Chapter 3

Modelling in the Frequency Domain

Outline

- Signals and systems
- Transfer function
- Developing a mathematical model by apply the fundamental physical laws of science end engineering in frequency domain

Introduction to Signals and Systems

Signal Taxonomy

๑ Deterministic signal

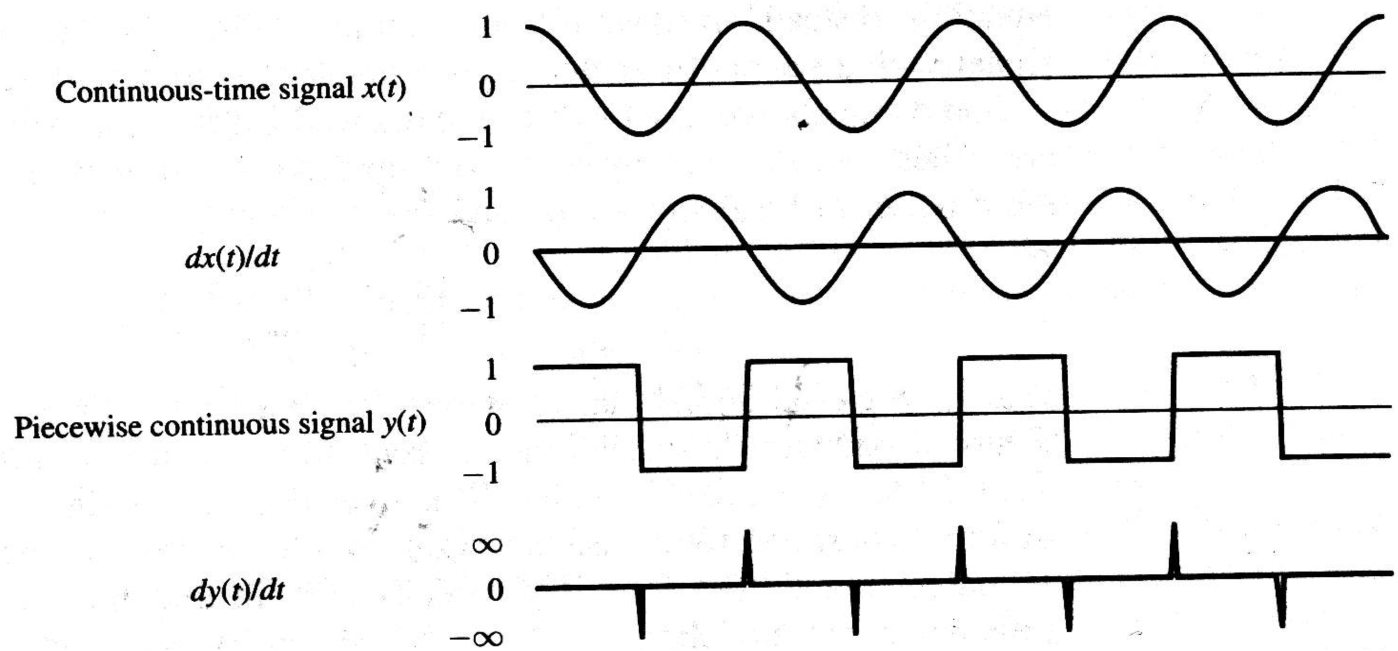
$x(t) = \sin(\omega t)$, $x(t)=$, $x(t)=t$

๒ Random signal

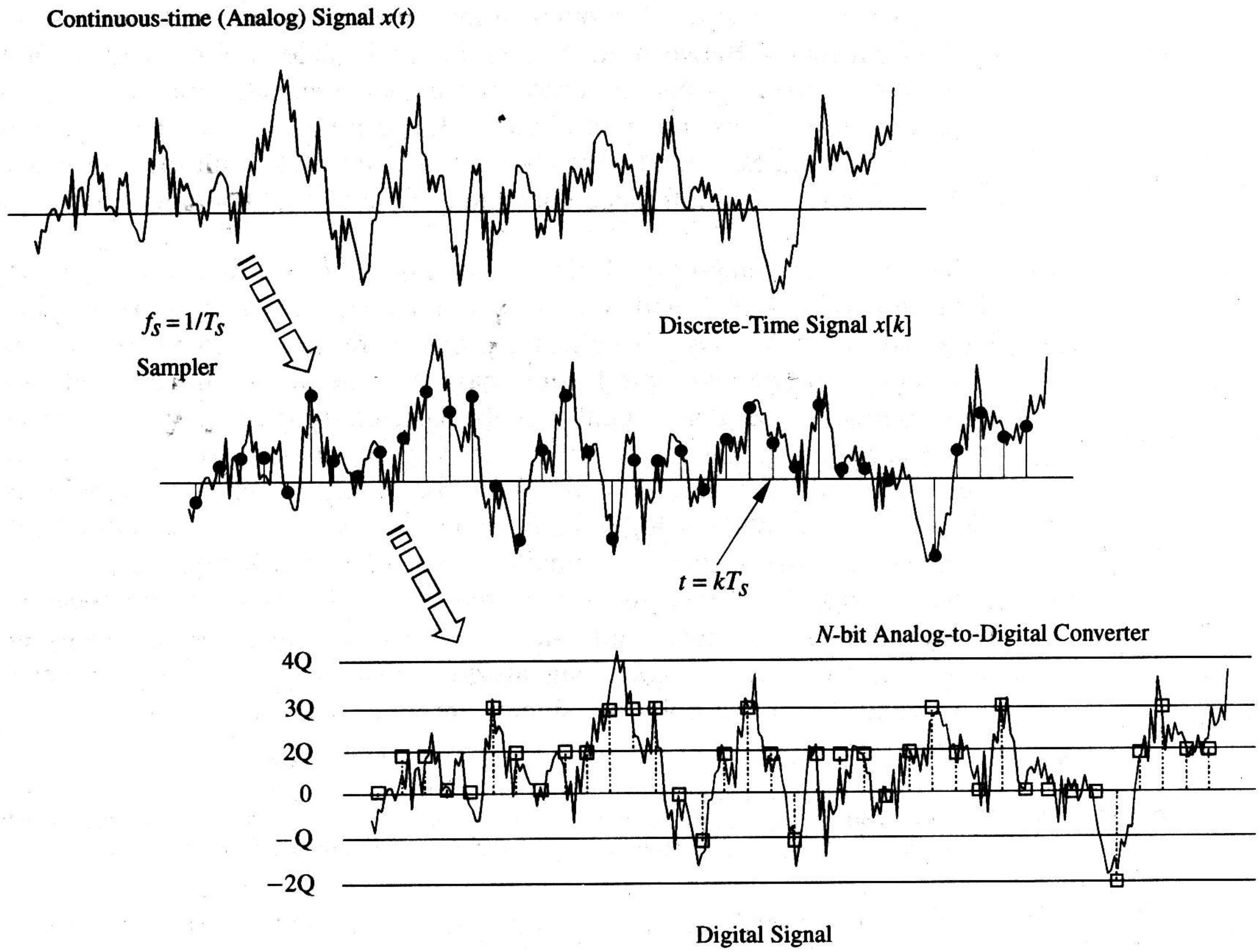
Causal system – physical signal generator is turned on at time t=๐ then the produced causal signal y(t) satisfies.

$$y(t) = \begin{cases} x(t) & for\ t \geq \text{๐} \\ \text{๐} & for\ t < \text{๐} \end{cases}$$

Continuous-Time Signals

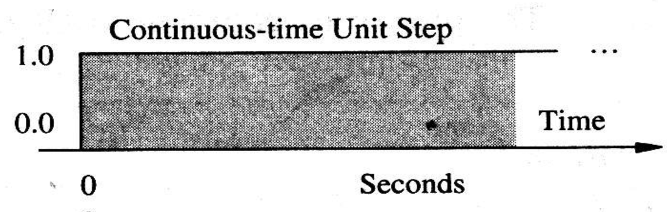


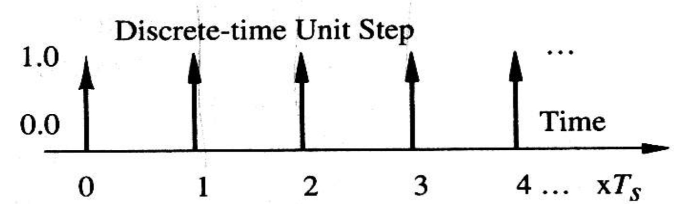
Discrete-Time Signals



Element signals

Unit step functions $$u\left(kT_s\right)=u\left(k\right)=\begin{cases}\text{๐} & for\ k<\text{๐}\\ \text{๑} & for\ k\geq\text{๐}\end{cases}$$

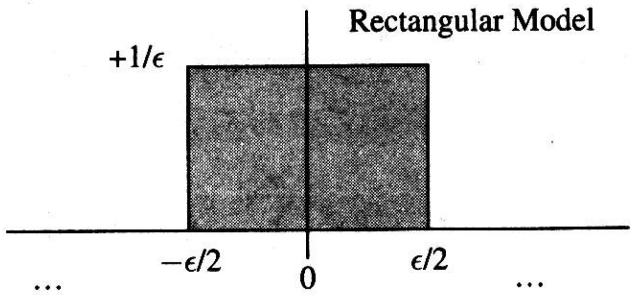


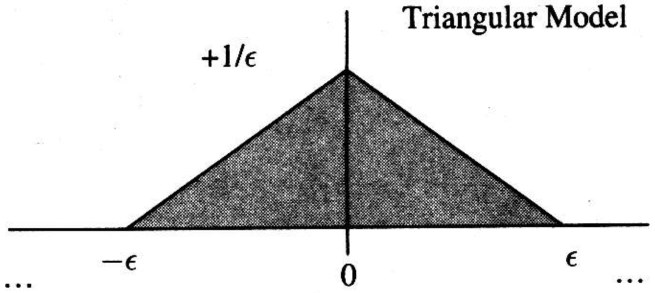


$$u\left(t\right)=\begin{cases}\text{๐} & for\ t<\text{๐}\\ \text{๑} & for\ t\geq\text{๐}\end{cases}$$

Unit impulse, Dirac impulse

$$\delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$





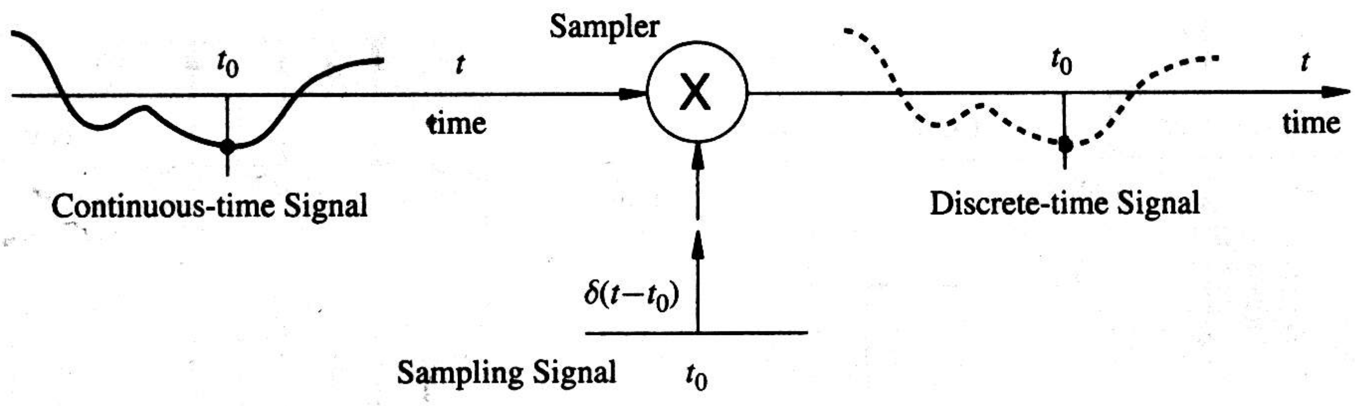
Properties of Dirac

$$๑.\int_{-\infty}^{\infty}\delta(\tau)\,d\tau = ๑$$

$$๒.\lim_{t\to ๐}\delta(t) = \infty$$

$$๓.\delta(t) = ๐ \qquad \text{for } t \neq ๐$$

$$๔.\delta(t) = \delta(-t)$$



Sampling property

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_{\text{๐}})\,dt=\int_{-\infty}^{\infty} x(t_{\text{๐}})\,\delta(t-t_{\text{๐}})\,dt=x(t_{\text{๐}})$$

Kronecker delta function

$$\delta(kT_s)=\delta(k)=\begin{Bmatrix} \text{๐} & \text{if } k\neq \text{๐} \\ \text{๑} & \text{for } k=\text{๐} \end{Bmatrix}$$

Property of discrete-time impulse

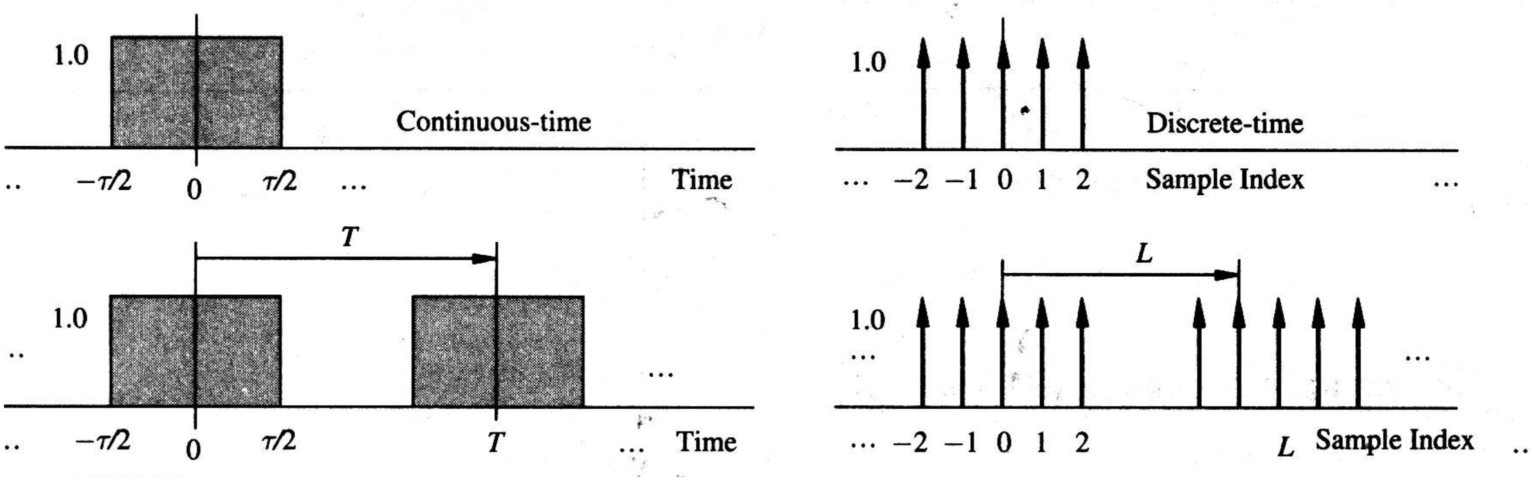
๑. $$\sum_{k=-\infty}^{\infty} \delta_K[k] = ๑$$

๒. $$\delta_K[k] = \delta_K[-k]$$

Sampling property of discrete-time impulse function

$$\sum_{k=-\infty}^{\infty} x[k]\delta_K[k-k_o] = \sum_{k=-\infty}^{\infty} x[k_o]\delta_K[k-k_o] = x[k_o]$$

Rectangular Pulse



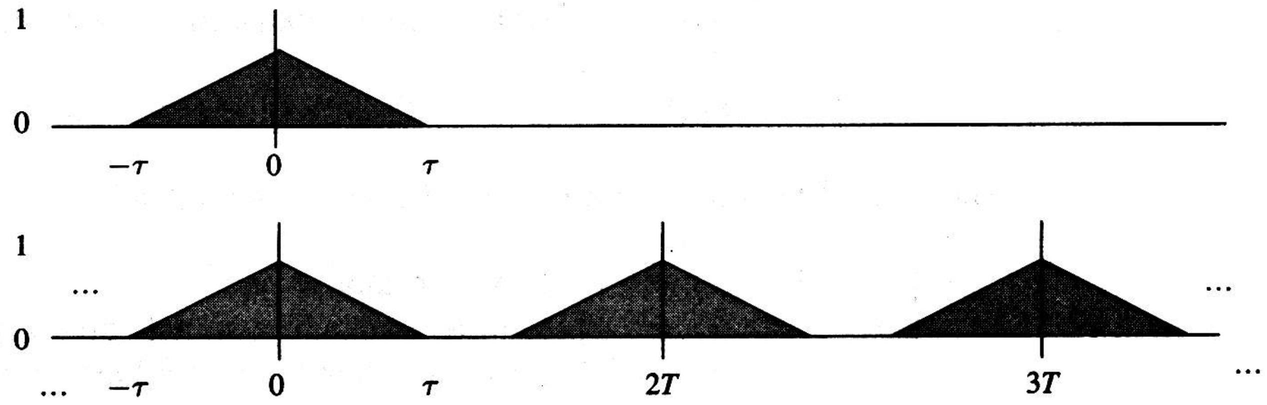
$$rect\left(\frac{t}{}\right) = \Big\{ \text{๑} \qquad if$$

Continuous-time

$$rect\left(\frac{k}{K}\right) = \begin{cases} \text{๑} & if\ |k| < K/\text{๒} \\ \text{๐} & otherwise \end{cases}$$

Discrete-time

Triangular Pulse



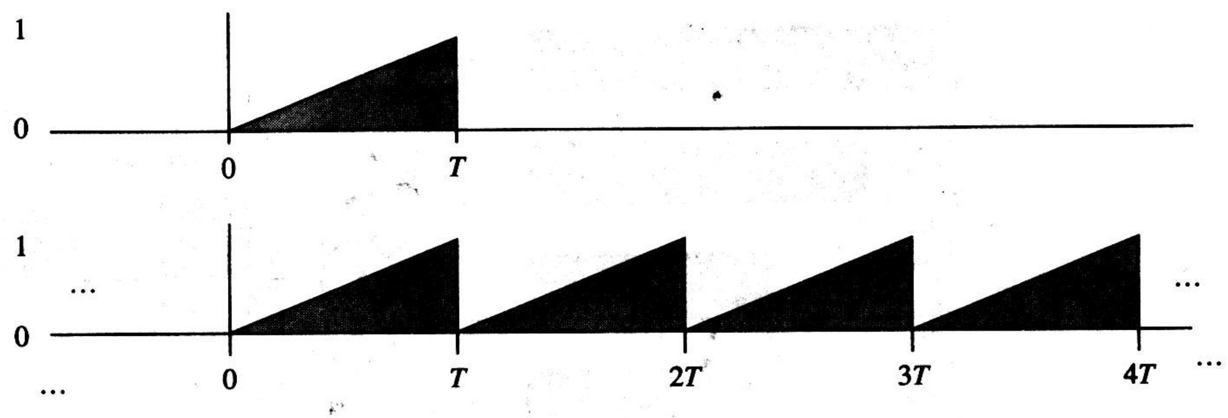
$$tri\left(\frac{t}{\tau}\right)=\begin{cases} ๑-\left|\frac{t}{\tau}\right| & if\ |t|<\tau \\ ๐ & otherwise \end{cases}$$

Continuous-time

$$tri\left(\frac{k}{K}\right)=\begin{cases} ๑-\left|\frac{k}{K}\right| & if\ |k|<K \\ ๐ & otherwise \end{cases}$$

Discrete-time

Ramp function



$$ramp\left(\frac{t}{T}\right)=\begin{cases}\dfrac{t}{T} & if\ 0\leq t<T\\ 0 & otherwise\end{cases}$$

Continuous-time

$$ramp\left(\frac{k}{K}\right)=\begin{cases}\dfrac{k}{K} & if\ 0\leq k<K\\ 0 & otherwise\end{cases}$$

Discrete-time

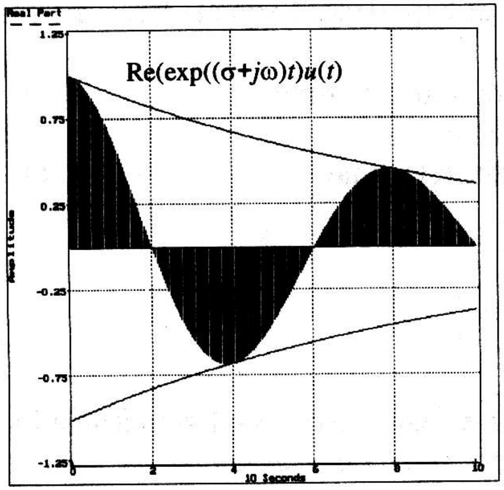
Complexe causal exponentials

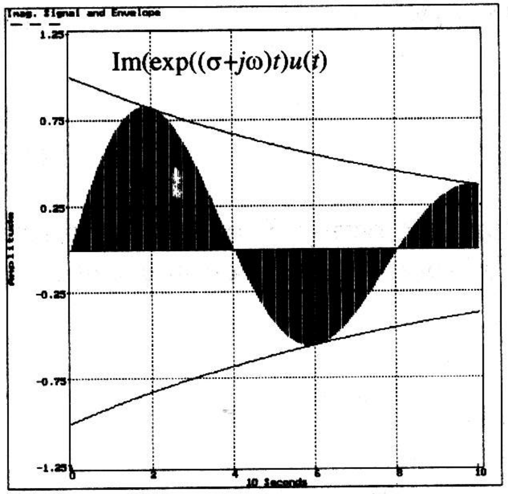
$$x(t) = Ae^{\lambda t} u(t)$$

$$x(k) = Ae^{\lambda k T_s} u(k)$$

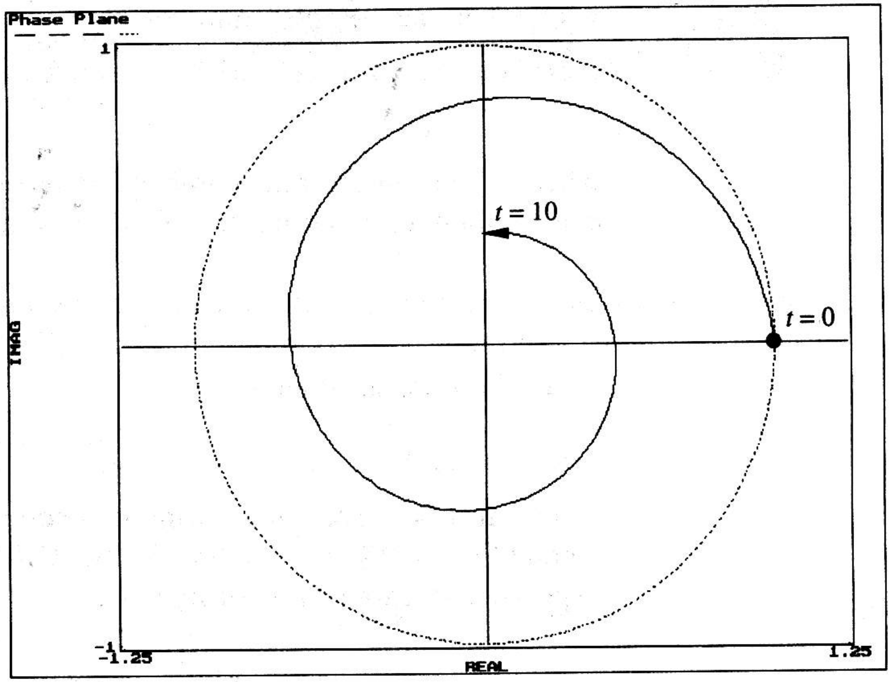
$$\lambda = \sigma + j\omega$$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$



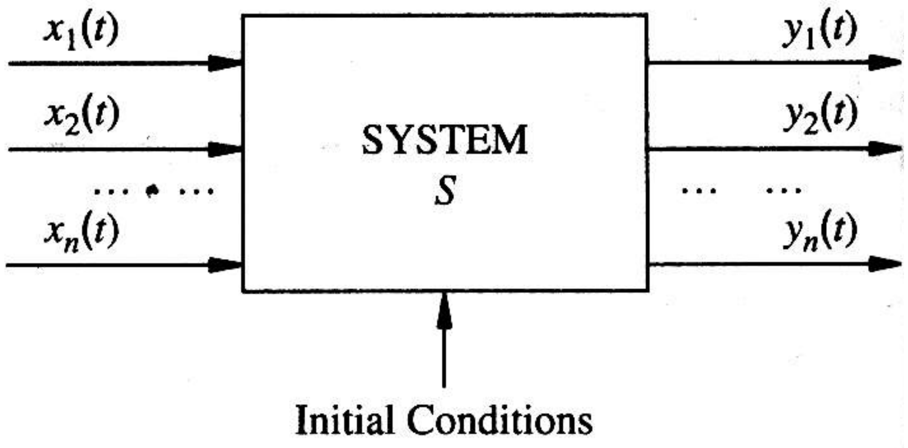


Polar Coordinates



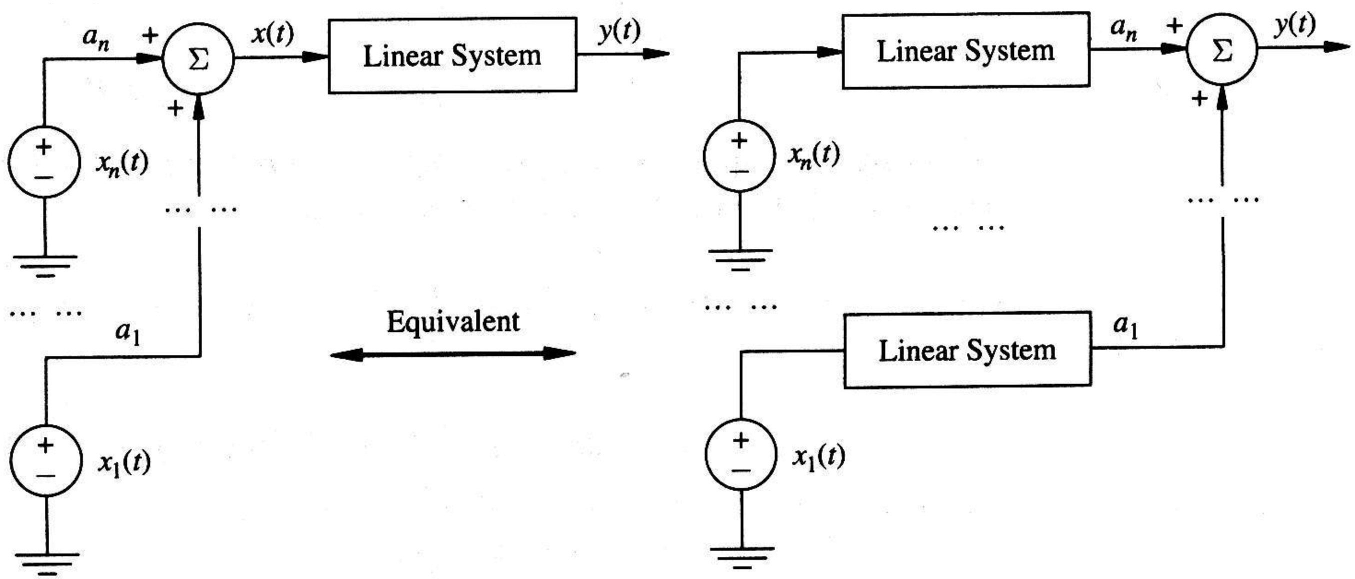
Phase-Plane

Linear system



$$Y(t) = (\mathcal{S}x)(t)$$

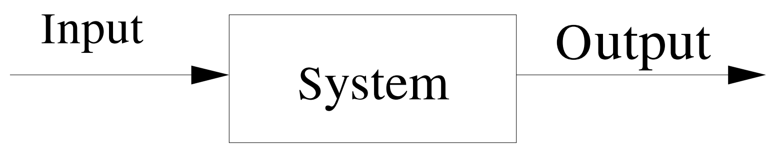
Superposition

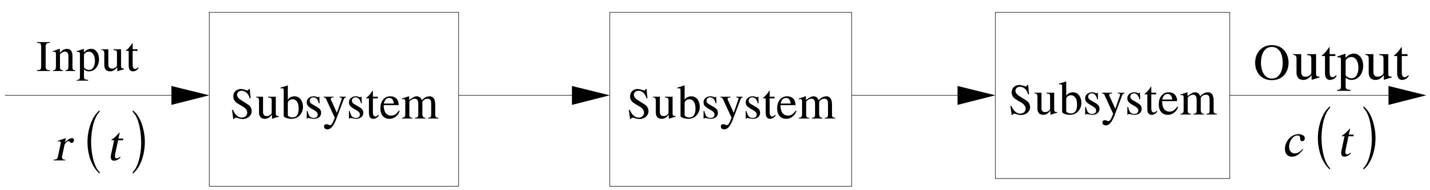


$$S(a_n x_n + \ldots + a_{๑} x_{๑})(t) = (S a_n x_n)(t) + \ldots + (S a_{๑} x_{๑})(t)$$

$$= a_n (S x_n)(t) + \ldots + a_{๑} (S x_{๑})(t)$$

Cascaded interconnections





$r(t)$ - Input (Reference input)

$c(t)$ - Output (Controlled variable)

Laplace Transform

- For solving linear differential equations by convert to complex plane.
  - s = $\sigma + j\omega$
  - s = complex variable
  - σ = real part
  - ω = imaginary part

Definition

Laplace Transform

$$L\left[f(t)\right]=F(s)=\int_{\text{๐}}^{\infty} f(t)\,\mathrm{e}^{-st}\,dt$$

Inverse Laplace Transform

$$L^{-\text{๑}}\left[F(s)\right]=\frac{\text{๑}}{\text{๒}\pi j}\int_{\varsigma-j\infty}^{\varsigma-j\infty} F(s)\,\mathrm{e}^{st}\,ds$$

LAPLACE TRANSFORM TABLE

$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt$
1	$\frac{1}{s}\,,\qquad s > 0$
$t^n\,,\qquad n$ an integer	$\frac{n!}{s^{n+1}}\,,\qquad s > 0$
e^{at}	$\frac{1}{s-a}\,,\qquad s > a$
$\sin bt$	$\frac{b}{s^2+b^2}\,,\qquad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}\,,\qquad s > 0$
$e^{at} f(t)$	$F(s-a)$
$e^{at}t^n\qquad n$ an integer	$\frac{n!}{(s-a)^{n+1}}\,,\qquad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}\,,\qquad s > a$
$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}\,,\qquad s > a$
$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}\qquad s > 0$
$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}{}^{*}\,,\qquad s > 0$
$u_c(t)f(t),\ c \geq 0$	$e^{-cs}\mathcal{L}\{f(t+c)\}(s)$
$u_c(t)f(t-c),\ c \geq 0^{**}$	$e^{-cs}\mathcal{L}\{f(t)\}(s)$
$y' = \dot{y} = \frac{dy}{dt}$	$sY(s) - y(0)$
$y'' = \ddot{y} = \frac{d^2y}{dt^2}$	$s^2Y(s) - sy(0) - \dot{y}(0)$

Inverse Laplace Transform

Partial-function expansion

๑ Roots of the Denominator of F(s) are Real and distinct

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{a_n s^n + a_{n-๑} s^{(n-๑)} + \ldots + a_๐}$$

$$=\frac{N(s)}{(s+P_๑)(s+P_๒)\ldots(s+P_m)(s+P_n)}$$

$$=\frac{K_๑}{s+p_๑}+\frac{K_๒}{s+p_๒}+\ldots+\frac{K_m}{s+p_m}+\frac{K_n}{s+p_n}$$

$$F(s)=\frac{N(s)}{(s+p_๑)(s+p_๒)\ldots(s+p_m)(s+p_n)}$$

$$=\frac{K_๑}{s+p_๑}+\frac{K_๒}{s+p_๒}+\ldots+\frac{K_m}{s+p_m}+\ldots+\frac{K_n}{s+p_n}$$

$$(s+p_m)F(s)=(s+p_m)\frac{K_๑}{s+p_๑}+(s+p_m)\frac{K_๒}{s+p_๒}+$$

$$\ldots+K_m+(s+p_m)\frac{K_n}{s+p_n}$$

$$K_m=\frac{N(p_m)}{(-p_m+p_๑)(-p_m+p_๒)\ldots(-p_m+p_n)}$$

$$F(s)=\frac{๑}{(s+๑)(s+๒)(s+๔)}$$

$$=\frac{K_{๑}}{s+๑}+\frac{K_{๒}}{s+๒}+\frac{K_{๓}}{s+}$$

$$K_{๑}=\frac{๑}{(-๑+๒)(-๑+๔)}=$$

$$K_{๒}=\frac{๑}{(-๒+๑)(-๒+๔)}=$$

$$K =$$

$$F(s)=\frac{๑/๓}{(s+๑)}+\frac{๑/๒}{(s+๒)}+\frac{๑/}{(s+}$$

$$f(t)=\frac{๑}{๓}e^{-t}+\frac{๑}{๒}e^{-๒t}+\frac{๑}{๖}e^{-๔t}$$

๒. Root of the Denominator of F(s) are real and repeated

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{(s+p_{๑})^{r}(s+p_{๒})\ldots+(s+p_{n})}$$

$$=\frac{K_{๑}}{(s+p_{๑})^{r}}+\frac{K_{๒}}{(s+p_{๑})^{r-๑}}+\ldots+\frac{K_{r}}{(s+p_{๑})^{+}}$$

$$\frac{K_{r+๑}}{(s+p_{๒})}+\ldots+\frac{K_{n}}{s+p_{n}}$$

$$F(s)=\frac{N(s)}{(s+p_{๑})^{r}(s+p_{๒})\ldots(s+p_{n})}$$

$$=K_{๑}+(s+p_{๑})K_{๒}+(s+p_{๑})^{๒}K_{๓}+\ldots+(s+p_{๑})^{r-๑}K_{r}+$$

$$\frac{(s+p_{๑})^{r}K_{r+๑}}{(s+p_{๒})}+\ldots+\frac{(s+p_{๑})^{r}K_{n}}{(s+p_{n})}$$

$$K_i=\left[\frac{๑}{(i-๑)!}\;\frac{d^{i-๑}F(s)}{ds^{i-๑}}\right]_{s\to p_๑}\qquad \begin{array}{l} i = ๑, ๒, \ldots, r \\ ๐! = ๑ \end{array}$$

Example

$$F(s)=\frac{๒}{(s+๑)(s+๒)^๒}$$

$$F(s)=\frac{K_๑}{(s+๒)^๒}+\frac{K_๒}{(s+๒)}+\frac{K_๓}{(s+๑)}$$

คูณด้วย $(s+๒)^๒$

$$\frac{๒}{s+๑}=K_๑+(s+๒)K_๒+\frac{(s+๒)^๒K_๓}{s+๑} \qquad \ldots\ldots\ldots (๑)$$

$$K_๑=-๒$$

First derivative of (๑)

$$\frac{-๒}{(s+๑)^{๒}} = K_{๒} - \left[(s+๑)๒(s+๒)K_{๓}\right] - \frac{(s+๒)^{๒}K_{๓}}{(s+๑)^{๒}}$$

$$= K_{๒} + (s+๒)K_{๓}\frac{\left[๒(s+๑)-(s+๒)\right]}{(s+๑)^{๒}}$$

$$\frac{-๒}{(s+๑)^{๒}} = K_{๒} + \frac{s(s+๒)K_{๓}}{(s+๑)^{๒}}$$

$$K_{๒} = -๒$$

$$\frac{๒}{(s+๒)^{๒}}=\frac{(s+๑)K_{๑}}{(s+๒)^{๒}}+\frac{(s+๑)K_{๒}}{s+๒}+K_{๓}$$

$$K_{๓}=๒$$

$$F(s)=\frac{-๒}{(s+๒)^{๒}}+\frac{-๒}{s+๒}+\frac{๒}{s+๑}$$

$$f(t)=-๒te^{-๒t}-๒e^{-๒t}+๒e^{-t}$$

๓. Root of the Denominator of F(s) are complex

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s+p_{๑})(s^{๒}+as+b)\ldots}$$

$$= \frac{K๑}{s+p_{๑}} + \frac{K_{๒}s+K_{๓}}{s^{๒}+as+b} + \ldots$$

$(s^{๒}+as+b)$ - complex or imaginary

Example

$$F(s)=\frac{๓}{s(s^{๒}+๒s+๕)}$$

$$\frac{๓}{s(s^{๒}+๒s+๕)}=\frac{K_{๑}}{s}+\frac{K_{๒}s+K_{๓}}{s^{๒}+๒s+๕}$$

$$K_{๑}=\frac{๓}{๕}$$

คูณด้วย $s(s^{๒}+๒s+๕)$ และจัดสมการใหม่

$$๓=(s^{๒}+๒s+๕)K_{๑}+K_{๒}s^{๒}+K_{๓}s$$

$$=(K_{๑}+K_{๒})s^{๒}+(๒K_{๑}+K_{๓})s+๕K_{๑}$$

แทนค่า $K_๑$

$$๓=\left(K_๒+\frac{๓}{๕}\right)s^๒+\left(K_๓+\frac{๖}{๕}\right)s+๓$$

$$K_๒+\frac{๓}{๕}=๐ \qquad K_๓+\frac{๖}{๕}=๐$$

$$K_๒=\frac{-๓}{๕} \qquad K_๓=\frac{-๖}{๕}$$

$$F(s)=\frac{๓}{s\left(s^๒+๒s+๕\right)}=\frac{๓/๕}{s}-\frac{๓}{๕}\,\frac{s+๒}{s^๒+๒s+๕}$$

$$L\left[Ae^{-\omega t}\cos(\omega t)\right]=\frac{A(s+a)}{(s+a)^{๒}+\omega^{๒}}$$

$$L\left[Be^{-at}\sin(\omega t)\right]=\frac{B\omega}{(s+a)^{๒}+\omega^{๒}}$$

$$L\left[Ae^{-at}\cos(\omega t)+Be^{-at}\sin(\omega t)\right]=\frac{A(s+a)+B\omega}{(s+a)^{๒}+\omega^{๒}}$$

$$F(s)=\frac{๓/๕}{s}-\frac{๓}{๕}\,\frac{(s+๑)+(๑/๒)\,๒}{(s+๑)^{๒}+๒^{๒}}$$

$$\boxed{f(t)=\frac{๓}{๕}-\frac{๓}{๕}e^{-t}\left(\cos(๒t)+\frac{๑}{๒}\sin(๒t)\right)}$$

Advantages and Disadvantages

Advantages : rapidly provide stability and transie nt response information.

Disadvantages : Limited applicability to linear to linear, time invariant system or system that can b e approximated as such.

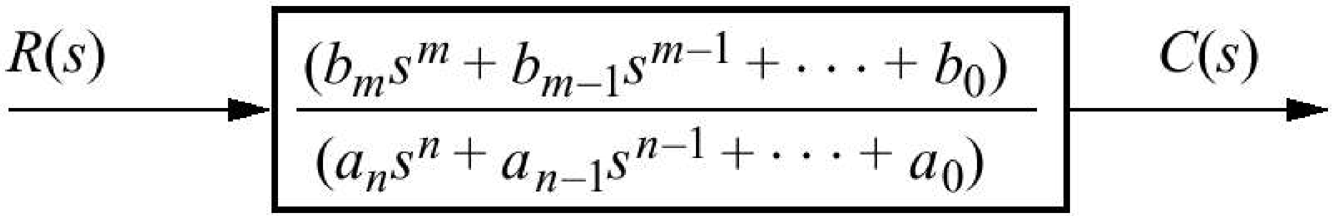
Transfer function

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-\text{๑}} \frac{d^{n-\text{๑}} c(t)}{dt^{n-\text{๑}}} + \cdots + a_\text{๐} c(t)$$

$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-\text{๑}} \frac{d^{m-\text{๑}} r(t)}{dt^{m-\text{๑}}} + \cdots + b_\text{๐} r(t)$$

$$a_n s^n C(s) + \ldots + a_\text{๐} C(s) = b_m s^m R(s) + \ldots + b_\text{๐} R(s)$$

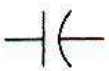
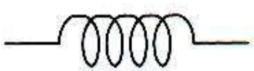
Transfer function



$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-\text{๑}} s^{m-\text{๑}} + \cdots + b_{\text{๐}}}{a_n s^n + a_{n-\text{๑}} s^{n-\text{๑}} + \cdots + a_{\text{๐}}}$$

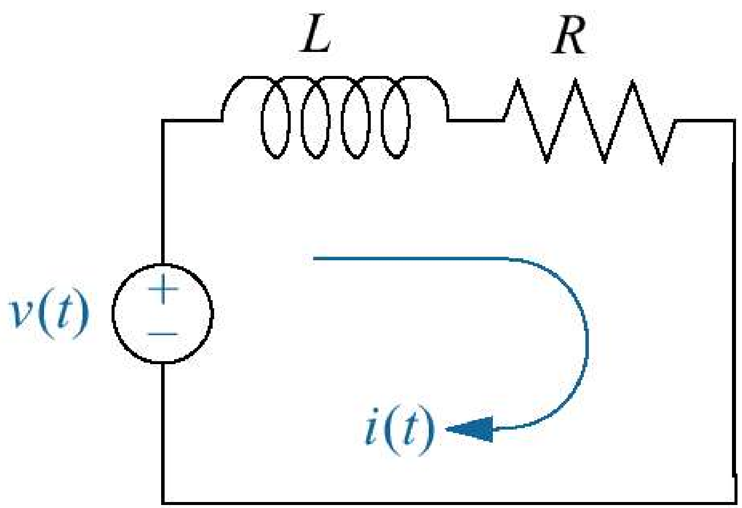
Electric Network Transfer Functions

Table 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Electric Network Transfer Functions



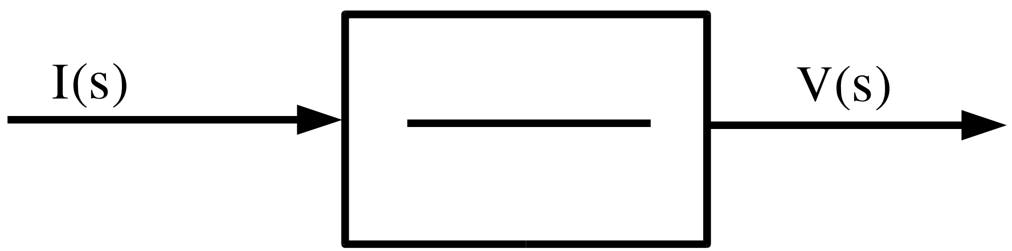
When v(t) is input and i(t) is output.

By Kirchhoff's voltage law, we have

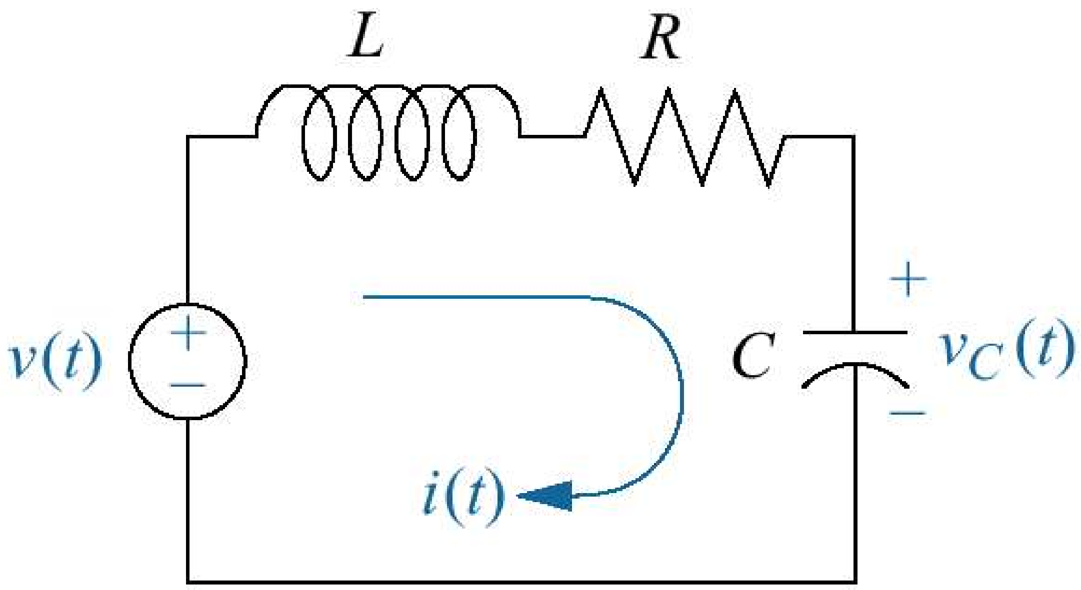
$$v(t) =$$

Transfer Function of RL circuit

$$\frac{I(s)}{V(s)} = \underline{\hspace{4cm}}$$



Electric Network Transfer Functions

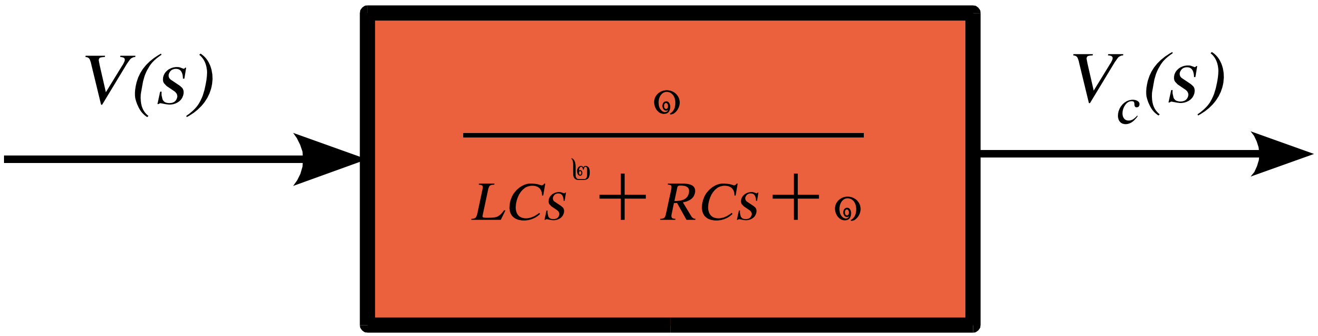


When we define : $v(t)$ is input and $v_c(t)$ is output.

By Kirchhoff's voltage law, we have

$$v(t) =$$

Block Diagram of Series RLC Network



When v(t) is input and $v_c(t)$ is output.

Block Diagram of Series RLC Network

$$V(s) \longrightarrow \boxed{\dfrac{Cs}{LCs^{๒}+RCs+๑}} \longrightarrow I(s)$$

When v(t) is input and i(t) is output.

Complex Circuits via Mesh Analysis

๑. Replace passive element values with their impedances.

๒. Replace all sources and time variables with their Laplace Transform

๓. Assume a transform current and a current direction in each mesh.

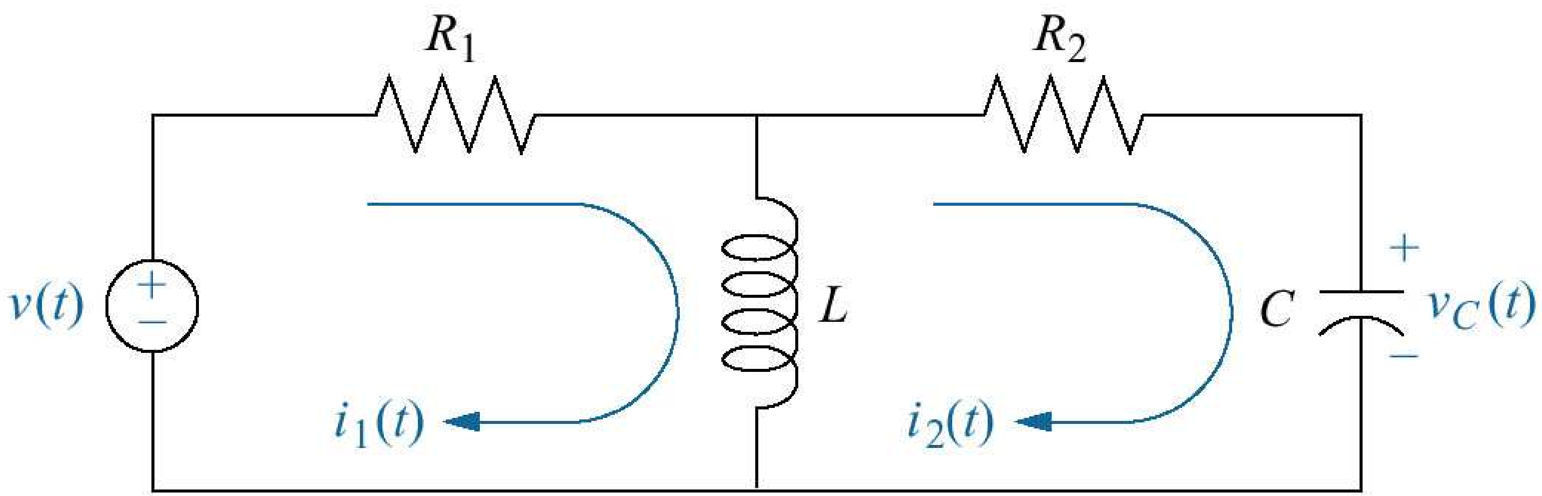
๔. Write Kirchhoff's voltage law around the output.

๕. Solve the simultaneous equations for the output.

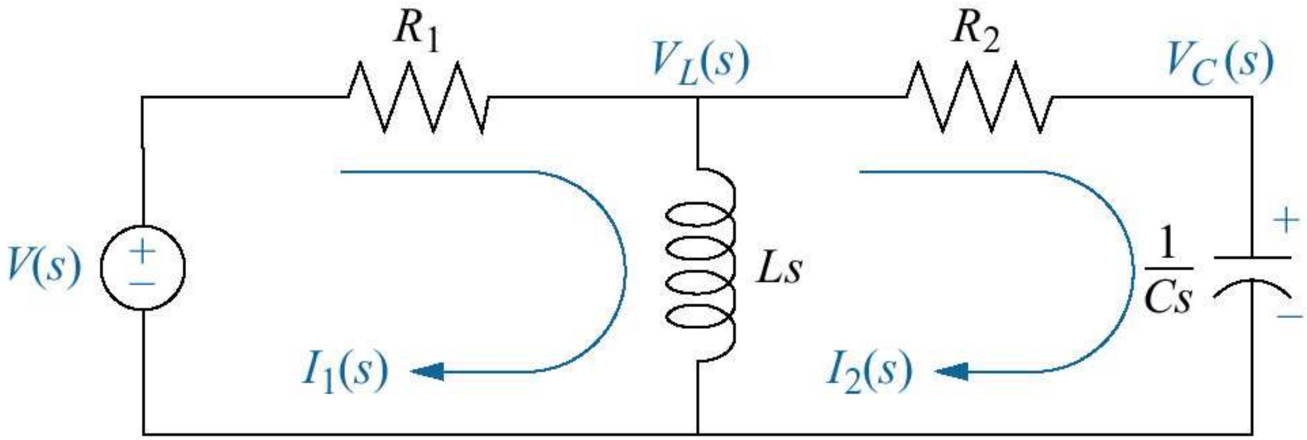
๖. Form the Transfer function.

Complex Circuits via Mesh Analysis

Example Find the transfer function of the circuit below when input is v(t) and output is i(t), $v_c(t)$



Replace passive element values with their impedances and Replace all sources and time variables with their Laplace Transform

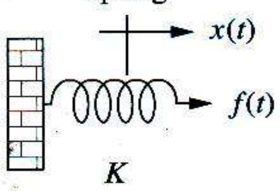
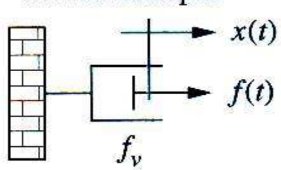
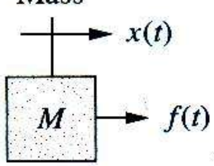


Transfer function of the circuit

$$V(s) \longrightarrow \boxed{\dfrac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}} \longrightarrow I_2(s)$$

Translational Mechanical System Transfer Functions

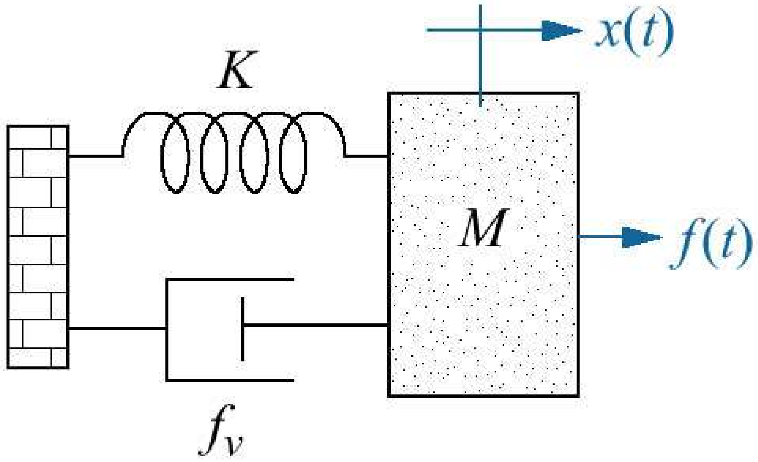
Table 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

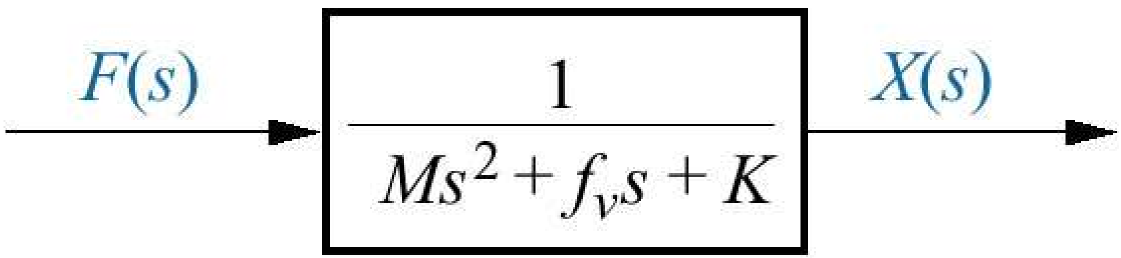
Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System Transfer Functions

Example Find Transfer function of the translational system below when input is f(t) and output is x(t)

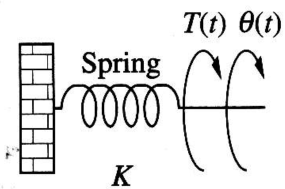
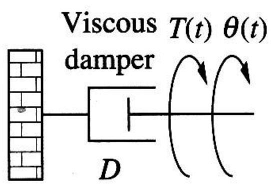
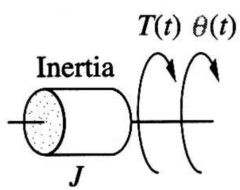


Transfer function of the translational system



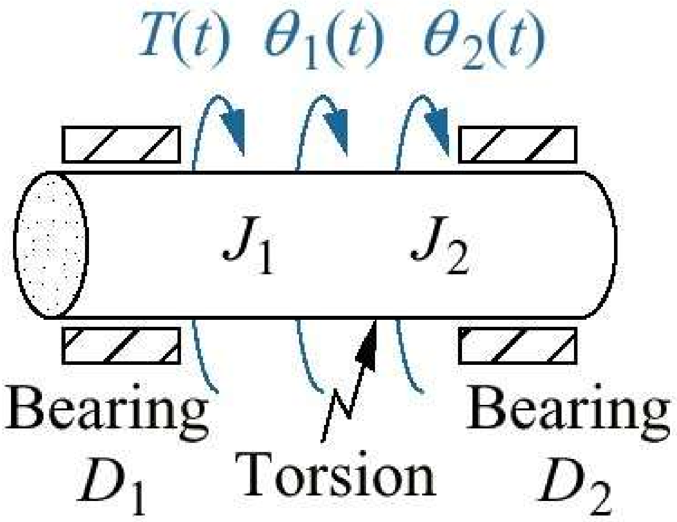
Rotational Mechanical System Transfer Functions

Table 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

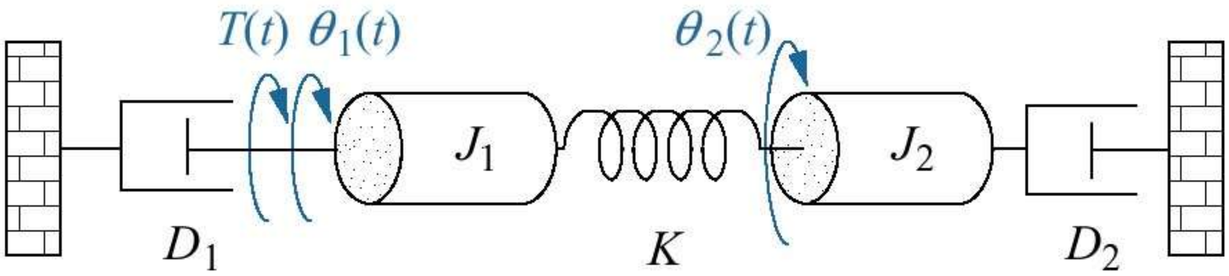
Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Example Find Transfer function of the rotational system below when input is T(t) and output is $\theta_{๒}$(t)



Schematic of the system



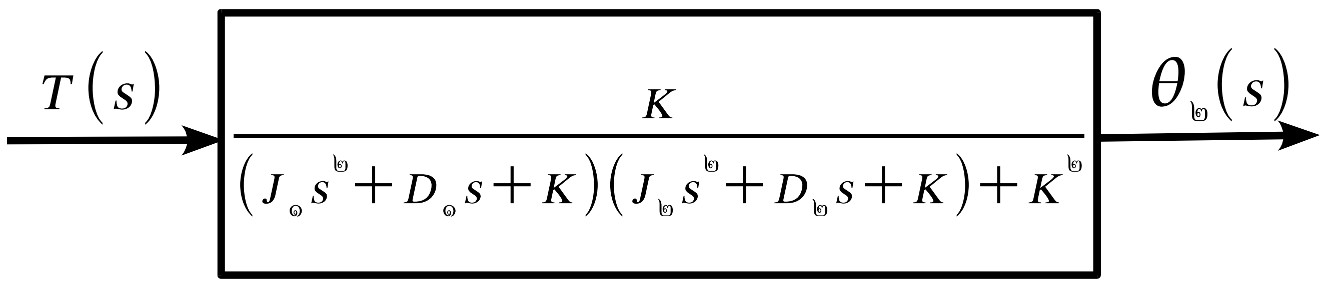
Equation on mass ๑ (M๑)

$$T(t) = J_{๑} \frac{d^{๒}\theta_{๑}(t)}{dt} + \qquad +$$

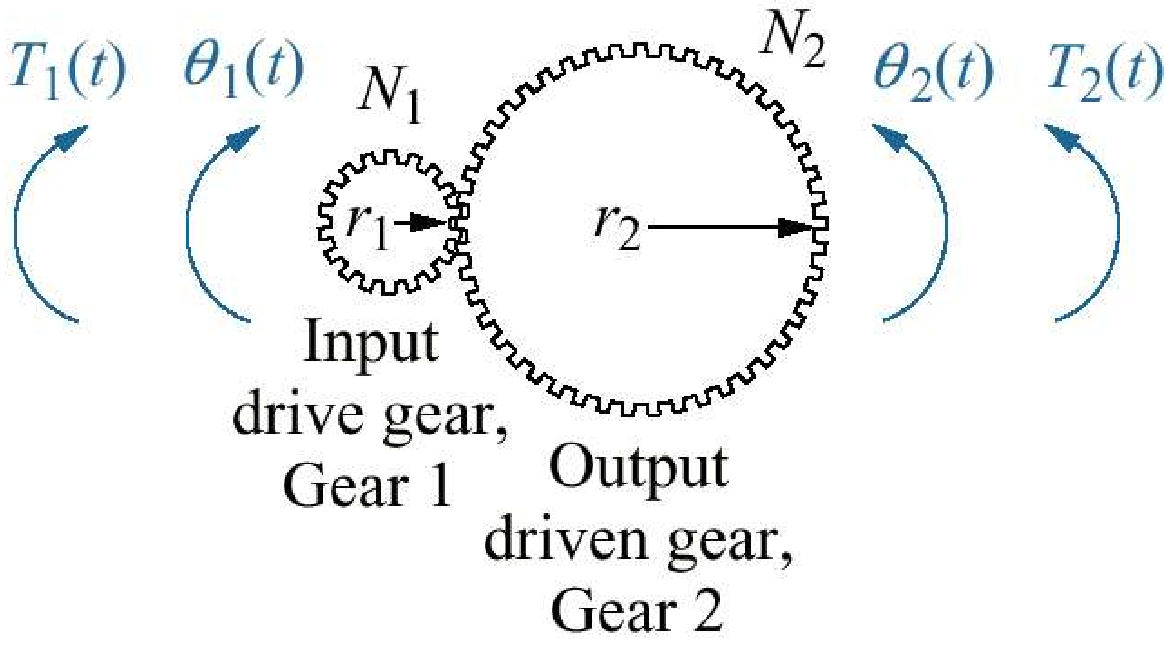
Equation on mass ๒ (M๒)

$$๐ = J_{๒} \frac{d^{๒}\theta_{๒}(t)}{dt} + \qquad +$$

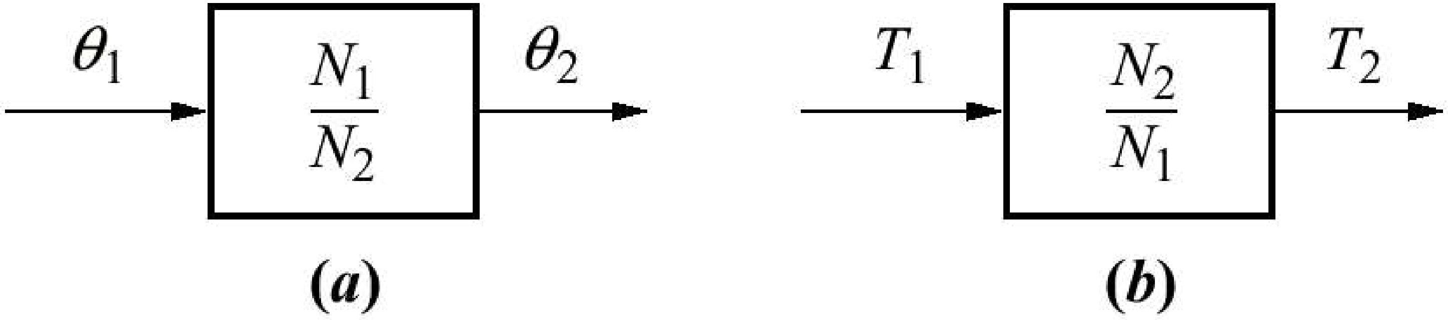
Transfer function of the system



Transfer Functions for Systems with Gears



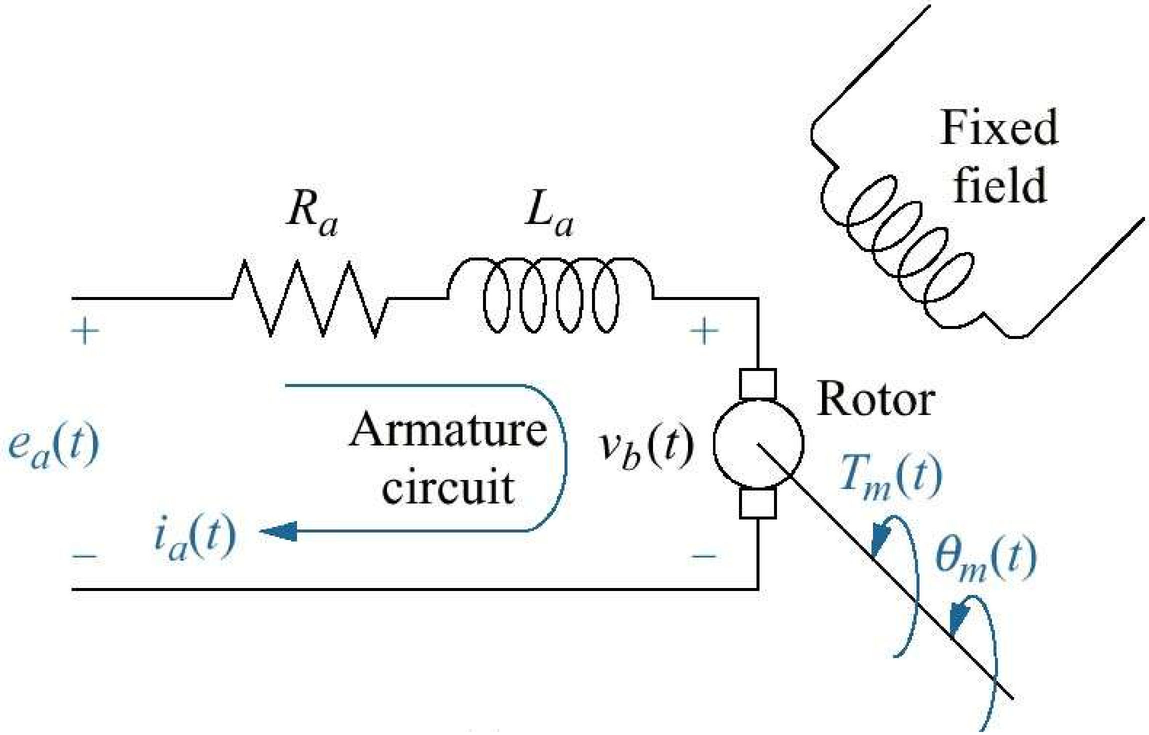
Transfer Functions for Systems with Gears



a. angular displacement inlossless gears.
b. torque in lossless gears.

Electromechanical System

Electromechanical System Transfer Functions



Schematic of DC motor

Motor Variables

$e_a(t)$ - Applied voltage

$i_a(t)$ - Armature current

$v_b(t)$ - Back emf

$T_m(t)$ - Motor torque

$\theta_m(t)$ - Rotor displacement

$\omega_m(t)$ - Rotor angular velocity

Motor parameters

L_a - Armature inductance

R_a - Armature resistance

J_m - Rotor inertia

D_m - Viscous-friction coefficient

k_T - Torque constant

k_b - Back-emf constant

Electromechanical System Transfer Functions

Position control transfer function

Input is $e_a(t)$ and output is $\theta_m(t)$

Electrical equation

$$e_a(t) = L_a \frac{d\,i_a(t)}{dt} + R_a i_a(t) + v_b(t)$$

Mechanical Equation

$$T_m(t) = J_m \frac{d^{๒}\theta_m(t)}{dt^{๒}} + D_m \frac{d\theta_m(t)}{dt}$$

Relation between Electrical and Mechanical

$$v_b(t) = k_b \frac{d\theta_m(t)}{dt}$$

$$T_m(t) = k_m i_a(t)$$

Position control transfer function

$$\frac{\theta_m(s)}{E_a(s)} = \frac{k_m}{(L_a s + R_a)(J_m s^2 + D_m s) + k_m k_b s}$$

$$E_a(s) \longrightarrow \boxed{\frac{k_m}{(L_a s + R_a)(J_m s^2 + D_m s) + k_m k_b s}} \longrightarrow \theta_m(s)$$