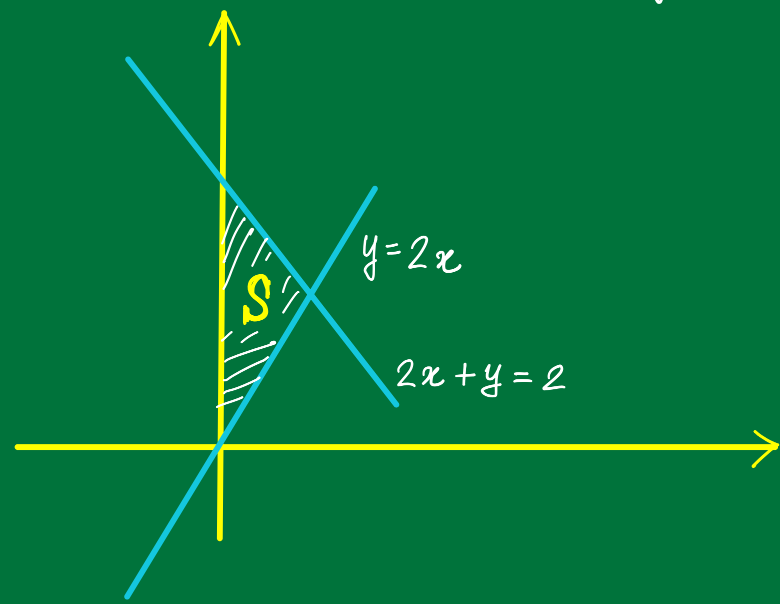
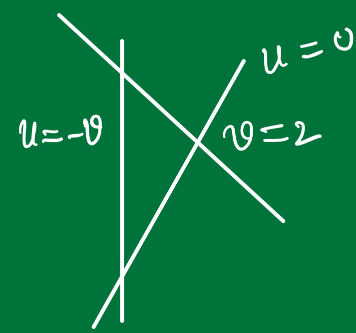
Example: Evaluate $\iint_S (4x^2-y^2)^4 \, dx\, dy$ where the region S is given by



Let $\left.\begin{aligned} u &= 2x-y \\ v &= 2x+y \end{aligned}\right\} \Rightarrow x = \dfrac{u+v}{4}$

$y = 2x - u = \dfrac{u+v}{2} - u = \dfrac{v-u}{2}$



$$\cancel{J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}}$$

$$0 \neq J(u,v) = \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix} = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix} \text{ also denoted by } \frac{\partial(x,y)}{\partial(u,v)}.$$

For $x = X(u,v)$

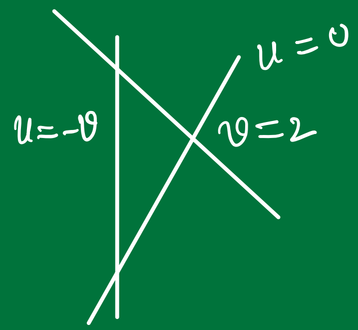
$y = Y(u,v)$

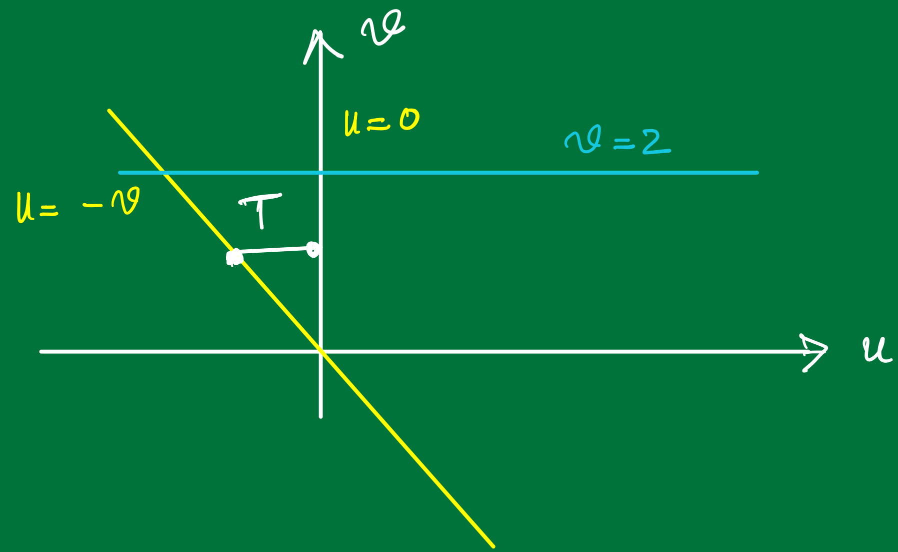
Note that $\dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(x,y)} = 1$ Where $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Let $\left.\begin{aligned} u &= 2x - y \\ v &= 2x + y \end{aligned}\right\}.$ Then $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$

Thus

$$\iint_S (4x^2 - y^2)^4 \, dx\, dy = \iint_T (uv)^4 \times \frac{1}{4} \times du\, dv$$





Thus

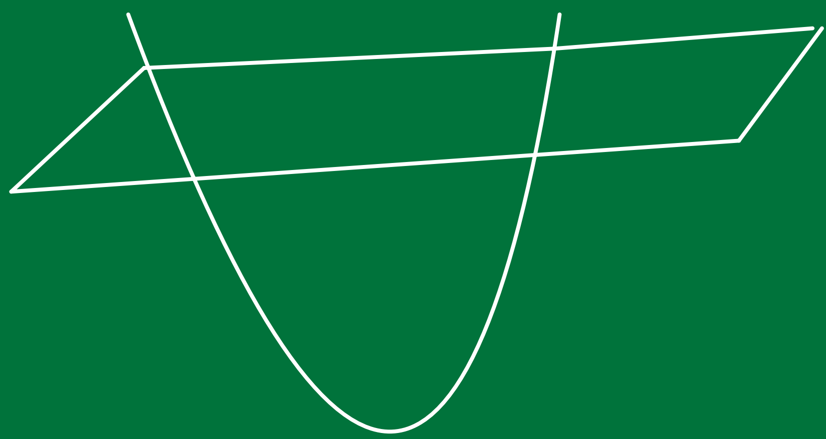
$$\iint_S (4x^2-y^2)^4\,dx\,dy \;=\; \int_0^2\int_{-v}^{0} (uv)^4 \cdot \frac{1}{4} \times du\,dv$$

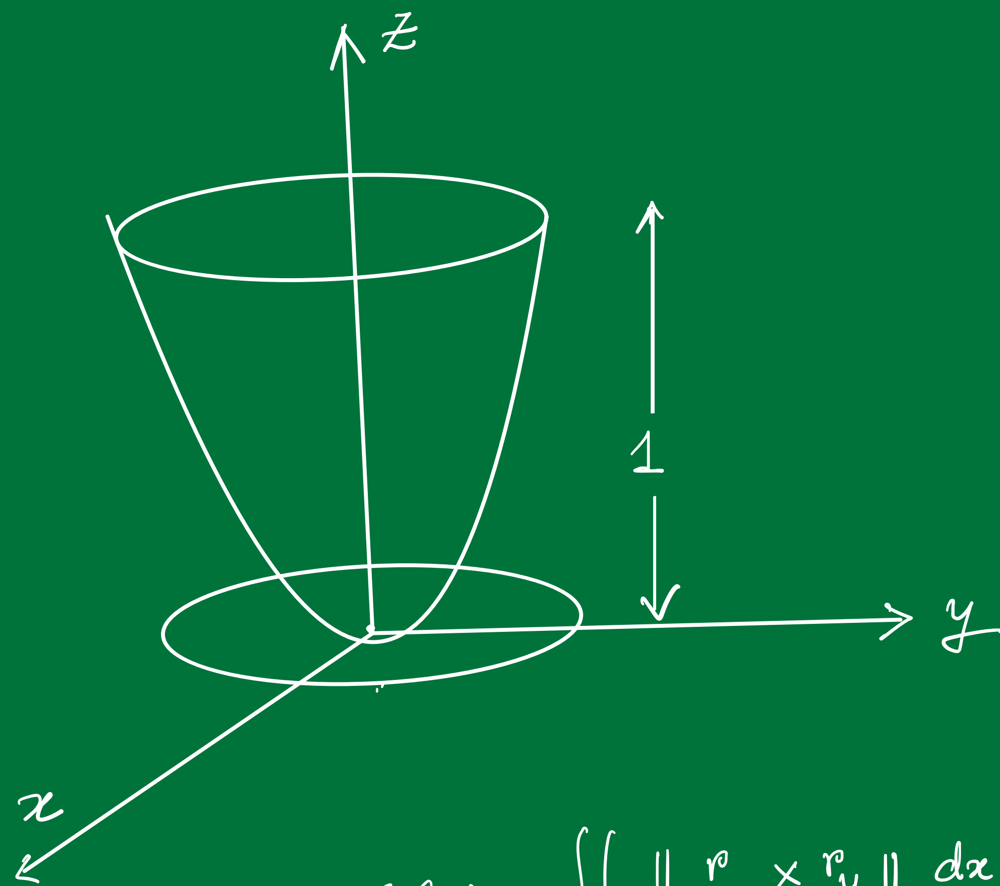
$$\vdots$$

$$= \frac{8}{50}$$

Example: Find the surface area $a(S)$ of the paraboloid

$S: \quad z = f(x,y) = x^2 + y^2$ that lies below the plane $z = 1$.





$$T = \{ (x,y) \ / \ x^2 + y^2 \leq 1 \}$$

$$r: T \longrightarrow \mathbb{R}^3$$
$$(x,y) \longmapsto (x, y, x^2 + y^2)$$

$$a(S) = \iint_T \| r_x \times r_y \| \, dx \, dy$$

$$a(S) = \iint_T \| r_x \times r_y \| \, dx\, dy$$

$$= \int\int_T \sqrt{1 + 4x^2 + 4y^2} \; dx\, dy$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \; r \; dr\, d\theta$$

$$= \frac{\pi}{6} \left(5\sqrt{5} - 1 \right)$$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix}$$

$$= -2x\, i - 2y\, j + k$$

$$\left\{ (r, \theta) \;/\; r \in [0,1] \text{ and } \theta \in [0, 2\pi] \right\}$$

We can rewrite the area of a parametric surface

$$a(S) = \iint_T \sqrt{EG - F^2}\, du\, dv$$

$$a(S) = \iint_T \| r_u \times r_v \| \, du\, dv$$

$$= \iint_T \sqrt{EG - F^2}\, du\, dv$$

$$\| r_u \times r_v \|^2$$

$$= \| r_u \|^2 \| r_v \|^2 \sin^2\theta$$

$$= \underline{\| r_u \|^2}\, \underline{\| r_v \|^2} - \underline{\| r_u \|^2 \| r_v \|^2 \cos^2\theta}$$

$$= EG - F^2$$

Where

$$E = r_u \cdot r_u$$

$$F = r_u \cdot r_v$$

$$G = r_v \cdot r_v$$

Surface integral $\iint_S g \, d\sigma$

Let $S = r(u,v)$ be a parametric surface defined on a parametric domain T.

Suppose $r_u : T \longrightarrow \mathbb{R}^3$, $r_v : T \longrightarrow \mathbb{R}^3$ are continuous and $g : S \longrightarrow \mathbb{R}$ be a bounded function.

The surface integral of g over S denoted by $\iint_S g \, d\sigma$ is defined as

$$\iint_S g \, d\sigma = \iint_T g(r(u,v)) \, \| r_u \times r_v \| \, du \, dv$$

$$= \iint_T g(r(u,v)) \sqrt{EG - F^2} \, du \, dv$$

provided the R.H.S double integral exists.

Remark. If the parametric surface S is defined by

$$f : T \to \mathbb{R}$$

$$(x,y) \longmapsto z = f(x,y)$$

i.e., $S = \{ (x, y, f(x,y)) \,/\, (x,y) \in T \}$

then $$\iint_S g \, d\sigma = \iint_T g(x, y, f(x,y)) \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$$

Where T is the projection of the surface S over the xy plane.

Example: S - ~~Unit~~ hemisphere

$$z = \sqrt{a^2 - x^2 - y^2} = f(x,y)$$

$$T = [0, 2\pi] \times [0, \tfrac{\pi}{2}]$$

$$r(\theta, \phi) = (a \sin\phi \cos\theta,\ a \sin\phi \sin\theta,\ a \cos\phi)$$

$$\iint_S \frac{d\sigma}{\sqrt{x^2 + y^2 + (z+a)^2}}$$

$$= \iint_T g(r(\theta,\phi)) \sqrt{EG - F^2}\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{a^{\cancel{2}} \sin\phi}{2\cancel{a} \cos\frac{\phi}{2}}\, d\theta\, d\phi = a\int_0^{2\pi}\int_0^{\pi/2} \sin\phi/2\, d\theta\, d\phi$$

$$g(r(\theta,\phi))$$

$$= \left[a^2 \sin^2\phi + a^2(1+\cos\phi)^2\right]^{-1/2}$$

$$= \left[a^2(\sin^2\phi + \cos^2\phi) + a^2 + 2a^2\cos\phi\right]^{-1/2}$$

$$= \left[2a^2(1+\cos\phi)\right]^{-1/2}$$

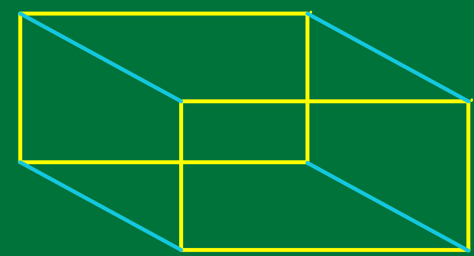
$$= 1/2a \cos\phi/2$$

Triple Integral

$$\iiint_Q f(x,y,z)\, dx\, dy\, dz \quad ;$$

where $Q = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subseteq \mathbb{R}^3$

is defined by extending the definition double integral defined on $[a_1, b_1] \times [a_2, b_2]$.



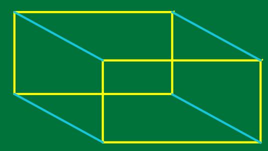
- f is a bounded function defined on Q;
- Partition P of Q; $\quad P = P_1 \times P_2 \times P_3$
- Definition of $L(P,f)$, $U(P,f)$ for a given partition of Q;
- Lower integral, upper integral and integral of f

Triple Integral

$$\iiint_Q f(x,y,z)\,dx\,dy\,dz \;; \qquad \text{or} \qquad \iiint_Q f(x,y,z)\,dV$$

where

$$Q = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subseteq \mathbb{R}^3$$



Remark. 1. The definition of triple integral can be extended to any bounded region as in the double integral.

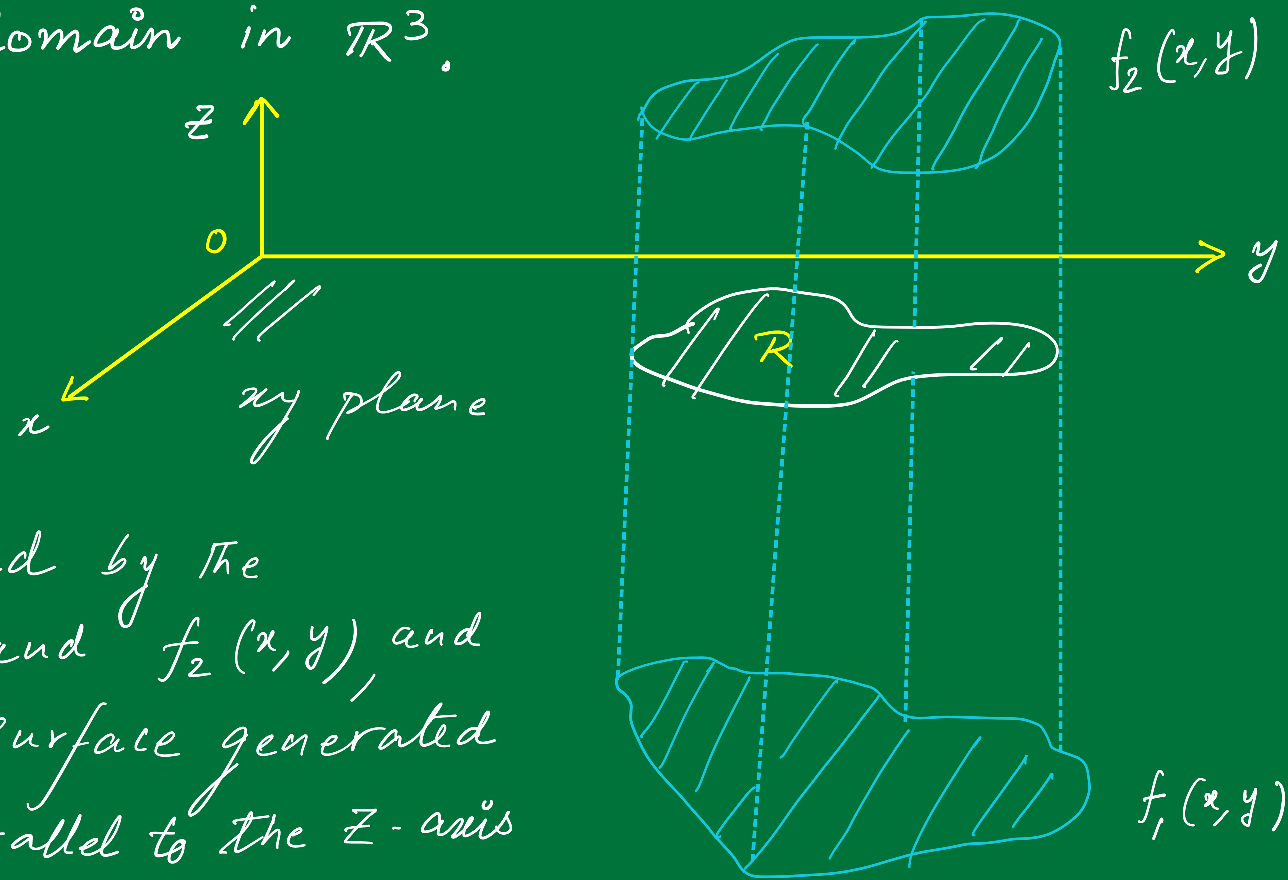
2. Every continuous function on Q is integrable.

- How to evaluate the triple integrals?

Fubini's theorem

Let $D = \{(x,y,z) \in \mathbb{R}^3 \,/\, (x,y) \in R \subseteq \mathbb{R}^2 \text{ and } f_1(x,y) \le z \le f_2(x,y)\}$

be a bounded domain in $\mathbb{R}^3$.

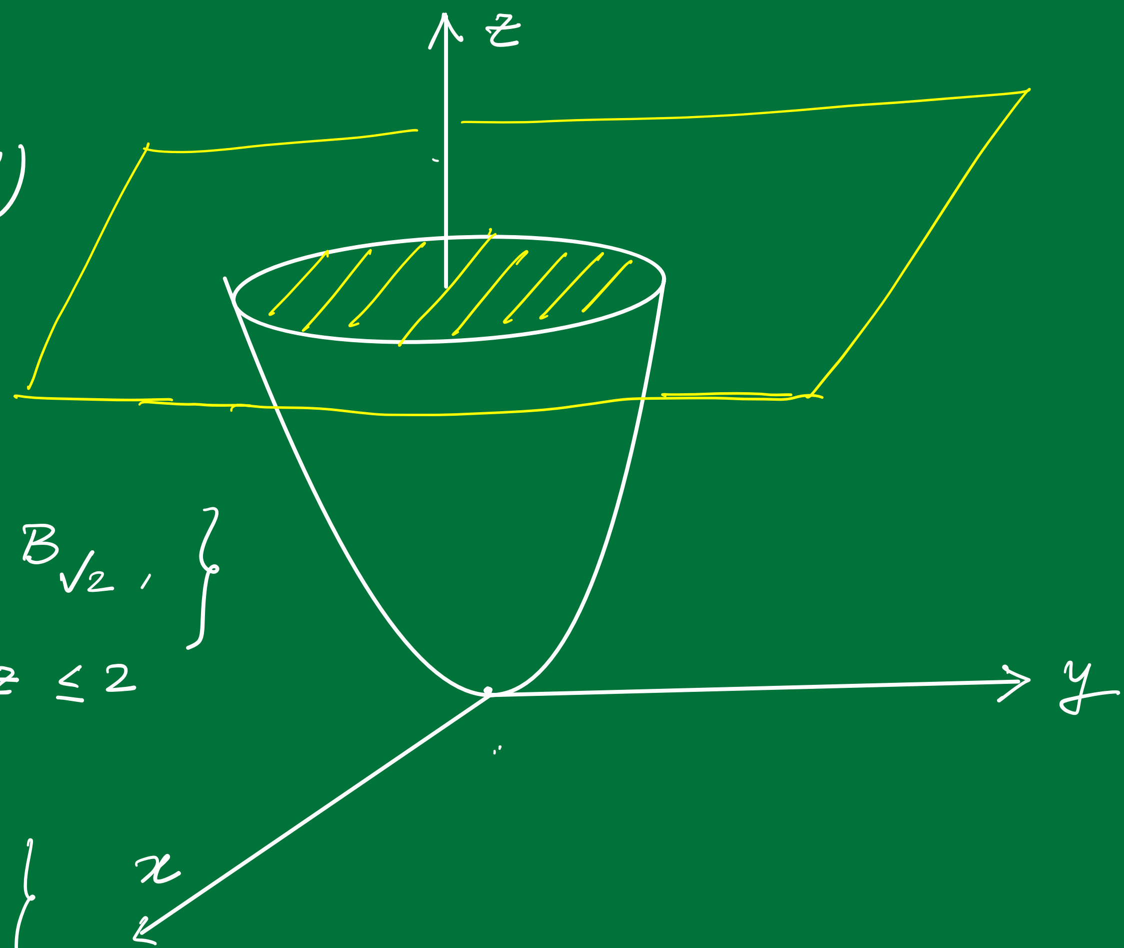


Here D is bounded by the Surfaces $f_1(x,y)$ and $f_2(x,y)$, and Cylindrical outer surface generated by line moving parallel to the Z-axis along the boundary of R (is the projection of D on the xy plane).

Example :

$$z = x^2 + y^2 = f_1(x,y)$$

$$z = 2 = f_2(x,y)$$



$$D = \{ (x,y,z) \in \mathbb{R}^3 / (x,y) \in B_{\sqrt{2}},\ x^2+y^2 \leq z \leq 2 \}$$

where

$$B_{\sqrt{2}} = \{ (x,y) \in \mathbb{R}^2 / 0 \leq x^2+y^2 \leq 2 \}$$

— circular disc of radius $\sqrt{2}$.

Theorem. If $f : D \to \mathbb{R}$ is a continuous function on

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle/\, \begin{array}{l} (x,y) \in R \subseteq \mathbb{R}^2 \text{ and} \\ f_1(x,y) \leq z \leq f_2(x,y) \end{array} \right\}$$

and $f_1, f_2 : R \to \mathbb{R}$ are continuous functions

then the triple integral

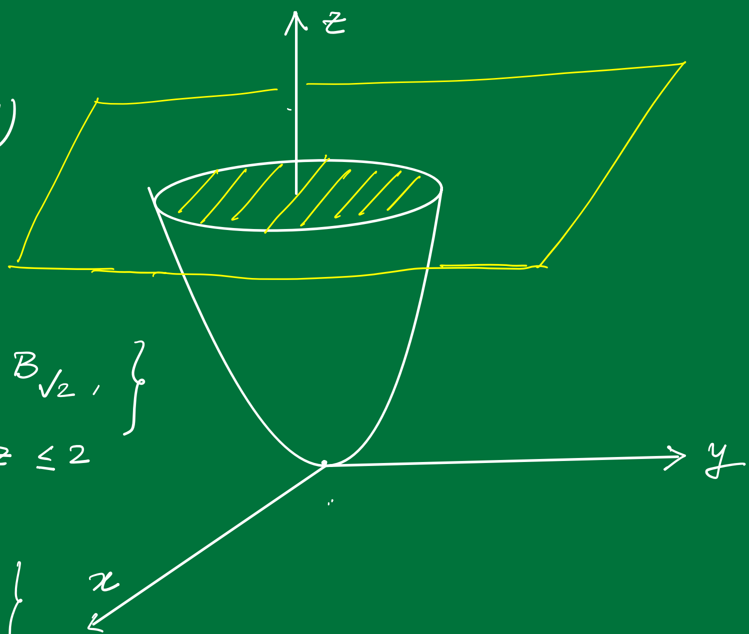
$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz \right) dA$$

$$= \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz \right) dx\, dy$$

Example :

$$z = x^2+y^2 = f_1(x,y)$$

$$z = 2 = f_2(x,y)$$



Consider

$$f(x,y,z) = x$$

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \middle/ \begin{array}{l}(x,y) \in B_{\sqrt{2}}, \\ x^2+y^2 \le z \le 2\end{array} \right\}$$

where

$$B_{\sqrt{2}} = \left\{ (x,y) \in \mathbb{R}^2 \middle/ 0 \le x^2+y^2 \le 2 \right\}$$

$$\iiint_D x \, dx\, dy\, dz = \iint_R \left(\int_{x^2+y^2}^{2} x \, dz \right) dx\, dy$$

Here,

$$R = \left\{ (x,y) \middle/ \begin{array}{l} 0 \le x \le \sqrt{2} \\ \text{and} \\ 0 \le y \le \sqrt{2-x^2} \end{array} \right\}$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x \, dz \right) dy\, dx = \frac{8\sqrt{2}}{15}.$$

Change of Variable in a triple integral

Formula under the similar assumptions as in the case of double integral:

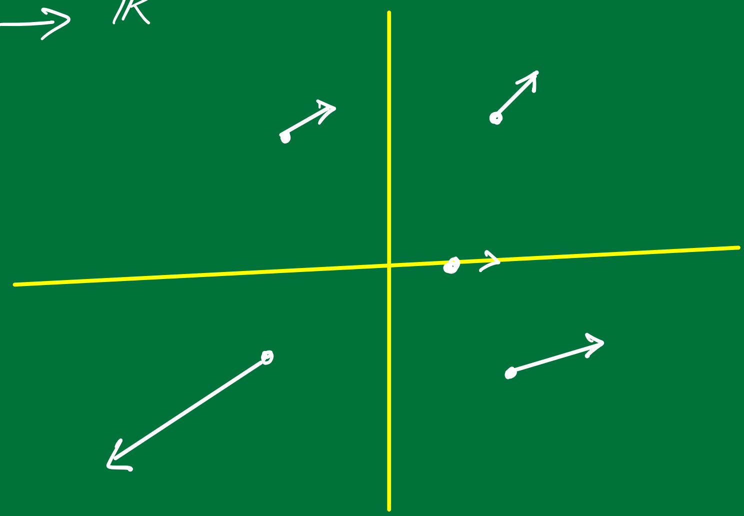
$$\iiint_S f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_T f\big(X(u,v,w),\, Y(u,v,w),\, Z(u,v,w)\big)\,\big|J(u,v,w)\big|\,du\,dv\,dw$$

where $J(u,v,w) = \begin{vmatrix} X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \\ X_w & Y_w & Z_w \end{vmatrix}.$

Vector valued functions

Let $C \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ and $f: C \to \mathbb{R}^3$



Examples. Vector fields

1. $(x, y) \longmapsto f(x,y) = x\vec{i} + y\vec{j}$

2. $(x, y) \longmapsto x\vec{i}$

3. $(x, y) \longmapsto -y\vec{i} + x\vec{j}$

$= (-y, x)$; $M(x,y) = -y$ and $N(x,y) = x$

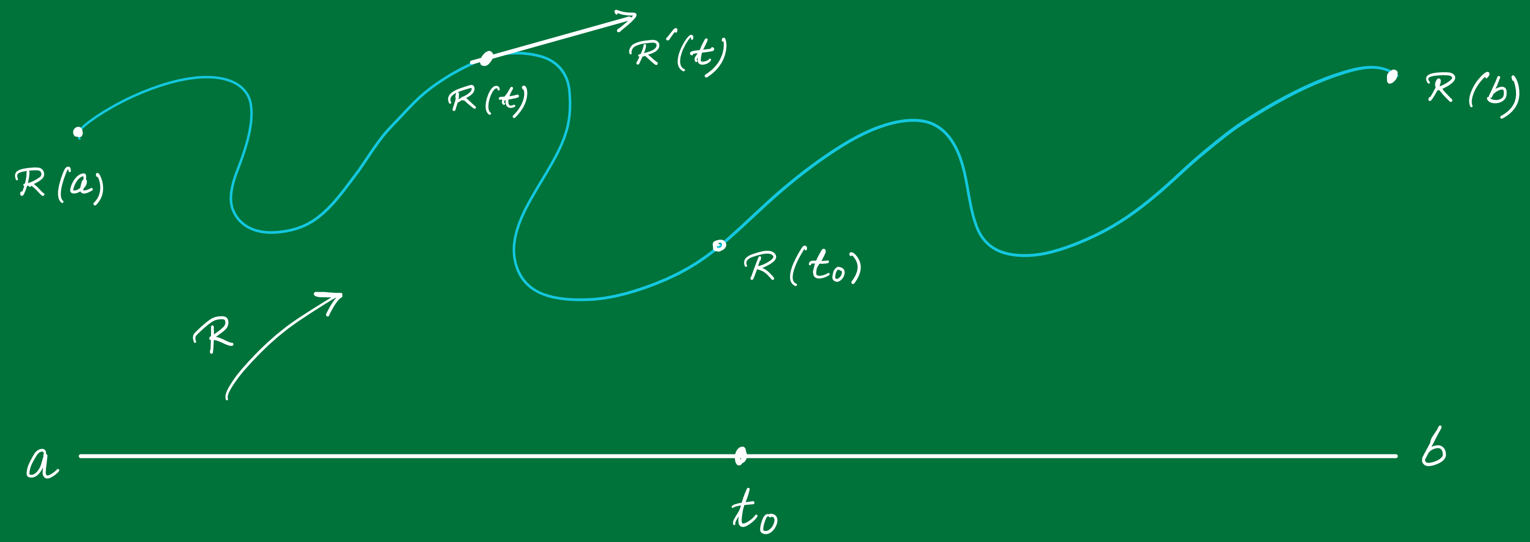
4. $(x, y) \longmapsto \big(M(x,y), N(x,y)\big)$

Suppose $R : [a, b] \to \mathbb{R}^3$

$$t \longmapsto R(t) = \big(x(t),\, y(t),\, z(t)\big)$$

is a differentiable function and

$C = \{ R(t) \,/\, t \in [a, b] \} \subseteq \mathbb{R}^3$ is a parametric curve.



□ For $f : C \to \mathbb{R}^3$, Consider $[a, b] \longrightarrow \mathbb{R}$

$$t \longmapsto f(R(t)) \cdot R'(t)$$

Definition.
The line integral of a vector valued function $f : C \to \mathbb{R}^3$ is denoted by $\int_C f \cdot dR$ and defined as

$$\int_C f \cdot dR = \int_a^b \left[f(R(t)) \cdot R'(t) \right] dt,$$

provided the R.H.S integral exists.

Notation:

- $f = (f_1, f_2, f_3) : C \to \mathbb{R}^3$
- $R(t) = (x(t), y(t), z(t))$ for $t \in [a, b]$
- $R'(t) = (x'(t), y'(t), z'(t))$ for $t \in [a, b]$

We have

$$\int_C f \cdot dR = \int_a^b f_1(R(t))\, x'(t)\, dt + \int_a^b f_2(R(t))\, y'(t)\, dt + \int_a^b f_3(R(t))\, z'(t)\, dt.$$

or $$\int_C f \cdot dR = \int_C f_1(x, y, z)\, dx + \int_C f_2(x, y, z)\, dy + \int_C f_3(x, y, z)\, dz.$$

Example: Let $f(x,y) = (-y, x) = -y\vec{i} + x\vec{j}$ and

$$C = \{(t, t^2) \,/\, t \in [0,1]\}.$$

Then $\int_C f.\,dR$

$$= \int_C (-y\,dx + x\,dy)$$

$$= \int_0^1 -t^2\,dt + \int_0^1 t(2t)\,dt$$

$$= -\int_0^1 t^2\,dt + 2\int_0^1 t^2\,dt$$

$$= \int_0^1 t^2\,dt$$

$$= \frac{1}{3}$$

Example: ∘ C – the line joining origin to $(1,2,4)$

$$\left[\quad \text{i.e.,} \quad R(t) = (x(t), y(t), z(t)) = (t, 2t, 4t). \quad\right]$$

∘ $f(x,y,z) = (x^2, y, xz - y)$

Evaluate $\int_C f \cdot dR$;

$$\int_C f \cdot dR$$

$$= \int_0^1 t^2 \, dt + \int_0^1 2t\,(2)\, dt + \int_0^1 (4t^2 - 2t)\, 4\, dt$$

$$= \int_0^1 (17t^2 - 4t)\, dt$$

$$= \frac{11}{3}$$

Recall : Let $f : [a, b] \longrightarrow \mathbb{R}$ with f' is continuous on $[a, b]$. By Second Fundamental Theorem of Calculus,

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Theorem. • Let $S \subseteq \mathbb{R}^3$, $f : S \to \mathbb{R}$ be differentiable function on S and $\nabla f : S \to \mathbb{R}^3$ be continuous.

• Let A, B be two points in S and $C = \{R(t) / t \in [a,b]\}$ be a curve joining A and B in S.

• Suppose $R'(t) : [a,b] \to \mathbb{R}^3$ is a continuous function.

Then $$\int_C \nabla f \cdot dR = f(B) - f(A).$$

Let $g(t) = f(R(t))$ for $t \in [a, b]$.

By chain rule, $g'(t) = \nabla f(R(t)) \cdot R'(t)$

$$\Rightarrow \int_a^b g'(t)\,dt = \int_a^b \nabla f(R(t)) \cdot R'(t)\,dt$$

or, $$\int_a^b \nabla f \cdot dR = g(b) - g(a)$$

$$= f(B) - f(A)$$

$\square$