PP 39 : Divergence Theorem

1. Let D be the solid bounded by $z = 0$ and the paraboloid $z = 4 - x^2 - y^2$. Let S be the boundary of D. If
$$F(x, y, z) = (x^3 + \cos(yz), y^3, x + \sin(xy)),$$
find $\iint_S F \cdot \hat{n} d\sigma$ where $\hat{n}$ is the unit outward normal to the surface S.

2. Let S be the sphere $x^2 + y^2 + z^2 = 1$. Evaluate the surface integral
$$\iint_S [x(2x + 3e^{z^2}) + y(-y - e^{x^2}) + z(2z + \cos^2 y)] d\sigma.$$

3. Let S be the sphere $x^2 + y^2 + (z - 1)^2 = 9$. Find the unit outward normal to the surface S and evaluate the surface integral
$$\iint_S [x^2 \sin y + y \cos^2 x + (z - 1)(y^2 - z \sin y)] d\sigma.$$

4. Let D be the region enclosed by the surfaces $x^2 + y^2 = 4$, $z = 0$ and $z = x^2 + y^2$. Let S be the boundary of D and $\hat{n}$ denote the unit outward normal vector to S. Suppose F is a vector field whose components have continuous first order partial derivatives. If $divF = \alpha(x - 1)$ for some $\alpha \in \mathbb{R}$ and $\iint_S F \cdot \hat{n} d\sigma = \pi$, evaluate α.

5. Let S be the sphere $x^2 + y^2 + z^2 = 1$. Suppose for some $\alpha \in \mathbb{R}$, $\iint_S [zx + \alpha y^2 + xz] d\sigma = \frac{4\pi}{3}$. Find α.

6. Let S be the hemisphere $x^2 + y^2 + z^2 = 1$ and $z \geq 0$. Evaluate $\iint_S [(z + \cos z)x + y^2 + xz] d\sigma$.

Practice Problems 39: Hints/Solutions

1. By divergence theorem
$$\iint_S F \cdot \hat{n} d\sigma = \iiint_D divF dV = \iiint_D 3(x^2 + y^2) dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3r^2 r dz dr d\theta = 32\pi.$$

2. Observe that the given surface integral is $\iint_S F \cdot \hat{n} d\sigma$ where $F(x, y, z) = (2x + 3e^{z^2}, -y - e^{x^2}, 2z + \cos^2 y)$ and $\hat{n} = (x, y, z)$ which is the unit outward normal to the sphere. By divergence theorem $\iint_S F \cdot \hat{n} d\sigma = \iiint_D divF dV = 3 \iiint_D dV = 4\pi$.

3. The given sphere S is $g(x, y, z) = 9$ where $g(x, y, z) = x^2 + y^2 + (z - 1)^2$. The unit normal vector $\hat{n}$ of S is $\frac{\nabla g}{\|\nabla g\|} = \frac{1}{3}(x, y, z - 1)$. Verify that $\hat{n}$ is the unit outward normal vector. The given surface integral is $\iint_S F \cdot \hat{n} d\sigma$ where $F(x, y, z) = (x \sin y, \cos x, y^2 - z \sin y)$. By divergence theorem, $\iint_S F \cdot 3\hat{n} d\sigma = 3 \iiint_D divF dV = 0$.

4. By divergence theorem $\iint_S F \cdot \hat{n} d\sigma = \iiint_D \alpha(x - 1) dV = \alpha \int_0^{2\pi} \int_0^2 \int_0^{r^2} (r \cos\theta - 1) dz r dr d\theta = -8\pi\alpha$. Therefore $\alpha = -\frac{1}{8}$.

5. Let D denote the solid enclosed by the surface S. By divergence theorem, $\iint_S (z, \alpha y, x) \cdot (x, y, z) d\sigma = \iiint_D \alpha dV = \alpha \frac{4\pi}{3}$. Hence $\alpha = 1$.

6. Let $F(x, y, z) = (z + \cos z, y, x)$ and S_1 be the disk $x^2 + y^2 \leq 1$, $z = 0$. Note that S is not a closed surface. Suppose D denotes the solid $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$. By divergence theorem, $\iint_S [(z + \cos z)x + y^2 + xz] d\sigma = \iiint_D divF dV - \iint_{S_1} (z + \cos z, y, x) \cdot (-\hat{k}) d\sigma = \frac{2\pi}{3}$.