ASSIGNMENT 6
MTH102A

(1) Describe all 2×2 orthogonal matrices. Prove that the action of an orthogonal matrix on $\mathbb{R}^2$ is composition of rotation and a reflection about a line.

Ans : Let A be a non-zero orthogonal matrix, then it a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ preserving the dot product. $A(0,0) = (0,0)$. There exists $(x_0, y_0) \neq (0,0)$ such that $A(x_0, y_0) \neq (0,0)$. Let θ be the angle between (x_0, y_0) and $A(x_0, y_0)$ Consider the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. R_θ is rotation about origin and it takes (x_0, y_0) to $A(x_0, y_0)$. $R_{-\theta}$ takes $A(x_0, y_0)$ to (x_0, y_0). Consider $R_{-\theta} \circ A$, it is again an orthogonal matrix which fixes origin and (x_0, y_0) i.e. $R_{-\theta} \circ A(0,0) = (0,0)$ and $R_{-\theta} \circ A(x_0, y_0) = (x_0, y_0)$.

Let L be the line passing through origin and (x_0, y_0). Being an orthogonal transformation, $R_{-\theta} \circ A$ preserves inner product hence it preserves distances between any two points $u, v \in \mathbb{R}^2$ (i.e. $||Au - Av|| = ||u - v||$ for all $u, v \in \mathbb{R}^2$). $R_{-\theta} \circ A$ fixes two points of L pointwise then by triangle inequality we have $R_{-\theta} \circ A$ fixes L pointwise.

Let $B = R_{-\theta} \circ A$. Let (x_1, y_1) be any point in $\mathbb{R}^2$ not in L. Now

$$||B(x_1, y_1) - B(p)||^2 = ||(x_1, y_1) - p||^2$$

for all $p \in L$. As $B(p) = p$, it implies $< B(x_1, y_1), p >=< (x_1, y_1) - p >$. So, $< B(x_1, y_1) - (x_1, y_1), p >= 0$ for all $p \in L$. Hence, either $B(x_1, y_1) = (x_1, y_1)$ or $B(x_1, y_1)$ is reflection of (x_1, y_1) about L. Let $P_L : \mathbb{R}^2 to \mathbb{R}^2$ be reflection about the line. Consider the composition $P_L \circ B$. Then either $B(x_1, y_1) = (x_1, y_1)$ or $P_L \circ B(x_1, y_1) = (x_1, y_1)$. Thus, we have a distance preserving map in $\mathbb{R}^2$ which fixes more than three non-collinear points, so it has to be identity map. Hence, either $B = Id$ or $P_L \circ B = Id$ i.e.

$$\text{either } R_{-\theta} \circ A = Id \text{ or } P_L \circ R_{-\theta} \circ A = Id.$$

Inverse of P_L is P_L, So, $A = R_\theta$ or $A = R_\theta \circ P_L$.

(2) Let $v, w \in \mathbb{R}^n$ with $n \geq 2$, with $||v|| = ||w||$. Prove that there exists an orthogonal matrix A of order n such that $A(v) = w$ and $det(A) = 1$.

Ans. If $v = w$, Then we can take A to Identitiy Matrix $I_{n\times n}$. Suppose $v \neq w$.Extend $\{v\}$ and $\{w\}$ to orthogonal basis of $\mathbb{R}^n$ say $\mathbb{B} = \{v_i\}_{1\leq i\leq n}$ and $\mathbb{B}' = \{w_i\}_{1\leq i\leq n}$, where $v_1 = \frac{v}{||v||}$ and $w_1 = \frac{w}{||w||}$. Now let T be linear transformation of $\mathbb{R}^n$ s.t.

$$T(v_i) = w_i, \quad for\ 1 \leq i \leq n\ ...(*)$$

Verify: T is an orthogonal transformation.

Now $T(v_1) = w_1 \implies ||w||T(v) = ||v||w$, as $v \neq w$ and $||v|| = ||w||$ therefore

T(v)=w.
Let A be the matrix of T, Then $det(A) = 1 or - 1$ If $det(A) = -1$,then change of order of basis will give desired matrix.

(3) Find the eigen values and eigen vectors of the matrix

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$$

Ans: characterisitic polynomial $= x(x+2)(x+3)$ so eigenvalues are $= 0, -2, -3$ eigenvector corresponding $\lambda = 0$ is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\lambda = -2$ is $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\lambda = -3$ is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(4) Let A be an $n \times n$ matrix.
(i) Prove that A is invertible if and only if all the eigen values of A are not zero.
(ii) Show if A is invertible the eigen values of A^{-1} are reciprocals of eigen values of A.
Ans:(i) $det(A)$ is product of eigen values . $det(A) \neq 0$ if and only if none of the eigen values are zero.

(ii)A invertible means all eigenvalues are non zero(by(i)). Let λ be an eigenvalue of A. Then $det(A^{-1} - \lambda^{-1}I) = \frac{1}{\lambda}det((\lambda I - A)A^{-1}) = \frac{1}{\lambda}det(\lambda I - A)det(A^{-1})$, as λ be an eigenvalue of A, therefore $det(A^{-1} - \lambda^{-1}I) = 0$

(5) Let A be an $n \times n$ matrix and α be a scalar. Find eigen values of $A - \alpha I$ in terms of eigen values of A. Show that A and $A - \alpha I$ and A have eigen vectors.
Ans: Let λ be an eigenvalue of A and v be an eigenvector corresponding to λ. Then $(A - \alpha I)(v) = Av - \alpha v = (\lambda - \alpha)v \implies (\lambda - \alpha)$ is an eigenvalue of $A - \alpha I$ $\forall$ eigenvalue λ of A.
From above it is clear that eigenvectors of A are eigenvectors of $A - \alpha I$. Now let u be an eigenvector of $A - \alpha I$ corresponding to eigenvalue β i.e. $A - \alpha I(u) = \beta u$
$\implies Au = (\alpha + \beta)u$
Therefore u is also an eigenvector of A.

(6) Let A be an $n \times n$ matrix. Show that A and A^T have same eigen values. Do they have the same eigen vectors ?
Ans: $det(A^T - \lambda I) = det(A^T - (\lambda I)^T) = det((A - \lambda I)^T) = det(A - \lambda I)$,

therefore A and A^T Have same eigenvalues.

Take Matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,then 1 is eigenvalue of A and A^T,but eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

(7) Let A be an $n \times n$ matrix,

(i) If A is idempotent i.e. $A^2 = A$, show that eigen values of A are either 0 or 1.

(ii) If A is nilpotent i.e. $A^m = 0$ for some $m \geq 1$, show that all eigen values of A are 0.

Ans:(i) Let λ be an eigenvalue of A and x be an eigenvector corresponding to λ

Then $Ax = \lambda x \implies A(Ax) = A(\lambda x) \implies A^2 x = \lambda Ax \implies \lambda x = \lambda^2 x$

As x is non zero, therefore $\lambda - \lambda^2 = 0$,hence λ is either 0 or 1.

(ii) Let λ be an eigenvalue of A.

So, $Ax = \lambda x \implies A(Ax) = \lambda(Ax) \implies A^2(x) = \lambda^2 x \; . \; . \; . \; A^m(x) = \lambda^m x = 0$,as $A^m = 0$. Now since x is non zero,hence $\lambda = 0$.

(8) Let A be a square matrix with an eigen value λ and let u be the eigen vector corresponding to λ.

(i) Show that λ^k is an eigen value of A^k $(k \geq 1)$ with eigen vector u,

(ii) Suppose $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ is polynomial. Define the matrix $p(A) := a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n$. Prove that $p(\lambda)$ is also an eigen value of $p(A)$ with eigen vector u.

Ans(i): Check that $A^k(u) = (\lambda)^k u$, therefore λ^k is an eigen value of A^k with eigen vector u

Ans(ii): let α be a scalar then check that $(\alpha A)(u) = \alpha \lambda u$

Verify: $p(A)(u) = p(\lambda)u$ (use Part(i) and above observation.).Hence the result.

(9) Let A and B be square matrices of same order. Prove that characteristic polynomials of AB and BA are same.

Ans: If any one of the matrix,say A is invertible then $A^{-1}(AB)A = BA$, so AB and BA being similar have same characteristic polynomial.

Suppose none of the matrix is invertible.

Define two matrix C and D of order $n \times n$ as follows,

$C = \begin{pmatrix} xI_n & A \\ B & I_n \end{pmatrix}$ and $D = \begin{pmatrix} I_n & 0 \\ -B & xI_n \end{pmatrix}$. where I_n is identity matrix of order n and x is inderminate.

Now check that ,

$$det(CD) = x^n det(xI_n - AB)$$

$$det(DC) = x^n det(xI_n - BA)$$

as det(CD)=det(DC) we get

$$det(xI_n - AB) = det(xI_n - BA)$$

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