MTH 114: ODE: Assignment-1

1. (**T**)Classify each of the following differential equations as linear, nonlinear and specify the order.

   (i) $y'' + (\cos x)\, y = 0$ (ii) $y'' + x \sin y = 0$ (iii) $y' = \sqrt{1+y}$
   (iv) $y'' + (y')^2 + y = x$ (v) $y'' + xy' = \sin y$ (vi) $(x\sqrt{1+x^2}\, y')' = e^x\, y$.

   Solution:

   A differential equation is linear if the dependant variable and all of its derivatives appear linearly in the equation. The order of a differential equation is the order of the highest derivative included in the equation.

   (i) Linear 2nd order ODE
   (ii) Nonlinear 2nd order ODE
   (iii) Nonlinear 1st order ODE
   (iv) Nonlinear 2nd order ODE
   (v) Nonlinear 2nd order ODE
   (vi) Linear 2nd order ODE

2. Find the differential equation of each of the following families of plane curves. Here $a, b, c \in \mathbb{R}$ denote arbitrary constants:
   (i) $xy^2 - 1 = cy$ (ii) $cy = c^2 x + 5$ (iii)(**T**) $y = ax^2 + be^{2x}$ (iv) $y = ax + b + c$
   (v)(**T**) Circles touching the x-axis with centres on y-axis.
   (vi) $y = a \sin x + b \cos x + b$,
   where a, b and c are arbitrary constants.

   Solution:

   Eliminate the constant(s) to find the differential equations:

   (i) Differentiating $(xy^2 - 1) = cy$ with respect to x, we get $cy' = (y^2 + 2xyy')$. Eliminating c we find $(xy^2 + 1)y' + y^3 = 0$

   (ii) Differentiating $cy = c^2 x + 5$ with respect to x we get, $cy' = c^2$. As $c \neq 0$ (since $c = 0$ can not satisfy the given equation), substituting $c = y'$ in the original equation gives $yy' = xy'^2 + 5$

   (iii) Differentiating $y = ax^2 + be^{2x}$ w.r.t. x gives $y' = 2ax + 2be^{2x}$, which on differentiation again w.r.t. x gives $y'' = 2a + 4be^{2x}$. Eliminating a and b from the three equations gives $x(x-1)y'' - (2x^2 - 1)y' + 2(2x-1)y = 0$

   (iv) Differentiating w.r.t. x gives $y' = a$. Differentiating again w.r.t. x gives $y'' = 0$. Note that the order of the ODE is two since b, c combine to make a single arbitrary constant.

(v) Circles touching the x axis with centre on y axis are given by $x^2+(y-c)^2 = c^2$ which on simplification gives $x^2 + y^2 = 2cy$. Differentiating w.r.t. x we get $x + yy' = cy'$. Eliminating c from the two equations gives $(x^2 - y^2)y' = 2xy$.

(vi) Proceeding as in (iii) above, we get $(1 + \cos x)y'' + \sin x\, y' + y = 0$

3. Verify that the given function in the left is a general solution to the corresponding differential equation in the right.

   (i) $x^3 + y^3 = 3cxy$ $\qquad x(2y^3 - x^3)y' = y(y^3 - 2x^3)$

   (**T**) (ii) $y = ce^{-x} + x^2 - 2x + 4$ $\qquad y' + y = x^2 + 2$

   (iii) $y = cx - c^2$ $\qquad y'^2 - xy' + y = 0$

   Solution:

   Show that the functions on the left satisfy the ODEs on the right.

   (i) Here the function is given implicitly. Differentiating the equation w.r.t. x gives $x^2 + y^2y' = c(xy' + y)$. Eliminating c gives

   $x^3 + y^3 = 3xy\frac{x^2+y^2y'}{xy'+y} \implies x(2y^3 - x^3)y' = y(y^3 - 2x^3)$.

   (ii) Differentiating w.r.t. x gives $y' = -ce^{-x} + 2x - 2 \implies y' + y = x^2 + 2$.

   (iii) Differentiating w.r.t. x gives $y' = c \implies y'^2 - xy' + y = c^2 - cx + cx - c^2 = 0$

4. Solve $\frac{dy}{dx} = y^2 - 2y + 2$ by separating variables.

   Solution:

   We can separate the variables:

   $\int dx = \int \frac{dy}{y^2-2y+2} = \int \frac{dy}{(y-1)^2+1} = \tan^{-1}(y - 1)$.

   So $y - 1 = \tan(x + c)$.

5. (**T**) Verify that $y = \frac{1}{x+c}$ is general solution of $y' = -y^2$. Find particular solutions such that (i) $y(0) = 5$, and (ii) $y(2) = -\frac{1}{5}$. In both the cases, find the largest interval I on which y is defined.

   Solution:

   $y = 1/(x + c) \implies y' = -1/(x + c)^2 \implies y' = -y^2$.

   (i) With $y(0) = 5$, the solution is $y = 5/(1 + 5x)$ and $I = (-1/5, \infty)$.

   (ii) With $y(2) = -1/5$, the solution is $y = 1/(x - 7)$ and $I = (-\infty, 7)$.

   (Note: The largest interval is determined by the fact that the solution must pass through the initial point and the solution must be continuous)

6. Solve the IVP $y\frac{dy}{dx} = e^x,\ y(0) = 1$. Find the largest interval of validity of the solution.

   Solution:

   Separating variable and integrating, we get $y^2 = 2e^x + c$. Using initial condition $c = -1$. Thus solution to the IVP is $y^2 = 2e^x - 1$, or $y = \sqrt{2e^x - 1}$.

Note that other root does not satisfy initial condition. The largest interval of validity is $x > -\ln 2$.

7. For each of the following differential equations, draw several isoclines with appropriate lineal elements. Solve the equations and sketch some solution curves.

   (**T**)(i) $\quad y' = x \qquad (ii) \quad y' = -\frac{x}{y}$.

   (http://mathlets.org/mathlets/isoclines/)

   Solution:

   [Graphs of solutions of a first order equation can be understood in terms of the slope field and isoclines. For $\frac{dy}{dx} = f(x, y)$ an *isocline* is a set of points in the xy-plane where all the solutions have the same slope $\frac{dy}{dx}$; thus, it is a level curve $f(x, y) = k$. The *lineal elements* are line segments of slope k drawn on Isocline where $k \in \mathbb{R}$ is constant. Isoclines and lineal elements are used to get an idea how the solutions curves look without solving the differential equations explicitly.]

   For the given odes, the isoclines are (i) $x = k$ and (ii) $-x/y = k$ and the lineal elements are line segments with slopes k. Note that the lineal elements are drawn with different lengths. You may draw them with equal lengths too. Also, you may not use arrows while drawing some solution curves.

   (a) $y' = x$ $\qquad$ (b) $y' = -\frac{x}{y}$

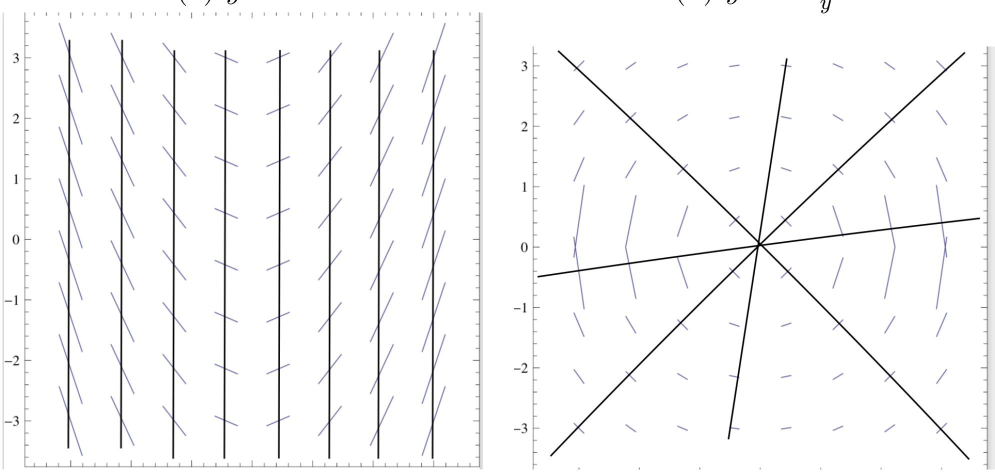


   Figure 1: Isoclines and Lineal elements

   Solution for first equation is $y = \frac{x^2}{2} + c$ and for the second one $x^2 + y^2 = c$.

8. Find the orthogonal trajectories of the following families of curves:

   (**T**) (i) $e^x \sin y = c \qquad$ (ii) $y^2 = cx^3$

   Solution:

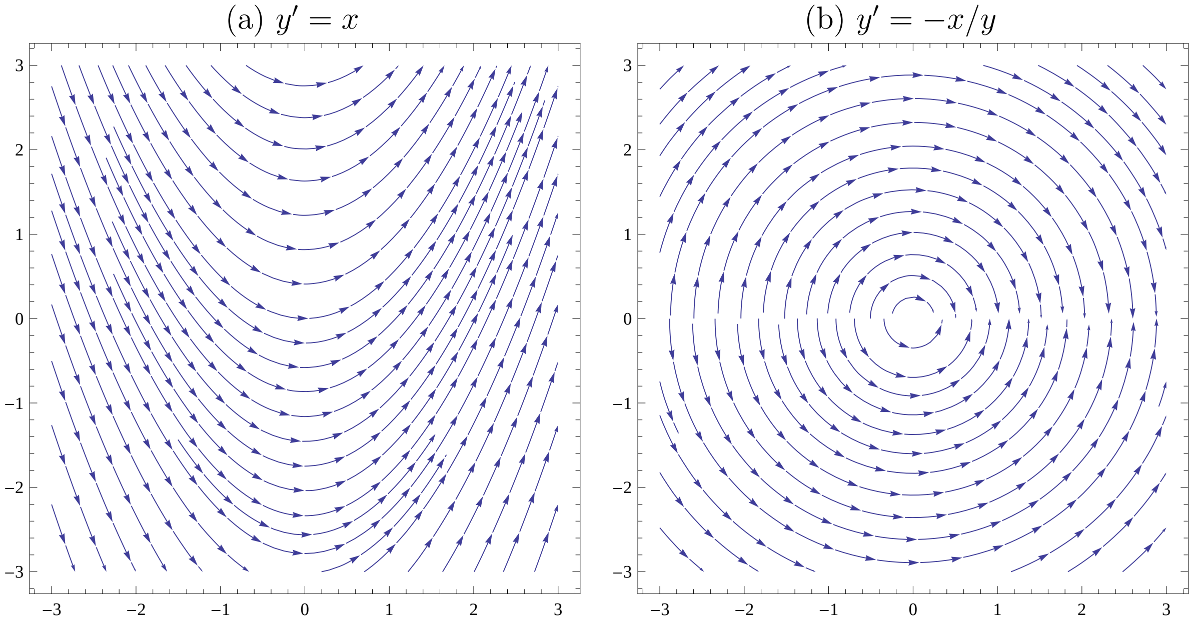


Figure 2: Some solution curves

(i) Differential equation for the family of curves $e^x \sin y = c$ is $\cos y y' + \sin y = 0$. The ODE of the orthogonal trajectories is obtained by replacing y' by $-\frac{1}{y'}$. So the ODE of the orthogonal trajectory is give by $-\frac{\cos y}{y'} + \sin y = 0$ i.e. $\cos y - (\sin y)y' = 0$. Solving this we get $e^x \cos y = C$.

(ii) Differentiating $y^2 = cx^3$ w.r.t. x we get $2yy' = 3cx^2$. Eliminating c, we obtain differential equation for the family of curves is $2xy' = 3y$. The ODE of the orthogonal trajectories is obtained by replacing y' by $-\frac{1}{y'}$. So the ODE of the orthogonal trajectory is give by $-\frac{2x}{y'} = 3y$ i.e. $2x\,dx + 3y\,dy = 0$. Solving this we get orthogonal trajectories are $x^2 + \frac{3y^2}{2} = C$.

9. Find the family of oblique trajectories which intersect the family of straight lines $y = cx$ at an angle of 45^o.

Solution:

Let ψ and ϕ are angles made by slopes at (x, y). If θ is the angle of intersection, then $\psi - \phi = \pm\theta$.

From $y = cx$ we find $y' = c = y/x$. Here $\tan\psi = y'$ and $\tan\phi = y/x$. From

$$\tan(\pm\theta) = \tan(\psi - \phi) = \frac{\tan\psi - \tan\phi}{1 + \tan\psi\tan\phi}$$

Thus $y' - y/x = \pm(1 + yy'/x)$. With plus and minus signs we find

$$y' = \frac{x+y}{x-y} \quad \text{and} \quad y' = \frac{y-x}{y+x}$$

With substitutions $y = vx$, these reduce to

$$\frac{dv}{1+v^2} - \frac{v\,dv}{1+v^2} - \frac{dx}{x} = 0 \quad \text{and} \quad \frac{dv}{1+v^2} + \frac{v\,dv}{1+v^2} + \frac{dx}{x} = 0$$

Solving we get oblique trajectories as

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln(x^2+y^2) = C \quad \text{and} \quad \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln(x^2+y^2) = C.$$

(The solutions are implicit solution.)

10. Show that the following families of curves are self-orthogonal:
(**T**) (i) $y^2 = 4c(x+c)$ (ii) $\frac{x^2}{c^2} + \frac{y^2}{(c^2-1)} = 1$

Draw the families.

Solution:

A family of curves is said to be self orthogonal if two curves of the family intersect, they intersect orthogonally. Thus the differential equations governing the curves remains unchanged if we replace dy/dx by $-dx/dy$.

(i) Differentiating w.r.t. x we find $c = yy'/2$. Eliminating c, the ODE governing the families of curves is $(yy')^2 + 2xyy' - y^2 = 0$. Replace y' by $-1/y'$ for the ODE governing orthogonal trajectories: $y^2/y'^2 - 2xy/y' - y^2 = 0$ which on simplification gives $(yy')^2 + 2xyy' - y^2 = 0$ (same as before).

(ii) Differentiating w.r.t. x we get $c^2 = x/(x+yy')$. Eliminating c, the ODE governing the family is $(x+yy')(xy'-y) = y'$. Replacing y' by $-1/y'$, we get orthogonal trajectories as $(x - y/y')(-x/y' - y) = -1/y'$ which on simplification gives the same ODE. Hence self orthogonal.

Geometrically:

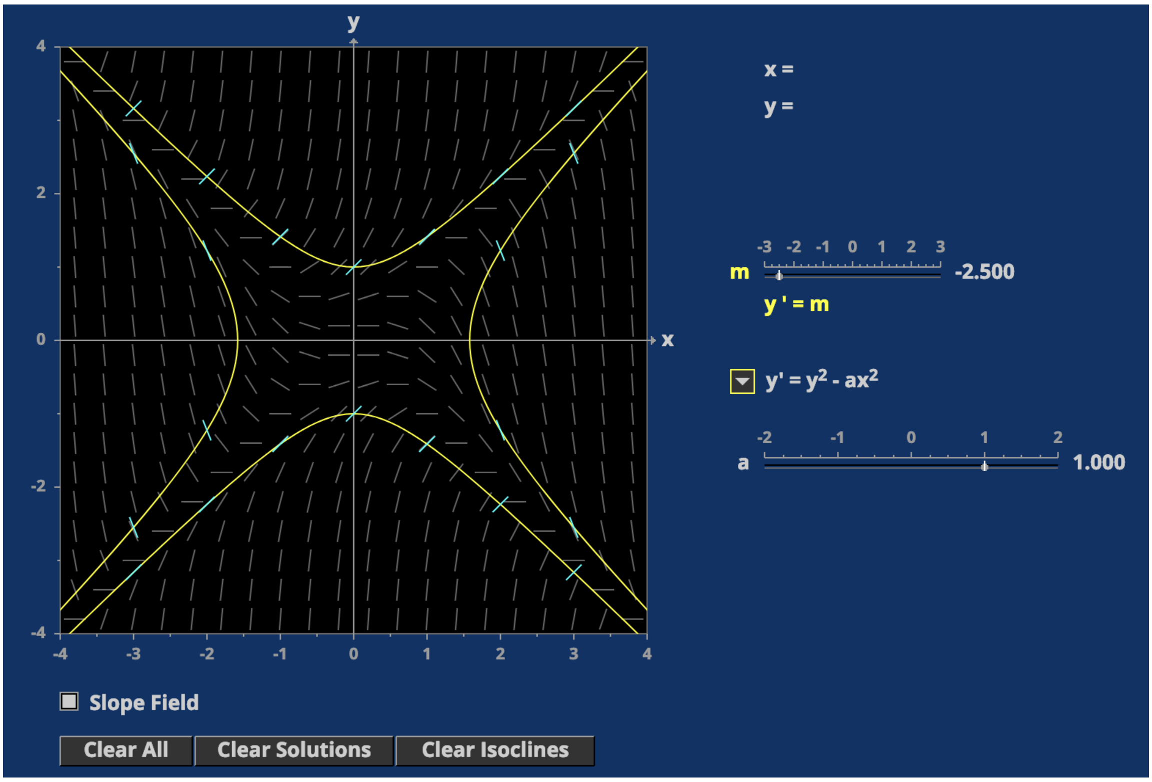
$y^2 = 4c(x+c)$: For $c > 0$ and $c < 0$ gives parabola in "opposite side" of the y-axis. They intersect orthogonally.

$x^2/c^2 + y^2/(c^2-1) = 1$: For $|c| > 1$ we get ellipse and for $|c| < 1$ we get hyperbolas. They intersect orthogonally.

11. Draw isoclines, lineal element (slope field) and use them to draw some solution curve of the equation $y' = y^2 - x^2$.

Solution:

Yellow lines are isoclines and other lines are lineal elements or slope field.



Blue lines are some of solution curves.