END-SEMESTER EXAMINATION
MTH-204, MTH-204A
ABSTRACT ALGEBRA
Spring 2023
Date: 27th April 2023

Time Allowed: 3 hrs Max. Marks: 40

1. State PRECISELY the following theorems. [7]

a. Sylow's theorems.
b. Fundamental theorem of finite abelian groups.
c. Burnside's lemma.
d. Chinese reminder theorem.
e. Gauss lemma.
f. Eisenstein's Criterion.
g. Jordan Holder's theorem.

Ans: Done in Class

2. Give an example of each of the following. [3]

a. A finite group with elements of order $1, 2, 3, 4, 5$ but has no element of order 6.

Ans: A_6.

b. A polynomial in $\mathbb{Z}[x]$ that is not irreducible in $\mathbb{Z}[x]$ but is irreducible in $\mathbb{Q}[x]$.

Ans: $2x$.

c. A non-commutative ring R for which the only ideals are the trivial ideals.

Ans: $M_n(F)$, the ring of $n \times n$ matrices with entries in F, where F is a field.

3. Prove Burnside's Lemma. [4]

Ans: Done in class.

4. Let G be a finite group, T an automorphism of G with the property that $T(x) = x$ if and only if $x = e$. Suppose further that $T^2 = I$. Prove that G must be abelian. [4]

Ans: Done in class. The idea is to show that $T(x) = x^{-1}$. Then the claim follows directly. To show that $T(x) = x^{-1}$, one needs to show that every element $x \in G$ is of the form $y^{-1}T(y)$ for some $y \in G$.

5. Let G be a simple group of order n. Show that if H is a subgroup of G with $[G : H] = k > 1$, then $k! \geq n$. [5]

Ans: Define an action of G on G/H by left multiplication. This induces a homomorphism $\phi : G \to S_k$ whose kernel is a normal subgroup of G. Since G is simple, either $Ker(\phi) = \{e\}$ or $Ker(\phi) = G$. In the latter case, it would follow that $g(xH) = xH$ for all $xH \in G/H$ and $g \in G$. In particular, $gH = H$ for all $g \in G$. Since $[G : H] > 1$ this cannot be, whence ϕ must be injective. This injectivity implies that $k! \geq n$, as was required.

6. Prove that a group of order 297 is nilpotent. [5]

Ans: $297 = 3^3 \times 11$. The number of sylow 3-subgroups is congruent to 1 mod 3 and divides 11. So $n_3 = 1$.

The number of sylow 11-subgroups is congruent to 1 mod 11 and divides 27. So $n_{11} = 1$.

Let H and K be the Sylow-3 and Sylow 11 subgroups respectively. Then $|H| = 27$ and $|K| = 11$ and both are normal. Again HK is a subgroup of order 297 and $H \cap K = \{e\}$. So $G = H \times K$. Since H and K are nilpotent (a p-group is nilpotent) we get that G is nilpotent.

7. Prove Chinese Reminder Theorem for Rings. [5]

Ans: Done in class.

8. Does the polynomial $f(x) = 2x^{11} - 98x^5 + 28x^2 + 35$ have any roots in $\mathbb{Z}$? Is the ring $\mathbb{Q}[x]/(f(x))$ a field ? Is the ring $\mathbb{C}[x]/(f(x))$ a field ? Justify your answer. (Here $(f(x))$ denotes the ideal generated by $f(x)$). [4]

Ans: $f(x) = 2x^{11} - 98x^5 + 28x^2 + 35 \in \mathbb{Z}[x]$. For $p = 7$, by Eisenstein's criterion $f(x)$ is irreducible in $\mathbb{Q}[x]$. Since $f(x)$ is primitive, by Gauss lemma $f(x)$ is also irreducible in $\mathbb{Z}[x]$. So $f(x)$ has no root in $\mathbb{Z}$.

$f(x)$ is irreducible in $\mathbb{Q}[x]$, and since $\mathbb{Q}[x]$ is a PID, $(f(x))$ is a maximal ideal and hence $\mathbb{Q}[x]/(f(x))$ is a field.

By fundamental theorem of algebra, $f(x)$ is reducible in $\mathbb{C}[x]$. So $(f(x))$ is not a maximal ideal and hence $\mathbb{C}[x]/(f(x))$ is not a field.

9. Let R be a finite commutative ring with identity. Prove that every prime ideal of R is a maximal ideal of R. [3]

Ans: Let I be a prime ideal of the ring R. Then the quotient ring R/I is an integral domain since I is a prime ideal. Since R is finite, R/I is also finite. Since a finite integral domain is a field, R/I is a field, and hence I is a maximal ideal.