Convergence Analysis

Definition **Condition Number)**

For a given non-singular matrix $A \in \mathbb{R}^{n\times n}$ and a given matrix norm $\|\cdot\|$, the condition number of A with respect to the given norm is defined by

$$\kappa(A) := \|A\|\|A^{-1}\|$$

Remark:

When the condition number of a matrix is very large, even a small variation in the RHS vector can lead to a drastic variation in the solution. Such matrices are called **ill-conditioned** matrices. The matrices with small condition number are called **well-conditioned** matrices.

Definition **(Diagonally Dominant Matrices).**

A matrix A is said to be **diagonally dominant**

if it satisfies the inequality

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \quad i = 1, 2, \cdots, n.$$

CONVERGENCE Of Jacobi Method

In the case of Jacobi method, we have

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{(m)} \right), \quad i = 1, \cdots, n \quad m \geq 0$$

Therefore, each component of the error satisfies

$$e_i^{(m+1)} = - \sum_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(m)}, \quad i = 1, \cdots, n \quad m \geq 0.$$

which gives

$$|e_i^{(m+1)}| \leq \sum_{j=1, j\neq i}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \|e^{(m)}\|_\infty.$$

Define

$$\mu = \max_{1\le i\le n} \sum_{j=1, j\ne i}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|.$$

Then

$$|e_i^{(m+1)}| \le \mu \|e^{(m)}\|_\infty,$$

which is true for all $i = 1, 2, \cdots, n$. Therefore, we have

$$\|e^{(m+1)}\|_\infty \le \mu \|e^{(m)}\|_\infty.$$

For $\mu < 1$, ie., when the matrix A is diagonally dominant, then Jacobi method converges.

Note that the

converse is not true. That is, the Jacobi method may converge for A not diagonally dominant.

CONVERGENCE Of Gauss-Seidel Method

We will now prove that the Gauss-Seidal method converges if the given matrix A is diagonally dominant. The Gauss-Seidal method reads

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)} \right\}, \quad i = 1, 2, \cdots, n.$$

Therefore, the error in each component is given by

$$e_i^{(m+1)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(m+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(m)}, \quad i = 1, 2, \cdots, n.$$

Define

$$\alpha_i = \sum_{j=1}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right|, \quad \beta_i = \sum_{j=i+1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|, \quad i = 1, 2, \cdots, n,$$

with $\alpha_1 = \beta_1 = 0$. Note that μ given can be written as

$$\mu = \max_{1 \le i \le n} (\alpha_i + \beta_i)$$

and since A is assumed to be diagonally dominant, we have $\mu < 1$. Now

$$|e_i^{(m+1)}| \le \alpha_i \|e^{(m+1)}\|_\infty + \beta_i \|e^{(m)}\|_\infty, \quad i = 1, 2, \cdots, n.$$

Let k be such that

$$\|e^{(m+1)}\|_\infty = |e_k^{(m+1)}|.$$

Then with $i = k$ in ,

$$\|e^{(m+1)}\|_\infty \le \alpha_k \|e^{(m+1)}\|_\infty + \beta_k \|e^{(m)}\|_\infty.$$

Since $\mu < 1$, we have $\alpha_k < 1$ and therefore the above inequality give

$$\|e^{(m+1)}\|_\infty \le \frac{\beta_k}{1 - \alpha_k} \|e^{(m)}\|_\infty.$$

Define

$$\eta = \max_{1 \le i \le n} \frac{\beta_k}{1 - \alpha_k}.$$

Then the above inequality takes the form

$$\|e^{(m+1)}\|_\infty \leq \eta \|e^{(m)}\|_\infty.$$

Since for each i,

$$(\alpha_i + \beta_i) - \frac{\beta_i}{1-\alpha_i} = \frac{\alpha_i[1-(\alpha_i+\beta_i)]}{1-\alpha_i} \geq \frac{\alpha_i}{1-\alpha_i}[1-\mu] \geq 0,$$

we have

$$\eta \leq \mu < 1.$$

Thus, Gauss-Seidal method converges more faster than the Jacobi method and also when the given matrix is diagonally dominant, then the Gauss-Seidal method converges.

Gauss-Seidel Method

Example $x + y + z = 7, x + 2y + 2z = 13, x + 3y + z = 13$

Solve Equations x+y+z=7,x+2y+2z=13,x+3y+z=13 using Gauss Seidel method

Solution:
Total Equations are 3

$$x + y + z = 7$$

$$x + 2y + 2z = 13$$

$$x + 3y + z = 13$$

The coefficient matrix of the given system is not diagonally dominant.

Hence, we re-arrange the equations as follows, such that the elements in the coefficient matrix are diagonally dominant.

$$x + y + z = 7$$

$$x + 3y + z = 13$$

$$x + 2y + 2z = 13$$

From the above equations

$$x_{k+1} = \frac{1}{1}\left(7 - y_k - z_k\right)$$

$$y_{k+1} = \frac{1}{3}\left(13 - x_{k+1} - z_k\right)$$

$$z_{k+1} = \frac{1}{2}\left(13 - x_{k+1} - 2y_{k+1}\right)$$

Initial gauss $(x, y, z) = (0, 0, 0)$

Initial gauss $(x, y, z) = (0, 0, 0)$

Solution steps are

1^{st} Approximation

$$x_1 = \frac{1}{1}[7 - (0) - (0)] = \frac{1}{1}[7] = 7$$

$$y_1 = \frac{1}{3}[13 - (7) - (0)] = \frac{1}{3}[6] = 2$$

$$z_1 = \frac{1}{2}[13 - (7) - 2(2)] = \frac{1}{2}[2] = 1$$

2^{nd} Approximation

$$x_2 = \frac{1}{1}[7 - (2) - (1)] = \frac{1}{1}[4] = 4$$

$$y_2 = \frac{1}{3}[13 - (4) - (1)] = \frac{1}{3}[8] = 2.6667$$

$$z_2 = \frac{1}{2}[13 - (4) - 2(2.6667)] = \frac{1}{2}[3.6667] = 1.8333$$

3^{rd} Approximation

$$x_3 = \frac{1}{1}[7 - (2.6667) - (1.8333)] = \frac{1}{1}[2.5] = 2.5$$

$$y_3 = \frac{1}{3}[13 - (2.5) - (1.8333)] = \frac{1}{3}[8.6667] = 2.8889$$

$$z_3 = \frac{1}{2}[13 - (2.5) - 2(2.8889)] = \frac{1}{2}[4.7222] = 2.3611$$

4^{th} Approximation

$$x_4 = \frac{1}{1}[7 - (2.8889) - (2.3611)] = \frac{1}{1}[1.75] = 1.75$$

$$y_4 = \frac{1}{3}[13 - (1.75) - (2.3611)] = \frac{1}{3}[8.8889] = 2.963$$

$$z_4 = \frac{1}{2}[13 - (1.75) - 2(2.963)] = \frac{1}{2}[5.3241] = 2.662$$

5^{th} Approximation

$$x_5 = \frac{1}{1}[7 - (2.963) - (2.662)] = \frac{1}{1}[1.375] = 1.375$$

$$y_5 = \frac{1}{3}[13 - (1.375) - (2.662)] = \frac{1}{3}[8.963] = 2.9877$$

$$z_5 = \frac{1}{2}[13 - (1.375) - 2(2.9877)] = \frac{1}{2}[5.6497] = 2.8248$$

6^{th} Approximation

$$x_6 = \frac{1}{1}[7 - (2.9877) - (2.8248)] = \frac{1}{1}[1.1875] = 1.1875$$

$$y_6 = \frac{1}{3}[13 - (1.1875) - (2.8248)] = \frac{1}{3}[8.9877] = 2.9959$$

$$z_6 = \frac{1}{2}[13 - (1.1875) - 2(2.9959)] = \frac{1}{2}[5.8207] = 2.9104$$

7^{th} Approximation

$$x_7 = \frac{1}{1}[7 - (2.9959) - (2.9104)] = \frac{1}{1}[1.0938] = 1.0938$$

$$y_7 = \frac{1}{3}[13 - (1.0938) - (2.9104)] = \frac{1}{3}[8.9959] = 2.9986$$

$$z_7 = \frac{1}{2}[13 - (1.0938) - 2(2.9986)] = \frac{1}{2}[5.909] = 2.9545$$

8^{th} Approximation

$$x_8 = \frac{1}{1}[7 - (2.9986) - (2.9545)] = \frac{1}{1}[1.0469] = 1.0469$$

$$y_8 = \frac{1}{3}[13 - (1.0469) - (2.9545)] = \frac{1}{3}[8.9986] = 2.9995$$

$$z_8 = \frac{1}{2}[13 - (1.0469) - 2(2.9995)] = \frac{1}{2}[5.954] = 2.977$$

9^{th} Approximation

$$x_9 = \frac{1}{1}[7 - (2.9995) - (2.977)] = \frac{1}{1}[1.0234] = 1.0234$$

$$y_9 = \frac{1}{3}[13 - (1.0234) - (2.977)] = \frac{1}{3}[8.9995] = 2.9998$$

$$z_9 = \frac{1}{2}[13 - (1.0234) - 2(2.9998)] = \frac{1}{2}[5.9769] = 2.9884$$

10^{th} Approximation

$$x_{10} = \frac{1}{1}[7 - (2.9998) - (2.9884)] = \frac{1}{1}[1.0117] = 1.0117$$

$$y_{10} = \frac{1}{3}[13 - (1.0117) - (2.9884)] = \frac{1}{3}[8.9998] = 2.9999$$

$$z_{10} = \frac{1}{2}[13 - (1.0117) - 2(2.9999)] = \frac{1}{2}[5.9884] = 2.9942$$

11^{th} Approximation

$$x_{11} = \frac{1}{1}[7 - (2.9999) - (2.9942)] = \frac{1}{1}[1.0059] = 1.0059$$

$$y_{11} = \frac{1}{3}[13 - (1.0059) - (2.9942)] = \frac{1}{3}[8.9999] = 3$$

$$z_{11} = \frac{1}{2}[13 - (1.0059) - 2(3)] = \frac{1}{2}[5.9942] = 2.9971$$

12^{th} Approximation

$$x_{12} = \frac{1}{1}[7 - (3) - (2.9971)] = \frac{1}{1}[1.0029] = 1.0029$$

$$y_{12} = \frac{1}{3}[13 - (1.0029) - (2.9971)] = \frac{1}{3}[9] = 3$$

$$z_{12} = \frac{1}{2}[13 - (1.0029) - 2(3)] = \frac{1}{2}[5.9971] = 2.9985$$

13^{th} Approximation

$$x_{13} = \frac{1}{1}[7 - (3) - (2.9985)] = \frac{1}{1}[1.0015] = 1.0015$$

$$y_{13} = \frac{1}{3}[13 - (1.0015) - (2.9985)] = \frac{1}{3}[9] = 3$$

$$z_{13} = \frac{1}{2}[13 - (1.0015) - 2(3)] = \frac{1}{2}[5.9985] = 2.9993$$

14^{th} Approximation

$$x_{14} = \frac{1}{1}[7 - (3) - (2.9993)] = \frac{1}{1}[1.0007] = 1.0007$$

$$y_{14} = \frac{1}{3}[13 - (1.0007) - (2.9993)] = \frac{1}{3}[9] = 3$$

$$z_{14} = \frac{1}{2}[13 - (1.0007) - 2(3)] = \frac{1}{2}[5.9993] = 2.9996$$

Iterations are tabulated as below

Iteration	x	y	z
1	7	2	1
2	4	2.6667	1.8333
3	2.5	2.8889	2.3611
4	1.75	2.963	2.662
5	1.375	2.9877	2.8248
6	1.1875	2.9959	2.9104
7	1.0938	2.9986	2.9545
8	1.0469	2.9995	2.977
9	1.0234	2.9998	2.9884
10	1.0117	2.9999	2.9942
11	1.0059	3	2.9971
12	1.0029	3	2.9985
13	1.0015	3	2.9993
14	1.0007	3	2.9996

Solution By Gauss Seidel Method.

$x = 1.0007 \cong 1$

$y = 3 \cong 3$

$z = 2.9996 \cong 3$