Frobenius method and Bessel function

1. For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:

   (a) $9x^2y'' + (9x^2 + 2)y = 0$
   (b) $x^2(x^2 - 1)y'' - x(1 + x^2)y' + (1 + x^2)y = 0$
   (T) (c) $xy'' + (1 - 2x)y' + (x - 1)y = 0$
   (d) $x(x - 1)y'' + 2(2x - 1)y' + 2y = 0$

   Solution:

   (a)

   The given ODE can be written as

   $$y'' + \frac{9x^2 + 2}{9x^2}y = 0$$

   Hence $x = 0$ is a regular singular point. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$. This gives

   $$\sum_{n=0} 9(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0} 9a_n x^{n+r+2} + \sum_{n=0} 2a_n x^{n+r} = 0$$

   which can be written as

   $$\sum_{n=0} 9(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=2} 9a_{n-2} x^{n+r} + \sum_{n=0} 2a_n x^{n+r} = 0$$

   This can be rearranged as (after canceling x^r)

   $$\left(9r(r-1)+2\right)a_0 + \left(9r(r+1)+2\right)a_1 x + \sum_{n=2} \left(9(n+r)(n+r-1)a_n + 9a_{n-2} + 2a_n\right)x^n = 0$$

   This implies

   $$\left(9r(r-1)+2\right)a_0 = 0, \quad \left(9r(r+1)+2\right)a_1 = 0 \quad \text{and} \quad a_n = -\frac{9a_{n-2}}{9(n+r)(n+r-1)+2}, \quad n \geq 2.$$

   Since $a_0 \neq 0$, we have $9r(r-1) + 2 = 0 \implies r = 2/3 = r_1$, $r = 1/3 = r_2$. Here $r_1 - r_2 = 1/3$ is not an integer and we have two independent Frobenius series solutions. With $r = r_1$ or $r = r_2$, $9r(r+1) + 2 \neq 0 \implies a_1 = 0$. This leads to $a_{2n+1} = 0$, $n \geq 0$. Also,

   $$a_n = -\frac{9a_{n-2}}{(3n+3r-2)(3n+3r-1)}, \quad n \geq 2.$$

   With $r = r_1 = 2/3$ we find

   $$y_1(x) = x^{2/3} \sum_{n=0} a_{2n} x^{2n}, \qquad a_0 = 1, \; a_{2n} = -\frac{3a_{2n-2}}{2n(6n+1)}, \; n \geq 1.$$

   With $r = r_1 = 1/3$ we find

   $$y_2(x) = x^{1/3} \sum_{n=0} a_{2n} x^{2n}, \qquad a_0 = 1, \; a_{2n} = -\frac{3a_{2n-2}}{2n(6n-1)}, \; n \geq 1.$$

(b)

The given ODE can be written as

$$y'' - \frac{1+x^2}{x(x^2-1)}y' + \frac{1+x^2}{x^2(x^2-1)} = 0$$

Hence $x = 0$ is a regular singular point. Let $y = \sum_{n=0} a_n x^{n+r}$, $a_0 \neq 0$. This gives

$$\sum_{n=0} \Big((n+r)(n+r-1)a_n(x^{n+r+2}-x^{n+r})-(n+r)a_n(x^{n+r}+x^{n+r+2})+a_n(x^{n+r}+x^{n+r+2})\Big) = 0$$

which can be written as

$$\sum_{n=2} \Big((n+r-2)(n+r-3)-(n+r-2)+1\Big)a_{n-2}x^{n+r} - \sum_{n=0} \Big((n+r)(n+r-1)+(n+r)-1\Big)a_n x^{n+r} = 0$$

This can be rearranged as (after canceling x^r)

$$-\Big(r^2-1\Big)a_0 - \Big((r+1)^2-1\Big)a_1 x + \sum_{n=2} \Big((n+r-3)^2 a_{n-2} - \big((n+r)^2-1\big)a_n\Big)x^n = 0$$

This implies

$$\Big(r^2-1\Big)a_0 = 0, \quad \Big((r+1)^2-1\Big)a_1 = 0, \quad \text{and} \quad a_n = \frac{(n+r-3)^2}{(n+r)^2-1}a_{n-2}, \ n \geq 2.$$

Since $a_0 \neq 0$, we have $r^2 - 1 = 0 \implies r = 1 = r_1$, $r = -1 = r_2$. Here $r_1 - r_2 = 2$ is an integer and we may or may not have two independent Frobenius series solutions.

With $r = r_1$, $(r+1)^2 - 1 \neq 0 \implies a_1 = 0$. Also,

$$a_n = \frac{(n-2)^2}{n(n+2)}a_{n-2}, \ n \geq 2 \implies a_n = 0, \ n \geq 1.$$

Hence

$$y_1(x) = x, \quad a_0 = 1.$$

For the other solution, let $y_2 = y_1 u(x) = xu$ (reduction of order technique)

$$x(x^2-1)u'' + (x^2-3)u' = 0 \implies \frac{u''}{u'} = \frac{1}{1+x} - \frac{1}{1-x} - \frac{3}{x} \implies u' = 1/x^3 - 1/x$$

which integrating again gives

$$u = -\log x - \frac{1}{2x^2}$$

Hence $y_2 = x \ln x + 1/(2x)$ (ignoring the negative sign)

(c)

The given ODE can be written as

$$y'' + \frac{1-2x}{x}y' + \frac{x-1}{x} = 0$$

Hence $x = 0$ is a regular singular point. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$. This gives

$$\sum_{n=0} \left((n+r)(n+r-1)a_n x^{n+r-1} + (n+r)a_n(x^{n+r-1} - 2x^{n+r}) + a_n(x^{n+r+1} - x^{n+r}) \right) = 0$$

which can be written as

$$\sum_{n=2} a_{n-2}x^{n+r-1} - \sum_{n=1} \left(2(n+r-1)+1 \right) a_{n-1} x^{n+r-1} + \sum_{n=0} \left((n+r)(n+r-1)+(n+r) \right) a_n x^{n+r-1} = 0$$

This can be rearranged as (after canceling x^{r-1})

$$r^2 a_0 + \left((r+1)^2 a_1 - (2r+1)a_0 \right) x + \sum_{n=2} \left((n+r)^2 a_n - \left(2(n+r-1)+1 \right) a_{n-1} + a_{n-2} \right) x^n = 0$$

This implies

$$r^2 a_0 = 0, \ (r+1)^2 a_1 = (2r+1)a_0, \ (n+r)^2 a_n = \left(2(n+r)-1 \right) a_{n-1} - a_{n-2}, \ n \geq 2$$

Now $a_0 \neq 0 \implies r = r_1 = 0, r = r_2 = 0$. Since the indicial equation has double roots, the given equation has only one independent Frobenius series solution. We take $r = 0$ and this gives $a_1 = a_0$. We also have

$$a_n = \frac{2n-1}{n^2} a_{n-1} - \frac{1}{n^2} a_{n-2}, \quad n \geq 2.$$

With $a_0 = 1$ we get $a_1 = 1$. This leads to $a_2 = 1/2!$, $a_3 = 1/3!$. We prove $a_n = 1/n!$ by induction. Clearly the induction hypothesis is true for $n = 1, 2, 3$. Let it be true for $n = k$. For $n = k+1$, we have

$$a_{k+1} = \frac{2k+1}{(k+1)^2} a_k - \frac{1}{(k+1)^2} a_{k-1} = \frac{1}{(k+1)^2(k-1)!} \left(\frac{2k+1}{k} - 1 \right) = \frac{1}{(k+1)!}$$

Hence

$$y_1(x) = \sum_{n=0} \frac{x^n}{n!} = e^x$$

For other solution let $y_2 = y_1 u(x) = e^x u$. This gives

$$xu'' + u' = 0 \implies u' = 1/x \implies u = \ln x$$

Hence $y_2(x) = e^x \ln x$

(d)

The given ODE can be written as

$$y'' + \frac{2(2x-1)}{x(x-1)} y' + \frac{2}{x(x-1)} y = 0$$

Hence $x = 0$ is a regular singular point. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$. This gives

$$\sum_{n=0} \left((n+r)(n+r-1)a_n(x^{n+r} - x^{n+r-1}) + (n+r)a_n(4x^{n+r} - 2x^{n+r-1}) + 2a_n x^{n+r} \right) = 0$$

which can be written as

$$\sum_{n=1}^{\infty}\Big((n+r-1)(n+r-2)+4(n+r-1)+2\Big)a_{n-1}x^{n+r-1}-\sum_{n=0}^{\infty}\Big((n+r)(n+r-1)+2(n+r)\Big)a_n x^{n+r-1}=0$$

This can be rearranged as (after canceling x^{r-1})

$$(r^2+r)a_0-\sum_{n=1}^{\infty}\Big((n+r)(n+r+1)a_n-\big((n+r-1)(n+r+2)+2\big)a_{n-1}\Big)x^n=0$$

This implies

$$(r^2+r)a_0=0,\ (n+r)(n+r+1)a_n-\big((n+r-1)(n+r+2)+2\big)a_{n-1}=0,\ n\geq 1$$

Now $a_0\neq 0\implies r=r_1=0, r=r_2=-1$. Hence $r_1-r_2=1$ is an integer and hence the ODE may or may not have two independent Frobenius series solution.

With $r=r_1=0$,

$$n(n+1)a_n=\big((n-1)(n+2)+2\big)a_{n-1}\implies a_n=a_{n-1}\implies a_n=a_0,\ n\geq 1.$$

Hence (with $a_0=1$)
$$y_1(x)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$$

For the other solution, let $y_2=y_1 u(x)$. This gives

$$xu''+2u'=0\implies u'=\frac{1}{x^2}\implies u=-1/x$$

Hence (neglecting the negative sign)

$$y_2(x)=\frac{1}{x(1-x)}$$

We can write
$$y_2(x)=\frac{1}{x}+\frac{1}{1-x}$$

Since the last term is $y_1(x)$, we can take $y_2(x)=1/x$

Note: If we continue the Frobenius series method with $r=r_2=-1$, then from the recurrence relation
$$n(n-1)a_n=n(n-1)a_{n-1},\ n\geq 1.$$

For $n=1$, the relation is automatically satisfied for any value of a_1. We may take $a_1=0$. This leads to $a_n=0$ for $n\geq 1$. Then we again get (taking $a_0=1$)

$$y_2(x)=\frac{1}{x}$$

2. Show that $2x^3y''+(\cos 2x-1)y'+2xy=0$ has only one Frobenius series solution.

   Solution:

We can write the ODE as

$$2x^2 y'' + \frac{\cos 2x - 1}{x^2} xy' + 2y = 0$$

Since

$$\lim_{x \to 0} \frac{\cos 2x - 1}{x^2} = -2,$$

the indicial equation is

$$2r(r-1) - 2r + 2 \implies r^2 - 2r + 1 \implies r = 1, 1.$$

Since the indicial equation has double roots, it has only one Frobenius series solution.

3. **(T)** Reduce $x^2 y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

Solution:

Suppose $y(x) = u(x)v(x)$. Hence

$$x^2(u''v + 2u'v' + uv'') + x(u'v + uv') + \left(x^2 - \frac{1}{4}\right)uv = 0$$

or

$$x^2 vu'' + (2x^2 v' + xv)u' + \left((x^2 v'' + xv' + \left(x^2 - \frac{1}{4}\right)v\right)u = 0.$$

To make the 2nd term vanish, we set

$$2x^2 v' + xv = 0 \implies 2xv' + v = 0 \implies v = \frac{1}{\sqrt{x}}$$

Using this transformation the given ODE reduces to

$$u'' + u = 0.$$

Thus general solution of the reduced equation is $u = A \sin x + B \cos x$. For the original equation, the general solution is

$$y = A \frac{\sin x}{\sqrt{x}} + B \frac{\cos x}{\sqrt{x}}.$$

4. Using recurrence relations, show the following for Bessel function J_n:

(i)**(T)** $J_0''(x) = -J_0(x) + J_1(x)/x$ \qquad (ii) $xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$

Solution:

Useful identities for problems with Bessel's functions:

$$\left(x^\nu J_\nu\right)' = x^\nu J_{\nu-1}, \quad \left(x^{-\nu} J_\nu\right)' = -x^{-\nu} J_{\nu+1},$$

$$J_{\nu-1} + J_{\nu+1} = 2\nu J_\nu/x, \quad J_{\nu-1} - J_{\nu+1} = 2J_\nu'.$$

(i)
$$2J_0'(x) = J_{-1}(x) - J_1(x) = -2J_1(x)$$
$$\implies 2J_0''(x) = -2J_1'(x) = J_2(x) - J_0(x) = 2J_1(x)/x - 2J_0(x)$$
$$\implies J_0''(x) = J_1(x)/x - J_0(x)$$

(ii)
$$\left(x^{n+1}J_{n+1}(x)\right)' = x^{n+1}J_n(x) \implies xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$$

5. Express
(i)(**T**) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
(iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

Solution:

(i) Using the identity $J_{\nu+1} = 2\nu J_\nu/x - J_{\nu-1}$ we have

$$\begin{aligned}
J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) = \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right) - J_1(x) \\
&= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)
\end{aligned}$$

(ii) Using identities involving Bessel's function, we get

$$2J_2'(x) = J_1(x) - J_3(x) = J_1(x) - \left(\frac{4}{x}J_2(x) - J_1(x)\right) = 2J_1(x) - \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right)$$

$$\text{Hence} \quad J_2'(x) = \frac{2}{x}J_0(x) + \left(1 - \frac{4}{x^2}\right)J_1(x)$$

(iii) Using the identity $J_{\nu+1} = 2\nu J_\nu/x - J_{\nu-1}$, we get

$$\begin{aligned}
J_4(ax) &= \frac{6}{ax}J_3(ax) - J_2(ax) = \frac{6}{ax}\left(\frac{4}{ax}J_2(ax) - J_1(ax)\right) - J_2(ax) \\
&= \left(\frac{24}{a^2x^2} - 1\right)J_2(ax) - \frac{6}{ax}J_1(ax) \\
&= \left(\frac{24}{a^2x^2} - 1\right)\left(\frac{2}{ax}J_1(ax) - J_0(ax)\right) - \frac{6}{ax}J_1(ax) \\
&= \frac{1}{ax}\left(\frac{48}{a^2x^2} - 8\right)J_1(ax) - \left(\frac{24}{a^2x^2} - 1\right)J_0(ax)
\end{aligned}$$

6. Prove that between each pair of consecutive positive zeros of Bessel function $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

Solution:

Let α and β be two consecutive positive zeros of $J_{\nu+1}$. Let $f(x) = x^{\nu+1}J_{\nu+1}$. Then $f(\alpha) = f(\beta) = 0$. Thus there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$. Taking $\gamma = \nu + 1$ in $[x^\gamma J_\gamma]' = x^\gamma J_{\gamma-1}$, we see that $J_\nu(c) = 0$. Thus there exists a zero of J_ν between

consecutive zeros of $J_{\nu+1}$. Similarly taking $\gamma = \nu$ in $[x^{-\gamma} J_\gamma]' = -x^{-\gamma} J_{\gamma+1}$, we conclude that there exists a zero of $J_{\nu+1}$ between consecutive positive zeros of J_ν. To prove uniqueness, let there exist two zero of J_ν between consecutive zeros α and β of $J_{\nu+1}$. This implies that there exist a zero of $J_{\nu+1}$ between α and β, which contradicts the fact that α and β are consecutive zeroes.

7. Show that the Bessel functions J_ν ($\nu \geq 0$) satisfy

$$\int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x) \, dx = \frac{1}{2} J_{\nu+1}^2(\lambda_n) \delta_{mn},$$

where λ_i are the positive zeros of J_ν.

Solution:

We know that $y(t) = J_\nu(t)$ satisfies

$$\ddot{y} + \frac{1}{t}\dot{y} + \left(1 - \frac{\nu^2}{t^2}\right) y = 0, \qquad \cdot \equiv \frac{d}{dt}$$

Let $t = \lambda x \implies y(t) = y(\lambda x) = u(x)$. Then $u'(x) = \lambda \dot{y}$ and $u''(x) = \lambda^2 \ddot{y}$. Hence $u(x) = J_\nu(\lambda x)$ satisfies

$$u'' + \frac{1}{x}u' + \left(\lambda^2 - \frac{\nu^2}{x^2}\right) u = 0, \tag{1}$$

Similarly, $v(x) = J_\nu(\mu x)$ satisfies

$$v'' + \frac{1}{x}v' + \left(\mu^2 - \frac{\nu^2}{x^2}\right) v = 0. \tag{2}$$

Multiplying (1) by v and (2) by u and subtracting, we find

$$\frac{d}{dx}\left[x(u'v - uv')\right] = \left(\mu^2 - \lambda^2\right) x u v.$$

Integrating from $x = 0$ to $x = 1$, we find

$$\left(\mu^2 - \lambda^2\right) \int_0^1 x u v \, dx = u'(1)v(1) - u(1)v'(1). \tag{3}$$

Now $u(1) = J_\nu(\lambda)$ and $v(1) = J_\nu(\mu)$. Let us choose $\lambda = \lambda_m$ and $\mu = \lambda_n$, where λ_m and λ_n are positive zeros of J_ν. Then $u(1) = v(1) = 0$ and thus find

$$(\lambda_n^2 - \lambda_m^2) \int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x) \, dx = 0.$$

If $n \neq m$, then

$$\int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x) \, dx = 0.$$

Now from (3), we find [since $u'(x) = \lambda J_\nu'(\lambda x)$ etc]

$$\begin{aligned}
\int_0^1 x J_\nu^2(\lambda x) \, dx &= \lim_{\mu \to \lambda} \frac{\lambda J_\nu'(\lambda) J_\nu(\mu) - \mu J_\nu(\lambda) J_\nu'(\mu)}{\mu^2 - \lambda^2} \\
&= \frac{\lambda\left(J_\nu'(\lambda)\right)^2 - J_\nu(\lambda) J_\nu'(\lambda) - \lambda J_\nu(\lambda) J_\nu''(\lambda)}{2\lambda}
\end{aligned}$$

Now if we take $\lambda = \lambda_n$, where λ_n is a positive zero of J_ν, then we find

$$\int_0^1 x J_\nu^2(\lambda_n x)\, dx = \frac{1}{2}\left(J_\nu'(\lambda_n)\right)^2.$$

Now from

$$\left(x^{-\nu} J_\nu(x)\right)' = -x^{-\nu} J_{\nu+1}(x) \implies J_\nu'(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x),$$

we find by substituting $x = \lambda_n$

$$J_\nu'(\lambda_n) = -J_{\nu+1}(\lambda_n).$$

Thus, finally we get

$$\int_0^1 x J_\nu^2(\lambda_n x)\, dx = \frac{1}{2} J_{\nu+1}^2(\lambda_n).$$

Laplace Transform

1. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at)$ $(a > 0)$.

   Solution:

   $$\mathcal{L}\big(f(at)\big) = \int_0^\infty e^{-st} f(at)\, dt = \frac{1}{a}\int_0^\infty e^{-(s/a)\tau} f(\tau)\, d\tau = \frac{1}{a} F(s/a)$$

2. Find the Laplace transforms:

   (a) $[t]$ (greatest integer function),   (b) $t^m \cosh bt$ $(m \in \text{non-negative integers})$,

   (**T**)(c) $e^t \sin at$,   (d) $\dfrac{e^t \sin at}{t}$,   (e) $\dfrac{\sin t \cosh t}{t}$,   (f) $f(t) = \begin{cases} \sin 3t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases}$

   Solution:

   (a)

   $$\begin{aligned}
   \mathcal{L}\big([t]\big) &= \int_1^2 e^{-st}\, dt + 2\int_2^3 e^{-st}\, dt + 3\int_3^4 e^{-st}\, dt + \cdots \\[2mm]
   &= \frac{e^{-s}}{s}(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots) = \frac{e^{-s}}{s(1 - e^{-s})} \quad (s > 0 \implies 0 < e^{-s} < 1)
   \end{aligned}$$

   (b)

   $$\begin{aligned}
   \mathcal{L}(t^m) &= \frac{m!}{s^{m+1}} \implies \mathcal{L}(t^m \cosh bt) = \frac{1}{2}\mathcal{L}(e^{bt}t^m + e^{-bt}t^m) \\[2mm]
   &= \frac{m!}{2}\left(\frac{1}{(s-b)^{m+1}} + \frac{1}{(s+b)^{m+1}}\right)
   \end{aligned}$$

   (c)

   $$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2} \implies \mathcal{L}(e^t \sin at) = \frac{a}{(s-1)^2 + a^2}$$

(d) Use $\mathcal{L}\big(f(t)/t\big) = \int_s^\infty F(s)\,ds$. Now

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}$$

$$\Longrightarrow \mathcal{L}\left(\frac{\sin at}{t}\right) = a\int_s^\infty \frac{ds}{s^2 + a^2} = \frac{\pi}{2} - \tan^{-1}(s/a)$$

$$\Longrightarrow \mathcal{L}\left(\frac{e^t \sin at}{t}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s-1}{a}\right)$$

(e) Using result of the previous question

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1}(s) \Longrightarrow \mathcal{L}\left(\frac{\cosh t \sin t}{t}\right) = \frac{1}{2}\left(\frac{e^t \sin t}{t} + \frac{e^{-t} \sin t}{t}\right)$$

$$= \frac{1}{2}\left(\pi - \tan^{-1}(s-1) - \tan^{-1}(s+1)\right)$$

(f)
$$\mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st} \sin 3t\,dt = \frac{3(1 + e^{-\pi s})}{s^2 + 9}$$

1. Find the Laplace transforms (Hint: use second shifting theorem):

(a) $f(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \cos t, & t > 2\pi, \end{cases}$

(b) $f(t) = \begin{cases} 0, & 0 < t < 1, \\ \cos(\pi t), & 1 < t < 2, \\ 0, & t > 2 \end{cases}$

Solution:

(a) Consider $g(t) = u(t) - u(t-\pi) + u(t-2\pi)\cos t = u(t) - u(t-\pi) + u(t-2\pi)\cos(t - 2\pi)$

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\big(g(t)\big) = \frac{1}{s} - e^{-\pi s}\frac{1}{s} + e^{-2\pi s}\frac{s}{s^2 + 1}$$

(b) Consider $g(t) = \big(u(t-1) - u(t-2)\big)\cos(\pi t) = -u(t-1)\cos\pi(t-1) - u(t-2)\cos\pi(t-2)$

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\big(g(t)\big) = -\left(e^{-s}\frac{s}{s^2 + \pi^2} + e^{-2s}\frac{s}{s^2 + \pi^2}\right)$$

2. Find the inverse Laplace transforms of

(a) $\tan^{-1}(a/s)$, (b)$\ln\dfrac{s^2 + 1}{(s+1)^2}$, (**T**)(c)$\dfrac{s+2}{(s^2 + 4s - 5)^2}$, (d) $\dfrac{se^{-\pi s}}{s^2 + 4}$, (e) $\dfrac{(1 - e^{-2s})(1 - 3e^{-2s})}{s^2}$.

Solution:

(a) Use $\mathcal{L}\big(-tf(t)\big) = F'(s)$. Thus,

$$F'(s) = -\frac{a}{s^2 + a^2} \Longrightarrow \mathcal{L}^{-1}\big(F'(s)\big) = -\sin at \Longrightarrow f(t) = \frac{\sin at}{t}$$

(b)

$$F'(s) = \frac{2s}{s^2 + 1} - \frac{2}{s+1} \implies \mathcal{L}^{-1}\big(F'(s)\big) = 2(\cos t - e^{-t}) \implies f(t) = \frac{2(e^{-t} - \cos t)}{t}$$

(c)

$$F(s) = \frac{s+2}{(s^2 + 4s - 5)^2} = \frac{1}{12}\left(\frac{1}{(s-1)^2} - \frac{1}{(s+5)^2}\right)$$

$$F'(s) = \frac{1}{12}\left(\frac{2}{(s+5)^3} - \frac{2}{(s-1)^3}\right) \implies \mathcal{L}^{-1}\big(F'(s)\big) = \frac{t^2 e^{-5t} - t^2 e^t}{12}$$

Thus,

$$f(t) = t\frac{e^t - e^{-5t}}{12}$$

(d)

$$\frac{se^{-\pi s}}{s^2 + 4} = e^{-\pi s}\mathcal{L}(\cos 2t) = \mathcal{L}\big(u(t - \pi)\cos 2(t - \pi)\big)$$

Thus,

$$\mathcal{L}^{-1}\left(\frac{se^{-\pi s}}{s^2 + 4}\right) = u(t - \pi)\cos 2t$$

(e)

$$\frac{(1 - e^{-2s})(1 - 3e^{-2s})}{s^2} = \frac{1}{s^2} - \frac{4e^{-2s}}{s^2} + \frac{3e^{-4s}}{s^2}$$

Thus,

$$f(t) = t - 4u(t - 2)(t - 2) + 3(t - 4)u(t - 4)$$

3. Using convolution, find the inverse Laplace transforms:

(T)(a) $\dfrac{1}{s^2 - 5s + 6}$,   (b) $\dfrac{2}{s^2 - 1}$,   (c) $\dfrac{1}{s^2(s^2 + 4)}$,   (d) $\dfrac{1}{(s-1)^2}$.

Solution:

(a)

$$F(s) = \frac{1}{s^2 - 5s + 6} = \frac{1}{(s-3)(s-2)}$$

Now

$$\mathcal{L}(e^{3t}) = \frac{1}{s-3}, \quad \mathcal{L}(e^{2t}) = \frac{1}{s-2}.$$

Hence,

$$f(t) = \int_0^t e^{3\tau}e^{2(t-\tau)}d\tau = e^{2t}\int_0^t e^{\tau}d\tau = e^{3t} - e^{2t}$$

(b)

$$F(s) = \frac{2}{s^2 - 1} = \frac{2}{(s+1)(s-1)}$$

Now

$$\mathcal{L}(e^t) = \frac{1}{s-1}, \quad \mathcal{L}(e^{-t}) = \frac{1}{s+1}.$$

Hence,

$$f(t) = 2\int_0^t e^{\tau}e^{-(t-\tau)}d\tau = 2e^{-t}\int_0^t e^{2\tau}d\tau = e^t - e^{-t} = 2\sinh t$$

(c)

$$F(s) = \frac{1}{s^2(s^2+4)} = \frac{1}{2}\frac{1}{s^2}\frac{2}{s^2+4}$$

Now

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad \mathcal{L}(\sin 2t) = \frac{2}{s^2+4}.$$

Hence,

$$f(t) = \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)d\tau = \frac{2t-\sin 2t}{8}$$

(d)

$$F(s) = \frac{1}{(s-1)^2} = \frac{1}{s-1}\frac{1}{s-1}$$

Now

$$\mathcal{L}(e^t) = \frac{1}{s-1}.$$

Hence,

$$f(t) = \int_0^t e^\tau e^{t-\tau}d\tau = e^t\int_0^t d\tau = te^t$$

6. Use Laplace transform to solve the initial value problems:

(a) $y'' + 4y = \cos 2t,$ $\quad y(0) = 0, y'(0) = 1.$

(**T**)(b) $y'' + 3y' + 2y = \begin{cases} 4t & \text{if } 0 < t < 1 \\ 8 & \text{if } t > 1 \end{cases}$ $\quad y(0) = y'(0) = 0$

(c) $y'' + 9y = \begin{cases} 8\sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases}$ $\quad y(0) = 0, y'(0) = 4$

(d) $y_1' + 2y_1 + 6\int_0^t y_2(\tau)\,d\tau = 2u(t), \quad y_1' + y_2' = -y_2, \quad y_1(0) = -5, y_2(0) = 6$

Solution:

(a) Taking Laplace Transform on both sides and simplifying (Y(s)=$\mathcal{L}$[y(t)])

$$Y(s) = s/(s^2+4)^2 + 1/(s^2+4)$$

Using convolution [or any other technique]

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^t \sin(2\tau)\cos(2(t-\tau))d\tau + \frac{\sin 2t}{2} \\ &= \frac{t\sin 2t}{4} + \frac{\sin 2t}{2} \end{aligned}$$

(b) Let $r(t) = 4(u(t) - u(t-1))t + 8u(t-1) = 4u(t-0)t + 4u(t-1)(1-(t-1))$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2 + 3s + 2)Y(s) = R(s) \implies Y(s) = \frac{4}{s^2(s+1)(s+2)} + e^{-s}\frac{4(s-1)}{s^2(s+1)(s+2)}$$

Using partial fraction and shifting theorem we get

$$Y(s) = \left(-\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}\right) + e^{-s}\left(\frac{5}{s} - \frac{2}{s^2} - \frac{8}{s+1} + \frac{3}{s+2}\right)$$

$$\implies y(t) = -3 + 2t + 4e^{-t} - e^{-2t} + u(t-1)\left(5 - 2(t-1) - 8e^{-(t-1)} + 3e^{-2(t-1)}\right)$$

(c) Let $r(t) = 8(u(t) - u(t-\pi))\sin t = 8u(t)\sin t + u(t-\pi)\sin(t-\pi)$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2 + 9)Y(s) = R(s) + 4 \implies Y(s) = \frac{4}{s^2+9} + \frac{R(s)}{s^2+9}$$

We can explicitly write $R(s)$ and then use partial fraction technique.

$$Y(s) = \frac{4}{s^2+9} + (1+e^{-\pi s})\frac{8}{(s^2+1)(s^2+9)} = \frac{4}{s^2+9} + (1+e^{-\pi s})\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

This gives

$$\begin{aligned}
y(t) &= \frac{4}{3}\sin 3t + \left(\sin t - \frac{1}{3}\sin 3t\right) + u(t-\pi)\left(\sin(t-\pi) - \frac{1}{3}\sin 3(t-\pi)\right) \\
&= \sin t + \sin 3t + u(t-\pi)\left(\frac{1}{3}\sin 3t - \sin t\right)
\end{aligned}$$

$\Bigg($ Otherwise, use convolution as follows

$$y(t) = \frac{4}{3}\sin 3t + \frac{1}{3}\int_0^t r(\tau)\sin 3(t-\tau)\,d\tau$$

Thus for $0 < t < \pi$, we get

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^t \sin\tau \sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t + \sin t - \frac{1}{3}\sin 3t = \sin 3t + \sin t$$

and for $t > \pi$, we get [since $r(t) = 0$]

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^\pi \sin\tau \sin 3(t-\tau)\,d\tau + \frac{1}{3}\int_\pi^t 0 \sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t$$

This solution matches with that obtained earlier. $\Bigg)$

(d) Taking Laplace transform, we get

$$\begin{aligned}
(s+2)Y_1 + \frac{6Y_2}{s} &= \frac{2}{s} - 5 \\
sY_1 + (s+1)Y_2 &= 1
\end{aligned}$$

Solving

$$\begin{aligned}
Y_1(s) &= \frac{1}{s} - \frac{12}{5}\frac{1}{s-1} - \frac{18}{5}\frac{1}{s+4} \\
Y_2(s) &= \frac{6}{5}\frac{1}{s-1} + \frac{24}{5}\frac{1}{s+4}
\end{aligned}$$

Thus,

$$y_1(t) = 1 - \frac{12}{5}e^t - \frac{18}{5}e^{-4t}$$

$$y_2(t) = \frac{6}{5}e^t + \frac{24}{5}e^{-4t}$$

7. Solve the integral equations:

(a) $y(t) + \int_0^t y(\tau)\, d\tau = u(t-a) + u(t-b)$

(b) $e^{-t} = y(t) + 2\int_0^t \cos(t-\tau)y(\tau)\, d\tau$

(c) $3\sin 2t = y(t) + \int_0^t (t-\tau)y(\tau)\, d\tau$

Solution:

(a) Taking Laplace Transform, we get

$$Y(s) = \frac{e^{-as}}{s+1} + \frac{e^{-bs}}{s+1} \implies y(t) = u(t-a)e^{-(t-a)} + u(t-b)e^{-(t-b)}$$

(b) Taking Laplace Transform, we get

$$Y(s) = \frac{s^2+1}{(s+1)^3} = \frac{1}{1+s} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

Thus,

$$y(t) = e^{-t}(t-1)^2$$

(c) Taking Laplace Transform, we get

$$Y(s) = -\frac{2}{s^2+1} + \frac{8}{(s^2+4)} \implies y(t) = -2\sin t + 4\sin 2t$$