365Days

English Study Plan

Introduction to ODE

$$t \mapsto x(t) = (x_1(t), x_2(t), x_3(t))$$

$$t \mapsto v(t) = (\dot{x}_t) = (\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t))$$

$$t \mapsto a(t) = \dot{v}(t) = \ddot{x}(t) = (\ddot{x}_1(t), \ddot{x}_2(t), \ddot{x}_3(t))$$

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

$$x \to (F_1(x), F_2(x), F_3(x))$$

$$\Rightarrow F(x(t)) = ma(t)$$
$$= m\ddot{x}(t)$$

$$\boxed{\ddot{x}(t) = \frac{1}{m} F(x(t))}$$

Autonomous ~~differential equation~~ force field doesn't change with time

Second order ODE

Example

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x_1, x_2, x_3) \to (\sin x_1, x_2^2, x_3^3)$$

$$(x_1, x_2, x_3(t)) = \sin x_1(t)$$

$$\ddot{x}_1(t) = \sin x_1$$

$$\ddot{x}_2(t) = \frac{1}{m} x_2^2$$

$$\ddot{x}_3(t) = \frac{1}{m}(x_3)^3$$

Convert 2D ODE into system of 1D ODE's

$$t \mapsto x(t) \in \mathbb{R}^3$$

$$t \mapsto (x(t), v(t)) \in \mathbb{R}^6$$

2 dependent variables

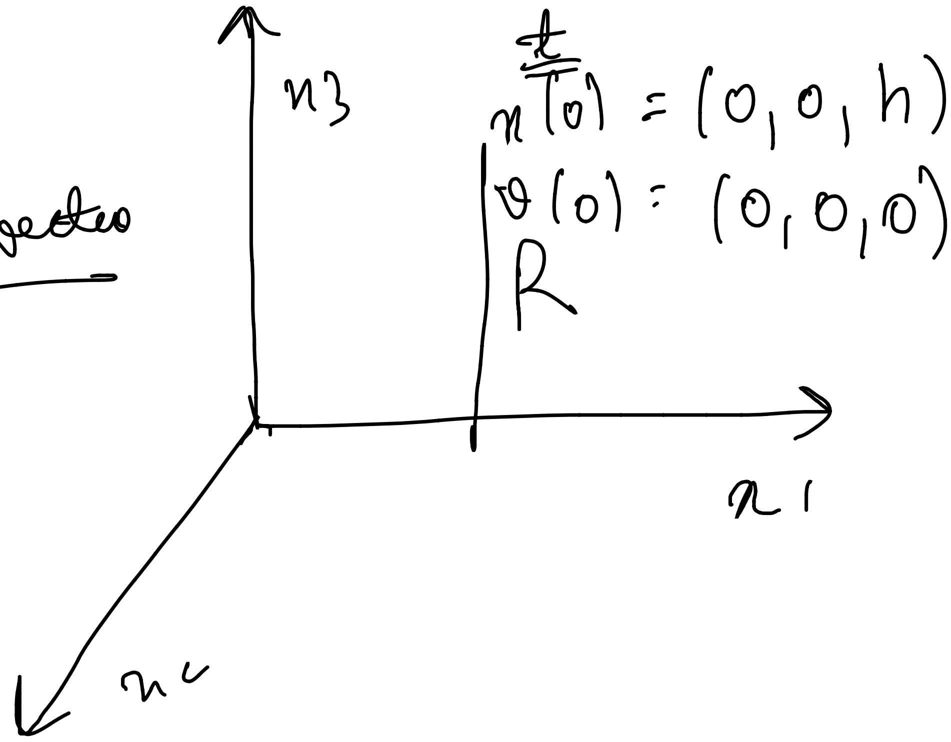
$$\dot{x}(t) = v(t) \quad \text{--- (1)}$$

$$\dot{v}(t) = \frac{1}{m} F(x(t)) \quad \text{--- (2)}$$

Example

$F = -mg\underbrace{(0,0,1)}_{\text{unit vector}}$

$= (0, 0, -mg)$



$\ddot{x}_1(t) = 0$

$\ddot{x}_2(t) = 0$

$\ddot{x}_3(t) = -g$

$x_1(t) = a_1 + a_2 t$ | $x_1(0) = a_1$, $a_2 = v_1(0)$

$x_2(t) = b_1 + b_2(t)$ | $x_2(0) = b_1$, $b_2 = v_2(0)$

$x_3(t) = c_1 + c_2 t + \frac{-gt^2}{2}$ | $x_3(0) = c_1$, $c_3 = v_3(0)$

$$x(t) = x(0) + t v(0) - \frac{1}{2} g t^2 (0,0,1)$$

2

$$F(x) = -\frac{g m M x}{\|x\|^3}$$

$$m\ddot{x}_j(t) = -\frac{g m M x_j(t)}{\left(|x_1(t)|^2 + |x_2(t)|^2 + (x_3(t)|^2\right)^{3/2}} \quad j = 1,2,3$$

Difficult to solve. Does solution exist?

Classification of differential equations

$$x^{(k)}(t) = f(t, x(t), x^{(1)}(t), \dots x^{(k-1)}(t)) \qquad f: \mathbb{R}^{k+1} \to \mathbb{R} \quad (1)$$

Here $x^{(k)}(t) = \dfrac{d^k x}{d t^{k}}$

Example

$f(x_1, x_2) = x_2^2$ Here $f: \mathbb{R}^2 \to \mathbb{R}$ $\boxed{k=1}$

$\dot{x}(t)$ $f(x_1, x_2) = x_2^2$

$(\dot{x}_1, x_2, \dots x_n)$

$$\left.\begin{aligned} x_1^{(k)}(t) &= f_1(t, x(t), x^2(t), \dots x^{(k-1)}(t)) \\ x_2^{(k)}(t) &= f_2(t, x(t), \dots\ x^{(k-1)}(t)) \\ &\vdots \\ x_n^{(k)}(t) &= f_n(t, x(t), \dots\ x^{(k-1)}(t)) \end{aligned}\right\} \text{System of } k \text{ order ODE} \quad (2)$$

$$t \longmapsto (x_1(t), x_2(t), \dots x_n(t)) \in \mathbb{R}^n$$

$$f \qquad (f_1, f_2, \dots f_n) \in \mathbb{R}^n$$

$\boxed{\text{Definition}}$

Solution for the differential equation (1) is a differentiable function φ s.t

$$\varphi^{k}(t) = f(t, \varphi(t), \varphi'(t) \dots \varphi^{k-1}(t))$$

(1) will be called linear if $x^{k}(t) = a_0(t) + a_1(t)x(t) \dots a_k(t)\, x^{(k-1)} t$

(2) will be called linear if for any i, $1 \le i \le n$

$$\underset{\text{component}}{x_i^{(k)}(t)} = g_i(t) + \sum_{i=1}^{n} \sum_{j=0}^{k-1} f_j(t)\, x_i^{(j)} \qquad x_i^{(j)}$$

(2) is called homogeneous if $g_i(t) = 0 \ \forall\ 1 \le i \le n$

Fri 1 August

① $y' = \sin(x)\, y + \cos y$ ✗ as ~~sin~~ $\cos y$

$y' = \sin y\, x + \cos x$ ✗ as $\sin y$

$y' = \sin x\, y + \cos x$ ✓ linear in y.

Which of them is linear?

② Find the most general form of a second order linear equation.

③ Transform them into a system of first order diff

(1) $x'' + t \sin x' = x$

(2) $\ddot{x} = -y$, $\ddot{y} = x$

$(x, \dot{x}) = (z_1, z_2)$

$q = (y, \dot{y}) = (q_1, q_2)$

$\dot{z}_2 = -q_1$ $\quad \dot{q}_2 = z_1$

$\dot{z}_1 = z_2$

$\dot{q}_1 = q_2$

2 $$\frac{d^2 x_1}{dt^2} = g_1(t) + (a_{11} x_1(t) + \dots a_{1n} x_n(t)) + (b_{11} \dot{x}_1(t) + \dots b_{1n} \dot{x}_n(t))$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix} + A(t) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + B(t) \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix}$$

3 Take $y = (y_1, y_2) = \boxed{(x, \dot{x})}$ = introduce 2 variables

① $\dot{y}_1 = y_2$

$\dot{y}_2 = -t \sin y_2 + y_1$

$\dot{y} = f(t, y)$

$(\dot{y}_1, \dot{y}_2)$ " y_2 & $-t \sin y_2 + y_1$

in vector format

Q) autonomous first order systems (transform)

$\ddot{x} + t\sin\dot{x} = x$

$\boxed{\text{Autonomous means } \dot{z}(t) = f(z) \text{ not } f(t,z)}$

Idea is put $(t, x, \dot{x})$ as dependent variable

not just $(x, \dot{x})$

$$\begin{cases} z = (t, x, \dot{x}) \\ \dot{z}_1 = 1 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = -z_1 \sin z_3 + z_2 \end{cases}$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = f(z_1, z_2, z_3) = \begin{pmatrix} 1 \\ z_3 \\ -z_1 \sin z_3 + z_2 \end{pmatrix}$$

Q) $x^k = f(x, x^{(1)}, \dots, x^{(k-1)})$ .. (1)

let ϕ be a solution of (1), then show that $\phi(t - t_0)$ is also a solution.

$\psi(t) = \phi(t - t_0)$

$\psi^{k}(t) = \phi^k(t - t_0)$

$= f(\phi(t-t_0), \dot{\phi}(t-t_0), \dots \phi^{k-1}(t-t_0))$

$\psi^k(t) = f(\psi(t), \dot{\psi}(t), \dots \psi^{k-1}(t))$

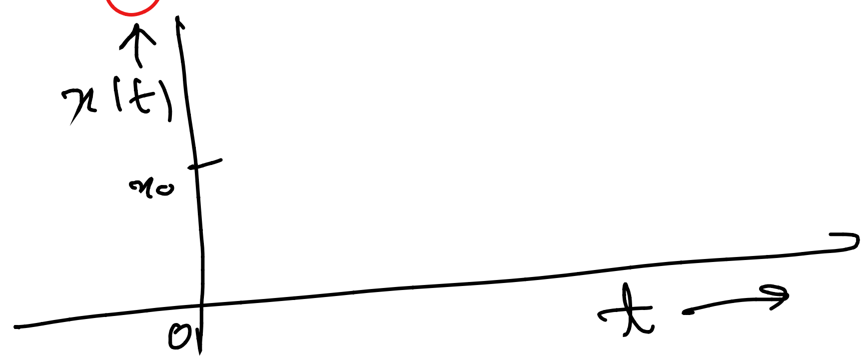
[class 2]

Consider an autonomous first order diff equation.

$\dot{x}(t) = f(x(t))$ $\quad x(0) = x_0$ — (1)

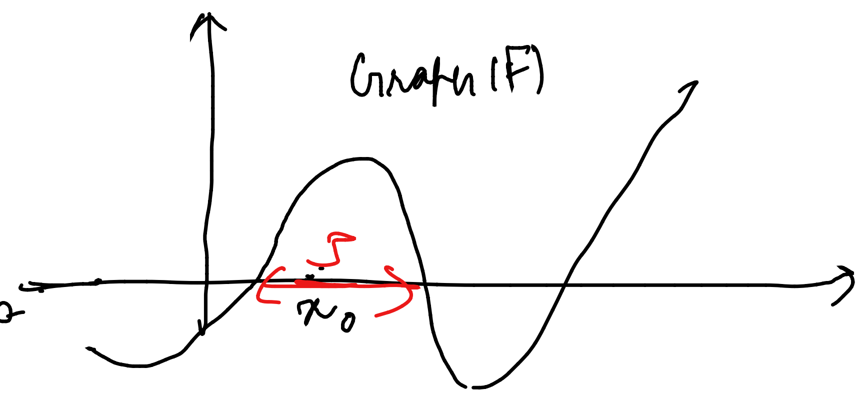
$\dot{x}(0) = f(x_0)$

$f \in C(I, \mathbb{R})$, $I \subset \mathbb{R}$, I is an interval



Example

i) $\dot{x}(t) = x(t)$, $x(0) = x_0$



Question: Does the solution of (1) always exist and if yes how many solutions are there.

Case I: $f(x_0) \neq 0$

Let $J \subset \mathbb{R}$ be the maximal interval containing x_0 such that $f \neq 0$ on J.

$\frac{1}{f} \in C(J)$
continuous function.

$\Rightarrow$ So we can define the function

$$F(y) = \int_{x_0}^{y} \frac{1}{f(s)}\,ds, \quad y \in J$$ as $\frac{1}{f}$ continuous so integral

By FTC, F is differentiable, infact $F \in C^1(J)$
differentiable & f' continuous

$F'(y) = \frac{1}{f(y)} \quad \forall y \in J$ — (2)

F is either strictly increasing or strictly decreasing depending on what sign in that interval?

Now let g be the inverse of F on J.

$F(g(y)) = y \quad \forall y \in J$

$F'(g(y))\,g'(y) = 1 \quad \forall y \in J$ — (3)

By (2), (3) we have

$\boxed{g'(y) = f(g(y)) \quad \forall y \in J}$ $\quad g(0) = F^{-1}(0) = x_0$

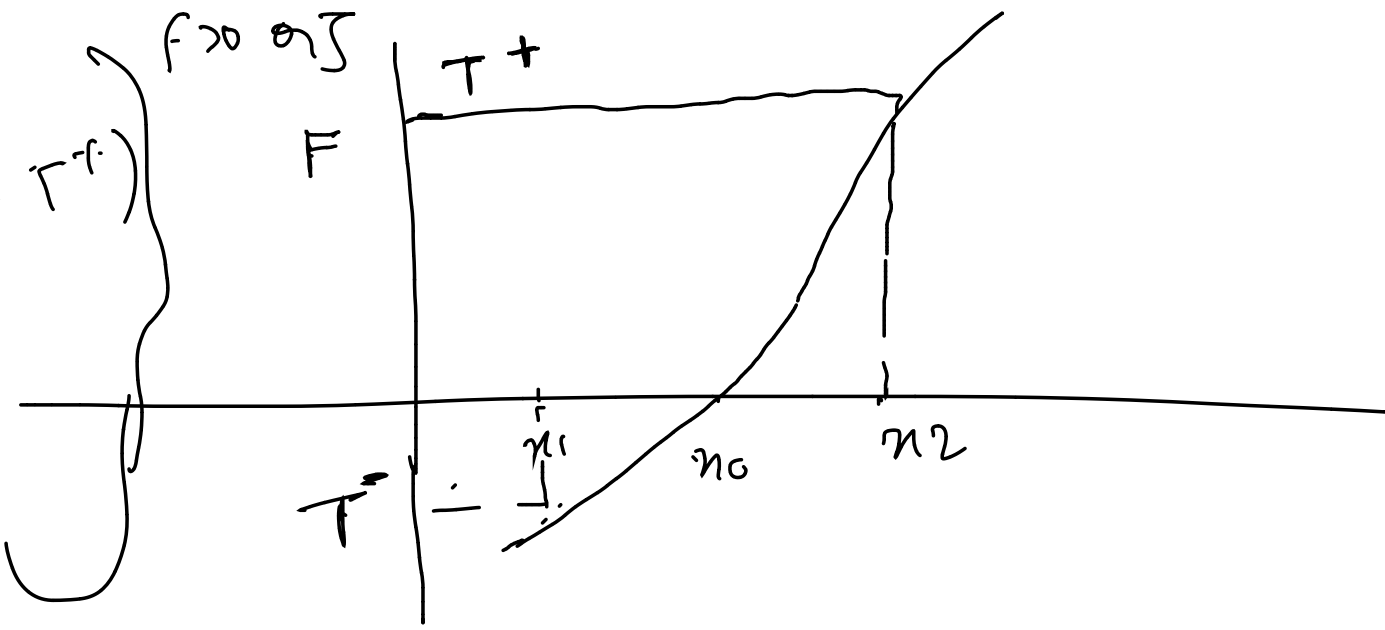
$J = (x_1, x_2)$

g is defined on (T^-, T^+)

$T_- = \lim_{x \downarrow x_1} F(x)$

$T_+ = \lim_{x \uparrow x_2} F(x)$

$f > 0$ on J



Conclusion

If you know $f(x_0) \neq 0$ then a small interval exists where the solution exists particularly $g(y)$.

$\boxed{\text{Case 2}}$: $f(x_0) = 0$ $\qquad \dot{x}(t) = |x(t)|^{1/2}$, $f(x) = |x|^{1/2}$

$x(0) = 0$

$x(t) = x_0$

$\dot{x}(t) = 0 = f(x(t)) = f(x_0) = 0$

So this is a trivial solution.

What if $x(t) = x_0$ not $x(0)$

$\dot{x}(t) = f(x(t))$,

$x(t_0) = x_0$, Suppose φ is a solution then $\varphi(t - t_0)$ is a solution of (1).

interestingly.

But we can check for other solutions

In the case when $\exists$ an interval $I \ni x_0$ such that $\frac{1}{f}$ is integrable on I & finite

$$\left| \int_I \frac{1}{f} \right| < +\infty$$

then we can define $F(y) = \int_{x_0}^{y} \frac{1}{f}$.

Example 1: $\dot{x}(t) = x(t)^2$, $x(0) = x_0 > 0$

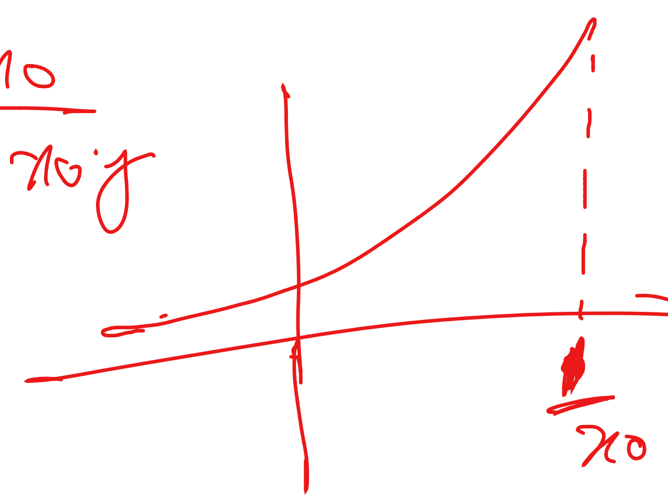
$f(s) = s^2$, $f(x_0) = x_0^2 \neq 0$

$J = (x_1, x_2) = (0, \infty)$

$$F(y) = \int_{x_0}^{y} \frac{1}{f(y)} ds = \int_{x_0}^{y} \frac{1}{s^2} ds = \frac{1}{x_0} - \frac{1}{y}, \quad y \in (0, \infty)$$

$\text{Range}(F) = \left(-\infty, \frac{1}{x_0}\right)$ = domain of my solution as solution is inverse.

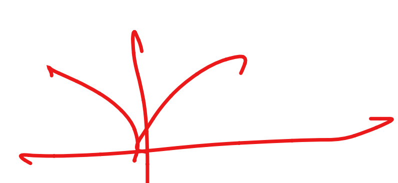
AND the solution is $g(y) = \frac{x_0}{1 - x_0 y}$



$\boxed{\text{we can see solution exists on a particular interval of } \mathbb{R}. \text{ not entire}}$

Example 2: $\dot{x}(t) = |x(t)|^{1/2}$ $x(0) = 0$

Here $f(s) = |s|^{1/2}$ $\forall s \in \mathbb{R}$



Clearly we have a trivial solution $x(t) = 0$

Here $f>0$ on $J=(0,\infty)$ and infact $\int_0^y \frac{1}{\sqrt{s}}\,ds = 2\sqrt{y}$

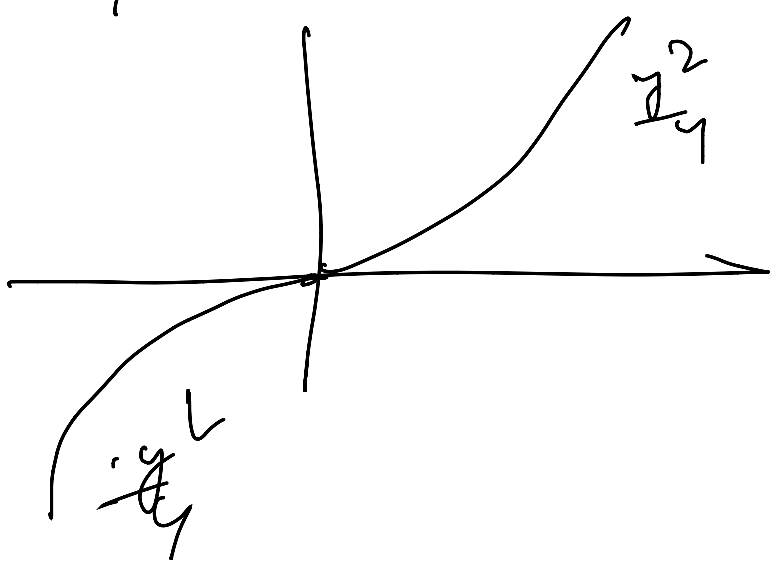
$F(y) = (2\sqrt{y})$ $y>0$

It's inverse is given by $g(y) = \frac{y^2}{4}$ on $\mathbb{R}_+$

$\Rightarrow$ If φ is a solution on $\mathbb{R}_+$ then $\psi(t)$ is a solution on $\mathbb{R}_-$, $\psi(t) = -\varphi(-t)$ $t \in \mathbb{R}$

$\therefore$ on $\mathbb{R}^-$, $-\frac{y^2}{4}$ is another solution.

Hence,



on full $\mathbb{R}$ there is a solution.

Wait other solution is there

Here so many solutions.

Class 4

$\Rightarrow$ $\dot{x}(t) = f(x(t))$, $x(0) = x_0$

$f \in C^1(\mathbb{R}, \mathbb{R})$ — $*$

Qualitative Analysis of solutions:

$\dot{x}(t) = f(x(t))$

$x(0) = x_0$

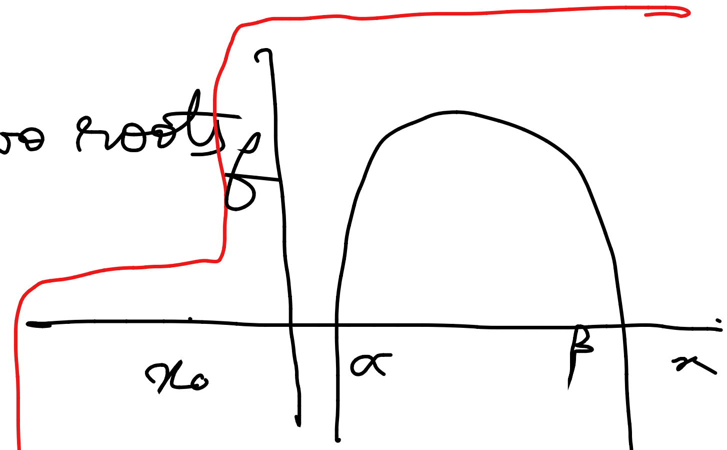
Assume that the solution is unique

$\Rightarrow$ $\dot{x}(t) = (1 - x(t))\, x(t) - h$ (Logistic growth model)

suppose $h \in \left(0, \frac{1}{4}\right)$

$f(s) = (1-s)\,s - h = s - s^2 - h$

The quadratic $s - s^2 - h$ has two roots



$\alpha = \dfrac{1 - \sqrt{1-4h}}{2}$

$\beta = \dfrac{1 + \sqrt{1-4h}}{2}$

Case 1: $x_0 \in (-\infty, \alpha)$ $f < 0$ on $(-\infty, \alpha)$

$F(y) = \int_{x_0}^{y} \frac{1}{f(s)}\, ds$ $y \in (-\infty, \alpha)$

F is strictly decreasing

Range $F = (T_-, T_f)$

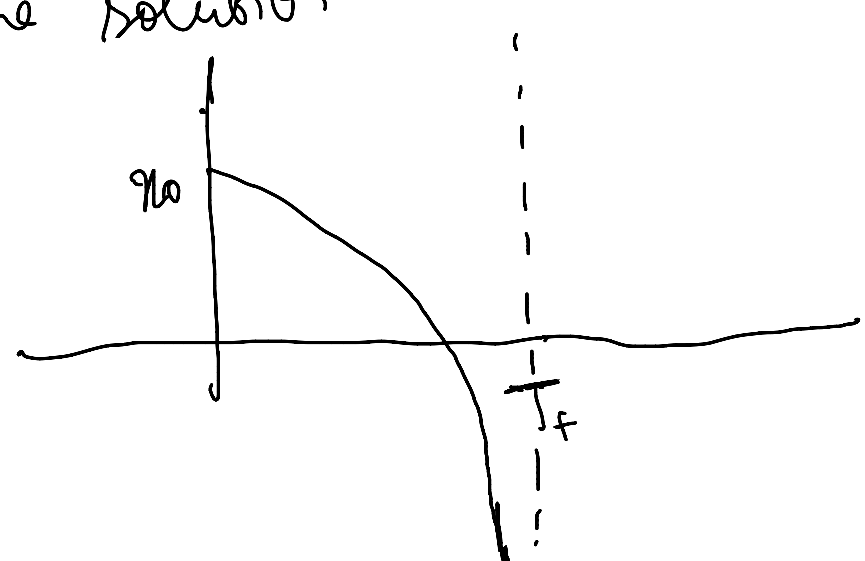
$T_f = \lim_{x \to -\infty} F(x)$

$T_- = \lim_{x \to \alpha} F(x)$

(T_-, T_f), F^{-1} is the solution

interval where sol exists.

$\lim_{y \to T_f} g(y) = -\infty$



Case II

Suppose $x_0 = \alpha$ or β

$x(t) = \alpha$ is a solution

Case III $x_0 \in (\alpha, \beta)$

$f|_{(\alpha,\beta)} > 0$ F is strictly increasing.

$\text{Range } F = (T_-, T_+)$

$T_+ = \lim_{x \to \beta} F(x(t))$ $\lim_{t \to T_+} g(t) = \beta$

$T_- = \lim_{x \to \alpha} F(x(t))$ $\lim_{t \to T_-} g(t) = \alpha$

Case IV

$x_0 \in (\beta, \infty)$

$f|_{(\beta,\infty)} < 0$ F is strictly decreasing

$\text{Range } F = (T_-, T_+)$ $\lim_{x \to \beta} F(x) = T_+$ $\lim_{t \to T_+} g(t) = \beta$

Lemma: Consider the first order autonomous initial value problem where $f \in C(\mathbb{R})$ is such that the solutions are unique.

1) if $f(x_0) = 0$ then the unique solution is $x(t) = x_0 \; \forall t$

2) if $f(x_0) \neq 0$ then $x(t)$ converges to the first zero of f left to x_0 ($f(x_0) < 0$) respect to the right zero of x_0 ($f(x_0) > 0$)

If there is no zero the solution converges to $-\infty$, resp ∞

Is logistic growth model an example of this?

② $\dot{x}(t) = x^2 - t^2$ —① (not autonomous) $\to$ unique solution exists

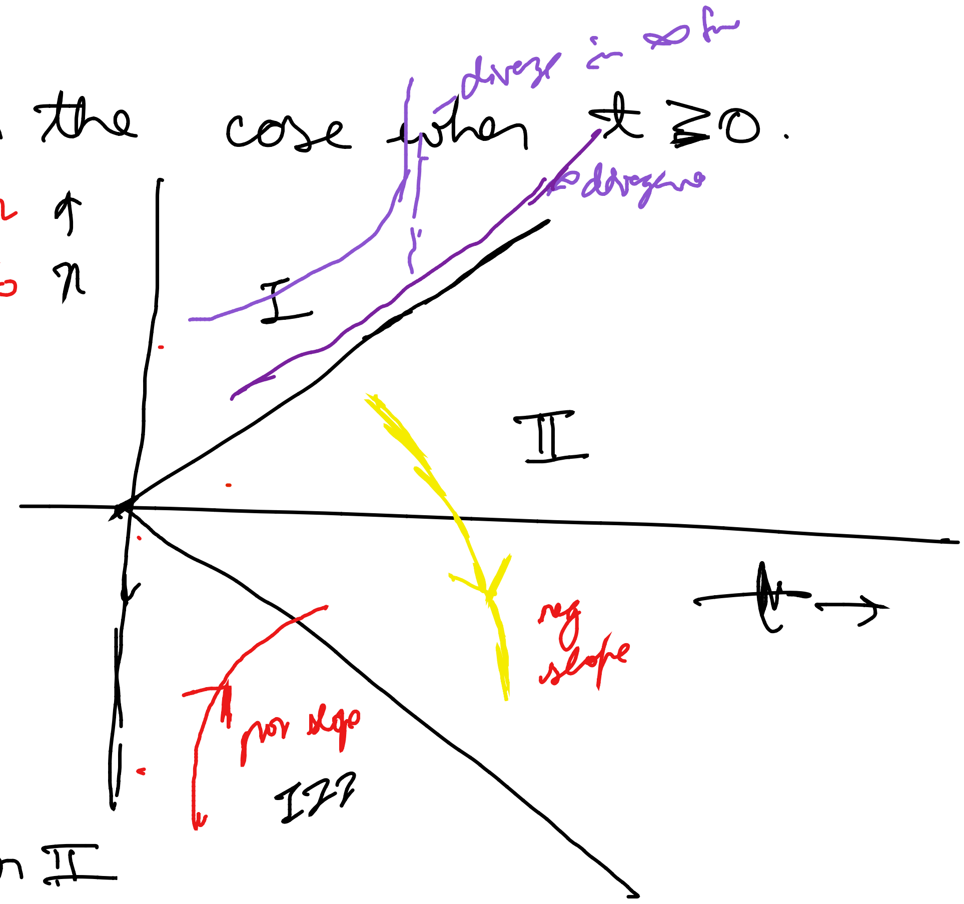
$f(x,t) = x^2 - t^2$

Note if ψ is a solution on $\mathbb{R}_+$ of ① then $-\psi(-t)$ is a solution on $\mathbb{R}_-$.

Therefore we can consider the case when $t \geq 0$.

$\dot{\psi}(t) = \psi(t)^2 - t^2$

$+\dot{\psi}(-t) = \psi(-t)^2 - t^2$, $t<0$



<u>Note:</u>

Region I: $f(t,x) > 0$ in Region I

Region II. $f < 0$

Region III: $f \geq 0$

$\Rightarrow$ If $(t_0, x_0) \in$ Region ① then either the solution remains in Region ① or it enters in Region II

Let's consider the case where it remains in Region I.

I claim it is always diverging to ∞ either in finite time or infinite time. Because if on the domain (T) it does not, then we can set another IVP with

time means

means it is bdd

$f(T^+) = M$ and extend the solution.

$\Rightarrow$ If $(t_0, x_0) \in$ Region II, then the solution decreases

$\Rightarrow$ $(x_0, t_0) \in$ Region III then the solution enters II after some time.

$$\boxed{\begin{array}{l}\text{Note the lemma in next page implies that if } (t_0,x_0) \in \text{Region II} \\ \text{then any solution of } \dot{x}(t) = x^2 - t^2 \text{ remains in between} \\ x = t,\ x = -t. \text{ In particular for any finite time, the} \\ \text{solution is bounded. This also shows that } x(t) \to -\infty \text{ as } t \to \infty\end{array}}$$

Why can't we do this in region II

Lecture 5

Consider $\dot{x}(t) = x^2 - t^2$

Note if φ is a solution of (1) on $I \subseteq \mathbb{R}_+$ then ψ is a solution of (1) on $-I$ where $\psi(t) := -\varphi(-t)$

Infact, $\dot{\psi}(t) = \dot{\varphi}(-t) = f\varphi(-t))^2 - t^2$

$$= (-\varphi(-t))^2 - t^2$$

$$= (\psi(t))^2 - t^2$$

Therefore for (1), we should focus on when $t \geq 0$

Definition :-

→ Consider a differential equation $\dot{x}(t) = f(t, x)$, $x(t_0) = x_0$ ... (2)

A differentiable function x_+ is called a super/upper sol. of (2) if $\dot{x}_+(t) > f(t, x_+(t))$ on an interval $[t_0, T)$

→ A differentiable function x_- is called a subsolution/lower solution of (2) if

$$\dot{x}_-(t) < f(t, x_-(t)) \text{ on a interval } [t_0, T)$$

Example : $f(t, x) = x^2 - t^2$

$x_+(t) = t$; $f(t, x_+(t)) = f(t, t) = t^2 - t^2 = 0$

$\dot{x}_+(t) = 1$ $\quad \dot{x}_+(t) > f(t, x_+(t))$

$x(t) = -t$

$\dot{x}(t) = -1$

$f(t, x_-(t)) = t^2 - t^2 = 0$ $\quad$ and $\dot{x}_-(t) < f(t, x_-(t))$

$\boxed{\text{Lemma}}$ Consider the diff Eq (2).

Suppose $x_+(t)$, $x_-(t)$, $x(t)$ are super solution, sub solution & solution respectively of (2). Then

$x_+(t) > x(t)$ $\forall$ $t \in (t_0, T)$ whenever $x(t_0) \leq x_+(t_0)$ and

$x_-(t) < x(t)$ $\forall$ $t \in (t_0, T)$ whenever $x_-(t_0) \leq x(t_0)$

Proof: $\Delta(t) = x_+(t) - x(t)$

$\Delta(t_0) \geq 0$

(3) $\Delta(t) = 0 \Rightarrow \dot{\Delta}(t) = \dot{x}_+(t) - \dot{x}(t) > f(x^+(t), t) - f(x, t)$

> 0

Suppose $\Delta(t_1) = 0$

$\dot{\Delta}(t_1) > 0$

$$\Delta'(t_1) = \lim_{h \to 0} \frac{\overbrace{\Delta(t_1)}^{0} - \overbrace{\Delta(t_1 - h)}^{> 0}}{\underbrace{h}_{> 0}}$$

< 0

Contradiction. $\blacksquare$

(1) f is continuous and $f(x_0) > 0$

$\exists$ a nhbd around x_0 s.t

$f(x) > 0$

$\exists \delta$ s.t $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

$f(x) = f(x_0) + \underset{\in \mathbb{R}}{\epsilon} \geq f(x_0) - |\epsilon|$

$\geq f(x_0) - |f(x) - f(x_0)|$

(2) $f(x) = 0$

$f'(x) > 0$

$\frac{f(x+h) - f(x)}{h} = f'(x)$

$f(x+h) > 0$

Lecture 6

Assume in definition of super/sub sol if we allow equality then the lemma is not true.

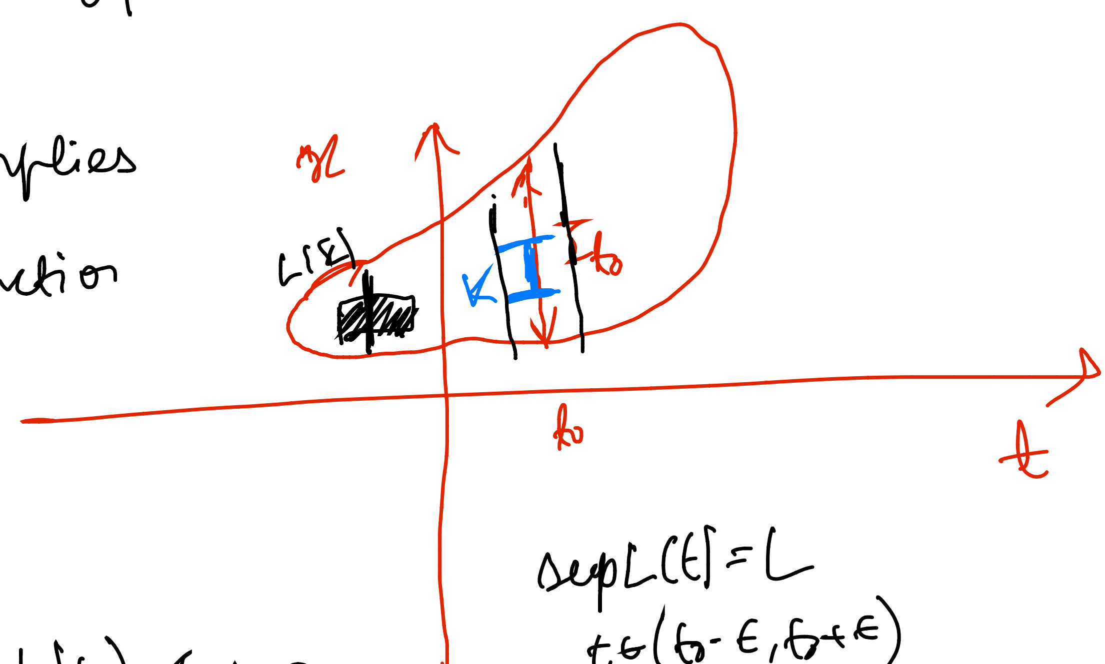
Definition

Let $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function.
open

We say f is locally Lipschitz in the second variable, uniformly w.r.t first variable if for any compact ~~set~~ $K \subset U$ cpt, $\exists L(K) \geq 0$ s.t

$$\text{s.t} \quad \sup_{\substack{(t,x) \neq (t,y) \\ \in K}} \frac{|f(t,x) - f(t,y)|}{|x-y|} = L(K) < +\infty$$

Note in particular, it implies



① for a fixed t_0 the function

$g_{t_0}(\cdot) = f(t_0, \cdot)$

$(t_0, x) \in U \;\; \forall x \in I_0$

$$\sup_{K \subset I_{t_0}} \frac{|f(t_0,x) - f(t_0,y)|}{|x-y|} = L(t_0) < +\infty$$

$$\sup_{t \in (t_0 - \epsilon, t_0 + \epsilon)} L(t) = L$$

② Moreover the constant $L(t_0)$ is uniform in a nbd of t_0.

$K \subset \mathbb{R}$ cpt $\quad g: I \to \mathbb{R}$

$$\boxed{\sup_{x \neq y \in I} \frac{|g(x) - g(y)|}{|x-y|} = L < +\infty} \quad \text{Lipschitz}$$

$$\boxed{\sup_{\substack{x \neq y \\ \in K}} \frac{|g(x) - g(y)|}{|x-y|} = L(K) < +\infty} \quad \text{locally Lipschitz}$$

implies continuity $\leftarrow$

Example: Let $f \in \underbrace{C'}_{\text{partial derivatives exist and are continuous}}(U, \mathbb{R})$. Then f is locally Lipschitz in the second variable, uniformly wrt to the first variable.

$\Rightarrow$ $f(t,x) = t^2 + x^2$

Fix (t_0, x_0) and take $\underbrace{(t_0 - r, t_0 + r)(x_0 - r, x_0 + r)}_{\text{product nbd}} = U(t_0, x_0)$

$$\sup_{(t,x) \in U(t_0,x_0)} \frac{|f(t,x) - f(t,y)|}{|x-y|} = L(t_0, x_0) < +\infty$$

$\Rightarrow$ $\boxed{\text{Theorem}}$: Suppose $f : \underset{\text{open}}{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be locally lipschitz in the second variable uniformly wrt first variable

Let $x(t)$ and $y(t)$ be two differentiable functions in $[t_0, T)$ st $\text{Graph}(x)$ and $\text{Graph}(y) \subset U$. Suppose $x(t_0) \leq y(t_0)$ and $x'(t) - f(t, x(t)) \leq y'(t) - f(t, y(t))$ $\forall t \in [t_0, T)$

Then we have
$$x(t) \leq y(t) \ \forall \ t \in [t_0, T)$$

Moreover if for some t_1, $x(t_0) < y(t_0)$ then $x(t) < y(t)$ $\forall t > t_1$.

We have uniqueness here as $x'(t) = f(x,t)$ and let x_1 & x_2 be 2 solutions and $x_1(t_0) = x_2(t_0)$ then $x_1(t) \leq x_2(t)$, $x_2(t) \leq x_1(t)$. Hence same

Proof: $\Delta(t) = x(t) - y(t)$, $t \in [t_0, T)$

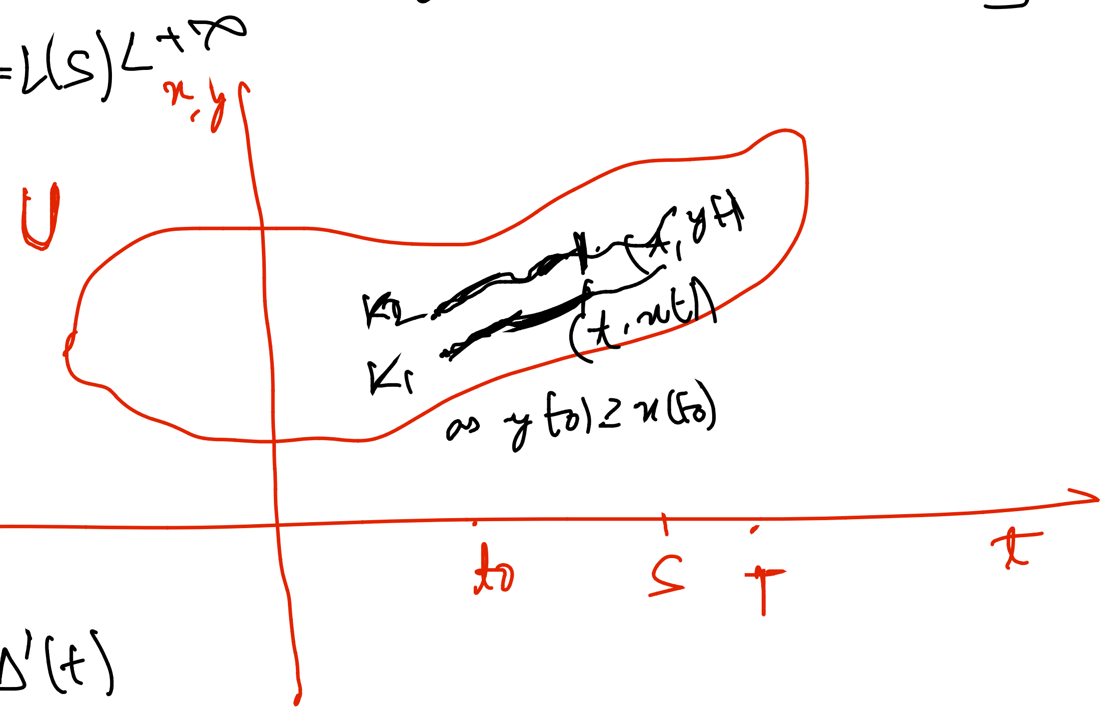
$\Delta(t_0) \le 0$ — (1)

$\dot{\Delta}(t) = \dot{x}(t) - \dot{y}(t)$

$\le f(t, x(t)) - f(t, y(t)) \quad \forall\, t \in [t_0, T)$

Fix a number $S < T$, then $K = \{(t, x(t)) : t \in [t_0, S]\}$

$$\sup_{K_1 \cup K_2} \frac{|f(t, x(t)) - f(t, y(t))|}{|x(t) - y(t)|} = L(S) < +\infty$$



$\dot{\Delta}(t) \le L(S)\,\Delta(t)$ — (2)

Define $\tilde{\Delta}(t) = \Delta(t) e^{-Lt}$

$\tilde{\Delta}'(t) = -L\Delta(t) e^{-Lt} + e^{-Lt}\Delta'(t)$

$= e^{-Lt}\left(\Delta'(t) - L\Delta(t)\right)$

≤ 0 — (3)

$\tilde{\Delta}(t_0) = \Delta(t_0) e^{-Lt_0}$

≤ 0 — (4)

M1 $\therefore \boxed{\tilde{\Delta}(t) \le 0}$ as initial value ≤ 0 value ≤ 0.

M2 Now, $\dfrac{\tilde{\Delta}(t) - \tilde{\Delta}(t_0)}{t - t_0} = \tilde{\Delta}(\xi)$ $\quad \xi \in [t_0, t)$ $\quad t < S$

$\tilde{\Delta}(t) - \tilde{\Delta}(t_0) \le 0 \Rightarrow \boxed{\tilde{\Delta}(t) \le \tilde{\Delta}(t_0)} \Rightarrow \boxed{\tilde{\Delta}(t) \le 0}$

$\Delta(t) \le \Delta(t_0) \le 0 \qquad t \in [t_0, S]$

$\Rightarrow x(t) \le y(t) \quad \forall\, t$

Lecture 7

$P_y(x,y) = y$

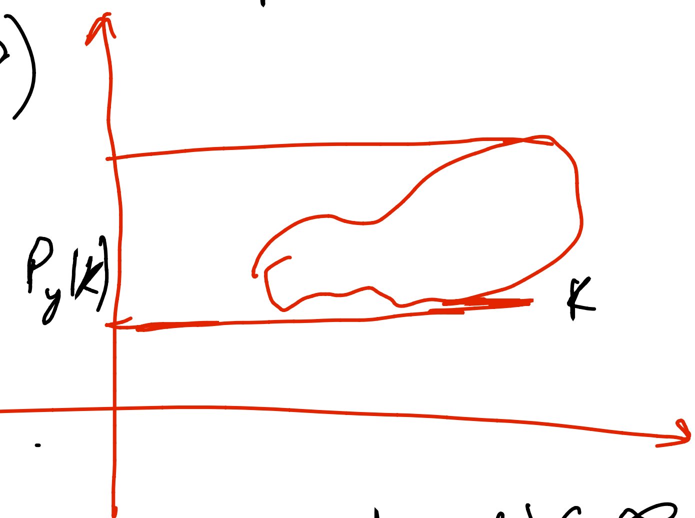
Example 1: $f(x,y) = x^2+y^2$, $U \subseteq \mathbb{R}^2$

Let K be a compact set on $\mathbb{R}^2$.

$$f(x,y_1) - f(x,y_2) = (x^2+y_1^2) - (x^2+y_2^2)$$

$$= y_1^2 - y_2^2$$

$$|f(x,y_1) - f(x,y_2)| = |y_1-y_2|\,|y_1+y_2|$$



$$\boxed{\sup_{(x_1,y_1)\neq(x,y_2)\in K} \frac{|f(x,y_1) - f(x,y_2)|}{|y_1-y_2|} = L(K) < \infty}$$

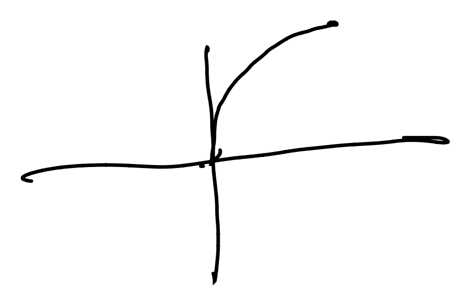
$$\sup_{y_1,y_2\in P_y(K)} |y_1+y_2| = L(K) < \infty$$

$$\le L(K)\,|y_1-y_2|$$

Example 2

$\Rightarrow$ Consider a compact set $[-1,1]\times[-1,1]$ $\quad f(x,y) = |y|^{1/2}$ on $\mathbb{R}^2$

Let $$\sup_{\substack{(x,y_1)\neq(x,y_2)\\ \in K}} \frac{|f(x,y_1) - f(x,y_2)|}{|y_1-y_2|} = L(K) < \infty$$



$$\frac{\left||y_1|^{1/2} - |y_2|^{1/2}\right|}{|y_1-y_2|} = \left|\frac{(y_1-y_2)}{(y_1-y_2)(\sqrt{y_1}+\sqrt{y_2})}\right|$$

$$\sup_{y_1>y_2>0} \frac{1}{\sqrt{y_1}+\sqrt{y_2}} = \infty$$

This is not locally lipschitz.

Corollary: Consider

$$\dot{x}(t) = f(t, x(t)) \quad , x(t_0) = x_0 \quad - ①$$

Assume f is locally Lipschitz in second variable, uniformly wrt first variable.

Let x_1, x_2 be solution of IVP ① then

$$x_1(t) = x_2(t)$$

$\boxed{\text{Exercise}}$

$$\left.\begin{aligned} f(x,y) &= x|y|^{1/2} \quad \forall (x,y) \in \mathbb{R}^2 \\ f(x,y) &= |x|^{1/2}|y|^{1/2} \quad \forall (x,y) \in \mathbb{R}^2 \\ f(x,y) &= xy \quad (x,y) \in \mathbb{R}^2 \end{aligned}\right\}$$

Show locally lipschitz in second variable uniformly wrt first variable

Chapter 2

Existence of solution for the first order differential equation

$$\dot{x}(t) = f(t, x(t)) \quad , x(t_0) = x_0$$

where $x : I \to \mathbb{R}^n$ is an unknown fn & $f : I \times U \to \mathbb{R}^n$ is a given continuous function.

$(t \quad x)$

$(f_1(t,x), \cdots f_n(t,x))$

$$\dot{x}_j(t) = f_j(t, x(t)) \quad , x(t_0) = y_0$$

component

→ Let's start with

$$\dot{x}(t) = f(t, x(t)) \quad x(t_0) = x_0 \quad - ①$$

$$f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(t_0, x_0)$$

$$x(t) = x_0 + \int_{t_0}^{t} f(s, \underline{x(s)})\, ds \quad - ②$$

unknown but

Consider the integral operator

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

Example $\dot{x}(t) = x(t)$ $\quad x(0) = 1$

$f(x, y) = y$

$$K(x)(t) = 1 + \int_0^t x(s)\, ds \Rightarrow$$ this tells give me a $x(t)$ I will compute this for you

like take $x(t) = 1$ or $x(t) = 1+t$

Lecture 8 — Photos

Lecture 9

$\Rightarrow$ F^n, $F = \mathbb{R}$ or $\mathbb{C}$

$p \geq 1$

Given $x \in F^n$, $x = (x_1, \ldots, x_n)$, define

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \cdots \quad (\star)$$

$$\|x\|_\infty = \sup\{ |x_i| : 1 \leq i \leq n \}$$

(1) $\|x\| \geq 0$

(2) $\|\alpha x\| = |\alpha| \|x\|$

(3) Triangle

Q $\rightarrow$ Verify that for $p = 1, 2, \infty$ $\|x\|_p$ defines a norm.

$\Rightarrow$ Fact:

On F^n, $n \geq 2$, for each $p \geq 1$, $(\star)$ defines a norm;

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \forall x, y \in F^n \rightarrow$$ Minkowski ineq

$\Rightarrow$ Young's Inequality

$a, b \geq 0$ and $p > 1$ & $q > 1$ s.t $\frac{1}{p} + \frac{1}{q} = 1$

Then $$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$f(a, b) = \frac{a^p}{p} + \frac{b^q}{q} - a \cdot b$

Fix $b_0 \geq 0$

$$g(a) = \frac{a^p}{p} + \frac{b_0^q}{q} - a \cdot b_0$$

g has global minima at the point $b_0^{1/p-1}$;

$\Rightarrow g(a) \geq g(b_0^{1/p-1}) = 0$

$\Rightarrow$ Holder's inequality

Let $x, y \in \mathbb{F}^n$, let $p \geq 1$ then

$$\sum_{i=1}^{n} |x_i||y_i| \leq \|x\|_p \|y\|_q$$

$$\boxed{\sum_{i=1}^{n} |x_i||y_i| \leq \left(\sum_{i=1}^{n} |x_i|\right)^{\dots}}$$

here $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_a^b |fg| \leq \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^q\right)^{1/q}$$

$\Rightarrow$ To show Minkowski's inequality:

we observe

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}$$

$\Rightarrow$ Norm 1 and Norm 2 are equivalent if $\exists c$ $\frac{1}{c}\|x\|_1 \leq \|\cdot\|_2 \leq c\|\cdot\|_1$

On $\mathbb{F}^n$, any two norms are equivalent.

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, $\|x\|_\infty = \sup\{|x_i| : 1 \leq i \leq n\}$

$$\|x\|_2 \leq \left[\sum_{i=1}^{n} (\|x\|_\infty)^2\right]^{1/2} = \sqrt{n}\|x\|_\infty$$

$\|x\|_2 \geq \|x\|_\infty$

$$\Rightarrow \boxed{\|x\|_\infty \leq \|x_2\| \leq \sqrt{n}\|x\|_\infty}$$

Example 2:

I a compact vector space of $\mathbb{R}$.

$I = [a,b]$

$C([a,b], \mathbb{R})$ = the set of real valued continuous functions on $[a,b]$

→ This is a vector space over $\mathbb{R}$.

Given $f \in C([a,b], \mathbb{R})$, define

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$$

① $\|f\|_\infty \geq 0$ and $\|f\|_\infty = 0$ iff $f = 0$

② $\|\alpha f\|_\infty = \sup\{|\alpha f(x)| : x \in [a,b]\}$
$= |\alpha| \|f\|_\infty$

③ $\|f+g\|_\infty = \sup\{|f(x) + g(x)| : x \in [a,b]\}$
$\leq \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)|$
$\leq \|f\|_\infty + \|g\|_\infty$ (triangular inequality.

Example 3

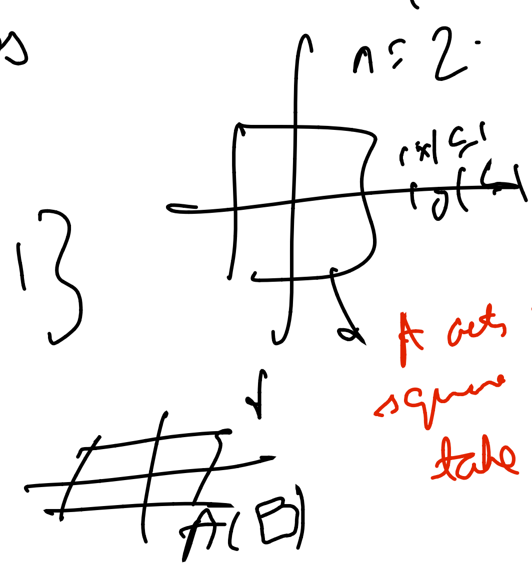
(2D case)

$n = 2$.

$M_n(F)$ = the set of all $n \times n$ matrices

Given $A \in M_n(F)$, we define;

$$\|A\|_{op} = \sup\{\|Ax\|_\infty : x \in F^n : \|x\|_\infty \leq 1\}$$



A acts on the square then take

① $\|A\|_{op} \geq 0$, $\|A\|_{op} = 0 \Rightarrow \|Ax\|_\infty = 0 \quad \forall \|x\|_\infty \leq 1$

$\Rightarrow \quad Ax = 0 \quad \forall x \in \mathbb{F}^n$

$\Rightarrow \quad A = 0$

② $\|\alpha A\|_{op} = \sup\{\|\alpha Ax\|_\infty : x \in \mathbb{F}^n : \|x\|_\infty \leq 1\}$

$= |\alpha| \, \|A\|_{op}$

③ $\|A_1 + A_2\|_{op} \leq \|A_1\|_{op} + \|A_2\|_{op}$

$\|A_1 + A_2\|_{op} = \sup\left\{\|(A_1 + A_2)(x)\|_\infty : x \in \mathbb{F}^n : \|x\|_\infty \leq 1\right\}$

$\leq \|A_1\|_{op} \times \|A_2\|_{op}$

④ $\|Ax\|_\infty \leq \|A\|_{op} \|x\|_\infty$

⑤ $\|A_1 A_2\|_{op} \leq \|A_1\|_{op} \|A_2\|_{op}$

Lecture 10

⇒ **Convergence in Normed Space**

Let $(V, F, \|\cdot\|)$ be a normed space

Recall the distance

$$d_{\|\cdot\|}(f,g) = \|f-g\|, \quad f,g \in V$$

① $d_{\|\cdot\|}(f,f) = 0, \quad d_{\|\cdot\|}(f,g) = 0$ iff $f = g$

② $d_{\|\cdot\|}(f,g) = d_{\|\cdot\|}(g,f)$

③ $d_{\|\cdot\|}(f,g) + d_{\|\cdot\|}(g,h) \geq d_{\|\cdot\|}(f,h)$

⇒ $(f_n) \subset (V, F, \|\cdot\|)$

We say (f_n) converges to $f \in V$ if $d(f_n, f) = \|f_n - f\| \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} f_n = f$$

⇒ $T : (V_1, F, \|\cdot\|_1) \to (V_2, F, \|\cdot\|_2)$

T will be called continuous if for any $(f_n) \subset V$, s.t.

$f_n \to f$ implies $T(f_n) \to T(f)$

⇒ **Exercise**

Check that the norm, vector addition & scalar multiplication are continuous.

$\|\cdot\| : V \to \mathbb{R}_+$

$(f_n) : f_n \to f$

$\|f_n\| \to \|f\|$ comes from triangular inequality $\big|\,\|f_n\| - \|f\|\,\big| \leq \|f_n - f\| \to 0$ as $n \to \infty$

to show

⇒ Definition (Cauchy sequences)

$(V, F, \|\cdot\|)$

Let $(f_n) \subset V$. We say (f_n) is Cauchy if given $\epsilon > 0, \exists N_0(\epsilon)$ such that $d(f_n, f_m) = \|f_n - f_m\| < \epsilon, \forall n, m > N_0(\epsilon)$

We say $(V, F, \|\cdot\|)$ is complete if each cauchy sequence in V ~~has a~~ limit point.

$\boxed{\text{Complete normed vector spaces: Banach spaces}}$

⇒ Example: $I = [a, b]$

$C(I, \mathbb{R})$ = the set of all cont fns on I

$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$

Let (f_n) be a sequence of cont fns such that (f_n) is Cauchy.

Given $\epsilon > 0, \exists N_0(\epsilon)$ s.t

$$\|f_n - f_m\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \epsilon \quad \forall n, m \geq N_0(\epsilon) \qquad (1)$$

(kind of uniform convergent)

⇒ Fix an $x \in [a, b]$ then (1) implies that $(f_n(x))$ is Cauchy.

$\forall x$ som N_0 cond as sup on left

$(f_n(x))$ is Cauchy sequence of real numbers ~~so~~ its has a limit

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \cdots (2)$$

We have shown pointwise convergence to f.
(i) To show uniform conv in that norm.
(ii) To show f is continuus

(ii) Again from (1) given $\epsilon > 0$, if we choose $n \geq N_0(\epsilon)$ and fix it and let $m \to \infty$ then we have

$$\sup_{x \in [a,b]} |f(x) - f_n(x)| < \epsilon \quad \forall\, n \geq N_0(\epsilon) \qquad -(*)$$

(iii) for continuous:

Fix $x_0 \in [a,b]$

$$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$$

$$\leq \underbrace{|f(x) - f_n(x)|}_{< \epsilon/3} + \underbrace{|f_n(x) - f_n(x_0)|}_{< \epsilon/3} + \underbrace{|f_n(x_0) - f(x_0)|}_{< \epsilon/3}$$

for large enough n and does not depend on x from $(*)$

using continuity at x_0, $x \in B_\delta(x_0)$ $< \epsilon/3$

$\Rightarrow$ $\boxed{\text{Banach Fixed Point Theorem}}$

Let $(V, \mathbb{F}, \|\cdot\|)$ be a Banach Space.

Let $C \subset V$. A map $T: C \to C$ is called a contraction with contraction constant $\theta \in [0,1)$ if

$$\|Tx - Ty\| \leq \theta \|x - y\| \quad \forall\, x, y \in C$$

$$d(Tx, Ty) \leq \theta\, d(x,y)$$

is norm a distance in a metric space?

yes.

$T : C \to C$ (not necessary contraction)

1. the n^{th} iterate of T is given by

$$T^n = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}$$

2. Given $x \in C$, the orbit of x is defined by

$$\{ T^n x : n \in \mathbb{N} \}$$

3. T is contraction with constant θ

$$\|Tx - Ty\| < \theta \|x - y\|$$

$$\left. \begin{aligned} \|T^n x - T^n y\| &\le \theta \|T^{n-1} x - T^{n-1} y\| \\ &\le \theta^2 \|T^{n-2} x - T^{n-2} y\| \\ &\vdots \\ &\le \theta^n \|x - y\| \end{aligned} \right\}$$

T^n is contraction with θ^n contraction constant.

$\boxed{\text{Theorem}}$.

Let $(V, \mathbb{F}, \|\cdot\|)$ be a Banach space & $C \subset V$ a closed subset of V. Let $T : C \to C$ be a contraction with constant θ. Then there exists a unique point $\bar{x} \in C$ s.t.

$T(\bar{x}) = \bar{x}$, i.e. T has a unique fixed point.

the limit of a cauchy seq also belongs to C.

Moreover for any $x \in C$, we have

$$\|T^n x - \bar{x}\| \le \frac{\theta^n}{1-\theta} \|x - Tx\|$$

Proof: Fix $x_0 \in C$ & consider

$$\{ T^n(x_0) : n \in \mathbb{N} \}$$

Claim: $(T^n(x_0))_{n \geq 1}$ is Cauchy.

Assume WLOG $n > m$

$$d(T^n(x_0), T^m(x_0)) \leq d(T^n(x_0), T^{n-1}(x_0)) + d(T^{n-1}(x_0), T^{n-2}(x_0)) + \dots + d(T^{m+1}(x_0), T^m(x_0))$$

$$\leq \theta^{n-1} d(x_0, Tx_0) + \theta^{n-2} d(T(x_0), x_0) + \dots \theta^m d(x_0, Tx_0)$$

$$\leq (\theta^{n-1} + \theta^{n-2} + \dots + \theta^m) d(x_0, T(x_0)) \quad \star$$

$$= \theta^m (1 + \theta + \theta^2 + \dots \theta^{n-m-1}) \| Tx_0 - x_0 \|$$

$$\leq \frac{\theta^m}{1-\theta} \| Tx_0 - x_0 \|$$

For $(n,m) \in \mathbb{N} \times \mathbb{N}$ with $n \geq m$ we have

$$\| T^n x_0 - T^m x_0 \| \leq \frac{\theta^m}{1-\theta} \| Tx_0 - x_0 \|$$

$\Rightarrow (T^n x_0)$ is Cauchy

$\frac{\theta^m}{1-\theta} \| Tx_0 - x_0 \| < \epsilon$

$\theta \in [0,1)$

$\Rightarrow (T^n x_0)_{n \geq 1} \subset C \subset V$

$\Rightarrow \exists \bar{x} \in C$ s.t $T^n x_0 \to \bar{x}$

Banach space so converges

$\boxed{\text{Claim 2:}}$ $\bar{x}$ is a fixed point of T.

$$T(\bar{x}) = \bar{x} \iff \| T\bar{x} - \bar{x} \| = 0$$

$\Rightarrow \quad \|T\bar{x} - \bar{x}\| = 0$

$$\Downarrow$$

$$\left\| T\left(\lim_{n\to\infty} T^n(x_0)\right) - \lim_{n\to\infty} T^n(x_0) \right\| = 0$$

$\|Tx - Ty\| \le \theta\|x-y\|$ implies continuity and continuity implies this

$$\Updownarrow$$

$$\left\| \lim_{n\to\infty} T(T^n x_0) - \lim_{n\to\infty} T^n(x_0) \right\| = 0$$

$$\left\| \lim_{n\to\infty} T^{n+1}(x_0) - \lim_{n\to\infty} T^n(x_0) \right\| = 0 \quad \therefore \text{True}$$

$$\bar{x} \qquad \bar{x}$$

This shows $\bar{x}$ is a fixed point.

$\Rightarrow$ let $\bar{y} \neq \bar{x}$ be another fixed point T.

$$T(\bar{y}) = \bar{y}$$

$$\|T\bar{x} - T\bar{y}\| = \|\bar{x} - \bar{y}\| \le \theta\|\bar{x} - \bar{y}\| \Rightarrow \theta \ge 1 \quad (\rightarrow\leftarrow)$$

$\Rightarrow$ Norm is a continuous function so

$$\therefore \lim_{n\to\infty} \|T^n(x_0) - T^m(x_0)\| \le \frac{\theta^m}{1-\theta} \|Tx_0 - x_0\|$$

$$\Downarrow$$

$$\therefore \left\| \lim_{n\to\infty} T^n(x_0) - T^m(x_0) \right\| \le \frac{\theta^m}{1-\theta} \|Tx_0 - x_0\|$$

$$\boxed{\|\bar{x} - T^m(x)\| \le \frac{\theta^m}{1-\theta} \|Tx - x\|}$$

Now let's get back to existence of solutions.

$\dot{x}(t) = f(t,x)$, $x(t_0)=x_0$ — (1) Here $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz in the second var wrt the first

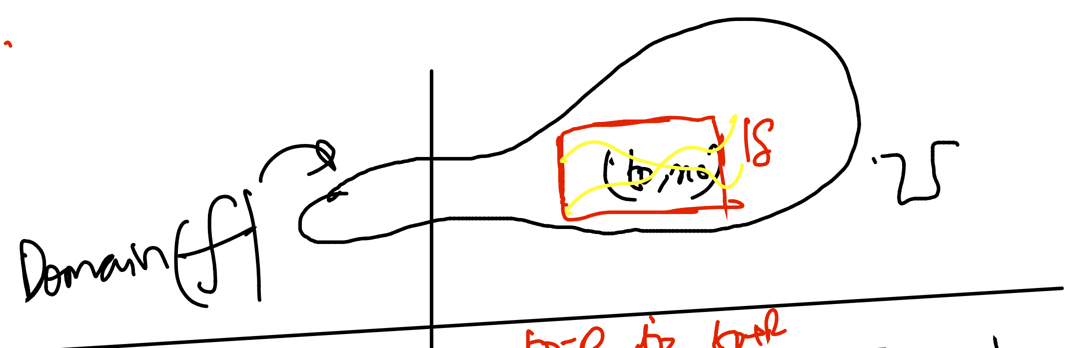
$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\,ds$ — (2)

$K(y)(t) = x_0 + \int_{t_0}^{t} f(s,y(s))\,ds$ — (3) (integral operator)

$\boxed{x(t) = K(x)(t)}$ → If this K has a fixed point then it follows 2.

Q But where are the ...tions for using Banach FPT?

$V = C(I,\mathbb{R})$, $I \ni t_0$



t_0-R to t_0+R

Suppose $y \in C(I,\mathbb{R})$
The graph of $(t,y(t))$ → curve may not lie on this domain as shown and here $f(s,y(s))$ won't make sense.
∴ We have to work locally on nbhd of (t_0,x_0)

⇒ Let $\delta, R > 0$ be such that
$K(\delta,R) = [t_0-R, t_0+R] \times \overline{B_\delta(x_0)} \subset U$ (closure)

Call it $K(\delta,R)$ its a compact set

⇒ Now consider the member

$$M = \sup_{K(\delta,R)} |f(t,x)| = 0$$

then trivial solution exists.

⇒ $V = C([t_0-R, t_0+R], \mathbb{R})$

$B = \{ y \in C([t_0-R, t_0+R],\mathbb{R}) : \mathrm{Im}(y) \subset \overline{B_\delta(x_0)} \}$

So we find a rectangle with centre (t_0,x_0) given by $[t_0-R, t_0+R] \times B_\delta(x_0)$ → so we can generalize to higher dimensions

$$B_\delta(x_0) = \{ x \in \mathbb{R} : |x - x_0| < \delta \}$$

Lecture 12

Consider the case $M = \sup_{K(R,\delta)} |f(t,x)| > 0$

$V \doteqdot C([t_0 - R, t_0+R], \mathbb{R})$ this is my normed space.

$B \doteqdot \{ y \in C([t_0-T, t_0+T], \mathbb{R}) : \text{Graph}(y) \subset \overline{K(R,\delta)},\ \text{Im}(y) \subset B_\delta(x_0) \}$

$\Rightarrow B$ is closed in $C([t_0-T, t_0+T], \mathbb{R})$ → Check this?

$\Rightarrow K : B \to C^1$ (continuously differentiable) (Looking at $K(y(t)) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds$

Question: Is $K(B) \subset B$?

Choose $y \in B$ and consider $K(y)(t)$

$$|K(y)(t) - x_0| = \left| \int_{t_0}^{t} f(s, y(s))\, ds \right|$$

$$\leq \int_{t_0}^{t} |f(s, y(s))|\, ds$$

$$\leq M|t - t_0| \; (\leq \delta) \to \boxed{R_1 = \min\{\tfrac{\delta}{M}, R\}} \quad \star$$

$\Rightarrow$ Now instead of having B, I will define

$C = \{ y \in C([t_0-R_1, t_0+R_1], \mathbb{R}) : \text{Im}(y) \subset \overline{B_\delta(x_0)} \}$

$\Rightarrow$ Now it follows from $\star$ that $K : C \to C$

Question: is K a contraction map?

Lecture 13

Since $K(T,\delta) = [t_0 - R, t_0 + R] \times \overline{B_\delta(x_0)} \subset U$

and let $M = \sup_{K(T,\delta)} |f(t,x)|$

(2) Since $K(T,\delta)$ is compact $\exists\, L(K) < +\infty$ s.t.

$$\sup_{(t,x) \neq (t,y)} \frac{|f(t,x) - f(t,y)|}{|x-y|} = C(K) < +\infty$$

(3) Consider $T_0 = \min\left\{ T, \delta/M, \frac{1}{L(K)+1} \right\}$

then if we let

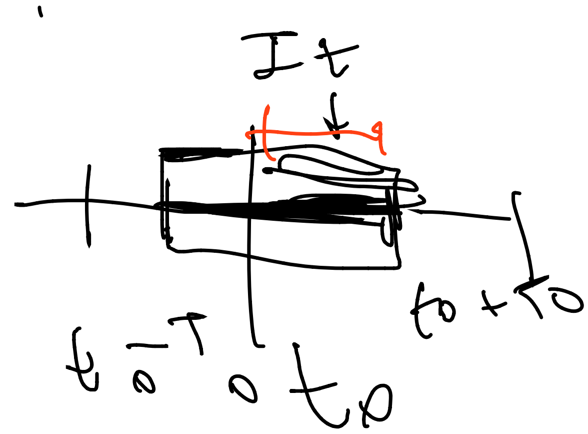
$V = C([t_0 - R, t_0 + R], \mathbb{R})$ and

$C = \left\{ y \in C([t_0 - R, t_0 + R], \mathbb{R}) : \operatorname{Img} y \subset \overline{B_\delta(x_0)} \right\}$

$K : C \to C$

(4) Consider $|K(y_1)(t) - K(y_2)(t)| = \left| \int_{t_0}^{t} f(s, y_1(s))\,ds - f(s, y_2(s))\,ds \right|$

$$\leq \left(|t - t_0| \right) \left(\sup_{I_t} |f(s, y_1(s)) - f(s, y_2(s))| \right)$$



$$\Rightarrow |f(s,y_1(s)) - f(s,y_2(s))| \leq L(K)\,|y_1(s)-y_2(s)| \quad \forall s \in I_t.$$

↳ actually $\forall t \in [t_0, T_0, t_0+T_0]$

$$\Rightarrow \sup_{I_t} |f(s,y_1(s)) - f(s,y_2(s))| \leq L(K) \sup_{I_t} |y_1(s)-y_2(s)|$$

$$\Rightarrow |K(y_1)(t) - K(y_2)(t)| \leq |I_t|\, L(K) \sup_{I_t} |y_1(s)-y_2(s)|$$

$$\forall t \in [t_0 - T_0, t_0 + T_0]$$

$$\leq L(K)\, T_0 \sup_{I_t} |y_1(s)-y_2(s)| \quad \forall t \in [t_0 - T_0, t_0+T_0]$$

$$\leq \frac{L(K)}{L(K)+1} \sup_{I_t} |y_1(s)-y_2(s)|$$

$$\Rightarrow \boxed{\|K(y_1)-K(y_2)\| \leq \frac{L(K)}{L(K)+1} \|y_1-y_2\|}$$

$\Rightarrow K$ is a contraction coeff $\dfrac{L(K)}{L(K)+1} < 1$

$\Rightarrow$ BFT $\quad \exists!\, y \in C$ s.t

$$K(y) = y$$

$$\Rightarrow K(y)(t) = y(t)$$

$$y(t) = x_0 + \int_{x_0}^{x} f(s,y(s))\,ds$$

Vector Valued Case

$\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0 = (x_0^1, \dots x_0^n)$ ↑ first component of x_0

$f : U \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ (open)

$(t, x) \longrightarrow (f_1(t,x), \dots f_n(t,x))$

$x = (x_1, \dots x_n)$

$x : I \longrightarrow \mathbb{R}^n$

$t \longrightarrow (x_1(t), \dots x_n(t))$

① $y : I \longrightarrow \mathbb{R}^n$

$$K(y)(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

$$\doteq \left(x_0^1 + \int_{t_0}^{t} f_1(s, y(s))\, ds, \dots\dots, x_0^n + \int_{t_0}^{t} f_n(s, y(s))\, ds \right)$$

② f is locally Lipschitz in x uniformly wrt t if for any compact set $K \subset U \subseteq \mathbb{R}^n$ we have

$$\sup_{\substack{(t,x) \neq (t,y) \\ \in K}} \frac{\|f(t,x) - f(t,y)\|_\infty}{\|x - y\|_\infty} = L(K) < \infty$$

Formulation

1. $(t_0, x_0) \in U \subseteq \mathbb{R}^{n+1}$

$T, \delta > 0$ s.t

- $K(T, \delta)$ $[t_0 - T, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$
- $B_\delta(x_0) = \{ x \in \mathbb{R}^n : \|x - x_0\|_\infty \leq \delta \}$
  $= \{ x \in \mathbb{R}^n : |x_j - x_0^j| \leq \delta \ \forall j \}$
- $M = \sup_{K(T,\delta)} |f(x,t)| = \max_{1 \leq j \leq n} \sup_{K(T,\delta)} |f_j(t,x)|$

$$C = \left\{ y \in C([t_0 - T_0, t_0 + T_0], \mathbb{R}^n) : \operatorname{Im} y \subset \overline{B_\delta(x_0)} \right\}$$

$\|y(t) - x_0\|_\infty \leq \infty$

Lecture 14

Q) $\dot{x}(t) = (x(t))^2 + \cos t^2$, $x(0) = 0$

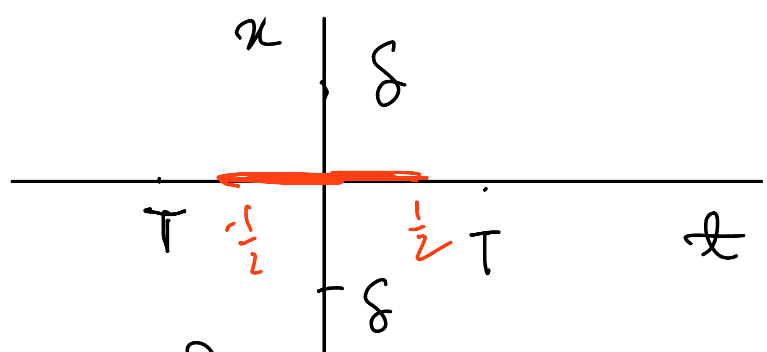
① $f(t,x) = \cos t^2 + x^2$ $\mathbb{R}^2$

$$|f(t,x_1) - f(t,x_2)| = |x_1^2 - x_2^2|$$
$$= |x_1 - x_2|\,|x_1 + x_2|$$
$$\leq L(K)\,|x_1 - x_2|$$

$\Rightarrow$ f is uniformly lipschitz in x, uniformly in t.

② $(t_0, x_0) = (0,0)$

Consider the rectangle



$K(T,\delta) = [-T,T] \times [-\delta,\delta] \subset \mathbb{R}^2$

$M = \sup_{K(T,\delta)} |f(t,x)| \Rightarrow |f(t,x)| = |x^2 + \cos t^2| \leq |x|^2 + 1 \leq \delta^2 + 1$

$M = \delta^2 + 1$

$$T_0 = \min\left\{ T, \frac{\delta}{M} \right\} = \min\left\{ T, \frac{\delta}{\delta^2+1} \right\} = \frac{\delta}{\delta^2+1}$$

$$T_0 = \frac{\delta}{\delta^2+1}$$

③ Find supremum of $\frac{\delta}{\delta^2+1} = \frac{1}{2}$

So Picards guarantees maximum interval where solution can exist.

Gronwall's Theorem

Suppose $\psi(t)$ satisfies;

$$\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)\,ds \quad ; t \in [0,T]$$

where α, β are continuous fns & $\beta \geq 0$. Then

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(\gamma)\,d\gamma\right) ds \, ; t \in [0,T]$$

Moreover, if α is increasing then

$$\psi(t) \leq \alpha(t)\exp\left(\int_0^t \beta(s)\,ds\right), \; t \in [0,T]$$

Proof: Define:

$$\phi(t) = \exp\left(-\int_0^t \beta(s)\,ds\right)$$

$\Rightarrow$ Consider $\frac{d}{dt}\left[\phi(t)\int_0^t \beta(s)\psi(s)\,ds\right] =$

$$\Rightarrow -\phi(t)\beta(t)\int_0^t \beta(s)\psi(s)\,ds + \phi(t)\beta(t)\psi(t)$$

$$\Rightarrow \phi(t)\beta(t)\left[\psi(t) - \int_0^t \beta(s)\psi(s)\,ds\right]$$

$$\leq \alpha(t)\beta(t)\phi(t)$$

$\Rightarrow$ Then $\int_0^t \frac{d}{dx}\left[\phi(x)\int_0^x \beta(s)\psi(s)\,ds\right] \leq \int_0^t \alpha(x)\beta(x)\phi(x)$

FTC $\quad \phi(t)\int_0^t \beta(s)\psi(s)\,ds \leq \int_0^t \alpha(x)\beta(x)\phi(x)\,dx$

$$\Rightarrow \int_0^t \beta(s)\psi(s)\,ds \le \int_0^t \alpha(x)\beta(x)\frac{\phi(x)}{\phi(t)}\,dx$$

$\Rightarrow$ Add $\alpha(t)$ both sides then

$$\psi(t) \le \alpha(t) + \int_0^t \beta(s)\psi(s)\,ds \le \alpha(t) + \int_0^t \alpha(x)\beta(x)\frac{\phi(x)}{\phi(t)}\,dx$$

$$\Rightarrow \frac{\phi(x)}{\phi(t)} = \exp\left(-\int_0^x \beta(s)\,ds + \int_0^t \beta(x)\,dx\right)$$

$$= \exp\left(\int_x^t \beta(x)\,dx\right) \quad \therefore \text{ proved}$$

$\Rightarrow$ Now if $\alpha(t)$ is increasing

$$\psi(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s)\left(\exp\int_s^t \beta(\gamma)\,d\gamma\right)ds$$

$$\le \alpha(t) + \alpha(t)\int_0^t \beta(s)\left\{\exp\int_s^t \beta(\gamma)\,d\gamma\right)ds$$

$$-1 + \exp\left(\int_0^t \beta(\gamma)\,d\gamma\right)$$

$$\psi(t) \le \alpha(t) - \alpha(t) + \alpha(t)\exp\int_0^t \beta(s)\,ds.$$

$$I = \int_0^t \beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds$$

$$\frac{d}{ds} \exp\left(\int_s^t \beta(r)dr\right) = \exp\left(\int_s^t \beta(r)dr\right)\left(-\beta(s)\right)$$

$$I = \int_0^t - \frac{d}{ds} \exp\left(\int_s^t \beta(r)dr\right) ds$$

$$= -\left[\exp\left(\int_s^t \beta(r)dr\right)\right]_0^t$$

$$= -1 + \exp \int_0^t \beta(r)dr$$

Tutorial

2. $y'(t) = y^2$; $y(0)=1$
Compute first three picard iterates

7. (1) $y''+y'-y=0$, $y(0)=1$, $y'(0)=0$

1. (4) $f(t,x) = \begin{cases} (x^2+t^2)^{1/3} & (x,t) \neq (0,0) \\ 0 & \end{cases}$

3. $f(t,x) = |x| \log(1+|\sin x|) + \frac{1}{1+t^2}$ $(t,x) \in \mathbb{R}\times I$

Solutions (2) $y_n(t) = y(t_0) + \int_{t_0}^{t} f(s, y_{n-1})\,dt$

$y_0 = 1$ (i) $y(t) = 1 + \int_0^t \cancel{y^2}\, 1\, dt = 1+t$ -(ii) $= y_1(t)$

$y(t) = 1 + \int_0^t (1+t)^2 dt = 1 + \frac{(1+t)^3}{3} - \frac{1}{3} = y_2(t)$

(7) $(y, y') = (x_1, x_2)$

$\dot{x}_1 = x_2$; $\dot{x}_1 = x_2$; $x_1(0)=1$

$\dot{x}_2 + x_2 - x_1 = 0$; $\dot{x}_2 = x_1 - x_2$; $x_2(0)=0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = b(t, x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 - x_2 \end{pmatrix}$

$x_n = x_0 + \int_0^t b(s, x_{n-1}(s))\,ds$

$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \begin{pmatrix} 1 \\ t \end{pmatrix}$

$x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} s \\ 1-s \end{pmatrix} ds = \begin{pmatrix} 1 + \frac{t^2}{2} \\ t - \frac{t^2}{2} \end{pmatrix}$

Q) $y'' = y + t^2$ $\quad y(0) = 1$, $y'(0) = 0$

$(y, y') = (x_1, x_2)$

$\dot{x}_1 = x_2, \quad x_1(0) = 1$

$\dot{x}_2 = x_1 + t^2, \quad x_2(0) = 0$

$$\boxed{x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

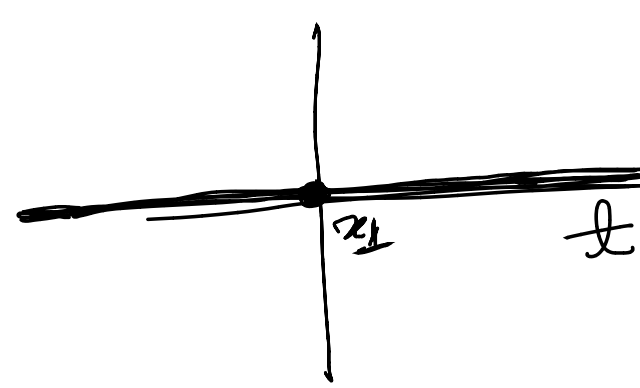
$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 1 + t^2 \end{pmatrix} dt$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t + \frac{t^3}{3} \end{pmatrix} = \boxed{x_1 = \begin{pmatrix} 1 \\ t + t^3/3 \end{pmatrix}}$$

$$x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} t + \frac{t^3}{3} \\ 1 + t^2 \end{pmatrix} dt$$

$$\boxed{x_2 = \begin{pmatrix} 1 + \frac{t^2}{2} + \frac{t^4}{12} \\ t + \frac{t^3}{3} \end{pmatrix}}$$

Q) $\sup\limits_{\substack{(t, x_1) \neq (t, x_2) \in K}} \dfrac{|f(t, x_1) - f(t, x_2)|}{|x_1 - x_2|} = L(K)$



$$= \frac{(x_1^2 + t^2)^{1/3} - 0}{x_1} \qquad \frac{t = 0}{x_2 = 0}$$

$$= \frac{|h^{2/3}|}{|h|} = |h|^{-1/3} \xrightarrow[h \to 0]{} \infty$$

not locally lipschitz.

Lecture 15

Another version of Gronwall's

$$\psi'(t) \le \beta(t)\psi(t) \quad \forall\, t \in [0,T]$$

$$\Rightarrow \psi(t) - \psi(0) \le \int_0^t \beta(s)\psi(s)\,ds$$

$$\Rightarrow \psi(t) \le \psi(0) + \int_0^t \beta(s)\psi(s)\,ds$$

$$\Rightarrow \boxed{\psi(t) \le \psi(0)\exp\left(\int_0^t \beta(s)ds\right)}$$

<u>Lemma</u> :- Suppose $\psi(t) \le \alpha + \int_0^t (\beta\psi(s) + \gamma)\,ds$, $t \in [0,T]$

for given constants, $\alpha \in \mathbb{R}$, $\beta \ge 0$, $\gamma \in \mathbb{R}$. Then

$$\boxed{\psi(t) \le \alpha \exp(\beta(t)) + \frac{\gamma}{\beta}\left(\exp \beta t - 1\right).}$$

<u>Proof</u> $\tilde{\varphi}(t) = \psi(t) + \frac{\gamma}{\beta}$

Then $\tilde{\varphi}(t) = \psi(t) + \frac{\gamma}{\beta} \le \alpha + \frac{\gamma}{\beta} + \int_0^t \left[\beta\left(\tilde{\varphi}(s) - \frac{\gamma}{\beta}\right) + \gamma\right] ds$

$$\le \alpha + \frac{\gamma}{\beta} + \int_0^t \beta\tilde{\varphi}(s)\,ds$$

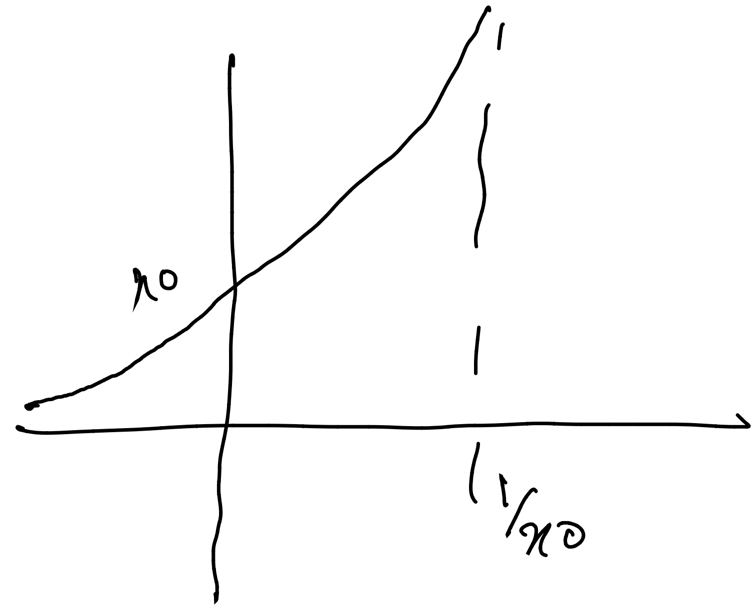
$\tilde{\varphi}(t) \le \left(\alpha + \frac{\gamma}{\beta}\right)\exp(\beta t)$ → <u>Gronwalls</u>

$$\psi(t) \le \left(\alpha + \frac{\gamma}{\beta}\right)\exp(\beta t) - \frac{\gamma}{\beta}$$

A simple example (Continuous departure of solution on initial value)

$\dot{x}(t) = (x(t))^2$, $x(0) = x_0 > 0$

$g_{x_0}(t) = \dfrac{x_0}{1-x_0 t}$ on $\left(-\infty, \dfrac{1}{x_0}\right)$



$g_1(t) = \dfrac{1}{1-t}$, $g_n(t) = \dfrac{n}{1-nt}$

$$|g_{x_0}(t) - g_{y_0}(t)| = \frac{|x_0 - y_0|}{|1-x_0 t|\,|1-y_0 t|}$$

For any $r < \min\left\{\dfrac{1}{x_0}, \dfrac{1}{y_0}\right\}$ $\exists$ $M(r)$ s.t.

$$|g_{x_0}(t) - g_{y_0}(t)| \le M(r)\,|x_0 - y_0|$$
$$\forall t \in (-\infty, r]$$

Theorem :- Let $f, g : U \underset{\text{open}}{\subseteq} \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions. Suppose f is locally Lipschitz in the second variable, unif wrt first variable. Suppose $x(t)$ & $y(t)$ are solutions of the initial value problems

$\dot{x}(t) = f(t,x)$, $x(t_0) = x_0$ & $\dot{y} = g(t,y)$, $y(t_0) = y_0$ on $[t_0, t_0+T]$

Then $\boxed{|x(t) - y(t)| \le |x_0 - y_0|\, e^{L|t-t_0|} + \dfrac{M}{L}\left(e^{L|t-t_0|} - 1\right)}$

$$L = \sup_{\substack{(t,x)\neq(t,y) \\ \in K = \text{graph}(x) \cup \text{graph}(y)}} \frac{|f(t,x) - f(t,y)|}{|x-y|} , \quad M = \sup_{(t,x)\in K} |f(t,x) - g(t,x)|$$

Lecture 16

Proof: $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$

$y(t) = y_0 + \int_{t_0}^{t} g(s, y(s))\,ds$

$$\Rightarrow |x(t) - y(t)| = \left| (x_0 - y_0) + \int_{t_0}^{t} \big(f(s, x(s)) - g(s, y(s))\big)\,ds \right|$$

$$\leq |x_0 - y_0| + \int_{t_0}^{t} |f(s, x(s)) - g(s, y(s))|\,ds$$

$$\leq |x_0 - y_0| + \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| + |f(s, y(s)) - g(s, y(s))|\,ds$$

$$\leq |x_0 - y_0| + \int_{t_0}^{t} \big(L|x(s) - y(s)| + M\big)\,ds$$

Use Gronwall's version II

$$\Rightarrow |x(t) - y(t)| \leq |x_0 - y_0| \exp\big(L(t - t_0)\big) + \frac{M}{L}\big(\exp L(t - t_0) - 1\big)$$

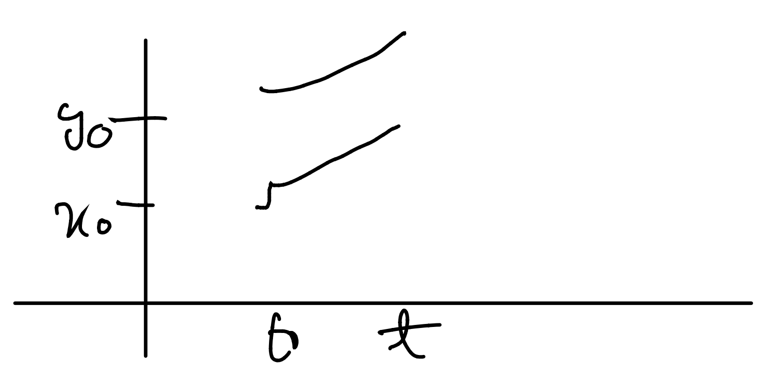
Corollary: Let $\phi(\cdot, t_0, x_0)$ be a solution of

$\dot{x}(t) = f(t, x)$

$x(t_0) = x_0$

Then,
$$\Big|\underbrace{\phi(t, t_0, x_0)}_{\text{notations not some functions}} - \phi(t, t_0, y_0)\Big| \leq |x_0 - y_0| e^{L|t - t_0|}$$

$\forall\, t \in [t_0, t_0 + T]$



Maximum Interval of existence & Extendibility of solution:-

$$\left.\begin{aligned} \dot{x}(t) &= f(t,x) \\ x(t_0) &= x_0 \end{aligned}\right\} \text{①} \qquad f : \underset{\text{open}}{U \subseteq \mathbb{R}^2} \xrightarrow[\text{continuous.}]{} \mathbb{R}$$

The IVP ① admits local unique solution (☆) important

(note we are not assuming locally lipschitz which talked about unique solution)

Consider $(\varphi_1, I_{\varphi_1})$ & $(\varphi_2, I_{\varphi_2})$ be two solutions defined on intervals I_{φ_1} and I_{φ_2}.

$\psi = \begin{cases} \varphi_1 & \text{on } I_{\varphi_1} \\ \varphi_2 & \text{on } I_{\varphi_2} \end{cases}$

Note that $\exists\, \epsilon > 0$ s.t on $(t_0 - \epsilon, t_0 + \epsilon)$ both φ_1 & φ_2 are same. Let (t_-, t_+) be the maximum interval on which $\varphi_1 = \varphi_2$

Claim : $t_+ = T_+$ and $t_- = T_-$ where $I_{\varphi_1} \cap I_{\varphi_2} = (T_-, T_+)$

$\Rightarrow$ Suppose $t_+ \neq T_+$ then $t_+ < T_+$. Now

$\varphi_1(t_+) = \varphi_2(t_+)$ (because they agree on (t_-, t_+) and are continuous, so they agree on t_+)

$\Rightarrow$ Now consider the IVP,

$$\dot{x}(t) = f(t,x)$$
$$x(t_+) = \varphi_1(t_+) = \varphi_2(t_+)$$

φ_1 and φ_2 are two solutions & therefore by local uniqueness we can find a $\delta > 0$ s.t

$\varphi_1 = \varphi_2$ on $(t_+ - \delta, t_+ + \delta)$.

But this will contradict the maximality.

$$\varphi_1|_{I_{\varphi_1} \cap I_{\varphi_2}} = \varphi_2|_{I_{\varphi_1} \cap I_{\varphi_2}}$$

Lemma: Consider the initial value problem (1). Let $(\varphi_1, I_{\varphi_1})$ & $(\varphi_2, I_{\varphi_2})$ be two solutions then φ defined on $I_{\varphi_1} \cup I_{\varphi_2}$ is also a solution.

Theorem: Consider the IVP (1). Then there exists a unique maximal solution defined on a maximum interval $I_{t_0, x_0} = (T_-(t_0, x_0), T_+(t_0, x_0))$

Lecture 17

Consider the IVP $\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0$ $\quad f : \underset{\text{open}}{U \subseteq} \mathbb{R}^2 \xrightarrow{\text{cnt}} \mathbb{R}$

Suppose the IVP admits locally unique solution. Then there exists a unique maximal interval $(T_-(x_0,t_0), T_+(x_0,t_0))$ such that each solution of IVP extends to a solution on the maximal interval.

Proof: ① $S \doteq \{ (\varphi, I_\varphi) : \varphi \text{ is a solution on } I_\varphi \ni t_0 \}$

$I = \bigcup I_\varphi$

I is an open set of $\mathbb{R}$ containing t_0.

Let $t_1 \in I$, $t_1 > t_0$. Note $t_1 \in I_\varphi$ for some φ. Since I_φ is an interval $[t_0, t_1] \subset I_\varphi \subset I$.

$\Rightarrow$ Similar argument for $t_1 < t_0 \Rightarrow I$ is an interval.

② Consider $\varphi_{max} : I \to \mathbb{R}$ defined by

$$\varphi_{max}(t) = \varphi(t) \text{ where } t \in I_\varphi \text{ for some } (\varphi, I_\varphi) \in S$$

Note φ_{max} is well defined

$$\text{if } t \in I_{\varphi_1} \cap I_{\varphi_2} \underset{\text{lemma}}{\Longrightarrow} \varphi_1(t) = \varphi_2(t)$$

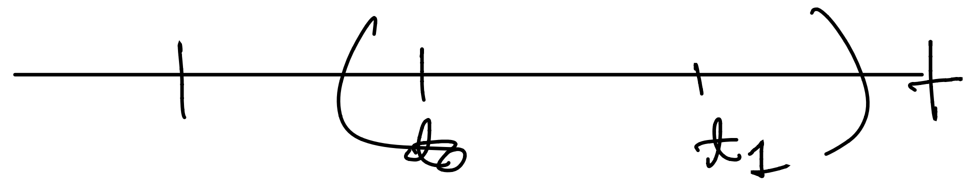
$$\varphi_{max}(t) = \varphi_1(t) = \varphi_2(t)$$

φ_{max} is well defined.

Continuous functions have solutions usually. that's why

(3) Note if $t_1 \in I$ then $t_1 \in I_\varphi$ for some φ s.t

$$(\varphi, I_\varphi) \in S$$



$\exists\, \epsilon > 0$ s.t

$$(t_0 - \epsilon, t_0 + \epsilon) \subset I_\varphi$$

$$\varphi_{max}(t) = \varphi(t) \text{ on } (t_0 - \epsilon, t_1 + \epsilon)$$

So any solution can be extended to φ_{max} and φ_{max} is also a local solution by point (3).

1. The solution and the interval in the above theorem are called maximal solution and maximal interval.
2. Suppose $\exists$? is a solution defined on all of $\mathbb{R}$, then such a solution is called a global solution.

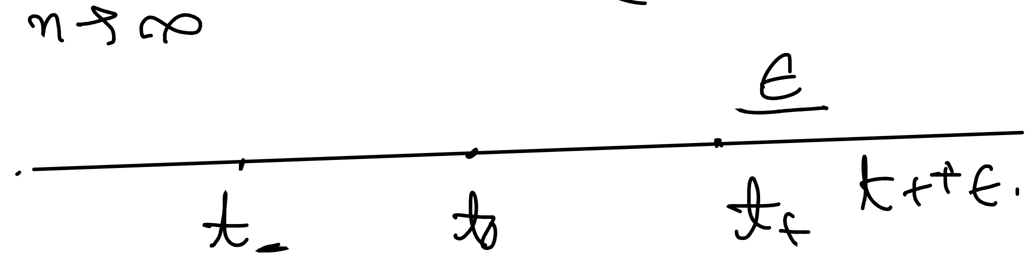
Lemma:

$$\left.\begin{aligned} \dot{x}(t) &= f(t, x(t)) \\ x(t_0) &= x_0 \end{aligned}\right\} \quad (1)$$

f is continuous.

$f : \underset{\text{open}}{U} \xrightarrow{\text{cont}} \mathbb{R}$

Assume it admits a local unique solution. Let φ be a solution of (1) on (t_-, t_+). Then $\exists\, \epsilon > 0$ s.t φ has an extension $\tilde{\varphi}$, a solution of (1), on $(t_-, t_+ + \epsilon)$ iff $\exists$ a sequence (t_n), $t_n \to t_+$ s.t $\lim_{n \to \infty} (t_n, \varphi(t_n)) = (t_+, y) \in U$



Proof: Suppose $\tilde{\varphi} : (t_- , t_+ + \epsilon) \to \mathbb{R}$ s.t

($\Rightarrow$) $\tilde{\varphi}$ is a solution of ① & $\tilde{\varphi} = \varphi$ on (t_-, t_+)

Note $\{(t, \tilde{\varphi}(t)) : t \in (t_-, t_+)\} \subset U$

$$\tilde{\varphi}(t_+) = \lim_{n\to\infty} \tilde{\varphi}(t_n) \in U \text{ for } (t_n) \text{ s.t } t_n \to t_+$$

by continuity

$$= \lim_{n\to\infty} \varphi(t_n) \quad (\text{if } (t_n) \subset (t_-, t_+))$$

($\Leftarrow$) Now suppose $\exists$ $(t_n) \subset (t_-, t_+)$ s.t $t_n \to t_+$ &

$$\lim_{n\to\infty} (t_n, \varphi(t_n)) = (t_+, y) \in U$$

We have to show that it extends continuously

Step 1: Let $(s_n) \subset (t_-, t_+)$ s.t $s_n \to t_+$ then

$$\varphi(s_n) \to y$$

$$|\varphi(s_n) - y| = |\varphi(s_n) - \underbrace{\varphi(t_n)}_{\text{subsequence}}| + |\varphi(t_n) - y|$$

$$\varphi(s_n) = x_0 + \int_{t_0}^{s_n} f(s, \varphi(s))\,ds \quad \text{because } \varphi \text{ is a soln}$$

$$\varphi(t_n) = x_0 + \int_{t_0}^{t_{k_n}} f(s, \varphi(s))\,ds$$

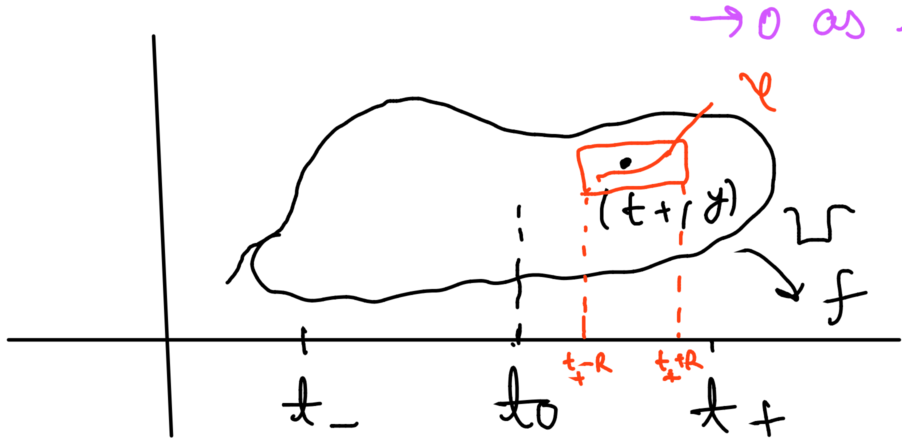
s.t $\quad n \geq k_n$

Lecture 18

$$\Rightarrow |\varphi(s_n) - \varphi(t_n)| = \left| \int_{t_0}^{s_n} f(s,\varphi(s))\,ds - \int_{t_0}^{t_n} f(s,\varphi(s))\,ds \right|$$

$$\le \underbrace{\ell(I(s_n,t_n))}_{\text{this will } \to 0 \text{ as } n\uparrow} \sup_{I(s_n,t_n)} |f(s,\varphi(s))|$$

$I(s_n,t_n)$ = interval



$\Rightarrow \exists\, R>0$ & $\delta>0$ s.t

$$K(R,\delta) = [t_+-R, t_++R] \times \overline{B_\delta(y)} \subset U$$

$\Rightarrow$ Let $M = \sup_{K(R,\delta)} |f(t,x)| < +\infty$ (as continuous fn on compact set)

$\Rightarrow$ When n is large, $I(s_n,t_n) \subset (t_+-R, t_+)$

$$|\varphi(s_n) - \varphi(t_n)| \le \ell(I(s_n,t_n))\,M \quad \text{when } n \ge N_0$$

$$|\varphi(s_n) - y| \le M\,\ell(I(s_n,t_n)) + |\varphi(t_n) - y| \Rightarrow 0$$

$$\boxed{\lim_{t \uparrow t^+} \varphi(t) = y}$$

Consider the IVP;

$$\left.\begin{aligned} \dot{x}(t) &= f(t, x(t)) \\ x(t_+) &= y \end{aligned}\right\} \text{(2)}$$

(By Peano's theorem it has a soln if f is continuous)

Let ψ be a solution of (2) on $(t_+ - \epsilon, t_+ + \epsilon)$.

Claim: $\tilde{\varphi} = \begin{cases} \varphi \text{ on } (t_-, t_+) \\ \psi \text{ on } [t_+, t_+ + \epsilon) \end{cases}$ then $\tilde{\varphi}$ is an extension and a solution of IVP (1)

The patching of solution can be done by continuity part.

Q: $x'(t) = a(t) F(\cos x(t)) + b(t) G(\sin x(t))$

$x(t_0) = x_0$

a, b are continuous on $\mathbb{R}$.

F, G are polynomials of fixed degree.

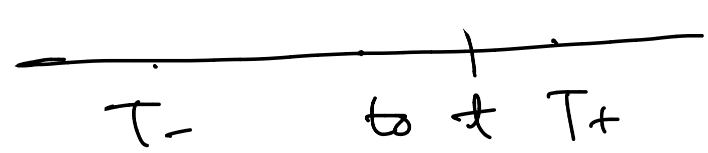
$f(t,x) = a(t) F(\cos x) + b(t) G(\sin x)$ → show locally lipschitz

Claim: local solution can be extended on $\mathbb{R}$.

Proof: (T_-, T_+) be the maximal interval.

Suppose $T_+ < +\infty$

Let φ be a solution on (T_-, T_+)



$$\varphi(t) - \varphi(t_0) = \int_{t_0}^{t} f(s, \varphi(s))\, ds$$

$$|\varphi(t) - x_0| = \left| \int_{t_0}^{t} f(s, \varphi(s))\, ds \right|$$

$$\Rightarrow \sup_{[T_-,T_+]} |f(t,x)| \le \sup_{[T_-,T_+]} |a(t)| \sup_{[-1,1]} |F| + \sup_{[T_-,T_+]} |b(t)| \sup_{[-1,1]} |G| \le C_{(T_-,T_+)}$$

" some quantity depending on (T_-,T_+)

$$\Rightarrow |\varphi(t) - x_0| \le |t-t_0|\, C_{(T_-,T_+)} \le T_+\, C(T_-,T_+)$$

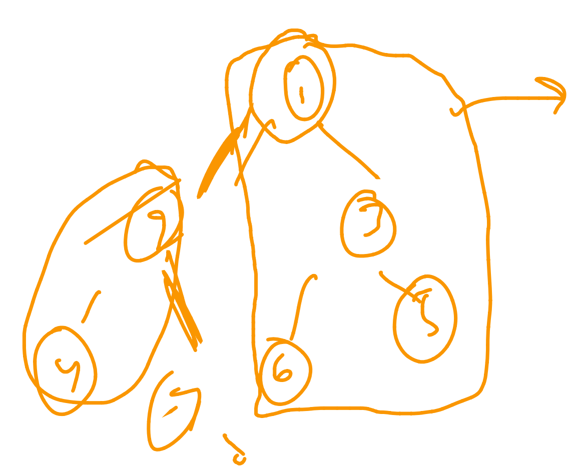
$\downarrow$

means if we take a sequence $(t_n) \subset [t_0, T_+]$), $\phi(t_n)$ is bdd.

So we can make a sub-sequence s.t

$\lim_{n\to\infty} (t_{n_k}, \phi(t_{n_k})) = (T_+, y) \in \mathbb{R}^2$ which is in domain of $f(t,x)$.

(Bolzano Weierstrass)



$\boxed{\text{Lecture 19}}$

$\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0$

$\varphi : (t_-, t_+) \to \mathbb{R}^n$

$f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$ $\quad (f_1, \cdots f_n)$

$\dot{x}_j(t) = f_j(t, x(t))$

φ has an extension at t_+ iff $\exists\ (t_n) \subset (t_-, t_+)$ s.t.

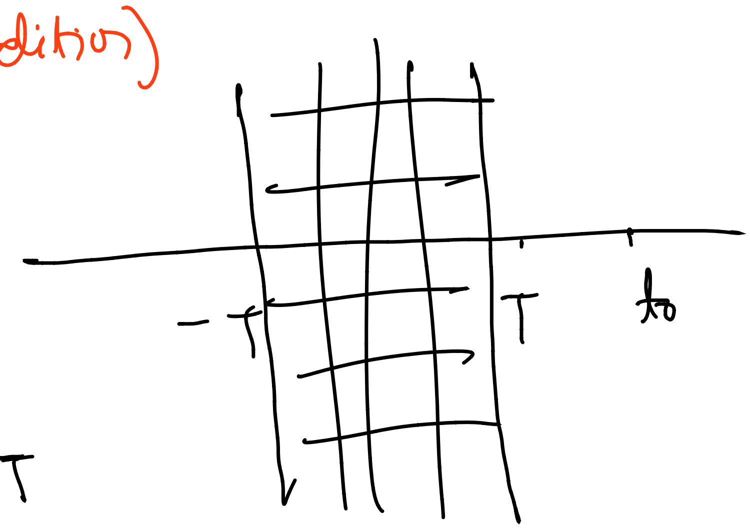
$t_n \to t_+$ & $\lim\limits_{n\to\infty} (t_n, \varphi(t_n)) = (t_+, y) \in U \subseteq \mathbb{R}^{n+1}$

<u>Theorem</u>: $\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0 \quad$ - ①

Let $f : \mathbb{R}^2 \to \mathbb{R}$ continuous. Suppose for each $T > 0$ there exist constants $M(T), L(T)$ s.t.

$$|f(t,x)| \le M(T) + L(T)|x| \qquad (t,x) \in [-T,T] \times \mathbb{R}$$

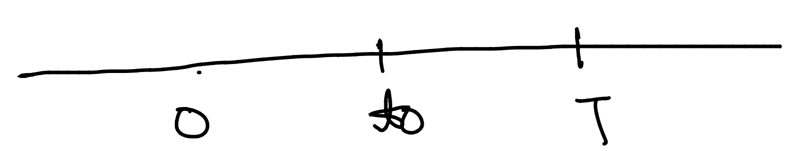
Then ~~the~~ solution of IVP -① can be extended to all of $\mathbb{R}$. (This is a sufficient condition)



<u>Ans</u>: $\sup\limits_{[-T,T]\times\mathbb{R}} |f(t,x)| \le \sup\limits_{[-T,T]} |a(t)| \sup\limits_{[-1,1]} |f|$ fixed

$\quad + \sup\limits_{[-T,T]} |b| \sup\limits_{[-1,1]} |G| = C_T$

<u>Proof</u>: If $\varphi(t)$ is a solution



$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds$

$T > t \ge t_0$

$|\varphi(t)| \le |x_0| + \int_{t_0}^{t} |f(s, \varphi(s))|\, ds \quad \forall t \in (t_0, T)$

Initially I have taken T s.t soln is defined till there. (T fixed)

$$|\varphi(t)| \le |x_0| + \int_{t_0}^{t} (M(T) + L(T)|\varphi(s)|)\,ds$$

By Gronwall's inequality;

$$|\varphi(t)| \le |x_0| \exp(L(T)t) + \frac{M(T)}{L(T)}(\exp L(T)t - 1) \quad t \in (t_0, T)$$

$$\le |x_0| \exp^{(L(T)T)} + \frac{M(T)}{L(T)}\left[\exp(L(T)T - 1)\right] \quad - (*)$$

Claim : $T_+ = +\infty$

Suppose $T_+ < +\infty$. Then from $(*)$ the solution remains bounded on (t_0, T_+). We can apply the lemma to extend φ in a nbhd of T_+, which is a contradiction.

System of Ordinary Differential Equations

$\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0$ where

$x : I \to \mathbb{R}^n$ & $f : U \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$

$\dot{x}_1(t) = f_1(t, x(t))$

$\vdots$

$\dot{x}_n(t) = f_n(t, x(t))$

$x(t_0) = x_0 = (x_{01}, \dots x_{0n})$

Linear Autonomous System

$\dot{x}(t) = Ax(t)$, $x(t_0) = x_0$

where $A \in M_n(\mathbb{R})$

$\dot{x}_j(t) = \sum_{k=1}^{n} A_{j,k} x_k(t)$ $\quad 1 \le j \le n$

$y = \frac{\ln x}{a\, x_0}$

$x_0 e^{ay}$

$\int_{x_0}^{x} \frac{1}{ax} dx = \frac{\ln x}{a\, x_0}$

(*) When $n=1$, $x(t) = x_0 e^{At}$

Question: Is there a general formula for the general case.

Exponential of a matrix

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$

$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \dots$

$x_n(A) = I + \frac{A}{1!} + \frac{A^2}{2!} + \dots \frac{A^n}{n!}$

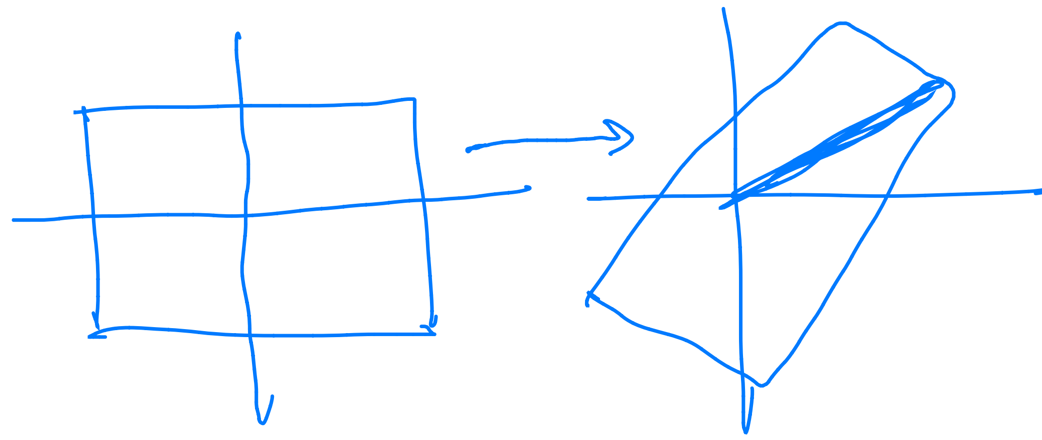
$\underbrace{(x_n(A))_{n \ge 1}}$

We have to show this sequence converges and then we define e^A.

$\Rightarrow M_n(F)$

Given $A \in M_n(F)$

$$T_A : F^n \to F^n$$
$$T_A(z) = Az$$



$$\|A\|_{op} = \sup\{\|Az\|_\infty : \|z\|_\infty \le 1\}$$

$\Rightarrow (M_n(\mathbb{C}), \|\cdot\|_{op})$ is a normed space.

① $\|A\|_{op} \ge 0$, $\|A\|_{op} = 0$ iff $A = 0$

② $\|\lambda A\|_{op} = |\lambda| \|A\|_{op}$ $\quad \lambda \in F$

③ $\|T_1 + T_2\|_{op} \le \|T_1\|_{op} + \|T_2\|_{op}$ $\forall T_1, T_2 \in M_n(\mathbb{C})$

④ $\max_{j,k} |A_{jk}| \le \|A\|_{op} \le n \max_{j,k} |A_{jk}|$ — ☆

$\hookrightarrow$ $(A_n) \to A$

If $\|A_n - A\|_{op} \to 0$ as $n \to \infty$

then $a_{11}(n) \to a_{11}$ (because of ☆) (all entries)

$\max_{i,j} |(A_n - A)_{ij}| \le \|A_n - A\|_{op}$

as RHS $\to 0$, LHS $\to 0$

$\downarrow$

using this $(M_n(F), \|\cdot\|_{op})$ is a complete normed space (Banach space)

(HINT: take $\|A_n - A_m\|_{op}$ as a cauchy seq. $\forall n, m \ge N$)

A is a fixed matrix

Claim: $(\eta_n(A))_{n\geq 1}$ is Cauchy in $(M_n(\mathbb{F}), \|\cdot\|_{op})$

$n \geq m$

$$\|\eta_n(A) - \eta_m(A)\|_{op} = \left\| \frac{A^{m+1}}{\lfloor m+1} + \cdots \frac{A^n}{\lfloor n} \right\|_{op}$$

$$\leq \left\| \frac{A^{m+1}}{\lfloor m+1} \right\|_{op} + \cdots + \left\| \frac{A^n}{\lfloor n} \right\|_{op}$$

(using $\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op}$) $$\leq \frac{\|A\|_{op}^{m+1}}{\lfloor m+1} + \cdots + \frac{\|A\|_{op}^n}{\lfloor n}$$

$$= |S_n(\|A\|_{op}) - S_m(\|A\|_{op})|$$ (using convergent sequence is Cauchy (e^x))

$$\left(S_k(\|A\|_{op}) = 1 + \frac{\|A\|_{op}}{1!} + \frac{\|A\|_{op}^2}{2!} + \cdots \frac{\|A\|_{op}^k}{k!} \right)$$

$\Rightarrow (S_k(\|A\|_{op}))_{k\geq 1}$ is Cauchy

$\Rightarrow (\eta_n(A))_{n\geq 1}$ is Cauchy $\qquad \lim_{n\to\infty} \eta_n(A) = B$

Then $\exists!\, B \in M_n(\mathbb{C})$ s.t

$$\eta_n(A) \to B \quad , B = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$$

$\Rightarrow \dot{x}(t) = Ax(t), \quad x(t_0) = x_0$

$$\boxed{x(t) = e^{tA} x_0}$$

$f(t,x) = Ax$

$\|f(t,x_1) - f(t,x_2)\|_\infty = \|Ax_1 - Ax_2\|_\infty = \|A(x_1 - x_2)\|_\infty$

$\leq \|A\|_{op} \|x_1 - x_2\|_\infty$

(globally Lipschitz)
so we can use picard iterates.

$\Rightarrow x_0(t) = x_0 = (x_{01}, x_{02}, \dots x_{0n})$

$$x_{1,j}(t) = x_{0,j} + \int_0^t \underbrace{\sum_{k=1}^n A_{jk} x_{0,k}}_{f_j(x_0(s),s)} ds$$

$$= x_{0,j} + \left(\sum_{k=1}^n A_{jk} x_{0,k} \right) t$$

$$\boxed{x_1(t) = x_0 + tAx_0}$$

$$\Rightarrow \boxed{x_2(t) = x_0 + tAx_0 + \frac{t^2}{2} A^2 x_0}$$

$\hookrightarrow x_2(t) = x_0 + \int_0^t A(x_0 + sAx_0)\, ds$

$\Rightarrow x_n(t) = \sum_{j=0}^n \frac{t^j A^j x_0}{j!}, \quad x_n(t) \to x(t)$

Tutorial

Q (3.3) $y' = \frac{y^3 e^t}{1+y^2} + t^2 \cos y \qquad y(t_0) = y_0$

Does this have a local solution, then show global.

$f(t,y) = \frac{y^3 e^t}{1+y^2} + t^2 \cos y \qquad f : \mathbb{R}^2 \to \mathbb{R}$

$\sup \frac{|f(t,y_1) - f(t,y_2)|}{|y_1 - y_2|} = \frac{y_1^3 e^t}{1+y_1^2} + t^2 \cos y_1 - y_2^3 e^t$

$\hookrightarrow C^1$ fn

$L(T) = \sup_{[-T,T]} |e^t| = e^T$

$M(T) = \sup_{[-T,T]} t^2$

$f(t,y) \le M(t)\,|y| \frac{|y|^2}{|1+y^2|} + L(t) \sup_{[-T,T]} \cos y$

$\frac{|y|^2}{|1+y^2|} \to$ bdd

C^1

Q (3.4) $y' = \underline{e^{-y^2}}\, \underline{\cos t} + \underbrace{\sin y}_{1}, \; y(t_0) = y_0$

- This is continuous
- Lipschitz condition

$y'(t) \le 2$

Q⑥ $t y''(t) + (\cos t) y'(t) + \frac{t}{1+t} y(t) = 2t$

$y\left(-\frac{1}{2}\right) = y_0 \quad , \quad y'\left(-\frac{1}{2}\right) = y_1$

$(y, \dot{y}) = (x_1, x_2)$

① $\dot{x}_1 = x_2 \qquad x_1\left(-\frac{1}{2}\right) = y_0$

$t\dot{x}_2 + \cos t \, x_2 + \frac{t}{1+t} x_1 = 2t$

② $\dot{x}_2 = \frac{2t}{t} - \frac{x_2 \cos t}{t} - \frac{x_1 t}{(1+t)t} \quad , \quad x_2\left(-\frac{1}{2}\right) = y_1$

$F : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$

① $F(t, x_1, x_2) = \left(x_2, -\frac{\cos t \, x_2}{t} - \frac{1}{1+t} x_1 + 2\right)$ should be

continuous in t (Picard Lindelöf)

$\boxed{t = 0, t = -1 \text{ are singularities}}$

$(-1, 0)$

② For any $(x_1, x_2) \in \mathbb{R}^2$, $(y_1, y_2) \in \mathbb{R}^2$, fixed $t \in K$

$\|F(t,x) - F(t,y)\|_\infty = \max\{|x_2 - y_2|, |a(t)(x_1 - y_1) + b(t)(x_2 - y_2)|\}$

$\leq \max\{|x_2 - y_2|, |a(t)| \|x - y\|_\infty + |b(t)| \|x - y\|_\infty\}$

$\leq \|x - y\|_\infty L$

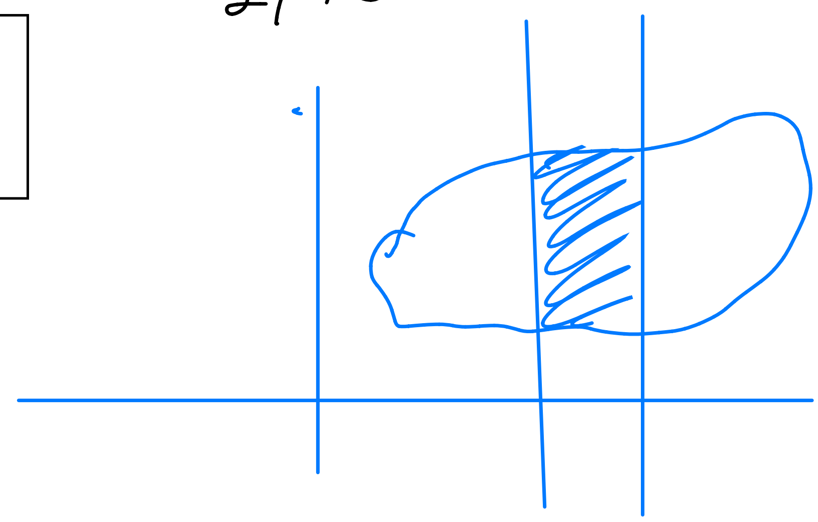
We had, $f: \mathbb{R}^2 \to \mathbb{R}$

$$|f(t,x)| \le M(T) + L(T)|x| \qquad (t,x) \in [-T,T] \times \mathbb{R}$$

$\Rightarrow$ $\dot{x}(t) = f(t, x(t))$ $\quad x(t_0) = x_0$ $\qquad f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$

Suppose for any compact interval I, let

$$\boxed{U_I = \{(t,x) \in U : t \in I\}}$$



$$|f(t,x)| \le M_I + L_I|x|$$
$$(t,x) \in U_I$$

$\Rightarrow$ $\dot{x}(t) = f(t, x(t))$

$(\dot{x}_1, \dot{x}_2) = (f_1(t, x(t)), f_2(t, x(t)))$ $\qquad f: \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$\boxed{|f_j(t,x)| \le M_T + L(T)\|x\|_\infty}$$

$\forall j$.

Choose $I \subset (-1,0)$, I is compact

$\Rightarrow$ $|f_1(t,x)| \le |x_2| \le \|x\|$

$\overset{\shortparallel}{\dot{x}_2}$

the constants depend on interval

Continuous functions on a compact interval so uniformly bounded by a constant

cst t

$\Rightarrow$ $|f_2(t,x)| \le M_1|x_2| + M_2|x_1| + 2$

$\le (M_1 + M_2)\|x\|_\infty + 2$

Then application of Gronwall's inequality tells we can extend soln.

$\hookrightarrow$ $\therefore$ The soln can be extended on $(-1, 0)$

Q: $y'' - a_1(t)y' + a_2(t)y(t) = b(t)$ $\quad y(0) = y_1$

$y'(0) = y_2$

a_1, a_2, b are continuous defined on $(-1,1)$

Show maximum interval is $(-1,1)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ b(t) - a_2(t)x_1 + a_1(t)x_2 \end{pmatrix}$$

$f : U \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, Choose $I \subseteq (-1,1)$

$|f_1(x,t)| \leq \|x\|_\infty$

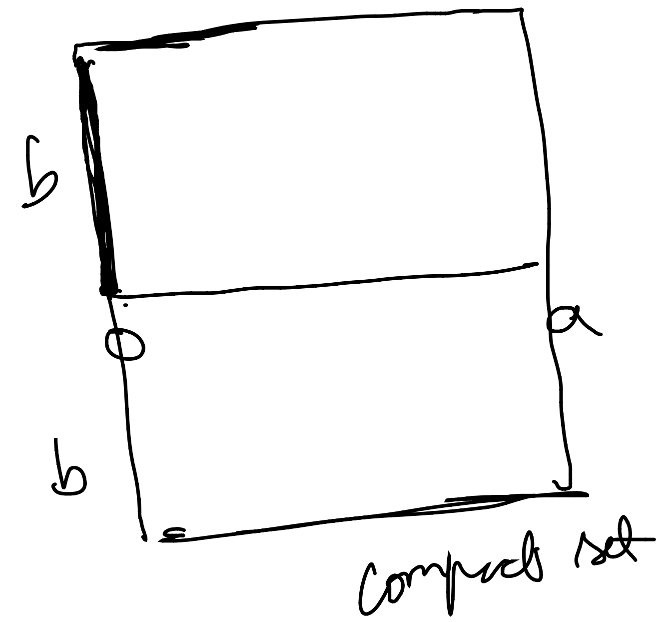
$|f_2(x,t)| \leq |a_1| \|x\|_\infty + |a_2| \|x\|_\infty + |b(t)|$

$\leq L\|x\|_\infty + M$

Q② $a,b>0$

$$R=\{(t,x)\in\mathbb{R}^2 : t\in[0,a],\ |x|\le b\}$$

$$f(t,x)=t^2+x^2$$



$$\begin{cases}\dot{x}(t)=f(t,x(t)), & t\in[0,a]\\ x(0)=0\end{cases}$$

$$h=\min\left(T,\frac{\delta}{M}\right)$$

$$=\min\left(a,\frac{b}{a^2+b^2}\right) \qquad M=\sup_{\underline{(t,x)\in R}} t^2+x^2$$

$$1(a^2+b^2)-b(2b)=0$$

$$2b^2=a^2+b^2$$

$$\underline{b^2=a^2},\ \boxed{b=a} \qquad \frac{a}{2a^2}=\frac{1}{2a}$$

Can you extend solution to $\mathbb{R}^+$?

$$|f(t,x)|\le M(T)+\underline{L(T)|x|}$$

Q 7

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\delta = \frac{a+d}{2}$, $\gamma = \frac{a-d}{2}$

$B = \begin{pmatrix} \gamma & b \\ c & -\gamma \end{pmatrix}$, $A = \delta I + B$

① $e^{A} = e^{\delta I + B} = e^{\delta I} e^{B} = e^{\delta} e^{B}$ (as I commutes with every matrix)

② $B^2 = \begin{pmatrix} \gamma & b \\ c & -\gamma \end{pmatrix}^2 = \begin{pmatrix} \gamma^2+bc & 0 \\ 0 & \gamma^2+bc \end{pmatrix} = (\gamma^2+bc) I$

$= \Delta I$

$B^{2k} = \Delta^k I$

$B^{2k+1} = \Delta^k B$

③ $e^{B} = \sum_{k\ge 0} \frac{B^{2k}}{(2k)!} + \sum_{k \ge 0} \frac{B^{2k+1}}{(2k+1)!}$

$= \underbrace{\left(\sum_{k\ge 0} \frac{\Delta^k}{(2k)!} \right)}_{\cosh(\sqrt{\Delta})} I + \underbrace{\left(\sum_{k \ge 0} \frac{\Delta^k}{(2k+1)!} \right)}_{\frac{\sinh \sqrt{\Delta}}{\sqrt{\Delta}}} B$

If $\Delta = 0$

$e^{B} = I + B$

$2\left[\left(\sqrt{\Delta}\right) + \frac{(\sqrt{\Delta})^3}{3!} \right]$

$\frac{e^{\sqrt{\Delta}} + e^{-\sqrt{\Delta}}}{2}$

$1 + \cancel{\sqrt{\Delta}} + \frac{(\sqrt{\Delta})^2}{2!} + \cancel{\frac{(\sqrt{\Delta})^3}{3!}} + \dots$

$+ 1 - \cancel{\sqrt{\Delta}} + \frac{(\sqrt{\Delta})^2}{2!} - \cancel{\frac{(\sqrt{\Delta})^3}{3!}}$

$$\boxed{e^{A} = e^{\delta}\left(\cosh(\sqrt{\Delta}) I + \frac{\sinh(\sqrt{\Delta})}{\sqrt{\Delta}} B \right)}$$

If $\Delta < 0$, $\sqrt{\Delta} = i\mu$

$e^{B} = \cos\mu\, I + \frac{\sin \mu}{\mu} B$

Q8 $\quad x'(t) = Ax(t)$; $x(0) = x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(i) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$e^{At} x_0$

$At = \begin{pmatrix} 2t & t \\ 0 & 2t \end{pmatrix}$, $\delta = \frac{4t}{2} = 2t$, $\gamma = \frac{a-d}{2} = 0$, $\Delta = \gamma^2 + bc = 0$

$B = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$

$e^{At} = e^{2t}\left(I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right)$ $\quad A'$

$= e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}_{2\times 2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{2\times 1}$

$= e^{2t} \begin{pmatrix} 1+t \\ 1 \end{pmatrix} = \boxed{\left[(1+t)e^{2t}, e^{2t}\right]^T}$

(ii) $At = \begin{pmatrix} -t & t \\ 0 & t \end{pmatrix}$; $x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\delta = \frac{a+d}{2} = 0$; $B = \begin{pmatrix} -t & t \\ 0 & t \end{pmatrix}$; $\Delta = \gamma^2 + bc = t^2$

$\gamma = \frac{a-d}{2} = \frac{-t-t}{2} = -t$

$e^{At} = e^{\delta} e^{B} = (\cosh t) I + \frac{\sinh t}{t} \begin{pmatrix} -t & t \\ 0 & t \end{pmatrix}$ $\quad A'$

$\begin{bmatrix} \cosh t - \sinh t & \sinh t \\ 0 & \cosh t + \sinh t \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftarrow \begin{bmatrix} \cosh t & 0 \\ 0 & \cosh t \end{bmatrix} + \begin{bmatrix} -\sinh t & \sinh t \\ 0 & \sinh t \end{bmatrix}$

$= (\cosh t - 2\sinh t, \ -\cosh t - \sinh t) = \text{Ans}$