Assignment 0

MTH 403, 2025

1 Warm Up Exercises

1.1 Basic Airthmetic

1. Express the following numbers in the form of $x + iy$.
   (i) $(1 + i)(1 - i)$
   (ii) $\frac{1}{1+4i}$
   (iii) $(1 \pm i)^2$

2. If $z = \bar{z}$ then show that $z \in \mathbb{R}$. If $z = -\bar{z}$, then z is purely imaginary.

3. If $z = x + iy$ then show that $|x| \leq |z|$ and $|y| \leq |z|$.

4. Find imaginary and real part of $(1 + \sqrt{3}i)^{99}$.

5. Prove the following

$$|z| - |\omega| \leq |z - \omega| \leq |z + \omega|.$$

When does the equality hold?

1.2 Polar Form

1. Assume arg is principal argument. Write $z = \frac{(1+i)^2}{(\sqrt{3}+i)^2}$ in polar form.

2. Show that a circle with centre at z_0 has an equation of the form

$$z\bar{z} - z_0\bar{z} - \bar{z}_0 z + k = 0 \quad \text{for some } k \in \mathbb{R}$$

3. Show that a necessary and sufficient condition for four points z_1, z_2, z_3, z_4 lie on a circle, in that order, is

$$|(z_1 - z_2)(z_3 - z_4)| + |(z_2 - z_3)(z_1 - z_4)| - |(z_3 - z_1)(z_4 - z_2)| = 0$$

4. Suppose $z_1, \ldots, z_5$ are roots of $(z+1)^5 + z^5 = 0$. Show that $\operatorname{Re} z_k = -\frac{1}{2}$.

5. Prove that $\operatorname{Re}\left(z^{\frac{1}{n}} + \bar{z}^{\frac{1}{n}}\right) = 2r^{\frac{1}{n}} \cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right), k = 0, 1, \ldots, n-1$, where $z = re^{i\theta}$.

1.3 Sketch

1. $\{z \in \mathbb{C} : |z| < 1\}$.
2. $\left\{z \in \mathbb{C} : 1 < |z| < 2, \quad \frac{\pi}{6} < \arg(z) < \frac{\pi}{3}\right\}$.
3. $\{z \in \mathbb{C} : |z| = 1 - \cos(\arg(z))\}$.
4. $\{z \in \mathbb{C} : |z^2 - 1| < 1\}$.

2 Linear Algebra

$\mathbb{C}$ is a vector space over $\mathbb{R}$ as well as $\mathbb{C}$. A mapping $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$ (or $\mathbb{R}$) linear if

$$T(z + \omega) = T(z) + T(\omega) \text{ and } T(\alpha z) = \alpha T(z)\ \forall \alpha \in \mathbb{C}(\text{ or } \mathbb{R}) \text{ and } \ \forall z, \omega \in \mathbb{C}.$$

1. Show that every $\mathbb{C}$-linear T is of the form $T(z) = \alpha z$ for some $\alpha \in \mathbb{C}$.
2. The map $z \mapsto \bar{z}$ is not $\mathbb{C}$ linear but $\mathbb{R}$ linear.
3. Show that $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear if and only if

$$T(z) = \lambda z + \mu \bar{z}, \text{ where } \lambda = \frac{1}{2}(T(1) - iT(i)), \mu = \frac{1}{2}(T(1) + iT(i)).$$

4. Can you find a sufficient condition in which an $\mathbb{R}$-linear map will be a $\mathbb{C}$-linear map.

5. Let $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear. Write the matrix representation of T as a 2×2 real matrix. Do the same exercise for T as $\mathbb{C}$-linear.

6. Let $z, \omega \in \mathbb{C}$. Define **inner product** as $\langle z, \omega \rangle = \mathrm{Re}(z\bar{\omega})$. Show that
(i) $\langle \omega, z \rangle^2 + \langle i\omega, z \rangle^2 = |\omega|^2 |z|^2$
(ii) (Cauchy-Schwartz) $|\langle \omega, z \rangle| \leq |\omega||z|$.
(iii) If $\langle \omega, z \rangle = 0$ then $|\omega + z|^2 = |\omega|^2 + |z|^2$.

3 Metric Properties

1. Prove that the function $d(z, w) = |z - w|$ defines a metric on $\mathbb{C}$. Verify the three axioms: positivity, symmetry, and the triangle inequality.

2. Let $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denote the open ball of radius $r > 0$ centered at $z_0 \in \mathbb{C}$. Sketch $B_1(0)$, $B_2(i)$, and $B_{1/2}(1 + i)$.

3. Prove that the open ball $B_r(z_0)$ is an open set in $\mathbb{C}$ with respect to the metric topology induced by $d(z, w) = |z - w|$.

4. Show that the complex field $\mathbb{C}$ is a complete metric space with respect to the metric $d(z, w) = |z - w|$.

5. Let $\{z_n\} \subset \mathbb{C}$ be a sequence such that $\lim_{n\to\infty} z_n = z \in \mathbb{C}$. Prove that $\lim_{n\to\infty} |z_n| = |z|$ and $\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z)$.

6. Prove that the set $\mathbb{Q}(i) := \{a + bi : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$.

7. Prove that the modulus function $z \mapsto |z|$ is continuous on $\mathbb{C}$.

8. Let $f : \mathbb{C} \to \mathbb{R}$ be defined by $f(z) = \mathrm{Re}(z^2)$. Is f continuous? Justify your answer.

9. Prove that any closed and bounded subset of $\mathbb{C}$ is compact.

10. Show that any open connected subset of $\mathbb{C}$ is path-connected.

11. Define $f : \mathbb{C} \to \mathbb{R}$ by $f(z) = \frac{|z|}{1+|z|}$. Show that f is uniformly continuous on $\mathbb{C}$, and that $f(z) \to 1$ as $|z| \to \infty$.

12. Let $A = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Show that the closure of A in $\mathbb{C}$ is $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, and determine its boundary.

13. Let $\{z_n\} \subset \mathbb{C}$ be such that $\sum_{n=1}^{\infty} |z_{n+1} - z_n| < \infty$. Show that $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}$ and hence converges.

14. Let $K \subset \mathbb{C}$ be compact. Prove that any continuous function $f : K \to \mathbb{C}$ is uniformly continuous.

15. Let $f(z) = \frac{1}{z}$. Show that f is continuous on $\mathbb{C} \setminus \{0\}$, but not uniformly continuous.

16. Prove that the function $d(z, w) = \sqrt{|z - w|}$ does not define a metric on $\mathbb{C}$. Which property fails?

17. Show that the space $\mathbb{C}$ with the usual metric is not compact.

18. Let $\mathbb{Q}(i) = \mathbb{Q} + i\mathbb{Q}$. Consider the function $f(z) = \mathrm{dist}(z, \mathbb{Q}(i))$, where $\mathrm{dist}(z, A) := \inf\{|z - a| : a \in A\}$. Prove that f is continuous and that $f(z) = 0$ for all $z \in \mathbb{C}$. What does this imply about $\mathbb{Q}(i) \subset \mathbb{C}$?

Assignment 1

MTH 403, 2025

1 Stereographic projection

(i) Prove that circle on $\mathbb{S}^2 = \{(\xi, \eta, \tau) \in \mathbb{R}^3 : \xi^2 + \eta^2 + \tau^2 = 1\}$, not passing through $(0, 0, 1)$ correspond under stereographic projection to circles in $\mathbb{C}$, and vice versa.

(ii) Let $z_1, z_2 \in \mathbb{C}$. Suppose it corresponds to $(\ \xi_i, \eta_i, \tau_i\ )$ on unit sphere $\mathbb{S}^2$ for $i = 1, 2$ respectively. Define $\rho(z_1, z_2)$ to be the usual Euclidean distance between (ξ_1, η_1, τ_1) and (ξ_2, η_2, τ_2). Show that

$$\rho(z_1, z_2) = \frac{2\left|z_2 - z_2\right|}{\sqrt{\left(\left|z_1\right|^2 + 1\right)\left(\left|z_2\right|^2 + 1\right)}}$$

This metric is called the Spherical metric.

(iii) Append a "point at infinity" in $\mathbb{C}$ corresponding to the point $(0, 0, 1)$ in $\mathbb{S}^2$. i.e. $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This is called extended complex plane. We can define $\rho(z, \infty$ to be the distance between corresponding point of z in $\mathbb{S}^2$ and $(0, 0, 1)$. Show that

$$\rho(z, \infty) = \frac{2}{\sqrt{|z|^2 + 1}}$$

(a) Show that $0 \leq \rho \leq 2$ and find out points where 2 is attended.

(b) Show that if $\phi(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$ then $\rho\left(\phi\left(z_1\right), \phi\left(z_2\right)\right) = \rho\left(z_1, z_2\right)$.

(c) Think about the the shortest path (geodesics) between two points in the complex plane!

(iv) Let ϕ be the stereographic projection. Show that from $\mathbb{S}^2 \backslash \{z \in \mathbb{C} : z \neq 0\}$ is not connected in $\mathbb{S}^2$.

2 Functions

1. Polynomials

   (i) Let $p(z) = \sum\limits_{k=0}^{n} a_k z^k$ be a polynomial of degree n. If $p(z_0) = 0$, then show that
   $$p(z) = (z - z_0)q(z),$$
   where q is a polynomial of degree $n-1$. In general, show that if $z_0, \ldots, z_{n-1}$ are roots of p then
   $$p(z) = a_n \prod_{k=0}^{n-1} (z - z_k).$$
   Show that
   $$\sum_{k=0}^{n-1} z_k = -\frac{a_{n-1}}{a_n}.$$

   (ii) If z_0 is a root, then show that $\bar{z}_0$ is also a root.

   (iii) Consider the polynomial $p(z) = z^4 + 2z^2 + z + 6$. Show that if z_0 is a root, then $|z_0| > 1$.

2. Continuity

   (i) Show that the function $F(z) = \int_0^1 \frac{dx}{x-z}$ is continuous on $\mathbb{C} \setminus [0,1]$.

   (ii) Let $s \in (0,1)$. For $\epsilon > 0$, find the real and imaginary parts of $F(s \pm i\epsilon)$. Show that
   $$\lim_{\epsilon \to 0} F(s \pm i\epsilon) = \ln \left| \frac{1-s}{s} \right| \pm i\pi.$$

(iii)

3. At which points are the following functions f differentiable?
(a) $f(z) = \bar{z}^2$
(b) $f(z) = \frac{z^3}{|z|^2}, z \neq 0$ and 0 when $z = 0$

4. Show that $f(z) = \sqrt{|xy|}$ is not differentiable at the origin. Does it satisfy Cauchy-Riemann equation at the origin?

5. Polar co-ordinates
(i) Prove that Cauchy Riemann equation in polar co-ordinates are

$$r\frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

(ii) Show that

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

6. Prove the sufficient condition for existence of $f'(z_0)$. i.e u_x, u_y, v_x, v_y are continuous.

7. Stein-Shakarchi - Page 27: 7,12,13,14,15

Assignment 2

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1. Write the first three terms in the power series expansion of the following functions:
   (i) $e^z \sin z$
   (ii) $\sin z \cos z$
   (iii) $\frac{1}{\cos z}$
   (iv) $\frac{\sin z}{\cos z}$

2. Let $0 < r < 1$. Show that $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}$.

3. Find the radius of convergence of the following power series:
   (i) $\sum\limits_{n=1}^{\infty} n^{(-1)^n} z^n$ (ii) $\sum_{n=0}^{\infty} n^n z^n$ (iii) $\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$ (iv) $\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^n$.

4. Let $\alpha, \beta \in \mathbb{C}$ and $|\alpha| < |\beta|$. Find the radius of convergence of $\sum_{n=0}^{\infty} (3\alpha^n - 4\beta^n) z^n$.

5. Let $r \in \mathbb{R}, |r| < 1$ and $\theta \in \mathbb{R}$. Compute
$$S(r, \theta) = \sum_{n=1}^{\infty} r^n \sin n\theta \text{ and } C(r, \theta) = \sum_{n=0}^{\infty} r^n \cos n\theta.$$

6. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$. Find f''.

7. Let $f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots$. Show that $f'(z) = \frac{1}{z^2+1}$.

8. Consider $f(z) = \sum_{n=1}^{\infty} \frac{z^2}{n}$. Find the radius of convergence. What can you say about the convergence for z to be the 2^k-th root of unit?

9. Stein & Shakarchi: Page 28-31: Exercises : 16,18, 20, 21,23,

10. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ converges on $\mathbb{D}$. If $|f(z)| \leq 1$, then show that $|a_n| \leq 1$.

Assignment 3

MTH 403, 2025

1 Practice Problem

1. In the following problems, all curves are oriented anti-clockwise.

   (i) Let $\gamma(t) = a + re^{it}$ for $0 \le t < 2\pi$. Compute $\int_\gamma (z-a)^m dz$ for every integer $m \in \mathbb{Z}$.

   (ii) Let γ be the line segment joining $z_1 = 0$ to $z_2 = 2 + i\frac{\pi}{4}$. Calculate $\int_\gamma e^z dz$.

   (iii) For a fixed $\delta > 0$ let $\gamma_r(t) = re^{it}$ for $\delta \le t \le \pi - \delta$. If $I(r) = \int_{\gamma_r} \frac{e^{iz}}{z} dz$, calculate $\lim_{r\to\infty} I(r)$.

   (iv) Let γ_1 be a semicircular path joining -1 and 1 with centre at 0 and γ_2 a rectangular path with vertices $-1, -1+i, 1+i$ and 1 . Find $\int_{\gamma_1} \bar{z} dz$ and $\int_{\gamma_2} \bar{z} dz$ (observe path dependence).

   (v) Let $f = u + iv$ be an entire function. Show that $\int_0^{2\pi} (u(r,\theta)\cos\theta - v(r,\theta)\sin\theta) d\theta = 0$.

   (vi) Show that

   (a) $\int_0^\infty e^{-x^2} \cos 2bx dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$ and

   (b) $\int_0^\infty e^{-x^2} \sin 2bx dx = e^{-b^2} \int_0^b e^{t^2} dt$ by integrating the differentiable function $f(z) := e^{-z^2}$ along the path $\gamma : [0,1] \to \mathbb{C}$ defined by $\gamma(t) := \begin{cases} 4tR & \text{if} \quad 0 \le t \le \frac{1}{4} \\ R + i(1-4t)b & \text{if} \quad \frac{1}{4} \le t \le \frac{1}{2} \\ (3-4t)R - ib & \text{if} \quad \frac{1}{2} \le t \le \frac{3}{4} \\ i4(t-1)b & \text{if} \quad \frac{3}{4} \le t \le 1, \end{cases}$ where $R > 0$ and $b > 0$.

   (vii) Show that $\int_\gamma \frac{e^{az}}{z^2+1} = 2\pi i \sin a$ if $\gamma(\theta) = 2e^{i\theta}$ for $0 \le \theta \le 2\pi$.

   (viii) Show that $\int_\gamma \frac{e^z}{z} = 2\pi i$ where $\gamma(\theta) = e^{i\theta}$ for $0 \le \theta \le 2\pi$. Hence or otherwise evaluate

   (a) $\int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta) d\theta$ and

   (b) $\int_0^{2\pi} e^{k\cos\theta} \sin(k\sin\theta) d\theta$.

   (ix) Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a polynomial and let $\gamma = \{z \in \mathbb{C} : |z| = R\}$. Evaluate $\int_\gamma \frac{P(z)}{z^k} dz$.

   (x) Let $\gamma = \{z \in \mathbb{C} : |z| = 2\}$. Find the values of $\int_\gamma z^n (1-z)^m dz$ for $m \in \mathbb{N} \cup \{0\}, n \in \mathbb{Z}$ and for $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}$.

   (xi) Evaluate
   a) $\int_{|z|=2} \frac{z}{(z^2-1)} dz$ b) $\int_{|z|=2} \frac{z}{(z^2-1)^2} dz$.

2. Find $\int_{|z|=1} \frac{1}{z} \left(z + \frac{1}{z}\right)^{2n} dz$. Conclude

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2n} x dx = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}$$

3. Let $D_R = \{z : |z - z_0| < R$ and for $0 < r < R, C_r = \{z : |z - z_0| = r\}$ with counterclockwise direction. If f is continuous on D_R, then show that

$$\lim_{r\to 0}\int_{C_r}\frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

4. Define the Hermite polynomials $H_n(x)$ as

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

and Hermite functions

$$\phi_n(x) = e^{-\frac{x^2}{2}}H_n(x).$$

(a) Show that H_n is a polinomial of degree n.
(b) Show that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_n(x)e^{-itx}dx = (-i)^n\phi_n(t), t\in\mathbb{R}$.
(c) Show that $\phi_n'' - x^2\phi_n + (2n+1)\phi_n = 0$.
(d) Show that $\int_{-\infty}^{\infty}\phi_n(x)\phi_m(x)dx = 0, n\neq m$.

5. Suppose f is holomorphic on $\mathbb{D} = \{z : |z| < 1\}$ and continuous on $\partial\mathbb{D} = C$. Show that $\int_C f(z)dz = 0$.

6. Stein & Shakarchi 1, 3, 4, 6,7 Page 64-65

2 Problems to be submitted before 31st August

7. Prove that the smooth paths defined by the parametrisations $\gamma_1(t) = e^{it}, t\in[0,2\pi]$ and $\gamma_2(t) = e^{it}, t\in[0,4\pi]$ are not equivalent.

8. Find $\int_\gamma e^z dz$ where where the graph γ is is parametrized by $y = \sin x, x\in[0,\pi]$.

9. Let f be a continuous function on $\mathbb{C}$ with $f(z_0)\neq 0$. Show that there exists a radius $R > 0$ such that for any two distinct points $z_1, z_2 \in D_R(z_0)$, where $D_R(z_0) = \{z : |z - z_0| < R\}$ $\int_{[z_1,z_2]} f(z)dz \neq 0$ and $[z_1, z_2]$ denotes the line joining z_1 and z_2.

10. Suppose f is a continuous function on $\mathbb{C}$. Suppose there exist $z_0\in\mathbb{C}, R > 0$ and $M > 0$ such that $|f(z)| \le M\forall z\in D(z_0, R)\setminus\{z_0\}$. Denote $I_r = \int_{C_r(z_0)} f(z)$. If I_r is same for all $0 < r < R$, then show that $I_r = 0$ for all $r\in(0, R)$.

11. Let C be a path parametrized by $\gamma(t) = x(t) + iy(t),\quad t\in[a,b]$. Define C^* to be the path parametrized by $\gamma^*(t) := x(t) - iy(t),\quad t\in[a,b]$. Assume that f is defined on the images of both C and C^*.

(a) Show that

$$\overline{\int_C f(z)\,dz} = \int_{C^*}\overline{f(\overline{z})}\,dz. \tag{1}$$

(b) Show that

$$\overline{\int_{|z|=1} f(z)\,dz} = -\int_{|z|=1}\overline{f(z)}\,\frac{dz}{z^2}. \tag{2}$$

(c) Let C be a smooth closed path, and let $z_0\notin C$. For $s\in[a,b]$ define

$$\phi(s) = (\gamma(s) - z_0)e^{-\int_a^s \frac{\gamma'(t)}{\gamma(t)-z_0}dt}.$$

Show that ϕ is constant and

$$W(C, z_0) := \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0} \tag{3}$$

is an integer.

12. Stein & Shakarchi 4, 6,7 Page 65

Assignment 4

MTH 403, 2025

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc.

1. Find the value
   (i) $\int_{|z|=1} \frac{\cos z}{z(z-4)} dz$ (ii) $\int_{|z|=2} \frac{e^z + z^2}{z-1}$ (iii) $\int_{|z|=1} \frac{\cos(e^z)}{z} dz$.

2. If $f \in \mathcal{H}(\mathbb{D})$ and $|f(z)| \leq 1 - |z|$ in $\mathbb{D}$. Show that $f \equiv 0$ on $\mathbb{D}$.

3. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that $|f'(z)| \leq \frac{1}{1-|z|}, \forall z \in \mathbb{D}$.

4. Write the power series around 0 of function
$$f(z) = \frac{1}{1 - z - z^2}$$
   in the appropriate domain.

5. Let $f, g \in \mathcal{H}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. If $(fg)(z) = \sum_{n=0}^{\infty} c_n z^n$, then using Cauchy integral formula find the value of c_n.

6. Suppose f is an entire function such that $\frac{f(z)}{z^n}$ is bounded for $|z| > R$ for some $R > 0$. Show that f is a polynomial of degree at most n. If $\frac{f(z)}{z^n}$ is bounded for entire $\mathbb{C}$ then what can you say about f.

7. Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$. Let f be holomorphic on disc $\{z : |z| < 2\}$. Show that $|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) p(e^{i\theta})| d\theta$.

8. Stein & Shakarchi : Page 65-67 : 8,9,10

9. Let $n \geq 3$ and $a, b \in \mathbb{C}$. Show that the sum of the roots of the polynomial equation $z^n + az + b = 0$ is zero.

Submit 2, 6, 7, SS 9,10 before 8th September

9. (For fun (Same proof without naming Liouville's Theorem))

Let $p(z) = \sum_{k=0}^{n} a_k z^k$, where $a_n = 1 (n \geq 1)$.
(a) For $|z| \geq 1$ show that $|p(z)| \geq |z|^n \left(1 - \frac{1}{|z|}\left(|a_0| + \cdots + |a_{n-1}|\right)\right)$.
(b) Show that $|p(z)| \to \infty$ as $|z| \to \infty$. Hence conclude there exists $R > 0$ such that $\left|p\left(re^{i\theta}\right)\right| > |p(0)|$ whenever $r > R$.
(c) Show that there exists a $z_0 \in \mathbb{C}$ such that $|p(z_0)| \leq |p(z)|, \forall z \in \mathbb{C}$.
(d) Let $p(z_0) \neq 0$ (z_0 as above). Define $q(z) = \frac{p(z+z_0)}{p(z_0)}, \forall z \in \mathbb{C}$. Then $q(z) = 1 + \sum_{k=0}^{n} b_k z^k$. Let m be the least integer for which $b_m \neq 0$. Denote $b_m = -|b_m| e^{-im\theta}$ for some θ. For $z = re^{i\theta}$. Show that $q(z) = 1 - |b_m| r^m + \sum_{k=m+1}^{n} b_k z^k$.
(e) Show that by taking r small we get that $\left|q\left(re^{i\theta}\right)\right| < 1$.
(f) Hence, conclude the Fundamental theorem of algebra.

Assignment 5

MTH043, 2025

1. Let $f \in \mathcal{H}(\mathbb{D})$ and we have $\int_{|z|=1} \frac{f(z)}{(n+1)z-1} dz = 0, \forall n \in \mathbb{N}$. Show that f is identically equal to 0 .

2. Suppose f is an entire function such that for each $z \in \mathbb{C}$ there exists one coefficient in the power series expansion is equal to 0 . Prove that f is a polynomial.

3. Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |a_n| < \infty$. If $\sum_{n=0}^{\infty} \frac{a_n}{k^n} = 0, \forall k \in \mathbb{N}$. Show that $a_n = 0, \forall n \in \mathbb{N}$.

4. Find all functions $f \in \mathcal{H}(\mathbb{D})$ such that $f\left(\frac{1}{n}\right) + f''\left(\frac{1}{n}\right) = 0, \forall n \in \mathbb{N}$.

5. Find all entire functions f such that
   (a) $|f(z)| \leq M(1 + \sqrt{|z-i|})$
   (b) $|f'(z)| \leq M(1 + \sqrt{|z|})$
   (c) $|f(z)| \leq M(1 + |z-i|)$
   (d) $|f(z)| \leq M(1 + |z|)$

6. Stein-Shakarchi: Exercise 8-15 Page 65-67

Assignment 6, MTH403, 2025

1. Show that $z = 0$ is a removable singularity of the function $f(z) = \frac{1}{\tan z} - \frac{1}{\sin z}$.
2. What are the singularities of the functions
   (a) $f(z) = \frac{z(z-1)^2}{\sin^2(\pi z)}$
   (b) $f(z) = \frac{z^4(z-1)}{\sin^2(\pi z)}$
   (c) $f(z) = \frac{z^3(z-1)^6}{\sin^5(\pi z)}$
   (d) $f(z) = \frac{\sin z}{z^4}$
   (e) $f(z) = \frac{\sin^3 z}{z}$
3. Assume that $z = a$ is a pole of order N of the function f. Show that it is a pole of order $N + 1$ of the function f'. .
4. Define ∞ an isolated singular point of $f(z)$ if 0 is an isolated singular point of $f(\frac{1}{z})$.

   Let f be an entire function and assume that ∞ is a pole of f. Find f.
5. Show that there is no function analytic in $\mathbb{C}\setminus\{0\}$ such that $(f(z))^2 = z$.
6. Determine the residue of the functions at their poles.
   $\frac{z^2+6z^3}{z^3}$, $\frac{\cos z}{1+z+z^2}$, $\frac{z^{99}}{z^{100}-1}$, $\frac{1}{\sin z}$, $\frac{z}{1-\cos z}$.
7. Let R be a rectangle oriented clockwise, with vertices $(0, 0)$, $(10, 0)$, $(10, 4)$, $(0, 4)$. Find the following integrals
$$\int_R \frac{1}{z^2 - 3z + 5}dz, \quad \int_R \frac{1}{z^2 + z + 1}dz, \quad \int_R \frac{1}{z^2 - z + 1}dz.$$
8. Let f be a function holomorphic on upper-half plane and on the real line. Suppose $|f(z)| \leq \frac{B}{|z|^a}$ $\forall z$ and for some $B, a > 0$. Prove that for any z in the upper-half plane
$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x - z}dx.$$
9. Show that $\int_0^\infty \frac{x^2}{x^6+1}dx = \frac{\pi}{6}$.
10. Stein and Shakarchi: 10, 11, 12, 13, 14 page 104-105.

Submit 7(b), and SS 10,12,13,14 by 7th October 2025 before 12 noon.

Assignment 7

MTH 403, 2025

1. If f is analytic in $|z| \leq 2$ and if $|f(z)| < 1$ on $|z| = 1$, the equation $f(z) = z^n$ has exactly n solutions in $|z| < 1$.

2. Let F be analytic in $|z| < 2$ and map the closed unit disk into the open unit disk. Show that F has a fixed point, i.e., there is z in the open unit disk such that $F(z) = z$.

3. How many roots has the equation $z^4 + z^3 - 4z + 1 = 0$ in the annulus $1 < |z| < 3$?

4. Let f be a holomorphic function function in an open set Ω containing a circle C and its interior. Let $z1, \ldots, z_n$ be zeroes of f in the the interior of C.If f never vanishes on C, then show that
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} g(z)dz = \sum_{k=1}^{n} g(z_k),\ \forall g \in \mathcal{H}(\Omega).$$

5. Let $\Omega = \mathbb{C} \setminus \{0\}$. Define $u(z) = \log|z|$. Show that u is harmonic but it is not real part of any holomorphic function.

6. Let Ω be a simply connected domain and u be a harmonic function on Ω.
   (i) The function $g(z) = u_x - iu_y$ is holomorphic in Ω.
   (ii) Find the the primitive of g.
   (iii) Conclude that u is the real part of a holomorphic function.

7. Let $f \in \mathcal{H}(D_R(z_0))$ with power series expansion $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. Show that
   (i) for all $n \geq 0$ and $0 < r < R$ we have $a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}d\theta$.
   (ii) for $n < 0$ we have $0 = \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}d\theta$.
   (iii) $f(z_0) = \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$.
   (iv) if $u = \operatorname{Re} f$, then $u(z_0) = \frac{1}{2\pi r^n}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$.

8. Stein & Shakarchi (Page 105-108): 15,16,17,18,19,20,21,22.

Submit 2, 4 SS: 15,16,20,21 before 10 AM on 15th OCTOBER

Assignment 8

MTH 403, 2025

1. Which of the following domains are simply connected
   (i) $\mathbb{C} \setminus [-1, 1]$ (ii) $\mathbb{C} \setminus \{-1, 0, 1\}$ (iii) Ω convex

   (iv) Ω is star shaped. i.e there exists a $z_0 \in \Omega$ such that for $z \in \Omega$ the line joining z_0 and z is in Ω.

2. Let $\mathbb{H} = \{z \in \mathbb{C} : \text{ Im } z > 0\}$. Define $f : \mathbb{H} \to \mathbb{C}$ by $f(z) = e^{2\pi i z}$. What is the image of f?

3. Compute the following values when the branch of log is defined on $\mathbb{C} \setminus (-\infty]$ and $\mathbb{C} \setminus [0, \infty)$.
   $\log i, \quad \log -i, \quad \log(-1 + i), \quad \log(-1 - i)$.

4. Consider the branch of log on $\mathbb{C}\setminus(-\infty, 0]$. Let $a > 0$ and $y > 0$. Find $\lim\limits_{y\to 0} (\log(a + iy) - \log(a - iy))$. Will limit change if $a < 0$?

5. Let γ be a closed curve such that $z_j \notin \gamma,\ j = 1, 2, \ldots, N$. If $P(z) = \prod\limits_{j=1}^{N} (z - z_j)$, then show that

$$\frac{1}{2\pi i} \int_\gamma \frac{P'(z)}{P(z)} dz = \sum_{j=1}^{N} W(\gamma, z_j).$$

6. (**Optional**)

   (i) Let Ω be a domain and $\mathbb{C}^*$ is the extended complex plane. Show that if $\mathbb{C}^* \setminus \Omega$ is not connected, then there exists a compact set E and $\delta > 0$ such that

$$d(z, w) \geq 4\delta,\ \forall z \in E \text{ and } w \in F = \mathbb{C} \setminus (\Omega \cup E).$$

   (ii) Fix a point $z_0 \in E$, cover $\mathbb{C}$ by grid of squares (interior are mutually disjoint) of side length δ so that z_0 lies in the interior of some square. Let K be the union of all squares in the grid that contain a point of distance at most δ from E. Show that

   (a) K is finite union of closed square.

   (b) $K \cap F = \emptyset$.

   (c) $\partial K \bigcap (E \cup F) = \emptyset$

   (d) ∂K is a finite union of straight line segments contained in Ω.

   (iii) Denote K_j the squares from the grid in K. Show that $\int_{\partial K} d(\text{ arg } (z - z_0)) = 2\pi$.

   (iv) Show that there exists a closed path γ such that $W(\gamma, z_0) \neq 0$.

   (v) Conclude that: If Ω is simply connected, then Ω^c in $\mathbb{C}^*$ is connected.

Assignment 9

MTH 403, 2025

1. Find the fractional linear transform which maps
   (a) $0, i, -i \mapsto 1, -1, 0$
   (b) $2, i, -2 \mapsto 1, i, -1$
   (c) $1, i, 0 \mapsto 0, i, 1$
   (d) $0, \infty, 1 \mapsto \infty, 0, i$.

2. Under the transformation $\omega = \frac{z}{z-1}$ find the image of the following :
   (i) the half plane $y \geq 0$.
   (ii) the half plane $x \geq 0$.
   (iii) the disc $|z| \leq 1$.
   (iv) the line $x = 1$

3. Find the image of the right half plane under the mapping $\phi(z) = i\frac{1-z}{1+z}$.

4. What is the image of the strip $S = \{z = x + iy : 0 < y < 2\}$ under the Möbius transformation $\phi(z) = \frac{z}{z-i}$?

5. Consider the transformation $\omega = \frac{1}{z}$.
   (i) Find out where the line $x = 1$ and $y = 2$ are mapped.
   (ii) Find out where the the set $A = \{z : a|z|^2 + b\bar{z} + \bar{b}z + c = 0$, where $0 \neq a, c \in \mathbb{R}$ and $b \in \mathbb{C}\}$.

6. The cross-ratio of the four points z_1, z_2, z_3, z_4 is denoted by (z_1, z_2, z_3, z_4) and is defined by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$$

   Show that the cross-ratio of four distinct points in the extended plane is invariant under Möbius transform

$$\omega = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

7. Find the image of $\mathbb{H}_+ = \{z : \text{ Im } z > 0\}$ under the maps (a) $f(z) = z^\alpha$ and (b) $f(z) = \log z$.

8. Stein & Shakarchi : Page 248-252; Q-1, 7, 9, 10, 11, 12, 13, 14,

Submit : Q2, SS- 7, 10, 11, 12, 13 before 5th November 10 AM

Assignment 10

MTH 403, 2025

1. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$ for $n = 2, 3, \ldots$ Show that the series $\sum_{n=2}^{\infty} a_n$ converges but infinite product $\prod_{n=2}^{\infty}(1 + a_n)$ diverges.

2. Let $a_n = \begin{cases} \frac{1}{\sqrt{k}} + \frac{1}{k} & \text{if } n = 2k \\ \frac{-1}{\sqrt{k}} & \text{if } n = 2k+1 \end{cases}, n = 2, 3, \ldots$

   Show that the series $\sum_{n=2}^{\infty} a_n$ diverges but $\prod_{n=2}^{\infty}(1 + a_n)$ converges.

3. Find the value of $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$.

4. By using product formula of sin, show that $\frac{\pi}{2} = \frac{2.2}{1.3}\frac{4.4}{3.5} \cdots \frac{2N.2N}{(2N-1)(2N+1)} \cdots$

5. Show that

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2\pi^2}\right) \text{ and } \cosh z = \prod_{n=0}^{\infty} \left(1 + \frac{4z^2}{(2n+1)^2\pi^2}\right).$$

6. Let Ω be an open connected set and $\mathbf{R} = \{z : \text{ Re } z > 0\}$. Show that

$$\mathcal{F} = \{f : \Omega \to \mathbf{R} : f \text{ is holomorphic}\}$$

   is a normal family.

7. Let Ω be a bounded, open and connected subset. Show that

$$\mathcal{F} = \{f \in \mathcal{H}(\Omega) : \iint_{\Omega} |f(x+iy)| dxdy \leq 1\}$$

   is a normal family.

8. Consider the vertical strip $S = \{z : |\text{ Re } z| < \frac{\pi}{2}\}$. Suppose $f \in \mathcal{H}(S)$ and $f \in C(\overline{S})$, with $|f(z)| \leq 1$ on ∂S. If $|f(z)| \leq Ae^{e^{\alpha|z|}}$ for some $\alpha \in (0, 1)$ and $0 < A < \infty$ for all $z \in S$, then show that $|f(z)| \leq 1$ for all $z \in S$.

9. Consider the vertical strip $S = \{z : a < \text{ Re } z < b\}$. Suppose $f \in \mathcal{H}(S)$ and $f \in C(\overline{S})$, with $|f(z)| \leq 1$ on ∂S. If $|f(z)| \leq Ae^{e^{\frac{\alpha\pi}{b-a}|z|}}$ for some $\alpha \in (0, 1)$ and $0 < A < \infty$ for all $z \in S$, then show that $|f(z)| \leq 1$ for all $z \in S$.

10. Consider the right-half plane $\mathbf{R} = \{z : 0 < \text{ Re } z\}$. Suppose $f \in \mathcal{H}(\mathbf{R})$ and $f \in C(\overline{\mathbf{R}})$, with $|f(z)| \leq 1$ on $\partial\mathbf{R}$. If $|f(z)| \leq Ae^{|z|^\alpha}$ for some $\alpha \in (0,1)$ and $0 < A < \infty$ for all $z \in \mathbf{R}$, then show that $|f(z)| \leq 1$ for all $z \in \mathbf{R}$.

11. Consider the sector $\Omega = \{z : |\arg z| < \frac{\pi}{2\beta},$ for some $\beta > \frac{1}{2}\}$. Suppose $f \in \mathcal{H}(\Omega)$ and $f \in C(\overline{\Omega})$, with $|f(z)| \leq 1$ on ∂S. If $|f(z)| \leq Ae^{|z|^\alpha}$ for some $\alpha \in (0,\beta)$ and $0 < A < \infty$ for all $z \in \Omega$, then show that $|f(z)| \leq 1$ for all $z \in \Omega$.