lecture 8 (Second order linear ODE)

Recall $y'' + p(x)y' + q(x)y = r(x)$.

(*) $\begin{cases} p, q, r \text{ are continuous functions on some interval } I. \quad x_0 \in I \\ \text{Initial condition } y(x_0) = \alpha_1, \; y'(x_0) = \alpha_2 \end{cases}$

Then (*) has unique solution on $\underline{I}$.

Homogeneous $r(x) = 0$

(**) $\{ y''(x) + p(x)y' + q(x)y = 0$

Set of solution of (**) forms a vector space.

$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

- $y_1, y_2$ are LD $\Rightarrow W(y_1, y_2) = 0$
- $W(y_1, y_2) = 0$ + $y_1, y_2$ are solutions of (**) $\Rightarrow y_1, y_2$ are LD.
- $\underline{y_1, y_2 \text{ sols of (**)}}$

$y_1, y_2$ LD on $I$ $\Leftrightarrow W(y_1, y_2) = 0$ on $I$

$y_1, y_2$ LI on $I$ $\Leftrightarrow W(y_1, y_2)$ never zero on $I$.

$W(y_1, y_2) = c\, e^{-\int p(x)\,dx}$

Corollary

(i) (**) admits two independent sols. $y_1$ & $y_2$.

(ii) Any solution $y(x)$ of (**) looks like $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for some constants $c_1$ & $c_2$.

Proof (i) Take $x_0 \in I$.

Let $y_1(x)$ be the solution of (**) satisfy.

$y_1(x_0) = 1$

$y_1'(x_0) = 0$

$y_2(x)$ - - - - - - - - - - - - -

$y_2(x_0) = 0$

$y_2'(x_0) = 1$

$$W(y_1, y_2)(x_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\Rightarrow$ $y_1, y_2$ are L.I.

(ii) Let $y(x)$ be any other solution of (**)

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$$

$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0)$$

These are two equations in variables $c_1, c_2$. The coefficient determinant

$$W(y_1, y_2)(x_0) \neq 0.$$

So $\exists!$ $(c_1, c_2)$ satisfying above.

So the solutions $y(x)$ & $c_1 y_1 + c_2 y_2$ have same initial conditions.
Hence they must be identical.

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \forall x \in I.$$

$\blacksquare$

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How to find independent sols of $(**)$?

- Suppose $y_1(x) \neq 0$ on $I$ is a solution of $(**)$

<u>claim</u> $\exists\, v(x)$ s.t $y_2(x) = v(x)\, y_1(x)$ is also a solution.

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2 v' y_1' + v y_1''$$

Substituting in the given equation $(**)$

$$v'' y_1 + v'(2y_1' + p y_1) = 0$$

$$\frac{v''}{v'} = -\frac{2y_1'}{y_1} - p$$

<u>integrating</u>

$$\log v' = -2\log y_1 - \int p(x)\,dx$$

$$v' = \frac{1}{y_1^2} e^{-\int p(x)dx}$$

$$v = \int \frac{1}{y_1^2} e^{-\int p(x)}$$

Then $y_2(x) = v\, y_1(x)$

$$W(y_1, y_2) = v' y_1^2 = e^{-\int p(x)dx} \neq 0$$

Then $y_1, y_2$ LI.

$\blacksquare$

- ~~Homo~~ constant coefficients

$$a y'' + b y' + c y = 0 \qquad a, b, c \in \mathbb{R}$$
$$a \neq 0$$

claim: $\exists$ suitable $m$ such that $e^{mx}$ is a solution.

Put $y = e^{mx}$

$$a m^2 + b m + c = 0$$
$$m = m_1, m_2$$

Case 1: $m_1, m_2$ are real distinct

$y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$ are LI sols

So general solution $c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Case 2: $m_1 = m_2 = m \in \mathbb{R}$

$$= -\frac{b}{2a}$$

$$y_1(x) = e^{mx}$$

Another solution $y_2 = v y_1$

$$v' = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

$$= e^{-2mx} e^{-\int \frac{b}{a}\,dx}$$

$$= e^{-2mx} e^{-\frac{b}{a}x} = e^0 = 1$$

$$v(x) = x.$$

Thus another solution $y_2 = x e^{mx}$

General solution $y(x) = c_1 e^{mx} + c_2 x e^{mx}$

Case 3 $m_1, m_2 = \alpha \pm i\beta$.

$\Rightarrow e^{m_1 x}, e^{m_2 x}$ are solutions,

$\Rightarrow \dfrac{e^{m_1 x} + e^{m_2 x}}{2}, \dfrac{e^{m_1 x} - e^{m_2 x}}{2i}$ — are soln

$\Rightarrow e^{\alpha x}\cos\beta x, \; e^{\alpha x}\sin\beta x$ are solutions

general sols: $y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$.

$\square$

Examples

① $y'' - 3y' + 2y = 0$

Characteristic equation $m^2 - 3m + 2 = 0$

$m = 1, 2$.

general sol: $y(x) = c_1 e^{x} + c_2 e^{2x}$.

② $y'' + 2y' + 6y = 0$

$m^2 + 2m + 6 = 0$

$m = -1 \pm \sqrt{5}\, i$

general soln

$y(x) = e^{-x}(c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x)$

$\square$

Example 3

$(1-x^2)y'' - 2xy' + 2y = 0$

Solve this equation, given that $y_1 = x$ is a solution.

Sol Let $y_2(x) = v(x)\, x$ another solution.

Then $v'(x) = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$ $\quad p(x) = \frac{-2x}{1-x^2}$

$- \quad - \quad - \quad -$

$= \frac{1}{x^2} + \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$

$v(x) = -\frac{1}{x} + \frac{1}{2}\left(-\log(1-x) + \log(1+x)\right)$

$y_2(x) = x\, v(x) = -1 + \frac{x}{2}\log\frac{1+x}{1-x}$ ∎

general sol: $y(x) = c_1 y_1 + c_2 y_2$

Example 4

$(x-1)y'' - 3xy' + 4y = 0 \quad y(-2) = 2$
$\quad y'(-2) = 1$

Find the largest interval in which the IVP has unique solution.

Sol $p(x) = -\frac{3x}{x-1} \quad q(x) = \frac{4}{x-1}$

These are continuous on $(-\infty, 1)$ & $(1, \infty)$.

Since the initial condition $-2 \in (-\infty, 1)$,

the answer is $(-\infty, 1)$. ∎

Example 5 $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

compute Wronskian of any two solution.

Sol $W(Y_1, Y_2) = c\, e^{-\int p(x)\,dx}$ $\quad p(x) = \dfrac{-2x}{1-x^2}$

$= \dfrac{C}{1-x^2}$

$c$ - const.

Example 6 $y(x) = c_1 e^x + c_2 e^{2x}$.

(i) By eliminating const, find the ODE $y(x)$ satisfy.

(ii) Verify that the ODE obtained is always admit unique solution for any initial condition

$y(x_0) = \alpha_1$

$y'(x_0) = \alpha_2$

Sol

$y(x) = c_1 e^x + c_2 e^{2x}$

$y' = c_1 e^x + 2c_2 e^{2x}$

$y'' = c_1 e^x + 4c_2 e^{2x}$.

Eliminating $c_1$ & $c_2$

$$\begin{vmatrix} y & e^x & e^{2x} \\ y' & e^x & 2e^{2x} \\ y'' & e^x & 4e^{2x} \end{vmatrix} = 0$$

$\rightsquigarrow y'' - 3y' + 2y = 0$

$\alpha_1 = y(x_0) = c_1 e^{x_0} + c_2 e^{2x_0}$

$\alpha_2 = y'(x_0) = c_1 e^{x_0} + 2c_2 e^{2x_0}$

Coeff. determinant $= \begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_0} & 2e^{2x_0} \end{vmatrix} = e^{3x_0} \neq 0$

$\Rightarrow \exists!\ c_1, c_2$

Example 7

Do the same as Example 6

$$y(x) = c_1 x + c_2 x^2$$

show that the ODE obtained has no solution for $y(0) = 1 \quad y'(0) = 1$

Sol ODE $\quad x^2 y'' - 2x y' + 2y = 0$

This has no soln for $y(0) = 1$

since $y(0) = c_1 0 + c_2 0^2 = 0$
$\neq 0$

$$p(x) = -\frac{2}{x}$$

$$q(x) = \frac{2}{x^2}$$