Theorem. Suppose a given function $f(x,y,z)$ of three variables is differentiable at $x_0 \in \mathbb{R}^3$.

Then the partial derivatives $\frac{\partial f}{\partial x}(x_0)$, $\frac{\partial f}{\partial y}(x_0)$ and $\frac{\partial f}{\partial z}(x_0)$ exists at x_0 and the derivative

$$\boxed{f'(x_0) = \left(\frac{\partial f}{\partial x}(x_0), \frac{\partial f}{\partial y}(x_0), \frac{\partial f}{\partial z}(x_0)\right).}$$

Let the derivative at X_0 be $f'(X_0) = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$.

We have the error term

$$E(H) = \frac{f(X_0+H) - f(X_0) - (\alpha_1,\alpha_2,\alpha_3)\cdot H}{\|H\|} \to 0 \text{ as } H \to \mathbf{0}$$

Now for $H = t\vec{i} = t(1,0,0)$, $H \to \mathbf{0}$ as $t \to 0$

$$\lim_{t\to 0} E(t\vec{i}) = \lim_{t\to 0} \frac{f(X_0+t\vec{i}) - f(X_0) - (\alpha_1,\alpha_2,\alpha_3)\cdot(t\vec{i})}{t}$$

$= 0$

$$\Rightarrow \lim_{t\to 0} \frac{f(X_0+t\vec{i}) - f(X_0)}{t} - \lim_{t\to 0} \frac{t\,\alpha_1}{t} = 0$$

or,

$$\frac{\partial f}{\partial x}(X_0) = \alpha_1$$

Similarly, for $H = t\vec{j}$ we find that $\frac{\partial f}{\partial y}(X_0) = \alpha_2$

and for $H = t\vec{k}$ we get $\frac{\partial f}{\partial z}(X_0) = \alpha_3$.

Thus the derivative

$$f'(X_0) = \left(\frac{\partial f}{\partial x}(X_0), \frac{\partial f}{\partial y}(X_0), \frac{\partial f}{\partial z}(X_0) \right).$$

Example.

Consider $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x, y, z) \longmapsto xyz$$

Is f differentiable at $X_0 = (a, b, c) \in \mathbb{R}^3$?

Here $\dfrac{\partial f}{\partial x}(x_0) = bc$

$$\frac{\partial f}{\partial y}(x_0) = ac$$

$$\frac{\partial f}{\partial z}(x_0) = ab$$

Q. Can we find that $f'(x_0) = \left(bc,\ ac,\ ab\right)$?

$X_0 = (x_0, y_0, z_0)$, $f : \mathbb{R}^3 \to \mathbb{R}$

$$f'(X_0) = ?$$

$$\frac{\partial f}{\partial x}(X_0), \ \frac{\partial f}{\partial y}(X_0), \ \frac{\partial f}{\partial z}(X_0) \quad \text{exist}$$

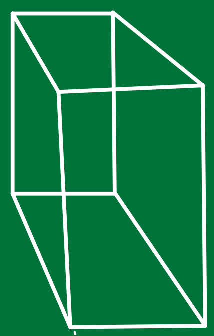
$+ \ ?$

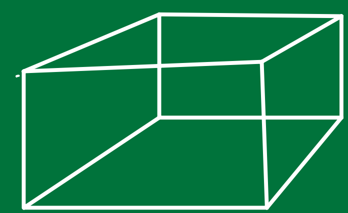
$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial f}{\partial z} \ : \ ?$$

Theorem. If $f: \mathbb{R}^3 \to \mathbb{R}$ is such that all its partial derivatives exists in a neighborhood of $X_0 = (x_0, y_0, z_0)$

$$\left\{ (x, y, z) \,\middle/\, \begin{array}{l} |x - x_0| < \varepsilon_1 \\ |y - y_0| < \varepsilon_2 \\ |z - z_0| < \varepsilon_3 \end{array} \right\}$$

and partial derivative functions are continuous at X_0, then f is differentiable at X_0.





Example.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is differentiable everywhere in $\mathbb{R}^2$.

True / False ?

Example:

(i) $f(x,y) = |x| + |y|$

(ii) $f(x,y) = |xy|$

- ☐ **Continuous everywhere**
- ☐ *differentiability?*

(i) $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{|h|}{h}$

does not exists

$\dfrac{\partial f}{\partial y}(0,0)$ —— — —

(ii) $\dfrac{\partial f}{\partial x}(0,0) = 0$

$\dfrac{\partial f}{\partial y}(0,0) = 0$

The error term

$$E(H) = \frac{|hk|}{\sqrt{h^2 + k^2}} \to 0 \text{ as } \|H\| \to 0$$

Thus f is differentiable at $(0,0)$.

For $b \neq 0$, $\dfrac{\partial f}{\partial x}(0,b) = \lim\limits_{h \to 0} \dfrac{|hb| - 0}{h}$ does not exist.

$\Rightarrow$ f is not differentiable at $(0, b)$ for $b \neq 0$.

f is differentiable at x_0

f is continuous at x_0

[If the partial derivative functions are continuous at x_0]

All the directional derivatives $D_{x_0}f(\vec{u})$ exist for $\vec{u} \in \mathbb{R}^2$ with $\|\vec{u}\| = 1$

[If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are bounded near x_0]

Increment theorem.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $X_0 = (a, b)$.

Then we have

$$f(X_0 + H) - f(X_0)$$

$$= f(a+h, b+k) - f(a,b)$$

$$= h \frac{\partial f}{\partial x}(X_0) + k \frac{\partial f}{\partial y}(X_0) + h\varepsilon_1(h,k) + k\,\varepsilon_2(h,k)$$

where $\varepsilon_1(h,k) \to 0$ and
$\varepsilon_2(h,k) \to 0$

as $H = (h,k) \to 0$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $X_0 = (a, b)$.

Then for $H = (h, k)$, we have

$$f(X_0 + H) - f(X_0)$$

$$= f'(X_0) \cdot H + \| H \| E(H)$$

$$= h \frac{\partial f}{\partial x}(X_0) + k \frac{\partial f}{\partial y}(X_0) + \| H \| E(H)$$

Now

$$\| H \| \; E(H)$$

$$= (h^2 + k^2) \frac{E(H)}{\| H \|} = h \left(h \frac{E(H)}{\| H \|} \right) + k \left(k \frac{E(H)}{\| H \|} \right)$$

$$= h \, \mathcal{E}_1(h, k) + k \, \mathcal{E}_2(h, k)$$

Then $\quad \|\varepsilon_1(H)\| \leq \|E(H)\|$,

$\|\varepsilon_2(H)\| \leq \|E(H)\|$

$\Rightarrow \quad \varepsilon_1(H) \to 0$ as $H \to 0$ and

$\varepsilon_2(H) \to 0$ as $H \to 0$.

Thus,

$$f(X_0 + H) - f(X_0)$$

$$= f(a+h, b+k) - f(a,b)$$

$$= h\frac{\partial f}{\partial x}(X_0) + k\frac{\partial f}{\partial y}(X_0) + h\varepsilon_1(h,k) + k\,\varepsilon_2(h,k)$$

where $\varepsilon_1(h,k) \to 0$ and $\varepsilon_2(h,k) \to 0$

as $H = (h,k) \to 0$

Application

$$\begin{tikzcd} t_0 \arrow[r, mapsto, "R"] \arrow[dr, dashed, mapsto, "W"'] & (x(t_0), y(t_0)) \arrow[d, mapsto, "f"] \\ & f(x(t_0), y(t_0)) \end{tikzcd}$$

$$\begin{tikzcd} t \arrow[d, mapsto] & \mathbb{R} \arrow[d] \\ f(x(t), y(t)) & \mathbb{R} \end{tikzcd}$$

$R(t) \in \mathbb{R}^2$

$f(x, y) \in \mathbb{R}$

Chain rule.

Let $f(x,y)$ be a differentiable function and if $x = x(t)$, $y = y(t)$ are differentiable function of t, then the composite function ω is differentiable and the derivative is given by

$$\begin{tikzcd} t_0 \arrow[r, mapsto] \arrow[d, dashed, mapsto, "\omega"'] & (x(t_0), y(t_0)) \arrow[dl, "f"] \\ f(x(t_0), y(t_0)) & \end{tikzcd}$$

$$\frac{d\omega}{dt}(t_0) = \left(\frac{\partial f}{\partial x}\right)(x(t_0), y(t_0))\, \frac{dx}{dt}(t_0) + \left(\frac{\partial f}{\partial y}\right)\left(x(t_0), y(t_0)\right) \frac{dy}{dt}(t_0).$$

i.e. $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\Delta f = f(X_0 + H) - f(X_0), \quad \Delta t = t - t_0$$

$$\lim_{t \to t_0} \frac{\Delta f}{\Delta t} = \lim_{t \to t_0} \frac{f(X_0 + H) - f(X_0)}{t - t_0}$$

For $H = (\Delta x, \Delta y)$ where $\Delta x = x(t) - x(t_0)$

$X_0 = (x(t_0), y(t_0))$ $\qquad \Delta y = y(t) - y(t_0)$

we have

$$\begin{aligned} X_0 + H &= (x(t_0), y(t_0)) + (\Delta x, \Delta y) \\ &= (x(t_0) + \Delta x, \ y(t_0) + \Delta y) \\ &= (x(t), \ y(t)). \end{aligned}$$

As $t \to t_0$, $\Delta x = x(t) - x(t_0) \to 0$
$\Delta y = y(t) - y(t_0) \to 0$

$$\Rightarrow \quad H = (\Delta x, \Delta y) \to (0,0)$$

Now

$$\lim_{t \to t_0} \frac{\Delta f}{\Delta t} = \lim_{t \to t_0} \frac{f(X_0 + H) - f(X_0)}{t - t_0}$$

$$\Rightarrow \quad \frac{df}{dt}(t_0) = \lim_{t \to t_0} \frac{\Delta x \frac{\partial f}{\partial x}(X_0) + \Delta y \frac{\partial f}{\partial y}(X_0) + \varepsilon_1(\Delta x, \Delta y) + \varepsilon_2(\Delta x, \Delta y)}{t - t_0}$$

By Increment theorem, where

$\varepsilon_1(\Delta x, \Delta y) \to 0$

$\varepsilon_2(\Delta x, \Delta y) \to 0$ as $H = (\Delta x, \Delta y) \to (0,0)$.

Thus,

$$\frac{df}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0) \lim_{t \to t_0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}(x_0) \lim_{t \to t_0} \frac{\Delta y}{\Delta t}$$

$$= \frac{\partial f}{\partial x}(x_0) \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0) \frac{dy}{dt}(t_0)$$

Example: 1.

$$f(x, y, z) = xy + y^2 + e^z x \quad \text{where}$$

$$x(t) = t$$
$$y(t) = \sin t$$
$$z(t) = \cos t$$

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (y + e^z) + (x + 2y)\cos t - e^z x \sin t$$

2. $f(x,y) = e^{x} \sin y$ where

$x = st^2$

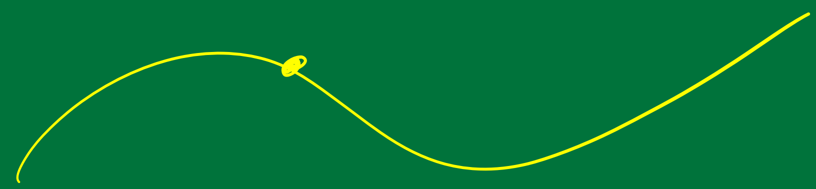
$y = s^2 t$

By chain rule,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Definition: The vector $\left(\frac{\partial f}{\partial x}(X_0), \frac{\partial f}{\partial y}(X_0), \frac{\partial f}{\partial z}(X_0)\right)$ is called the gradient vector of f at X_0 and it is denoted by $\nabla f(X_0)$.



Theorem: If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable at X_0 then all the directional derivatives $D_{X_0} f(\vec{u})$ exist for all unit vector $\vec{u}$ in $\mathbb{R}^3$. Moreover,

$(\vec{u} \in \mathbb{R}^3, \|\vec{u}\| = 1)$

$$
\begin{aligned}
D_{X_0} f(\vec{u}) &= f'(X_0) \cdot \vec{u} \\
&= \nabla f(X_0) \cdot \vec{u}
\end{aligned}
$$

Note: $D_{x_0} f(\vec{u}) = \nabla f(x_0) \cdot \vec{u}$

$= \| \nabla f(x_0) \| \cos\theta$ where $\theta \in [0, \pi]$ is the angle between the gradient vector $\nabla f(x_0)$ and $\vec{u}$.

$\Rightarrow$ $D_{x_0} f(\vec{u})$ is maximum when $\theta = 0$ and is minimum when $\theta = \pi$

$\Rightarrow$ f increases most rapidly around x_0 in the direction

$$\vec{u} = \frac{\nabla f(x_0)}{\| \nabla f(x_0) \|}$$ and

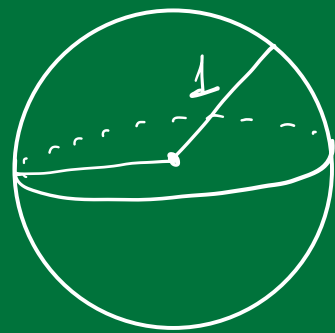
f decreases most rapidly around x_0 in the direction

$$\vec{u} = -\frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$

$$(x, y, z) \longmapsto f(x, y, z) = x^2 + y^2 + z^2$$

Then $f^{-1}(\{1\}) = \{ (x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1 \}$



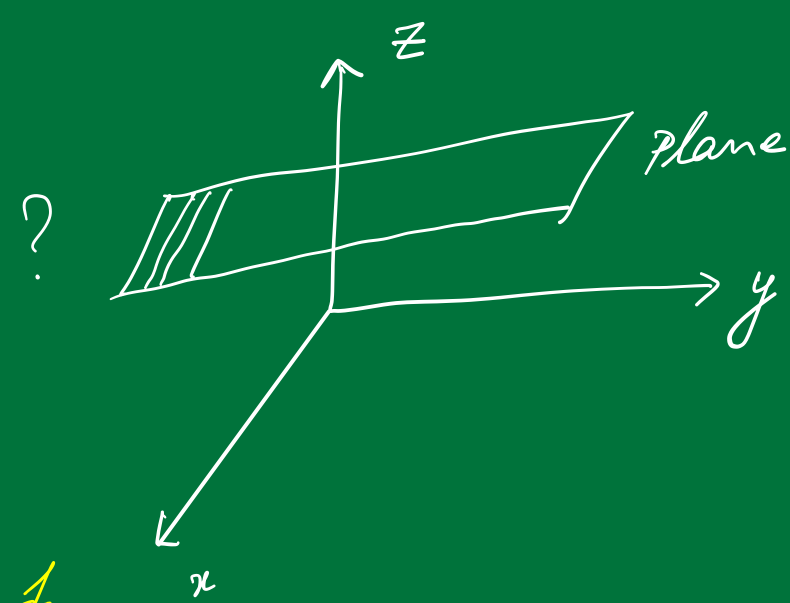
Unit Sphere

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and $c \in \mathbb{R}$.

$$(x,y,z) \longmapsto f(x,y,z)$$

Then $S = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = c \} = f^{-1}(\{c\})$

is called a level surface (for the function f) at the height c.

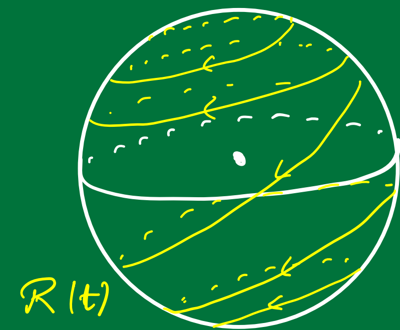


Ex. Unit Sphere is a level surface at height 1 of the function $f(x,y,z) = x^2+y^2+z^2$

Suppose S is a level surface and $P = (x_0, y_0, z_0) \in S$ and

$R(t) = \big(x(t),\ y(t),\ z(t)\big)$

is a differentiable curve lying on S and passing through P.

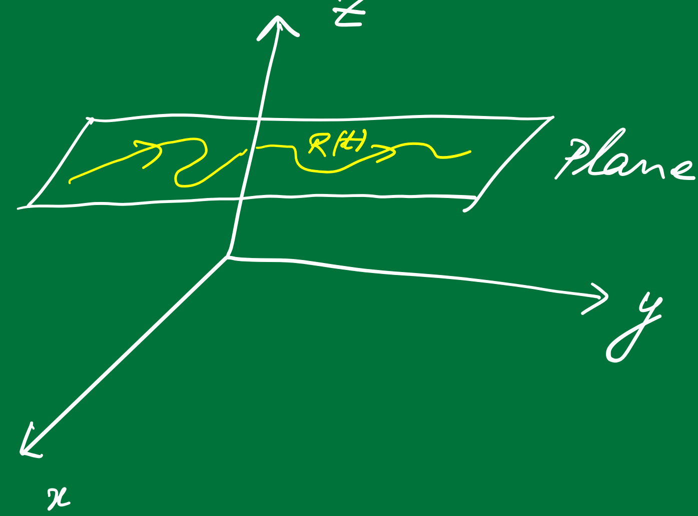


Let

$\vec{v} = R'(t_0)$

be the tangent vector to $R(t)$?

at $P = (x_0, y_0, z_0) = R(t_0)$.



Then we have $\frac{d}{dt} f(x(t), y(t), z(t)) = 0$

or, $\frac{\partial f}{\partial x}(R(t_0)) \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(R(t_0)) \frac{dy}{dt}(t_0) + \frac{\partial f}{\partial z}(R(t_0)) \frac{dz}{dt}(t_0) = 0$

or, $\nabla f \, (R(t_0)) \cdot \left(x'(t_0), \; y'(t_0), \; z'(t_0) \right) = 0$

or, $\nabla f \cdot \vec{v} = 0$

Tangent vector

A vector $v \in \mathbb{R}^3$ is said to be a tangent vector to the level surface S at $P(x_0, y_0, z_0)$ if v is a tangent vector to a smooth curve $R(t)$ passing through the point P.

The collection of all such tangent vectors at P of the level surface constitute tangent space of S at the point P.

The tangent space of S at P :

Let $v \perp \nabla f(P)$

$\Rightarrow$ The tangent plane.

$\vec{n} = \nabla f(P)$

$P = (x_0, y_0, z_0)$