Perturbation Analysis

for the problem of solving linear systems $Ax = b$.

we obtain not the exact solution x but

an approximate computed solution x.

The difference $e = x - \widehat{x}$ is called the error vector

one can test the accuracy of $\widehat{x}$ by forming $A\widehat{x}$ to see whether it is close to b.

Definition *Let $\widehat{x}$ be the computed solution to the linear system of equations $Ax = b$. Then the vector*

$$r = b - A\widehat{x}$$

is called the residual vector.

Then we can derive the residual equation

$$Ae = Ax - A\widehat{x} = b - A\widehat{x} = r$$

between the error vector and the residual vector.

Theorem $$\frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|} \leq \frac{\|x-\widehat{x}\|}{\|x\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}.$$

Proof:

Notice that $\widehat{x}$ is the exact solution of the linear system

$$A\widehat{x} = \widehat{b},$$

which has a perturbed right-hand side

$$\widehat{b} = b - r.$$

Then

$$\begin{aligned}
\|x - \widehat{x}\| &= \|A^{-1}b - A^{-1}\widehat{b}\| = \|A^{-1}(b - \widehat{b})\| \\
&\leq \|A^{-1}\|\|b - \widehat{b}\| = \|A^{-1}\|\|b\|\frac{\|b-\widehat{b}\|}{\|b\|} = \|A^{-1}\|\|Ax\|\frac{\|b-\widehat{b}\|}{\|b\|} \\
&\leq \|A^{-1}\|\|A\|\|x\|\frac{\|b-\widehat{b}\|}{\|b\|}
\end{aligned}$$

Therefore

$$\frac{\|x-\widehat{x}\|}{\|x\|} \le \kappa(A)\frac{\|b-\widehat{b}\|}{\|b\|} = \kappa(A)\frac{\|r\|}{\|b\|},$$

where

$$\kappa(A) = \|A\|\|A^{-1}\|$$

is called the condition number of A.

On the other hand, by the residual vector, we have

$$\|r\|\|x\| = \|Ae\|\|A^{-1}b\| \le \|A\|\|A^{-1}\|\|e\|\|b\| = \kappa(A)\|x-\widehat{x}\|\|b\|.$$

Hence

$$\frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|} \le \frac{\|x-\widehat{x}\|}{\|x\|}.$$

Therefore we have established relationships between the relative error in x and b.

Matrix Norm Definition and Properties

Definition *A matrix norm is a function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying the following conditions for all $A, B \in \mathbb{R}^{m\times n}$ and $\alpha \in \mathbb{R}$.*

1. $\|A\| \ge 0 \quad (\|A\| = 0 \Leftrightarrow A = 0)$;
2. $\|A+B\| \le \|A\| + \|B\|$;
3. $\|\alpha A\| = |\alpha|\|A\|$.

Definition *Some of the most frequently used matrix norms are*

- **Frobenius norm:**

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}.$$

- **2-norm:**

$$\|A\|_2 = \max_{\substack{x\in\mathbb{R}^n \\ x\ne 0}} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1}\|Ax\|_2.$$

- **1-norm:**

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|.$$

- **∞-norm:**

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

- **p-norm:**

$$\|A\|_p = \max_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p.$$

Definition **(submultiplicative property)** *A matrix norm $\|\cdot\|$ is said to have the submultiplicative property if for any matrices $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{p\times n}$ such that*

$$\|AB\| \le \|A\|\|B\|.$$

Property *All matrix norms satisfying the submultiplicative property are equivalent.*

$$\|A\|_2 \le \|A\|_F \le \sqrt{n}\|A\|_2.$$

$$\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty.$$

$$\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1.$$

$$\max_{i,j} |a_{ij}| \le \|A\|_2 \le \sqrt{mn}\max_{i,j} |a_{ij}|.$$

Theorem 1.15 *Suppose that $A \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ is a submultiplicative matrix norm. If $\|A\| < 1$, then $I - A$ is nonsingular and*

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$$

with

$$\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}.$$

Theorem 1.16 *If A is nonsingular and $\|A^{-1}E\| < 1$, then $A + E$ is nonsingular and*

$$\|(A+E)^{-1} - A^{-1}\| \le \frac{\|E\|\|A^{-1}\|^2}{1-\|A^{-1}E\|}.$$

Lemma *Suppose that x and $\widetilde{x}$ satisfy*

$$Ax = b \qquad \text{and} \qquad (A + \triangle A)\widetilde{x} = b + \triangle b,$$

where $A \in \mathbb{R}^{n\times n}$, $\triangle A \in \mathbb{R}^{n\times n}$, $0 \neq b \in \mathbb{R}^n$, and $\triangle b \in \mathbb{R}^n$, with

$$\frac{\|\triangle A\|}{\|A\|} \leq \delta \qquad \text{and} \qquad \frac{\|\triangle b\|}{\|b\|} \leq \delta.$$

If $\kappa(A) \cdot \delta < 1$, then $A + \triangle A$ is nonsingular and

$$\frac{\|\widetilde{x}\|}{\|x\|} \leq \frac{1 + \kappa(A)\delta}{1 - \kappa(A)\delta}.$$

Proof: Since $\|A^{-1}\triangle A\| \leq \|A^{-1}\|\|\triangle A\| \leq \delta\|A^{-1}\|\|A\| = \delta\kappa(A) < 1$, it follows from Theorem 1.16 that $A + \triangle A$ is nonsingular. Now $(A + \triangle A)\widetilde{x} = b + \triangle b$,

$$(I + A^{-1}\triangle A)\widetilde{x} = A^{-1}b + A^{-1}\triangle b = x + A^{-1}\triangle b,$$

and so by taking norms and using Theorem 1.15 we find

$$\begin{aligned}
\|\widetilde{x}\| &\leq \|(I + A^{-1}\triangle A)^{-1}\| \left(\|x\| + \|A^{-1}\|\|\triangle b\|\right) \\
&\leq \|(I + A^{-1}\triangle A)^{-1}\| \left(\|x\| + \delta\|A^{-1}\|\|b\|\right) \\
&\leq \frac{1}{1 - \|A^{-1}\triangle A\|} \left(\|x\| + \delta\|A^{-1}\|\|b\|\right) \\
&\leq \frac{1}{1 - \delta\kappa(A)} \left(\|x\| + \delta\|A^{-1}\|\|b\|\right) \\
&= \frac{1}{1 - \delta\kappa(A)} \left(\|x\| + \delta\|A^{-1}\|\|Ax\|\right) \\
&\leq \frac{1}{1 - \delta\kappa(A)} \left(\|x\| + \delta\|A^{-1}\|\|A\|\|x\|\right) \\
&= \frac{1}{1 - \delta\kappa(A)} \left(\|x\| + \delta\kappa(A)\|x\|\right) \\
&= \frac{1}{1 - \delta\kappa(A)} \left(1 + \delta\kappa(A)\right)\|x\|.
\end{aligned}$$

Therefore

$$\frac{\|\widetilde{x}\|}{\|x\|} \leq \frac{1 + \delta\kappa(A)}{1 - \delta\kappa(A)}.$$

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Theorem *If the conditions of Lemma hold then*

$$\frac{\|x-\widetilde{x}\|}{\|x\|} \leq \frac{2\delta}{1-\kappa(A)\delta}\kappa(A)$$

Proof: Since $\widetilde{x}$ satisfies $(A+\triangle A)\widetilde{x} = b+\triangle b$, $A\widetilde{x} = b+\triangle b - \triangle A\widetilde{x}$. Then we have

$$A\widetilde{x} - Ax = \triangle b + \triangle A\widetilde{x}$$

and

$$\widetilde{x} - x = A^{-1}\left(\triangle b + \triangle A\widetilde{x}\right).$$

Hence

$$\begin{aligned} \|\widetilde{x}-x\| &\leq \|A^{-1}\|\left(\|\triangle b\| + \|\triangle A\|\|\widetilde{x}\|\right) \\ &\leq \|A^{-1}\|\left(\delta\|b\| + \delta\|A\|\|\widetilde{x}\|\right) \\ &= \delta\|A^{-1}\|\left(\|Ax\| + \|A\|\|\widetilde{x}\|\right) \\ &\leq \delta\|A\|\|A^{-1}\|\left(\|x\| + \|\widetilde{x}\|\right), \end{aligned}$$

which gives

$$\frac{\|\widetilde{x}-x\|}{\|x\|} \leq \delta\kappa(A)\left(1+\frac{\|\widetilde{x}\|}{\|x\|}\right) \leq \delta\kappa(A)\left(1+\frac{1+\kappa(A)\delta}{1-\kappa(A)\delta}\right) = \frac{2\delta\kappa(A)}{1-\delta\kappa(A)}.$$

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