Assignment 1
Non-Linear Regression Analysis

1. Suppose $\{x_1, \ldots, x_n\}$ is a set of n real numbers. Find a such that $\sum_{i=1}^{n}(x_i - a)^2$ is minimized.

2. Suppose $\{x_1, \ldots, x_n\}$ is a set of n real numbers. Find a such that $\sum_{i=1}^{n}|x_i - a|$ is minimized.

3. Suppose $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ is a set of n pairs. Find a such that $\sum_{i=1}^{n}|y_i - ax_i|$ is minimized.

4. Consider the following simple linear regression model:

$$y_i = a^0 + b^0 x_i + \epsilon_i; \qquad i = 1, \ldots, n.$$

Here ϵ_i's are i.i.d. normal random variables with mean 0 and variance 1. (a) Find $\widehat{a}$ and $\widehat{b}$, the least squares estimators of a^0 and b^0. (b) As $n \to \infty$, where these quantities $\widehat{a}$ and $\widehat{b}$ will converge? Under what conditions on x_i's $\widehat{a}$ and $\widehat{b}$ will converge to a^0 and b^0 respectively.

5. Consider the same problem as in Problem 4. But it is known that $0 \le a \le 1$ and $0 \le b \le 1$. Answer (a) and (b).

6. Consider the following simple linear regression model

$$y_i = b^0 x_i + \epsilon_i; \qquad i = 1, \ldots, n.$$

Suppose ϵ_i's are i.i.d. normal random variables with mean 0 and variance 1. Find the least absolute deviation estimator of b^0. Find a sufficient condition so that it is consistent, i.e. as $n \to \infty$, $\widehat{b}$ converges to b^0.

7. Consider the following simple linear regression model:

$$y_i = a^0 + b^0 x_i + \epsilon_i; \qquad i = 1, \ldots, n.$$

Here ϵ_i's are i.i.d. random variables with mean 0 and variance 1, it is known that $0 \le a \le 1$ and $0 \le b \le 1$. Let us assume you have enough computer power in your disposal. Can you suggest a method to compute the least absolute deviation estimators of a^0 and b^0.

8. Consider the following simple linear regression model:

$$y_i = a^0 + b^0 x_i + \epsilon_i; \qquad i = 1, \ldots, n.$$

Here ϵ_i's are i.i.d. random variables with mean 0 and variance 1. It is known that $a^0 = 2b^0$, find the least absolute deviation estimator of a.

9. Suppose we have the following observations:

$$y_i = \alpha + \beta i + \epsilon_i; \qquad i = 1, \ldots, 50.$$

(a) Suppose $E(\epsilon_i) = 0$ and $\text{Var}(\epsilon_i) = i^2$, find the least squares estimators of α and β.

(b) Under the same conditions as in (a), find the generalized least squares estimators of α and β. Will you prefer the least squares estimators or the generalized least squares estimators and why?

(c) Suppose you know that $\alpha + \beta = 2$, how do you find the least squares estimators of α and β under the assumptions that $E(\epsilon_i) = 0$ and $\text{Var}(\epsilon_i) = 1$.

10. Consider the following linear model (here ω is a known constant $\in (0, \pi)$)

$$y_t = A\cos(\omega t) + B\sin(\omega t) + \epsilon_t; \qquad t = 1, \ldots N$$

(a) Suppose, $E(\epsilon_t) = 0$ and $\text{Var}(\epsilon_t) = \sigma^2$, find the least squares estimators of A and B.

(b) Find an estimator of σ^2.

(c) Under the assumptions that for large N, $\frac{1}{N}\sum_{t=1}^{N}\cos^2(\omega t) \approx \frac{1}{2}$, $\frac{1}{N}\sum_{t=1}^{N}\sin^2(\omega t) \approx \frac{1}{2}$, $\frac{1}{N}\sum_{t=1}^{N}\sin(\omega t)\cos(\omega t) \approx 0$, find the approximate values of the least squares estimators of A and B for large N.

11. Consider the following linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}.$$

Here $\mathbf{Y}$ is a $2n \times 1$ vector, $\mathbf{X}$ is a $2n \times 3$ matrix, $\boldsymbol{\beta}$ is a 3×1 unknown parameter vector, $\boldsymbol{\epsilon}^T = (\epsilon_1, \ldots, \epsilon_{2n})$ and ϵ_i's are *i.i.d.* $N(\mathbf{0}, \sigma^2)$. The matrix $\mathbf{X}$ is as follows;

$$\mathbf{X}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 2 & -2 & 2 & -2 & \ldots & 2 & -2 & 2 & -2 \\ 3 & 3 & -3 & -3 & \ldots & 3 & 3 & -3 & -3 \end{bmatrix}$$

(a) Find the least squares estimates of $\boldsymbol{\beta}$ and σ^2.

(b) Find the distributions of the least squares estimates of $\boldsymbol{\beta}$ and σ^2.

(c) Construct a 100(1-α)% confidence set of $\boldsymbol{\beta}$.

(d) Show that the least squares estimate of $\boldsymbol{\beta}$ is a consistent estimate of $\boldsymbol{\beta}$.

(e) Find the asymptotic distribution of the least squares estimate of $\boldsymbol{\beta}$.

12. Consider the following regression model:

$$y_t = f_t(a_0, a_1, b_0, b_1) + \epsilon_t, \quad t = 1, 2, \ldots, n,$$

here ϵ_t's are independent and identically distributed random variables with mean zero and variance 1 and

$$f_t(a_0, a_1, b_0, b_1) = \begin{cases} a_0 + a_1 t & \text{if} \quad t < t_0 \\ b_0 + b_1 t & \text{if} \quad t \geq t_0 \end{cases}$$

Here $1 < t_0 < n$, is an integer

(a) If t_0 is known find the least squares estimators of a_0, a_1, b_0 and b_1.

(b) If t_0 is known, and it is known that $a_0 + a_1 t_0 = b_0 + b_1 t_0$, find the least squares estimators of a_0, a_1, b_0 and b_1.

(c) If t_0 is unknown find the least squares estimators of a_0, a_1, b_0, b_1 and t_0.

(d) If t_0 is unknown, but it is known that $a_0 + a_1 t_0 = b_0 + b_1 t_0$, find the least squares estimators of a_0, a_1, b_0, b_1 and t_0.

(e) I want to construct 95% confidence interval of a_0/b_0 how will you do it?

Assignment 2
Non-Linear Regression Analysis

1. Consider the following non-linear regression model;

$$y(t) = \frac{\alpha_1 + \alpha_2 t}{\beta_1 + \beta_2 t} + e(t); \qquad t = 1, \ldots, n.$$

Here $e(t)$'s are independent identically distributed normal random variables with mean zero and finite variance σ^2.
(a) If it is known that $\beta_1 = 1$ and $\beta_2 = 2$, find the maximum likelihood estimators of α and α_2
(b) Provide an algorithm to find the maximum likelihood estimators of α_1, α_2, β_1, and β_2 if all are unknown.
(c) Find the maximum likelihood estimators of σ^2.
(d) Do you think the maximum likelihood estimators will be consistent estimators of the unknown parameters or not?

2. Consider the following non-linear regression model:

$$y_t = \frac{1}{\theta_0 + \theta_1 t} + \epsilon_t; \quad t = 1, \ldots, n.$$

Here θ_0 and θ_1 are unknown parameters and ϵ_t's are *i.i.d.* random variables with mean zero and variance σ^2.

(a) Provide the Newton-Raphson algorithm to compute the least squares estimate of θ_0 and θ_1.
(b) Suppose $\theta_0 = 1$ (known) and we want to estimate θ_1 only. We have only two observations 0.51 and 0.31 at $t = 1$ and at $t = 2$ respectively. We are using the Newton-Raphson algorithm to compute the least squares estimate of θ_1. If at the $i - th$ iterate the estimated value is 1.0, what will be the value at the $(i + 1)$-th iterate. Explain it graphically also.
(c) If both θ_0 and θ_1 are unknown, are the least squares estimates consistent? Justify your answer.
(d) If $\theta_1 = 0$, find a consistent estimate of θ_0 (not necessarily least squares estimate) and show that it is consistent.
(e) If $\theta_0 = 0$, find a consistent estimate of θ_1 (not necessarily least squares estimate) and show that it is consistent.

3. Consider the following non-linear regression model;

$$y_t = \cos(\omega_0 t) + \sin(\omega_0 t) + e_t; \qquad t = 1, \ldots N$$

Here e_t's are i.i.d. normal random variables with mean zero and variance 1. Find the least squares estimator of ω_0 using Newton-Raphson and also Gauss-Newton method.

4. Consider the following linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}.$$

Here $\mathbf{Y} = (1, \ldots, 1)$ is a $2n \times 1$ vector, $\mathbf{X}$ is a $2n \times 2$ matrix, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is a 2×1 unknown parameter vector, $\boldsymbol{\epsilon}^T = (\epsilon_1, \ldots, \epsilon_{2n})$ and ϵ_i's are i.i.d. $N(0, \sigma^2)$ random variables. The matrix $\mathbf{X}$ is as follows;

$$\mathbf{X}^T = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ 2 & 2^2 & \ldots & 2^{2n-1} & 2^{2n} \end{bmatrix}$$

(a) Find explicitly the maximum likelihood estimates of $\boldsymbol{\beta}$ and σ^2.

(b) Find the distributions of the maximum likelihood estimates of $\boldsymbol{\beta}$ and σ^2.

(c) Construct a 95% confidence set of $\boldsymbol{\beta}$.

(d) Show that the maximum likelihood estimate of $\boldsymbol{\beta}$ is a consistent estimate of $\boldsymbol{\beta}$.

(e) Find the asymptotic distribution of the maximum likelihood estimate of $\boldsymbol{\beta}$.

(f) If we know $\beta_1 = \beta_2 = \beta$, find the maximum likelihood estimate of β.

(g) Will it be consistent?

5. Suppose we have the following observations:

$$y_i = \alpha + \beta i + \epsilon_i; \qquad i = 1, \ldots, 50.$$

(a) Suppose, $E(\epsilon_i) = 5$ and $\text{Var}(\epsilon_i) = 1$, find the least squares estimators of α and β.

(b) Suppose $E(\epsilon_i) = 0$ and $\text{Var}(\epsilon_i) = i^2$, find the least squares estimators of α and β.

(c) Under the same conditions as in (b), find the generalized least squares estimators of α and β. Will you prefer the least squares estimators or the generalized least squares estimators and why?

(d) Suppose you know that $\alpha + \beta = 2$, how do you find the least squares estimators of α and β under the assumptions that $E(\epsilon_i) = 0$ and $\text{Var}(\epsilon_i) = 1$.

6. Consider the following linear model (here ω is a known constant $\in (0, \pi)$)

$$y_t = A\cos(\omega t) + B\sin(\omega t) + \epsilon_t; \qquad t = 1, \ldots N$$

(a) Suppose, $E(\epsilon_t) = 0$ and $\text{Var}(\epsilon_t) = \sigma^2$, find the least squares estimators of A and B.

(b) Find an estimator of σ^2.

(c) Under the assumptions that for large N, $\frac{1}{N}\sum_{t=1}^{N}\cos^2(\omega t) \approx \frac{1}{2}$, $\frac{1}{N}\sum_{t=1}^{N}\sin^2(\omega t) \approx \frac{1}{2}$, $\frac{1}{N}\sum_{t=1}^{N}\sin(\omega t)\cos(\omega t) \approx 0$, find the approximate values of the least squares estimators of A and B for large N.

7. Consider the following non-linear regression model;

$$y(x_t) = e^{Ax_t} + \epsilon_t; \qquad t = 1, \ldots, N$$

Explain how do you find the the least squares estimators of A under the assumptions that $E(\epsilon_t) = 0$ and $\text{Var}(\epsilon_t) = 1$.

Assignment 3
Non-Linear Regression Analysis

1. Consider the following non-linear regression model;

$$y(x_t) = e^{Ax_t} + \epsilon_t; \qquad t = 1, \ldots, N$$

Explain how do you find the the least squares estimators of A under the assumptions that $E(\epsilon_t) = 0$ and $\mathrm{Var}(\epsilon_t) = 1$.

2. Consider the following non-linear regression model:

$$y(t) = \frac{2}{1 + \theta x_t} + e(t); \qquad t = 1, \ldots, N.$$

Here $e(t)$'s are i.i.d. random variables with the following probability density function

$$f(x) = c|x|^2 e^{-|x|^3}; \quad -\infty < x < \infty, c > 0.$$

(a) Find c, so that $f(x)$ becomes a proper probability density function.

(b) Based on the observed sample $\{y(1), \ldots, y(n)\}$, find the likelihood function of θ.

(c) Provide an explicit method to compute the maximum likelihood estimator of θ, starting from an initial guess value of θ, say $\theta^{(0)}$.

(d) Construct 95% confidence interval of e^{θ}.

3. Consider the following general non-linear regression model

$$y_t = Af_t(\theta) + \epsilon_t; \qquad t = 1, \ldots n.$$

Here A and θ are scalars, and $f_t(\theta)$ is a twice differentiable function for all t. The error random variables ϵ_t are *i.i.d.* with the probability density function $g(x)$, where

$$g(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad -\infty < x < \infty.$$

(a) If θ is known, find a method to compute the maximum likelihood estimate of A.

(b) If both are unknown find a method to compute the maximum likelihood estimates of A and θ.

(c) In case of (b), how to construct 95% confidence intervals for A and θ?

4. Consider the following non-linear regression model:

$$y(t) = \frac{\alpha t}{1 + \beta t} + \epsilon_t; \qquad t = 1, \ldots N.$$

Here ϵ_t's are normally distributed with mean 0 and variance t^2.

(a) Find the least squares estimator of α when β is known.

(b) Show that when both α and β are unknown the least squares estimators of α and β can be obtained by solving a one dimensional optimization problem.

(c) Suggest an suitable algorithm to compute the least squares estimators of α and β.
on β.

Assignment 4
Non-Linear Regression Analysis

In all the following questions we are considering the non-linear/ linear regression model of the form:

$$y_t = f_t(\theta^0) + \epsilon_t; \quad t = 1, \ldots, n.$$

Here $f_t(\theta)$ is continuous in $\theta \in \Theta \subset R^p$, θ^0 is an interior point of Θ, ϵ_t's are i.i.d. random variables with mean 0 and finite variance. Further

$$D_n(\theta_1, \theta_2) = \frac{1}{n}\sum_{t=1}^{n}(f_t(\theta_1) - f_t(\theta_2))^2 \overset{uniformly}{\longrightarrow} D(\theta_1, \theta_2) = 0 \text{ iff } \theta_1 = \theta_2.$$

1. Consider the following non-linear regression model;

$$y(x_t) = e^{A^0 t} + \epsilon_t; \quad t = 1, \ldots, N$$

Here $A^0 \in (-1, 0)$, ϵ_t's are i.i.d. random variables as above. Does this model satisfy the above conditions?

2. Consider the following non-linear regression model:

$$y(t) = \frac{2}{1 + \theta^0 t} + e(t); \quad t = 1, \ldots, N.$$

Here $e(t)$'s are i.i.d. random variables with the following probability density function

$$f(x) = c|x|^2 e^{-|x|^3}; \quad -\infty < x < \infty, c > 0.$$

Here $\theta^0 \in (0, 100)$, ϵ_t's are i.i.d. random variables as above. Does this model satisfy the above conditions?

3. Consider the following general non-linear regression model

$$y_t = A^0 \cos(\theta^0 t) + B^0 \sin(\theta^0 t) + \epsilon_t; \quad t = 1, \ldots n.$$

Here $A^0, B^0 \in (-10, 10)$ and $\theta^0 \in (0, \pi)$. ϵ_t's are i.i.d. random variables as above. Does this model satisfy the above conditions?

4. Consider the following linear regression model:

$$y(t) = \alpha^0 + \beta^0 t + \epsilon_t; \quad t = 1, \ldots N.$$

Here $\alpha^0, \beta^0 \in (-10, 10)$, ϵ_t's are i.i.d. normally distributed with mean 0 and variance 1. Does this model satisfy the above conditions? Is it possible to obtain consistent estimators of α^0 and β^0 in this case?