## lecture 2

1st order ODE: $\frac{dy}{dx} = f(x,y)$

- **Seperable ODE**

In this case, the variables $x, y$ seperate.

**Example** Solve $\frac{dy}{dx} = \frac{y\cos x}{1+y^2}$ $\quad y(0)=1$

**Sol** $\frac{1+y^2}{y} dy = \cos x \, dx$

Integrating $\log y + \frac{y^2}{2} = \sin x + c$

$y(0)=1 \Rightarrow c = 1/2$

**Definition**

$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$

We will call this an Initial Value Problem (IVP)

**Homogenous ODE**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substitute : $y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx} x + v$$

Putting in the given eq.

$\frac{dv}{dx} x + v = f(v) \Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$

— Variables seperated.

**Example 1**

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} = \frac{2\,y/x}{1-\left(\frac{y}{x}\right)^2}$$

Put $y = vx$

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$$\frac{1-v^2}{v(1+v^2)}\,dv = \frac{dx}{x}$$

$$\left(-\frac{2v}{1+v^2} + \frac{1}{v}\right)dv = \frac{dx}{x}$$

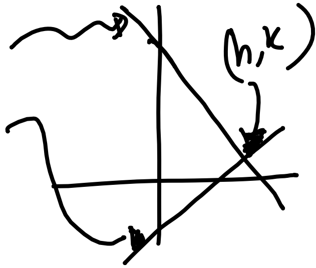
$$-\log(1+v^2) + \log v = \log x + c$$

$$\frac{1+v^2}{v}\cdot x = c' \qquad c' = \text{const}$$

Put $v = y/x$ $\qquad x^2 + y^2 = c'y$

**Example 2**

$$\frac{dy}{dx} = \frac{x+y-3}{x-y-1}$$




Substitute

$$x = X + h \qquad h, k - \text{consts}$$

$$y = Y + k$$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{X+Y+\underline{h+k-3}}{X-Y+\underline{h-k-1}}$$

Choose $h, k$ such that

$$\left.\begin{aligned} h+k-3 &= 0 \\ h-k-1 &= 0 \end{aligned}\right\} \quad \begin{aligned} h &= 2 \\ k &= 1 \end{aligned}$$

With this choice

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \quad \text{— homogeneous ODE}$$

Example $\dfrac{dy}{dx} = \dfrac{x+y-3}{2x+2y-1}$

Put $x+y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} - 1 = \frac{v-3}{2v-1}$$

$$\frac{dv}{dx} = \frac{v-3}{2v-1} + 1 = \frac{3v-4}{2v-1}$$

$$\frac{2v-1}{3v-4}\, dv = dx$$

— Variables separated.

## Exact ODE

$$(2x+y^2) + 2xy\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x+y^2}{2xy}$$

- Variable not separation
- not homogeneous.

$$(2x+y^2)\,dx + 2xy\,dy = 0$$

$$\underbrace{2x\,dx} + \underbrace{y^2\,dx + 2xy\,dy} = 0$$

$$d(x^2) + d(xy^2) = 0$$

$$\left[du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad u(x,y)\right]$$

Integrating $x^2 + xy^2 = \text{const}.$

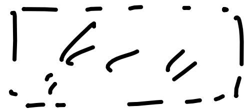
Definition

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{— } (*)$$

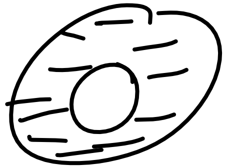
$M, N : D \longrightarrow \mathbb{R}$

where $D \subseteq \mathbb{R}^2$

$D =$ 

$D =$ 

$D = \mathbb{R}^2$

$D =$ 

Then the ODE

(*) is exact on $D$

if $\exists$ a function

$u : D \longrightarrow \mathbb{R}$ such that

$$\frac{\partial u}{\partial x} = M \quad \& \quad \frac{\partial u}{\partial y} = N$$

In this case

$$M\,dx + N\,dy = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$
$$= d(u).$$

So solution of (*) is $u = \text{const}.$

Q) How to check if ~~some~~ an ODE is exact.

**Theorem** $M(x,y)\,dx + N(x,y)\,dy = 0$

$M, N : D \longrightarrow \mathbb{R}$ are $C^1$-functions

$D =$  $\mathbb{R}^2$

Then $M\,dx + N\,dy = 0$ is exact on $D$

$$\iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ on } D.$$

(Recall $C^1$-functions: all partial derivative of order 1 are continuous functions)

**Example**

$$\underbrace{\left(y\cos x + 2xe^{y}\right)}_{M}dx + \underbrace{\left(\sin x + x^2e^{y} - 1\right)}_{N}dy = 0$$

$D = \mathbb{R}^2$

$$M_y = \frac{\partial M}{\partial y} = \cos x + 2xe^{y}$$

$$N_x = \frac{\partial N}{\partial x} = \cos x + 2xe^{y}$$

So by the theorem the given ODE is exact. So $\exists\, u$ s.t

$$\frac{\partial u}{\partial x} = M \quad \Big| \quad \frac{\partial u}{\partial y} = N$$

$$= y\cos x + 2xe^{y}$$

integrating w.r.t $x$: $u = y\sin x + x^2e^{y} + \varphi(y)$

Putting in $\frac{\partial u}{\partial y} = N$

$$\sin x + x^2 e^y + \varphi'(y) = \sin x + x^2 e^y - 1$$

$$\varphi'(y) = -1$$

$$\varphi(y) = -y$$

So $u = y \sin x + x^2 e^y - y$

So Solution is

$$u(x,y) = \text{const}. \quad \blacksquare$$

Example 2

$$M(x,y) = -\frac{y}{x^2+y^2} \qquad N(x,y) = \frac{x}{x^2+y^2}$$

$$D = \mathbb{R}^2 \setminus \{(0,0)\}$$

Check that, $M, N$ are $C^1$-functions

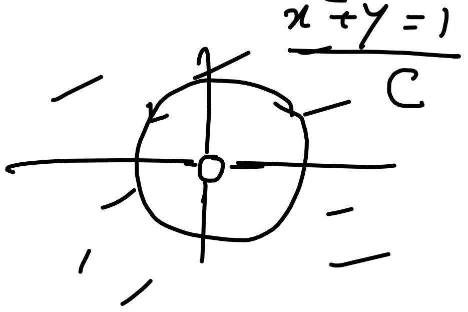
& $M_y = N_x$

But $M\,dx + N\,dy$ is NOT exact.

Assume $M\,dx + N\,dy = du$

Integrate both side over $C$.

$$\int_C du = 0$$



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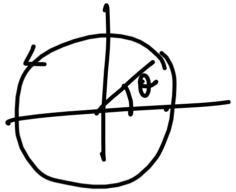


$$\int_C M\,dx + N\,dy$$

$$= \int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$C: x^2+y^2=1$

$x = \cos\theta$

$y = \sin\theta$

$0 \le \theta \le 2\pi$

$$= \int_{\theta=0}^{2\pi} -\frac{\sin\theta}{1}(-\sin\theta)\,d\theta + \frac{\cos\theta}{1}(\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} 1\,d\theta = 2\pi$$

This is a contradiction.

So the ODE is not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$. ∎

## Integrating factor

$$\underbrace{x}_{N}\, dy - \underbrace{y}_{M}\, dx = 0$$

$M = -y \qquad N = x$

$M_y = -1 \qquad N_x = 1$

So the ODE is NOT exact.

But $\dfrac{x\,dy - y\,dx}{x^2} = 0$

$$\overset{\parallel}{d\left(\frac{y}{x}\right)}$$

So $\frac{1}{x^2}$ is called an integrating factor.

**Definition.**
A function $\mu(x,y)$ is called an IF for $M\,dx + N\,dy = 0$ if

$\mu M\,dx + \mu N\,dy = 0$ is exact.

How to find integrating factor?

**Theorem.**

(i) $\frac{1}{N}(M_y - N_x) = f(x)$ is a function of $x$ only, then

$e^{\int f(x)\,dx}$ is an IF.

(ii) $\frac{1}{M}(M_y - N_x) = g(y)$ then

$e^{-\int g(y)\,dy}$ is an IF.

Exmple

$\underbrace{(8xy - 9y^2)}_{M} dx + \underbrace{(2x^2 - 6xy)}_{N} dy = 0$

$M_y = 8x - 18y$

$N_x = 4x - 6y.$

$\frac{1}{N}(M_y - N_x) = \frac{4x - 12y}{2x^2 - 6xy}$

$= \frac{2}{x}$

So $IF = e^{\int \frac{2}{x} dx} = x^2$

Multipy y by $x^2$

$(8x^3y - 9x^2y^2)dx + (2x^4 - 6x^3y)dy = 0$

$d(2x^4y) - d(3x^3y^2) = 0$

So soln $2x^4y - 3x^3y^2 = const$