Theorem Divergence Theorem

- Let D be a solid in $\mathbb{R}^3$ bounded by a piecewise smooth orientable surface S.
- Let $F(x,y,z) = f_1(x,y,z)\,\vec{i} + f_2(x,y,z)\,\vec{j} + f_3(x,y,z)\,\vec{k}$ be a vector field such that f_1, f_2, f_3 are continuous and have continuous partial derivatives in an open set containing D.
- Suppose $\vec{n}$ is the unit outward normal to the surface S. Then

$$\underbrace{\iiint_D \operatorname{div}(F)\, dv}_{} = \underbrace{\iint_S (F.\vec{n})\, d\sigma}_{}.$$

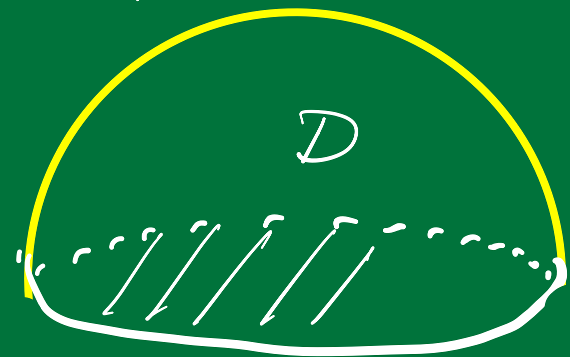
Example:

Triple integral   Surface integral

$F(x,y,z)$
$$= \left(xz \sin yz + x^3,\ \cos(yz),\ 3zy^2 - e^{x^2+y^2} \right)$$

$S_1 : z = 4 - x^2 - y^2,\ z \in [0,4]$

$$\iint_{S_1} (F \cdot \vec{n})\, d\sigma = ?$$



S_2

By divergence theorem

$$\iiint_D \operatorname{div}(F)\, dV = \iint_{S = S_1 \cup S_2} (F \cdot \vec{n})\, d\sigma$$

$$\iint_{S_1} (F \cdot \vec{n})\, d\sigma = \underbrace{\iiint_D \operatorname{div}(F)\, dV}_{?} - \underbrace{\iint_{S_2} (F \cdot \vec{n})\, d\sigma}_{?}$$

$$\iiint_D \operatorname{div}(F)\, dV = \iint_{x^2+y^2 \leq 4} \left(\int_{z=0}^{4-x^2-y^2} 3(x^2+y^2)\, dz \right) dx\, dy$$

$$= \dots$$

$$= 32\pi$$

$$\iint_{S_2} (F \cdot \vec{n})\, d\sigma = \iint_{\{(x,y)\,:\, x^2+y^2 \le 4\}} e^{x^2+y^2}\, dx\, dy$$

$$= \cdots$$

$$= \pi\,(e^4 - 1)$$

Therefore, $\displaystyle\iint_{S_1} (F \cdot \vec{n})\, d\sigma = 32\pi - \pi(e^4 - 1).$

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1 \right\}$$

Evaluate $\displaystyle\iint_S x\left(2x + 3e^{z^2}\right) + y\left(-y - e^{z^2}\right) + z\left(2z + \cos^2 y\right) \Big]\, d\sigma$

by Divergence Thm.

D? 

$$\iiint_D \operatorname{div}(F)\, dv = \iint_S F \cdot \vec{n}\, d\sigma$$

$F = (f_1, f_2, f_3)$

Consider

- $F(x, y, z) = \left(\underbrace{2x + 3e^{z^2}}, \underbrace{-y - e^{x^2}}, \underbrace{2z + \cos^2 y} \right)$

- $\vec{n} = ?$ $\qquad$ $\underline{(x, y, z)}$

- $\operatorname{div}(F) = 2 - 1 + 2 = 3$

By Divergence Theorem.

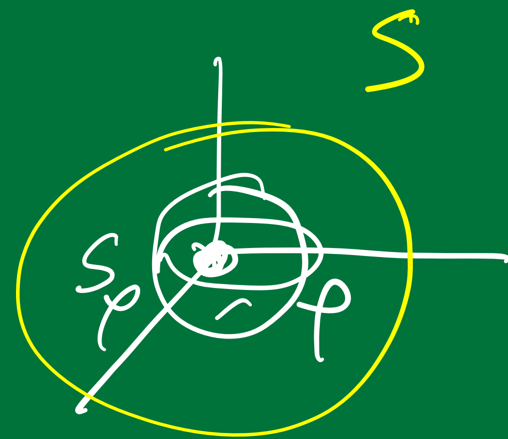
$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \mathrm{div}(F)\, dv$$

$= 3 \times$ Volume of the unit sphere

$= 4\pi$

4 $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Prove that $\displaystyle\iint_S \vec{F}\cdot\vec{n}\, d\sigma = 4\pi$



Here

$$F(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

$\Rightarrow \operatorname{div}(F) = 0$ for $(x,y,z) \neq (0,0,0)$

and $\operatorname{div}(F)$ is not defined at $(0,0,0)$.

Boundary surface of D = $S \cup S_a$

$$\iiint_D \operatorname{div}\vec{F}\, dv = 0$$

$$= \iint_S \vec{F}\cdot\vec{n}\, d\sigma - \iint_{S_\rho} \vec{F}\cdot\vec{n}\, d\sigma$$

$$= \frac{1}{\rho^2}\iint_{S_\rho} d\sigma = 4\pi$$