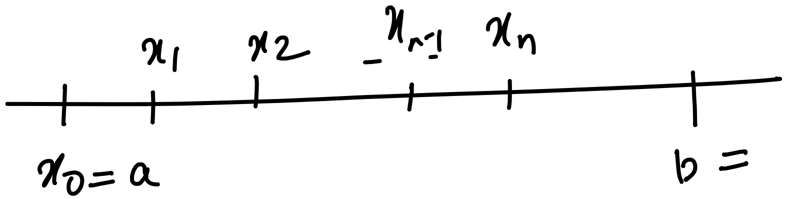
lecture 6 Numerical methods

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0 \quad \text{—— } (*)$$



Suppose $x_0 = a$ & we want to find the value of $y(b)$.

Divide $[a \;\; b]$ into N-equal intervals

$$h = \frac{a-b}{N}$$

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h$$

We know $y(x)$ is a solutn of $(*)$

$\Leftrightarrow$ $y(x)$ satisfy the integral equatn

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y_0 = y(x_0)$$

$$y_1 = y(x_1) = y_0 + \int_{x_0}^{x_1} f(t, y(t))\, dt$$

$$y_2 = y(x_2) = y_0 + \int_{x_0}^{x_2} f(t, y(t))\, dt$$

$$= y_1 + \int_{x_1}^{x_2} f(t, y(t))\, dt$$

$$y_n = y(x_n) = y_0 + \int_{x_0}^{x_n} f(t, y(t))\, dt$$

$$= y_{n-1} + \int_{x_{n-1}}^{x_n} f(t, y(t))\, dt$$

$$y_n = y_{n-1} + \int_{x_{n-1}}^{x_n} f(t, y(t))\, dt$$

$$y_n = y_{n-1} + \underbrace{(x_n - x_{n-1})} f(\xi, y(\xi))$$

where $\xi \in (x_{n-1}, x_n)$

(Recall MVT of integral calculus

$$\int_a^b \varphi(x)\, dx = (b-a)\, \varphi(\xi)$$

where $\xi \in (a\ b)$. assuming

$\varphi(x)$ is contiuous on $[a\ b]$

Quick proof

Since φ is continuous on $[a\ b]$,

$\exists\ m, M$ s.t

$$\inf \varphi(x) = m \le \varphi(x) \le M = \sup \varphi(x)$$

integrate from a to b

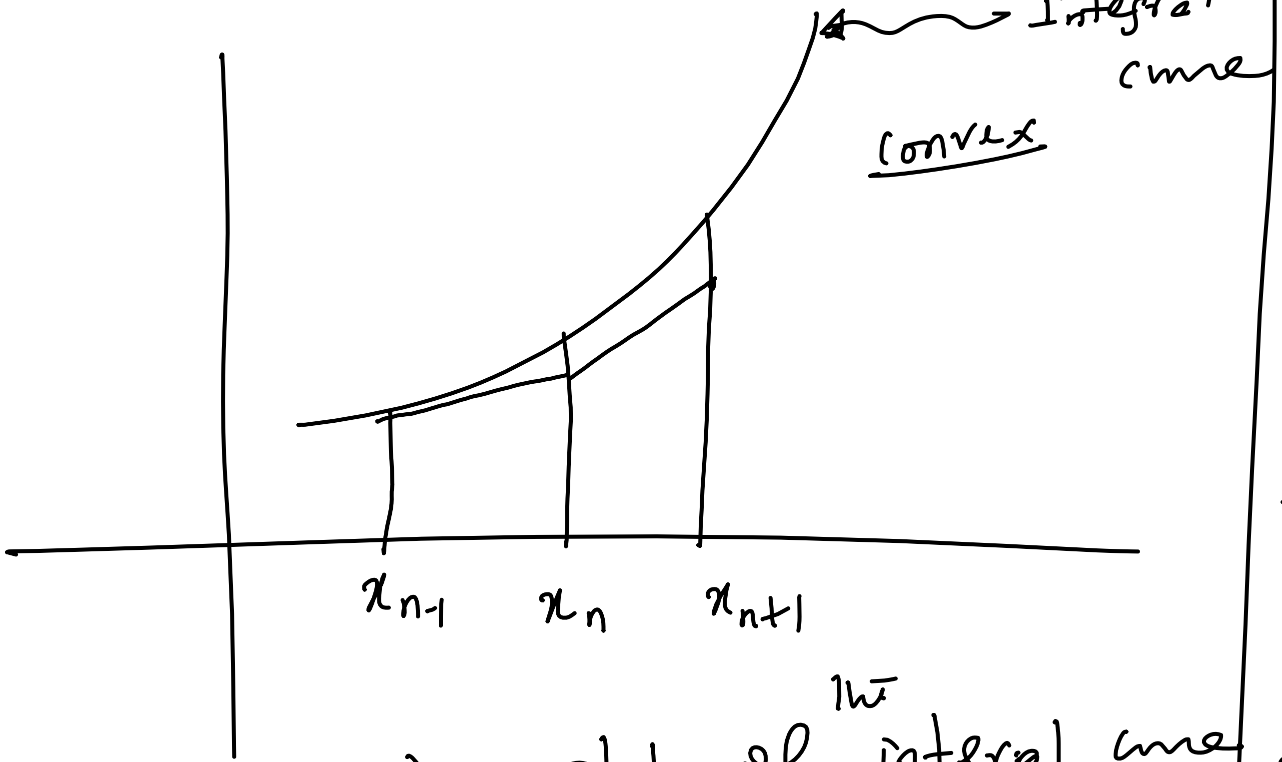
$$m(b-a) \le \int_a^b \varphi(x)\, dx \le M(b-a)$$

$$m \le \frac{1}{b-a} \int_a^b \varphi(x)\, dx \le M$$

By Intermediate Value Property of continuous function, $\exists\ \xi \in (a\ b)$

s.t $\varphi(\xi) = \frac{1}{b-a} \int_a^b \varphi(x)\, dx$]

$$y_n = y_{n-1} + (x_n - x_{n-1})\, f(\xi, y(\xi))$$



$f(\xi, y(\xi))$ = slope of the integral curve at some point between x_{n-1} & x_n.

Euler Approximation

Approximate $f(\xi, y(\xi))$ by $f(x_{n-1}, y_{n-1})$

$$\boxed{y_n \approx y_{n-1} + (x_n - x_{n-1})\, f(x_{n-1}, y_{n-1})}$$

Remark The errors in the Euler method will accumulate if the integral curve is convex or concave

In convex case, the approx. value will be lower than actual value

In concave case, the approx. value will be higher than actual value.

Modifield/Improved Euler method

$$y_n = y_{n-1} + (x_n - x_{n-1})\left[\frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2}\right]$$

$$y_n^* = y_{n-1} + (x_n - x_{n-1})\, f(x_{n-1}, y_{n-1})$$

Solution by Euler method

Example

$y' = x - y \quad y(0) = 1.$

(i) Find $y(0.1)$ $y(0.2)$ $y(0.3)$ by Euler method.

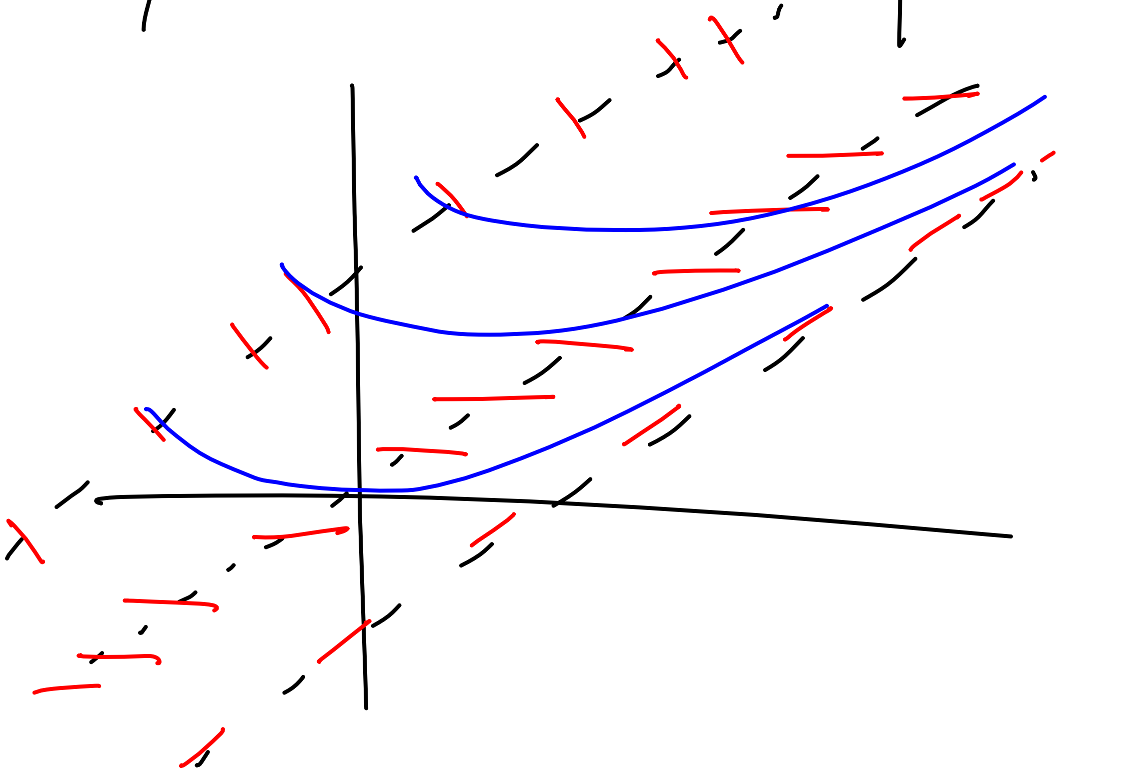
Is your answer $y(0.1)$ is higher or lower than actual value?

(ii) Do the same with Modified Euler method?

Sol By Euler method

$$y_n = y_{n-1} + (0.1)\, f(x_{n-1}, y_{n-1})$$

x	0	0.1	0.2	0.3
y	1	0.9	0.82	0.758



From graphical method, we see the integral curves are convex & the approx value is lower than the actual.

(ii) Using Modified Euler meth.

$$y_1 = y_0 + \frac{0.1}{2}\left[f(0,1) + f(0.1, y_1^*)\right]$$

$$y_1^* = 0.9$$

$$= 0.91$$

Example (1) $\frac{dy}{dx} = |y|$ $\quad y(x_0) = y_0$

~~For~~ Discuss existence & uniqueness for different (x_0, y_0).

Sol $f(x,y) = |y|$ — continuous on $\mathbb{R}^2$

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{\big||y_1| - |y_2|\big|}{|y_1 - y_2|} \le \frac{|y_1 - y_2|}{|y_1 - y_2|} = 1$$

$$|f(x,y_1) - f(x,y_2)| \le |y_1 - y_2|$$

So LC satisfied on $\mathbb{R}^2$.

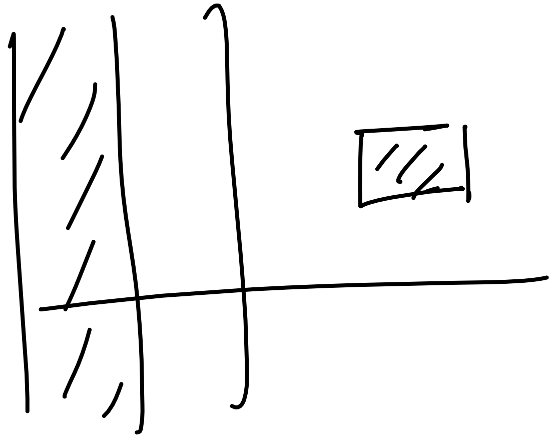
Conclusion Unique sol. for any (x_0, y_0).

Example $f(x,y) = xy$.

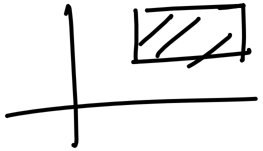
(i) Show that f satisfy LC on any $[a,b] \times [c,d]$

(ii) - - - - - - - - - $[a,b] \times \mathbb{R}$

(iii) - - - - .. DOES NOT - - - - on $\mathbb{R}^2$



Sol
(i) On $[a\ b] \times [c\ d]$ $\quad f(x,y) = xy$



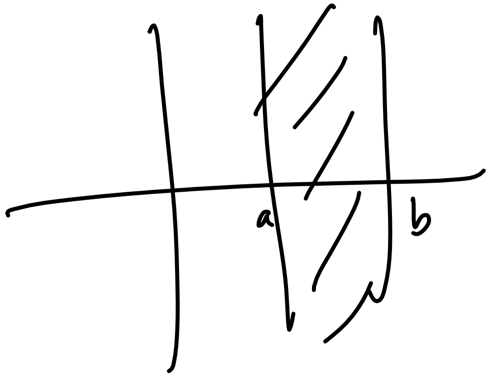
$$|f(x,y_1) - f(x,y_2)|$$

$$= |xy_1 - xy_2| = |x|\,|y_1 - y_2|$$

$$\leq L\,|y_1 - y_2|$$

$$L = \max(|a|, |b|)$$

Also works for $[a\ b] \times \mathbb{R}$



(iii) On $\mathbb{R}^2$

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = |x|$$

— unbonded on $\mathbb{R}^2$.

So $\nexists L$ for LC for $\mathbb{R}^2$ $\blacksquare$