Chapter 5

Time Response

Outline

1 Poles, Zeros, and System Response

2 First order systems

3 Second order systems

4 Time domain Solution of State equations

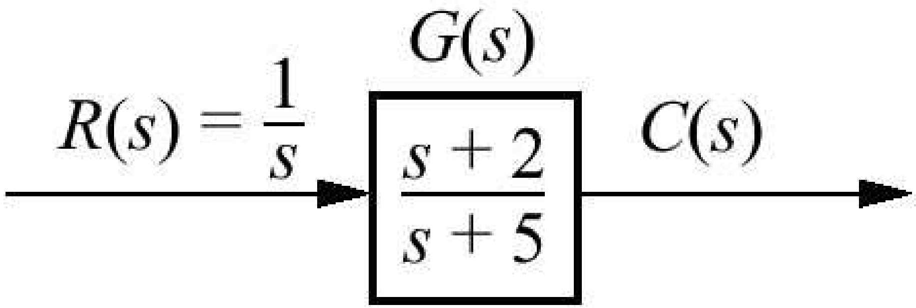
Poles of a Transfer Function

- The values of the Laplace transform variable,s, that cause the transfer function to infinite.
- Any roots of the denominator of transfer function that are common to roots of the numerator.

Zeros of a Transfer Function

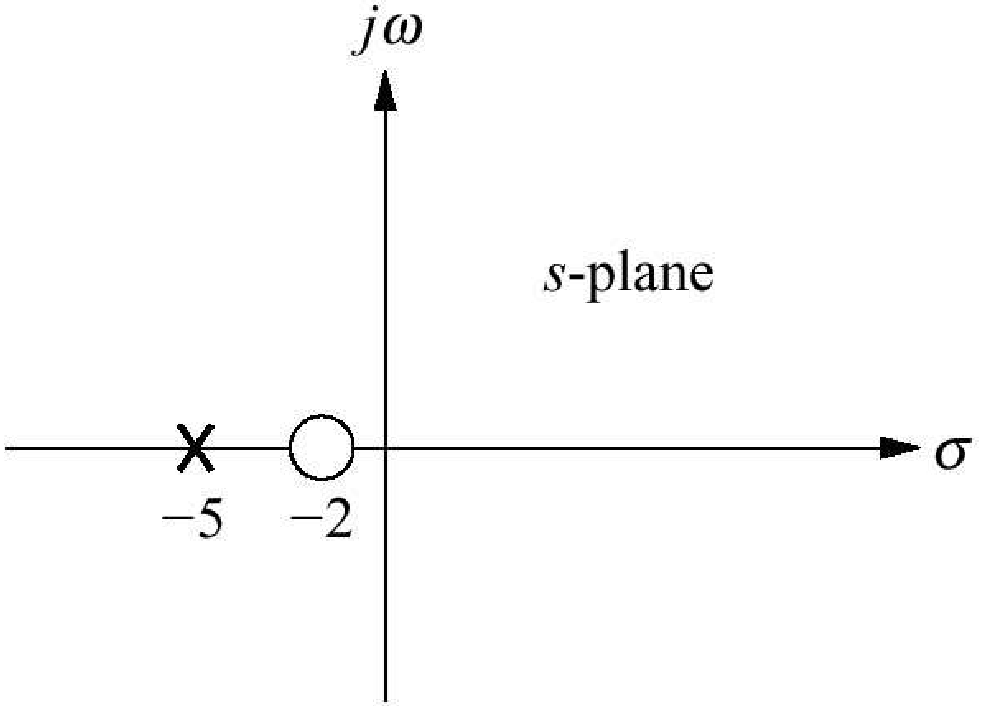
- The values of the Laplace transform variable,s, that cause the transfer function to zero.
- Any roots of the numerator of transfer function that are common to roots of the denominator.

Unit step response of systems



$$C(s)=\frac{(s+๒)}{s(s+๕)}$$

Pole–Zero plot of the system on s-plane



$$C(s)=\frac{(s+๒)}{s(s+๕)}=\frac{A}{s}+\frac{B}{s+๕}$$

By using Partial Fraction Expansion

$$A=\left[\frac{(s+๒)}{(s+๕)}\right]_{s\rightarrow ๐}=\frac{๒}{๕}$$

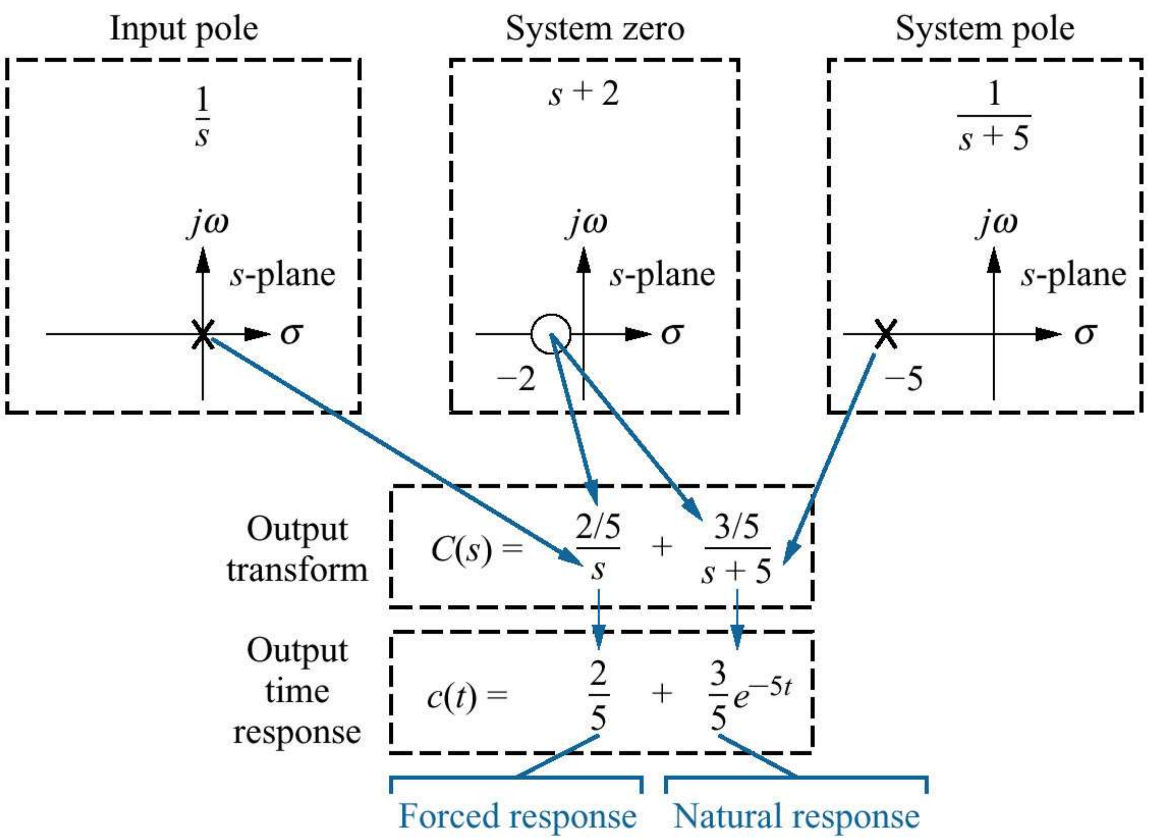
$$B=\left[\frac{(s+๒)}{s}\right]_{s\rightarrow ๕}=\frac{๓}{๕}$$

$$c(s)=\frac{๒/๕}{s}+\frac{๓/๕}{s+๕}$$

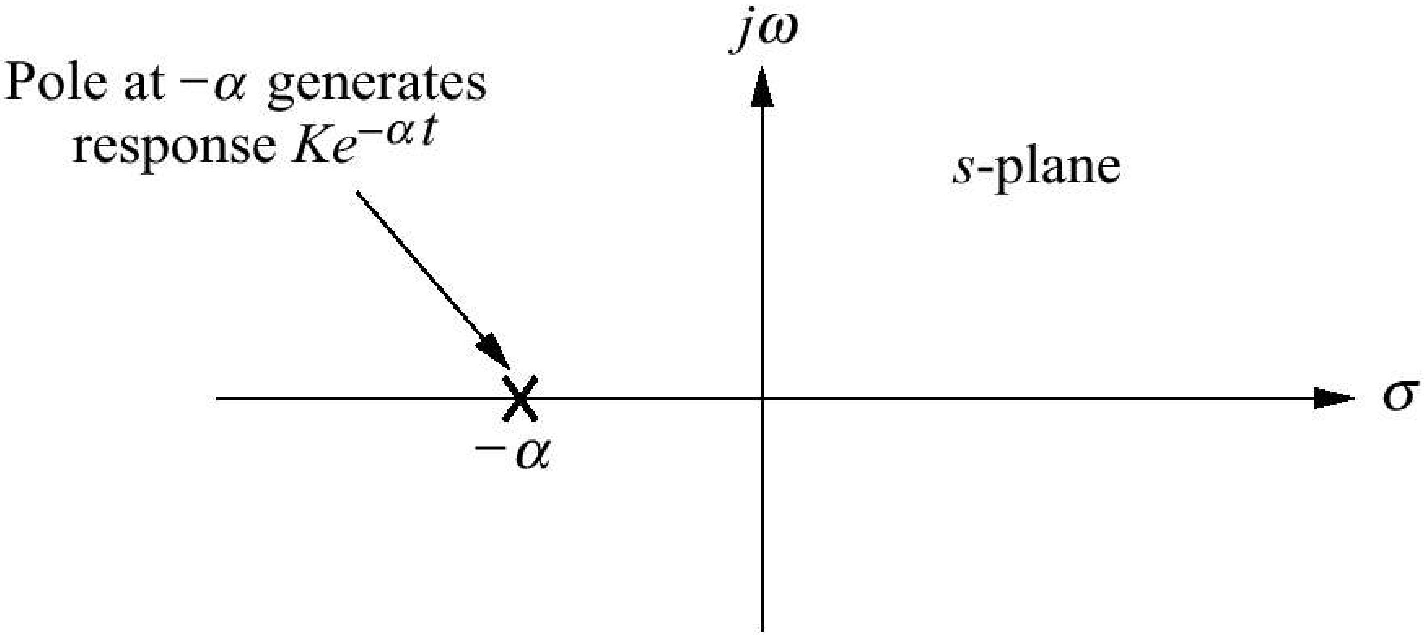
By Inverse Laplace transform

$$c(t)=\frac{๒}{๕}+\frac{๓}{๕}e^{-๕t}$$

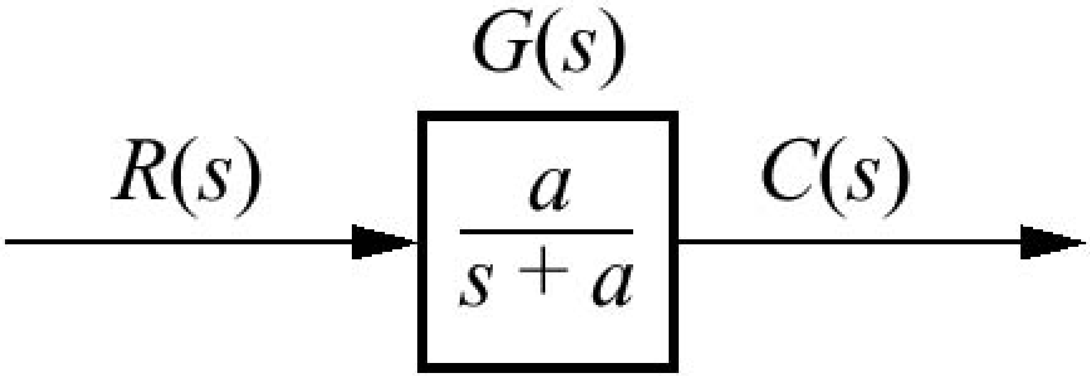
System response and response component generated by the pole or zero.



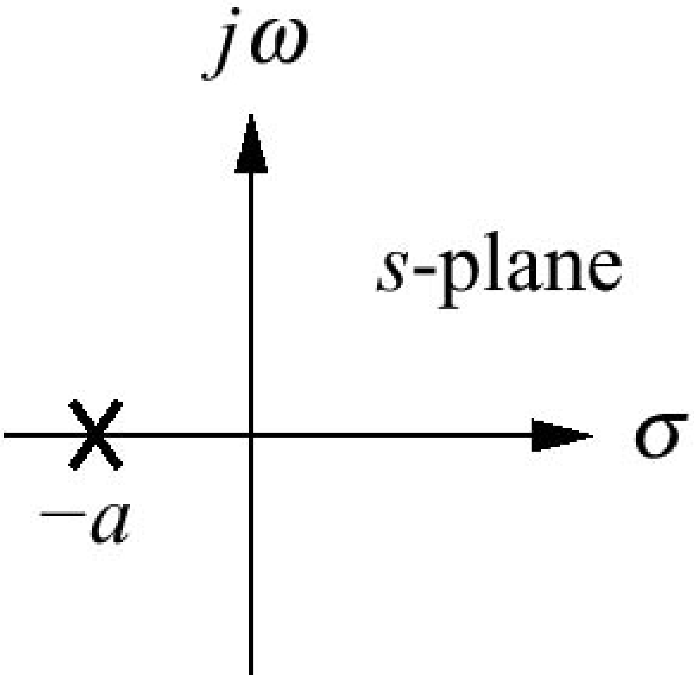
Effect of a real-axis pole upon transient response



First-Order Systems



Pole plot of first order system



Unit-step response of First-Order System

$$C(s)=R(s)G(s)=\frac{a}{s(s+a)}$$

By Inverse Laplace Transform

$$c(t)=c_f(t)+c_n(t)=๑-e^{-at}$$

Unit-step response of First-Order System

$$c(t) = ๑ - e^{-at}$$

At $t = ๑/a$, $๒/a$ and $๓/a$

$$c(t) = ๐.๖๓,\ ๐.๘๖ \text{ and } ๐.๙๕$$

Time response definition

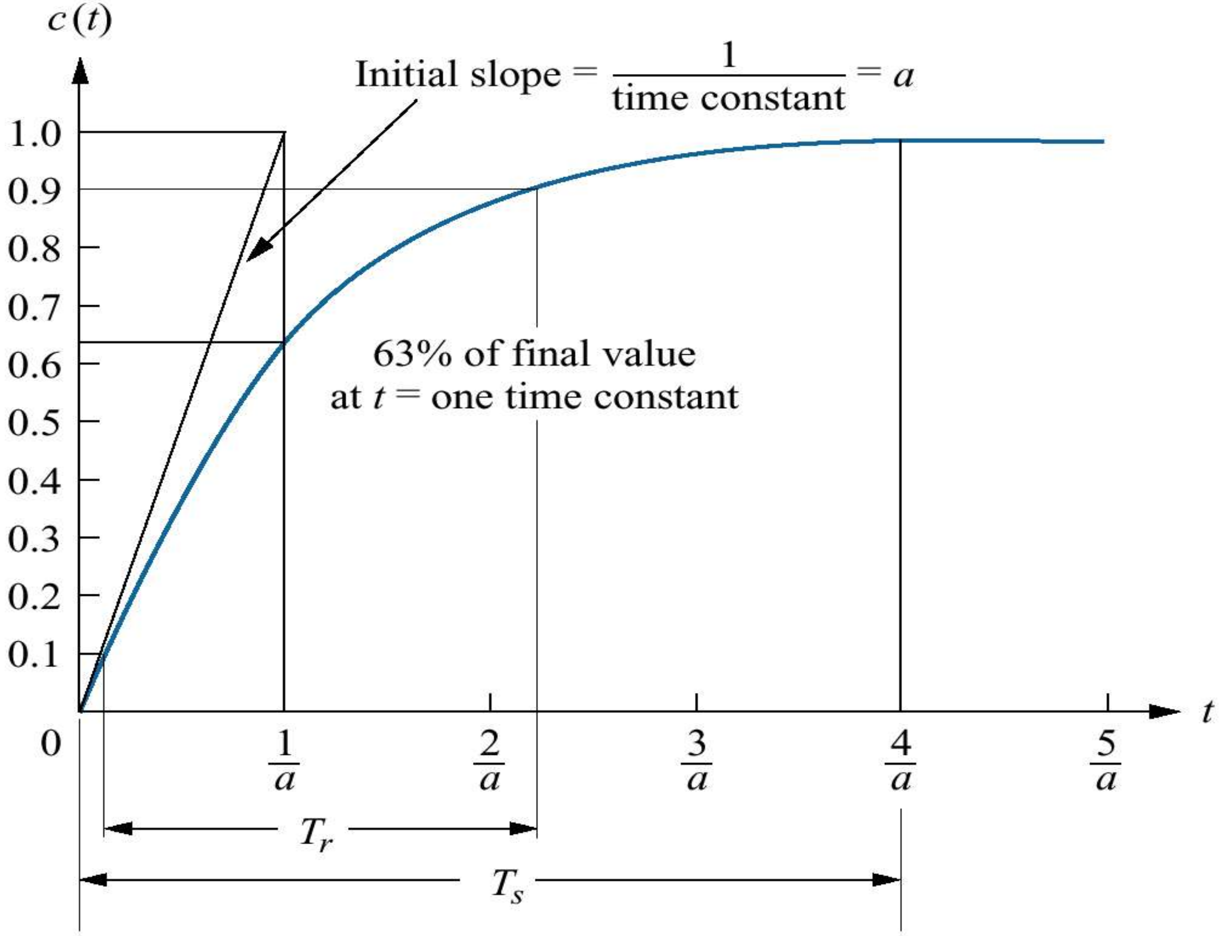
Rise time is define as the time for the waveform to go from ๐.๑ to ๐.๙ of its final value.

$$T_r = ๒.๓๑/a - ๐.๑๑/a$$

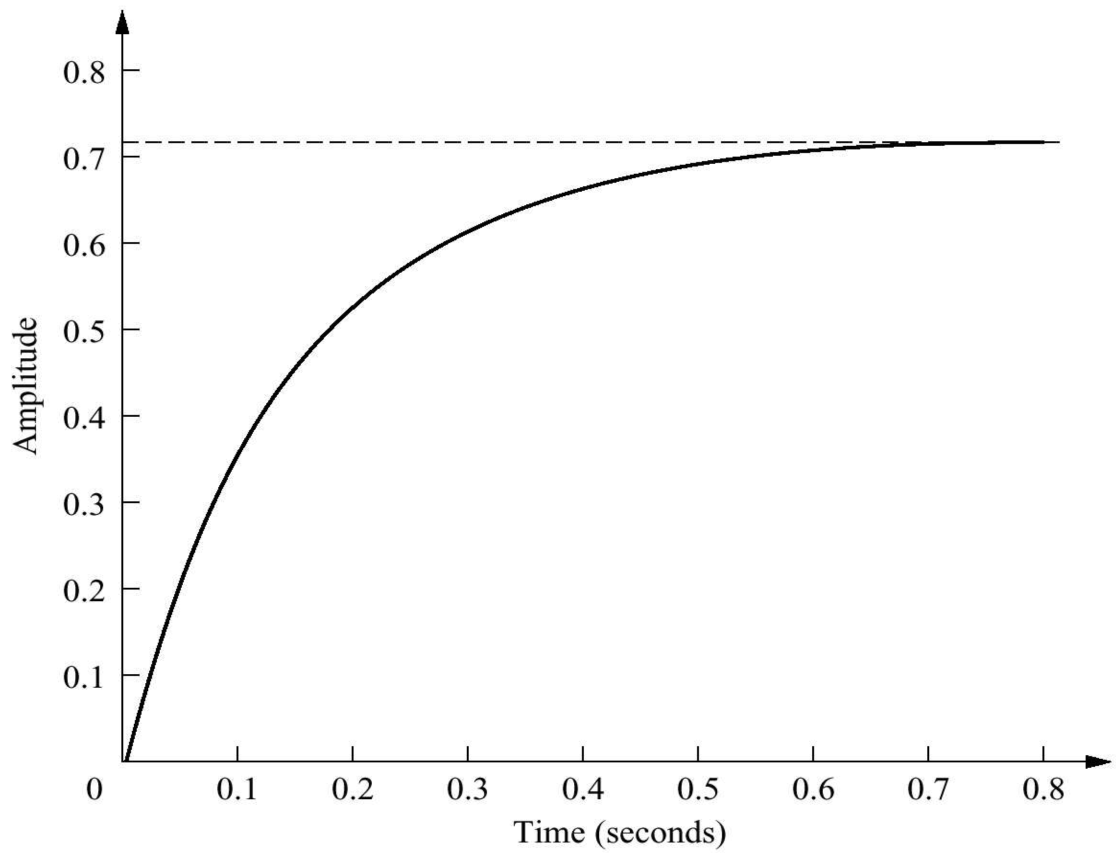
Settling time is the time for the response to reach, and stay within ๒% of its final value.

$$T_s = ๔/a$$

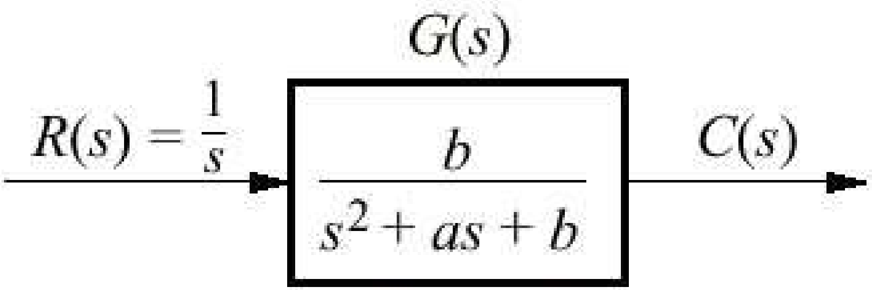
First-order system response to a unit step



Laboratory results of a system unit step response test

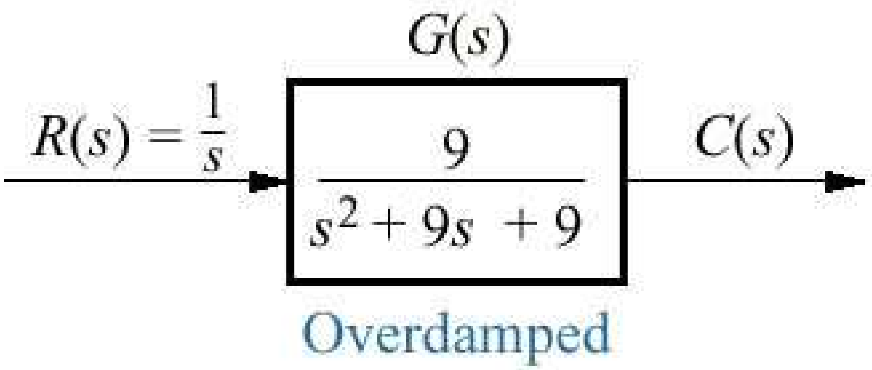


Second-Order Systems

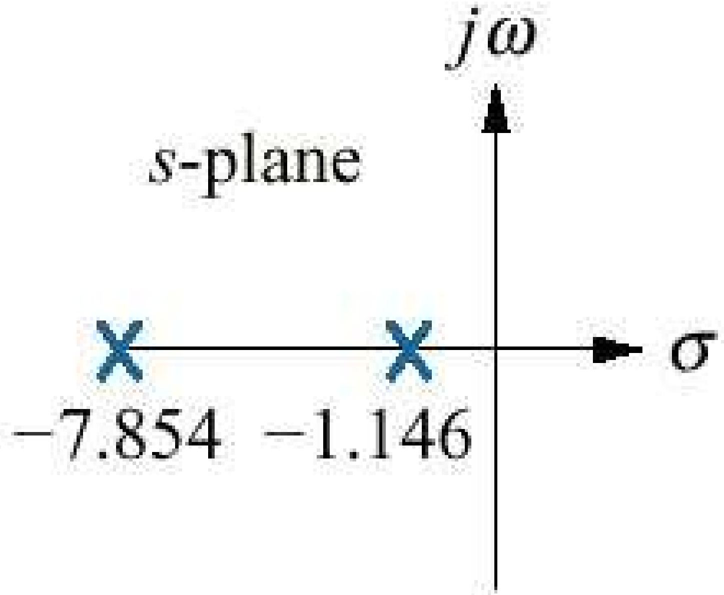


General

Overdamped Response



Poles plot of Overdamped System



Overdamped Response

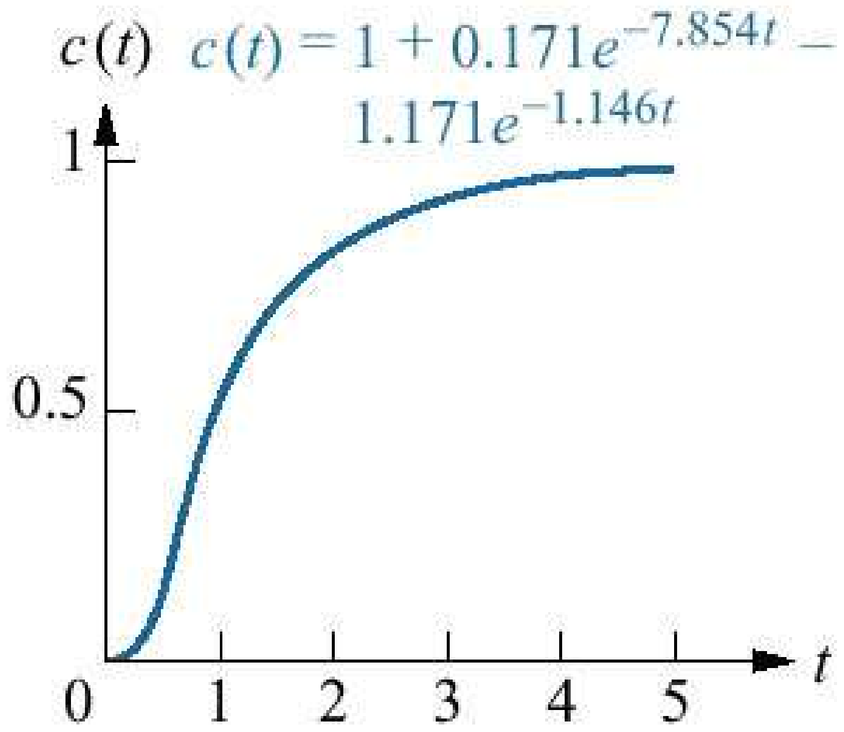
$$C(s)=\frac{๙}{s\left(s^{๒}+๙s+๙\right)}$$

$$=\frac{๙}{s\left(s+๗.๘๕๔\right)\left(s+๑.๑๔๖\right)}$$

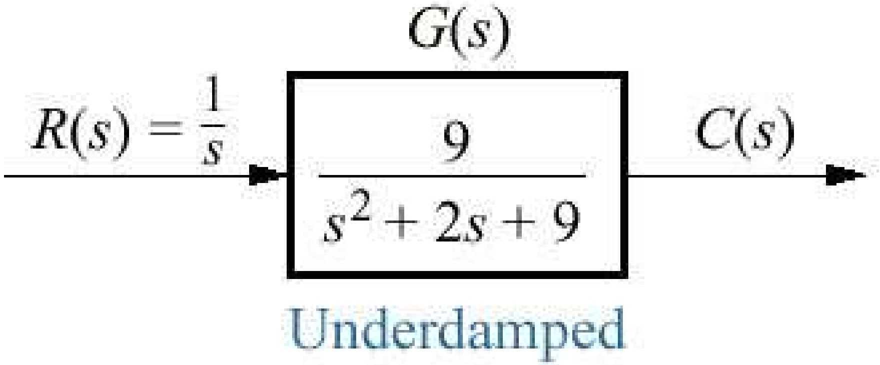
By Inverse Laplace Transform

$$c(t)=K_{๑}+K_{๒}e^{-๗.๘๕๔t}+K_{๓}e^{-๑.๑๔๖t}$$

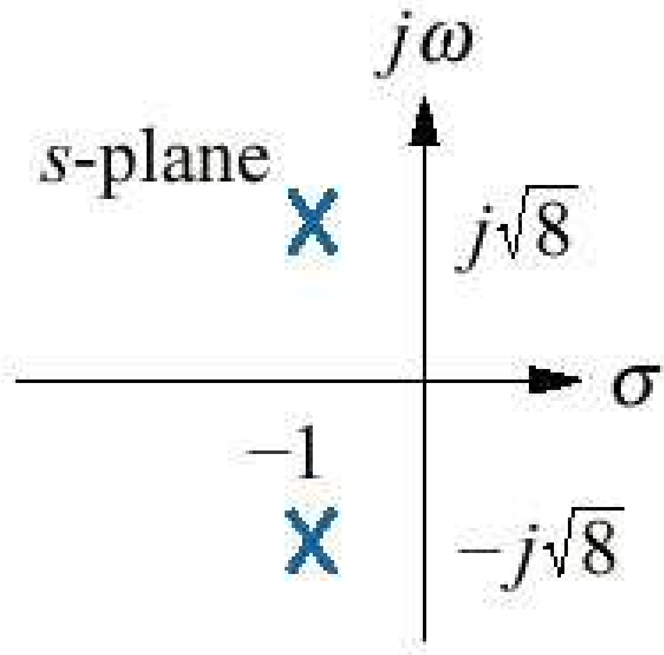
Overdamped Response



Underdamped Response



Poles plot of Underdamped Response



Underdamped Response

$$C(s)=\frac{๙}{s(s^{๒}+๒s+๙)} \qquad s=-๑\pm j\sqrt{๘}$$

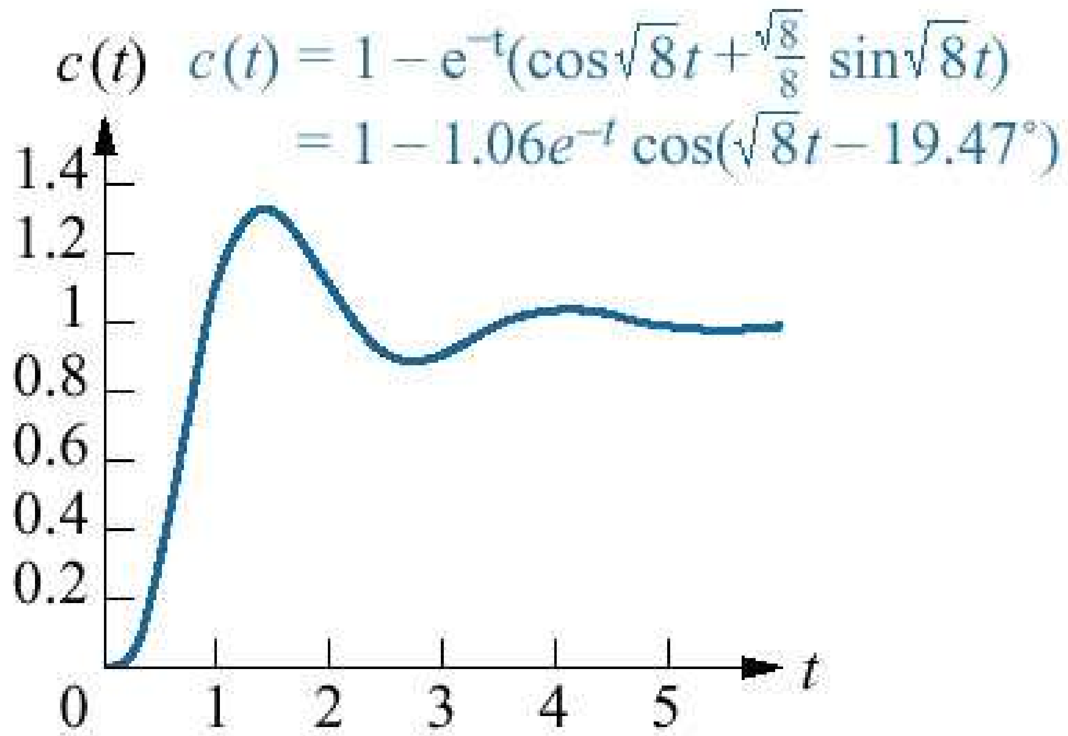
$$C(s)=\frac{K_{๑}}{s}+\frac{K_{๒}}{s-(๑+j\sqrt{๘})}+\frac{K_{๓}}{s-(๑-j\sqrt{๘})}$$

By Inverse Laplace Transform

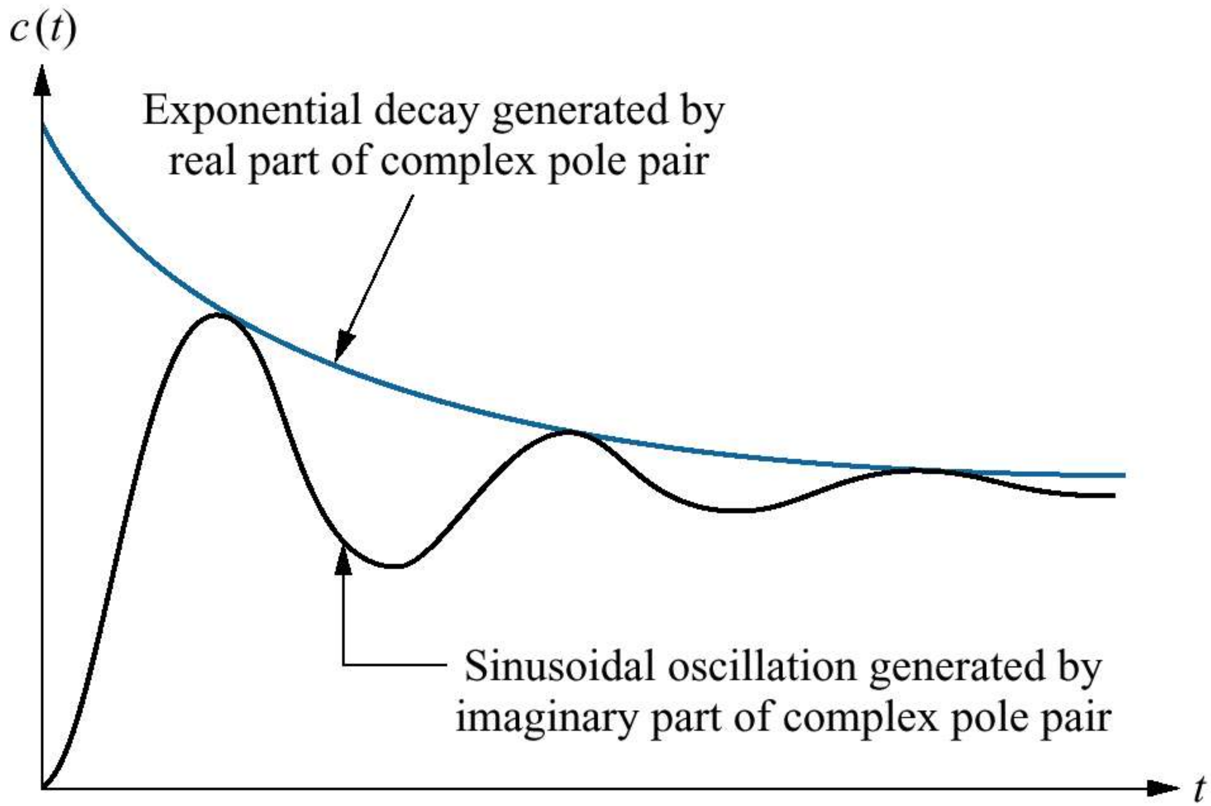
$$c(t)=๑-e^{-t}\left(\cos\sqrt{๘}t+\frac{\sqrt{๘}}{๘}\sin\sqrt{๘}t\right)$$

$$c(t)=๑-๑.๐๖e^{-t}\cos(\sqrt{๘}t-๑๙.๔๗^{\circ})$$

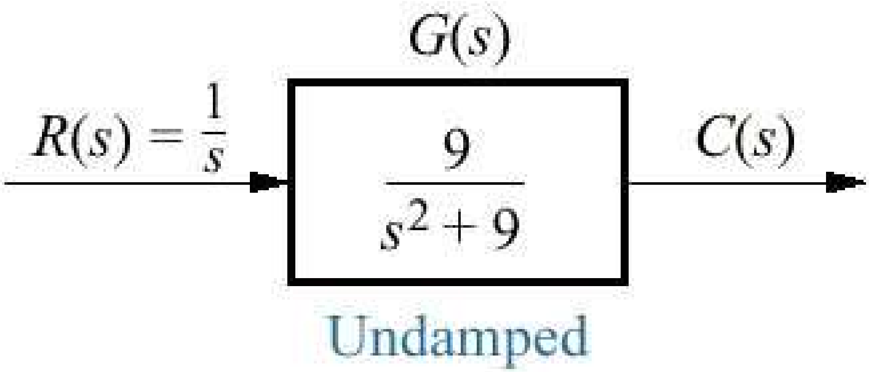
Underdamped Response



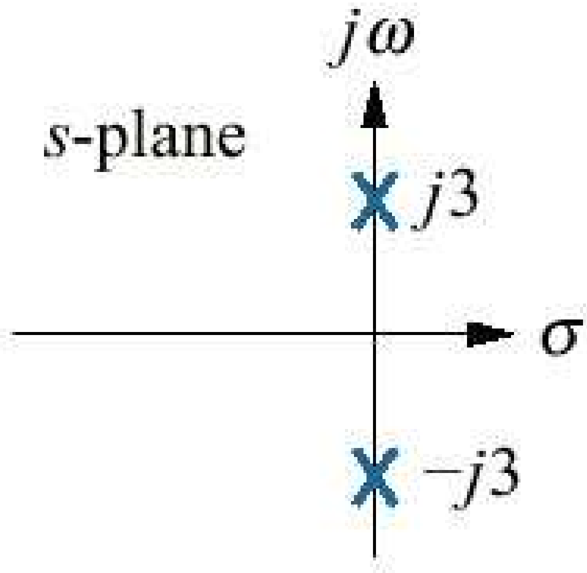
Second-order unit step response components generated by complex poles



Undamped Response



Poles plot of Undamped Response



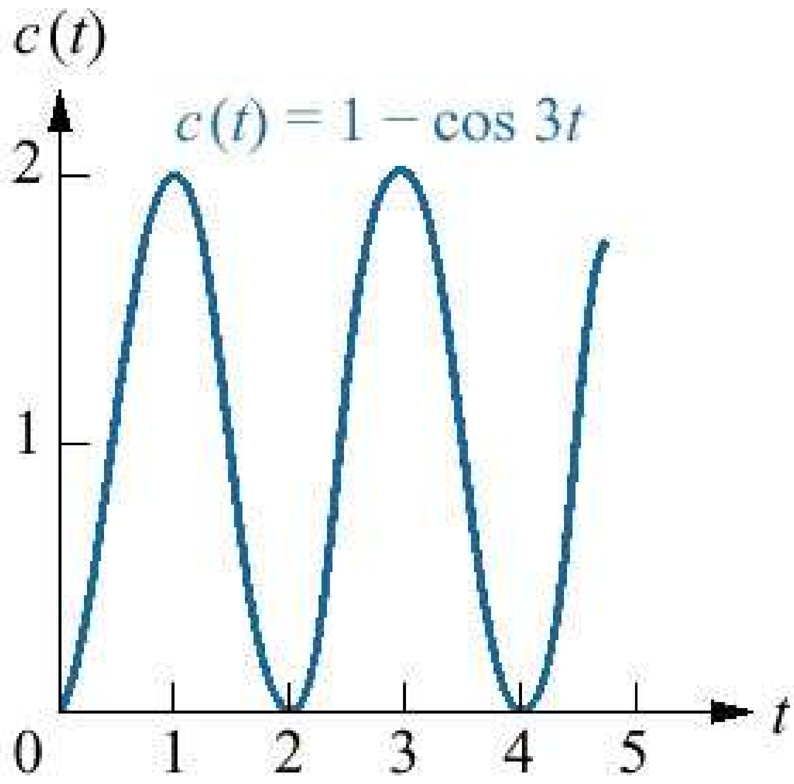
Undamped Response

$$C(s)=\frac{๙}{s(s^{๒}+๙)}$$

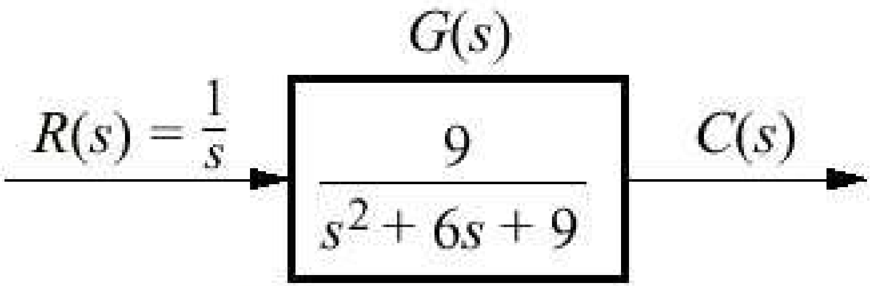
$$s=\pm j๓$$

$$c(t)=K_{๑}+K_{๔}\cos(๓t-\phi)$$

Undamped Response

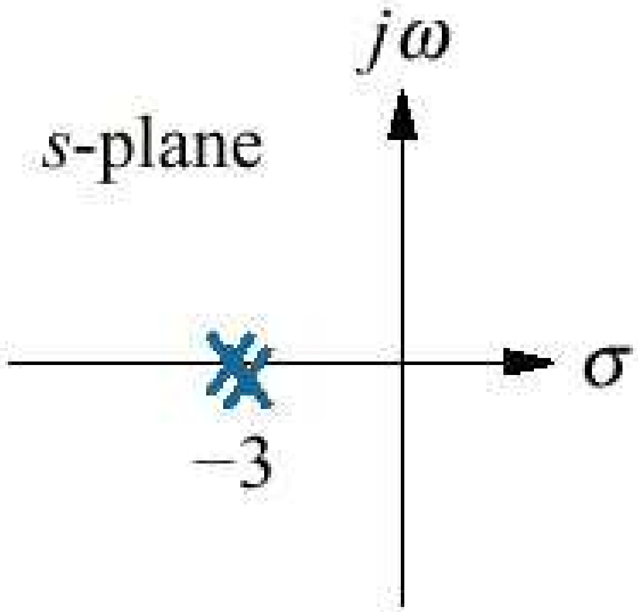


Critically Damped Response



Critically damped

Critically Damped Response



Critically Damped Response

$$C(s)=\frac{๙}{s(s^{๒}+๖s+๙)}$$

$$s=-๓$$

$$c(t)=K_{๑}+K_{๒}e^{-๓t}+K_{๓}te^{-๓t}$$

$$K_{๑}=๑$$

Critically Damped Response

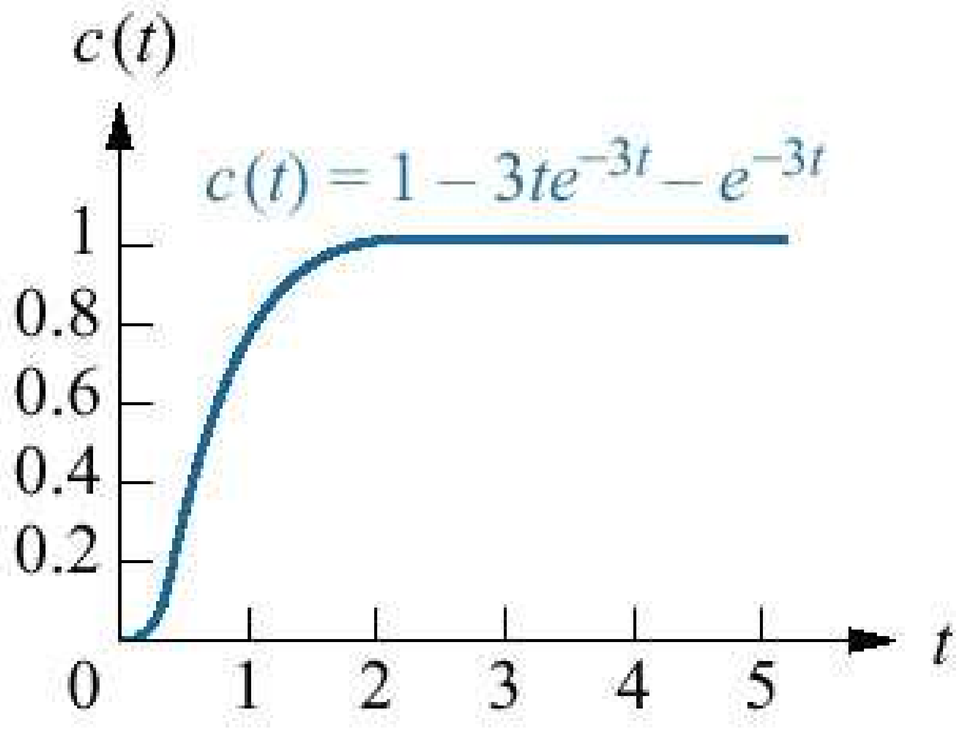
Two real poles at same point

$$c(t) = K_{๑} e^{-\sigma_{๑} t} + K_{๒} t\, e^{-\sigma_{๒} t}$$

Then

$$c(t) = ๑ - ๓ t e^{-๓t} - e^{-๓t}$$

Critically Damped Response



Natural responses of second order system

๑. Overdamped response

Poles : two real at $-\sigma_{๑}, \sigma_{๒}$

Natural response :

$$\boxed{c(t) = K_{๑} e^{-p_{๑} t} + K_{๒} e^{-p_{๒} t}}$$

๒. Underdamped response

Poles : two real at $-\sigma_d \pm j\omega_d$

Natural response :

$$\boxed{c(t) = K_{๑} e^{-\sigma_d t} \cos(\omega_d t - \phi)}$$

๓. Undamped response

Poles : two real at $\pm j\omega_n$

Natural response :

$$\boxed{c(t) = A\cos(\omega_๑ t - \phi)}$$

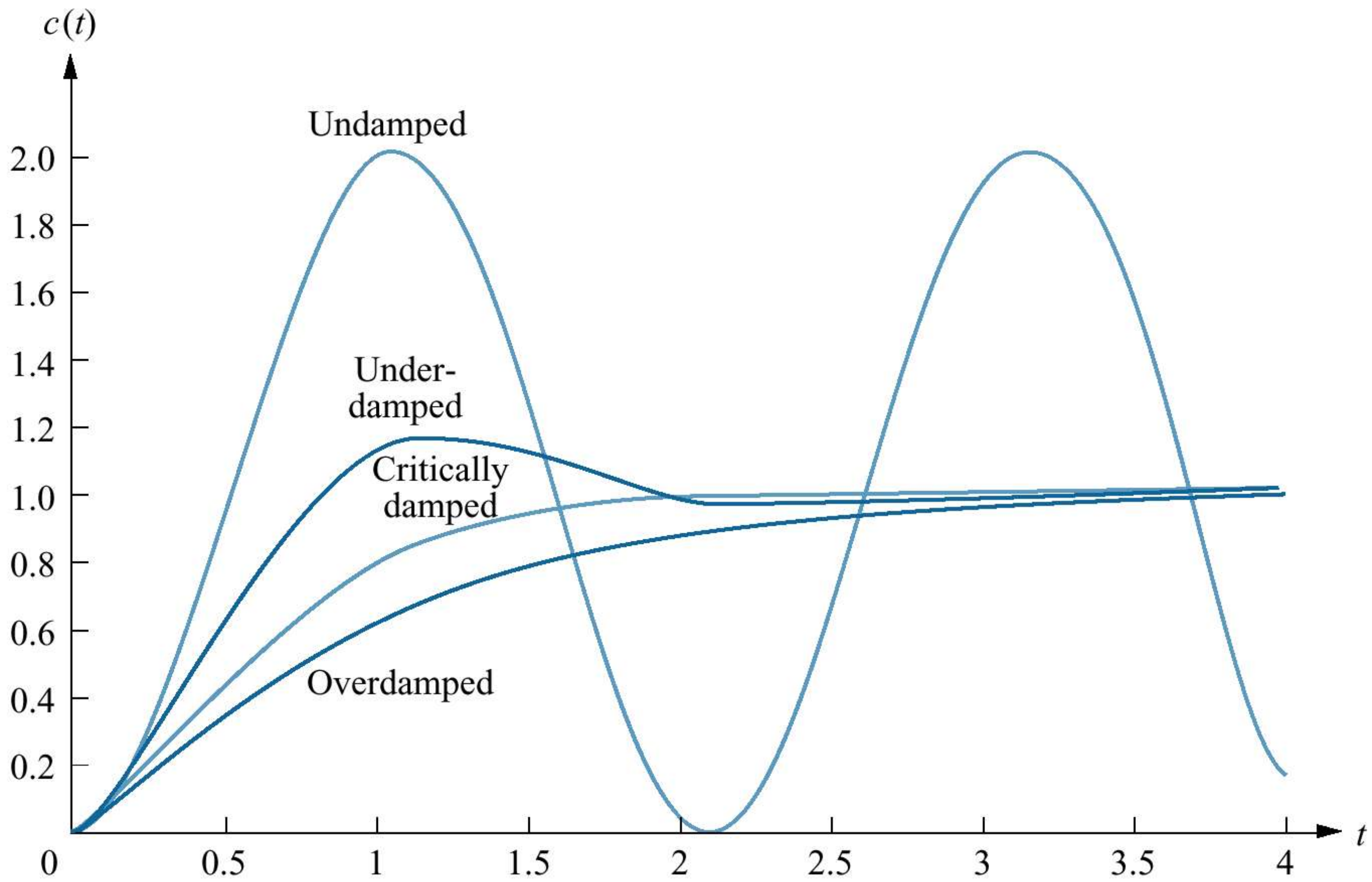
๔. Critically damped responses

Poles : two real at $-\sigma_{๑}$

Natural response :

$$\boxed{c(t) = K_{๑} e^{-p_{๑} t} + K_{๒} t e^{-p_{๑} t}}$$

Step responses for second-order system damping cases



The General Second-Order Systems

Transfer function

$$\boxed{G(s)=\frac{\omega_n^{๒}}{s^{๒}+๒\zeta\omega_n s+\omega_n^{๒}}}$$

ω_n - Natural Frequency

ζ - Damping Ratio

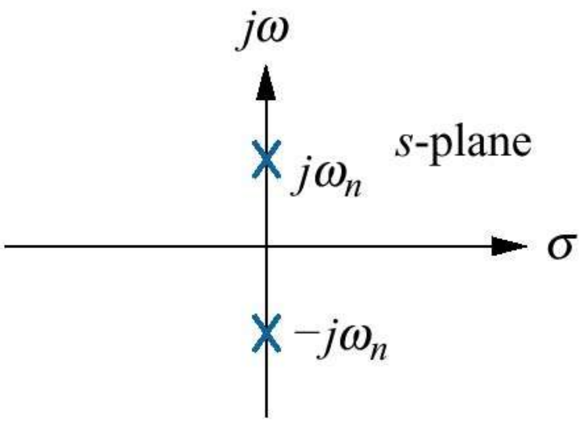
Poles of the Transfer function

$$s_{๑,๒} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^{๒} - ๑}$$

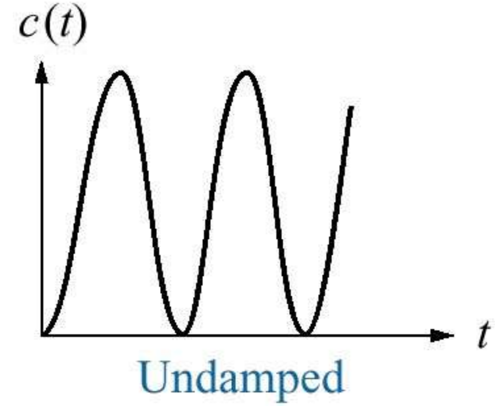
The General Second-Order Systems

Second-order response as a function of damping ratio.

$$\zeta = ๐$$

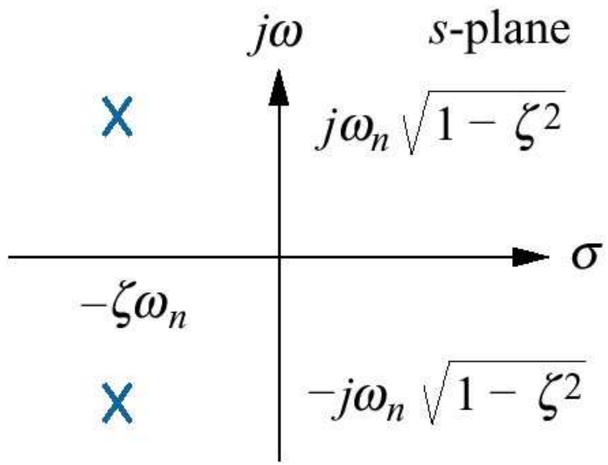


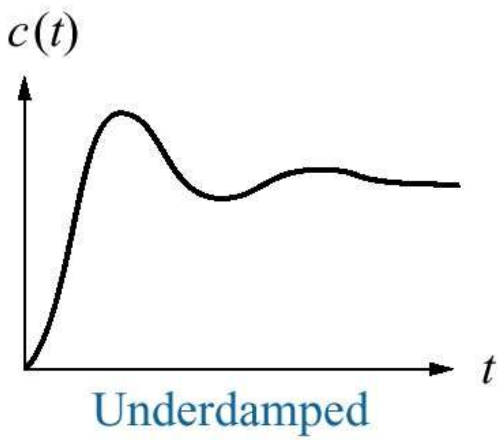
Poles



Response

$$๐ < \zeta < ๑$$

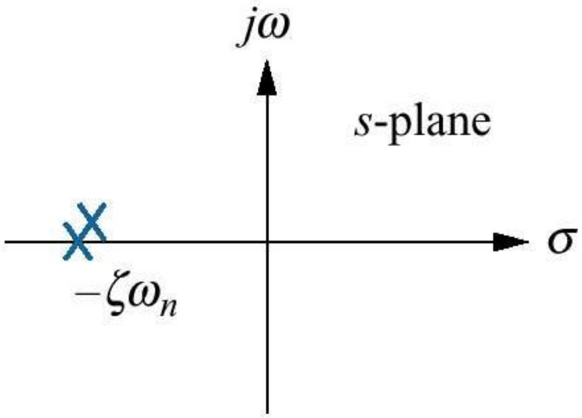




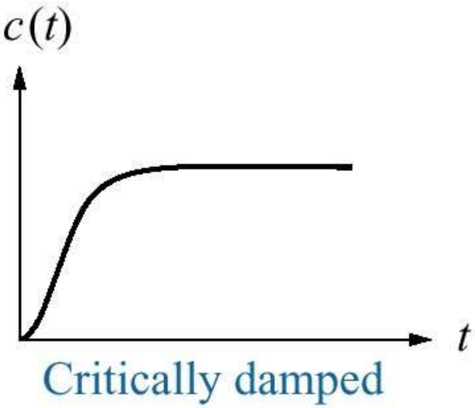
Poles

Response

$$\zeta = ๑$$

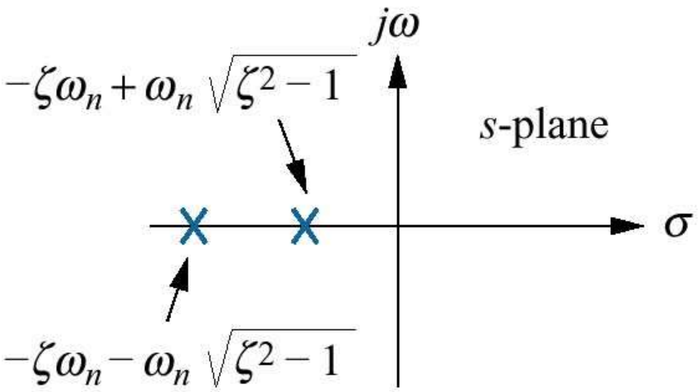


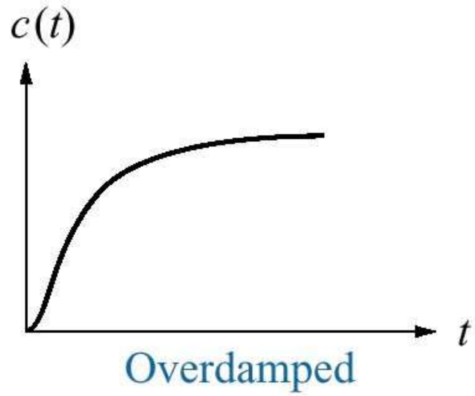
Poles



Response

$\zeta >$ ๑



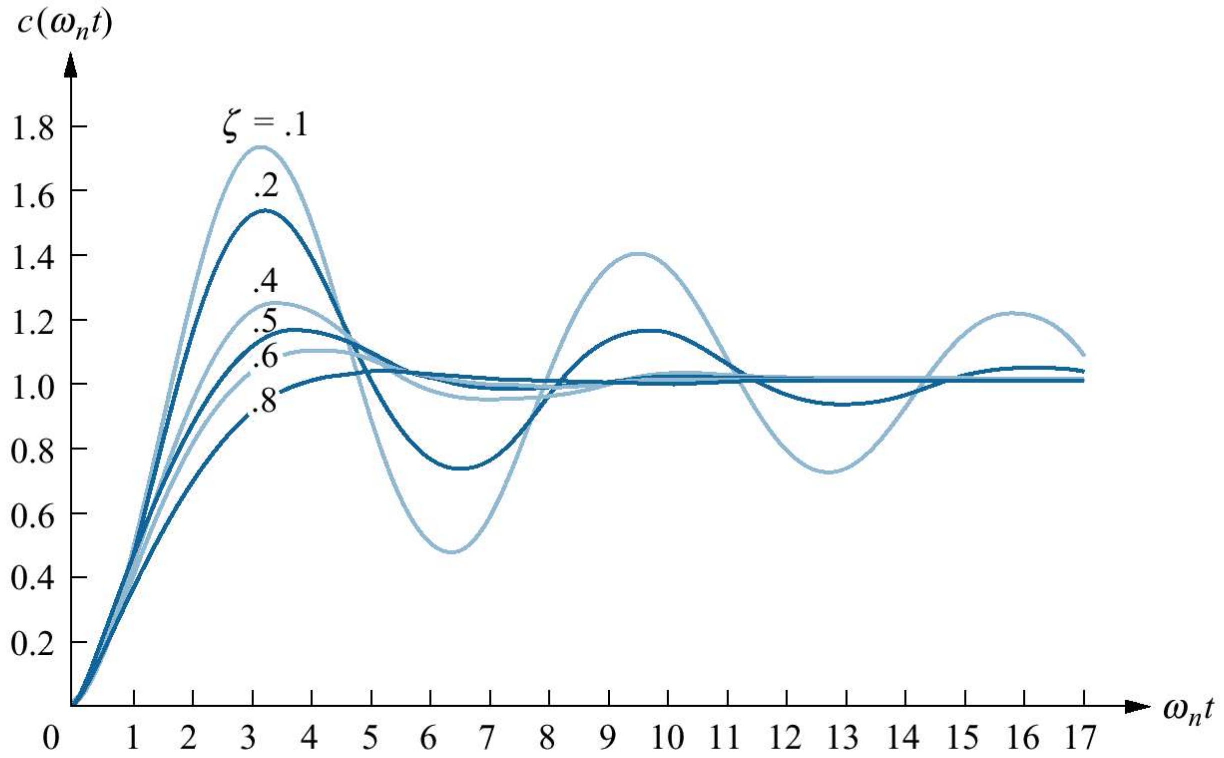


Poles

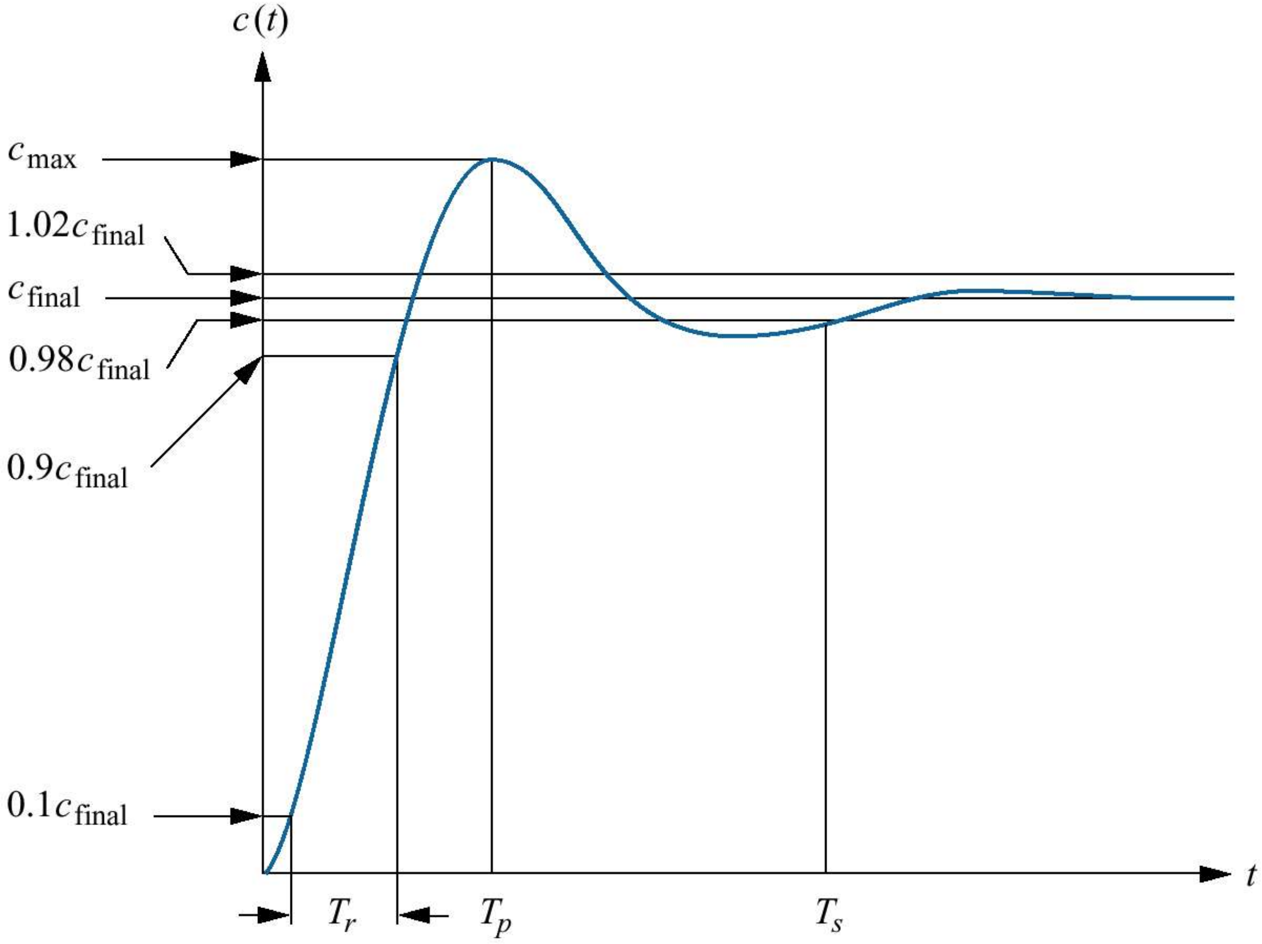
Response

Underdamped Second-Order Systems

Second-order underdamped responses for damping ratio values



Second-order underdamped response specifications



Natural Frequency of a second order system is the frequency of oscillation of the system without damp ing.

Damping ratio is the proportion between exponentia l decay frequency and natural frequency (rad/second)

$$\zeta = \frac{Exponential\ decay\ frequency}{Natural\ frequency\left(rad / second\right)}$$

$$= \frac{\text{๑}}{\text{๒}\pi} \frac{natural\ period\left(seconds\right)}{Exponential\ time\ constant}$$

Percent overshoot is a ratio between the different between system time response and final value.

$$OS = \frac{c_{max} - c_{final}}{c_{final}} \times ๑๐๐$$

Delay time is the time require for the system time response y(t) to reach ๕๐% of final value.

Peak time is the time require to reach the first, or maximum, peak.

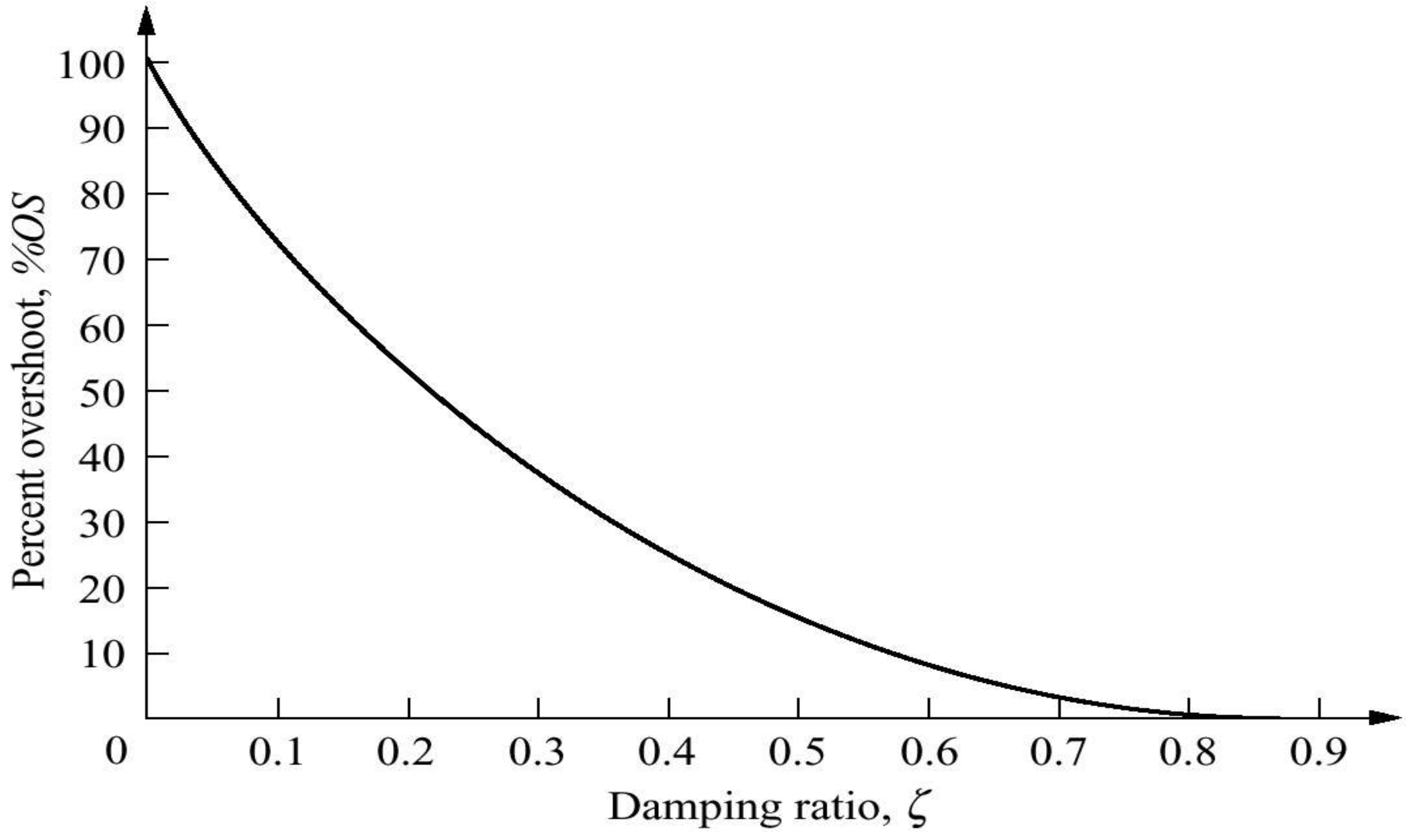
Time constant is the time require for step response t o rise to ๖๓% of its final value.

Time constant has the units (๑ seconds), or frequency, thus we call the parameter *a* the exponential frequency.

$$c(t) = ๑ - e^{-at}$$

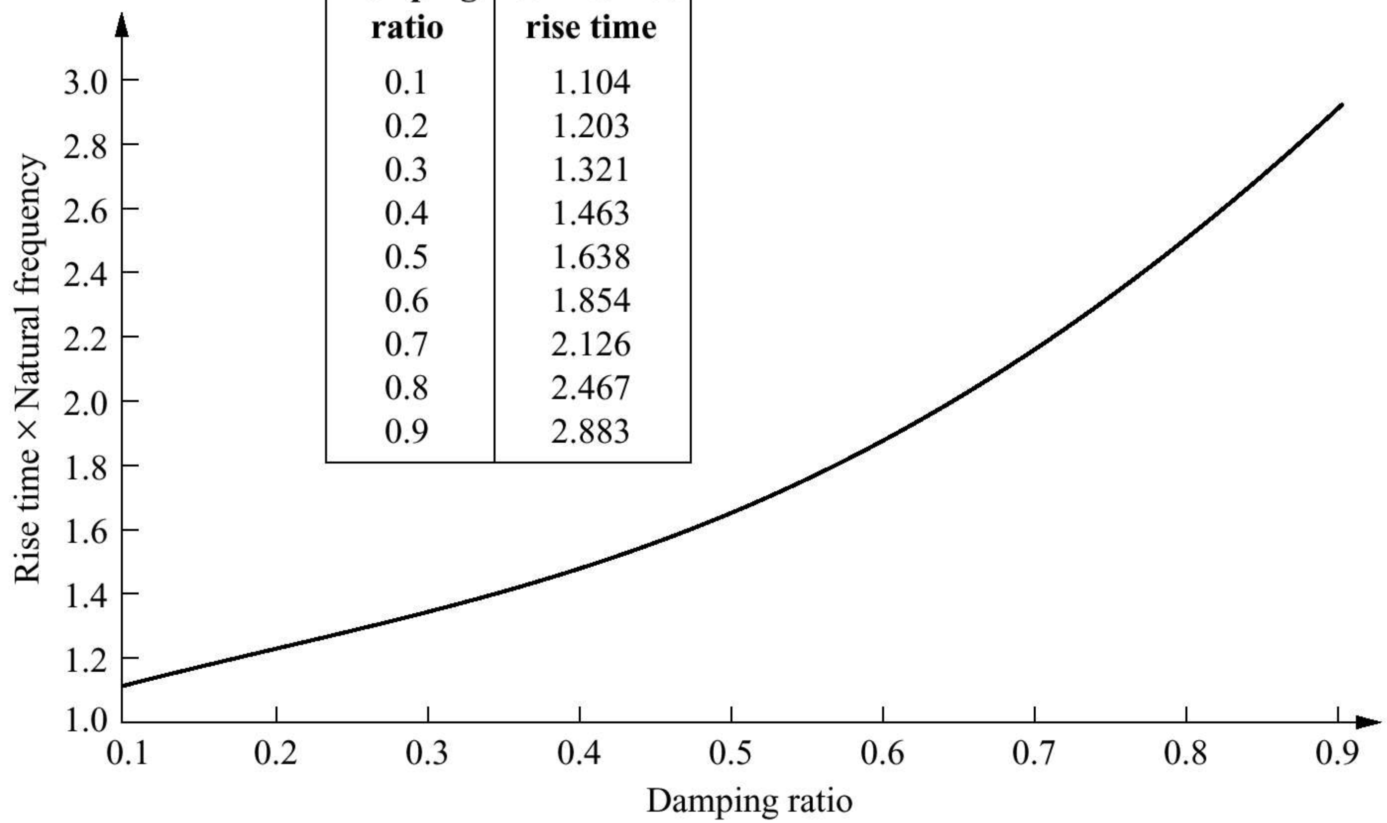
$$t = \frac{๑}{a}$$

Percent overshoot vs. damping ratio

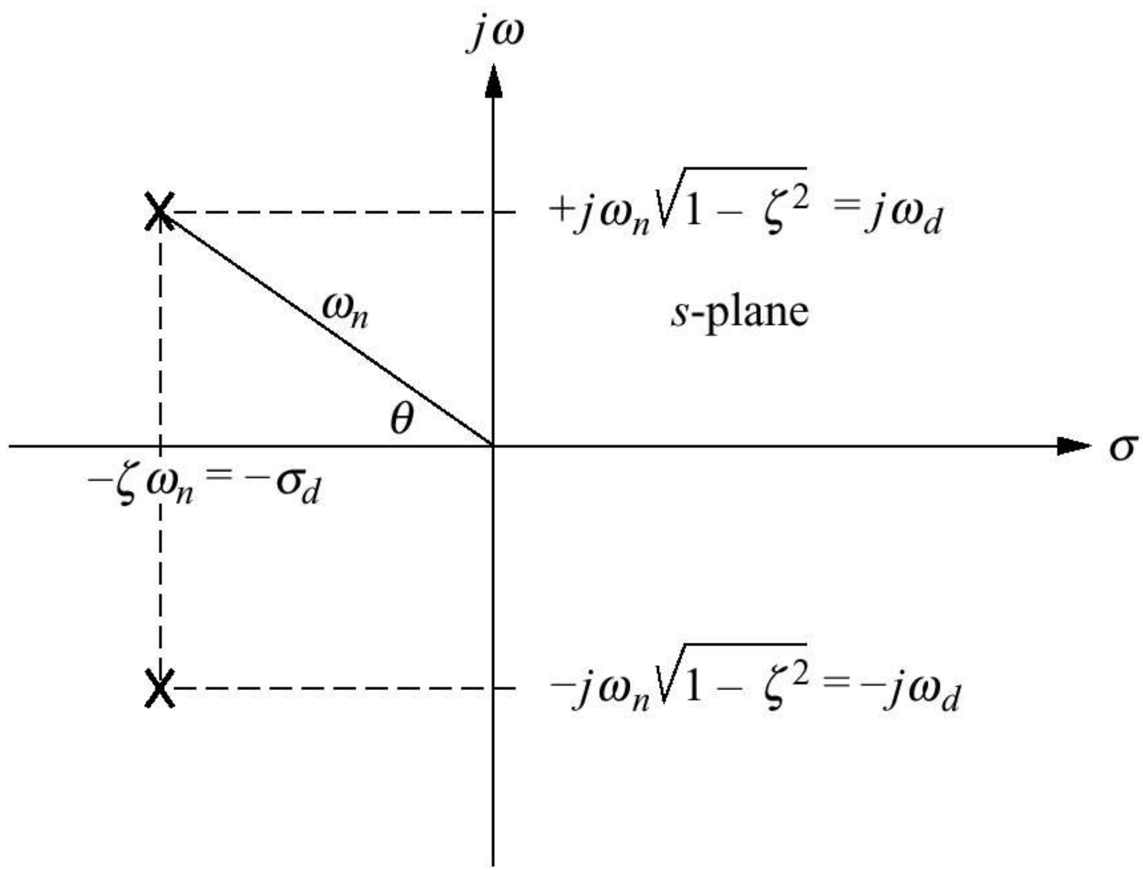


Normalized risetime vs. damping ratio for a second-order underdamped response

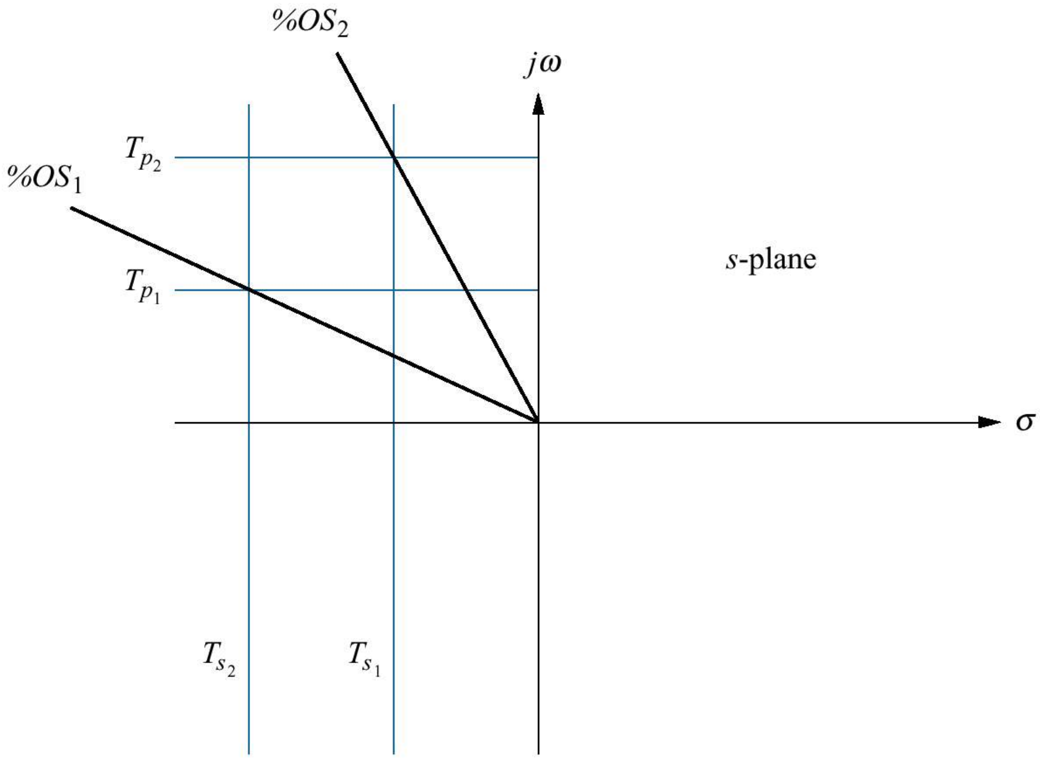
Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



Pole plot for an underdamped second-order system



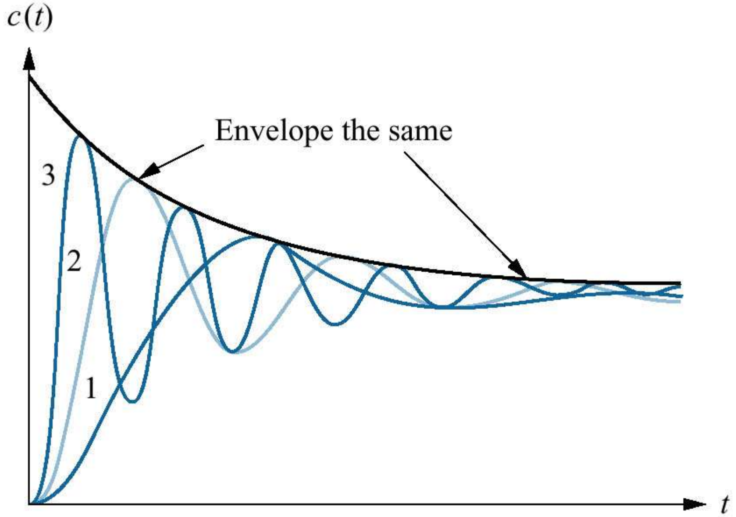
Lines of constant peak time,T_p , settlingtime,T_s , and percent overshoot, %OS

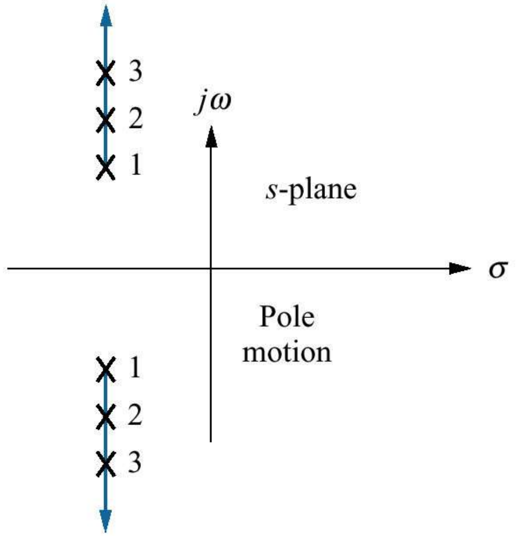


Note: $T_{s_๒} < T_{s_๑}$; $T_{p_๒} < T_{p_๑}$; $\%OS_๑ < \%OS_๒$

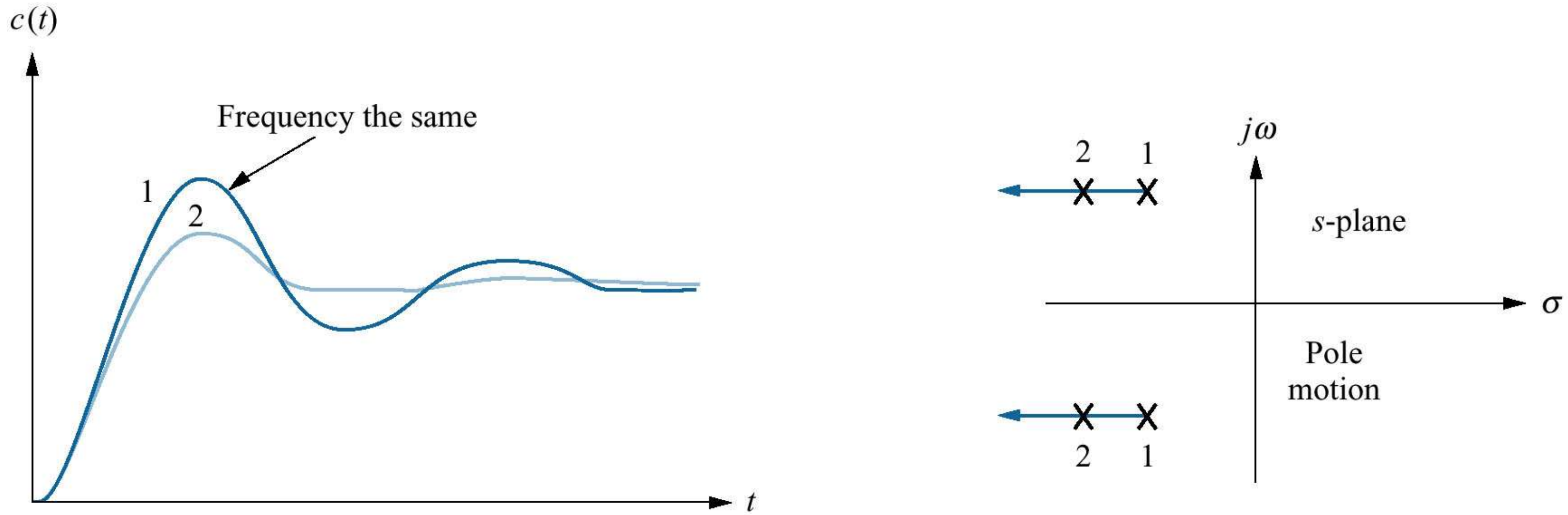
Step responses of second-order underdamped systems as poles move

Constant real part

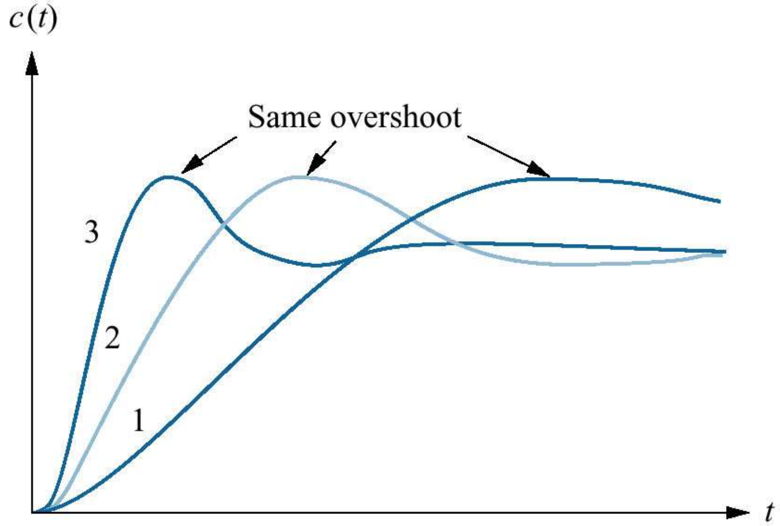


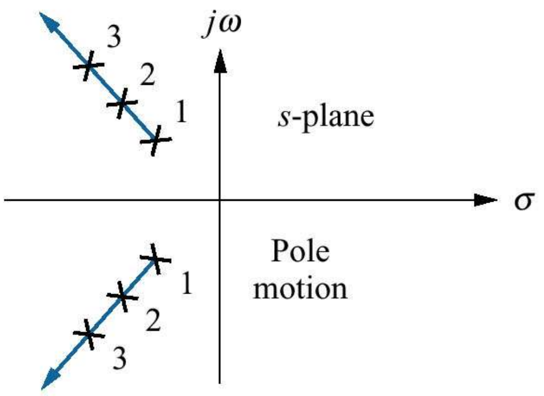


Constant imaginary part



Constant damping ratio





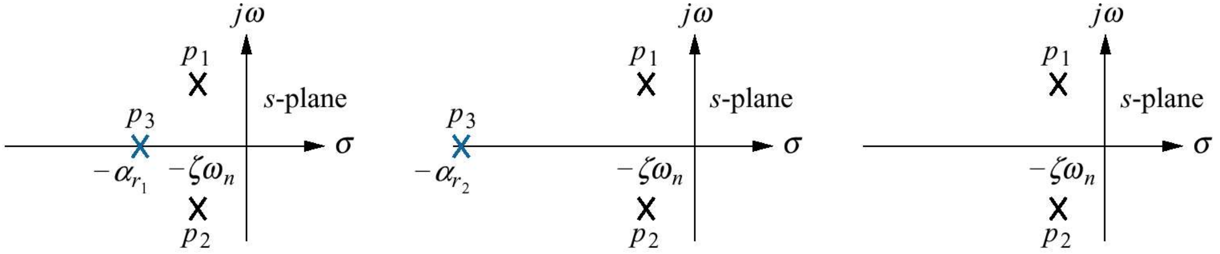
System Response with Additional Poles

Component responses of a three-pole system :

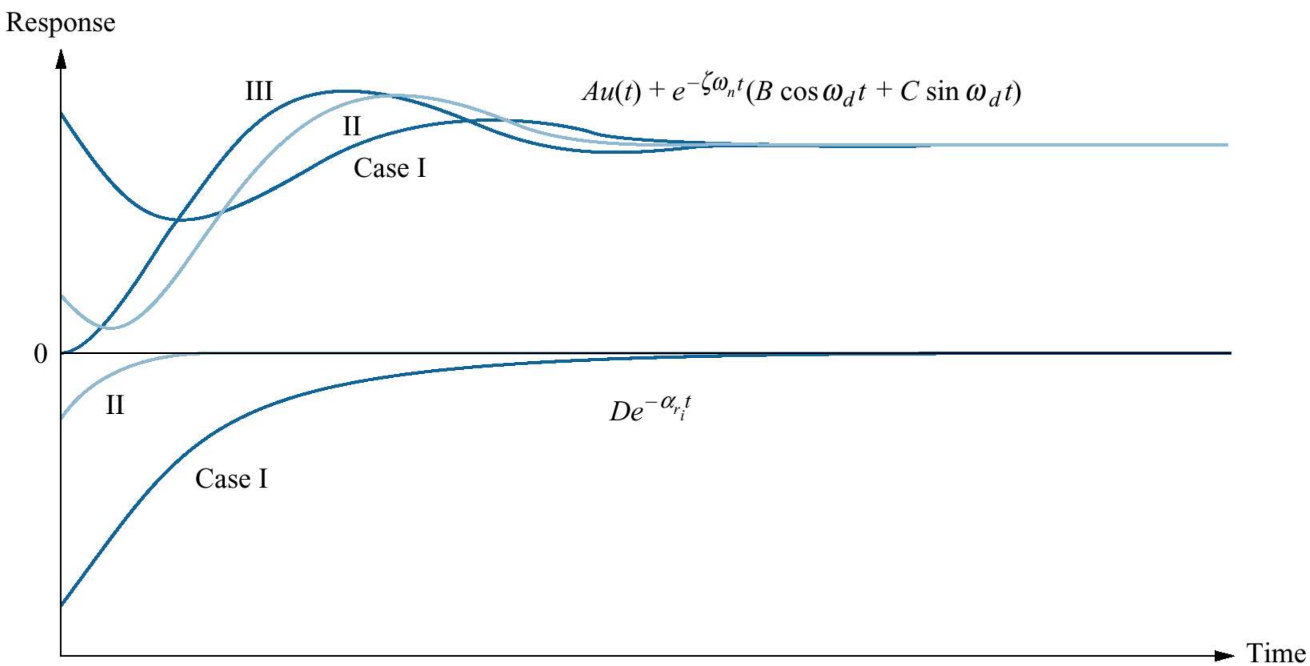
Case I : non-dominant pole is near dominant second-order poles pair.

Case II : non-dominant pole is far from the pair.

Case III : non-dominant pole is at infinity.


$j\omega$
p_1
s-plane
p_3
σ
$-\alpha_{r_1}$
$-\zeta\omega_n$
p_2
Case I
$j\omega$
p_1
s-plane
p_3
σ
$-\alpha_{r_2}$
$-\zeta\omega_n$
p_2
Case II
$j\omega$
p_1
s-plane
σ
$-\zeta\omega_n$
p_2
Case III

(a)


Response
III
II
Case I
$Au(t) + e^{-\zeta\omega_n t}(B \cos \omega_d t + C \sin \omega_d t)$
0
II
$De^{-\alpha_{r_i} t}$
Case I
Time

(b)

Example ๔.๘ (Nise)

$$T_{๑}(s)=\frac{๒๔.๕๔๒}{s^{๒}+๔s+๒๔.๕๔๒}$$

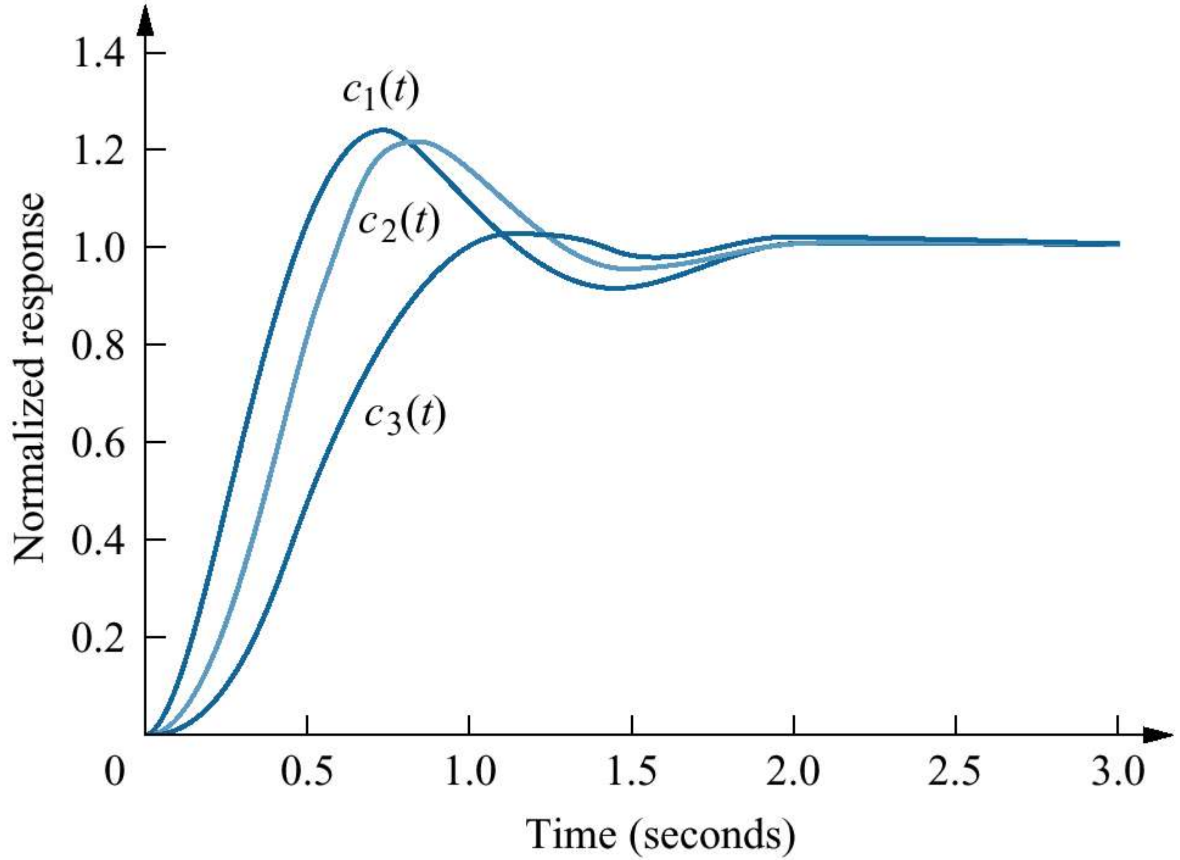
$$T_{๒}(s)=\frac{๒๔.๕๔๒}{(s+๑๐)(s^{๒}+๔s+๒๔.๕๔๒)}$$

$$T_{๓}(s)=\frac{๒๔.๕๔๒}{(s+๓)(s^{๒}+๔s+๒๔.๕๔๒)}$$

$$c_{๑}(t) = ๑ - ๑.๐๙e^{-๒t}\cos(๔.๕๓๒t - ๒๓.๘)$$

$$c_{๒}(t) = ๑ - ๐.๒๙e^{-๑๐t} - ๑.๑๘๙e^{-๒t}\cos(๔.๕๓๒t - ๕๓.๓๔)$$

$$c_{๓}(t) = ๑ - ๑.๑๔e^{-๓t} - ๐.๗๐๓e^{-๒t}\cos(๔.๕๓๒t + ๗๘.๖๗)$$



Step responses of system $T_๑(s)$, system $T_๒(s)$, and system $T_๓(s)$

System Response with Zeros

$$T(s)=\frac{(s+a)}{(s+b)(s+c)}=\frac{A}{s+b}+\frac{B}{s+c}$$

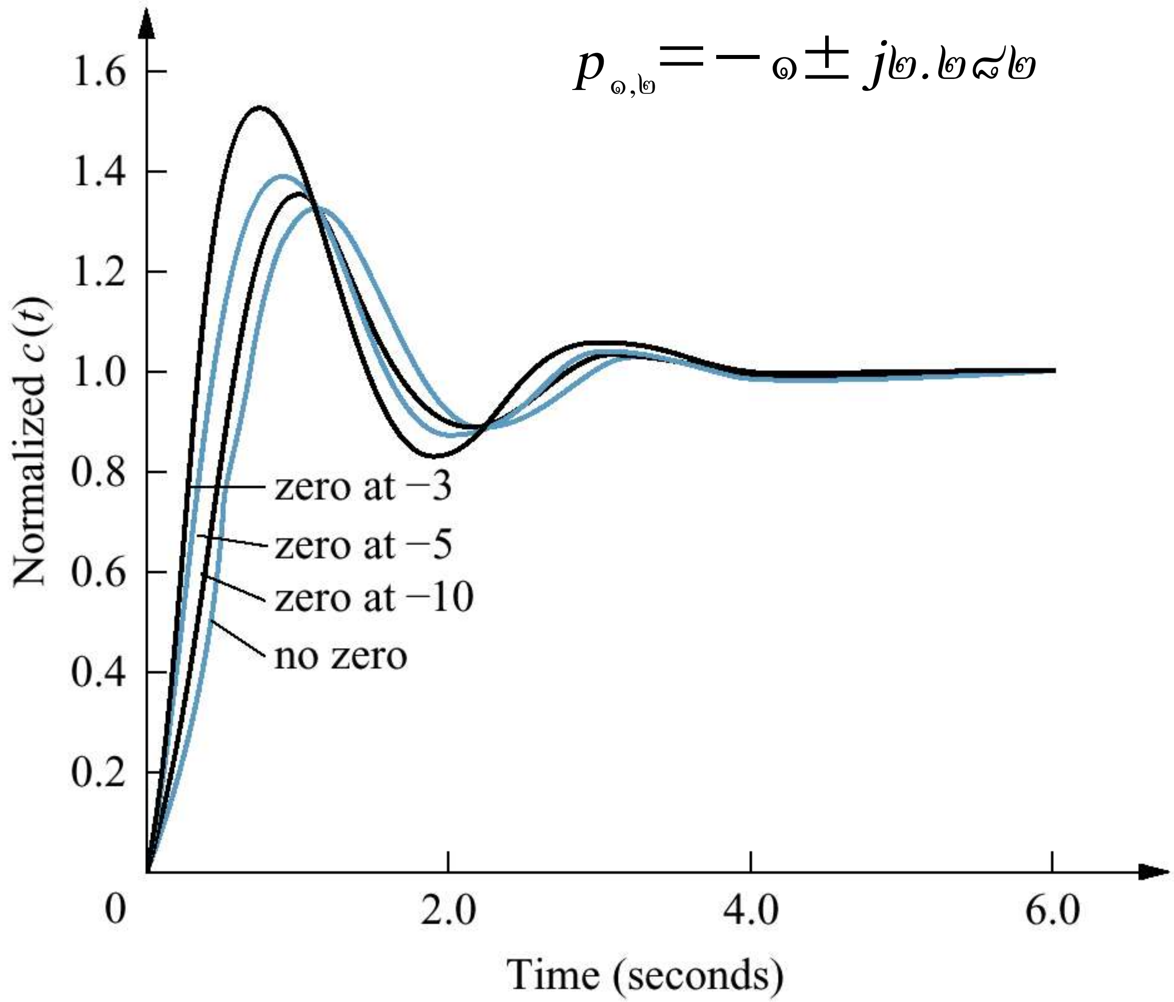
$$T(s)=\frac{(-b+a)/(-b+c)}{s+b}+\frac{(-c+a)/(-c+b)}{s+c}$$

If zero is far from poles, then *a* is large compared to *b* and *c*.

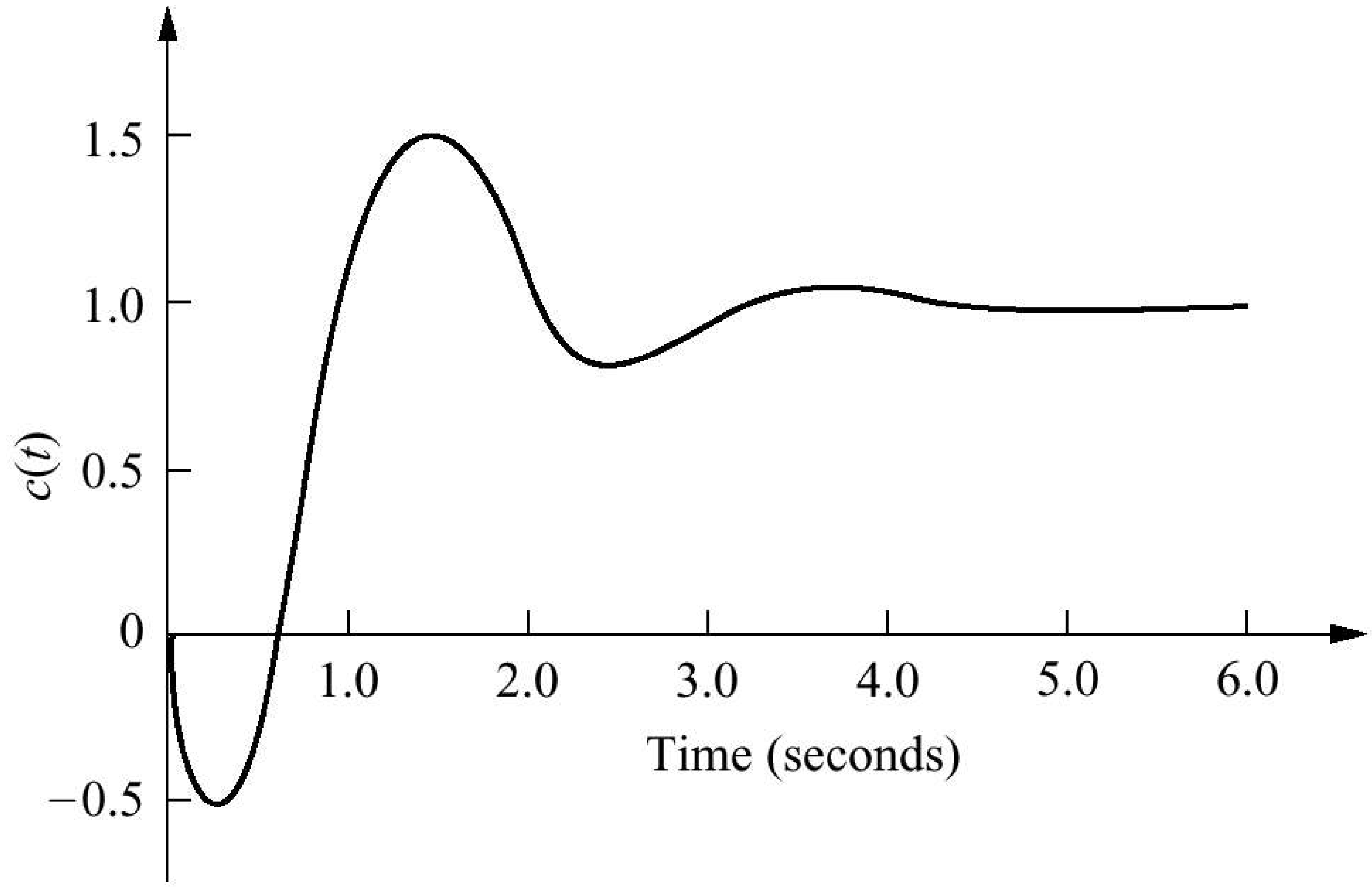
$$T(s)=\frac{a}{(s+b)(s+c)}$$

System Response with Zeros

$$p_{๑,๒} = -๑ \pm j๒.๒๘๒$$



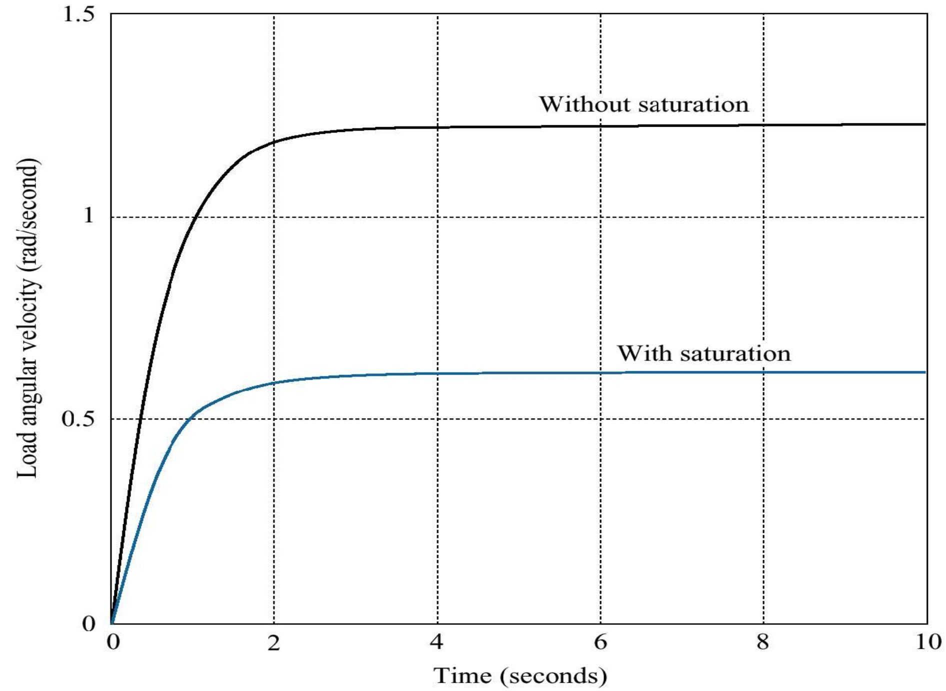
Effect of adding a zero to a two-pole system



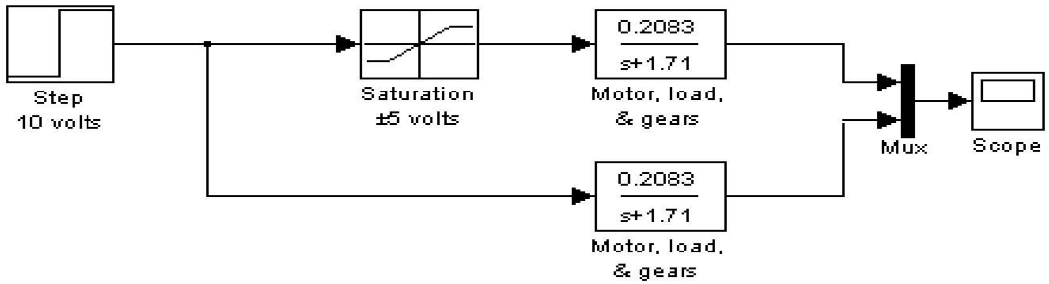
Step response of a nonminimum-phase system

Effects of Non-linearities Upon Time Response

a. Effect of amplifier saturation on load angular velocity response;
b. Simulink block diagram

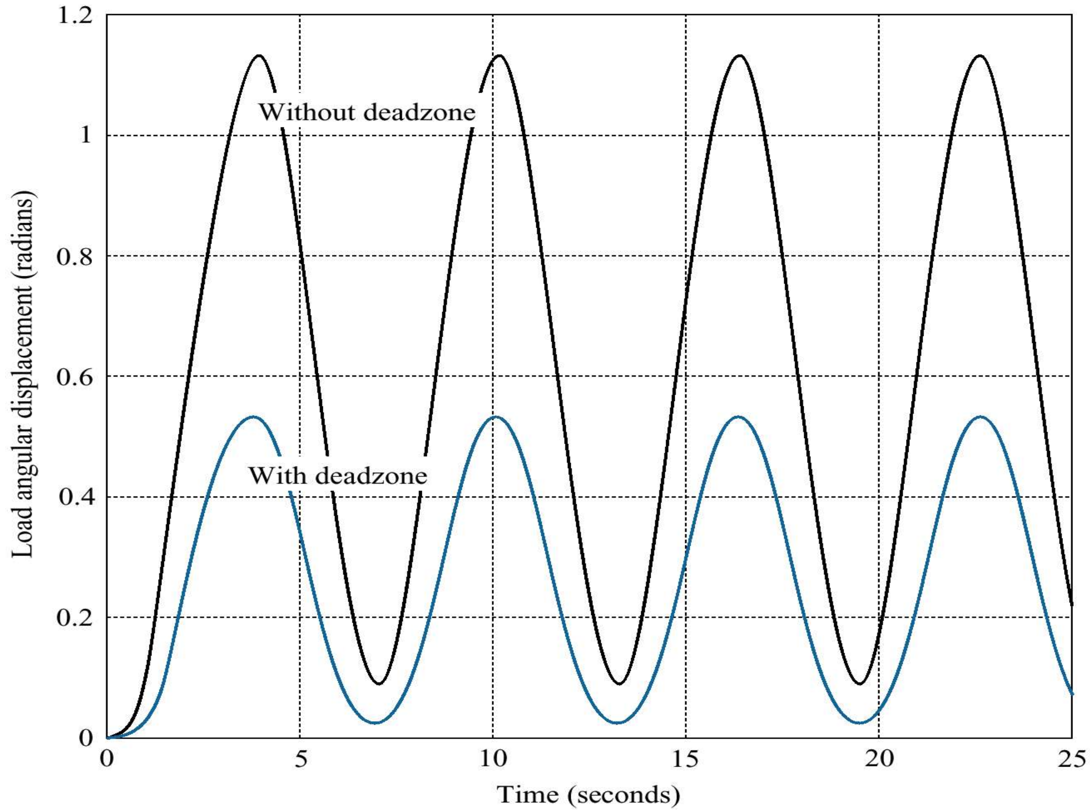


(*a*)

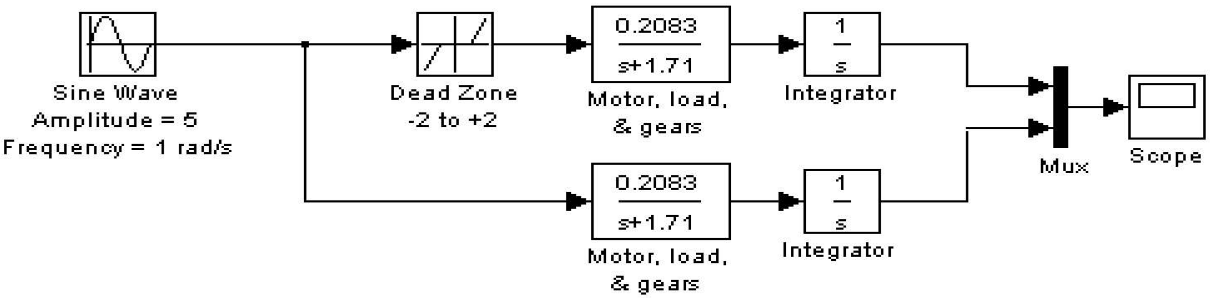


(*b*)

a. Effect of deadzone on load angular displacement response;
b. Simulink block diagram

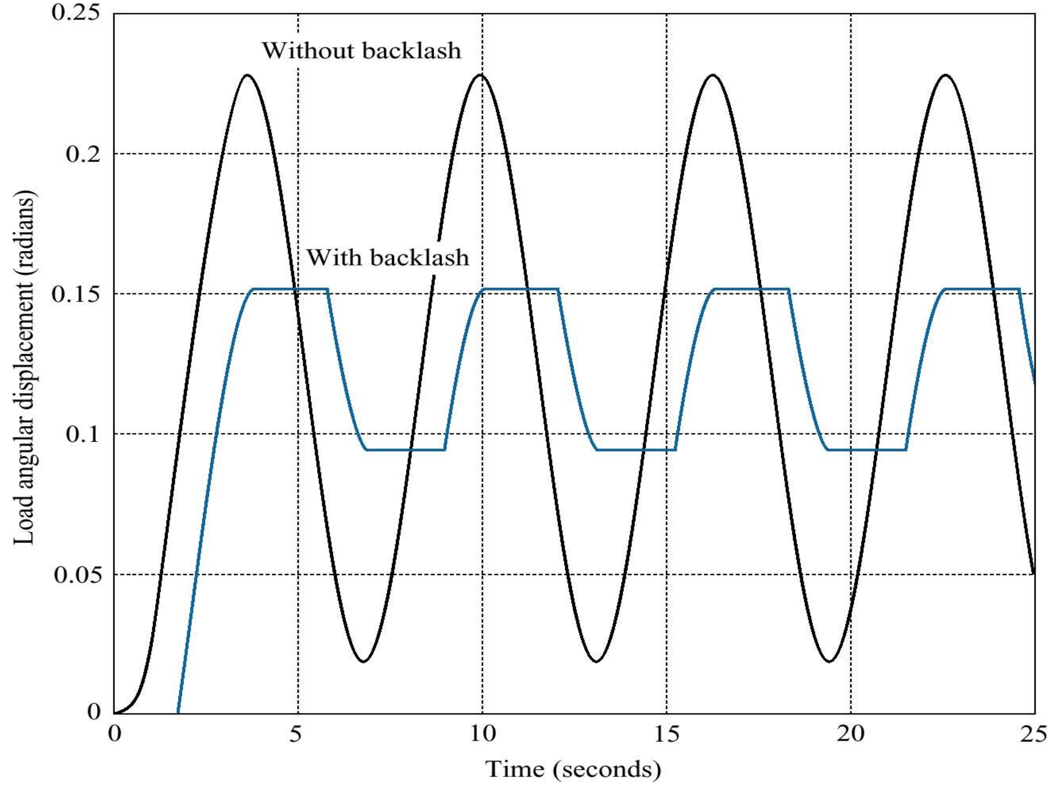


(*a*)

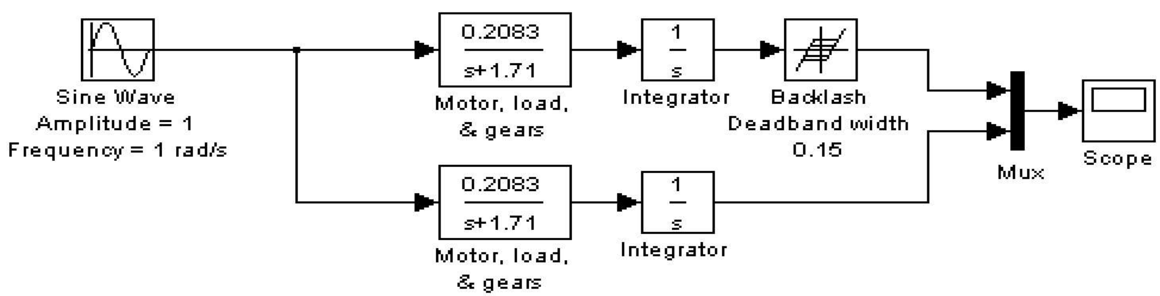


(*b*)

a. Effect of backlash on load angular displacement response;
b. Simulink block diagram



(*a*)



(*b*)

Time domain Solution of State equations

$$e^{at} = ๑ + at + \frac{๑}{๒!} a^{๒} t^{๒} + \ldots + \frac{๑}{k!} a^{k} t^{k} + \ldots$$

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{๑}{๒!} \boldsymbol{A}^{๒} t^{๒} + \ldots + \frac{๑}{k!} \boldsymbol{A}^{\boldsymbol{k}} t^{k} + \ldots$$

Scalar differential equation

$$\dot{x} = ax \rightarrow x(t) = e^{at} x(๐)$$

Vector-matrix differential equation

$$\dot{x} = Ax \rightarrow x(t) = e^{At} x(๐)$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

Rearrange and multiply both sides by e^{-At}

$$e^{-At}\left[\dot{\boldsymbol{x}}(t) - \boldsymbol{A}\boldsymbol{x}(t)\right] = e^{-At}\boldsymbol{B}\boldsymbol{u}(t)$$

Derivative of the product $e^{-At}\boldsymbol{x}(t)$

$$\frac{d}{dt}\left[e^{-At}\boldsymbol{x}(t)\right] = e^{-At}\dot{\boldsymbol{x}}(t) - Ae^{-At}\boldsymbol{x}(t)$$

Integrating both sides yields

$$\left[e^{-At} \boldsymbol{x}(t) \right]_0^t = e^{-At} \boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-A\tau} Bu(\tau)\, d\tau$$

We obtain

$$\boldsymbol{x}(t) = e^{At} \boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$= \Phi(t)\boldsymbol{x}(0) + \int_0^t \Phi(t-\tau) Bu(\tau)\, d\tau$$

$\Phi(t) = e^{At}$ - state-transition matrix

$(s\boldsymbol{I}-\boldsymbol{A})^{-๑}$ is the Laplace transform of the state-transtion matrix $\boldsymbol{\Phi}(t)$

$$L^{-๑}\left[(s\boldsymbol{I}-A)^{-๑}\right]=\boldsymbol{\Phi}(t)$$

$det(s\boldsymbol{I}-A)=๐$ - characteristic equation