MTH 114: ODE: Assignment-2

1. Find the solution of the initial value problem

$$xy' = y + \frac{2x^4}{y}\cos(x^2), \quad y(\sqrt{\pi/2}) = \sqrt{\pi}.$$

2. Reduce the differential equation

$$y' = f\left(\frac{ax+by+m}{cx+dy+n}\right), \; ad - bc \neq 0$$

to a separable form. Also discuss the case of $ad = bc$.

3. Show that if the differential equation is of the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0,$$

where $a, b, c, d, m, n, p, q \in \mathbb{R}$ $(mq \neq np)$ are constants, then there exits suitable $h, k \in \mathbb{R}$ such that $x^h y^k$ is an integrating factor. Hence find a general solution of $(x^{1/2}y - xy^2) + (x^{3/2} + x^2y)y' = 0$.

4. (**T**) Given that the equation $(3y^2 - x) + 2y(y^2 - 3x)y' = 0$ admits an integrating factor which is a function of $(x + y^2)$. Hence solve the differential equation.

5. Consider first order ODE $M(x,y)dx + N(x,y)dy = 0$ with M, N are C^1 functions on $\mathbb{R}^2$. Show that

   (**T**)(i) If $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/N = f(x)$ depends on x only then, $\exp(\int f(x)dx)$ is an integrating factor for the given ODE.

   (ii) If $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/M = g(y)$ depends on y only then, $\exp(-\int g(y)dy)$ is an integrating factor for the given ODE.

6. Find integrating factor and solve the following.

   (**T**) (i) $2\sin(y^2) + xy\cos(y^2)y' = 0$.

   (ii) $xy - (x^2 + y^4)y' = 0$.

7. (**T**) Show that the set of solutions of the homogeneous linear equation, $y' + P(x)y = 0$ on an interval $I = [a, b]$ form a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

8. Solve the linear first order linear IVP $y' + y\tan x = \sin 2x, \quad y(0) = 1$.

9. (**T**) Let ϕ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$. Show that $\phi_1 + \phi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$.

   Solve $y' + y = x + 1$, $y' + y = \cos 2x$. Hence solve $y' + y = 1 + x/2 - \cos^2 x$

10. Using appropriate substitution, reduce the following differential equations into linear form and solve:
(**T**) (i) $y^2y' + y^3/x = x^{-2}\sin x$ (ii) $y'\sin y + x\cos y = x$ (iii) $y' = y(xy^3 - 1)$

11. (**T**) A radioactive substance A decays into B, which then further decays to C.

    a) If the decay constants of A and B are respectively λ_1 and λ_2, and the initial amounts are respectively A_0 and B_0, set up an ODE for determining $B(t)$, the amount of B present at time t, and solve it. (Assume $\lambda_1 \neq \lambda_2$.)

    b) Assume $\lambda_1 = 1,\ \lambda_2 = 2$. When B(t) reaches a maximum?

12. According to Newton's Law of Cooling, the rate at which the temperature T of a body changes is proportional to the difference between T and the external temperature. At time $t = 0$, a pot of boiling water is removed from the stove. After five minutes, the water temperature is $80C$. If the room temperature is $20C$, when will the water have cooled to $60C$?