Plan:

I. Vector valued function
Limit, continuity and differentiability.

II. Tangent vector (tangent line)

III. Reparametrization

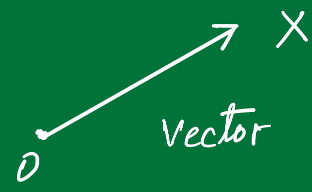
IV. Application.

# Vector-valued functions

$X \in \mathbb{R}^3$

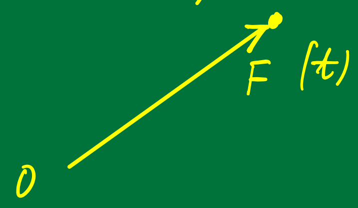
$(x, y, z) \in \mathbb{R}^3$

Let $I \subset \mathbb{R}$ and $F : I \to \mathbb{R}^3$




is a vector valued function

$t \mapsto F(t)$




$$F(t) = \left(f_1(t), f_2(t), f_3(t)\right) \in \mathbb{R}^3$$

$\Rightarrow$ Each vector valued function $F : I \to \mathbb{R}^3$ determine three real valued functions $f_1, f_2, f_3 : I \to \mathbb{R}$ such that $F(t) = \left(f_1(t), f_2(t), f_3(t)\right)$ for all $t \in I$.

Here $f_1(t)$ is the $x$- coordinate (Componet) of $F(t)$

$f_2(t)$ is the $y$ - coordinate (Componet) of $F(t)$ and

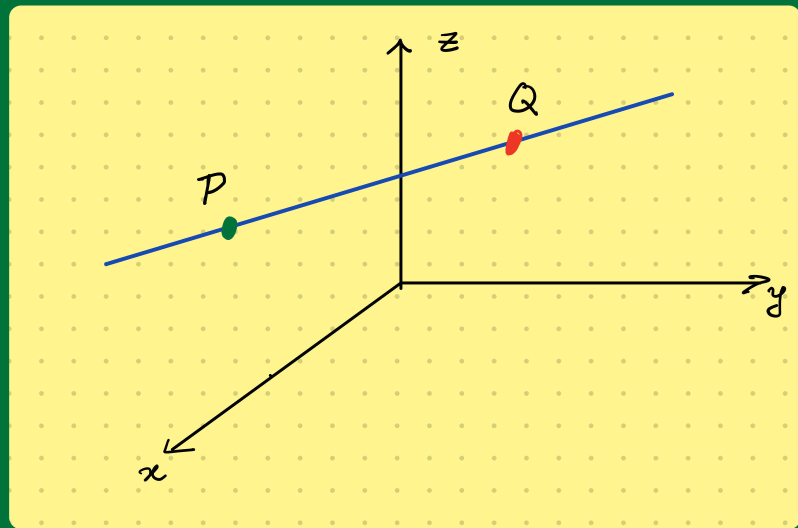
$f_3(t)$ is the $Z$ - coordinate (Componet) of $F(t)$.

## Example ①

Let $P, Q \in \mathbb{R}^3$ and $Q \neq 0$.

Consider $F(t) = P + tQ$

for all $t \in \mathbb{R}$.




- $F: \mathbb{R} \to \mathbb{R}^3$ is a vector valued function and the range (or the image set of $F$ in $\mathbb{R}^3$) of $F$ is the line through the point $P$ and parallel to the vector $Q$.

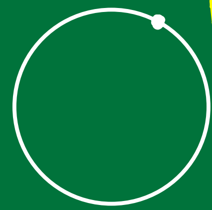
(2)

$F_1 : I \longrightarrow \mathbb{R}^2$

$t \longmapsto (\cos t, \sin t)$

$F_2 : \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto (\cos t, \sin t, t)$



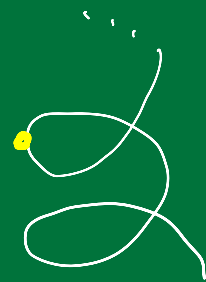
Circle

$F_1(t) = (\cos t, \sin t)$

$t \in [0, 2\pi]$

?

$F_2(t)$



$(\cos t, \sin t, t)$

helix $\quad t \in \mathbb{R}$

Limit

Continuity and

derivative of vector valued

functions

**Aim:**

- Study curves in space
- Calculus of vector valued functions

Let $I \subset \mathbb{R}$ and $F : I \longrightarrow \mathbb{R}^3$ be a vector valued function.

$$t \longmapsto (f_1(t), f_2(t), f_3(t))$$

Let $L = (l_1, l_2, l_3) \in \mathbb{R}^3$.

We say $F(t) \longrightarrow L$ as $t \to t_0$ and write it as $\lim_{t \to t_0} F(t) = L$ if $\lim_{t \to t_0} \| F(t) - L \| = 0$

Note that

$$I \longrightarrow \mathbb{R}$$

$$t \longmapsto \| F(t) - L \|$$

is a real valued function and

$$= \| (f_1(t) - l_1, f_2(t) - l_2, f_3(t) - l_3) \|$$

$$= \sqrt{(f_1(t) - l_1)^2 + (f_2(t) - l_2)^2 + (f_3(t) - l_3)^2}$$

$$\lim_{t \to t_0} \| F(t) - L \| = 0$$

is a limit computation of real valued function.

Remark: $\lim_{t \to t_0} \| F(t) - L \| = 0$

$\iff \lim_{t \to t_0} \sqrt{(f_1(t) - l_1)^2 + (f_2(t) - l_2)^2 + (f_3(t) - l_3)^2} = 0$

where $f_i : I \to \mathbb{R}$
$t \mapsto f_i(t)$
for $i = 1, 2, 3$

$\iff \lim_{t \to t_0} (f_i(t) - l_i)^2 = 0$ for $i = 1, 2, 3$

$\iff \lim_{t \to t_0} f_i(t) = l_i$ for $i = 1, 2, 3$

Proposition: With the above notations, we get

$\lim_{t \to t_0} F(t) = L$ if and only if $\lim_{t \to t_0} f_i(t) = l_i$

for $i = 1, 2, 3$.

Thus if $F(t) = \left(f_1(t), f_2(t), f_3(t)\right)$

then $\lim_{t \to t_0} F(t) = \left(\lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \lim_{t \to t_0} f_3(t)\right)$

Whenever the right hand is defined for The component function $f_1$, $f_2$ and $f_3$ respectively.

The vector valued function $F : I \to \mathbb{R}^3$

$$t \mapsto \left(f_1(t), f_2(t), f_3(t)\right)$$

is said to be continuous at $t_0 \in I$

if $\lim\limits_{t \to t_0} F(t) = F(t_0) = \left(f_1(t_0), f_2(t_0), f_3(t_0)\right)$.

Remark: The vector valued function $F : I \to \mathbb{R}^3$

$$t \mapsto \left(f_1(t), f_2(t), f_3(t)\right)$$

is continuous at $t_0 \in I$

if and only if all the three real valued

functions $f_1$, $f_2$ and $f_3$ are continuous at $t_0$.

The vector valued function $F : I \to \mathbb{R}^3$

$$t \mapsto (f_1(t), f_2(t), f_3(t))$$

is said to be differentiable at $t_0$

if $\displaystyle\lim_{h \to 0} \frac{F(t_0+h) - F(t_0)}{h}$ exists.

The limit is called the derivative of $F$ at $t_0$ and it is denoted by $F'(t_0)$.

**Remark:** The vector valued function $F : I \to \mathbb{R}^3$

$$t \mapsto (f_1(t), f_2(t), f_3(t))$$

is differentiable at $t_0 \in I$ if and only if all the three real valued functions $f_1$, $f_2$ and $f_3$ are differentiable at $t_0$. Moreover, $F'(t_0) = (f_1'(t_0), f_2'(t_0), f_3'(t_0))$.

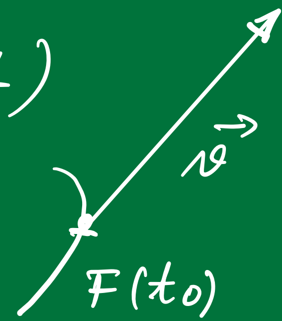
**Example:**

$$F : \mathbb{R} \to \mathbb{R}^3$$

$$t \mapsto (\cos t, \sin t, t)$$

$$F'(t_0) = (-\sin t_0,\ \cos t_0,\ 1) \neq 0$$

Consider $\vec{v} = F'(t_0) = (-\sin t_0, \cos t_0, 1)$

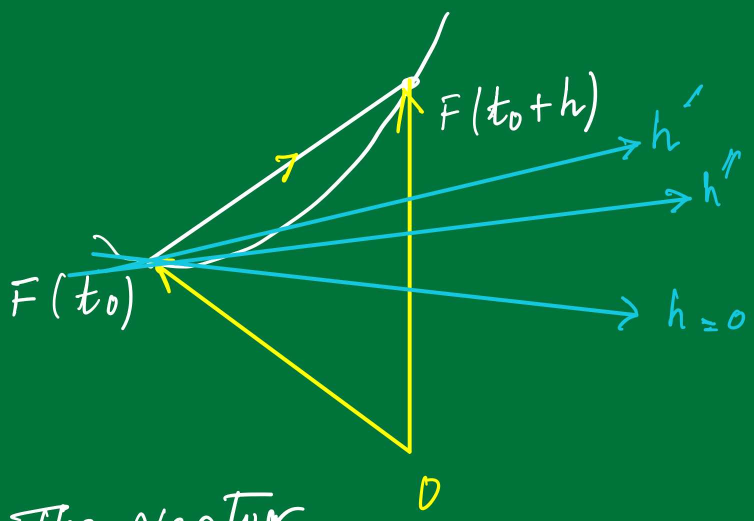



Equation of the line through $F(t_0)$ in the direction of $\vec{v}$ is

$$\vec{X}(t) = F(t_0) + t\,\vec{v}$$
$$= F(t_0) + t\,F'(t_0)$$

— the tangent line to the curve $C$ (for $F$) at $F(t_0)$.

Remark: The vector $F'(t_0)$ and condition of tangency.




The vector

$\frac{1}{h}\left(F(t_0+h) - F(t_0)\right)$ is parallel to $F(t_0+h) - F(t_0)$ and

moves to be a tangent vector as $h \to 0$.

Definition. Suppose $\mathcal{C}$ is a curve defined by a differentiable vector valued function $R : I \to \mathbb{R}^3$.

Suppose the derivative $R'(t_0) \neq 0$. The vector $R'(t_0)$ is called a tangent vector to $\mathcal{C}$ at $R(t_0)$ and the line

$$X(t) = R(t_0) + t\,R'(t_0)$$

is called the tangent line to the curve $\mathcal{C}$ at $R(t_0)$.

**Example:** Take

$$R(t) = (\cos t, \sin t, t), \quad t_0 = \frac{\pi}{3}.$$

Find the equation of the plane perpendicular to (the curve $\mathcal{C}$ given by) $R$ at $t_0$.

We need to find the plane passing through $R(\pi/3)$ and perpendicular to $R'(\pi/3)$.

$\square$ Let $X = (x, y, z)$ be an arbitrary point on the required plane.

Then the direction of $\vec{X} - R(\pi/3)$ is perpendicular to $R'(\pi/3)$.

So, $$R'(\pi/3) \cdot \left(X - R(\pi/3)\right) = 0$$

or, $$\boxed{R'(\pi/3)\,(x, y, z) = R'(\pi/3) \cdot R(\pi/3).}$$

## Arc Length for space curves:

– Plane curve?

Let $C$ be a curve in $\mathbb{R}^3$ defined by

$$R(t) = \big(x(t), y(t), z(t)\big) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

for $t \in [a, b]$.

Assume that $R$ is differentiable and $R'$ is continuous.

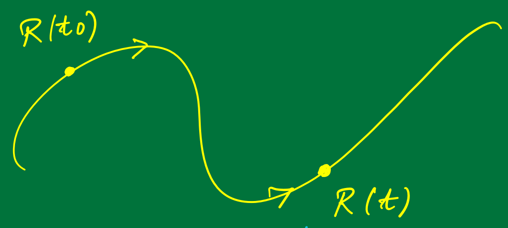
Then the length of the curve $C$ is defined to be

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

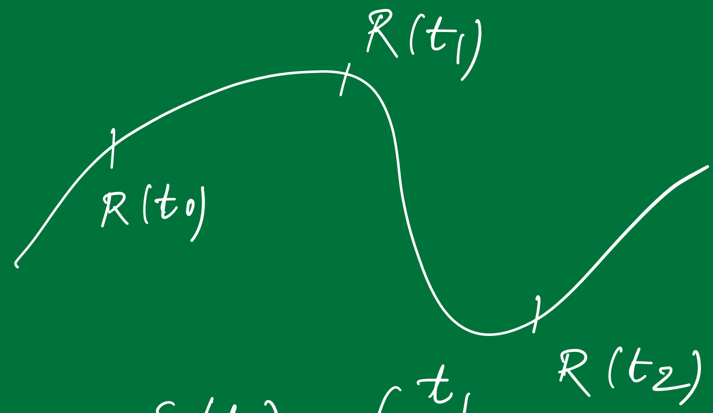
$$= \int_a^b \| R'(t) \|\, dt$$

or $$\int_a^b \left\| \frac{dR}{dt} \right\| dt$$

New parameter — Arc length parameter s:




$$S(t) = \int_{t_0}^{t} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\, d\tau$$




$$S(t_1) = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

$$= \int_{t_0}^{t_1} \left\| \frac{dR}{dt} \right\| dt$$

We can reparametrize the curve $C$ with the arc-length parameter $s$.

We express $t$ in terms of the arc-length parameter $s$.

Example: Consider $R(t) = (a\cos t,\ a\sin t,\ bt)$.

$$R(s) = ?$$

Here,

$$R'(t) = (-a\sin t,\ a\cos t,\ b) \text{ and } \|R'(t)\| = \sqrt{a^2+b^2}.$$

So, $$s = \int_0^t \|R'(u)\|\,du$$

or $$s = \left(\sqrt{a^2+b^2}\right)t \Rightarrow t = \frac{s}{\sqrt{a^2+b^2}},$$

* the parameter $s$ as a function of $t$

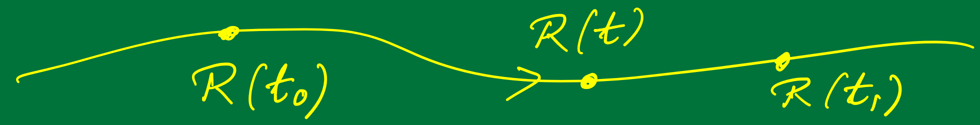
* the parameter $t$ as a function of $s$

Changing the parameter to $s$, we get

$$R(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right)$$

— a unit speed curve!

$C$ :




Let $R(t)$ be the position vector of a particle moving along the curve $C$.

The directed distance measured along the curve $C$ from $R(t_0)$ to $R(t_1)$ is

$$S(t_1) = \int_{t_0}^{t_1} \left\| \frac{dR}{dt} \right\| dt .$$

Then by Fundamental theorem of calculus (FTC),

$$\frac{ds}{dt} = \left\| \frac{dR}{dt} \right\| .$$

$\Rightarrow$ $\frac{ds}{dt}$ is the speed with which the particle moves along $R(t)$

— the magnitude of the velocity vector

$$V(t) = \frac{dR}{dt} = R'(t)$$

Recall:

Definition. Suppose $C$ is a curve defined by a differentiable vector valued function $R: I \to \mathbb{R}^3$. Suppose the derivative $R'(t_0) \neq 0$. The vector $R'(t_0)$ is called a tangent vector to $C$ at $R(t_0)$ and the line

$$X(t) = R(t_0) + t\, R'(t_0)$$

is called the tangent line to the curve $C$ at $R(t_0)$.

Unit tangent vector: The unit tangent vector of the curve $R(t)$ is

$$T(t) = \frac{R'(t)}{\| R'(t) \|} \quad \text{whenever } R'(t) \neq 0.$$

Considering the expression

$$\frac{ds}{dt} = \| R'(t) \|$$

we have

$$T(t) = \frac{\frac{dR}{dt}}{\frac{ds}{dt}}$$

**Theorem**: Let $f:[a,b] \to \mathbb{R}$ be differentiable function with $f'(x) \neq 0$ for all $x \in [a,b]$. Then $f^{-1}: f([a,b]) \to [a,b]$ is continuous, differentiable and the derivative

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

$$\Rightarrow \quad T(t) = \frac{dR}{dt} \times \left(\frac{ds}{dt}\right)^{-1}$$

$$= \frac{dR}{dt} \times \left(\frac{dt}{ds}\right)$$

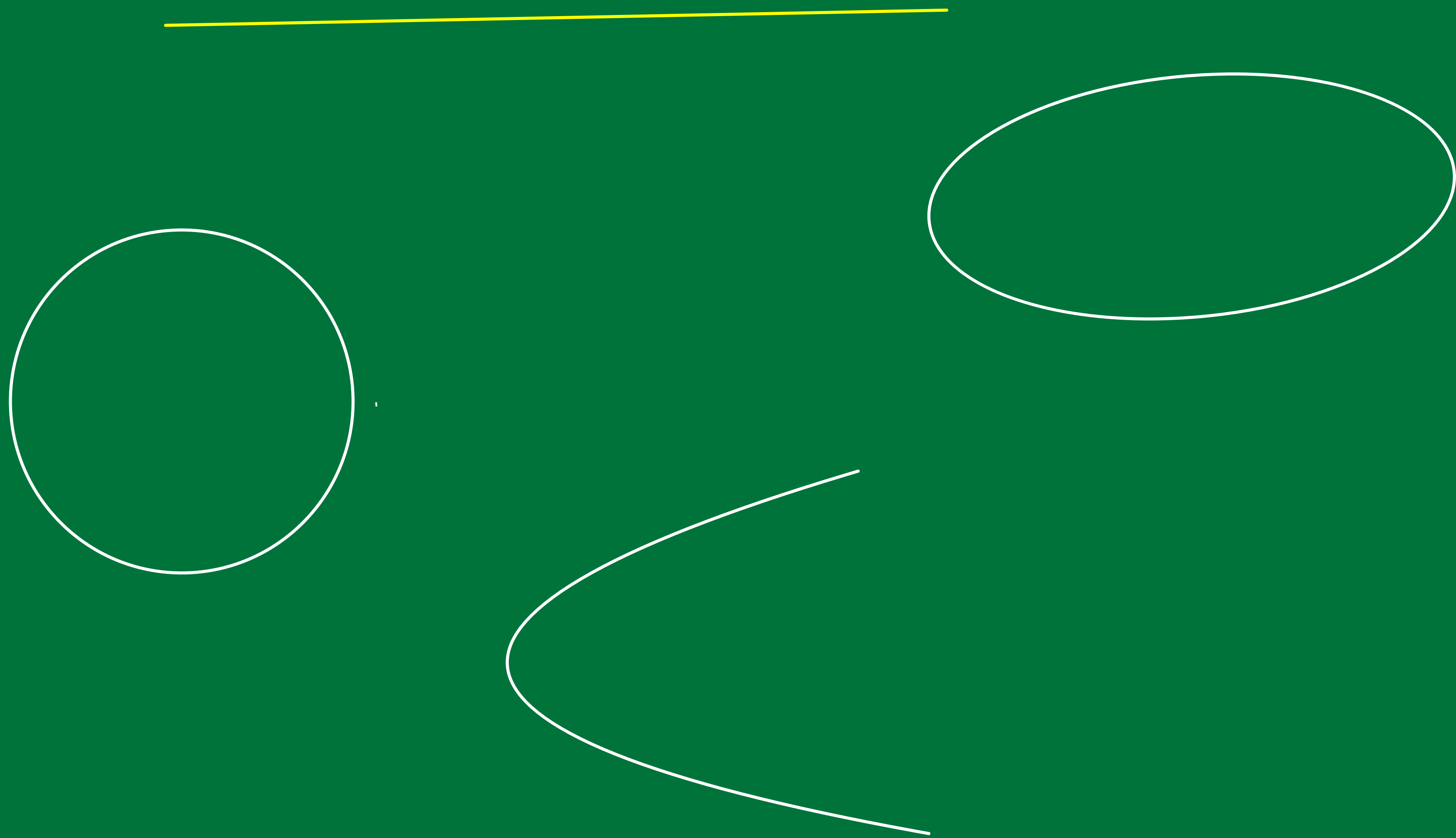
$$\begin{tikzcd} {[a,b]} \arrow[r, bend left, "s"] & R \arrow[l, bend left, "t"] \end{tikzcd}$$

$$= \frac{dR}{ds} \qquad (\text{By chain rule}).$$

Consequently, $\left\| \frac{dR}{ds} \right\| = \| T(t) \| = 1.$

This shows that $R(s)$ is a unit speed curve.

How a given curve is curved?



Curvature of a curve: measures the rate of change of the unit tangent vector $T$ with respect to the arc length.

The curvature of a curve $R(t)$ is

$$K = \left\| \frac{dT}{ds} \right\|$$

Note that by chain rule, $\frac{dT}{ds} = \frac{dT}{dt} \frac{dt}{ds}$

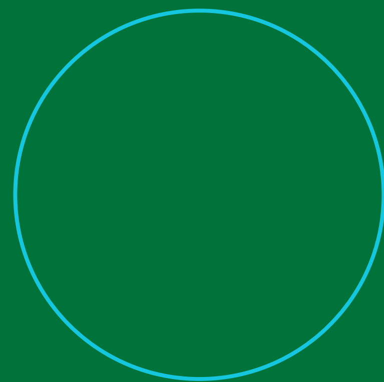
$$= \frac{dT}{dt} \times \frac{1}{\frac{ds}{dt}}$$

$$= \frac{dT}{dt} \times \frac{1}{\left\| \frac{dR}{dt} \right\|}$$

$$\Rightarrow K(t) = \frac{\| T'(t) \|}{\| R'(t) \|}.$$

Example: Circle of radius $a$:

1. $$R(t) = (a\cos t,\ a\sin t)$$



$$T(t) = \frac{R'(t)}{\|R'(t)\|}$$

$$= (-\sin t,\ \cos t)$$

$$T'(t) = (-\cos t,\ -\sin t)$$

$$|K| = \frac{1}{a}$$

Example : A straight line :

2. $R(t) = X_0 + tA \quad ; \quad A \neq 0$

$= (x_0, y_0, z_0) + t(a, b, c)$

$$R'(t) = A$$

$$T(t) = \frac{R'(t)}{\| R'(t) \|}$$

$$= \frac{A}{\| A \|}$$ – constant function

(a vector value function)

$$K = \left\| \frac{dT}{ds} \right\|$$

$$= 0.$$