Lecture 10 : Linear transformation & Rank - Nullity theorem

Consider a matrix A of order $m \times n$.
The matrix A induces a map $T_A : \mathbb{R}^n \to \mathbb{R}^m$
as follows : $T_A(x) = Ax \ \forall \ x \in \mathbb{R}^n$, here
x is treated as $n \times 1$ matrix (or column vector)

T_A satisfies the following properties :

$\star \quad T_A(X+Y) = T_A(X) + T_A(Y)$

$\star \quad T_A(\lambda x) = \lambda T_A(x)$

$\forall \ x, Y \in \mathbb{R}^n, \ \lambda \in \mathbb{R}$.

These properties leads to notion of linear transformation.

Linear transformation : Let V and W be two vector spaces over K $(K = \mathbb{R}$ or $\mathbb{C})$
A map $T : V \to W$ is said to be linear transformation if

(i) $T(x+y) = T(x) + T(y)$, $\forall \ x, y \in V$ &

(ii) $T(\lambda x) = \lambda T(x) \ \forall \ \lambda \in K, \ x \in V$.

Examples :- (1) As discussed earlier, for a matrix A of order $m \times n$, the map

$$T_A : K^n \to K^m \text{ given by}$$

$$T_A(X) = AX, \quad X \in K^n$$ is a linear transformation, where $K = \mathbb{R}$ or $\mathbb{C}$.

For instance, let us take $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

& $K = \mathbb{R}$, $m, n = 2$, $T_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$

or written as $T_A(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$

(2) Let $P(x)$ be the vectorspace of all polynomial of single variable with coeffecients from $\mathbb{R}$.

Consider the derivative map

$$\frac{d}{dx} : P(x) \to P(x)$$

$$\frac{d}{dx}(f(x)) = f'(x)$$

Check that $\frac{d}{dx}$ is linear.

(iii) The map $T: \mathbb{R}^{n+1} \to P(x)$

$$T(a_0, a_1, \cdots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$$

is linear.

(iv) The map $T: P(x) \to P(x)$

$$T(f(x)) = \int_0^1 f(x)\,dx$$

is linear.

Non-Example: (i) The map $T: \mathbb{R} \to \mathbb{R}$

$$T(x) = x^2$$

is not linear.

(ii) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (x, y^3)$$

is not linear.

Proposition: Let $T: V \to W$ be a linear map then $T(0) = 0$

Proof: $T(0) = T(0+0) = T(0) + T(0)$

$\Rightarrow T(0) = 0$. $\square$

Problem: Every linear transformation $T: \mathbb{R} \to \mathbb{R}$ is of the form

$$T(x) = \lambda x \text{ for some } \lambda \in \mathbb{R}$$

Solution: Here $\mathbb{R}$ is vector space over itself. $\{1\}$ is a basis of $\mathbb{R}$, any element $x \in \mathbb{R}$ can be written as $x = x \cdot 1$

$$T(x) = T(x \cdot 1) = x \cdot T(1)$$

Let $\lambda = T(1)$, $T(x) = x \cdot \lambda = \lambda x$

Kernel of a linear transformation

Let $T: V \to W$ be a linear transformation. Kernel of T is defined as $\{u \in V : T(u) = 0\}$. It is denoted as $\ker(T)$.

Proposition: $\ker(T)$ is a subspace of V.

Proof: Left as an exercise

Example: (1) Consider the linear map $T: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$T(x, y) = x - y.$$

$$\begin{aligned} \ker(T) &= \{(x, y) \in \mathbb{R}^2 : T(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 : y = x\} \\ &= (\text{line passing through origin with slope } 1) \end{aligned}$$

(2) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, y)$ is linear & $\ker(T) = \{(x,y) \in \mathbb{R}^2 : T(x,y) = (0,0)\}$

$$= \{(x,y) \in \mathbb{R}^2 ; (x+y, y) = (0,0)\}$$

$$= \{(0,0)\}$$

(3) Consider the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$

$T_A(x) = Ax$ where A is a matrix of order $m \times n$.

$$\ker(T_A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

Finding a point in $\ker(T_A)$ means finding a solution of the homogenous system of linear equations $AX = 0$.

Proposition: A linear map $T: V \to W$ is injective if and only if $\ker(T) = \{0\}$

Proof: Let T be injective. Let $x \in \ker(T)$

$\Rightarrow T(x) = 0 = T(0) \Rightarrow x = 0$ (as T is injective)

Conversely, let $\ker(T) = \{0\}$

Let $T(x) = T(y) \Rightarrow T(x-y) = 0$

$\Rightarrow x - y \in \ker T = \{0\}$

$\Rightarrow x = y$. ☐

Range of a Linear Map :- Let $T : V \to W$ be a linear transformation. Range of T, denoted by $R(T)$, is the set

$$\{ T(u) : u \in V \}.$$

Proposition: $R(T)$ is a subspace of W.

Proof: Left as an exercise.

Example: Consider the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (x-y,\ x-2y+z,\ 2x-y-z)$$

$$T(x, y, z) = (0, 0, 0) \iff x = y = z$$

$$\therefore \ \operatorname{Ker}(T) = \{ (x, x, x) : x \in \mathbb{R} \}$$

Its basis is $\{(1,1,1)\}$ & $\dim(\operatorname{Ker}(T)) = 1$.

$$R(T) = \{ (x-y,\ x-2y+z,\ 2x-y-z) : x, y, z \in \mathbb{R} \}$$

$$\begin{aligned}
&(x-y,\ x-2y+z,\ 2x-y-z) \\
&= x(1,1,2) - y(1,2,1) + z(0,1,-1) \\
&= x(1,1,2) - y\{(1,1,2) + (0,1,-1)\} + z(0,1,-1) \\
&= (x-y)(1,1,2) + (z-y)(0,1,-1)
\end{aligned}$$

$$R(T) = L(\{(1,1,2),\ (0,1,-1)\})$$

& $\{(1,1,2),\ (0,1,-1)\}$ is L.I.

$\dim(R(T)) = 2$. So,

$\dim(\ker(T)) + \dim(R(T)) = 3 = \dim(\mathbb{R}^3)$.

This formula is true in general & it is called Rank - Nullity theorem

Theorem: Let $T: V \to W$ be a linear trans-formation between finite dimensional vector spaces V & W. Then

$$\dim(\ker(T)) + \dim(R(T)) = \dim(V).$$

Proof: Let $n = \dim(V)$ & $\{u_1, \dots, u_m\}$ be a basis of $\ker(T)$. Extend this basis to a basis $\{u_1, \dots, u_m, u_{m+1}, \dots, u_n\}$ of V. We will prove that $\{T(u_{m+1}), \dots, T(u_n)\}$ is a basis of $R(T)$.

Let $y \in R(T) \ni \exists\ x \in V$ such that

$$y = T(x), \quad x = \sum_{i=1}^{n} \alpha_i u_i$$

$$y = T(x) = \sum_{i=1}^{n} \alpha_i T(u_i) = \sum_{i=m+1}^{n} \alpha_i T(u_i)$$

as $T(u_i) = 0 \ \forall\ 1 \le i \le m$.

$$\Rightarrow R(T) = L(\{T(u_{m+1}), \dots, T(u_n)\})$$

$$\sum_{i=1}^{n} \lambda_i T(u_{m+i}) = 0 \Rightarrow T\left(\sum_{i=1}^{n} \lambda_i u_{m+i}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i u_{m+i} \in \ker(T) = L(\{u_1, \ldots, u_m\})$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i u_{m+i} = \sum_{j=1}^{m} \beta_j u_j$$

$$\Rightarrow \beta_1 u_1 + \cdots + \beta_m u_m - \lambda_1 u_{m+1} - \cdots - \lambda_n u_{m+n} = 0$$

$$\Rightarrow \beta_1 = \cdots = \beta_m = \lambda_1 = \cdots = \lambda_n = 0$$

as $\{u_1, \ldots, u_n\}$ is L.I.

So, $\lambda_1 = \cdots = \lambda_n = 0$

So $\{T(u_{m+1}), \ldots, T(u_n)\}$ is a basis of $R(T)$

$\Rightarrow \dim(R(T)) = n-m$.

$$\therefore \dim(\ker(T)) + \dim(R(T)) = m + (n-m)$$
$$= n$$
$$= \dim(V) \quad \boxtimes$$

Rank of a linear transformation Let $T: V \to W$ be a linear transformation between finite dimensional vector spaces V & W. By rank of T we mean $\dim(R(T))$. Thus, the terminology 'Rank-Nullity' theorem makes sense.