Design and Analysis of Algorithms (CS345)

Practice-sheet : **Maximum Flow**

1. (**Flow fundamentals**)

   Suppose you are given a directed graph $G = (V, E)$ with a positive capacity c_e on each edge $e \in E$, a designated source $s \in V$, and a designated sink $t \in V$. You are also given a flow f. Design the fastest algorithm to determine if f is a maximum (s, t)-flow.

2. (**Flow fundamentals**)

   Suppose you are given a directed graph $G = (V, E)$ with a positive capacity c_e on each edge $e \in E$, a designated source $s \in V$, and a designated sink $t \in V$. Prove that there always exists a maximum (s, t)-flow that is acyclic. That is, there is no cycle in the subgraph induced by the edges carrying non-zero flow.

3. (**Flow fundamentals**)

   Suppose you are given a directed graph $G = (V, E)$ with a positive integer capacity c_e on each edge $e \in E$, a designated source $s \in V$, and a designated sink $t \in V$. You are also given an integer maximum (s, t)-flow in G, defined by a flow value f_e on each edge $e \in E$. Now suppose we pick a specific edge $e \in E$ and increase its capacity by one unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of vertices in G.

   Hint: Work on the residual network. How much time does it take to build it for a given flow f ?

4. (**Max-damage to network**)

   You are given a flow network with unit capacity edges: It consists of a directed graph $G = (V, E)$, a source $s \in V$, and a sink $t \in V$; and $c_e = 1$ for every $e \in E$. You are also given a parameter k.

   The goal is to delete k edges so as to reduce the maximum-flow on G by as much as possible. In other words, you should find a set of edges $F \subseteq E$ so that $|F| = k$ and the maximum $s - t$ flow in $G' = (V, E - F)$ is as small as possible subject to this. Give a polynomial time algorithm to solve this problem.

   Hint: For maximum damage, which cut should we pick ?

5. (**unique min-cut**)

   Let $G = (V, E)$ be a directed graph with source $s \in V$, sink $t \in V$ and nonnegative edge capacities $\{c_e\}$. Give a polynomial time algorithm to decide whether G has a unique minimum s-t cut (i.e., an s-t cut of capacity strictly less than that of all other s-t cuts.)

   Hint: Recall the proof we gave for Maxflow-Mincut theorem. Which mincut did we analyse ? What if we carried out analysis from side of t instead of s ?

6. (**Applicants and jobs with multiple vacancies**)
   There are n applicants $A_1, \ldots, A_n$ and m jobs $J_1, \ldots, J_m$. For each $1 \leq i \leq m$, there is a positive integer v_i which denotes the vacancies in job J_i, that is, J_i can be assigned to at most v_i applicants. You are also given a set S of pairs of applicants and jobs such that a pair $(A_i, J_k) \in S$ means that applicant A_i is eligible for job J_k. Note that a job can be given to an applicant if and only if the applicant is eligible for that job. Moreover, an applicant can be assigned to at most one job. You need to design an algorithm that assigns jobs to applicants so that the maximum number of applicants get employed. Instead of designing any arbitrary algorithm for this problem and provide its proof of correctness, you must do the following.

   (a) Reduce the given problem to max-flow problem on a suitably defined graph. For this clearly describe the corresponding graph (vertices, edges, edge capacities).

   (b) State and prove the theorem which connects the instance of the given problem with the instance of the max-flow problem.

   Hint: We analysed this problem during bipartite matching. You will have to update capacities of certain edges in the graph we built.

7. (**Blood bank problem**)

   We all know the basic rule for blood donation: A patient of blood group A can receive only blood of group A or O. A patient of blood group B can receive only blood of group A or O. A patient of blood group O can receive only blood of group O. A patient of blood group AB can receive blood of any group.

   Let s_O, s_A,s_B, s_{AB} denote the supply in whole units of the different blood types on hand. Assume that the hospital knows the projected demand for each blood type d_O,d_A,d_B, and d_{AB} for the coming week. Give a polynomial time algorithm to evaluate if the blood on hand would suffice for the projected need. You should formulate this problem as a max-flow problem, establish a relation between the two problems by stating a theorem, and then you should prove the theorem.

   Hint: Form a bipartite graph suitably, add edges and assign them capacities suitably. Add source and sink along with edges and their capacities suitably.