Name ____________________

Roll Number _______________

Quiz 2

MTH302: Set Theory and Mathematical Logic

(Odd Semester 2024/25, IIT Kanpur)

Question 1. [3 × 1 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) If $T \vdash (\phi \vee \psi)$, then either $T \vdash \phi$ or $T \vdash \psi$.

- **False**. Consider $T = \emptyset$, $\phi \equiv \exists_{\geq 2}$ and $\psi \equiv \neg\phi$.

(ii) $(\mathbb{Z}, +)$ is an elementary submodel of $(\mathbb{Q}, +)$.

- **False**. Let $\psi \equiv (\forall x)(\exists y)(y + y = x)$. Then $(\mathbb{Q}, +) \models \psi$ and $(\mathbb{Z}, +) \models \neg\psi$.

(iii) If $A \subseteq \omega$ and $\omega \setminus A$ are both c.e., then A is computable.

- **True**. See Slide 158.

Question 2. [7 Points]

(a) [**2 Points**] Let $\mathcal{L}$ be a first order language and T be an $\mathcal{L}$-theory.

(i) Define what it means for T to be **consistent**.

Solution: There is no $\mathcal{L}$-sentence ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$. $\square$

(ii) Define what it means for T to be **complete**.

Solution: For every $\mathcal{L}$-sentence ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$. $\square$

(b) [**2 Points**] Let $\mathcal{L}$ be the empty language. Show that $T = \{\exists_{\geq n} : n \geq 2\}$ is a complete $\mathcal{L}$-theory.

Solution: Note that $M \models T$ iff $|M| \geq \omega$. So T is consistent and it does not have a finite model. Also T is κ-categorical for every infinite cardinal κ. By the theorem on Slide 141, T is complete. $\square$

(c) [**3 Points**] Describe all elementary submodels of $(\mathbb{Q}, +)$.

Solution 1: Suppose $M \subseteq \mathbb{Q}$ and $(M, +)$ is an elementary submodel of $(\mathbb{Q}, +)$. First note that $0 \in M$ since $(\exists x)(x + x = x)$ is true in $(\mathbb{Q}, +)$. Since $(\mathbb{Q}, 0, +) \models$TFDAG, by elementarity, $(M, 0, +) \models$TFDAG. As models of TFDAG are $\mathbb{Q}$-linear vector spaces, M must be closed under multiplication by rationals. Also, M is infinite because $\mathbb{Q} \models \exists_{\geq n}$ and therefore $(M, +) \models \exists_{\geq n}$ for every $n \geq 2$. Pick any $a \neq 0$ in M. Then the product $r \cdot a \in M$ for every $r \in \mathbb{Q}$. It follows that $\mathbb{Q} \subseteq M$. So $M = \mathbb{Q}$. Hence the only elementary submodel of $(\mathbb{Q}, +)$ is $(\mathbb{Q}, +)$. $\square$