MTH-686

-Debasis Kundu

21/10/24

* Prony's equation

→ Suppose $u_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t}$, $t = 1, 2, \cdots$

and $\beta_1 \neq \beta_2$.

then $\exists\ g_0, g_1, g_2$ S.T $g_0^2 + g_1^2 + g_2^2 = 1$ &

$$g_0 u_1 + g_1 u_2 + g_2 u_3 = 0$$

$$g_0 u_2 + g_1 u_3 + g_2 u_4 = 0$$

$$\vdots$$

$$g_0 u_{n-2} + g_1 u_{n-1} + g_2 u_n = 0.$$

$$\Rightarrow \begin{bmatrix} u_1 & u_2 & u_3 \\ \vdots & \vdots & \vdots \\ u_{n-2} & u_{n-1} & u_n \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ We can prove this as follows

Note that this following quadratic equation:

$$h(x) = (x - e^{\beta_1})(x - e^{\beta_2}) = 0$$

has roots e^{β_1} & e^{β_2}.

$$\Rightarrow h(x) = x^2 - (e^{\beta_1} + e^{\beta_2})x + e^{\beta_1 + \beta_2}$$

$$= g_0 + g_1 x + g_2 x^2$$

→ If we take $g_0 = e^{\beta_1 + \beta_2}$, $g_1 = -(e^{\beta_1} + e^{\beta_2})$

& $g_2 = 1$ then $h(x) = g_0 + g_1 x + g_2 x^2 = 0$

has roots e^{β_1} & e^{β_2}.

$\Rightarrow h(e^{\beta_1}) = 0 = h(e^{\beta_2})$

$$h(e^{\beta_1}) = g_0 + g_1 e^{\beta_1} + g_2 e^{2\beta_1} = 0 \quad -(*)$$
$$h(e^{\beta_2}) = g_0 + g_1 e^{\beta_2} + g_2 e^{2\beta_2} = 0 \quad -(**)$$

From $(*)$: $g_0 + g_1 e^{\beta_1} + g_2 e^{2\beta_1} = 0$

$$\Rightarrow g_0 e^{k\beta_1} + g_1 e^{(k+1)\beta_1} + g_2 e^{(k+2)\beta_1} = 0$$

$$\Rightarrow g_0 e^{k\beta_2} + g_1 e^{(k+1)\beta_2} + g_2 e^{(k+2)\beta_2} = 0$$

$k = 0, 1, 2, \ldots\ldots$

$\rightarrow \underset{n-2\times n}{G}\,\underset{n\times 2}{A} = \underset{n-2\times 2}{0}$

$\Rightarrow$ We want to show:

$N_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t}$; $t = 1, 2, \ldots\ldots, n$

$\Rightarrow \exists\ g_0, g_1, g_2$ s.t

$h(x) = g_0 + g_1 x + g_2 x^2 = 0$,

$$\rightarrow \underset{n-2\times 3}{M}\ \underset{3\times 1}{g} = 0 \quad ; \quad g = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix}$$

$\text{Rank}(M) = 2$

$$Mg = 0 \Rightarrow (\underset{3\times n-2}{M^T}\ \underset{n-2\times 3}{M})\ \underset{3\times 1}{g} = 0.$$

$\rightarrow \text{Rank}(M) = \text{Rank}(M^T M) = 2$

$\Rightarrow g$ is an eigenvector corresponding to eigenvalue 0.

$$\rightarrow y_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + \varepsilon_t \ ; \ t = 1, \cdots, n$$

$E(\varepsilon_t) = 0$; $V(\varepsilon_t) = \sigma^2$; $\varepsilon_1, \cdots, \varepsilon_n$ iid.

$$\rightarrow \tilde{M} = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ y_{n-2} & y_{n-1} & y_n \end{bmatrix}$$

$\underset{3\times 3}{\tilde{M}^T \tilde{M}}$

$\rightarrow$ Suppose $\tilde{g}$ is the eigenvector corresponding to 0 eigenvalue

$$\rightarrow \nu_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} \ ; \ t = 1, \cdots, n-$$

$\Rightarrow \exists\ g_0, g_1, g_2$ s.t.

$$\tilde{h}(x) = \tilde{g}_0 + \tilde{g}_1 x + \tilde{g}_2 x^2 = 0.$$

→ If σ^2 is not very small than the roots of

$$\tilde{h}(x) = \tilde{g}_0 + \tilde{g}_1 x + \tilde{g}_2 x^2 = 0$$

can be: (i) Both +ve — we can recommend β_1/β_2

(ii) Both −ve — we cannot

(iii) One +ve / One −ve — we cannot

(iv) Complex conjugate — we cannot

The results are true for general p, i.e

$$u_t = \alpha_1 e^{\beta_1 t} + \cdots + \alpha_p e^{\beta_p t}; \; t = 1, \ldots, n.$$

$\exists\, g_0, g_1, \ldots g_p$ s.t:

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_{p+1} \\ u_2 & u_3 & \cdots & u_{p+2} \\ \vdots & & & \\ u_{n-p} & \cdots & \cdots & u_p \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

→ $h(x) = g_0 + g_1 x + \cdots$ has roots $\{e^{\beta_1}, \ldots, e^{\beta_p}\}$

Model: $y_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + \varepsilon_t$ 22/10/24

$\beta_1 \neq \beta_2$

$\alpha_1, \alpha_2 \neq 0$

ε_t are iid $\sim (0, \sigma^2)$

If we take

$N_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t}$

$\Rightarrow \exists\, g_0, g_1, g_2$ s.t. $g_0 N_k + g_1 N_{k+1} + g_2 N_{k+2} = 0$

for $k = 1, 2, \cdots, n-2$

However $h(x) = g_0 + g_1 x + g_2 x^2 = 0$ has ~~roots~~ e^{β_1}, e^{β_2}

\# Prony's equation

→ We use this to find out LSE of $\underset{\sim}{\alpha}, \underset{\sim}{\beta}$ we will provide an algo to compute LSE.

$\Theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)$

LSE: minimising $Q(\Theta) = \sum_{t=1}^{n} (y_t - \alpha_1 e^{\beta_1 t} - \alpha_2 e^{\beta_2 t})^2$ $\quad \cdots \quad$ — (*)

We expand in matrix form:

$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, the matrix $X(\underset{\sim}{\beta}) = \begin{bmatrix} e^{\beta_1} & e^{\beta_2} \\ \vdots & \vdots \\ e^{n\beta_1} & e^{n\beta_2} \end{bmatrix}$

, $\underset{\sim}{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$; $\underset{\sim}{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

→ $Q(\Theta) = (Y - X(\beta)\alpha)^T (Y - X(\beta)\alpha)$ (*)

Note that for a given β:

$$\hat{\alpha}(\beta) = [X^T(\beta) X(\beta)]^{-1} X^T(\beta) Y$$

minimizes (*).

→ The LSE can be obtained by minimizing:

$$Q(\Theta) = \sum_{t=1} (y_t - \alpha_1 e^{\beta_1 t} - \alpha_2 e^{\beta_2 t})^2 \quad (*)$$

We expand in matrix form:

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \text{ the matrix } X(\underset{\sim}{\beta}) = \begin{bmatrix} e^{\beta_1} & \cdots & e^{\beta_2} \\ \vdots & & \vdots \\ e^{n\beta_1} & & e^{n\beta_2} \end{bmatrix}$$

$$, \underset{\sim}{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$Q(\Theta) = \|Y - X_{(\beta)}\alpha\|^2$$

$$= (Y - X_{(\beta)}\alpha)^T (Y - X_{(\beta)}\alpha)$$

$$= Y^T Y - 2(X_{(\beta)}\alpha)^T Y + \alpha^T X_{(\beta)}^T X_{(\beta)}\alpha$$

Advantage! If β is known we know exactly what value of α minimizes the given value.

$$\hat{\alpha}_\beta = (X^T X)^{-1} X^T y \text{ minimizes } (*)$$

$$Q(\Theta) = \|Y - X_{(\beta)}[X^T X]^{-1} X^T y\|^2$$

$\hat{\alpha}_1, \beta_1, \hat{\alpha}_2, \beta_2$

$$R(\beta_1, \beta_2) = \|(I - X(X^T X)^{-1} X^T) y\|^2$$

$$R(\underset{\sim}{\beta}) = y^T (I - P_{X(\beta)}) y$$

$$= Y^T [I - X_{(\beta)} [X_{(\beta)}^T X_{(\beta)}]^{-1} X^T_{(\beta)}] Y$$

We can obtain LSE of β_1, β_2 by minimizing $R(\underset{\sim}{\beta})$ wrt β

→ Suppose $\hat{\beta}_1$ & $\hat{\beta}_2$ minimize $R(\beta_1, \beta_2)$ then:

$$\hat{\alpha} = [X^T(\hat{\beta})X(\hat{\beta})]^{-1} X^T(\hat{\beta})Y$$

→ Minimising $R(\beta_1, \beta_2)$ wrt β is = maximizing $Y^T X(X^TX)^{-1}X^T Y = Y^T P_X(\beta) \cdot Y = S(g_0, g_1, g_2)$

$$S(\beta) = Y^T P_{X(\beta)} Y \quad \text{wrt } \beta$$

$$\frac{\partial S}{\partial \beta} = \frac{\partial P_X(\beta)}{\partial \beta} \times \frac{\partial S(\beta)}{\partial P_X(\beta)}$$ ↓ too long

→ We will convert the problem in terms of g_0, g_1, g_2.

$$\underset{n \times 2}{X(\beta)} = \begin{bmatrix} e^{\beta_1} & \cdots & e^{\beta_2} \\ \vdots & & \vdots \\ e^{n\beta_1} & & e^{n\beta_2} \end{bmatrix}$$

$$\underset{n-2 \times n}{G^T} = \begin{bmatrix} g_0 g_1 g_2 & & & \\ 0 & \ddots & & \\ & & \ddots & \\ & & & g_0 g_1 g_2 \end{bmatrix}$$

$$\boxed{\text{Note: } G^T X(\beta) = 0}$$

$$P_{X(\beta)} = X(\beta)(X^T(\beta)X(\beta))^{-1} X^T(\beta)$$

$$P_{G(g)} = G(g)(G^T(g)G(g))^{-1} G^T(g)$$

Note that: $P_{G(g)} = P_{G(cg)}$.

→ I need to maximise:

$$Y^T(I - P_{G(g)})Y \quad \text{s.t.} \quad g_0^2 + g_1^2 + g_2^2 = 1$$

→ I need to minimise:

$$Y^T P_{G(g)} Y \quad \text{ST} \quad g^T g = 1$$

→ we write $G^T(g)$ as follows:

$$\underset{n-2 \times n}{U_0^T} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & & & \vdots \\ \vdots & & 1 & \vdots \end{bmatrix} ; \quad \underset{n-2 \times n}{U_1^T} = \begin{bmatrix} 0 & 1 & & \\ 0 & 0 & 1 & \\ 1 & & & \\ \vdots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

→ $G^T(\underset{\sim}{g}) = (g_0 U_0^T + g_1 U_1^T + g_2 U_2^T)$

We need to minimizes:

$$Y^T P_G Y = Y^T G (G^T G)^{-1} G^T Y.$$

$$\rightarrow Y^T P_G Y = Y^T \left(\sum_{i=0}^{2} g_i U_i\right) \left(\left(\sum_0^2 g_i U_i^T\right) \left(\sum_0^2 g_i U_i\right) \right)^{-1} \left(\sum_{i=1}^{2} g_i U_i^T\right) Y.$$

$$\frac{\partial Y}{\partial g_k} = Y^T U_k \left[\left(\sum g_i U_i^T\right) \left(\sum g_i U_i\right) \right]^{-1} \left(\sum g_i U_i^T\right) y$$

$$+ y^T \left(\sum g_i U_i\right) \left[G^T G \right]^{-1} \left(U_k^T\right) y$$

$$+ y^T G \left[\frac{\partial}{\partial g_k} (G^T G)^{-1} \right] G^T y$$

* Model

29/10/24

$$y_t = A^0 \cos(\omega^0 t) + B^0 \sin(\omega^0 t) + \varepsilon_t$$

A^0, B^0 & ω^0 are the true values, and we don't know these values. ε_t's are an iid with mean 0 and variance σ^2.

We want to estimate A^0, B^0 & ω^0 based on the sample $\{y_1, \ldots, y_n\}$.
We would like to look at the properties of these estimators.

• Properties

→ $\hat{A}, \hat{B}$ & $\hat{\omega}$ are estimators of A^0, B^0 and ω^0, then what will happen to $\hat{A}, \hat{B}$ & $\hat{\omega}$ as sample size increases!

- Note that $\hat{A}, \hat{B}$ & $\hat{\omega}$ are all R.Vs.

→ We will be using the following facts:

(I) $\frac{1}{n}\sum_{t=1}^{n} \cos^2(\omega t) = \lim_{n\to\infty} \frac{1}{n} \sum \sin^2(\omega t) = \frac{1}{2}$

(II) $\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \cos(\omega t) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \sin(\omega t) = 0$

(III) $\lim_{n\to\infty} \frac{1}{n} \sum \sin(\omega t)\cos(\omega t) = 0$.

→ $y_t = A^0 \cos(\omega^0 t) + B^0 \sin(\omega^0 t) + \varepsilon_t$.

$Q(A, B, \omega) = \sum (y_t - A\cos(\omega t) - B\sin(\omega t))^2$

LSE of A, B, ω can be obtained by minimizing $Q(A, B, \omega)$ wrt A, B, ω.

→ Note that exactly same as before:

$$Y = X(\omega^0)\beta^0 + e$$

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} ; X = \begin{bmatrix} \cos(\omega^0) & \cdots & \sin(\omega^0) \\ \vdots & & \vdots \\ \cos(n\omega^0) & \cdots & \sin(n\omega^0) \end{bmatrix}$$

$$\beta^0 = \begin{bmatrix} A^0 \\ B^0 \end{bmatrix} ; e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$$Q(A, B, \omega) = [Y - X(\omega)\beta]^T [Y - X(\omega)\beta]$$

$$\hat{\beta}(\omega) = [X^T(\omega) X(\omega)]^{-1} X^T(\omega) Y.$$

$$Q(\hat{A}(\omega), \hat{B}(\omega), \omega) = Y^T [I - X(\omega)[X^T(\omega)X(\omega)]^{-1} X^T(\omega)] Y.$$

→ We can obtain the least squares estimator of ω by maximizing:

$$R(\omega) = Y^T X(\omega)[X^T(\omega) X(\omega)]^{-1} X^T(\omega) Y.$$

$$= \frac{1}{n} Y^T X(\omega) \left[\frac{1}{n} X^T(\omega) X(\omega)\right]^{-1} X^T(\omega) Y.$$

→ $R(\omega) = \frac{1}{n} Y^T X(\omega) \underbrace{\left[\frac{1}{n} X^T(\omega) X(\omega)\right]}^{-1} X^T(\omega) Y.$

$$X^T(\omega) X(\omega) = \begin{bmatrix} \cos(\omega) & \cdots & \cos(n\omega) \\ \sin(\omega) & \cdots & \sin(n\omega) \end{bmatrix} \begin{bmatrix} \cos(\omega) & \sin(\omega) \\ \vdots & \vdots \\ \cos(n\omega) & \sin(n\omega) \end{bmatrix}$$

$$\frac{1}{n} X^T(\omega) X(\omega) = \begin{bmatrix} \frac{1}{n}\sum_{t=1}^{n} \cos^2(\omega t) & \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t)\sin(\omega t) \\ \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t)\sin(\omega t) & \frac{1}{n}\sum_{t=1}^{n} \sin^2(\omega t) \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{1}{n} X^T(\omega) X(\omega) \longrightarrow \frac{1}{2} I.$$

For large n:

$$R(\omega) \approx I(\omega) = \frac{2}{n} Y^T X(\omega) X^T(\omega) Y.$$

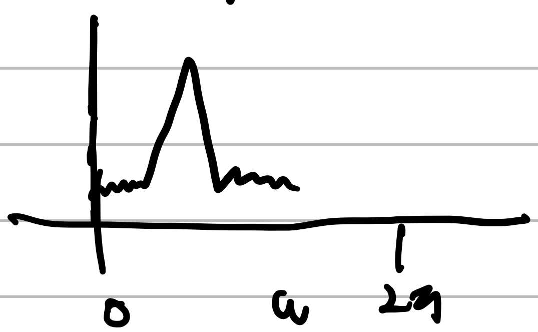
We obtain the approximate least squares estimator of ω by maximizing $I(\omega) = \frac{2}{n} Y^T X(\omega) X^T(\omega) Y$.

$$I(\omega) = \frac{2}{n} \begin{bmatrix} \sum y_t \cos(\omega t) & \sum y_t \sin(\omega t) \end{bmatrix} \begin{bmatrix} \sum y_t \cos(\omega t) \\ \sum y_t \sin(\omega t) \end{bmatrix}$$

$$= \frac{2}{n} \left[\left(\sum y_t \cos(\omega t)\right)^2 + \left(\sum y_t \sin(\omega t)\right)^2 \right]$$

Periodogram function.

→ Suppose we plot ω vs $I(\omega)$



→ Let's look at the behaviour of $I(\omega)$ for large n.

$$y_t = A^0\cos(\omega^0 t) + B^0\sin(\omega^0 t) + \varepsilon_t.$$

$$\frac{2}{n}\left[\sum y_t\cos(\omega t)\right]^2 + \frac{2}{n}\left[\sum_{t=1}^{n} y_t\sin(\omega t)\right]^2$$

$$= \frac{2}{n}\left\{\left[\sum (A^0\cos(\omega^0 t) + B^0\sin(\omega^0 t) + \varepsilon_t)\cos(\omega t)\right]^2 + \left[\sum (A^0\cos(\omega^0 t) + B^0\sin(\omega^0 t) + \varepsilon_t)\sin(\omega t)\right]^2\right\}$$

$$\Rightarrow \frac{2}{n}\left(\sum_{t=1}^{n} (A^0\cos(\omega^0 t) + B^0\sin(\omega^0 t))\cos(\omega^0 t) + \sum_{t=1}^{n} \varepsilon_t\cos(\omega t)\right)^2$$

→ Note that the approx LSE can be obtained by maximizing $I(\omega)$ $\Leftrightarrow$ maximizing $\frac{1}{n}I(\omega)$.

$$\Rightarrow 2\Big[\underbrace{\frac{1}{n}\sum y_t\cos(\omega t)}_{J_1(\omega)}\Big]^2 + 2\Big[\underbrace{\frac{1}{n}\sum y_t\sin(\omega t)}_{J_2(\omega)}\Big]^2$$

$$\rightarrow \frac{1}{n}\sum y_t\cos(\omega t) = \frac{1}{n}\sum (A^0\cos(\omega^0 t) + B^0\sin(\omega^0 t) + \varepsilon_t)\cos(\omega t)$$

$$\Rightarrow A^0\frac{1}{n}\underbrace{\sum \cos(\omega^0 t)\cos(\omega t)}_{\frac{1}{2}} + \frac{B^0}{n}\underbrace{\sum \sin(\omega^0 t)\cos(\omega t)}_{0} + \underbrace{\frac{1}{n}\sum \varepsilon_t\cos(\omega t)}_{0}.$$

$$\rightarrow E\left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t\cos(\omega^0 t)\right) = 0$$

$$V\left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t\cos(\omega^0 t)\right) = \frac{1}{n^2}V\left(\sum_{t=1}^{n}\varepsilon_t\cos(\omega^0 t)\right)$$

$$= \frac{B^2}{n^2} \sum_{t=1}^{n} \cos^2(w^0 t) \longrightarrow 0$$

$$J_1(w) = \begin{cases} (A^0/2) \, ; & \text{if } w = w^0 \\ 0 \, ; & \text{if } w \neq w^0 \end{cases}$$

$$J_2 = \begin{cases} B_0/2 & \text{if } w = w^0 \\ 0 & \text{if } w \neq w^0 \end{cases}$$

My

* Sinusoidal model

→ $y_t = A^0 \cos(\omega^0 t) + B^0 \sin(\omega^0 t) + \varepsilon_t; t = 1, \ldots, n.$

$\{y_1, \ldots, y_n\}$ $\varepsilon_t \sim iid(0, \sigma^2)$

→ Least square estimators of A^0, B^0, ω^0.

* Multicomponent Sinusoidal model

→ $y_t = \sum_{k=1}^{p} \{A_k^0 \cos(\omega_k^0 t) + B_k^0 \sin(\omega_k^0 t)\} + \varepsilon_t$
$; t = 1, \ldots, n.$

$\{y_1, \ldots, y_n\}$; $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$

→ We want to estimate $\{(A_k^0, B_k^0, \omega_k^0); k = 1, \ldots, p\}$
often in practice we need to estimate p also.

→ We have $3p$ unknown parameters.
We will consider the most intuitive LSE.

→ For notational purposes we take $p = 2$.

$$y_t = \sum_{k=1}^{2} \{A_k^0 \cos(\omega_k^0 t) + B_k^0 \sin(\omega_k^0 t)\} + \varepsilon_t.$$

→ We will write it in matrix notation:

$$Y = X(\omega_1^0, \omega_2^0) \beta^0 + \varepsilon$$

$$Q(A_1, B_1, \omega_1, A_2, B_2, \omega_2) = [Y - X(\omega_1, \omega_2)\beta]^T [Y - X(\omega_1, \omega_2)\beta] \quad (*)$$

$$\underset{n\times 1}{Y} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} ; \quad X(\omega_1, \omega_2) = \begin{bmatrix} \cos(\omega_1) & \sin(\omega_1) & \cdots & \sin(\omega_2) \\ \cos(2\omega_1) & & & \sin(2\omega_2) \\ \vdots & & & \vdots \\ \cos(n\omega_1) & & & \sin(n\omega_2) \end{bmatrix}$$

$$\beta = \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix}$$

For fixed ω_1, ω_2:

$$\hat{\beta}(\omega_1, \omega_2) = [X^T(\omega_1,\omega_2) X(\omega_1,\omega_2)]^{-1} X^T(\omega_1,\omega_2) Y.$$

minimize (*) wrt β.

Then for the LSE of ω_1, ω_2 can be obtained by minimizing:

$$Q(\hat{A}_1(\omega_1,\omega_2), \hat{B}_1(\omega_1,\omega_2), \omega_1, \hat{A}_2(\omega_1,\omega_2), \hat{B}_2(\omega_1,\omega_2), \omega_2)$$

$$= [Y - X(\omega_1,\omega_2)[X^T(\omega_1,\omega_2) X(\omega_1,\omega_2)]^{-1} X^T(\omega_1,\omega_2) Y]^T \; [-]$$

$$I(\omega_1,\omega_2) = Y^T[I - P_{X(\omega_1,\omega_2)}]^T[I - P_{X(\omega_1,\omega_2)}]Y$$
$$= Y^T[I - P_{X(\omega_1,\omega_2)}]Y.$$

$\rightarrow$ First we maximize $Y^T P_{X(\omega_1,\omega_2)} Y$ w.r.t ω_1, ω_2 if $\hat{\omega}_1, \hat{\omega}_2$ maximize $Y^T P_{X(\omega_1,\omega_2)} Y$ then Least square Estimator $\hat{\beta}(\hat{\omega}_1, \hat{\omega}_2)$.

→ In this particular case the LSEs can be obtained by maximising a 2D function.
In general we need to solve a p-dimensional optim. problem. We also need a $2p \times 2p$ matrix inverse computation.

$$P_X(\omega_1, \omega_2) = X(\omega_1, \omega_2) \left[X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) \right]^{-1} X^T(\omega_1, \omega_2)$$

$$\underset{n \times 4}{X(\omega_1, \omega_2)} = \begin{bmatrix} \cos(\omega_1) & \sin(\omega_1) & \cos(\omega_2) & \sin(\omega_2) \\ \vdots & & & \vdots \\ \cos(n\omega_1) & & & \sin(n\omega_2) \end{bmatrix}$$

→ We observe again the matrix $X^T(\omega_1, \omega_2) X(\omega_1, \omega_2)$

$$\frac{1}{n} \left[X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) \right] = \begin{bmatrix} \frac{1}{n} \sum \cos^2(\omega_1 t) & \frac{1}{n} \sum \cos(\omega_1 t) \sin(\omega_1 t) \\ & \ddots \end{bmatrix}$$

$$\lim_{n \to \infty} \frac{1}{n} X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) = \frac{1}{2} I_{4 \times 2p}.$$

$$\Rightarrow Y^T P_X Y$$

for large n: $\approx 2 Y^T X X^T Y$.

We need to maximise $I(\omega_1, \omega_2) = 2 \| X^T Y \|_2^2$

$$\hat{\omega} = \underset{\omega}{\arg\max}\, I(\omega)$$

$$= \arg\max \frac{1}{n} Y^T X (X^T X)^{-1} X^T Y$$

$$= \arg\max \; Y^T X \left[\frac{X^T X}{n}\right]^{-1} X^T Y.$$

→ so for large n:

$$(\hat{\omega}_1, \hat{\omega}_2) = \arg\max \frac{1}{n^2} Y^T X(\omega_1, \omega_2) X^T(\omega_1, \omega_2) Y.$$

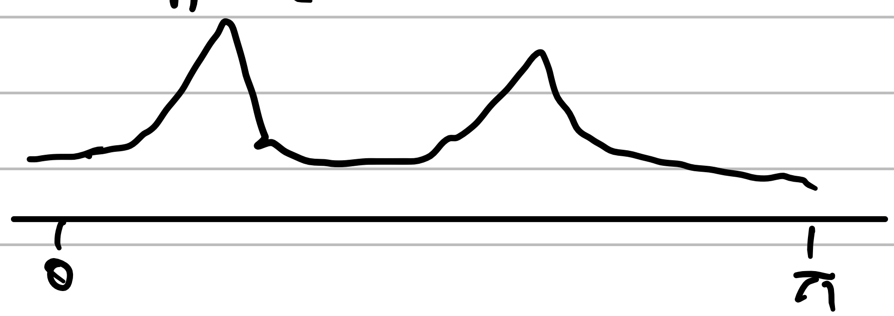
$$= \frac{1}{n^2}\left[\left(\sum y_t \cos(\omega_1 t)\right)^2 + \left(\sum y_t \sin(\omega_1 t)\right)^2 + \left(\sum y_t \cos(\omega_2 t)\right)^2 + \left(\sum y_t \sin(\omega_2 t)\right)^2\right]$$

$$\hat{\omega}_1 = \arg\max \left\{ \frac{1}{n^2}\left\{\left(\sum y_t \cos(\omega_1 t)\right)^2 + \left(\sum y_t \sin(\omega_1 t)\right)^2\right\}\right.$$

$$\hat{\omega}_2 = \arg\max \left\{ \frac{1}{n^2}\left[\left(\sum y_t \cos \omega_2 t\right)^2 + \left(\sum y_t \sin \omega_2 t\right)^2\right]\right.$$

→ therefore instead of solving a 2-D optim problem, we need to solve 2 1-D optim problem.

$$J(\omega) = \frac{1}{n^2}\left\{\left(\sum y_t \cos(\omega t)\right)^2 + \left(\sum y_t \sin(\omega t)\right)^2\right\}$$



5/11/24

→ We will show that

$$\lim_{n\to\infty} I(\omega_1^0) > \sigma^2 \quad ; \lim_{n\to\infty} I(\omega_2^0) > \sigma^2$$

$$\lim_{n\to\infty} I(\omega) = \sigma^2 \text{ if } \omega \neq \omega_1^0 \text{ or } \omega_2^0.$$

Note that σ^2 is the error variance.

$$\frac{1}{n}\sum y_t \cos(\omega t) = \frac{1}{n}\Big[\sum\big\{(A_1^0\cos(\omega_1^0 t) + B_1^0 \sin(\omega_1^0 t) + A_2^0\cos(\omega_2^0 t) + B_2^0 \sin(\omega_2^0 t) + \varepsilon_t\big\}\Big]\cos(\omega t)$$

$$= \frac{1}{n}\sum_{j=1}^{2}\sum_{t=1}^{n}\Big[A_j^0\cos(\omega_j^0 t)\cos(\omega t) + B_j^0\sin(\omega_j^0 t)\cos(\omega t) + \varepsilon_t\cos(\omega t)\Big]$$

$$= A_1^0\frac{1}{n}\sum\cos(\omega_1^0 t)\cos(\omega t) + B_1^0\frac{1}{n}\sum\sin(\omega_1^0 t)\cos(\omega t)$$
$$+ A_2^0\frac{1}{n}\sum\cos(\omega_2^0 t)\cos(\omega t) + B_2^0\frac{1}{n}\sum\sin(\omega_2^0 t)\cos(\omega t)$$
$$+ \frac{1}{n}\sum\varepsilon_t\cos(\omega t)$$

$\rightarrow$ Suppose $\omega = \omega_1^0$, $\lim_{n\to\infty} I(\omega) = \frac{A_1^0}{2} + \underbrace{\lim_{n\to\infty} \frac{\varepsilon_t \cos(\omega t)}{n}}_{??}$

$$\eta = \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t \cos(\omega t)$$

$$E(\eta) = 0 \; ; \; var(\eta) = \frac{1}{n^2}\sum \sigma^2 \cos^2(\omega t)$$

as $n \to \infty$.

$$\frac{1}{n}\sum_{t=1}^{\infty} \varepsilon_t \cos(\omega_1^0 t) \xrightarrow{p} 0$$

$$P\left[\left|\frac{1}{n}\sum \varepsilon_t \cos(\omega_1^0 t)\right| > \varepsilon\right] \to 0 \quad \forall \varepsilon > 0.$$

$\rightarrow$ suppose $\omega = \omega_1^0$.

$\left(\frac{A_1^0}{2}\right)^2$; $\left(\frac{1}{n}\sum y_t \cos(\omega_1^0 t)\right)^2 \to \left(\frac{A_1^{02}}{4}\right)$

$\left(\frac{1}{n}\sum y_t \sin(\omega_1^0 t)\right)^2 \to \left(\frac{B_1^{02}}{4}\right)$

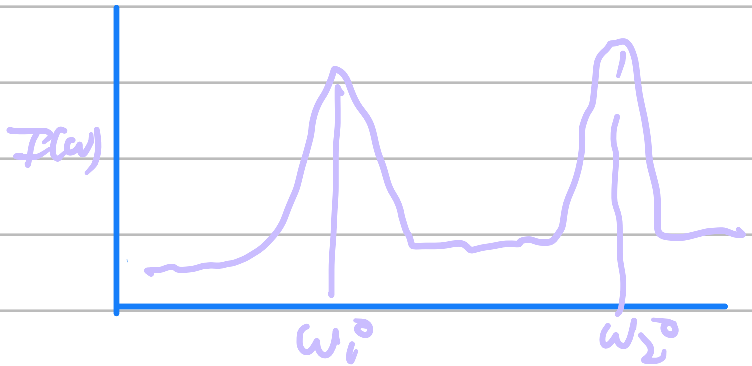
$$\rightarrow I(\omega) = \left(\frac{1}{n}\sum y_t \cos(\omega t)\right)^2 + \left(\frac{1}{n}\sum y_t \sin(\omega t)\right)^2$$

$$I(\omega_1^0) \rightarrow \left(\frac{A_1^{02}}{4} + \frac{B_1^{02}}{4}\right)$$

$$I(\omega_2^0) \rightarrow \left(\frac{A_2^{02}}{4} + \frac{B_2^{02}}{4}\right)$$

$$I(\omega) \rightarrow 0$$

$$\lim I(\omega) = \begin{cases} \frac{1}{4}(A_1^{0\,2}+B_1^{0\,2}) & \text{if } \omega=\omega_1^0 \\ \frac{1}{4}(A_2^{0\,2}+B_2^{0\,2}) & \text{if } \omega=\omega_2^0 \\ 0 & \text{o.w.} \end{cases}$$

→ 

Periodogram

→ In practice we choose initial values at:
$$\left(0, \frac{\pi}{n}, \ldots\ldots, \frac{n\pi}{n}\right)$$

→ You compute $I\left(\frac{i\pi}{n}\right)$ for $i=0,1,\ldots\ldots,n$.

Choose that i for which $I(i\pi/n)$ is the maximum. You start your initial guess from that $i\pi/n$.

Problem: (i) If sample size is not large, then we do not know how $I(\omega)$ behaves??
(we need to do simulation)
(ii) If ω_1^0 & ω_2^0 are very close.

→ Suppose I have
$$y_t = \sum_{k=1}^{p} \left(A_k^0 \cos(\omega_k^0 t) + B_k^0 \sin(\omega_k^0 t)\right) + \varepsilon_t.$$

Here p is also unknown.

$\rightarrow$ $R(A, B, \omega) = \sum_{t=1}^{n} (y_t - A\cos(\omega t) - B\sin(\omega t))^2$.

We minimise $R(A, B, \omega)$ wrt A, B & ω.

$R(A, B, \omega) = [Y - X(\omega)\beta]^T [Y - X(\omega)\beta]$

$\hat{\beta} = [X^T(\omega) X(\omega)]^{-1} X^T(\omega) Y$.

$\rightarrow$ We can estimate ω by maximising:

$Y^T X(\omega) (X^T(\omega) X(\omega))^{-1} X^T(\omega) Y$.

7/11/24

$\rightarrow Q(A,B,\omega) = \sum_{t=1}^{n} \left(y_t - A\cos(\omega t) - B\sin(\omega t)\right)^2$ $(*)$

We want to minimise $Q(A,B,\omega)$ wrt A, B, ω.

$\rightarrow (\hat{A}, \hat{B}, \hat{\omega}) = \operatorname{argmin} Q(A,B,\omega)$
$= \operatorname{argmin} \frac{1}{n} Q(A,B,\omega).$

$$\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega) = \frac{1}{n}\sum_{t=1}^{n}\left(y_t - A\cos(\omega t) - B\sin(\omega t)\right)^2$$

$$= \frac{1}{n}\sum_{t=1}^{n}\left(\sum_{k=1}^{p}\left\{A_k^{\circ}\cos(\omega_k^{\circ} t) + B_k^{\circ}\sin(\omega_k^{\circ} t)\right\} + \varepsilon_t - A\cos(\omega t) - B\sin(\omega t)\right)^2$$

$$\lim \frac{1}{n} Q(A,B,\omega) = \frac{1}{n}\left[\sum_{t=1}^{n} A_1^{\circ 2}\cos^2(\omega_k^{\circ} t) + \cdots + \sum_{t=1}^{n} A_p^{\circ 2}\cos^2(\omega_k^{\circ} t) + \sum_{t=1}^{n} B_1^{\circ 2}\sin^2(\omega_1^{\circ} t) + \cdots \right.$$
$$\left. \cdots \sum_{t=1}^{n} B_p^{\circ 2}\sin^2(\omega_k^{\circ} t) + \frac{1}{n}\sum \varepsilon_t^2 + \frac{1}{n}\sum_{t=1}^{n} A^2\cos^2(\omega t) + \frac{1}{n}\sum_{t=1}^{n} B^2\sin^2(\omega t)\right]$$

$+$ cross-product term.

$$\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega) = \frac{1}{2}\left[\sum_{k=1}^{p}\left(A_k^{\circ 2} + B_k^{\circ 2}\right)\right] + \frac{1}{2}(A^2 + B^2) + \sigma^2.$$

For $K = 1, \cdots, p$

$$\frac{1}{n}\sum_{t=1}^{n} AA_1^0 \cos(w_k^0 t)\cos(wt)$$

$$= AA_k^0 \frac{1}{n}\sum \cos(w_k^0 t)\cos(wt) = \begin{cases} 0 & \text{if } w \neq w_k^0 \\ \frac{1}{2}AA_k^0 & \text{if } w = w_k^0 \end{cases}$$

$$\frac{1}{n}\sum BB_k^0 \sin(w_k^0 t)\sin(wt) = \begin{cases} 0 & \text{if } w \neq w_k^0 \\ \frac{1}{2}BB_k^0 & \text{if } w = w_k^0 \end{cases}$$

$\rightarrow$ If $w \neq w_k^0$ for $K = 1, \cdots, p$.

$$\lim_{n\to\infty} \frac{1}{n} Q(A, B, w) = \frac{1}{2}\left[\sum_{k=1}^{p} (A_k^{0^2} + B_k^{0^2})\right] + \frac{1}{2}(A^2 + B^2) + \sigma^2$$

If $w = w_k^0$ for some K:

$$\lim \frac{1}{n} Q(A, B, w) = \frac{1}{2}\left(\sum_{k=1}^{p} (A_k^{0^2} + B_k^{0^2})\right) + \frac{1}{2}(A^2 + B^2) + \sigma^2 - (AA_k^0 + BB_k^0)$$

$\rightarrow$ WLOG: $(A_1^{0^2} + B_1^{0^2}) > \cdots\cdots \geq (A_p^{0^2} + B_p^{0^2})$

$$y_t = \sum_{k=1}^{p} \left(A_k^0 \cos(w_k^0 t) + B_k^0 \sin(w_k^0 t)\right) + \varepsilon_t.$$

$\rightarrow$ Observe: If we choose $w = w_1^0$, $A = A_1^0$, $B = B_1^0$.

$$\lim_{n\to\infty} \frac{1}{n} Q(A_1^0, B_1^0, w_1^0) = \frac{1}{2}\left(\sum_{k=1}^{p} (A_k^{0^2} + B_k^{0^2})\right) + \frac{1}{2}(A_1^{0^2} + B_1^{0^2}) + \sigma^2.$$

$$\leq \lim_{n\to\infty} \frac{1}{n} Q(A, B, w)$$

Claim: $\frac{1}{2}(A^2+B^2) - (AA_i^0 + BB_i^0) \geq -\frac{1}{2}(A_i^{0^2}+B_i^{0^2})$

$\Leftrightarrow (A^2+B^2+A_i^{0^2}+B_i^{0^2}) \geq 2(AA_i^0+BB_i^0)$

$(A-A_i^0)^2+(B-B_i^0)^2 \geq 0.$

$\rightarrow$ If we minimize $\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega)$, then the minimum occurs at $(A_1^0, B_1^0, \omega_1^0)$.

Then value of $\lim_{n\to\infty} Q(A,B,\omega)$ will be

$$\frac{1}{2}\left[\sum_{k=1}^{p}(A_k^{0^2}+B_k^{0^2}) - \frac{1}{2}(A_1^{0^2}+B_1^{0^2})+\sigma^2\right]$$

$$= \frac{1}{2}\left[\sum_{k=2}^{p}(A_k^{0^2}+B_k^{0^2})\right]+\sigma^2.$$

$\tilde{y}_t = y_t - \hat{A}\cos(\hat{\omega}t) - \hat{B}\sin(\hat{\omega}t)$

$$Q_2(A,B,\omega) = \sum_{t=1}^{n}(\tilde{y}_t - A\cos(\omega t) - B\sin(\omega t))^2$$

$\lim_{n\to\infty} \frac{1}{n} Q_2(A,B,\omega)$ (**)

The minimum of ( ) occurs at A_2^0, B_2^0, ω_2^0.
The value of min $\frac{1}{2}\left(\sum_{k=2}^{p}(A_k^{0^2}+B_k^{0^2})\right)^2+\sigma^2$.

$\rightarrow Q_{p+1}(A,B,\omega) = \sum_{t=1}^{n}(\varepsilon_t - A\cos\omega t - B\sin\omega t)$

QUIZ #4 14/11/24

1. $x_1, \cdots, x_n \sim f(x|\mu,\lambda) = \lambda e^{-\lambda(x-\mu)}$; $x>\mu$, $\lambda>0$

We know $\mu=10$, $\pi(\lambda) = 10e^{-10\lambda}$.

(i) Bayes estimator wrt Squared Error Loss function.

(ii) " " " absolute " " ".

$\rightarrow \pi(\lambda|\underline{x}) = \dfrac{f(x_1,\cdots\cdots,x_n|\lambda,10)\,\pi(\lambda)}{\int_0^\infty f(x_1,\cdots,x_n|\lambda,10)\,\pi(\lambda)\,d\lambda}$

$= \dfrac{\lambda^n e^{-\lambda\sum x_i-10} \times 10 e^{-10\lambda}}{10\int_0^\infty \lambda^n e^{-\lambda(\sum(x_i-10)+10)}\,d\lambda}$

$= K\lambda^n e^{-(\sum(x_i-10)+10)} \sim \text{Gamma}\left(n+1, \sum(x_i-10)+10\right)$

$K = \dfrac{[\sum(x_i-10)+10]^{n+1}}{\Gamma n+1}$

(i) Bayes estimator : $\dfrac{n+1}{\sum(x_i-10)+10}$

(ii) Median of $\text{gamma}(n+1, \sum(x_i-10)+10)$

2.) x_i's $\sim f(x|\mu,\lambda) = \lambda e^{-\lambda(x-\mu)}$; $x>\mu$, $\lambda>0$.

Here μ & λ are both unknown.

$\pi(\lambda) = 10e^{-10\lambda}$. $P[\mu=0] = \frac{1}{4}$, $P[\mu=1] = \frac{3}{4}$.

$$\pi(\lambda,0|\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}|\lambda,0)\pi(0)\pi(\lambda)}{\sum_{i=0}^{1}\int_0^\infty f(x|\lambda,i)\pi(i)\pi(\lambda)\,d\lambda}.$$

$$\pi(\lambda,1|\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}|\lambda,1)\pi(\lambda)\pi(1)}{\sum_{i=0}^{1}\int_0^\infty f(x|\lambda,i)\pi(i)\pi(\lambda)\,d\lambda}$$

$$\pi(\lambda,0|\underset{\sim}{x}) = \frac{\frac{1}{4} \times \lambda^n e^{-\lambda(\Sigma x_i - 0)}\ 10e^{-10\lambda}.}{10 \times \frac{1}{4} \times \int \lambda^n e^{-\lambda(\Sigma x_i + 10)}\,d\lambda + \frac{3}{4}\int_0^\infty \lambda^n e^{-\lambda(\Sigma(x_i-1)+10)}\,d\lambda}$$

$$\pi(\lambda,1|x) \propto \frac{3}{4} \times \lambda^n e^{-\lambda[\Sigma(x_i-1)+10)}$$

$$\begin{aligned}\pi(\lambda|\underset{\sim}{x}) &= \pi(\lambda,1|\underset{\sim}{x}) + \pi(\lambda,0|\underset{\sim}{x})\\ &= K\left\{\left[\frac{1}{4}\lambda^n e^{-\lambda(\Sigma x_i+10)}\right] + \left[\frac{3}{4}\lambda^n e^{-\lambda(\Sigma(x_i-1)+10)}\right]\right.\end{aligned}$$

$$\underline{\underline{\lambda \geq 0}}$$

$$\rightarrow \pi(\lambda,i|\underset{\sim}{x}) = \frac{\pi(\underset{\sim}{x}|\lambda,i)\pi(\lambda)\pi(i)}{\sum_{i=0}^{1}\int \pi(x|\lambda,i)\pi(\lambda)\pi(i)\,d\lambda}\ ;\ i=0,1.$$

$$= K\pi(\underset{\sim}{x}|\lambda,i)\,\pi(\lambda)\,\pi(i).$$

$$\pi(\lambda, 0) = K \frac{1}{4} \lambda^n e^{-\lambda \sum x_i} 10 e^{-10\lambda}$$

$$\pi(\lambda, 1) = K \frac{3}{4} \lambda^n e^{-\lambda(\sum x_i - 1)} 10 e^{-10\lambda}.$$

$$\pi(0 | \underset{\sim}{x}) = \frac{1}{4} \times K \times \frac{\Gamma(n+1)}{(\sum x_i + 10)^{n+1}}$$

$$\pi(1 | \underset{\sim}{x}) = \frac{3}{4} \times K \times \frac{\Gamma(n+1)}{(\sum (x_i - 1) + 10)^{n+1}}$$

$$K^{-1} = \frac{1}{4} \frac{\Gamma(n+1)}{\sum (x_i + 10)^{n+1}} + \frac{3}{4} \frac{\Gamma(n+1)}{(\sum (x_i - 1) + 10)^{n+1}}$$

$$\hat{\theta} = \frac{\dfrac{3 \times \Gamma(n+1)}{(\sum (x_i - 1) + 10)^{n+1}}}{\dfrac{\Gamma(n+1)}{(\sum x_i + 10)^{n+1}} + \dfrac{\Gamma(n+1)}{(\sum (x_i - 1) + 10)^{n+1}}}.$$

* FIM

$$y_i = f(\theta^{\sim k}; t_i) + \varepsilon_i$$

→ 2^{nd} derivative of neg-log-likelihood.
we need to know the dist. of ε_i.

$$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2).$$

→ $$\prod_{i=1}^{n} f(y_i | \theta) = \frac{1}{(\sqrt{2\pi\sigma^2})} e^{-\frac{1}{2\sigma^2} \sum (y_i - f(\theta, t_i))^2}$$

$$\ell(\theta) = -n \frac{\ln(2\pi\sigma^2)}{2} - \frac{1}{2\sigma^2} \sum (y_i - f(\theta; t_i))^2$$

→ FIM can generate CIs.

$\underset{5\times5}{I(\theta)}$ $\quad$ $I^{-1}(\theta)$ The diagonal elements will provide you approx variances of $\hat{\theta}_1, \dots, \hat{\theta}_p$.