Question 11. (1 mark) Consider the functions $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_1(x) := \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}, f_2(x) := \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

$$f_3(x) := \begin{cases} 1, & \text{if } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases}, f_4(x) := \begin{cases} 2, & \text{if } x \in (0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

Which of the function(s) is/are a p.d.f. of $Uniform(0,1)$ distribution? <u>Underline</u> the correct answer(s).

$$(a) f_1 \quad (b) f_2 \quad (c) f_3 \quad (d) f_4$$

Question 12. (3 marks) Which of the following RV(s) Y is/are symmetric about 0? Put a tick (✓) beside all correct statement(s) to get credit.

(i) $Y = |X - 3|$, where $X \sim N(3,4)$

(ii) $Y = 2X - 1$, where $X \sim Uniform(0,1)$

(iii) $Y = -2X$, where $X \sim t_4$

(iv) $Y = X - 4$, where $X \sim Binomial(7, 0.5)$

Question 13. (2 + 1 marks) Let $X \sim N(1,2), Y \sim Uniform(0,1), Z \sim N(3,9)$ be independent RVs defined on the same probability space.

(i) Take $W = -2 \ln Y$. Then $W \sim$ $\boxed{}$

(ii) Take $V = \left(\frac{X-1}{\sqrt{2}}\right)^2 + \left(\frac{Z-3}{3}\right)^2$. Then $\frac{W}{V} \sim$ $\boxed{}$

Question 14. (2 marks) An RV X has the characteristic function given by

$$\Phi_X(t) = \frac{\frac{1}{3}}{1 - \frac{2}{3}e^{it}}, \forall t \in \mathbb{R}.$$

Then, $\boxed{\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) = }$

MSO205A: Introduction to Probability Theory

2022-23-I Semester

End Semester Examination (November 24, 2022)

Duration: 5:30pm - 8:00pm

Maximum Marks: 40

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7. You may write your answers as fractions or roots and include e or π, if required.
8. $\Phi(1.645) = 0.95$, $\Phi(1.96) = 0.975$ and $\Phi(2.575) = 0.995$, Φ being the DF of $N(0,1)$ distribution.

Name:

Roll No.:

Question 1. (1 + 1 marks) A box of 100 mangoes contains exactly 10 unripe mangoes. Suppose we successively draw the mangoes one after the other, check for ripeness and set them aside. Let X be the number of fruits drawn until the 4th unripe mango is found. Then X follows [ ] (write the name of the distribution) with parameters [ ]

Question 2. (1 + 2 marks) Let $Z = (X, Y)$ be a 2-dimensional continuous random vector with the joint p.d.f. of the form

$$f_Z(x, y) = \begin{cases} \alpha, & \text{if } x^2 + y^2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\alpha \in \mathbb{R}$. Then, [$f_{X|Y}(x \mid \frac{\sqrt{3}}{2}) =$]

and [$Var(Y \mid X = \frac{\sqrt{3}}{2}) =$]

Question 3. (1 + 2 marks) For $\alpha \in \mathbb{R}$, consider the functions $f_\alpha, g_\alpha : \mathbb{R} \to \mathbb{R}$ below.

$$f_\alpha(x) := \begin{cases} \alpha\, 3^{-x}, \forall x \in \{-1, 0, 1, 2, 3, \cdots\}, \\ 0, \text{ otherwise.} \end{cases}$$

$$g_\alpha(x) := \alpha \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right) + (1-\alpha) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x+1)^2}{2}\right), \forall x.$$

If f_α is a p.m.f., then [$\alpha =$]. For this α, consider the continuous RV Y with the p.d.f. g_α. Then [$Var(Y^2) =$]

Question 4. (2 marks) The coefficient of skewness for $Y = 1 + X$ with $X \sim Exponential(1)$ is given by [ ].

Question 5. (3 marks) Let X_1, X_2, X_3, X_4 be independent $Bernoulli(\frac{1}{3})$ RVs. Then, [$X_{(2)} \sim$]

Question 6. (1.5 + 1.5 + 2 marks) Let (X, Y) be a bi-variate random vector such that $X \sim N(2, 1)$ and $X - Y \sim N(1, 4)$ are independent. Then [$Y \sim$], [$\mathbb{E}(X + Y)^2 =$] and for $(x, y) \in \mathbb{R}^2$, the joint p.d.f. [$f_{X,Y}(x, y) =$]

Question 7. (3 marks) Let $\{X_n\}_n$ be a sequence of i.i.d. $Gamma(2, 3)$ RVs. For each $n = 1, 2, \cdots$, write $\bar{X}_n := \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. Given that $\sqrt{n}((\bar{X}_n)^2 - 36) \xrightarrow[n\to\infty]{d} Y$ for some RV Y. Then, [$Y \sim$]

Question 8. ((1 + 1) + 1 + 1 + 1) Let X be an RV with the DF F_X given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ (\alpha+1)x + \frac{1}{5}, & \text{if } 0 \le x \le 1, \\ (\beta+1)x^2, & \text{if } 1 < x \le 2, \\ 1, & \text{if } x > 2 \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$. A possible value of (α, β) for which F is a DF is [$(\alpha, \beta) =$]. For this (α, β), let γ denote the number of discontinuity point(s) of F. Then, [$\alpha + 2\beta + 2\gamma =$] Write down a median of X [ ]. Is it unique? [Yes/No]

Question 9. (2 marks) Let X be a discrete RV with support $S_X = \{1, 2, \cdots\}$, the set of non-negative integers. Suppose $\mathbb{E}X^2 < \infty$. Which of the following statement(s) is/are necessarily true? Put a tick (✓) beside all correct statement(s) to get credit.

(a) $\lim_{n\to\infty} n\mathbb{P}(X \ge n) = 0$ (b) $\lim_{n\to\infty} n^2\mathbb{P}(X \ge n) = 0$ (c) $\lim_{n\to\infty} n^3\mathbb{P}(X \ge n) = 0$

Question 10. (3 marks) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $Binomial(4, 0.5)$ distribution. Define the RVs $Y_1, Y_2, \cdots, Y_n$ as follows: for $j = 1, 2, \cdots, n$, set $Y_j = 1$ if $X_j \le \sqrt{5}$ and 0 otherwise. Then, [$\frac{1}{n}\sum_{j=1}^{n} Y_j^2 \xrightarrow[n\to\infty]{P}$]

Question 11. (1 mark) Consider the functions $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$ defined by

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$$f_3(x) := \begin{cases} 1, & \text{if } x \in (0,\frac{1}{2}) \cup (\frac{1}{2},1) \\ 0, & \text{otherwise.} \end{cases}, \quad f_4(x) := \begin{cases} 2, & \text{if } x \in (0,\frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

Which of the function(s) is/are a p.d.f. of $Uniform(0,1)$ distribution? Underline the correct answer(s).

$$(a) f_1 \quad (b) f_2 \quad (c) f_3 \quad (d) f_4$$

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(i) $Y = |X - 3|$, where $X \sim N(3,4)$

(ii) $Y = 2X - 1$, where $X \sim Uniform(0,1)$

(iii) $Y = -2X$, where $X \sim t_4$

(iv) $Y = X - 4$, where $X \sim Binomial(7, 0.5)$

Question 13. (2 + 1 marks) Let $X \sim N(1,2), Y \sim Uniform(0,1), Z \sim N(3,9)$ be independent RVs defined on the same probability space.

(i) Take $W = -2 \ln Y$. Then $W \sim$

(ii) Take $V = \left(\frac{X-1}{\sqrt{2}}\right)^2 + \left(\frac{Z-3}{3}\right)^2$. Then $\frac{W}{V} \sim$

Question 14. (2 marks) An RV X has the characteristic function given by

$$\Phi_X(t) = \frac{\frac{2}{3}}{1 - \frac{1}{3}e^{it}}, \forall t \in \mathbb{R}.$$

Then, $\mathbb{P}(X > \frac{1}{2} \mid X \le \frac{5}{2}) =$

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8. $\Phi(1.645) = 0.95$, $\Phi(1.96) = 0.975$ and $\Phi(2.575) = 0.995$, Φ being the DF of $N(0,1)$ distribution.

Name:

Roll No.:

Question 1. (1 + 1 marks) A box of 100 mangoes contains exactly 20 unripe mangoes. Suppose we successively draw the mangoes one after the other, check for ripeness and set them aside. Let X be the number of fruits drawn until the 5th unripe mango is found. Then X follows [ ] (write the name of the distribution) with parameters [ ]

Question 2. (1 + 2 marks) Let $Z = (X, Y)$ be a 2-dimensional continuous random vector with the joint p.d.f. of the form

$$f_Z(x,y) = \begin{cases} \alpha, \text{ if } x^2 + y^2 < 1, \\ 0, \text{ otherwise.} \end{cases}$$

for some $\alpha \in \mathbb{R}$. Then, $f_{X|Y}(x \mid \frac{\sqrt{3}}{2}) =$

and $Var(Y \mid X = -\frac{\sqrt{3}}{2}) =$

Question 3. (1 + 2 marks) For $\alpha \in \mathbb{R}$, consider the functions $f_\alpha, g_\alpha : \mathbb{R} \to \mathbb{R}$ below.

$$f_\alpha(x) := \begin{cases} \alpha\, 3^{-x}, \forall x \in \{-1, 0, 1, 2, 3, \cdots\}, \\ 0, \text{ otherwise.} \end{cases}$$

$$g_\alpha(x) := \alpha \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right) + (1-\alpha) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x+1)^2}{2}\right), \forall x.$$

If f_α is a p.m.f., then $\alpha =$ [ ]. For this α, consider the continuous RV Y with the p.d.f. g_α. Then $Var(Y^2) =$

Question 4. (2 marks) The coefficient of skewness for $Y = 1 + X$ with $X \sim Exponential(1)$ is given by [ ].

Question 5. (3 marks) Let X_1, X_2, X_3, X_4 be independent $Bernoulli(\frac{2}{3})$ RVs. Then, $X_{(3)} \sim$

Question 6. (1.5 + 1.5 + 2 marks) Let (X, Y) be a bi-variate random vector such that $X \sim N(-2, 1)$ and $X - Y \sim N(1, 4)$ are independent. Then $Y \sim$ [ ], $\mathbb{E}(X + Y)^2 =$ [ ]

and for $(x, y) \in \mathbb{R}^2$, the joint p.d.f.

$f_{X,Y}(x, y) =$

Question 7. (3 marks) Let $\{X_n\}_n$ be a sequence of i.i.d. $Gamma(3, 2)$ RVs. For each $n = 1, 2, \cdots$, write $\bar{X}_n := \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. Given that $\sqrt{n}((\bar{X}_n)^2 - 36) \xrightarrow[n\to\infty]{d} Y$ for some RV Y. Then,

$Y \sim$

Question 8. ((1 + 1) + 1 + 1 + 1) Let X be an RV with the DF F_X given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ (\alpha + 1)x + \frac{1}{5}, \text{ if } 0 \leq x \leq 1, \\ (\beta - 1)x^2, \text{ if } 1 < x \leq 2, \\ 1, \text{ if } x > 2 \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$. A possible value of (α, β) for which F is a DF is $(\alpha, \beta) =$ [ ]. For this (α, β), let γ denote the number of discontinuity point(s) of F. Then, $\alpha + 2\beta + 2\gamma =$ [ ] Write down a median of X [ ]. Is it unique? Yes/No

Question 9. (2 marks) Let X be a discrete RV with support $S_X = \{1, 2, \cdots\}$, the set of non-negative integers. Suppose $\mathbb{E}X^2 < \infty$. Which of the following statement(s) is/are necessarily true? Put a tick (✓) beside all correct statement(s) to get credit.

(a) $\lim_{n\to\infty} n\mathbb{P}(X \geq n) = 0$ (b) $\lim_{n\to\infty} n^2\mathbb{P}(X \geq n) = 0$ (c) $\lim_{n\to\infty} n^3\mathbb{P}(X \geq n) = 0$

Question 10. (3 marks) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $Binomial(4, 0.5)$ distribution. Define the RVs $Y_1, Y_2, \cdots, Y_n$ as follows: for $j = 1, 2, \cdots, n$, set $Y_j = 1$ if $X_j \leq \sqrt{5}$ and 0 otherwise. Then, $\frac{1}{n}\sum_{j=1}^n Y_j^3 \xrightarrow[n\to\infty]{P}$

Question 11. (1 mark) Consider the functions $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_1(x) := \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}, \quad f_2(x) := \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

$$f_3(x) := \begin{cases} 1, & \text{if } x \in (0,\frac{1}{2}) \cup (\frac{1}{2},1) \\ 0, & \text{otherwise.} \end{cases}, \quad f_4(x) := \begin{cases} 2, & \text{if } x \in (0,\frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

Which of the function(s) is/are a p.d.f. of $Uniform(0,1)$ distribution? Underline the correct answer(s).

$(a) f_1 \quad (b) f_2 \quad (c) f_3 \quad (d) f_4$

Question 12. (3 marks) Which of the following RV(s) Y is/are symmetric about 0? Put a tick (✓) beside all correct statement(s) to get credit.

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(iii) $Y = -2X$, where $X \sim t_4$

(iv) $Y = X - 4$, where $X \sim Binomial(7, 0.5)$

Question 13. (2 + 1 marks) Let $X \sim N(1, 2), Y \sim Uniform(0, 1), Z \sim N(3, 9)$ be independent RVs defined on the same probability space.

(i) Take $W = -2 \ln Y$. Then $W \sim$ [ ]

(ii) Take $V = \left(\frac{X-1}{\sqrt{2}}\right)^2 + \left(\frac{Z-3}{3}\right)^2$. Then $\frac{W}{V} \sim$ [ ]

Question 14. (2 marks) An RV X has the characteristic function given by

$$\Phi_X(t) = \frac{\frac{1}{4}}{1 - \frac{3}{4}e^{it}}, \forall t \in \mathbb{R}.$$

Then, $\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) =$

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Name:

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Question 1. ($1+1$ marks) A box of 100 mangoes contains exactly 15 unripe mangoes. Suppose we successively draw the mangoes one after the other, check for ripeness and set them aside. Let X be the number of fruits drawn until the 6th unripe mango is found. Then X follows $\boxed{}$ (write the name of the distribution) with parameters $\boxed{}$

Question 2. ($1+2$ marks) Let $Z=(X,Y)$ be a 2-dimensional continuous random vector with the joint p.d.f. of the form

$$f_Z(x,y)=\begin{cases}\alpha, & \text{if } x^2+y^2<1,\\ 0, & \text{otherwise.}\end{cases}$$

for some $\alpha\in\mathbb{R}$. Then, $\boxed{f_{X|Y}(x\mid\frac{1}{\sqrt{2}})=}$

and $\boxed{Var(Y\mid X=\frac{1}{\sqrt{2}})=}$

Question 3. ($1+2$ marks) For $\alpha\in\mathbb{R}$, consider the functions $f_\alpha,g_\alpha:\mathbb{R}\to\mathbb{R}$ below.

$$f_\alpha(x):=\begin{cases}\alpha\,3^{-x}, \forall x\in\{-1,0,1,2,3,\cdots\},\\ 0, \text{ otherwise.}\end{cases}$$

$$g_\alpha(x):=\alpha\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+1)^2}{2}\right)+(1-\alpha)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-1)^2}{2}\right), \forall x.$$

If f_α is a p.m.f., then $\boxed{\alpha=}$. For this α, consider the continuous RV Y with the p.d.f. g_α. Then $\boxed{Var(Y^2)=}$

Question 4. (2 marks) The coefficient of skewness for $Y=1+X$ with $X\sim$ *Exponential*(1) is given by $\boxed{}$.

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Question 6. ($1.5+1.5+2$ marks) Let (X,Y) be a bi-variate random vector such that $X\sim N(2,1)$ and $X-Y\sim N(-1,4)$ are independent. Then $\boxed{Y\sim}$, $\boxed{\mathbb{E}(X+Y)^2=}$

and for $(x,y)\in\mathbb{R}^2$, the joint p.d.f.

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Question 7. (3 marks) Let $\{X_n\}_n$ be a sequence of i.i.d. *Gamma*$(2,3)$ RVs. For each $n=1,2,\cdots$, write $\bar{X}_n:=\frac{1}{n}(X_1+X_2+\cdots+X_n)$. Given that $\sqrt{n}((\bar{X}_n)^3-216)\xrightarrow[n\to\infty]{d}Y$ for some RV Y. Then,

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for some $\alpha,\beta\in\mathbb{R}$. A possible value of (α,β) for which F is a DF is $\boxed{(\alpha,\beta)=}$. For this (α,β), let γ denote the number of discontinuity point(s) of F. Then, $\boxed{\alpha+2\beta+2\gamma=}$ Write down a median of X $\boxed{}$. Is it unique? $\boxed{\text{Yes/No}}$

Question 9. (2 marks) Let X be a discrete RV with support $S_X=\{1,2,\cdots\}$, the set of non-negative integers. Suppose $\mathbb{E}X^2<\infty$. Which of the following statement(s) is/are necessarily true? Put a tick (✓) beside all correct statement(s) to get credit.

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Question 10. (3 marks) Let $X_1,X_2,\cdots,X_n$ be a random sample of size n from *Binomial*$(4,0.5)$ distribution. Define the RVs $Y_1,Y_2,\cdots,Y_n$ as follows: for $j=1,2,\cdots,n$, set $Y_j=1$ if $X_j\le\sqrt{5}$ and 0 otherwise. Then, $\boxed{\frac{1}{n}\sum_{j=1}^n Y_j^2\xrightarrow[n\to\infty]{P}}$

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(iii) $Y = -2X$, where $X \sim t_4$

(iv) $Y = X - 4$, where $X \sim Binomial(7, 0.5)$

Question 13. (2 + 1 marks) Let $X \sim N(1,2), Y \sim Uniform(0,1), Z \sim N(3,9)$ be independent RVs defined on the same probability space.

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Then, $\mathbb{P}(X > \frac{1}{2} \mid X \leq \frac{5}{2}) =$

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3. You may use books, notebooks, handwritten/photocopied notes of classroom lectures, printouts of supplementary lecture materials (and solutions of practice problem sets) and writing instruments during the quiz. Do not share these materials with other students.
4. Usage of internet or any e-material (including e-books) is prohibited. Electronic communication devices like mobile phones must be switched off and kept in the place designated by the invigilator(s). The invigilators will not be responsible for loss of such a device. If such a device is found on person during the quiz, appropriate action shall be taken. Usage of calculators is not allowed.
5. DO NOT do any rough work on this sheet. If required, do it in your notebook or additional sheets (which may be provided to you). DO NOT attach any additional sheet to this page.
6. Write your answers in the designated place. Any statements written outside the designated place will be taken as rough work and no credit will be provided for such statements. If your answer is not legible, you shall not get credit.
7. You may write your answers as fractions or roots and include e or π, if required.
8. $\Phi(1.645) = 0.95$, $\Phi(1.96) = 0.975$ and $\Phi(2.575) = 0.995$, Φ being the DF of $N(0,1)$ distribution.

Name:

Roll No.:

Question 1. (1 + 1 marks) A box of 100 mangoes contains exactly 25 unripe mangoes. Suppose we successively draw the mangoes one after the other, check for ripeness and set them aside. Let X be the number of fruits drawn until the 7th unripe mango is found. Then X follows $\boxed{}$ (write the name of the distribution) with parameters $\boxed{}$

Question 2. (1 + 2 marks) Let $Z = (X, Y)$ be a 2-dimensional continuous random vector with the joint p.d.f. of the form

$$f_Z(x,y) = \begin{cases} \alpha, \text{ if } x^2 + y^2 < 1, \\ 0, \text{ otherwise.} \end{cases}$$

for some $\alpha \in \mathbb{R}$. Then, $\boxed{f_{X|Y}(x \mid -\frac{1}{\sqrt{2}}) =}$ and $\boxed{Var(Y \mid X = -\frac{1}{\sqrt{2}}) =}$

Question 3. (1 + 2 marks) For $\alpha \in \mathbb{R}$, consider the functions $f_\alpha, g_\alpha : \mathbb{R} \to \mathbb{R}$ below.

$$f_\alpha(x) := \begin{cases} \alpha\, 3^{-x}, \forall x \in \{-1, 0, 1, 2, 3, \cdots\}, \\ 0, \text{ otherwise.} \end{cases}$$

$$g_\alpha(x) := \alpha \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x+1)^2}{2}\right) + (1-\alpha)\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right), \forall x.$$

If f_α is a p.m.f., then $\boxed{\alpha =}$. For this α, consider the continuous RV Y with the p.d.f. g_α. Then $\boxed{Var(Y^2) =}$

Question 4. (2 marks) The coefficient of skewness for $Y = 1 + X$ with $X \sim$ $Exponential(1)$ is given by $\boxed{}$.

Question 5. (3 marks) Let X_1, X_2, X_3, X_4 be independent $Bernoulli(\frac{2}{3})$ RVs. Then, $\boxed{X_{(2)} \sim}$

Question 6. (1.5 + 1.5 + 2 marks) Let (X, Y) be a bi-variate random vector such that $X \sim N(-2, 1)$ and $X - Y \sim N(-1, 4)$ are independent. Then $\boxed{Y \sim}$, $\boxed{\mathbb{E}(X+Y)^2 =}$ and for $(x, y) \in \mathbb{R}^2$, the joint p.d.f. $\boxed{f_{X,Y}(x,y) =}$

Question 7. (3 marks) Let $\{X_n\}_n$ be a sequence of i.i.d. $Gamma(3, 2)$ RVs. For each $n = 1, 2, \cdots$, write $\bar{X}_n := \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. Given that $\sqrt{n}((\bar{X}_n)^3 - 216) \xrightarrow[n\to\infty]{d} Y$ for some RV Y. Then, $\boxed{Y \sim}$

Question 8. ((1 + 1) + 1 + 1 + 1) Let X be an RV with the DF F_X given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ (\alpha - 1)x + \frac{1}{5}, \text{ if } 0 \le x \le 1, \\ (\beta - 1)x^2, \text{ if } 1 < x \le 2, \\ 1, \text{ if } x > 2 \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$. A possible value of (α, β) for which F is a DF is $\boxed{(\alpha, \beta) =}$. For this (α, β), let γ denote the number of discontinuity point(s) of F. Then, $\boxed{\alpha + 2\beta + 2\gamma =}$ Write down a median of X $\boxed{}$. Is it unique? $\boxed{\text{Yes/No}}$

Question 9. (2 marks) Let X be a discrete RV with support $S_X = \{1, 2, \cdots\}$, the set of non-negative integers. Suppose $\mathbb{E}X^2 < \infty$. Which of the following statement(s) is/are necessarily true? Put a tick (✓) beside all correct statement(s) to get credit.

(a) $\lim_{n\to\infty} n\mathbb{P}(X \ge n) = 0$ (b) $\lim_{n\to\infty} n^2\mathbb{P}(X \ge n) = 0$ (c) $\lim_{n\to\infty} n^3\mathbb{P}(X \ge n) = 0$

Question 10. (3 marks) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $Binomial(4, 0.5)$ distribution. Define the RVs $Y_1, Y_2, \cdots, Y_n$ as follows: for $j = 1, 2, \cdots, n$, set $Y_j = 1$ if $X_j \le \sqrt{5}$ and 0 otherwise. Then, $\boxed{\frac{1}{n}\sum_{j=1}^n Y_j^3 \xrightarrow[n\to\infty]{P}}$