Zorn's Lemma for subsets of sets:

Let X be a non-empty set and $\mathcal{F}$ be a non-empty collection of subsets of X.

Let $\mathcal{S}$ be a chain in $\mathcal{F}$ i.e.

if $A, B \in \mathcal{S}$ then either $A \subseteq B$ or $B \subseteq A$.

Example :- Consider $\mathcal{F}$ to be the power set $\mathcal{P}(\mathbb{N})$ i.e. $\mathcal{F}$ is the collection of all subsets of natural numbers $\mathbb{N}$.

Let $\mathcal{S} = \{\{1\}, \{1,2\}, \{1,2,3\}, \dots, \{1,2,\dots n\}\}$

Then $\mathcal{S}$ is a chain.

If $\mathcal{S} = \{\{1,2\}, \{1,3\}, \{1,2,3\}\}$ then $\mathcal{S}$ is not a chain as $\{1,2\}$ & $\{1,3\}$ are not subset of each other.

Upper bound of a Chain :- Let $\mathcal{S}$ be a chain in $\mathcal{F}$. An element $P \in \mathcal{F}$ is said to be an upper bound for $\mathcal{S}$ if

$\forall A \in \mathcal{S}$, $A \subseteq P$.

Zorn's lemma: Let $\mathcal{F}$ be a non-empty collection of subsets of X. If every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$ then there exists a maximal element of $\mathcal{F}$ i.e. $\exists$ an element $M \in \mathcal{F}$, s.t $\forall A \in \mathcal{F}$,

$M \not\subset A$.

Theorem: Every non-zero vector space has a basis.

Proof:- Let V be a vector space & $V \neq \{0\}$

Let $\mathcal{F}$ be a collection of all linearly independent subsets of V.

$\mathcal{F} \neq \phi$ as $\exists$ $v \in V$ s.t $v \neq 0$ & $\{v\}$ is a linearly independent set & hence $\{v\} \in \mathcal{F}$.

Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Then for any two elements $A, B \in \mathcal{F}$ either $A \subseteq B$ or $B \subseteq A$.

Let $P = \bigcup_{A \in \mathcal{C}} A$

Claim:- P is a linearly independent set.

Proof of claim:- Let F be any finite subset of P. As F is finite there exists finitely many elements $A_1, A_2, \dots, A_n$ in $\mathcal{C}$ such that $F \subseteq \bigcup_{i=1}^{n} A_i$.

As $\mathcal{C}$ is a chain there exists A_K such that $A_1, \dots, A_{K-1}, A_{K+1}, \dots, A_n \subseteq A_K$.
Hence $\bigcup_{i=1}^{n} A_i = A_K$ & this implies $F \subseteq A_K$.

As $A_K \in \mathcal{C} \subseteq \mathcal{F}$. A_K is a linearly independent set & hence F is linearly independent. So, any finite subset of P is linearly independent. Hence, P is linearly independent.

Thus, $P = \bigcup_{A \in \mathcal{C}} A$ is an upper bound of the chain $\mathcal{C}$ and $P \in \mathcal{F}$.

By Zorn's lemma, $\exists$ a maximal element $M \in \mathcal{F}$.

So, M is linearly independent.

<u>Claim:-</u> Linear span of M is whole of V

i.e $L(M) = V$

<u>Pf. of claim:-</u> Suppose $L(M) \subsetneq V$.

then $\exists$ $x \in V$ such that $x \notin L(M)$

Let $Q = M \cup \{x\}$, then Q is Linearly independent, $M \subsetneq Q$ & $Q \in \mathcal{F}$.

This contradicts maximality of M.

Hence, $L(M) = V$

So, M is a basis of V. $\boxed{}$