MTH114: ODE Assignment-6

1. Consider $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$. Then:

(a) Calculate $f', \quad f'', \quad f'''$.

(b) Prove derivative of $\frac{c}{x^p}e^{-1/x^2}$ consists of sum of terms of similar form. Hence deduce that $f^{(n)}(x)$ consists of sum terms of the form $\frac{c}{x^p}e^{-1/x^2}$ for different $c, p \in \mathbb{N}$.

(c) Prove that

$$\lim_{x\to 0} \frac{c}{x^p}e^{-1/x^2} = 0, \quad c, p \in \mathbb{N}.$$

(d) Deduce that $f^{(n)}(0) = 0$ for all n.

(e) Thus conclude that f is infinitely differentiable but f is not analytic at 0.

[Recall: A real valued function is said to be analytic at x_0 if $f(x)$ can be written as a convergent power series $\sum a_n(x - x_0)^n$ on $|x - x_0| < R$ for some $R > 0$. A function is analytic on a domain Ω if it is analytic at each $x_0 \in \Omega$. We know that any analytic function is infinitely differentiable BUT there exists infinitely real differentiable functions which are not analytic.]

Solution:

(a)

$$f'(x) = \frac{2}{x^3}e^{-1/x^2}, \; f''(x) = \frac{4}{x^6}e^{-1/x^2} - \frac{6}{x^4}e^{-1/x^2}, \; f'''(x) = \frac{8}{x^9}e^{-1/x^2} - \frac{36}{x^7}e^{-1/x^2} + \frac{24}{x^5}e^{-1/x^2}.$$

(b)

$$\frac{d}{dx}(\frac{c}{x^p}e^{-1/x^2}) = -\frac{pc}{x^{p+1}}e^{-1/x^2} + \frac{2c}{x^{p+3}}e^{-1/x^2}.$$

Clearly, by induction, $f^{(n)}(x)$ consists of sum terms of the form $\frac{c}{x^p}e^{-1/x^2}$ for different $c, p \in \mathbb{N}$.

(c)

$$\lim_{x\to 0} \frac{c}{x^p}e^{-1/x^2} = \lim_{u\to\infty} cu^p e^{-u^2} = \lim_{u\to\infty} \frac{cu^p}{e^{u^2}} = 0. \;\; c, p \in \mathbb{N}.$$

(d) Combining (b) and (c) we conclude that $f^{(n)}(0) = 0$ for all n.

(e) If $f(x) = \sum a_n x^n$ on a nbd of 0, then $a_n = f^{(n)}(0)/n! = 0$. Hence $f = 0$ on a nbd of 0. This is a contradiction. So f is not analytic at 0.

2. Prove that if f, g are analytic at x_0 and $g(x_0) \neq 0$ then f/g is analytic at x_0.

Solution:

Assume $f(x) = \sum a_n(x - x_0)^n$ and $g(x) = \sum b_n(x - x_0)^n$ with $g(x_0) = b_0 \neq 0$.

Claim: We can find $c_n \in \mathbb{R}$ such that $f/g = \sum c_n(x - x_0)^n$ i.e.

$$\sum a_n(x - x_0)^n = \sum b_m(x - x_0)^m \sum c_k(x - x_0)^k.$$

Equating coefficients of different x^n:

$a_0 = b_0c_0 \implies c_0 = a_0/b_0$.

$a_1 = b_0c_1 + b_1c_0 \implies c_1$ can be found using known value ofc_0.

$a_2 = b_0c_2 + b_1c_1 + b_2c_0 \implies c_2$can be found using known values of c_0, c_1.

Thus inductively we can solve for all $c_k's$.

3. Is x_0 is an ordinary point of the ODE? If so expand $p(x), q(x)$ in power series about x_0. Find a minimum value for the radius of convergence of a power series solution about x_0.

   (a) $(x+1)y'' - 3xy' + 2y, \quad x_0 = 1$

   (**T**)(b) $(1+x+x^2)y'' - 3y = 0, \quad x_0 = 1$.

   Solution:

   (a) Here $p(x) = -3x/(x+1), \quad q(x) = 2/(x+1)$. Clearly $x_0 = 1$ is an ordinary point.

   Now $x/(x+1) = x/(2+x-1) = \frac{x}{2}\frac{1}{1+(x-1)/2} = \frac{1}{2}(x-1+1)\sum[(1-x)/2]^n$ valid for $|1-x| < 2$.

   The only singular point is $x = -1$. Thus the minimum radius of convergence of the solution is the distance between $x_0 = 1$ and -1, which is 2.

   (b) Here $p(x) = 0, \quad q(x) = -3/(x^2+x+1)$. Clearly $x_0 = 1$ is an ordinary point.

   The singular points are $x = (-1 \pm \sqrt{3}i)/2$. Thus the minimum radius of convergence of the solution is the distance between $x_0 = 1$ and $(-1 \pm \sqrt{3}i)/2$, which is $\sqrt{3}$.

   Now for $t = x - 1$

   $$\frac{1}{x^2+x+1} = \frac{1}{3+3t+t^2} = \frac{1}{3(1+[t^2+3t)/3])} = \frac{1}{3}\sum[-(t^2+3t)/3]^n$$

   valid for $|(t^2+3t)/3| < 1$ that is $|t| < \sqrt{3}$.

4. Locate and classify the singular points in the following:
   (**T**)(i) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$ (ii) $(3x+1)xy'' - xy' + 2y = 0$

   Solution:

   (i) The given ODE can be written as

   $$y'' - \frac{2}{x^3}y' + \frac{3}{x^2(x-1)}y = 0$$

   Hence, $x = 1$ regular and $x = 0$ irregular singular points

   (ii) The given ODE can be written as

   $$y'' - \frac{1}{3x+1}y' + \frac{2}{x(3x+1)}y = 0$$

   Hence, both $x = 0, x = -1/3$ are regular singular points

5. Consider the equation $y'' + y' - xy = 0$.

   (i) Find the power series solutions $y_1(x)$ and $y_2(x)$ such that $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$.

   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

   Solution:

   (i) Substituting $y = \sum_{n=0} a_n x^n$ into $y'' + y' - xy = 0$, we get

$$\sum_{n=0}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}(n+1)a_{n+1}x^n - \sum_{n=1} a_{n-1}x^n = 0$$

   Rearranging, we find

$$(2a_2 + a_1) + \sum_{n=1}\left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_{n-1}\right]x^n = 0$$

   Hence,

$$2a_2 + a_1 = 0,\ a_{n+2} = -\frac{a_{n+1}}{n+2} + \frac{a_{n-1}}{(n+1)(n+2)}, \qquad n \geq 1.$$

   Iterating we get

$$a_2 = -a_1/2,\ a_3 = a_1/(2\cdot 3) + a_0/(2\cdot 3),\ a_4 = a_1/(2\cdot 3\cdot 4) - a_0/(2\cdot 3\cdot 4), \cdots .$$

   Thus,

$$\begin{aligned} y &= a_0\left[1 + \frac{x^3}{2\cdot 3} - \frac{x^4}{2\cdot 3\cdot 4} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} + \cdots\right] + a_1\left[x - \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} - \frac{4x^5}{2\cdot 3\cdot 4\cdot 5} + \cdots\right] \\ &= a_0 y_1(x) + a_1 y_2(x). \end{aligned}$$

   Now, y_1 and y_2 have the desired properties.

   (ii) For the given ODE, $p(x) = 1$ and $q(x) = -x$ both of which have radius of convergence $R = \infty$. Hence, both y_1 and y_2 have radius of convergence $R = \infty$.

6. (**T**) Consider the equation $(1 + x^2)y'' - 4xy' + 6y = 0$.

   (i) Find its general solution in the form $y = a_0 y_1(x) + a_1 y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.

   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

   Solution:

   (i) Substituting $y = \sum_{n=0} a_n x^n$ into $(1 + x^2)y'' - 4xy' + 6y = 0$, we get

$$\sum_{n=0}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2} n(n-1)a_n x^n - \sum_{n=1} 4na_n x^n + \sum_{n=0} 6a_n x^n = 0$$

   Rearranging we find

$$(2a_2+6a_0)+(6a_3-4a_1+6a_1)x+\sum_{n=2}\left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n\right]x^n = 0$$

Hence,

$$a_2 = -3a_0,\; a_3 = -\frac{a_1}{3},\; a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n, \qquad n \geq 2.$$

Iterating we get

$$a_2 = -3a_0,\; a_3 = -\frac{a_1}{3},\; a_n = 0, \quad n \geq 4.$$

Thus,

$$\begin{aligned} y &= a_0\left(1-3x^2\right) + a_1\left(x - \frac{x^3}{3}\right) \\ &= a_0 y_1(x) + a_1 y_2(x) \end{aligned}$$

(ii) Both the series are polynomials and hence converges for all x. Note that here $p(x) = -4x/(1+x^2)$ and $q(x) = 6/(1+x^2)$ are analytic at $x = 0$ and have radius convergence $R = 1$. Thus the existence and uniqueness theorem for the ordinary point guarantees existence of unique solution in $|x| < 1$ but actually we find the existence of unique solution for all x.

7. Find the first three non zero terms in the power series solution of the IVP

$$y'' - (\sin x)y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0.$$

Solution: As the initial values are given at π, the expansion should be about $x_0 = \pi$. The best way to do this is to first shift x_0 to 0. To do this, let $t = x - \pi$. Then $t_0 = x_0 - \pi = 0$. The equation becomes

$$y'' + (\sin t)y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Assuming $y = \sum a_n t^n$ and using $\sin t = \sum \frac{(-1)^n}{(2n+1)!}t^{2n+1}$ we get

$$0 = y'' + (\sin t)y = 2a_2 + (6a_3 + a_0)t + (12a_4 + a_1)t^2 + (20a_5 + a_2 - a_0/6) + \cdots .$$

From initial conditions $a_0 = 1, a_1 = 0$. So $a_2 = 0, a_3 = -1/6, a_4 = 0, a_5 = 1/120$.

8. Using Rodrigues' formula for $P_n(x)$, show that

(**T**)(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n'(-x) = (-1)^{n+1}P_n'(x)$

(iii) $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$ (iv) $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $n > m$.

Solution:

(i) Replace x in $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left((x^2-1)^n\right)$ by $-z$ to get (using $d/dx = -d/dz$)

$$P_n(-z) = (-1)^n \frac{1}{2^n n!}\frac{d^n}{dz^n}\left((z^2-1)^n\right) = (-1)^n P_n(z)$$

(ii) By differentiating (i) w.r.t. x, we get

$$-P_n'(-x) = (-1)^n P_n(x) \implies P_n'(x) = (-1)^{n+1}P_n(x).$$

(iii) Let $f(x)$ be any function with at least n continuous derivatives in $[-1,1]$. Consider the integral

$$I = \int_{-1}^{1} f(x)P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx.$$

Repetition of integration by parts repeatedly gives

$$I = (-1)^n \frac{1}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,dx.$$

If $m \neq n$, without any loss of generality we take $f = P_m$, $m < n$ and then $f^{(n)}(x) = 0$ (since P_m is a polynomial of degree $m < n$) and thus $I = 0$.

If $f(x) = P_n(x)$, then

$$f^{(n)}(x) = \frac{1}{2^n n!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = \frac{2n!}{2^n n!}.$$

Thus,

$$I = \frac{2n!}{2^{2n}(n!)^2)}\int_{-1}^{1}(1-x^2)^n\,dx = \frac{2(2n!)}{2^{2n}(n!)^2)}\int_{0}^{1}(1-x^2)^n\,dx.$$

Substitute $x = \sin\theta$ to get

$$I = \frac{2(2n!)}{2^{2n}(n!)^2)}\int_{0}^{\pi/2}\cos^{2n+1}\theta\,d\theta = \frac{2(2n!)}{2^{2n}(n!)^2}J_n.$$

Using integration by parts

$$\int \cos^{2n+1}\,d\theta = \sin\theta\cos^{2n}\theta + 2n\int \sin^2\theta\,\cos^{2n-1}\theta\,d\theta = \sin\theta\cos^{2n}\theta + 2n\int(1-\cos^2\theta)\cos^{2n-1}\theta\,d\theta$$

This leads to

$$J_n = \int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta = \frac{2n}{2n+1}J_{n-1} = \frac{2n}{2n+1}\frac{2(n-1)}{2n-1}\cdots\frac{2}{3}J_0.$$

Now

$$J_0 = \int_0^{\pi/2}\cos\theta\,d\theta = 1.$$

Hence,

$$J_n = \frac{2^n n!}{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)} = \frac{2^{2n}(n!)^2}{(2n!)(2n+1)}$$

Thus,

$$I = \frac{2}{2n+1}$$

(iv) Follows from (iii) by taking $f(x) = x^m$ where $m < n$.

9. Expand the following functions in terms of Legendre polynomials over $[-1,1]$:

(i) $f(x) = x^3 + x + 1$ (**T**)(ii) $f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x \le 1 \end{cases}$ (first three nonzero terms)

Solution:

We know from Legendre Expansion Theorem that any continuous function $f(x)$ on $[-1,1]$, has Legendre series expansion as

$$f(x)=\sum_{n=0} a_n P_n(x), \qquad \text{with } a_n=\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx; \quad x\in[-1,1].$$

(See N. N. Lebedev, Special Functions and Their Applications, pp. $53-58$, Prentice-Hall, Englewood Cliffs, N.J. , 1965.)

(i) We can use the above formula to find a_n. Alternately, we know that

$$P_0(x), P_1(x)=x, P_3(x)=\frac{5x^3-3x}{2}.$$

So we find

$$1=P_0(x), \quad x=P_1(x), \quad x^3=\frac{2P_3(x)+3P_1(x)}{5}.$$

Hence,

$$f(x)=P_0(x)+P_1(x)+\frac{2P_3(x)+3P_1(x)}{5}=P_0(x)+\frac{8}{5}P_1(x)+\frac{2}{5}P_3(x)$$

(Remark: Note that, if f has derivatives of all order then, $\int_{-1}^{1} f(x)P_n(x)dx=\frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n dx$. In particular, if $f(x)$ is a polynomial of degree n then $a_m=0$ for all $m>n$.)

(ii) Using the above formula,

$$a_0=\frac{1}{4},\ a_1=\frac{1}{2},\ a_2=\frac{5}{16}.$$

Thus,

$$f(x)=\frac{1}{4}P_0(x)+\frac{1}{2}P_1(x)+\frac{5}{16}P_2(x)+\cdots$$

10. Suppose $m>n$. Show that $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx=0$ if $m-n$ is odd. What happens if $m-n$ is even?

Solution:

Proceeding as in 4(iii), we get (taking $f(x)=x^m$)

$$I=\int_{-1}^{1} x^m P_n(x)\,dx=\frac{m(m-1)\cdots(m-n+1)}{2^n n!}\int_{-1}^{1} x^{m-n}(1-x^2)^n\,dx$$

If $m-n$ is odd, then $I=0$, since the integrand then becomes an odd function.

If $m-n=2k$ is even, then

$$\begin{aligned} I &= \frac{2m(m-1)\cdots(m-n+1)}{2^n n!}\int_0^{\pi/2}\sin^{2k}\theta\cos^{2n+1}\theta\,d\theta \\ &= \frac{2m(m-1)\cdots(m-n+1)}{2^n n!} I_{k,n} \end{aligned}$$

where

$$I_{k,n} = \int_0^{\pi/2} \sin^{2k}\theta \cos^{2n+1}\theta \, d\theta = \frac{2n}{2k+1} I_{k+1,n-1}$$

By repeated application of this relation, the last subscript becomes zero. Then the resulting integral can be evaluated by substitution:

$$I_{k+n,0} = \int_0^{\pi/2} \sin^{2(k+n)}\theta \cos\theta \, d\theta = \frac{1}{2(k+n)+1}$$

Thus,

$$\begin{aligned} I_{k,n} &= \frac{2n \cdot 2(n-1) \cdots 2.1}{(2k+1)(2k+3)\cdots\{2(k+n-1)+1\}} I_{k+n,0} \\ &= \frac{2^n n!}{(2k+1)(2k+3)\cdots\{2(k+n-1)+1\}\{2(k+n)+1\}} \end{aligned}$$

Substituting $I_{k,n}$ into the expression of I gives the value of the integral when $m-n$ is even.

11. The function on the left side of

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

is called the generating function of the Legendre polynomial P_n. Assuming this, show that

(a) (**T**) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

(b) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(c) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$

(d) $P_n(1) = 1,\ P_n(-1) = (-1)^n$

(e) $P_0(0) = 1,\ P_{2n+1}(0) = 0,\ P_{2n}(0) = (-1)^n \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!},\ n \geq 1.$

Solution:

(i) Differentiating both sides w.r.t. t, we get

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=1} nP_n(x) t^{n-1}$$

which gives

$$(x-t)\sum_{n=0} P_n(x)t^n = (1-2xt+t^2)\sum_{n=0}(n+1)P_{n+1}(x)t^n$$

Equating the coefficient of t^n from both sides, we get

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1},$$

which on simplification yields

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

(ii) Differentiating both sides w.r.t. x, we get

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0} P_n'(x)t^n$$

which gives

$$(1-2xt+t^2)\sum_{n=0} P_n' t^n = t\sum_{n=0} P_n t^n$$

Equating the coefficient of t^n from both sides, we get

$$P_n' - 2xP_{n-1}' + P_{n-2}' = P_{n-1}$$

which on replacing n by $n+1$ gives

$$P_{n+1}' - 2xP_n' - P_n + P_{n-1}' = 0. \qquad (*)$$

Differentiating the relation in (i) w.r.t. x, we get

$$(n+1)P_{n+1}' - (2n+1)\Big(P_n + xP_n'\Big) + nP_{n-1}' = 0. \qquad (**)$$

Elimination of P_{n+1}' between (*) and (**) gives

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

(iii) Proceeding as in (ii) we arrive in relation given in (*) and (**). Eliminate p_{n-1}' between (*) and (**) to find

$$P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$$

(iv) Substituting $x=1$ into the relation we find

$$\sum_{n=0} P_n(1)t^n = \frac{1}{1-t} = \sum_{n=0} t^n$$

Equating coefficients of t^n, we get $P_n(1) = 1$.

Similarly, substituting $x=-1$ into the relation we find

$$\sum_{n=0} P_n(-1)t^n = \frac{1}{1+t} = \sum_{n=0} (-1)^n t^n$$

Equating coefficients of t^n, we get $P_n(-1) = (-1)^n$.

(v) Substitute $x=0$ into the relation we get

$$\sum_{n=0} P_n(0)t^n = \frac{1}{\sqrt{1+t^2}} = 1 + \sum_{n=1} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} t^{2n}$$

or

$$P_0(0) + \sum_{n=1} P_{2n}(0)t^{2n} + \sum_{n=1} P_{2n+1}(0)t^{2n+1} = 1 + \sum_{n=1} (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} t^{2n}$$

Equating the coefficients of t^n we get

$$P_0(0) = 1,\ P_{2n+1}(0) = 0,\ P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}, \quad n \geq 1$$