PP 38 : Stokes' Theorem

1. Let F, S and $\hat{n}$ be as in the statement of Stokes' theorem. Show that

   (a) $\iint_S (curlF) \cdot \hat{n}\ d\sigma = \iint_T (curlF) \cdot (r_u \times r_v)\ dudv$ if S is the parametric surface defined by $r(u,v), (u,v) \in T$.

   (b) $\iint_S (curlF) \cdot \hat{n}\ d\sigma = \iint_D (curlF) \cdot (-f_x\hat{i} - f_y\hat{j} + \hat{k})\ dxdy$ if S is the graph defined by $z = f(x,y), (x,y) \in D$.

2. Consider the surfaces S_1 and S_2 as given below. Let C be the curve of intersection of S_1 and S_2. Suppose C is oriented counterclockwise when viewed from above. Parameterize C.

   (a) S_1 is $x^2 + y^2 + z^2 = 4$ and S_2 is $x^2 + y^2 = 1$.

   (b) S_1 is $y + z = 2$ and S_2 is $x^2 + y^2 = 1$.

   (c) S_1 is $x^2 + y^2 + z^2 = 25$ and S_2 is $z = 4$.

3. Let C be the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Suppose C is oriented counterclockwise when viewed from above.

   (a) If $F(x,y,z) = (z,x,y)$, find $\oint_C F \cdot dR$.

   (b) If $F(x,y,z) = (z, x + e^{y^2}, y + e^{z^2})$, find $\oint_C F \cdot dR$.

   (c) If $F(x,y,z) = -\alpha y^2\hat{i} + \alpha x\hat{j} + (z + \cos z)^2\hat{k}$ for some $\alpha \in \mathbb{R}$ and $\oint_C F \cdot dR = 2\pi$, find α.

   (d) If $curlF = \alpha\hat{k}$ for some $\alpha \in \mathbb{R}$ and $\oint_C F \cdot dR = 2\pi$, find α.

4. Let $F(x,y,z) = (z,x,y)$ and S be the surface as described below. Let C be the boundary of the surface which is oriented counterclockwise when viewed from above. Evaluate $\oint_C F \cdot dR$ using Stokes' theorem.

   (a) S is the part of the plane $z = x + 4$ that lies inside the cylinder $x^2 + y^2 = 4$.

   (b) S is the part of the surface $2x^2 + 2y^2 + z^2 = 9$ that lies above the surface $z = \frac{1}{2}\sqrt{x^2 + y^2}$.

   (c) S is the part of the plane that lies inside the triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$.

   (d) $S = S_1 \cup S_2$ where S_1 is the part of the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 4$ and S_2 is the disk $x^2 + y^2 \leq 1, z = 4$.

5. Let S be the upper hemi-sphere $x^2 + y^2 + z^2 = 1,\ z \geq 0$.

   (a) Find F such that $curlF = xe^y\hat{i} - e^y\hat{j}$.

   (b) Evaluate $\iint_S (x^2e^y - ye^y)d\sigma$.

6. Let C be the parameteric curve $R(t) = (\cos t, \sin t, \cos t + 4), 0 \leq t \leq 2\pi$ and

$$F(x,y,z) = (z^2 + e^z, 4x, e^z\cos^2 z).$$

   Evaluate $\oint_C F \cdot dR$.

7. Let $F(x,y,z) = (y, -x, 2z^2 + x^2)$ and S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$. Evaluate $\iint_S curlF \cdot \hat{n}d\sigma$ where $\hat{n}$ is the unit outward normal of S.

Practice Problems 38: Hints/Solutions

1. (a) Note that $\hat{n} = \frac{r_u \times r_v}{\|r_u \times r_v\|}$.

   (b) Note that $\hat{n} = \frac{-f_x\hat{i} - f_y\hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$.

2. (a) $R(\theta) = (\cos\theta, \sin\theta, \sqrt{3}), \quad 0 \le \theta \le 2\pi$.

   (b) $R(\theta) = (\cos\theta, \sin\theta, 2 - \sin\theta), \quad 0 \le \theta \le 2\pi$.

   (c) $R(\theta) = (3\cos\theta, 3\sin\theta, 4), \quad 0 \le \theta \le 2\pi$.

3. (a) Note that $curlF = (1,1,1)$, $z = 2 - y = f(x,y)$ and $(-f_x, -f_y, 1) = (0,1,1)$. By Stokes' theorem and Problem 1(b), $\oint_C F \cdot dR = \iint_D (1,1,1) \cdot (0,1,1) dxdy$ where D is the disk $x^2 + y^2 \le 1$.

   (b) $curlF = (1,1,1)$ and the rest is similar to the solution to Problem 3(a).

   (c) Since $curlF = \alpha(1 + 2y)\hat{k}$, by Stokes' theorem, $\oint_C F \cdot dR = \iint_R (0, 0, \alpha(1 + 2y)) \cdot (0,1,1) dxdy = \alpha \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) r dr d\theta = \alpha\pi$. Hence $\alpha = 2$.

   (d) Let D be the disk $x^2 + y^2 \le 1$ and $z = 0$. By Stokes' theorem, $\oint_C F \cdot dR = \iint_D (0,0,\alpha) \cdot (0,1,1) dxdy = \alpha \iint_D dxdy$. Hence $\alpha = 2$.

4. (a) Since $curlF = (1,1,1)$ and $(-f_x, -f_y, 1) = (-1,0,1)$, by Stokes' theorem $\oint_C F \cdot dR = 0$.

   (b) Observe that C is the circle $x^2 + y^2 = 4$ and $z = 1$. Moreover, C is also the boundary for the surface S_1 which is the part of the plane $z = 1$ that lies inside the cylinder $x^2 + y^2 = 4$. By Stokes' theorem, $\oint_C F \cdot dR = \iint_{S_1} (curlF) \cdot \hat{n} d\sigma = \iint_D (1,1,1) \cdot (0,0,1) dxdy$ where D is the disk $x^2 + y^2 \le 4$.

   (c) The equation of the plane is $z = 1 - x - y$ and hence $(-f_x, -f_y, 1) = (1,1,1)$. By Stokes' theorem, $\oint_C F \cdot dR = \iint_D 3 dxdy$ where D is the triangular region whose vertices are $(0,0), (0,1), (1,0)$.

   (d) The solution to this problem is similar to that of Problem 4(b). Note that $\oint_C F \cdot dR = \iint_{S_3} curlF \cdot \hat{n} d\sigma$ where S_3 is the disk $x^2 + y^2 \le 1$ and $z = 0$. Hence $\oint_C F \cdot dR = \iint_D (1,1,1) \cdot (0,0,1) dxdy$ where $D = S_3$.

5. (a) By observation $F(x,y,z) = (0,0,xe^y)$.

   (b) Observe that $\iint_S (x^2 e^y - ye^y) d\sigma = \iint_S curlF \cdot (x,y,z) d\sigma = \iint_S curlF \cdot \hat{n} d\sigma$. By Stokes' theorem, $\iint_S (x^2 e^y - ye^y) d\sigma = \oint_C F \cdot dR$ where C is the unit circle in the xy-plane. Hence $\oint_C F \cdot dR = \oint_C xe^y dz = 0$.

6. Observe that C is the boundary of the part of the surface $z = x + 4$ that lies inside the cylinder $x^2 + y^2 = 1$. Note that $curlF = (0, 2z + e^z, 4)$. By Stokes' theorem $\oint_C F \cdot dR = \iint_D (0, 2z + e^z, 4) \cdot (-1, 0, 1) dxdy$ where D is the unit circle in the xy-plane.

7. The boundary C of the surface S is defined by $(3\sin\theta, 3\cos\theta, 4)$. Note that C is oriented clockwise when viewed from above. By Stokes' theorem, $\iint_S curlF \cdot \hat{n} d\sigma = \oint_C F \cdot dR = \int_0^{2\pi} (3\cos\theta, -3\sin\theta, 32 + 9\sin^2\theta) \cdot (3\cos\theta, -3\sin\theta, 0) d\theta = 18\pi$.