Practice Problems 23 : Review of Vectors, Equations of lines and planes, Quadric surfaces

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. Let S denote the plane containing $\mathbf{u}, \mathbf{v}$ and $(0, 0, 0)$.

   (a) Show that every element $\mathbf{p}$ in S can be expressed as $\mathbf{p} = \alpha\mathbf{u} + \beta\mathbf{v}$ for some $\alpha, \beta \in \mathbb{R}$. Further show that $\alpha = \mathbf{p} \cdot \mathbf{u}$ and $\beta = \mathbf{p} \cdot \mathbf{v}$.

   (b) Suppose $\mathbf{w}$ is not in S. Show that there exists an element $\mathbf{q}$ in S such that $\mathbf{w} - \mathbf{q}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

2. Find a parametric equation of the line passing through $(5, 2, 0)$ and that is perpendicular to the plane $4x - 2y + z = 2$.

3. Find a parametric equation of the line of intersection of $x - 2z = 3$ and $y + 2z = 5$.

4. Find an equation of the plane that contains the line $x = 1 + 2t, y = t, z = 3 - t$ and is parallel to the plane $2x + 4y + 8z = 1$.

5. Find an equation of the plane that passes through the point $(6, 0, 0)$ and contains the line $x = 4 - 2t, y = 2 + 3t, z = 3 + 5t$.

6. Evaluate the distance between the lines $\frac{x-2}{4} = \frac{y-7}{-4} = \frac{z+2}{3}$ and $\frac{x+1}{1} = \frac{y+2}{4} = \frac{z-1}{-3}$.

7. Show that the distance between the point $Q = (x_0, y_0, z_0)$ and the plane $(x, y, z) \cdot \mathbf{n} = d$ is $\frac{|n \cdot (x_0, y_0, z_0) - d|}{\|\mathbf{n}\|}$.

8. Consider the plane $x - 2y + 3z = 6$. Show that the plane is expressed parametrically (with parameters s and t) as $X = P_0 + s(X_0 - P_0) + t(Y_0 - P_0)$ where $X = (x, y, z), P_0 = (6, 0, 0), X_0 = (0, -3, 0)$ $Y_0 = (0, 0, 2)$ and $s, t \in \mathbb{R}$.

9. Let $\mathbf{r}$ denote (x, y, z). Suppose that $r \cdot \mathbf{n}_1 = d_1$, $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ and $\mathbf{r} \cdot \mathbf{n}_3 = d_3$ are three distinct planes. Show that their intersection is a line if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{n}_3 = \alpha\mathbf{n}_1 + \beta\mathbf{n}_2$ and $d_3 = \alpha d_1 + \beta d_2$.

10. Find an equation for the surface consisting of all points P such that the distance from P to the x-axis is twice the distance from P to the yz-plane.

11. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$.

12. Find an equation for the cylinder generated by a line through the curve $x^2 + y^2 = 4x, z = 0$ moving parallel to the vector $i + j + k$.

13. Find an equation for the surface generated by revolving the curve $4x^2 + 9y^2 = 36, z = 0$ around the y-axis.

14. (*) Sketch the surfaces by sketching the cross sections cut from the surfaces by the planes $x = 0, y = 0$ and $z = 1$.

    (a) $z = x^2$ (Cylinder)

    (b) $x^2 + y^2 = 4$ (Circular Cylinder)

    (c) $4z = x^2 + y^2$ (Paraboloid)

    (d) $4z^2 = x^2 + y^2$ (Circular cone(s))

    (e) $z = 5 - \sqrt{x^2 + y^2}$ (Circular cone)

15. (*) Sketch the surface $4z = y^2 - x^2$ by sketching the cross sections cut from the surface by the planes $x = 0, y = -1, y = 0$ and $y = 1$.

Practice Problems 23 : Hints/Solutions

1. (a) Let $\mathbf{a}$ be the projection of $\mathbf{p}$ on to the line joining $\mathbf{u}$ and $(0,0,0)$. Write $\mathbf{p} = \mathbf{a} + (\mathbf{p} - \mathbf{a})$. Observe that $\mathbf{a} = (\mathbf{p} \cdot \mathbf{u})\mathbf{u}$ and $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{u} = 0$.

   (b) Note that $(\mathbf{w} \cdot \mathbf{u})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{v}$ lies in S and that $\mathbf{w} - [(\mathbf{w} \cdot \mathbf{u})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{v}]$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

2. The line is parallel to the normal vector of the plane. A parametric equation of the line is $(x, y, z) = (5, 2, 0) + t(4, -2, 1)$.

3. Note that a point (x, y, z) lies on the line $\Leftrightarrow$ $(x, y, z) = (3 + 2z, 5 - 2z, z) = (3, 5, 0) + z(2, -2, 1)$. Therefore a parametric equation of the line is $(x, y, z) = (3, 5, 0) + t(2, -2, 1), t \in \mathbb{R}$. One can also obtain the direction of the line $(2, -2, 1)$ by taking the cross product of the normals of the planes.

4. Note that the line passes through $(1, 0, 3)$ and is parallel to the vector $\mathbf{P} = (2, 1, -1)$. Observe that the normal $\mathbf{n}$ of the required plane is $(2, 4, 8)$ and $\mathbf{n}.\mathbf{P} = 0$. Therefore an equation of the required plane is $(x, y, z) \cdot (2, 4, 8) = (1, 0, 3) \cdot (2, 4, 8)$.

5. Note that $(6, 0, 0)$ does not lie on the line as $t = -1$ gives $(6, -1, -2)$. The line passes through $(4, 2, 3)$ and is parallel to the vector $(-2, 3, 5)$. So a normal vector $\mathbf{n}$ for the plane is $[(6, 0, 0) - (4, 2, 3)] \times (-2, 3, 5)$. An equation of the plane is $(x, y, z) \cdot \mathbf{n} = (6, 0, 0) \cdot \mathbf{n}$.

6. Both the lines are perpendicular to the vector $(4, -4, 3) \times (1, 4, -3) = (0, 15, 20) = 5(0, 3, 4)$. The vector $(3, 9, -3)$ joins the points $(2, 7, -2)$ and $(-1, -2, 1)$ which lie on the first and the second lines respectively. The required distance is $(3, 9, -3) \cdot \frac{1}{5}(0, 3, 4) = 3$.

7. Let P be any point on the plane. Let Q' be the point of intersection of the plane and the line passing through Q and parallel to $\mathbf{n}$. The required distance is obtained by projecting of the vector $\overrightarrow{QP}$ on to $\overrightarrow{QQ'}$. The required distance is equal to $\|\frac{(Q-P)\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\| = \frac{|Q\cdot\mathbf{n} - P\cdot\mathbf{n}|}{\|\mathbf{n}\|}$.

8. Observe that the points P_0, X_0 and Y_0 lie on the plane. If X is any point on the plane then $X - P_0 = s(X_0 - P_0) + t(Y_0 - P_0)$ for some $s, t \in \mathbb{R}$.

9. Observe that the planes intersect in a line if and only if their normal vectors lie on a plane.

10. The distance from P to the x-axis is $\sqrt{y^2 + z^2}$ and distance from P to the yz-plane is $|x|$. An equation of the surface is $y^2 + z^2 = 4x^2$.

11. Let $P = (x_0, y_0, z_0)$ be any point on the surface and $Q = (-1, 0, 0)$. Since the distance between P and Q is equal to the distance from P to the plane $x = 1$, we have $(x_0 + 1)^2 + y_0^2 + z_0^2 = (x_0 - 1)^2$. Therefore an equation of the surface is $y^2 + z^2 = -4x$.

12. The line passing through a point on the curve $(x_0, y_0, 0)$ and parallel to the vector $(1, 1, 1)$ lie on the cylinder. The equation of the line is $\frac{x-x_0}{1} = \frac{y-y_0}{1} = \frac{z}{1}$. Therefore $x_0 = x - z$ and $y_0 = y - z$. Since $(x_0, y_0, 0)$ satisfies the equation $(x - 2)^2 + y^2 = 4$, an equation for the surface is $(x - z - 2)^2 + (y - z)^2 = 4$.

13. Let $P = (x, y, z)$ be a point on the surface. Consider the point $Q = (x_0, y, 0)$ on the curve. Note that the distance from Q to the y-axis and the distance from P to the y-axis are same. Therefore we get $x_0^2 = x^2 + z^2$. An equation of the surface is $4(x^2 + z^2) + 9y^2 = 36$.

14. See Figure 1-5 for (a)-(e).

15. See Figure 6.