Chapter 6

Reduction of Multiple Sub-Systems

Outline

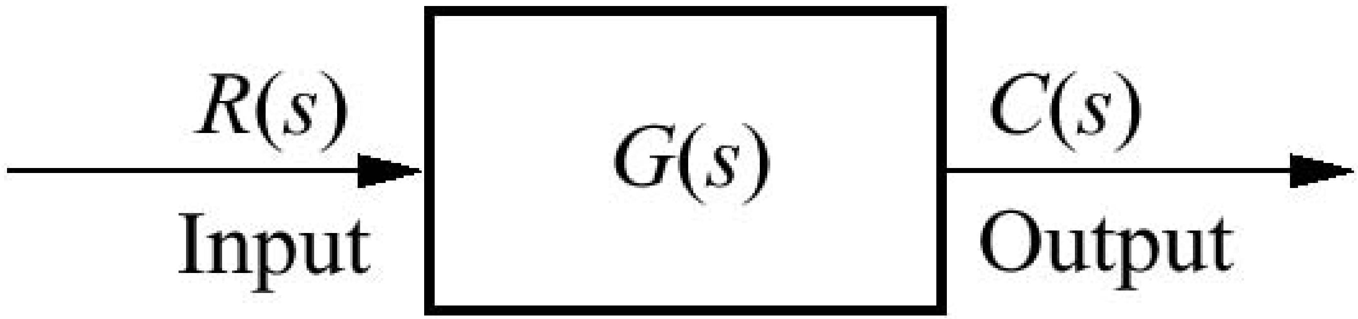
- Block diagram
- Signal flow graph

Components of a block diagram for a linear, time-invariant system

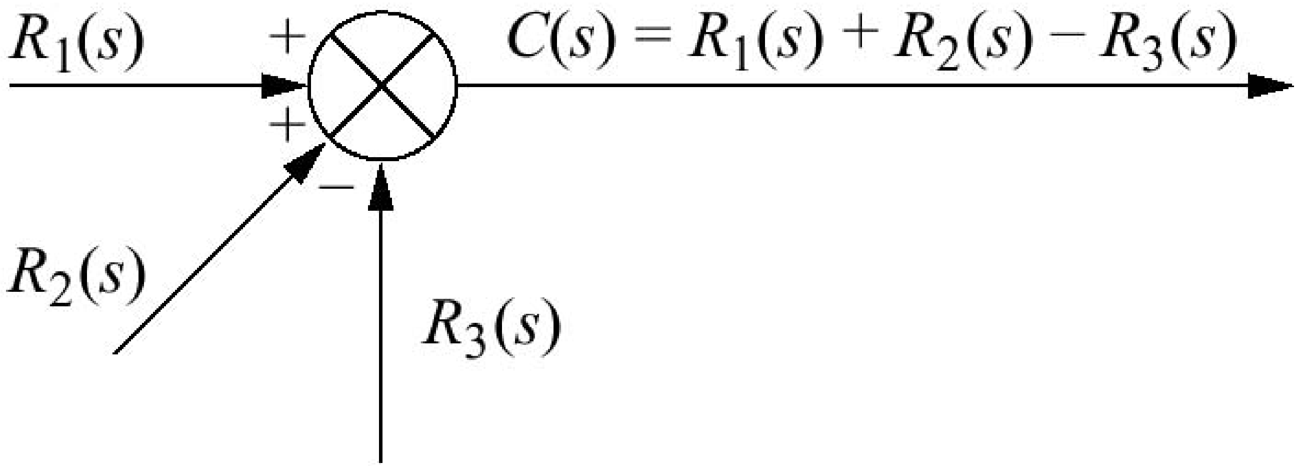
Signals

$$\xrightarrow{\quad R(s) \quad} \qquad \xrightarrow{\quad C(s) \quad}$$

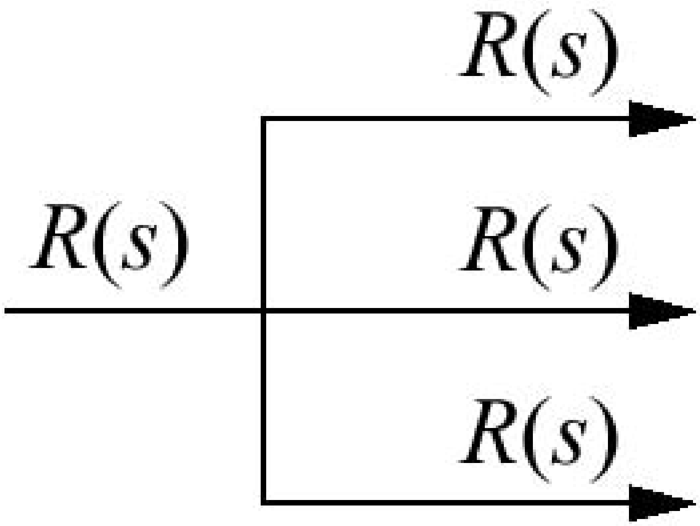
System

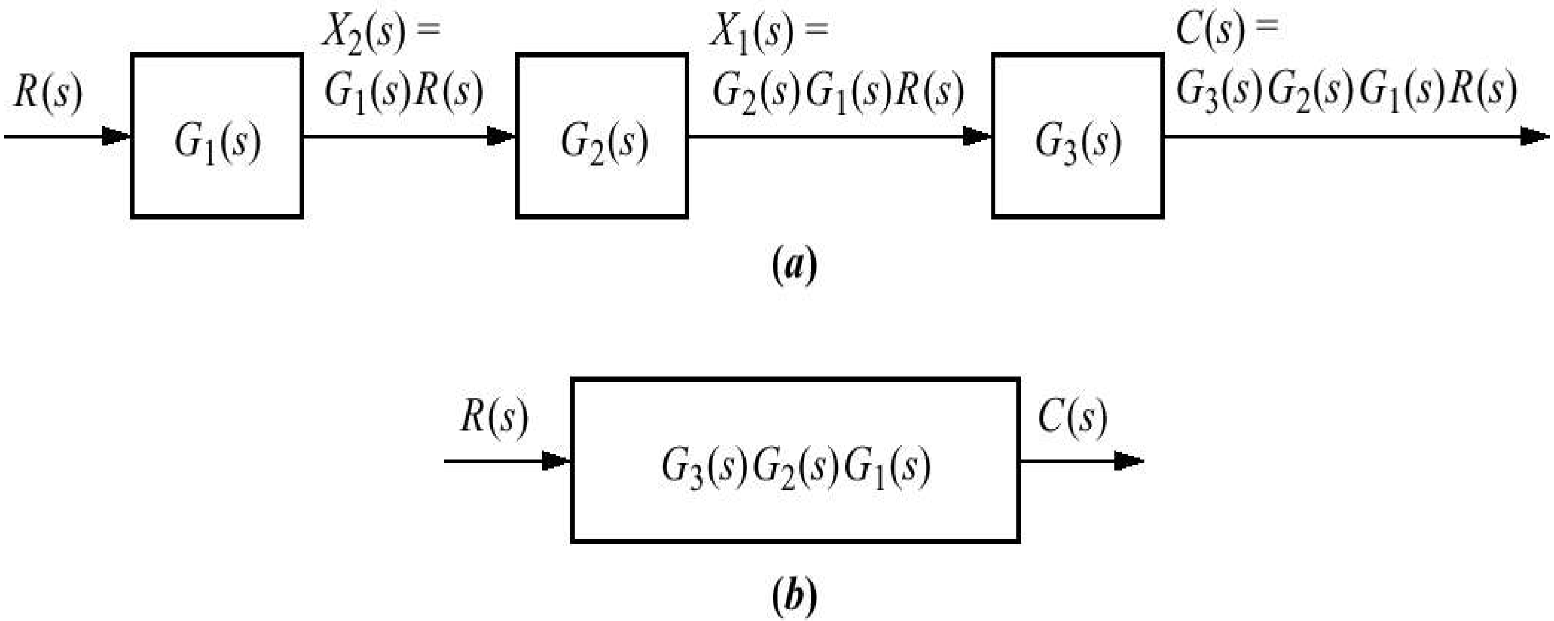


Summing Junction



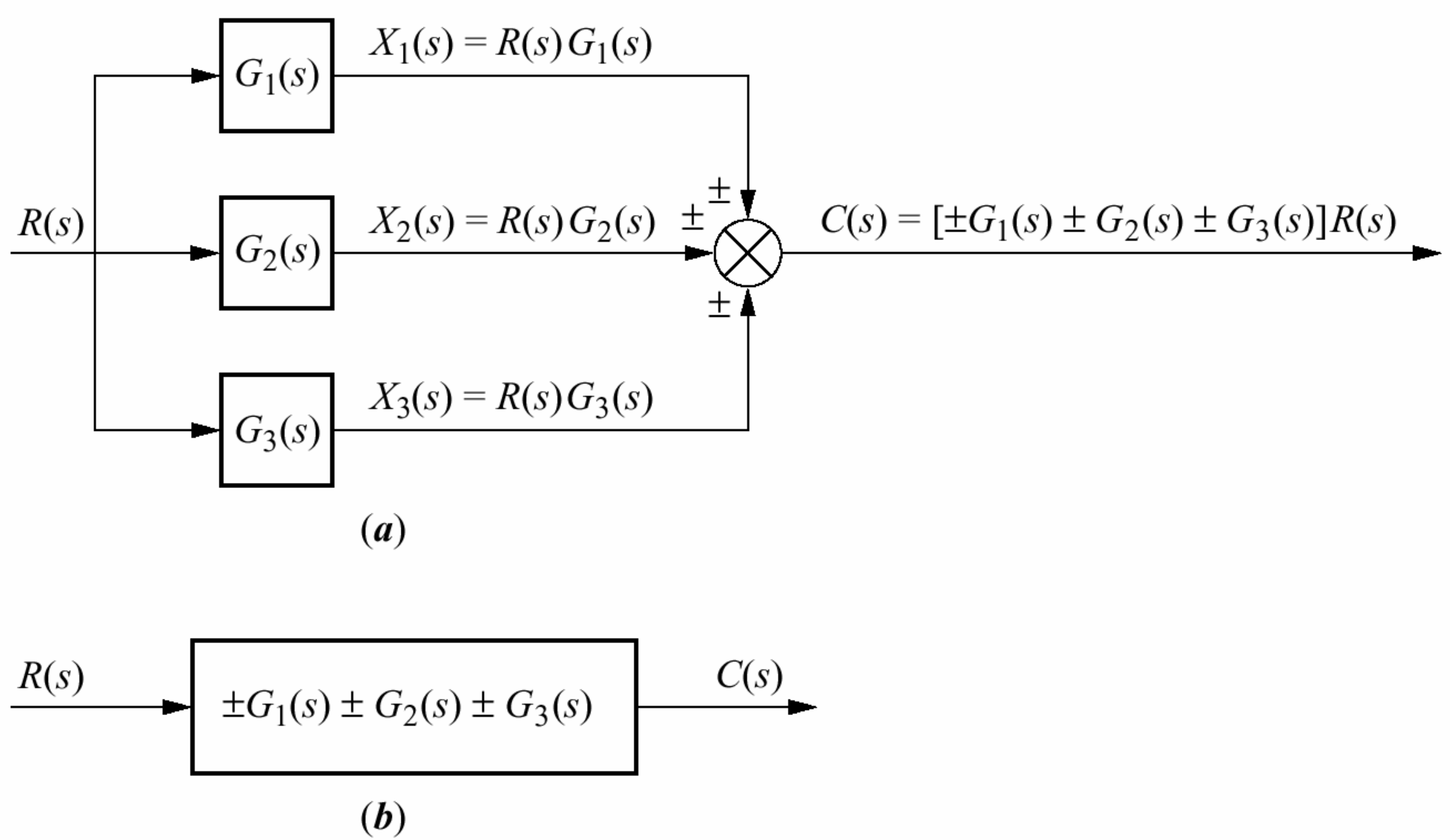
Pickoff point





a. Cascaded subsystems

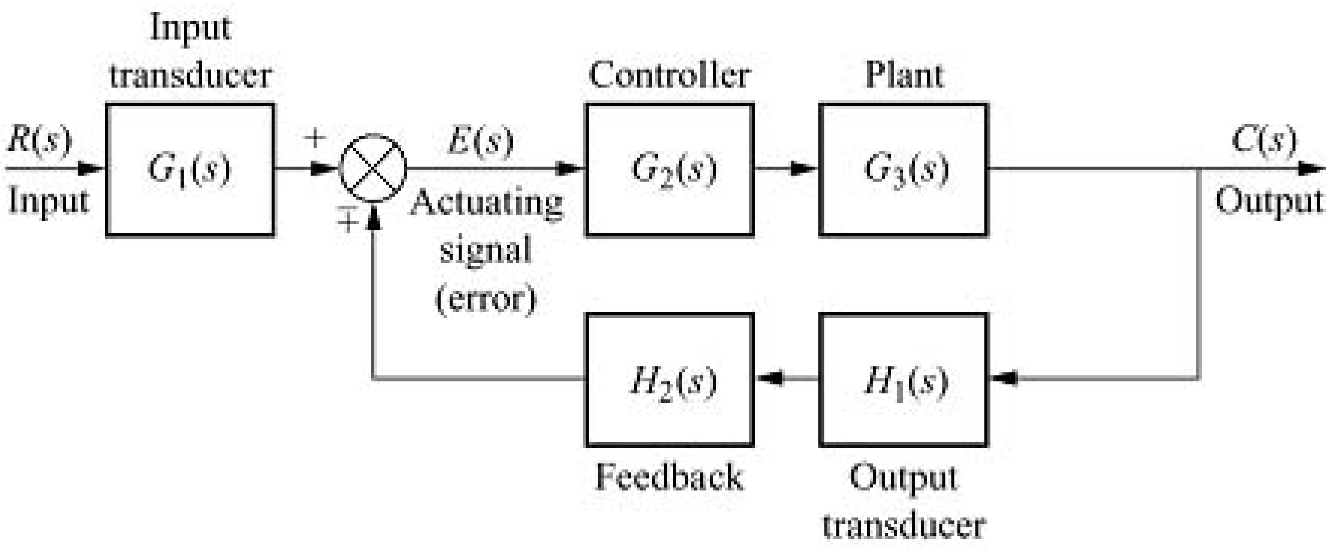
b. equivalent transfer function



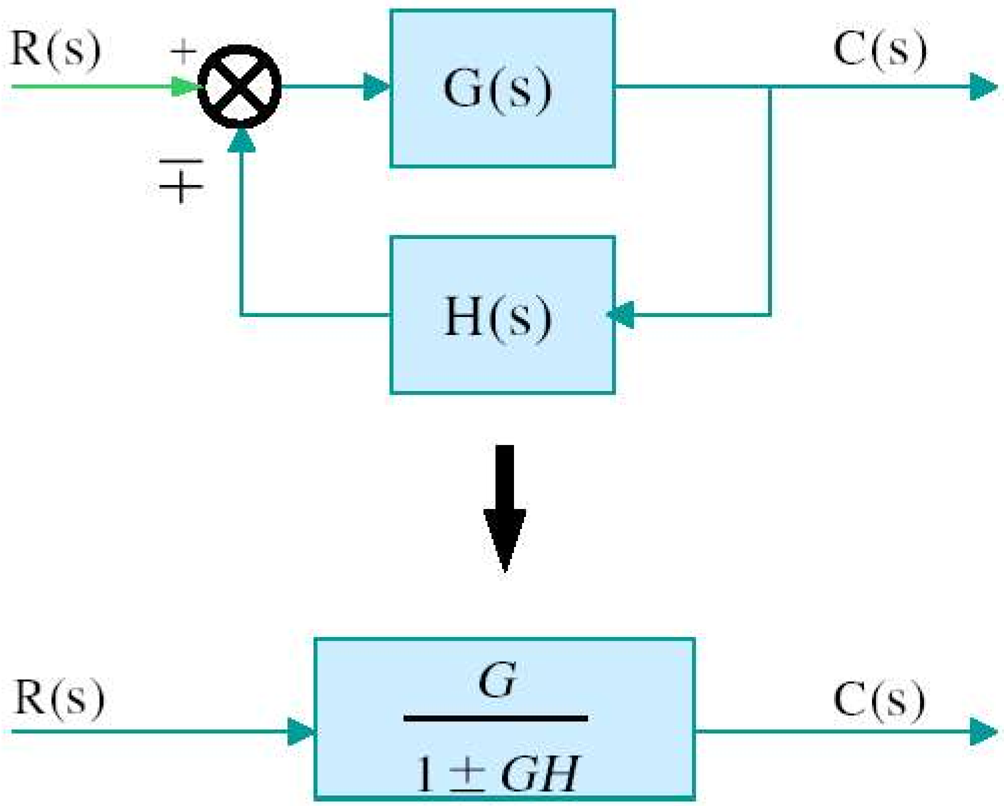
a. Parallel subsystems

b. equivalent transfer function

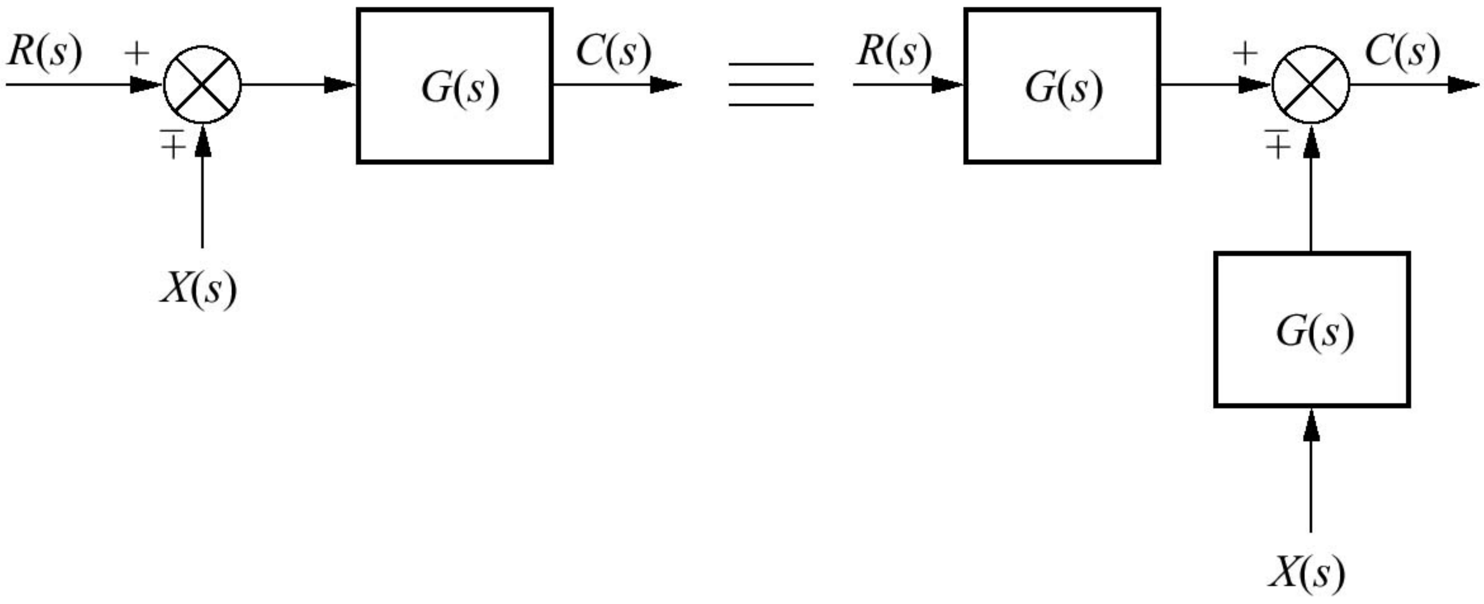
Feedback Control System



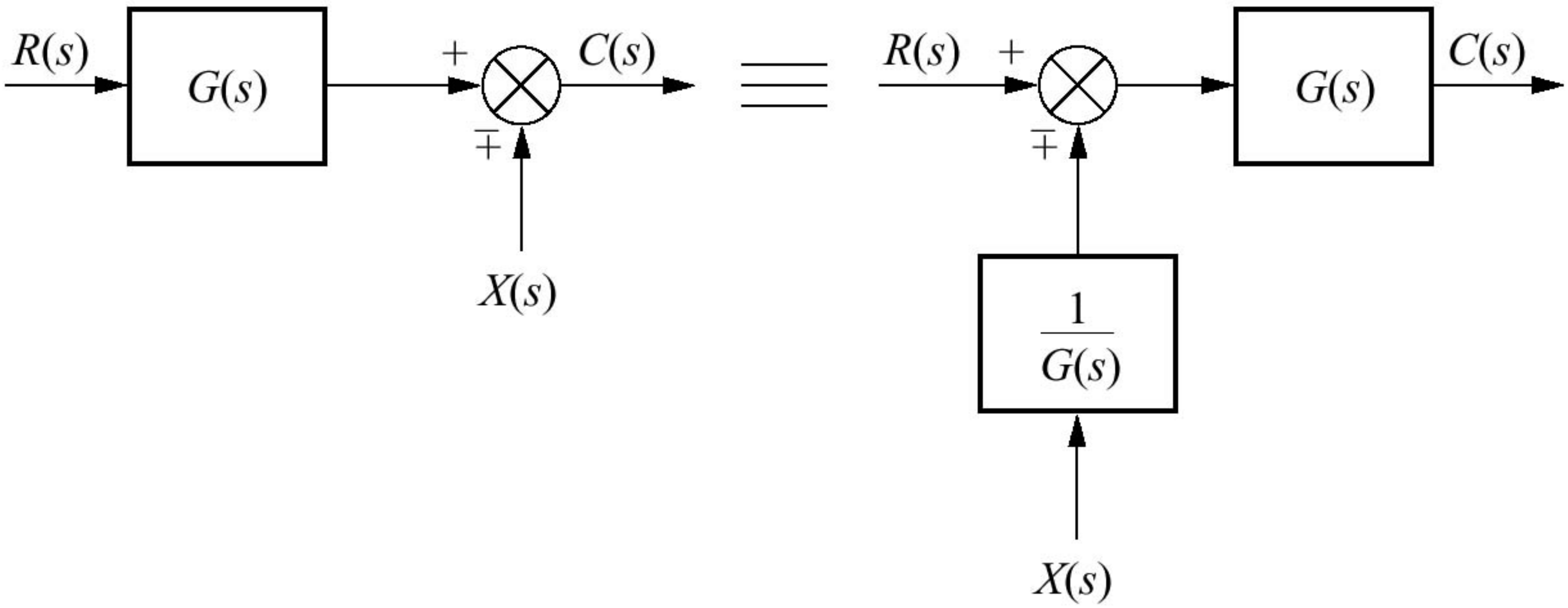
Eliminating single feedback loops



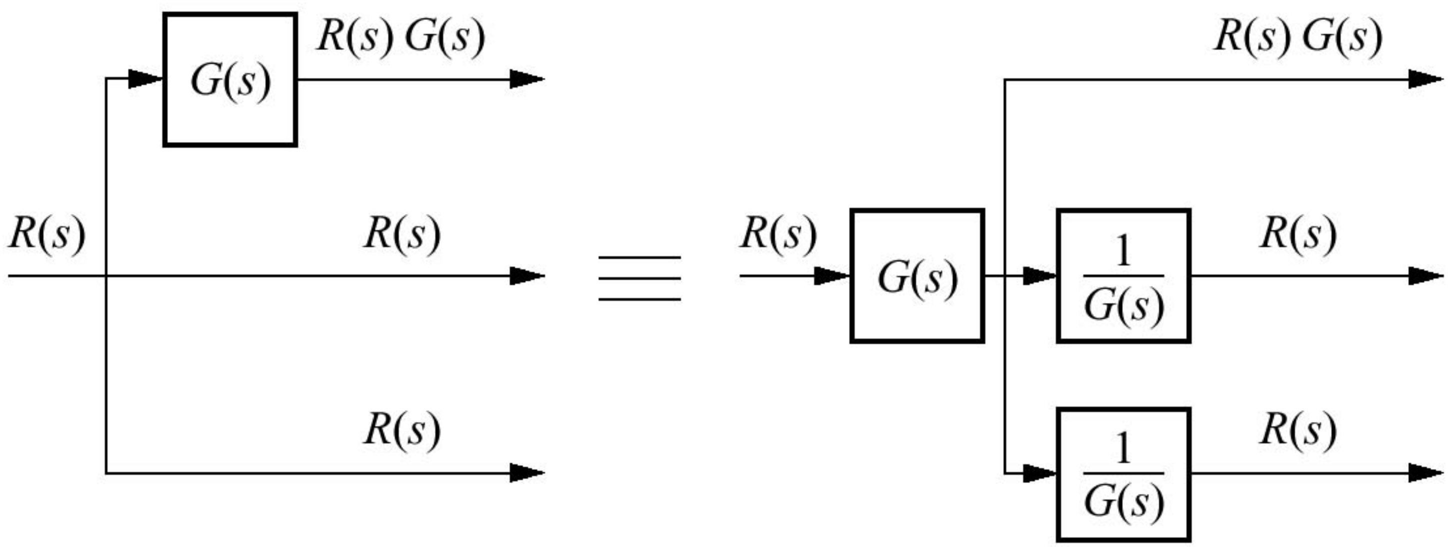
Moving a block to the left past a summing junction



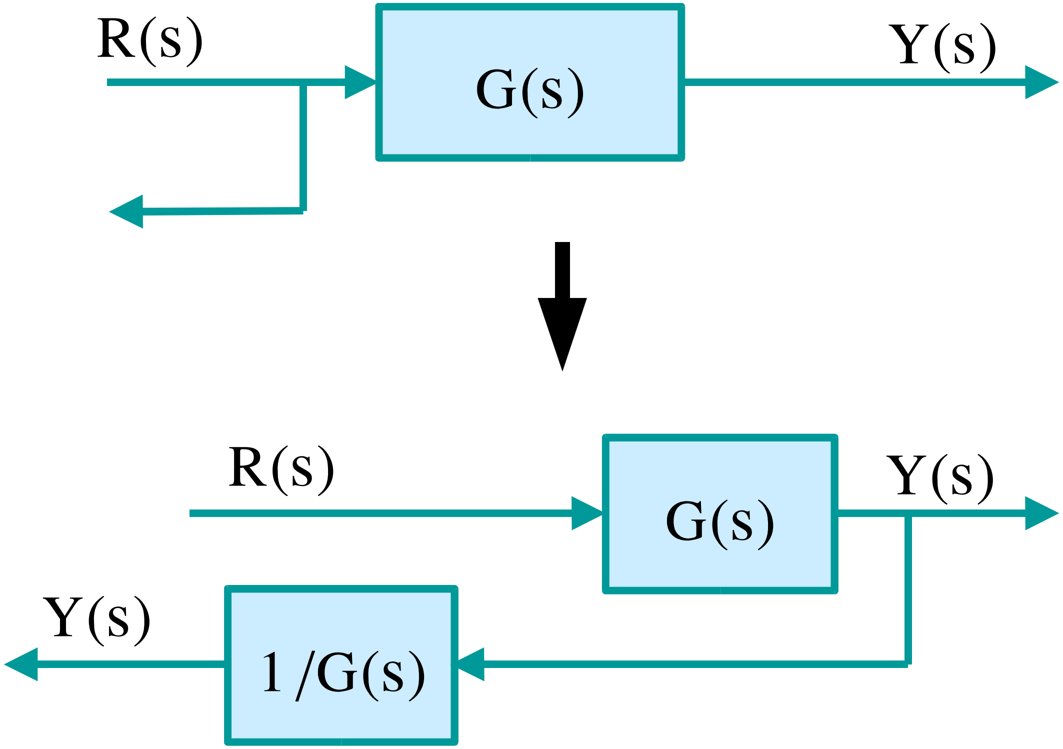
Moving a block to the right past a summing junction



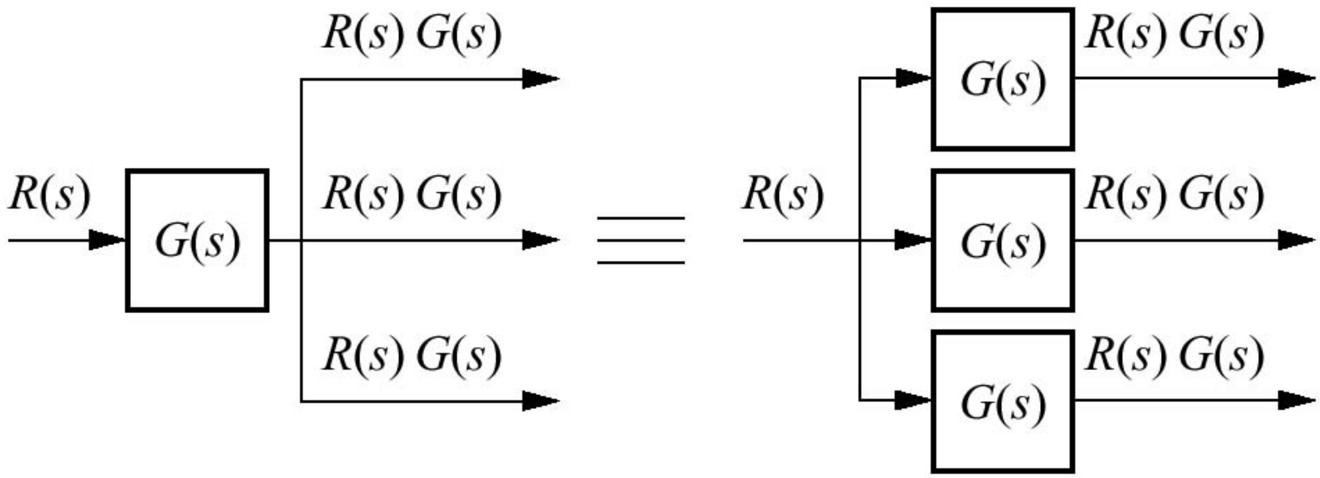
Moving a block to the left past a pickoff point



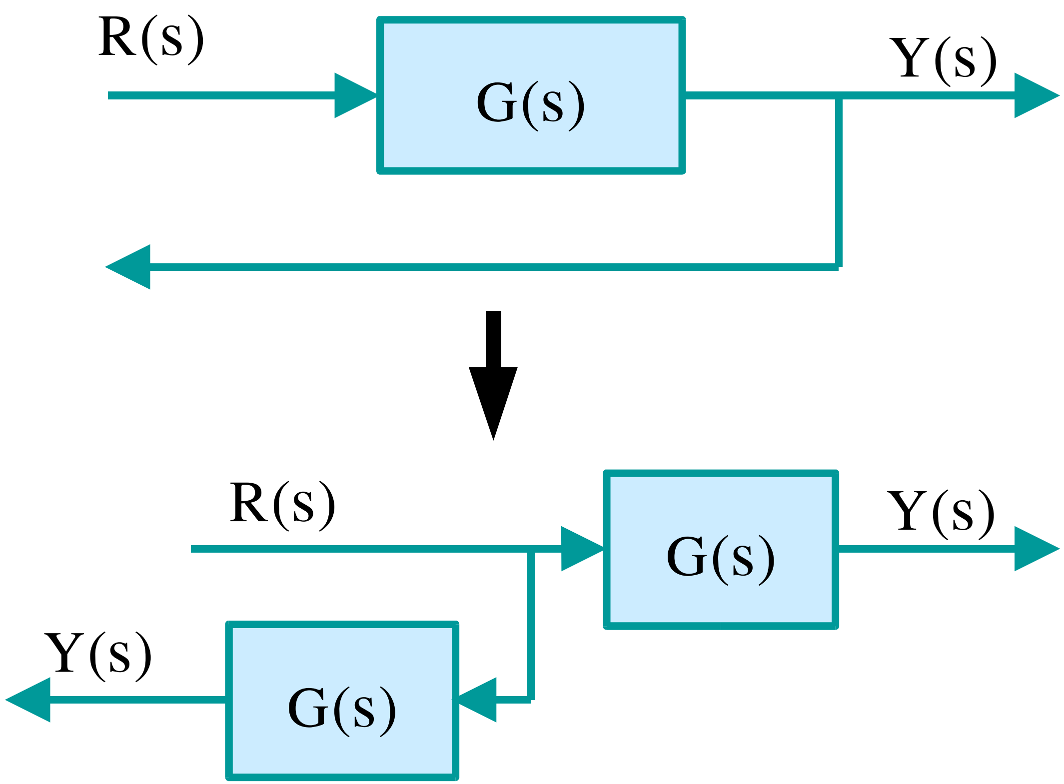
Moving a block to left past a pickoff point



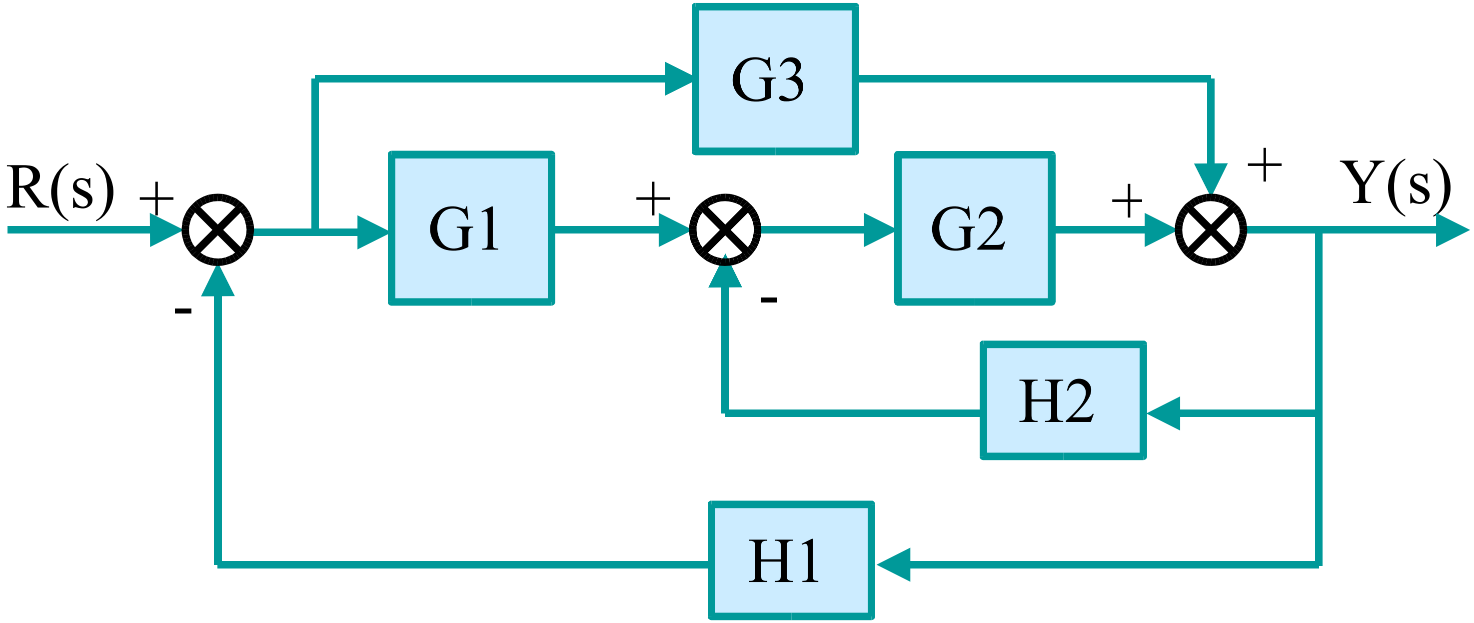
Moving a block to the right past a pickoff point



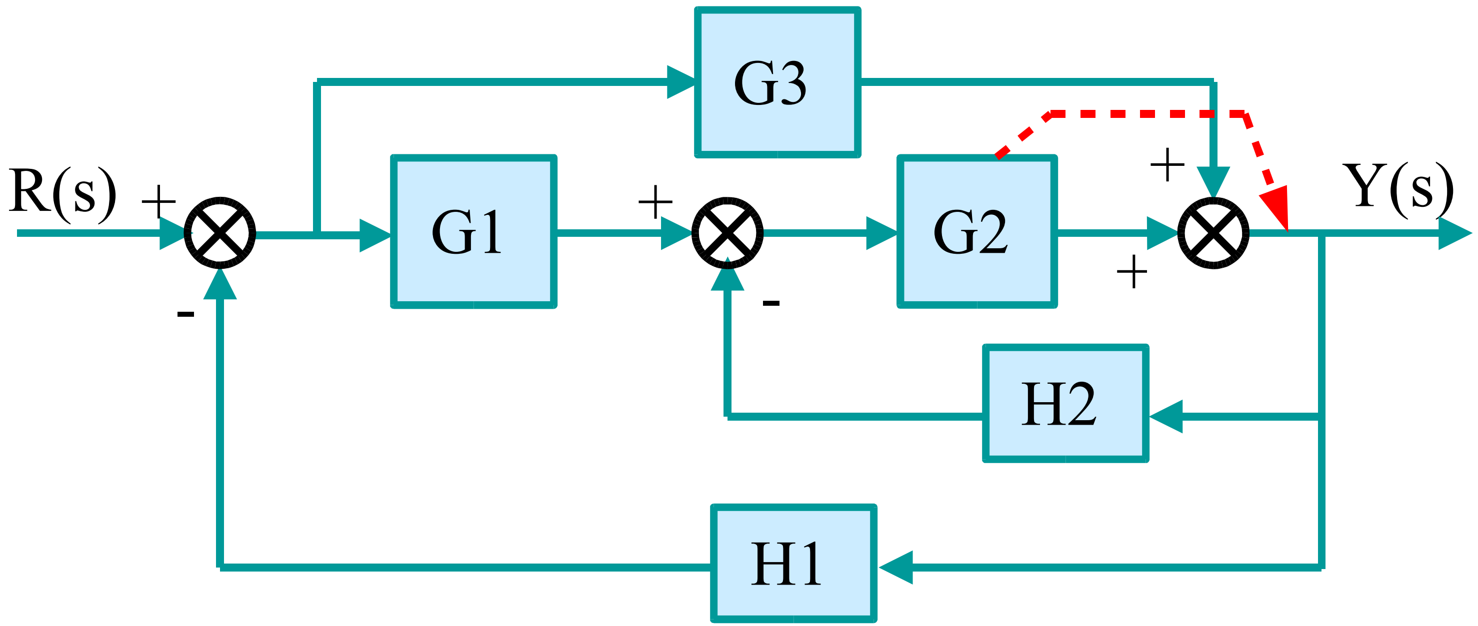
Moving a block to right past a pick-off point



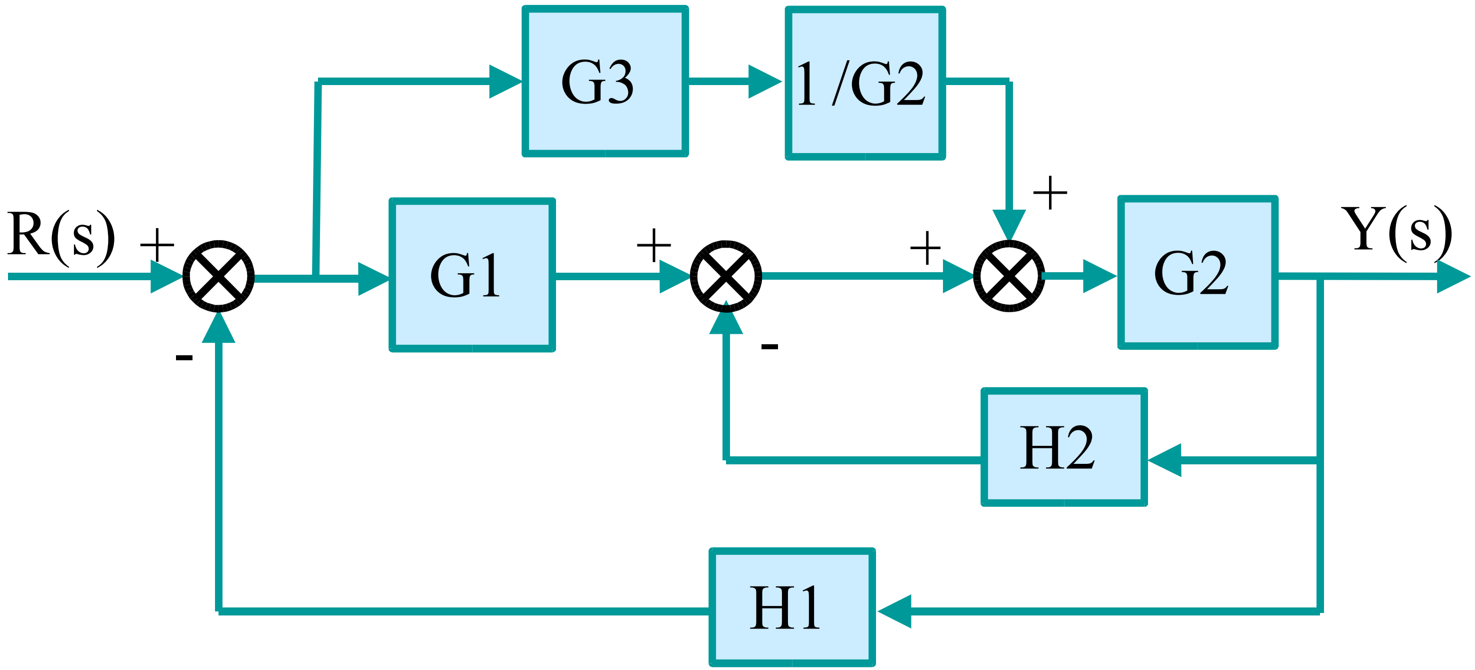
Example



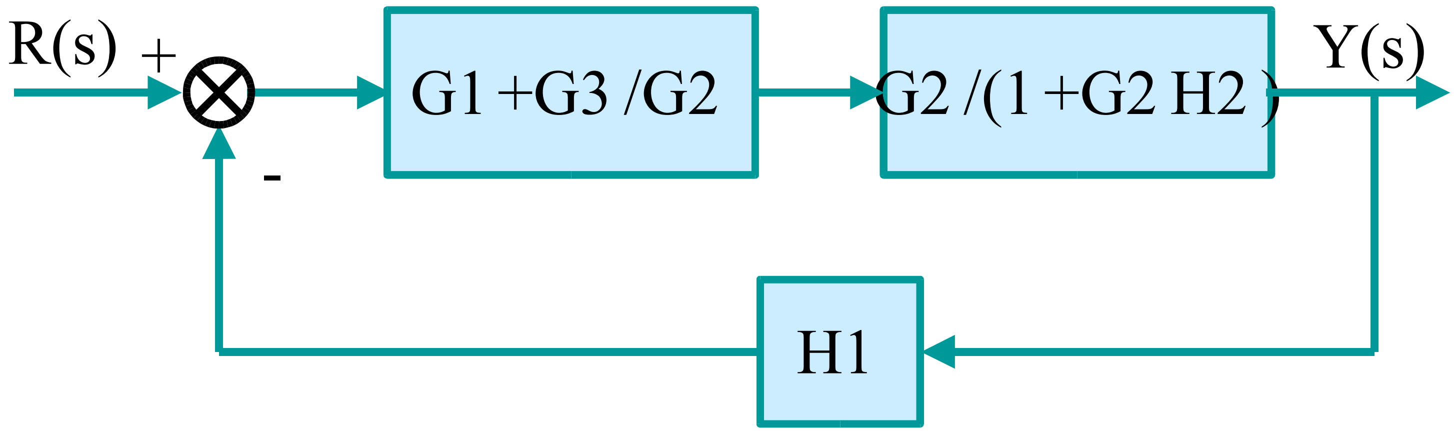
Step 1 : move block to summing junctio n



Result



Step 2 : summing each block



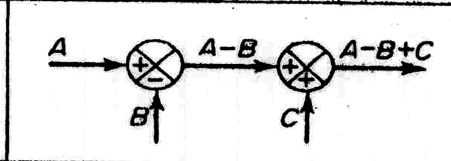
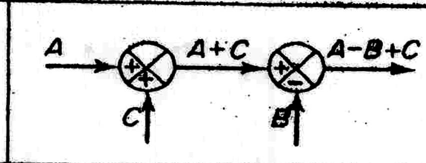
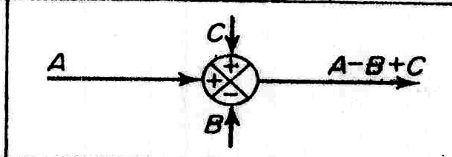
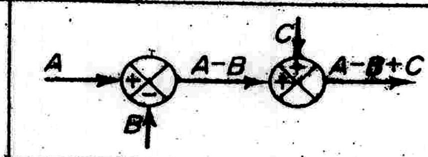
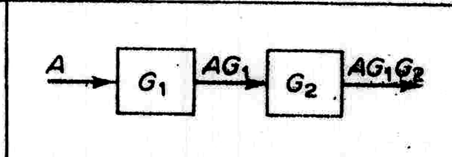
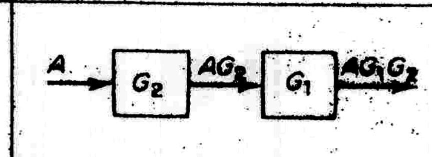
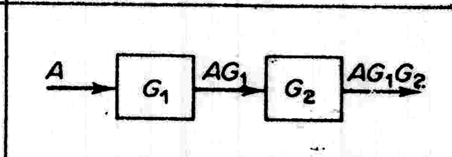
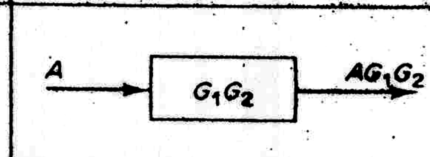
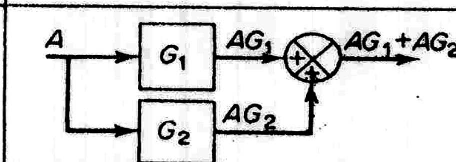
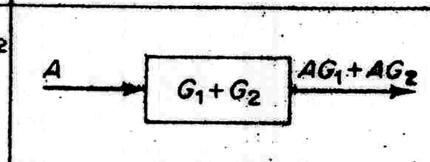
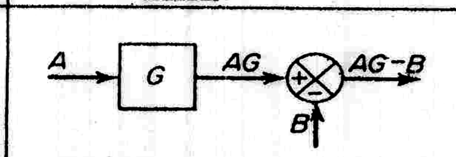
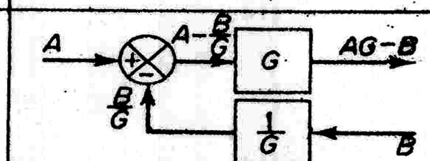
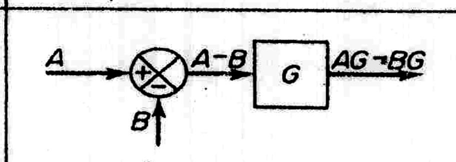
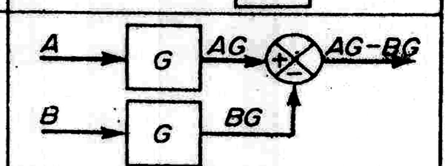
Step ๓: Answer

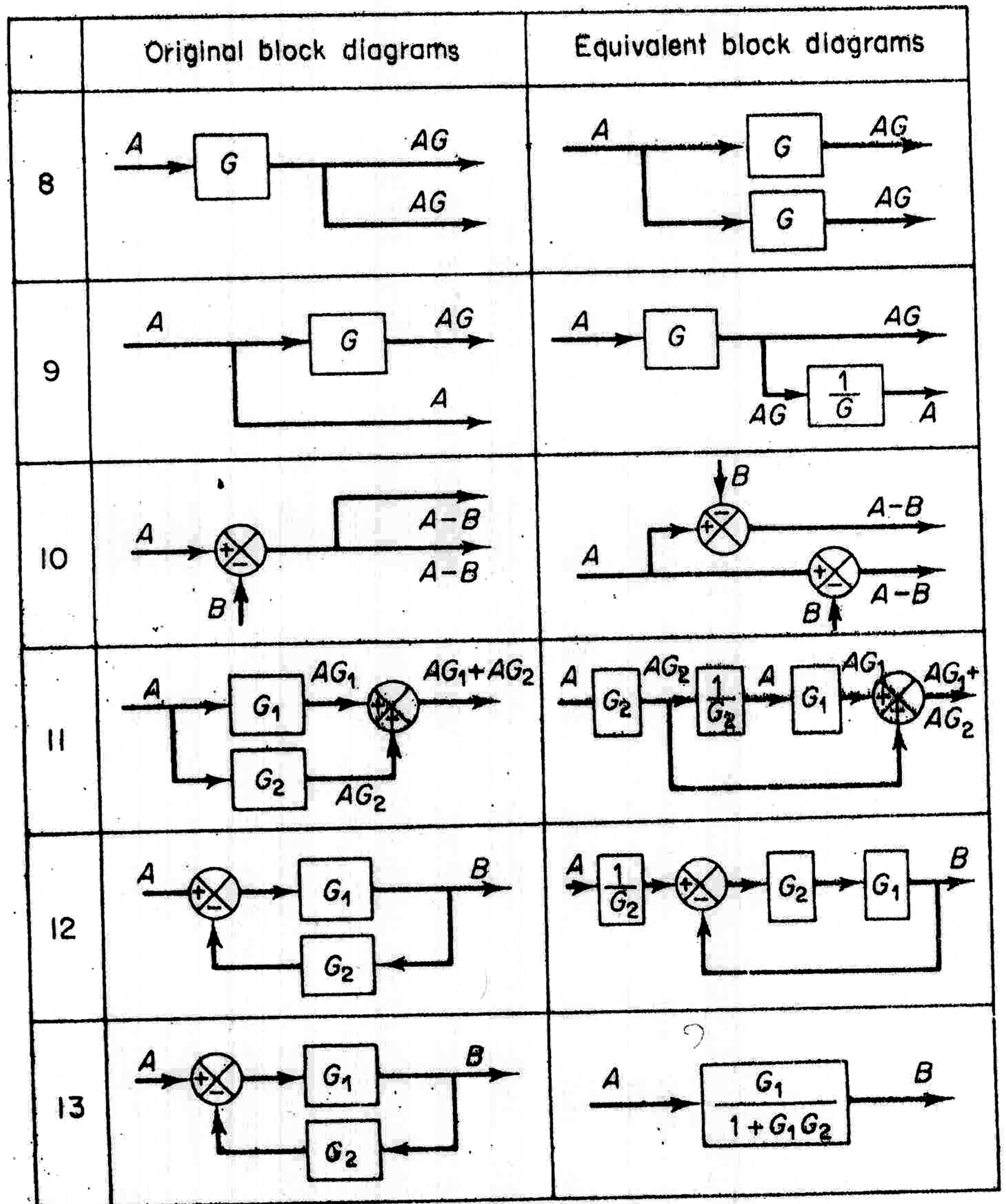
R(s)

$$\frac{G_{๑}G_{๒}+G_{๓}}{๑+G_{๒}H_{๒}+G_{๑}G_{๒}H_{๑}+G_{๓}H_{๑}}$$

Y(s)

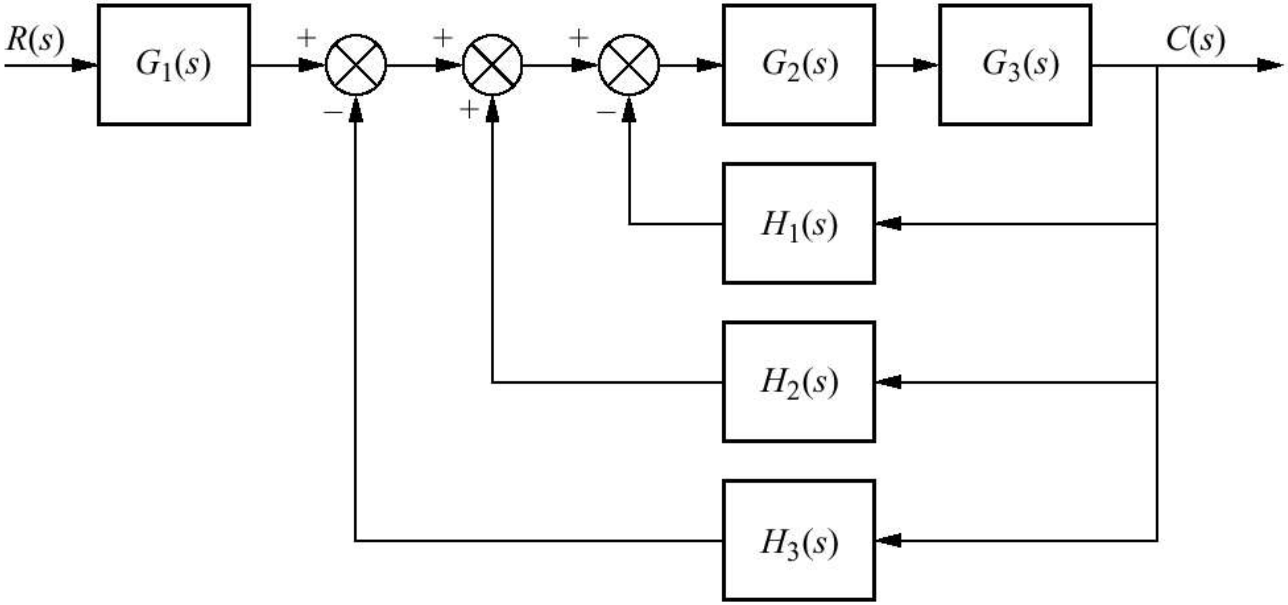
Table 4-3. Rules of Block Diagram Algebra

	Original block diagrams	Equivalent block diagrams
1		
2		
3		
4		
5		
6		
7		

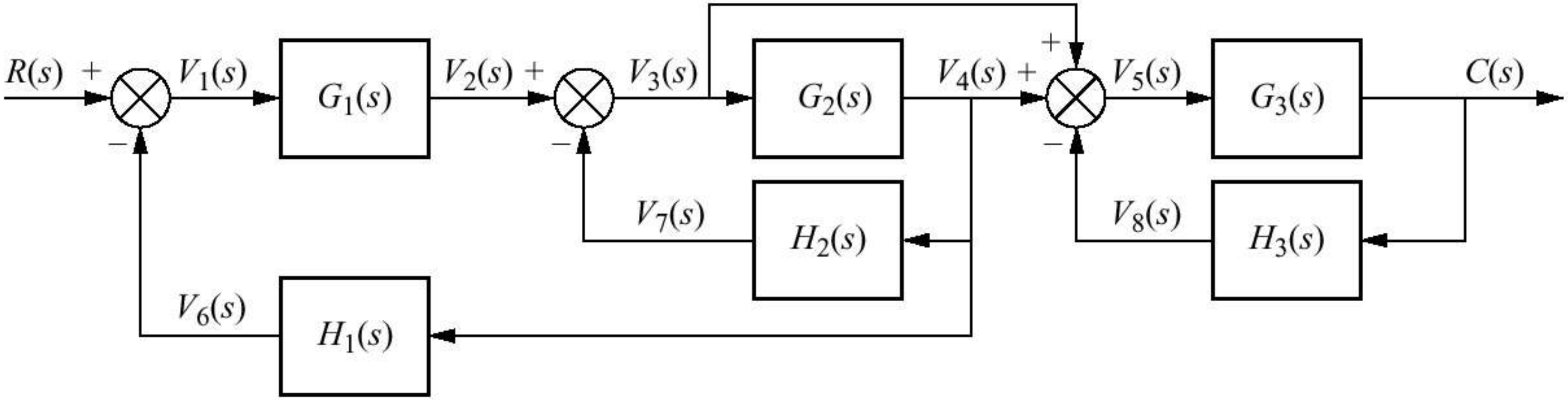


Original block diagrams
Equivalent block diagrams
8
A
G
AG
AG
9
A
G
AG
A
1/G
10
A
B
A−B
A−B
11
A
G1
G2
AG1
AG2
AG1+AG2
1/G2
12
A
G1
G2
B
1/G2
13
A
G1
G2
B
G1/(1+G1G2)

Example ๕.๑ (Nise)

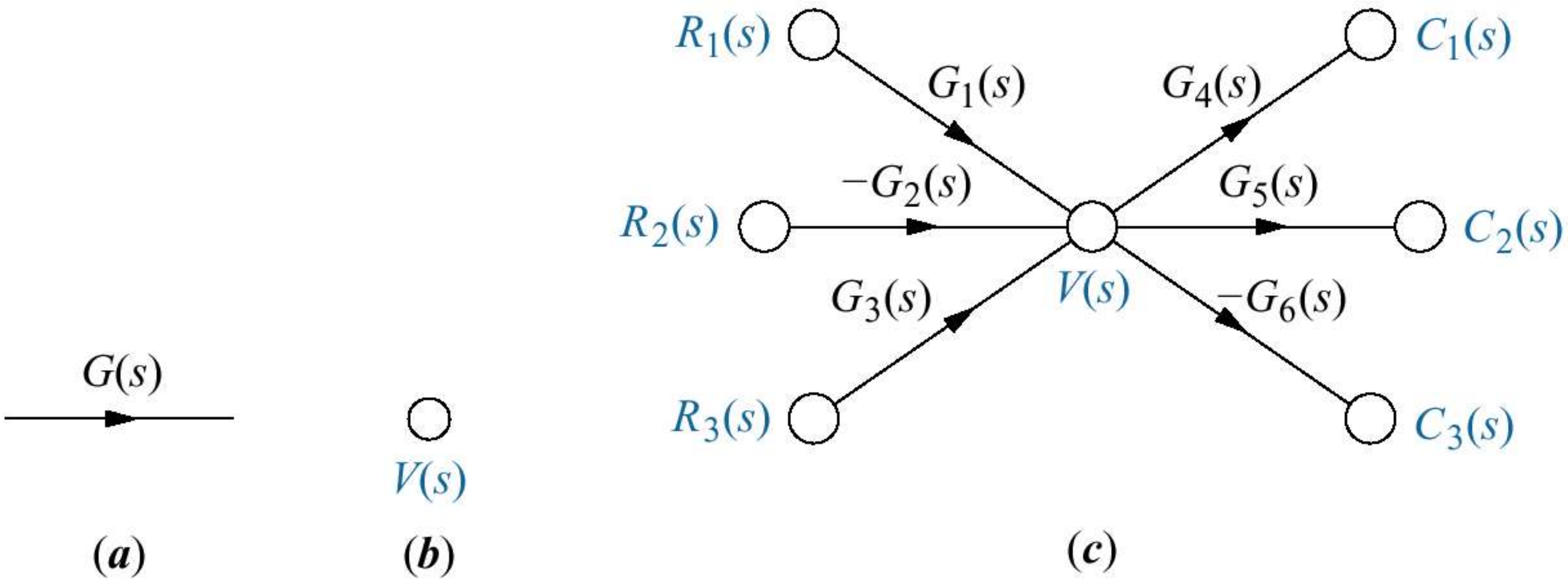


Example ๕.๒ (Nise)



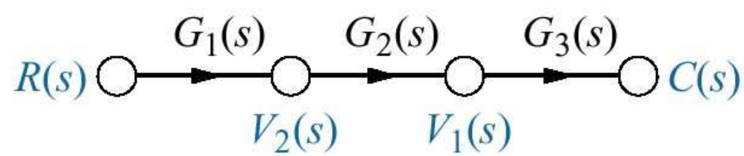
Signal-Flow Graphs

Signal-flow graph components: **a**. System; **b**. signal; **c**. interconnection of systems and signals



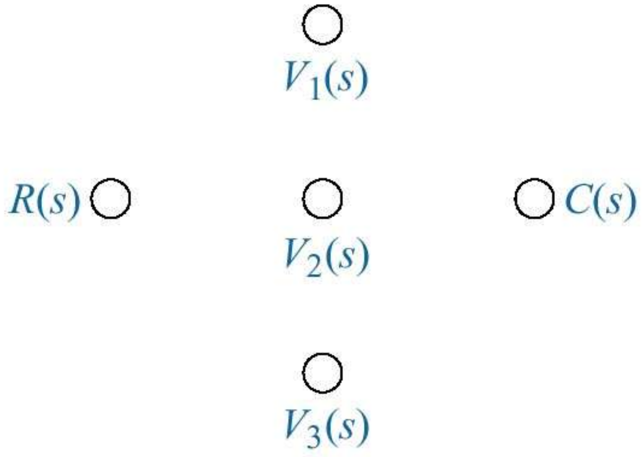
Building signal-flow graphs

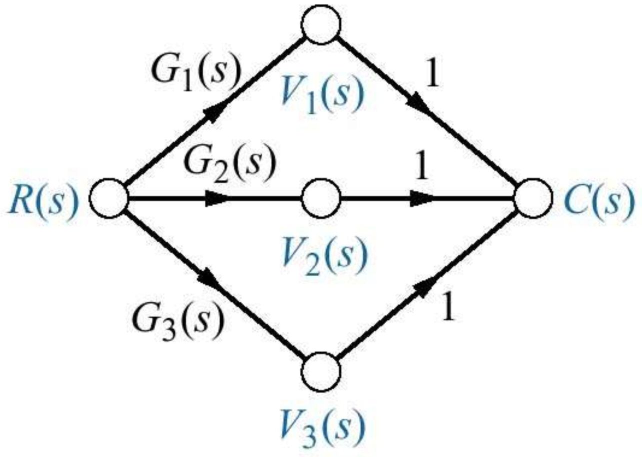
Cascaded system nodes and cascaded system signal-flow graph



Building signal-flow graphs

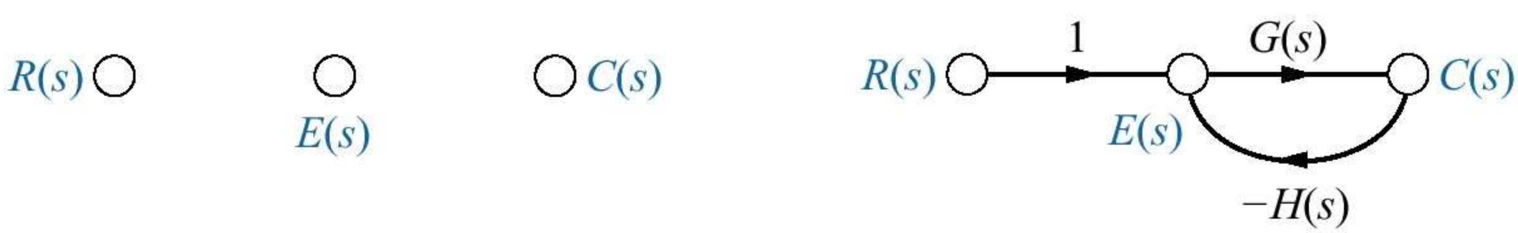
Parallel system nodes and parallel system signal-flow graph



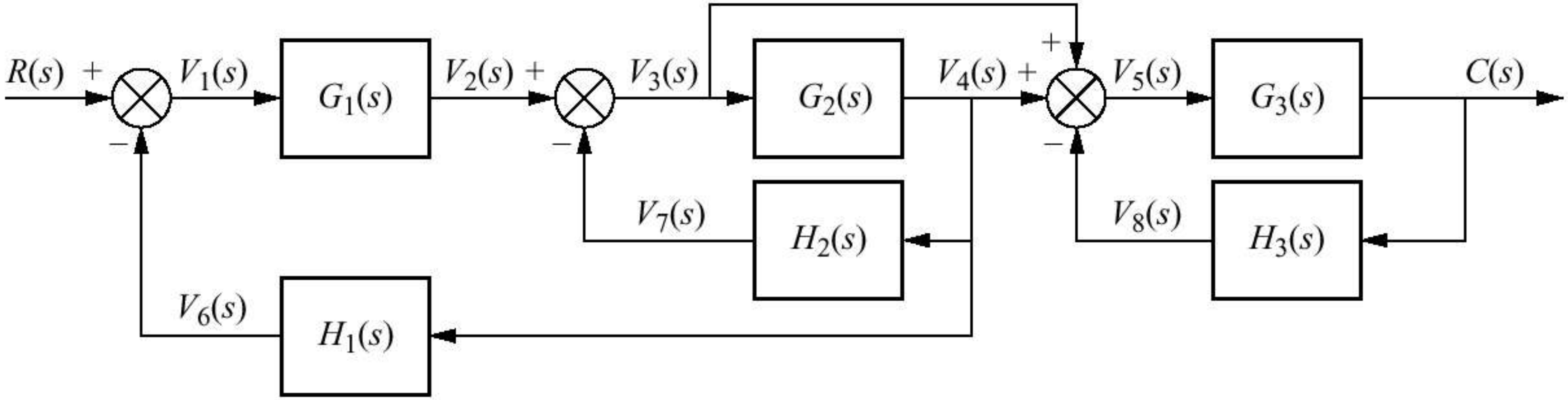


Building signal-flow graphs

Feedback system nodes feedback system signal-flow graph

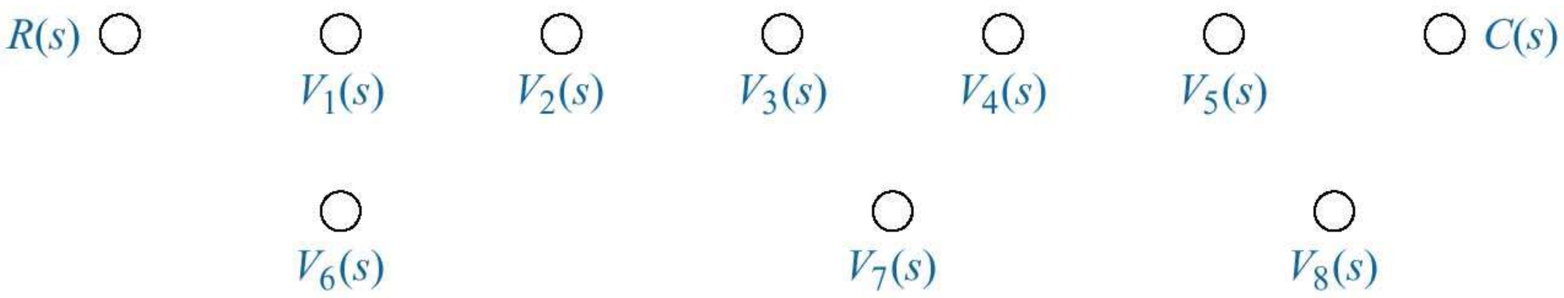


Example ๕.๓ Building signal-flow graphs from the system below



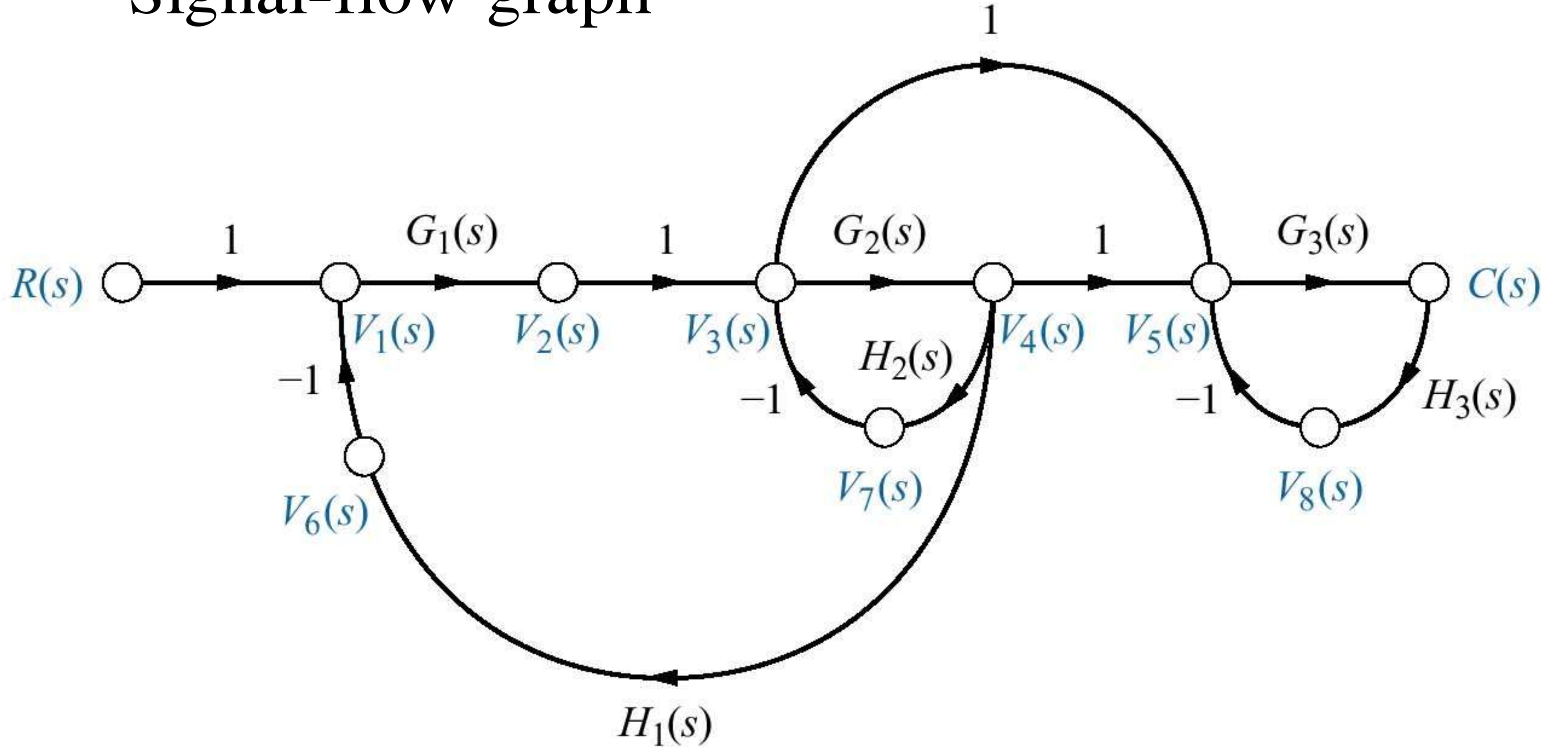
Signal-flow graph development of Ex. ๕.๒

Signal nodes



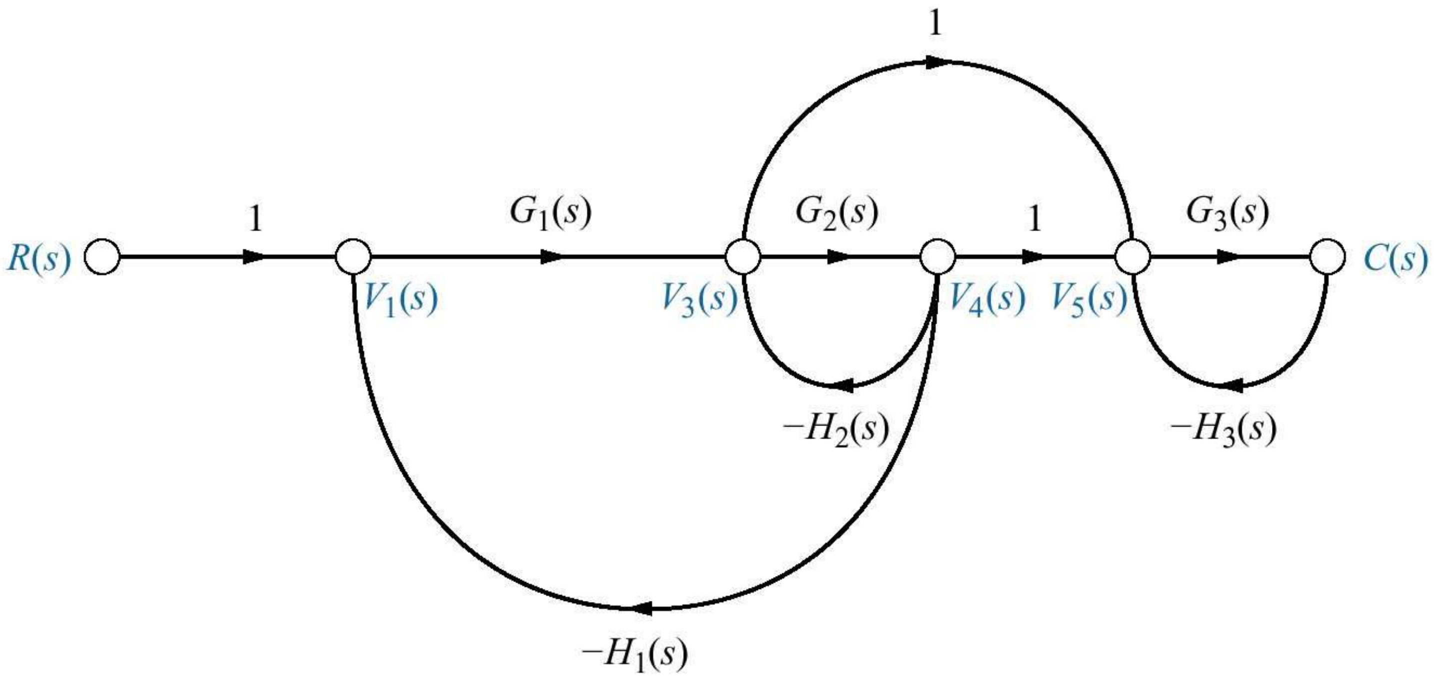
Signal-flow graph development of Ex. ๕.๒

Signal-flow graph



Signal-flow graph development of Ex. ๕.๒

Simplified signal-flow graph



Single Flow Graph Definitions

Node : Nodes on a signal flow graph represent system variables.

Branch : Branches are unidirectional paths that connect the nodes. An arrow is assigned to indicate the direction of cause and effect.

Input node : An input node has only out going branches.

Single Flow Graph Definitions

Output node : An output node has only incoming branches.

Path : A path is continuous connection of branches with arrows in the same direction.

Loop : A loop is path that starts and ends on the same node with all other nodes in the loop touched only once.

Common node : A common node is a node that is contained in two or more loops.

Single Flow Graph Definitions

Non-touching loop : Non-touching loops are loops that no common nodes

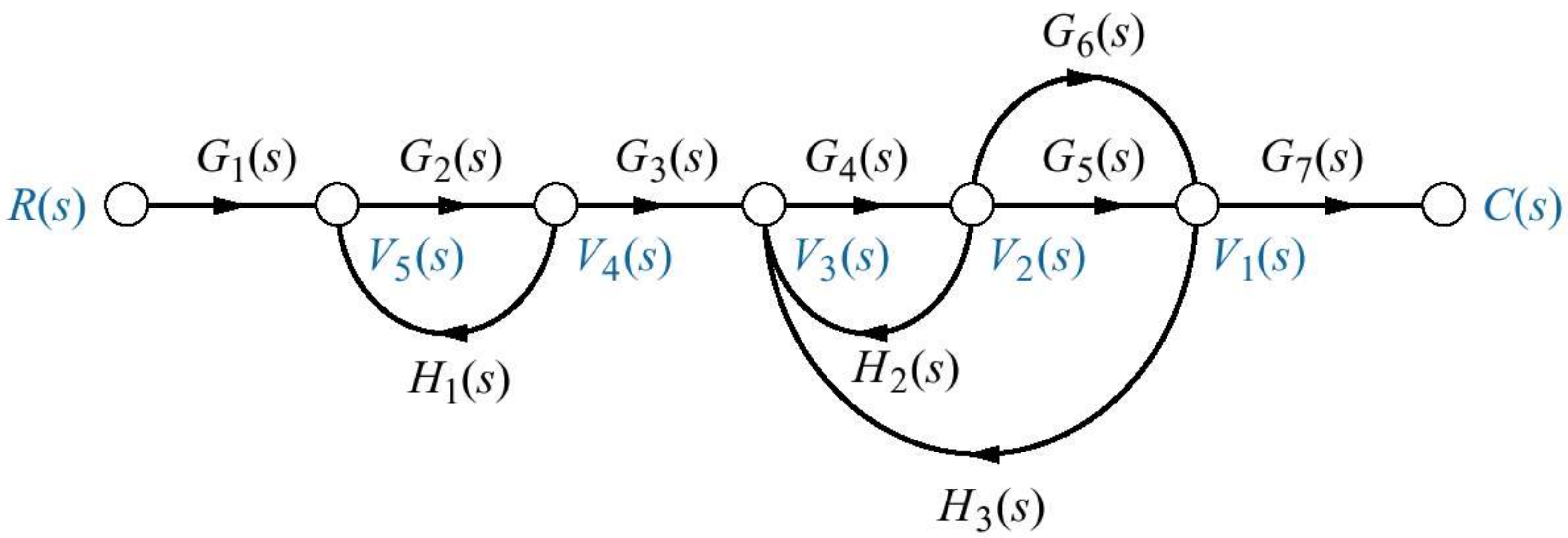
Forward path : A forward path starts at an input node, ends at an output node, and touches no node more than once. A forward path may traverse one or more feedback branches in proceeding from input to output node.

Single Flow Graph Definitions

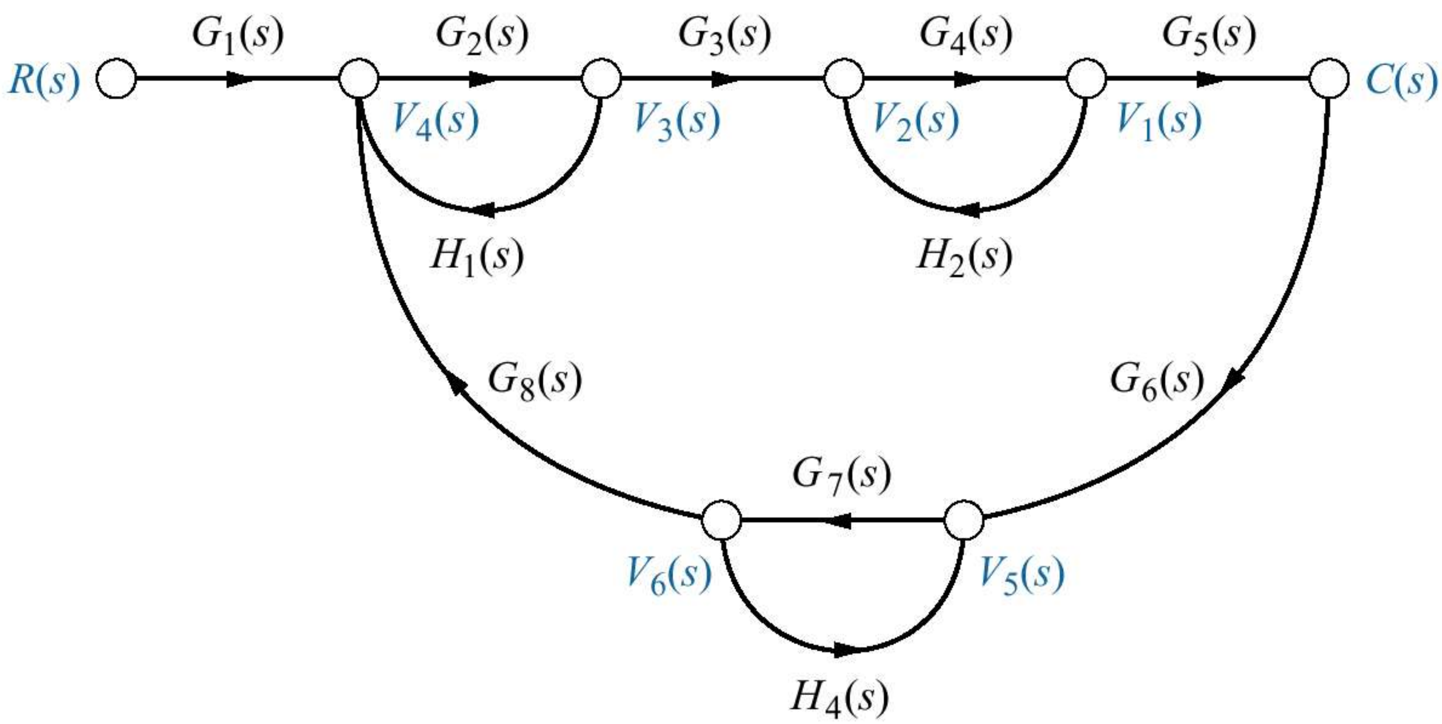
Gains : Gains for paths and loops are defined as the products of branch gains for the paths or loops.

Mason's Rule

Signal-flow graph for demonstrating Mason's rule



Signal-flow graph for an example



Mason's formula for transfer function

$$\frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

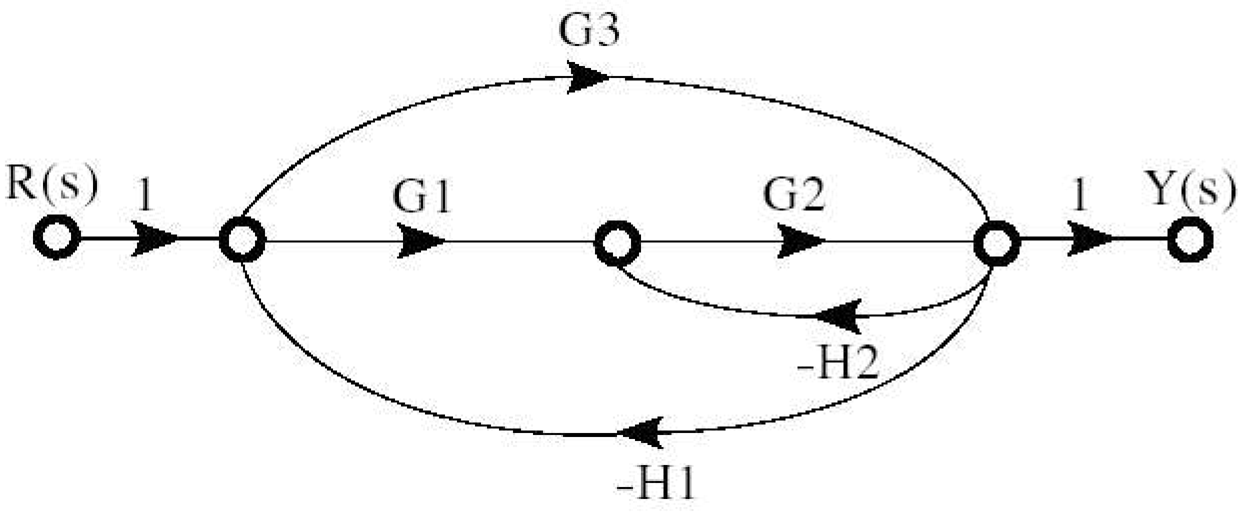
$$k = number\ of\ forward\ path$$

$$T_k = the\ kth\ forward - path\ gain$$

$$\Delta = ๑ - \Sigma \; loop\ gain + \Sigma \; nontouching - loop\ gains\ taken\ two\ at\ a\ time - \Sigma \; nontouching - loop\ gains\ taken\ three\ at\ a\ time + \Sigma \; nontouching - loop\ gains\ taken\ four\ at\ a\ time - \cdots$$

$$\Delta_k = \Delta - \Sigma \; loop\ gain\ terms\ in\ \Delta\ that\ touch\ the\ kth\ forward\ path.\ In\ other\ word\ ,\ \Delta_k\ is\ formed\ by\ eliminating\ form\ \Delta\ those\ loop\ gain\ that\ touch\ the\ kth\ forward\ path$$

Example ๑



Loop gain

$$L_{๑} = -G_{๒}H_{๒}$$

$$L_{๒} = -G_{๑}G_{๒}H_{๑}$$

$$L_{๓} = -G_{๓}H_{๑}$$

Forward path gain

$$T_{๑} = G_{๑}G_{๒}$$

$$T_{๒} = G_{๓}$$

$$
\begin{aligned}
\Delta &= ๑ - \left[L_{๑} + L_{๒} + L_{๓} \right] \\
&= ๑ + G_{๒}H_{๒} + G_{๑}G_{๒}H_{๓} + G_{๓}H_{๑}
\end{aligned}
$$

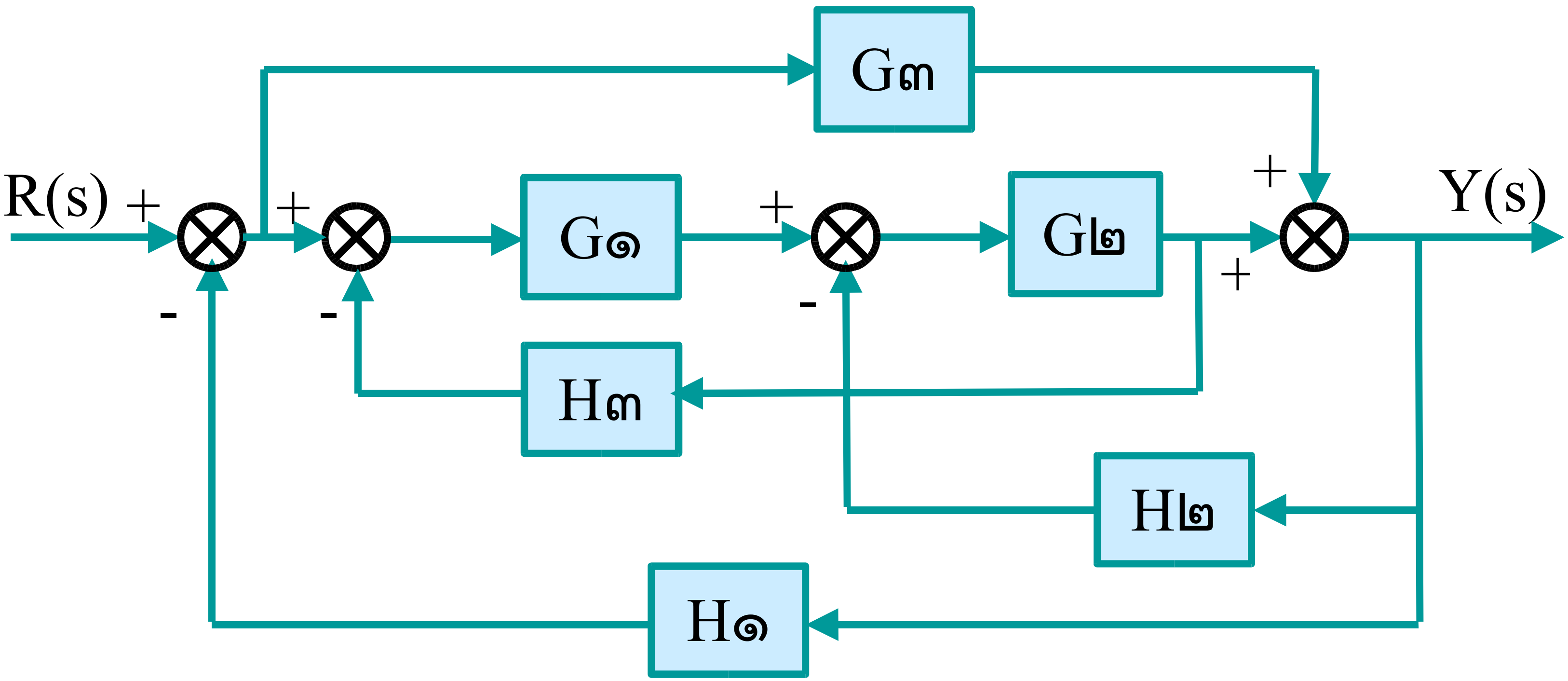
$$
\frac{Y(s)}{R(s)} = \frac{T_{๑}\Delta_{๑} + T_{๒}\Delta_{๒}}{\Delta}
$$

$$
G(s) = \frac{G_{๑}G_{๒} + G_{๓}}{๑ + G_{๒}H_{๒} + G_{๑}G_{๒}H_{๓} + G_{๓}H_{๑}}
$$

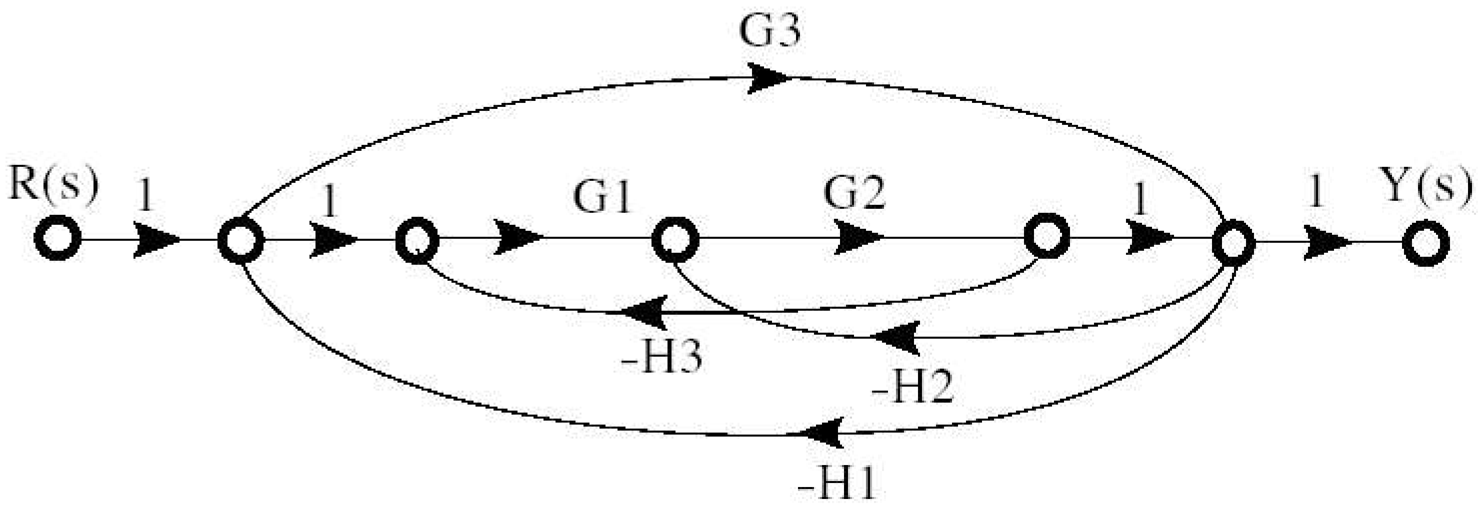
To find Δ_k

Using same formula for Δ but excluding single loops (and combinations of there) that touch the *k*th path.

Example ๒



Signal flow graph of Example ๒



SOLUTION

Loop gain

$$L_{๑} = -G_{๑}G_{๒}H_{๑}$$

$$L_{๒} = -G_{๑}G_{๒}H_{๖}$$

$$L_{๓} = -G_{๒}H_{๒}$$

$$L_{๔} = -G_{๓}H_{๑}$$

Forward path gain

$T_{๑} = G๑G๒$

$T_{๒} = G๓$

$\Delta_{๑} = ๑$

$\Delta_{๒} = ๑ - L๒$

$= ๑ + G๑G๒H๓$

$$\Delta = ๑ - [L๑ + L๒ + L๓ + L๔] + [L๒L๔]$$
$$= ๑ + G๑G๒H๑ + G๑G๒H๓ + G๒H๒ + G๓H๑ + G๑G๒G๓H๑H๓$$

Transfer function

$$\frac{Y(s)}{R(s)} = \frac{\sum T_k \Delta_k}{\Delta}$$

$$\frac{Y(s)}{R(s)} = \frac{G_{๑}G_{๒} + G_{๓}\left(๑ + G_{๑}G_{๒}H_{๓}\right)}{๑ + G_{๑}G_{๒}H_{๑} + G_{๑}G_{๒}H_{๓} + G_{๒}H_{๒} + G_{๓}H_{๑} + G_{๑}G_{๒}H_{๑}H_{๓}}$$

Signal-Flow Graphs of State Equation

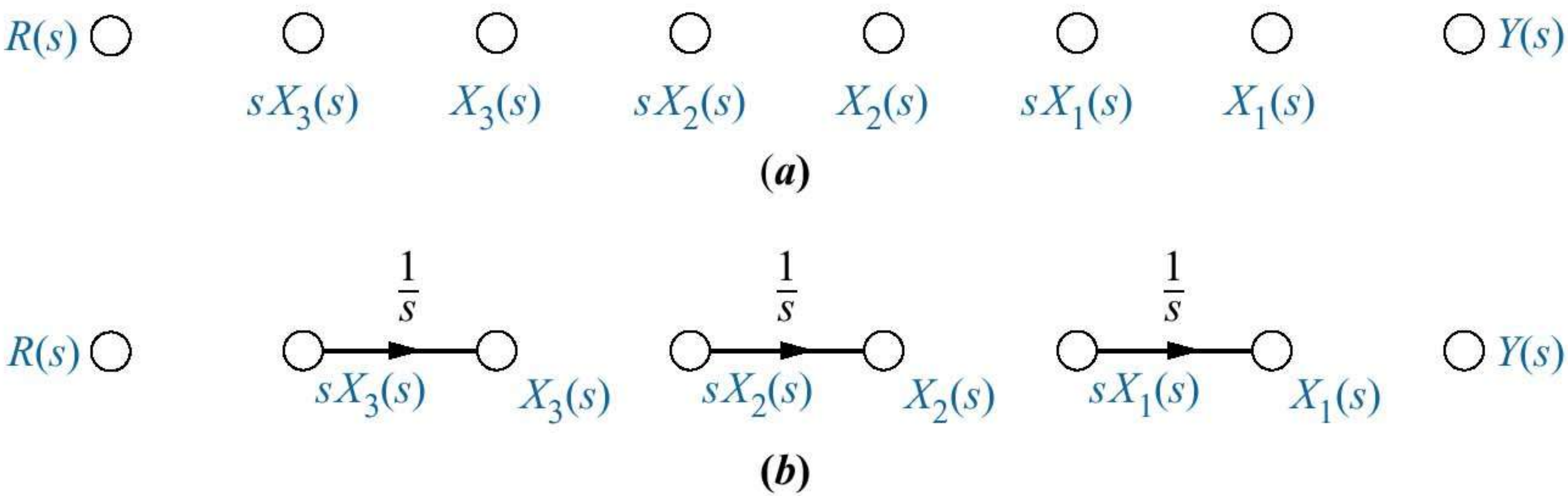
$$\dot{x}_{๑} = ๒x_{๑} - ๕x_{๒} + ๓x_{๓} + ๒r$$

$$\dot{x}_{๒} = -๖x_{๑} - ๒x_{๒} + ๒x_{๓} + ๕r$$

$$\dot{x}_{๓} = x_{๑} - ๓x_{๒} - ๔x_{๓} + ๗r$$

$$y = -๔x_{๑} + ๖x_{๒} + ๙x_{๓}$$

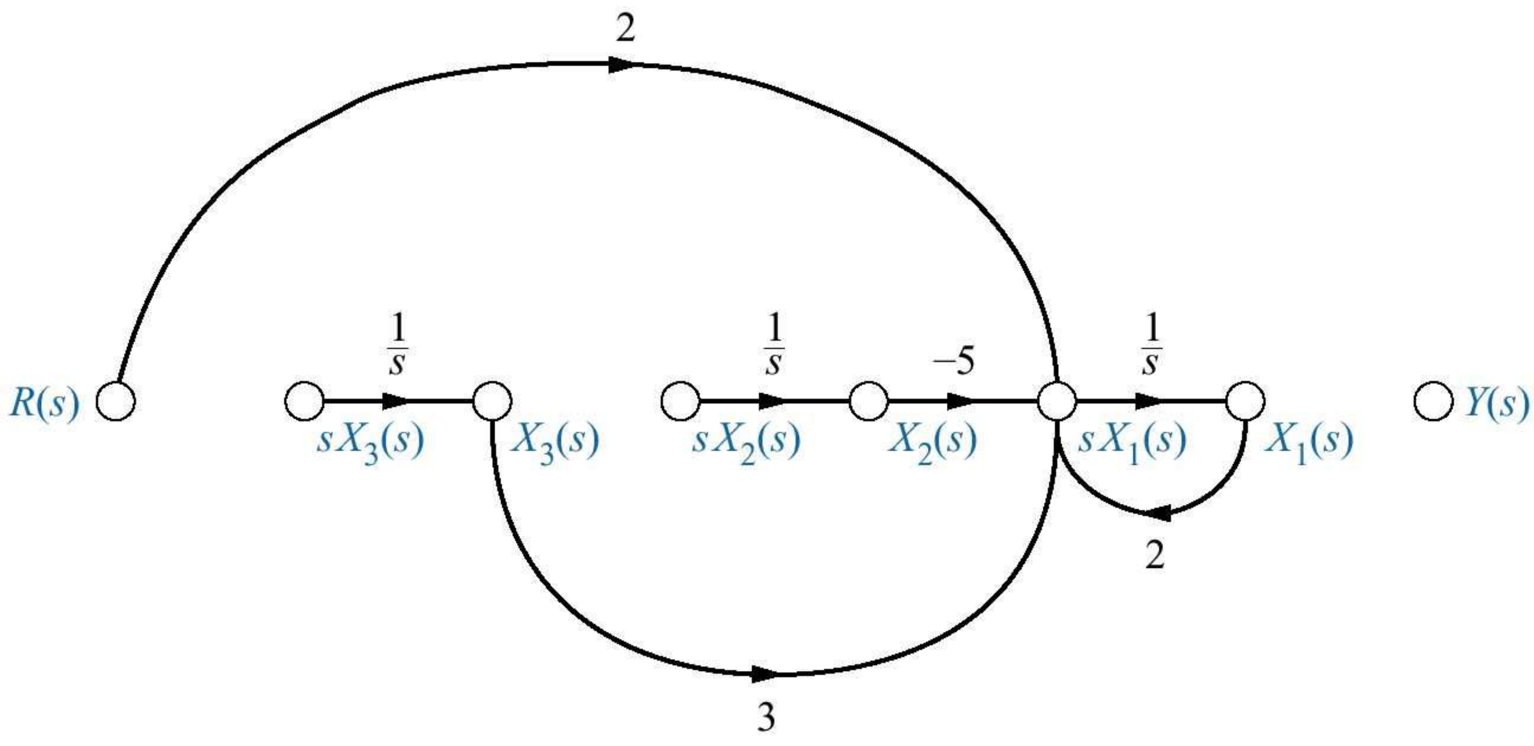
Stages of development of a signal-flow graph



(a) Place nodes

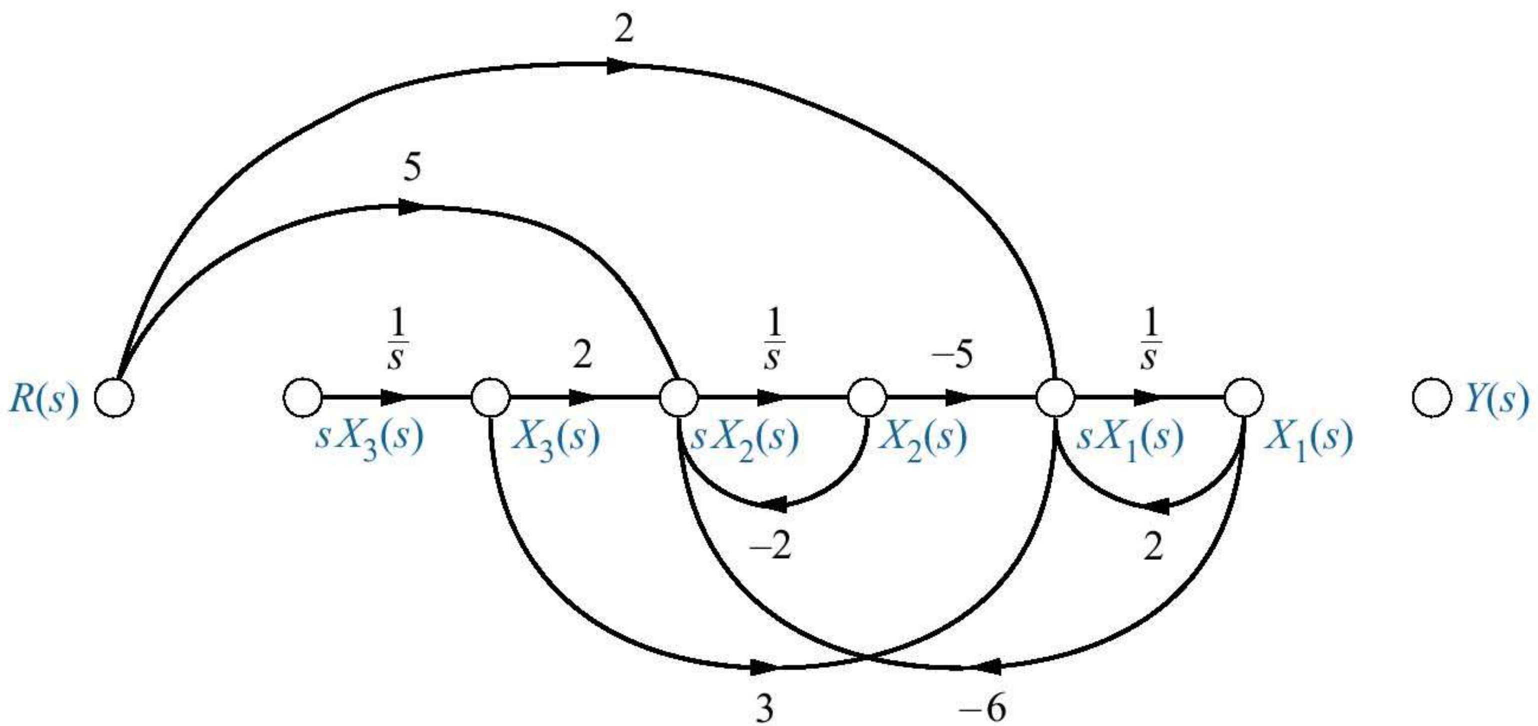
(b) Interconnect state variables and derivatives

Stages of development of a signal-flow graph



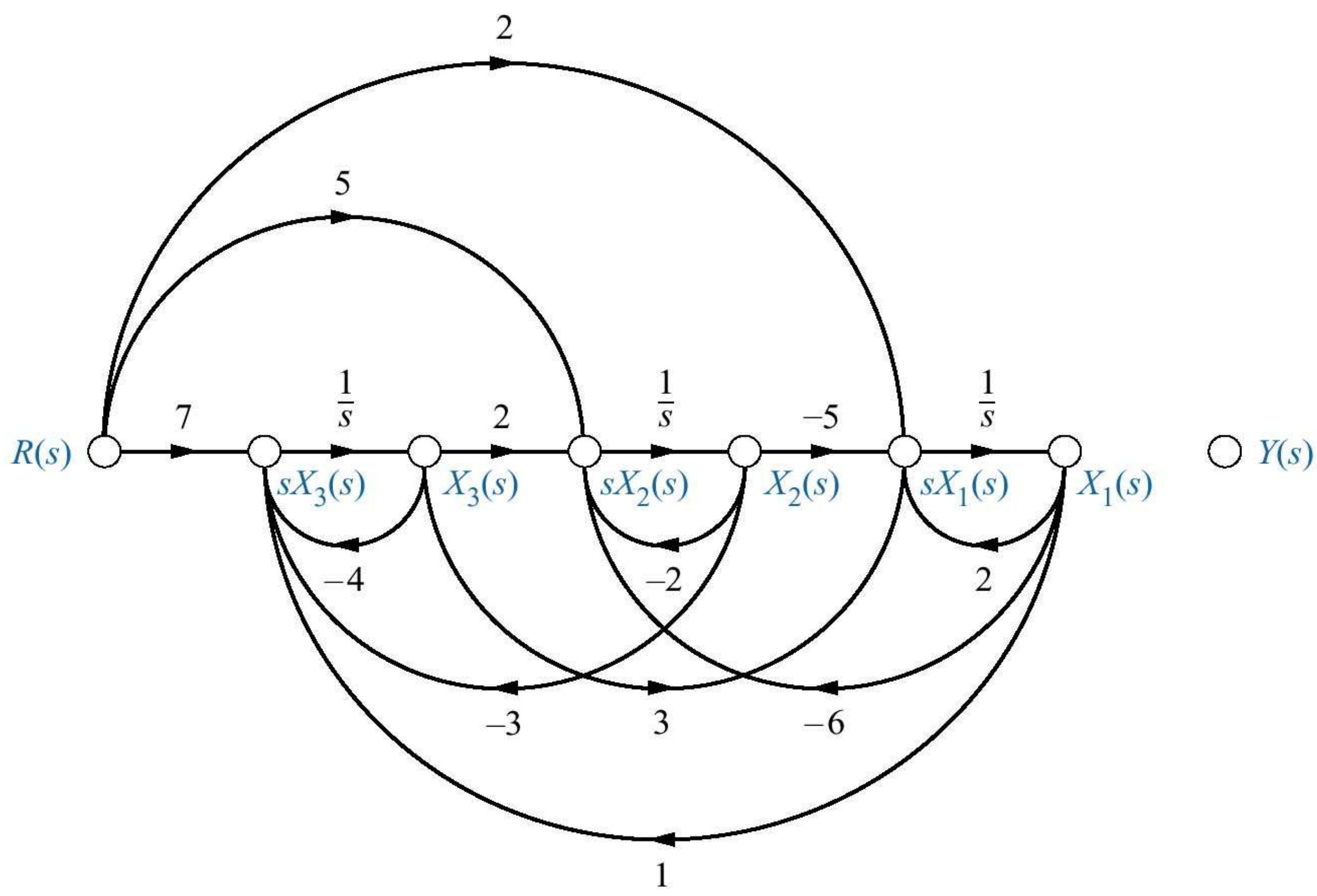
(c) Form $dx_{๑}/dt$

Stages of development of a signal-flow graph



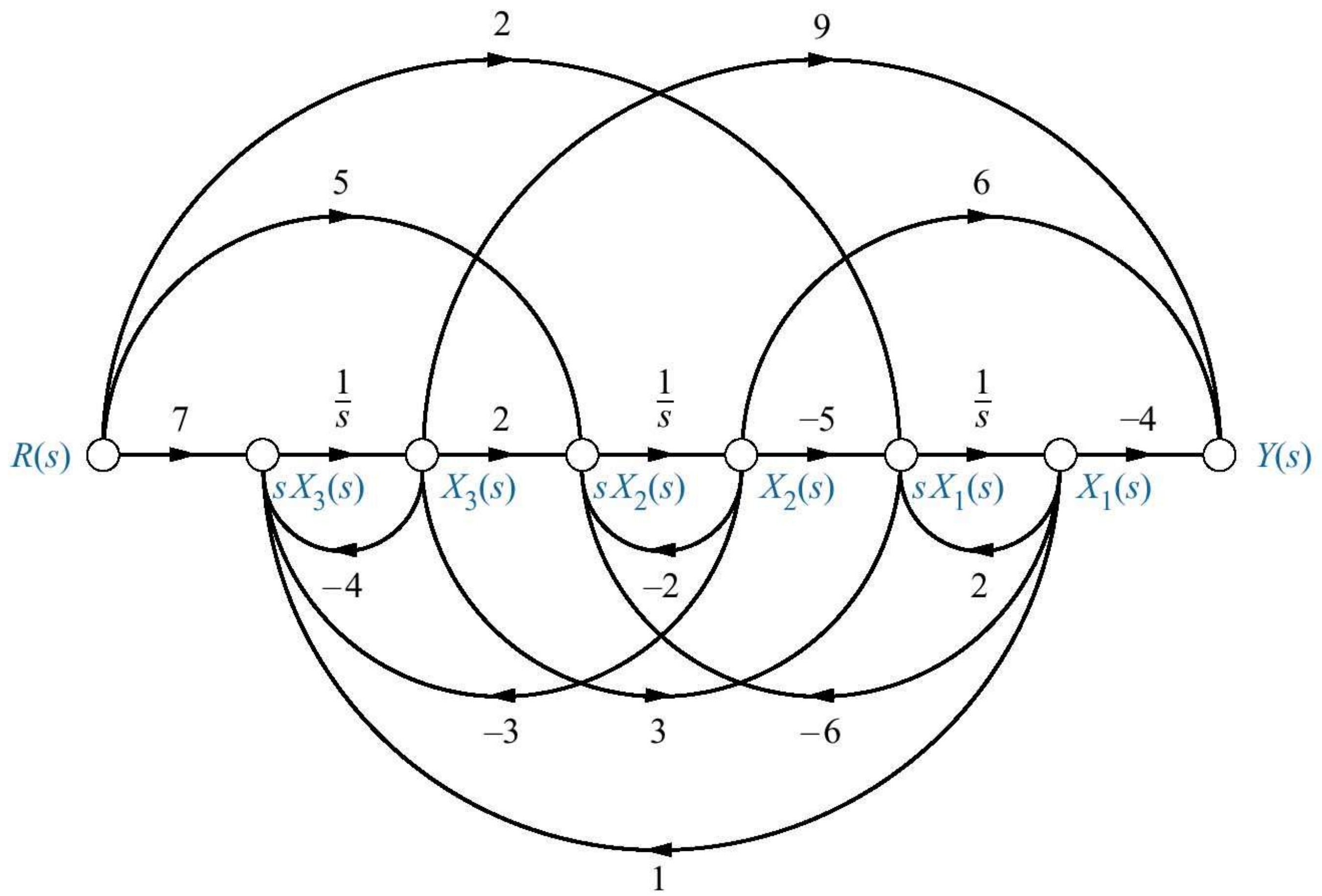
(d) form dx๒/dt

Stages of development of a signal-flow graph



(e) form $dx_{๓}/dt$

Stages of development of a signal-flow graph

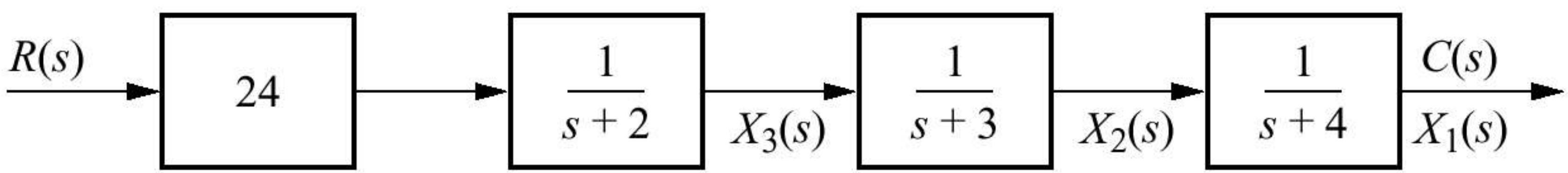


(f) form output

Alternative Representations in State Space

Representation of system as cascaded first-order systems

$$\frac{C(s)}{R(s)} = \frac{๒๔}{(s+๒)(s+๓)(s+๔)}$$



$$\frac{C_i(s)}{R_i(s)} = \frac{๑}{(s+a_i)}$$

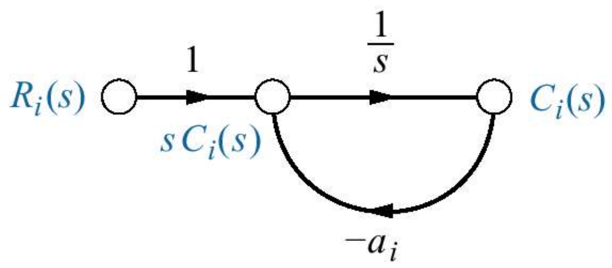
$$(s+a_i)\,C_i(s) = R_i(s)$$

$$\frac{dc_i(t)}{dt} + a_i c_i(t) = r_i(t)$$

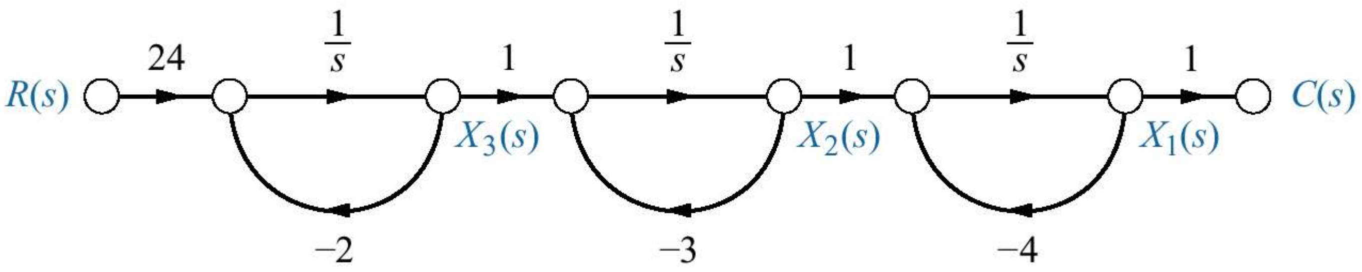
$$\frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t)$$

a. First-order subsystem;

b. signal-flow graph for a system



(a)



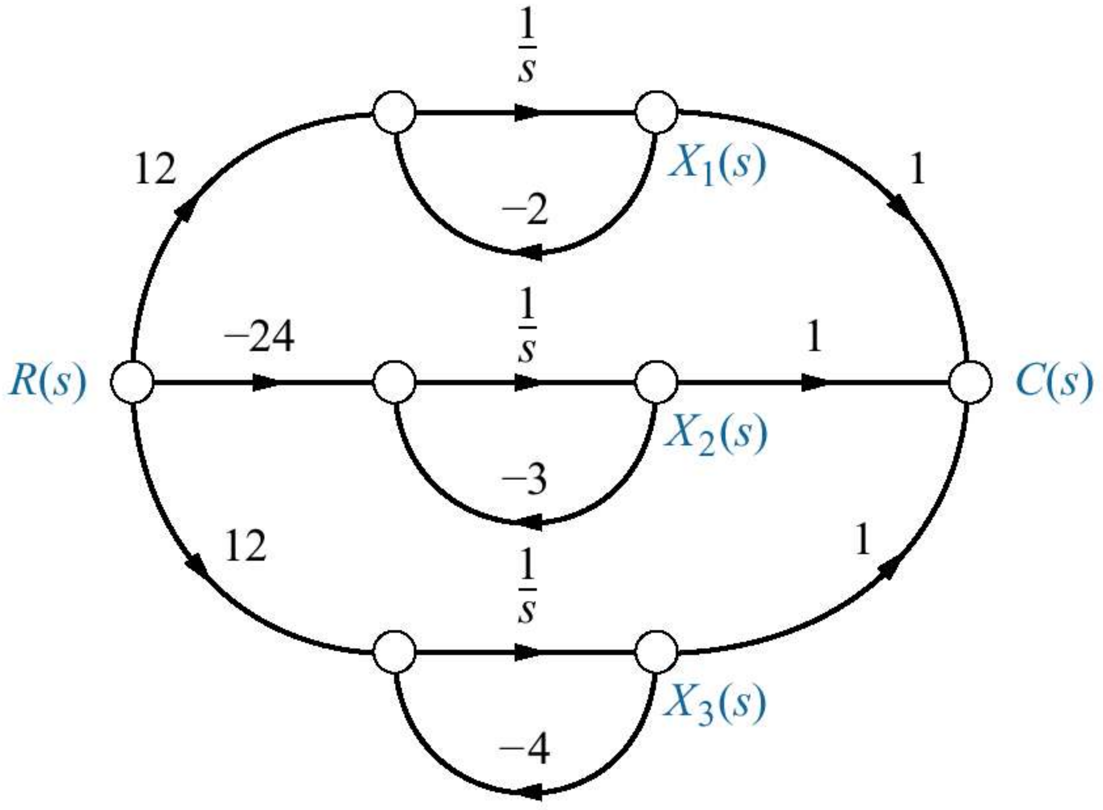
(b)

Signal-flow representation for parallel form

$$\frac{C(s)}{R(s)} = \frac{๒๔}{(s+๒)(s+๓)(s+๔)}$$

$$= \frac{๑๒}{(s+๒)} - \frac{๒๔}{(s+๓)} + \frac{๑๒}{(s+๔)}$$

Signal-flow representation for parallel form



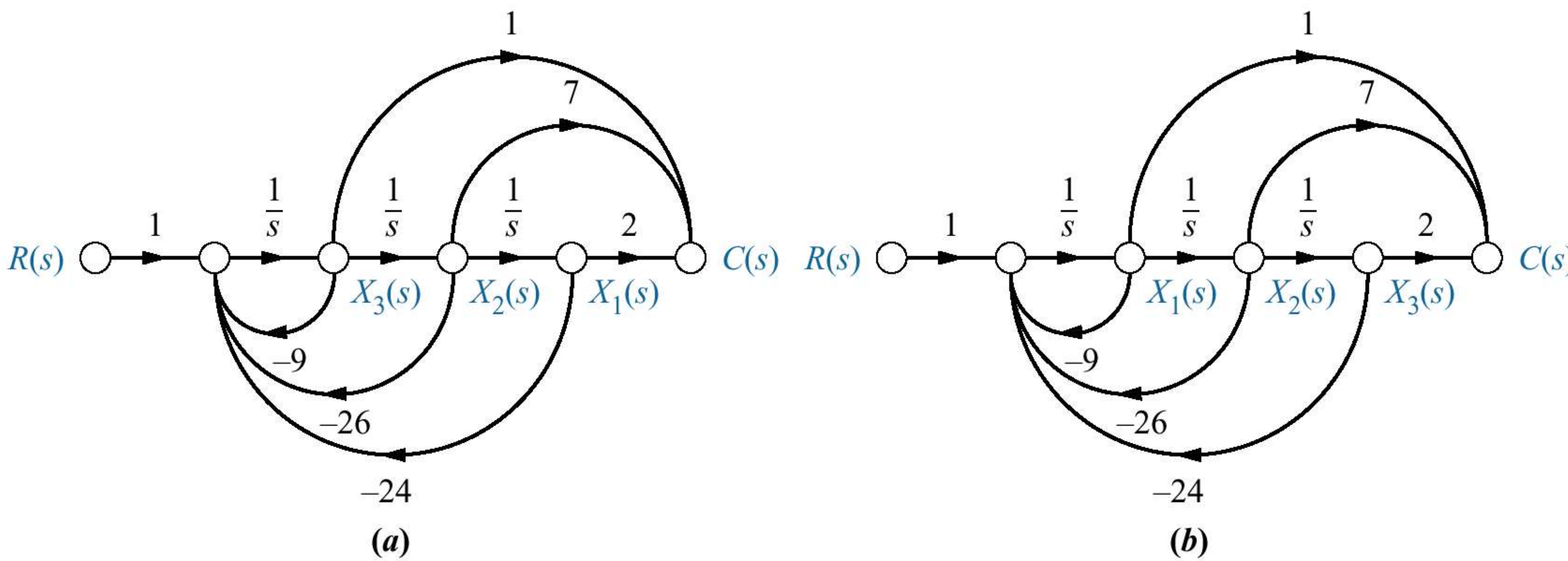
Controller Canonical Form

$$\frac{C(s)}{R(s)}=\frac{s^{๒}+๗s+๒}{s^{๓}+๙s^{๒}+๒๖s+๒๔}$$

$$\begin{bmatrix}\dot{x}_{๑}\\ \dot{x}_{๒}\\ \dot{x}_{๓}\end{bmatrix}=\begin{bmatrix}๐ & ๑ & ๐\\ ๐ & ๐ & ๑\\ -๒๔ & -๒๖ & -๙\end{bmatrix}\begin{bmatrix}x_{๑}\\ x_{๒}\\ x_{๓}\end{bmatrix}+\begin{bmatrix}๐\\ ๐\\ ๑\end{bmatrix}$$

$$y=\begin{bmatrix}๒ & ๗ & ๑\end{bmatrix}\begin{bmatrix}x_{๑}\\ x_{๒}\\ x_{๓}\end{bmatrix}$$

Signal-flow graph for controller canonical form variables



(a) phase-variable form

(b) controller canonical form

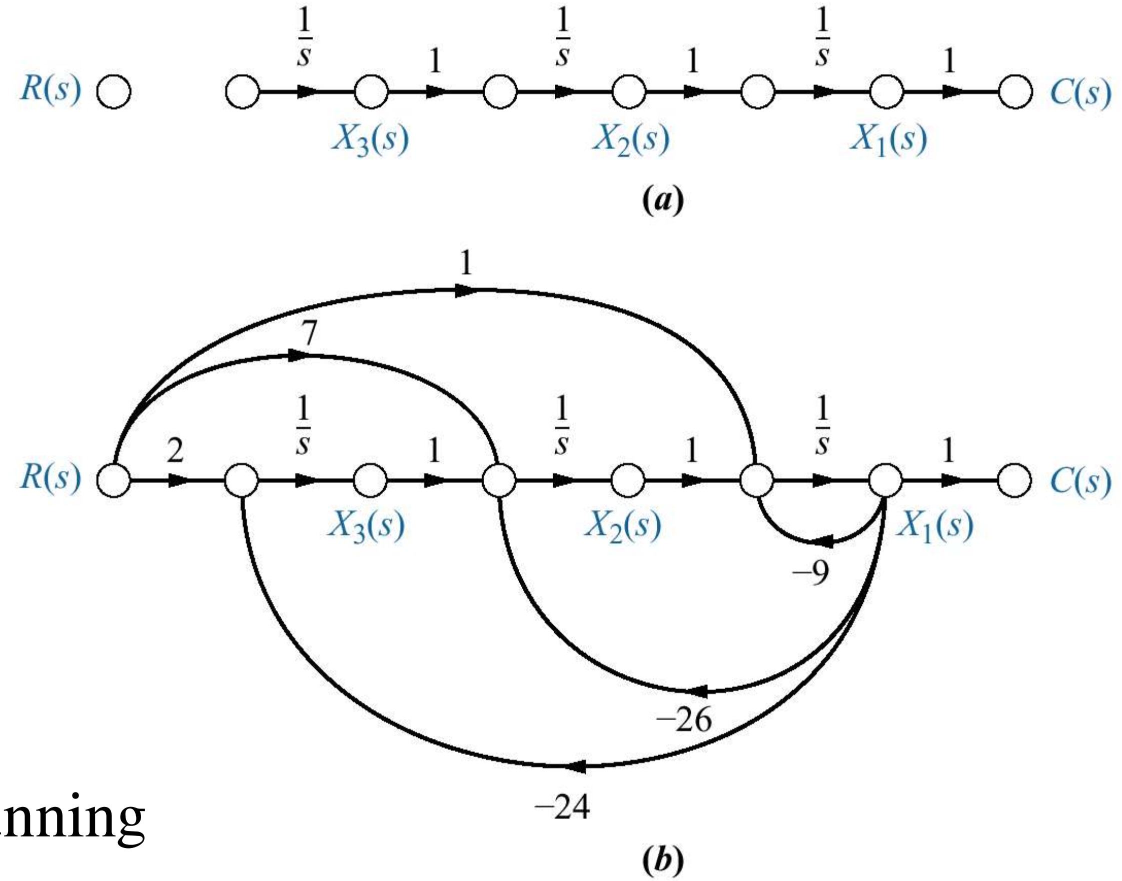
Observer Canonical Form

$$\frac{C(s)}{R(s)}=\frac{\frac{๑}{s}+\frac{๗}{s^{๒}}+\frac{๒}{s^{๓}}}{๑+\frac{๙}{s}+\frac{๒๖}{s^{๒}}+\frac{๒๔}{s^{๓}}}$$

$$\dot{\boldsymbol{x}}=\begin{bmatrix} -๙ & ๑ & ๐ \\ -๒๖ & ๐ & ๑ \\ -๒๔ & ๐ & ๐ \end{bmatrix}\boldsymbol{x}+\begin{bmatrix} ๑ \\ ๗ \\ ๒ \end{bmatrix}r$$

$$y=\begin{bmatrix} ๑ & ๐ & ๐ \end{bmatrix}\boldsymbol{x}$$

Signal-flow graph for observer canonical form variables



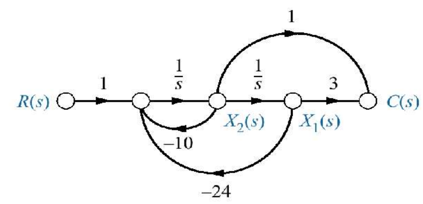
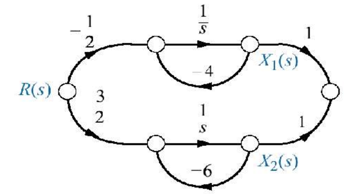
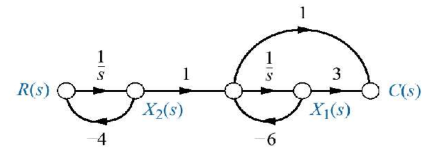
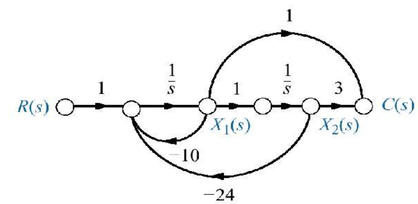
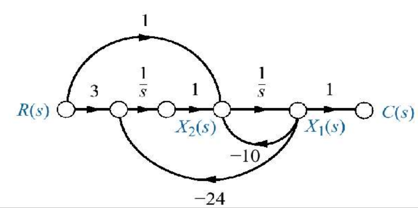
(a) planning

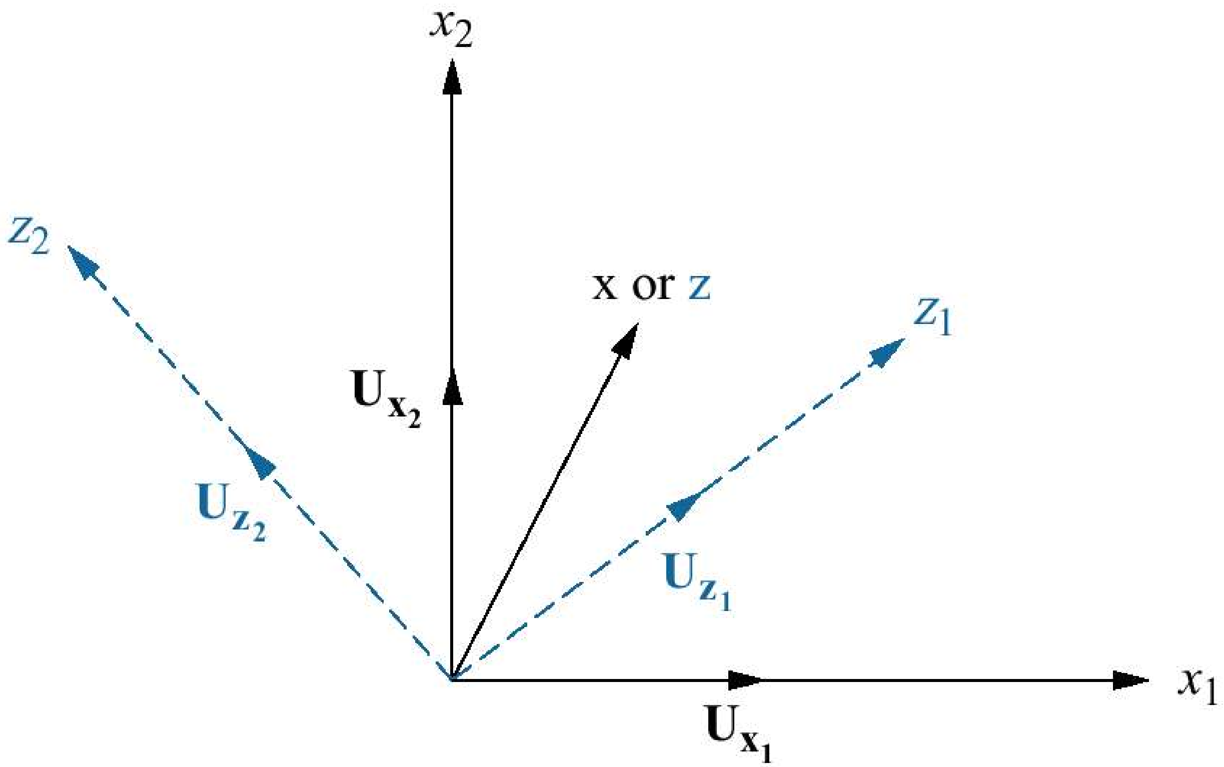
(b) implementation

Similarity Transformations

State-space forms for C(s)/R(s) = (s+ ๓)/[(s+ ๔)(s+ ๖)].

Note: y = c(t)

Form	Transfer Function	Signal-Flow Diagram	State Equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$



State-space transformations