ASSIGNMENT 5
MTH102A

(1) ***Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y) = (ax + by, cx + dy)$. Find the matrix of T with respect to standard basis of $\mathbb{R}^2$. Now do the same by considering the basis $\{(1.0), (1, 1)\}$ on domain and co-domain of T.

Soln. First part :- $T(1,0) = (a,c) = a(1,0) + c(0,1)$

$T(0,1) = (b,d) = b(1,0) + d(0,1)$

Matrix of T with respect to standard basis of $\mathbb{R}^2$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2nd Part :- $T(1,0) = (a,c)$, $T(1,1) = (a+b, c+d)$

Let $(a,c) = p(1,0) + q(1,1) \Rightarrow p+q = a$, $q = c$

$\Rightarrow p = a-c$, $q = c$

$(a+b, c+d) = r(1,0) + s(1,1)$

$\Rightarrow r + s = a+b$, $s = c+d$

$\Rightarrow r = a+b-c-d$, $s = c+d$

$T(1,0) = p(1,0) + q(1,1)$ & $T(1,1) = r(1,0) + s(1,1)$

Matrix of T is $\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} a-c & a+b-c-d \\ c & c+d \end{pmatrix}$

(2) Let V be a finite dimensional vector space. Using Rank-Nullity theorem, a linear transformation $T : V \to V$ is onto if and only if it is injective.

Solution:

$$\dim \ker T + \dim R(T) = \dim V$$

$$T \text{ injective} \iff \ker T = \{0\}$$

$$\iff \dim \ker T = 0$$

$$\iff \dim R(T) = \dim V$$

$$\iff R(T) = V \quad (\text{as } R(T) \subseteq V \text{ also})$$

$$\iff T \text{ is surjective}$$

(3) *** Consider the linear map $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = iz$. By considering the basis $\{1, i\}$ of $\mathbb{C}$ (over $\mathbb{R}$) on domain and co-domain of T, find the matrix of T.

Solution:-

$$T(z) = iz$$

$$T(1) = i = 0 \cdot 1 + 1 \cdot i$$

$$T(i) = -1 = -1 \cdot 1 + 0 \cdot i$$

matrix of T is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(4) *** Let $T : V \to V$ be a linear transformation with $Ker(T) = R(T)$, $R(T)$ is range of T. Show that $T^2 = 0$. Give example of such a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$.

Soln. $Ker(T) = R(T)$ (Given)

Let $x \in V \Rightarrow T(x) \in R(T) = Ker(T)$

$\Rightarrow T(T(x)) = 0$

$\Rightarrow T^2 = 0$

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= (y, 0)$

T is a linear map

$T(T(x, y)) = T(y, 0) = (0, 0)$

$\ker T = \{(x, 0) : x \in \mathbb{R}\}$

$R(T) = \{T(x, y) : y \in \mathbb{R}\}$

$= \{(y, 0) : y \in \mathbb{R}\}$

$\Rightarrow \ker T = R(T)$.

(5) ***Does there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that range of T, $R(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$.

Soln. Suppose $\exists$ such a linear transformation T. By Rank-Nullity theorem,

$$\dim(\ker(T)) + \dim R(T) = 2 \quad \text{---} (*)$$

$$\dim R(T) = 3$$

$(x_1, x_2, x_3, x_4) \in R(T)$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 0$

$\Rightarrow x_1 = -x_2 - x_3 - x_4$

$$(x_1, x_2, x_3, x_4) = x_2(-1, 1, 0, 0) + x_3(-1, 0, 1, 0) + x_4(-1, 0, 0, 1)$$

& $\{(-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1)\}$ is Linearly independent

So, $\dim R(T) = 3$

L.H.S of $(*)$ is ≥ 3

whereas R.H.S of $(*) = 2$

This is a contradiction.

So $\nexists$ a linear transformation.

(6) *** Let V be a vector space of dimension n and $\{u_1, u_2, ..., u_n\}$ be a basis of V. Suppose $w_1, w_2, ..., w_n$ are n-elements of V with $w_j = a_{1j}u_1 + a_{2j}u_2 + ... + a_{nj}u_n$ ($(a_{1j}, a_{2j}, ..., a_{nj})$ is said to be coordinates of w_j with respect to basis $\{u_1, ..., u_n\}$). Let $A = (a_{ij})$ then show that $\{w_1, w_2, ..., w_n\}$ is a basis of V if and only if A is invertible.

Solution:- Suppose $\{w_1, \ldots, w_n\}$ is a basis.

Suppose A is not invertible then $\det(A) = 0 \Rightarrow$ Columns of A are linearly dependent $\Rightarrow \exists\, c_1, \ldots, c_n$ not all zero such that
$$c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} = 0 \quad -(*)$$

Consider
$$\sum_{j=1}^{n} c_j w_j = \left(\sum_{j=1}^{n} c_j a_{1j}\right) u_1 + \cdots + \left(\sum_{j=1}^{n} c_j a_{nj}\right) u_n \quad -(**)$$
$$= 0 \quad (\text{from } (*))$$

$\Rightarrow \{w_1, \ldots, w_n\}$ is L.D, contradiction.

Conversely, suppose A is invertible

If $\{w_1, \ldots, w_n\}$ is not a basis then it is L.D.

$\Rightarrow \exists\, c_1, \ldots, c_n$ not all zero s.t. $\sum_{j=1}^{n} c_j w_j = 0$.

From $(**)$ & $\{u_1, \ldots, u_n\}$ is L.I implies that
$$c_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = 0$$

$\Rightarrow$ Column vectors of A are L.D

$\Rightarrow \det A = 0 \Rightarrow A$ is not invertible $\rightarrow\leftarrow$

(7) Find the kernel and range of $T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$.

Solution:- $T(x,y,z) = (0,0,0)$

$\Rightarrow (x+z,\ x+y+2z,\ 2x+y+3z) = (0,0,0)$

$x+z=0, \quad x+y+2z=0, \quad 2x+y+3z=0$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \xrightarrow[R_{31}(-2)]{R_{21}(-1)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$\downarrow R_{32}(-1)$

$$(\text{say}),\ B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x+z=0, \quad y+z=0$

$\Rightarrow x=-z, \quad y=-z$

$\therefore \ker(T) = \{x \in \mathbb{R}^3 : Ax = 0\} = \{(-c,-c,c) : c \in \mathbb{R}\}$

$(p,q,r) = T(x,y,z) = (x+z,\ x+y+2z,\ 2x+y+3z)$

$= x(1,1,2) + y(0,1,1)$
$+ z(1,2,3)$

$= x(1,1,2) + y(0,1,1)$
$+ z((1,1,2)+(0,1,1))$

$= (x+z)(1,1,2) + (y+z)(0,1,1)$

$\{(1,1,2),(0,1,1)\}$ is L.I

So, $R(T) = L(\{(1,1,2),(0,1,1)\})$.

(8) *** Let $<,>$ be an inner product on $\mathbb{R}^n$. Prove that there exists a symmetric matrix A of order n such that $< u, v >= u^T Av$ for all $u, v \in \mathbb{R}^n$.

Solution:- Let $\{e_1, \dots, e_n\}$ be standard basis of $\mathbb{R}^n$.

Let $a_{ij} = \langle e_i, e_j \rangle$

$e_i = (0, \dots, \underset{\substack{\uparrow \\ i\text{th position}}}{1}, \dots 0)$, $e_j = (0, \dots \underset{\substack{\uparrow \\ j\text{th position}}}{1}, \dots 0)$

Let $A = a_{ij}$, A is symmetric.

$u = \sum_{i=1}^{n} \alpha_i e_i$, $v = \sum_{j=1}^{n} \beta_j e_j$

$$\langle u, v \rangle = \left\langle \sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \langle e_i, e_j \rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j a_{ij}$$

$u = (\alpha_1, \dots, \alpha_n)$, $v = (\beta_1, \dots, \beta_n)$

$$\langle u, v \rangle = (\alpha_1 \dots \alpha_n) A \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$= u A v^T.$$

(9) *** Equip $\mathbb{R}^3$ with usual standard inner product. Using Gram-Schimdt process, transform the set of vectors $\{(1,1,1),(1,0,2),(0,1,2)\}$ into an orthonormal basis of $\mathbb{R}^3$.

Solution :- Let $u_1 = (1,1,1)$, $u_2 = (1,0,2)$, $u_3 = (0,1,2)$

Let $w_1 = u_1$, $w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$$w_3 = u_3 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$\{w_1, w_2, w_3\}$ is orthogonal basis of $\mathbb{R}^3$.

$\left\{\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}\right\}$ is orthonormal basis of $\mathbb{R}^3$.

(Computation part is left to students)