4. Week 4

Theorem 4.1. *Let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ and DF F_X. Then*

(a) F_X is non-decreasing, i.e. $F_X(x) \leq F_X(y), \forall x < y$.
(b) F_X is right continuous, i.e. $\lim_{h\downarrow 0} F_X(x+h) = F_X(x), \forall x \in \mathbb{R}$.
(c) $F_X(-\infty) := \lim_{x\to-\infty} F_X(x) = 0$ and $F_X(\infty) := \lim_{x\to\infty} F_X(x) = 1$.

Proof. For all $x < y$, observe that $(-\infty, x] \subsetneq (-\infty, y]$. Since $\mathbb{P}_X$ is a probability measure, we have $\mathbb{P}_X((-\infty, x]) \leq \mathbb{P}_X((-\infty, y])$. The statement (a) follows.

By definition, F_X takes values in $[0, 1]$ and hence it is bounded. Since F_X is non-decreasing, the limit $F_X(x+) = \lim_{h\downarrow 0} F_X(x+h)$ exists for all $x \in \mathbb{R}$. Using the non-decreasing property, we use the following fact from real analysis that $F_X(x+) = \lim_{n\to\infty} F_X(x + \frac{1}{n})$. By Proposition 3.17, we have

$$F_X(x+) = \lim_{n\to\infty} F_X\left(x + \frac{1}{n}\right) = \lim_{n\to\infty} \mathbb{P}_X(\left(-\infty, x + \frac{1}{n}\right]) = \mathbb{P}_X((-\infty, x]) = F_X(x).$$

This proves statement (b). Here, we use the fact that $(-\infty, x + \frac{1}{n}] \downarrow (-\infty, x]$.

Similar to the proof of statement (b), we have

$$F_X(-\infty) = \lim_{n\to\infty} F_X(-n) = \lim_{n\to\infty} \mathbb{P}_X((-\infty, -n]) = \mathbb{P}_X(\emptyset) = 0,$$

and

$$F_X(\infty) = \lim_{n\to\infty} F_X(n) = \lim_{n\to\infty} \mathbb{P}_X((-\infty, n]) = \mathbb{P}_X(\mathbb{R}) = 1.$$

Here, we use that facts that $(-\infty, -n] \downarrow \emptyset$ and $(-\infty, n] \uparrow \mathbb{R}$. This proves statement (c). $\square$

The next theorem is stated without proof. The arguments required to prove this statement is beyond the scope of this course.

Theorem 4.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing and right continuous function such that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. Then there exists an RV X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $F = F_X$, i.e. $F(x) = F_X(x), \forall x$.*

Remark 4.3. Given any function $F : \mathbb{R} \to \mathbb{R}$, as soon as we check the relevant conditions, we can claim that it is the DF of some RV by Theorem 4.2.

Example 4.4. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

The function is a constant on $(-\infty, 0)$ and on $[1, \infty)$. Moreover, it is non-decreasing in the interval $[0, 1)$. Further for $x < 0, y \in (0, 1), z > 1$, we have

$$F(x) = F(0) < F(y) < F(1) = F(z).$$

Hence, F in non-decreasing over $\mathbb{R}$. Again, by definition F is continuous on the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. We check for right continuity at the points 0 and 1. We have

$$\lim_{h \downarrow 0} F(0 + h) = \lim_{h \downarrow 0} h = 0 = F(0), \quad \lim_{h \downarrow 0} F(1 + h) = \lim_{h \downarrow 0} 1 = 1 = F(1).$$

Hence, F is right continuous on $\mathbb{R}$. Finally, $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} 0 = 0$ and $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 = 1$. Hence, F is the DF of some RV. Later on, we shall identify the corresponding RV.

Proposition 4.5 (Further properties of a DF)**.** *Let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}_X$ and DF F_X.*

(a) For all $x \in \mathbb{R}$, the limit $F_X(x-) = \lim_{h \downarrow 0} F_X(x - h)$ exists and equals $\mathbb{P}_X((-\infty, x)) = \mathbb{P}(X < x)$.

Proof. Since F_X in non-decreasing and bounded, as argued in Theorem 4.1, the limit $F_X(x-) = \lim_{h \downarrow 0} F_X(x - h)$ exists and moreover, by Proposition 3.17 we have

$$F_X(x-) = \lim_{n \to \infty} F_X\left(x - \frac{1}{n}\right) = \lim_{n \to \infty} \mathbb{P}_X(\left(-\infty, x - \frac{1}{n}\right]) = \mathbb{P}_X((-\infty, x)) = \mathbb{P}(X < x).$$

Here, we use the fact that $(-\infty, x - \frac{1}{n}] \uparrow (-\infty, x)$. $\square$

(b) For all $x \in \mathbb{R}$, $\mathbb{P}(X \geq x) = 1 - F_X(x-)$.

Proof. We have, $\mathbb{P}(X \geq x) = \mathbb{P}_X([x, \infty)) = \mathbb{P}_X((-\infty, x)^c) = 1 - \mathbb{P}_X((-\infty, x)) = 1 - F_X(x-)$. $\square$

(c) *For any $x \in \mathbb{R}$, $F_X(x-) \leq F_X(x+)$.*

Proof. By the non-decreasing property of F_X, for all $x \in \mathbb{R}$ and positive integers n, we have, $F_X(x - \frac{1}{n}) \leq F_X(x + \frac{1}{n})$. Letting n go to infinity in this inequality, we get the result. $\square$

(d) *F_X is continuous at x if and only if $F_X(x) = F_X(x-)$.*

Proof. A real valued function is continuous at a point x if and only if the function is both right continuous and left continuous at the point x. Now, by construction, F_X is right continuous on $\mathbb{R}$. Hence, F_X is continuous at x if and only if F_X is left continuous at x. The last statement is exactly the statement to be proved. $\square$

(e) *Only possible discontinuities of F_X are jump discontinuities.*

Proof. As discussed in Theorem 4.1 and in part (a), for any $x \in \mathbb{R}$, both the limits $F_X(x+)$ and $F_X(x-)$ exist and $F_X(x+) = F_X(x)$. Since $F_X(x-) \leq F_X(x+)$, the only possible discontinuity appears if and only if $F_X(x-) < F_X(x+)$. These discontinuities are jump discontinuities. This completes the proof. $\square$

(f) *For all $x \in \mathbb{R}$, we have $F_X(x+) - F_X(x-) = \mathbb{P}(X = x)$.*

Proof. By the finite additivity of $\mathbb{P}_X$, we have $F_X(x+) - F_X(x-) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = \mathbb{P}_X((-\infty, x]) - \mathbb{P}_X((-\infty, x)) = \mathbb{P}_X(\{x\}) = \mathbb{P}(X = x)$. $\square$

(g) *If F_X has a jump at x, then the jump is given by $F_X(x+) - F_X(x-) = \mathbb{P}(X = x)$.*

Proof. If F_X has a jump at x, then the jump is given by $F_X(x+) - F_X(x-)$. The result follows from statement (f). $\square$

(h) *F_X is continuous at x if and only if $\mathbb{P}(X = x) = 0$.*

Proof. Recall that $F_X(x+) = F_X(x)$. Then by statement (d) and (f), we have F_X is continuous at x if and only if $F_X(x+) = F_X(x-)$ and hence, if and only if $\mathbb{P}(X = x) = 0$. $\square$

(i) *Consider the set $D := \{x \in \mathbb{R} : F_X \text{ is discontinuous at } x\} = \{x \in \mathbb{R} : F_X(x-) < F_X(x+)\} = \{x \in \mathbb{R} : \mathbb{P}(X = x) > 0\}$. Then D is either finite or countably infinite. (Note that if F_X is continuous on $\mathbb{R}$, then $D = \emptyset$.)*

Proof. Left as an exercise in practice problem set 4. □

(j) For all $x < y$, we have

$$\mathbb{P}(x < X \leq y) = F_X(y) - F_X(x),$$

$$\mathbb{P}(x < X < y) = F_X(y-) - F_X(x),$$

$$\mathbb{P}(x \leq X < y) = F_X(y-) - F_X(x-),$$

$$\mathbb{P}(x \leq X \leq y) = F_X(y) - F_X(x-).$$

Proof. We prove the first two equalities. Proof of the last two equalities are similar.

By the finite additivity of $\mathbb{P}_X$, we have $F_X(y) - F_X(x) = \mathbb{P}_X((-\infty, y]) - \mathbb{P}_X((-\infty, x]) = \mathbb{P}_X((x, y]) = \mathbb{P}(x < X \leq y)$.

Again, $F_X(y-) - F_X(x) = \mathbb{P}_X((-\infty, y)) - \mathbb{P}_X((-\infty, x]) = \mathbb{P}_X((x, y)) = \mathbb{P}(x < X < y)$.

This completes the proof. □

Example 4.6. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

Assume that F is the DF of some RV X (left as an exercise in practice problem set 4). Since F is continuous on the intervals $(-\infty, 0), (0, 1), (1, 2)$ and $(2, \infty)$, discontinuities may arise only at the points $0, 1, 2$.

We have $F(0-) = \lim_{h \downarrow 0} F(0 - h) = 0$ and $F(0) = \frac{1}{4}$. Therefore F is discontinuous at 0 with jump $F(0) - F(0-) = \frac{1}{4}$.

We have $F(1-) = \lim_{h \downarrow 0} F(1 - h) = \lim_{h \downarrow 0}[\frac{1}{4} + \frac{1-h}{2}] = \frac{3}{4}$ and $F(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. Therefore F is continuous at 1.

We have $F(2-) = \lim_{h \downarrow 0} F(2 - h) = \lim_{h \downarrow 0}[\frac{1}{2} + \frac{2-h}{4}] = 1$ and $F(2) = 1$. Therefore F is continuous at 2.

Only discontinuity of F is at the point 0. In particular, $\mathbb{P}(X = 0) = F(0) - F(0-) = \frac{1}{4}$. At all other points F is continuous and hence $\mathbb{P}(X = x) = 0, \forall x \neq 0$.
Observe that $\mathbb{P}(0 \leq X < 1) = F(1-) - F(0-) = \frac{3}{4}$. Again, $\mathbb{P}(\frac{3}{2} < X \leq 2) = F(2) - F(\frac{3}{2}) = 1 - [\frac{1}{2} + \frac{3}{8}] = \frac{1}{8}$.

We now discuss special classes of RVs defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that $\mathbb{P}_X$ and F_X denote the law/distribution and the distribution function (DF) of an RV X, respectively.

Definition 4.7 (Discrete RV)**.** An RV X is said to be a discrete RV if there exists a finite or countably infinite set $S \subsetneq \mathbb{R}$ such that

$$1 = \mathbb{P}_X(S) = \mathbb{P}(X \in S) = \sum_{x \in S} \mathbb{P}_X(\{x\}) = \sum_{x \in S} \mathbb{P}(X = x)$$

and $\mathbb{P}(X = x) > 0, \forall x \in S$. In this situation, we refer to the set S as the support of the discrete RV X.

Remark 4.8. Let X be a discrete RV with DF F_X and support S. Then we have the following observations.

(a) $\mathbb{P}_X(S^c) = 1 - \mathbb{P}_X(S) = 0$. In particular, for any $x \in S^c$, $0 \leq \mathbb{P}(X = x) = \mathbb{P}_X(\{x\}) \leq \mathbb{P}_X(S^c) = 0$ and hence $\mathbb{P}(X = x) = 0, \forall x \in S^c$.

(b) Since $\mathbb{P}_X(S) = 1$, for any $A \subseteq \mathbb{R}$, we have $\mathbb{P}_X(A) = \mathbb{P}_X(A \cap S)$ (see problem set 1). Moreover,
$$\mathbb{P}_X(A) = \mathbb{P}(X \in A) = \mathbb{P}_X(A \cap S) = \sum_{x \in A \cap S} \mathbb{P}(X = x).$$

(c) Recall that F_X is right continuous, i.e. $F_X(x+) = F_X(x), \forall x \in \mathbb{R}$. Moreover, $F_X(x) - F_X(x-) = \mathbb{P}(X = x)$. From the discussion above, we conclude that
$$F_X(x) - F_X(x-) = \mathbb{P}(X = x) \begin{cases} > 0, & \text{if } x \in S, \\ = 0, & \text{if } x \in S^c. \end{cases}$$
Hence, the set of discontinuities of F_X is exactly the support S.

(d) Note that

$$1 = \sum_{x \in S} \mathbb{P}(X = x) = \sum_{x \in S} [F_X(x) - F_X(x-)].$$

Hence, the sum of the jumps of F_X is exactly 1.

Example 4.9. Consider the DF $F : \mathbb{R} \to [0, 1]$ considered in Example 4.6 given by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

As discussed earlier, F only has a discontinuity at the point 0. If an RV X has this F as the DF, then

$$\sum_{x \in D} \mathbb{P}(X = x) = \mathbb{P}(X = 0) = \frac{1}{4} \neq 1,$$

with $D = \{0\}$ as the set of discontinuities of F. This RV X is not discrete.

Example 4.10. Let X denote the number of heads in tossing a fair coin twice independently. As computed earlier in Example 3.23, the DF F_X is given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4}, \text{ if } 0 \leq x < 1, \\ \frac{3}{4}, \text{ if } 1 \leq x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

Clearly, the set D of discontinuities of F_X is $\{0, 1, 2\}$ with

$$\mathbb{P}(X = x) = F_X(x) - F_X(x-) = \begin{cases} \frac{1}{4} - 0 = \frac{1}{4}, \text{ if } x = 0, \\ \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \text{ if } x = 1, \\ 1 - \frac{3}{4} = \frac{1}{4}, \text{ if } x = 2. \end{cases}$$

Since $\sum_{x \in D} \mathbb{P}(X = x) = 1$, the RV X is discrete with support D.

Definition 4.11 (Probability Mass Function (p.m.f.))**.** Let X be a discrete RV with DF F_X and support S. Consider the function $f_X : \mathbb{R} \to \mathbb{R}$ defined by

$$f_X(x) := \begin{cases} F_X(x) - F_X(x-) = \mathbb{P}(X = x), \text{ if } x \in S, \\ 0, \text{ if } x \in S^c. \end{cases}$$

This function f_X is called the probability mass function (p.m.f.) of X.

Example 4.12. Continuing with the Example 4.10, the p.m.f. f_X is given by

$$f_X(x) = \begin{cases} \frac{1}{4}, \text{ if } x = 0, \\ \frac{1}{2}, \text{ if } x = 1, \\ \frac{1}{4}, \text{ if } x = 2., \\ 0, \text{ otherwise.} \end{cases}$$

Remark 4.13. Let X be a discrete RV with DF F_X, p.m.f. f_X and support S. Then we have the following observations.

(a) Continuing the discussion from Remark 4.8, we have for all $A \subseteq \mathbb{R}$,

$$\mathbb{P}_X(A) = \mathbb{P}(X \in A) = \sum_{x \in A \cap S} f_X(x).$$

(b) As a special case of the previous observation, note that for $A = (-\infty, x], x \in \mathbb{R}$, we obtain

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \in (-\infty, x]) = \sum_{t \in (-\infty, x] \cap S} f_X(t).$$

Therefore, the p.m.f. f_X is uniquely determined by the DF F_X and vice versa.

(c) To study a discrete RV X, we may study any one of the following three quantities, viz. the law/distribution $\mathbb{P}_X$, the DF F_X or the p.m.f. f_X. Given any one of these quantities, the other two can be obtained using the relations described above.

(d) By Definition 4.7 and Definition 4.11, we have that the p.m.f. $f_X : \mathbb{R} \to \mathbb{R}$ is a function such that

$$f_X(x) = 0, \forall x \in S^c, \quad f_X(x) > 0, \forall x \in S, \quad \sum_{x \in S} f_X(x) = 1.$$

Remark 4.14. Let $\emptyset \neq S \subset \mathbb{R}$ be a finite or countably infinite set and let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = 0, \forall x \in S^c, \quad f(x) > 0, \forall x \in S, \quad \sum_{x \in S} f(x) = 1.$$

Then by an argument similar to Proposition 1.44, we conclude that $\mathbb{P}$ as defined below is a probability function/measure on $\mathbb{B}$, where $\mathbb{B}$ denotes the power set of $\mathbb{R}$. For all $A \subseteq \mathbb{R}$, consider

$$\mathbb{P}(A) := \sum_{x \in A \cap S} f(x).$$

By an argument similar to Theorem 4.1, we can then show that the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) := \mathbb{P}((-\infty, x]), \forall x \in \mathbb{R}$ is non-decreasing, right continuous with $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. By Theorem 4.2, this F is the DF of some RV Y, i.e. $F_Y = F$ and by construction, Y must be discrete with support S and p.m.f. $f_Y = f$.

Example 4.15. Take S to be the set of natural numbers $\{1, 2, \cdots\}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \frac{1}{2^x}, & \text{if } x \in S, \\ 0, & \text{if } x \in S^c. \end{cases}$$

Then f takes non-negative values with $\sum_{x \in S} f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Therefore f is the p.m.f. of some RV X with DF F_X given by

$$\begin{aligned} F_X(x) = \mathbb{P}(X \leq x) &= \sum_{t \in (-\infty, x] \cap S} f_X(t) \\ &= \begin{cases} 0, & \text{if } x < 1, \\ \sum_{n=1}^{m} \frac{1}{2^n}, & \text{if } x \in [m, m+1), m \in S. \end{cases} = \begin{cases} 0, & \text{if } x < 1, \\ 1 - \frac{1}{2^m}, & \text{if } x \in [m, m+1), m \in S. \end{cases} \end{aligned}$$

Definition 4.16 (Continuous RV and its Probability Density Function (p.d.f.))**.** An RV X is said to be a continuous RV if there exists an integrable function $f : \mathbb{R} \to [0, \infty)$ such that

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \forall x \in \mathbb{R}.$$

The function f is called the probability density function (p.d.f.) of X.

Remark 4.17*.* Let X be a continuous RV with DF F_X and p.d.f. f_X. Then we have the following observations.

(a) Since f_X is integrable, from the relation $F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt, \forall x \in \mathbb{R}$, we have F_X is continuous on $\mathbb{R}$. In particular, F_X is absolutely continuous. Moreover, for all $a < b$, we have

$$F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(t)\, dt - \int_{-\infty}^{a} f_X(t)\, dt = \int_{a}^{b} f_X(t)\, dt.$$

(b) Since F_X is continuous, we have

(i) $F_X(x-) = F_X(x) = F_X(x+), \forall x \in \mathbb{R}$.

(ii) $\mathbb{P}(X = x) = \mathbb{P}_X(\{x\}) = F_X(x) - F_X(x-) = 0, \forall x \in \mathbb{R}$.

(iii) $\mathbb{P}(X < x) = F_X(x-) = F_X(x) = \mathbb{P}(X \leq x), \forall x \in \mathbb{R}$.

(iv) For all $a < b$,

$$\begin{aligned}\mathbb{P}(a < X < b) &= \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b)\\ &= F_X(b) - F_X(a) = \int_{a}^{b} f_X(t)\, dt.\end{aligned}$$

(c) If $A \subset \mathbb{R}$ is finite or countably infinite, then by the finite/countable additivity of $\mathbb{P}_X$, we have

$$\mathbb{P}(X \in A) = \mathbb{P}_X(A) = \sum_{x \in A} \mathbb{P}_X(\{x\}) = 0.$$

(d) By definition, we have $f_X(x) \geq 0, \forall x \in \mathbb{R}$ and

$$1 = \lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f_X(t)\, dt = \int_{-\infty}^{\infty} f_X(t)\, dt.$$

Remark 4.18*.* Let $f : \mathbb{R} \to [0, \infty)$ be an integrable function with $\int_{-\infty}^{\infty} f(t)\, dt = 1$. Then the function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) := \int_{-\infty}^{x} f(t)\, dt, \forall x \in \mathbb{R}$ is non-decreasing and continuous with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. By Theorem 4.2, this F is the DF of some RV Y, i.e. $F_Y = F$ and by construction, Y must be continuous with p.d.f. $f_Y = f$.

Example 4.19. Let X be an RV with the DF $F_X : \mathbb{R} \to \mathbb{R}$ as discussed in Example 4.4. Here,

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ x, \text{ if } 0 \leq x < 1, \\ 1, \text{ if } x \geq 1. \end{cases}$$

Then the function $f : \mathbb{R} \to [0, \infty)$ defined by

$$f(x) := \begin{cases} 1, \text{ if } 0 \leq x \leq 1, \\ 0, \text{ otherwise} \end{cases}$$

is an integrable function with $F_X(x) = \int_{-\infty}^{x} f(t)\, dt, \forall x \in \mathbb{R}$. Therefore, X is a continuous RV with p.d.f. f.

Example 4.20. Consider the DF $F : \mathbb{R} \to [0, 1]$ considered in Example 4.6 given by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

As discussed earlier, F has a discontinuity at the point 0. Therefore, an RV X with DF F is not a continuous RV.

Note 4.21. Given a continuous RV X with p.d.f. f_X, the DF F_X is computed by the formula $F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt, \forall x \in \mathbb{R}$.

Example 4.22. Consider a function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \begin{cases} \alpha x, \text{ if } x \in [-1, 0), \\ \frac{x^2}{8}, \text{ if } x \in [0, 2], \\ 0, \text{ otherwise} \end{cases}$$

for some $\alpha \in \mathbb{R}$. For this f to be a p.d.f. of a continuous RV, two conditions need to be satisfied, viz. $f(x) \geq 0, \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
The first condition is satisfied on $(-\infty, -1) \cup [0, \infty)$. For $x \in [-1, 0)$, we must have $\alpha x \geq 0$, which implies $\alpha \leq 0$.
From the second condition, we have $\int_{-1}^{0} \alpha x\, dx + \int_0^2 \frac{x^2}{8}\, dx = 1$. This yields $\alpha = -\frac{4}{3}$, which satisfies $\alpha \leq 0$.
Therefore, for f to be a p.d.f. we must have $\alpha = -\frac{4}{3}$.

In what follows, we consider the question of computing f_X from the DF F_X.

Remark 4.23 (Is the p.d.f. of a continuous RV unique?). Let X be a continuous RV with DF F_X and p.d.f. f_X. Fix any finite or countably infinite set $A \subset \mathbb{R}$ and fix $c \geq 0$. Consider the function $g : \mathbb{R} \to [0, \infty)$ defined by

$$g(x) := \begin{cases} f_X(x), \text{ if } x \in A^c, \\ c, \text{ if } x \in A. \end{cases}$$

Then g is integrable and $F_X(x) = \int_{-\infty}^{x} g(t)\, dt, \forall x \in \mathbb{R}$. Hence, g is also a p.d.f. for X. Therefore, the RV X with DF F_X is a continuous RV with p.d.f. f (or g). For example,

$$g(x) := \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise} \end{cases}$$

is a p.d.f. for X as in Example 4.19. More generally, we may also consider

$$g(x) := \begin{cases} f_X(x), \text{ if } x \in A^c, \\ c_x, \text{ if } x \in A \end{cases}$$

as a p.d.f., where $c_x \geq 0, \forall x \in A$.

Note 4.24. In fact, a p.d.f. f_X for a continuous RV X is determined uniquely on the complement of sets of 'length 0', such as sets which are finite or countably infinite. We do not make a precise statement – this is beyond the scope of this course. However, we consider the deduction of p.d.f.s from the DFs.

The next result is stated without proof.

Theorem 4.25. *Let X be an RV with DF F_X.*

(a) If F_X is differentiable on $\mathbb{R}$ with $\int_{-\infty}^{\infty} F_X'(t)\,dt = 1$, then X is a continuous RV with p.d.f. F_X'.

(b) If F_X is differentiable everywhere except on a finite or a countably infinite set $A \subset \mathbb{R}$ with $\int_{-\infty}^{\infty} F_X'(t)\,dt = 1$, then X is a continuous RV with p.d.f. f given by

$$f(x) := \begin{cases} F_X'(x), & \textit{if } x \in A^c, \\ 0, & \textit{if } x \in A. \end{cases}$$

Note 4.26. Continuing the discussion from Note 4.21, the DF F_X of a continuous RV X may be used to compute the p.d.f. f_X. In Example 4.19, the DF F_X is given by

$$F_X(x) := \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

It is differentiable everywhere except at the points 0 and 1. Using Theorem 4.25, we have the p.d.f. given by

$$f(x) := \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note 4.27. To study a continuous RV X, we may study any one of the following three quantities, viz. the law/distribution $\mathbb{P}_X$, the DF F_X or the p.d.f. f_X. Given any one of these quantities, the other two can be obtained using the relations described above.

Remark 4.28 (Identifying discrete/continuous RVs from their DFs). Suppose that the distribution of an RV X is specified by a given DF F_X. In order to check if X is a discrete/continuous RV, we use the following steps.

(a) Identify the set $D = \{x \in \mathbb{R} : F_X(x-) < F_X(x+)\} = \{x \in \mathbb{R} : \mathbb{P}(X = x) > 0\}$ of discontinuities of F_X. Recall that D is a finite or a countably infinite set.

(b) If D is empty, then F_X is continuous on $\mathbb{R}$. By verifying the hypothesis of Theorem 4.25 or otherwise, check if there exists a p.d.f.. If a p.d.f. exists, then X is a continuous RV. Otherwise, X is not a continuous RV.

(c) If F_X has at least one discontinuity, then F_X is not continuous on $\mathbb{R}$ and hence X cannot be a continuous RV. For X to be a discrete RV X, we must have

$$\sum_{x \in D} [F_X(x+) - F_X(x-)] = \sum_{x \in D} \mathbb{P}(X = x) = 1.$$

If the above condition is satisfied, X is a discrete RV. Otherwise, X is not a discrete RV.

Note 4.29. Consider the DF $F : \mathbb{R} \to [0, 1]$ considered in Example 4.6 given by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

As discussed in Example 4.9 and Example 4.20, an RV with DF F is neither discrete nor continuous.