Definition.
The line integral of a vector valued function $f : C \to \mathbb{R}^3$ is denoted by $\int_C f . dR$ and defined as

$$\int_C f . dR = \int_a^b \left[f(R(t)) \cdot R'(t) \right] dt ,$$ provided the R.H.S integral exists.

Notation:

- $f = (f_1, f_2, f_3) : C \to \mathbb{R}^3$
- $R(t) = (x(t), y(t), z(t))$ for $t \in [a, b]$
- $R'(t) = (x'(t), y'(t), z'(t))$ for $t \in [a, b]$

We have

$$\int_C f \cdot dR = \int_a^b f_1(R(t))\, x'(t)\, dt + \int_a^b f_2(R(t))\, y'(t)\, dt + \int_a^b f_3(R(t))\, z'(t)\, dt.$$

or $$\int_C f \cdot dR = \int_C f_1(x, y, z)\, dx + \int_C f_2(x, y, z)\, dy + \int_C f_3(x, y, z)\, dz.$$

Example: Let $f(x,y) = (-y, x) = -y\vec{i} + x\vec{j}$ and

$$C = \{ (t, t^2) \,/\, t \in [0,1] \}.$$

Then $\int_C f \cdot dR$

$$= \int_C (-y\,dx + x\,dy)$$

$$= \int_0^1 -t^2\,dt + \int_0^1 t(2t)\,dt$$

$$= -\int_0^1 t^2\,dt + 2\int_0^1 t^2\,dt$$

$$= \int_0^1 t^2\,dt$$

$$= \frac{1}{3}$$

Example:

- C - the line joining origin to $(1,2,4)$

$$\left[\text{ i.e., } \quad R(t) = (x(t), y(t), z(t)) = (t, 2t, 4t). \right]$$

- $f(x,y,z) = (x^2, y, xz - y)$

Evaluate $\int_C f \cdot dR$;

$$\int_C f \cdot dR$$

$$= \int_0^1 t^2\,dt + \int_0^1 2t\,(2)\,dt + \int_0^1 (4t^2 - 2t)\,4\,dt$$

$$= \int_0^1 (17t^2 - 4t)\,dt$$

$$= \frac{11}{3}$$

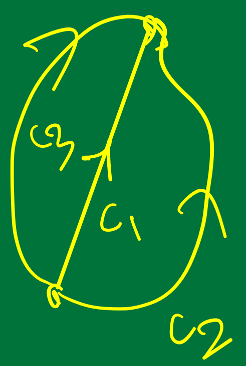
Recall: Let $f : [a, b] \longrightarrow \mathbb{R}$ with f' is continuous on $[a, b]$. By Second Fundamental Theorem of Calculus,

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Theorem. • Let $S \subseteq \mathbb{R}^3$, $f : S \to \mathbb{R}$ be differentiable function on S and $\nabla f : S \to \mathbb{R}^3$ be continuous.

• Let A, B be two points in S and $C = \{R(t) / t \in [a, b]\}$ be a curve joining A and B in S.

• Suppose $R'(t) : [a, b] \to \mathbb{R}^3$ is a continuous function.



Then $\displaystyle\int_C \nabla f \cdot dR = f(B) - f(A)$.

Let $g(t) = f(R(t))$. for $t \in [a, b]$.

By chain rule, $g'(t) = \nabla f(R(t)) \cdot R'(t)$

$$\Rightarrow \int_a^b g'(t)\,dt = \int_a^b \nabla f(R(t)) \cdot R'(t)\,dt$$

or, $$\int_a^b \nabla f \cdot dR = g(b) - g(a)$$

$$= f(B) - f(A)$$

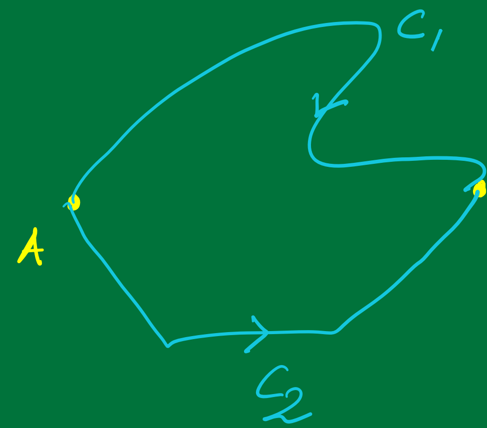
$\square$

We have the line integral

$$\int_C \nabla f \cdot dR = f(B) - f(A).$$

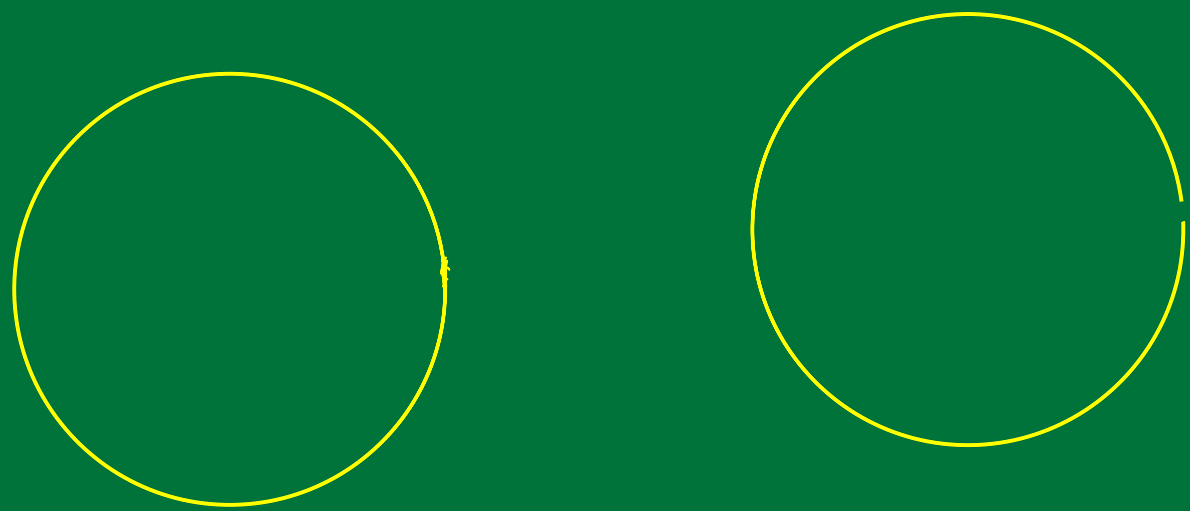
Remark. Let C_1 and C_2 be two curves joining the points A and B in $\mathbb{R}^3$. We have the value $f(B) - f(A)$ for both line integrals:

$$\int_{C_1} \nabla f \cdot dR = f(B) - f(A) = \int_{C_2} \nabla f \cdot dR$$

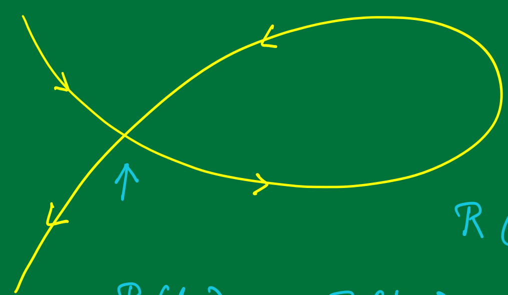


☒ We can express a double integral of some function over a plane region R to line integral for some function along the boundary of the region R. — Green's Theorem.

- $R(t) = (\cos t, \sin t)$ for $t \in (0, 2\pi)$

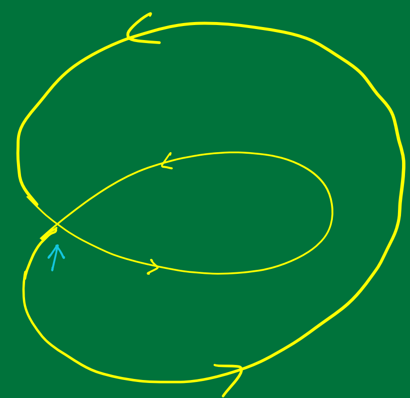


- $R(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$

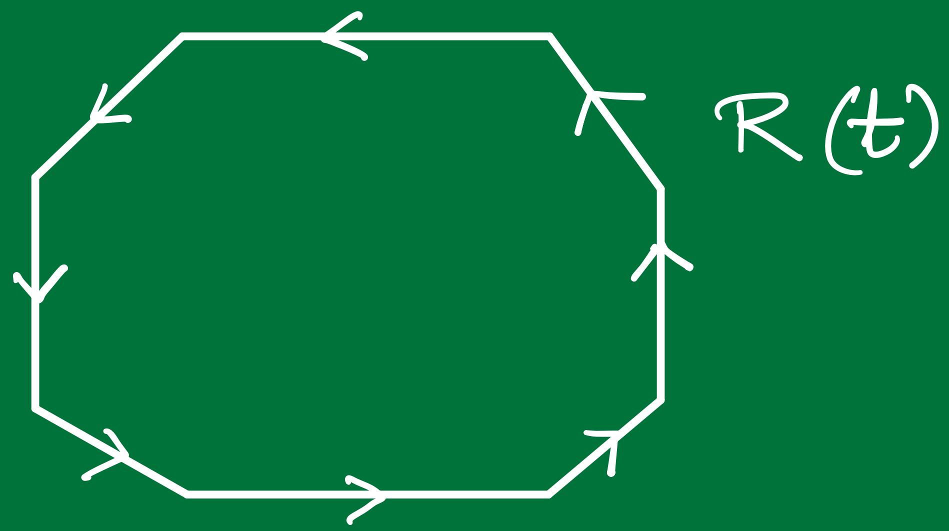


$R(t) = (x(t), y(t))$

$R(t_1) = R(t_2)$




R(t)


R(t)

Let $C = \{ R(t) / t \in [a,b] \}$ be a parametric curve defined by

$$R : [a,b] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto (x(t), y(t), z(t))$$

<u>Closed curve</u>: The curve C is said to be a closed curve if $R(a) = R(b)$.

<u>Simple curve</u>: The curve C is said to be simple curve if $R(t_1) \neq R(t_2)$ for $t_1, t_2 \in [a,b)$ and $t_1 \neq t_2$.

Piecewise smooth curve :

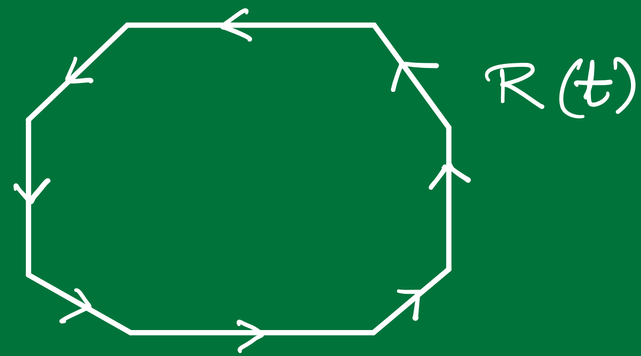
Consider

$R' : [a,b] \to \mathbb{R}^3$

$t \mapsto (x'(t), y'(t), z'(t))$.

The curve C is called piecewise smooth if the interval $[a, b]$ is divided into finite number of subintervals on each of which R' is continuous.

(on each of which R is smooth)

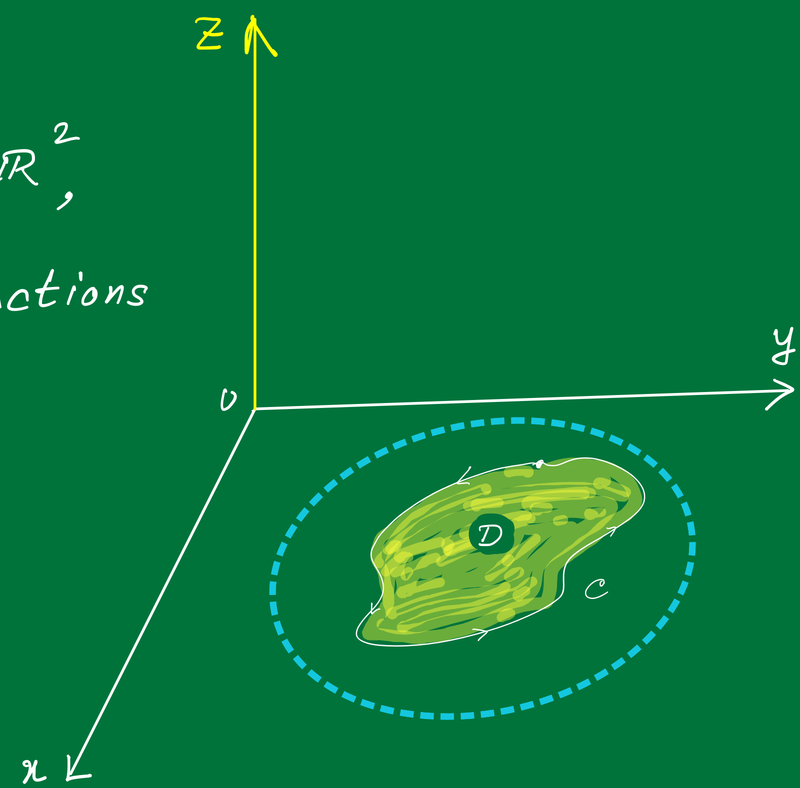


Green's Theorem:

- Let C be a piecewise smooth, simple closed curve (described by $R : [a, b] \to \mathbb{R}^3$) in the xy plane and let D be the closed region enclosed by C.

- Let $(x, y) \xmapsto{f} (M(x,y), N(x,y)) \in \mathbb{R}^2$,

  $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are continuous functions in an open set containing D.



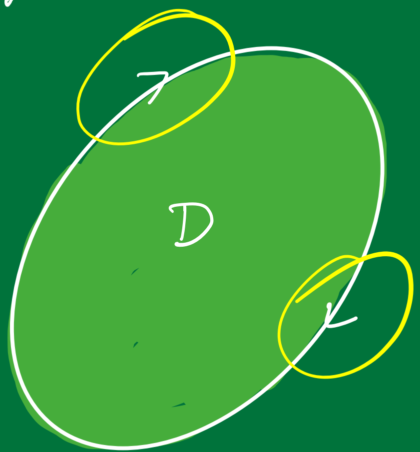
Then $\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$

$= \int_C f \cdot dR$, where the line integral is taken in the counter clockwise direction.

Application. Area expressed as a line integral

(I)

As in the Green's theorem, let C be a piecewise smooth simple closed curve and D be a region bounded by the curve C.



Now, the area of the region D is

$$= \iint_D dx\,dy$$

Consider $M(x,y) = -\frac{y}{2}$ and $N(x,y) = \frac{x}{2}$

Then $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$ and

$$\iint_D dx\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$= \oint_C \left(-\frac{y}{2}, \frac{x}{2}\right) dR$$

$$= \oint_C \left(-\frac{y}{2}\,dx + \frac{x}{2}\,dy\right)$$

$$= \frac{1}{2} \oint_C (-y\,dx + x\,dy)$$

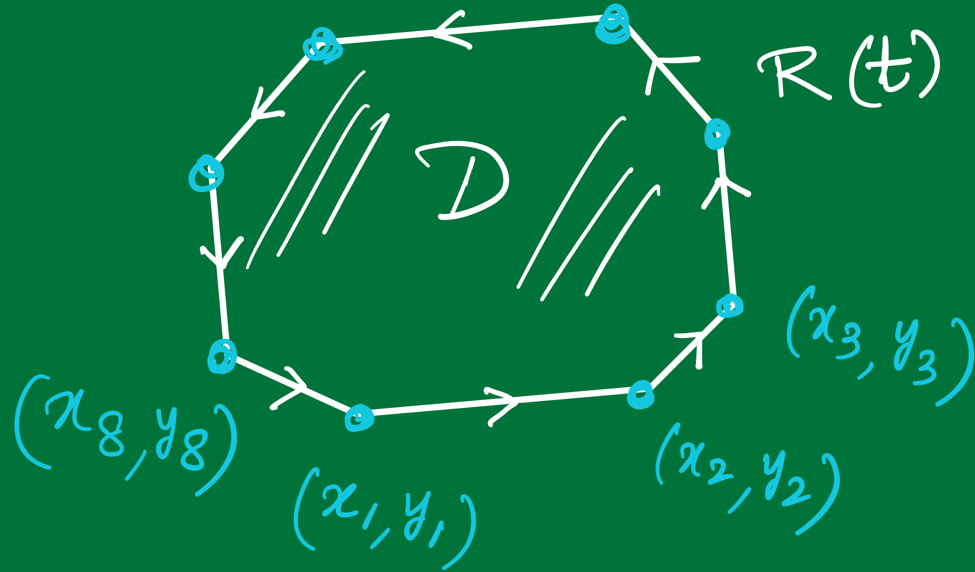
$f(x,y)$

$= (M(x,y), N(x,y))$

$= \left(-\frac{y}{2}, \frac{x}{2}\right)$

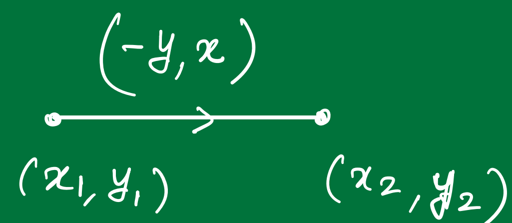
$$\oint_C f \cdot dR = \oint \left(-\frac{y}{2}, \frac{x}{2}\right) \cdot dR$$

(II). Area bounded by a polygon with vertices $(x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)$ ordered in the anticlockwise direction.



$$\text{Area}(D) = \frac{1}{2} \oint_C -y\,dx + x\,dy$$

$$= \frac{1}{2} \int_{C_1} -y\,dx + x\,dy + \frac{1}{2} \int_{C_2} -y\,dx + x\,dy \;\vdots\; + \frac{1}{2} \int_{C_n} -y\,dx + x\,dy$$



$$R_1(t) = \left((1-t)x_1 + t x_2, \; (1-t) y_1 + t y_2 \right)$$

for $t \in [0, 1]$

$$\int_{C_1} f\cdot dR = \int_0^1 f(R_1(t))\cdot R_1'(t)\,dt$$

$$= \int_0^1 \left(-(1-t)y_1 - t y_2,\ (1-t)x_1 + t x_2\right)\cdot\left(x_2 - x_1,\ y_2 - y_1\right)dt$$

$$= (x_2-x_1)\int_0^1 \left[-(1-t)y_1 - t y_2\right]dt + (y_2-y_1)\int_0^1 \left[(1-t)x_1 + t x_2\right]dt$$

$$= (x_2-x_1)\left[-y_1 + \frac{1}{2}y_1 - \frac{1}{2}y_2\right] + (y_2-y_1)\left[x_1 - \frac{1}{2}x_1 + \frac{1}{2}x_2\right]$$

$$= \frac{1}{2}(x_1-x_2)(y_2+y_1) + \frac{1}{2}(y_2-y_1)(x_1+x_2)$$

$$= \frac{1}{2}\left[x_1y_2 + \cancel{x_1y_1} - \cancel{x_2y_2} - x_2y_1 + x_1y_2 - \cancel{x_1y_1} + \cancel{x_2y_2} - x_2y_1\right]$$

$$= \left[x_1y_2 - x_2y_1\right]$$

Thus,

Area (D)

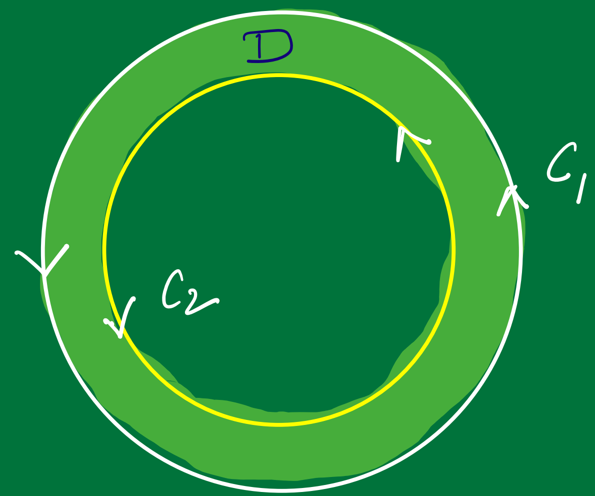
$$= \frac{1}{2} \oint_C -y\,dx + x\,dy$$

$$= \frac{1}{2} \Big[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_n y_1 - x_1 y_n) \Big]$$

Green's theorem for annular region

Consider

$$f(x,y) = \big(M(x,y), N(x,y)\big).$$



Now

$$\iint_D (N_x - M_y)\, dx\, dy$$

$$= \iint_{D_1 \cup D_2} (N_x - M_y)\, dx\, dy$$

$$= \oint_{C_1} M\,dx + N\,dy - \oint_{C_2} M\,dx + N\,dy$$

$f_1\, dx + f_2\, dy$

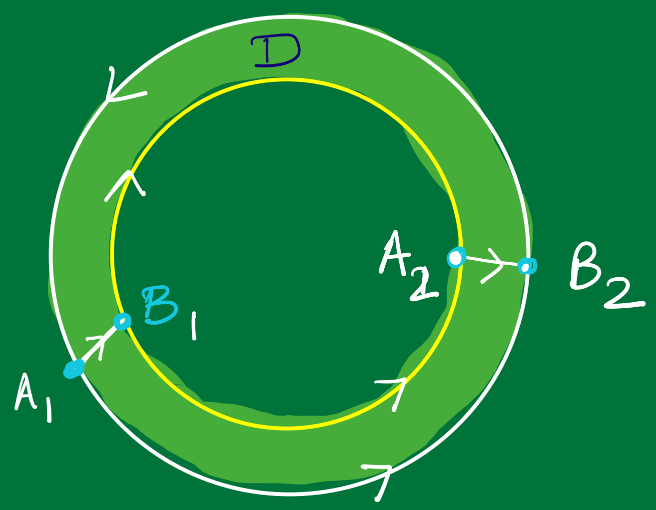
Recall:

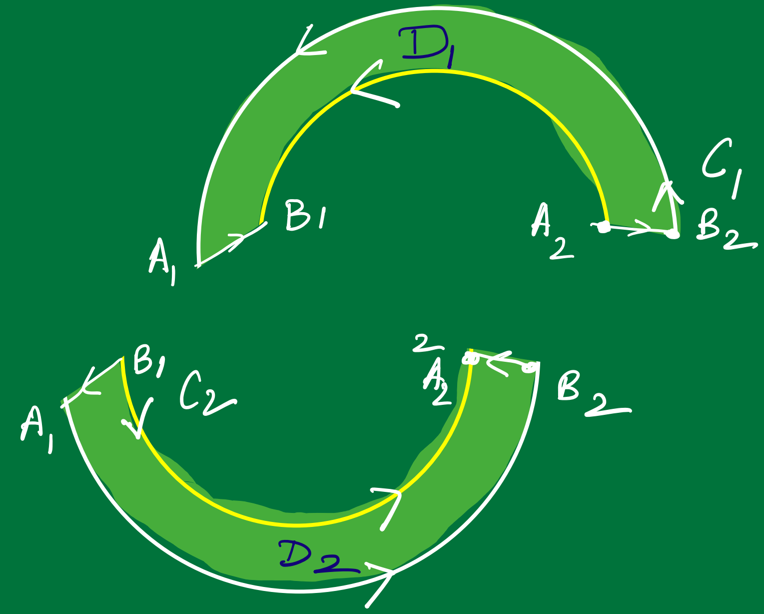
- Let $(x,y) \xmapsto{f} \big(M(x,y), N(x,y)\big) \in \mathbb{R}^2$,

  $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are continuous functions in an open set containing D.

  Then $\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

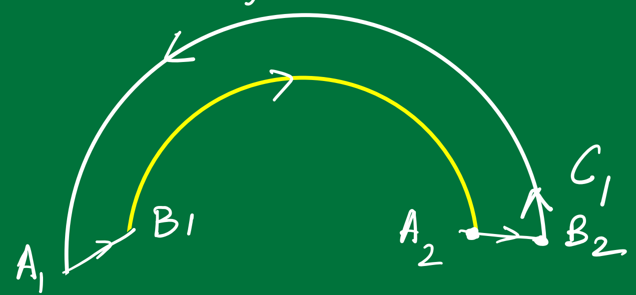
  $= \int_C f \cdot dR$, where the line integral is taken in the counter clockwise direction.

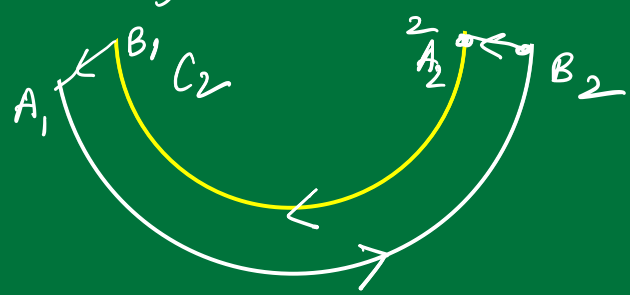




$$\iint_{D_1} (N_x - M_y)\, dx\, dy$$

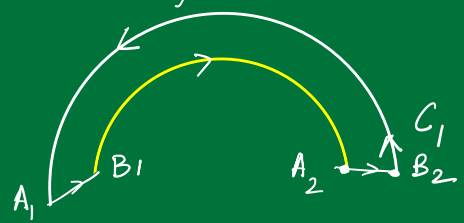
$$= \oint M\,dx + N\,dy \qquad + \quad \oint M\,dx + N\,dy$$

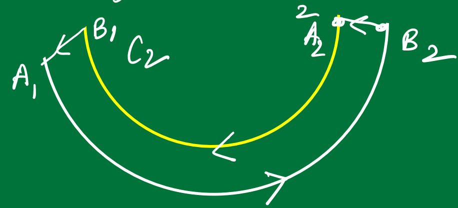




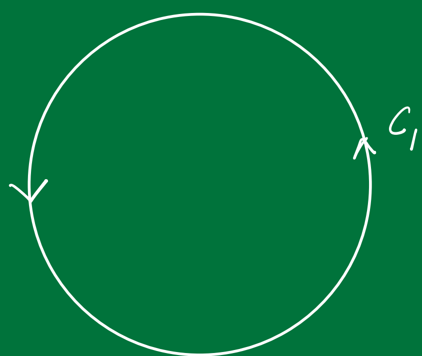
$$\iint_{D_1} (N_x - M_y)\,dx\,dy + \iint_{D_2} \cdots$$

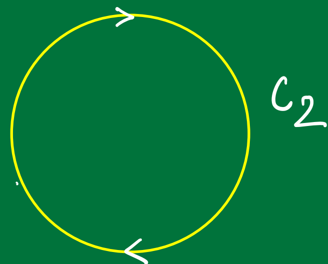
$$= \oint M\,dx + N\,dy \quad + \quad \oint M\,dx + N\,dy$$





$$= \oint M\,dx + N\,dy \quad + \quad \oint M\,dx + N\,dy$$

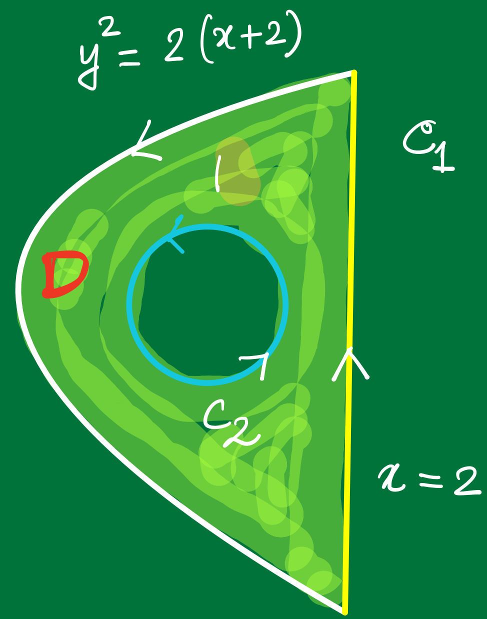




$$= \oint_{C_1} M\,dx + N\,dy - \oint_{C_2} M\,dx + N\,dy$$

$$\int_a^b F(t)\,dt = -\int_b^a F(t)\,dt$$

Example: Consider the region D bounded by the curves C_1 and C_2.



Q. Compute the value of

$$\oint_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2}.$$

Hint: Check that

$$\oint_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2} = \oint_{C_2} \frac{x\,dy - y\,dx}{x^2+y^2}$$

Take $M(x,y) = \dfrac{-y}{x^2+y^2}$ and $N(x,y) = \dfrac{x}{x^2+y^2}$

So, $N_x = \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$M_y = \dfrac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$$\Rightarrow \iint_D (N_x - M_y)\, dx\, dy = 0$$

Following the Green's theorem and the previous computation for annular region we get,

$$\iint_D (N_x - M_y)\, dx\, dy = \oint_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2} - \oint_{C_2} \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$\Rightarrow \oint_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2} = \oint_{C_2} \frac{x\,dy - y\,dx}{x^2+y^2}, \quad \text{since } \iint_D (N_x - M_y)\,dx\,dy = 0$$

$$= \int_0^{2\pi} \frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\,dt = 2\pi$$

*

Recall:

$R(t) = (\cos t, \sin t)$

for $t \in [0, 2\pi]$

- $x(t) = \cos t$
  $y(t) = \sin t$

- $x'(t)\,dt = dx$
  $y'(t)\,dt = dy$

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field

$$(x, y, z) \longmapsto F(x, y, z) = \big(P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\big)$$

Definition: Curl (F):

The Curl (F) is another vector field defined as

Curl F or Curl (F) : $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longmapsto \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

- Here, we think ∇ as a vector composed of all partial derivatives that we use just to help us remember the formulas:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

- We interpret $\frac{\partial}{\partial x} \cdot P = \frac{\partial P}{\partial x}$ and so on.

$$\frac{\partial}{\partial y} \cdot P = \frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial z} \cdot P = \frac{\partial P}{\partial z}$$

Divergence F or $\operatorname{div}(F)$ or $\operatorname{div} F : \mathbb{R}^3 \longrightarrow \mathbb{R}$

defined as $\operatorname{div}(F) : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x, y, z) \longmapsto \nabla \cdot F$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Remark. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \mapsto \big(M(x,y),\ N(x,y)\big)$$

Then $\operatorname{Curl}(F) := \nabla \times F$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (M, N, 0)$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

We may relook the expressions in Green's theorem as follows:

$$\underbrace{\iint_D \operatorname{Curl}(F)\cdot\vec{k}\ dx\,dy}_{\text{surface integral ?}} = \underbrace{\oint_C F\cdot dR}_{\text{line integral}}$$