Lecture 4 : Determinant & its properties

We will define determinant of a square matrix with the help of notion of permutation.

Permutation: Let A be a finite set. By a permutation on A we mean a bijection from A to A i.e. a map $\sigma : A \to A$ which is injective (1-1) and surjective (Onto).

We will be interested in permutations on the set $\{1, 2, 3, \dots, n\}$. A permutation σ on $\{1, 2, \dots, n\}$ will be written as $\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

Example: Suppose $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ is given by $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. σ is written as $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

The set of all permutations on the set $\{1, 2, \dots, n\}$ is denoted by S_n. Note that cardinality of S_n is $n!$. The set S_n is also called symmetric group on $\{1, 2, \dots, n\}$.

★ The product $\sigma\tau$ of 2 permutations σ & τ is just the composition $\sigma\circ\tau$ of 2 maps σ and τ. For example, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ & $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ then

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \quad \& \quad \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

★ As permutations are bijections, inverse of each permutation exists and is also a permutation. Inverse of a permutation σ is denoted by σ^{-1} & it satisfies $\sigma\circ\sigma^{-1} = \sigma^{-1}\circ\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$

Inverse of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ is $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

Transposition A permutation is called transposition if it permutes only two elements of $\{1, 2, \ldots, n\}$ and fixes all other elements. A transposition $\sigma(i) = j$, $\sigma(j) = i$ & $\sigma(k) = k$ $\forall\, k \neq i, j$ where $i, j, k \in \{1, 2, \ldots, n\}$ is denoted by $(i\ j)$.

Example: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ is $(2\ 4)$

Cyclic Permutation: A permutation σ is said to be cycle of length s if there exists $1 \leq a_1 < a_2 < \dots < a_s \leq n$ such that
$$\sigma(a_1) = a_2, \dots, \sigma(a_i) = a_{i+1}, \sigma(a_s) = a_1$$
& σ fixes rest of the elements.

It is written as $(a_1\, a_2 \cdots a_s)$.

A cycle of length 2 is a transposition.

Proposition: Every permutation is product of transpositions.

Proof: First note that every permutation is product of disjoint cycles. Let $(a_1\, a_2 \dots a_s)$ be a cycle of length s.

Then $(a_1\, a_2 \cdots a_s) = (a_1 a_2)(a_2 a_3) \dots (a_{s-1} a_s)$

Hence we have the required proposition. ∎

Example: Consider the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 1 & 6 & 7 & 6 \end{pmatrix}$

It is equal to
$$(1\,3\,4)(2\,5\,6\,7) = (1\,3)(3\,4)(2\,5)(5\,6)(6\,7)$$

Even and Odd Permutation: A permutation is said to be even (respectively odd) permutation if it can be written as even (respectively odd) transpositions.

Sign of a permutation: Let σ be a permutation.

$$\operatorname{sign}(\sigma) := \begin{cases} +1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

Determinant: Let $A = (a_{ij})$ be a square matrix of order n. Determinant of A is defined as:

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

It is denoted by $\det(A)$ or $|A|$.

Example:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{\sigma \in S_2} a_{1\sigma(1)} a_{2\sigma(2)}$$

$$= \operatorname{sign}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} a_{11} a_{22}, \quad S_2 = \{id, (1\,2)\}$$

$$+ \operatorname{sign}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} a_{12} a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

★ Identity permutation is even & $(1\,2)$ is odd permutation.

This definition of determinant is not very convenient for computation but is very useful in proving some important things.

Properties of Determinant

Property 1: Let $A = (a_{ij})$ & $B = (b_{ij})$ be two matrices of order $n \times n$ such that B is obtained from A by interchanging two rows of A. Then $|A| = -|B|$

Proof: Suppose pth row of A is interchanged with qth row then $b_{pj} = a_{qj}$, $b_{qj} = a_{pj}$ & $b_{ij} = a_{ij}$ $\forall\, i \neq p, q$ $\forall\, j = 1, 2, \ldots, n$

Consider the transposition $\tau = (p\ q)$

Note that $S_n = \{ \sigma \circ \tau : \sigma \in S_n \}$

$$|B| = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma \circ \tau)\, b_{1\sigma\tau(1)} \cdots b_{p\sigma\tau(p)} \cdots b_{q\sigma\tau(q)} \cdots b_{n\sigma\tau(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \operatorname{sign}(\tau)\, b_{1\sigma(1)} \cdots b_{p\sigma(q)} \cdots b_{q\sigma(p)} \cdots b_{n\sigma(n)}$$

$$= -\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{q\sigma(q)} \cdots a_{p\sigma(p)} \cdots a_{n\sigma(n)}$$

$$= -|A| \qquad \boxtimes$$

Property 2: Let A be a square matrix with two identical rows then $|A| = 0$.

Proof: Let B be the matrix obtained from A by interchanging two identical rows of A, then $B = A$ but by property 1, $|B| = -|A|$. This implies $|A| = 0$.

Property 3: Let B be the matrix obtained from a square matrix A by multiplying a row with constant c. Then $|B| = c|A|$.

Proof: Let $B = (b_{ij})$, $A = (a_{ij})$, $b_{kj} = c a_{kj} \; \forall j$ & $b_{ij} = a_{ij} \; \forall i \neq k, \forall j$

$$|B| = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, b_{1\sigma(1)} \cdots b_{k\sigma(k)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sign}(\sigma)\, a_{1\sigma(1)} \cdots (c\, a_{k\sigma(k)}) \cdots b_{n\sigma(n)}$$

$$= c\, |A| \qquad \square$$

Similarly, we can prove the following property.

Property 4 Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ be 3 matrices of same order such that $c_{kj} = a_{kj} + b_{kj}$ for some k & $c_{ij} = a_{ij} = b_{ij} \; \forall i \neq k \; \forall j = 1, 2, \ldots, n$

Then $|C| = |A| + |B|$.

Property 5: Let $B = E_{ij}(c)A$, where A is a square matrix. Then $|B| = |A|$
(Recall B is obtained from A by adding ith row of A to c-multiple of jth row)

Proof of this follows from previous two properties. It is left as an exercise.

Using above properties, we have
$$|E_{ij}| = -1, \quad |E_i(c)| = c, \quad |E_{ij}(c)| = 1$$
Thus, we have the following property:

Property 6: Let $B = EA$, where E is an elementary matrix, then
$$|B| = |E||A|$$

Property 7: Square matrix A is invertible if and only if $|A| \neq 0$.

Proof: Suppose A is invertible then there exists elementary matrices $E_1, E_2, \ldots, E_r$ such that
$$A = E_1 E_2 \cdots E_r.$$ Determinant of R.H.S. is not equal to zero $\Rightarrow |A| \neq 0$

Conversely, suppose A is not invertible.

Then there exists elementary matrices $E_1, \dots E_p$ & $E_1', E_2', \dots, E_q'$ such that

$$E_1 \cdots E_p \, A \, E_1' E_2' \cdots E_q' = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}_{n \times n}$$

where $n = \text{order}(A)$ & $k < n$.

$$A = E_1^{-1} \cdots E_p^{-1} \underbrace{\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} (E_1')^{-1} \cdots (E_q')^{-1}}_{D}$$

$A = E_1^{-1} \cdots E_p^{-1} D$, where D is a matrix whose last row is zero. Hence, $|D| = 0$

This implies $|A| = 0$. ☐

Proposition: Let A and B be square matrices of order n such that $AB = I$. Then A is invertible.

Proof: Suppose A is not invertible then there exists elementary matrices $E_1, \dots E_p$ such that $E_1 E_2 \cdots E_p A$ is a matrix with last row zero.

So, $E_1 E_2 \cdots E_p = E_1 E_2 \cdots E_p AB$

Elementary matrices are invertible & product of invertible matrices are invertible. Hence L.H.S is invertible. R.H.S is a matrix whose last row is zero. Determinant of this matrix is 0, hence it is not invertible. This is a contradiction. Hence, A is invertible.

Property 8 Let A, B be square matrices of same order, then $|AB| = |A||B|$.

Proof: Case (i) Suppose A is not invertible.

If AB is invertible then $\exists$ a matrix Q such that $(AB)Q = I \Rightarrow A(BQ) = I$. By previous proposition, A is invertible which violates our assumption, so AB is not invertible.

By Property 7, $|A| = 0$ & $|AB| = 0$

so $|AB| = 0 = |A||B|$

Case (ii) Suppose A is invertible. Then there exists elementary matrices $E_1, \ldots, E_p$ such that

$$A = E_1 E_2 \ldots E_p.$$

By property 6, $|AB| = |E_1||E_2 \ldots E_p B| = |E_1||E_2||E_3 \ldots E_p B|$

$$= |E_1| \ldots |E_p||B| = |A||B|$$

$$(\text{Since } |A| = |E_1| \cdots |E_p|) \qquad \boxtimes$$

Property 9: Let A be a square matrix then

$$|A| = |A^T|$$

Proof: Note that $S_n = \{\sigma^{-1} : \sigma \in S_n\}$

Let $A = (a_{ij})_{n\times n}$

$$|A| = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma^{-1}(\sigma(1))\sigma(1)}\, a_{\sigma^{-1}(\sigma(2))\sigma(2)} \cdots a_{\sigma^{-1}(\sigma(n))\sigma(n)}$$

Suppose $\sigma(k) = 1$, then $a_{\sigma^{-1}(1)1}$ is in the above expression, similarly $a_{\sigma^{-1}(2)2}, \cdots, a_{\sigma^{-1}(n)n}$ are in the above expression.

So
$$|A| = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma^{-1}(1)1} \cdots a_{\sigma^{-1}(n)n}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma^{-1})\, a_{\sigma^{-1}(1)1} \cdots a_{\sigma^{-1}(n)n}$$

$(\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma^{-1}))$

$$= |A^T|$$

∎