

Problem 1. Let y be the solution of the initial value problem

$$y'(t) = t y(t)^2 + \sin t, \quad y(0) = 1,$$

and let $[0, T)$ be the maximal interval of existence of y . Show that $T \leq \sqrt{2}$.

Solution. For $t \in [0, \pi]$, we have $\sin t \geq 0$. Hence, on $[0, \pi]$, we have

$$(0.1) \quad y'(t) \geq t y(t)^2.$$

Now consider the comparison IVP

$$u'(t) = t u(t)^2, \quad u(0) = 1.$$

This is separable:

$$\frac{du}{u^2} = t dt \implies -\frac{1}{u(t)} = \frac{t^2}{2} + C.$$

Using $u(0) = 1$ gives $-1 = C$, hence

$$\frac{1}{u(t)} = 1 - \frac{t^2}{2}, \quad u(t) = \frac{1}{1 - \frac{t^2}{2}}.$$

So u blows up at $t = \sqrt{2}$.

The right-hand side $f(t, y) = t y^2$ is locally Lipschitz in y , uniformly with respect to t . Since $y(0) = u(0) = 1$ and we have (0.1), the standard comparison theorem for scalar ODE implies

$$y(t) \geq u(t) = \frac{1}{1 - \frac{t^2}{2}} \quad \text{for } 0 \leq t < \pi.$$

As $t \uparrow \sqrt{2}$, $u(t) \rightarrow +\infty$, so y cannot remain finite beyond $t = \sqrt{2}$. Therefore the maximal existence time satisfies

$$\boxed{T \leq \sqrt{2}}.$$

Problem 2. Consider the initial value problem

$$y'(t) = F(t, y(t)), \quad y(0) = 0,$$

where

$$F(t, y) = |y| (1 + \cos t) + \frac{|t|}{1 + y^2}.$$

(a) Show that $F(t, y)$ is locally Lipschitz in y , uniformly with respect to t . Hence the IVP has a unique local solution through $(0, 0)$.

(b) Show that every solution is global (i.e. it exists for all $t \in \mathbb{R}$).

Solution. (a) It is sufficient to consider the rectangle $|t| \leq T$ and $|y| \leq M$ instead of a general compact set. Let $y_1, y_2 \in [-M, M]$. Then

$$\begin{aligned} |F(t, y_1) - F(t, y_2)| &= \left| |y_1|(1 + \cos t) - |y_2|(1 + \cos t) + \frac{|t|}{1 + y_1^2} - \frac{|t|}{1 + y_2^2} \right| \\ &\leq |1 + \cos t| \left| |y_1| - |y_2| \right| + |t| \left| \frac{1}{1 + y_1^2} - \frac{1}{1 + y_2^2} \right|. \end{aligned}$$

For the first term, $|1 + \cos t| \leq 2$ and $||y_1| - |y_2|| \leq |y_1 - y_2|$, so

$$|1 + \cos t| \left| |y_1| - |y_2| \right| \leq 2 |y_1 - y_2|.$$

For the second term,

$$\left| \frac{1}{1 + y_1^2} - \frac{1}{1 + y_2^2} \right| = \frac{|y_1^2 - y_2^2|}{(1 + y_1^2)(1 + y_2^2)} \leq |y_1^2 - y_2^2| = |y_1 - y_2| |y_1 + y_2|.$$

On $|y_1|, |y_2| \leq M$, we have $|y_1 + y_2| \leq 2M$, hence

$$|t| \left| \frac{1}{1+y_1^2} - \frac{1}{1+y_2^2} \right| \leq |t| \cdot 2M |y_1 - y_2| \leq 2MT |y_1 - y_2|.$$

Thus

$$|F(t, y_1) - F(t, y_2)| \leq (2 + 2MT) |y_1 - y_2| \quad \text{for all } |t| \leq T, |y_i| \leq M.$$

Therefore F is locally Lipschitz in y , uniformly in t . By Picard–Lindelöf, the IVP has a unique local solution through $(0, 0)$.

(b) On the strip $|t| \leq T$,

$$|F(t, y)| = |y|(1 + \cos t) + \frac{|t|}{1+y^2} \leq 2|y| + |t| \leq 2|y| + T.$$

Thus we can take $A(T) = T$, $B(T) = 2$, and the hypothesis of the theorem proved in class is satisfied. Therefore every maximal solution is global, i.e. defined for all $t \in \mathbb{R}$.

Problem 3. Let $P \in C^1([a, b])$ with $P(t) > 0$ and $Q \in C([a, b])$. Consider the self-adjoint equation

$$(P(t)u'(t))' + Q(t)u(t) = 0.$$

Let u be a nontrivial solution and let $p \in (a, b)$ be a point where u has an extremum. Let $\theta_u(t)$ denote the corresponding phase variable. Show that

$$\frac{d\theta_u}{dt}(p) = Q(p).$$

Solution. Let (r_u, θ_u) be the Prüfer variables associated to u . The Prüfer substitution gives

$$u(t) = r_u(t) \sin \theta_u(t), \quad P(t)u'(t) = r_u(t) \cos \theta_u(t).$$

Since u is a nontrivial solution we get $r_u(t) > 0$ for all t . Since p is a point of extremum we have $u'(p) = 0$ and therefore we have

$$r_u(p) \cos \theta_u(p) = 0 \implies \cos \theta_u(p) = 0.$$

Now we know that

$$\begin{aligned} \frac{d\theta_u}{dt}(p) &= Q(p) \sin^2 \theta_u(p) + \frac{1}{P(p)} \cos^2 \theta_u(p) \\ &= Q(p) \end{aligned}$$

Problem 4. Consider the Floquet system $x'(t) = A(t)x(t)$ where

$$A(t) = \begin{pmatrix} \cos t - \sin t & \sin t \\ 0 & 1 + \cos t \end{pmatrix},$$

which is 2π -periodic.

(a) Find the Floquet multipliers of the system.

(b) Determine whether the system admits a nontrivial 2π -periodic solution. Justify your answer.

Solution. (a) Let $\Phi(t)$ be the fundamental solution matrix of the system with $\Phi(0) = I$, i.e. Φ is the Principal Matrix Solution. Note that, as discussed in the class, the computation of Floquet multipliers does not depend on the chosen fundamental solution matrix. In this case, the Floquet multipliers are the eigenvalues of $\Phi(2\pi)$.

The columns of Φ are the solutions of the system with initial conditions $x(0) = e_j$, $j = 1, 2$. In the case $x(0) = (1, 0)$, observe linearity and uniqueness implies that $\Phi_{21}(t) \equiv 0$. This says that $\Phi(t)$

is upper-triangular whence to compute the eigenvalues of $\Phi(2\pi)$, we only need to compute $\Phi_{11}(2\pi)$ and $\Phi_{22}(2\pi)$. Observe that $\Phi_{11}(t)$ is the solution of the (IVP)

$$x'_1(t) = (\cos t - \sin t)x_1(t), \quad x_1(0) = 1.$$

Solving the above we get $\Phi_{11}(t) = \exp(\sin t + \cos t)$. Similarly $\Phi_{22}(t)$ is the solution of the (IVP)

$$x'_2(t) = (1 + \cos t)x_2(t), \quad x_2(0) = 1.$$

Solving this we get $\Phi_{22}(t) = e^{t+\sin t}$. Clearly, the Floquet multipliers are given by the diagonal entries of $\Phi(2\pi)$, i.e. 1 and $\exp(2\pi)$.

(b) Since one of the eigenvalue is 1, by the theorem done in the class the system admits a periodic solution. In this case, we see that $(\exp(\sin t + \cos t), 0)$ is a solution that is periodic.

Problem 5. Let $A(t)$ be a continuous $n \times n$ real matrix on an interval I and consider

$$x'(t) = A(t)x(t).$$

Let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^n .

(a) Show that

$$\frac{d}{dt} \|x(t)\|^2 = x(t)^\top (A(t) + A(t)^\top) x(t).$$

(b) Deduce that the Euclidean length $\|x(t)\|$ of every solution $x(t)$ is constant in t if and only if $A(t)$ is skew-symmetric for all $t \in I$, i.e. $A(t)^\top = -A(t)$.

(c) Let $S(t) := A(t) + A(t)^\top$, and let $\lambda_{\max}(S(t))$ denote the largest eigenvalue of the symmetric matrix $S(t)$. Show that, for any $t, t_0 \in I$, every solution satisfies

$$\|x(t)\| \leq \|x(t_0)\| \exp\left(\frac{1}{2} \int_{t_0}^t \lambda_{\max}(S(\tau)) d\tau\right).$$

Solution. (a) We have

$$\|x(t)\|^2 = x(t)^\top x(t).$$

Differentiate:

$$\frac{d}{dt} \|x(t)\|^2 = \frac{d}{dt} (x^\top x) = x'(t)^\top x(t) + x(t)^\top x'(t).$$

Using $x'(t) = A(t)x(t)$, we get $x'(t)^\top = x(t)^\top A(t)^\top$, hence

$$\frac{d}{dt} \|x(t)\|^2 = x(t)^\top A(t)^\top x(t) + x(t)^\top A(t)x(t) = x(t)^\top (A(t) + A(t)^\top) x(t).$$

(b) Set $S(t) := A(t) + A(t)^\top$. Then

$$\frac{d}{dt} \|x(t)\|^2 = x(t)^\top S(t)x(t).$$

If $A(t)^\top = -A(t)$ for all t : then $S(t) = 0$ for all t , so

$$\frac{d}{dt} \|x(t)\|^2 \equiv 0,$$

and thus $\|x(t)\|$ is constant in t for every solution.

Conversely, assume that for every solution $x(t)$, the length $\|x(t)\|$ is constant in t . Then $\|x(t)\|^2$ is constant, so

$$\frac{d}{dt} \|x(t)\|^2 = x(t)^\top S(t)x(t) = 0 \quad \text{for all } t, \text{ all solutions } x.$$

Fix $t_0 \in I$. For each $v \in \mathbb{R}^n$, consider the unique solution with $x(t_0) = v$. At $t = t_0$ we have

$$0 = x(t_0)^\top S(t_0)x(t_0) = v^\top S(t_0)v \quad \text{for all } v \in \mathbb{R}^n.$$

For the symmetric matrix $S(t_0)$, the identity $v^\top S(t_0)v = 0$ for all v implies $S(t_0) = 0$. Since t_0 was arbitrary, $S(t) = 0$ for all $t \in I$, i.e. $A(t)^\top = -A(t)$ for all $t \in I$. Thus $A(t)$ is skew-symmetric.

(c) Again $S(t) = A(t) + A(t)^\top$ is symmetric. For any symmetric S , we have

$$v^\top S v \leq \lambda_{\max}(S) \|v\|^2 \quad \text{for all } v.$$

Applying this with $v = x(t)$ and $S = S(t)$, we get

$$\frac{d}{dt} \|x(t)\|^2 = x(t)^\top S(t) x(t) \leq \lambda_{\max}(S(t)) \|x(t)\|^2.$$

Let $y(t) := \|x(t)\|^2$. Then $y(t) \geq 0$ and

$$y'(t) \leq \lambda_{\max}(S(t)) y(t).$$

Assuming $x \not\equiv 0$, we have $y(t) > 0$ on I , so

$$\frac{y'(t)}{y(t)} \leq \lambda_{\max}(S(t)).$$

Integrating from t_0 to t ,

$$\int_{t_0}^t \frac{y'(s)}{y(s)} ds \leq \int_{t_0}^t \lambda_{\max}(S(\tau)) d\tau.$$

The left-hand side is $\ln y(t) - \ln y(t_0)$, so

$$\ln y(t) - \ln y(t_0) \leq \int_{t_0}^t \lambda_{\max}(S(\tau)) d\tau.$$

Exponentiating,

$$y(t) \leq y(t_0) \exp \left(\int_{t_0}^t \lambda_{\max}(S(\tau)) d\tau \right).$$

Recalling $y(t) = \|x(t)\|^2$, we obtain

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 \exp \left(\int_{t_0}^t \lambda_{\max}(S(\tau)) d\tau \right).$$

Taking square roots,

$$\boxed{\|x(t)\| \leq \|x(t_0)\| \exp \left(\frac{1}{2} \int_{t_0}^t \lambda_{\max}(S(\tau)) d\tau \right)}.$$

Problem 6. Let P and Q be continuous functions on the interval $[a, b]$. Consider the differential equation

$$x''(t) + P(t)x'(t) + Q(t)x(t) = 0.$$

(a) Let $x(t)$ be a nontrivial solution. Show that x has only finitely many zeros in $[a, b]$.

(b) Assume that $Q(t) < 0$ for all $t \in [a, b]$. Show that the differential equation is non-oscillatory on $[a, b]$, i.e. no nontrivial solution has more than one zero in $[a, b]$.

Solution. (a) Let x be a nontrivial solution. Suppose $x(t_0) = 0$ and $x'(t_0) = 0$ for some $t_0 \in [a, b]$. Then the IVP $x'' + Px' + Qx = 0$, $x(t_0) = 0$, $x'(t_0) = 0$ has the trivial solution $x \equiv 0$. By uniqueness, this would force $x \equiv 0$ on $[a, b]$, contradicting nontriviality. Hence no zero of x can have $x'(t_0) = 0$. Thus every zero is simple (i.e. $x'(t_0) \neq 0$).

Now suppose x had infinitely many zeros (t_n) in the compact interval $[a, b]$, they would accumulate at some point $c \in [a, b]$. Without loss of generality, we can assume that $t_n \rightarrow c$. By continuity, $x(c) = 0$. Note that using Rolle's Theorem, we can find a sequence (s_n) such that $x'(s_n) = 0$ and $s_n \rightarrow c$. Since x is continuously differentiable, we get $x'(c) = 0$, a contradiction that all zeros are simple.

(b) Assume $Q(t) < 0$ for all $t \in [a, b]$. Note that from part (a), we know that every nontrivial solution has only finitely many zeros in $[a, b]$. Suppose, for contradiction, that a nontrivial solution x has two distinct zeros $\alpha < \beta$ in $[a, b]$ and $x(t) \neq 0$ on (α, β) . Note that this implies that either

$x \geq 0$ or $x \leq 0$ on $[\alpha, \beta]$, i.e. it cannot change sign. By continuity, x attains a strict maximum or minimum at some $c \in (\alpha, \beta)$ with $x(c) \neq 0$. Then:

$$x'(c) = 0, \quad \text{and} \quad x(c)x''(c) \leq 0$$

(because at a local maximum with $x(c) > 0$, we have $x''(c) \leq 0$, and at a local minimum with $x(c) < 0$, we have $x''(c) \geq 0$, in both cases $x(c)x''(c) \leq 0$).

Evaluate the equation at $t = c$:

$$x''(c) + P(c)x'(c) + Q(c)x(c) = 0.$$

Since $x'(c) = 0$, this gives

$$x''(c) = -Q(c)x(c).$$

Thus

$$x(c)x''(c) = -Q(c)x(c)^2 > 0,$$

contradicting $x(c)x''(c) \leq 0$. Therefore x cannot have two distinct zeros in $[a, b]$.

Problem 7. Consider the Euler–Cauchy equation

$$t^2 x''(t) + 2tx'(t) + bx(t) = 0, \quad t \geq 1,$$

where $b \in \mathbb{R}$ is a constant. Show that the equation is oscillatory on $[1, \infty)$ if and only if $b > \frac{1}{4}$.

Solution. We use the standard change of variables

$$t = e^s, \quad s \in [0, \infty), \quad x(t) = u(s).$$

Then

$$\frac{dx}{dt} = \frac{du}{ds} \cdot \frac{ds}{dt} = \frac{1}{t} u'(s),$$

and

$$\frac{d^2x}{dt^2} = \frac{d}{dt} \left(\frac{1}{t} u'(s) \right) = -\frac{1}{t^2} u'(s) + \frac{1}{t} u''(s) \frac{ds}{dt} = -\frac{1}{t^2} u'(s) + \frac{1}{t^2} u''(s) = \frac{1}{t^2} (u''(s) - u'(s)).$$

Substitute into the given equation:

$$t^2 x'' + 2tx' + bx = 0$$

becomes

$$t^2 \cdot \frac{1}{t^2} (u'' - u') + 2t \cdot \frac{1}{t} u' + bu = 0,$$

i.e.

$$u''(s) - u'(s) + 2u'(s) + bu(s) = 0,$$

or

$$u''(s) + u'(s) + bu(s) = 0.$$

Next, remove the first-derivative term by setting

$$u(s) = e^{-s/2} z(s).$$

Then

$$u'(s) = e^{-s/2} (z'(s) - \frac{1}{2} z(s)), \quad u''(s) = e^{-s/2} (z''(s) - z'(s) + \frac{1}{4} z(s)).$$

Substitute into $u'' + u' + bu = 0$:

$$e^{-s/2} (z'' - z' + \frac{1}{4} z) + e^{-s/2} (z' - \frac{1}{2} z) + be^{-s/2} z = 0.$$

Factor $e^{-s/2}$:

$$z'' - z' + \frac{1}{4} z + z' - \frac{1}{2} z + bz = 0 \quad \Rightarrow \quad z'' + (b - \frac{1}{4}) z = 0.$$

Thus the transformed equation is

$$z''(s) + \omega^2 z(s) = 0, \quad \text{where } \omega^2 := b - \frac{1}{4}.$$

Case 1: $b > 1/4$ (i.e. $\omega^2 > 0$). Then $\omega = \sqrt{b - 1/4} > 0$, and

$$z'' + \omega^2 z = 0$$

has general solution

$$z(s) = C_1 \cos(\omega s) + C_2 \sin(\omega s).$$

If $z \not\equiv 0$, then z has infinitely many zeros as $s \rightarrow \infty$. For instance, with $C_1 = 0$, $C_2 \neq 0$,

$$z(s) = C_2 \sin(\omega s)$$

has zeros at $s_k = k\pi/\omega$, $k \in \mathbb{Z}$. Correspondingly,

$$u(s) = e^{-s/2} z(s), \quad x(t) = u(\ln t) = t^{-1/2} z(\ln t).$$

Zeros of z give zeros of x . Specifically, $s_k = k\pi/\omega$ gives $t_k = e^{s_k} = \exp(k\pi/\omega) \rightarrow \infty$ as $k \rightarrow \infty$. Thus there exists a nontrivial solution x with infinitely many zeros on $[1, \infty)$, and in fact every nontrivial solution oscillates. So the equation is oscillatory on $[1, \infty)$ for $b > 1/4$.

Case 2: $b = 1/4$ (i.e. $\omega^2 = 0$). Then $z''(s) = 0$, so

$$z(s) = C_1 + C_2 s.$$

Thus

$$u(s) = e^{-s/2}(C_1 + C_2 s), \quad x(t) = u(\ln t) = t^{-1/2}(C_1 + C_2 \ln t).$$

A nontrivial affine function $C_1 + C_2 s$ has at most one real zero, so z has at most one zero on $[0, \infty)$. Hence $x(t)$ has at most one zero on $[1, \infty)$. Therefore the equation is non-oscillatory for $b = 1/4$.

Case 3: $b < 1/4$ (i.e. $\omega^2 < 0$). Write $\omega^2 = -\alpha^2$ with $\alpha = \sqrt{1/4 - b} > 0$. Then

$$z'' - \alpha^2 z = 0,$$

so

$$z(s) = C_1 e^{\alpha s} + C_2 e^{-\alpha s}.$$

This function has at most one real zero (unless $z \equiv 0$). Hence $x(t) = t^{-1/2} z(\ln t)$ has at most one zero in $[1, \infty)$, and the equation is again non-oscillatory.

Combining the three cases, we conclude that the Euler–Cauchy equation is oscillatory on $[1, \infty)$ if and only if $b > 1/4$.

Oscillatory on $[1, \infty)$ $\iff b > \frac{1}{4}$.
