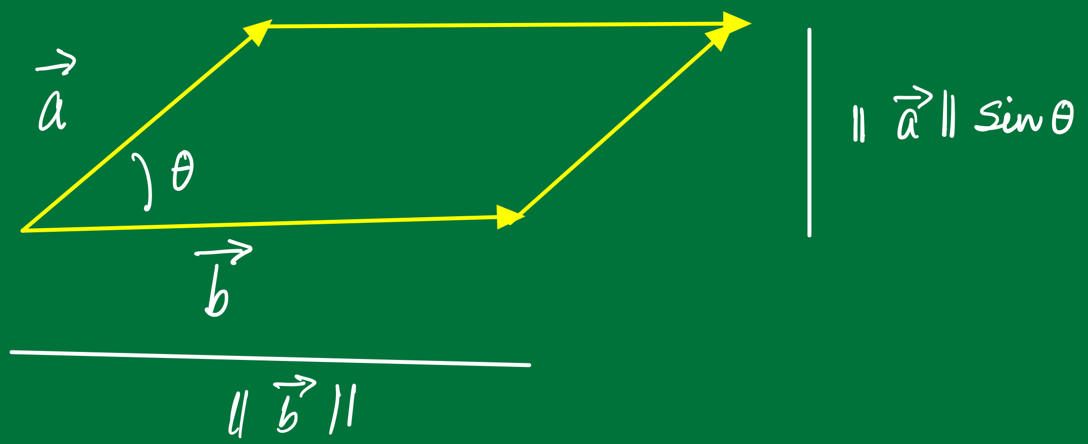
Cross product

The cross product is defined only for three-dimensional vectors. If $\vec{a}$ and $\vec{b}$ are two three dimensional vectors, then their cross product $\vec{a} \times \vec{b}$ is defined by the following three requirements:

1. $\vec{a} \times \vec{b}$ is a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$.

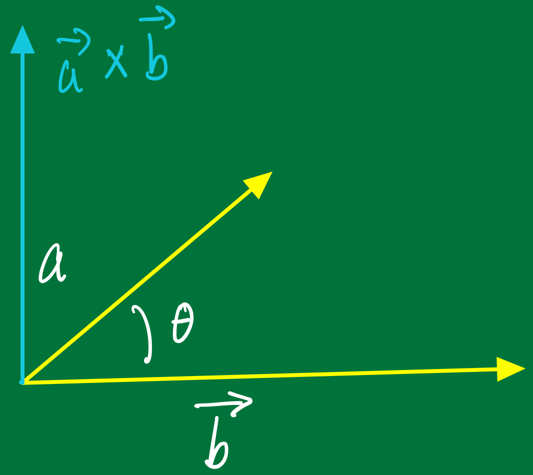
2. The magnitude (or length) of $\vec{a} \times \vec{b}$, written as $\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ (the parallelogram whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$).

3. The direction of $\vec{a} \times \vec{b}$ is determined by the right-hand rule. If we curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ then the thumb points in the direction of $\vec{a} \times \vec{b}$.



Area of the parallelogram is

$\|\vec{a}\| \, \|\vec{b}\| \sin\theta$



- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
- $\vec{a}\times\vec{a} = 0$

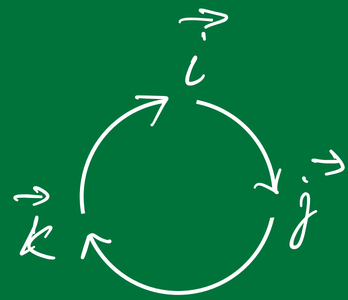
- $(\lambda\vec{a})\times\vec{b} = \lambda(\vec{a}\times\vec{b}) = \vec{a}\times(\lambda\vec{b})$
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
- $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$

Formula for cross product

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$



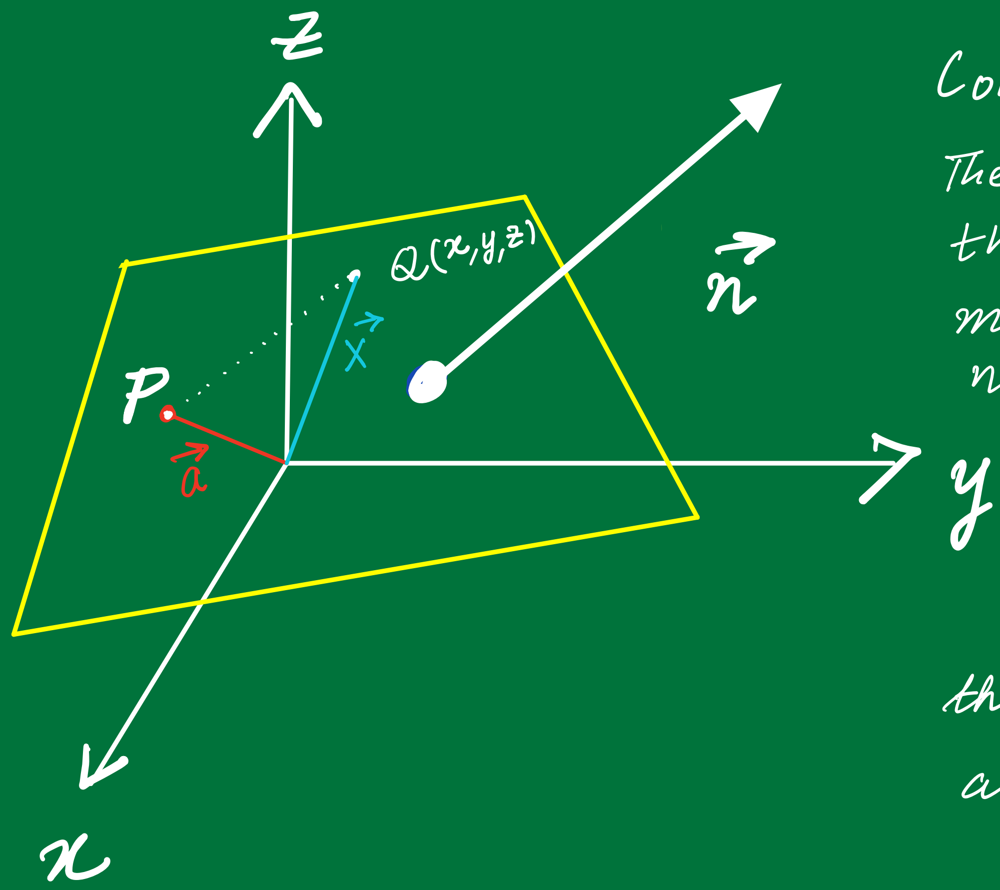
For $\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Given a point P, there are many planes that contain P.

However, in three-dimensional space ($\mathbb{R}^3$), a plane is uniquely determined if we specify a normal vector $\vec{n}$ (is a vector perpendicular to the plane).



Consider $\vec{x} - \vec{a}$ and $\vec{n}$;

The point represented by $\vec{x}$ is in the plane, so the vector $\vec{x} - \vec{a}$ must be perpendicular to the normal vector $\vec{n}$.

Thus $\vec{n} \cdot (\vec{x} - \vec{a}) = 0.$

This is the equation of the plane that goes through the point $P(\vec{a})$ and perpendicular to $\vec{n}$.

For $\vec{n} = (a, b, c)$, $P = (x_0, y_0, z_0)$ the equation of above plane is :

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

or, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

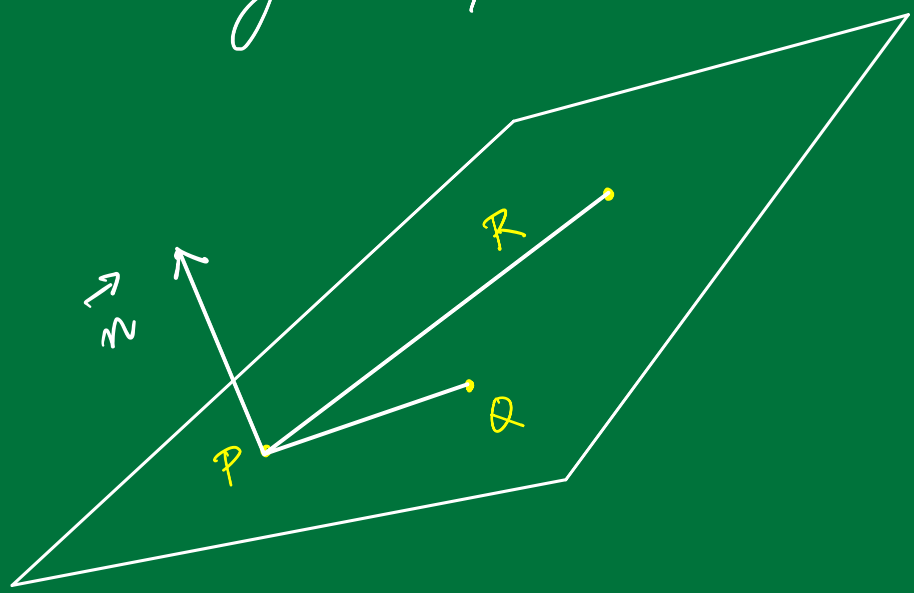
or, $ax + by + cz = ax_0 + by_0 + cz_0$.

Example: Let P, Q, R be three (non-collinear) points on a plane. How to describe an arbitrary point $\vec{X}$ on the plane?

Hint: • Get $\vec{n}$ by considering cross product of vectors between P, Q and P, R.

• Let $\vec{a}$ be the vector representing one of the given points. Then any arbitrary point $\vec{X}$ we have

$$\vec{n} \cdot (\vec{X} - \vec{a}) = 0 .$$



1. Find the equation for the plane through the point $(0,1,-7)$ perpendicular to the vector $(4,-1,6)$

$\vec{a} = (0,1,-7)$

$\vec{n} = (4,-1,6)$

For $\vec{x} = (x,y,z)$; $\vec{n} \cdot (\vec{x} - \vec{a}) = 0$

$\Rightarrow \underline{4x - y + 6z} + 43 = 0$

Coefficients!

2. Find the equation for the plane through the points $(0,1,-7)$, $(3,1,-9)$, and $(0,-5,-8)$.

Let $P = (0,1,-7)$ and $\vec{b} = P - Q = (-3,0,2)$

$Q = (3,1,-9)$ $\qquad \vec{c} = P - R = (0,6,1)$.

$R = (0,-5,-8)$

Now $\vec{n} = \vec{b} \times \vec{c}$ – normal vector

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 2 \\ 0 & 6 & 1 \end{vmatrix} = (-12, 3, -18).$$

Let $\vec{a} = (0, 1, -7)$. Then the equation for the plane becomes

$$(-12, 3, -18) \cdot (x-0, y-1, z+7) = 0$$

or, $$-12x + 3y - 18z - 129 = 0$$

Notice that both planes of the examples passing through same point $(0, 1, -7)$ and their normal vectors are parallel to each other.

Q. What is the relationship between these planes?

Example: Can you find the equation for the plane through the points $(1, 2, 3)$, $(2, 4, 6)$ and $(-3, -6, -9)$?

Hint:

$P \leftrightarrow (1,2,3)$

$Q \leftrightarrow (2,4,6)$

$R \leftrightarrow (-3,-6,-9)$

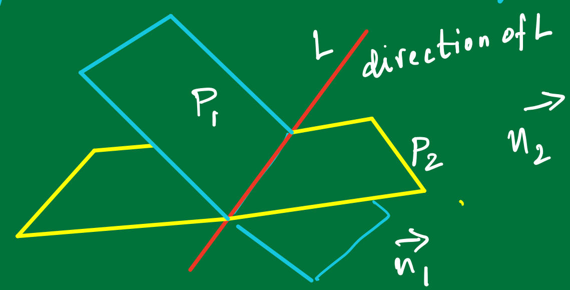
$\vec{b} = Q - P = (1,2,3)$

$\vec{c} = P - R = (4,8,12)$

Check $\rightarrow$ $\vec{n} = \vec{b} \times \vec{c}$

$= (0,0,0)$!

Exc: Equation of line of intersection of two planes;



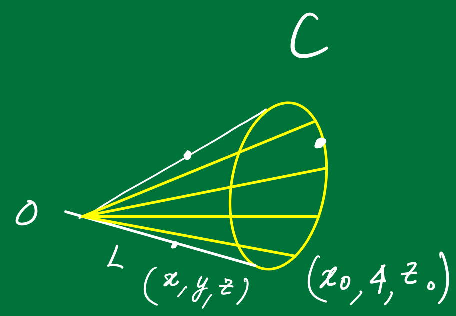
direction of L is same as direction of $\vec{n_1} \times \vec{n_2}$

Try for
$$x - 2z = 3$$
$$y + 2z = 5$$

Equation of Surfaces in $\mathbb{R}^3$ from the description of generating Lines:

Example: Surface generated by lines joining origin $(0,0,0)$ and a point on the circle

$$C: x^2+z^2=25,\ y=4.$$



- Equation of the line L passing through $(0,0,0)$ and $(x_0, 4, z_0)$ is

$$\frac{x}{x_0} = \frac{y}{4} = \frac{z}{z_0}$$

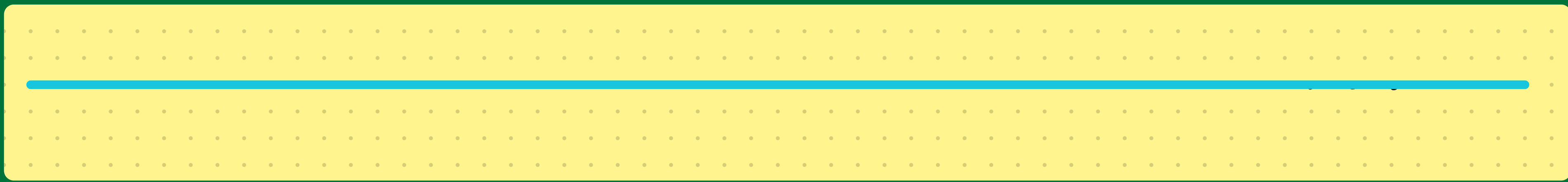
So, $x_0 = \frac{4x}{y}$, $z_0 = \frac{4z}{y}$ and (x_0, z_0) lies on the circle C implies that

$$\left(\frac{4x}{y}\right)^2 + \left(\frac{4z}{y}\right)^2 = 25$$

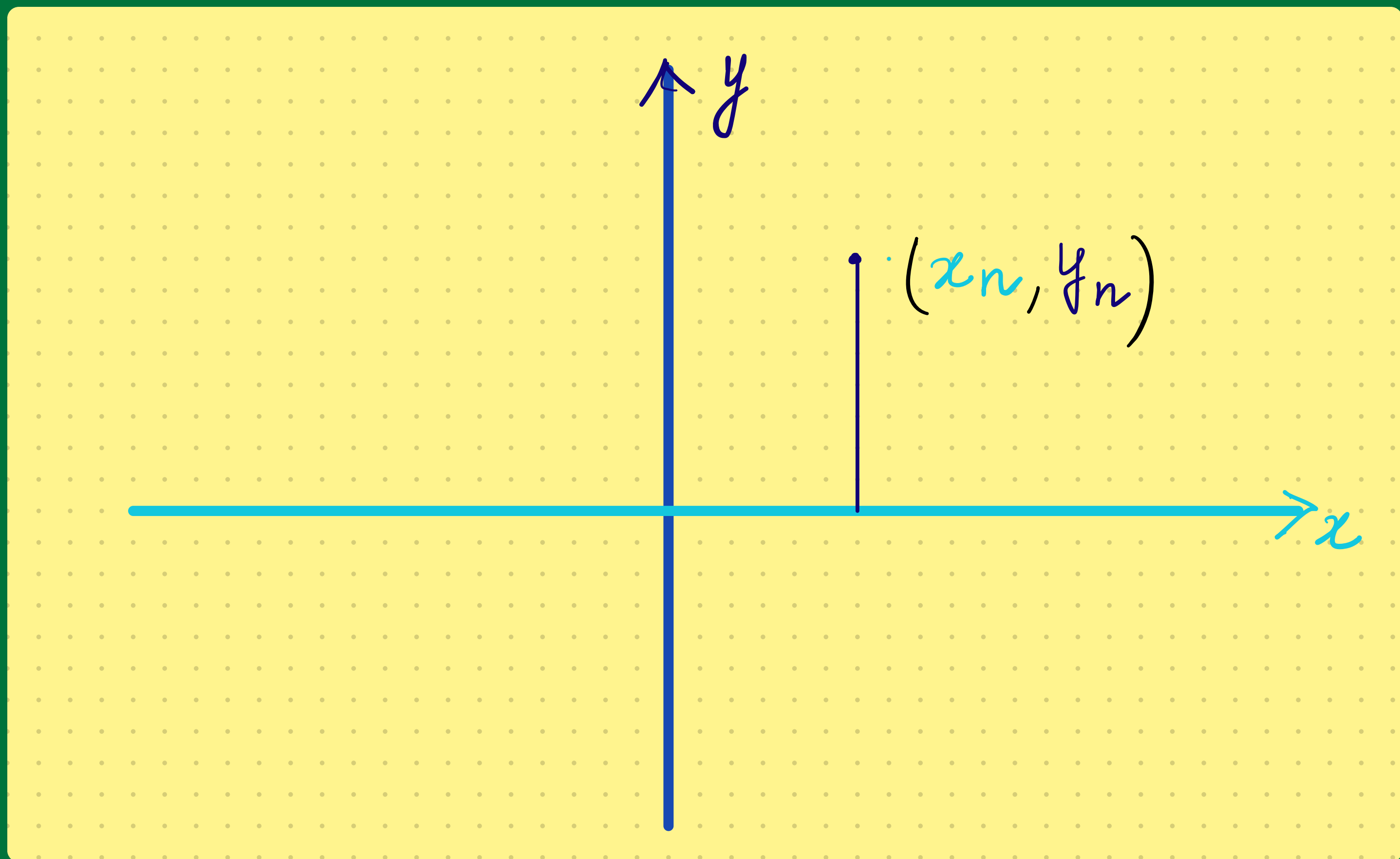
or, $16(x^2+z^2) = 25y^2$.

If $\vec{X} = (x, y, z)$ is an arbitrary point on the surface then the components satisfy: $16(x^2+z^2) = 25y^2$ $\square$

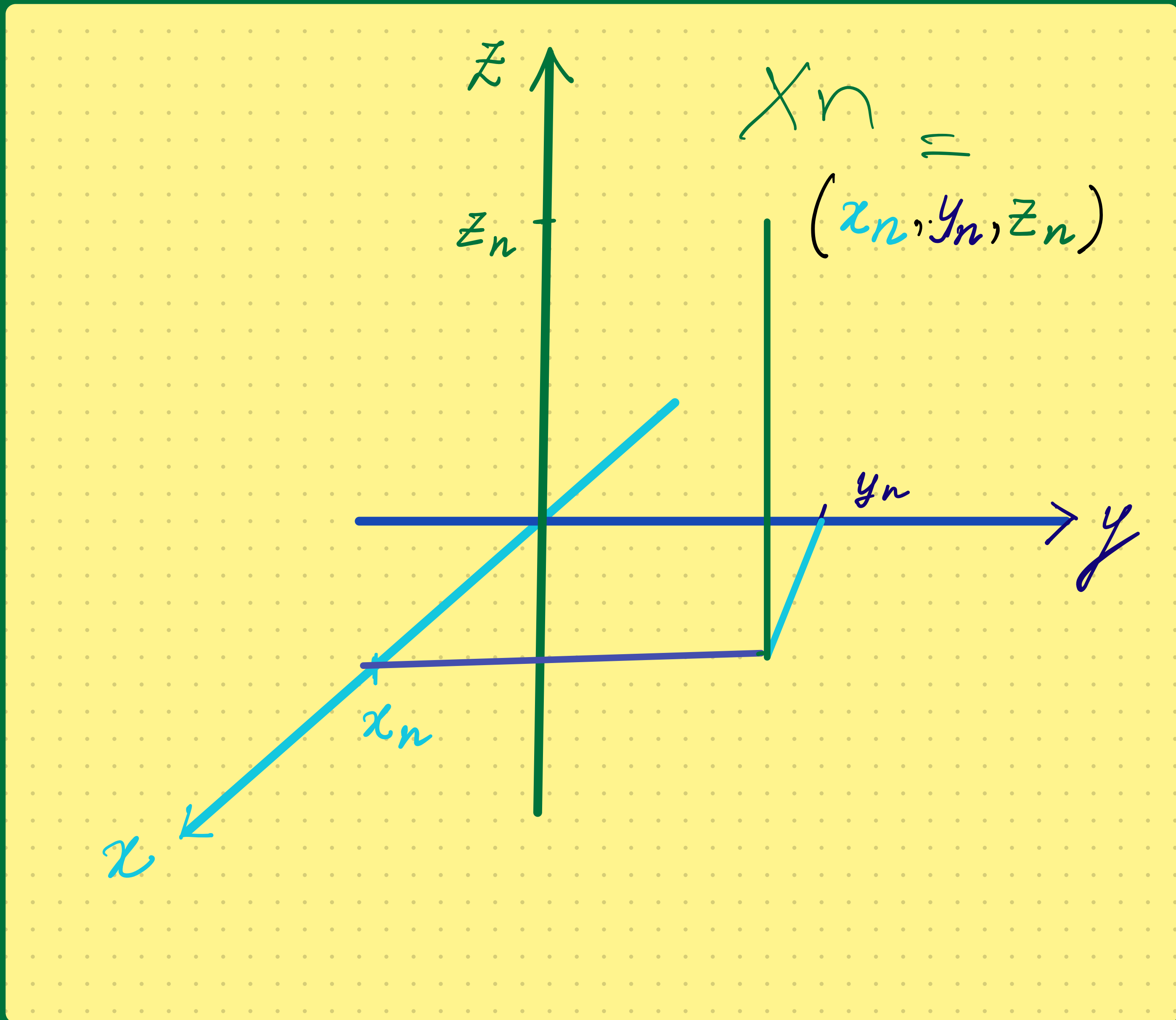
Sequence in $\mathbb{R}$ - the real line :



Sequence in $\mathbb{R}^2$ - the x-y plane :



Sequence in $\mathbb{R}^3$ - three-dimensional space:



Let $X_n = (x_n, y_n, z_n) \in \mathbb{R}^3$ for $n = 1, 2, 3, 4, \dots$

A sequence $X_1, X_2, \dots, X_n, \dots$

or (X_n) is convergent if there exist $X_0 = (a, b, c) \in \mathbb{R}^3$

such that $\| X_n - X_0 \| \to 0$ as $n \to \infty$.

In this case we say that the sequence X_n converges to X_0 and we write $X_n \to X_0$

or, $\lim_{n \to \infty} X_n = X_0$.

Remark: Note that a sequence (X_n) where $X_n = (x_n, y_n, z_n)$ determine three sequences of real numbers

$$x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$$

$$y_1, y_2, \ldots, y_n, y_{n+1}, \ldots \quad \text{and}$$

$$z_1, z_2, \ldots, z_n, z_{n+1}, \ldots,$$

and vice-versa.

□ Let

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots$$

$$-1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}, -\frac{1}{n+1}, \ldots$$

$$1, -1, 1, \quad , (-1)^{n+1}, (-1)^{n+2}, \ldots$$

be three sequences of real numbers.

Then $\left(X_n = \left(\frac{1}{n}, -\frac{1}{n}, (-1)^{n+1}\right)\right)$ is a sequence in $\mathbb{R}^3$.

— NOT CONVERGENT

Theorem:

$$X_n = (x_n, y_n, z_n) \to X_0 = (a, b, c) \text{ in } \mathbb{R}^3$$

if and only if

$$x_n \to a$$
$$y_n \to b$$
$$z_n \to c.$$

Suppose $X_n = (x_n, y_n, z_n) \to X_0 = (a, b, c)$

$\Leftrightarrow \quad \| X_n - X_0 \| \to 0$ as $n \to \infty$

or, $(x_n - a)^2 + (y_n - b)^2 + (z_n - c)^2 \to 0$

as $n \to \infty$

$\Leftrightarrow$

$x_n - a \to 0$ as $n \to \infty$

$y_n - b \to 0$ as $n \to \infty$

$z_n - c \to 0$ as $n \to \infty$

$\Leftrightarrow \quad x_n \to a, \quad y_n \to b, \quad z_n \to c$ as $n \to \infty$

Theorem: A sequence $(X_n = (x_n, y_n, z_n))$ is bounded if and only if each sequence (x_n), (y_n), and (z_n) is bounded.

$(X_n = (x_n, y_n, z_n))$ is bounded means that there exists a real number $M > 0$ such that $\|X_n\| \leq M$ for all n.

$$\Rightarrow \quad x_n^2 + y_n^2 + z_n^2 \leq M^2$$

$$\Rightarrow \quad |x_n| \leq M, \quad |y_n| \leq M \text{ and } |z_n| \leq M$$

$\Rightarrow$ (x_n), (y_n) and (z_n) are bounded sequences of real numbers.

Problem: Every Convergent sequence is bounded.

$\underline{\qquad\qquad} / \underline{\qquad\qquad}$ $x_n \to 0$ as $n \to \infty$

Note that $x_n \to x_0$

$\Rightarrow \|x_n - x_0\| \to 0$ as $n \to \infty$

$\Rightarrow (\|x_n - x_0\|)$ is a bounded sequence of real numbers

$\therefore$

$\Rightarrow (\|x_n\|)$ is a bounded sequence of real numbers

Bolzano-Weierstrass Theorem:

Every bounded sequence in $\mathbb{R}^2$ $((x_n, y_n))$ has a convergent subsequence.

Example. $(1, -1), (-1, -\frac{1}{2}), (1, -\frac{1}{3}), \dots ((-1)^n, -\frac{1}{n}), \dots$

Let $X_n = (x_n, y_n)$ and (X_n) be a bounded sequence in $\mathbb{R}^2$.

So, (x_n) is a bounded sequence of real numbers and by Bolzano - Weierstrass theorem for sequences in $\mathbb{R}$, we find $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ such that

$$x_{n_1}, x_{n_2}, \ldots, x_{n_k}, x_{n_{k+1}}, \ldots$$ is a convergent sequence in $\mathbb{R}$. Let $x_{n_k} \to \alpha$ as $n_k \to \infty$.

Now consider the sequence $\left(X_{n_k} = (x_{n_k}, y_{n_k})\right)$ in $\mathbb{R}^2$.

(y_{n_k}) is a bounded sequence of real numbers and by Bolzano - Weierstrass theorem for sequences in $\mathbb{R}$, we find $n_1' < n_2' < \cdots < n_k' < n_{k+1}' < \cdots$ such that

$y_{n_1'}, y_{n_2'}, \ldots, y_{n_k'}, y_{n_{k+1}'}, \ldots$ is a convergent sequence in $\mathbb{R}$. Let $y_{n_k'} \to \beta$ as $n_k' \to \infty$.

Then $X_{n_k'} = (x_{n_k'}, y_{n_k'}) \to (\alpha, \beta)$ as $n_k' \to \infty$ and $(X_{n_k'})$ is a convergent subsequence of the bounded sequence (X_n).

$$2 < 4 < 6 < \dots < 2n < 2n+2 \quad \dots$$

$$\left(-1, -\frac{1}{2}\right), \left(-1, -\frac{1}{4}\right), \left(-1, -\frac{1}{6}\right), \left(-1, -\frac{1}{2n}\right), \left(-1, -\frac{1}{2n+2}\right), \dots$$

$$\left(-1, -\frac{1}{2n}\right)$$

$$\rightarrow (-1, 0) \quad \text{in } \mathbb{R}^2$$

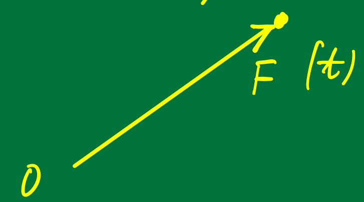
Vector-valued functions

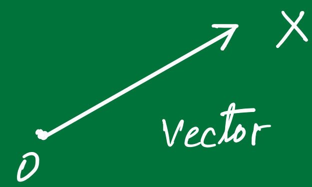
$X \in \mathbb{R}^3$

$(x, y, z) \in \mathbb{R}^3$

Let $I \subset \mathbb{R}$ and $F : I \to \mathbb{R}^3$ is a vector valued function

$t \mapsto F(t)$





$$F(t) = \left(f_1(t), f_2(t), f_3(t) \right) \in \mathbb{R}^3$$

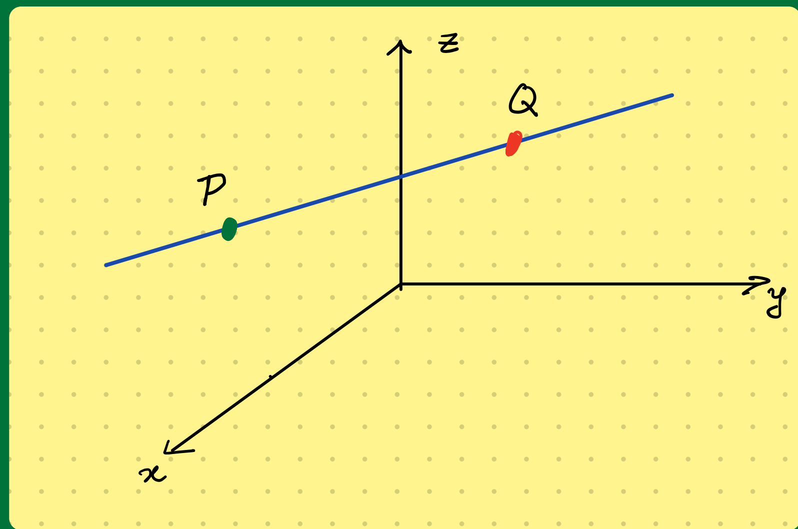
$\Rightarrow$ Each vector valued function $F : I \to \mathbb{R}^3$ determine three real valued functions $f_1, f_2, f_3 : I \to \mathbb{R}$ such that $F(t) = \left(f_1(t), f_2(t), f_3(t) \right)$ for all $t \in I$.

Here $f_1(t)$ is the x - coordinate (Componet) of $F(t)$
$f_2(t)$ is the y - coordinate (Componet) of $F(t)$ and
$f_3(t)$ is the z - coordinate (Componet) of $F(t)$.

Example ①

Let $P, Q \in \mathbb{R}^3$ and $Q \neq 0$.

Consider $F(t) = P + tQ$ for all $t \in \mathbb{R}$.

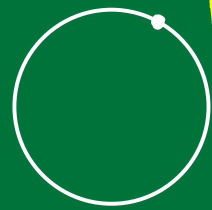


- $F : \mathbb{R} \to \mathbb{R}^3$ is a vector valued function and the range (or the image set of F in $\mathbb{R}^3$) of F is the line through the point P and parallel to the vector Q.

(2)

$F_1 : I \to \mathbb{R}^2$
$t \mapsto (\cos t, \sin t)$

$F_2 : \mathbb{R} \to \mathbb{R}^3$
$t \mapsto (\cos t, \sin t, t)$



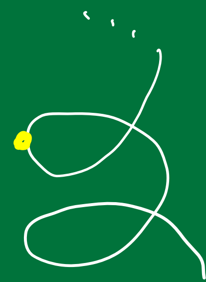
$F_1(t) = (\cos t, \sin t)$

$t \in [0, 2\pi]$

Circle

?

$F_2(t)$



$(\cos t, \sin t, t)$

helix $\quad t \in \mathbb{R}$

Limit
Continuity and
derivative of vector valued
functions

Aim:

- Study curves in space
- Calculus of vector valued functions

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