Lecture 3: Invertible Matrices, Determinants and its properties.

Invertible Matrix: Let A be a square matrix of order n. The matrix A is said to be invertible if there exists a matrix B of order n such that

$$AB = I_n = BA$$

Where I_n is identity matrix of order n. We say B is inverse of A and is denoted by A^{-1}.

Examples: (1) Identity Matrix is invertible.

(2) $$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is invertible.

Non-Example

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Proposition (1) Inverse of a matrix is unique.

Proof: Let A be an invertible matrix with two inverses B & C. Then

$$AB = I = BA \ \&$$
$$AC = I = CA.$$

$$B = BI = B(AC) = (BA)C = IC = C$$

(2) $(AB)^{-1} = B^{-1}A^{-1}$, if A, B invertible.

Proof: $(B^{-1}A^{-1})(AB) = I = (AB)(B^{-1}A^{-1})$

Lemma: Elementary matrices are invertible.

Proof: (i) For $c \neq 0$, $E_i(c)\,E_i(1/c) = I$

(ii) Check $E_{ij}\,E_{ij} = I$

(iii) Check $E_{ij}(c)\,E_{ij}(-c) = I$.

Theorem: Every invertible matrix is a product of elementary matrices.

Proof: Let A be an invertible matrix of order n. Then by a previous theorem, there exist elementary matrices $E_1, E_2, \dots E_p, F_1, F_2, \dots, F_q$ such that

$$E_1 E_2 \dots E_p A F_1 F_2 \dots F_q = \begin{pmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r} \end{pmatrix}$$

By previous lemma, E_i's & F_j's are invertible. L.H.S of the above equation is product of invertible matrices, hence L.H.S is invertible. R.H.S is invertible if and only if $r=n$. Thus,

$$A = E_p^{-1} \dots E_2^{-1} E_1^{-1} F_q^{-1} \dots F_2^{-1} F_1^{-1}.$$

By previous lemma, inverse of an elementary matrix is also elementary. Hence, A is product of elementary matrices. $\square$

Theorem: A square matrix is invertible if and only if there exists a sequence of elementary row operations which reduces the matrix to identity matrix.

Proof: Let A be a square matrix which is invertible, then by previous theorem, there exists a finite sequence $E_1, E_2, \ldots, E_r$ of elementary matrices such that

$$A = E_1 E_2 \cdots E_r$$

Then, $$E_r^{-1} E_{r-1}^{-1} \cdots E_2^{-1} E_1^{-1} A = I$$

Inverse of elementary matrix is also elementary. Thus, we have the required result.

Conversly, let $EA = I$ where E is finite product of elementary matrices, which is invertible.

So, $$A = E^{-1} \Rightarrow AE = I$$

Hence, A is invertible. $\square$

Gauss-Jordan Method to find Inverse

Let A be an invertible matrix. Then by previous theorem, there exists a finite sequence of elementary matrices $E_1, E_2, \ldots, E_r$ such that

$E_1 E_2 \cdots E_r A = I$, This implies

$$A^{-1} = E_1 E_2 \cdots E_r I. \quad \square$$

Algorithm: Take a square matrix A & take the augmented matrix $(A \mid I)$. Apply row operations on the augmented matrix to reduce A into RREF. If A is invertible, A will reduce to identity matrix. The effect of row operations on I will give inverse of A.

Example: Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 4 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 4R_1]{R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -3 & 1 & 0 \\ 0 & -6 & -1 & -4 & 0 & 1 \end{array}\right)$$

$$\downarrow R_3 - \tfrac{6}{5} R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & -6 & 5 \end{array}\right) \xleftarrow{5R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -3 & 1 & 0 \\ 0 & 0 & \tfrac{1}{5} & -\tfrac{2}{5} & -\tfrac{6}{5} & 1 \end{array}\right)$$

$$\xrightarrow[R_1 - R_3]{R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 6 & -5 \\ 0 & -5 & 0 & -5 & -5 & 5 \\ 0 & 0 & 1 & -2 & -6 & 5 \end{array}\right) \xrightarrow{-\tfrac{1}{5} R_2}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 6 & -5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 & -6 & 5 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & -3 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 & -6 & 5 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} 1 & 4 & -3 \\ 1 & 1 & -1 \\ -2 & -6 & 5 \end{pmatrix}$