MSO205A End Semester Examination Hints to the Answers

1. Question 1

Set 1: Negative Hypergeometric distribution with parameters $100, 10, 4$

Set 2: Negative Hypergeometric distribution with parameters $100, 20, 5$

Set 3: Negative Hypergeometric distribution with parameters $100, 15, 6$

Set 4: Negative Hypergeometric distribution with parameters $100, 25, 7$

2. Question 2

We have $\alpha = \frac{1}{\pi}$. Standard calculations yield, for any $-1 < \beta < 1$,

$$X|Y = \beta \sim Uniform(-\sqrt{1-\beta^2}, \sqrt{1-\beta^2}), \quad Y|X = \beta \sim Uniform(-\sqrt{1-\beta^2}, \sqrt{1-\beta^2}).$$

Note that for $Uniform(-\sqrt{1-\beta^2}, \sqrt{1-\beta^2})$, the variance is $\frac{1}{3}(1-\beta^2)$.

Set 1:

$$f_{X|Y}(x \mid \tfrac{\sqrt{3}}{2}) = \begin{cases} 1, \text{ if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0, \text{ otherwise.} \end{cases},$$

$Var(Y \mid X = \frac{\sqrt{3}}{2}) = \frac{1}{12}$

Set 2:

$$f_{X|Y}(x \mid -\tfrac{\sqrt{3}}{2}) = \begin{cases} 1, \text{ if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0, \text{ otherwise.} \end{cases},$$

$Var(Y \mid X = -\frac{\sqrt{3}}{2}) = \frac{1}{12}$

Set 3:

$$f_{X|Y}(x \mid \tfrac{1}{\sqrt{2}}) = \begin{cases} \frac{1}{\sqrt{2}}, \text{ if } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, \\ 0, \text{ otherwise.} \end{cases},$$

$Var(Y \mid X = \frac{1}{\sqrt{2}}) = \frac{1}{6}$

Set 4:

$$f_{X|Y}(x \mid -\tfrac{1}{\sqrt{2}}) = \begin{cases} \frac{1}{\sqrt{2}}, \text{ if } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, \\ 0, \text{ otherwise.} \end{cases},$$

$Var(Y \mid X = -\frac{1}{\sqrt{2}}) = \frac{1}{6}$

3. Question 3

Standard calculations yield $\alpha = \frac{2}{9}$. Now,

$$Var(Y^2) = \mathbb{E}Y^4 - (\mathbb{E}Y^2)^2.$$

For $X_1 \sim N(1,1), X_2 \sim N(-1,1)$, we have (Practice Problem set 8)

$$\mathbb{E}X_1^2 = 2, \mathbb{E}X_1^4 = 10, \mathbb{E}X_2^2 = 2, \mathbb{E}X_2^4 = 10$$

Sets 1 & 2:

$$\mathbb{E}Y^2 = \alpha\mathbb{E}X_1^2 + (1-\alpha)\mathbb{E}X_2^2 = 2,$$

$$\mathbb{E}Y^4 = \alpha\mathbb{E}X_1^4 + (1-\alpha)\mathbb{E}X_2^4 = 10$$

and

$$Var(Y^2) = 6$$

Sets 3 & 4:

$$\mathbb{E}Y^4 = \alpha\mathbb{E}X_2^4 + (1-\alpha)\mathbb{E}X_1^4 = 2,$$

$$\mathbb{E}Y^4 = \alpha\mathbb{E}X_2^4 + (1-\alpha)\mathbb{E}X_1^4 = 10$$

and

$$Var(Y^2) = 6$$

4. Question 4

For $Y = 1 + X$, $\mathbb{E}Y = 1 + \mathbb{E}X$ and

$$Var(Y) = Var(X), \mathbb{E}(Y - \mathbb{E}Y)^3 = \mathbb{E}(X - \mathbb{E}X)^3$$

Therefore, the coefficient of skewness for Y is the same as X, which is 2 (Refer to supplementary materials).

5. Question 5

For X_1, X_2, X_3, X_4 be independent $Bernoulli(p)$ RVs,

$$X_{(2)} \sim Bernoulli(4p^3 - 3p^4), X_{(3)} \sim Bernoulli(1 - (1-p)^4 - 4p(1-p)^3)$$

Set 1: $p = \frac{1}{3}, X_{(2)} \sim Bernoulli(\frac{1}{9})$
Set 2: $p = \frac{2}{3}, X_{(3)} \sim Bernoulli(\frac{8}{9})$
Set 3: $p = \frac{1}{3}, X_{(3)} \sim Bernoulli(\frac{11}{27})$
Set 4: $p = \frac{2}{3}, X_{(2)} \sim Bernoulli(\frac{16}{27})$

6. Question 6

If $X \sim N(\mu_1, \sigma_1^2)$ and $X - Y \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$Y = X - (X - Y) \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2).$$

Using the given independence, we have

$$Cov(X, X - Y) = 0$$

and

$$Cov(X, Y) = Var(X) = \sigma_1^2, \quad \rho = \rho(X, Y) = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}$$

Using the given independence, we have

$$\begin{pmatrix} X \\ X - Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right).$$

Then, (Practice problem set 12)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ X-Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_1-\mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_1^2 \\ \sigma_1^1 & \sigma_1^2+\sigma_2^2 \end{pmatrix}\right).$$

and

$$\mathbb{E}(X+Y)^2 = \mathbb{E}\left(2X-(X-Y)\right)^2 = 4(\mu_1^2+\sigma_1^2) + (\mu_2^2+\sigma_2^2) - 4\mu_1\mu_2$$

$$f_{X,Y}(x,y)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_2^2}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_1+\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) + \left(\frac{y-\mu_1+\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2\right\}\right],$$

Set 1: $X \sim N(2,1)$ and $X-Y \sim N(1,4)$ are independent and hence

$$\mu_1 = 2, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 2$$

$$Y = X-(X-Y) \sim N(1,5)$$

$$\mathbb{E}(X+Y)^2 = 17$$

$$f_{X,Y}(x,y) = \frac{1}{4\pi}\exp\left[-\frac{5}{8}\left\{(x-2)^2 - \frac{2}{5}(x-2)(y-1) + \frac{1}{5}(y-1)^2\right\}\right]$$

Set 2: $X \sim N(-2,1)$ and $X-Y \sim N(1,4)$ are independent and hence

$$\mu_1 = -2, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 2$$

$$Y = X-(X-Y) \sim N(-3,5).$$

$$\mathbb{E}(X+Y)^2 = 33$$

$$f_{X,Y}(x,y) = \frac{1}{4\pi}\exp\left[-\frac{5}{8}\left\{(x+2)^2 - \frac{2}{5}(x+2)(y+3) + \frac{1}{5}(y+3)^2\right\}\right]$$

Set 3: $X \sim N(2,1)$ and $X-Y \sim N(-1,4)$ are independent and hence

$$\mu_1 = 2, \sigma_1 = 1, \mu_2 = -1, \sigma_2 = 2$$

$$Y = X-(X-Y) \sim N(3,5).$$

$$\mathbb{E}(X+Y)^2 = 33$$

$$f_{X,Y}(x,y) = \frac{1}{4\pi}\exp\left[-\frac{5}{8}\left\{(x-2)^2 - \frac{2}{5}(x-2)(y-3) + \frac{1}{5}(y-3)^2\right\}\right]$$

Set 4: $X \sim N(-2,1)$ and $X-Y \sim N(-1,4)$ are independent and hence

$$\mu_1 = -2, \sigma_1 = 1, \mu_2 = -1, \sigma_2 = 2$$

$$Y = X-(X-Y) \sim N(-1,5).$$

$$\mathbb{E}(X+Y)^2 = 17$$

$$f_{X,Y}(x,y) = \frac{1}{4\pi}\exp\left[-\frac{5}{8}\left\{(x+2)^2 - \frac{2}{5}(x+2)(y+1) + \frac{1}{5}(y+1)^2\right\}\right]$$

7. Question 7

For $X_n \sim Gamma(\alpha, \beta)$ i.i.d.,

$$\mathbb{E}X = \alpha\beta, Var(X) = \alpha\beta^2$$

by CLT,

$$\sqrt{n}\frac{\bar{X}_n - \alpha\beta}{\sqrt{\alpha\beta^2}} \xrightarrow[n\to\infty]{d} N(0,1)$$

and by Delta Method applied to $g(x) = x^p, \forall x$ with $p = 2, 3$

$$\sqrt{n}\frac{(\bar{X}_n)^p - (\alpha\beta)^p}{\sqrt{\alpha\beta^2}} \xrightarrow[n\to\infty]{d} p(\alpha\beta)^{p-1}N(0,1).$$

Hence,

$$\sqrt{n}((\bar{X}_n)^p - (\alpha\beta)^p) \xrightarrow[n\to\infty]{d} N(0, \alpha\beta^2 p^2 (\alpha\beta)^{2(p-1)})$$

Set 1: $\alpha = 2, \beta = 3, p = 2, N(0, 72 \times 36) = N(0, 2592)$

Set 2: $\alpha = 3, \beta = 2, p = 2, N(0, 48 \times 36) = N(0, 1728)$

Set 3: $\alpha = 2, \beta = 3, p = 3, N(0, 162 \times 1296) = N(0, 209952)$

Set 4: $\alpha = 3, \beta = 2, p = 3, N(0, 108 \times 1296) = N(0, 139968)$

8. Question 8

For

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ \alpha x + \frac{1}{5}, \text{ if } 0 \le x \le 1, \\ \beta x^2, \text{ if } 1 < x \le 2, \\ 1, \text{ if } x > 2 \end{cases}$$

we have $\alpha = \frac{1}{20}, \beta = \frac{1}{4}, \gamma = 1$, the unique median is at $\sqrt{2}$.

Set 1: $(\alpha + 1) = \frac{1}{20}, (\beta + 1) = \frac{1}{4}, \alpha = -\frac{19}{20}, \beta = -\frac{3}{4}, \alpha + 2\beta + 2\gamma = -\frac{9}{20}$

Set 2: $(\alpha + 1) = \frac{1}{20}, (\beta - 1) = \frac{1}{4}, \alpha = -\frac{19}{20}, \beta = \frac{5}{4}, \alpha + 2\beta + 2\gamma = \frac{71}{20}$

Set 3: $(\alpha - 1) = \frac{1}{20}, (\beta + 1) = \frac{1}{4}, \alpha = \frac{21}{20}, \beta = -\frac{3}{4}, \alpha + 2\beta + 2\gamma = \frac{31}{20}$

Set 4: $(\alpha - 1) = \frac{1}{20}, (\beta - 1) = \frac{1}{4}, \alpha = \frac{21}{20}, \beta = \frac{5}{4}, \alpha + 2\beta + 2\gamma = \frac{111}{20}$

9. Question 9

Correct options: (a), (b)

10. Question 10

$\mathbb{P}(X_j \le \sqrt{5}) = \mathbb{P}(X_j \le 2) = \frac{11}{16}$

$Y_j \sim Bernoulli(\mathbb{P}(X_j \le \sqrt{5})) = Bernoulli(\frac{11}{16})$. Hence, $Y_j^2 \sim Bernoulli(\frac{11}{16})$ and $Y_j^3 \sim Bernoulli(\frac{11}{16})$.

By WLLN, answer is $\frac{11}{16}$.

11. Question 11

Correct options: (a), (b), (c)

12. Question 12

Correct options: (ii), (iii)

13. Question 13

$$W \sim \chi^2_2 = Gamma(1,2) = Exponential(2)$$

$$V \sim F_{2,2}$$

14. Question 14

For $X \sim Geometric(p)$,

$$\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) = \frac{\mathbb{P}(\frac{1}{2} \leq X \leq \frac{5}{2})}{\mathbb{P}(X \leq \frac{5}{2})} = \frac{\mathbb{P}(X = 1 \text{ or } 2)}{\mathbb{P}(X \leq 2)} = \frac{2 - 3p + p^2}{3 - 3p + p^2}$$

Set 1: $X \sim Geometric(\frac{1}{3})$

$$\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) = \frac{10}{19}$$

Set 2: $X \sim Geometric(\frac{2}{3})$

$$\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) = \frac{4}{13}$$

Set 3: $X \sim Geometric(\frac{1}{4})$

$$\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) = \frac{21}{37}$$

Set 4: $X \sim Geometric(\frac{3}{4})$

$$\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) = \frac{5}{21}$$