A - $\underline{m \times n}$ matrix

$\underline{Ax = b,}$

$[A \mid b]_{m \times (n+1)}$

$\downarrow$

$\downarrow$

$\downarrow$

$\underline{\text{RREF of } [A \mid b]} \longrightarrow \underbrace{\begin{bmatrix} A' & \bigg| & b' \end{bmatrix}}_{m \times (n+1)}$

$A'x = b'$

$A' \qquad b'$

$\left[\qquad \qquad \Bigg| \quad \right]$

$\underline{Q}$: Is A' an RRE matrix?

A: $\underline{\text{YES}}$

CASE 1:- The last coln. of $[A'|b']$ has a pivot

$$
\begin{array}{c}
\\
\\
r \\
\\
\end{array}
\left[\begin{array}{ccc|c}
 & * & & * \\
 & & & \vdots \\
0 & \cdots & 0 & 1 \\
 & 0 & & 0
\end{array}\right]
$$

A - coeff. matrix

$[A|b]$ - aug. "

$\text{rank } [A|b] = r$

$\text{rank } A = r-1$

i.e., $\text{rank } A < \text{rank } [A|b]$

CONCLUSION:- The system $Ax=b$ does not have a soln.

CASE 2:- The last coln of $[A'|b']$ does not have a pivot

$$
\begin{array}{c}
\\
\\
m
\end{array}
\left[\begin{array}{cccc|c}
 & & & & \\
 & & & 1 & \\
0 & \cdots & \cdot & 1 & *
\end{array}\right]
$$

$\text{rank } A = r = \text{rank } [A|b]$

i.e.: $\left[\begin{array}{c|c} I_m & \begin{matrix} * \\ \vdots \\ * \end{matrix} \end{array}\right]$

In this case let $x_{j_1}, \dots, x_{j_r}$ be the dep. variables, $j_1 < \dots < j_r$ &
$x_{k_1}, \dots, x_{k_{n-r}}$ be the free variables, $k_1 < \dots < k_{n-r}$

$$x_{j_1} + \sum_{s=1}^{n-r} c_{1k_s} x_{k_s} = b_1'$$

$$x_{j_2} + \sum_{s=1}^{n-r} c_{2k_s} x_{k_s} = b_2'$$

$$\vdots$$

$$x_{j_r} + \sum_{s=1}^{n-r} c_{rk_s} x_{k_s} = b_r'$$

$\therefore$ $Ax = b$ has a soln.

Thus we have proved

Ax = b has a soln if and only if the last coln of the RREF of [A|b] does not have a pivot.

<u>RANK OF A MATRIX</u> :- # nonzero rows in its RREF.

Ax = b has a soln if and only if rank A = rank [A|b]. (i)

<u>REM</u>: Row equiv. matrices have same rank.

<u>COR</u>: Ax = b has a soln if rank A = m (ii')

— <u>Obs</u> - Suppose Ax = b has a soln.
Then w.l.g. we can assume $\boxed{\text{rank } A = m}$

Subcase 1 :- Suppose rank $A (= m) = n$ (iii)

$\therefore$ All coln contain pivots

$\therefore$ There is a unique soln

Subcase 2: rank $A (= m) < n$ (iv)

$\therefore$ ∞-ly soln

Case 2. Assume rank $A < m$.

$$Ax = 0$$

If rank $A < n$.

$$\begin{bmatrix} \\ \\ 0 & \cdots & 0 & 1 & x & \cdots \end{bmatrix}$$

CONCL. If $Ax = b$ has a soln then it has ∞-ly many soln

Ex. rank $A = n$ ..

DETERMINAT :-

PERMUTATION :-

$$\begin{array}{cccccc} A & B & C & D & E & F \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ C & D & A & E & B & F \end{array}$$

DEF:- Given a set X, a permutation on X is an 1-1 & onto mapping from X to X.

NOTATION:- $\{1, 2, 3, 4, 5\}$

$$\left\{ \begin{array}{l} 1 \mapsto 3 \\ 2 \mapsto 4 \\ 4 \mapsto 2 \\ 3 \mapsto 1 \\ 5 \mapsto 5 \end{array} \right.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$