7/10/25 –

Consider the (LH) system –

$$x'(t) = A(t)x(t)$$

Suppose $f_1, \dots, f_n$ be solns. of (LH). Consider

$$U(t) = [f_1(t), \dots, f_n(t)]$$

then $U(t)$ satisfies $\qquad f_j'(t) = A(t) f_j(t)$

$$U'(t) = A(t)U(t)$$

Consider $(A(t)U(t))_{i,j} = \sum_{k=1}^{n} (A(t))_{i,k} (U(t))_{k,j}$

$$= \sum_{k=1}^{n} (A(t))_{i,k} f_{j,k}(t)$$

Here $f_j(t) = (f_{j,1}(t), \dots, f_{j,n}(t))$

$$= (A(t) f_j(t))_{i,1} = (\dot{f_j}(t))_{i,1} = \dot{f}_{j,i}(t) = (\dot{U}(t))_{i,j}$$

$U(t)$ is a solution matrix.

- Note that the principal soln. matrix at a point t_0, $\Phi(\cdot, t_0)$ is a solution matrix.

Def^n:- Given homogeneous system,

$$x'(t) = A(t)x(t)$$

$A : I \xrightarrow{\text{cont.}} M_n(\mathbb{R})$. A matrix valued func^n $\psi : I \to M_n(\mathbb{R})$ is called a fundamental soln. matrix if $\psi(t)$ satisfies the following two conditions

① $\psi'(t) = A(t)\psi(t) \ \forall t \in I$

② the columns of ψ are linearly independent in $C(I, \mathbb{R}^n)$.

Note the principal soln. matrix is a fundamental matrix.

THEOREM: A necessary & sufficient condition that a solution matrix $\psi : I \to M_n(\mathbb{R})$ be a fundamental soln. matrix is that $\det(\psi(t)) \neq 0$ for some $t \in I$.

PROOF: First suppose $\psi : I \to M_n(\mathbb{R})$ is a fundamental soln. matrix.

$$\psi(t) = [\psi_1(t) \dots \psi_n(t)] \quad t \in I$$

Claim: Fix $t_0 \in I$ then the vector $(\psi_1(t_0), \dots, \psi_n(t_0))$ are lin. ind.

Suppose not. $\sum_{i=1}^{n} c_i \psi_i(t_0) = 0$, where not all c_i's are zero.

Let $\varphi = \sum_{i=1}^{n} c_i \psi_i$, then φ is also a soln. of (LH).

$$\varphi(t_0) = \left(\sum_{i=1}^{n} c_i \psi_i\right) t_0 = \sum_{i=1}^{n} c_i \psi_i(t_0)$$

$$\Rightarrow \varphi \equiv 0$$

$\sum_{i=1}^{n} c_i \psi_i \equiv 0 \Rightarrow \{\psi_1, \dots, \psi_n\}$ are lin. dep. which is a contradic^n.

—X—

Now suppose $\quad \psi'(t) = A(t)\psi(t)$

Suppose for some $t_0 \in I$, $\det(\psi(t_0)) \neq 0$

Claim: $(\psi_1, \dots, \psi_n)$ are lin. ind.

Suppose not. Then $\sum_{i=1}^{n} c_i \psi_i \equiv 0$, where not all c_i's are zero.

$\Rightarrow \sum_{i=1}^{n} c_i \psi_i(t_0) = 0 \qquad$ Not all c_i's are zero.

$$\psi(t_0) C = 0 \qquad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\Rightarrow \det(\psi(t_0)) = 0 \ \text{(Contradiction)}$$

—X—

$U(t) \leftarrow$ Solution matrix

$W(t) = \det(U(t)) \qquad t \in I$

Wrenski detor.

If for some t_0, $W(t_0) \neq 0$ then for any other t
$$W(t) \neq 0$$

Exercise: $U(t) \leftarrow$ solution matrix

① $U(t)C$, $C \in M_n(\mathbb{R})$ is a soln. matrix.

② ψ_1 & ψ_2 are two FSM then for any $t_0 \in I$

$\psi_1(t)(\psi_1(t_0))^{-1} = \psi_2(t)(\psi_2(t_0))^{-1} = \Phi(t, t_0)$

$$\dot{x}(t) = A(t)x(t) + \varphi(t) \quad \text{①}$$

φ_1, φ_2 then $\varphi_1 - \varphi_2$ is a solution of
$$\dot{x}(t) = A(t)x(t)$$

This implies that if we know a particular solution of ①, φ_0 then all other solns. are given by φ_0 + soln. set of (LH)
$$\varphi_0 + \psi$$
$$\dot{x}(t) = A(t)x(t) + \varphi(t)$$
$$x(t_0) = x_0$$

Suppose the soln. to be of the form
$$\varphi(t) = \Phi(t, t_0)C(t)$$

13/10/2025 $\qquad \dot{x}(t) = A(t)x(t) \qquad x(t_0) = x_0$

Soln. is given by $x(t) = \Phi(t, t_0)x_0$

Recall $\Phi(\cdot, t_0)$ is the principal soln. matrix.

(1) $\Phi'(t, t_0) = A(t)\Phi(t, t_0)$

(2) $\Phi(t_0, t_0) = I_n$

Inhomogeneous System

$$\dot{x}(t) = A(t)x(t) + g(t) \qquad \text{(ILS)}$$
$$A: I \xrightarrow{\text{cont.}} M_n(\mathbb{R})$$
$$g: I \xrightarrow{\text{cont.}} \mathbb{R}^n$$

Let φ_1 & φ_2 be two soln. of (ILS). Then $\varphi_1 - \varphi_2$ is a soln. of (LH)

Hence, if we know a particular soln. φ of (ILS) then any other soln. is of the type $\psi + \varphi$ where ψ is a soln. of (LH).

2. Consider (ILS) with the initial condition $x(t_0) = x_0$.

The idea is to guess that the soln. is of the form
$$\varphi(t) = \Phi(t, t_0)c(t)$$

Observe that $\varphi(t_0) = x_0 = c(t_0)$
$$\begin{aligned}\varphi'(t) &= \Phi'(t, t_0)c(t) + \Phi(t, t_0)c'(t) \\ &= A(t)\Phi(t, t_0)c(t) + \Phi(t, t_0)c'(t) \\ &= A(t)\varphi(t) + \Phi(t, t_0)c'(t) \\ &= A(t)\varphi(t) + g(t) \quad \text{[By assumption]}\end{aligned}$$

$$\Phi(t,t_0)c'(t) = g(t)$$
$$\Rightarrow c'(t) = \Phi(t,t_0)^{-1} g(t) = \Phi(t_0,t)\, g(t)$$
$$\Rightarrow c(t) - c(t_0) = \int_{t_0}^{t} \Phi(t_0,s) g(s)\,ds$$

$$\boxed{c(t) = x_0 + \int_{t_0}^{t} \Phi(t_0,s) g(s)\,ds}$$

Our soln. :- $\varphi(t) = \Phi(t,t_0)x_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi(t_0,s) g(s)\,ds$

$$\boxed{\varphi(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s) g(s)\,ds}$$

1-d case: $\boxed{x'(t) = a(t)x(t) + b(t)}$

$$\Phi'(t,t_0) = a(t)\Phi(t,t_0)$$
$$\Phi(t_0,t_0) = 1$$
$$\Phi(t,t_0) = \exp\left(\int_{t_0}^{t} a(s)\,ds\right)$$

Linear Differential Eqns. of order n:

- HOMOGENEOUS CASE:

Consider $a_0, a_1, \dots, a_n$, cnt. fnc^{n}s on I, assume $a_0(t) \neq 0$ for any $t \in I$. Consider the differential operator.

$$L_n = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \dots + a_n(t)$$

acts on all fnc^n on I that are at least n times differentiable.

$$L_n(g) = a_0(t)g^{(n)}(t) + a_1(t)g^{(n-1)}(t) + \dots + a_{n-1}(t)g^{(1)}(t) + a_n(t)g(t)$$

The linear diff. eqn. of order n is the eqn.

$$L_n x = 0 \Leftrightarrow a_0(t)x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \dots + a_{n-1}(t)x^{(1)}(t) + a_n(t)x(t) = 0$$

$$x^{(n)}(t) + \frac{a_1(t)}{a_0(t)}x^{(n-1)}(t) + \dots + \frac{a_{n-1}(t)}{a_0(t)}x^{(1)}(t) + \frac{a_n(t)}{a_0(t)}x(t) = 0 \quad \text{— ①}$$

* Introduce $(y_1, y_2, \dots, y_n) = (x(t), x^{(1)}(t), \dots, x^{(n)}(t))$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= -\frac{a_1(t)}{a_0(t)}y_{n-1} - \frac{a_2(t)}{a_0(t)}y_{n-2} - \dots - \frac{a_n(t)}{a_0(t)}y_1 \end{aligned}}$$

$y'(t) = A(t)y(t)$ — ②

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & & 1 \\ -\frac{a_n(t)}{a_0(t)} & -\frac{a_{n-1}(t)}{a_0(t)} & \dots & & & -\frac{a_1(t)}{a_0(t)} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$y(t) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

1) Observe that if φ is a soln. of

① i.e. $L_n \varphi = 0$.

Then $(\varphi, \varphi^{(1)}, \dots, \varphi^{(n-1)})$ is a soln. of ②

2) Suppose $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$ is a soln. of ②. Then

$$\varphi_2 = \varphi_1'$$
$$\varphi_3 = \varphi_2' = \varphi_1''$$
$$\vdots$$
$$\varphi_{n-1} = \varphi_1^{(n-2)}$$
$$\varphi_n = \varphi_1^{(n-1)} \text{ \& } \varphi_1 \text{ satisfies}$$
$$L_n \varphi_1 = 0$$

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$$L_n x = 0 \quad \text{— ①}$$
$$L_n = a_0 \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \dots + a_{n-1} \frac{d}{dt} + a_n$$
$$L_n x \equiv 0 \iff y' = A(t) y \quad \text{— ②}$$
$$\varphi \iff (\varphi, \varphi', \dots, \varphi^{(n-1)})$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ \frac{-a_n(t)}{a_0(t)} & \frac{-a_{n-1}(t)}{a_0(t)} & - & - & - & \frac{-a_1(t)}{a_0(t)} \end{pmatrix}$$

1) Notice that given $t_0 \in I$ & $y \in \mathbb{R}^n$, we know that there is a unique soln. φ of $y' = A(t) y$. WHY?

If we write $\varphi = (\varphi_1, \dots, \varphi_n)$ then

$$\varphi(t_0) = x_0 = (x_{01}, x_{02}, \dots, x_{0n})$$

$$\varphi(t_0) = (\varphi_1(t_0), \varphi_1'(t_0), \dots, \varphi_1^{(n-1)}(t_0))$$
$$= (x_{01}, x_{02}, \dots, x_{0n})$$
$$L_n \varphi_1 = 0$$

2) Now let $\varphi_1, \dots, \varphi_n$ be soln. of

$$L_n x = 0$$

Consider the matrix

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1^{(1)} & \varphi_2^{(1)} & \dots & \varphi_n^{(1)} \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{pmatrix} = [\Phi_1, \dots, \Phi_n]$$

This is a soln. matrix for the linear system

$$y' = A(t)y$$

The determinant of the soln. matrix is called the Wronskian associated to $\varphi_1, \dots, \varphi_n$ denoted by

$$W(\varphi_1, \dots, \varphi_n)(t) = |\Phi|$$

LEMMA: $x'(t) = A(t)x(t)$. Let $U(t)$ be a soln. matrix i.e.

$U'(t) = A(t)U(t)$. Then

$$|U(t)| = |U(t_0)| \exp\left(\int_{t_0}^{t} \mathrm{trace}(A(s))\,ds\right) \leftarrow \text{Abel's Identity}$$

$t \in I$

$$|U(t)|' = \mathrm{trace}(A(t))\,|U(t)|$$

THEOREM: Every soln. of $L_n x = 0$ is a linear combination of any n-linearly independent solutions. Given $\varphi_1, \dots, \varphi_n$, solution of $L_n x = 0$ they are lin. ind. iff

$$W(\varphi_1, \dots, \varphi_n) \neq 0 \text{ on } I$$

PROOF: Soln. set $L_n x = 0 \xleftrightarrow[\text{onto}]{\text{one-one}}$ Soln. set of $y' = A(t)y(t)$

$$L: \varphi \longmapsto (\varphi, \varphi^{(1)}, \dots, \varphi^{(n-1)})$$

$L(\varphi) = (\varphi, \varphi^{(1)}, \dots, \varphi^{(n-1)})$ is a vector space isomorphism i.e.

① L is one-one and onto.

② $L(\alpha\varphi + \beta\psi) = \alpha L(\varphi) + \beta L(\psi)$

$\alpha, \beta \in \mathbb{R}$, φ, ψ solns. of $L_n x = 0$

The soln. space of $L_n x = 0$ is n-dim.

Suppose $\varphi_1, \dots, \varphi_n$ are lin. ind. solns. of $L_n x = 0$. Then the matrix Φ is a fundamental soln. matrix.

$$W(\varphi_1, \dots, \varphi_n)(t) \neq 0 \ \forall t \in I$$

THEOREM: Let $\varphi_1, \dots, \varphi_n$ be functions on I such that $W(\varphi_1, \dots, \varphi_n)(t) \neq 0$ on I. Then there exists a unique DE of order n* for which these functions form a fundamental set, namely:

$$\frac{W(x, \varphi_1, \dots, \varphi_n)}{W(\varphi_1, \dots, \varphi_n)} = 0$$

DEF.: A set of n lin. ind. solns. of $L_n x = 0$ is called a fundamental set for $L_n x = 0$.

*where coefficients of higher order is 1.

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EXAMPLE: Consider fnc^{n}s $I = (a, b)$ not containing 0 then

$$f_1(t) = t, \quad f_2(t) = te^t$$

$$W(f_1, f_2)(t) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2 e^t$$

$W(f_1, f_2)(t) \neq 0$ for any $t \in (a, b)$

$$\Rightarrow \frac{W(x, f_1, f_2)}{W(f_1, f_2)} = 0$$

$$\begin{vmatrix} x & t & te^t \\ \dot{x} & 1 & te^t + e^t \\ \ddot{x} & 0 & te^t + 2e^t \end{vmatrix} = 0$$

$$\boxed{x(te^t + 2e^t) - \dot{x}(t^2e^t + 2te^t) + \ddot{x}(t^2e^t) = 0}$$

Reduction of order of a DE :-

$$L_n x = 0$$

$$a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_{n-1}(t)x^{(1)} + a_n(t)x = 0 \quad \text{—} \; ①$$

Say, $f(t)$ is a soln.

We write $x(t) = f(t)y(t)$

$$\frac{dx}{dt} = f'(t)y(t) + f(t)y'(t)$$

$$\vdots$$

$$\frac{d^n x}{dt^n} = f^{(n)}(t)y(t) + C_1^n f^{(n-1)}(t)y'(t) + \ldots + C_{n-1}^n f'(t)y^{(n-1)}(t) + f(t)y^{(n)}(t)$$

$$a_0(t)\left(f(t)y^{(n)}(t) + C_1^n f'(t)y^{(n-1)} + \ldots + f^{(n)}(t)y^{(1)}\right) + a_1(t)\left(f(t)y^{(n-1)}(t) + C_1^{n-1} f^{(1)}(t)y^{(n-2)}(t) + \ldots + f^{(n-1)}(t)y(t)\right) + \ldots + a_n(t)f(t)y(t) = 0$$

$$\Leftrightarrow a_0(t)f(t)y^{(n)} + \left(na_0(t)f^{(1)}(t) + (n-1)a_1(t)f(t)\right)y^{(n-1)} + \ldots + y(t)\left(a_0(t)f^{(n)}(t) + a_1(t)f^{(n-1)}(t) + \ldots + a_n(t)f(t)\right) = 0 \quad \text{—} \; ②$$

② reduces to

$$A_0(t)y^{(n)}(t) + A_1(t)y^{(n-1)}(t) + \ldots + A_{n-1}(t)y^{(1)}(t) = 0 \quad \text{—} \; ③$$

Put $w = y'(t)$

$$A_0(t)w^{(n-1)}(t) + A_1(t)w^{(n-2)}(t) + \ldots + A_{n-1}(t)w(t) = 0 \quad \text{—} \; ④$$

$\Rightarrow$ Now suppose $w_1, \ldots, w_{n-1}$ is a known fundamental set of solutions for ④. Then $v_j(t) = \int w_j(t)$, $j = 1, \ldots, n-1$ & $v_n \equiv 1$ is a fundamental set of solutions for ③

$$\Leftrightarrow \begin{vmatrix} 1 & v_1 & v_2 & \cdots & v_{n-1} \\ 0 & w_1 & w_2 & & \\ 0 & w_1^{(1)} & w_2^{(1)} & & \\ \vdots & \vdots & \vdots & & \\ 0 & w_1^{(n-1)} & w_2^{(n-1)} & & \end{vmatrix}$$

notice non-zero as $w_1, \ldots, w_{n-1}$ [I

Finally for fundamental set of ①, multiply by $f(t)$ by (*)

Inhomogeneous

$$\Rightarrow L_n(x) = g(t) \qquad x(t_0) = x_0 = (x(t_0), x'(t_0), \dots, x^{(n-1)}(t_0))$$

$$\Leftrightarrow x^{(n)}(t) + \frac{a_1(t)}{a_0(t)} x^{(n-1)}(t) + \dots + \frac{a_n(t)}{a_0(t)} x = \frac{g(t)}{a_0(t)}$$

$$\Leftrightarrow y'(t) = A(t) y(t) + \tilde{g}(t) \qquad (y_1, \dots, y_n) = (x, x', \dots, x^{(n-1)})$$

$$\tilde{g}(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{g(t)}{a_0(t)} \end{pmatrix} \qquad \begin{gathered} y(t_0) = x_0 \\ (x(t_0), x'(t_0), \dots, x^{(n-1)}(t_0)) = x_0 \end{gathered}$$

The solution is given by

$$y(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, s) \tilde{g}(s) \, ds$$

where $\Phi(t, t_0)$ is the principal solution matrix.

$$\Phi'(t, t_0) = A(t) \Phi(t, t_0) \text{ and } \Phi(t_0, t_0) = \mathbb{I}$$

$\varphi_j(t, t_0)$ is the solution of $L_n x = 0$ s.t.

$$(\varphi_j(t_0, t_0), \varphi_j^{(1)}(t_0, t_0), \dots, \varphi_j^{(n-1)}(t_0, t_0)) = e_j$$

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Class Cancelled

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Order n differential equation

$$L_n = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t)$$

$$L_n x = g(t)$$

$a_0, a_1, \dots, a_n$ are cts. func^n on $I \subset \mathbb{R}$

HOMOGENEOUS CASE

$$L_n x = 0 \Leftrightarrow \dot{y}(t) = A(t)y(t)$$

INHOMOGENEOUS CASE

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x(t) = g(t) \qquad (*)$$

By considering the change of variable

$$y = (y_1, \dots, y_n) = (x, x^{(1)}, \dots, x^{(n-1)})$$

The associated linear system is given by

$$y'(t) = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & & \cdots & & -\frac{a_1}{a_0} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \overset{\tilde{g}(t)}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \frac{g(t)}{a_0(t)} \end{pmatrix}}$$

This linear system has a unique soln., given $y(t_0) = y_0 \in \mathbb{R}^n$

$\Rightarrow$ This implies that given $t_0 \in I$, $y_0 \in \mathbb{R}^n$ $\exists$ unique soln. with $(x(t_0), x^{(1)}(t_0), \dots, x^{(n-1)}(t_0)) = y_0$

The unique soln. of $(**)$ is given by

$$y(t) = \underbrace{\Phi(t,t_0)}_{\text{Principal soln. matrix}} y_0 + \int_{t_0}^{t} \Phi(t,s)\,\tilde{g}(s)\,ds$$

First coordinate of ① given by

$$x(t) = \sum_{j=1}^{n} \Phi_{1j}(t,t_0)\, y_{0,j} + \int_{t_0}^{t} \Phi_{1n}(t,s)\, \frac{g(s)}{a_0(s)}\, ds$$

$$\Phi(t,s) = [\Phi_1(t,s) \; \ldots\ldots \; \Phi_n(t,s)]$$

$$\Phi_1'(t,s) = A(t)\,\Phi_1(t,s)$$

$$\Phi_1(s,s) = (1,0,\ldots,0)$$

$$(\Phi_{ij}(t_0,t_0), \Phi_{ij}'(t_0,t_0), \ldots, \Phi_{ij}^{(n-1)}(t_0,t_0)) = (0,\ldots,1,\ldots,0)$$

Linear Equations of order n with constant coefficients :-

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = g(t)$$

Here $a_1, \ldots, a_n$ are constants

HOMOGENEOUS CASE

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = 0 \quad \text{—— ①}$$

$y' = Ay$ Associated linear system

LEMMA: The characteristic polynomial of matrix A is given by:

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$$

THEOREM:

Consider ①. Let $\lambda_1, \ldots, \lambda_s$ be distinct eigenvalues of matrix A. Let m_i be the multiplicity of λ_i as a zero of $f(\lambda)$. Then the set of functions $t^k e^{t\lambda_i}$, $k = 0, \ldots, m_i - 1$ form a fundamental set of solns.

PROOF: $L_n(e^{t\lambda}) = f(\lambda)e^{t\lambda}$ —— ①

$$L_n(t^k e^{t\lambda}) = L_n\left(\frac{\partial^k}{\partial\lambda^k} e^{t\lambda}\right)$$

$$= \frac{\partial^k}{\partial\lambda^k}\left(L_n(e^{t\lambda})\right) = \frac{\partial^k}{\partial\lambda^k}\left(f(\lambda)e^{t\lambda}\right)$$

$$= \left(f^{(k)}(\lambda) + k f^{(k-1)}(\lambda)t + \frac{k(k-1)}{2} f^{(k-2)}(\lambda)t^2 + \ldots + f(\lambda)t^k\right)e^{t\lambda} \quad \text{—— ②}$$

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Ordinary differential equation of order n with constant coefficients

$$L_n x = x^{(n)} + a_1 x^{(n-1)} + \dots + a_n x = 0$$
$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$$
$$L_n = \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \dots + a_n$$

THEOREM: Let $\lambda_1, \dots, \lambda_s$ be distinct zeros of the polynomial $f(\lambda)$. Also let m_i be the multiplicity of λ_i as a zero of f.
$$(f(\lambda) = (\lambda - \lambda_i)^{m_i} g(\lambda))$$
Then $t^k e^{t\lambda_i}$, $k = 0, \dots, m_i - 1$ are fundamental solutions of ①.

FACT: λ_i has multiplicity m_i if the derivatives of f at λ_i vanish upto order $m_i - 1$.

PROOF: $\frac{d^k}{dt^k}(e^{t\lambda}) = \lambda^k e^{t\lambda}$

$$\boxed{L_n(e^{t\lambda}) = f(\lambda)e^{t\lambda}} \text{ — ①}$$

$$L_n(t^k e^{t\lambda}) = L_n\left(\frac{\partial^k}{\partial\lambda^k} e^{t\lambda}\right) = \frac{\partial^k}{\partial\lambda^k} L_n(e^{t\lambda}) = \frac{\partial^k}{\partial\lambda^k}(f(\lambda)e^{t\lambda})$$

Periodic Linear Systems :-

$$\dot{x}(t) = A(t)x(t) \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$
$$A(t+\omega) = A(t) \ \forall t \ \& \ \omega > 0$$

EXAMPLE: $\dot{x} = \sin^2 t \, x$ $\qquad x(t) = c \exp(t/2) \exp\left(-\frac{\sin 2t}{4}\right)$

$A(t) = \sin^2 t$ $\qquad c \neq 0$

$A(t+\pi) = A(t)$ $\qquad \dot{x} = \sin t \, x(t)$

THEOREM (FLOQUET):- Consider the linear system
$$\dot{x}(t) = A(t)x(t) \text{ — ①}$$
where $A: \mathbb{R} \to M_n(\mathbb{R})$ continuous with the least positive period ω, i.e,
$$A(t) = A(t+\omega) \ \forall t \in \mathbb{R}.$$
Let ψ be a fundamental solution matrix of ① then
$$\psi_\omega(t) = \psi(\omega + t)$$
is a fundamental solution matrix. Furthermore,
$$\psi(t) = P(t)e^{Bt}$$
where $P: I \to M_n(\mathbb{R})$ is a continuously differentiable periodic function with period ω and B is a complex matrix (constant)

THEOREM: Let C be a nonsingular matrix. Then $\exists B \in M_n(\mathbb{C})$ s.t.
$$e^B = C$$

PROOF $n=2$ $C=PJP^{-1}$ $J=\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}$

Case 1: Say, C diagonalisable; $J=\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}$

Take $B=\begin{pmatrix}\log\mu_1 & 0\\ 0 & \log\mu_2\end{pmatrix}$

or

$J=\begin{pmatrix}\mu_1 & 1\\ 0 & \mu_2\end{pmatrix}$

also $P^{-1}e^{B}P=e^{P^{-1}BP}$ Take $\widetilde{B}=P^{-1}BP$

$C=PJP^{-1}$

$C=Pe^{K}P^{-1}=e^{PKP^{-1}}$ $B=PKP^{-1}=P\begin{pmatrix}\log\mu_1 & 0\\ 0 & \log\mu_2\end{pmatrix}P^{-1}$

Case 2: $J=\begin{pmatrix}\mu & 1\\ 0 & \mu\end{pmatrix}$

$K=\begin{pmatrix}a_1 & a_2\\ 0 & a_1\end{pmatrix}$ $e^{K}=\begin{pmatrix}e^{a_1} & a_2e^{a_1}\\ 0 & e^{a_1}\end{pmatrix}$

$K=\begin{pmatrix}\log\mu & 1/\mu\\ 0 & \log\mu\end{pmatrix}$ $\mu\neq 0$

$B=PKP^{-1}$

Jordan Canonical Form

$$J=\begin{bmatrix}J_1 & 0 & & \\ 0 & J_2 & & \\ & & \ddots & 0\\ & & 0 & J_k\end{bmatrix}$$

where for each i

① $J_i=[\lambda]$ ② $J_i=\begin{bmatrix}\lambda_1 & 1 & 0 & 0\\ 0 & \lambda_2 & 1 & 0\\ \vdots & & \ddots & 1\\ 0 & 0 & 0 & \lambda_n\end{bmatrix}$

$J=\bigoplus_{i=1}^{k}J_i$ $K=\bigoplus_{i=1}^{k}K_i$

$$\exp\begin{bmatrix}K_1 & & \\ & K_2 \ddots & \\ & & K_k\end{bmatrix}=\begin{bmatrix}\exp(K_1) & 0 & 0 & 0\\ 0 & \exp(K_2) & 0 & 0\\ & & \ddots & 0\\ 0 & 0 & 0 & \exp(K_k)\end{bmatrix}$$

PROOF OF FLOUQUET'S THEOREM:-

$$\psi_\omega'(t)=\psi'(t+\omega)=A(t+\omega)\psi(t+\omega)$$
$$=A(t)\psi(t+\omega)=A(t)\psi_\omega(t)$$

$\Rightarrow \psi_\omega$ is a soln. matrix

$\det(\psi_\omega(t))=\det(\psi(\omega+t))\neq 0$

$\Rightarrow \psi(t+\omega)$ is a FSM.

Fix a point $t_0\in\mathbb{R}$.

$C_0=\psi^{-1}(t_0)\psi_\omega(t_0)$

Note C_0 is a non-singular matrix.

$\psi_\omega(t_0)=\psi(t_0)C$

Now observe $\psi(t)C$ is a FSM and agrees with $\psi_\omega(t)$ at t_0. Hence,

$\psi_\omega(t)=\psi(t)C$

$\psi(t+\omega)=\psi(t)C$ $\forall t\in\mathbb{R}$

By the existence of log for $M\in GL_n(\mathbb{R})$

$\exists B\in M_n(\mathbb{C})$ s.t. $C=\exp(B\omega)$

Consider $P(t)=\psi(t)\exp(-Bt)$ $\psi(t)=P(t)\exp(Bt)$

$$P(t+\omega) = \psi(t+\omega)\exp(-B(t+\omega)) = \psi(t)\exp(-Bt)\exp(-B\omega)\, C$$
$$= \psi(t)\exp(-Bt) = P(t)$$

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Stability of (LHS):-

$$\dot{x}(t) = A(t)x(t)$$

In the 2×2 case, A is a constant matrix.

$$\dot{x}(t) = A(t)x(t)$$

Case 1. A is diagonalizable with eigenvalues α_1 & α_2.

$$x(t) = ae^{\alpha_1 t}u_1 + be^{\alpha_2 t}u_2$$

$$\|x(t)\|^2 = e^{2\alpha_1 t}\|u_1\|^2 a^2 + e^{2\alpha_2 t}\|u_2\|^2 |b|^2 + 2e^{(\alpha_1+\alpha_2)t} ab\langle u_1, u_2\rangle$$

$$\leq \left(e^{\alpha_1 t}|a|\,\|u_1\| + e^{\alpha_2 t}|b|\,\|u_2\|\right)^2 < \epsilon$$

$$e^{\alpha_1 t}|a|\,\|u_1\| + e^{\alpha_2 t}|b|\,\|u_2\| < \sqrt{\epsilon}$$

Suppose $\alpha_1, \alpha_2 < 0$,
then $|a|\,\|u_1\| + |b|\,\|u_2\| < \sqrt{\epsilon}$

THEOREM: Consider $\dot{x}(t) = A(t)x(t)$ where A is an $n \times n$ real matrix. Then

a) Suppose $\exists \lambda$, an eigenvalue of A such that $Re(\lambda) > 0$ then the trivial soln. is unstable

b) Suppose all the eigenvalues with zero real part are simple, and all other eigenvalues have negative real part, then the solution is stable.

c) If all eigenvalues of A have negative real part then the solution is asymptotically stable.

Periodic System:-

$$\dot{x}(t) = A(t)x(t) \qquad A(t+\omega) = A(t)$$

THEOREM: Let $u_1, \ldots, u_n$ be the Floquet multipliers of the system. Then the following holds true.

① If $|u_i| < 1 \ \forall\, 1 \leq i \leq n$ then the trivial soln. is asymptotically stable.

② Stable if $\forall i, |u_i| \leq 1$

③ Unstable if $\exists i, |u_i| > 1$

Kellert & Peterson

Boundary Value Problems :

$$\boxed{\frac{\partial^2 u(t,x)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(t,x)}{\partial t^2}}$$ One-dim. wave eqn.

$u(t,x)$:= disp. of the start at x, and time t,
c := propogation speed

$$u(t,0)=0, \quad u(t,1)=0$$
$$u(0,x)=u(x), \quad \frac{\partial u}{\partial t}(0,x)=V(x)$$
$$u(t,x)=W(t)y(x)$$

$$\frac{1}{c^2}\frac{W''(t)}{W(t)} = \frac{y''(x)}{y(x)} = -\lambda$$
$$\Rightarrow y''(x) = -\lambda y(x) - ① \quad y(0)=0, \ y(1)=0$$
$$W''(t) = -c^2\lambda W(t) - ②$$

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Stability of Autonomous System

THEOREM : Consider $x'(t)=Ax(t)$. Then the trivial soln.

Stable $\Leftrightarrow$ for any $\lambda \in \sigma(A)$, $Re(\lambda) \le 0$ and $m_g(\lambda) = m_a(\lambda) \ \forall \lambda$ with $Re(\lambda)=0$.

Unstable $\Leftrightarrow$ $\exists \lambda \in \sigma(A)$ s.t. $Re(\lambda)>0$ or $\exists \lambda \in \sigma(A)$ s.t. $Re(\lambda)=0$ & $m_a(\lambda) > m_g(\lambda) \geq 1$

Asymptotic Stable $\Leftrightarrow$ $Re(\lambda)<0 \ \forall \lambda \in \sigma(A)$

Here, $A \in M_n(\mathbb{R})$
$\sigma(A)$:= the set of all eigenvalues of A
$\lambda \in \sigma(A)$
$m_g(\lambda)$:= the geometric multiplicity of λ
$= \dim(\ker A - \lambda \mathbb{I})$
$m_a(\lambda)$:= algebraic multiplicity of λ

Sturm Boundary Value Problems :-

$$\frac{\partial^2}{\partial x^2}u(t,x) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(t,x)$$ One-dim.

$u(0,x)=u(x)$
$\frac{\partial}{\partial x}u(0,x)=v(x)$
$u(t,0)=0$, $u(t,1)=0$

$$u(t,x) = W(t)y(x)$$
$$\Rightarrow \frac{1}{c^2}\frac{W''(t)}{W(t)} = \frac{y''(x)}{y(x)} = -\lambda \text{ (a constant)}$$

$$\Rightarrow W''(t) + c^2 \lambda W(t) = 0$$

$$\boxed{y''(x) + \lambda y(x) = 0 \quad , y(0) = 0, \; y(1) = 0} \qquad (1)$$

$$\varsigma^2 + c^2 \lambda = 0$$

Case (1) : $\lambda > 0$

$$\pm i c \sqrt{\lambda}$$

$$\alpha e^{ic\sqrt{\lambda}} + \beta e^{-ic\sqrt{\lambda}}$$

$$W(t) = \alpha \cos c\sqrt{\lambda} t \; + \beta \sin c\sqrt{\lambda} t$$

$$\varsigma^2 + \lambda = 0$$

$$y(x) = \gamma \cos \sqrt{\lambda} x + \delta \sin \sqrt{\lambda} x$$

$y(0) = 0 \Rightarrow \gamma = 0$

$$y(1) = 0$$

$$\delta \sin \sqrt{\lambda} = 0$$

$$\delta = 0 \text{ or } \sin \sqrt{\lambda} = 0$$

$$\Rightarrow \sqrt{\lambda} = n\pi \Rightarrow \lambda = n^2 \pi^2$$

$$\boxed{U_n(t, x) = (\alpha \cos cn\pi t + \beta \sin cn\pi t) \sin n\pi x}$$

DEFINITION (ADJOINT):-

Consider second order ODE,

$$a_0(t) \frac{d^2 x}{dt^2} + a_1(t) \frac{dx}{dt} + a_2(t) x(t) = 0 \quad (1)$$

Here: a_0 is twice continuously differentiable & $a_0(t) \neq 0$ on I.
a_1 is continuously differentiable.
a_2 is continuous.

The adjoint of (1) is the differential equation:

$$\frac{d^2}{dt^2}(a_0(t) x(t)) - \frac{d}{dt}(a_1(t) x(t)) + a_2(t) x(t) = 0$$

$$\Leftrightarrow a_0(t) \frac{d^2 x}{dt^2} + [2a_0'(t) - a_1(t)] \frac{dx}{dt} + (a_0''(t) - a_1'(t) + a_2(t)) x(t) = 0$$

DEFINITION: (1) is called *self-adjoint* if (1) and its adjoint eqns. are identical. $\Leftrightarrow 2a_0'(t) - a_1(t) = a_1(t)$

$$\Leftrightarrow a_0'(t) = a_1(t)$$

$\Leftrightarrow$ (1) can be written in the following form:-

$$\frac{d}{dt}\left(a_0(t) \frac{d}{dx}\right) + a_2(t) x(t) = 0$$

EXAMPLE: $(1 - t^2) \frac{d^2 x}{dt^2} - 2t \frac{dx}{dt} + n(n+1) x(t) = 0$

HOMEWORK:

$$a_0(t)\frac{d^2x}{dt^2}+a_1(t)\frac{dx}{dt}+a_0(t)x(t)=0 \quad \text{———} (*)$$

$\frac{d}{dt}\left(p(t)\frac{dx}{dt}\right)+Q(t)x(t)=0$ by multiplying $(*)$ by

$\frac{1}{a_0(t)}\exp\int\frac{a_1(t)}{a_0(t)}dt$

$$P(t)=\exp\left(\int\frac{a_1(t)}{a_0(t)}dt\right)$$

$$Q(t)=\frac{a_2(t)}{a_0(t)}\exp\left(\int\frac{a_1(t)}{a_0(t)}dt\right)$$

From now on the general, second-order, self-adjoint ordinary DE that we will consider:

$$(p(t)x')'+q(t)x(t)=h(t)$$

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Consider the self-adjoint equation:

$$(p(t)x')'+q(t)x(t)=h(t)$$

Assuming that $p(t), q(t), h(t)$ are continuous on I.

$\mathcal{D}:=\{x\in C(I):$

DEFINITION: $x(t)$ & $y(t)$ are differentiable fncs, then we define

$$W(x(t),y(t))=\det\begin{pmatrix}x(t) & y(t)\\ x'(t) & y'(t)\end{pmatrix}$$

Lagrange's Identity: Suppose $x,y\in\mathcal{D}$ Then

$$x(t)L(y(t))-y(t)L(x(t))=[p(t)W(x(t),y(t))]' \quad \forall t\in I$$

Corollary: If x & y are solns of $L(x(t))\equiv 0$

then $W(x(t),y(t))=\frac{C}{p(t)}$, where C is a constant $\forall t\in I$.

Strum-Liouville Boundary Problems:-

$$(p(t)x')'+(\lambda r(t)+q(t))x(t)=0$$

together with boundary conditions

$\alpha x(a)-\beta x'(a)=0$

$\gamma x(b)-\delta x'(b)=0$

(CSLP) $\quad p(t)>0 \quad r(t)\geq 0 \quad I=[a,b] \quad \alpha^2+\beta^2>0 \quad \gamma^2+\delta^2>0$

$$\boxed{L(x(t)) = (p(t)x')' + q(t)x(t)}$$

DEFINITION: $\lambda_0 \in \mathbb{R}$ is called an eigenvalue of (SLP) if $\exists x_0$, a nontrivial fnc^n s.t.

$$Lx_0(t) = -\lambda\gamma(t)x_0(t)$$

satisfying the boundary condition.

(λ_0, x_0) an eigenpair, x_0 is called the eigen function corresponding to λ_0.

Suppose, given a λ_0 an eigenvalue of (SLP) if any two eigenfnc^{n}s corresponding to λ_0 are lin. dependent, then λ_0 is called simple.

EXAMPLE: $x'' = -\lambda x$, $x(0)=0$, $x(\pi)=0$

Case I: $\lambda > 0$, $\lambda = \mu^2$

$$x'' + \mu^2 x = 0$$

$$t^2 + \mu^2 = 0 \qquad t = \pm\mu$$

$$x(t) = \alpha e^{i\mu t} + \beta e^{-i\mu t}$$

$$= \alpha\cos\mu t + \beta\sin\mu t$$

$$0 = \alpha, \quad 0 = \beta\sin\pi\mu$$

$$\beta \neq 0, \quad \sin\pi\mu = 0 \qquad \pi\mu = \pm n\pi$$

$$\mu = \pm n$$

$$\lambda = \mu^2 = n^2$$

$$\lambda = n^2, \qquad n_\lambda(t) = \pm\sin nt$$

Case II: $\lambda = 0$.

$$x'' = 0 \qquad x(t) = \alpha + \beta t$$

$$x(0) = 0 \Rightarrow \alpha = 0$$

$$x(\pi) = 0, \ \beta\pi = 0, \ \beta = 0$$

Case III: $\lambda = -\mu^2$

$$\begin{pmatrix} 1 & 1 \\ e^{\pi\mu} & e^{-\pi\mu} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$x'' - \mu^2 x = 0$$

$$x(t) = \alpha e^{\mu t} + \beta e^{-\mu t}$$

$$x(0) = 0$$

$$\alpha + \beta = 0$$

$$\alpha e^{\pi\mu} + \beta e^{-\pi\mu} = 0$$

DEFINITION: Let $\gamma \in C([a,b], \mathbb{R})$ & assume $\gamma \geq 0$ & not identically equal to zero.

Then on $C([a,b], \mathbb{R})$, we introduce the inner product, defined by $\langle x(t), y(t)\rangle_\gamma = \int_a^b x(t)y(t)\gamma(t)\,dt$

x & y are orthogonal w.r.t. γ if $\langle x(t), y(t)\rangle_\gamma = 0$

THEOREM: $Lx(t) = -\lambda\gamma(t)x(t)$

$$\alpha x(a) - \beta x'(a) = 0 \qquad \alpha^2 + \beta$$

$$\gamma x(b) - \delta x'(b) = 0$$

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THEOREM: Consider the (SLP)

$$(px')' + (\lambda r(t) + q(t))\,x = 0$$

$$\left.\begin{aligned} \alpha x(a) - \beta x'(a) &= 0 \\ \delta x(b) - \delta x'(b) &= 0 \end{aligned}\right\} \text{(BV)}$$

When $p > 0$, $r \geqslant 0$ and $q \in C[a,b]$

The eigenfunctions corresponding to distinct eigen values are orthogonal.
Each eigen value is simple.

PROOF: (λ_1, x_1) (λ_2, x_2)

$$(\lambda_1 - \lambda_2)\langle x_1, x_2\rangle_r = \left[p(t)\,W(x_1(t), x_2(t))\right]_a^b$$

$$W(x_1(t), x_2(t))\Big|_{t=a} = x_1(a)\,x_2'(a) - x_2(a)\,x_1'(a)$$

$$= x_1(a)\cancel{\frac{\alpha x_2(a)}{\beta}} - x_2(a)\cancel{\frac{\alpha x_1(a)}{\beta}} = 0$$

$$\alpha x_j(a) - \beta x_j'(a) = 0$$

$$\beta = 0,\; W(x_1(t), x_2(t))\Big|_{t=a} = 0$$

$(\lambda_1 - \lambda_2)\langle x_1, x_2\rangle_\gamma = 0$

$\lambda_1 \neq \lambda_2 \Rightarrow \langle x_1, x_2\rangle_\gamma = 0$

$(\lambda_1, x_1), (\lambda_2, x_2)$

$(\lambda_1 - \lambda_2)\,\gamma(t) x_1(t) x_2(t) = [p(t) W(x_1, x_2)]$

$\Rightarrow p(t) W(x_1, x_2) = C$

From boundary condi^n, $C = 0 \Rightarrow W(x_1, x_2) = 0$

$\Rightarrow x_1$ and x_2 are l.d.

STRUM-LOUIVILLE THEOREM:

Then the (SLP) has an infinite seq. of eigenvalues (λ_n) with $\lambda_n < \lambda_{n+1}$ & $\lim_{n\to\infty} \lambda_n = \infty$. The eigenfunc^n corresponding to each λ_n has precisely $(n-1)$ zeros in $[a, b]$.

Oscillation Theory:

Consider

$$(Px')' + Q(t)x = 0$$

Prüfer Substitution:

We write

$$px' = \gamma(t)\cos\theta(t)$$

$$x = \gamma(t)\sin\theta(t)$$

$$(px')^2 + x^2 = (\gamma(t))^2$$

$$\tan\theta(t) = \frac{x}{px'}$$

$$\cot\theta(t) = \frac{px'}{x}$$

$$-\operatorname{cosec}^2\theta(t)\frac{d\theta}{dt} = \frac{(Px')'}{x} - \frac{Px'^2}{x^2}$$

$$= -Q(t) - \frac{1}{P}\frac{\gamma(t)^2\cos^2\theta(t)}{\gamma(t)^2\sin^2\theta(t)}$$

$$2\gamma(t)\gamma'(t) = 2Px'(Px')' + 2xx'$$

$$= 2Px'(-Q(t)x(t)) + 2xx'$$

$$= 2\gamma^2(t)\cos\theta(t) - Q(t)\sin\theta(t) + \frac{2\gamma^2(t)}{P}\sin\theta(t)\cos\theta(t)$$

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= Q(t)\sin^2\theta + \frac{1}{P(t)}\cos^2\theta \\ &= F(t,\theta) \quad \text{— ①} \\ \frac{d\gamma}{dt} &= \frac{1}{2}\gamma\sin 2\theta\left(\frac{1}{P(t)} - Q(t)\right) \quad \text{— ②} \end{aligned}}$$

PRÜFER SYSTEM

LEMMA: If (SLP) has non-trivial soln., $r(t)$ never vanishes

Consider the self-adjoint equation

$$(Px')' + Q(t)x(t) = 0 \quad \text{— (SA)}$$

We say (SA) is oscillatory if it admits a soln. that has more than one zero.

OBSERVATION: If x is a soln. of (SA) on $[a,b]$ and let $t_0 \in [a,b]$ s.t. $x(t_0) = 0$ then $\theta(t_0) = m\pi, m \in \mathbb{Z}$

PROPOSITION: For each $m \in \mathbb{Z}$, $\exists$ atmost one point $t_m \in [a,b]$ s.t. $\theta(t_m) = m\pi$. Moreover if $t < t_m$ then $\theta(t) < m\pi$ and if $t > t_m$, $\theta(t) > m\pi$

PROOF: Consider the constant fnc^n

$$y(t) = m\pi$$

$$y'(t) = 0 \qquad F(t, m\pi) = \frac{1}{P(t)} > 0 = y'(t)$$

Suppose $t_m \in [a,b]$ is such that

$$\theta(t_m) = m\pi = y(t_m)$$
$$\theta(t) > m\pi \ \forall t > t_m$$

Say, $\tilde{t}_m$ s.t. $\theta(\tilde{t}_m) = m\pi$, $\tilde{t}_m < t_m \Rightarrow \theta(t_m) > m\pi$ (*)

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Consider θ-equations corresponding to two Prüfer systems:

$$\hat{\theta}' = \hat{Q}\sin^2\hat{\theta} + \frac{1}{\hat{P}}\cos^2\hat{\theta} \qquad \hat{\theta}(a) = \hat{\theta}_0$$

$$\theta' = Q\sin^2\theta + \frac{1}{P}\cos^2\theta = F(t,\theta) \quad \theta(a) = \theta_0$$

PROPOSITION 1: Suppose $Q(t) \geq \hat{Q}(t) \ \forall t \in (a,b)$ and $\frac{1}{P(t)} \geq \frac{1}{Q(t)} \ \forall t \in (a,b)$ and assume that $\theta_0 \geq \hat{\theta}_0$. Then $\theta(t) \geq \hat{\theta}(t) \ \forall t \in [a,b]$. Moreover, if $\exists t_* (a,b]$ s.t. $\theta(t_*) = \hat{\theta}(t_*)$ then $\theta(t) = \hat{\theta}(t) \ \forall t \in [a, t_*]$

PROOF: $\hat{\theta}' - F(t,\hat{\theta}) = \hat{Q}\sin^2\hat{\theta} + \frac{1}{\hat{P}}\cos^2\hat{\theta} - \left(Q\sin^2\hat{\theta} + \frac{1}{P}\cos^2\hat{\theta}\right)$

$$= \sin^2\hat{\theta}(\hat{Q} - Q) + \cos^2\hat{\theta}\left(\frac{1}{\hat{P}} - \frac{1}{P}\right) \leq 0 = \theta' - F(t,\theta)$$

Since $\hat{\theta}(a) \leq \theta(a) \Rightarrow \theta(t) \geq \hat{\theta}(t) \ \forall t \in [a,b]$

COROLLARY: Suppose in Proposition 1, we have either $Q(t) > \hat{Q}(t) \ \forall t \in (a,b)$ or $\frac{1}{P(t)} > \frac{1}{\hat{P}(t)} \ \forall t \in (a,b)$. Then $\theta(t) > \hat{\theta}(t) \ \forall t \in (a,b)$

PROOF: Suppose $Q(t) > \hat{Q}(t) \ \forall t \in (a,b)$. Suppose $\exists t_* \in (a,b]$ s.t. $\theta(t_*) = \hat{\theta}(t_*)$

$\theta(t) = \hat{\theta}(t) \ \forall t \in [a, t_*]$,

$$\underbrace{(Q-\hat{Q})}_{\geq 0}\sin^2\theta + \underbrace{\left(\frac{1}{P} - \frac{1}{\hat{P}}\right)}_{\geq 0}\cos^2\theta = 0 \quad \text{on } [a, t_*]$$

$$\Rightarrow \sin^2\theta(t) = 0, \ \cos^2\theta(t) = 1$$

$\Rightarrow \theta$ is constant

$\theta' = \frac{1}{P}$, θ is strictly increasing (*)

STURM COMPARISON THEOREM:-

$$\underbrace{\left(\hat{P}\frac{d\hat{x}}{dt}\right)' + \hat{Q}\hat{x} = 0}_{\mathrm{I}} \ \& \ \underbrace{\left(P\frac{dx}{dt}\right)' + Qx = 0}_{\mathrm{II}}$$

Suppose $Q(t) \geq \hat{Q}(t)$ & $P(t) \leq \hat{P}(t) \ \forall t \in [a,b]$. Now, let $\hat{x}$ be a nontrivial soln. of I that has zeros at $c, d \in [a,b]$. Let x be a soln. of II then x_2 has a zero in the interval $[c,d]$. This implies if $\hat{x}$ has n zeros in $[a,b]$ then x has atleast $(n-1)$ zeros in the interval $[a,b]$.

$$\hat{x}(c) = 0 = \hat{r}(t)\sin\hat{\theta}(c) \Rightarrow \hat{\theta}(c) = m\pi, m \in \mathbb{Z}.$$

$$\hat{x}(d) = 0 = \hat{r}(t)\cos\hat{\theta}(d) \Rightarrow \hat{\theta}(d) = n\pi, n \in \mathbb{Z}$$

We will assume $\hat{\theta}(c) = 0$, $\hat{\theta}(d) = n\pi$, $n \in \mathbb{Z}_+$.

Consider the $\hat{\theta}$ and θ eqns. with initial condi^{n}s

$$\hat{\theta}(c) = 0$$

$$\theta(c) \in [0, \pi)$$

Then by Proposition 1, $\theta(t) \geq \hat{\theta}(t) \ \forall t \in [c,b]$

$$\theta(d) \geq \hat{\theta}(d) = n\pi, \ n \in \mathbb{Z}^+$$

$$\Rightarrow \exists t \in [c,d] \text{ s.t. } \theta(t) = n\pi$$

$$x(t) = \sin\theta(t) = 0$$

First part