MTH302: *Set Theory and Mathematical Logic*
Homework: These will be periodically updated. Odd Semester, 2024/24

(1) Show that there is no set V such that every set is a member of V.

Solution: Suppose not and let V be a set such that every set is a member of V. Define $W = \{x \in V : x \notin x\}$. Then W is a set by the axiom of comprehension. Since W is a set, either $W \in W$ or $W \notin W$. If $W \in W$, then since $W \in V$, we must have $W \notin W$. Similarly, if $W \notin W$, then $W \in W$. In either case we get a contradiction. Hence V does not exist. □

(2) Show that $(x, y) = (a, b)$ iff $x = a$ and $y = b$.

Solution: The right to left implication is obvious. So assume $(x, y) = (a, b)$ and we'll show $x = a$ and $y = b$. We consider two cases.

Case $x = y$: In this case, $(x, y) = \{\{x\}, \{x, y\}\} = \{\{x\}\}$. Hence $(a, b) = \{\{a\}, \{a, b\}\} = \{\{x\}\}$. It follows that $\{x\} = \{a\} = \{a, b\}$. So $x = a = b$. Hence $x = y = a = b$.

Case $x \neq y$: In this case, $\{\{x\}, \{x, y\}\}$ is a set with two distinct members. It follows that $\{\{a\}, \{a, b\}\}$ is also a set with two distinct members. So $a \neq b$. Now $\{x\} \in \{\{a\}, \{a, b\}\}$ implies $\{x\} = \{a\}$ since $\{x\} \neq \{a, b\}$ (as the latter has two distinct members). Similarly, $\{x, y\} = \{a, b\}$. As $x = a$, and $y \neq x$, we get $y = b$. □

(3) Suppose R is an equivalence relation on A. For each $a \in A$, define the R-equivalence class of a by $[a] = \{b \in A : aRb\}$. Show that $\{[a] : a \in A\}$ is a partition of A. Furthermore, show that for every partition $\mathcal{F}$ of A, there is an equivalence relation S on A such that $\mathcal{F}$ is the set of all S-equivalence classes.

Solution: To show that $\{[a] : a \in A\}$ is a partition of A, we need to show that $\bigcup\{[a] : a \in A\} = A$ and for any two distinct R-equivalence classes $[a], [b]$, we must have $[a] \cap [b] = \emptyset$.

Since R is a reflexive relation on A, for every $a \in A$, $a \in [a]$. Hence $A \subseteq \bigcup\{[a] : a \in A\}$. As $[a] \subseteq A$ for every $a \in A$, we also have $\bigcup\{[a] : a \in A\} \subseteq A$. Thus $\bigcup\{[a] : a \in A\} = A$.

Next, towards a contradiction, suppose $a, b \in A$, $[a] \neq [b]$ and $[a] \cap [b] \neq \emptyset$. Fix $c \in [a] \cap [b]$. Since $c \in [a]$, we get aRc. Similarly, bRc. Since R is symmetric, it follows that cRb. Since aRc and cRb, using the fact that R is transitive, we get aRb and hence also bRa (as R is symmetric). We now claim the following.

$[a] \subseteq [b]$: Fix $x \in [a]$. Then aRx. As bRa, by transitivity of R, we get bRx. Hence $x \in [b]$. So $[a] \subseteq [b]$.

$[b] \subseteq [a]$: Fix $y \in [b]$. Then bRy. As aRb, by transitivity of R, we get aRy. Hence $y \in [a]$. So $[b] \subseteq [a]$.

It follows that $[a] = [b]$ which contradicts our assumption that $[a] \neq [b]$. This finishes the proof that $\{[a] : a \in A\}$ is a partition of A.

Now fix a partition $\mathcal{F}$ of A and define a relation S on A as follows. For $a, b \in A$, aSb iff there exists $E \in \mathcal{F}$ such that both a and b are members of E.

Let us first check that S is an equivalence relation on A. It is clear that S is a symmetric relation on A. Since $\bigcup \mathcal{F} = A$, it follows that S is reflexive. Next suppose aSb and bSc. Fix E, F in $\mathcal{F}$ such that $a, b \in E$ and $b, c \in F$. Since $\mathcal{F}$ has pairwise disjoint members and since $E \cap F \neq \emptyset$, we must have $E = F$. Hence aSc. So S is transitive. It follows that S is an equivalence relation on A.

Finally, let us check that the set $\{[a] : a \in A\}$ of S-equivalence classes is equal to $\mathcal{F}$. Let $[a]$ be an S-equivalence class. Fix $E \in \mathcal{F}$ such that $a \in E$. Then by the definition of S, it follows that $[a] = \{b \in A : aSb\} = \{b \in A : b \in E\} = E$. Conversely, if $E \in \mathcal{F}$, then for every $a \in E$, $[a] = E$. Hence $\{[a] : a \in A\} = \mathcal{F}$. □

(4) Suppose $X \subseteq \omega$ satisfies $0 \in X$ and $(\forall y \in X)(y \cup \{y\} \in X)$. Show that $X = \omega$.

Solution. By assumption, X is inductive and ω is the intersection of all inductive sets, so $\omega \subseteq X$. As $X \subseteq \omega$, $X = \omega$. □

(5) Let $(L, \prec)$ be a linear ordering. Prove the following.

(a) $(L, \prec)$ is a well-ordering iff there is no sequence $\langle x_n : n < \omega \rangle$ in L such that $(\forall n < \omega)(x_{n+1} \prec x_n)$.

(b) $(L, \prec)$ is a well-ordering iff for every $A \subseteq L$, $(A, \prec)$ is isomorphic to an initial segment of $(L, \prec)$.

Solution: (a) First suppose that $(L, \prec)$ is a well-ordering. We'll show that there is no $\prec$-decreasing sequence in L. Towards a contradiction, suppose there is a sequence $\langle x_n : n < \omega \rangle$ in L such that for every $n < \omega$, $x_{n+1} \prec x_n$. Let $A = \{x_n : n < \omega\}$ be the range of this sequence. Then A has no $\prec$-least member which contradicts the fact that $(L, \prec)$ is a well-ordering.

Now suppose $(L, \prec)$ is not a well-ordering and fix a nonempty $A \subseteq L$ such that A does not have a $\prec$-least member. We'll construct a $\prec$-decreasing sequence $\langle x_n : n < \omega \rangle$ in L. Using the axiom of choice, fix a choice function $F : \mathcal{P}(A) \setminus \{\emptyset\} \to A$. So for every nonempty $W \subseteq A$, $F(W) \in W$. By recursion on $n < \omega$, define $\langle x_n : n < \omega \rangle$ as follows. $x_0 = F(A)$ and for every $n < \omega$,

$$x_{n+1} = F\left(\{x \in A : x \prec x_n\}\right)$$

Note that this is well-defined since $\{x \in A : x \prec a_n\}$ is nonempty (as A has no $\prec$-least member). It is clear that $\langle x_n : n < \omega \rangle$ is as required. □

(b) First suppose $(L, \prec)$ is a well-ordering. Fix $A \subseteq L$. We'll construct an isomorphism from $(A, \prec)$ to an initial segment of $(L, \prec)$. Define

$$f = \{(x, a) \in L \times A : (\mathsf{pred}(L, \prec, x), \prec) \cong (\mathsf{pred}(A, \prec, a), \prec)\}$$

(i) f is a function: Clearly, f is a relation. To see that it is a function, fix $(x, a), (x, b) \in f$ and we'll show that $a = b$. Towards a contradiction suppose $a \neq b$. Without loss of generality suppose $a \prec b$. Since (x, a) and (x, b) are both in f, we get $(\mathsf{pred}(L, \prec, x), \prec) \cong (\mathsf{pred}(A, \prec, a), \prec)$ and $(\mathsf{pred}(L, \prec, x), \prec) \cong (\mathsf{pred}(A, \prec, b), \prec)$. Hence $(\mathsf{pred}(A, \prec, a), \prec) \cong (\mathsf{pred}(A, \prec, b), \prec)$. But this means that $(\mathsf{pred}(A, \prec, b), \prec)$ is a well-ordering that is isomorphic to a proper initial segment of itself. Contradiction. So f is a function.

(ii) f is injective: The proof is similar to (i) above.

(iii) $\mathsf{dom}(f)$ is an initial segment of $(L, \prec)$: Suppose $x \in \mathrm{dom}(f)$ and $y \prec x$. We need to show that $y \in \mathrm{dom}(L)$. Let $f(x) = a$. Fix an isomorphism $h : (\mathsf{pred}(L, \prec, x), \prec) \to (\mathsf{pred}(A, \prec, a), \prec)$. Note that $y \in \mathrm{dom}(h)$. Let $h(y) = b$. It is clear that $h \restriction \mathsf{pred}(L, \prec, y)$ is an isomorphism from $(\mathsf{pred}(L, \prec, y), \prec)$ to $(\mathsf{pred}(A, \prec, b), \prec)$. Hence $(y, b) \in f$ and so $y \in \mathrm{dom}(f)$.

(iv) $\mathsf{range}(f)$ is an initial segment of $(A, \prec)$: The proof is similar to (iii) above.

(v) f is an isomorphism from $(\mathsf{dom}(f), \prec)$ to $(\mathsf{range}(f), \prec)$: Suppose $x \prec y$ are in $\mathrm{dom}(f)$. Put $a = f(x)$ and $b = f(y)$. Using the definition of f, it follows that $(\mathsf{pred}(L, \prec, x), \prec) \cong (\mathsf{pred}(A, \prec, a), \prec)$ and $(\mathsf{pred}(L, \prec, y), \prec) \cong (\mathsf{pred}(A, \prec, b), \prec)$. As $x \prec y$, it follows that $(\mathsf{pred}(A, \prec, a), \prec)$ is isomorphic to an initial segment of $(\mathsf{pred}(A, \prec, b), \prec)$. Since no well-ordering can be isomorphic to a proper initial-segment of itself, it follows that $a \prec b$. So f is an isomorphism from $(\mathrm{dom}(f), \prec)$ to $(\mathrm{range}(f), \prec)$.

(vi) $\mathsf{range}(f) = A$: Suppose not. Let $a = \min(A \setminus \mathrm{range}(f))$. Since $\mathrm{range}(f)$ is an initial segment of $(A, \prec)$, it follows that $\mathrm{range}(f) = \mathsf{pred}(A, \prec, a)$. We claim that $\mathrm{dom}(f) = L$. For suppose not and let $x = \min(L \setminus \mathrm{dom}(f))$. Then $\mathrm{dom}(f) = \mathsf{pred}(L, \prec, x)$. But this implies that $(a, b) \in f$ using (i)-(v) above which is a contradiction. So $\mathrm{dom}(f) = L$. Hence $a \in \mathrm{dom}(f)$. Now observe that $f(a) \prec a$ (since $\mathrm{range}(f) = \mathsf{pred}(A, \prec, a)$) and iteratively applying f, we get $a \succ f(a) \succ f(f(a)) \succ \dots$. But this means that $(L, \prec)$ has an infinite $\prec$-descending sequence which is impossible by part (a).

(i)-(vi) imply that $(A, \prec)$ is isomorphic (via f^{-1}) to an initial segment of $(L, \prec)$ (namely $\mathrm{dom}(f)$).

Next we show the converse. Suppose for every $A \subseteq L$, $(A, \prec)$ is isomorphic to an initial segment of $(L, \prec)$. We'll show that $(L, \prec)$ must be a well-ordering. We can assume that $L \neq \emptyset$. Let $a \in L$. Then $(\{a\}, \prec)$ is isomorphic to an initial segment f $(L, \prec)$. This implies that L has a $\prec$-least element, say x. Now let $A \subseteq L$ be nonempty and fix an isomorphism $f : (A, \prec) \to (W, \prec)$ where W is an initial segment of $(L, \prec)$. Note that $x \in W$. Put $a = f^{-1}(x)$. Then a is the $\prec$-least element of A. It follows that $(L, \prec)$ is a well-ordering. $\square$

(6) Suppose $(X, \prec_1)$ and $(Y, \prec_2)$ are well-orderings. Then exactly one of the following holds.

(a) $(X, \prec_1) \cong (Y, \prec_2)$.

(b) For some $x \in X$, $(\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (Y, \prec_2)$.

(c) For some $y \in Y$, $(\mathsf{pred}(Y, \prec_2, y), \prec_2) \cong (X, \prec_1)$.

Furthermore, in each of the three cases, the isomorphism is unique.

Solution: Define

$$f = \{(a, b) \in X \times Y : (\mathsf{pred}(X, \prec_1, a), \prec_1) \cong (\mathsf{pred}(Y, \prec_2, b), \prec_2)\}$$

(i) f is a function: Clearly, f is a relation. To see that it is a function, fix $(a, b), (a, c) \in f$ and we'll show that $b = c$. Towards a contradiction suppose $b \neq c$. Without loss of generality suppose $b \prec_2 c$. Since (a, b) and (a, c) are both in f, we get $(\mathsf{pred}(X, \prec_1, a), \prec_1) \cong (\mathsf{pred}(Y, \prec_2, b), \prec_2)$ and $(\mathsf{pred}(X, \prec_1, a), \prec_1) \cong (\mathsf{pred}(Y, \prec_2, c), \prec_2)$. Hence $(\mathsf{pred}(Y, \prec_2, c), \prec_2) \cong (\mathsf{pred}(Y, \prec_2, b), \prec_2)$. But this means that $(\mathsf{pred}(Y, \prec_2, c), \prec_2)$ is a well-ordering that is isomorphic to a proper initial segment of itself. Contradiction. So f is a function.

(ii) f is injective: The proof is similar to (i) above.

(iii) $\mathsf{dom}(f)$ is an initial segment of $(X, \prec_1)$: Suppose $x \in \mathrm{dom}(f)$ and $y \prec_1 x$. We need to show that $y \in \mathrm{dom}(X)$. Let $f(x) = a$. Fix an isomorphism $h : (\mathsf{pred}(X, \prec_1, x), \prec_1) \to (\mathsf{pred}(Y, \prec_2, a), \prec_2)$. Note that $y \in \mathrm{dom}(h)$. Let $h(y) = b$. It is clear that $h \restriction \mathsf{pred}(X, \prec_1, y)$ is an isomorphism from $(\mathsf{pred}(X, \prec_1, y), \prec_1)$ to $(\mathsf{pred}(Y, \prec_2, b), \prec_2)$. Hence $(y, b) \in f$ and so $y \in \mathrm{dom}(f)$.

(iv) $\mathsf{range}(f)$ is an initial segment of $(A, \prec)$: The proof is similar to (iii) above.

(v) f is an isomorphism from $(\mathsf{dom}(f), \prec_1)$ to $(\mathsf{range}(f), \prec_2)$: Suppose $x \prec_1 y$ are in $\mathrm{dom}(f)$. Put $a = f(x)$ and $b = f(y)$. Using the definition of f, it follows that $(\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (\mathsf{pred}(Y, \prec_2, a), \prec_2)$ and $(\mathsf{pred}(X, \prec_1, y), \prec_1) \cong (\mathsf{pred}(Y, \prec_2, b), \prec_2)$. As $x \prec_1 y$, it follows that $(\mathsf{pred}(Y, \prec_2, a), \prec_2)$ is isomorphic to an initial segment of $(\mathsf{pred}(Y, \prec_2, b), \prec_2)$. Since no well-ordering can be isomorphic to a proper initial-segment of itself, we must have $a \prec_2 b$. So f is an isomorphism from $(\mathrm{dom}(f), \prec_1)$ to $(\mathrm{range}(f), \prec_2)$.

(vi) Either $\mathsf{dom}(f) = X$ or $\mathsf{range}(f) = A$: Suppose not. Let x be the $\prec_1$-least member of $X \setminus \mathrm{dom}(f)$ and let a be the $\prec_2$-least member of $Y \setminus \mathrm{range}(f)$. Since $\mathrm{range}(f)$ is an initial segment of $(Y, \prec_2)$, it follows that $\mathrm{range}(f) = \mathsf{pred}(Y, \prec_2, a)$. Similarly, $\mathrm{dom}(f) = \mathsf{pred}(X, \prec_1, x)$. But now $(x, a) \in f$ using (i)-(v) above which is a contradiction.

If $\mathrm{dom}(f) = X$ and $\mathrm{range}(f) = Y$, we get clause (a). If $\mathrm{dom}(f) \neq X$ and $\mathrm{range}(f) = Y$, we get clause (b). If $\mathrm{dom}(f) = X$ and $rng(f) \neq Y$, we get clause (c).

The uniqueness part follows from the fact that the only isomorphism from a well-ordering to itself is the identity function. $\square$

(7) Suppose X is a nonempty set and F is a choice function on $\mathcal{P}(X) \setminus \{\emptyset\}$. Let $(X, \prec_1)$ and $(X, \prec_2)$ be two well-orderings satisfying $(\forall y \in X)(F(X \setminus \mathsf{pred}(X, \prec_k, y))) = y$ for $k \in \{1, 2\}$. Show that $(\forall a, b \in X)(a \prec_1 b \iff a \prec_2 b)$.

Solution. Observe that both $(X, \prec_1)$ and $(X, \prec_2)$ are F-directed well-orderings. So by Claim (2) in the proof of Zermelo's well-ordering theorem (finish that proof), either $(X, \prec_1) = (X, \prec_2)$ or one of these is a proper initial segment of the other. Since the latter case is impossible, we must have $\prec_1 = \prec_2$. $\square$

(8) Let $f : \mathcal{P}(\omega) \setminus \{\emptyset\} \to \omega$ be defined by $f(X) = \min(X)$. Call a well-orderings $(A, \prec)$ f-directed iff $A \subseteq \omega$ and for every $x \in A$,

$$f(\omega \setminus \mathsf{pred}(A, \prec, x)) = x$$

Describe all f-directed well-orderings.

Solution: It is clear that each well-ordering in $\{(\alpha, <) : \alpha \leq \omega\}$ is f-directed. Let us show that there is no other f-directed well-ordering. Suppose $(A, \prec)$ is an f-directed well-ordering. First suppose that A is finite (and nonempty) and let $x_0 \prec x_1 \prec \cdots \prec x_n$ list the members of A where $n < \omega$. Then an easy induction on $k \leq n$ shows that $x_k = k$. Next suppose that A is infinite. Let $\mathsf{type}(A, \prec) = \alpha$. So $\alpha \geq \omega$. Let $\langle x_\beta : \beta < \alpha \rangle$ be an order isomorphism from α to $(A, \prec)$. Once again by induction on $n < \omega$, we get $x_n = n$. Since $A \subseteq \omega$, it follows that $\alpha = \omega$ and hence $(A, \prec) = (\omega, <)$. $\square$

(9) Show that if $\alpha < \beta$ are ordinals, then there is an ordinal γ such that $\alpha + \gamma = \beta$. (**Hint**: $\gamma = \mathsf{type}(\beta \setminus \alpha, \in)$).

Solution: Following the hint, put $\gamma = \mathsf{type}(\beta \setminus \alpha, \in)$. Note that

$$(\beta, \in) \cong (\alpha, \in) \oplus (\beta \setminus \alpha, \in)$$

Hence

$$\alpha + \gamma = \alpha + \mathsf{type}((\beta \setminus \alpha, \in)) = \mathsf{type}((\alpha, \in) \oplus (\beta \setminus \alpha, \in)) = \beta$$

where we used the fact that isomorphic well-orderings have the same type.

(10) Suppose α, β, γ are ordinals and $\alpha + \beta = \alpha + \gamma$. Show that $\beta = \gamma$.

Solution Suppose $\alpha + \gamma_1 = \alpha + \gamma_2 = \beta$. We'll show that $\gamma_1 = \gamma_2$. Suppose not and without loss of generality assume $\gamma_1 < \gamma_2$. Then $\gamma_1 + 1 \leq \gamma_2$ since $\gamma_1 + 1$ is the smallest ordinal strictly bigger than γ_1. Now

$$\beta = \alpha + \gamma_2 \geq \alpha + (\gamma_1 + 1) = (\alpha + \gamma_1) + 1 > \alpha + \gamma_1 = \beta$$

So $\beta > \beta$ which is impossible. $\square$

(11) Suppose $\alpha \cdot \alpha = \beta \cdot \beta$. Show that $\alpha = \beta$.

Solution: If α or β is 0, then this is clear. So assume $\alpha \geq 1$ and $\beta \geq 1$. Towards a contradiction, suppose $\alpha \neq \beta$ and without loss of generality say $\alpha < \beta$. Then $\alpha + 1 \leq \beta$. Now

$$\beta \cdot \beta \geq \alpha \cdot (\alpha + 1) = (\alpha \cdot \alpha) + \alpha \geq (\alpha \cdot \alpha) + 1 > \alpha \cdot \alpha$$

which contradicts $\alpha \cdot \alpha = \beta \cdot \beta$. □

(12) Show that there is an uncountable chain in $(\mathcal{P}(\omega), \subseteq)$. (**Hint**: Identify ω with the set of rationals $\mathbb{Q}$ and for each real number x, consider $\{r \in \mathbb{Q} : r < x\}$).

Solution: Let $\mathbb{Q}^+$ be the set of positive rational numbers and $\mathbb{R}^+$ be the set of positive real numbers. Define $h : \mathbb{Q}^+ \to \omega$ by

$$h\left(\frac{m}{n}\right) = 2^m 3^n$$

where n, m are coprime. Note that h is injective and hence a bijection from $\mathbb{Q}^+$ to range$(h) \subseteq \omega$. For each $x \in \mathbb{R}^+$, let $A_x = \{r \in \mathbb{Q}^+ : r < x\}$. Then $x < y$ implies $A_x \subsetneq A_y$. Hence $\{A_x : x \in \mathbb{R}^+\}$ is an uncountable chain in $(\mathcal{P}(\mathbb{Q}^+), \subseteq)$. It follows that $\{h[A_x] : x \in \mathbb{R}^+\}$ is an uncountable chain in $(\mathcal{P}(\omega), \subseteq)$. □

(13) Suppose $f : \mathbb{R} \to \mathbb{R}$ is additive and $a = f(1)$.

(a) Show that $f(0) = 0$.

(b) Show that for every $x \in \mathbb{R}$, $f(-x) = -f(x)$.

(c) Show that for every $x \in \mathbb{Q}$, $f(x) = ax$.

Solution: (a) Taking $x = y = 0$, we get $f(0 + 0) = f(0) + f(0)$. So $f(0) = 0$.

(b) Taking $y = -x$, we get $f(x + (-x)) = f(x) + f(-x)$. So $f(x) + f(-x) = f(0) = 0$. Hence $f(-x) = -f(x)$.

(c) For each $m, n \geq 1$, $f(m) = f(n(m/n)) = f(m/n + m/n + \cdots + m/n) = nf(m/n)$. So $f(m/n) = f(m)/n$. Next $f(m) = f(1 + 1 + \cdots + 1) = mf(1) = ma$. So $f(m/n) = a(m/n)$. Also $f(-m/n) = -f(m/n) = a(-m/n)$. It follows that for each nonzero $x \in \mathbb{Q}$, $f(x) = ax$. Since $f(0) = 0$, part (c) follows. □

(14) Let $H \subseteq \mathbb{R}$ be a Hamel basis.

(a) Show that every nonzero $x \in \mathbb{R}$ can be uniquely written as

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

where $x_1 < x_2 < \cdots < x_n$ are in H and $a_1, a_2, \ldots a_n$ are nonzero rational numbers. Uniqueness means the following: Suppose

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b_1 y_1 + b_2 y_2 + \cdots + a_m y_m$$

where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ are in H and $a_1, \dots a_n, b_1, \dots, b_m$ are nonzero rationals. Show that $m = n$ and for every $1 \le k \le n$, $x_k = y_k$ and $a_k = b_k$.

(b) Let $f : H \to \mathbb{R}$. Show that there is a unique additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f \subseteq g$.

Solution: (a) First let us check uniqueness: Suppose

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1y_1 + b_2y_2 + \cdots + a_my_m$$

where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ are in H and $a_1, \dots a_n, b_1, \dots, b_m$ are nonzero rationals. We must show that $m = n$ and for every $1 \le k \le n$, $x_k = y_k$ and $a_k = b_k$. Note that

$$(a_1x_1 + a_2x_2 + \cdots + a_nx_n) - (b_1y_1 + b_2y_2 + \cdots + b_my_m) = 0$$

After collecting like terms this boils down to showing the following. If $w_1 < w_2 < \cdots < w_p$ are in H, $c_1, c_2, \dots c_p$ are rationals and

$$c_1w_1 + c_2w_2 + \cdots + c_pw_p = 0$$

then $c_1 = c_2 = \cdots = c_p = 0$. But this is true because H is $\mathbb{Q}$-linearly independent.

Next suppose $x \in \mathbb{R}$ is nonzero. We must show that x is a finite $\mathbb{Q}$-linear combination of members of H. If $x \in H$, then $x = 1 \cdot x$ hence this is clear. So assume $x \notin H$. As H is a maximal $\mathbb{Q}$-linearly independent subset of $\mathbb{R}$, it follows that $H \cup \{x\}$ is not $\mathbb{Q}$-linearly independent. As H is $\mathbb{Q}$-linearly independent, this means that there are $x_1 < x_2 < \cdots < x_n$ in H and nonzero rationals $a_1, a_2, \dots a_n, b$ such that

$$a_1x_1 + a_2x_2 + \dots a_nx_n + bx = 0$$

Therefore,

$$x = -\frac{a_1}{b}x_1 - \frac{a_2}{b}x_2 - \cdots - \frac{a_n}{b}x_n$$

(b) Define $g(x)$ as follows. If $x = a_1x_1 + \dots a_nx_n$ where $x_1 < \cdots < x_n$ are in H and $a_1, \dots, a_n$ are rationals, then

$$g(x) = a_1f(x_1) + \cdots + a_nf(x_n)$$

g is well-defined by part (a). That g is additive is clear from its definition. To see uniqueness suppose $g' : \mathbb{R} \to \mathbb{R}$ is another additive extension of f. Then for every $r \in \mathbb{Q}$, $g'(rx) = rg'(x)$. Hence if $x = a_1x_1 + \dots a_nx_n$ where $x_1 < \cdots < x_n$ are in H and $a_1, \dots, a_n$ are rationals, then

$$g'(x) = a_1g'(x_1) + \cdots + a_ng'(x_n) = a_1f(x_1) + \cdots + a_nf(x_n) = g(x)$$

So $g' = g$. $\square$

(15) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$.

(a) Show that either f is identically zero or range$(f) \subseteq \mathbb{R}^+$.

(b) Suppose f is continuous and not identically zero. Show that $f(x) = a^x$ for some $a > 0$.

Solution: (a) Suppose there is some $a \in \mathbb{R}$ such that $f(a) = 0$. Then for every $x \in \mathbb{R}$, $f(x+a) = f(x)f(a) = f(x) \cdot 0 = 0$. Hence f is identically zero. Next suppose $f(a) \neq 0$ for every $a \in \mathbb{R}$. Then $f(x) = f(x/2 + x/2) = (f(x/2))^2 > 0$. So either f is identically zero or range$(f) \subseteq \mathbb{R}^+$.

(b) By part (a), range$(f) \subseteq \mathbb{R}^+$ so we can define $g(x) = \ln(f(x))$. Then g is a continuous additive function and hence $g(x) = bx$ where $b = g(1)$. It follows that $f(x) = e^{g(x)} = e^{bx} = a^x$ where $a = e^b > 0$. $\square$

(16) Show that there is a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

Solution: Let $g : \mathbb{R} \to \mathbb{R}$ be any discontinuous additive function and define $f(x) = e^{g(x)}$. $\square$

(17) Prove the following.

(a) For every ordinal α, $|\alpha| \leq \alpha$.

(b) If κ is a cardinal and $\alpha < \kappa$, then $|\alpha| < \kappa$.

(c) There is an injection from X to Y iff $|X| \leq |Y|$.

(d) There is a surjection from X to Y iff $|Y| \leq |X|$.

(e) There is a bijection from X to Y iff $|X| = |Y|$.

Solution: Let us write $X \preceq Y$ iff there is an injection from X to Y and $X \sim Y$ iff there is a bijection from X to Y.

(a) Let $|\alpha| = \beta$. Then for every γ, if $\gamma \sim \alpha$, then $\beta \leq \gamma$. Since $\alpha \sim \alpha$, it follows that $\alpha \leq \beta = |\alpha|$.

(b) By part (a), $|\alpha| \leq \alpha < \kappa$.

(c) Since $X \sim |X|$ and $Y \sim |Y|$, we get $X \preceq Y$ iff $|X| \preceq |Y|$. So it suffices to show that if κ, λ are cardinals, then $\kappa \preceq \lambda$ iff $\kappa \leq \lambda$. It is clear that if $\kappa \leq \lambda$, then $\kappa \preceq \lambda$. Next suppose $\lambda < \kappa$. Since $|\kappa| = \kappa$ and $\lambda < \kappa$, $\lambda \nsim \kappa$. Since $\lambda \preceq \kappa$, by the Schröder-Bernstein theorem, it follows that $\kappa \npreceq \lambda$.

(d) By part (c), it suffices to show that for any X and Y, there is a surjection from X to Y iff $Y \preceq X$. We can assume that X, Y are nonempty. Suppose $Y \preceq X$. Fix an injective function $f : Y \to X$. Then $f : Y \to$ range(f) is a bijection. Fix $y_0 \in Y$. Define $g : X \to Y$ as follows: If $x \in$ range(f), then $g(x) = f^{-1}(x)$, otherwise $g(x) = y_0$. Clearly, range$(g) = Y$.

Next suppose $f : X \to Y$ and range$(f) = Y$. Let $\mathcal{F} = \{f^{-1}[\{y\}] : y \in Y\}$. Then $\mathcal{F}$ is a partition of Y into nonempty sets. Using the axiom of choice let $h : \mathcal{F} \to Y$ be a choice function. Define $g : Y \to X$ by $g(y) = h(f^{-1}[\{y\}])$. Then $g : Y \to X$ is injective.

(e) Use part (c). $\square$

(18) Prove the following.

(a) $|\mathbb{R}^\omega| = \mathfrak{c}$.

(b) $|C(\mathbb{R})| = \mathfrak{c}$ where $C(\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(c) Let A be the set of all real numbers which are roots of some polynomial equation with rational coefficients. Show that $|A| = \omega$.

Solution: (a) Let us write $X \sim Y$ iff there is a bijection from X to Y. Then it is easy to check that for any set A,

$$(A^\omega)^\omega \sim A^{\omega\times\omega} \sim A^\omega$$

Taking $A = 2 = \{0, 1\}$ and using the fact that $|\mathbb{R}| = |2^\omega| = \mathfrak{c}$, we get $|\mathbb{R}^\omega| = |(2^\omega)^\omega| = |2^\omega| = \mathfrak{c}$.

(b) It is clear that $|C(\mathbb{R})| \geq |\mathbb{R}| = \mathfrak{c}$ since every constant function is continuous. To show that $|C(\mathbb{R})| \leq \mathfrak{c}$, we'll construct an injective function from $C(\mathbb{R})$ to $\mathbb{R}^\omega$. This suffices since by part (a), $|\mathbb{R}^\omega| = \mathfrak{c}$. Since $|\mathbb{Q}| = \omega$, it is enough to construct an injective function $H : C(\mathbb{R}) \to \mathbb{R}^\mathbb{Q}$ where $\mathbb{R}^\mathbb{Q}$ is the set of all functions from $\mathbb{Q}$ to $\mathbb{R}$. Given $f \in C(\mathbb{R})$, define $H(f) = f \restriction \mathbb{Q}$. We claim that H is injective. To see this assume that $H(f) = H(g)$ and we'll show that $f = g$. Let $x \in \mathbb{R}$. Let $\langle a_n : n < \omega\rangle$ be a sequence of rationals converging to x. Since f, g are continuous, $f(a_n)$ converges to $f(x)$ and $g(a_n)$ converges to $g(x)$. As $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$, for every $n < \omega$ we must have $f(a_n) = g(a_n)$. Hence $f(x) = g(x)$. So $f = g$ and H is injective.

(c) For each $1 \leq n \leq \omega$, let P_n be the set of all polynomials of degree n with rational coefficients. Any polynomial $f \in P_n$ is of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_0$ are in $\mathbb{Q}$ and $a_n \neq 0$. It follows that $|P_n| \leq |\mathbb{Q}^n| = \omega$. Let $A_n = \{a \in \mathbb{R} : (\exists p \in P_n)(p(a) = 0)\}$. Since each polynomial in P_n has $\leq n$ real roots, it follows that A_n is a countable union of finite sets. So each A_n is countable. Finally, $A = \bigcup\{A_n : 1 \leq n < \omega\}$ is a countable union of countable sets. Hence A is also countable. As every rational is in A, $|A| \geq \omega$. Hence $|A| = \omega$. $\square$

(19) Suppose $(P, \preceq)$ is a partial ordering in which every countable chain has an upper bound. Must P have a maximal element?

Solution: No. Let $P = \{X \subseteq \mathbb{R} : |X| \leq \omega\}$. Then every countble chain in $(P, \subseteq)$ has an upper bound in P (namely its union, since a countable union of countable sets is countable). Let$X \in P$ be arbitrary. We will show that X is not maximal in $(P, \subseteq)$.

As X is countable and $\mathbb{R}$ is uncountable, $\mathbb{R} \setminus X \neq \emptyset$. Let $Y = X \cup \{a\}$ where $a \in \mathbb{R} \setminus X$. Then Y is countable and X is a proper subset of Y. So X is not a $\subseteq$-maximal element of P. $\square$

(20) Use Zorn's lemma to show that $\mathbb{R}^+ = (0, \infty)$ can be partitioned into two nonempty sets that are both closed under addition.

Solution: See Problem 6(a) in Chapter 14 of Komjath-Totik book `https://doi.org/10.1007/0-387-36219-3`.

(21) Is there a subset $X \subseteq \mathbb{R}^2$ that meets every circle at 2 points?

Solution: Suppose X is such a set. First observe that all the points in X must be collinear otherwise we can draw a circle C through 3 non-collinear points in X so that $|X \cap C| \geq 3$. Let ℓ be a line such that $X \subseteq \ell$. Choose a circle C such that $C \cap \ell = \emptyset$. Then $C \cap X = \emptyset$ which is a contradiction. $\square$

(22) Show that there is a subset $X \subseteq \mathbb{R}^2$ that meets every circle at 3 points.

Solution: Let $\mathcal{E}$ be the set of all circles in $\mathbb{R}^2$. Since each circle is uniquely determined by its center and radius, $|\mathcal{E}| \leq |\mathbb{R}^2 \times \mathbb{R}^+| = |\mathfrak{c} \times \mathfrak{c}| = \mathfrak{c}$. Also there are at least $\mathfrak{c}$ distinct circles so $|\mathcal{E}| = \mathfrak{c}$. Let $\langle C_\alpha : \alpha < \mathfrak{c} \rangle$ be an injective sequence with range $\mathcal{E}$. Using transfinite recursion, we are going to construct a $\subseteq$-increasing sequence $\langle S_\alpha : \alpha < \mathfrak{c} \rangle$ of subsets of $\mathbb{R}^2$ such that the following hold for every $\alpha < \mathfrak{c}$.

1. $S_0 = \emptyset$.
2. $|S_\alpha| \leq \max(|\alpha|, \omega) < \mathfrak{c}$.
3. No 4 points in S_α are concyclic.
4. $|S_{\alpha+1} \cap C_\alpha| = 3$.
5. If α is a limit ordinal, then $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$.

Limit stage: First observe that at any limit stage α, defining $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ does not violate Clause 3. For suppose S_α does contain 4 concylic points. Then all of these 4 points must appear at some stage $\beta < \alpha$ which is impossible.

Successor stage: Having constructed S_α, $S_{\alpha+1}$ is obtained as follows. By Clause 3, $|S_\alpha \cap C_\alpha| = n \leq 3$. We would like to add $n - 3$ points from C_α to S_α to get $S_{\alpha+1}$ while ensuring that no 4 points in $S_{\alpha+1}$ are concylic. We have the following cases.

Case (a): $n = 3$. Define $S_{\alpha+1} = S_\alpha$.

Case (b): $n = 2$. Let $\mathcal{T}$ be the set of circles that pass through 3 points in S_α. Then $|\mathcal{T}| \leq |S_\alpha \times S_\alpha \times S_\alpha| \leq \max(|\alpha|, \omega) < \mathfrak{c}$. Let B be the set of points of intersection of C_α with the circles in $\mathcal{T}$. Since each circle in $\mathcal{T}$ can intersect C_α in at most 2 points, we get $|B| \leq \max(|\alpha|, \omega) < \mathfrak{c}$. Choose any point $p \in C_\alpha \setminus B$ and define $S_{\alpha+1} = S_\alpha \cup \{p\}$.

Case (c): $n = 1$. First choose $p \in C_\alpha \setminus B$ and define $S' = S_\alpha \cup \{p\}$. Let $\mathcal{T}'$ be the set of all circles that pass through 3 points in S'. As before, $|\mathcal{T}'| < \mathfrak{c}$. Let B' be the set of points of intersection of C_α with the circles in $\mathcal{T}'$. Then $|B'| < \mathfrak{c}$. Choose $q \in C_\alpha \setminus (B' \cup \{p\})$ and define $S_{\alpha+1} = S_\alpha \cup \{p, q\}$.

Case (d): $n = 0$. Proceed as in Case (c).

In each of the Cases (a)-(d), check that $S_{\alpha+1}$ does not contain 4 concylic points and $|S_{\alpha+1} \cap C_\alpha| = 1$. This concludes the description of $\langle S_\alpha : \alpha < \mathfrak{c} \rangle$.

Define $S = \bigcup_{\alpha<\mathfrak{c}} S_\alpha$. It should be clear that S meets every circle at exactly 3 points. $\square$

(23) Suppose that the set of propositional variables $\mathcal{V}ar$ is uncountable. Use Zorn's lemma to show the following: Let S be a set of propositional formulas such that every finite subset of S is satisfiable. Then S is satisfiable. **Hint**: Apply Zorn's lemma to $(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is the set of all functions h such that $\mathrm{dom}(h) \subseteq \mathcal{V}ar$, $\mathrm{range}(h) \subseteq \{0, 1\}$ and for every finite $F \subseteq S$, there exists a valuation $val : \mathcal{V}ar \to \{0, 1\}$ such that $h \subseteq val$ and every formula in F is true under val.

Solution: Let $\mathcal{F}$ be the set of all functions h such that $\mathrm{dom}(h) \subseteq \mathcal{V}ar$, $\mathrm{range}(h) \subseteq \{0, 1\}$ and for every finite $F \subseteq S$, there exists a valuation $val : \mathcal{V}ar \to \{0, 1\}$ such that $h \subseteq val$ and every formula in F is true under val.

We claim that every chain in $(\mathcal{F}, \subseteq)$ has an upper bound. To see this, fix an arbitrary chain $\mathcal{C} \subseteq \mathcal{F}$ and define $g = \bigcup \mathcal{C}$. Since $\mathcal{C}$ is a chain, it is easy to see that g is a function. Clearly, $\mathrm{dom}(g) \subseteq \mathcal{V}ar$ and $\mathrm{range}(g) \subseteq \{0, 1\}$. So it would be sufficient to show that $g \in \mathcal{F}$ since then g is an upper bound of $\mathcal{C}$ in $(\mathcal{F}, \subseteq)$. Towards a contradiction, suppose $g \notin \mathcal{F}$. Fix a finite $F \subseteq S$ such that there is no valuation $val : \mathcal{V}ar \to \{0, 1\}$ satisfying: $g \subseteq val$ and every formula in F is true under val. Choose a finite $V \subseteq \mathcal{V}ar$ that contains every propositional variable that occurs in a formula in F. Put $W = V \cap \mathrm{dom}(g)$. Since $\mathcal{C}$ is a chain, we can find an $h \in \mathcal{C}$ such that $W \subseteq \mathrm{dom}(h)$. Since $h \in \mathcal{F}$, there exists a valuation $val' : \mathcal{V}ar \to \{0, 1\}$ such that $h \subseteq val'$ and every formula in F is true under val'. Define another valuation $val : \mathcal{V}ar \to \{0, 1\}$ as follows:

$$val(p) = \begin{cases} g(p) & \text{if } p \in \mathrm{dom}(g) \\ val'(p) & \text{otherwise} \end{cases}$$

Observe that val and val' agree on every propositional variable in V. Hence every formula in F is true under val. But $g \subseteq val$ so we have a contradiction. So $g \in \mathcal{F}$ is an upper bound of $\mathcal{C}$.

Using Zorn's lemma, fix a $\subseteq$-maximal f in $\mathcal{F}$. We claim that $\mathrm{dom}(f) = \mathcal{V}ar$. This will complete the proof since it implies that f is a valuation under which every formula in $\mathcal{F}$ is true. Towards a contradiction, assume some propositional variable $p \notin \mathrm{dom}(f)$. Define $f_0 = f \cup \{(p, 0)\}$ and $f_1 = \{(p, 1)\}$. By the Lemma on Lecture slide no. 89, it follows that one of f_0, f_1 is in $\mathcal{F}$. But this contradicts the maximality of g. Hence $\mathrm{dom}(f) = \mathcal{V}ar$ and the proof is complete. $\square$

(24) Let ϕ be a propositional formula in which $\neg$ does not occur. Show that ϕ is satisfiable.

Solution. Define $val(p) = 1$ for every propositional variable p. By induction on the length of formula ϕ, show that ϕ is true under val. $\square$

(25) Show that the axiom of extensionality is not a logically valid $\mathcal{L}_{ZFC}$-sentence.

Solution: Recall that the axiom of extensionality is the following $\mathcal{L}_{ZFC}$-sentence

$$\psi \equiv (\forall x)(\forall y)[(\forall v)(v \in x \iff v \in y) \implies (x = y)].$$

Let $\mathcal{M} = (M, \in^{\mathcal{M}})$ where $M = \{0, 1, 2\}$ and $\in^{\mathcal{M}} = \{(0, 1), (0, 2)\}$. Now $(\forall v \in M)(v \in^{\mathcal{M}} 1 \iff v \in^{\mathcal{M}} 2) \wedge 1 \neq 2$. So $\mathcal{M} \models \neg\psi$. Since we have shown that there is an $\mathcal{L}_{ZFC}$-structure (namely $\mathcal{M}$) in which ψ is false, it follows that ψ is not logically valid. $\square$

(26) Show that $(\forall x)(\forall y)(x + y = y + x)$ is not a logically valid $\mathcal{L}_{PA}$-sentence.

Solution: Consider $\mathcal{M} = (M, +^{\mathcal{M}})$ where $M = \mathbb{R}$ and $x +^{\mathcal{M}} y = x - y$. $\square$

(27) Let $\mathcal{L}$ be a first order language and T be an $\mathcal{L}$-theory. Assume $T \vdash \phi$ and $T \vdash (\phi \implies \psi)$. Show that $T \vdash \psi$.

Solution: Let $\phi_1, \cdots, \phi$ be a proof of ϕ in T and $\psi_1, \cdots, (\phi \implies \psi)$ be a proof of $(\phi \implies \psi)$ in T. Check that

$$\phi_1, \cdots, \phi, \psi_1, \cdots, (\phi \implies \psi), \psi$$

is a proof of ψ in T. $\square$

(28) Let $\mathcal{L}$ be a first order language and T be an $\mathcal{L}$-theory. Assume $T \vdash \phi_1$ and $T \vdash \phi_2$. Show that $T \vdash (\phi_1 \wedge \phi_2)$.

Solution: Use the fact that $\phi_1 \implies (\phi_2 \implies (\phi_1 \wedge \phi_2))$ is a propositional tautology. $\square$

(29) Let $\mathcal{L}$ be a first order language. Call a set S of $\mathcal{L}$-sentences deductively closed if S contains every logical axiom of $\mathcal{L}$ and whenever ϕ and $(\phi \implies \psi)$ are in S, ψ is also in S. Show that for any $\mathcal{L}$-theory T, the set of theorems of T is the smallest set of $\mathcal{L}$-sentences that contains T and is deductively closed.

Solution: Let S be the set of all $\mathcal{L}$-sentences ψ such that $T \vdash \psi$. So S is the set of all theorems of T.

We first show that S is deductively closed. If ψ is a logical axiom of $\mathcal{L}$, then $T \vdash \psi$, so we get $\psi \in S$. Next assume ϕ and $(\phi \implies \psi)$ are both in S. We need to show that $\psi \in S$. But this follows from problem (27). Thus S is deductively closed.

Next. to show that S is the smallest deductively closed set of $\mathcal{L}$-sentences containing T, suppose S' is another deductively closed set of $\mathcal{L}$-sentences such that $T \subseteq S'$. We will show that $S \subseteq S'$. Let $\psi \in S$. Then $T \vdash \psi$. Fix a proof $\psi_1, \cdots, \psi_n \equiv \psi$ of ψ in T. By induction on $k \leq n$, we will show that each $\psi_k \in S'$. It will follows that $\psi \in S'$ and hence $S \subseteq S'$.

If $k = 1$, it must be the case that either $\psi_1 \in T$ or ψ_1 is a logical axiom of $\mathcal{L}$. If $\psi_1 \in T$, then $\psi_1 \in S'$ since $T \subseteq S'$. If ψ_1 is a logical axiom, then as S' is deductively closed, we must have $\psi_1 \in S'$. So in both cases $\psi_1 \in S'$.

For the inductive step, assume $\psi_j \in S$ for each $j \leq k < n$. We will show that $\psi_{k+1} \in S'$. First suppose that either ψ_{k+1} is a logical axiom of $\mathcal{L}$ or $\psi_{k+1} \in T$. Then $\psi_{k+1} \in S'$ as argued above. Finally suppose ψ_{k+1} was deduced from some ψ_i and $\psi_j \equiv (\psi_i \implies \psi_{k+1})$ (where $i, j \leq k$) via modus ponens. Then using the inductive hypothesis we get that $\psi_i, \psi_j \in S'$ and by using the fact that S' is deductively closed, we can conclude that $\psi_{k+1} \in S'$. This completes the inductive step. $\square$

(30) Describe all substructures of $(\mathbb{Z}, <)$ and $(\mathbb{Z}, +)$.

Solution: A substructure of $(\mathbb{Z}, <)$ is of the form $\mathcal{M} = (M, <)$ where $M \subseteq \mathbb{Z}$ is any nonempty subset and $<$ is the usual order on M. A substructure of $(\mathbb{Z}, +)$ is of the form $\mathcal{M} = (M+)$ where $M \subseteq \mathbb{Z}$ is nonempty and closed under addition. We leave it as an exercise to describe such subsets of $\mathbb{Z}$. $\square$

(31) Describe all elementary submodels of $(\mathbb{Z}, <)$ and $(\mathbb{Z}, +)$.

Solution: The only elementary submodel of $(\mathbb{Z}, <)$ is $(\mathbb{Z}, <)$. To see this, suppose $(M, <)$ is an elementary submodel of $(\mathbb{Z}, <)$ and towards a contradiction, assume $M \neq \mathbb{Z}$. Fix some $n \in \mathbb{Z} \setminus M$. Since

$$(\mathbb{Z}, <) \models (\forall x)(\exists y)(y < x) \wedge (\forall x)(\exists y)(x < y)$$

we must have

$$(M, <) \models (\forall x)(\exists y)(y < x) \wedge (\forall x)(\exists y)(x < y)$$

So $(M, <)$ does not have a largest or smallest integer. Using this fact, we can find $a, b \in M$ such that a is the largest integer in M strictly below n and b is the smallest integer in M strictly above b. Observe that there is no integer in M that is strictly between a and b. So $(M, <) \models \neg(\exists x)(a < x \wedge x < b)$. As $(M, <)$ is an elementary submodel of $(\mathbb{Z}, <)$, we get $(\mathbb{Z}, <) \models \neg(\exists x)(a < x \wedge x < b)$ which is false since $x = n$ is strictly between a and b. This contradiction shows that $M = \mathbb{Z}$.

Next, we will show that $(\mathbb{Z}, +)$ is the only elementary submodel of $(\mathbb{Z}, +)$. Suppose $(M, +)$ is an elementary submodel of $(Z, +)$. Then M is closed under $+$. Moreover, since $(\mathbb{Z}, +) \models (\forall x)(\forall y)(\exists z)(x = y + z)$. it follows that M is also closed under $-$. We will show that $1 \in M$. Since M is closed under both $+$ and $-$, it will follow that $M = \mathbb{Z}$.

First note that since $\mathbb{Z}$ is infinite, M must also be infinite (use $(\mathbb{Z},<) \models \exists_{\geq n}$ for every $n \geq 1$, Slide 126). So M must have both positive and negative integers. Let p be any positive integer in M. If $p = 1$, then we are done. So suppose $p > 1$. Observe that $(\mathbb{Z},+) \models (\exists x)(px = p)$ where px is the term $(x + x + \cdots + x)$ (p times). Hence $(M,+) \models (\exists x)(px = p)$. Therefore $1 \in M$ and we are done. $\square$

(32) For each $n \geq 1$ and $\mathcal{L} = \emptyset$. Give an example of a consistent $\mathcal{L}$-theory T such that for every model $\mathcal{M} \models T$, $|M| = n$.

Solution: Let $T = \{\exists_{\geq n}, \neg(\exists_{n+1})\}$ (Slide 126). $\square$

(33) Let $\mathcal{L} = \mathcal{L}_{PA} \cup \{c\}$ where c is a new constant symbol. Let $\mathsf{Primes} = \{2, 3, 5, 7, \dots\}$ be the set of all primes numbers. For each $p \in \mathsf{Primes}$, let "p divides c" denote the $\mathcal{L}$-sentence $(\exists y)(S^p(0) \cdot y = c)$. For each $X \subseteq \mathsf{Primes}$, let T_X be the $\mathcal{L}$-theory

$$T_X = TA \cup \{(p \text{ divides } c) : p \in X\} \cup \{\neg(p \text{ divides } c) : p \in \mathsf{Primes} \setminus X\}$$

where $TA = Th(\omega, 0, S, +, \cdot)$ denotes true arithmetic.

(a) Show that T_X is consistent for every $X \subseteq \mathsf{Primes}$.

(b) Show that TA has continuum many pairwise non-isomorphic countable models.

Solution: (a) We will show that every finite subset of T_X has a model. This suffices since then, by compactness theorem, it will follows that T_X has a model and therefore T_X is consistent.

Let F be a finite subset of T_X. We will construct a model of F. Let W be the set of all primes p such that $(p \text{ divides } c) \in F$. Note that W is a finite subset of X. Let $\mathcal{M} = (\omega, 0, S, +, \cdot, c^{\mathcal{M}})$ where $(\omega, 0, S, +, \cdot)$ is the standard model of arithmetic and $c^{\mathcal{M}}$ is the product of all the primes in W (If $W = \emptyset$, then define $c^{\mathcal{M}} = 1$). Then a prime p divides $c^{\mathcal{M}}$ iff $p \in W$. It follows that $\mathcal{M} \models F$.

(b) Using part (a), we can fix a family $\{\mathcal{M}'_X : X \subseteq \mathsf{Primes}\}$ such that for each $X \subseteq \mathsf{Primes}$, $\mathcal{M}'_X = (M_X, 0^{\mathcal{M}_X}, S^{\mathcal{M}_X}, +^{\mathcal{M}_X}, \cdot^{\mathcal{M}_X}, c^{\mathcal{M}_X})$ is a countable $\mathcal{L}$-structure such that $\mathcal{M}'_X \models T_X$. Let $\mathcal{M}_X = (M_X, 0^{\mathcal{M}_X}, S^{\mathcal{M}_X}, +^{\mathcal{M}_X}, \cdot^{\mathcal{M}_X})$. Then $\mathcal{M}_X$ is an $\mathcal{L}_{PA}$-structure such that $\mathcal{M}_X \models TA$.

We claim that for any $X \subseteq \mathsf{Primes}$, $\{Y \subseteq \mathsf{Primes} : \mathcal{M}_X \cong \mathcal{M}_Y\}$ is countable. Since $|\{X : X \subseteq \mathsf{Primes}\}| = \mathfrak{c}$, it will follow that there are continuum many pairwise non-isomorphic models of TA in $\{\mathcal{M}_X : X \subseteq \mathsf{Primes}\}$.

Let $X \subseteq \mathsf{Primes}$. For each prime p, let ϕ_p denote the formula "p divides x" where x is a variable. For each $a \in M_X$, let $T_a = \{p \in \mathsf{Primes} : \mathcal{M}_X \models \phi_p(a/x)\}$. Then $\mathcal{T}_X = \{T_a : a \in M_X\}$ is a countable family of subsets of Primes.

Now observe that if $Y \subseteq \mathsf{Primes}$ and $Y \notin \mathcal{T}_X$, then $\mathcal{M}_X$ cannot be isomorphic to $\mathcal{M}_Y$. This is because there exists a member $a \in M_Y$ (namely, $a = c^{\mathcal{M}_Y}$) such that $Y = \{p \in \mathsf{Primes} : \mathcal{M}_Y \models \phi_p(a/x)\}$ while there is no such member in M_X. So $\{Y \subseteq \mathsf{Primes} : \mathcal{M}_X \cong \mathcal{M}_Y\}$ is countable and we are done. $\square$

(34) Show that $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

Solution: Put $\mathcal{L} = \{<\}$. By the Tarski-Vaught criterion, it suffice to show that for every $\mathcal{L}$-formula $\phi(x, y_1, \cdots, y_n)$ and $a_1, \cdots, a_n \in \mathbb{Q}$, if there exists $b \in \mathbb{R}$ such that $(\mathbb{R}, <) \models \phi(b, a_1, \cdots, a_n)$, then there exists $c \in \mathbb{Q}$ such that $(\mathbb{R}, <) \models \phi(c, a_1, \cdots, a_n)$. Fix some $b \in \mathbb{R}$ such that $(\mathbb{R}, <) \models \phi(b, a_1, \cdots, a_n)$. If $b \in \mathbb{Q}$, then we are done. So we can assume that $b \in \mathbb{R} \setminus \mathbb{Q}$. In particular, $b \notin \{a_1, \cdots, a_n\}$. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, choose $c \in \mathbb{Q} \setminus \{a_1, \cdots, a_n\}$ such that for every $1 \leq k \leq n$, $(a_n < c) \iff (a_n < b)$. Now check that there is a (piecewise linear) order-preserving bijection $h : \mathbb{R} \to \mathbb{R}$ such that $h(a_k) = a_k$ for every $1 \leq k \leq n$ and $h(b) = c$ (Draw a picture on the number line and use the fact that for any two bounded closed intervals in $\mathbb{R}$, there is an order-preserving linear function between them). Since h is an isomorphism from $(\mathbb{R}, <) \to (\mathbb{R}, <)$, by the Lemma on Slide 125, it follows that $(\mathbb{R}, <) \models \phi(h(b), h(a_1), \cdots, h(a_n))$. As $h(b) = c$ and $h(a_k) = a_k$ for every $1 \leq k \leq n$, we get $(\mathbb{R}, <) \models \phi(c, a_1, \cdots, a_n)$. $\square$

(35) Show that every countable linear ordering $(L, <_L)$ is isomorphic to $(A, <)$ for some $A \subseteq \mathbb{Q}$.

Solution: If L is finite, this is clear – Say $L = \{x_0 <_L x_1 <_L x_2 <_L \cdots <_L x_k\}$. Take $A = \{0, 1, 2, \cdots, k\}$. Then $f(x_i) = i$ is an order isomorphism from $(L, <_L)$ to $(A, <)$.

So assume $|L| = \omega$ and fix an injective enumeration $\langle x_k : k < \omega \rangle$ of L. By induction on $n < \omega$, define $f_n : \{x_k : k \leq n\} \to \mathbb{Q}$ as follows. Start by defining $f_0(x_0) = 0$. Suppose $f_n : \{x_k : k \leq n\} \to \mathbb{Q}$ has been defined such that for every $i < j \leq n$, $x_i <_L x_j \iff f(x_i) < f(x_j)$.

Define $f_{n+1} : \{x_k : k \leq n+1\} \to \mathbb{Q}$ as follows. Define $f_{n+1}(x_k) = f_n(x_k)$ for every $k \leq n$. Next, define $f_{n+1}(x_{n+1}) = r \in \mathbb{Q}$ that satisfies: For every $i \leq n$, $x_i <_L x_{n+1} \iff f_n(x_i) < r$. This is always possible because $(\mathbb{Q}, <) \models DLO$.

Having defined f_n for every $n < \omega$, put $f = \bigcup_{n<\omega} f_n$ and note that $f : L \to \mathbb{Q}$ is an order preserving function. Put $A = \text{range}(f)$. Then $(L, <_L) \cong (A, <)$. $\square$

(36) Let $\mathcal{L} = \{\prec\}$ where $\prec$ is a binary relation symbol. Let T be the $\mathcal{L}$-theory obtained by replacing Axiom (6) of DLO (Slide 142) with the following

$$(\exists y)(\forall x)(y \prec x \vee y = x)$$

Show that T is ω-categorical and therefore complete. Use this to also show that for every $\mathcal{L}$-sentence ϕ

$$T \vdash \phi \iff (\mathbb{Q}_{\geq 0}, <) \models \phi$$

where $\mathbb{Q}_{\geq 0}$ is the set of all non-negative rationals.

Solution: Let $(L_1, <_1)$ and $(L_2, <_2)$ be two models of T such that $|L_1| = |L_2| = \omega$. Then both L_1, L_2 have a least member since the sentence $(\exists y)(\forall x)(y \prec x \vee y = x)$ is true in both of them. Let a_1 be the $<_1$-least member of L_1 and a_2 be the $<_2$-least member of L_2. Put $L_1' = L_1 \setminus \{a_1\}$ and $L_2' = L_2 \setminus \{a_2\}$. Then $(L_1', <_1) \models DLO$ and

$(L_2', <_2) \models DLO$. As DLO is ω-categorical, it follows that $(L_1', <_1) \cong (L_2' <_2)$. Let $f : L_1' \to L_2'$ be an isomorphism. Extend f to L_1 be defining $f(a_1) = a_2$. It is easy to see that f is an isomorphism from L_1 to L_2.

Note that T does not have any finite model since any linear ordering that models T does not have a largest member. So by the theorem on slide 141, as T is ω-categorical, it must be complete. Since $(\mathbb{Q}_{\geq 0}, <) \models T$, by soundness theorem, it follows that if $T \vdash \phi$, then $(\mathbb{Q}_{\geq 0}, <) \models \phi$. Finally, suppose $(\mathbb{Q}_{\geq 0}, <) \models \phi$ and towards a contradiction, assume $T \nvdash \phi$. Then since T is complete, we must have $T \vdash \neg\phi$ and so $(\mathbb{Q}_{\geq 0}, <) \models \neg\phi$ which is impossible. □

(37) Let $W \subseteq \omega$ be nonempty. Show that W is c.e. iff there exists a computable function $f : \omega \to \omega$ such that $\text{range}(f) = W$.

Solution: First assume that W is c.e. Fix a program P such that for each $n < \omega$, P halts on input n iff $n \in W$.

Define a program Q as follows. On input n, Q runs P on each one of the inputs $0, 1, \ldots, n$ for n steps. Let S_n be the set of those $k \leq n$ such that P halts on input k in at most n steps. Let W_n be the set of outputs of Q on inputs $0, 1, \ldots, n-1$. If $S_n \setminus W_n \neq \emptyset$, then Q outputs $\min(S_n \setminus W_n)$. Otherwise, Q outputs $\min(W)$.

It is clear that Q halts on every input. Let $f : \omega \to \omega$ be the function computed by Q. We claim that $\text{range}(f) = W$. That $\text{range}(f) \subseteq W$ is obvious. For the other inclusion, towards a contradiction, suppose $W \setminus \text{range}(f) \neq \emptyset$ and let $n_\star = \min(W \setminus \text{range}(f))$. Choose $m > n_\star$ large enough such that $W \cap n_\star \subseteq \text{range}(f \restriction m)$ and for every $n \leq n_\star$, if $n \in W$, then P halts on input n is less than m steps. Now observe that on input m, Q must output $n_\star$: A contradiction. So we must have $W \subseteq \text{range}(f)$. It follows that $W = \text{range}(f)$.

Next assume that $f : \omega \to \omega$ is computable. Put $W = \text{range}(f)$. Let P be a program that on input n starts computing $f(0), f(1), f(2), \ldots$ and halts iff n appears in this list. Then P witnesses that W is c.e. □

(38) Let $W \subseteq \omega$ be infinite. Show that W is c.e. iff there exists an injective computable function $f : \omega \to \omega$ such that $\text{range}(f) = W$.

Solution: Assume $W \subseteq \omega$ is c.e. and infinite. By the previous problem, we can fix a computable $g : \omega \to \omega$ such that $\text{range}(g) = W$. Define $f : \omega \to \omega$ as follows. $f(0) = g(0)$ and $f(n+1)$ is the first number that appears in the list $g(0), g(1), g(2), \cdots$, that is not in $\{f(0), f(1), \cdots, f(n)\}$ (such a number exists since $\text{range}(g)$ is infinite). It is clear that f is computable, injective and $\text{range}(f) = \text{range}(g) = W$. The other implication follows from the previous problem. □

(39) Let $W \subseteq \omega$ be an infinite c.e. set. Show that W has an infinite computable subset.

Solution: By the previous problem, we can fix an injective computable $f : \omega \to \omega$ such that $\text{range}(f) = W$. Define $g : \omega \to \omega$ as follows. $g(0) = f(0)$ and $g(n+1)$ is the first entry in the list $f(0), f(1), f(2), \cdots$, that is strictly bigger than $g(n)$. Then $g : \omega \to \omega$ is computable and strictly increasing. Put range$(g) = A$. Then $A \subseteq W$ is infinite. To see that A is computable, note that to check whether $n \in A = \text{range}(g)$, we only need to check if $n \in \{g(0), g(1), \cdots, g(n)\}$ (because g is strictly increasing). □

(40) Let $W \subseteq \omega$ be nonempty. Show that W is c.e. iff there exists a computable $A \subseteq \omega^2$ such that $W = \{n \in \omega : (\exists m)((n, m) \in A)\}$.

Solution: First assume W is c.e. and fix a program P that halts on input n iff $n \in W$. Define A to be the set of $(n, m) \in \omega^2$ such that on input n, P halts in $\leq m$ steps. Then A is computable and $W = \{n \in \omega : (\exists m)((n, m) \in A)\}$.

Next suppose $A \subseteq \omega^2$ is computable and $W = \{n \in \omega : (\exists m)((n, m) \in A)\}$. Consider a program Q that on input n, starts checking if $(n, m) \in A$ for $m = 0, 1, 2, 3, \cdots$ and halts as soon as it finds some m for which $(n, m) \in A$. Then Q halts on input n iff $(\exists m)((n, m) \in A)$ iff $n \in W$. So W is c.e. □

(41) Suppose $X \subseteq \omega$ is numeralwise representable in PA. Show that X is computable.

Solution: Fix an $\mathcal{L}_{PA}$ formula $\phi(x)$ such that for every $n < \omega$, if $n \in A$, then $PA \vdash \phi(\overline{n})$ and if $n \notin A$, then $PA \vdash \neg\phi(\overline{n})$. Since the set of theorems in PA is c.e. (see Slides 159-160), we can fix a program P such that for any $\mathcal{L}_{PA}$-sentence ψ, P halts on input ψ iff $PA \vdash \psi$. Consider the program Q which on input n, runs P with inputs $\phi(\overline{n})$ and $\neg\phi(\overline{n})$. If P halts on input $\phi(\overline{n})$, then Q returns 1. If P halts on input $\neg\phi(\overline{n})$, then Q returns 0. It is easy to see that Q computes X. □

(42) Let $E \subseteq \omega$ be a non-computable c.e. set. Show that E is definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$ but not numeralwise representable in PA.

Solution: By problem (40), we can fix a computable $A \subseteq \omega^2$ such that $E = \{n : (\exists m)((n, m) \in A)\}$. Since A is computable, it is definable in $\mathcal{N}$. So there is an $\mathcal{L}_{PA}$-formula $\phi(y, x)$ such that for every $(n, m) \in \omega^2$, $(n, m) \in A$ iff $\mathcal{N} \models \phi(n, m)$. Let $\psi(y)$ be the formula $(\exists x)(\phi(y, x))$. Then for every $n < \omega$, $n \in E$ iff $(\exists m)((n, m) \in A)$ iff $\mathcal{N} \models \psi(n)$. Hence E is definable in $\mathcal{N}$ via $\psi(y)$. That E is not numeralwise representable in PA follows from problem (41) and the fact that E is non-computable. □

(43) Do the Exercise on the last slide (180) of notes.

Solution: Let $m < \omega$. We must show that if $m \in H$, then Q returns 1 on input m and if $m \notin H$, then Q returns 0 on input m.

First suppose $m \in H$. Then for some $n < \omega$, $f(n) = m$. By Clause 1, $PA \vdash \psi(\overline{m}, \overline{n})$. Note that $\psi(\overline{m}, \overline{n}) \implies (\exists x)(\psi(\overline{m}, x))$ is a logical axiom of type 5. So by Modus Ponens, $PA \vdash (\exists x)(\psi(\overline{m}, x))$. Hence Q returns 1 on input m.

Next suppose $m \notin H$. We must show that $PA \nvdash (\exists x)(\psi(\overline{m}, x))$. Towards a contradiction, suppose $PA \vdash (\exists x)(\psi(\overline{m}, x))$. Since $\mathcal{N}$ is a model of PA, it follows that

$\mathcal{N} \models (\exists x)(\psi(m, x))$. Fix $n < \omega$ such that $\mathcal{N} \models \psi(m, n)$. Since $m \notin H = \text{range}(f)$, we must have $f(n) \neq m$. By Clause 2, this implies that $PA \vdash \neg\psi(\overline{m}, \overline{n})$. As $\mathcal{N}$ models PA, we get $\mathcal{N} \models \psi(m, n)$. So $\mathcal{N} \models \psi(m, n)$ and $\mathcal{N} \models \neg\psi(m, n)$: A contradiction.

It follows that Q computes H. $\square$