

MTH 114: ODE: Assignment-4

In this assignment, we will denote:

$$y'' + p(x)y' + q(x)y = r(x), \quad x \in I \quad (*)$$

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I \quad (**)$$

where $I \subset \mathbb{R}$ is an interval and $p(x), q(x), r(x)$ are continuous functions on I .

1. (T) Let y_1 be the solution of the IVP

$$y'' + (2x - 1)y' + \sin(e^x)y = 0, \quad y(0) = 1, y'(0) = -1;$$

and y_2 be the solution of the IVP

$$y'' + (2x - 1)y' + \sin(e^x)y = 0, \quad y(0) = 2, y'(0) = -1.$$

Find the Wronskian of y_1, y_2 . What is the general solution of $y'' + (2x - 1)y' + \sin(e^x)y = 0$?

2. Can x^3 be a solution of (**) on $I = [-1, 1]$? Find two 2nd order linear homogeneous ODE with x^3 as a solution.
3. (T) Can $x \sin x$ be a solution of a second order linear homogeneous equation with constant coefficients?
4. (T) Find the largest interval on which a unique solution is guaranteed to exist of the IVP. $(x + 2)y'' + xy' + \cot(x)y = x^2 + 1, \quad y(2) = 11, \quad y'(2) = -2.$
5. Without solving determine the largest interval in which the solution is guaranteed to uniquely exist of the IVP $ty'' - y' = t^2 + t, \quad y(1) = 1, y'(1) = 5$. Verify your answer by solving it explicitly.
6. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
(i) $y = \cos(ax + b)$ (ii) $y = ax + \frac{b}{x}$ (iii) $y = ae^x + bxe^x$
7. Find general solution of the following differential equations given a known solution y_1 :
(i) (T) $x(1 - x)y'' + 2(1 - 2x)y' - 2y = 0$ $y_1 = 1/x$
(ii) $(1 - x^2)y'' - 2xy' + 2y = 0$ $y_1 = x$
8. Verify that $\sin x/\sqrt{x}$ is a solution of $x^2y'' + xy' + (x^2 - 1/4)y = 0$ over any interval on the positive x -axis and hence find its general solution.
9. Reduce the following second order differential equation to first order differential equation and hence solve.
(i) $xy'' + y' = y'^2$ (ii) (T) $yy'' + y'^2 + 1 = 0$ (iii) $y'' - 2y' \coth x = 0$

10. Find the curve $y = y(x)$ which satisfies the ODE $y'' = y'$ and the line $y = x$ is tangent at the origin.

11. Are the following functions **linearly dependent** on the given intervals?

(i) $\sin 4x, \cos 4x$ $(-\infty, \infty)$ (ii) $\ln x, \ln x^3$ $(0, \infty)$

(iii) $\cos 2x, \sin^2 x$ $(0, \infty)$ (iv)(T) $x^3, x^2|x|$ $[-1, 1]$

12. By using the method of variation of parameters, find the general solution of:

(i) $y'' + 4y = 2\cos^2 x + 10e^x$

(ii) (T) $y'' + y = x \sin x$

(iii) $y'' + y = \cot^2 x$

(iv) $x^2 y'' - x(x+2)y' + (x+2)y = x^3, \quad x > 0.$

[Hint. $y = x$ is a solution of the homogeneous part]

Initial Value Problem vs. Boundary Value Problem

A second-order *initial value problem* consists of a second-order ordinary differential equation $y''(t) = F(t, y(t), y'(t))$ and initial conditions $y(t_0) = y_0, \quad y'(t_0) = y'_0$ where t_0, y_0, y'_0 are numbers.

It might seem that there are more than one ways to present the initial conditions of a second order equation. Instead of locating both initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$ at the same point t_0 , couldn't we take them at different points, for examples $y(t_0) = y_0$ and $y(t_1) = y_1$; or $y'(t_0) = y'_0$ and $y'(t_1) = y'_1$? The answer is NO. **All the initial conditions in an initial value problem must be taken at the same point t_0 .** The sets of conditions above where the values are taken at different points are known as *boundary conditions*. A boundary value problem does not have the existence and uniqueness guaranteed.

Example: Every function of the form $y = C \sin(t)$, where C is a real number satisfies the boundary value problem $y'' + y = 0, y(0) = 0, y(\pi) = 0$. Therefore, the problem has infinitely many solutions, even though $p(t) = 0, \quad q(t) = 1, \quad r(t) = 0$ are all continuous everywhere.