# MTH114: ODE: Assignment-3

1. (**T**) A surface $z = y^2 - x^2$ in the shape of a saddle is lying outdoors in a rainstorm. Find the paths along which raindrops will run down the surface.

2. (**T**) Does $f(x, y) = xy^2$ satisfies Lipschitz condition (LC) on any rectangle $[a, b] \times [c, d]$? What about on an infinite strip $[a, b] \times \mathbb{R}$?

   [A function $f(x, y)$ is said to satisfy Lipschitz condition on a domain $D \subseteq \mathbb{R}^2$, if there exists $L > 0$ such that $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ for all $(x, y_1), (x, y_2) \in D$.]

3. (**T**) Consider the IVP $y' = 2\sin(3xy),\ \ y(0) = y_0$. Show that it has unique solution in $(-\infty, \infty)$.

4. Consider the ODE $y' = \frac{2xy}{y^2 - x^2}$. Solve it. Sketch the solutions. Verify Picard theorem for initial values in $\mathbb{R}^2 - \{(x, y) :\ \ x^2 = y^2\}$. What is your solution passing through $(1, 0)$?

5. (**T**) What does Picard theorem says about existence and uniqueness of solution of the IVP $y' = (3/2)y^{1/3},\ \ y(0) = 0$? Show that it has uncountably many solutions.

6. Consider the IVP $y' = \sqrt{y} + 1,\ \ y(0) = 0,\ \ x \in [0, 1]$. Show that $f(x, y) = \sqrt{y} + 1$ does not satisfy Lipschitz condition in any rectangle containing origin, but still the solution is unique.

   (Remark: It is fact that if an IVP, with $f$ is continuous (not necessarily Lipschitz), has more than one solution, then it has uncountably many solutions. This is known as Kneser's Theorem. The previous exercise illustrates this phenomenan.)

7. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions:
   (i) (**T**) $y' = 2\sqrt{x},\ y(0) = 1$ (ii) $y' + xy = x,\ y(0) = 0$ (iii) $y' = 2\sqrt{y}/3,\ y(0) = 0$

8. Solve $y' = (y - x)^{2/3} + 1$. Show that $y = x$ is also a solution. What can be said about the uniqueness of the initial value problem consisting of the above equation with $y(x_0) = y_0$, where $(x_0, y_0)$ lies on the line $y = x$.

9. Discuss the existence and uniqueness of the solution of the initial value problem
$$(x^2 - 2x)y' = 2(x - 1)y, \qquad y(x_0) = y_0.$$

10. (**T**) Consider the IVP $y' = x - y,\ \ y(0) = 1$. Show that for Euler method, $y_n = 2(1-h)^n - 1 + nh$ where $h$ is the step size. ($x_n = nh$ with $x_0 = 0,\ y_0 = y(0) = 1$). Deduce that if we take $h = 1/n$, then the limit of $y_n$ converges to actual value of $y(1)$.

11. Use Euler method and step size .1 on the IVP $y' = x + y^2,\ \ y(0) = 1$ to calculate the approximate value for the solution $y(x)$ when $x = .1, .2, .3$. Is your answer for $y(.3)$ is higher or lower than the actual value ?

12. Verify that $y = x^2 \sin x$ and $y = 0$ are both solution of the initial value problem (IVP)
$$x^2y'' - 4xy' + (x^2 + 6)y = 0,\ \ y(0) = y'(0) = 0.$$
    Does it contradict uniqueness of solution of IVP?