lecture 10

(Second order non-homogeneous linear ODE)

Recall $y'' + p(x)y' + q(x)y = r(x)$ — (*)

$y'' + p(x)y' + q(x)y = 0$ — (**).

Any solution of (*) looks like

$y = c_1 y_1 + c_2 y_2 + y_p$

y_1, y_2 are two LI sols of (**)

y_p = some particular solution of (*).

Q) How to find y_p?

Variation of Parameters

Suppose we know two LI solutions y_1, y_2 of (**)

Claim $\exists$ $u(x), v(x)$ such that

$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$.

$y_p' = (\underbrace{u'y_1 + v'y_2}_{=0}) + (uy_1' + vy_2')$

$y_p'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$

Then

$y_p'' + p(x)y_p' + q(x)y_p$

$= \underbrace{u'y_1' + v'y_2'}_{r(x)}$

Thus we want

$$u' y_1 + v' y_2 = 0$$

$$u' y_1' + v' y_2' = r(x)$$

Solving:

$$u'(x) = -\frac{r(x)\, y_2(x)}{W(y_1, y_2)} \Rightarrow u = \int \cdots$$

$$v'(x) = +\frac{r(x)\, y_1(x)}{W(y_1, y_2)} \Rightarrow v = \int \cdots$$

Example Solve $y'' + y = \tan x$.

Sol Homogeneous equation

$$y'' + y = 0$$

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$W(y_1, y_2) = 1$$

So particular soln $y_p = u y_1 + v y_2$

where $u' = -\frac{y_2\, r(x)}{W} = -\tan x \sin x$

$$v' = \frac{r(x)\, y_1}{W} = \sin x$$

So $u = -\log(\sec x + \tan x) + \sin x$

$$v = -\cos x$$

Hence the general soln of the given ODE

$$y = c_1 \cos x + c_2 \sin x + u \cos x + v \sin x$$

Exmple

$$x^2 y'' - x y' - 3y = \frac{\log x}{x} \qquad x>0.$$

Sol $\quad x = e^t$

$$\ddot{y} - 2\dot{y} - 3y = t e^{-t}$$

Homogeon prt: $\qquad r(t) = t e^{-t}$

$$\ddot{y} - 2\dot{y} - 3y = 0$$

$$y_1 = e^{-t} \qquad y_2 = e^{3t}$$

$$\boxed{W(y_1, y_2) = 4 e^{2t}}$$

$$u' = -\frac{r y_2}{W} = -t/4 \Rightarrow u = -t^2/8.$$

$$v' = \frac{r y_1}{W} = \frac{t}{4} e^{-4t} \Rightarrow v = -\frac{t e^{-4t}}{16} - \frac{1}{64} e^{-4t}$$

General Solutn

$$y = c_1 e^{-t} + c_2 e^{3t} + \left(-\frac{t^2}{8}\right) e^{-t} + v e^{3t}$$

$$= \frac{c_1}{x} + c_2 x^3 - \frac{(\log x)^2}{8x} - \frac{\log x}{16 x}.$$

$\blacksquare$

Zeros of Solution

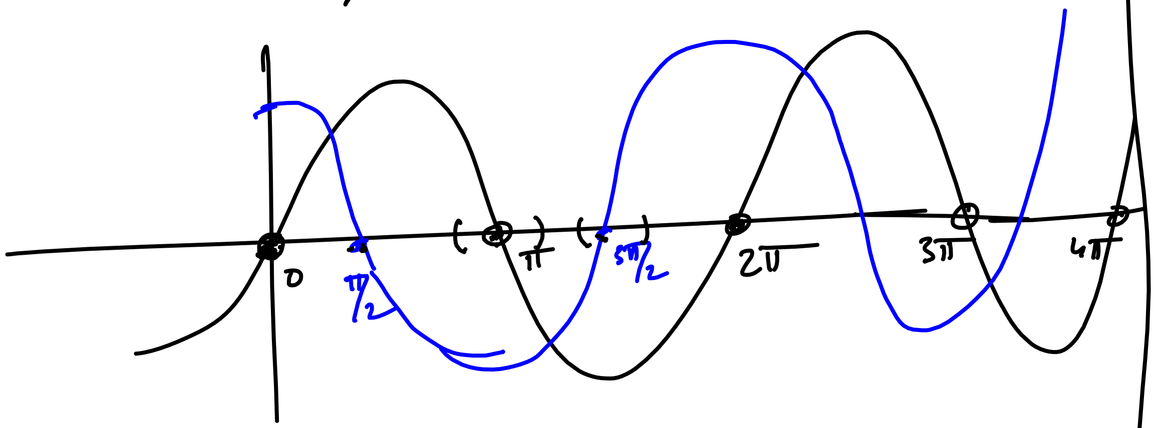
$$y'' + p(x)y' + q(x)y = 0 \quad \text{--- } (**)$$

Let $y(x)$ be a non-trivial solution of $(**)$.

$y(\alpha)=0$: α is a zero of y.

Example $y = \sin x$ $\quad y = \cos x.$

$$y'' + y = 0$$



(i) The zeros of $y(x)$ are isolated points.

Proof Assume $y(\alpha)=0$

If α is not isolated $\quad y(x_n)=0$

i.e. $\exists \{x_n\} \longrightarrow \alpha$

$y(\alpha)=0$

$$y'(\alpha) = \lim_{x\to\alpha} \frac{y(x) - \cancel{y(\alpha)}^{\,0}}{x-\alpha}$$

$$= \lim_{x\to\alpha} \frac{y(x)}{x-\alpha}$$

$$= \lim_{x_n\to\alpha} \frac{\cancel{y(x_n)}^{\,0}}{x_n-\alpha}$$

$$= 0.$$

Hence by uniqueness $y \equiv 0 \rightarrow\leftarrow$ ▢

(ii) If y_1 & y_2 are two independent solutions of (**), then they DO NOT have a common zero.

If $y_1(\alpha) = 0 = y_2(\alpha)$.

$$W(y_1, y_2)(\alpha) = 0.$$

Since y_1, y_2 are LI $\rightarrow\leftarrow$

Theorem (Sturm Seperation Theorem)

Let y_1, y_2 be two LI sol of (**). Let α, β be two consecutive zeros of y_1. Then $\exists$ exactly one zero of y_2 in $(\alpha\ \beta)$.

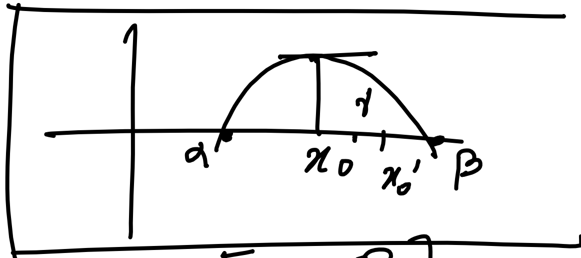
Proof Given $y_1(\alpha) = y_1(\beta) = 0$

To show $\exists\ \alpha < x_0 < \beta$ s.t. $y_2(x_0) = 0$

If $y_2(x) \neq 0\ \forall\ \alpha < x < \beta$

$\Rightarrow y_2(x) \neq 0\ \forall\ \alpha \leq x \leq \beta$.

$$\varphi(x) = \frac{y_1(x)}{y_2(x)}$$



- $\varphi(x)$ is continuous on $[\alpha\ \beta]$
- $\varphi(x)$ is differentiable on $(\alpha\ \beta)$.
- $\varphi(\alpha) = 0$, $\varphi(\beta) = 0$

Hence by Rolle's Th. $\exists\ \alpha < x_0 < \beta$

$\varphi'(x_0) = 0 \Rightarrow W(y_1, y_2)(x_0) = 0$ $\rightarrow\leftarrow$

Example

① Zeros of $2\sin x + 5\cos x = y_1$
$3\sin x + 2\cos x = y_2$

appear alternatively.

$$y'' + y = 0$$

② $y_1 = \cos(\log x)$ $y_2 = \sin(\log x)$.

They satisfy Cauchy-Euler eqn

$$x^2 y'' + x y' + y = 0 \quad x > 0$$

Sturm Comparison Theorem

$$y'' + Q_1(x)\, y = 0$$
$$y'' + Q_2(x)\, y = 0$$

where Q_1, Q_2 are continuous fns on some interval I.

Assume $\underline{Q_1(x) \leq Q_2(x)}$ $\forall x \in I$.

Let φ_1 be a non-trivial sol of 1st eqn

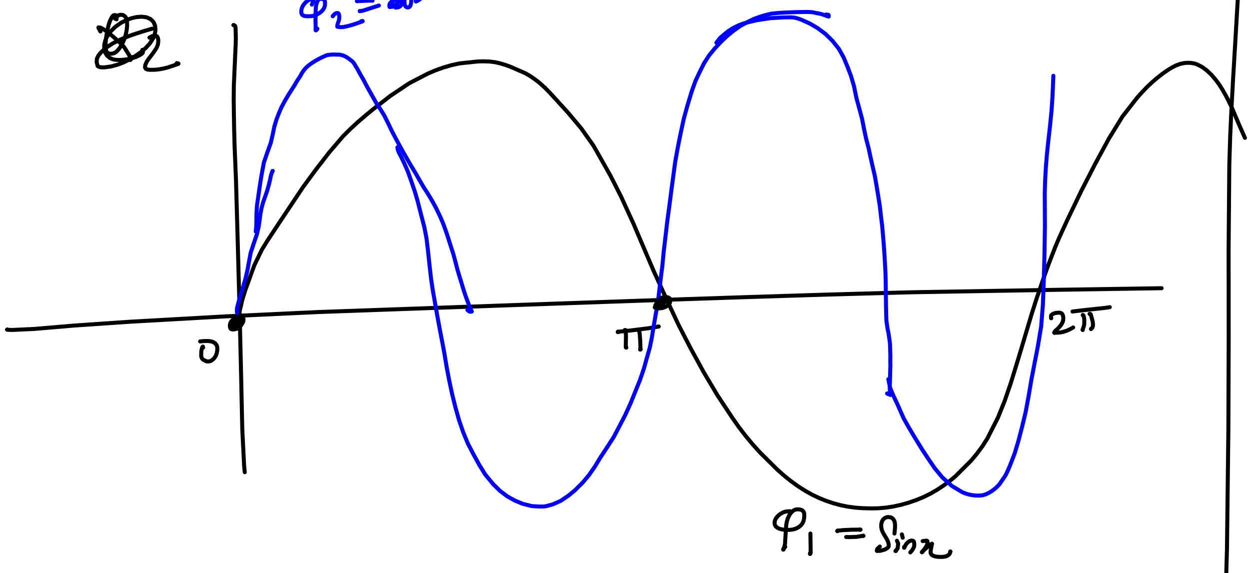
φ_2 - - - - - - - 2nd - - -

Then between two consecutive zeros of φ_1 $\exists$ at least one zero of φ_2, unless $Q_1 \equiv Q_2$

Example 1 $y'' + y = 0 \qquad \varphi_1 = \sin x$

$y'' + 4y = 0 \qquad \varphi_2 = \sin 2x$

$Q_1(x) = 1 \le Q_2(x) = 4.$



Example 2

$y'' + (1 + \sin^2 x)y = 0.$

show that any non-trivial sol has infinitely many zero.

Sol compare with

$y'' + y = 0$

$Q_1 = 1 \qquad Q_2 = 1 + \sin^2 x$

$Q_1 \le Q_2$

$\varphi_1 = \sin x.$