Lecture 16 : Eigen Values, Eigen vectors & Diagonalization

Here, we introduce two concepts eigen values and eigen vectors of a square matrix which has wide applications in Mathematics as well as various field of science & engineering.

Eigen vector: Let A be a square matrix of order n. Then $A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ is a linear map. $\mathbb{R}$ is included in $\mathbb{C}$ in a natural way $j : \mathbb{R} \hookrightarrow \mathbb{C}$

$x \mapsto x + i0$ (i is imaginary number)

$$A : \mathbb{C}^n \to \mathbb{C}^n$$

$$U \mapsto AU, \quad U = (z_1, \ldots, z_n) = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

A vector $U \in \mathbb{C}^n$ is said to be an eigen vector of A if there exists a scalar $\lambda \in \mathbb{C}$ such that $AU = \lambda U$.

Eigen Value: The scalar λ is called an eigen value of A.

Example (1) $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. $A\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

& $A\begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$(2,0)$ & $(3,0)$ are eigen vectors corresponding to eigen values 2 & 3 respectively.

(2) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A : \mathbb{C}^2 \to \mathbb{C}^2$ is $U \mapsto AU$

a linear map.

$A\begin{pmatrix} 1 \\ i \end{pmatrix} = i\begin{pmatrix} 1 \\ i \end{pmatrix}$, i is imaginary number $i^2 = -1$.

☆ Eigen vector is called real if all the entries of the eigen vector is real (imaginary part is zero)

★ If U is an eigen vector A then αU is also an eigen vector of A for all scalar α.

<u>Proof:</u> $AU = \lambda U$ for some scalar λ.

$$A(\alpha U) = \alpha A(U) = \alpha A U = \alpha \lambda U$$
$$= \lambda(\alpha U)$$

$\Rightarrow$ αU is an eigen vector with same eigen value λ. $\boxtimes$

Suppose U is an eigen vector of A & the corresponding eigen value is λ. $\boxed{1}$

$$AU = \lambda U \Rightarrow (A - \lambda I)U = 0$$

where I is identity matrix of order n.

U is a non-zero vector $\Rightarrow$ the system of linear equation

$$(A - \lambda I)x = 0$$ has a non-zero solution $\Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow$ λ is a solution of the equation

$$\det(A - xI) = 0.$$

Note that $\det(A - xI)$ is a polynomial of degree n.

Characteristic Polynomial: Let A be a square matrix of order n, the polynomial $\det(A - xI)$ is called characteristic polynomial.

Proposition: The roots of $\det(A - xI) = 0$ are eigen values of A.

Proof: Let λ be a root of $\det(A - xI) = 0$

$\Rightarrow \det(A - \lambda I) = 0$

Consider the linear system of equations

$$(A - \lambda I)X = 0$$

$\det(A - \lambda I) = 0 \Rightarrow \operatorname{rank}(A - \lambda I) < n$

$\Rightarrow (A - \lambda I)X = 0$ has a non-zero solution, say U.

$(A - \lambda I)U = 0$

$\Rightarrow AU = \lambda U$

Converse is shown earlier. $\square$

Example (1) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Characteristic polynomial is $(x-1)^2$.

Eigen values of A are $1, 1$.

$$Ax = x \Rightarrow (A-I)x = 0 \quad , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_2 = 0$$

Eigen space of A is $\{(x, 0) : x \in \mathbb{C}\}$
$(= X\text{-axis})$

(2) $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$ $\quad \det(A - xI) = 0$

$$\Rightarrow (4-x)(-3-x) + 10 = 0$$

$$\Rightarrow -12 + 3x - 4x + x^2 + 10 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$\Rightarrow$ A has distinct eigen values -1 & 2

$$Ax = -x \Rightarrow \begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 5x_1 - 5x_2 = 0 \ \& \ 2x_1 - 2x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

Eigen space corresponding to -1 is $\{c(1,1) : c \in \mathbb{C}\}$

$$Ax = 2x \Rightarrow \begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad 2x_1 = 5x_2 \quad \Rightarrow \quad x_1 = \frac{5}{2} x_2$

Eigen space corresponding to eigen value 2 is $\{ c(5/2, 1) : c \in \mathbb{C} \}$

$= \{ c(5,2) : c \in \mathbb{C} \}$

Note that two eigen vectors $(1,1)$ & $(5,2)$ are L.I.

(3) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\det(A - xI) = x^2+1$

$x^2+1=0$ has no real solutions.

Roots of $x^2+1=0$ are i & $-i$.

$AX = iX \Rightarrow A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow x_2 = ix_1$ & $-x_1 = ix_2$

Eigen space corresponding to i is

$\{c(1, i) : c \in \mathbb{C}\}$

$AX = -iX \Rightarrow x_2 = -ix_1, \; -x_1 = -ix_2$

Eigen space corresponding to $-i$ is

$\{ c(1, -i) : c \in \mathbb{C}\}$

A square matrix is said to be triangular if either it is upper triangular or lower triangular matrix.

<u>Proposition:</u> Eigen values of a triangular matrix are the diagonal elements.

<u>Proof:</u> Let A be an upper triangular matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ & & & \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} \qquad a_{ij} = 0 \quad i > j$$

$$|A - xI| = (a_{11} - x)(a_{22} - x) \cdots (a_{nn} - x)$$

$\Rightarrow$ $a_{11}, a_{22}, \ldots, a_{nn}$ are eigen values.

<u>Proposition:</u> Let A be an $n \times n$ matrix with eigen values $\lambda_1, \ldots, \lambda_n$ (repetition may occur).

(1) $\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$

(2) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{trace}(A)$ (sum of diagonal elements of A)

Proof $\det(A - xI) = (-1)^n \det(xI - A)$

$= (-1)^n (x-\lambda_1) \cdots (x - \lambda_n)$

$= (-1)^n (x^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) x^{n-1} + \cdots + (-1)^n \lambda_1 \cdots \lambda_n)$

Put $x = 0$

$\Rightarrow \det(A) = (-1)^{2n} \lambda_1 \cdots \lambda_n = \lambda_1 \cdots \lambda_n$

coefecient of x^{n-1} in the exbansion of $\det(A - xI)$ is $(-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$

Therefrre,

$(-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) = (-1)^{n+1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

$\Rightarrow$ $\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. $\square$

Diagonalization of Matrices:

A square matrix is said to be diagonal matrix if apart of diagonal elements all the entries are zero.

A square matrix 'B' is said to be diagonaliz-able if there exists an invertible matrix P such that $P^{-1} B P$ is a diagonal matrix.

Example (1) Diagonal matrix is diagonalizable

(2) $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$ is diagonalizable

Let $P = \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix}$ (columns are L.I. eigen vectors)

$$P^{-1} = -\frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 1 \end{pmatrix}$$

$$P^{-1}AP = -\frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} -2 & 5 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Theorem Let A be a square matrix then $P^{-1}AP$ and A have same eigen values for all invertible matrix P of order same as order of A

Proof:

$$\det(P^{-1}AP - xI) = \det\left(P^{-1}(A - xI)P\right)$$

$$= \det P^{-1} \det(A - xI) \det P$$

$$= \det(A - xI) \det P^{-1} \det P$$

$$= \det|A - xI| \det(P^{-1}P)$$

$$= \det(A - xI) \quad \blacksquare$$

Corollary: Let A be a diagonalizable matrix & A be diagonalized to a diagonal matrix D. Eigen values of A are the diagonal entries of D.