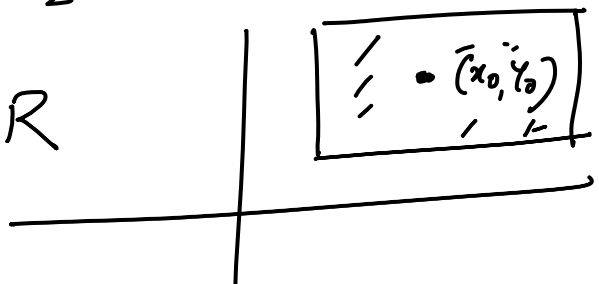
lecture 5

Recall (Picard's Theorem)

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0 \qquad \text{———} (*)$$

$$R = [x_0 - a \quad x_0 + a] \times [y_0 - b, \ y_0 + b]$$



- f continuous on R
  $\Rightarrow$ IVP $(*)$ has a solution near x_0.

- $f, f_y = \frac{\partial f}{\partial y}$ continuous on R
  $\Rightarrow$ IVP $(*)$ has unique solution near x_0.

Remark

f_y continuous can be replaced with Lipschitz condition (LC)

ie $\exists$ $L > 0$ s.t

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

$$\forall (x, y_1) (x, y_2) \in R.$$

f_y continuous on R

$\Rightarrow$ f satisfy LC on R

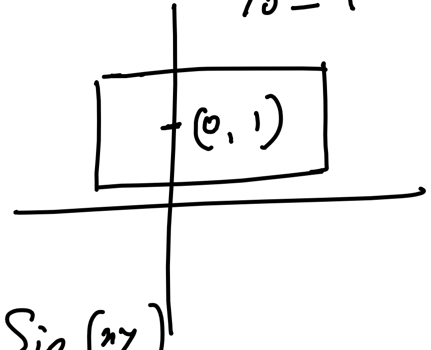
$\not\Leftarrow$

Example 1

$$\frac{dy}{dx} = \frac{\sin(xy)}{x^2+y^2} \qquad y(0)=1$$

$x_0 = 0$
$y_0 = 1$

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}$$



$$f_y = \frac{x\cos(xy)}{x^2+y^2} - \frac{2y\sin(xy)}{(x^2+y^2)^2}$$

f, f_y are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

So by Picard theorem, $\exists!$ solution.

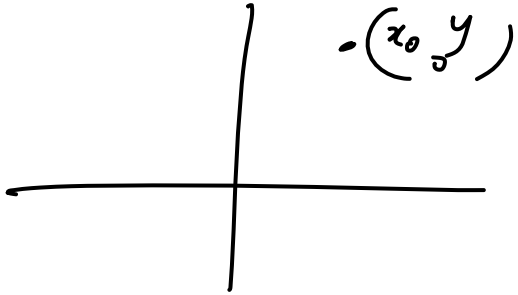
Example 2

Discuss the existence & uniqueness of solution of IVP for different (x_0, y_0).

$$\frac{dy}{dx} = \frac{2y}{x} \qquad y(x_0) = y_0.$$

Sol
$$\left.\begin{aligned} f(x,y) &= \frac{2y}{x} \\ f_y &= \frac{2}{x} \end{aligned}\right\} \text{continuous on } \mathbb{R}^2 \setminus y\text{-axis}$$

- $\exists!$ solution for $x_0 \neq 0$



- For $x_0 = 0$

Solving $y = cx^2 \qquad y(0) = y_0$

- $y_0 \neq 0 \quad \nexists$ a solution.
- $y_0 = 0 \quad \exists$ infinitely many solution $y = cx^2$

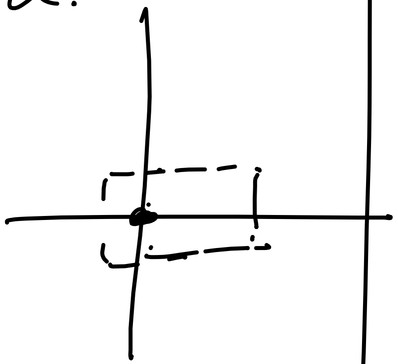
Example

$$\frac{dy}{dx} = \sqrt{y} + 1 \qquad y(0) = 0$$
$$x \in [0\ 1]$$

$$f(x, y) = \sqrt{y} + 1. \qquad \text{———} (*)$$

- f is continuous.
- f is not Lipschitz around origin.
- But solution is unique.

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|}$$



$$= \left| \frac{\sqrt{y_1} - \sqrt{y_2}}{y_1 - y_2} \right| = \left| \frac{1}{\sqrt{y_1} + \sqrt{y_2}} \right|$$

Choose $y_1 = \delta > 0$
$y_2 = 0$

$$= \frac{1}{\sqrt{\delta}} \to \infty \text{ as } \delta \to 0$$

Thus $\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|}$ is

NOT bounded in any rectangle containing $(0, 0)$.

Thus f does not satisfy LC on any rectangle containing $(0, 0)$.

- But solution is unique

Assume $y_1(x)$ & $y_2(x)$ are two solutions of (*)

$$Z(x) = \left(\sqrt{Y_1(x)} - \sqrt{Y_2(x)}\right)^2$$

$$Z'(x) = 2\left(\sqrt{Y_1} - \sqrt{Y_2}\right)\left(\frac{1}{2\sqrt{Y_1}}Y_1' - \frac{1}{2}\frac{Y_2'}{\sqrt{Y_2}}\right)$$

$$= \cancel{2}\left(\sqrt{Y_1} - \sqrt{Y_2}\right)\frac{1}{\cancel{2}}\left(\frac{\sqrt{Y_1}+1}{\sqrt{Y_1}} - \frac{\sqrt{Y_2}+1}{\sqrt{Y_2}}\right)$$

. . . ------

$$= -\frac{\left(\sqrt{Y_1} - \sqrt{Y_2}\right)^2}{\sqrt{Y_1}\sqrt{Y_2}} \leq 0$$

$$0 \leq Z(x) \leq Z(0) = 0 \implies Z(x) = 0$$

$Y_1(x) = Y_2(x)$ ∎

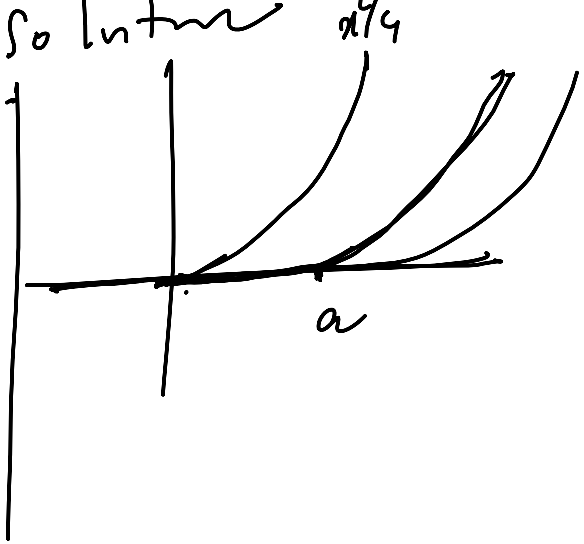
Example $\frac{dy}{dx} = \sqrt{y}$ $y(0) = 0$

- LC is not satisfied around $(0,0)$

- Solving: $y = \frac{x^2}{4}$, $y \equiv 0$ — more than one solution

It has more solution

$$y(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \frac{(x-a)^2}{4} & x > a \end{cases}$$



Idea of Proof of Picard Theorem

step 1 $\frac{dy}{dx} = f(x,y) \quad y(x_0) = y_0$ —— (*)

$y(x)$ is a solution of (*)

$\iff y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$ —— (*)

(Integrate from x_0 to x)

step 2 $y_0(x) = y_0$

Picard Iterates

$$\begin{cases} y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt \\ y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \\ y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))\, dt \\ \quad - \quad - \quad - \quad - \\ y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \end{cases}$$ —— (**)

Step 3 $\{y_n(x)\}$ converge uniformly.

$\varphi(x) = \lim_n y_n(x)$.

Then take limit $n \to \infty$ in (**)

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt \quad \blacksquare$$

Example Solve by Picard iteration method

$$\frac{dy}{dx} = xy \quad y(0) = 1 \quad x_0 = 0$$

Sol: $y_0(x) = y_0 = 1.$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt$$

$$= 1 + \frac{x^2}{2}$$

$$y_2(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4\cdot 2}$$

$$y_3(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4\cdot 2} + \frac{x^6}{6\cdot 4\cdot 2}$$

$$= 1 + \left(\frac{x^2}{2}\right) + \left(\frac{x^2}{2}\right)^2 \frac{1}{\lfloor 2} + \left(\frac{x^2}{2}\right)^3 \frac{1}{\lfloor 3}$$

$$\lim y_n(x) =$$

$$y_4(x) = 1 + \left(\frac{x^2}{2}\right) + \left(\frac{x^2}{2}\right)^2 \frac{1}{\lfloor 2} + \left(\frac{x^2}{2}\right)^3 \frac{1}{\lfloor 3} + \left(\frac{x^2}{2}\right)^4 \frac{1}{\lfloor 4}$$

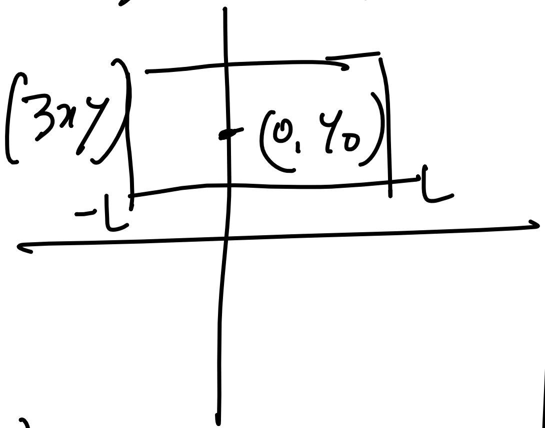
$$\lim_{n} y_n(x) = e^{x^2/2} \quad \boxed{\ }$$

Example $\dfrac{dy}{dx} = 2\sin(3xy)$

$y(0) = y_0.$

show that this IVP has unique solution over $(-\infty\ \infty)$. $x_0 = 0$

Sol $f(x,y) = 2\sin(3xy)$



$f_y = 6x\cos(xy)$

To show that it has unique solution on $(-\infty\ \infty)$, it is enough to show that $\exists!$ solution on $(-L, L)$ for any $L > 0$.

Fix L choose interval

$[-L\ \ L] \times [y_0 - b,\ \ y_0 + b]$

By Picard theorem $!$ solution exist for $(-\alpha\ \ \alpha)$ where

$\alpha = \min\left\{ a = L,\ \dfrac{b}{M} \right\} = \min\left\{L, \dfrac{b}{2}\right\} = L$ if $b > 2L$

$M = \sup_{R} f(x,y) = 2$

Thus $\exists!$ solution for $(-L\ \ L)$ for any $L > 0$.