Name: ____________________

Roll Number: ______________

Practice Midsem

MTH302 - Set Theory and Mathematical Logic

(Odd Semester 2024/25, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.
2. This exam contains **4 + 1** questions and is worth **30%** of your grade.
3. Answer **ALL** questions.

Question 1. [5 × 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) For every countable infinite limit ordinal α, there is an ordinal β such that $\alpha = \beta + \omega$.

(ii) There is a subset of plane that intersects every circle at exactly 2 points.

(iii) Every chain in $(\mathcal{P}(\omega), \subseteq)$ is countable.

(iv) There is an injective function $f : \mathbb{R}^3 \to \mathbb{R}$ such that for every $\vec{x}, \vec{y} \in \mathbb{R}^3$, $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$.

(v) $(((\neg p) \vee q) \implies (p \implies q))$ is a tautology.

Solution.

(i) False. Consider $\alpha = \omega \cdot \omega$.

(ii) False. See homework.

(iii) False. See homework.

(iv) True. Both are isomorphic as vector spaces over $\mathbb{Q}$.

(v) True.

Question 2. [7 Points]

(a) [**1 Point**] State Schrőder-Bernstein theorem.

(b) [**2 Points**] Show that there is a bijection from $[0,1]$ to $\mathbb{R}$.

(a) [**4 Points**] Let $\mathcal{B}$ be the set of all bijections from $\mathbb{Q}$ to $\mathbb{Q}$. Show that $|\mathcal{B}| = \mathfrak{c}$.

Solution.

(a) If there are injective functions from A to B and from B to A, then there is a bijection from A to B. $\square$

(b) By Schrőder-Bernstein theorem, it suffices to show that there are injections from $[0,1]$ to $\mathbb{R}$ and from $\mathbb{R}$ to $[0,1]$. The identity function on $[0,1]$ is clearly an injection from $[0,1]$ to $\mathbb{R}$. Next check that $f(x) = (\pi/2 + \arctan x)/\pi$ is a bijection from $\mathbb{R}$ to $(0,1)$ and hence also an injection from $\mathbb{R}$ to $[0,1]$. $\square$

(a) Let $\mathcal{F}$ be the set of all bijection from ω to ω. We first claim that $|\mathcal{F}| = |\mathcal{B}|$. Since $|\mathbb{Q}| = \omega$, we can fix a bijection $h : \omega \to \mathbb{Q}$. Now check that $H : \mathcal{B} \to \mathcal{F}$ defined by $H(f)(x) = f(h(x))$ for every $f \in \mathcal{F}$ and $x \in \omega$ is well-defined (i.e. $H(f) \in \mathcal{F}$) and a bijection. So $|\mathcal{F}| = |\mathcal{B}|$.

Since $\mathcal{F} \subseteq \omega^\omega \subseteq \mathcal{P}(\omega \times \omega)$, we get $|\mathcal{F}| \leq |\mathcal{P}(\omega \times \omega)| = |\mathcal{P}(\omega)| = \mathfrak{c}$.

So it suffices to show $|\mathcal{F}| \geq \mathfrak{c}$. As $|\mathcal{P}(\omega)| = \mathfrak{c}$, it is enough to find an injection $G : \mathcal{P}(\omega) \to \mathcal{F}$. Define $H(A) = f_A$ where $f_A : \omega \to \omega$ is defined by: (i) If $k \in A$, then $f_A(2k) = 2k+1$ and $f_A(2k+1) = 2k$. (ii) If $k \notin A$, then $f_A(2k) = 2k$ and $f_{2k+1} = 2k+1$. Now check that f_A is a bijection from ω to ω and $A \neq B \implies f_A \neq f_B$. Therefore, G is an injection and we are done. $\square$

Question 3. [6 Points]

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$,

$$f(x + y) = f(x) + f(y) + f(x)f(y).$$

(a) [**2 Points**] Define $h = 1 + f$. Show that $h(x + y) = h(x)h(y)$.

(b) [**4 Points**] Suppose f is continuous and not identically equal to -1. Show that $f(x) = a^x - 1$ for some $a > 0$.

Solution.

(a) $h(x + y) = 1 + f(x) + f(y) + f(x)f(y) = (1 + f(x))(1 + f(y)) = h(x)h(y)$. $\square$

(b) By Homework problem 15, either h is identically zero or $h(x) = a^x$ where $a = h(1) > 0$. In the former case, f is identically -1 and in the latter case, $f(x) = a^x - 1$ where $a = 1 + f(1) > 0$. $\square$

Question 4. [7 Points]

(a) [**5 Point**] Using transfinite recursion, show that there exists $X \subseteq \mathbb{R}^2$ such that for every line $\ell \subseteq \mathbb{R}^2$, $|X \cap \ell| = 5$.

(b) [**2 Points**] Let $\mathcal{L} = \{\prec\}$ where $\prec$ is a binary relation symbol. Show that there is an $\mathcal{L}$-theory T such that for every $\mathcal{L}$-structure $\mathcal{M} = (M, \prec^{\mathcal{M}})$,

$$\mathcal{M} \models T \text{ iff "}(M, \prec^{\mathcal{M}}) \text{ is a linear ordering".}$$

Solution.

(a) Let $\langle \ell_\alpha : \alpha < \mathfrak{c} \rangle$ enumerate all lines in $\mathbb{R}^2$. Using transfinite recursion, we are going to construct $\langle X_\alpha : \alpha < \mathfrak{c} \rangle$ such that the following hold for every $\alpha < \mathfrak{c}$.

(1) $X_\alpha \subseteq \mathbb{R}^2$ and $X_0 = \emptyset$.
(2) For every $\beta < \alpha$, $X_\beta \subseteq X_\alpha$.
(3) If α is a limit ordinal, then $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$.
(4) $|X_\alpha| \leq \max(\omega, |\alpha|) < \mathfrak{c}$.
(5) $|X_{\alpha+1} \cap \ell_\alpha| = 5$.
(6) No 6 points in X_α are collinear.

Suppose $\langle X_\beta : \beta < \alpha \rangle$ has been defined. We define X_α as follows.

Case (i) $\alpha = 0$. Define $X_0 = \emptyset$.

Case (ii) α is a limit ordinal. Define $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$.

Case (iii) α is a successor ordinal Say $\alpha = \beta + 1$. Let $\mathcal{T}$ be the set of all lines that pass through 2 points in X_β. Let B be the set of points of intersection of ℓ_β with all the lines in $\mathcal{T} \setminus \{\ell_\beta\}$. Since two distinct lines can intersect at at most 1 point, we must have $|B| \leq \max(\omega, |\mathcal{T}|) < \mathfrak{c}$. Put $n = |X_\beta \cap \ell_\beta|$. Then $n \leq 5$ (by clause (6)). If $n = 5$, define $X_\alpha = X_\beta$. Otherwise, choose $5 - n$ distinct points from $\ell_\beta \setminus B$ and add them to X_β to get X_α.

It is easy to check that clauses (1)-(10) are satisfied. The main point is that Clause (6) continues to hold by out choices in Case (iii). Define $X = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$. Let us check that X is as required. Let $\ell \subseteq \mathbb{R}^2$ be any line,. Fix $\alpha < \mathfrak{c}$ such that $\ell_\alpha = \ell$. Then by Clause (5), $|X_{\alpha+1} \cap \ell| = 5$. Since $X_{\alpha+1} \subseteq X$, we get $|X \cap \ell| \geq 5$.

To get the other inequality, towards a contradiction, assume $|X \cap \ell| \geq 6$ and fix $\{p_i : 1 \leq i \leq 6\} \subseteq X \cap \ell$. Choose $\alpha_1, \cdots, \alpha_6 < \mathfrak{c}$ such that $p_i \in X_{\alpha_i}$ for each $1 \leq i \leq 6$. Let $\alpha = \max(\alpha_1, \cdots, \alpha_6)$. Then by Clause (2), $|X_\alpha \cap \ell| \geq 6$. But this violates Clause (6). Hence $|X \cap \ell| = 5$. □

(b) Take $T = \{\phi_1, \phi_2, \phi_3\}$ where

$$\phi_1 \equiv (\forall x)(\neg(x \prec x))$$

$$\phi_2 \equiv (\forall x)(\forall y)(\forall z)(((x \prec y) \wedge (y \prec z)) \implies (z \prec z))$$

$$\phi_3 \equiv (\forall x)(\forall y)((x = y) \vee (x \prec y) \vee (y \prec x))$$

□

Bonus Question [5 Points]

Let $\mathcal{F}$ be an uncountable family of finite subsets of $\mathbb{R}$. Show that there exists an uncountable $\mathcal{A} \subseteq \mathcal{F}$ and a finite set $W \subseteq \mathbb{R}$ such that for every $X, Y \in \mathcal{A}$,

$$X \neq Y \implies X \cap Y = W.$$

Solution

For each $1 \leq n < \omega$, define $\mathcal{F}_n = \{X \in \mathcal{F} : |X| = n\}$. Then $\mathcal{F} = \bigcup\{\mathcal{F}_n : 1 \leq n < \omega\}$. Since the union of a countable family of countable sets is countable and $\mathcal{F}$ is uncountable, we can fix some n such that $\mathcal{F}_n$ is uncountable.

By induction on n, we will show that for every uncountable family $\mathcal{G}$ of sets each of size n, there exist an uncountable $\mathcal{A} \subseteq \mathcal{G}$ and a finite set $W \subseteq \mathbb{R}$ such that for every $X, Y \in \mathcal{A}$, $(X \neq Y \implies X \cap Y = W)$. This clearly suffices.

If $n = 1$, then $\mathcal{A} = \mathcal{G}$ and $W = \emptyset$ are as required.

Next assume that $\mathcal{G}$ is an uncountable family of sets each of size $n + 1$. For each $A \in \mathcal{G}$, fix some $x_A \in A$ (using AC). Define $\mathcal{G}' = \{A \setminus \{x_A\} : A \in \mathcal{G}\}$. Since each set in $\mathcal{G}'$ has size n, by the inductive hypothesis, we can find an uncountable $\mathcal{A}' \subseteq \mathcal{G}'$ and $W' \subseteq \mathbb{R}$ such that for every $X \neq Y$ in $\mathcal{A}'$, $X \cap Y = W'$. We now have the following two cases.

Case (1). $\{x_A : A \in \mathcal{A}'\}$ is uncountable. Choose, using Zorn's lemma, a maximal subfamily $\mathcal{A} \subseteq \mathcal{A}'$ such that for every $A, B \in \mathcal{A}$, $(A \neq B \implies x_A \neq x_B)$. Observe that $\mathcal{A}$ must be uncountable since $\{x_A : A \in \mathcal{A}'\}$ is uncountable. It is clear that $\mathcal{A}$, $W = W'$ are as required.

Case (2). $\{x_A : A \in \mathcal{A}'\}$ is countable. Since $\mathcal{A}'$ is uncountable, we can fix some x and an uncountable $\mathcal{A} \subseteq \mathcal{A}'$ such that for every $A \in \mathcal{A}$, $x_A = x$. Then $\mathcal{A}$ and $W = W' \cup \{x\}$ are as required. □

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End of Exam