I. Shell Method

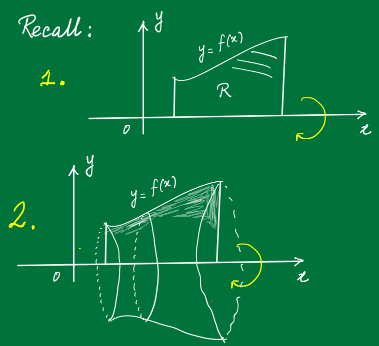
II. Areas of surfaces of revolution.

III. Pappus's Theorems

Shell Method – slices parallel to the axis of revolution which look like shells.

Recall:

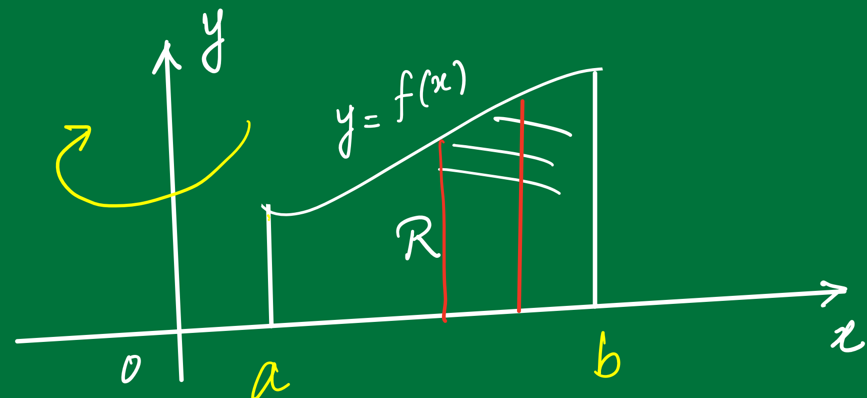
1.



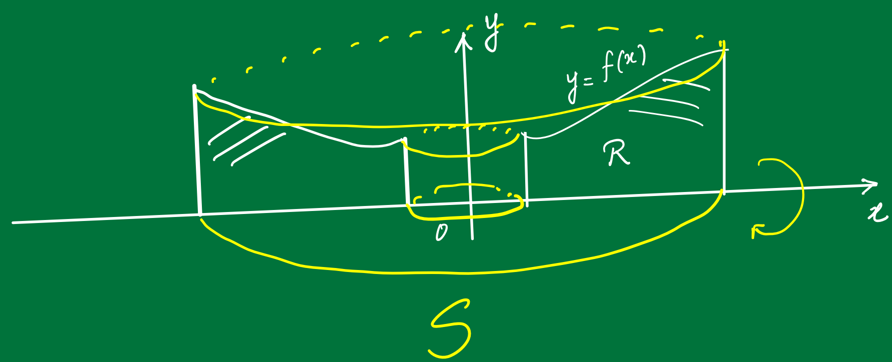
2.

Washer method

1'.



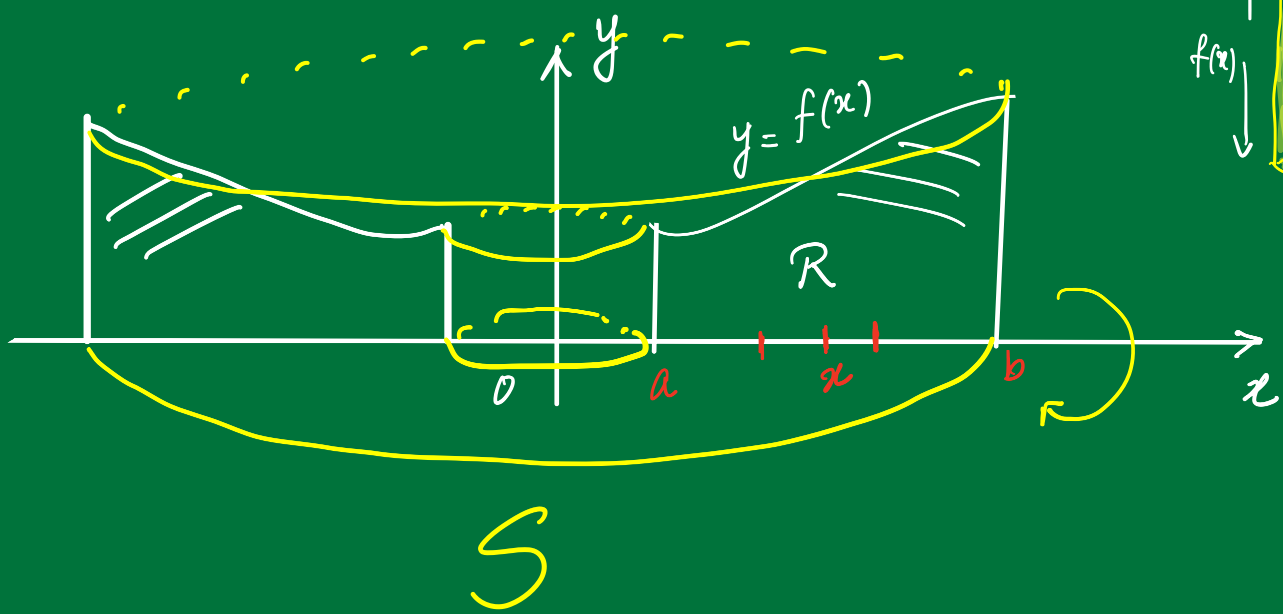
2'.

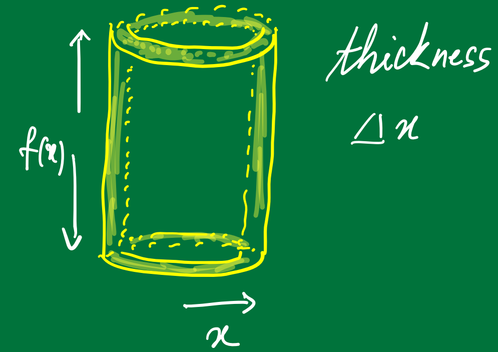


$\Rightarrow$ find cylindrical shells and the solid S is a union of these shells

Let us consider circular cylindrical slices parallel to the axis of revolution in finding volume of the solid.

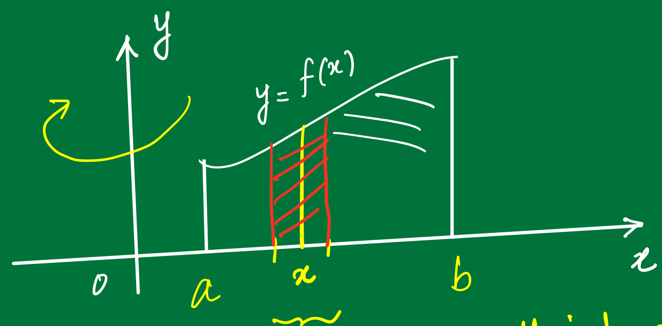
Now the solid S is viewed as the union of circular cylinders of increasing radii, the axis of each cylinder is parallel to the axis of revolution.



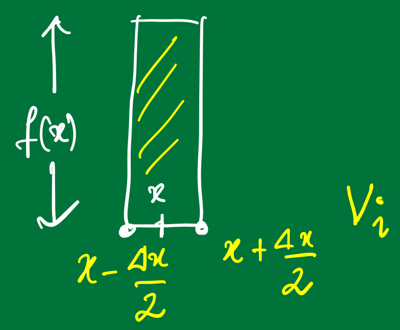


We define the volume of the solid S generated by revolving R about the y-axis to be

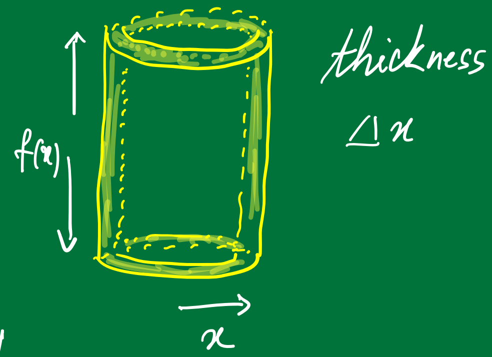
$$V = \int_a^b 2\pi x f(x)\, dx$$



I — a small interval



$2\pi x f(x)$



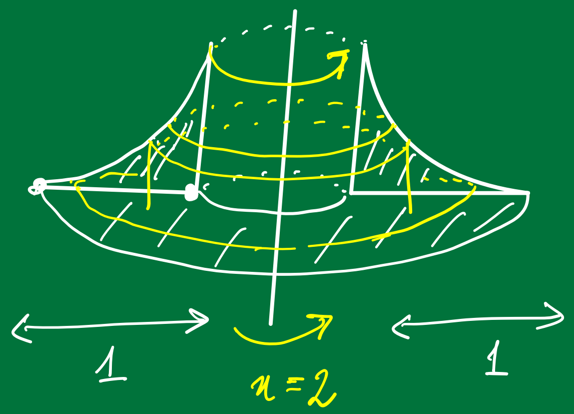
revolving the rectangle ⇒ Cylindrical Shell of volume = $\underline{2\pi x f(x)\,\Delta x}$

Example 1. Consider the solid S obtained by revolving the region bounded by the graph of functions

$$y = x^2 + x + 1,$$

$$y = 1 \text{ and } x = 1$$

about the line $x = 2$.



By the shell method, the volume is

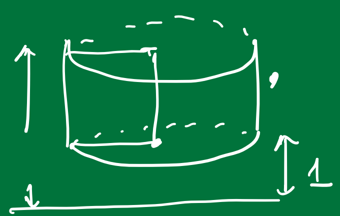
$$V = \int_a^b 2\pi \,(\text{shell radius}) \times (\text{shell height})\, dx$$

$$= \int_0^1 2\pi \,(2-x)\,(x^2+x)\, dx$$

The shell at $x \in [0,1]$ is of radius $(2-x)$ and the shell height is

$$(x^2+x+1-1)$$

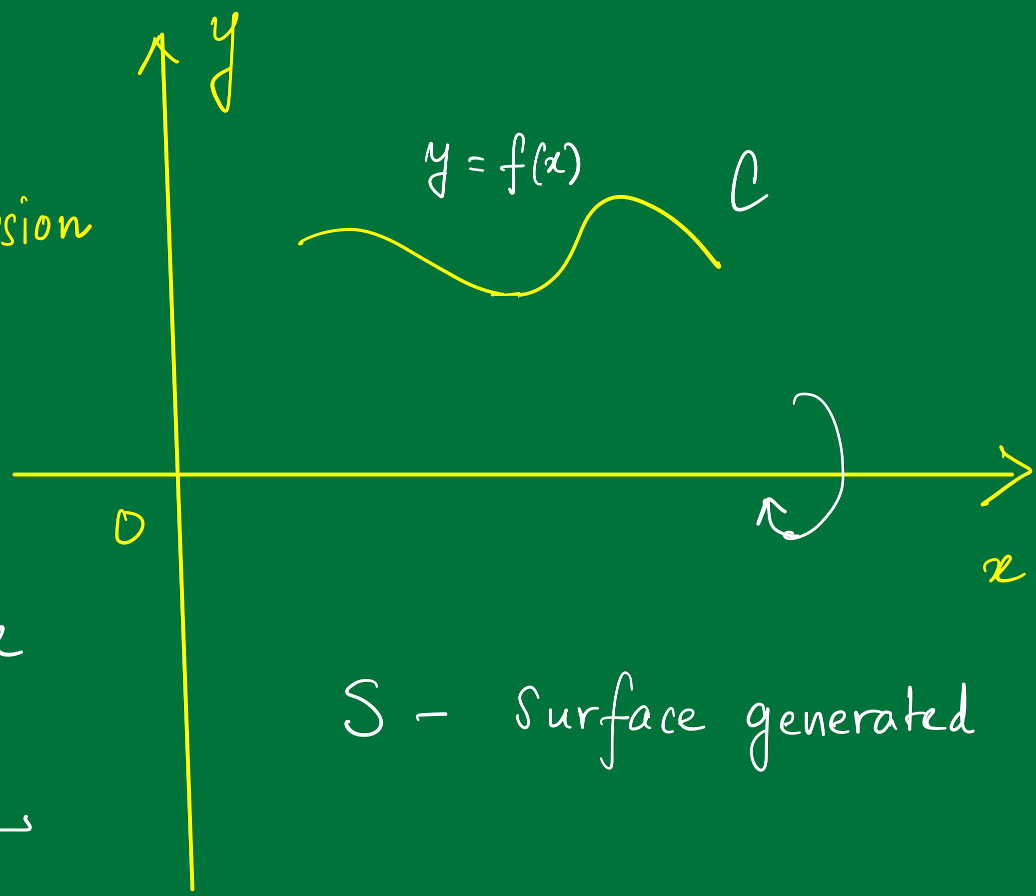
$$= x^2+x$$



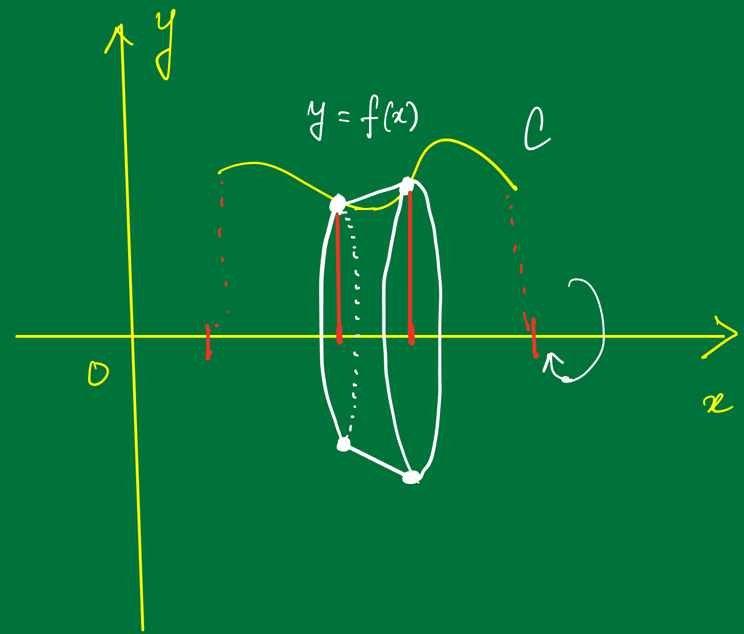
Areas of surfaces of revolution, Pappus's Theorems

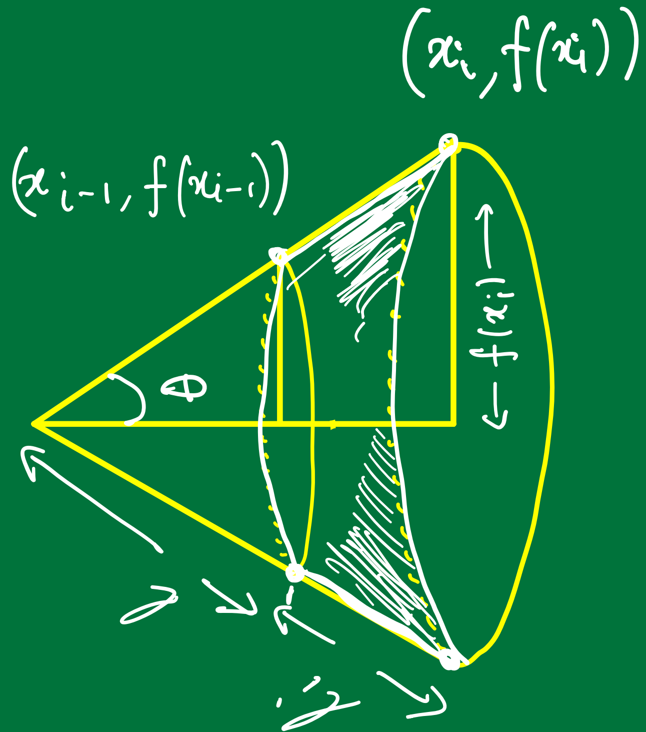
- Surface area of S in terms of an integral expression

$$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$



S - Surface generated



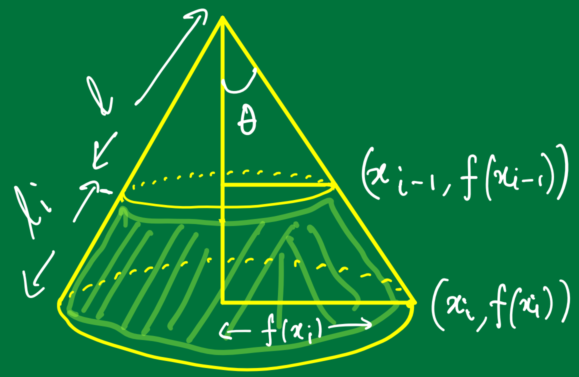


$$A_i = \pi f(x_i)\,(l + l_i)$$

$$A_{i-1} = \pi f(x_{i-1})\,l$$

$$A_i = \pi f(x_i)(l + l_i)$$

$$A_{i-1} = \pi f(x_{i-1})\, l$$



$$\frac{1}{\sin\theta} = \frac{l}{f(x_{i-1})} = \frac{l+l_i}{f(x_i)} = \frac{l_i}{f(x_i) - f(x_{i-1})}$$

$= \alpha$ (say).

$$A_{i-1}^{i} = \left(A_i - A_{i-1}\right) = \pi f(x_i)(l + l_i) - \pi f(x_{i-1})\, l$$

$$= \pi f(x_i)\, \alpha f(x_i) - \pi f(x_{i-1})\, \alpha f(x_{i-1})$$

$$= \pi \alpha \left(f(x_i) + f(x_{i-1})\right)\left(f(x_i) - f(x_{i-1})\right)$$

$$= \pi \left(f(x_i) + f(x_{i-1})\right) l_i$$

$$\sum_{i=1}^{n} A_{i-1}^{i} = \sum_{i=1}^{n} \pi\Big(f(x_i) + f(x_{i-1})\Big) l_i$$

$; \ l_i =$ distance between $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$

$$= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \ \Delta x_i$$

$$\Downarrow$$

$$\sqrt{1 + f'(x)^2}$$

$$= \sum_{i=1}^{n} \pi f(x_i) \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$+ \sum_{i=1}^{n} \pi f(x_{i-1}) \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

If f' is continuous, each sum on the RHS converges to

$$\int_a^b \pi f(x) \sqrt{1 + f'(x)^2}\, dx$$

as $\|P\| \to 0$.

$\Rightarrow$ Surface area generated by revolving the curve about x-axis to be

$$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx.$$

Remark: If $f(x) \le 0$ then the formula for area is

$$\int_a^b 2\pi \underbrace{|f(x)|}\sqrt{1+f'(x)^2}\,dx.$$

Parametric case

If the curve is given in the parametric form

$$[a,b] \xrightarrow{f} \mathbb{R} \times \mathbb{R}$$
$$t \longmapsto (x(t), y(t))$$

and x', y' are continuous functions on $[a,b]$,

then the surface area generated is

$$\int_a^b 2\pi \rho(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

where $\rho(t)$ is the distance between the axis of revolution and the curve.

Example: The curve

$$[0,4] \xrightarrow{f} \left(t+1,\ \frac{t^2}{2}+t\right)$$

is revolved about the y-axis. Find the area of the surface generated.

Here $\rho(t) = t+1$, $x'(t) = 1$, $y'(t) = t+1$

$$\text{Area} = \int_0^4 2\pi\,(t+1)\sqrt{1+(t+1)^2}\,dt$$

$$= \int_1^5 2\pi\, z\sqrt{1+z^2}\,dz$$

Polar case :

Consider the curve $r = f(\theta)$.

The surface generated by revolving the curve about the x-axis (or the initial ray) is $\displaystyle\int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Example :

$$r^2 = 2a^2 \cos 2\theta$$

$$S = 2\int_0^{\pi/4} 2\pi r \sin\theta \left(\frac{2a^2}{r}\right) d\theta \quad ; \quad \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \frac{2a^2}{r}$$

$$= 8\pi a^2 \left(1 - \frac{1}{\sqrt{2}}\right)$$

Check !

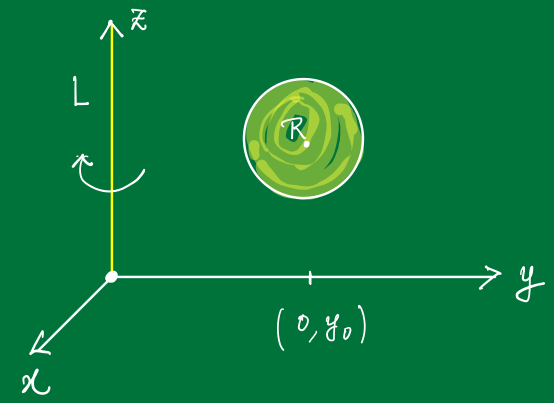
In the fourth century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids* to surfaces and solid of revolutions.

Many structures and mechanical systems behave as if there masses were concentrated at a single point, called the center of mass.

* When the density function is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the centroid of the shape.

How to locate this point — Center of mass or Centroid ?

Pappus's theorem 1.

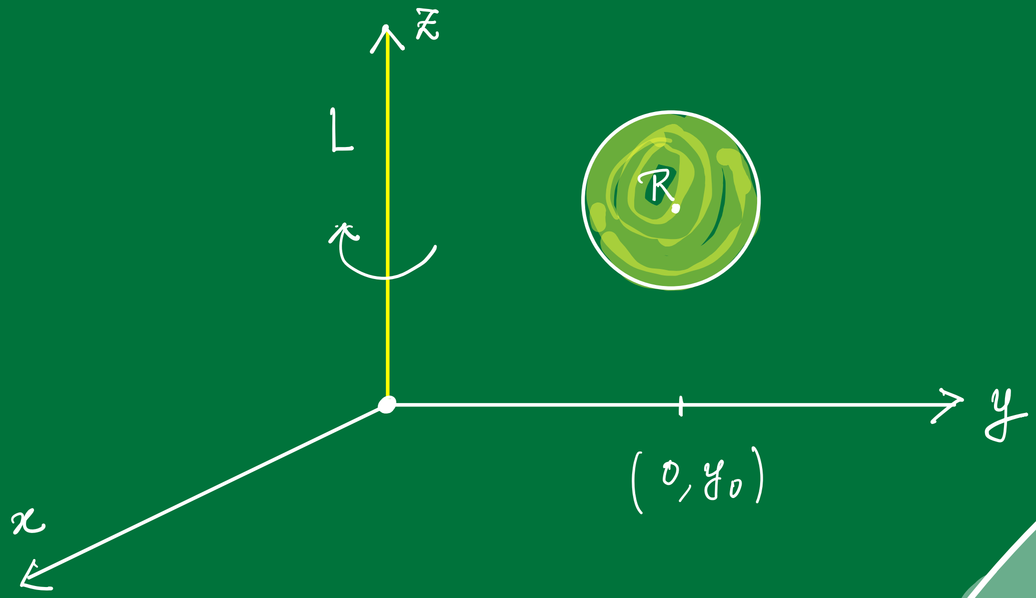


Suppose a plane region R is revolved about a line L which does not cut through the interior of R. Then the volume of the solid generated is

$$V = 2\pi \rho A$$

where ρ = the distance of centroid from the axis of revolution

and A = area of the region R.

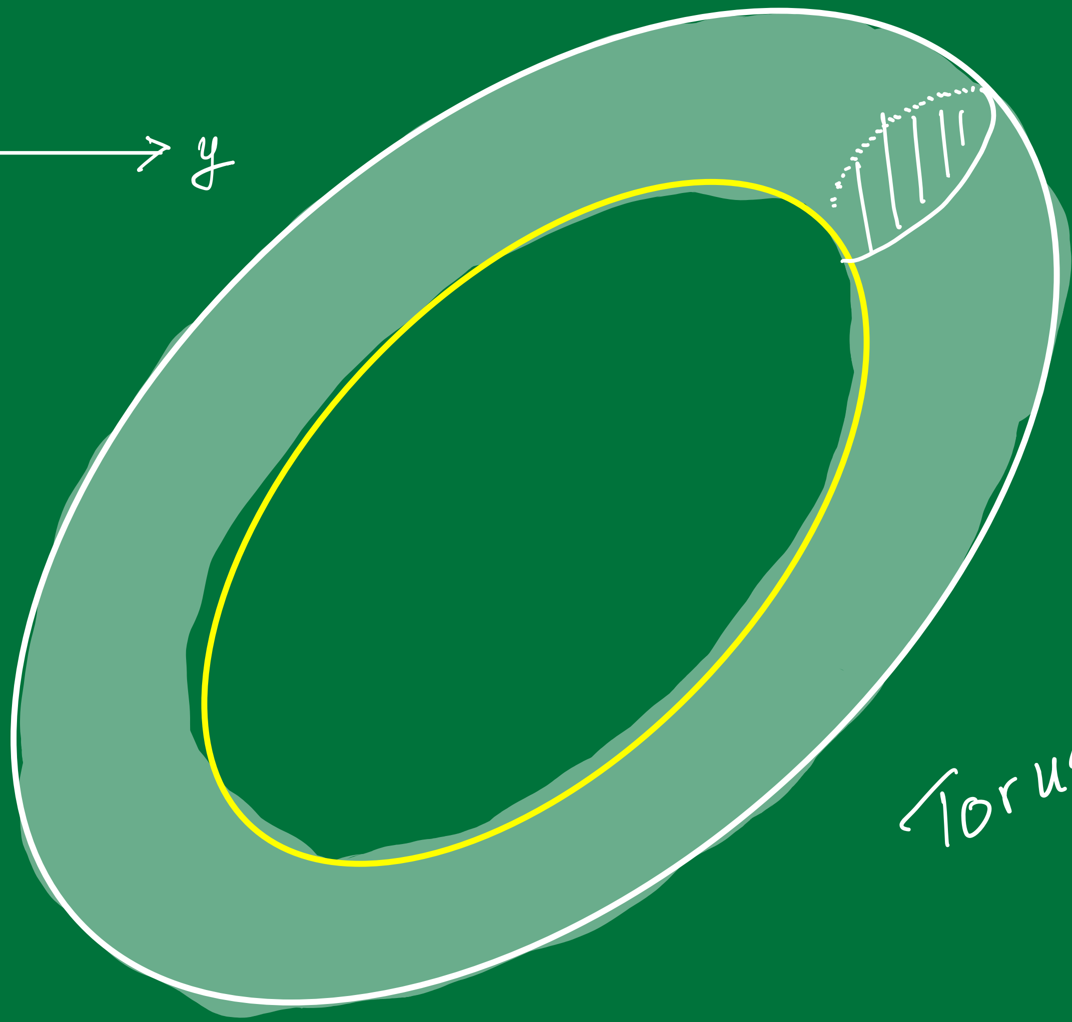


$$\rho = y_0$$

$$A = \pi r^2$$

$$V = 2\pi\, y_0\, \pi r^2$$

$$= 2\pi^2 r^2 y_0$$



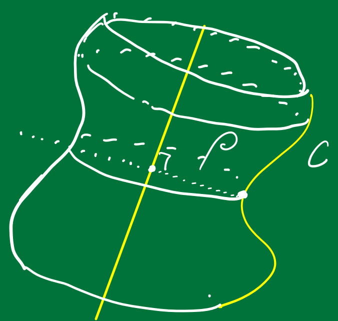
Torus

Remark: Circumference of the circle described by the centroid as a result of revolution is $2\pi\rho$.

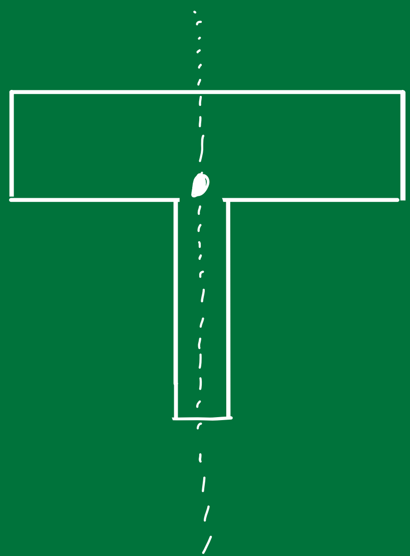
Theorem 2. If a (smooth) plane curve C is revolved about the line which does not cut through the interior of C, then the area of the surface generated is

$$S = 2\pi\rho L,$$

where ρ is the distance from the axis of revolution to the centroid of C and L is the length of the curve C.



Remark: If we have a line of symmetry, then the centroid lies on that line.



Example

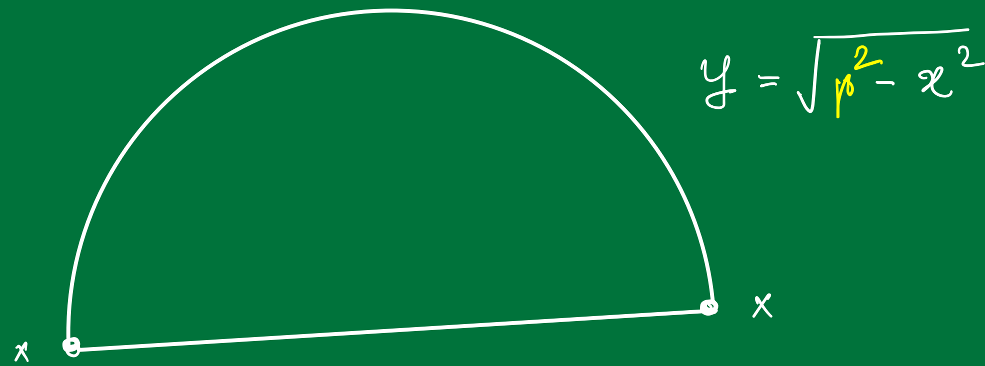
$y = \sqrt{r^2 - x^2}$

Semi circular arc

Example 1.

Centroid is at $(0, \bar{y})$

$\bar{y} = ?$



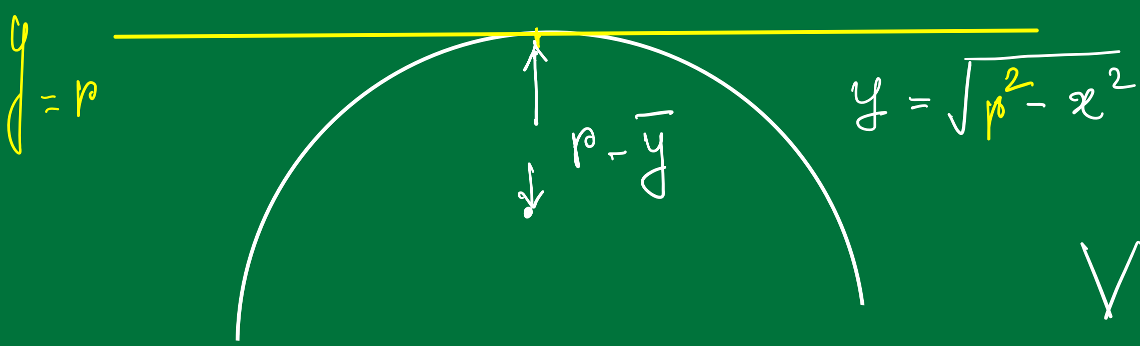
Semi circular arc

$$\left.\begin{aligned} &2\pi \rho L \\ &= 2\pi \bar{y}\, \pi r \\ &= 2\pi^2 \bar{y} \end{aligned}\right\} = 4\pi r^2$$

$$\Rightarrow \pi \bar{y} = 2r$$

or $\bar{y} = 2r/\pi$

Example 2.

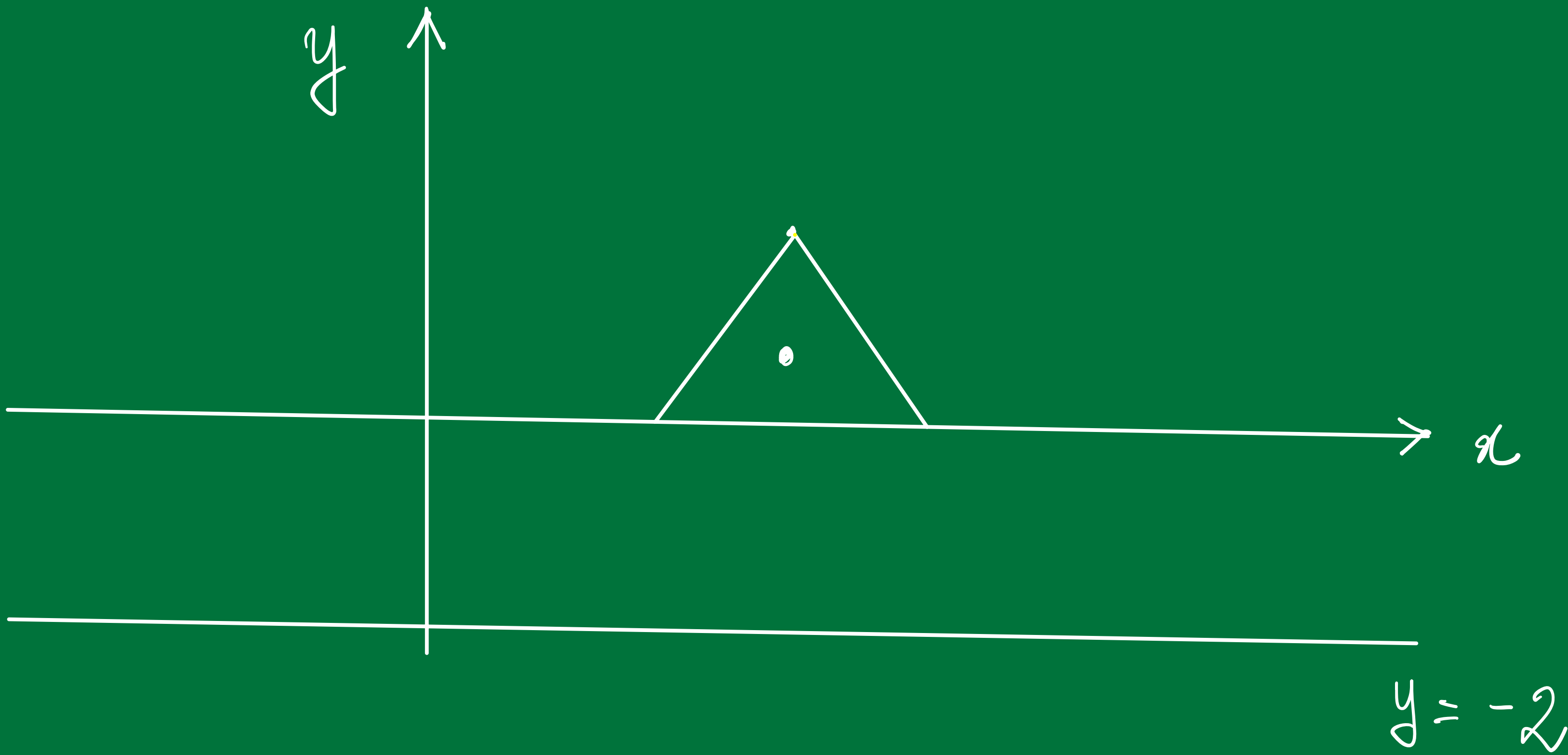


ρ $\quad$ A

$$V = 2\pi (r - \bar{y})\pi r$$

$$= 2\pi \left(r - \frac{2r}{\pi}\right)\pi r$$

$$= 2\pi(\pi - 2) r^2$$



$$P = 2 + \left(\frac{1}{3} \times \frac{\sqrt{3}}{2} \times 2\right) = 2 + \frac{1}{\sqrt{3}}$$

$$A = \frac{1}{\cancel{2}} \times \cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} \times \cancel{2} = \sqrt{3}$$

$$V = 2\pi\left(2 + \frac{1}{\sqrt{3}}\right)\sqrt{3}$$

$$= 2\pi\left(2\sqrt{3} + 1\right)$$