MTH 201 2024-2025 (1)

Quiz 1

Date: 02/09/2024
Time: 05:10 pm to 6:00 pm
Max marks: 15

There are 4 questions and all are compulsory. Write your answers as clearly and as completely as possible.

1. (a) Does there exist a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $R(T) = N(T)$, where $R(T)$ and $N(T)$ are range and null spaces of T respectively ? Justify your answer (by giving an example of T in case if it exists and if your answer is no then give the correct reasoning). (2 marks only after correct justification)

   (b) Does there exist a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $R(T) = N(T)$, where $R(T)$ and $N(T)$ are range and null spaces of T respectively? Justify your answer (by giving an example of T in case if it exists and if your answer is no then give the correct reasoning). (2 marks only after correct justification)

2. (a) Define row reduced echelon matrix. (2 marks)

   (b) Define similar matrices. (1 mark)

3. Let A be a 12×12 matrix with real entries and the property that $A^{97} = 0$, i.e., A^{97} is a zero 12×12 matrix. Prove or disprove $A^{12} = 0$. (4 marks)

4. Let $\mathbb{C}^{2\times2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}.$$

Let $T : \mathbb{C}^{2\times2} \to \mathbb{C}^{2\times2}$ be a linear map defined by $T(A) = AB$. Write down the matrix of T with respect to a basis $\{E_{ij} : 1 \leq i, j \leq 2\}$ of $\mathbb{C}^{2\times2}$ and determine its rank. (4 marks)

$A \begin{bmatrix} 1 & 1 \\ -4 & 4 \end{bmatrix}$

$[\alpha]_B = P[\alpha]_{B'}$.

$T[\alpha]_B = TP[\alpha]_{B'}$ $\quad P^{-1} B P^{-1}$

$[T]_B = [\alpha]_B [B]_B$

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Mid Semester Exam

Date: 22/09/2024
Time: 06:00 pm to 8:00 pm
Max mark: 30

There are 6 questions and all are compulsory and carry equal marks. Write your answers as clearly and as completely as possible.

1. Let V be an n dimensional vector space over a field F. Let T be a linear operator on V. Let B and B' be two ordered bases of V and let $A_1 = [T]_B$ and $A_2 = [T]_{B'}$ be $n \times n$ matrices with entries in F. Prove that there exists an $n \times n$ invertible matrix P such that $A_1 = P^{-1}A_2P$. Using this result or otherwise prove that the following two matrices are similar.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \cdots & & n-1 \\ 0 & 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ n-1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & n-2 & 0 & \cdots & & 0 \\ 0 & 0 & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined as

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_2 + x_1, -x_1 - 2x_2 + 2x_3).$$

Let $R(T)$ = range of T and $N(T)$ = null space of T. Then

(a) Determine the condition on $a, b, c \in \mathbb{R}$ such that $(a, b, c) \in R(T)$.

(b) Determine the condition on $a, b, c \in \mathbb{R}$ such that $(a, b, c) \in N(T)$.

3. Let A and B be $n \times n$ matrices over a field F such that $A^2 = A$ and $B^2 = B$. Assume that the matrix $I - (A + B)$ is invertible, where I is an $n \times n$ identity matrix. Prove that rank A = rank B.

4. Define the following terms:

(a) The transpose of a linear map $T : V \to W$, where V and W are vector spaces over a field F.

(b) A row reduced $m \times n$ matrix with entries in F.

(c) A linear functional on a vector space V over a field F.

(d) A determinant function on $n \times n$ matrices with entries in K, where K is a commutative ring with identity.

(e) An ideal of the polynomial algebra $F[x]$, where F is a field.

5. Let V be the vector space of all $n \times n$ matrices over a field F. Let f be a functional on V such that $f(AB) = f(BA)$ for all $A, B \in V$. Show that $f = r\mathrm{Tr}$, where Tr denotes trace function on V and $f(I) = r$, where I is the identity matrix in V.

6. State and prove the rank-nullity theorem.

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Quiz 2

Date: 25/10/2024
Time: 05:10 pm to 6:00 pm
Max marks: 15

There are 4 questions and all are compulsory. Write your answers as clearly and as completely as possible.

1. Let $A = (a_{ij})$ be an $n \times n$ matrix with all its entries equal to 1, i.e., $a_{ij} = 1$ for all $1 \leq i, j \leq n$. Prove or disprove A is diagonalisable. 4 marks

2. Let T be a linear operator on a finite dimensional vector space V over a field F. Define the following terms:

   (a) The T-cyclic subspace $Z(v; T)$ generated by v.

   (b) A T-admissible subspace of W of V.

   (c) A T-conductor of v into W, where v is a vector in V and W is a T-invariant subspace of V. 3 marks (1 mark each)

3. Let T be an operator on a finite dimensional vector space V over a complex field $\mathbb{C}$. Let $g(x)$ be any polynomial in the polynomial algebra $\mathbb{C}[x]$. Let λ be a characteristic value of the operator $g(T)$. Prove or disprove that there exist $c \in \mathbb{C}$ such that $g(c) = \lambda$ and c is a characteristic value of T. 3 marks

4. Notation: For $n \times m$ matrix $A = [A_{ij}]$, and $n \times r$ matrix $B = [B_{ij}]$, $(A|B)$ denotes the $n \times (m + r)$

$$\begin{bmatrix} A_{11} & \dots & A_{1m} & B_{11} & \dots & B_{1r} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \dots & A_{nm} & B_{n1} & \dots & B_{nr} \end{bmatrix}.$$

Let C be an $n \times n$ symmetric matrix of rank ℓ, i.e., $C^t = C$, and D is an $n \times (n - \ell)$ matrix such that the rank of an $n \times (2n - \ell)$ matrix $(C|D)$ is n. Prove that the following $(2n - \ell) \times (2n - \ell)$ matrix is invertible

$$\begin{bmatrix} C & D \\ D^t & \mathbf{0} \end{bmatrix}.$$

5 marks

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End Semester Exam

Date: 22/11/2024
Time: 05:00 pm to 8:00 pm
Max mark: 40

There are 8 questions and all are compulsory. Write your answers as clearly and as completely as possible.

1. Notation: For an $n \times m$ matrix $A = [A_{ij}]$, and an $n \times r$ matrix $B = [B_{ij}]$, $(A|B)$ denotes the $n \times (m + r)$

$$(A|B) := \begin{bmatrix} A_{11} & \dots & A_{1m} & B_{11} & \dots & B_{1r} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \dots & A_{nm} & B_{n1} & \dots & B_{nr} \end{bmatrix}.$$

Let C be an $n \times n$ symmetric matrix of rank ℓ, i.e., $C^t = C$, and D is an $n \times (n - \ell)$ matrix such that the rank of an $n \times (2n - \ell)$ matrix $(C|D)$ is n. Prove that the following $(2n - \ell) \times (2n - \ell)$ matrix is invertible

$$\begin{bmatrix} C & D \\ D^t & \mathbf{0} \end{bmatrix}.$$

4 marks

2. Let V be a finite dimensional vector space over a field F and $T : V \to V$ be a linear transformation. State following theorems:

   (a) Primary decomposition theorem of operator T on V.

   (b) Cyclic decomposition theorem of operator T on V.

   (c) Cayley-Hamilton theorem.

   (d) Generalised Cayley-Hamilton theorem.

8 marks, 2 marks each

3. Let A and B be two $n \times n$ matrices over a field F. Let m_A and f_A denote minimal polynomial and characteristic polynomial of A respectively and let m_B and f_B denote minimal polynomial and characteristic polynomial of B respectively. Are the following statements true or false? Justify your answers. (2 marks each, full marks will be given only after correct justification)

   (a) If $m_A = m_B$ and $f_A = f_B$, then A and B are similar.

   (b) If $m_A = m_B = f_A = f_B$, then A and B are similar.

   (c) If A and B are diagonalisable and $m_A = m_B$ and $f_A = f_B$, then A and B are similar.

8 marks

(d) If A and B are similar then $m_A = m_B = f_A = f_B$.

4. Let N be a nilpotent 11×11 matrix with the minimal polynomial x^5. Let dimension of Kernel of $N = 4$, dimension of Kernel of $N^2 = 6$, dimension of Kernel of $N^4 = 10$. Determine dimension of Kernel of N^3. Also determine the sizes of all the Jordan blocks of N in non-increasing order. 5 marks

5. Let A be a $n \times n$ matrix over a field F with minimal polynomial $m(x)$. Let $c \in F$. Prove that $m(c) = 0$ iff c is a characteristic value of the matrix A. 4 marks

6. State and prove the Rank-Nullity theorem. 4 marks

7. Show that the system

$$x_1 - 2x_2 + x_3 + 2x_4 = 1$$

$$x_1 + x_2 - x_3 + x_4 = 2$$

$$x_1 + 7x_2 - 5x_3 - x_4 = 3$$

has no solution. 4 marks

8. Define a non-singular linear transformation on a vector space V. Let T be a linear transformation from a vector space V into a vector space W. Prove that if T is non-singular then T carries each linearly independent subset of V onto a linearly independent subset of W. 3 marks