Lecture 13 : Inner Product Space & Cauchy-Schwarz Inequality

In $\mathbb{R}^3$, the angle between two vectors (a_1, b_1, c_1) & (a_2, b_2, c_2) is given by

$$\theta = \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

& distance between two points (x_1, y_1, z_1) & (x_2, y_2, z_2) is given by

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

For arbitrary vector space, we will introduce 'Inner Product' to define angle & distances.

Inner Product: Let V be a vector space over K (where $K = \mathbb{R}$ or $\mathbb{C}$). An inner product, denoted by $\langle , \rangle$, is a map $\langle , \rangle : V \times V \to K$ satisfying the following properties :

(i) (Positive Definiteness) $\langle x,x\rangle \in \mathbb{R}$ & $\langle x,x\rangle \geq 0 \ \forall x \in V$ & $\langle x,x\rangle = 0$ if and only if $x=0$.

(ii) (Conjugate Symmetry) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ $\forall x,y \in V$.

(If $K=\mathbb{R}$, $\langle x,y\rangle = \langle y,x\rangle \ \forall x,y \in V$)

(iii) (Linearity in first coordinate)

$$\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$$

$$\& \quad \langle \alpha x,y\rangle = \alpha\langle x,y\rangle$$

$$\forall x,y,z \in V \ \& \ \alpha \in K \qquad \square$$

Note that $\langle x,\alpha y\rangle = \overline{\langle \alpha y,x\rangle} = \overline{\alpha\langle y,x\rangle}$
$= \bar{\alpha}\,\overline{\langle y,x\rangle}$

& $\langle x,y+z\rangle = \overline{\langle y+z,x\rangle} = \overline{\langle y,x\rangle + \langle z,x\rangle}$
$= \overline{\langle y,x\rangle} + \overline{\langle z,x\rangle}$
$= \langle x,y\rangle + \langle x,z\rangle$

If $K=\mathbb{R}$, $\langle\ ,\ \rangle$ is also linear in 2nd coordinate.

Example: (1) In $\mathbb{R}^n$, the dot product is an inner product i.e.

$$\langle (a_1,a_2,\dots,a_n),(b_1,b_2,\dots,b_n)\rangle = a_1b_1+a_2b_2+\dots+a_nb_n.$$

(2) In $\mathbb{C}^n$ (treated as vector space over $\mathbb{C}$)

$\langle (z_1, z_2, \ldots, z_n), (w_1, w_2, \ldots, w_n) \rangle$

$= z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$ is an inner product.

(3) Consider $C([0,1])$ = the set of all continuous function on $[0,1]$. It is a real vector space under the following addition & scalar multiplication:

$$(f+g)(x) := f(x) + g(x), \quad f, g \in C[0,1]$$

$$\& \quad (\lambda \cdot f)(x) := \lambda f(x) \quad \forall \lambda \in \mathbb{R} \ \& \ f \in C[0,1]$$

$\langle \, , \rangle : C[0,1] \times C[0,1] \to \mathbb{R}$ defined by

$$\langle f, g \rangle := \int_0^1 f(x) g(x) \, dx$$

is an inner product on $C([0,1])$.

<u>Norm of a vector</u>: Norm of a vector $v \in V$, denoted by $\|v\|$, defined to be

$$\|v\| := \sqrt{\langle v, v \rangle}$$

☆ $\|v\| = 0$ iff $v = 0$

Orthogonal vectors: Two vectors v and w are orthogonal to each other if $\langle v, w \rangle = 0$.

Cauchy-Schwarz Inequality: Let V be an inner product space i.e. V is endowed with inner product $\langle\ ,\ \rangle$. For all $v, w \in V$,

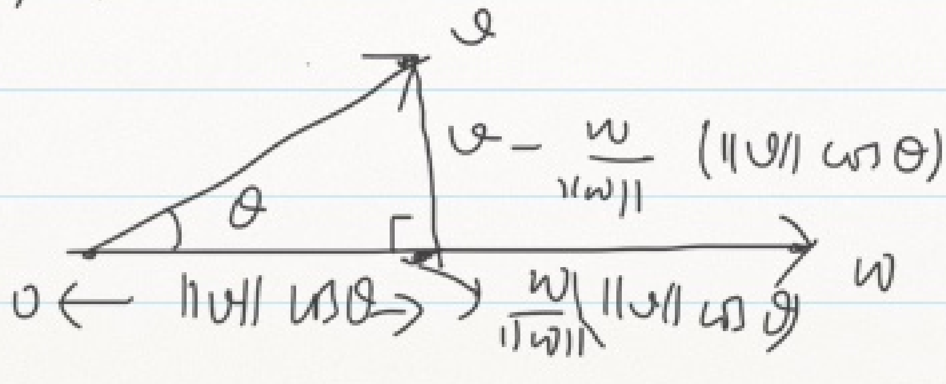
$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

the equality holds if and only if v or w is multiple of the other.

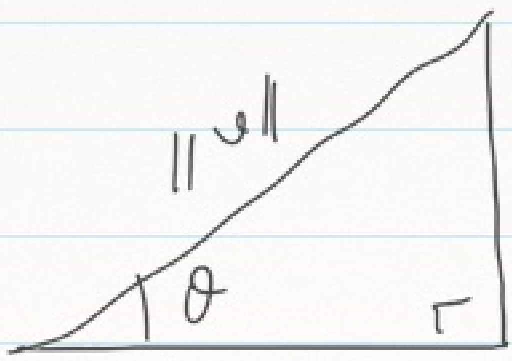
☆ We will first check this for $\mathbb{R}^2$ geometrically. In $\mathbb{R}^2$, the usual inner product is the dot product of two vectors. Let $v, w \in \mathbb{R}^2$.

$$\langle v, w \rangle = v \cdot w = \|v\| \|w\| \cos\theta$$

Let $w \neq 0$



Thus, we have the following right angle triangle



By Pythagoras theorem,

$$\|v\|^2 \cos^2\theta + \left\|v - \frac{w}{\|w\|} \|v\| \cos\theta\right\|^2 = \|v\|^2$$

i.e. $$\|v\|^2 \|w\|^2 \cos^2\theta + \|v\|w\| - w\|v\|\cos\theta\|^2 = \|v\|^2$$

i.e. $$\langle v, w\rangle^2 + \left\|v\|w\| - \frac{w}{\|w\|}\langle v, w\rangle\right\|^2 = \|v\|^2$$

i.e. $$\left\langle v, \frac{w}{\|w\|}\right\rangle^2 + \left\|v - \frac{w}{\|w\|}\left\langle v, \frac{w}{\|w\|}\right\rangle\right\|^2 = \|v\|^2$$

$$\Rightarrow \left\langle v, \frac{w}{\|w\|}\right\rangle^2 - \|v\|^2 \leq 0$$

$$\Rightarrow |\langle v, w\rangle| \leq \|v\| \|w\|.$$

Proof of Cauchy-Schwarz inequality:

If $w=0$, $\langle v, 0\rangle = \langle v, 0+0\rangle$

$= \langle v, 0\rangle + \langle v, 0\rangle$

$\Rightarrow \langle v, 0\rangle = 0$

Let w be a non-zero vector & $u = \frac{w}{\|w\|}$

Check $\|v - u\langle v,u\rangle\|^2 = \|v\|^2 - |\langle v,u\rangle|^2$

$\Rightarrow \|v\|^2 - |\langle v,u\rangle|^2 \geq 0$

Equality holds iff $v - u\langle v,u\rangle = 0$

$\Rightarrow |\langle v,w\rangle| \leq \|v\|^2\|w\|^2$

& equality holds iff $v - \frac{w}{\|w\|}\langle v, \frac{w}{\|w\|}\rangle = 0$

(i.e. v is scalar multiple of w) $\square$

Angle: Angle between two vectors u & v in a inner product space V with inner product $\langle\, ,\rangle$ is

$$\cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\,\|v\|}\right).$$

The definition of angle makes sense as by Cauchy-Schwarz inequality,

$$|\langle u, v\rangle| \le \|u\| \, \|v\|$$

$$\Rightarrow \quad -1 \le \frac{\langle u, v\rangle}{\|u\| \, \|v\|} \le 1 .$$

Orthogonal Projection: Let v & w be two vectors in an inner product space with $w \neq 0$. The vector $\frac{\langle v, w\rangle}{\langle w, w\rangle} w$

is orthogonal projection of v on the linear span $L(w) = \{\lambda w : \lambda \in K\}$
(It represents a line in V)

Orthogonal Subspaces: Two subspaces P and Q of an inner product space are orthogonal to each other if $\langle x, y\rangle = 0 \ \forall x \in P \ \& \ y \in Q$.

Orthogonal Complement: Let W be a subspace of inner product space V.
The orthogonal complement of W, denoted by $W^{\perp}$, is $W^{\perp} = \{v \in V : \langle v, w\rangle = 0 \ \forall w \in W\}$

Proposition: (i) $W^\perp$ is a subspace of V,
(ii) $W \cap W^\perp = \{0\}$

Proof: (i) Let $w_1, w_2 \in W^\perp$, then for any $v \in V$, $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle = 0$
& $\langle v, \lambda w_1 \rangle = \bar{\lambda} \langle v, w_1 \rangle = 0$

(ii) Let $v \in W \cap W^\perp$ then $\langle v, v \rangle = 0$
$\Rightarrow \|v\| = 0$
$\Rightarrow v = 0$. $\square$

Triangle Inequality Let V be an inner product space.

$\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V$.

Proof: $\|v + w\|^2 = \langle v + w, v + w \rangle$
$= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle$
$= \|v\|^2 + \langle v, w \rangle + \langle w, v \rangle + \|w\|^2$
$\leq \|v\|^2 + \|v\|\|w\| + \|w\|\|v\| + \|w\|^2$
(by Cauchy-Schwarz Inequality)
$\leq (\|v\| + \|w\|)^2$

$\Rightarrow \|v + w\| \leq \|v\| + \|w\|$

The equality holds if v or w is scalar multiple of the other. $\square$