MTH113 QAM Class

Problem 15

Prove that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} j {a_j}^2\right)\left(\sum_{j=1}^{n} \frac{{b_j}^2}{j}\right)$$

for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

Proof. Let

$$u = \left(a_1, \sqrt{2}a_2, \ldots, \sqrt{n}a_n\right) \quad \text{and} \quad v = \left(b_1, \frac{1}{\sqrt{2}}b_2, \ldots, \frac{1}{\sqrt{n}}b_n\right).$$

Since $\langle u, v\rangle = \sum_{k=1}^{n} a_k b_k$, the Cauchy-Schwarz Inequality yields

$$\begin{aligned}(a_1b_1 + \cdots + a_nb_n)^2 &\leq \|u\|^2\|v\|^2 \\ &= \left(a_1{}^2 + 2a_2{}^2 + \cdots + na_n{}^2\right)\left(b_1{}^2 + \frac{b_2{}^2}{2} + \cdots + \frac{b_n{}^2}{n}\right),\end{aligned}$$

as desired. $\square$

Problem 9

Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1-|\langle u, v\rangle|.$$

Proof. By the Cauchy-Schwarz Inequality, we have $|\langle u, v\rangle| \leq \|u\|\|v\|$. Since $\|u\| \leq 1$ and $\|v\| \leq 1$, this implies

$$0 \leq 1 - \|u\|\|v\| \leq 1 - |\langle u, v\rangle|,$$

and hence it's enough to show

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \leq 1 - \|u\|\|v\|.$$

Squaring both sides, it suffices to prove

$$\left(1 - \|u\|^2\right)\left(1 - \|v\|^2\right) \leq (1 - \|u\|\|v\|)^2. \tag{3}$$

Notice

$$\begin{aligned}
(1 - \|u\|\|v\|)^2 - \left(1 - \|u\|^2\right)\left(1 - \|v\|^2\right) &= \|u\|^2 - 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| - \|v\|)^2 \\
&\geq 0,
\end{aligned}$$

and hence inequality (3) holds, completing the proof. $\square$

Problem 5

Let V be finite-dimensional. Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Let $v \in \operatorname{null}(T - \sqrt{2}I)$, and suppose by way of contradiction that $v \neq 0$. Then

$$\begin{aligned} Tv - \sqrt{2}v = 0 &\implies Tv = \sqrt{2}v \\ &\implies \|\sqrt{2}v\| \leq \|v\| \\ &\implies \sqrt{2} \cdot \|v\| \leq \|v\| \\ &\implies \sqrt{2} \leq 1, \end{aligned}$$

a contradiction. Hence $v = 0$ and $\operatorname{null}(T - \sqrt{2}I) = \{0\}$, so that $T - \sqrt{2}I$ is injective. Since V is finite-dimensional, this implies $T - \sqrt{2}I$ is invertible, as desired. $\square$

Problem 13

Find $p \in \mathcal{P}_5(\mathbb{R})$ that makes

$$\int_{-\pi}^{\pi} \left|\sin x - p(x)\right|^2 dx$$

as small as possible.

Proof. Let $\mathcal{C}_{\mathbb{R}}[-\pi, \pi]$ denote the real inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx,$$

and let U denote the subspace of $\mathcal{C}_{\mathbb{R}}[-\pi, \pi]$ consisting of the polynomials with real coefficients and degree at most 5. In this inner product space, observe that

$$\|\sin x - p(x)\| = \sqrt{\int_{-\pi}^{\pi} (\sin x - p(x))^2 \, dx} = \sqrt{\int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, dx}.$$

Notice also that $\sqrt{\int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, dx}$ is minimized if and only if $\int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, dx$ is minimized. Thus by 6.56, we may conclude $p(x) = P_U(\sin x)$ minimizes $\int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, dx$. To compute $P_U(\sin x)$, we first find an orthonormal basis of $\mathcal{C}_{\mathbb{R}}[-\pi, \pi]$ by applying the Gram-Schmidt Procedure to the basis $1, x, x^2, x^3, x^4, x^5$ of U. A lengthy computation yields the orthonormal basis

$$
\begin{aligned}
e_1 &= \frac{1}{\sqrt{2\pi}} \\
e_2 &= \frac{\sqrt{\frac{3}{2}}x}{x^{3/2}} \\
e_3 &= -\frac{\sqrt{\frac{5}{2}}\left(\pi^2 - 3x^2\right)}{2\pi^{5/2}} \\
e_4 &= -\frac{\sqrt{\frac{7}{2}}\left(3\pi^2 x - 5x^3\right)}{2\pi^{7/2}} \\
e_5 &= \frac{3\left(3\pi^4 - 30\pi^2 x^2 + 35x^4\right)}{8\sqrt{2}\pi^{9/2}} \\
e_6 &= -\frac{\sqrt{\frac{11}{2}}\left(15\pi^4 x - 70\pi^2 x^3 + 63x^5\right)}{8\pi^{11/2}}.
\end{aligned}
$$

Now we compute $P_U(\sin x)$ using 6.55(i), yielding

$$P_U(\sin x) = \frac{105\left(1485 - 153\pi^2 + \pi^4\right)}{8\pi^6}x - \frac{315\left(1155 - 125\pi^2 + \pi^4\right)}{4\pi^8}x^3 + \frac{693\left(945 - 105\pi^2 + \pi^4\right)}{8\pi^{10}}x^5,$$

which is the desired polynomial. $\square$

15 Suppose $C_{\mathbf{R}}([-1,1])$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

for $f, g \in C_{\mathbf{R}}([-1,1])$. Let φ be the linear functional on $C_{\mathbf{R}}([-1,1])$ defined by $\varphi(f) = f(0)$. Show that there does not exist $g \in C_{\mathbf{R}}([-1,1])$ such that

$$\varphi(f) = \langle f, g\rangle$$

for every $f \in C_{\mathbf{R}}([-1,1])$.

15. Solution: Suppose there exists g such that $\varphi(f) = \langle f, g\rangle$ for all $f \in C_{\mathbb{R}}[-1,1]$. We would like to show a contradiction.

For any positive integer n and integer $-n \leqslant i \leqslant n-1$, define

$$f_{n,i}(x) = \begin{cases} 4n^2(x - i/n), & \text{if } x \in [i/n, i/n + 1/(2n)] \\ 4n^2((i+1)/n - x), & \text{if } x \in [i/n + 1/(2n), (i+1)/n] \\ 0, & \text{otherwise ,} \end{cases}$$

then $f_{n,i}(x) \in C_{\mathbb{R}}[-1,1]$ and $f_{n,i}(0) = 0$.

Given any $\epsilon > 0$, since $g \in C_{\mathbb{R}}[-1,1]$, by the fact that a continuous function on a closed interval is uniformally continuous, there exists N such that for any $n \geqslant N$, we have

$$|g(x) - g(y)| \leqslant \epsilon \tag{1}$$

if $|x-y| \leqslant 1/n$.

Note that

$$\int_{-1}^{1} f_{n,i}(x)dx = \int_{i/n}^{(i+1)/n} f_{n,i}(x)dx = 1, \tag{2}$$

for any $y \in [i/n, (i+1)/n]$ we have

$$\begin{aligned} &\left| g\left(y\right) - \int_{-1}^{1} f_{n,i}(x) g(x) dx \right| \\ = &\left| \int_{i/n}^{(i+1)/n} f_{n,i}(x) \left(g\left(y\right) - g(x)\right) dx \right| \end{aligned}$$

for any $y \in [i/n, (i+1)/n]$ we have

$$\begin{aligned}
&\left|g(y) - \int_{-1}^{1} f_{n,i}(x)g(x)dx\right| \\
&= \left|\int_{i/n}^{(i+1)/n} f_{n,i}(x)\left(g(y) - g(x)\right)dx\right| \\
&\leqslant \int_{i/n}^{(i+1)/n} f_{n,i}(x)\left|g(y) - g(x)\right|dx \\
\text{by (1) and (2)} \quad &\leqslant \int_{i/n}^{(i+1)/n} f_{n,i}(x)\epsilon dx = \epsilon.
\end{aligned}$$

On the other hand, we also have

$$0 = f_{n,i}(0) = \varphi(f_{n,i}) = \langle f_{n,i}, g\rangle = \int_{-1}^{1} f_{n,i}(x)g(x)dx.$$

Hence we have

$$|g(y)| = |g(y) - f_{n,i}(0)| \leqslant \epsilon$$

for any $y \in [i/n, (i+1)/n]$. Thus $|g(x)| \leqslant \epsilon$ by taking all $-n \leqslant i \leqslant n-1$ with $n \geqslant N$. Since ϵ is chosen arbitrarily, we have $g(x) \equiv 0$. Hence $\varphi f \equiv 0$ for all $f \in C_{\mathbb{R}}[-1,1]$, which is impossible. Therefore the proof is complete.

. (a) Describe the orthogonal complement of $U = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = 0,\ z + w = 0\}$ (denoted $U^\perp$) with respect to the standard inner product of $\mathbb{R}^4$.

(b) Consider $V = \mathrm{Span}\{(1, 1, 0), (-1, 1, 1)\}$ as a subspace of $\mathbb{R}^3$. Find the orthogonal projection of $(1, 0, 1)$ onto V. (6+6)

Solution: (a) (i) Basis of U is given by $\{(1,-1,0,0),(0,0,1,-1)\}$.

(ii) $U^{\perp}=\{(x,y,z,w) \mid \langle(x,y,z,w),(1,-1,0,0)\rangle=\langle(x,y,z,w),(0,0,1,-1)\rangle=0\}$.

(iii)Therefore

$$U^{\perp}=\{(x,x,y,y) \mid x,y\in\mathbb{R}\}$$

(b) (i) Orthonormal basis of V is given by $\{\frac{1}{\sqrt{2}}(1,1,0),\frac{1}{\sqrt{3}}(-1,1,1)\}$.

(ii) Orthogonal projection of $(1,0,1)$ onto V is

$$\langle(1,0,1),\frac{1}{\sqrt{2}}(1,1,0)\rangle\frac{1}{\sqrt{2}}(1,1,0)+\langle(1,0,1),\frac{1}{\sqrt{3}}(-1,1,1)\rangle\frac{1}{\sqrt{3}}(-1,1,1)=(\frac{1}{2},\frac{1}{2},0).$$

Example 5. Find an orthogonal basis for the space of solutions of the homogeneous equations

$$\begin{aligned} 3x - 2y + z + w &= 0, \\ x + y + 2w &= 0. \end{aligned}$$

Let W be the space of solutions in $\mathbf{R}^4$. Then W is the space orthogonal to the two vectors

$$(3, \ -2, \ 1, \ 1) \qquad \text{and} \qquad (1, \ 1, \ 0, \ 2).$$

These are obviously linearly independent (by any number of arguments, you can prove at once that the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$$

has rank 2, for instance). Hence

$$\dim W = 4 - 2 = 2.$$

Next we find a basis for the space of solutions. Let us put $w = 1$, and solve

$$\begin{aligned} 3x - 2y + z &= -1, \\ x + \ \ y \quad\ \ &= -2, \end{aligned}$$

by ordinary elimination. If we put $y = 0$, then we get a solution with $x = -2$, and

$$z = -1 - 3x + 2y = 5.$$

If we put $y = 1$, then we get a solution with $x = -3$, and

$$z = -1 - 3x + 2y = 10.$$

Thus we get the two solutions

$$A = (-2, \ 0, \ 5, \ 1) \qquad \text{and} \qquad B = (-3, \ 1, \ 10, \ 1).$$

Question 3: 6.4.13 For the space $\mathbb{R}^4$, let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}$,

$y = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and let $W = sp\{w_1, w_2\}$. (a) Find a basis for W consisting of two orthogonal vectors. (b) express y as the sum of a vector in W and a vector in $W^{\perp}$.

Solution (a) Apply step 1 of Gram-Schmidt:

$$v_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$v_2 = w_2 - \frac{(w_2, v_1)}{(v_1, v_1)} v_1 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix} - \frac{\left(\begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)}{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}.$$

This gives us an orthogonal basis $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$ for W.

(b) We must find vectors $w \in W$ and $w' \in W^\perp$ such that $y = w + w'$. Using our orthogonal basis from (a) and the Second Projection Theorem, we get

$$w = \frac{\left(\begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)}{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\left(\begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} \right)}{\left(\begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} \right)} \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$ Then w'

$$w' = y - w = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix}.$$

Compute $W^\perp$, where

$$W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Solution

According to the proposition, we need to compute the null space of the matrix

$$\begin{pmatrix} 1 & 7 & 2 \\ -2 & 3 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1/17 \\ 0 & 1 & 5/17 \end{pmatrix}.$$

The free variable is x_3, so the parametric form of the solution set is $x_1 = x_3/17$, $x_2 = -5x_3/17$, and the parametric vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1/17 \\ -5/17 \\ 1 \end{pmatrix}.$$

Scaling by a factor of 17, we see that

$$W^{\perp} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -5 \\ 17 \end{pmatrix} \right\}.$$

We can check our work:

$$\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 17 \end{pmatrix} = 0 \qquad \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 17 \end{pmatrix} = 0.$$

QAM CLASS-TOPIC:8

Eigen-values, Eigen-Vectors

made by:Harsh agrawal

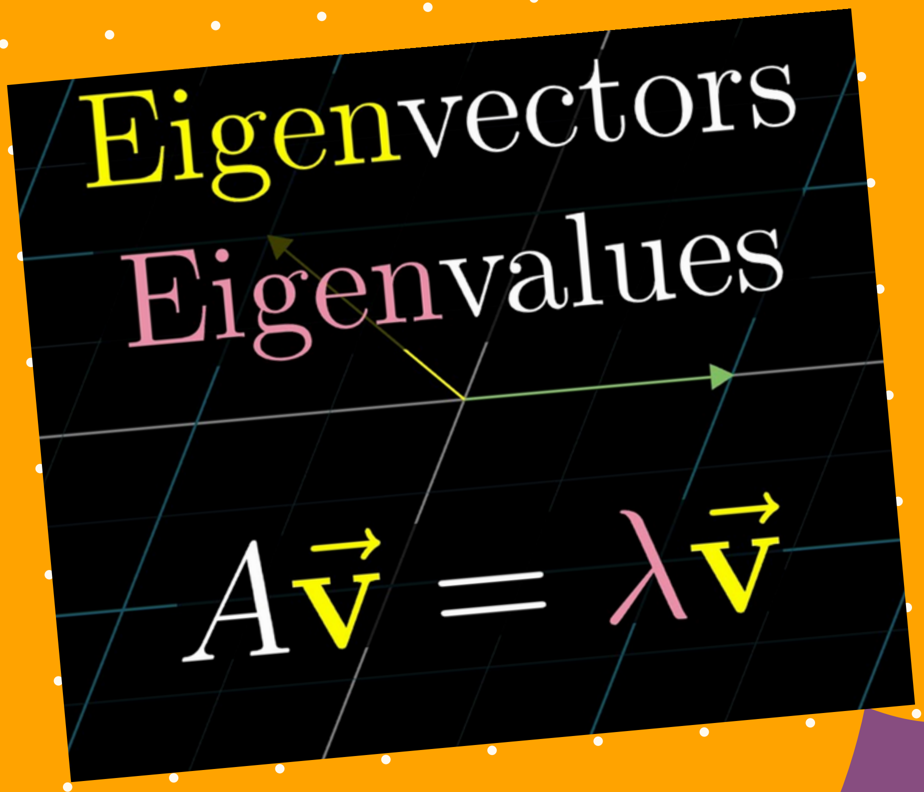
QUESTION -01

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

A) DETERMINE THE EIGENVALUES OF A BY COMPUTING THE CHARACTERISTIC POLYNOMIAL OF A.

B) OBTAIN EIGEN VECTORS FOR EACH OF THE EIGENVALUES OF A.

RECALL THE STEPS



WE SUMMARIZE THE COMPUTATIONAL APPROACH FOR DETERMINING EIGENPAIRS (λ, x) (EIGENVALUES AND EIGENVECTOR) AS A TWO-STEP PROCEDURE:

Step I. To find the eigenvalues of **A** compute the roots of the characteristic equation $\mathbf{det(A - \lambda I_n)} = 0$.

Step II. To find an eigenvector corresponding to an eigenvalue μ, compute a nontrivial solution to the homogeneous linear system $\mathbf{(A - \mu I_n)x = 0}$.

SOLUTION-01

(A)

Proof: (a) The characteristic polynomial of A is $|\lambda - IA| = \begin{vmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 = \lambda^2 - 3\lambda - \lambda + 3 = (\lambda - 3)(\lambda - 1)$. So the eigenvalues are $3, 1$. **(1 Marks)**

(B)

(b) **Step 1:** Let eigenvector corresponding to 1 be $v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\begin{aligned} Av_1 = v_1 &\Rightarrow \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ &\Rightarrow 2x - y = x \text{ and } -x + 2y = y \\ &\Rightarrow x = y \\ &\Rightarrow v_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

In particular, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1.

Step 2: Let an eigenvector corresponding to 3 be $v_3 = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\begin{aligned} Av_3 = v_3 &\Rightarrow 2x - y = 3x \text{ and } -x + 2y = 3y \\ &\Rightarrow v_3 = x \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

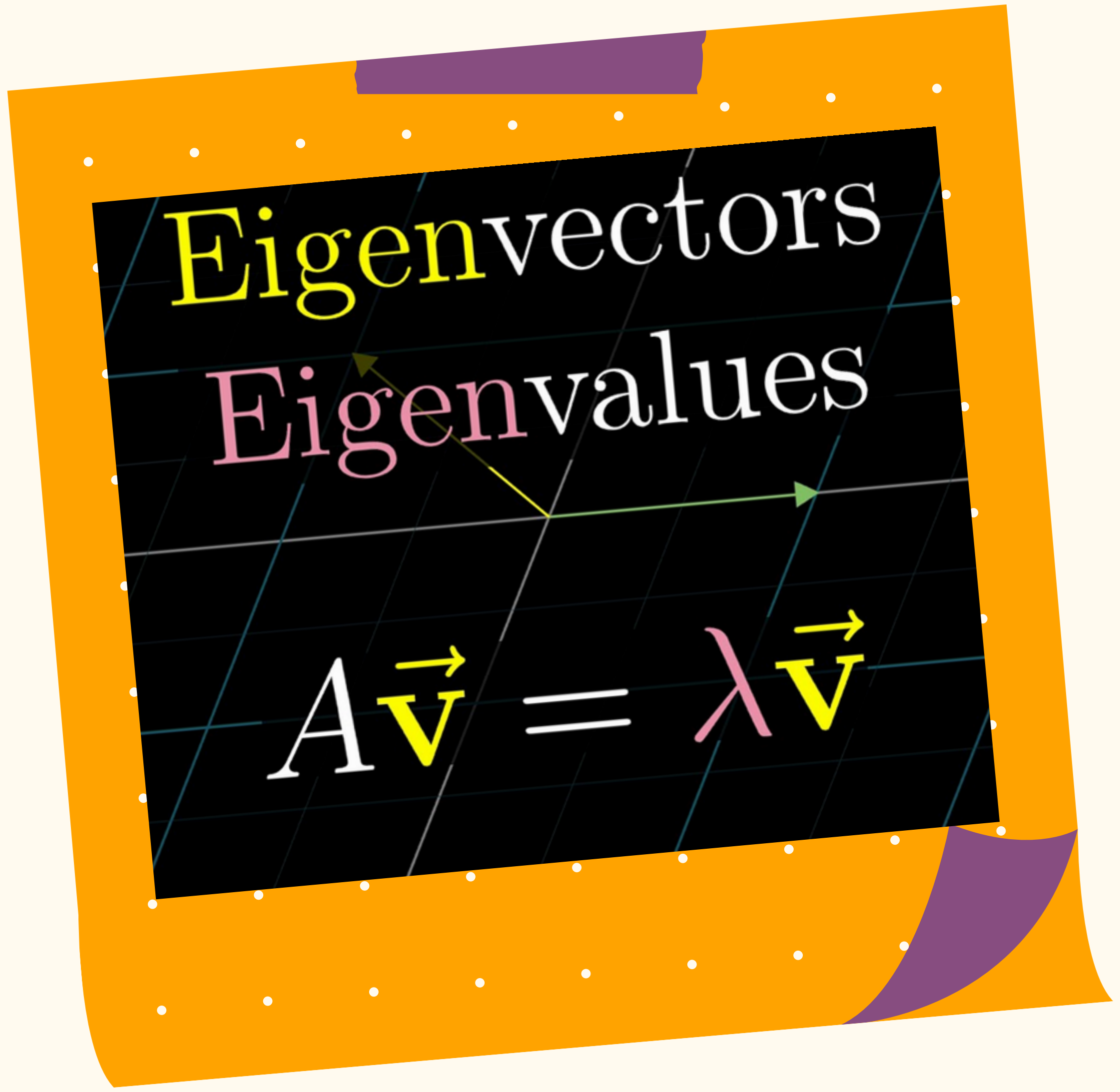
In particular, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to 3.

QUESTION -02

(a) Show that $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable.

(b) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$. Find a real matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. Justify your answer. (5 + 7)

RECALL THE STEPS



DIAGONALIZATION OF MATRICES

- LET A BE A N*N MATRIX, THEN A IS DIAGONALIZABLE IF AND ONLY IF A HAS N LINEARLY INDEPENDENT EIGEN VECTORS

SOLUTION-01

(A)

(i) $\det(A - \lambda I) = (3 - \lambda)(1 - \lambda)^2$. Therefore eigenvalues of A are $1, 3$.

(ii) We first note that $A - 3I = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ and it has rank two. Hence $N(A - 3I)$ is one dimensional. (2)

(iii) Next, $A - I = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is also of rank two, hence $N(A - I)$ also has dimension one. (1)

(iv) Therefore there exists only two L.I. eigenvectors of A. This gives that A is not diagonalizable. (1)

(B) (b) Here eigenvalues are $1, 2, 3$. (1)

We have $A - I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$ and (x, y, z) is in the kernel of $A - I$ if and only if $x = y$ and $2x + 2y + z = 0$. Hence it is the span of $(1, 1, -4)$. (2)

We also have $A - 3I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix}$ and (x, y, z) is in the kernel of $A - 3I$ if and only if $x + y = 0$ and $2x + 2y - z = 0$. Hence an eigenvector corresponding to the eigenvalue 3 is $(1, -1, 0)$. (2)

We also have $A - 2I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$ and (x, y, z) is in the kernel of $A - 2I$ if and only if $x = y = 0$. Hence an eigenvector corresponding to the eigenvalue 2 is $(0, 0, 1)$. (1) (basically 2+2+1 marks for finding three L.I. eigenvectors)

For $Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$ (1)

we have

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(1)

TAKE HOSTEL PROBLEM

$$A = \begin{bmatrix} 4 & 10 & 3 \\ 4 & -2 & -3 \\ 0 & 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

1. If the eigenvalues of A are $\lambda 1 = -4$, $\lambda 2$, $\lambda 3$, determine $\lambda 2$, $\lambda 3$ and the eigenvectors corresponding to $\lambda 2$ and $\lambda 3$.
2. Among A and B, which one is diagonalizable?

ANSWERS

1. (0,8).
   For 0, an eigenvector is (1, -1, 2) AND For 8, an eigenvector is (9, 3, 2)
2. A is diagonalizable as it has 3 distinct eigenvalues. B is NOT diagonalizable as it has only two linearly independent eigenvectors.

MTH 113 QAM CLASS
Lecture 9

Question 1

(3) Let W_1 and W_2 be subspaces of $M_{2\times 2}(\mathbb{R})$ (the space of 2×2 real matrices) given by

$$W_1 = \{A \in M_2(\mathbb{R}) \mid A = A^T\}$$

and

$$W_2 = \{A \in M_2(\mathbb{R}) \mid EAE = A^T\},$$

where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find a basis of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

What if we were only asked to find dimension instead of basis?

Solution

Here $W_1 = \{\begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R}\}$ and basis of $W_1 = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

$W_2 = \{\begin{bmatrix} a & b \\ c & a \end{bmatrix} \mid a, b, d \in \mathbb{R}\}$ and a basis of $W_2 = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$

basis of $W_1 \cap W_2 = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$

Solution

Hence we get:

dimension of W1=3
dimension of W2=3
dimension of $W1 \cap W2=2$

Solution

In order to find dim W1+W2, we can use the following theorem:

$$\dim(P+Q) = \dim(P) + \dim(Q) - \dim(P \cap Q)$$

where, P and Q are subspaces of a finitely-dimensional vector space

Solution

Thus, dim $W1+W2$= dim $W1$+ dim $W2$– dim $W1 \cap W2$
$=3+3-2=4$

dim $W1+W2=4$

Thus, using the theorem, we were able to calculate dimension of $W1+W2$, without actually finding basis for it!

Question 2

Let W be a subspace of V. Show that there is no subspace U such that $U \cap W = \{0\}$ and $\dim U + \dim W > \dim V$.

This can also be done by previous theorem!

Solution

Since $U \cap W = \{0\}$, $\dim U \cap W = 0$

Since, $\dim U+W = \dim U + \dim V - \dim U \cap W$

and $\dim U \cap W = 0$,

$\dim U+W = \dim U + \dim V$

Solution

Since U and W are subspaces of V, thus for every $a \in U$, $b \in W$, $a+b \in V$

Thus, $\dim(U+W) <= \dim V$

Solution

Thus,

$$\dim U + \dim W = \dim (U+W) <= \dim V$$

$$\dim U + \dim W <= \dim V$$

Solution

Thus, it contradicts the fact that
$\dim U + \dim W > \dim V$

Thus, there is no such subspace U!