ASSIGNMENT 4
MTH102A

(1) Let $\{w_1, w_2, ..., w_n\}$ be a basis of a finite dimensional vector space V. Let v be a non zero vector in V. Show that there exists w_i such that if we replace w_i by v in the basis it still remains a basis of V.

Solution. Let $v = \sum_1^n a_i w_i$ for some $a_1, ..., a_n \in \mathbb{F}$, since v is non-zero at least one $a_i \neq 0$ for some $1 \leq i \leq n$. Assume $a_1 \neq 0$. Write $w_1 = \frac{1}{a_1}v - \sum_2^n \frac{a_i}{a_1} w_i$. Replace w_1 by v. Clearly $\{v, w_1, ..., w_n\}$ spans V.
Now we show that this set is L.I.. Let $b_1, ..., b_n$ be such that $b_1 v + \sum_2^n b_j w_j = 0, \Rightarrow b_1 \sum_1^n a_i w_i + \sum_2^n b_j w_j = 0 \Rightarrow b_1 a_1 w_1 + \sum_2^n (b_1 a_j + b_j) w_j = 0, \Rightarrow b_1 a_1 = 0, b_1 a_j + b_j = 0$ for $2 \leq j \leq n$. Since $a_1 \neq 0 \Rightarrow b_j = 0$ for all $1 \leq j \leq n$. Hence done.

(2) Find the dimension of the following vector spaces :
(i) $X = \{A : A$ is $n \times n$ real upper triangular matrices$\}$,
(ii) $Y = \{A : A$ is $n \times n$ real symmetric matrices$\}$,
(iii) $Z = \{A : A$ is $n \times n$ real skew symmetric matrices$\}$,
(iv) $W = \{A : A$ is $n \times n$ real matrices with $Tr(A) = 0\}$

Solution. Let E_{ij} be the matrix with ij^{th} entry one and others are zero , F_{ij} be matrix with ij^{th} and ji^{th} entries are 1 and others are zero and for $i \neq j$ define D_{ij} be the matrix with ij^{th} entry is 1, ji^{th} entry is -1 and others are zero.
(i) Set $\{E_{ij}; i \leq j\}$ forms a basis for X. Hence $dim(X) = n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$.
(ii) Set $\{Fij; i \leq j\}$ is a basis for Y, hence $dim(Y) = \frac{n(n+1)}{2}$.
(iii) For a real skew-symmetric matrix all diagonal entries are zero. Then the set $\{D_{ij}; i < j\}$. Hence $dim(Z) = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$.
(iv) Let A be any matrix with trace zero, then $\sum_1^n a_{ii} = 0, \Rightarrow a_{11} = -(a_{22} + ... + a_{nn})$.Hence Set $dim(W) = n^2 - 1$.

(3) Let $\mathcal{P}(X, \mathbb{R})$ be vector space of all single variable polynomials with real coefficients and $\mathcal{P}_n(X, \mathbb{R})$ be the subspace of all polynomials with degree less or equal to n. Find a basis of $\mathcal{P}_n(X, \mathbb{R})$. Prove that $S = \{X + 1, X^2 - X + 1, X^2 + X - 1\}$ is a basis of $\mathcal{P}_2(X, \mathbb{R})$. Hence, determine the coordinates of following elements: $2X - 1, 1 + X^2, X^2 + 5X - 1$.

Solution. $P = \{1, X, X^2, ..., X^n\}$ is a basis(every polynomial is a linear combination of elements of P and the set is L.I.).
First we show that the set S is L.I..Let $a_0, a_1, a_2 \in \mathbb{R}$ such that $a_0(X+1)+a_1(X^2-X+1)+a_2(X^2+X-1) = 0, \Rightarrow a_0+a_1-a_2+(a_0-a_1+a_2)X+(a_1+a_2)X^2 = 0$. We get $a_0+a_1-a_2 = 0, a_0-a_1+a_2 = 0, a_1+a_2 = 0$, solving this system of equation we get $a_0, a_1, a_2 = 0$.
Let $p(X) = a_0 + a_1X + a_2X^2$ be any element in $\mathcal{P}_2(X, \mathbb{R})$. Let $b_0, b_1, b_2 \in \mathbb{R}$ such that $p(X) = a_0 + a_1X + a_2X^2 = b_0(X+1) + b_1(X^2+X-1) + b_3(X^2-X+1)$ then we get $b_0 = \frac{a_0+a_1}{2}, b_1 = \frac{a_0-a_1+2a_2}{4}, b_2 = \frac{a_1-a_0+2a_2}{4}$. Hence S spans $\mathcal{P}_2(X, \mathbb{R})$.
$2X-1 = \frac{1}{2}(X+1) - \frac{3}{4}(X^2-X+1) + \frac{3}{4}(X^2+X-1)$.
$1+X^2 = \frac{1}{2}(X+1) + \frac{3}{4}(X^2-X+1) + \frac{1}{4}(X^2+X-1)$.
$X^2+5X-1 = 2(X+1) - 1(X^2-X+1) + (X^2+X-1)$.

(4) Let W be a subspace of a finite dimensional vector space V
(i) Show that there is a subspace U of V such that $V = W+U$ and $W \cap U = \{0\}$,
(ii) Show that there is no subspace U of V such that $W \cap U = \{0\}$ and $dim(W) + dim(U) > dim(V)$.

Solution.
(i) Let $\dim(V) = n$, since V is finite dimensional W is also finite dimensional. Let $\dim(W) = k$ and $B_w = \{w_1, ..., w_k\}$ be a basis for W.In case $k = n$ nothing to prove, so assume $k < n$. Now we can extend B_w to a basis B for V. Let $B = \{w_1, ..., w_k, v_{k+1}, ..., v_n\}$. Let U be a subspace of V generated by $\{v_{k+1}, ..., v_n\}$.

Let $v \in V$ be any. Then there exist scalars $a_1, ..., a_n \in \mathbb{F}$ such that $v = a_1w_1 + ... + a_kw_k + a_{k+1}v_{k+1} + ... + a_nv_n = (a_1w_1 + ... + a_kw_k) + (a_{k+1}v_{k+1} + ... + a_nv_n) \in W+U$.

Now we show that $W \cap U = \{0\}$. Let $v \in W \cap U, \Rightarrow v \in W$ and $v \in U$. Then there exist scalars $a_1, ..., a_k$ and $b_{K+1}, ..., b_n$ such that $a_1w_1 + ... + a_kw_k = v = b_{k+1}v_{k+1} + ... + b_nv_n, \Rightarrow a_1w_1 + ... + a_kw_k - b_{k+1}v_{k+1} - .... - b_nv_n = 0, \Rightarrow a_1, ..., a_k, b_{k+1}, ..., b_n = 0$, as B is L.I.. Hence $W \cap U = \{0\}$.
(ii) Let $W \cap U = \{0\}$ and $dim(W) + dim(U) > dim(v), \Rightarrow dim(W) + dim(U) - 0 > dim(V), \Rightarrow dim(W) + dim(U) - dim(W \cap U) > dim(V), \Rightarrow dim(W+U) > dim(V)$, which is a contradiction as $W+U$ is a subspace of V so its dimension has to be less or equal to n.

(5) Let $W_1 = L(\{(1,0,-1), (1,0,1)\})$ and $W_2 = L(\{(0,1,2), (0,1,-1)\})$ be two subspaces of $\mathbb{R}^3$. Prove that $W_1 + W_2 = \mathbb{R}^3$. Given an example $v \in \mathbb{R}^3$ such that v can be written in two different ways of the form $v = w_1 + w_2$ where $w_1 \in W_1, w_2 \in W_2$.

Solution. Let $(x, y, z) \in \mathbb{R}^3$ be any. Let $a, b, c, d \in \mathbb{R}$ be such that $(x, y, z) =$

$a(1,0,-1)+b(1,0,1)+c(0,1,2)+d(0,1,-1)$. First assume that $c = 0, \Rightarrow (x,y,z) = a(1,0,-1)+b(1,0,1)+d(0,1,-1) = (a, b+c, -a-c), \Rightarrow a = \frac{x-y-z}{2}, b = \frac{x+y+z}{2}$ and $d = y$, $\Rightarrow (x,y,z) \in W_1 + W_2$. So $\mathbb{R}^3 = W_1 + W_2$.

Now $(x,y,z) = p(1,0,-1) + q(1,0,1) + r(0,1,2) + s(0,1,-1)$, assume $s = 0$, then $(x,y,z) = p(1,0,-1)+q(1,0,1)+r(0,1,2), \Rightarrow p = \frac{x+2y-z}{2}, q = \frac{x-2y+z}{2}$ and $r = y$. Let $v = (1,2,3)$. Write $(1,2,3) = a(1,0,-1) + b(1,0,1) + d(0,1,-1), \Rightarrow (1,2,3) = -2(1,0,-1) + 3(1,0,1) + 2(0,1,-1) = (1,0,5) + (0,2,-2) \in W_1 + W_2$. Let $(1,2,3) = p(1,0,-1)+q(1,0,1)+r(0,1,2) = (1,0,-1)+0(1,0,1)+2(0,1,2) = (1,0,-1) + (0,2,4) \in W_1 + W_2$.

(6) Decide which of the followings are linear transformation:
(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+2y, z, |x|)$,
(ii) Let $M_n(\mathbb{R})$ be set of all $n \times n$ real matrices. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by
(a) $T(A) = A^T$,
(b) $T(A) = I + A$, where I is identity matrix of order n,
(c) $T(A) = BAB^{-1}$, where $B \in M_n(\mathbb{R})$ is an invertible matrix.

Solution.
(i) Not a linear transformation.
Take $x = (1,0,1)$ and $a = -1 \in \mathbb{R}$. Then $T(ax) = T(-1,0,-1) = (-1,-1,1) \neq aT(x) = -1(1,1,1) = (-1,-1,-1)$.
(ii) (a) Linear Transformation.
Let $A, B \in M_n(\mathbb{R})$ and $a \in \mathbb{R}$. Then $T(A+aB) = A + aB^T = A^T + aB^T$.
(b) Not a linear transformation.
Let O be zero matrix, then $T(o) = I \neq O$.
(c) Linear transformation.
Let $P, Q \in M_n(\mathbb{R})$ and $a \in \mathbb{R}$, then $T(P+aQ) = B(P+aQ)B^{-1} = BPB^{-1} + aBQB^{-1} = T(P) + aT(Q)$.

(7) Let $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = \overline{z}$, is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Solution. Let $a+ib, c+id \in \mathbb{C}$ and $\alpha \in \mathbb{R}$,then $T((a+ib)+\alpha(c+id)) = T(a+\alpha c+i(b+\alpha d)) = a+\alpha c-i(b+\alpha d) = (a-ib)+\alpha(c-id) = T(a+ib)+\alpha T(c+id)$, hence T is $\mathbb{R}$-linear.
Let $T(i(i)) = T(-1) = -1 \neq iT(i) = i(-i) = 1$, hence not $\mathbb{C}$-linear.

(8) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (1,0,0), T(1,1,0) = (1,1,1), T(1,1,1) = (1,1,0)$. Find $T(x,y,z)$, $Ker(T)$, $R(T)$ (Range of T). Prove that $T^3 = T$.

Solution. Let $(x, y, z) \in \mathbb{R}^3$. Then $(x, y, z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1), \Rightarrow T((x,y,z)) = (x-y)T(1,0,0) + (y-z)T(1,1,0) + zT(1,1,1) = (x-y)(1,0,0) + (y-z)(1,1,1) + z(1,1,0) = (x, y, y-z)$.

Let $x = (a, b, c) \in Ker(T)$, then $T(x) = (a, b, b-c) = (0,0,0), \Rightarrow (a, b, c) = (0,0,0)$. Hence $Ker(T) = \{(0,0,0)\}$.

$R(T) = \{(x, y, y-z); x, y, z \in \mathbb{R}\} =< \{(1,0,0), (0,1,1), (0,0,-1)\} >$.

$T^3((x,y,z)) = T^2((x, y, y-z)) = T((x, y, y-(y-z))) = T((x,y,z))$.

(9) Find all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$.

Solution. For all set of $\{a_1, ..., a_n \in \mathbb{R}\}$, there exist a linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ such that $T((x_1, ..., x_n)) = a_1x_1 + ... + a_nx_n$.

(10) Let V and W be two finite dimensional vector spaces over k, where $k = \mathbb{R}$ or $\mathbb{C}$. Prove that set $L(V, W)$ of all linear transformations from V to W is a vector space over k of dimension $dim(V).dim(W)$.

Solution. Let $dim(V) = n, dim(W) = m$ and Let $B_v = \{v_1, ..., v_n\}, B_w = \{w_1, ..., w_m\}$ be ordered basis for V and W respectively. Define a map $\phi : L(V, W) \to M_{m\times n}$ as $\phi(T) = [T]_{B_v}^{B_w}$, where $[T]_{B_v}^{B_w}$ is matrix of linear transformation T with respect to B_v and B_w. We show that ϕ is an isomorphism. Let $T, U \in L(V, W)$, then $\phi(T+U) = [T+U]_{B_v}^{B_w} = [T]]_{B_v}^{B_w} + [U]_{B_v}^{B_w} = \phi(T) + \phi(U)$ and for $\alpha \in k, \phi(\alpha T) = [\alpha T]_{B_v}^{B_w} = \alpha[T]_{B_v}^{B_w}$. So ϕ is linear.

To show ϕ to be one-one and onto, we show that for each $A \in M_{m\times n}$ there exist a unique $T \in L(V, W)$ such that $\phi(T) = A$. Consider

$$T_A(v_j) = \sum_{i=1}^{m} a_{ij} w_i \qquad 1 \le i \le n.$$

where a_{ij} is ij^{th} entry of A. This T_A is linear and $\phi(T_A) = A$. We now need to show that T_A is unique. Let $U \in L(V, W)$ such that $U(v_j) = \sum_{i=1}^{m} a_{ij} w_i$. Let $x \in V$ be any, then there exist $b_1, ..., b_n \in k$ such that $v = b_1v_1 + ... + b_nv_n$ Then $U(x) = U(b_1v_1 + ... + b_nv_n) = {}_{i=1}^{m} a_{ij} w_i b_j = T_A(x)$. Hence $T_A = U$.

Hence $L(V, W) \cong M_{m\times n}, \Rightarrow L(V, W)$ is a vector space of dimension mn as $M_{m\times n}$ is a vector space of dimension of $mn = dim(W).dim(V)$.