Name: ______________________

Roll Number: ________________

Practice final

MTH302 - Set Theory and Mathematical Logic

(Odd Semester 2024/25, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.
2. This exam contains **5 + 1** questions and is worth **50%** of your grade.
3. Answer **ALL** questions.

Question 1. [5 $\times$ 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) There exists a sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ such that $\alpha_{n+1} < \alpha_n$ for every $n < \omega$.

(ii) $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

(iii) Every consistent first order theory has an infinite model.

(iv) $\mathsf{Th}([0,1], <)$ is ω-categorical.

(v) If $A \subseteq \omega$ is c.e., then $A + A = \{m + n : m, n \in A\}$ is c.e.

Question 2. [10 Points]

Call a subset $E \subseteq \mathbb{R}$ mid-point free if there do not exist $x < y < z$ in E such that $x + z = 2y$.

(a) [**2 Point**] State Zorn's lemma.

(b) [**4 Points**] Show that there is a maximal mid-point free set $E \subseteq \mathbb{R}$. This means that E is mid-point free and for every $x \in \mathbb{R} \setminus E$, $E \cup \{x\}$ is not mid-point free.

(c) [**4 Points**] Show that every maximal mid-point free set has cardinality $\mathfrak{c}$.

Question 3. [10 Points]

Let $\mathcal{L}$ be the empty language. For each $n \geq 2$, recall that $\exists_{\geq n}$ denotes the following $\mathcal{L}$-sentence:

$$(\exists x_1)(\exists x_2)\dots(\exists x_n)\left(\bigwedge_{i<j\leq n} \neg(x_i = x_j)\right)$$

Define $T = \{\exists_{\geq n} : n \geq 2\}$. Prove the following.

(a) [**2 Points**] T does not have a finite model.

(b) [**2 Points**] T is consistent.

(c) [**2 Points**] T is ω-categorical.

(d) [**2 Points**] T is complete.

(e) [**2 Points**] $\{\phi : T \vdash \phi\}$ is computable.

Question 4. [10 Points]

Let $T = \mathsf{Th}(\mathbb{Z}, <)$. So T is an $\mathcal{L}$-theory where $\mathcal{L} = \{<\}$.

(a) [**2 Point**] Is T a complete theory? Justify your response.

(b) [**2 Point**] State the compactness theorem for first order logic.

(c) [**6 Points**] Show that T is not ω-categorical. **Hint:** Put $\mathcal{L}_1 = \{<, c, d\}$ where c, d are constant symbols and consider the $\mathcal{L}_1$-theory $T_1 = T \cup \{\psi_n : n \geq 1\}$ where ψ_n is says "there are at least n elements between c and d".

Question 5 [10 Points]

Let $\mathcal{N} = (\omega, 0, S, +, \cdot)$ be the standard model of PA. Define

$$\mathsf{True}_{\mathcal{N}} = \{\ulcorner\psi\urcorner : \mathcal{N} \models \psi\} \text{ and } \mathsf{False}_{\mathcal{N}} = \{\ulcorner\psi\urcorner : \mathcal{N} \models \neg\psi\}$$

(a) [**3 Points**] Let $E \subseteq \omega$. What does it mean to say that E is definable in $\mathcal{N}$?

(b) [**3 Points**] State Tarski's theorem on undefinability of truth in arithmetic.

(c) [**4 Points**] Suppose $S \subseteq \omega$, $\mathsf{True}_{\mathcal{N}} \subseteq S$ and $\mathsf{False}_{\mathcal{N}} \subseteq (\omega \setminus S)$. Show that S is not computable.

Bonus Question [5 Points]

Recall that φ_e is the eth partial computable function on ω. Define $A = \{e : e \in \mathsf{dom}(\varphi_e) \text{ and } \varphi_e(e) = 0\}$ and $B = \{e : e \in \mathsf{dom}(\varphi_e) \text{ and } \varphi_e(e) = 1\}$. Show that $A, B \subseteq \omega$ are disjoint c.e. sets and there is no computable $R \subseteq \omega$ such that $A \subseteq R$ and $B \subseteq \omega \setminus R$.

End of Exam