Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field

$$(x, y, z) \longmapsto F(x, y, z) = \big(P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\big)$$

Definition: Curl (F):

The Curl (F) is another vector field defined as

$$\text{Curl } F \text{ or } \text{Curl}(F) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longmapsto \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

- Here, we think ∇ as a vector composed of all partial derivatives that we use just to help us remember the formulas:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

- We interpret $\frac{\partial}{\partial x} \cdot P = \frac{\partial P}{\partial x}$ and so on.

$$\frac{\partial}{\partial y} \cdot P = \frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial z} \cdot P = \frac{\partial P}{\partial z}$$

Divergence F or $\operatorname{div}(F)$ or $\operatorname{div} F : \mathbb{R}^3 \longrightarrow \mathbb{R}$

defined as $\operatorname{div}(F) : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x, y, z) \longmapsto \nabla \cdot F$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Remark. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \mapsto \big(M(x,y),\ N(x,y) \big)$$

Then

$$\operatorname{Curl}(F) := \nabla \times F$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (M, N, 0)$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \vec{k}$$

We may relook the expressions in Green's theorem as follows:

$$\underbrace{\iint_D \operatorname{Curl}(F) \cdot \vec{k} \; dx\, dy}_{\text{surface integral ?}} = \underbrace{\oint_C F \cdot dR}_{\text{line integral}}$$

☐ We may relook the expressions in Green's theorem as follows:

$$\underbrace{\iint_D \operatorname{Curl}(F)\cdot \vec{K}\, dx\, dy}_{\text{surface integral ?}} = \underbrace{\oint_C F\cdot dR}_{\text{line integral}}$$

☐ We now extend this form of Green's theorem to a vector field **F** defined on a surface **S** having the boundary curve **C**.

Recall:

1. Let $T \subseteq \mathbb{R}^2$ be a region of $\mathbb{R}^2$ and $r: T \to \mathbb{R}^3$ be given by $r(u,v) = \big(X(u,v), Y(u,v), Z(u,v)\big) \in \mathbb{R}^3$ be a continuous function on T.

The range of r, i.e., the subset

$$S = \{ r(u,v) \in \mathbb{R}^3 \,/\, (u,v) \in T \}$$

is called a parametric surface with the parameter domain T and parameters u and v.

2. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and $c \in \mathbb{R}$.

$$(x,y,z) \longmapsto f(x,y,z)$$

Then $S = \{ (x,y,z) \in \mathbb{R}^3 \,/\, f(x,y,z) = c \} = f^{-1}(\{c\})$ is called a level surface (for the function f) at the height c.

Smooth Surfaces

(I). Smooth Parametric surface

Let $S = r(u,v)$ be a parametric surface defined on a parametric domain T. We say that S is a smooth surface if the functions $r_u : T \to \mathbb{R}^3$, $r_v : T \to \mathbb{R}^3$ are continuous and $r_u \times r_v : T \to \mathbb{R}^3$ is never zero on T.

(II). Smooth level surface

Let $S = \{(x,y,z) \in \mathbb{R}^3 / f(x,y,z) = c\}$ be a level surface.

We say that S is a smooth level surface if ∇f is continuous and never zero on S.

Remark:

(1) For a smooth parametric surface $S = r(u,v)$, the function $r_u \times r_v$ provides normal direction to the surface S.

Here, the unit normal: $\dfrac{r_u \times r_v}{\| r_u \times r_v \|}$

(2) For a smooth level surface $S = \{ (x,y,z) \in \mathbb{R}^3 \,/\, f(x,y,z) = c \}$ the function ∇f provides normal direction to the surface S.

Here, the unit normal: $\dfrac{\nabla f}{\| \nabla f \|}$

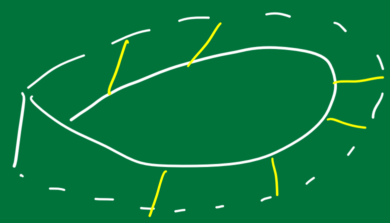
Definition: An orientable surface is a smooth surface S (Parametric or level surface) with a continuous unit normal vector function defined at each point of S.

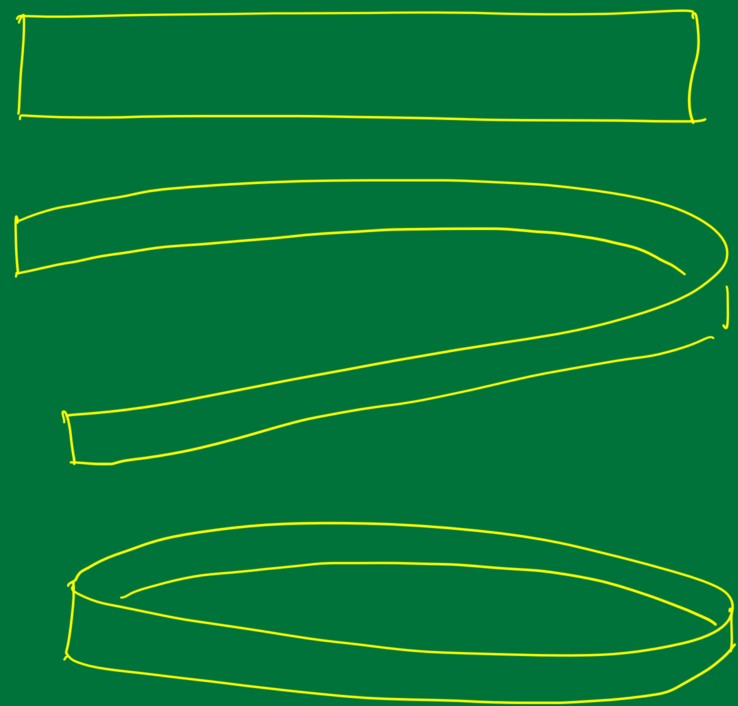
In practice, we consider an orientable surface as a smooth surface with two sides.

For example,

- Spheres
- planes
- Cylinders
- Paraboloids

are examples of orientable surfaces.





Möbious strip

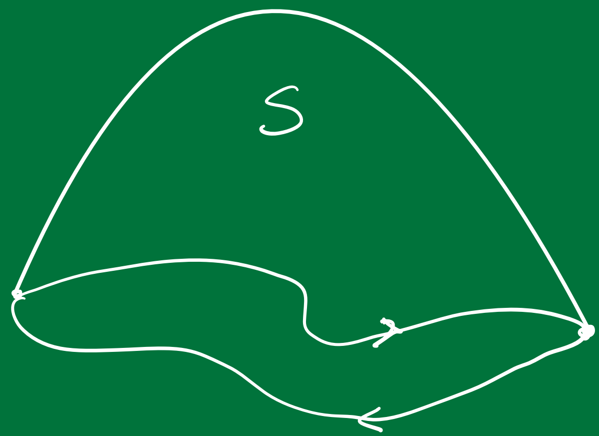
or

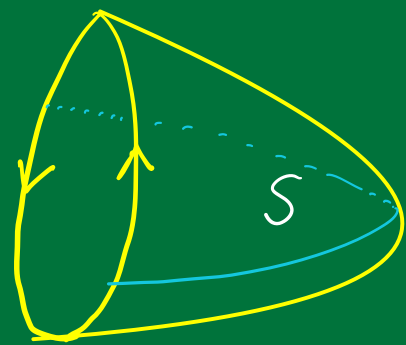
Möbious band

We will be dealing only orientable surfaces.

Let S be an orientable surface with the boundary curve C.

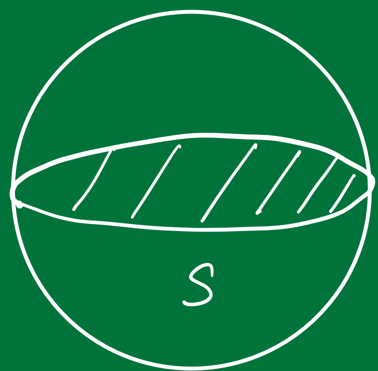
I

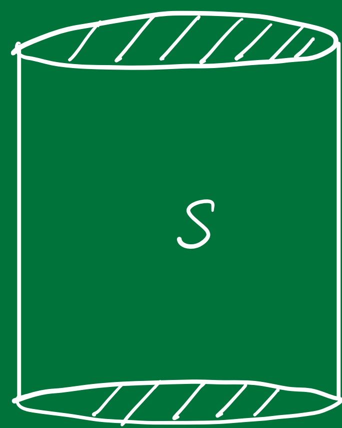




II.

?





I. Since S has two sides, we consider a side of the surface, that is, we take a unit normal $\vec{n}$ to the surface S. With respect to this normal (that is corresponding to a side of the surface S), we define an orientation on C (the boundary of S).

The orientation on the curve C will be useful for computation of line integral involved in Stokes' theorem.

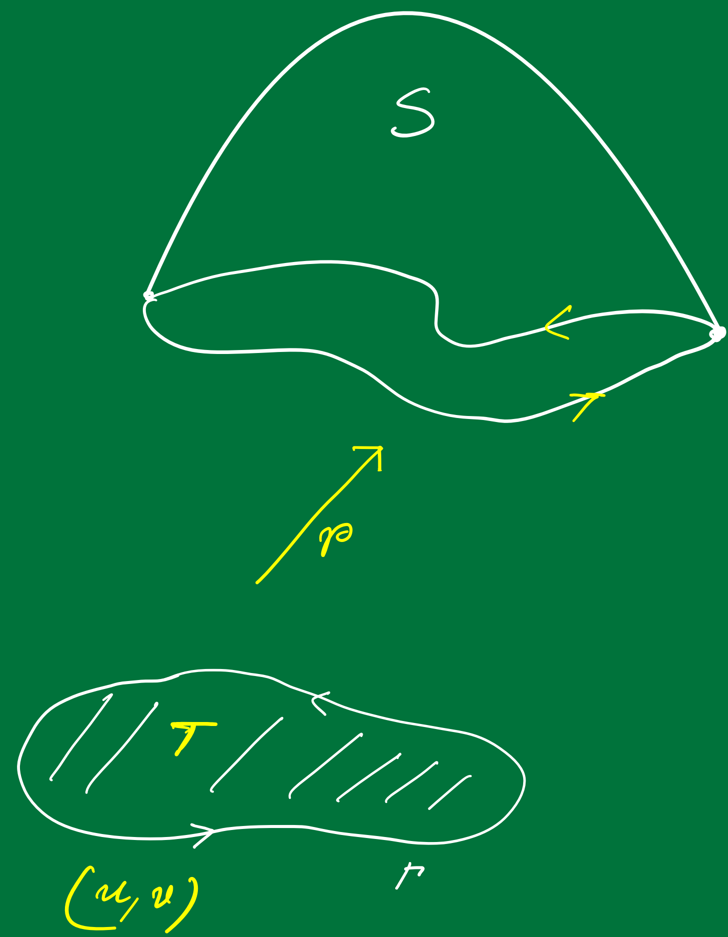
Suppose $S = r(u, v)$ is a smooth parametric surface which is orientable and defined by a one-to-one map $r: T \to \mathbb{R}^3$.

Consider the unit normal vector function

$$\vec{n} = \frac{r_u \times r_v}{\| r_u \times r_v \|}$$ of the surface S.

Let $r(\Gamma) = C$,

where Γ is the boundary of the parametric domain T.



Theorem. • Let S be a smooth orientable surface and a piecewise smooth simple oriented closed curve C be its boundary.

• Let
$$F(x,y,z) = \left(f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \right)$$
$$= f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$$

be a vector field such that f_1, f_2, f_3 are continuous and have continuous partial derivatives in an open set containing the surface S.

• Then
$$\iint_S (\text{Curl } F) \cdot \vec{n} \, d\sigma = \oint_C F \cdot dR$$

Where the line integral is taken around C in the direction of the orientation of C w.r.t $\vec{n}$.

Remark ① Here, in

$$\iint_S (\text{Curl } F)\cdot \vec{n}\, d\sigma = \oint_C F\cdot dR,$$

the value of of surface integral depends only on the boundary C. The shape of the surface is irrelevent. If other requirements are satisfied with same boundary curve C then the surface integral is always same as in the 2nd FTC.

Remark 2. Stokes' theorem is a direct extension of Green's theorem.

If S is a plane region in the xy plane; then from Stokes' theorem we get the identity in Green's theorem.

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

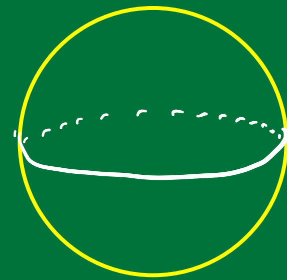
$$(x,y) \mapsto \big(M(x,y), N(x,y) \big)$$

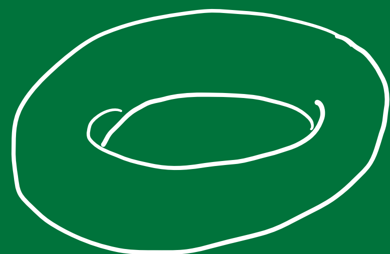
Then $\operatorname{Curl}(F) := \nabla \times F$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (M, N, 0)$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \vec{k}$$

Remark ③. For a closed oriented surface like

Sphere 

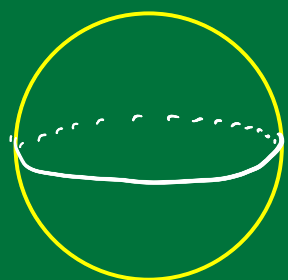
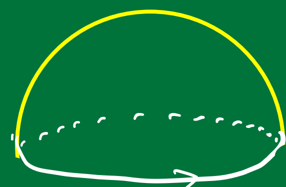
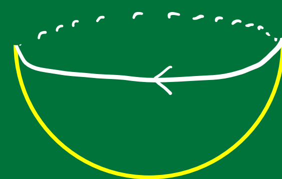
or torus, 

there is no boundary

and in this case

$$\iint_S \operatorname{Curl} F \cdot \vec{n} \, d\sigma = 0$$

*

 =  $\cup$ 

Upper hemisphere lower hemisphere

Remark ④. Stokes' theorem can also be extended to a smooth surface which has more than one simple closed curve forming the boundary of the surface.

Example:

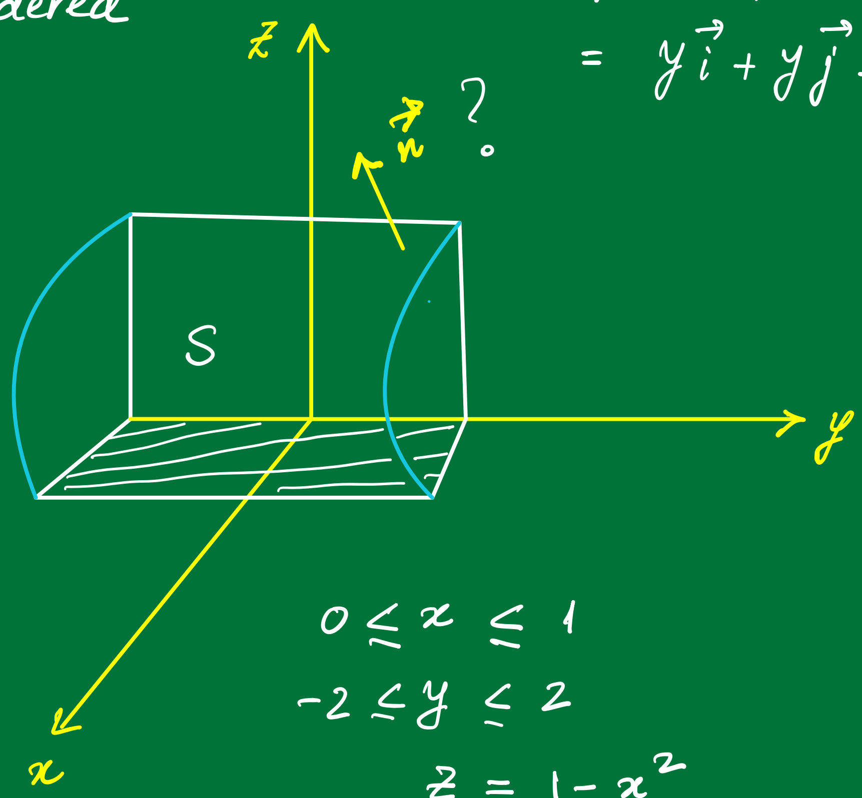
- The surface can be considered as a graph of the function

$$f(x,y) = 1 - x^2$$

or a parametric surface

$$r(x,y) = (x, y, f(x,y)).$$

- $F(x,y,z) = y\vec{i} + y\vec{j} + z\vec{k}$



$0 \le x \le 1$

$-2 \le y \le 2$

$z = 1 - x^2$

- Unit normal is given by $\dfrac{r_x \times r_y}{\| r_x \times r_y \|} = \dfrac{-f_x - f_y + \vec{k}}{\sqrt{1+f_x^2+f_y^2}} = \dfrac{2x\vec{i} + \vec{k}}{\sqrt{1+4x^2}}$

at $(0,1)$ the direction is along $\vec{k}$, the outer normal.

- $\text{Curl } F = -\vec{k}$

- $\text{Curl } F \cdot \vec{n} = \dfrac{-1}{\sqrt{1+4x^2}}$

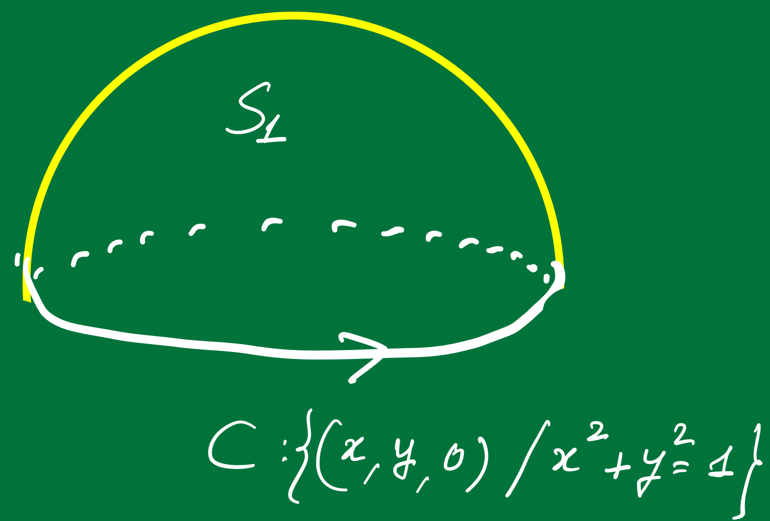
- By Stokes' theorem $\displaystyle\oint_C F \cdot dR = \iint_S \frac{-1}{\sqrt{1+4x^2}}\, d\sigma$

$$\oint_C F \cdot dR = \iint_S \frac{-1}{\sqrt{1+4x^2}}\, d\sigma = \iint_D \frac{-1}{\sqrt{1+4x^2}} \sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

$$= \int_0^1 \int_{-2}^{2} -1 \, dx\, dy$$

Verify the stokes' theorem

Consider $F(x,y,z) = (y, xz, 1)$ and



Now,

- $\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & 1 \end{vmatrix}$

$$= -x\,\vec{i} + (z-1)\,\vec{k}$$

- Unit normal vector $\vec{n} =$?

Here,

$$S_1 = \left\{ r(u,v) \;\middle|\; \begin{array}{l} r : [0,1] \times [0, 2\pi] \to \mathbb{R}^3 \text{ and} \\ r(u,v) = (u\cos v,\ u\sin v,\ 0) \end{array} \right\}$$

So, $r_u = (\cos v,\ \sin v,\ 0)$ and

$$r_v = (-u\sin v,\ u\cos v,\ 0)$$

Thus,

$$r_u \times r_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= u\vec{k}$$

$$\Rightarrow \vec{n} = \vec{k}$$

The surface S_1 is bounded by the smooth and simple closed curve $C = \{ R(t) = (\cos t, \sin t) \,/\, t \in [0, 2\pi] \}$.

Thus, the surface integral,

$$\iint_{S_1} \operatorname{Curl}(F) \cdot \vec{n} \; d\sigma$$

$$= \iint_{S_1} (-x, 0, z-1) \cdot (0, 0, 1) \; d\sigma$$

$$= \iint_{S_1} (z-1) \; d\sigma$$

$$= \int_0^{2\pi}\int_0^1 (-1)\ \|r_u \times r_v\|\ du\,dv$$

$$= -\int_0^{2\pi}\left(\int_0^1 u\,du\right) dv$$

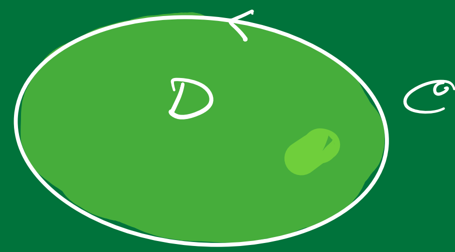
$$= -\pi$$

And the line integral,

$$\oint_C F.dR = \oint_C y\,dx + xz\,dy + dz = \int_0^{2\pi} -\sin^2 t\,dt = -\pi$$

Green's theorem to a form which will be generalized to solids.

Let D be a region (in xy plane) enclosed by a simple smooth curve C.



Suppose $F(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ and

$$C = \left\{ R(t) = x(t)\,\vec{i} + y(t)\,\vec{j} \right\}$$

satisfy the condition in Green's theorem.

Here, $R'(t) = \cancel{x'(t)\,\vec{i} + y'(t)\,\vec{j}}$ gives the tangent vector and $\vec{n}$ is ~~determined by~~ $y'(t)\,\vec{i} - x'(t)\,\vec{j}$

The vector $T = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$ is a unit tangent vector valued function to the curve C.

So, $\vec{n} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$ is a unit normal to the curve C.

By Green's theorem,

$$\oint_C F \cdot dR = \iint_D (N_x - M_y)\, dx\, dy$$

$$(F.\vec{n})\,ds = \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$$

$$= M\,y'(s)\,ds - N\,x'(s)\,ds$$

$$\Rightarrow \oint_C (F.\vec{n})\,ds = \oint_C (-N, M)\,.\,dR$$

Remark: $\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C (P,Q).\,dR$

$$\Rightarrow \oint_C -N\,dx + M\,dy$$

$$= \oint_C (-N, M)\,.\,dR$$

$$= \iint_D \left[M_x - (-N)_y\right]dx\,dy$$

Thus, by applying Green's theorem we have

$$\oint_C (F.\vec{n})\,ds = \oint_C -N\,dx + M\,dy$$

$$= \iint_D (M_x + N_y)\,dx\,dy$$

$$= \iint_D \operatorname{div}(F)\,dx\,dy$$

$$\Rightarrow \iint_D \operatorname{div}(F)\, dx\, dy = \oint_C (F \cdot \vec{n})\, ds$$

Theorem Divergence Theorem

- Let D be a solid in $\mathbb{R}^3$ bounded by a piecewise smooth orientable surface S.
- Let $F(x,y,z) = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$ be a vector field such that f_1, f_2, f_3 are continuous and have continuous partial derivatives in an open set containing D.
- Suppose $\vec{n}$ is the unit outward normal to the surface S. Then

$$\underbrace{\iiint_D \operatorname{div}(F)\, dv}\; = \;\underbrace{\iint_S (F\cdot\vec{n})\, d\sigma}.$$