$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \end{aligned}\right\}$$

RRE MATRIX (MODIFIED)

(4) If i-th row is zero then $\forall j > i$ $R_j = 0$.

<u>THEOREM</u> :- Given any A - $m \times n$ matrix $/\mathbb{F}$, there is a unique $m \times n$ matrix B s.t.

(i) B is a RRE matrix

(ii) B can be obtained from A by ele. row. op.

1. Suppose A is an $n \times n$ matrix. Then

(i) A' has zero bottom row

(ii) $A' = I_n$.

$\Rightarrow \exists$ elementary matrices $E_1, \ldots, E_k$ of $n \times n$

s.t. $E_k E_{k-1} \cdots E_1 A = \underline{\underline{A'}}$.

$$A' = \begin{bmatrix} & & \\ & & \\ & & 1 \end{bmatrix}$$

n — k_n -th col.

$n-1$ $k_{n-1} < k_n \leq n-1$

$$1 \leq \underbrace{k_1 < k_2 < \cdots < k_{n-1} < k_n}_{} \leq n-1 \rightarrow \leftarrow$$

$\therefore k_n = n$

$CD = I_n$

2. If A is inv. in addition then $\underbrace{A' = I_n}$

$$\underbrace{(E_k E_{k-1} \cdots E_1)}_{\text{Inv.}} A = \underline{\underline{A'}}$$

3. A is inv.

$\Rightarrow \quad (E_k E_{k-1} \cdots E_1) A = I_n \quad \Rightarrow \quad E_k \cdots E_1 = A^{-1}$

$\Rightarrow \quad (E_k \cdots E_1) I_n = A^{-1}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & \cancel{-2}\ 1 & \cancel{-3}\ \frac{3}{2} & \cancel{-1}\ -\frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

4. A - $n \times n$ matrix. Suppose B - $n \times n$ is s.t. $AB = I_n$ $(BA = I_n)$

$\Rightarrow \quad (E_k \cdots E_1) AB = (E_k \cdots E_1)$

$\Rightarrow \quad A'B = \quad$ "

$\Rightarrow$ A' cannot have zero bottom row

$\Rightarrow A' = I_n$, i.e.,

$$(E_k \cdots E_1)A = I_n$$

$$\Rightarrow \quad A = E_1^{-1} \cdots E_k^{-1}.$$

5. A - $n \times n$ matrix, inv

$$(E_k \cdots E_1)A = I_n \Rightarrow A = E_1^{-1} \cdots E_k^{-1}.$$

$\therefore$ Every inv. matrix is a product of elementary matrices.

HOMOGENEOUS SYSTEM :- A - $m \times n$ matrix, assume $m < n$.

$$Ax = 0$$

$A'x = 0$, , A' is the RREF of A.

$$A' = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{m \times n}$$

$\Rightarrow$ There exists at least one free variable.

- THEOREM :-

$Ax = b$

$Ax = 0$

Let $y \in \mathbb{F}^n$ be a soln of $Ax = b$

Then, the soln of $Ax = b$ are precisely of the form $y + z$ where z is a soln of $Ax = 0$

PROOF:- $A(y+z) = Ay + Az = Ay + 0 = b$

If $y' \in \mathbb{F}^n$ is a soln of $Ax = b$

$$\begin{aligned} Ay' &= b \\ Ay &= b \end{aligned} \quad \Rightarrow \quad A(\underline{\underline{y - y'}}) = b - b = 0$$

$$\therefore \quad y' = y + (y - y')$$