Lecture 16 (Laplace Transform)

Recall

- $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} \, dt, \quad p > 0$
- $\Gamma(p+1) = p\,\Gamma(p)$
- $\Gamma(1) = 1, \quad \Gamma(n+1) = \lfloor n \quad n \in \mathbb{N}$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(p)$ for $p < 0$ (& $p \notin \mathbb{Z}$) are defined using the relation $\Gamma(p) = \frac{\Gamma(p+1)}{p}$
- $\Gamma(0)$ & $\Gamma(-n)$ $\underline{\text{not defined}}$

  n, +ve integer.

- $\Gamma\left(-\frac{1}{2}\right) = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$
- $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} \, dt$

$t = u^2$

$$= 2\int_0^\infty e^{-u^2} \, du = \sqrt{\pi}$$

$$I = \int_0^\infty e^{-x^2} \, dx \qquad I = \int_0^\infty e^{-y^2} \, dy$$

$$I^2 = I \cdot I = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r \, dr \, d\theta = \frac{\pi}{4}$$

Power Series

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a(n)\, x^n = A(x)$$

Thm $\left\{ a(n) \right\} \rightsquigarrow A(x)$

$a(n) = 1 \rightsquigarrow \frac{1}{1-x}$ $\quad |x| < 1$

$a(n) = \frac{1}{\lfloor n} \rightsquigarrow e^x$ $\quad x \in \mathbb{R}$

Continuous analog

$\{ a(t) \}$ $0 \leq t < \infty$

$$A(x) = \int_0^{\infty} a(t)\, x^t \, dt$$

$$= \int_0^{\infty} a(t)\, e^{t \log x} \, dt$$

$$= \int_0^{\infty} a(t)\, e^{-st} \, dt$$

$x = e^{\log x}$

$x^t = e^{t \log x}$

$0 < x < 1$

Thm for.

$f(t) : [0 \;\; \infty) \to \mathbb{R}$

$\log x < 0$

$\log x = -s$

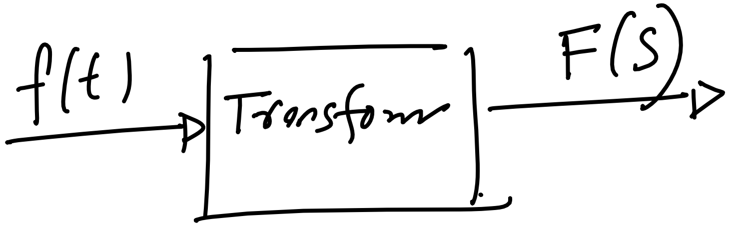
define

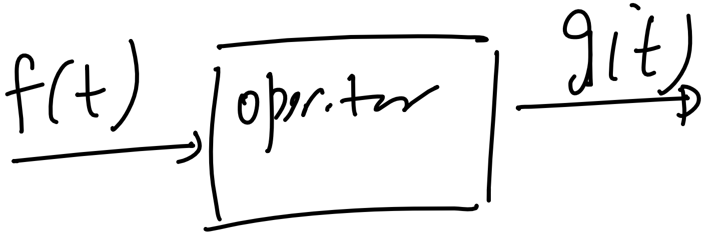
$$\boxed{F(s) = \int_0^{\infty} f(t)\, e^{-st} \, dt}$$

whenever the R.H.S integral converge.

LAPLACE TRANSFORM.

$F(s) = \mathcal{L}(f(t))$.





$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}(f(t))$$

- <u>Linear transformation</u>

$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$$

$$\mathcal{L}(cf) = c\,\mathcal{L}(f).$$

<u>Example</u>

① $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st} \underbrace{f(t)}_{1}\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

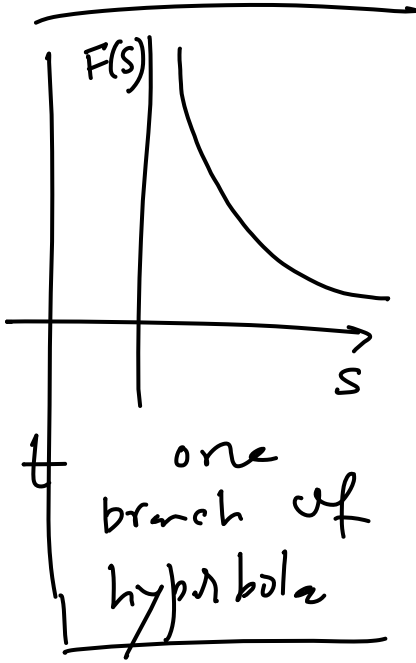
$$= \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{-st}}{-s}\right]_{t=0}^{R}$$

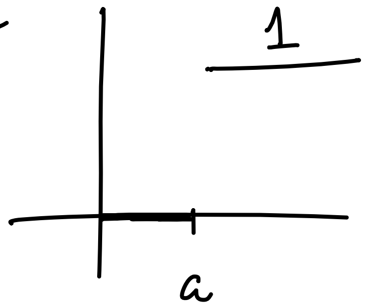
$$= \lim_{R\to\infty} \left[\frac{1 - e^{-sR}}{s}\right]$$

$$= \frac{1}{s} \quad \text{for } s>0$$

$$\mathcal{L}(1) = \frac{1}{s} \quad \text{for } s>0$$



② $f(t) = \begin{cases} 0 & 0 \le t \le a \\ 1 & t > a \end{cases}$



$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_{t=a}^{\infty} = \frac{e^{-as}}{s} \quad \text{for } s>0$$

③ $f(t) = t$

$$F(s) = \frac{1}{s^2} \quad s>0$$

$f(t) = t^2$

$$F(s) = \frac{\lfloor 2}{s^3} \quad s>0$$

$f(t) = t^n \qquad F(s) = \frac{\lfloor n}{s^{n+1}} \quad s>0.$

④ More generally

$f(t) = t^a \qquad a > -1$

$$F(s) = \int_0^\infty e^{-st}\, t^a\, dt$$

$st = u$

$$= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s}$$

$$= \frac{1}{s^{a+1}} \underbrace{\int_0^\infty e^{-u} u^a\, du}$$

$$= \frac{\Gamma(a+1)}{s^{a+1}} \quad \text{for } \underline{s>0}$$

⑤ $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-t(s-a)} \, dt$$

$$= -\frac{1}{s-a} \left[e^{-t(s-a)} \right]_{t=0}^{\infty}$$

$$= \frac{1}{s-a} \quad \text{for } \underline{s > a}$$

⑥ • $f(t) = \cos(at)$

$$F(s) = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \frac{s}{s^2 + a^2} \quad \text{for } s > 0$$

$$\left(\int \cos at \, e^{bt} \, dt = \frac{e^{bt}}{a^2 + b^2} \left(b \cos(at) + a \sin(at) \right) \right)$$

• $f(t) = \sin(at)$

$$F(s) = \frac{a}{s^2 + a^2} \quad \text{for } s > 0$$

Existence of Laplace Transform

Theorem.

- f is piecewise continuous on $[0, \infty)$
- f is of 'exponential order' $\exists\, M, c, t_0$ s.t.

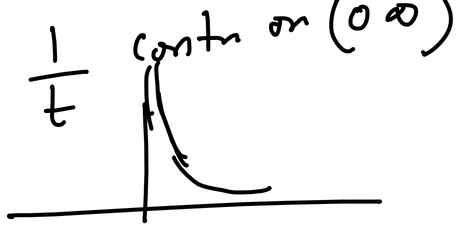
$$|f(t)| \leq M e^{ct} \quad \text{for } \forall\, t \geq t_0.$$

$\Rightarrow$ The Laplace transform $F(s)$ of f exists for $s > s_0$ where s_0 depends on f.

Remark The above conditions are sufficient but <u>not</u> necessary.

$f(t) = \dfrac{1}{\sqrt{t}}$ — we know $F(s)$ exists for $s > 0$ from example 4.

But $f(t)$ is <u>NOT</u> piecewise continuous.



Thm If $\mathcal{L}(F) = f$, then

$$F(s) \longrightarrow 0 \text{ as } s \to \infty$$

Example $\dfrac{s}{s-1}$ or $\dfrac{s^2}{1+s^2}$ are NOT Laplace transform of any function f.

First shift formula

$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)$$

$$\Rightarrow \mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

Proof

$$\mathcal{L}\left(e^{at} f(t)\right) := \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a) \quad \blacksquare$$

Example

$$\mathcal{L}\left(\cos(4t)\right) = \frac{s}{s^2+16}$$

$$\mathcal{L}\left(e^{-5t}\cos 4t\right) = \frac{s+5}{(s+5)^2+16} \quad \blacksquare$$