lecture 9 (Second order liner ODE).

Recall. $y'' + p(x) y' + q(x) y = 0$

p, q are continuous functions on some interval. If we know one solution $y_1(x)$ ($\neq 0 \ \forall x \in I$) then we can find another solution $y_2(x) = v(x) y_1(x)$

- $a y'' + b y' + c y = 0$

a, b, c constants $a \neq 0$.

We know how to solve it.

$(y = e^{mx})$

Today (Cauchy - Euler equation)

$$a x^2 y'' + b x y' + c y = 0$$

a, b, c consts

$a \neq 0$

$x \in (0, \infty)$.

- $x = e^t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = e^{-t} \dot{y}$$

$$\boxed{\dot{y} = \frac{dy}{dt} \quad \ddot{y} = \frac{d^2 y}{dt^2}}$$

$$x y' = \dot{y}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dt}(e^{-t} \dot{y}) \Big/ \frac{dx}{dt}$$

$$= (e^{-t} \ddot{y} - e^{-t} \dot{y}) e^{-t}$$

$$= e^{-2t} (\ddot{y} - \dot{y})$$

$$\boxed{x^2 y'' = \ddot{y} - \dot{y}}$$

Substituting in the given equation

$$a(\ddot{y} - \dot{y}) + b\dot{y} + cy = 0$$

$$a\ddot{y} + (b-a)\dot{y} + cy = 0$$

— Second order, linear const coff

Characteristic equation

$$am^2 + (b-a)m + c = 0$$

Q. Roots $m = m_1, m_2$

Case 1 m_1, m_2 are real & distinct

$$y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$= c_1 x^{m_1} + c_2 x^{m_2}$$

Case 2 $m_1 = m_2 = m$

$$y = c_1 e^{mt} + c_2 t e^{mt}$$

$$= c_1 x^{m} + c_2 x^{m} \log(x)$$

Case 3 $m_1, m_2 = \alpha \pm j\beta$

$$y = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

$$= x^{\alpha}\left(c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)\right)$$

Example

(1) $x^2 y'' - x y' - 3y = 0.$

$$x = e^t$$

$$x y' = \dot{y}$$

$$x^2 y'' = \ddot{y} - \dot{y}$$

Substituting in the given equation.

$$\ddot{y} - 2\dot{y} - 3y = 0$$

$$m = +3, -1.$$

Solutions $y = c_1 e^{+3t} + c_2 e^{-t}$

$$= c_1 x^3 + \frac{c_2}{x}$$

(2) $x^2 y'' - 3x y' + 5y = 0$

$$x = e^t$$

Putting in the given equation

$$(\ddot{y} - \dot{y}) - 3\dot{y} + 5y = 0$$

$$\ddot{y} - 4\dot{y} + 5y = 0$$

Characteristic equation

$$m^2 - 4m + 5 = 0$$

$$m = 2 \pm i$$

Solution $y = e^{2t}\left(c_1 \cos(t) + c_2 \sin(t)\right)$

$$= x^2 \left(c_1 \cos(\log x) + c_2 \sin(\log x) \right).$$

(3) $x^2 y'' - 3xy' + 4y = 0$

characteristic equation

$m^2 - 4m + 4 = 0$

$m = 2, 2.$

Solutions $y = c_1 e^{2t} + c_2 t e^{2t}$

$= c_1 x^2 + c_2 x^2 (\log x).$

Higher order linear homogeneous

const coeff

$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$

$a_0, a_1 \ldots a_n$ consts.

$a_0 \neq 0.$

Put $y = e^{mx}$

characteristic equation

$a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$

Case 1 A real root m is repeated K- times.

Then $e^{mx}, x e^{mx}, x^2 e^{mx}, \ldots x^{k-1} e^{mx}$

are L.I solutions.

Justification:

$$D = \frac{d}{dx} \qquad D^2 = \frac{d^2}{dx^2}$$

$$(D - mI)\,y = y' - my$$

$$(D - mI)^2 = (D - mI)(D - mI)$$

$K = 3$ Our ODE has factor of the form $(D - mI)^3$

- $(D - mI)^3 (e^{mx}) = (D - mI)(D - mI)\underline{(D - mI)e^{mx}}$

$$= (D - mI)^2 (me^{mx} - me^{mx})$$

$$= 0.$$

- $(D - mI)^3 (xe^{mx}) = (D - mI)(D - mI)(D - mI)(xe^{mx})$

$$= (D - mI)\underline{(D - mI)}(e^{mx} + x\cancel{m}e^{mx} - m\cancel{x}e^{x})$$

$$= 0.$$

- $(D - mI)^3 (x^2 e^{mx})$

$$= (D - mI)(D - mI)(D - mI)(x^2 e^{mx})$$

$$= (D - mI)^2 (\underline{2xe^{mx}} + x^2\cancel{m}e^{mx} - mx^2 e^{mx})$$

$$= 0.$$

Case 2 $m = \alpha \pm i\beta$ repeated K- times.

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x$$

$$x e^{\alpha x}\cos\beta x, \quad x e^{\alpha x}\sin\beta x$$

$$- - - - -$$

$$x^{k-1} e^{\alpha x}\cos\beta x, \quad x^{k-1} e^{\alpha x}\sin(\beta x).$$

Example

(1) $y^{(5)} + y^{(4)} - 2y^{(3)} - 2y^{(2)} + y^{(1)} + y = 0$

Charectaristic eqution.

$$m^5 + m^4 - 2m^3 - 2m^2 + m + 1 = 0$$

$$(m+1)^3 (m-1)^2 = 0.$$

Solution

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + c_4 e^{x} + c_4 x e^{x}.$$

(2) $y^{(6)} + 8y^{(5)} + 25y^{(4)} + 32y^{(3)} - y^{(2)} - 40y^{(1)} - 25y = 0$

Charectaristic eqution.

$$(m+1)(m-1)(\underbrace{m^2 + 4m + 5}_{-2 \pm i})^2 = 0$$

Solution

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x}\cos x + c_4 e^{-2x}\sin x + c_5 x e^{-2x}\cos x + c_6 x e^{-2x}\sin x$$

(3) $x^3 y''' + x^2 y'' - 2x y' + 2y = 0$

Cauchy- Euler

$x = e^t$

$x y' = \dot{y} = \frac{d}{dt}(y)$

$x^2 y'' = \ddot{y} - \dot{y} = \frac{d}{dt}\left(\frac{d}{dt} - I\right) y.$

$x^3 y''' = \frac{d}{dt}\left(\frac{d}{dt} - I\right)\left(\frac{d}{dt} - 2I\right) y.$

$x^4 y'''' = - \; - \; -$

Putting in the given eqn.

. . . . .

Characteristic equation

$m(m-1)(m-2) + m(m-1) - 2m + 2 = 0$

$(m^2 - 1)(m-2) = 0$

$m = \pm 1, 2$

$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}$

$= c_1 x + \frac{c_2}{x} + c_3 x^2$

Non-Homogeneous 2nd order ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x) \quad \text{———} \quad (*)$$

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad \text{———} \quad (**)$$

- Assume that y_p is SOME particular solution of $(*)$

Let $y(x)$ be any ~~at~~ solution of $(*)$.

Then $(y - y_p)$ solution of $(**)$.

$$y - Y_p = c_1 Y_1 + c_2 Y_2 \quad \text{where}$$

y_1 & Y_2 are two LI solutions of $(**)$.

Summary

Any solution of $(*)$ looks like

$$\boxed{y(x) = c_1 Y_1 + c_2 Y_2 + Y_p.}$$

Y_p = some particular solution of $(*)$

Y_1, Y_2 are two LI solution of $(**)$

How to find y_p. ?

Variation of Parameter.

Q)

True / False

- f, g are LI over some interval I

$\overset{?}{\Rightarrow}$ f, g LI over subinterval $J \subseteq I$

FALSE $\quad f = x|x| \quad g = x^2 \quad [-1, 1)$

- f, g are LI over I

$\overset{?}{\Rightarrow}$ f, g are LI over $J \supseteq I$.

True

$$c_1 f(x) + c_2 g(x) = 0 \quad \forall x \in I$$

$$\Rightarrow c_1 = c_2 = 0$$

- y_1, y_2 solution of $(\cancel{**}) + y_1, y_2$ LI over I

$\Rightarrow y_1, y_2$ LI over $J \subseteq I$. Q