MTH 201 2024-2025 (1)

Tutorial 2

13/08/2024

1. Determine which of the following subsets of $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$:

   (a) $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$

   (b) $\{x \in \mathbb{R}^n : x_1 + x_n = 2x_3\}$, where x_i denotes the i-th coordinate of $x \in \mathbb{R}^n$ and $n \geq 4$.

   (c) $W = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1\}$.

   (d) $W = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n \geq 0\}$.

   (e) $W = \{x \in \mathbb{R}^n : x_{i+1} - x_i = 1, i = 1, 2, \ldots, n-1\}$.

   (f) $W = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$.

2. Which of the following subsets of $\mathbb{R}[x]$ are subspaces?

   (a) $\{p(x) \in \mathbb{R}[x] : p(x) = p(1-x)\}$.

   (b) $\{p(x) \in \mathbb{R}[x] : p(x) \geq 1\}$.

   (c) $\{p(x) \in \mathbb{R}[x] : p(x) = x^2 p^{(2)}(x) + 4p(x) = 0\}$, where $p^{(2)}(x)$ denotes the second order derivative of $p(x)$.

3. Prove that $\mathbb{R}$ is not a finite dimensional vector space over $\mathbb{Q}$. What is the dimension of $\mathbb{R}$ over $\mathbb{Q}$? Is it countable?

4. Prove that each $m \times n$ matrix A is row equivalent to one and only one row reduced echelon matrix.

5. Let $V = \{p(x) \in \mathbb{R}[x] : \deg p(x) \leq 2n\}$. Compute the dimension of V_0, where $V_0 := \{p(x) \in V : p(x)$ only has even degree terms and $p(1) + p(-1) = 0\}$.

6. Find three vectors u, v, w of $\mathbb{R}^4$ such that the set $\{u, v, w\}$ is linearly dependent whereas each set $\{u, v\}, \{v, w\}$ and $\{u, w\}$ is linearly independent. Extend each linearly independent set to a basis of $\mathbb{R}^4$.

7. Show that the vectors $(1+i, 2i)$ and $(1, 1+i)$ in $\mathbb{C}^2$ are linearly independent over $\mathbb{R}$. Are they linearly independent over $\mathbb{C}$ as well ?

8. Let V be a vector space over $\mathbb{C}$ of all functions from $\mathbb{R}$ to $\mathbb{C}$. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$ and $f_2(x) = e^{-ix}$.

   (a) Prove that f_1, f_2, f_3 are linearly independent (over $\mathbb{C}$).

(b) Let $g_1 = 1, g_2 = \sin x$ and $g_3 = \cos x$. Find an invertible 3×3 matrix P such that $g_i = \sum_{i=1}^{3} P_{ij} f_i$.

9. Let V be a vector space of 2×2 matrices over a field F. Find a basis A_1, A_2, A_3, A_4 of V such that $A_i^2 = A_i$ for $1 \leq i \leq 4$.