Lecture 11 : Isomorphism theorem + Rank of a matrix

Isomorphism between Vector spaces:

A linear transformation $T: V \to W$ is said to be an isomorphism if T is injective and surjective i.e. T is a bijection.

Proposition: A linear transformation $T: V \to W$ is an isomorphism if and only if $\ker(T) = \{0\}$ & $R(T) = W$

Proof: Exercise.

Example (1) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x+y, y)$ is an isomorphism.

We have seen $\ker(T) = \{0\}$

By Rank-Nullity theorem,

$$\dim(\ker(T)) + \dim(R(T)) = \dim(\mathbb{R}^2)$$

$$\Rightarrow \quad 0 + \dim(R(T)) = 2$$

$$\Rightarrow \quad \dim(R(T)) = 2 \ \& \ R(T) \subseteq \mathbb{R}^2$$

$$\Rightarrow \quad R(T) = \mathbb{R}^2.$$

(2) Any linear transformation $T : V \to V$ (V finite dimensional) with $\ker(T) = \{0\}$ is an isomorphism.

First Isomorphism Theorem:

Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces V & W. $\ker(T)$ is a subspace of V. We can think about the quotient space

$$V/\ker(T) \underset{\text{defn.}}{:=} \{ u + \ker(T) : u \in V \}$$

where $u + \ker(T) = \{ u + x : x \in \ker(T) \}$

If $\{u_1, \ldots, u_m\}$ is a basis of $\ker(T)$, we can extend it to a basis $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ of V, $n = \dim(V)$. From Rank-Nullity theorem, we have seen that $\{T(u_{m+1}), \ldots, T(u_n)\}$ is a basis of $R(T)$. Check that there is a bijective correspondce between the sets $\{u_{m+i} + \ker(T) : 1 \leq i \leq n\}$ & $\{T(u_{m+i}) : 1 \leq i \leq n\}$

Thus, we have the following theorem:

Theorem: Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces V & W. T induces a map

$$\overline{T} : V/_{\ker(T)} \longrightarrow R(T)$$

$$\overline{T}\,(u + \ker(T)) := T(u), \text{ where } u \in V$$

Then $\overline{T}$ is an isomorphism.

Example: $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x, y, z) = x + y + z$ is a linear map.

$$\ker(T) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

(the plane $P : x + y + z = 0$)

Let $r \in \mathbb{R}$ then $T(r, 0, 0) = r$ so T is surjective. By 1st theorem,

$\mathbb{R}^3/_P$ is isomorphic to $\mathbb{R}$.

☆ There are other Isomorphism theorems, for this course we deal with this one only.

Row space, Column Space & Rank of a matrix

Let $A = (a_{ij})$ be a matrix of order $m \times n$.

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & & C_n \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \\ \\ R_m \rightarrow \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

$R_i : (a_{i1}, a_{i2}, \cdots, a_{in})$ is ith row vector.

$C_j : (a_{1j}, a_{2j}, \cdots, a_{mj})$ is jth column vector.

Row space : Linear span of $R_1, \cdots, R_m$

$$L(R_1, \cdots, R_m) = \left\{ \sum_{i=1}^{m} \lambda_i R_i : \lambda_1, \cdots \lambda_m \in \mathbb{R} \right\}$$

Column space : Linear span of $C_1, \cdots C_n$

$$L(C_1, \cdots, C_n) = \left\{ \sum_{j=1}^{n} \beta_j C_j : \beta_1, \cdots, \beta_j \in \mathbb{R} \right\}$$

Row space is subspace of $\mathbb{R}^n$ &
Column space is subspace of $\mathbb{R}^m$.

Let us denote row space of A as $RS(A)$
& column space as $CS(A)$
Note that $\dim(RS(A)) \leq \min\{m, n\}$
& $\dim(CS(A)) \leq \min\{m, n\}$.

Row Rank of a matrix A is $\dim(RS(A))$
Column Rank of a matrix A is $\dim(CS(A))$

Theorem: Row rank of a matrix A
= Column rank of A.

Proof: Let A be a matrix of order $m \times n$
& k = row rank of A = $\dim(RS(A))$
Let $\{u_1, \ldots, u_k\}$ be a basis of $RS(A)$.
& $u_r = (b_{r1}, \ldots, b_{rn})$

$$R_i = \sum_{r=1}^{k} \alpha_{ir} u_r \quad , \quad i \in \{1, \ldots, m\}$$

$$(a_{i1}, a_{i2}, \ldots, a_{in}) = \sum_{r=1}^{k} \alpha_{ir} (b_{r1}, b_{r2}, \ldots, b_{rn})$$

$$= \left(\sum_{r=1}^{k} \alpha_{ir} b_{r1}, \ldots, \sum_{r=1}^{k} \alpha_{ir} b_{rn} \right)$$

$$\Longrightarrow \quad a_{ij} = \sum_{r=1}^{k} \alpha_{ir} b_{rj} \quad , \quad 1 \leq j \leq n.$$

Thus, $a_{1j} = \sum_{r=1}^{k} \alpha_{1r} b_{rj}, \ldots, a_{mj} = \sum_{r=1}^{k} \alpha_{mr} b_{rj}$

& $c_j = (a_{1j}, \ldots, a_{mj})$

$$= \left(\sum_{r=1}^{k} \alpha_{1r} b_{rj}, \ldots, \sum_{r=1}^{k} \alpha_{mr} b_{rj} \right)$$

$$= b_{1j} (\alpha_{11}, \alpha_{21}, \ldots, \alpha_{m1})$$
$$+ b_{2j} (\alpha_{12}, \alpha_{22}, \ldots, \alpha_{m2}) + \ldots$$
$$+ b_{kj} (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk})$$

Let $w_p = (\alpha_{1p}, \alpha_{2p}, \ldots, \alpha_{mk})$

where $p \in \{1, 2, \ldots, k\}$

$$\Rightarrow c_j = \sum_{p=1}^{k} b_{pj} w_p$$

$c_j \in L(\{w_1, \ldots, w_k\})$ (span of $w_1, \ldots, w_k$)

$\forall j \in \{1, \ldots, n\}$

$\therefore CS(A) = L(\{c_1, \ldots, c_n\}) \subseteq L(\{w_1, \ldots, w_k\})$

$\Rightarrow \dim(CS(A)) \leq \dim(L(\{w_1, \ldots, w_k\}))$

$\leq k = \dim(RS(A))$

Similarly, $\dim(RS(A)) \leq \dim(CS(A))$ $\boxtimes$

Rank of a matrix is $\dim(RS(A))$
$(\text{or } \dim(CS(A)))$

Example: we will compute the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

Applying row operations $R_2 - 2R_1$, $R_3 - 3R_1$ & then $(-1)R_2$, we get RREF of A

$$B = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that each row of RREF of A is in $RS(A) \Rightarrow RS(B) \subseteq RS(A)$

Also, $E_2(-1)\, E_{31}(-3)\, E_{21}(-2)\, A = B$

Elementary matrices are invertible

$$A = E_{21}^{-1}(-2)\, E_{31}^{-1}(-3)\, E_2^{-1}(-1)\, B$$

Inverses of elementary matrices are again elementary.

A can be obtained from B by applying row operations on B. So, $RS(A) \subseteq RS(B)$

So, $RS(A) = RS(B)$

$\dim RS(B) = 2$, as $(1, 0, -1, -2)$ & $(0, 1, 2, 3)$ is L.I.

So, $\dim RS(A) = 2$

$\Rightarrow \text{rank}(A) = 2$.

Proposition: Let A be a matrix in RREF then $\text{rank}(A)$ = number of non-zero rows.

Proof: Let A have k non-zero rows $A_1, A_2, \dots, A_k$.

consider $c_1 A_1 + c_2 A_2 + \dots + c_k A_k = 0$

The leading coeffecient of A_i is 1 & in that column rest of the elements are zero. This implies $c_i = 0$

$\Rightarrow A_1, A_2, \dots, A_k$ are L.I.

So, $\text{rank}(A) = \dim(L(A_1, \dots, A_k))$

$= k$ ▫

A matrix $A=(a_{ij})$ of order $m\times n$ induces a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$,

$$T_A(x_1, x_2, \ldots, x_n) = A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

$$= \begin{pmatrix} a_{11}x_1 \cdots a_{1n}x_n \\ - - - - - \\ a_{i1}x_1 - - \cdot a_{in}x_n \\ - - - - - \\ a_{m1}x_1 \quad a_{mn}x_n \end{pmatrix}$$

$$= x_1 C_1 + \cdots + x_n C_n$$

$$C_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

So, $(x_1, \ldots, x_n) \in R(T_A) \iff (x_1, x_2, \ldots, x_n) \in CS(A)$.

Therefore, $R(T_A) = CS(A)$

$\Rightarrow \dim(R(T_A)) = \operatorname{rank}(A)$. So we have the following theorem

Theorem: Let A be a matrix of order $m\times n$. Then $\dim(R(T_A)) = \operatorname{rank}(A)$ □