Lecture Notes: Periodogram Estimates

We are considering the following sinusoidal model:

$$y_t = A^0 \cos(\omega^0 t) + B^0 \sin(\omega^0 t) + \epsilon_t; \quad t = 1, \ldots, n.$$

It is assumed that ϵ_t's are i.i.d. random variables with mean zero and variance σ^2. The least squares estimators of the unknown parameters can be obtained by minimizing the function $Q(A, B, \omega)$, where

$$Q(A, B, \omega) = \sum_{t=1}^{n} \left(y_t - A\cos(\omega t) - B\sin(\omega t)\right)^2,$$

with respect to A, B and ω. By using the facts that for $0 < \omega < \pi$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos^2(\omega t) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin^2(\omega t) = \frac{1}{2} \text{ and } \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t)\sin(\omega t) = 0$$

it has been shown in the class that if $\widehat{\omega}$ is the least squares estimator of ω^0, then for large n it can be obtained by maximizing

$$I(\omega) = \left(\frac{1}{n}\sum_{t=1}^{n} y_t \cos(\omega t)\right)^2 + \left(\frac{1}{n}\sum_{t=1}^{n} y_t \sin(\omega t)\right)^2.$$

It has been shown in the class that $\widehat{\omega} \to \omega^0$ in probability. Now we would like to obtain the asymptotic distribution of $\widehat{\omega}$. The idea is as follows. First we expand

$$I'(\widehat{\omega}) - I'(\omega^0) = (\widehat{\omega} - \omega^0) I''(\overline{\omega}).$$

Here $\overline{\omega}$ is a point on the line joining $\widehat{\omega}$) and ω^0. Since $\widehat{\omega} \to \omega^0$, hence $\overline{\omega} \to \omega^0$. Furthse note that $I'(\widehat{\omega}) = 0$. Let us look at $I''(\omega^0)$.

$$\begin{aligned} I'(\omega) &= -\frac{2}{n^2}\left(\sum_{t=1}^{n} y_t \cos(\omega t)\right)\left(\sum_{t=1}^{n} y_t t \sin(\omega t)\right) + \\ &\quad \frac{2}{n^2}\left(\sum_{t=1}^{n} y_t \sin(\omega t)\right)\left(\sum_{t=1}^{n} y_t t \cos(\omega t)\right) \end{aligned}$$

$$
\begin{aligned}
I''(\omega) \;=\; & \frac{2}{n^2}\left(\sum_{t=1}^{n} y_t t \sin(\omega t)\right)^2 - \frac{2}{n^2}\left(\sum_{t=1}^{n} y_t \cos(\omega t)\right)\left(\sum_{t=1}^{n} y_t t^2 \cos(\omega t)\right) + \\
& \frac{2}{n^2}\left(\sum_{t=1}^{n} y_t t \cos(\omega t)\right)^2 - \frac{2}{n^2}\left(\sum_{t=1}^{n} y_t \sin(\omega t)\right)\left(\sum_{t=1}^{n} y_t t^2 \sin(\omega t)\right).
\end{aligned}
$$

We use the following facts:

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n^2}\sum_{t=1}^{n} t\cos^2(\omega t) = \lim_{n\to\infty} \frac{1}{n^2}\sum_{t=1}^{n} t\sin^2(\omega t) = \frac{1}{4} \\
& \lim_{n\to\infty} \frac{1}{n^2}\sum_{t=1}^{n} t\cos(\omega t)\sin(\omega t) = 0 \\
& \lim_{n\to\infty} \frac{1}{n^3}\sum_{t=1}^{n} t^2\cos^2(\omega t) = \lim_{n\to\infty} \frac{1}{n^3}\sum_{t=1}^{n} t^2\sin^2(\omega t) = \frac{1}{6}.
\end{aligned}
$$

First we will show that

$$\lim_{n\to\infty} \frac{1}{n^2} I''(\omega^0) = c \neq 0.$$

Let us look at

$$
\begin{aligned}
\frac{1}{n^2}\left(\sum_{t=1}^{n} y_t t \sin(\omega t)\right) \;&=\; \frac{1}{n^2}\left(\sum_{t=1}^{n} t(A^0\cos(\omega^0 t) + B^0 \sin(\omega^0 t) + \epsilon_t)\sin(\omega^0 t)\right) \\
&\longrightarrow\; \frac{B^0}{4}.
\end{aligned}
$$

Similarly, considering the other it can be seen that

$$\lim_{n\to\infty} I''(\omega^0) = -\frac{A^2+B^2}{24}$$

By using Central limit theorem it can be shown that

$$\frac{1}{\sqrt{n}} I'(\omega^0) \to N\left(0, \frac{A^2+B^2}{24}\right).$$

Hence; it follows that

$$n^{3/2}(\widehat{\omega} - \omega^0) \to N\left(0, \frac{24}{A^2+B^2}\right)$$