ASSIGNMENT 3
MTH102A

Question 1.
Let A be a square matrix with entries from the set of complex numbers $\mathbb{C}$. A is said to be Hermitian if $A = \overline{A}^T$ (A is equal to its conjugate transpose). A is said to be skew-Hermitian if $A = -\overline{A}^T$. Which of the following statements are true. Justify your answer.
(a) set of Hermitian matrices of order n is a vector space over $\mathbb{C}$ under usual matrix addition and scalar multiplication;
(b) set of Hermitian matrices of order n is vector space over set of real numbers $\mathbb{R}$ under usual matrix addition and scalar multiplication;
(c) set of skew-Hermitian matrices of order n is a vector space over $\mathbb{C}$ under usual matrix addition and scalar multiplication;
(d) set of skew-Hermitian matrices of order n is a vector space over $\mathbb{R}$ under usual matrix addition and scalar multiplication;

Solution 1.
(a). False.
Take $A =$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

and scalar $\alpha = i$, then A is Hermitian matrix but αA is not hermitian.

(b). True.

(1) Zero matrix of order n is additive identity.
(2) Identity matrix of order n is multiplicative identity.
(3) Let A and B are two Hermitian complex matrices and $a, b \in R$ then :
$\overline{(A + aB)}^T = \overline{A}^T + a\overline{B}^T = A + aB$.
(4) Distributivity and Associativity are obvious.

Alternatively, prove that it is a subspace and a subspace is a vector space in its own right.
(c). False.
Take $A =$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is skew-Hermitian matrix, but iA is not.

(d).True.

(1) Zero matrix of order n is additive identity.

(2) Identity matrix of order n is multiplicative identity.

(3) Let A and B are two Hermitian complex matrices and $a, b \in R$ then :
$\overline{(A + aB)}^T = \overline{A}^T + a\overline{B}^T = -A - aB = -(A + aB)$.

(4) Distributivity and Associativity are obvious.

Alternatively, prove that it is a subspace and a subspace is a vector space in its own right.

Question 2.
In $\mathbb{R}$, consider the addition $x \oplus y = x + y - 1$ and the scalar multiplication $\lambda.x = \lambda(x-1)+1$. Prove that $\mathbb{R}$ is a vector space with respect to these operations and the additive identity is 1.

Solution 2.
Let $x, y, \lambda_1, \lambda_2 \in \mathbb{R}$.

(1) 1 is additive identity, as $1 \oplus x = 1 + x - 1 = x$ and $x \oplus 1 = x + 1 - 1 = x$.
(2) 1 is multiplicative identity, as $1.x = 1.(x - 1) + 1 = x$.
(3) Let $x, y, z \in \mathbb{R}$, then

$$(x \oplus y) \oplus z = (x + y - 1) \oplus z = x + y - 1 + z - 1 = x + (y + z - 1) - 1 = x \oplus (y \oplus z).$$

(4) For $x, y, \lambda \in \mathbb{R}$,

$$\lambda.x \oplus \lambda.y = \lambda(x - 1) + 1 + \lambda(y - 1) + 1 - 1 = \lambda(x + y - 2) + 1$$

and

$$\lambda.(x \oplus y) = \lambda.(x + y - 1) = \lambda(x + y - 1 - 1) + 1 = \lambda(x + y - 2) + 1$$

$\Rightarrow \lambda.x \oplus \lambda.y = \lambda.(x \oplus y)$

(5)

$$(\lambda_1 + \lambda_2).(x) = (\lambda_1 + \lambda_2)(x - 1) + 1$$

and

$$\lambda_1.x \oplus \lambda_2.x = \lambda_1.x + \lambda_2.x - 1 = \lambda_1(x - 1) + 1 + \lambda_2(x - 1) + 1 - 1 = (\lambda_1 + \lambda_2)(x - 1) + 1$$

$\Rightarrow (\lambda_1 + \lambda_2).(x) = \lambda_1.x \oplus \lambda_2.x$

Question 3.
Which of the followings are true:
(i) $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ is a subspace of $\mathbb{R}^2$,
(ii) $X = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is a subspace of $\mathbb{R}^2$,
(iii) the set $\mathcal{P}_n(X)$ of all single variable polynomials of degree at most n is a subspace of vector space $\mathcal{P}(X)$ of all single variable polynomials.
(iv) $X = \{(x, y, z); x + y + z = 1\}$ is a subspace of $\mathbb{R}^3$.
(v) $\{A \in M_2(\mathbb{R}) : det(A) = 0\}$ is a subspace of $M_2(\mathbb{R})$, where $M_2(\mathbb{R})$ is vector space of real 2×2 matrices under usual matrix addition and scalar multiplication,

(vi) Let $C([0,1])$ be vector space of real valued continuous functions on $[0,1]$ and let $a \in [0,1]$. The set $M_a = \{f \in C([0,1]) : f(a) = 0\}$ is a subspace of $C([0,1])$.
(vii) the space of all upper triangular matrices of order n is a subspace of $M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is vector space of real $n \times n$ matrices under usual matrix addition and scalar multiplication
(viii) the set of all orthogonal matrices of order 2 is a subspace of $M_2(\mathbb{R})$. (A square matrix A is said to be orthogonal if $AA^T = I$

Solution 3.
(i) False.
As $(1,0) \in X$, but $-1.(1,0) = (-1,0) \notin X$.

(ii) False.
$(1,1),(2,4) \in X$ but $(1,1) + (2,4) = (3,5) \notin X$.

(iii) True.

(1) Zero polynomial is additive identity and $p(x) = 1$ is multiplicative identity.
(2) Sum of two polynomial of degree less then equal to n is a polynomial of degree less than or equal to n.
(3) Let $p(x) \in P_n(X)$ then for any $\alpha \in \mathbb{R}$, $\alpha p(x) \in P_n(X)$.

(iv) False.
Take $(1,0,0),(0,1,0) \in X$ but $(1,0,0) + (0,1,0) = (1,1,0) \notin X$.
(v) False.
Take matrices
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
with determinant zero but their sum is a non-singular matrix.

(vi)True.

(1) $f \equiv 0$ is additive identity.
(2) Let $f,g \in M_a$ then $f(a) + g(a) = 0, \Rightarrow f + g \in M_a$.
(3) Let $\alpha \in \mathbb{F}$, then $\alpha f = \alpha.0 = 0, \Rightarrow \alpha f \in M_a$.

(vii)True.

(1) Zero matrix and identity matrix of order n is upper triangular matrix.
(2) Sum of two upper triangular is upper triangular.
(3) Scalar multiplication of a scalar with an upper triangular matrix is again upper triangular .

(viii)False.
Zero matrix is not orthogonal, and then there is no additive identity.

Question 4.
Show that $W = \{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\}$ is a subspace of $\mathbb{R}^4$ spanned by vectors $(1, 0, 0, -1), (0, 1, 0, 1), (0, 0, 1, 1)$.

Solution 4.

(1) $(0, 0, 0, 0)$ is additive identity and $(1, 1, 1, 1)$ is multiplicative identity.

(2) Let $(x_1, x_2, x_3, x_4), (a_1, a_2, a_3, a_4) \in W$
$\Rightarrow x_4 - x_3 = x_2 - x_1$ and $a_4 - a_3 = a_2 - a_1$
$\Rightarrow (x_4 + a_4) - (x_3 + a_3) = (x_2 + a_2) - (x_1 + a_1)$
$\Rightarrow (x_1 + a_1, x_2 + a_2.x_3 + a_3, x_4 + a_4) \in W$.

(3) Let $a \in \mathbb{R}$, then $a(x_1, x_2, x_3, x_4) \in W$.

(4) Distributivity and associativity is easy to check.

Hence W is a subspace of $\mathbb{R}^4$.

Let $(x_1, x_2, x_3, x_4) \in W$ be any. Then $x_4 - x_3 = x_2 - x_1 \Rightarrow x_4 = x_2 - x_1 + x_3$,
$\Rightarrow (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_2 - x_1 + x_3) = x_1(1, 0, 0, -1) + x_2(0, 1, 0, 1) + x_3(0, 0, 1, 1)$
Hence W is spanned by these three vectors.

Question 5.
Let W_1 and W_2 be two subspaces of a vector space V such that $W_1 \cup W_2$ is a subspace of V. Prove that either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Solution 5.
Let $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$, then there exist $w_1 \in W_1, w_2 \in W_2$ such that $w_1 \notin W_2$ and $w_2 \notin W_1$. Then $w_1 + w_2 \in W_1 \cup W_2$, as $w_1, w_2 \in W_1 \cup W_2$.
$\Rightarrow$ either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$.
$\Rightarrow$ either $w_2 \in W_1$ or $w_1 \in W_2$.
Which is a contradiction to our assumption.

Question 6.
Find all subspaces of $\mathbb{R}^3$.

Solution 6.
$\{0\}$ and $\mathbb{R}^3$ are trivial subspaces of $\mathbb{R}^3$
Any line passing through origin is one-dimensional subspace of $\mathbb{R}^3$
Any plane passing through origin is two dimensional subspace of $\mathbb{R}^3$
Let W be a non-trivial proper subspace of $\mathbb{R}^3$. $dim(W) = 1$ or 2. Suppose $dim(W) = 1$ and $\{e\}$ be basis of W, $r \in \mathbb{R}^3$ then $W = \{\lambda e : \lambda \in \mathbb{R}\}$ (This is a line passing through origin in the direction e). Suppose $dim(W) = 2$ and $\{e_1, e_2\}$ be a basis of W. The linear span of $\{e_1, e_2\}$ is W and it is a plane passing through origin with normal vector $e_1 \times e_2$.

Question 7.
Let V be a vector space over $\mathbb{C}$ and $\{u_1, u_2, ..., u_n\}$ be a linearly independent set of vectors in V. Prove that $\{u_1, u_2, ..., u_n, iu_1, iu_2, ..., iu_n\}$ is a set of linearly independent vectors when considered V as a vector space over $\mathbb{R}$.

Solution 7.
Let $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}$.
Let $\sum_1^n a_j u_j + \sum_1^n b_j(\iota u_j) = 0$
$\Rightarrow \sum_1^n (a_j + b_j \iota) u_j = 0$
$\Rightarrow a_j + b_j \iota = 0$ for all $j = 1, 2, ...n$,as $\{u_1, ..., u_n\}$ are linearly independent in $\mathbb{C}$.
$\Rightarrow a_j = 0 = b_j$ for all $j = 1, 2, ..., n$.
Hence $\{u_1, ..., u_n, \iota u_1, ..., \iota u_n\}$ is linearly independent.

Question 8.
Discuss the linear dependence/independence of following set of vectors:
(i) $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ in $\mathbb{R}^3$,
(ii) $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (3, 2, 1, 0)\}$ in $\mathbb{R}^4$,
(iii) $\{(1, i, 0), (1, 0, 1), (i + 2, -1, 2)\}$, in $\mathbb{C}^3(\mathbb{C})$ ($\mathbb{C}^3$ considered as a vector space over $\mathbb{C}$),
(iv) $\{(1, i, 0), (1, 0, 1), (i + 2, -1, 2)\}$, in $\mathbb{C}^3(\mathbb{R})$ ($\mathbb{C}^3$ considered as a vector space over $\mathbb{R}$),
(v) $\{u + v, v + w, w + u\}$ in a vector space V given that $\{u, v, w\}$ is linearly independent.

Solution 8.
(i) Linearly independent.
Determinant of matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

is not zero.

(ii) Linearly dependent.
As $(3, 2, 1, 0) = (1, 0, 0, 0) + (1, 1, 0, 0) + (1, 1, 1, 0)$

(iii) Linearly dependent.
Let $a_1, b_1, a_2, b_2, c_1, c_2 \in \mathbb{R}$ such that $(a_1 + ib_1)(1, i, 0) + (a_2 + ib_2)(1, 0, 1) + (c_1 + ic_2)(i + 2, -1, 2) = (0, 0, 0)$
$\Rightarrow (a_1 + ib_1, ia_1 - b_1, 0) + (a_2 + ib_2, 0, a_2 + ib_2) + (ic_1 - c_2 + 2c_1 + 2ic_2, -c_1 - ic_2, 2c_1 + 2ic_2) = (0, 0, 0)$
$\Rightarrow (a_1 + ib_1 + a_2 + ib_2 + ic_1 - c_2 + 2c_1 + 2ic_2, ia_1 - b_1 - c_1 - ic_2, a_2 + ib_2 + 2c_1 + 2ic_2) = (0, 0, 0)$
$\Rightarrow$ We get six equation by comparing real and imaginary parts to 0. By solving these equation we get a non-zero solution. Hence linearly dependent.

(iv) Linearly independent.
Let $a, b, c \in \mathbb{R}$ such that $a(1, i, 0) + b(1, 0, 1) + c(i + 2, -1, 2) = (0, 0, 0)$
$\Rightarrow (a + b + ci + 2c, ai - c, b + 2c) = (0, 0, 0)$
$\Rightarrow a + b + 2c = 0, c = 0, a = 0, c = 0, b + 2c = 0$
$\Rightarrow a, b, c = 0$, Hence linearly independent.

(v) Linearly independent.
Let $a, b, c \in \mathbb{F}$ such that $a(u+v)+b(v+w)+c(w+u)=0$
$\Rightarrow (a+c)u+(b+a)v+(b+c)w=0$
$\Rightarrow a+c=0, a+b=0, b+c=0$, as $\{u, v, w\}$ is L.I.
Solving these three equation we get $a, b, c=0$.