CLASSIFICATION OF QUADRICS

Let $X^t A X + b X + c = 0$ be a real quadric in n variables, where A is a real symmetric matrix and b be a row vector.

There exists an orthogonal matrix P s.t. $P^t A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$. Applying the linear change $X = PY$, the equation of the quadric becomes

$$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 + b_1' y_1 + \cdots + b_n' y_n + c = 0.$$

Suppose that precisely r of $\lambda_1, \ldots, \lambda_n$ are $\neq 0$. Then after permuting the variables, if necessary, and relabelling, the equation can be brought to $\lambda_1 z_1^2 + \cdots + \lambda_r z_r^2 + b_1'' z_1 + \cdots + b_n'' z_n + c = 0$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, $\lambda_1, \ldots, \lambda_r \neq 0$.

Observe that $\lambda_i z_i^2 + b_i'' z_i = \lambda_i \left(z_i + \dfrac{b_i''}{2\lambda_i} \right)^2 - \dfrac{b_i''^2}{4\lambda_i}$, $\forall\, i = 1, \ldots, r$.

Hence putting $w_i = z_i + \dfrac{b_i''}{2\lambda_i}$, $i = 1, \ldots, r$ &

$$w_{r+1} = z_{r+1}, \ldots, w_n = z_n,$$

the equation changes to

$$\lambda_1 w_1^2 + \cdots + \lambda_r w_r^2 + b_{r+1}'' w_{r+1} + \cdots + b_n'' w_n + c' = 0.$$

<u>Remark</u>: Geometrically, the origin has now been shifted to the pt. $\left(-\dfrac{b_1''}{2\lambda_1}, \ldots, -\dfrac{b_r''}{2\lambda_r}, 0, 0, \ldots, 0 \right)$.

Now if $\exists\, j \in \{r+1, \dots, n\}$ s.t. $b_j'' \neq 0$, then we write

$$b_j'' w_j + c' = b_j''\left(w_j + \frac{c'}{b_j''}\right).$$

Let $\left.\begin{array}{l} \eta_i := w_i, \quad i \neq j \\ \& \ \eta_j = w_j + \dfrac{c'}{b_j''}. \end{array}\right\}$ Note :- We are applying a translation once again.

Then the equation becomes

$$\cancel{\lambda_1 w_1^2 + \dots + \lambda_r w_r^2 + b_{r+1}'' \eta_{r+1} + \dots + b_n'' \eta_n = 0}$$

$$\shortparallel$$

$$\lambda_1 \eta_1^2 + \dots + \lambda_r \eta_r^2 + b_{r+1}'' \eta_{r+1} + \dots + b_n'' \eta_n = 0$$

$$\shortparallel$$

$$\eta^t \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix} \eta + v\eta$$

where $v = (0, \dots, 0, b_{r+1}'', \dots, b_n'')$

Clearly $\langle v, e_i \rangle = 0, \ \forall\, i = 1, \dots, r.$

Extend $\left\{ e_1, \dots, e_r, \frac{v}{\|v\|} \right\}$ to an ON basis of $\mathbb{R}^n$, say $\left\{ e_1, \dots, e_r, \frac{v}{\|v\|}, u_1, \dots, u_\ell \right\}$.

Consider $P = \begin{bmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_r & \frac{v}{\|v\|} & u_1 & \cdots & u_\ell \\ | & & | & | & | & & | \end{bmatrix}$

Apply the change of variables $\eta = P\zeta$:

$$\xi^t \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & & & \\ 0 & 1 & \ddots & \vdots & & 0 & \\ \vdots & & \ddots & 1 & & & \\ 0 & \cdots & \cdots & 0 & \frac{b''_{r+1}}{\|v\|} & \cdots & \frac{b''_n}{\|v\|} \\ & 0 & & & & * & \end{bmatrix}}_{P^t} \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \cdots & 0 & 0 & \\ \vdots & 1 & & \vdots & 0 \\ 0 & & 1 & 0 & \\ & & & \frac{b''_{r+1}}{\|v\|} & \\ & 0 & & \vdots & * \\ & & & \frac{b''_n}{\|v\|} & \end{bmatrix}}_{P} \xi$$

$$+ \begin{bmatrix} 0 & \cdots & 0 & b''_{r+1} & \cdots & b''_n \end{bmatrix} P \qquad = 0$$

$$\Rightarrow \lambda_1 \xi_1^2 + \cdots + \lambda_r \xi_r^2 + \|v\| \xi_{r+1} = 0 .$$

CONCLUSION :- Every real quadric in n-variable can be brought to one of the following forms:

$$\lambda_1 \xi_1^2 + \cdots + \lambda_r \xi_r^2 + c = 0 \quad (r \leq n) \quad \text{or}$$

$$\lambda_1 \xi_1^2 + \cdots + \lambda_r \xi_r^2 + d \xi_{r+1} = 0 \quad (r < n)$$