Chapter 3

OBJECT LOCATION

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1 Cartisian coordinates

2 Two-dimensional transformation

3 Three-dimensional transformation matrices

4 Coordinate frames

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Notation

Point vector : $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$

Coordinates : $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1$

Matrices : $\mathbf{A}, \mathbf{T}, \mathbf{R}$

Frames : W, R, H

Points : x, y, z, x_1, y_1, z_1

Derivatives : dx, d^2x

Notation (Cont.)

Increment : Δx

Transforms : ${}^{R}\mathbf{T}_{N}$: R=reference frame,

N=new frame

${}^{R}\mathbf{Trans}$: translation transform

${}^{N}\mathbf{Rot}$: rotation transform

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Notation (Cont.)

Vectors : ${}^{R}\boldsymbol{p}_{n}$: position

${}^{N}\boldsymbol{v}_{p}$: linear velocity

${}^{R}\boldsymbol{\omega}_{p}$: angular velocity

${}^{N}\boldsymbol{a}_{q}$: linear acceleration

${}^{R}\boldsymbol{\alpha}_{q}$: angular acceleration

Notation (Cont.)

$^{R}\boldsymbol{p}_{n-๑,n}$: position of vector from frame n-๑ to frame n with respect to frame R

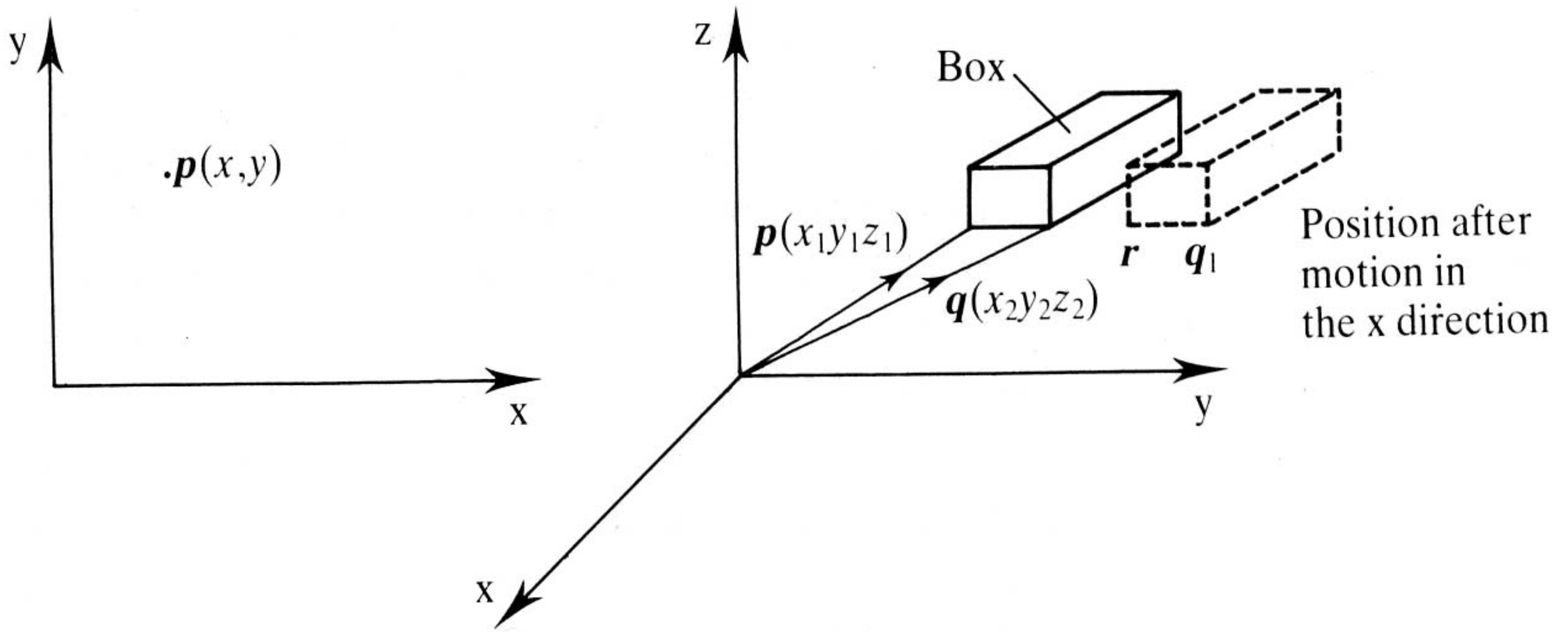
$^{R}\boldsymbol{f}_{n-๑,n}$: force vector from frame n-๑ to frame n with respect to frame R

$^{R}\boldsymbol{Z}_{n}$: Torque at joint n as seen from frame R

๑ Cartesian Coordinate

๑ Two-dimensional system
๒ Three-dimension system

- Uses to describe the position of object in space.

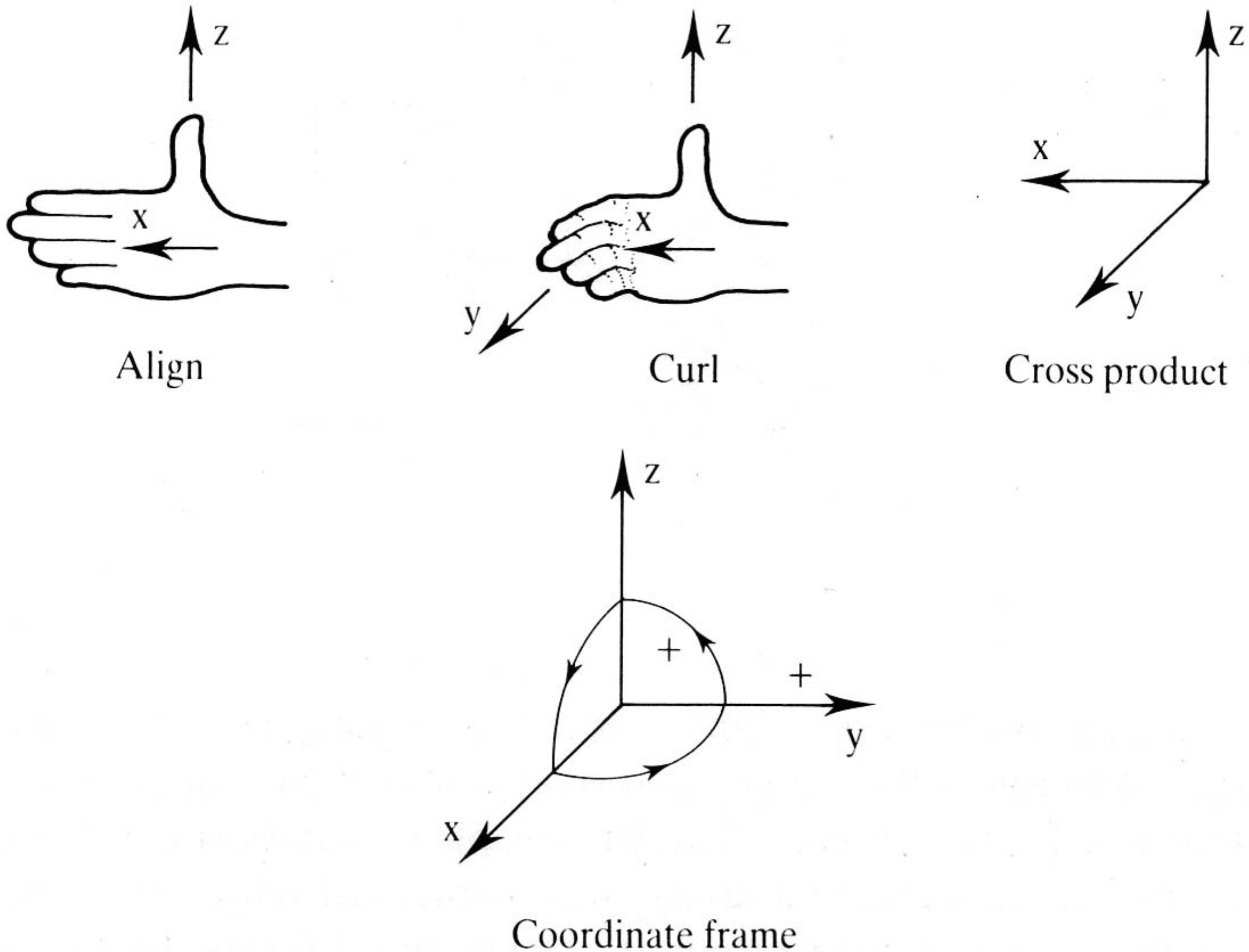


Location of a point in a two-dimensional coordinate system, and an object in a three-dimensional coordinate system.

Right hand rule

Positive direction of rotation is chosen accordance with the right hand rule

Hold right hand open, with the thumb pointing in the direction of the axis of rotation, and fingers pointing in the direction of the second axis. Curl the fingers through ๙๐ degree until they point in the direction of the third axis- This is the direction of positive rotation.



Coordinate frame showing positive directions of axes according to the right-hand rule.

The positive directions of the axes are the direction in which the thumb, the hand and the fingers are pointing

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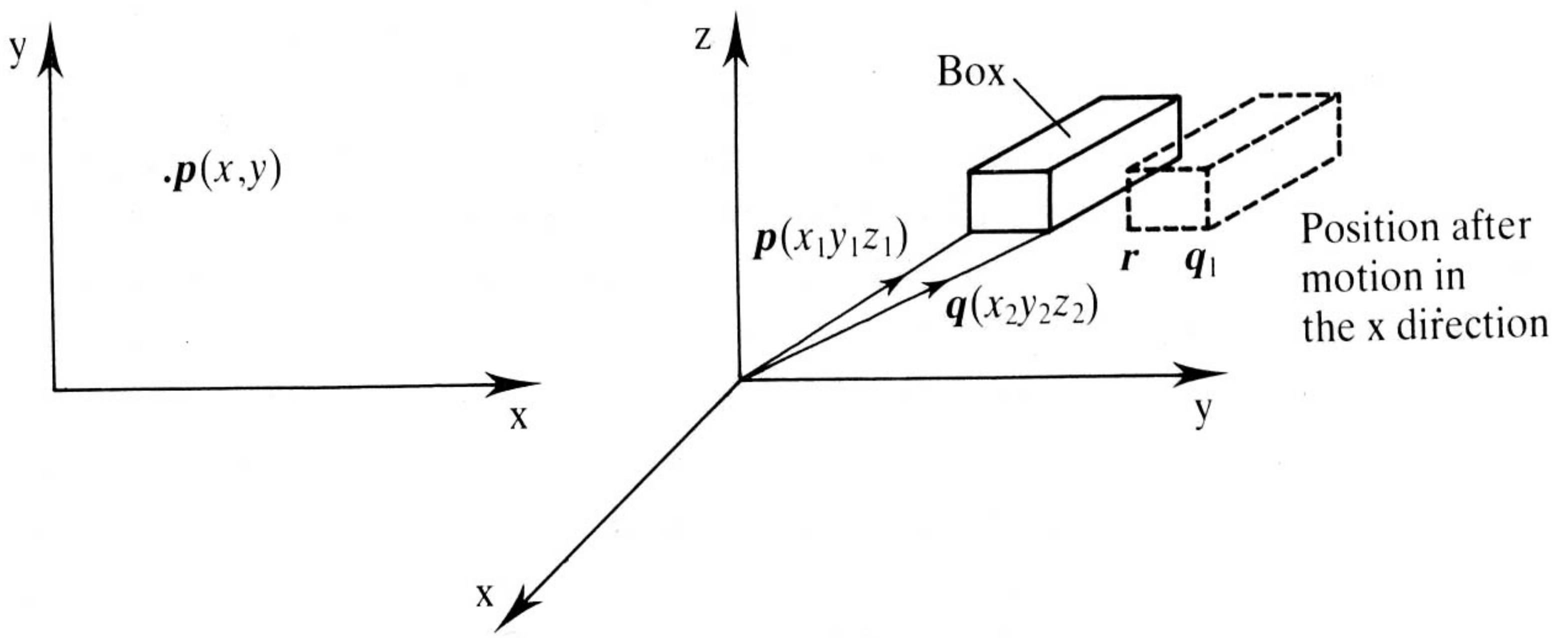


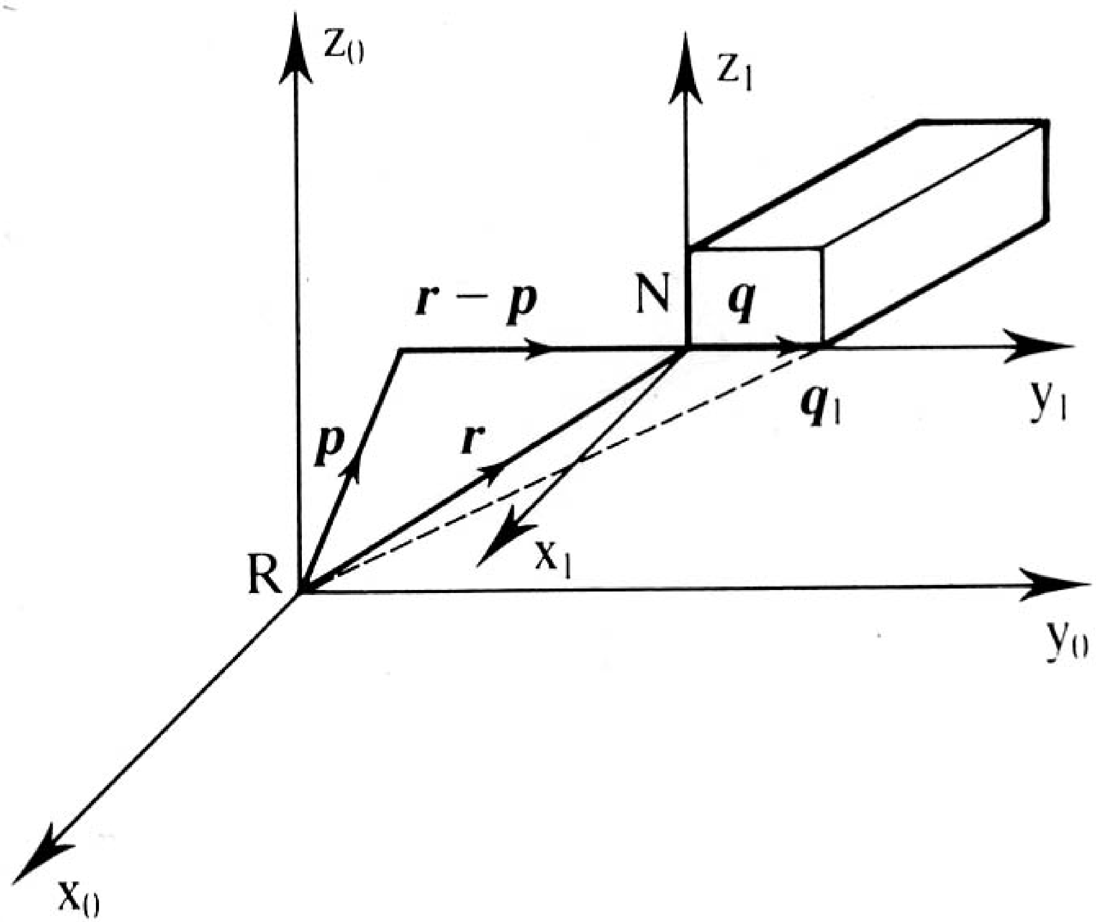
Location of an object in a three-dimensional coordinate system.

If the box is moved in x direction from $\boldsymbol{p}(x_{๑},y_{๑},z_{๑})$ to $\boldsymbol{r}(x_{๓},y_{๓},z_{๓})$, eight new vectors must be calculated to define the new position of the box.

Example bottom right hand corner

$$q_{๑} = q + r - p$$

If the movement is complex, the calculation time is required.



Object location described with a coordinate system located in the object.

Location of an object can be describe by using the origin of coordinate frame ($x_{๑}$,$y_{๑}$,$z_{๑}$) or a point vector (***p***) with respect to the reference coordinate frame ($x_{๐}$,$y_{๐}$,$z_{๐}$).

The position of points on the object are defined with respect to the object coordinate frame.

The movement of the origin of object coordinate frame from vector ***p*** to vector ***r***.

$$\boldsymbol{r} = \boldsymbol{p} + (\boldsymbol{r} - \boldsymbol{p})$$

Location of any points on the object can be calculated with respect to reference coordinate frame.

Example. to fine the vector $\boldsymbol{q}_{๑}$, We add the vector which describes the object with respect to the origin of the object coordinate frame (${}^{R}\boldsymbol{q}$ or $\boldsymbol{q}$) to the vector which describes the origin of the object coordinate frame with respect to reference coordinate frame (${}^{R}\boldsymbol{r}$ or $\boldsymbol{r}$).

$$\boldsymbol{q}_{๑} = \boldsymbol{r} + \boldsymbol{q} \quad \text{or} \quad {}^{R}\boldsymbol{q}_{๑} = {}^{R}\boldsymbol{r} + {}^{R}\boldsymbol{q}$$

If the object is rotated, the axes of the new coordinate frame are no longer parallel to the axes of the original coordinate frame.

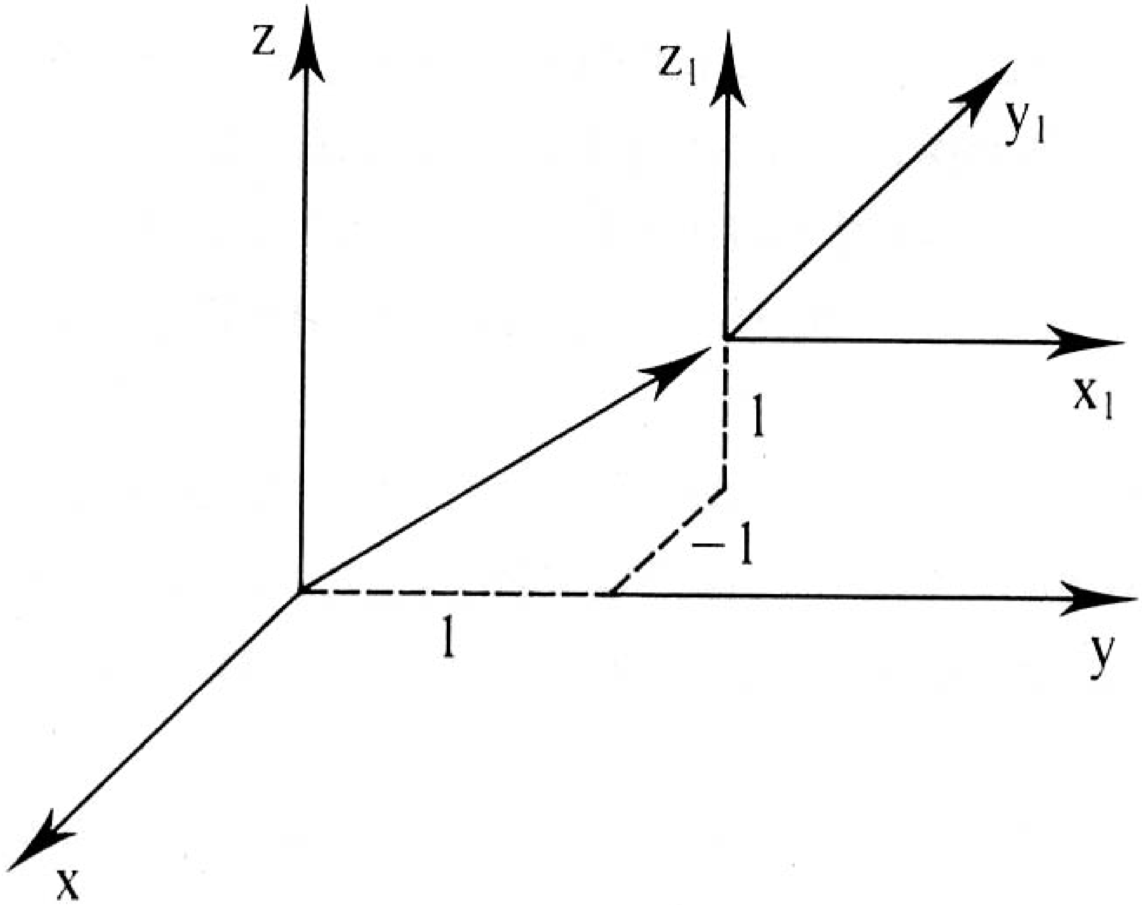
After rotation, the new coordinate frame is represented by four vectors :

๑ The relocation of the origin.

๒ The direction of x axis.

๓ The direction of y axis.

๔ The direction of z axis.



Relationship between two coordinate frames.
The new frame has been rotated by ๙๐ degrees
around the z axis, and then translated.

In conclusion, an object has ๖ **degrees of freedom** three degrees of translation and three degrees of rotation :

๑ Translation in x direction.
๒ Translation in y direction.
๓ Translation in z direction.
๔ Rotation about the x axis.
๕ Rotation about the y axis.
๖ Rotation about the z axis.

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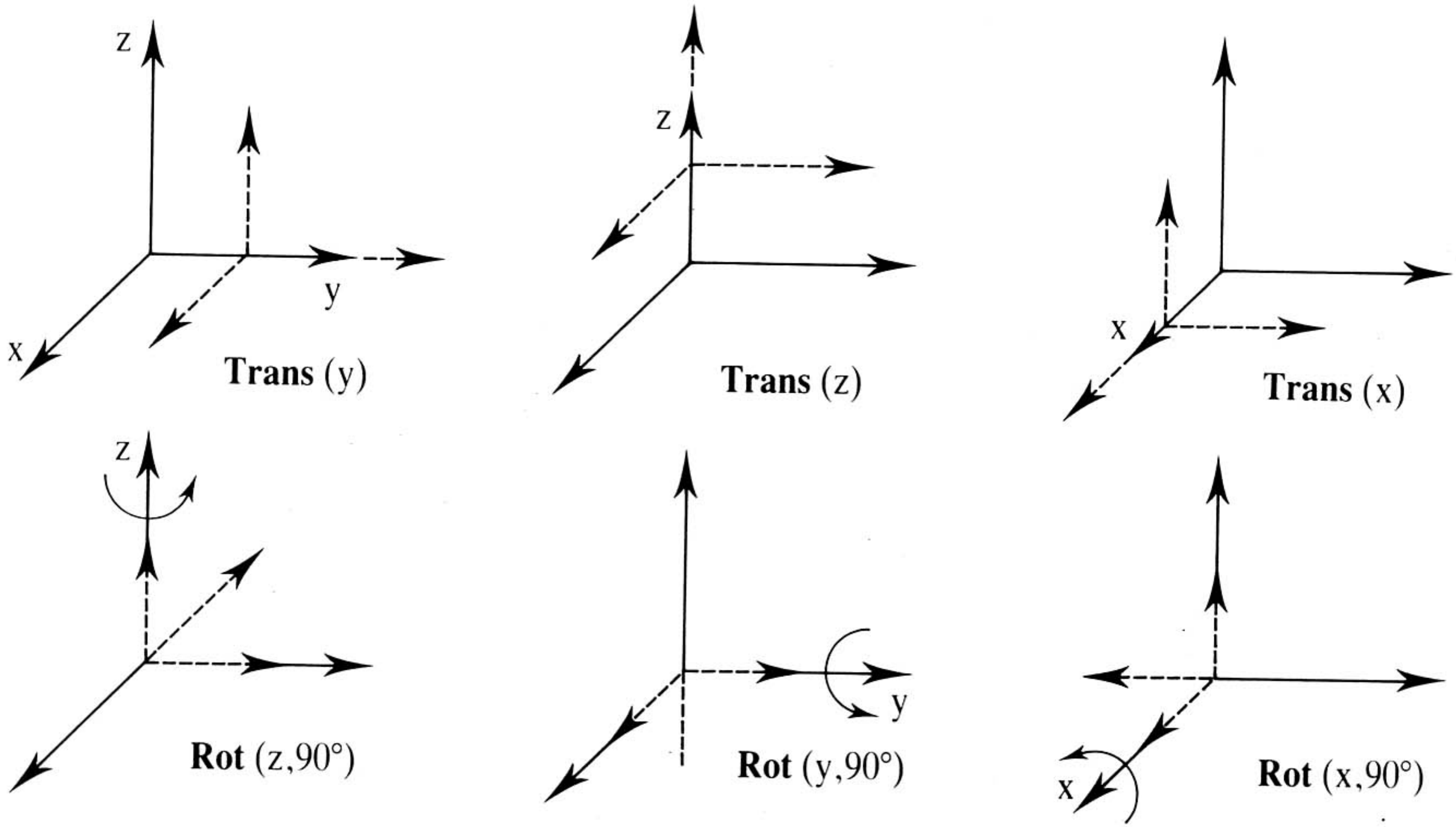

The six degrees of freedom of a coordinate frame, three rotations and three translations, direction according to right hand rule.

๒ Two-dimensional transformations

To concentrate on concepts, the two-dimension transformations are examined.

In two-dimension system, one point is described with a two-component vector ***p*** [*x*,*y*].

A set of transformed coordinates $\boldsymbol{p}_{๑}[x_{๑},y_{๑}]$ can be obtain by multiply the point $\boldsymbol{p}[x,y]$ with a general ๒*๒ transformation matrix.

$$\begin{bmatrix} x_{๑} \\ y_{๑} \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (ax+cy) \\ (bx+dy) \end{bmatrix}$$

Where $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is a general ๒*๒ transformation matrix.

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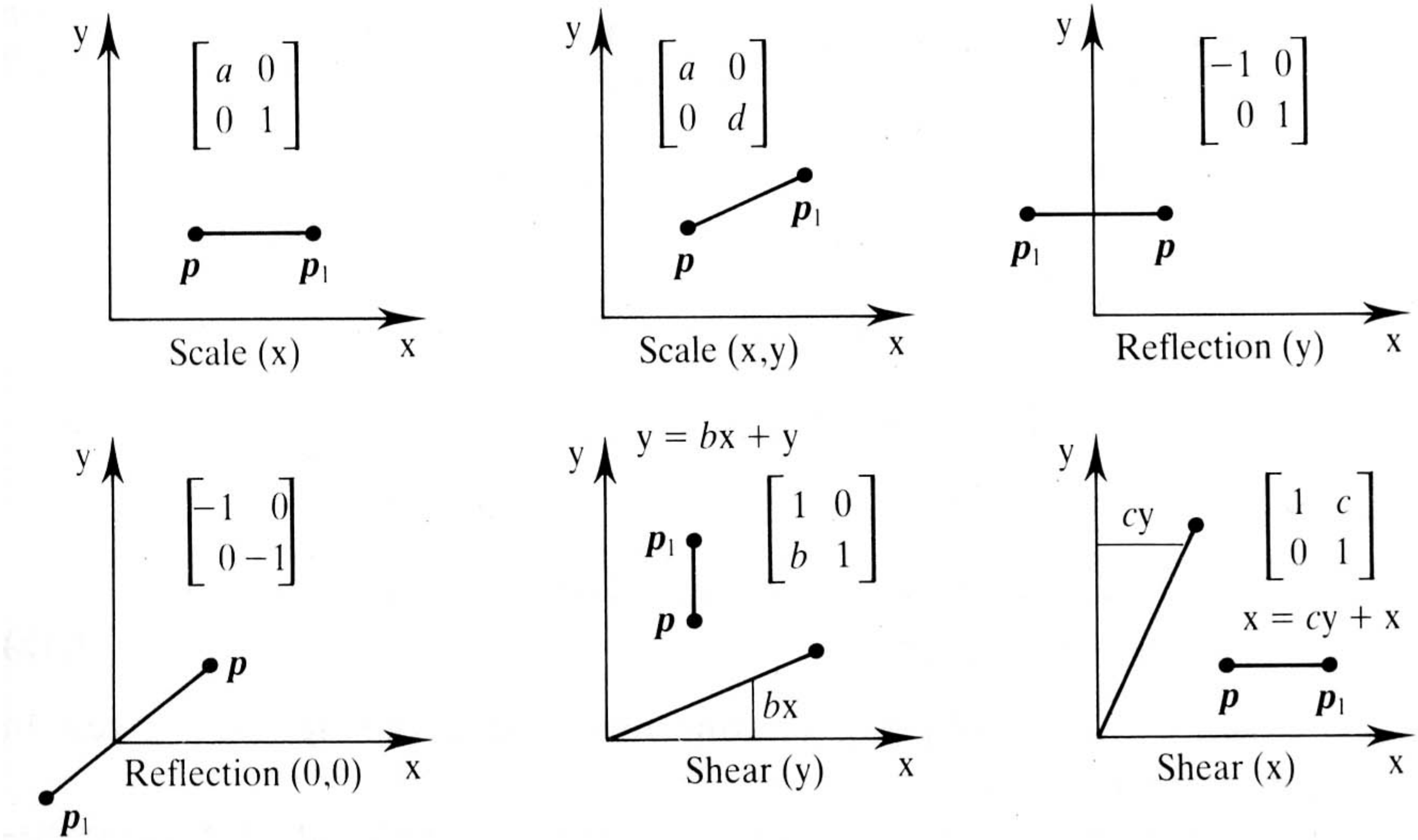
Identity matrix

$$\begin{bmatrix} ๑ & ๐ \\ ๐ & ๑ \end{bmatrix}$$

Scale change in x direction

$$\begin{bmatrix} x_{๑} \\ y_{๑} \end{bmatrix} = \begin{bmatrix} a & ๐ \\ ๐ & ๑ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$$



Effect of various terms in the ๒*๒ general transformation matrix.

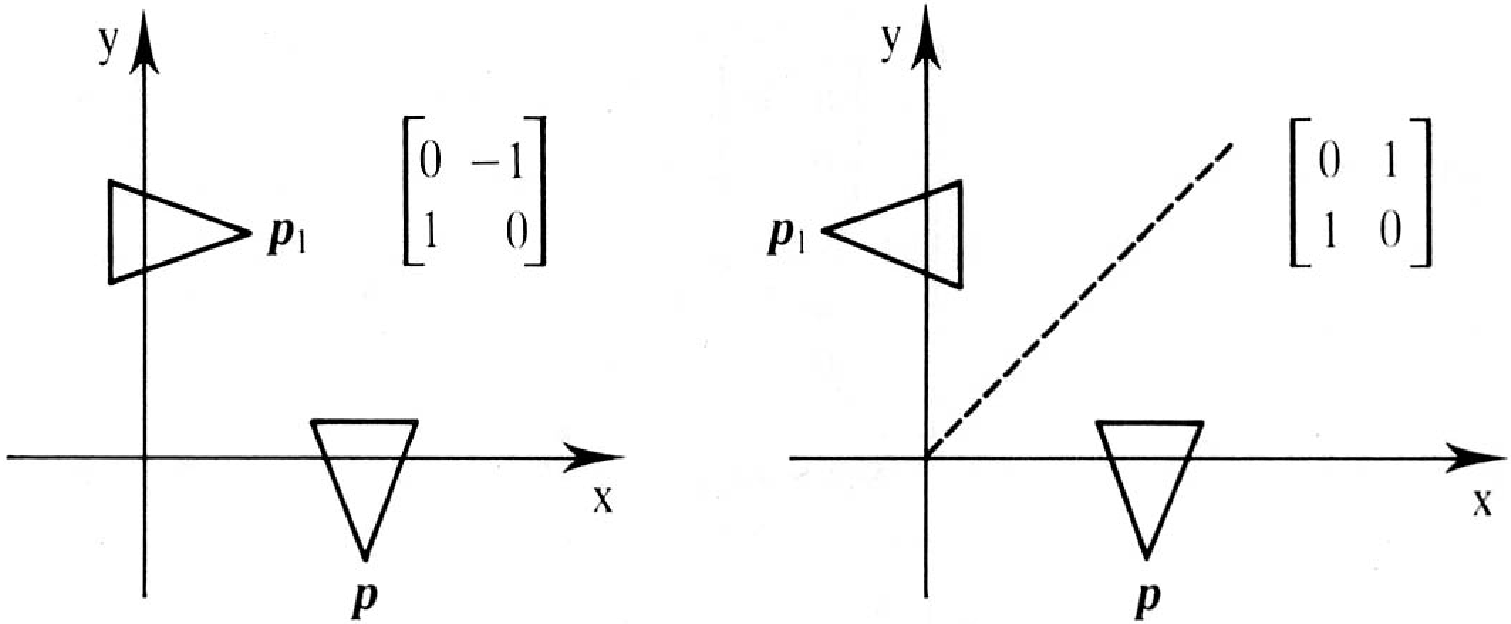
A rotation in two-dimension, the triangle is rotated through ๙๐ degree about origin in counter clockwise direction :

$$\begin{bmatrix} x_{๑} \\ y_{๑} \end{bmatrix} = \begin{bmatrix} ๐ & -๑ \\ ๑ & ๐ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

where

$$\begin{bmatrix} ๐ & -๑ \\ ๑ & ๐ \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

when $\phi = ๙๐^{\circ}$



Rotation about (0,0)

Reflection about y = x

Rotation and reflection

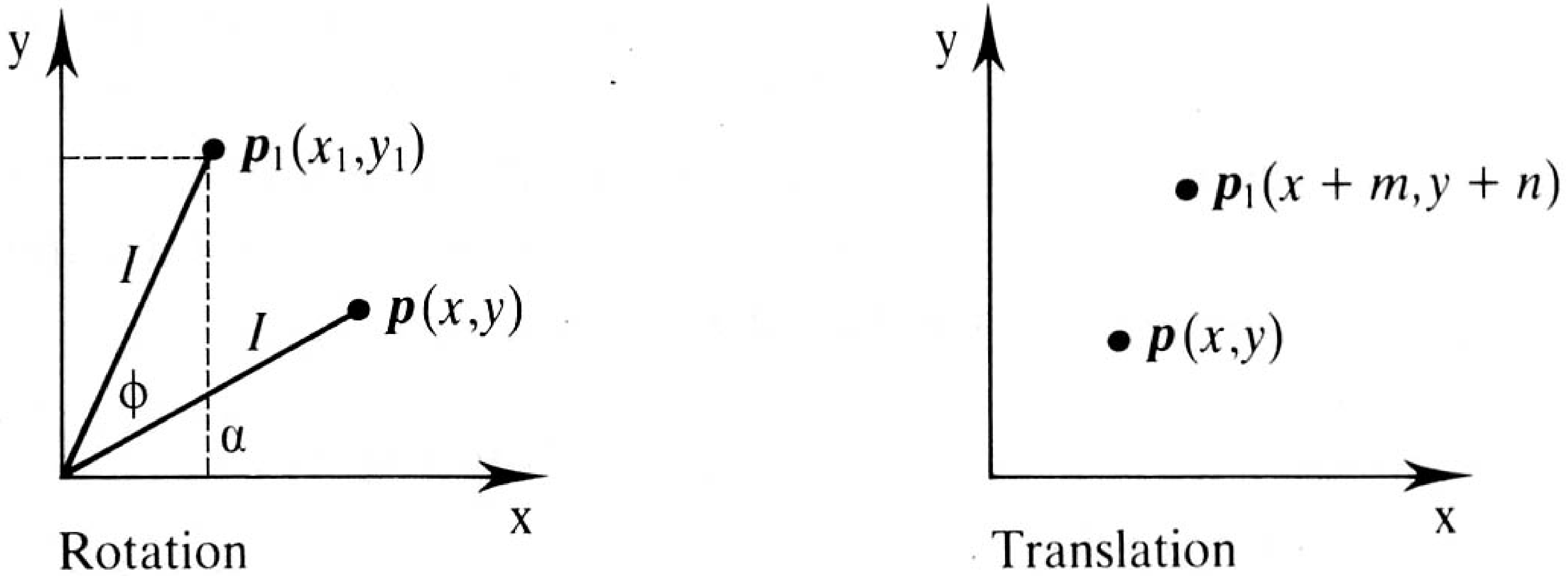
The ๒*๒ transformation matrix which produces rotation about the origin is

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Vector rotation

The vector ***p***, which is at α° to the x axis and pass through the origin. If it is rotated through ϕ° to become $\boldsymbol{p}_{๑}$ the coordinates of the point $\boldsymbol{p}_{๑}$ are :

$$
\begin{aligned}
x_{๑} &= l\cos(\alpha+\phi) \\
&= l\cos(\alpha)\cos(\phi) - l\sin(\alpha)\sin(\phi) \\
&= x\cos(\phi) - y\sin(\phi)
\end{aligned}
$$



Rotation of a vector and translation of a point, using general transformation matrix.

$$
\begin{aligned}
y_{๑} &= l\sin(\alpha+\phi) \\
&= l\sin(\alpha)\cos(\phi) - l\cos(\alpha)\sin(\phi) \\
&= y\cos(\phi) + x\sin(\phi)
\end{aligned}
$$

In rotation transformation :

$$
\begin{bmatrix} x_{๑} \\ y_{๑} \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

Homogeneous coordinate

To assign the translation to the transformation matrix, we introduce the third component to the point vector $\boldsymbol{p}[x\ \ y\ \ ๑]^T$ and $\boldsymbol{p}_๑[x_๑\ \ y_๑\ \ ๑]^T$. The new coordinates are known as *Homogeneous coordinate.*

The transformation equation becomes :

$$\begin{bmatrix} x_{๑} \\ y_{๑} \\ ๑ \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & m \\ \sin(\phi) & \cos(\phi) & n \\ ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} x \\ y \\ ๑ \end{bmatrix}$$

Where m and n are the translation constants in the x and y directions respectively.

Principle of Robotics

๓ Three-dimensional transformations

The general, ๔*๔, three-dimensional transformation matrix :

$$T=\left[\begin{array}{c|c} \text{๓} * \text{๓} & \text{๓} * \text{๑} \\ \hline \text{๑} * \text{๓} & \text{๑} * \text{๑} \end{array}\right]=\left[\begin{array}{c|c} \text{rotation} & \text{translation} \\ \hline \text{๐} & \text{๑} \end{array}\right]$$

The general transformation matrix, corresponding to a translation by a vector $p_x i + p_y j + p_z k$:

$$Trans(p_x, p_y, p_z) = \begin{bmatrix} ๑ & ๐ & ๐ & p_x \\ ๐ & ๑ & ๐ & p_y \\ ๐ & ๐ & ๑ & p_z \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

Three transformation matrix, corresponding to a rotation about the x, y and z axis by angle ϕ :

$$Rot(x,\phi)=\begin{bmatrix} \text{๑} & \text{๐} & \text{๐} & \text{๐} \\ \text{๐} & \cos(\phi) & -\sin(\phi) & \text{๐} \\ \text{๐} & \sin(\phi) & \cos(\phi) & \text{๐} \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix}$$

$$Rot(y,\phi)=\begin{bmatrix} \cos(\phi) & ๐ & \sin(\phi) & ๐ \\ ๐ & ๑ & ๐ & ๐ \\ -\sin(\phi) & ๐ & \cos(\phi) & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

$$Rot(z,\phi)=\begin{bmatrix} \cos(\phi) & -\sin(\phi) & ๐ & ๐ \\ \sin(\phi) & \cos(\phi) & ๐ & ๐ \\ ๐ & ๐ & ๑ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

A translation and a rotation can be combined by multiplying the transformation matrices.

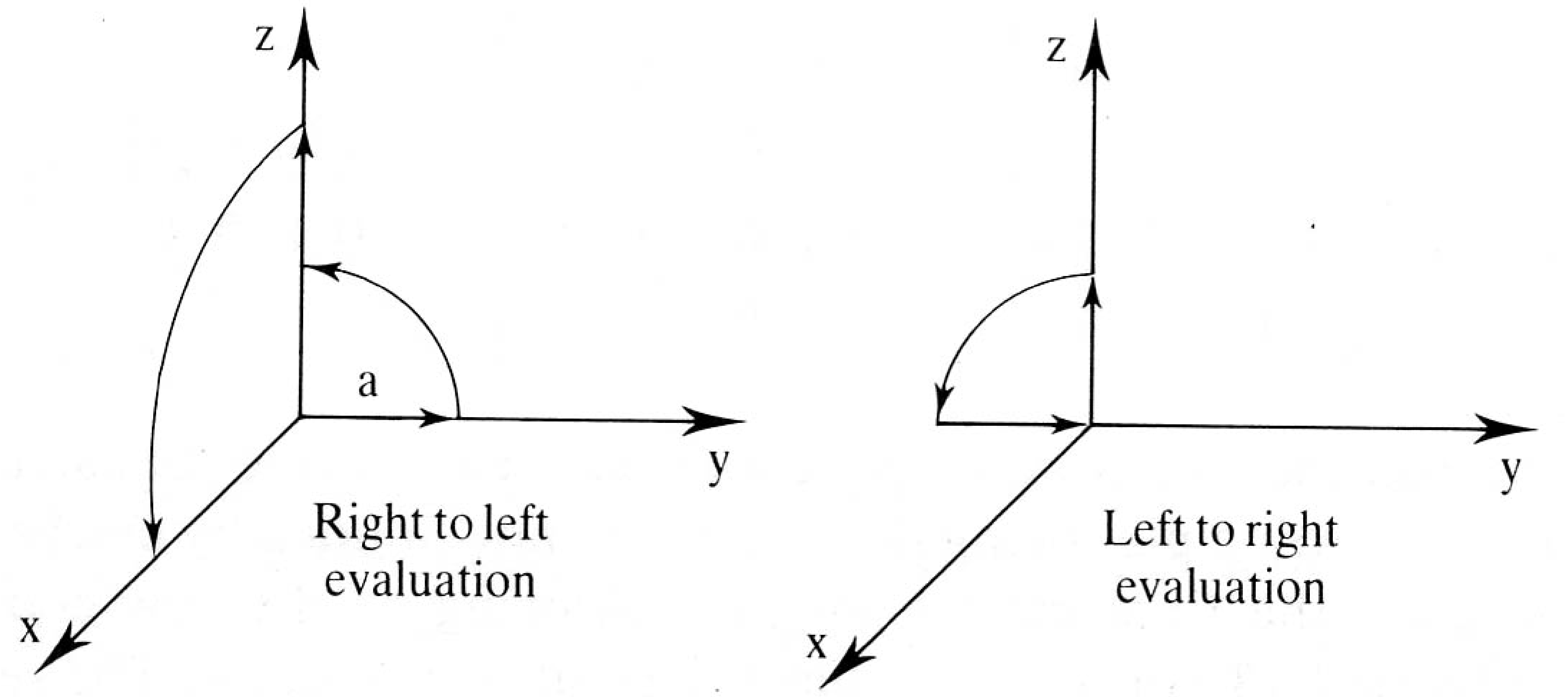
$$Trans(p_x, p_y, p_z)\,Rot(z,\phi) = \begin{bmatrix} \text{๑} & \text{๐} & \text{๐} & p_x \\ \text{๐} & \text{๑} & \text{๐} & p_y \\ \text{๐} & \text{๐} & \text{๑} & p_z \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & \text{๐} & \text{๐} \\ \sin(\phi) & \cos(\phi) & \text{๐} & \text{๐} \\ \text{๐} & \text{๐} & \text{๑} & \text{๐} \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\phi) & -\sin(\phi) & \text{๐} & p_x \\ \sin(\phi) & \cos(\phi) & \text{๐} & p_y \\ \text{๐} & \text{๐} & \text{๑} & p_z \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix}$$

A point located at the origin of coordinate frame is translated in the y direction, rotated around the x axis, translated in z direction and rotated around the y axis.

$$^{R}T_{N} = Rot(y, \text{๙})\, Trans(z, a)\, Rot(x, \text{๙})\, Trans(y, a)$$

<u>Order of evaluation(multiplication) is important.</u>



Combined rotation and translation, showing the effect of the order of evaluation of the transform equation.

๔ Coordinate frames

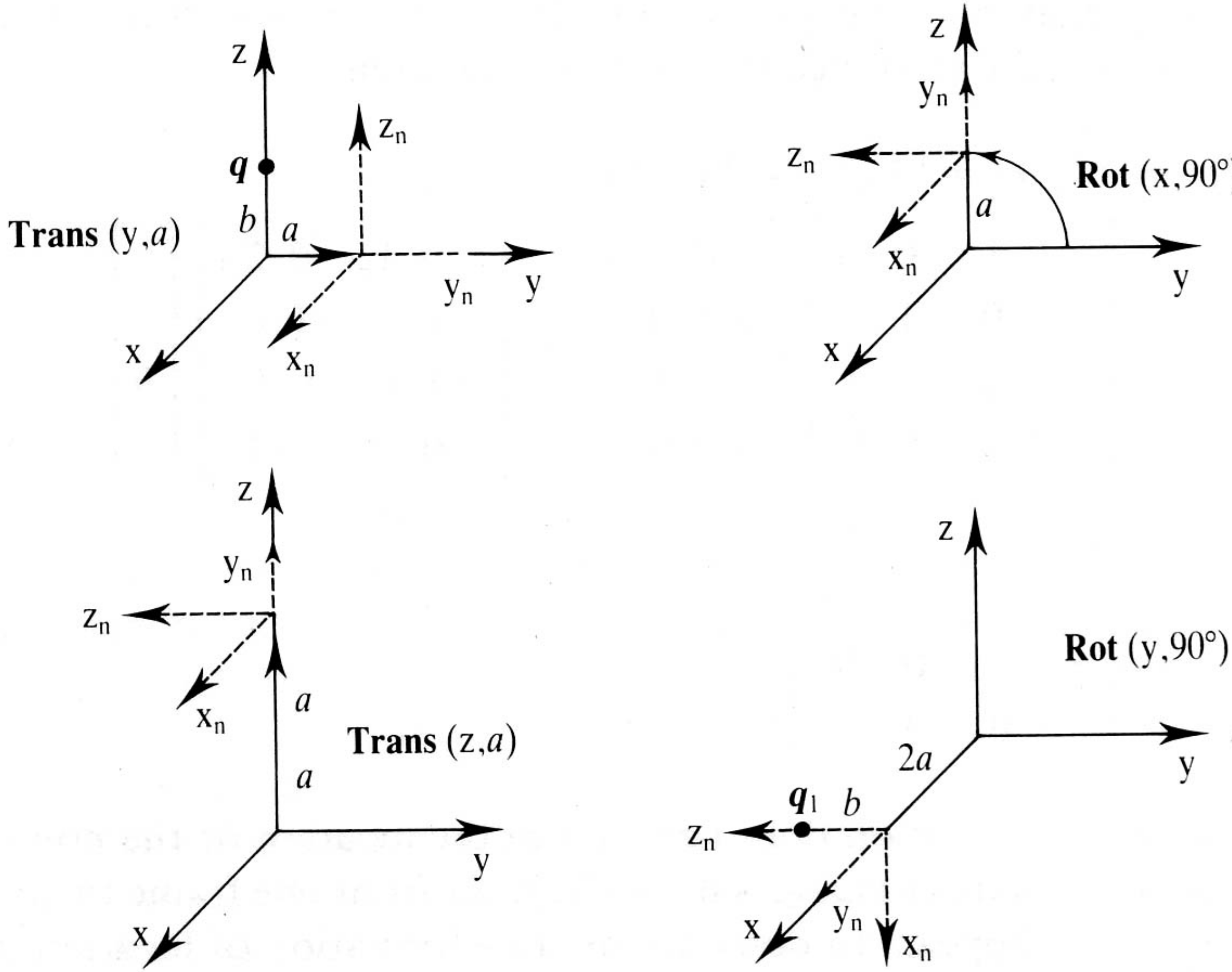
A coordinate frame located at the origin of the reference coordinate frame is translated a unit in the y direction, rotated ๙๐° around the x axis, translated a unit in z direction and rotated around ๙๐° the y axis.

$${}^{R}T_{N} = Rot(y, \text{๙๖})\, Trans(z, a)\, Rot(x, \text{๙๖})\, Trans(y, a)$$

$$ {}^{R}T_{N} = Rot(y, \text{๙๐})\, Trans(z, a)\, Rot(x, \text{๙๐})\, Trans(y, a) $$

$$ = \begin{bmatrix} \text{๐} & \text{๑} & \text{๐} & \text{๒}a \\ \text{๐} & \text{๐} & -\text{๑} & \text{๐} \\ -\text{๑} & \text{๐} & \text{๐} & \text{๐} \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix} $$

$$ = \begin{bmatrix} x & y & z & p \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix} $$



Interpreting the transform as a coordinate frame by evaluating the transformation equation right to left with respect to the reference frame.

๑ The first column is a vector describing the direction of the new x axis : x is in the negative z direction.

- ๒ The second column is a vector describing the direction of the new y axis: y is in the x direction.

๓ The third column is a vector describing the direction of the new z axis : z is in the negative y direction.

- ๔ The fourth column is a vector describing the displacement of the origin of the new coordinate frame from the origin of the reference coordinate frame: ***p*** is a translation of ๒a units in the positive x direction.

The resultant transformation matrix effectively transforms the origin of the reference coordinate frame to the origin of the new coordinate frame. It describes the position and orientation of the new coordinate frame in terms of the reference frame.

The transformation of any point that has been defined with respect to the new frame to the reference frame can be done by pre-multiplying the vector describing the point with the transformation matrix.

Example the point ${}^{N}\boldsymbol{q}_{\text{๑}}(\text{๐},\text{๐},b)$ in the new frame becomes ${}^{N}\boldsymbol{q}_{\text{๑}}(\text{๒}a,-b,\text{๐})$ in the reference frame.

$$^{R}q_{๑} = {}^{R}T_{N}\,{}^{N}q_{๑}$$

$$= \begin{bmatrix} ๐ & ๑ & ๐ & ๒a \\ ๐ & ๐ & -๑ & ๐ \\ -๑ & ๐ & ๐ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} ๑ & ๐ & ๐ & ๐ \\ ๐ & ๑ & ๐ & ๐ \\ ๐ & ๐ & ๑ & b \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

$$= \begin{bmatrix} ๐ & ๑ & ๐ & ๒a \\ ๐ & ๐ & -๑ & -b \\ -๑ & ๐ & ๐ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

๔ Relative transformations

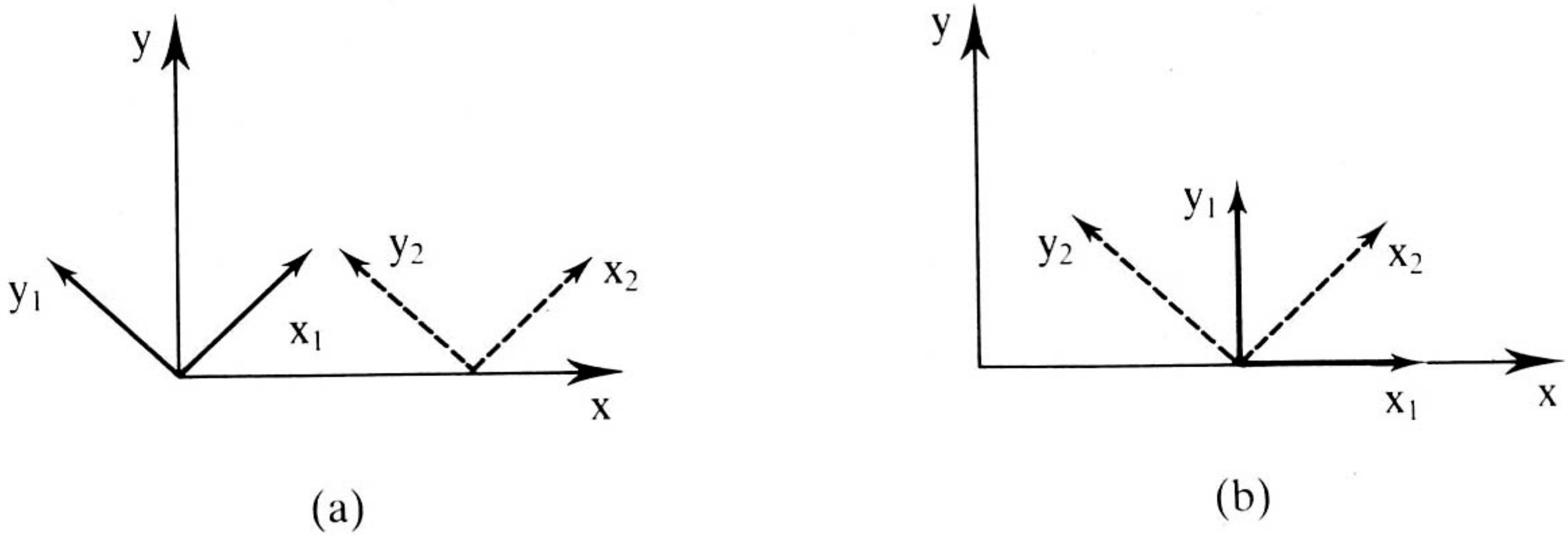
The translation and rotation with respect to the reference coordinate frame is known as the *absolute* or *forward transformation.*

The translation and rotation with respect to the new coordinate frame is known as the *relative transformation.*

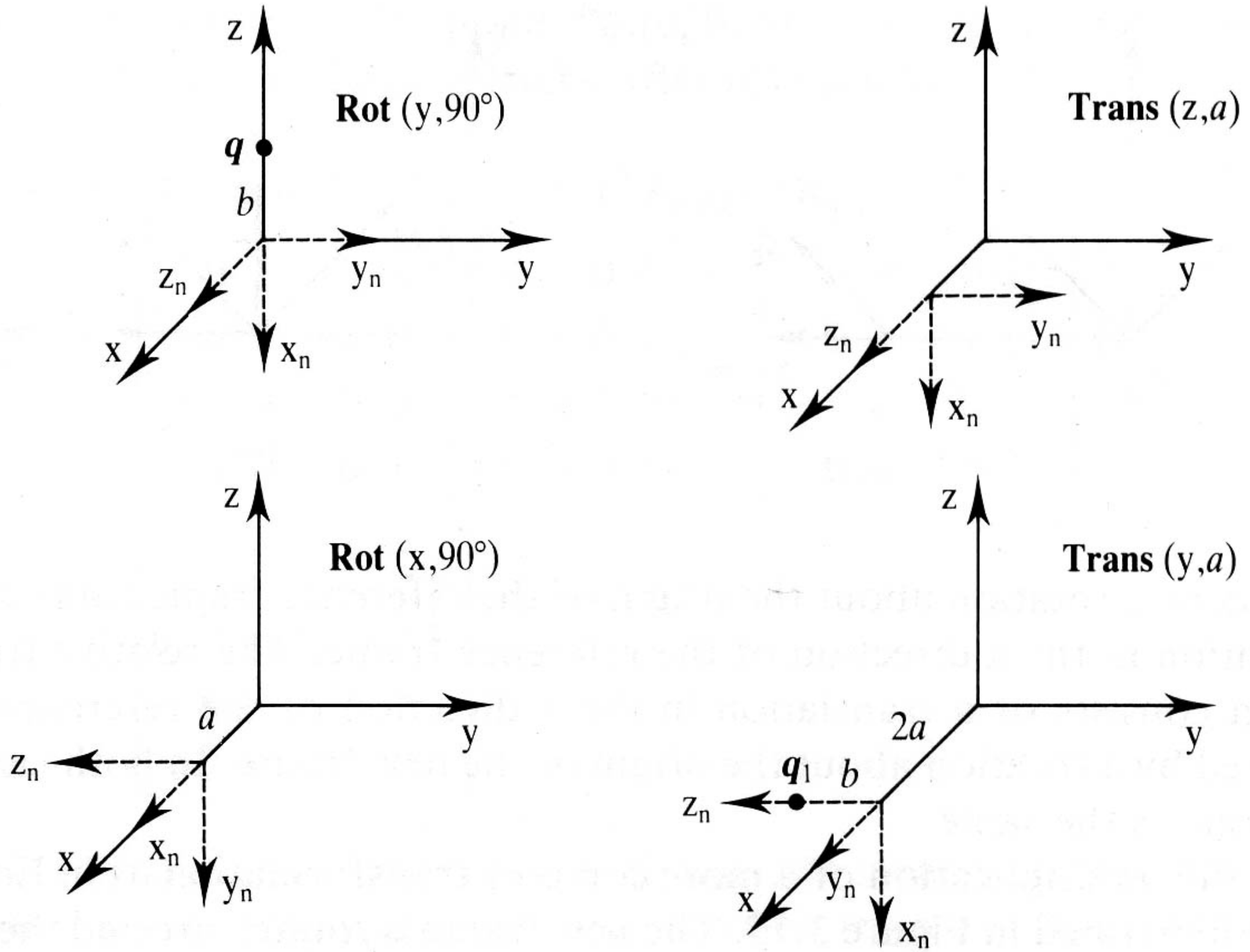
The two methods for processing a transformation equation :

๑ Execute the transformations right to left with respect to the reference frame (premultiply).

๒ Execute the transformations left to right with respect to the new frame (postmultiply).



Absolute (a) transformation versus relative transformation for the equation (b) Trans(x,a)Rot(๔๕)



Interpreting the transform as a coordinate frame by evaluating the transformation matrix from left to right with respect to the new coordinate frames.

๖ General transformations

A transformation can be thought about in a number of way :

- As a transformation from one coordinate frame to another.
- As a description of the origin and axes of a new coordinate frame in terms of reference frame.

- As a description of the motion of an object from one location (reference frame) to another (new frame).
- As a means of calculating the location of a point on an object with respect to the reference frame from its location with respect to the new frame.

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$$ {}^{R}\mathbf{T}_{N}\,{}^{N}\boldsymbol{q} = {}^{R}\boldsymbol{q} $$

${}^{R}\mathbf{T}_{N}$ is the transform of the mew frame with respect to the reference frame.

The transformation made on frame R to get frame N is known as ***forward transform***.

The general transformation matrix is :

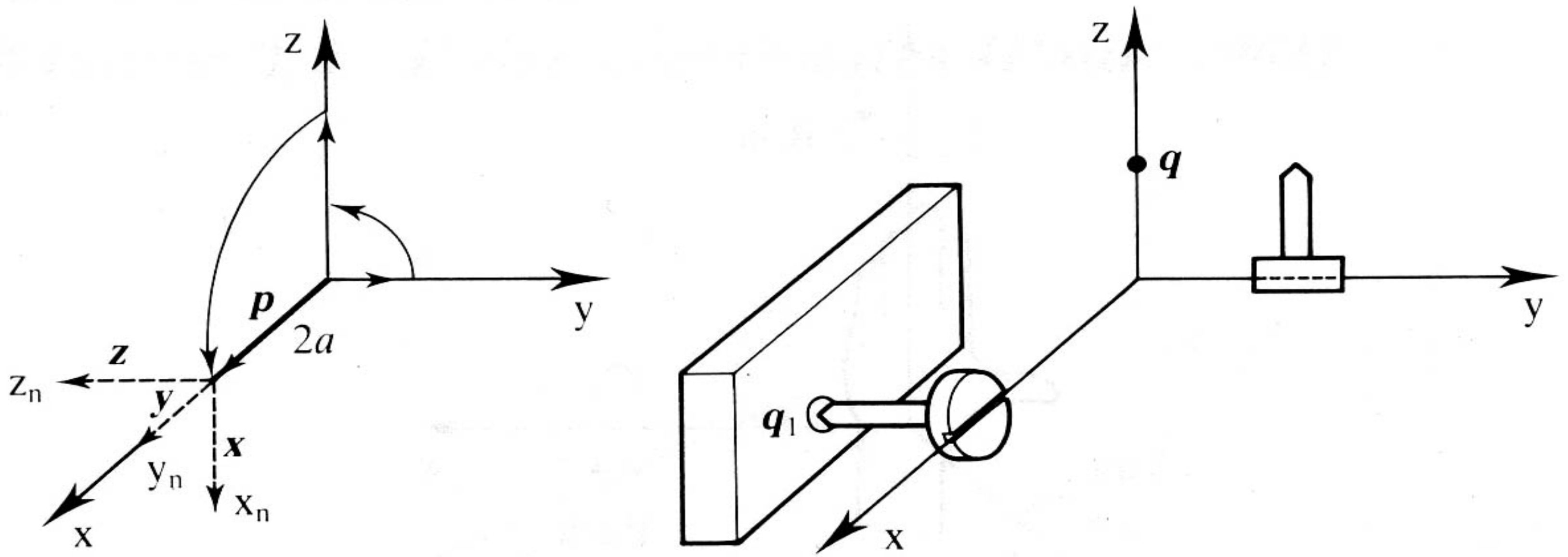
$$
{}^{\mathbf{R}}\mathbf{T}_{\mathbf{N}} = \begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} & \boldsymbol{p} \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix}
$$

$\boldsymbol{p} = p_x \boldsymbol{i} + p_y \boldsymbol{j} + p_z \boldsymbol{k}$ - location of the origin of the new frame.

$\boldsymbol{x} = x_x \boldsymbol{i} + x_y \boldsymbol{j} + x_z \boldsymbol{k}$ - direction of the x axis of the new frame.

$\boldsymbol{y} = y_x \boldsymbol{i} + y_y \boldsymbol{j} + y_z \boldsymbol{k}$ - direction of the y axis of the new frame.

$\boldsymbol{z} = z_x \boldsymbol{i} + z_y \boldsymbol{j} + z_z \boldsymbol{k}$ - direction of the z axis of the new frame.



Interpreting the transform as a coordinate frame, general transformation and movement of an object.

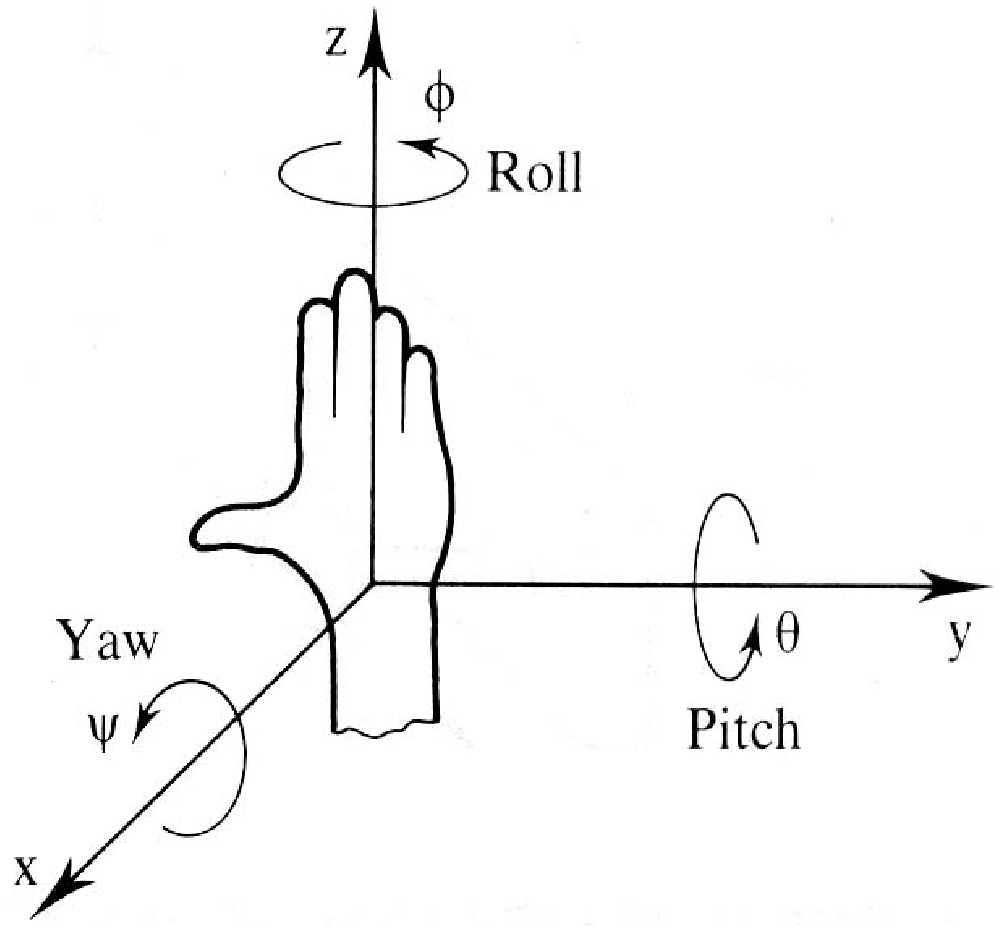
๗ General orientation transformations

The roll-pitch-yaw transformation :

Stretch your arm out in front of you with your palm facing down. First, rotate your hand about an axis along your arm-this is roll; second tilt your hand up and down about your wrist joint-this is pitch; third, turn your hand side by side about your wrist joint-this is yaw.

$$RPY(\phi,\theta,\psi)=Rot(z,\phi)\,Rot(y,\theta)\,Rot(x,\psi)$$

$$=\begin{bmatrix} C(\phi)C(\theta) & C(\phi)S(\theta)S(\psi)-S(\phi)C(\psi) & C(\phi)S(\theta)C(\psi)+S(\phi)S(\psi) & \text{๐} \\ S(\phi)C(\theta) & S(\phi)S(\theta)S(\psi)+C(\phi)C(\psi) & S(\phi)S(\theta)C(\psi)-C(\phi)S(\psi) & \text{๐} \\ -S(\theta) & C(\theta)S(\psi) & C(\theta)C(\psi) & \text{๐} \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix}$$



Roll, Pitch and Yaw orientation angles

Principle of Robotics

Euler transformation

$$Euler(\phi,\theta,\psi)=Rot(z,\phi)\,Rot(y,\theta)\,Rot(z,\psi)$$

$$
{}^{R}\mathbf{T}_{N} = \begin{bmatrix} x & y & z & p \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix} = \begin{bmatrix} \text{translation} \\ \text{transform} \end{bmatrix} \begin{bmatrix} \text{orientation} \\ \text{transform} \end{bmatrix}
$$

$$
= \begin{bmatrix} \text{๑} & \text{๐} & \text{๐} & p_x \\ \text{๐} & \text{๑} & \text{๐} & p_y \\ \text{๐} & \text{๐} & \text{๑} & p_z \\ \text{๐} & \text{๐} & \text{๐} & \text{๑} \end{bmatrix} \begin{bmatrix} \text{Orientation} \\ \text{transform} \\ \text{Roll} - \text{pitch} - \\ \text{yaw} \end{bmatrix}
$$

$$ {}^{R}q = {}^{R}p_{q} = {}^{R}p_{N,q} + {}^{R}p_{N} = {}^{R}T_{N}\, {}^{N}q = {}^{R}T_{N}\, {}^{N}p_{q} $$

Where

$\boldsymbol{p}$ is position vector.

${}^{R}p_{N,q} = {}^{R}\mathbf{Rot}_{N}\, {}^{R}p_{q}$ is the vector from the origin of frame N to point q as seen from frame R.

${}^{R}\mathbf{Rot}_{N}$ is the rotation transform from frame R to N.

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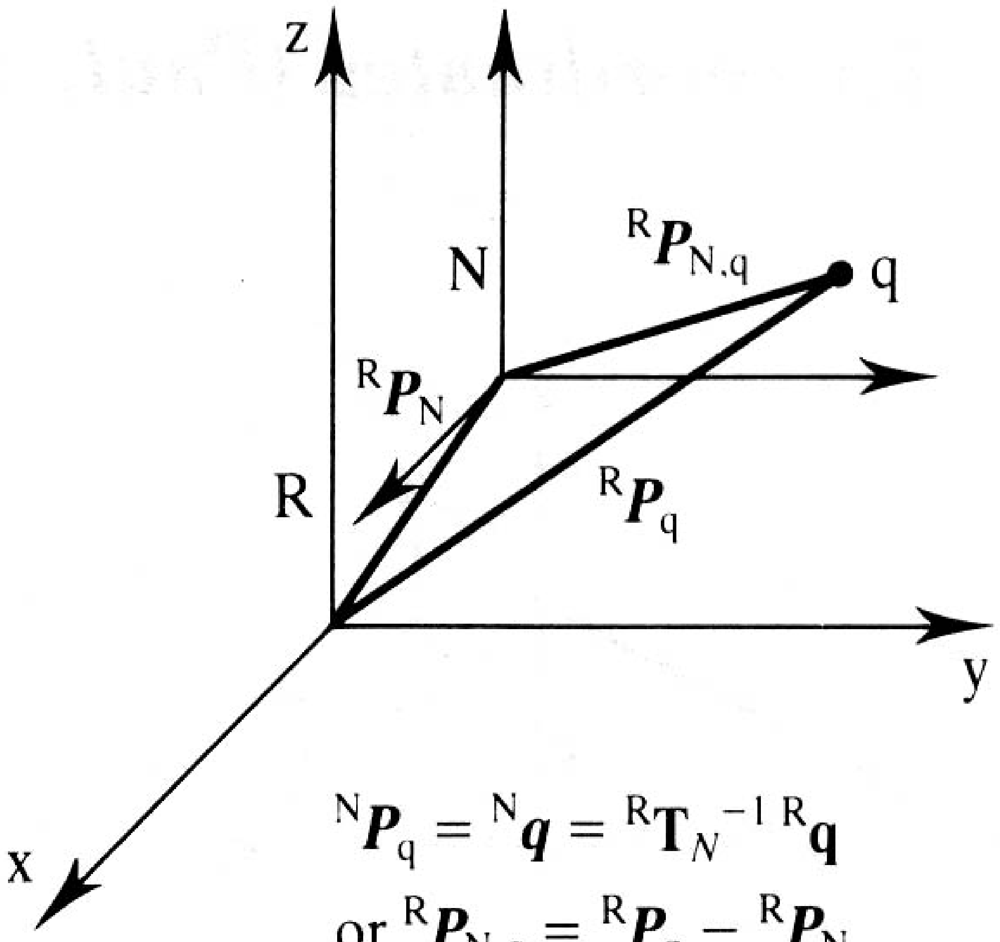
๘ Inverse transformations

Inverse transformation is the transformation from the new coordinate frame in the hand back to the reference frame.

$$^{R}\mathbf{T}_{\mathbf{N}}^{-๑}\,{}^{R}\mathbf{T}_{N}\,{}^{N}q = {}^{R}\mathbf{T}_{N}^{-๑}\,{}^{R}q$$

$$^{N}q = {}^{R}\mathbf{T}_{N}^{-๑}\,{}^{R}q$$



$${}^{N}\boldsymbol{P}_{q} = {}^{N}\boldsymbol{q} = {}^{R}\mathbf{T}_{N}^{-1}\, {}^{R}\mathbf{q}$$

$$\text{or } {}^{R}\boldsymbol{P}_{N,q} = {}^{R}\boldsymbol{P}_{q} - {}^{R}\boldsymbol{P}_{N}$$

Inverse transformation.

$$^{R}p_{N,q} = {}^{R}p_{q} - {}^{R}p_{N}$$

$$T^{-๑} = \begin{bmatrix} x_x & x_y & x_z & -p\cdot x \\ y_x & y_y & y_z & -p\cdot y \\ z_x & z_y & z_z & -p\cdot z \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

$$= \begin{bmatrix} x_x & x_y & x_z & -p_x x_x - p_y x_y - p_z x_z \\ y_x & y_y & y_z & -p_x y_x - p_y y_y - p_z y_z \\ z_x & z_y & z_z & -p_x z_x - p_y z_y - p_z z_z \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

$$TT^{-๑} = \begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} x_x & x_y & x_z & -p_x x_x - p_y x_y - p_z x_z \\ y_x & y_y & y_z & -p_x y_x - p_y y_y - p_z y_z \\ z_x & z_y & z_z & -p_x z_x - p_y z_y - p_z z_z \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

$$= \begin{bmatrix} ๑ & ๐ & ๐ & ๐ \\ ๐ & ๑ & ๐ & ๐ \\ ๐ & ๐ & ๑ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}$$

$$ {}^{R}\mathbf{T}_{N}^{-๑} = \begin{bmatrix} ๐ & ๐ & -๑ & ๐ \\ ๑ & ๐ & ๐ & -๒a \\ ๐ & -๑ & ๐ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} $$

$$ {}^{N}q_{๑} = {}^{R}\mathbf{T}_{N}^{-๑}\, {}^{R}q_{๑} $$

$$ = \begin{bmatrix} ๐ & ๐ & -๑ & ๐ \\ ๑ & ๐ & ๐ & -๒a \\ ๐ & -๑ & ๐ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} ๒a \\ -b \\ ๐ \\ ๑ \end{bmatrix} = \begin{bmatrix} ๐ \\ ๐ \\ b \\ ๑ \end{bmatrix} $$

๙ Object location

The transformation of coordinate frames was introduced to make modelling the relocation of objects easier. An object is described with respect to a frame located in the object, and this frame is relocated with a transformation.

Example We wish to use a robot to pick up a screw located at (๐,a,๐), relocate it to (๒a,๐,๐) and change its orientation so that it points at the negative z direction, with slot of the screw head parallel to the x axis.

The two translations serve separate purposes : first to relocate the new frame from the origin to the base of the screw, and the second to relocate the screw to a position where it can be screwed into another object.

The two rotations align the screw with the axis of the hole in the object, and align the slot in the screw with the x-axis.

๑๐ Transform graphs

The problem of locating an object has been decomposed into two steps :

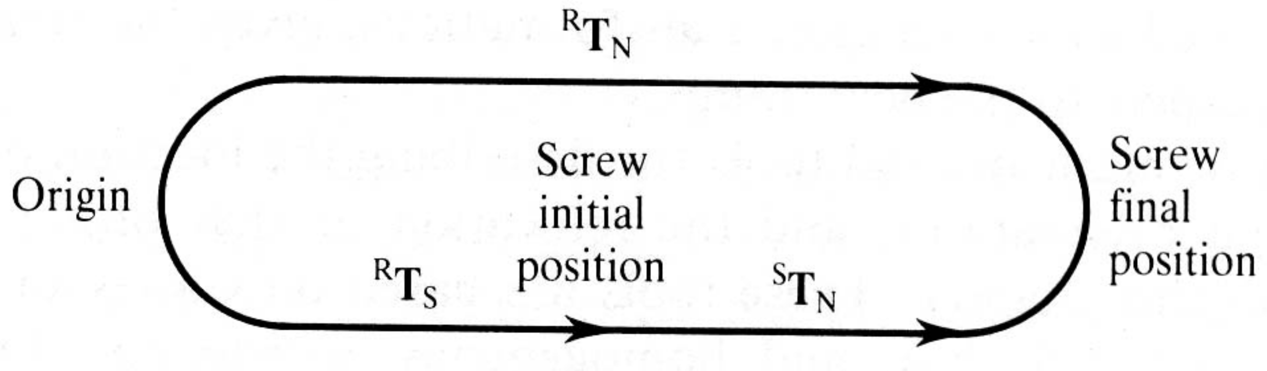
First, described the object with respect to a coordinate frame located within it.

Second, described the object frame with a transformation of the reference frame.

We described the relocation of this object as a series of transformation that relocate the object frame to the new location of the object. This series of transformation can be illustrated with a ***transform graph***.

$$^{R}\mathbf{T}_{N} = {}^{R}\mathbf{T}_{S}\,{}^{S}\mathbf{T}_{N}$$

Principle of Robotics



Transformation graph for transformation

The transform which describes the relocation of the screw (${}^{S}T_{N}$) is found either by reading the transform equation directly from the graph or by premuliplying both side of the above equation by the inverse of the transform which describes the screw frame with respect to the reference frame.

$$ {}^{R}\mathbf{T}_{S}^{-๑}\,{}^{R}\mathbf{T}_{N} = {}^{S}\mathbf{T}_{N} $$

Example we wish to grasp the box with the hand on the manipulator.

To do this, the robot has to move the hand until it is positioned around the box, and then close it to grasp the box.

$$^{W}\mathbf{T}_{O}\,^{O}\mathbf{T}_{G} = {}^{W}\mathbf{T}_{R}\,^{R}\mathbf{T}_{H}\,^{H}\mathbf{T}_{G}$$

Where

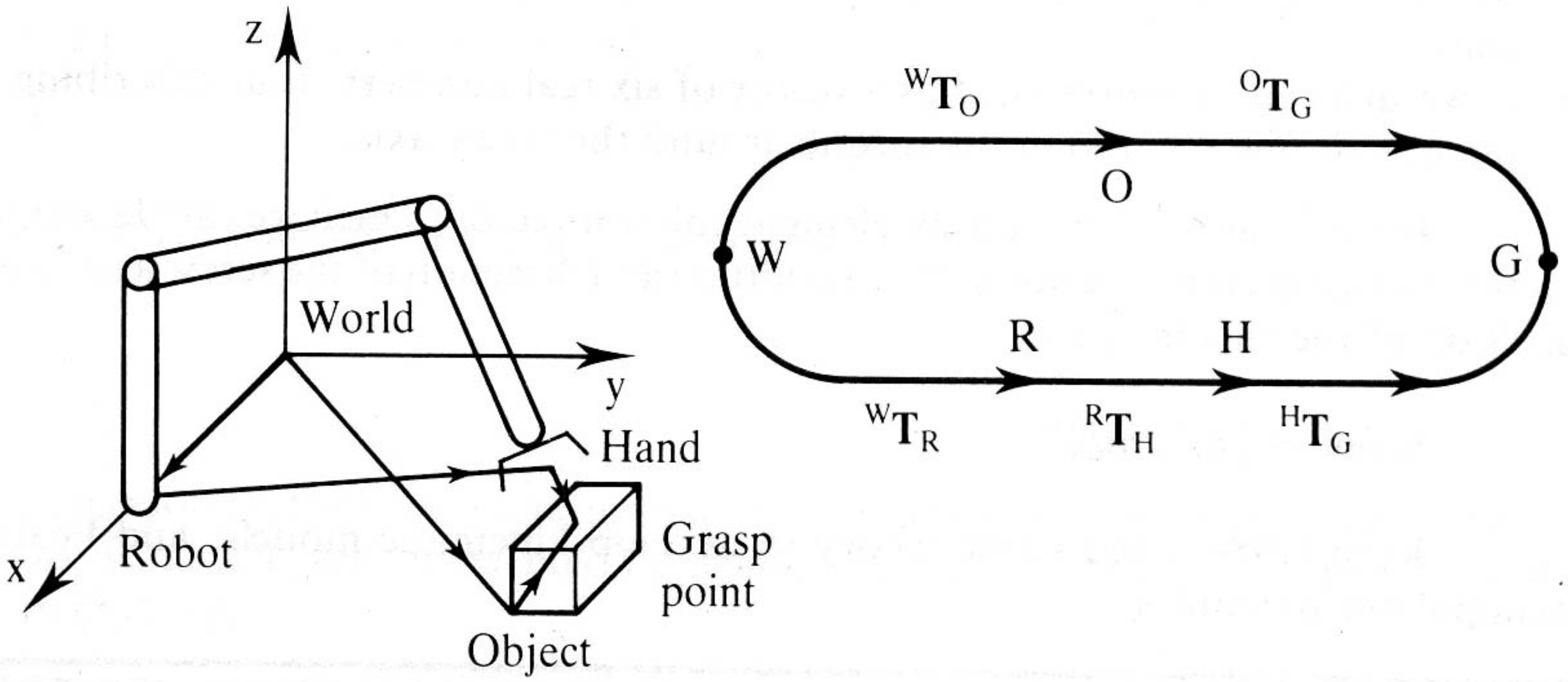
W is the origin of the world coordinate system.

O is the origin of the coordinate frame in the object.

G is grasp point.

H is a point half way between the fingers of the hand.

R is the base of the robot.



Transform graph for an object about to be grasped by manipulator

The transform from the hand to the grasp point

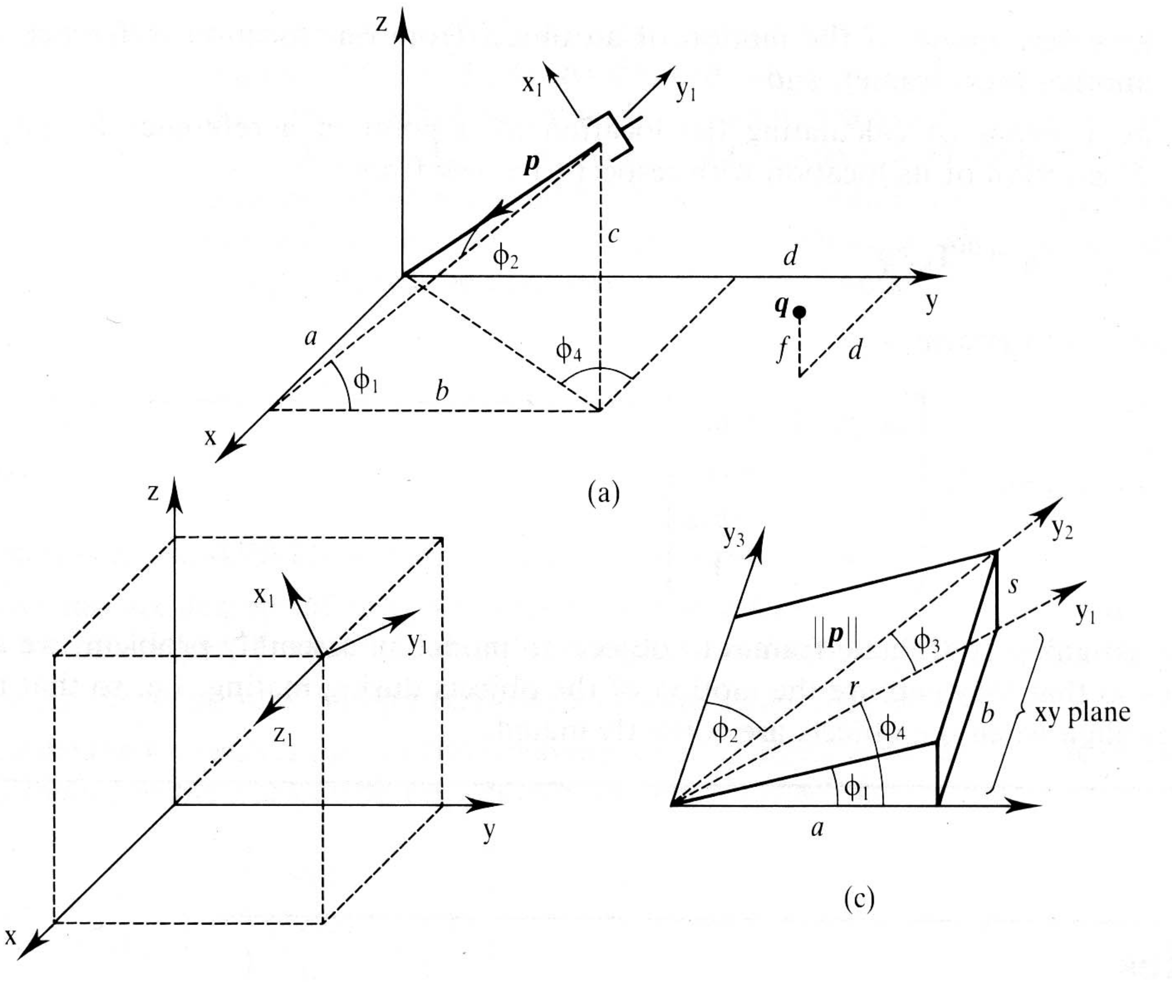
$$ {}^{W}\mathbf{T}_{R}^{-๑}\,{}^{R}\mathbf{T}_{H}^{-๑}\,{}^{W}\mathbf{T}_{O}\,{}^{O}\mathbf{T}_{G} = {}^{H}\mathbf{T}_{G} $$

๑๑ Programming

๑๒ Example

Example ๑ Consider that the hand on a robot is to be located at p(a,b,c), pointing towards the origin. What is the transformation matrix from the origin of the reference frame to the hand?

Principle of Robotics



Location of the hand in Example ๑ (a) Problem description; (b) Coordinate frame; (c) Orientation transform.

Principle of Robotics

The transformation of axes can be done in the following sequence :

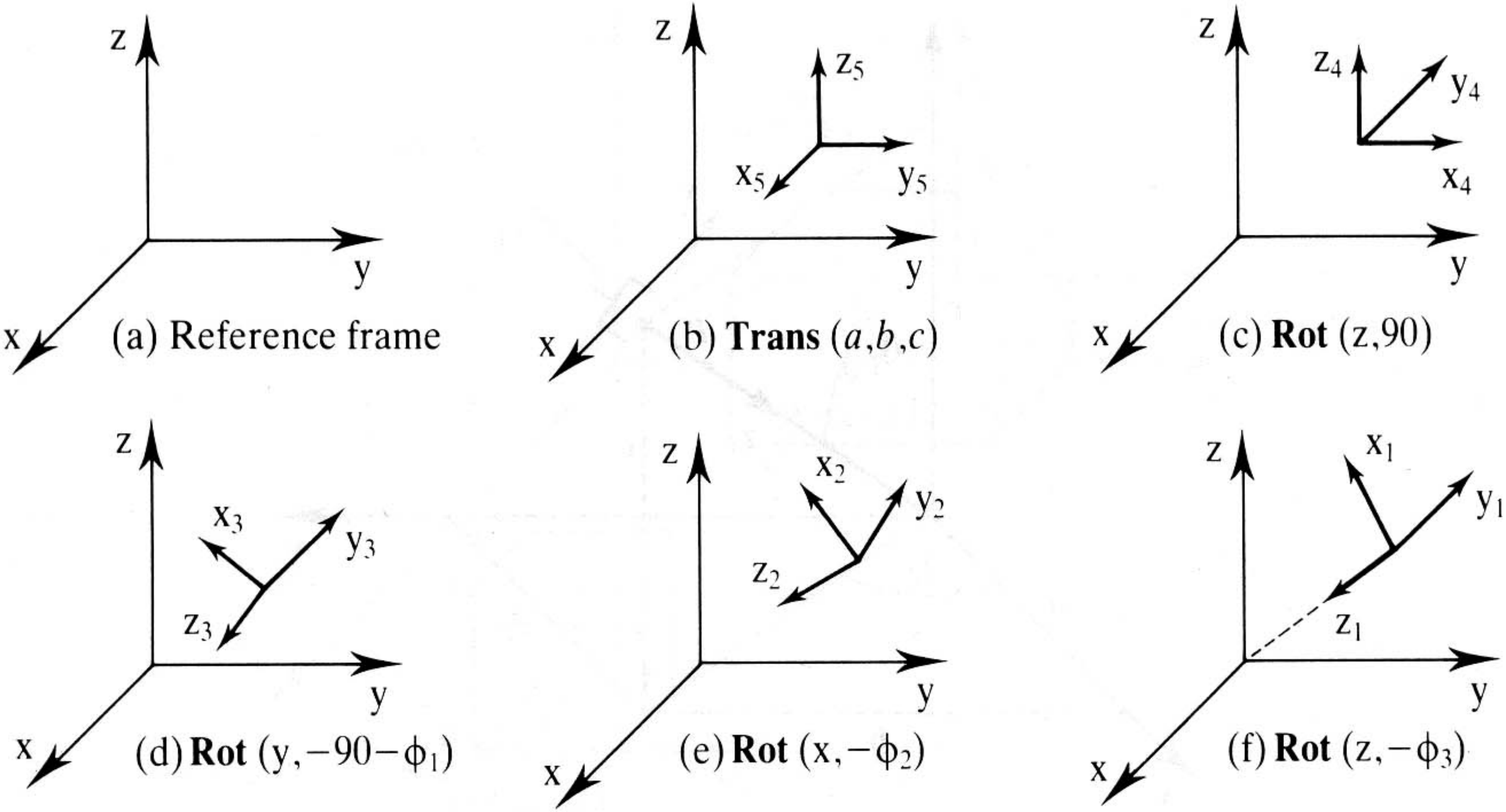
(a) translate to $\boldsymbol{p}(a,b,c)$

(b) rotate around the new z axis by ๙๐$^\circ$

(c) rotate around the new y axis by -๙๐-$\phi_{๑}^\circ$

(d) rotate around the new x axis by -$\phi_{๒}^\circ$

(e) rotate around the new z axis by -$\phi_{๓}^\circ$



Sequence of transforms and resulting
frames to solve Example ๑

The transformation equation :

$$^{R}\mathbf{T}_{H} = \mathbf{Trans}(a,b,c)\mathbf{Rot}(z,๙๐)\mathbf{Rot}(y,๙๐-\phi_{๑})$$

$$\mathbf{Rot}(x, -\phi_{๒})\mathbf{Rot}(z, -\phi_{๓})$$

$$= {}^{R}\mathbf{T}_{A}\ {}^{A}\mathbf{T}_{H}$$

$$= \text{Approach transform} \times \text{Orientation transform}$$

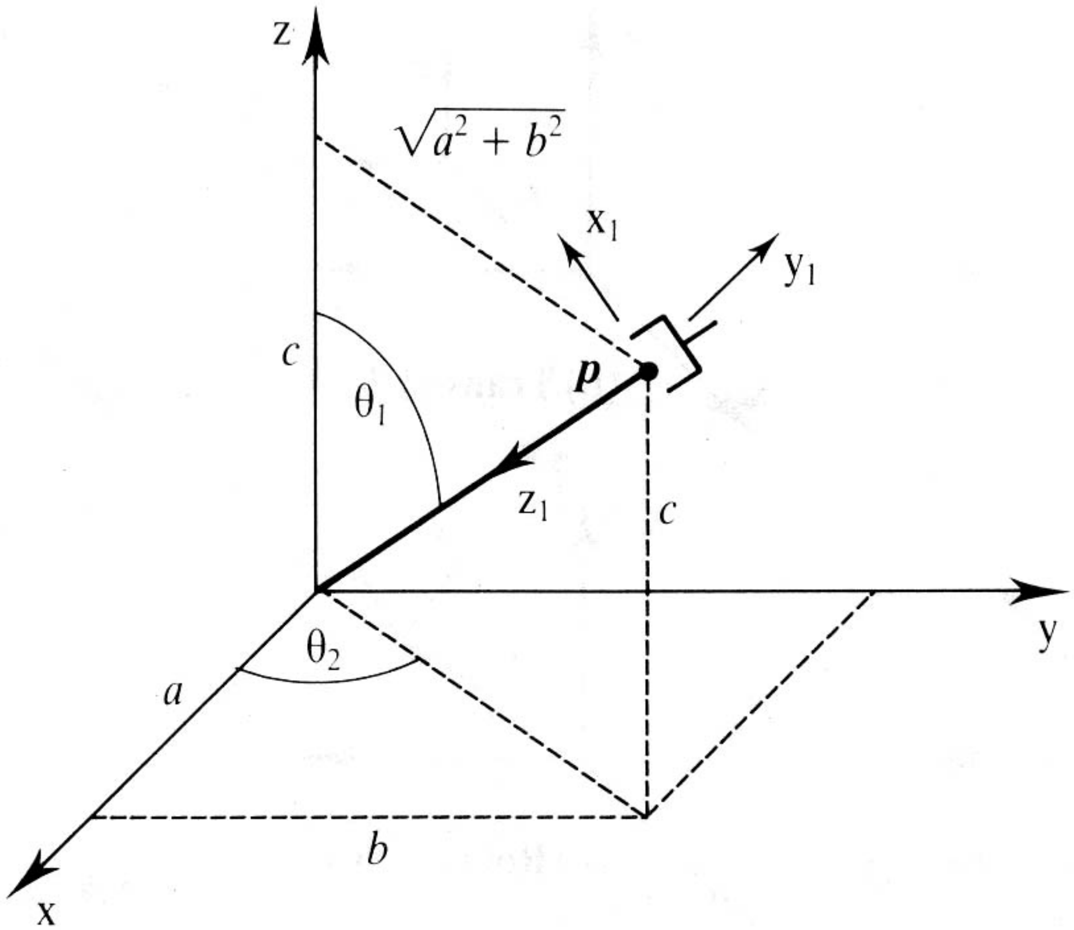
$$\phi_{๑} = \arctan\left(\frac{c}{b}\right)$$

$$\phi_{๒} = \arctan\left(\frac{a}{\sqrt{(b^{๒}+c^{๒})}}\right)$$

$$\phi_{๓} = \arctan\left(\frac{a\,c}{b\sqrt{(a^{๒}+b^{๒})}}\right) = \arctan\left(\frac{s}{r}\right)$$

We can simplify the transform calculation by constraining y axis to remain in the xy plane during the transformation.

$$^{R}\mathbf{T}_{H} = \mathbf{Trans}(a,b,c)\mathbf{Rot}(z,\theta_{๒})\mathbf{Rot}(y,๑๘๐+\theta_{๑})$$



Alternative transformation sequence which maintains the new y axis in the correct orientation.

$$ {}^{R}\mathbf{T}_{H} = \mathbf{Trans}(a,b,c)\mathbf{Rot}(z,\theta_{๒})\mathbf{Rot}(y,๑๘๐+\theta_{๑}) $$

$$ = \begin{bmatrix} ๑ & ๐ & ๐ & a \\ ๐ & ๑ & ๐ & b \\ ๐ & ๐ & ๑ & c \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} C_{๒} & -S_{๒} & ๐ & ๐ \\ S_{๒} & C_{๒} & ๐ & ๐ \\ ๐ & ๐ & ๑ & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} -C_{๑} & ๐ & -S_{๑} & ๐ \\ ๐ & ๑ & ๐ & ๐ \\ S_{๑} & ๐ & -C_{๑} & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} $$

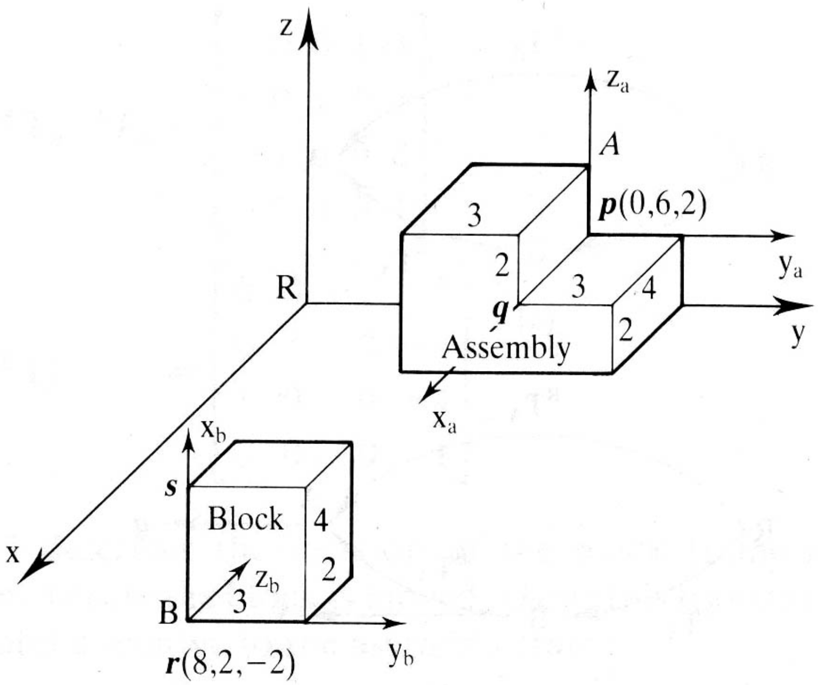
$$ = \begin{bmatrix} -C_{๑}C_{๒} & -S_{๒} & -S_{๑}C_{๒} & a \\ -C_{๑}S_{๒} & C_{๒} & -S_{๒}S_{๑} & b \\ -S_{๑} & ๐ & -C_{๑} & ๐ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} $$

$$\cos(\phi_x) = \frac{a}{\|\boldsymbol{p}\|}$$

$$\cos(\phi_y) = \frac{b}{\|\boldsymbol{p}\|}$$

$$\cos(\phi_z) = \frac{c}{\|\boldsymbol{p}\|}$$

Example* ๒** We wish to placed the block on the assembly so that line ***rs is coincident with line ***pq***.

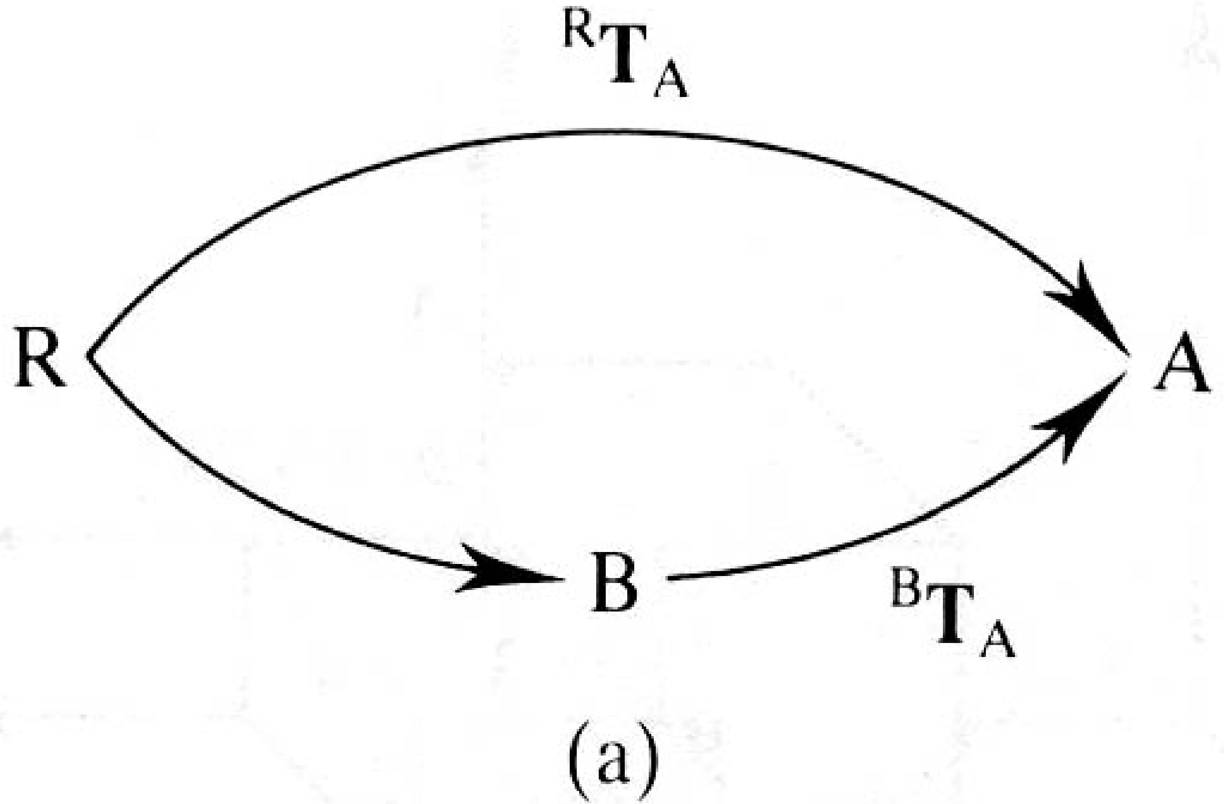


To achieve this problem :

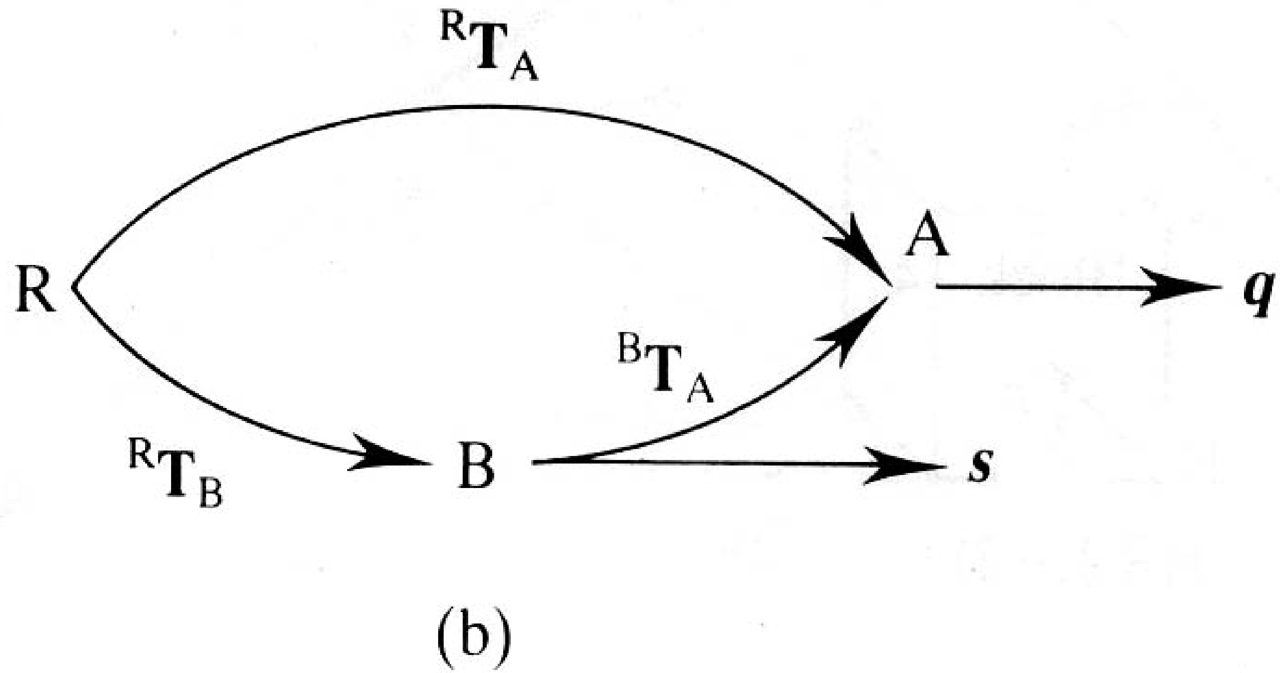
First, we place frames at ***r*** and ***p***

Second, we draw the transform graph for the problem

Third, we calculate the transforms



Transform graphs for assembly problem (a) frame graph



(b) original location of points

$$
{}^{R}\mathbf{T}_{A} = \begin{bmatrix} ๑ & ๐ & ๐ & ๐ \\ ๐ & ๑ & ๐ & ๖ \\ ๐ & ๐ & ๑ & ๒ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}
$$

$$
{}^{R}\mathbf{T}_{B} = \mathbf{Trans}(๘,๒,-๒)\mathbf{Rot}(y,-๙๐)
$$

$$
= \begin{bmatrix} ๐ & ๐ & -๑ & ๘ \\ ๐ & ๑ & ๐ & ๒ \\ ๑ & ๐ & ๐ & -๒ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}
$$

Equation for the motion of the block :

$$ {}^{B}\mathbf{T}_{A} = {}^{R}\mathbf{T}_{B}^{-๑}\,{}^{R}\mathbf{T}_{A} = \begin{bmatrix} ๐ & ๐ & ๑ & ๔ \\ ๐ & ๑ & ๐ & ๔ \\ -๑ & ๐ & ๐ & ๘ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} $$

$$ {}^{A}\mathbf{T}_{B} = {}^{B}\mathbf{T}_{A}^{-๑} = \begin{bmatrix} ๐ & ๐ & -๑ & ๘ \\ ๐ & ๑ & ๐ & -๔ \\ ๑ & ๐ & ๐ & -๔ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} $$

Location of point s relative to assembly frame

$$A_S = {}^{A}\mathbf{T}_B\, B_S = \begin{bmatrix} ๐ & ๐ & -๑ & ๘ \\ ๐ & ๑ & ๐ & -๔ \\ ๑ & ๐ & ๐ & -๔ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix}\begin{bmatrix} ๔ \\ ๐ \\ ๐ \\ ๑ \end{bmatrix} = \begin{bmatrix} ๘ \\ -๔ \\ ๐ \\ ๑ \end{bmatrix}$$

Original location of point s is

$${}^{R}\boldsymbol{s} = {}^{R}\mathbf{T}_B\, {}^{B}\boldsymbol{s}$$

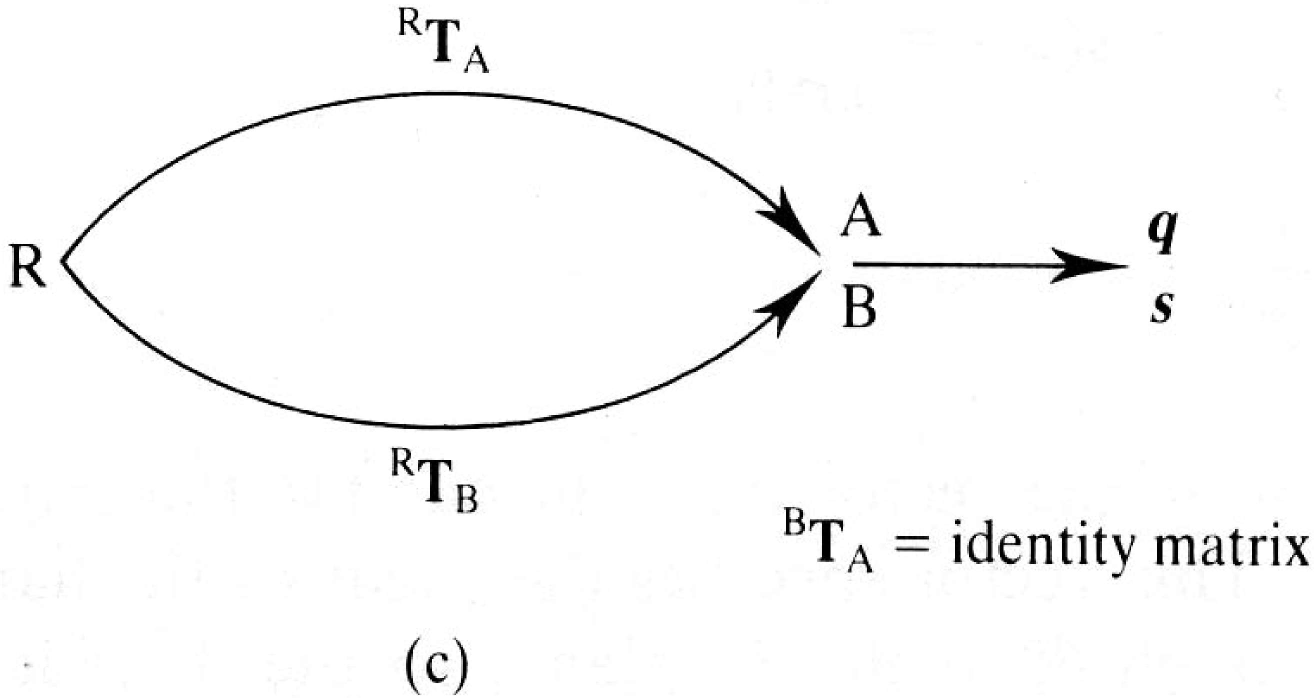
$${}^{B}\boldsymbol{s} = {}^{R}\mathbf{T}_B^{-๑}\, {}^{R}\boldsymbol{s}$$

$${}^{A}\boldsymbol{q} = {}^{B}\boldsymbol{s} \text{ (after motion)}$$

$${}^{A}\boldsymbol{q} = {}^{R}\mathbf{T}_{B}^{-๑}\,{}^{R}\boldsymbol{s}$$

$${}^{R}\boldsymbol{q} = {}^{R}\mathbf{T}_{A}\,{}^{A}\boldsymbol{q} = {}^{R}\mathbf{T}_{A}\,{}^{R}\mathbf{T}_{B}^{-๑}\,{}^{R}\boldsymbol{s}$$

$$= \begin{bmatrix} ๑ & ๐ & ๐ & ๐ \\ ๐ & ๑ & ๐ & ๖ \\ ๐ & ๐ & ๑ & ๒ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} ๐ & ๐ & ๑ & ๒ \\ ๐ & ๑ & ๐ & -๒ \\ -๑ & ๐ & ๐ & ๘ \\ ๐ & ๐ & ๐ & ๑ \end{bmatrix} \begin{bmatrix} ๘ \\ ๒ \\ ๒ \\ ๑ \end{bmatrix} = \begin{bmatrix} ๔ \\ ๖ \\ ๒ \\ ๑ \end{bmatrix}$$



(c) final location of points