Lecture 15: Direct Sum of Subspaces, Fundamental Subspaces, Least Square Solutions

Let V be a vector space over K ($K = \mathbb{R}$ or $\mathbb{C}$)

Let P and Q be two subspaces of V.

Direct Sum of P and Q, denoted by $P \oplus Q$, is the subspace $P + Q$ with $P \cap Q = \{0\}$.

Proposition: Each element x of $P \oplus Q$ can be written uniquely as $x = p + q$.

Proof: Let $x \in P \oplus Q$, then $\exists\, p \in P$ & $q \in Q$ such that $x = p + q$.

If $\exists\, p' \in P$ & $\exists\, q' \in Q$ such that

$$x = p' + q'$$

then
$$p + q = p' + q'$$
$$\Rightarrow \quad p - p' = q - q' \in P \cap Q = \{0\}$$
$$\Rightarrow \quad p = p' \ \& \ q = q' \qquad \square$$

Proposition: Let W be a subspace of an inner product space V, then $V = W \oplus W^{\perp}$.

Proof: We have seen $V = W + W^{\perp}$ & $W \cap W^{\perp} = \{0\}$

Fundamental Subspaces :

Let $A = (a_{ij})$ be a matrix of order $m \times n$. A induces a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(x) = Ax$, x is treated as $n \times 1$ matrix.

Notation:- (i) $\ker(T_A) \equiv N(A)$, also called null space of A

(ii) Range of T_A, $R(T_A) \equiv R(A)$,

(iii) $RS(A)$ = Row space of A (Subspace generated by row vectors)

(iv) $CS(A)$ = Column space of A (subspace generated by column vectors)

Note that $N(A) \subseteq \mathbb{R}^n$, $R(A) \subseteq \mathbb{R}^m$

$RS(A) \subseteq \mathbb{R}^n$, $CS(A) \subseteq \mathbb{R}^m$.

$A^T = (a_{ji})_{n \times m}$, $A^T : \mathbb{R}^m \to \mathbb{R}^n$

$N(A^T) \subseteq \mathbb{R}^m$, $R(A^T) \subseteq \mathbb{R}^n$.

Fundamental Theorem of Linear Algebra :

Theorem (i) $N(A) \oplus RS(A) = \mathbb{R}^n$

(ii) $N(A^T) \oplus CS(A) = \mathbb{R}^m$.

Proof (i) Let $w \in N(A)$ & $Aw = 0$

$\Rightarrow \quad a_{i1}w_1 + \cdots + a_{in}w_n = 0$ ☆

$\forall\ 1 \le i \le m$.

Let $v \in RS(A)$

$$v = \sum_{i=1}^{m} \lambda_i R_i \quad , \lambda_i \in \mathbb{R}$$

& $R_i = (a_{i1}, \cdots, a_{in})$

R_i is the row vector.

☆ implies $\langle (a_{i1}, \cdots, a_{in}), (w_1, \cdots, w_n) \rangle = 0$

$\Rightarrow \langle R_i, w \rangle = 0$

$\Rightarrow \langle \sum_{i=1}^{m} \lambda_i R_i, w \rangle = 0$

$\Rightarrow \langle v, w \rangle = 0$

Therefore, $N(A)$ is orthogonal to $RS(A)$. So, $N(A) \subseteq (RS(A))^{\perp}$

By Rank - Nullity theorem,

$\dim(N(A)) + \dim(R(A)) = n$

& $\dim(RS(A)) = \dim(CS(A)) = \dim(R(A))$

$\mathbb{R}^n = RS(A) \oplus (RS(A))^{\perp}$

$\Rightarrow n = \dim(RS(A)) + \dim(RS(A)^{\perp})$

$\Rightarrow \dim(N(A)) = \dim(RS(A)^{\perp})$

So, $N(A) = RS(A)^{\perp}$

So, $N(A) \oplus RS(A) = \mathbb{R}^n$.

(ii) Proof of (ii) is similar.

A system of linear equations $AX = B$ is said to be **consistent** if there exists a solution of $AX = B$, otherwise it is called **inconsistent**.

Let $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ & $b = (b_1, b_2, \ldots, b_m)$

* If $b \in CS(A)$, then the system $AX = B$ is consistent & if $b \notin CS(A)$ then the system $AX = B$ is inconsistent. We want to find a "pseud-solution" i.e. a point x_0 such that distance of Ax_0 and b is minimum.

Least Square Solution

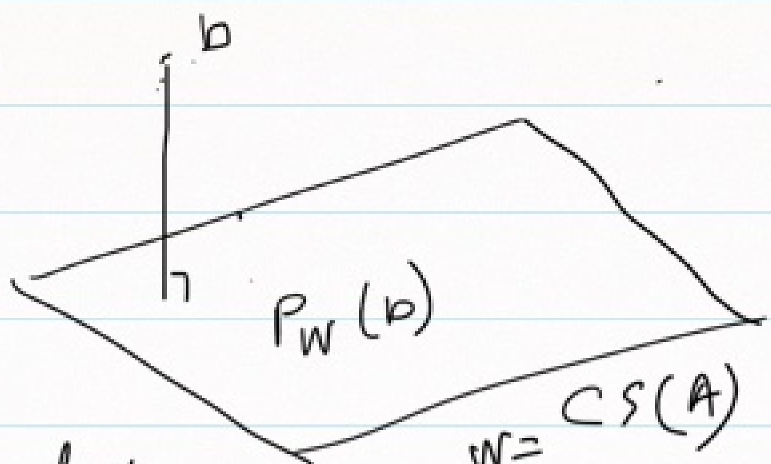
Let $A = (a_{ij})$ be a matrix of order $m \times n$, it induces a linear map

$$A : \mathbb{R}^n \to \mathbb{R}^m, \quad A(X) = AX.$$

Let $b \in \mathbb{R}^m$, we want to find $x_0 \in \mathbb{R}^n$ such that

$$\|Ax_0 - b\|^2 = \min_{x \in \mathbb{R}^n} \|Ax - b\|^2 .$$

Existence of x_0:



Let $W = CS(A)$.
Take orthogonal projection $P_W(b)$ of b on W. $P_W(b) \in CS(A)$

$\Rightarrow \exists\, x_0 \in \mathbb{R}^n$ s.t

$P_W(b) = Ax_0$ & we know that $\|b - P_W(b)\| \leq \|b - y\| \ \forall y \in W$.

Such a solution x_0 need not be unique, if we take $z \in N(A)$ then $A(x_0 + z) = Ax_0$ hence $x_0 + z$ is also a solution.

So, any element of $x_0 + N(A)$ is a solution. As $P_W(b)$ is unique such point on W, the solution set is $x_0 + N(A)$.

What is x_0? To find $P_W(b)$, we need to find an orthonormal basis of $CS(A)$ & then find the point x_0. The whole process may be cumbersome. We will use fundamental theorem of linear algebra to avoid it.

$$\mathbb{R}^m = N(A^T) \oplus CS(A) \quad , W = CS(A).$$

$b \in \mathbb{R}^m$, $\quad b = (b - P_W(b)) + P_W(b)$

$$P_W(b) \in CS(A)$$

$$\Rightarrow \quad b - P_W(b) \in N(A^T)$$

$$\Rightarrow \quad A^T(b - P_W(b)) = 0$$

$$\Rightarrow \quad A^T(P_W(b) - b) = 0$$

$$\Rightarrow \quad A^T(Ax_0 - b) = 0$$

$$\Rightarrow \quad A^T A x_0 = A^T b$$

$\Rightarrow$ x_0 is a solution of $A^T A X = A^T b$.

Conversely, if y_0 is a solution of $A^T A X = A^T b$

$$\Rightarrow \quad A^T(Ay_0 - b) = 0$$

$$\Rightarrow \quad Ay_0 - b \in N(A^T), \quad Ay_0 \in CS(A)$$

& $b = (b - Ay_0) + Ay_0$

b can be written uniquely as $x + y$, where

$x \in N(A^T)$ & $y \in CS(A)$.

So, $Ay_0 = P_W(b)$

Thus, we have the following theorem

Theorem: x_0 is a solution of $A^TAx = A^Tb$ if and only if $\|Ax_0 - b\|^2 = \min_{x \in \mathbb{R}^n} \|Ax - b\|^2$

The system of linear equation

$$A^TAX = A^TB$$ is called

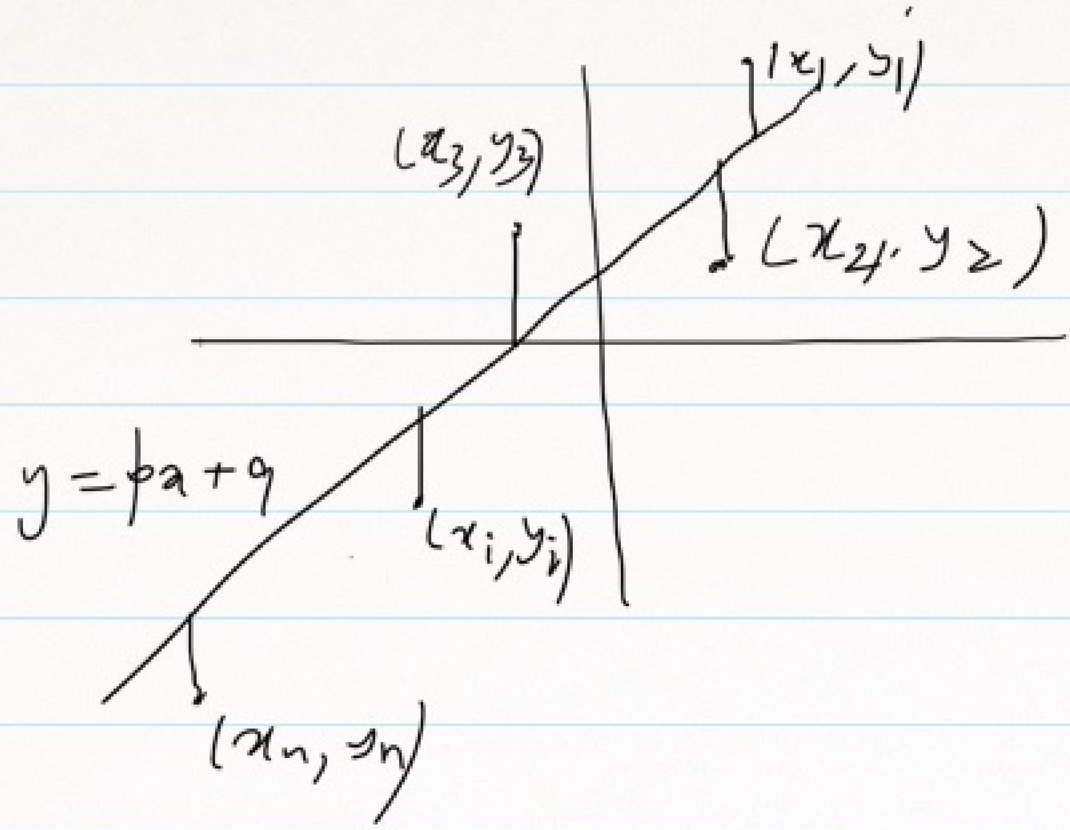
Normal system of equations.

Least square fittings:

Suppose $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are finitely many points in $\mathbb{R}^2$. Least square fittings is to find a straight line $y = px + q$ such that

$$\sum_{i=1}^{n} |y_i - (px_i + q)|^2$$ is

minimum. We will apply least square solution to find the straight line $y = px + q$.



Consider

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \quad X = \begin{pmatrix} q \\ p \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

The least square fittings is to find $X_0 = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$ such that,

$$\|AX_0 - b\|^2 = \min_{X \in \mathbb{R}^2} \|AX - b\|^2 \quad — ☆$$

Where $b = (b_1, \ldots, b_n)$.

By least square solution, solution of ☆ exists & it is a solution of

$$A^T A X = A^T B$$

Example : $(x_1, y_1) = (1, 0),\ (x_2, y_2) = (2, 3),$
$(x_3, y_3) = (3, 4),\ (x_4, y_4) = (4, 4)$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} q \\ p \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{20} \begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix}$$

$$A^T B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ 34 \end{pmatrix}$$

$$A^T A X = A^T B$$

$$\Rightarrow X = \begin{pmatrix} q \\ p \end{pmatrix} = (A^T A)^{-1} A^T B = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix} \begin{pmatrix} 11 \\ 34 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 13/10 \end{pmatrix}$$

$y = \frac{13}{10} x - \frac{1}{2}$ is the required

straight line.