End-Semster Exam MTH113M/MTH102A : SET B

Date : 19/02/2024 | Time : 6:00- 8:00 pm

NAME : SOLUTION	ROLL : END - SEM

Question B.1. Consider the vector space $C[-1, 1]$ of all real valued continuous functions defined over $[-1, 1]$. Define the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, where $f, g \in C[-1, 1]$. Let $W = \{f \in C[-1, 1] : f$ is an odd function$\}$.
(i) Show that the orthogonal complement $W^{\perp}$ of W is the subspace of all even functions in $C[-1, 1]$.
(ii) Let $f(x) = e^x$, where $x \in [-1, 1]$. Find $\inf\{\|f - g\| : g \in W\}$. [4+3=7 Marks]

Answer B.1.: (i) $W^{\perp} = \{ f \in C[-1,1] : \langle f, g\rangle = 0 \ \forall g \in W \}$

Let f be an even function & g be an odd function.

$\langle f, g\rangle = \int_{-1}^{1} f(x) g(x)\, dx$ Put $y = -x$

$= \int_{1}^{-1} f(-y) g(-y)\, d(-y) = -\int_{-1}^{1} f(y) g(y)\, dy$

$= -\langle f, g\rangle$

$\Rightarrow \langle f, g\rangle = 0$

Let E denote the subspace of even functions in $C[-1,1]$

Then $E \subseteq W^{\perp}$. $C[-1,1] = W \oplus W^{\perp}$.

Let $h \in C[-1,1]$

$h(x) = h_o(x) + h_e(x)$, where $h_o(x) \overset{\text{defn.}}{:=} \frac{1}{2}(h(x) - h(-x))$

& $h_e(x) \overset{\text{defn.}}{:=} \frac{1}{2}(h(x) + h(-x))$

$h_o \in W$ & $h_e \in E$.

Let $h \in W^{\perp} \Rightarrow h = h_o + h_e \Rightarrow h_o = h - h_e \in W^{\perp}$

as $h \in W^{\perp}$ & $h_e \in E \subseteq W^{\perp}$

$h_o \in W \Rightarrow h_o \in W \cap W^{\perp} = \{0\}$

$\Rightarrow h_o = 0 \Rightarrow h = h_e \in E$

Thus $W^{\perp} = E$

(ii) Let the orthogonal projection of f on W be $P_W(f)$, $f(x) = e^x$

Then $P_W(f) = f_o$, where $f_o(x) = \frac{e^x - e^{-x}}{2}$

$\inf\{\|f - g\| : g \in W\} = \|f - f_o\|$.

$\|f - f_o\|^2 = \langle f - f_o, f - f_o\rangle = \int_{-1}^{1} (f(x) - f_o(x))^2 dx = \int_{-1}^{1} \left(e^x - \frac{e^x - e^{-x}}{2}\right)^2 dx$

$= \int_{-1}^{1} \left(\frac{e^x + e^{-x}}{2}\right)^2 dx$

$= \frac{1}{4}(e^2 - e^{-2}) + 1$

$\Rightarrow \|f - f_o\| = \sqrt{\frac{1}{4}(e^2 - e^{-2}) + 1}$

Question B.2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$. [4+3=7 Marks]

Answer B.2.: Let $v_1 = (1,0,1,0)$, $v_2 = (0,1,0,1)$, $v_3 = (1,-1,1,1)$, $v_4 = (2,1,1,1)$

Apply Gram-Schmidt orthonormalization process on $\{v_1, v_2, v_3, v_4\}$.

Let $w_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{2}}(1,0,1,0)$

$\langle w_1, v_2\rangle = 0$. Let $w_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{1}{\sqrt{2}}(0,1,0,1)$

Let $u_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$

$= v_3 - \langle (1,-1,1,1), \frac{1}{\sqrt{2}}(1,0,1,0)\rangle w_1$
$\quad - \langle (1,-1,1,1), \frac{1}{\sqrt{2}}(0,1,0,1)\rangle w_2$

$= v_3 - \sqrt{2}\, w_1 = (1,-1,1,1) - \frac{\sqrt{2}}{\sqrt{2}}(1,0,1,0) = (0,-1,0,1)$

Let $w_3 = \dfrac{u_3}{\|u_3\|} = \dfrac{1}{\sqrt{2}}(0,-1,0,1)$

Let $v_4 = (2,1,1,1)$

Let $u_4 = v_4 - \langle v_4, w_1\rangle w_1 - \langle v_4, w_2\rangle w_2 - \langle v_4, w_3\rangle w_3$

$= v_4 - \langle (2,1,1,1), \frac{1}{\sqrt{2}}(1,0,1,0)\rangle w_1 - \langle (2,1,1,1), \frac{1}{\sqrt{2}}(0,1,0,1)\rangle w_2$
$\quad - \langle (2,1,1,1), \frac{1}{\sqrt{2}}(0,-1,0,1)\rangle w_3$

$= v_4 - \frac{3}{\sqrt{2}} w_1 - \sqrt{2}\, w_2 - 0\cdot w_3$

$= (2,1,1,1) - \frac{3}{2}(1,0,1,0) - (0,1,0,1) = \frac{1}{2}(1,0,-1,0)$

Let $w_4 = \dfrac{u_4}{\|u_4\|} = \sqrt{2}\, u_4$. Let $Q = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \end{pmatrix}$

Note

$v_1 = \sqrt{2}\, w_1$

$v_2 = 0 w_1 + \sqrt{2}\, w_2$

$v_3 = \sqrt{2}\, w_1 + 0 w_2 + \sqrt{2}\, w_3$

$v_4 = \frac{3}{\sqrt{2}} w_1 + \sqrt{2}\, w_2 + 0\cdot w_3 + \frac{1}{\sqrt{2}} w_4$

$\Rightarrow A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} & 3/\sqrt{2} \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix}}_{R}$

$\Rightarrow A = QR.$

Question B.3. Consider the point $b = (1, 2, 3, 4) \in \mathbb{R}^4$. Let W be the linear span of the set $\{(1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 1, 1)\}$ in $\mathbb{R}^4$. Find the orthogonal projection $P_W(b)$ of b on the subspace W. [6 Marks]

Answer B.3.:

Method 1: Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. $P_W(b) = Ax_0$ where x_0 is a solution of $A^TAx = A^Tb$.

Let $B = A^TA$

Then $B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ & $A^Tb = A^T\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$

Let $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The system $A^TA\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^Tb$

is $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$. $\det B = 7$

$$B^{-1} = \frac{1}{7}\begin{pmatrix} 5 & -2 & -1 \\ -2 & 5 & 1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$x_0 = B^{-1}\begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -2 \\ 5 \\ 20 \end{pmatrix}$$

Thus, $P_W(b) = Ax_0 = \frac{1}{7}A\begin{pmatrix} -2 \\ 5 \\ 20 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \\ 20 \end{pmatrix}$

$$= \frac{1}{7}\begin{pmatrix} 3 \\ 18 \\ 25 \\ 20 \end{pmatrix}$$

Method 2: Note $(1,-1,-1,2)$ is orthogonal to $(1,1,0,0)$, $(1,0,1,0)$ & $(0,1,1,1)$.

Let $(x,y,z,w) \in LS(\{(1,1,0,0),(1,0,1,0),(0,1,1,1)\}$.

Then $\langle (1,-1,-1,2),(x,y,z,w)\rangle = 0$

$\Rightarrow$ $x-y-z+2w=0$

Thus,

$$P_W(b) = (1,2,3,4) - \left\langle (1,2,3,4), \frac{(1,-1,-1,2)}{\|(1,-1,-1,2)\|}\right\rangle \frac{(1,-1,-1,2)}{\|(1,-1,-1,2)\|}$$

$$= (1,2,3,4) - \frac{4}{7}(1,-1,-1,2) = \frac{1}{7}(3,18,25,20).$$

Let $u_1 = (1,1,0,0)$, $u_2 = (1,0,1,0)$, $u_3 = (0,1,1,1)$

Apply Gram-Schmidt on $\{u_1, u_2, u_3\}$

Let $w_1 = \frac{1}{\sqrt{2}} u_1$ Let $v_2 = u_2 - \langle u_2, w_1\rangle w_1$

$$= u_2 - \tfrac{1}{\sqrt{2}} w_1$$
$$= (1,0,1,0) - \tfrac{1}{2}(1,1,0,0)$$
$$= (1/2, -1/2, 1, 0)$$

$\langle u_2, w_1\rangle = \frac{1}{\sqrt{2}}\langle (1,0,1,0), (1,1,0,0)\rangle = 1/\sqrt{2}$

Let $w_2 = \frac{v_2}{\|v_2\|} = \sqrt{2/3}\,(1/2, -1/2, 1, 0)$

$\langle u_3, w_2\rangle = \sqrt{2/3}\langle (0,1,1,1), (1/2,-1/2,1,0)\rangle = \sqrt{2/3}\,(-1/2+1) = \frac{1}{2}\sqrt{2/3}$

$\langle u_3, w_1\rangle = \frac{1}{\sqrt{2}}\langle (0,1,1,1), (1,1,0,0)\rangle = \frac{1}{\sqrt{2}}$

Let $v_3 = u_3 - \langle u_3, w_2\rangle w_2 - \langle u_3, w_1\rangle w_1$

$$= u_3 - \frac{1}{2}\sqrt{2/3}\, w_2 - \frac{1}{\sqrt{2}} w_1$$
$$= (0,1,1,1) - \frac{1}{2}\times\frac{2}{3}(1/2, -1/2, 1, 0) - \frac{1}{2}(1,1,0,0)$$
$$= (-1/6, 1+1/6, 1-1/3, 1) - (1/2, 1/2, 0, 0)$$
$$= (-2/3, 2/3, 2/3, 1).$$

Let $w_3 = \frac{v_3}{\|v_3\|} = \sqrt{3/7}\,(-2/3, 2/3, 2/3, 1)$. $3 - 1/2$

$b = (1,2,3,4)$

$P_W(b) = \langle b, w_1\rangle w_1 + \langle b, w_2\rangle w_2 + \langle b, w_3\rangle w_3$

$\langle b, w_1\rangle = \langle (1,2,3,4), \frac{1}{\sqrt{2}}(1,1,0,0)\rangle = 3/\sqrt{2}$

$\langle b, w_2\rangle = \langle (1,2,3,4), \sqrt{2/3}\,(1/2, -1/2, 1, 0)\rangle = \sqrt{2/3}\,(1/2 - 1 + 3) = 5/\sqrt{6}$

$\langle b, w_3\rangle = \langle (1,2,3,4), \sqrt{3/7}\,(-2/3, 2/3, 2/3, 1)\rangle = \sqrt{3/7}\,(-2/3 + 4/3 + 6/3 + 4) = \frac{20}{\sqrt{21}}$

$$P_W(b) = \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}}(1,1,0,0) + \frac{5}{\sqrt{6}}\sqrt{2/3}\,(1/2, -1/2, 1, 0) + \frac{20}{\sqrt{21}}\sqrt{3/7}\,(-2/3, 2/3, 2/3, 1)$$
$$= (3/2, 3/2, 0, 0) + (5/6, -5/6, 5/3, 0) + \left(\frac{-40}{21}, \frac{40}{21}, \frac{40}{21}, 20/7\right)$$
$$= \frac{1}{7}(3, 18, 25, 20)$$

Question B.4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Show the following:

(i) All eigenvalues of A are real.

(ii) Eigenvectors of A corresponding to distinct eigenvalues are orthogonal to each other. [3+3=6 Marks]

Answer B.4. : (i) Let λ be an eigenvalue of A with non-zero eigenvector u.

$$Au = \lambda u \quad \Rightarrow \quad \bar{u}^T \bar{A}^T = \bar{\lambda} \bar{u}^T$$

$$\Rightarrow \bar{u}^T A^T = \bar{\lambda} \bar{u}^T \quad \text{as } \bar{A} = A$$

$$\Rightarrow \bar{u}^T A = \bar{\lambda} \bar{u}^T \quad \text{as } A^T = A$$

$$\Rightarrow \bar{u}^T A u = \bar{\lambda} \bar{u}^T u$$

$$\bar{u}^T \lambda u = \bar{\lambda} \bar{u}^T u \quad \text{as } Au = \lambda u$$

$$\Rightarrow (\lambda - \bar{\lambda}) \bar{u}^T u = 0$$

$$u \neq 0, \; \bar{u}^T u \neq 0 \Rightarrow \lambda = \bar{\lambda}. \text{ Hence } \lambda \text{ is real.}$$

(ii) Let λ, μ be two distinct eigen values of A.

Let u, v be eigenvectors of A corresponding to λ, μ respectively.

$$Au = \lambda u, \quad Av = \mu v$$

Let $u = (u_1, \ldots, u_n)$
$v = (v_1, \ldots, v_n)$
& $A = (a_{ij})$

$$\langle Au, v \rangle = A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot (v_1, \ldots, v_n)$$

$$= \left(\sum_{j=1}^{n} a_{1j} u_j, \ldots, \sum_{j=1}^{n} a_{nj} u_j \right) \cdot (v_1, \ldots, v_n)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} v_i u_j$$

$$= (u_1, \ldots, u_n) \cdot \left(\sum_{i=1}^{n} a_{i1} v_i, \ldots, \sum_{i=1}^{n} a_{in} v_i \right)$$

$$= u \cdot A^T v = \langle u, A^T v \rangle$$

$$A^T = A \Rightarrow \langle Au, v \rangle = \langle u, Av \rangle$$

$$\langle \lambda u, v \rangle = \langle u, \mu v \rangle$$

$$\Rightarrow \lambda \langle u, v \rangle = \bar{\mu} \langle u, v \rangle = \mu \langle u, v \rangle \quad \text{as eigenvalues of } A \text{ are real}$$

$$\lambda \neq \mu \Rightarrow \langle u, v \rangle = 0$$ i.e. u & v are orthogonal to each other.