Chapter 4

Modelling in the Time Domain

Outline

1 The Geneneral State-Space representation.
2 Applying the state-space representation.
3 Converting a transfer function to state-space function.
4 Converting from state-space to a transfer function.

Avantage

- used with multiple-inputs, multiple-outputs system.
- can be used for the same class of system modeled by the classical approach.
- can be used to represent nonlinear systems that have backlash, saturation and dead zone.

Some Observations

1. We select a particular subset of all possible system variables and call the variables in this subset state variables.

2. For the *n*th order system, we write *n* simultaneous, first order differential equations in terms of the state variables. We call this system of simultaneous differential equations state equations.

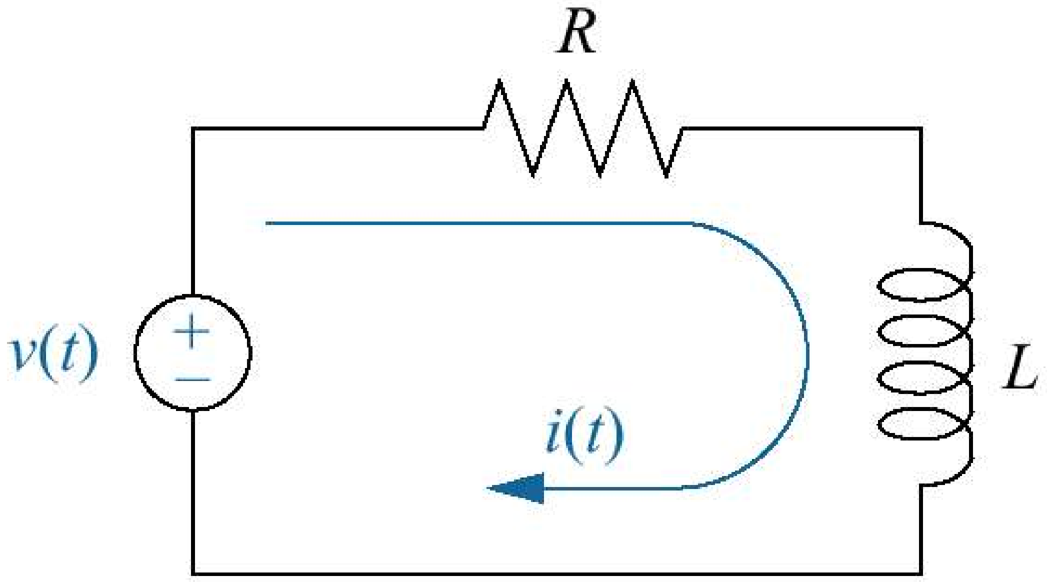
3. If we have the initial condition of all of the state variables at t_0 as well as the system input for $t \geq t_0$. We can solve the simultaneous differential equations for the state variables for $t \geq t_0$.

4. We algebraically combine the state variables with the system's input and find all of the other system variables for $t \geq t_0$. We call this algebraic equation the output equation.

5. We consider the state equations and the output equations a variable represent of the system. We call this representation of the system a state-space representation.

Example

Consider the value changes at each step. We have to find input and output equation.



RL network

๑. We select the current, i(t), for which we will write and solve a differential equation using Laplace Transforms.

๒. We write the loop equation.

$$L\frac{di}{dt}+Ri=v(t)$$

๓. Taking the Laplace transformation and including the initial conditions, yield.

$$L\left[sI(s)-i(๐)\right]+RI(s)=V(s)$$

Assuming the input, v(t), to be unit step, u(t), where Laplace Transform is V(s) = ๑/s, so we solve for I(s) and get

$$I(s)=\frac{๑}{R}\left(\frac{๑}{s}-\frac{๑}{s+\frac{R}{L}}\right)+\frac{i(๐)}{s+\frac{R}{L}}$$

๔. We can now solve of all other network variables algebraically in term of i(t) and the applied voltage, v(t). The voltage across the resistor is :

$$v_R(t) = Ri(t)$$

The voltage across the resistor is

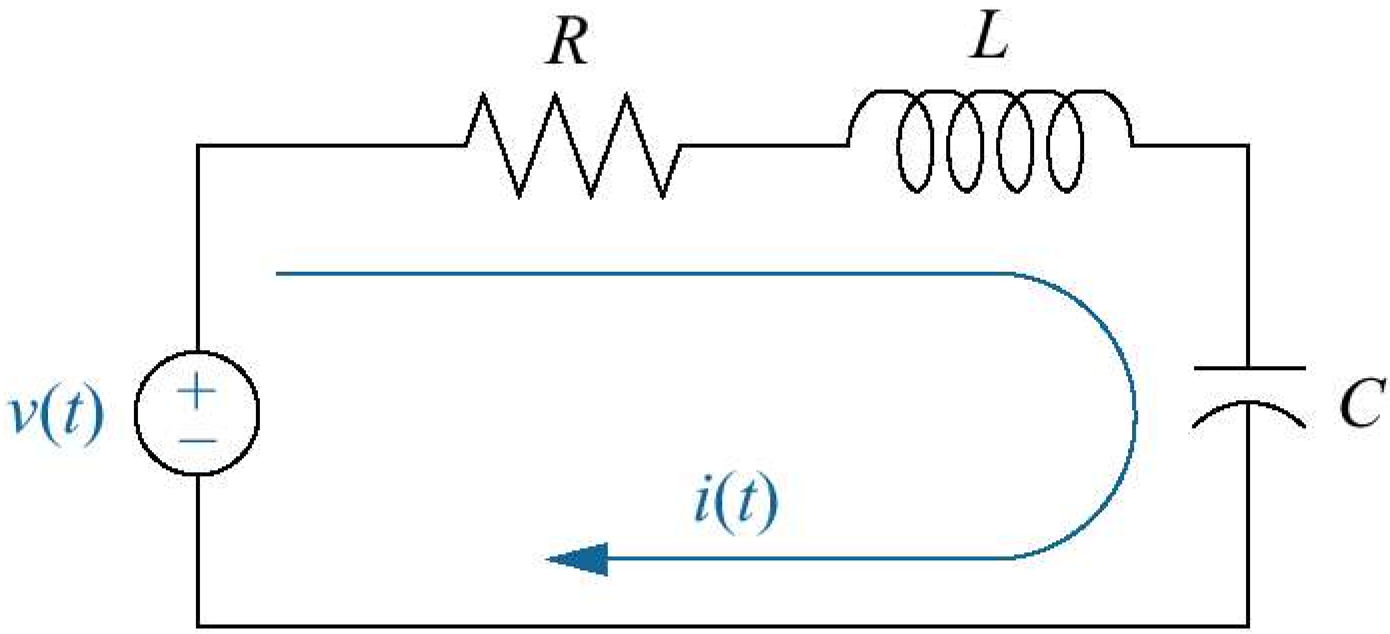
$$v_L(t) = v(t) - Ri(t)$$

The derivative of the current is

$$\frac{di}{dt} = \frac{\text{๑}}{L}\left[v(t) - Ri(t)\right]$$

Thus, knowing the state variable, i(t), and the input, v(t), we can find the value or state, of any network variable at any time, $t \geq t_o$. Hence, the algebraic equations presented are output equations.

RLC network (second order)



Second order system

๑. Since the network is of second order, two simultaneous, first order differential equations are needed to solve for two state variables. We select i(t) and q(t), the change on the capacitor, as the two state variables.

๒. Writing the loop equation yields

$$L\frac{di}{dt}+Ri+\frac{๑}{C}\int idt=v(t)$$

Converting to charge, using i(t) = dq/dt, we get :

$$L\frac{d^{๒}q}{dt^{๒}}+R\frac{dq}{dt}+\frac{๑}{C}q=v(t)$$

*n*th-order differential equation can be converted to *n* simultaneous first-order differential equation of the form :

$$\frac{dx_i}{dt}=a_{i๑}x_{๑}+a_{i๒}x_{๒}+\ldots+a_{in}x_n+b_i f(t)$$

If $\frac{dq}{dt} = i$ **Then**

$$\frac{di}{dt} = \frac{-๑}{LC} q - \frac{R}{L} i + \frac{๑}{L} v(t)$$

๓. These equations are the state equations and can be solved in simultaneously for the state variables, q(t) and i(t), using the Laplace transform, if we known the initial condition for i(t) and q(t) and if we know v(t), the input.

๔. From equation in ๒. We can solve for all other network variables. Example, voltage across inductance as :

$$v_L(t) = -\frac{๑}{C} q(t) - Ri(t) + v(t)$$

๕. The combined state equation for input(๒) and output(๔), we call a state-space representation.

We can write in vector-matrix form as follow :

$$\dot{x} = Ax + Bu$$

$$\dot{x} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}; \qquad A = \begin{bmatrix} ๐ & ๑ \\ -๑/LC & -R/L \end{bmatrix}$$

$$x = \begin{bmatrix} q \\ i \end{bmatrix}; \qquad B = \begin{bmatrix} ๐ \\ ๑/L \end{bmatrix}; \quad u = v(t)$$

$$y = Cx + Du$$

$$y = v_L(t); \quad C = \begin{bmatrix} -๑/C & -R \end{bmatrix}; \qquad x = \begin{bmatrix} q \\ i \end{bmatrix}; \quad D = ๑; \quad u = v(t)$$

If we select V_R and V_C as state variables then we have state equation :

$$\frac{dv_R}{dt} = \frac{-R}{L} v_R - \frac{R}{L} v_C \frac{R}{L} v(t)$$

$$\frac{dv_C}{dt} = \frac{\text{๑}}{RC} v_R$$

The General State-Space Representation

Linear combination : A linear combination of *n* variables, x_i, for i=๑ to *n* is given by the following sum S :

$$S = K_n x_n + K_{n-๑} x_{n-๑} + \ldots + K_๑ x_๑$$

Linear independence : A set of variables is said to be linearly independent if none of variables can be written as a linear combination of the other.

System variable : Any variable that responds to an input or initial condition in a system.

State variable : The smallest set of linearly independent system variables such that the values of the members of the set at time t_o along with known forcing functions completely determine the value of all system variables for all $t \geq t_o$

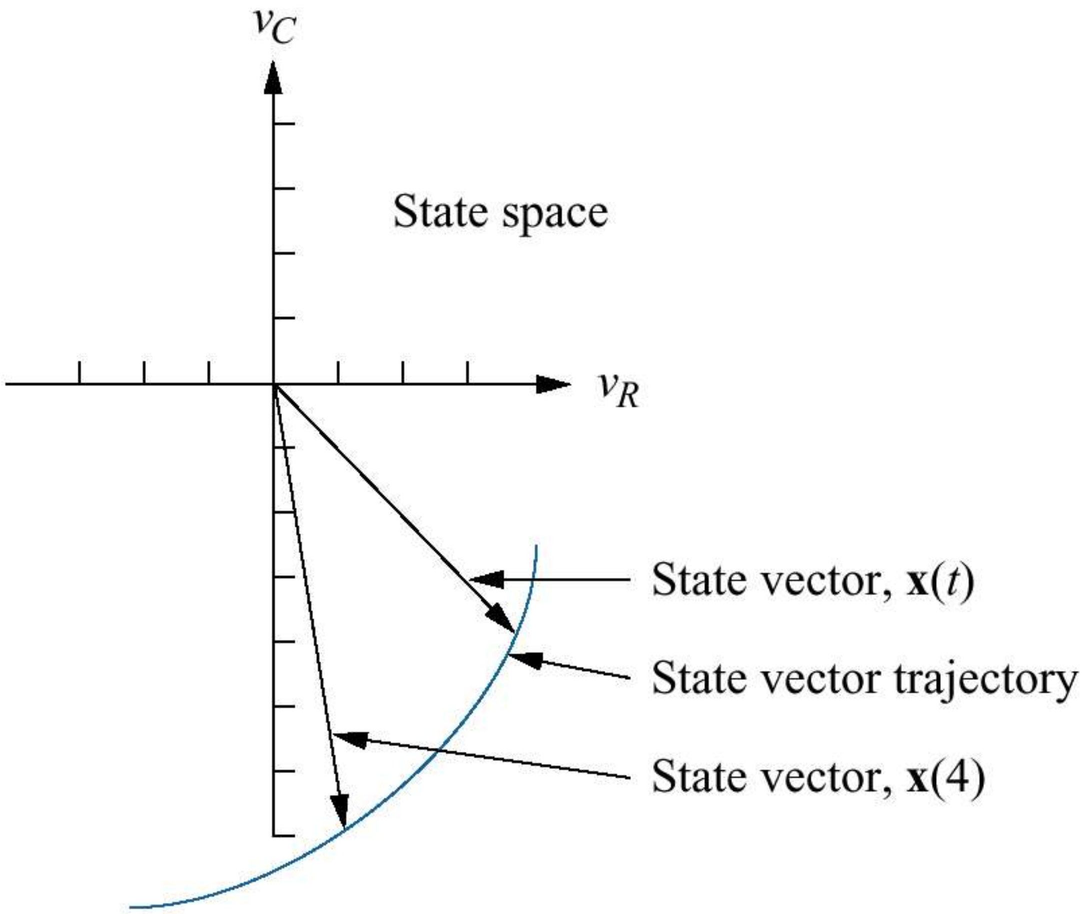
State vector : A vector whose elements are the state variables.

State space : The *n*-dimentional space whose axes are the state variables.

State equation : A set of *n* simultaneous, first-order differential equations with *n* variables, where the n variables to be solved are the state variables.

Output equation : The algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

Graphic representation of state space and a state vector



Applying the State Space Representation

The state of a system is define as the minimum number of initial number of the initial conditions that must be specified at some initial time t_o to determine completely the dynamic behavior of the system for $t \geq t_o$ when given its input r(t) for $t \geq t_o$.

The state variable must be linearly independent.

The number and choice of state variables depend on the level of detail desire in the dynamic model selected to describe a given physical system.

In general, an nth-order model has n state variables with n corresponding first-order differential equations express terms of these state variables.

$$\frac{dx_{๑}}{dt} = a_{๑๑}x_{๑} + \ldots + a_{๑n}x_n + b_{๑๑}u_{๑} + \ldots + b_{๑r}u_r$$

$$\frac{dx_{๒}}{dt} = a_{๒๑}x_{๑} + \ldots + a_{๒n}x_n + b_{๒๑}u_{๑} + \ldots + b_{๒r}u_r$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = a_{n๑}x_{๑} + \ldots + a_{nn}x_n + b_{n๑}u_{๑} + \ldots + b_{nr}u_r$$

$$\boldsymbol{A}=\begin{bmatrix} a_{๑๑} & a_{๑๒} & \cdots & a_{๑n} \\ a_{๒๑} & a_{๒๒} & \cdots & a_{๒n} \\ \vdots & & \ddots & \vdots \\ a_{n๑} & a_{n๒} & \cdots & a_{nn} \end{bmatrix} \qquad \boldsymbol{x}=\begin{bmatrix} x_{๑} \\ x_{๒} \\ \vdots \\ x_{n} \end{bmatrix}$$

$$\boldsymbol{B}=\begin{bmatrix} b_{๑๑} & b_{๑๒} & \cdots & b_{๑r} \\ b_{๒๑} & b_{๒๒} & \cdots & b_{๒r} \\ \vdots & & \ddots & \vdots \\ b_{n๑} & b_{n๒} & \cdots & b_{nr} \end{bmatrix} \qquad \boldsymbol{u}=\begin{bmatrix} u_{๑} \\ u_{๒} \\ \vdots \\ u_{r} \end{bmatrix}$$

$$
\begin{aligned}
y_{\text{๑}} &= c_{\text{๑๑}} x_{\text{๑}} + \ldots + c_{\text{๑}n} x_n + d_{\text{๑๑}} u_{\text{๑}} + \ldots + d_{\text{๑}r} u_r \\
y_{\text{๒}} &= c_{\text{๒๑}} x_{\text{๑}} + \ldots + c_{\text{๒}n} x_n + d_{\text{๒๑}} u_{\text{๑}} + \ldots + d_{\text{๒}r} u_r \\
&\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
y_m &= c_{m\text{๑}} x_{\text{๑}} + \ldots + c_{mn} x_n + d_{m\text{๑}} u_{\text{๑}} + \ldots + d_{mr} u_r
\end{aligned}
$$

$$\boldsymbol{y} = \begin{bmatrix} y_{๑} \\ y_{๒} \\ \vdots \\ y_{n} \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} c_{๑๑} & c_{๑๒} & \cdots & c_{๑n} \\ c_{๒๑} & c_{๒๒} & \cdots & c_{๒n} \\ \vdots & & \ddots & \vdots \\ c_{m๑} & c_{m๒} & \cdots & c_{mn} \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} d_{๑๑} & d_{๑๒} & \cdots & d_{๑r} \\ d_{๒๑} & d_{๒๒} & \cdots & d_{๒r} \\ \vdots & & \ddots & \vdots \\ d_{m๑} & d_{m๒} & \cdots & d_{mr} \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

x = state vector

$\dot{x}$ = derivative of the state vector with respect to time

y = output vector

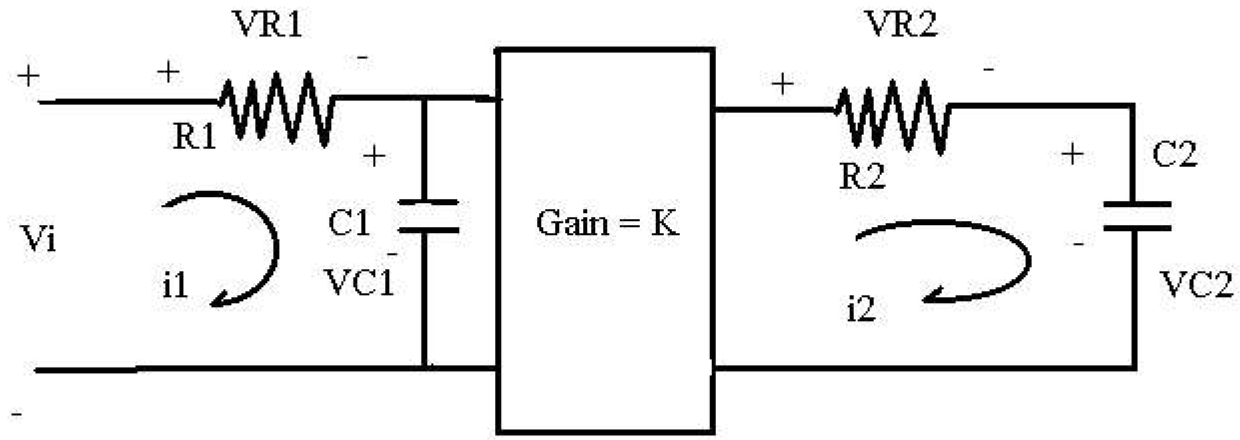
u = input or control vector

A = system matrix

B = input matrix

C = output matrix

D = feedforward matrix



Example

$$v_i = R๑ \times i๑ + VC๑ \qquad i๑ = C๑ \frac{dVC๑}{dt} = C๑ \dot{VC}๑$$

$$KVC๑ = R๒ \times i๒ + VC๒ \qquad i๒ = C๒ \frac{dVC๒}{dt} = C๒ \dot{VC}๒$$

We select the state variables $x_๑ = VC_๑$ and $x_๒ = VC_๒$. Substitute into the equations and simplifying gives the component equation :

$$\dot{x}_๑ = -\frac{๑}{R_๑C_๑}x_๑ + \frac{Vi}{R_๑C_๑}$$

$$\dot{x}_๒ = \frac{K}{R_๒C_๒}x_๑ - \frac{๑}{R_๒C_๒}x_๒$$

$$y = x_๒$$

$$A=\begin{bmatrix} -\dfrac{๑}{R๑C๑} & ๐ \\ \dfrac{K}{R๒C๒} & -\dfrac{๑}{R๒C๒} \end{bmatrix}$$

$$B=\begin{bmatrix} \dfrac{๑}{R๑C๑} \\ ๐ \end{bmatrix}$$

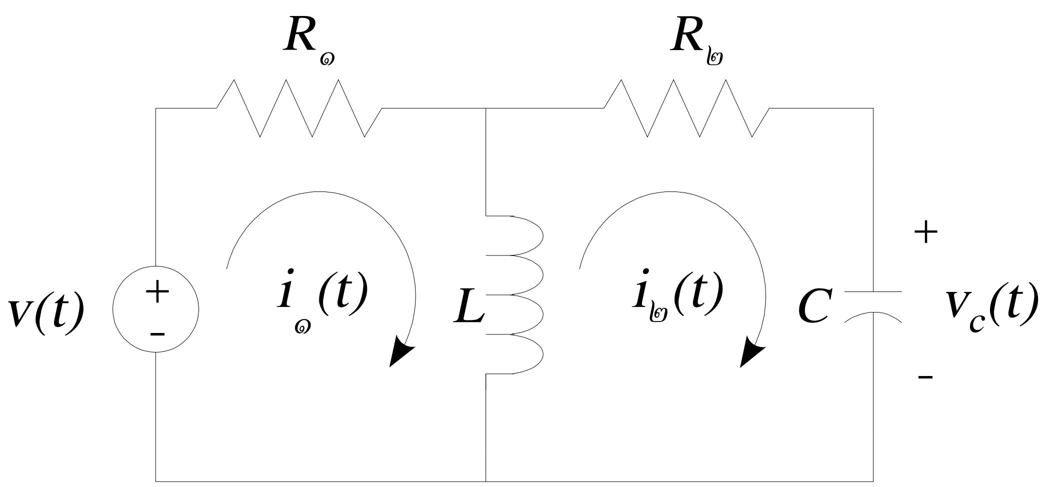
$$C=\begin{bmatrix} ๐ & ๑ \end{bmatrix}$$

$$D=๐$$

$$
\begin{bmatrix} \dot{x}๑ \\ \dot{x}๒ \end{bmatrix} = \begin{bmatrix} \dfrac{-๑}{R๑C๑} & ๐ \\ \dfrac{K}{R๒C๒} & \dfrac{-๑}{R๒C๒} \end{bmatrix} \begin{bmatrix} x๑ \\ x๒ \end{bmatrix} + \begin{bmatrix} \dfrac{๑}{R๑C๑} \\ ๐ \end{bmatrix} u
$$

$$
\boldsymbol{Y} = \begin{bmatrix} ๐ & ๑ \end{bmatrix} \boldsymbol{x} = x๒
$$

Example Find state equation and output equation of the circuit below when $v(t)$ is input and $v_C(t)$ is output



Laplace transform solution of State equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$s\,X(s) - X(\text{๐}) = AX(s) + BU(s)$$

$$(sI - A)\,X(s) = X(\text{๐}) + BU(s)$$

$$\boxed{X(s) = (sI - A)^{-\text{๑}}\left(X(\text{๐}) + BU(s)\right)}$$

Converting a Transfer Function to State Space

At the beginning, there are two methods of repre senting systems :

๑. The transfer function.

๒. The state-space representation.

These two methods can convert to each other.

Converting a Transfer Function to State Space

$$\frac{d^n y}{dt^n} + a_{n-๑}\frac{d^{n-๑} y}{dt^{n-๑}} + \ldots + a_{๑}\frac{dy}{dt} + a_{๐}y = b_{๐}u$$

$$x_{๑} = y \qquad \dot{x}_{๑} = \frac{dy}{dt}$$

$$x_{๒} = \frac{dy}{dt} \qquad \dot{x}_{๒} = \frac{d^{๒} y}{dt^{๒}}$$

$$\vdots \qquad \vdots$$

$$x_n = \frac{d^{n-๑} y}{dt^{n-๑}} \qquad \dot{x}_n = \frac{d^n y}{dt^n}$$

$$\dot{x}_๑ = x_๒$$

$$\dot{x}_๒ = x_๓$$

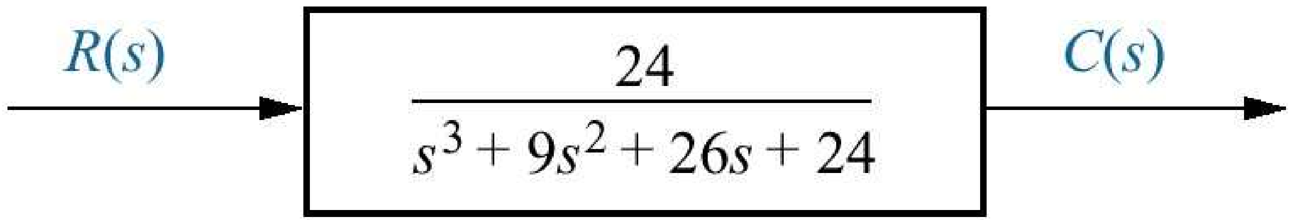
$$\vdots$$

$$\dot{x}_n = -a_๐ x_๑ - a_๑ x_๒ - \ldots - a_{n-๑} x_n + b_๐ u$$

$$\begin{bmatrix} \dot{x}_{๑} \\ \dot{x}_{๒} \\ \vdots \\ \dot{x}_{n} \end{bmatrix} = \begin{bmatrix} ๐ & ๑ & ๐ & \ldots ๐ \\ ๐ & ๐ & ๑ & \ldots ๐ \\ -a_{๐} & -a_{๑} & -a_{๒} & \ldots -a_{n-๑} \end{bmatrix} \begin{bmatrix} x_{๑} \\ x_{๒} \\ \vdots \\ x_{n} \end{bmatrix} + \begin{bmatrix} ๐ \\ \vdots \\ b_{๐} \end{bmatrix} u$$

$$y = \begin{bmatrix} ๑ & ๐ & ๐ & \cdots & ๐ \end{bmatrix} \begin{bmatrix} x_{๑} \\ x_{๒} \\ x_{๓} \\ \vdots \\ x_{n} \end{bmatrix}$$

Example Find the state space representation of the system below



Step ๑.

$$\frac{C(s)}{R(s)} = \frac{๒๔}{(s^{๓} + ๙s^{๒} + ๒๖s + ๒๔)}$$

Cross-multiplying yields

$$(s^{๓} + ๙s^{๒} + ๒๖s + ๒๔)C(s) = ๒๔R(s)$$

The corresponding differential equation is found by taking inverse Laplace transform, assuming zero initial conditions:

$$\dddot{c}+๙\ddot{c}+๒๖\dot{c}+๒๔c=๒๔r$$

Step ๒. Select the state variables.

$$x_๑ = c$$

Then compose state input and output equations:

Differential equation

$$\dot{x}_๑ = x_๒$$

$$\dot{x}_๒ = x_๓$$

$$\dot{x}_๓ = -๒๔x_๑ - ๒๖x_๒ - ๙x_๓ + ๒๔r$$

Output equation

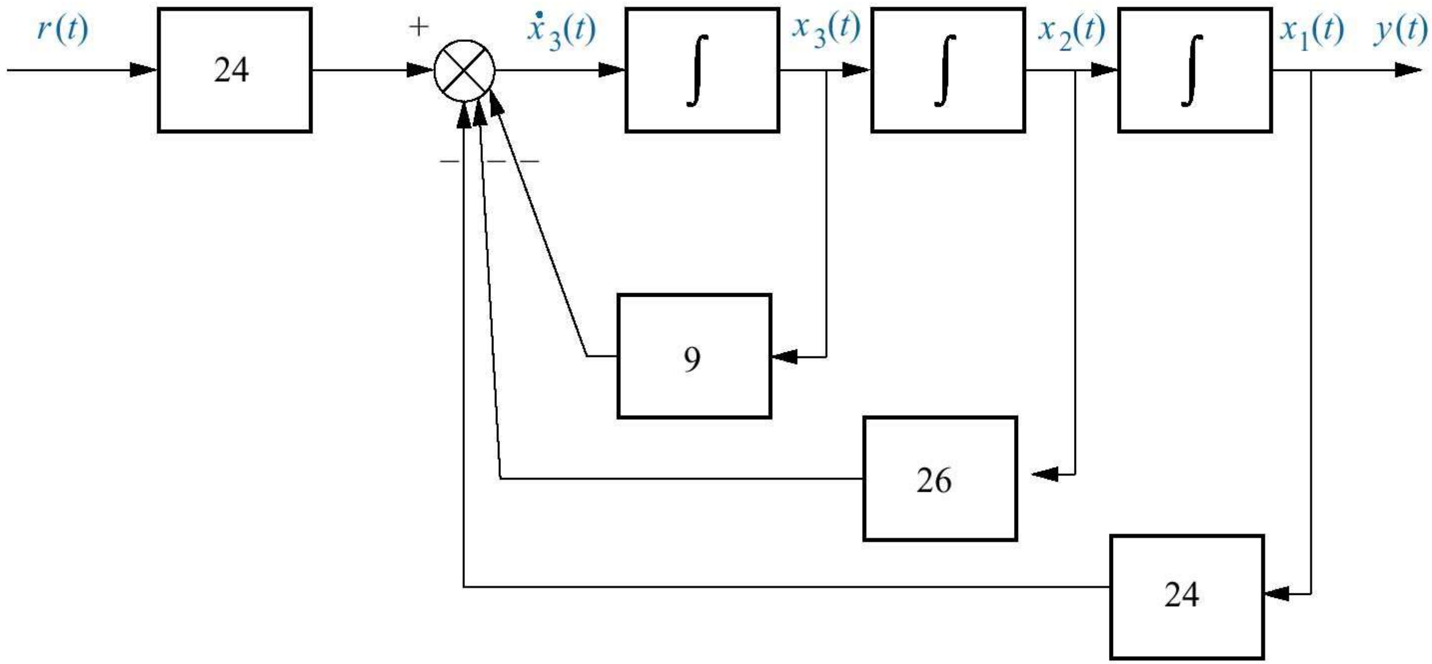
$$y = c = x_{๑}$$

Matrix form

$$\begin{bmatrix} \dot{x}_{๑} \\ \dot{x}_{๒} \\ \dot{x}_{๓} \end{bmatrix} = \begin{bmatrix} ๐ & ๑ & ๐ \\ ๐ & ๐ & ๑ \\ -๒๔ & -๒๖ & -๙ \end{bmatrix} \begin{bmatrix} x_{๑} \\ x_{๒} \\ x_{๓} \end{bmatrix} + \begin{bmatrix} ๐ \\ ๐ \\ ๒๔ \end{bmatrix} r$$

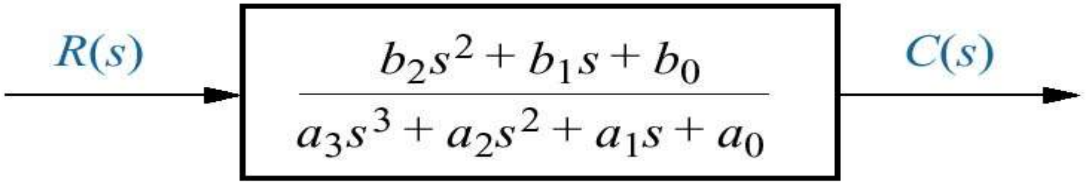
$$y = \begin{bmatrix} ๑ & ๐ & ๐ \end{bmatrix} \begin{bmatrix} x_{๑} \\ x_{๒} \\ x_{๓} \end{bmatrix}$$

Block Diagram of the system

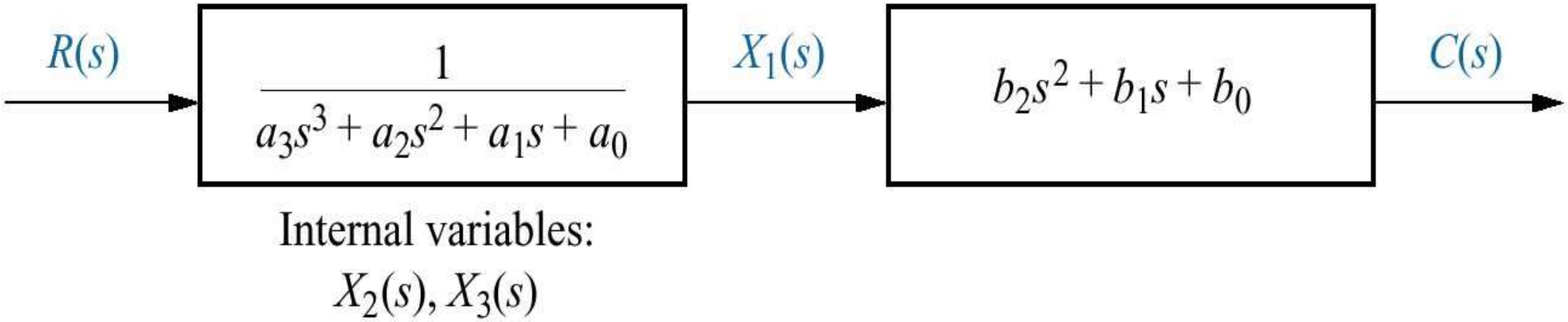


Decomposing a transfer function

When the transfer function has nominator



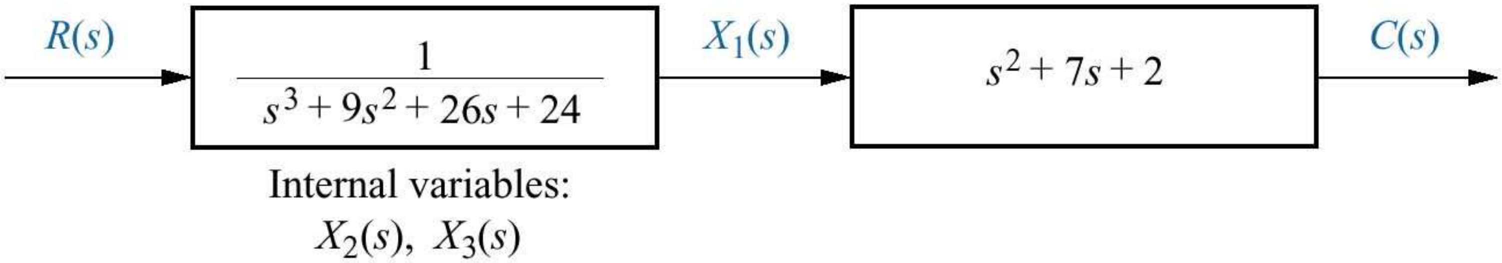
Transfer function can be decompose to ๒ transfer function



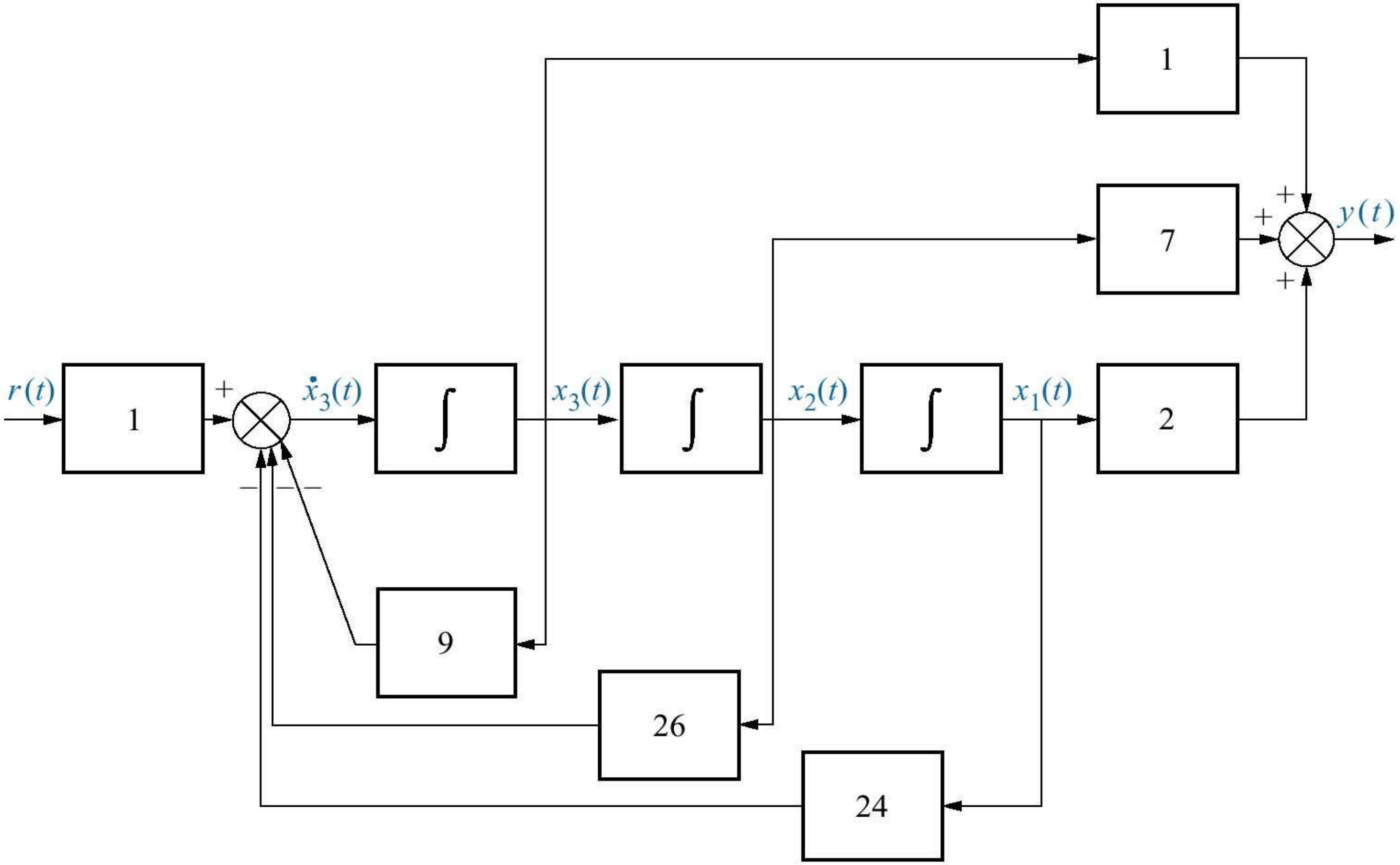
Example Find the state space representation of the system below

$$R(s) \longrightarrow \boxed{\dfrac{s^2+7s+2}{s^3+9s^2+26s+24}} \longrightarrow C(s)$$

Transfer function can be decompose to :



Equivalent block diagram



Converting From State-Space to a Transfer Function

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Using Laplace Transform

$$s\,\boldsymbol{X}(s)=\boldsymbol{A}\boldsymbol{X}(s)+\boldsymbol{B}\boldsymbol{U}(s)$$

$$\boldsymbol{Y}(s)=\boldsymbol{C}\boldsymbol{X}(s)+\boldsymbol{D}\boldsymbol{U}(s)$$

Rearrange the state equation

$$(s\,\boldsymbol{I}-\boldsymbol{A})\,\boldsymbol{X}(s)=\boldsymbol{B}\boldsymbol{U}(s)$$

$$\boldsymbol{X}(s)=(s\,\boldsymbol{I}-\boldsymbol{A})^{-๑}\boldsymbol{B}\boldsymbol{U}(s)$$

$$Y(s)=C(sI-A)^{-๑}BU(s)+DU(s)$$

$$=\left(C(sI-A)^{-๑}B+D\right)U(s)$$

$$\boxed{T(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-๑}B+D}$$