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Quiz 1

MTH302: Set Theory and Mathematical Logic

(Odd Semester 2024/25, IIT Kanpur)

Question 1. [**3** $\times$ **1 Points**]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) For every uncountable linear ordering $(L, \prec)$, there exists an infinite $X \subseteq L$ such that $(X, \prec)$ is a well-ordering.

Answer False. Consider $(\omega_1, >)$. $\square$

(ii) The set of all irrationals numbers has the same cardinality as the set of all real numbers.

Answer True. Since $|\mathbb{R}| = |\mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q}| = \max(|\mathbb{Q}|, |\mathbb{R} \setminus \mathbb{Q}|) = \max(\omega, |\mathbb{R} \setminus \mathbb{Q}|)$. Since $\mathbb{R}$ is uncountable, it follows that $|\mathbb{R}| = |\mathbb{R} \setminus \mathbb{Q}|$. $\square$

(iii) There exists a sequence of sets $\langle X_n : n < \omega \rangle$ such that $|X_{n+1}| < |X_n|$ for every $n < \omega$.

Answer False. Otherwise $\{|X_n| : n < \omega\}$ does not have smallest ordinal. $\square$

Question 2. [**7 Points**]

(a) [**1 Point**] State Schrőder-Bernstein theorem.

Answer If there are injective functions from A to B and from B to A, then there is a bijection from A to B. $\square$

(b) [**2 Points**] State Zorn's lemma.

Answer Let $(P, \preceq)$ be a partial ordering such that every chain in P has an upper bound in P. Then $(P, \preceq)$ has a maximal element. $\square$

(c) [**2 Points**] Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function with $f(1) = 7$. Assume f is continuous at 0. Show that $f(x) = 7x$ for all $x \in \mathbb{R}$.

Answer It suffices to show that f is everywhere continuous. Let $x \in \mathbb{R}$. Then

$$\lim_{h \to 0} f(x+h) = \lim_{h \to 0} (f(x) + f(h)) = \lim_{h \to 0} f(x) + \lim_{h \to 0} f(h) = f(x) + f(0) = f(x) + 0 = f(x)$$

where in the third equality we used the fact that f is continuous at 0. $\square$

(d) [**2 Points**] Show that the set of all bijections from ω to ω is uncountable.

Answer Suppose not and towards a contradiction let $\langle f_k : k < \omega \rangle$ list all bijections from ω to ω. Define $d : \omega \to \omega$ as follows.

If $f_k(2k) = 2k + 1$, then $d(2k) = 2k$ and $d(2k + 1) = 2k + 1$.

If $f_k(2k) \neq 2k + 1$, then $d(2k) = 2k + 1$ and $d(2k + 1) = 2k$.

It is clear that d is a bijection from ω to ω distinct from each f_k (as $d(2k) \neq f_k(2k)$). This contradicts our assumption that $\langle f_k : k < \omega \rangle$ lists all bijections from ω to ω. $\square$