lecture 11 (Zeros of solutions)

Recall: (Sturm Comparison Theorem)

$$y'' + Q_1(x)\, y = 0$$
$$y'' + Q_2(x)\, y = 0$$

Q_1, Q_2 are continuous functions on some interval I

Assume $Q_1(x) \le Q_2(x) \ \forall\ x \in I$.

& $Q_1 \not\equiv Q_2$

Let φ_1 be a non-trivial sol of 1st eq.

φ_2 . - - - - - - - 2nd - - .

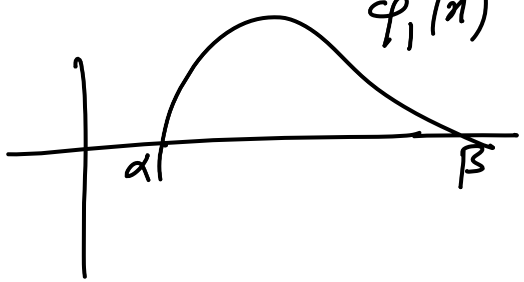
Then between two consecutive zeros of φ_1, $\exists$ at least one zero of φ_2.

Proof Let α, β be two consecutive zeros of φ_1

WLOG assume $\varphi_1(x) > 0 \ \forall\ \alpha < x < \beta$

$\varphi_1'(\alpha) > 0$

$\varphi_1'(\beta) < 0$



Assume $\varphi_2(x) \neq 0 \ \forall\ \alpha < x < \beta$

$\varphi_2(x) > 0 \ \forall\ \alpha < x < \beta$

(if not then replace φ_2 by $-\varphi_2$)

$$\varphi_1'' + Q_1 \varphi_1 = 0 \quad — \quad (1)$$

$$\varphi_2'' + Q_2 \varphi_2 = 0 \quad — \quad (2)$$

$(1) \times \varphi_2 - (2) \times \varphi_1$

$$\underline{(\varphi_1''\varphi_2 - \varphi_1\varphi_2'')} + (Q_1 - Q_2)\varphi_1\varphi_2 = 0$$

$$-\frac{dW}{dx} + (Q_1 - Q_2)\varphi_1\varphi_2 = 0$$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$\frac{dW}{dx} = \underbrace{(Q_1 - Q_2)}_{\leq 0}\underbrace{\varphi_1}_{>0}\underbrace{\varphi_2}_{>0}$$

for $\alpha < x < \beta$

$$\frac{dW}{dx} \leq 0 \quad \forall \; \alpha < x < \beta.$$

with strict inequality for some point.

integrating from α to β

$$W(\beta) - W(\alpha) < 0$$

$$W(\beta) < W(\alpha)$$

$$W(\alpha) = -\underbrace{\varphi_2(\alpha)}_{\geq 0}\underbrace{\varphi_1'(\alpha)}_{>0} \leq 0$$

$$W(\beta) = -\underbrace{\varphi_2(\beta)}_{\geq 0}\underbrace{\varphi_1'(\beta)}_{<0} \geq 0$$

This contracts ~~the~~ ~~above~~ the inequality

$$W(\beta) < W(\alpha)$$

$\blacksquare$

Normal form

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad \text{---} \quad (**)$$

Can be transformed into

$$u'' + Q(x)\, u = 0 \quad \text{---} \quad (***)$$

$\underbrace{\qquad\qquad}$ normal form of $(**)$.

To do this put $y = u \cdot v$. in $(**)$

$$u''v + u'\underbrace{(2v' + pv)}_{0} + u\,\underbrace{\left(v'' + p(x)v' + q\,v\right)}_{Q} = 0$$

$$\frac{v'}{v} = -\frac{p}{2} \Rightarrow \boxed{v = e^{-\int \frac{p(x)}{2}\,dx} > 0}$$

$$\boxed{Q = \frac{v'' + p\,v' + qv}{v} = q - \frac{p'}{2} - \frac{p^2}{4}}$$

Example (Bessel equation)

$$x^2 y'' + x y' + (x^2 - \nu^2)\, y = 0 \qquad x > 0.$$

$$p = \frac{1}{x} \qquad q = \frac{x^2 - \nu^2}{x^2}$$

Normal form

$$u'' + \left(1 + \frac{\frac{1}{4} - \nu^2}{x^2}\right) u = 0$$

Claim Any non-trivial solution of the Bessel equation has infinitely many positive zero.

$\exists\, x_0 > 0$ s.t

$$\underbrace{1 + \frac{\frac{1}{4} - \nu^2}{x^2}}_{Q_2} > \underbrace{\frac{1}{4}}_{Q_1} \quad \forall\, x \geq x_0$$

$$u'' + \frac{1}{4} u = 0$$

$\varphi_1 = \sin(x/2)$

$\varphi_2 = J_\gamma(x)$ – solntn of Bessel eqntn.

$\varphi_1(x)$ vanish infinitely times for $x > x_0$.

So by Sturm comparison $J_\gamma(x)$ also will have infinitely many positive zeros

Remark

$$y'' + p(x)\, y' + q_1(x)\, y = 0$$

$$y'' + p(x)\, y' + q_2(x)\, y = 0$$

with $q_1 \leq q_2$ and $q_1 \not\equiv q_2$

Then φ_1 non-trivial solntn of 1st eqntn

φ_2 " - - - - 2nd.

Then between two consecutive zeros of φ_1, $\exists$ at least one zero of φ_2.

$$Q_1 = q_1 - \frac{p'}{2} - \frac{p^2}{4}$$

$$Q_2 = q_2 - \frac{p'}{2} - \frac{p^2}{4}$$

$$Q_1 \leq Q_2$$

Boundary Value problem

$$y'' + p(x)y' + q(x)y = r(x)$$

$$a \le x \le b.$$

Boundary conditions

- $y(a) = \eta_1 , \; y(b) = \eta_2$ — Dirichlet
- $y'(a) = \eta_1 \quad y'(b) = \eta_2$ — Neuman
- $\alpha_1 y(a) + y'(a) = \eta_1$
  $\beta_1 y(b) + y'(b) = \eta_2$ — mixed
- $y(a) = y(b) \quad y'(a) = y'(b)$ — Periodic

No Existance & Uniqness for BVP

Example

$$y'' + y = 0 \qquad 0 \le x \le \pi$$

$$y = A\cos x + B\sin x$$

- $y(0) = 1 \quad y'(\pi) = 1$ → Unique solution $y = \cos x + \sin x$
- $y'(0) = 1 \quad y'(\pi) = -1$ → Infinite solution
- $y'(0) = 1 \quad y'(\pi) = 1$ → no - solution.

Sturm-Liouville BVP

$$\underbrace{\frac{d}{dx}\left(p(x)y'\right) + q(x)y}_{L(y)} + \lambda r(x) y = 0$$

$a \le x \le b.$

boundary condition

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

$(a_1, a_2) \neq (0\ 0)$ $\quad (b_1, b_2) \neq (0\ 0)$

The problem is to find the values of λ for which the BVP has non-triv) solutions. ~~Suc~~

Defn Such a λ is called an eigenvalue & the corresponding non-trivial solutions are called eigen-function.

Example $y'' + \lambda y = 0.$ $\quad y(0) = 0$, $y'(\pi) = 0$

- $\lambda < 0$ $\quad \lambda = -\mu^2$ $\quad \mu > 0$

$$y(x) = A e^{\mu x} + B e^{-\mu x}$$

$y(0) = 0 \Rightarrow A + B = 0$

$y'(\pi) = 0 \Rightarrow A\mu e^{\mu\pi} - B\mu e^{-\mu\pi} = 0$

$$A\mu\left(e^{\mu\pi} + e^{-\mu\pi}\right) = 0.$$

$$A\left(\underbrace{e^{\mu\pi} + e^{-\mu\pi}}_{\neq 0}\right) = 0$$

$A = 0$ i.e. $y \equiv 0$

Thus $\lambda < 0$ is not an eigenvalue.

- $\lambda = 0$ $\quad y'' = 0$

$y = Ax + B$ $\quad \left.\begin{array}{l} y(0) = 0 \\ y'(\pi) = 0 \end{array}\right\} \Rightarrow A = B = 0$

So $\lambda = 0$ is NOT an eigenvalue.

- $\lambda > 0$ $\quad \lambda = \mu^2$ — $y'' + \mu^2 y = 0$

$y = A\cos(\mu x) + B\sin(\mu x)$

$0 = A \quad (\Leftarrow y(0) = 0)$

$y'(\pi) = 0 \Rightarrow \quad B\cos(\mu\pi) = 0$

To have non-trivial solution, we must have $\cos(\mu\pi) = 0$

$\mu\pi = \frac{2n+1}{2}\pi \qquad n = 0, \pm 1, \pm 2 \ldots$

$\mu = \frac{2n+1}{2}$

Then $\lambda_n = \left(\frac{2n+1}{2}\right)^2$ — are eigenvalues

and Eigen functions are

$\varphi_n(x) = B\sin\left(\frac{2n+1}{2}x\right)$. $\blacksquare$

Example: $y'' + \lambda y = 0$ $\qquad y(0) = y(\pi)$

$y'(0) = y'(\pi).$

- $\lambda < 0$ $\qquad \lambda = -\mu^2 \Rightarrow$ trial solution — not eigenvalue.

- $\lambda = 0$ $\qquad y = Ax + B$

  $y = A$ — eigen function.

  $\lambda = 0$ is an eigenvalue

- $\lambda > 0$ $\qquad \lambda = \mu^2.$

  $\lambda_n = 4n^2$ are eigenvalues

  Eigen function: $\begin{cases} \varphi_n(x) = \cos(2nx) \\ \psi_n(x) = \sin(2nx) \end{cases}$

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