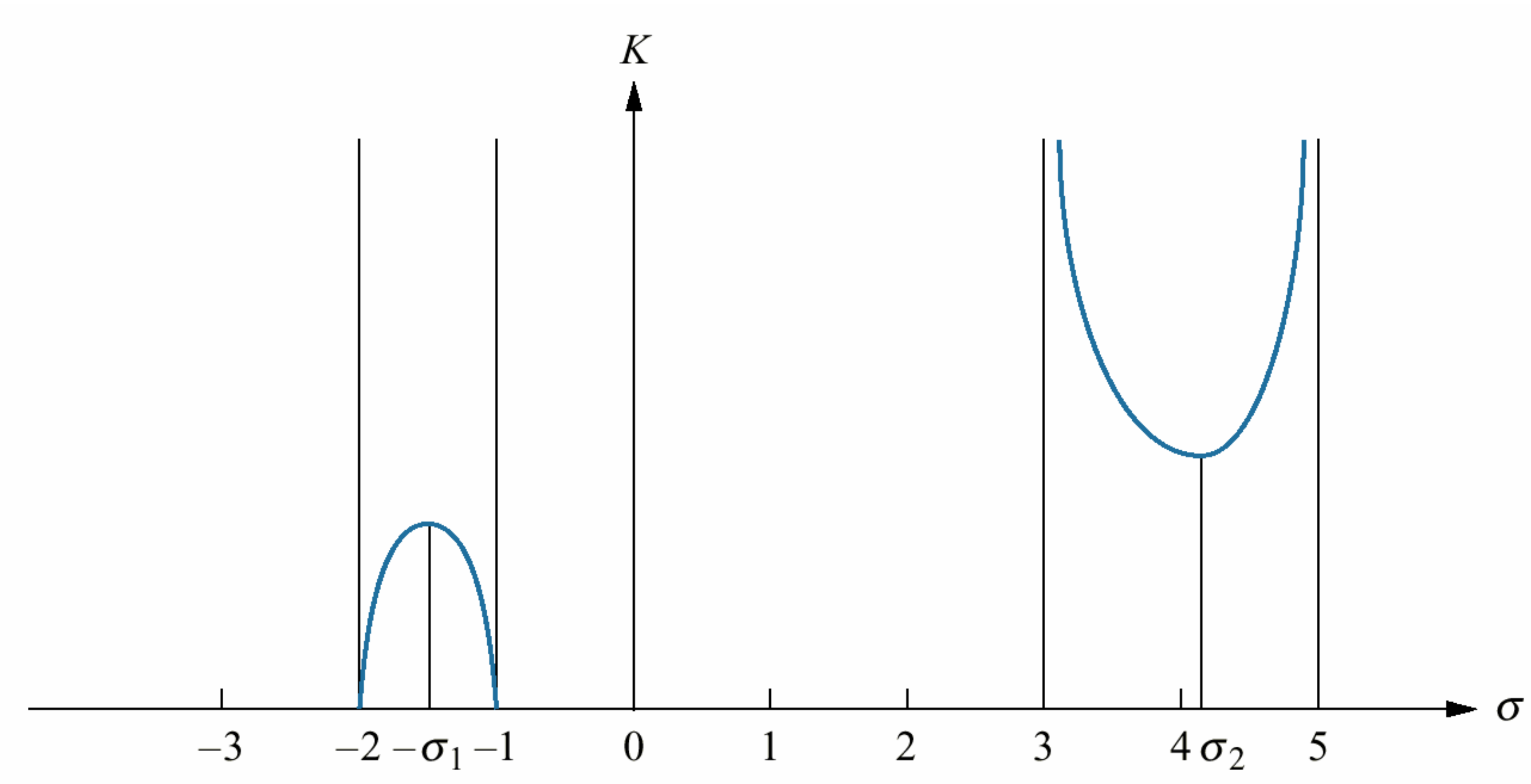
where z_i and p_i are the negative of the zeros and poles values of G(s)H(s).

6.2 Maximize and minimize the gain K, using differential calculus.

$$K = -\frac{1}{G(s)H(s)}$$



Root locus example showing real- axis breakaway (σ_1) and break-in points (σ_2)



Variation of gain along the real axis for the root locus of

Example (Nise) Find the breakaway and break-in point for the root locus of slide 36

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

$$= \frac{K(s^2 - 8s + 15)}{(s^2 + 3s + 2)}$$

For all point along the roots locus

$$KG(s)H(s) = -1$$

And all point along the real axis s=σ

$$K = \frac{-(\sigma^2 + 3\sigma + 2)}{(\sigma^2 - 8\sigma + 15)}$$

Differentiating K respect to σ and setting the derivative equal to zero

$$\frac{dK}{d\sigma}=\frac{\left(11\sigma^{2}-26\sigma-61\right)}{\left(\sigma^{2}-8\sigma+15\right)^{2}}=0$$

σ=-1.45 is breakaway point and

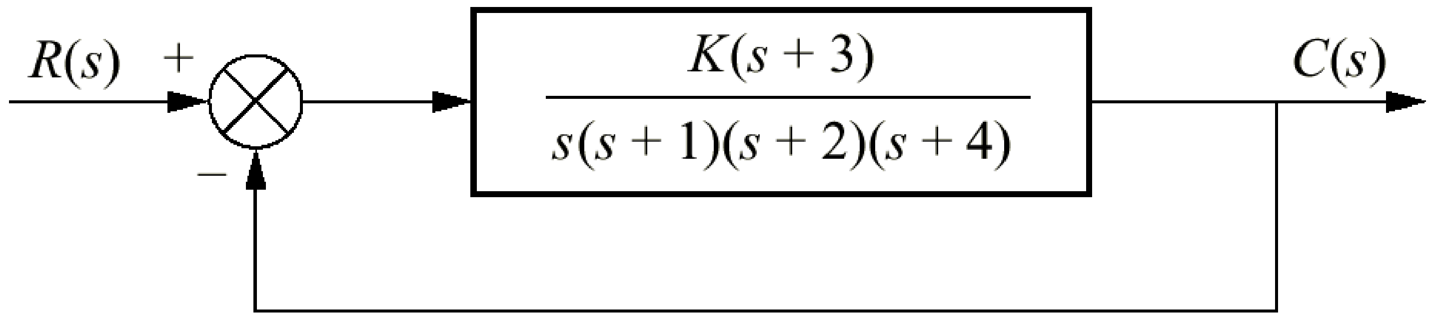
σ=3.82 is break-in point

7. The $j\omega$-axis crossing

7.1 Let $s=j\omega$ in the characteristic equation equate both real part and imaginery part to zero and then solve for ω and K

7.2 Use of Routh's stability criterion.

Example (Nise) Sketch the root locus for the system below.



Closed-loop transfer function is

$$T(s)=\frac{K(s+3)}{s^4+7s^3+14s^2+(8+K)s+3K}$$

Complex number

Characteristic Equation

$$s^4 + 7s^3 + 14s^2 + (8+K)s + 3K = 0$$

Let s= jω

$$\omega^4 - j7\omega^3 - 14\omega^2 + j(8+K)\omega + 3K = 0$$

$$(\omega^4 - 14\omega^2 + 3K) + j(-7\omega^3 + (8+K)\omega) = 0$$

$$\omega^4 - 14\omega^2 + 3K = 0$$

$$-7\omega^3 + (8 + K)\omega = 0$$

K=9.65 $\omega = \pm 1.59$

Routh table

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\dfrac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

From s^1 row

$$-K^2 - 65K + 720 = 0$$

$$K = 9.65$$

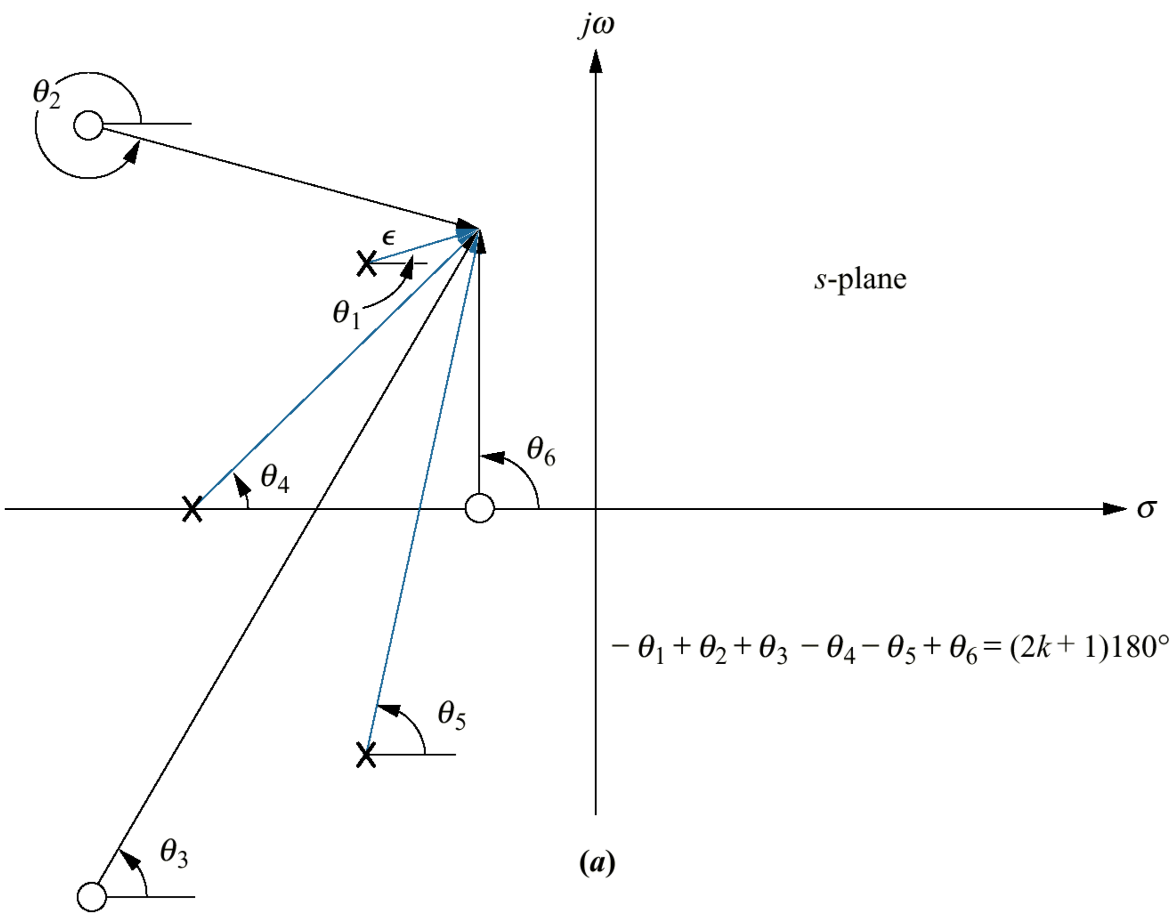
Form s^2 row with K=9.65

$$(90 - K)s^2 + 21K = 80.35s^2 + 202.7 = 0$$

$$S = \pm j9.65$$

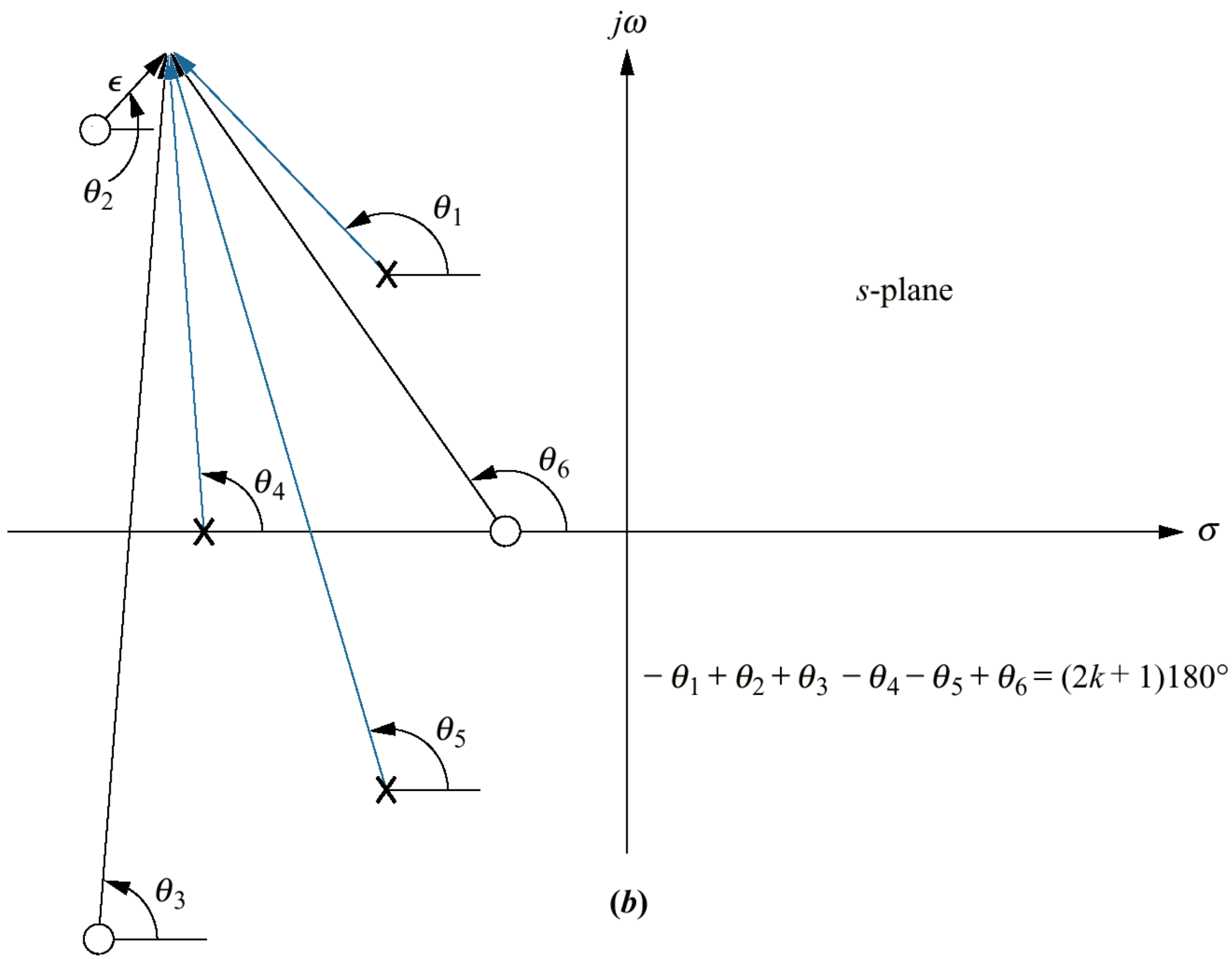
Angles of Departure and Arrival

For complex poles and zeros : Open-loop poles and zeros and **calculation of** angle of departure



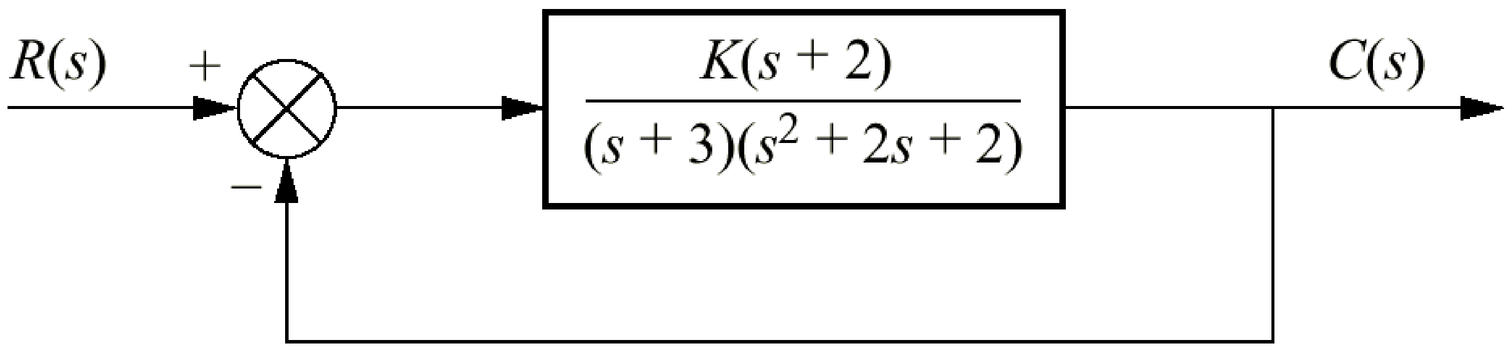
(a)

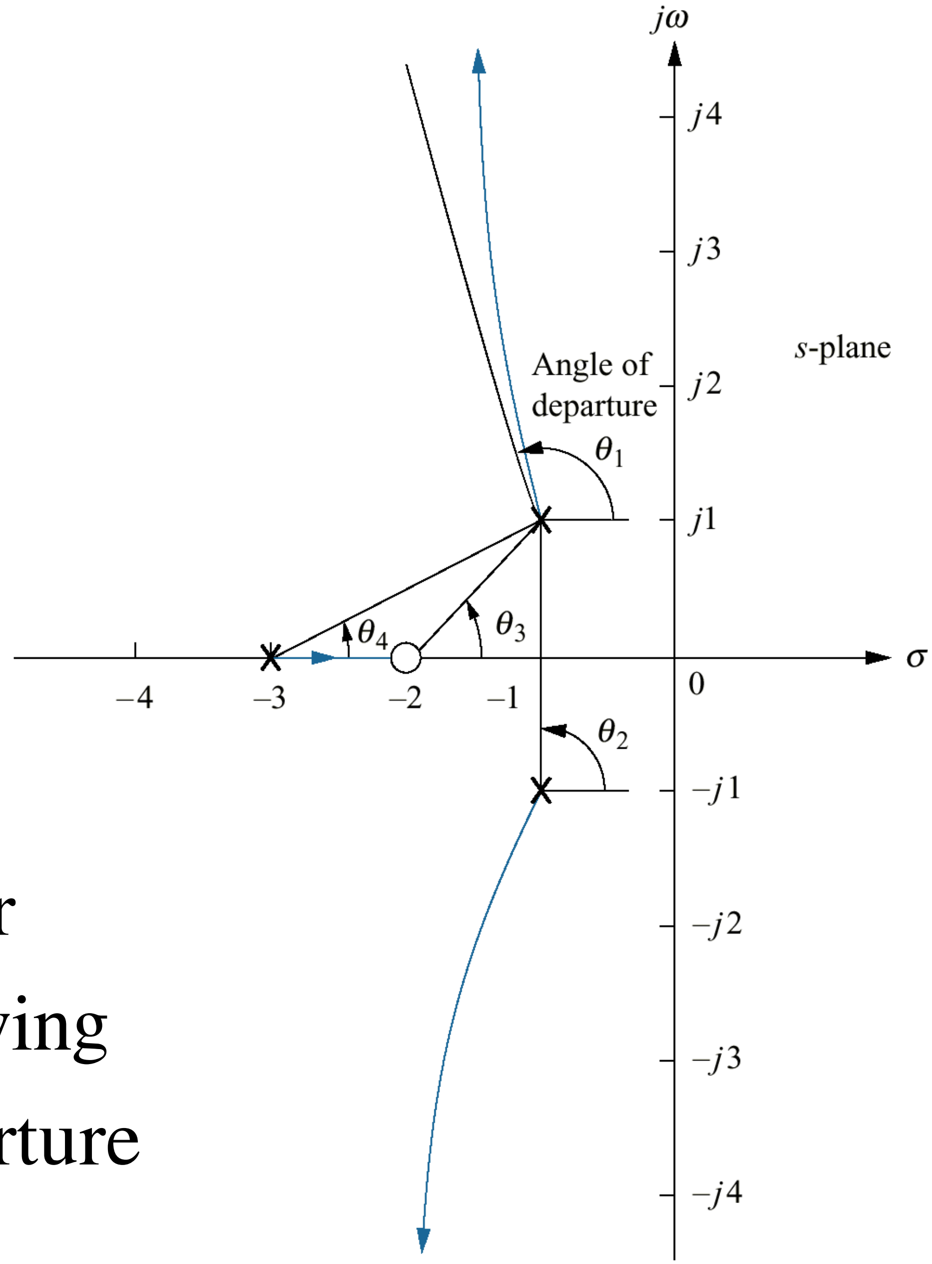
Open- loop poles and zeros and calculation of angle of arrival



(b)

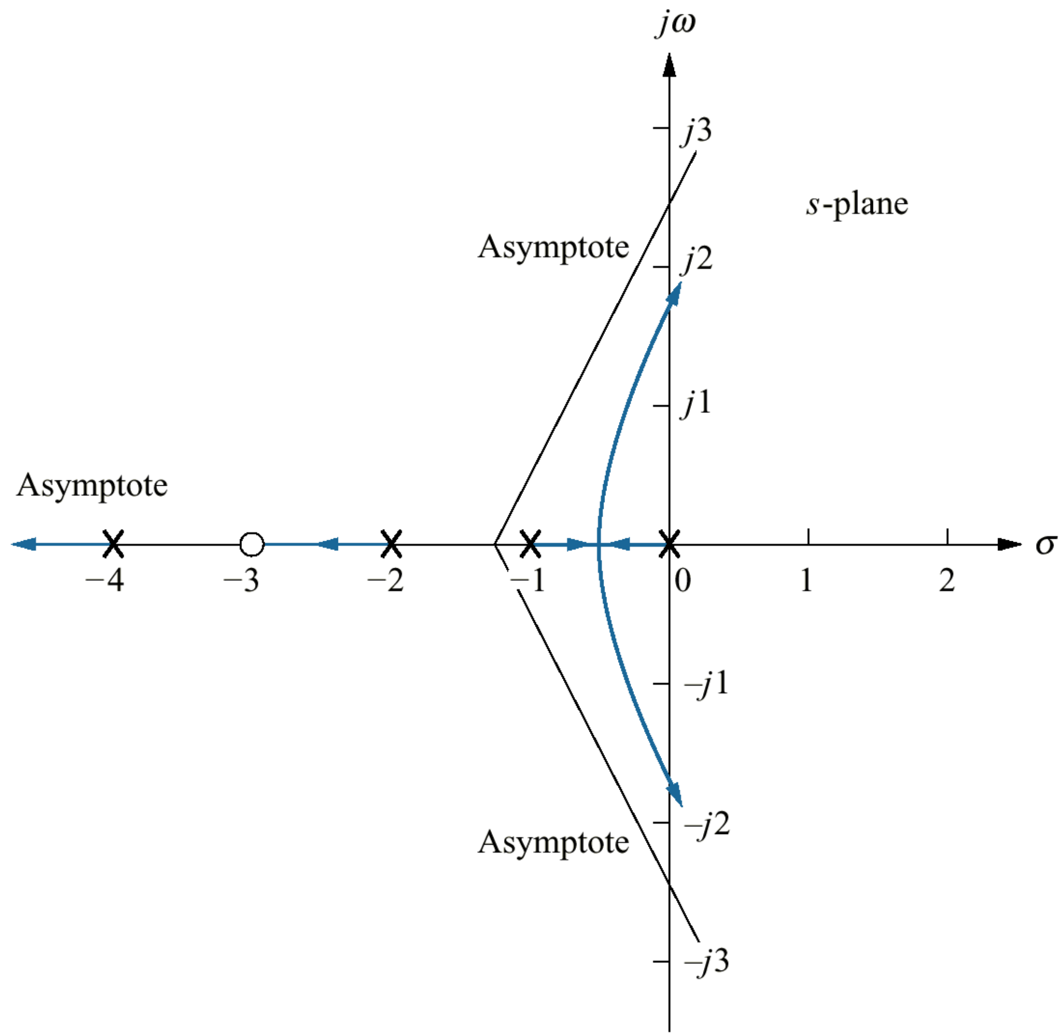
Example (Nise)

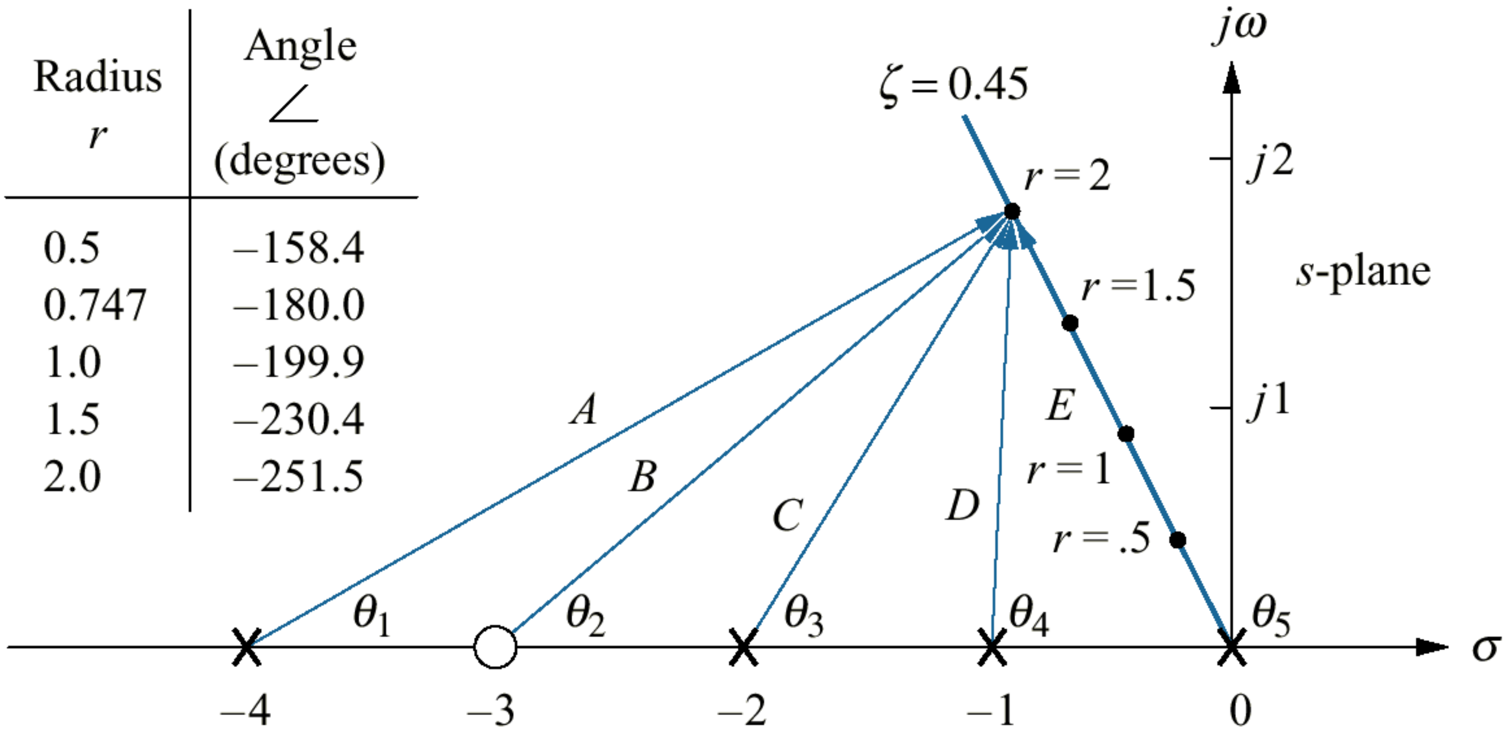




Root locus for example showing angle of departure

Plotting and Calibrating the Root Locus





Radius r	Angle $\angle$ (degrees)
0.5	–158.4
0.747	–180.0
1.0	–199.9
1.5	–230.4
2.0	–251.5

Finding and calibrating exact points on the root locus

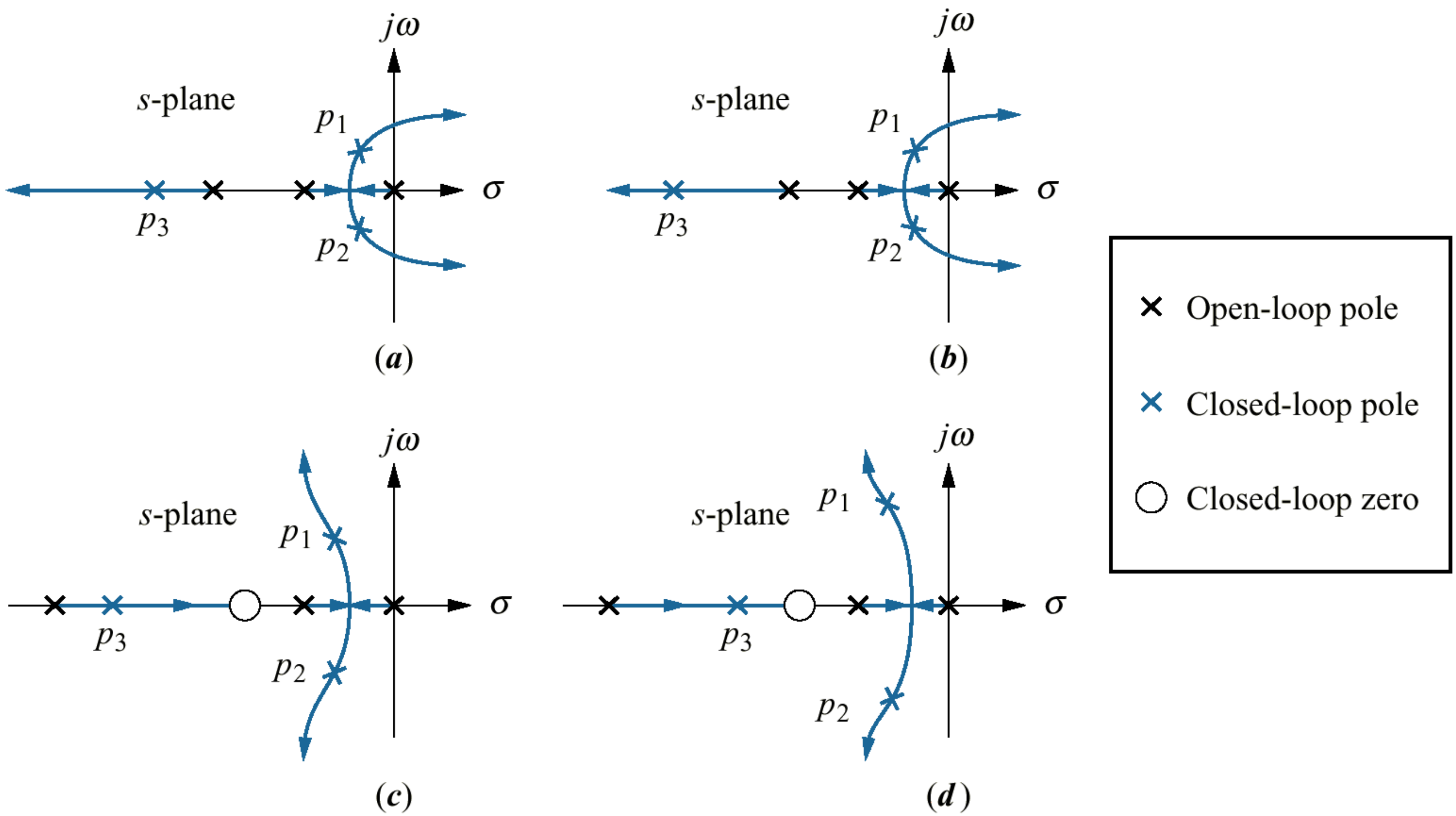
Transient Response Design via Gain Adjustment

The conditions justifying a second-order approximation are :

1 High order poles are much farther into the left half of the s-plane than the dominant second-order pair of poles. The response that results from a higher-order pole does not appreciably change the transient response expected from the dominant second-order poles.

2 Closed-loop zeros near the closed-loop second-order pole pair are nearly canceled by the close proximity of higher-order closed-loop poles.

3 Closed-loop zeros not canceled by the close proximity of higher-order closed-loop poles are far removed from the closed-loop second-order pole pair



Making second-order approximations

Design procedure for higher-order system

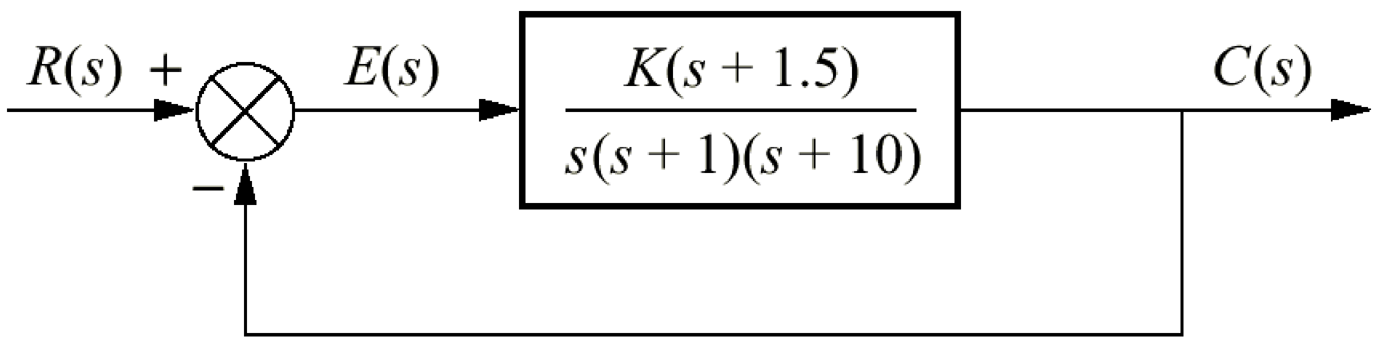
1 Sketch the root locus for the given system

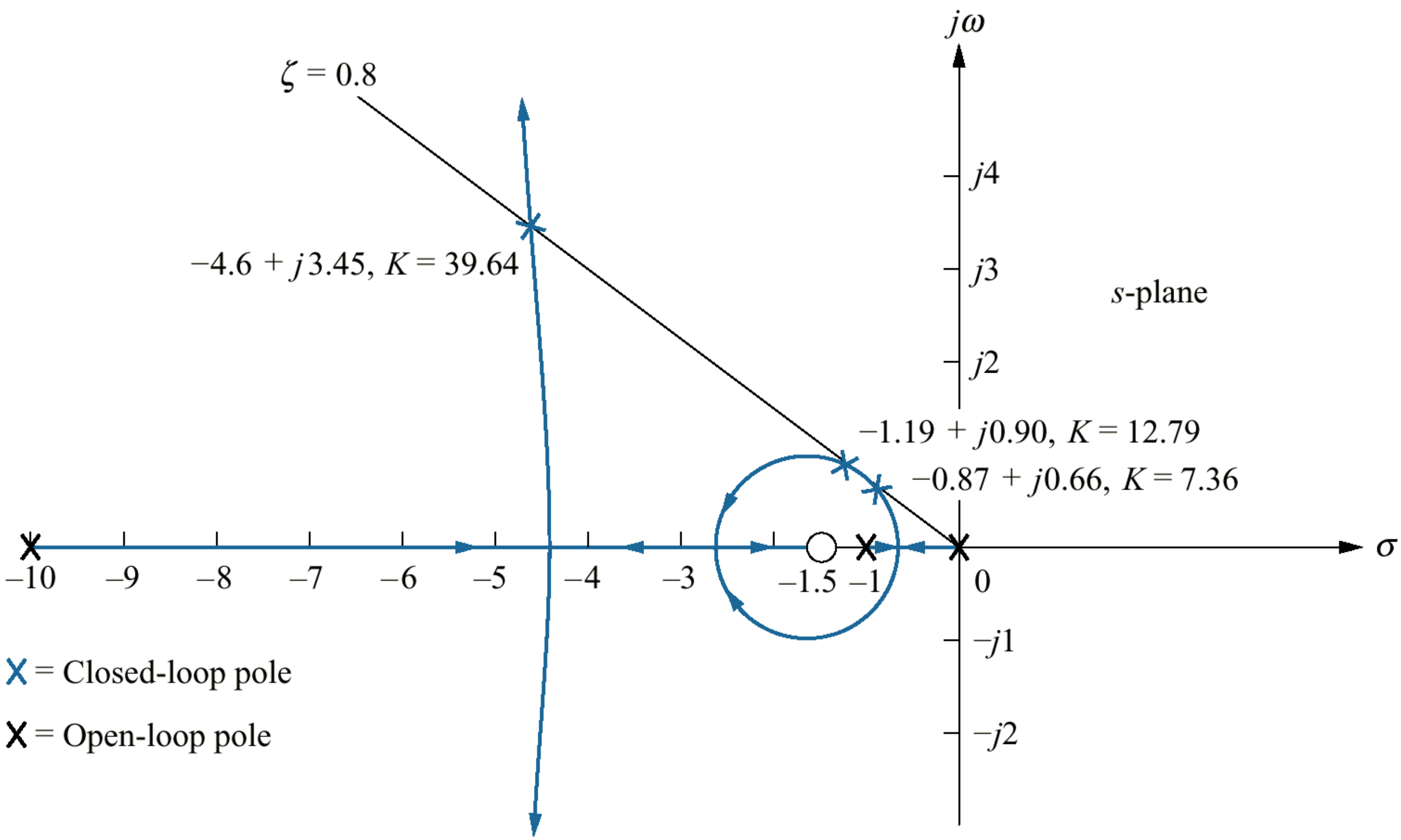
2 Assume the system is a second-order system without any zeros and then find the gain to meet the meet the transient response specification.

3 Justify the second-order assumption by finding the location of all higher-order poles and evaluating the fact that they are much farther from the $j\omega$-axis than the dominant second-order pair.

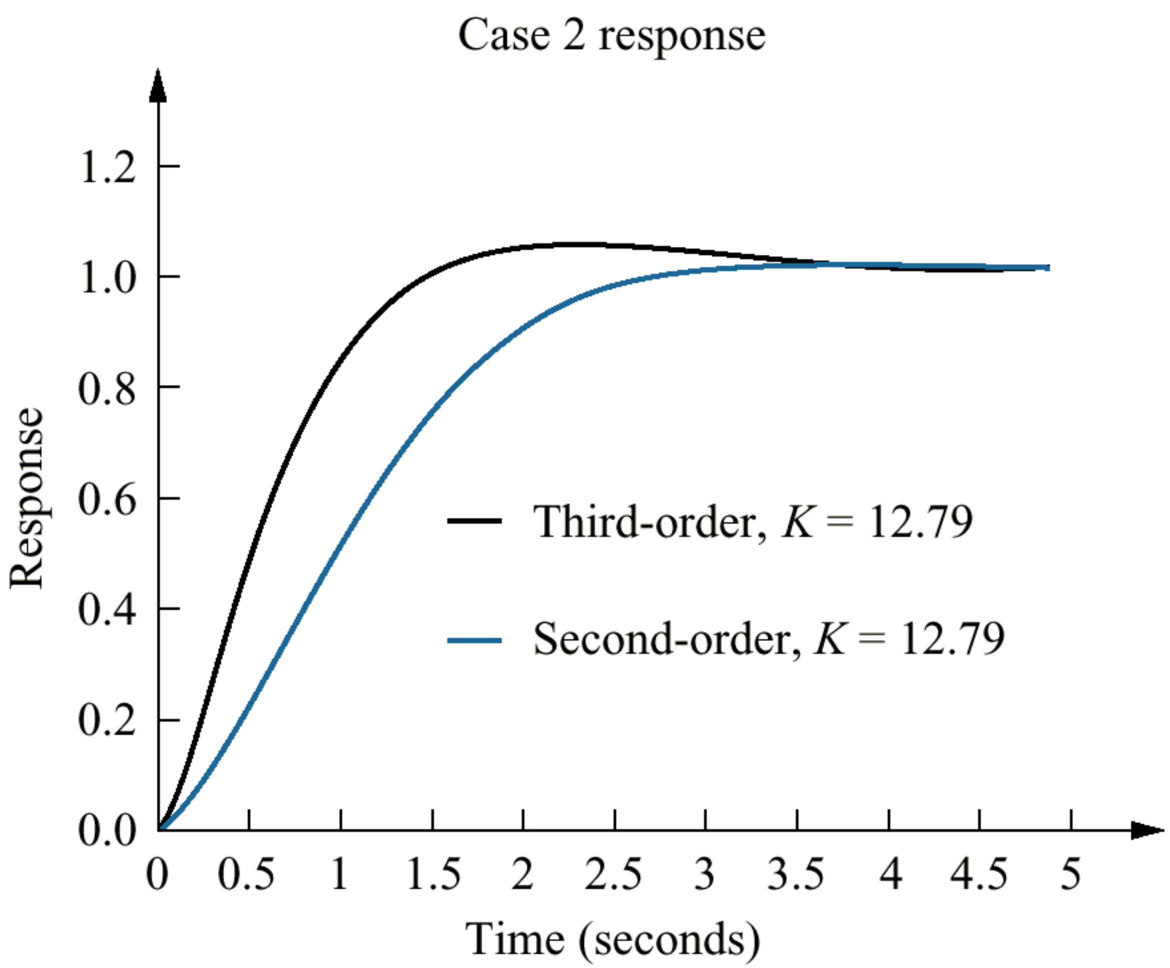
4 If the assumptions cannot be justified, you solution will have to be simulated in order to be sure it meets the transient response specification.

Example (Nise) Design the value of gain K, to yield 1.52% overshoot for the system below. Also estimate the settling time, peak time, and steady-state error.

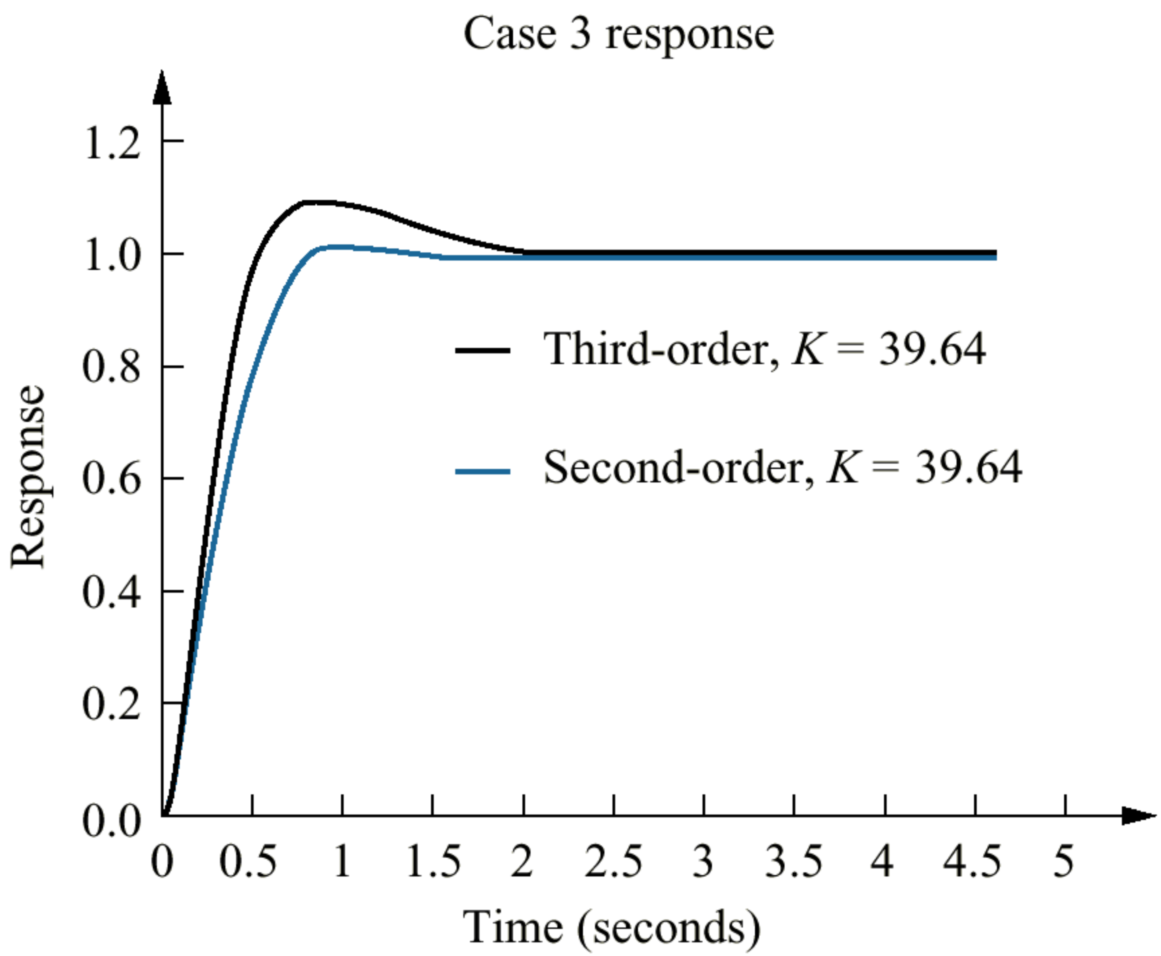




Root locus of the system in slide 54



Second- and third-order responses for case 2



Second- and third-order responses for case 3