Chapter 11

Oscillation theory and the spectra of eigenvalues

The basic problems of the Sturm-Liouville theory are two: (1) to establish the existence of eigenvalues and eigenfunctions and describe them qualitatively and, to some extent, quantitatively and (2) to prove that an "arbitrary" function can be expressed as an infinite series of eigenfunctions. In this chapter we take up the first of these problems. The principal tool for this is the oscillation theory. It has extensions to other, related eigenvalue problems that are of interest in applications, and we shall take up two of these as well.

11.1 The Prüfer substitution

For a fixed value of the parameter λ the Sturm-Liouville differential equation takes the form

$$\frac{d}{dx}\left(P\left(x\right)\frac{du}{dx}\right)+Q\left(x\right)u=0. \tag{11.1}$$

We temporarily put aside the Sturm-Liouville problem and investigate this equation, under the assumptions

$$P\in C^{1} \text{ and } P>0,\ Q\in C \text{ on the interval } [a,b]. \tag{11.2}$$

Equations for which some solutions have two or more zeros in the open interval (a,b) will be called *oscillatory*[1]. Our first results will concern conditions

[1] One zero will not do: *any* equation of the form (11.1) has a solution with one zero.

guaranteeing that equation (11.1) is oscillatory and estimating the number of zeros that its solutions have. This is facilitated by a change of variables.

Define, in place of u and u' variables r and θ through the equations

$$u = r\sin\theta, \quad Pu' = r\cos\theta. \tag{11.3}$$

This substitution, which is called the *Prüfer* substitution after its discoverer, represents a legitimate change of variables provided r is never zero. Since we shall always assume that equation (11.1) is solved with initial data

$$u(a) = u_0, \; u'(a) = u_0', \; u_0, u_0' \text{ not both zero,} \tag{11.4}$$

and since $r = \sqrt{u^2 + P^2 u'^2}$, it is indeed never zero on the interval $[a, b]$. Differentiating the equations (11.3) and using the differential equation, we find the system

$$r' = \left(\frac{1}{P} - Q\right) r \sin\theta\cos\theta, \; \theta' = \frac{1}{P}(\cos\theta)^2 + Q(\sin\theta)^2. \tag{11.5}$$

This system is to be solved with initial data r_0, θ_0 such that

$$u_0 = r_0 \sin\theta_0, \; P(a)\, u_0' = r_0 \cos\theta_0. \tag{11.6}$$

For $\theta_0 \in [0, 2\pi)$ there is a unique solution of these equations for r_0 and θ_0. Likewise, given a solution of the system (11.5), one can easily check that equation (11.1) is satisfied. There is therefore a complete equivalence of these two systems; we'll use whichever is the more convenient in a given context.

The advantage of the Prüfer system for the study of the zeros of u are that (1) $u = 0$ whenever θ is a multiple of π and (2) the equation for θ is independent of r, i.e., is a first-order equation for θ alone. It is further true that for the study of zeros it is immaterial whether we consider u or $-u$, and for that reason we may assume that $u_0 \geq 0$. It is easy to verify that $\theta(a)$ can then be restricted to either of the half-open intervals $[0, \pi)$ and $(0, \pi]$; we choose the first of these[2] and can learn about the zeros of u by studying the first-order initial-value problem

$$\theta' = f(x, \theta) = \frac{1}{P}(\cos\theta)^2 + Q(\sin\theta)^2, \; \theta(a) = \gamma \in [0, \pi). \tag{11.7}$$

[2]We'll choose the second in connection with the right-hand boundary condition, to be considered later.

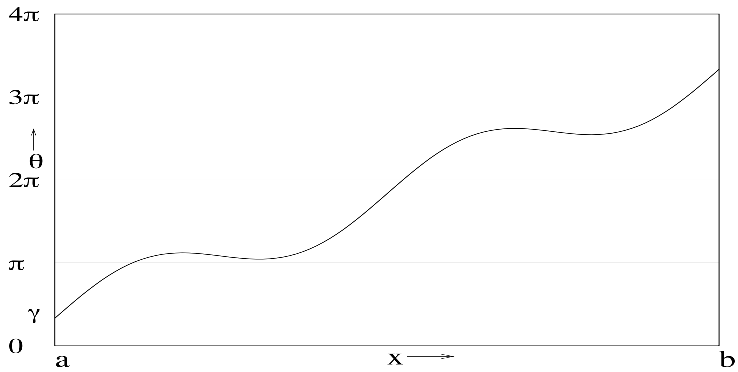

Figure 11.1: The behavior of the θ variable of the Prüfer substitution is illustrated here. It is not in general monotone-increasing, but once it reaches the value $n\pi$ at a certain point x_n, it remains greater than $n\pi$ for $x > x_n$.

The r-equation can be solved by quadratures if the solution to this equation is assumed known.

Since the solutions of equation (11.1) exist on the entire interval $[a, b]$, the same is true of equation (11.7) and it is not difficult to establish this directly (see the next problem set).

The solution θ of equation (11.7) need not be uniformly increasing on $[a, b]$, but it has the following similar property regarding points where θ is an integer multiple of π.

Proposition 11.1.1 *There is at most one value of $x \in [a, b]$ such that $\theta(x) = m\pi$, where m is an integer, say $x = x_m$. If $x < x_m$ we have $\theta(x) < m\pi$ and if $x > x_m$ we have $\theta(x) > m\pi$.*

Proof: Observe that if $x_m < b$ exists then

$$\theta'(x_m) = 1/P(x_m) > 0,$$

so that $\theta > m\pi$ at least on some sufficiently small interval to the right of x_m. Suppose for some $c > x_m$ that $\theta(c) \leq m\pi$. Consider the set $S = \{x \in (x_m, c] | \theta(x) \leq m\pi\}$. S has a greatest lower bound $x_* > x_m$ satisfying the conditions that $\theta - m\pi > 0$ on (x_m, x_*) and $\theta(x_*) - m\pi = 0$. It follows that $\theta'(x_*) \leq 0$. But the inequality above holds also at x_*: $\theta'(x_*) > 0$. This contradiction shows that there is no such point c and therefore that $\theta > m\pi$ for $x > x_m$. If there were another value $\tilde{x}$ where $\theta = m\pi$ then necessarily

$\tilde{x} < x_m$ and , on applying the reasoning above to $\tilde{x}$ we would arrive at the contradiction $\theta(x_m) > m\pi$. $\square$

This result is illustrated in Figure 11.1

11.2 Comparison theorems

In this section we present comparison theorems under assumptions appropriate to equations (11.7) and (11.1) above.

Theorem 11.2.1 *Suppose F and G are defined and continuous in a region $\Omega \subset R^2$ and $F(x,y) \geq G(x,y)$ there, and suppose further that each satisfies a Lipschitz condition with respect to y on Ω, with Lipschitz constant L. If there are functions $y(x)$ and $z(x)$ such that*

$$y' = F(x,y) \quad \text{and} \quad z' = G(x,z) \quad \text{on } (a,b), \tag{11.8}$$

and if, further, $y(a) \geq z(a)$, then $y(x) \geq z(x)$ on $[a,b]$.

Proof: Let $g(x) = z(x) - y(x)$. Then $g(a) \leq 0$ and the object is to show that $g \leq 0$ on $[a,b]$. Suppose to the contrary that for some $c \in (a,b)$ $g(c) > 0$. Consider the set

$$S = \{x \in [a,c] |\ g(x) \leq 0\}.$$

This set is bounded above by c and is not empty, so it has a least upper bound x_1. We must have $g(x_1) = 0$, and $g > 0$ on $(x_1, c]$. We'll consider the two equations on this interval.

The equivalent integral-equation formulations of equations (11.8) give

$$z(x) - y(x) = \int_{x_1}^{x} \{G(s, z(s)) - F(s, y(s))\}\ ds.$$

Rewrite the integrand:

$$G(s,z) - F(s,y) = (G(s,z) - G(s,y)) + (G(s,y) - F(s,y)).$$

The last terms on the right make a negative (or at least non-positive) contribution, so

$$z(x) - y(x) \leq \int_{x_1}^{x} \{G(s, z(s)) - G(s, y(s))\}\ ds.$$

The integrand is dominated by $L\left|z\left(s\right)-y\left(s\right)\right|$ where L is the Lipschitz constant. On the interval $\left(x_1,c\right]$ $g\left(s\right)=z\left(s\right)-y\left(s\right)>0$ so the absolute-value sign is not needed and the preceding equation reads

$$g\left(x\right)\leq L\int_{x_1}^{x}g\left(s\right)\,ds.$$

It now follows from Gronwall's lemma (Lemma 1.4.1) that $g\left(x\right)\leq 0$ on $\left(x_1,c\right]$, which is a contradiction. Consequently there is no such point c. □

This theorem can be strengthened by "running it backwards:"

Corollary 11.2.1 *Suppose that for some $x_*>a$ we have $y\left(x_*\right)=z\left(x_*\right)$. Then $y\left(x\right)=z\left(x\right)$ on $\left[a,x_*\right]$.*

Proof: Let $x=x_*-\xi$ so that for $0\leq\xi\leq x_*-a$ we have $x\in\left[a,x_*\right]$. Set $y\left(x\right)=\eta\left(\xi\right)$ and $z\left(x\right)=\zeta\left(\xi\right)$. The differential equations become

$$\eta'=\Phi\left(\xi,\eta\right),\quad \zeta'=\Psi\left(\xi,\zeta\right)$$

with initial data $\eta\left(0\right)=\zeta\left(0\right)$, $\Phi\left(\xi,\eta\right)=-F\left(x_*-\xi,\eta\right)$ and $\Psi\left(\xi,\zeta\right)=-G\left(x_*-\xi,\zeta\right)$. We now have $\Psi\geq\Phi$ and conclude on applying the theorem to these initial-value problems that $\zeta\geq\eta$ on $\left[0,x_*-a\right]$, i.e., that $y\left(x\right)\leq z\left(x\right)$ on $\left[a,x_*\right]$. But we know from the theorem that $y\left(x\right)\geq z\left(x\right)$ on this interval. □

We can apply these results to the θ equation of the Prüfer system. Consider the initial-value problems

$$\hat{\theta}' = \hat{Q}\left(\sin\hat{\theta}\right)^2+\frac{1}{\hat{P}}\left(\cos\hat{\theta}\right)^2\equiv G\left(x,\hat{\theta}\right),\quad \hat{\theta}\left(a\right)=\hat{\theta}_0, \tag{11.9}$$

$$\theta' = Q\left(\sin\theta\right)^2+\frac{1}{P}\left(\cos\theta\right)^2\equiv F\left(x,\theta\right),\quad \theta\left(a\right)=\theta_0. \tag{11.10}$$

Here the region $\Omega=\left[a,b\right]\times R$ and the coefficients all satisfy the conditions (11.2).

Proposition 11.2.1 *Suppose that in equations (11.9) and (11.10)*

$$Q\geq\hat{Q}\text{ and }P\leq\hat{P}.$$

If $\theta_0\geq\hat{\theta}_0$, then $\theta\left(x\right)\geq\hat{\theta}\left(x\right)$ on $\left[a,b\right]$. Moreover, if for some $x_*>a$ we have $\theta(x_*)=\hat{\theta}(x_*)$, then $\theta(x)=\hat{\theta}(x)$ on $\left[a,x_*\right]$.

Proof: The assumptions imply that $F \geq G$, so the conclusion follows from Theorem 11.2.1 above and its corollary. □

This result can be strengthened in a manner that will be useful below.

Corollary 11.2.2 *Suppose that one of the assumptions of Proposition 11.2.1 is strengthened: $Q > \hat{Q}$ on (a, b). Then $\theta(x) > \hat{\theta}(x)$ on $(a, b]$.*

Proof: Suppose, to the contrary, that $\theta(x_*) = \hat{\theta}(x_*)$ for some $x_* \in (a, b]$. By the Proposition $\theta = \hat{\theta}$ on $[a, x_*]$. It follows that

$$\left(Q - \hat{Q}\right)(\sin\theta)^2 + \left(\frac{1}{P} - \frac{1}{\hat{P}}\right)(\cos\theta)^2 = 0$$

on that interval. Each of the two terms is non-negative so each must vanish separately. Since $Q > \hat{Q}$ on (a, b) this is only possible if $\sin\theta = 0$ on (a, x_*), and hence by continuity on the interval $[a, x_*]$, and therefore θ is constant there. Equation (11.10) then leads to a contradiction. □

Remark: An alternative hypothesis leading to the same conclusion is that $P < \hat{P}$ and Q does not vanish identically on any open subinterval of $[a, b]$.

Proposition 11.2.1 provides a comparison theorem for solutions of equation (11.1).

Theorem 11.2.2 *(Sturm Comparison Theorem) Let $\hat{u}$ be a nontrivial solution of the equation*

$$\frac{d}{dx}\left(\hat{P}\frac{d\hat{u}}{dx}\right) + \hat{Q}\hat{u} = 0$$

vanishing at points x_1 and x_2 in $[a, b]$. Let u be any solution of equation (11.1) on this interval, and suppose that $Q \geq \hat{Q}$ and $P \leq \hat{P}$ on $[x_1, x_2]$. Then u vanishes at least once on this interval. If $\hat{u}$ has k zeros on this interval then u has at least $k - 1$ zeros there.

Proof: Consider the corresponding Prüfer equations (11.10) and (11.9). Since $\sin\hat{\theta}(x_1) = 0$, we may assign the value $\hat{\theta}(x_1) = 0$ without loss of generality. Similarly, at x_2, $\sin\hat{\theta} = 0$ and we have $\hat{\theta}(x_2) = m\pi$ where the integer m is at least equal to 1, by Proposition 11.1.1. For θ, the Prüfer variable corresponding to u, we may choose $\theta(x_1) \in [0, \pi)$. The conditions of Proposition 11.2.1 are satisfied and we infer that $\theta(x_2) \geq m\pi$. Therefore θ takes on the value π somewhere on the interval, and at that point $u = 0$. If $\hat{u}$ has k

zeros, including those at the endpoints, then u has at least $k-1$ zeros on $[x_1, x_2]$. □

This comparison theorem provides the key to showing that equation (11.1) is oscillatory and to estimating the number of zeros of its solutions. For the comparison equation choose the constant coefficients

$$\hat{P} = P_M \quad \text{and} \quad \hat{Q} = Q_m$$

where P_M is the maximum value of P on $[a, b]$ and Q_m is the minimum value of Q. The comparison equation is therefore

$$\hat{u}'' + \frac{Q_m}{P_M}\hat{u} = 0.$$

If Q_m/P_M is positive, this equation has the solution $\hat{u} = \sin(\kappa(x-a))$ on $[a, b]$, where $\kappa = \sqrt{Q_m/P_M}$. If $\kappa(b-a) > (k-1)\pi$, this has at least k zeros on $[a, b]$ and consequently any solution u of equation (11.1) must have at least $k-1$ zeros.

We need to be able to choose Q to be large. In the Sturm-Liouville theory the means for doing this is the presence of the parameter λ. We now turn to this.

11.3 The Sturm-Liouville Theorem

Recall the Sturm-Liouville system:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \{\lambda\rho(x) - q(x)\}u = 0, \quad a < x < b, \tag{11.11}$$

$$A[u] \equiv \alpha u(a) + \alpha' u'(a) = 0, \quad B[u] \equiv \beta u(b) + \beta' u'(b) = 0. \tag{11.12}$$

The conditions on the coefficients are now familiar and are given precisely in statement (10.12), but we remind the reader of two of these:

$$p(x) > 0 \text{ for each } x \text{ in the closed interval } [a, b] \tag{11.13}$$

and

$$\rho(x) > 0 \text{ for each } x \text{ at least in the open interval } (a, b). \tag{11.14}$$

Our objective is to prove the following

Theorem 11.3.1 *(Sturm-Liouville) The boundary-value problem (11.11), (11.12) has an infinite sequence of eigenvalues* $\{\lambda_n\}_0^\infty$ *with* $\lambda_n < \lambda_{n+1}$ *and* $\lambda_n \to +\infty$ *as* $n \to \infty$*. The eigenfunction* u_n *associated to* λ_n *has precisely* n *zeros in the open interval* (a, b)*.*

We shall obtain this result through consideration of the θ-equation of the corresponding Prüfer system,

$$\theta' = (\lambda\rho(x) - q(x))(\sin\theta)^2 + \frac{1}{p(x)}(\cos\theta)^2, \quad \theta(a) = \gamma, \tag{11.15}$$

where $\gamma \in [0, \pi)$ is chosen so that the right-hand boundary condition $A[u] = 0$ is satisfied, i.e., γ is the unique number in $[0, \pi)$ such that

$$\alpha \sin\gamma + \alpha' (P(a))^{-1} \cos\gamma = 0. \tag{11.16}$$

The solution $\theta = \theta(x, \lambda)$ is a continuous function of λ for each $x \in [a, b]$, by the discussion in §6.3.

Lemma 11.3.1 *The solution* $\theta(x, \lambda)$ *of the initial-vlaue problem (11.15) is, for each fixed* $x \in (a, b]$*, a continuous, monotonically increasing function of* λ*.*

Proof: The functions P, Q of the preceding sections take the values $P = p$ and $Q = \lambda\rho - q$, respectively, and a choice of comparison equation has coefficients $\hat{P} = p$ and $\hat{Q} = \hat{\lambda}\rho - q$. Consequently, if $\lambda > \hat{\lambda}$ then $Q > \hat{Q}$ on (a, b), and Corollary 11.2.2 implies that $\theta(x, \lambda) > \theta\left(x, \hat{\lambda}\right)$ for any $x > a$. In other words, for any $x > a$, θ is a strictly increasing function of λ. □

We denote by $x_n(\lambda)$ the nth zero of the solution u of equation (11.11) in (a, b), if it exists. The next lemma guarantees that it does exist and describes its behavior as function of λ.

Lemma 11.3.2 *Let* $u(x, \lambda)$ *be any solution of equation (11.11). For any* $n \geq 1$ *and sufficiently large* λ*,* $x_n(\lambda)$ *is defined and* $x_n(\lambda) \to a$ *as* $\lambda \to \infty$*.*

Proof: We first show that x_n exists if λ is large enough. Take for a comparison equation one with coefficients

$$\hat{P} = p_M, \quad \hat{Q} = \lambda\rho_m - q_M$$

where p_M, q_M denote the maximum values of p, q on $[a, b]$, and ρ_m the minimum value of ρ. Assume provisionally that $\rho_m > 0$. Then for a given positive value of λ the conditions of the Sturm Comparison Theorem hold. The comparison equation is

$$\hat{u}'' + \kappa^2 \hat{u} = 0, \quad \kappa^2 = \frac{\lambda \rho_m - q_M}{p_M},$$

which has a solution $\hat{u} = \sin(\kappa[x - a])$. If λ is large enough, κ^2 is not only positive but as large as we please. Choose λ so that $\kappa[b-a] > (n+1)\pi$. Then $\hat{u}$ has at least $n + 1$ zeros in (a, b) and, by the Sturm Comparison Theorem, u has at least n zeros in this interval. This shows that $x_n(\lambda)$ exists for large enough λ. It also establishes the final statement that $x_n(\lambda) \to a$, for the k-th zero of the comparison equation in (a, b) is $\hat{x}_k = a + k\pi/\kappa$. For any k this tends to a as λ tends to infinity. But $x_n(\lambda) \leq \hat{x}_{n+1}(\lambda)$, again by the Sturm Comparison Theorem.

If ρ vanishes at one or both endpoints, the analysis above must be modified. Choose ϵ, δ to be arbitrarily small positive numbers and consider the comparison equation on the interval $[a_1, b_1]$, where $a_1 = a + \epsilon$ and $b_1 = b - \delta$; if ϵ, δ are chosen small enough, this is an interval slightly smaller than $[a, b]$. On this modified interval the minimum $\rho_m > 0$. Repeating the analysis above we find that $\hat{u}$ has $n + 1$ zeros in (a_1, b_1) if λ is large enough, and that any solution of equation (11.11) has at least n zeros on this interval and therefore also on (a, b). This shows that $x_n(\lambda)$ exists. It also tends to a_1 as $\lambda \to \infty$, so for sufficiently large λ, $x_n(\lambda) - a_1 < \epsilon$ and therefore $x_n(\lambda) - a < 2\epsilon$. Since ϵ is arbitrary, this shows that $x_n \to a$. $\square$

Remark: We emphasize the phrase *any solution* in the statement of this lemma: irrespective of initial conditions, and for any $n \geq 1$, there exists a number Λ_n such that, if $\lambda > \Lambda_n$, $x_n(\lambda)$ exists and therefore $\theta(b, \lambda) > n\pi$.

We now obtain the decisive result, the asymptotic behavior of θ as $\lambda \to \pm\infty$.

Proposition 11.3.1 *For any* $x > a$, $\theta(x, \lambda) \to +\infty$ *as* $\lambda \to +\infty$ *and* $\theta(x, \lambda) \to 0$ *as* $\lambda \to -\infty$.

Proof: The first statement can be inferred from the preceding lemma, as follows. Since for any fixed n we have $x_n(\lambda) \to a$, for any $x > a$ we have $x_n(\lambda) < x$ for all sufficiently large λ. This means that $\theta(x, \lambda) > n\pi$, from which the first statement follows.

To prove the second statement we need to show that, given $\epsilon > 0$, there is a number Λ, depending in general on ϵ and on the choice of $x \in (a, b]$, such that $\theta(x, \lambda) < \epsilon$ provided $\lambda < \Lambda$. We proceed in stages.

1. Consider equation 11.15 for θ. The terms that are not multiplied by λ are bounded by $K = |q|_M + 1/p_m$, where the subscripts M and m denote the maximum and minimum values, respectively, that these functions take on the interval $[a, b]$. Therefore we have

$$\theta' \leq \lambda \rho(x) \left(sin\theta\right)^2 + K. \tag{11.17}$$

2. We first establish the result on the open interval (a, b) and return to a consideration of the right-hand point b later. We pick a point x_1 in this open interval.

3. The initial value $\theta(a, \lambda) = \gamma \in [0, \pi)$. For negative values of λ, $\theta' \leq K$ by equation (11.17) above. From this one infers via the mean-value theorem that $\theta(x, \lambda) \leq \gamma + K(x - a)$ and therefore, for sufficiently small values of $x - a$, $\theta < \pi$. In particular, given the value of ϵ, we can choose $a_1 \in (a, x_1)$ such that

$$\theta(a_1, \lambda) < \gamma_1 = \gamma + \epsilon < \pi - \epsilon.$$

For this we may first have to rechoose ϵ to be smaller than the originally prescribed value: this is permitted. We choose it so that $\gamma + 2\epsilon < \pi$. Since the estimate K for θ' is independent of λ, so also is the choice of a_1, provided only that $\lambda \leq 0$.

4. The straight line $\theta = s(x)$ connecting the points a_1, γ_1 and x_1, ϵ in the $x - \theta$ plane has slope

$$m = -\left(\gamma_1 - \epsilon\right) / \left(x_1 - a_1\right),$$

negative unless $\gamma = \gamma_1 - \epsilon = 0$. Let ρ_1 be the minimum value of ρ on $[a_1, x_1]$ and choose

$$\Lambda_1 = \left(m - K\right) / \rho_1 \left(\sin \epsilon\right)^2;$$

we shall suppose $\lambda < \Lambda_1$. At $x = a_1, \theta(x) < s(x)$. If for some value of x in the interval $[a_1, x_1]$ $\theta(x) > s(x)$ then there must be a first such value x_*, at which $\theta(x_*) = s(x_*)$ and

$$\theta'(x_*) \geq m.$$

But then $\theta(x_*) \in [\epsilon, \pi - \epsilon]$ so that $\sin\theta(x_*) \geq \sin\epsilon$ and

$$m < \lambda\rho(x_*)\left(\sin\theta(x_*)\right)^2 + K \leq \lambda\rho_1\left(\sin\epsilon\right)^2 + K < m$$

if $\lambda < \Lambda_1$ as given above. This contradiction shows that indeed $\theta(x_1, \lambda) < \epsilon$ if $\lambda < \Lambda_1$.

5. We have thus far excluded the right-hand endpoint b from consideration because it is possible that $\rho(b) = 0$. Choose $b_1 < b$ so that $\theta(b) \leq \theta(b_1) + \epsilon$. This can be done, independently of λ for $\lambda < 0$, because of the estimate $\theta'(x, \lambda) < K$. Then applying the reasoning above at b_1, we infer that $\theta(b_1, \lambda) < \epsilon$ – and therefore $\theta(b, \lambda) < 2\epsilon$ – provided $\lambda < \Lambda$. This completes the proof. □

Proof of Theorem 11.3.1: The initial condition on the function θ ensures that the left-hand boundary condition is satisfied. The corresponding solution u of equation (11.11) will be an eigenfunction if the second boundary condition is also satisfied. This will be so if $\theta(b, \lambda) = \delta + n\pi$ for $n = 0, 1, \ldots$, provided δ is such that

$$\beta\sin\delta + \beta'\left(p\left(b\right)\right)^{-1}\cos\delta = 0. \tag{11.18}$$

There is a unique value $\delta \in (0, \pi]$ satisfying this equation. Choosing this value, we now ask: can we find λ such that $\theta(b, \lambda) = \delta$? The answer is yes, because of Proposition 11.3.1; call this value λ_0 and the corresponding eigenfunction u_0. Since $\delta \leq \pi$, $\theta < \pi$ in (a, b) and therefore u does not vanish in this interval. Can we find λ such that $\theta(b, \lambda) = \delta + \pi$? Yes again, by virtue of Proposition 11.3.1, and the corresponding eigenfunction u_1 has a single zero in (a, b) since θ takes the value π there. Continuing in this way, we see that Proposition 11.3.1, in conjunction with Proposition 11.1.1, proves the Sturm-Liouville theorem. □

PROBLEM SET 11.3.1

1. Show that for any real k, the equation $u'' - k^2u = 0$ is non-oscillatory, i.e., has at most one zero on any interval.
2. Verify that any solution of the Prüfer system (11.5) provides a solution of equation (11.1) through the relations (11.3).

3. Let u be a solution of equation (11.1) that has zeros at points x_1 and x_2 in (a, b). Suppose v is any linearly independent solution. Show that v has a zero in the interval (x_1, x_2).

4. Let P be as described in the conditions (11.2) and suppose $Q = 1/P$. Solve equation (11.1) in quadratures (i.e., find a formula requiring only the integration of known functions).

5. Show that the function f defined by equation (11.7) satisfies a Lipschitz condition in the region $\Omega = [a, b] \times R$ and estimate the Lipschitz constant. Argue that equation (11.3) has a solution on $[a, b]$ without referring to equation (11.1).

6. In Theorem 11.2.1 it is assumed that each of the functions F and G satisfies a Lipschitz condition.

   (a) Show that it suffices if one them satisfies a Lipschitz condition.

   (b) Show that some condition stronger than continuity must be imposed on at least one of the functions by producing a counterexample otherwise (Hint: consider equations with non-unique solutions like $y' = \sqrt{y}$).

7. Prove the remark following Corollary 11.2.2.

8. Suppose that $q > 0$ in equation (11.11), and that the boundary conditions are $u\,(a) = u\,(b) = 0$. Show that all eigenvalues are positive.

9. Suppose that ρ is a positive, decreasing function, q a positive, strictly increasing function on (a, b), i.e., if $x_2 > x_1$ then

$$\rho(x_2) \leq \rho(x_1),\ q(x_2) > q(x_1),\ \rho > 0, q > 0 \text{ on } (a, b).$$

   Let u be an eigenfunction of the Sturm-Liouville problem (11.11), (11.12) with consecutive zeros at a_1 and a_2, and suppose that $u > 0$ on the interval (a_1, a_2). Show that u has a single maximum on that interval (Hint: if it has two, then there are three points of the interval at which $u' = 0$; examine the sign of pu''/u at these points).

10. For the eigenvalues $\{\lambda_k\}_0^\infty$ of the Sturm-Liouville problem, show that there exists a number $h > 0$ such that $\lambda_{k+1} - \lambda_k \geq h$ for all $k = 0, 1, 2, \ldots$ (Hint: assume the contrary and use the fact that the continuous function $\theta(b, \lambda)$ satisfies the condition $\theta(b, \lambda_{k+1}) - \theta(b, \lambda_k) = \pi$).