lecture 12 (Power Series solution of ODE)

Power Series

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$a_i \in \mathbb{R}$.

Example:

(i) $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ $\quad |x|<1$ $\quad R=1$

(ii) $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots = e^x$ $\quad x \in \mathbb{R}$ $\quad R=\infty$

(iii) $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots = \cos x$ $\quad R=\infty$

(iv) $x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \cdots = \sin x$ $\quad R=\infty$

(v) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$= \log(1+x)$

$R = 1$ $\quad -1 < x \leq 1$

Remark (1) $\sum_{n\geq 0} a_n x^n$ - conv for $x=0$

Question For which values of x does the power series converge?

Remark 2

$= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$

$\sum_{n\geq 0}^{\infty} a_n (x-x_0)^n$

Power Series around x_0

This convs for $x = x_0$.

Theorem

Given a power series, $\exists\,!$ real number R $(0 \le R \le \infty)$ such that

(i) The power series conv for $|x| < R$

(ii) - - - - - - - divg - - $|x| > R$

(Remark: The theorem does not say anything for $|x| = R$)

Definition This real number R is called the radius of convergence of the power series.

Recall $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - -$ conv.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - -$

— divg.

How to find R?

Theorem (Ratio test)

- $\lim \frac{|a_{n+1}|}{|a_n|} = L$

then $R = \frac{1}{L}$ $\left(\begin{array}{l} L = 0 \Rightarrow R = \infty \\ L = \infty \Rightarrow R = 0 \end{array} \right)$

- Root test

$$\frac{1}{R} = \limsup \left(\sqrt[n]{|a_n|} \right)$$

Example Find R for $\sum \frac{(-2)^n}{n+1}(x-3)^n$.

Sol $a_n = \frac{(-2)^n}{n+1}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_n \frac{2^{n+1}/n+2}{2^n/n+1}$$

$$= \lim 2\left(\frac{n+1}{n+2}\right)$$

$$= 2$$

$R = \frac{1}{2}$

Thus the power series convg for $|x-3| < \frac{1}{2}$ & divg for $|x-3| > \frac{1}{2}$ ∎

Remark For $|x-3| = \frac{1}{2}$

$x-3 = \pm\frac{1}{2}$

$x-3 = \frac{1}{2}$ $\quad \sum \frac{(-2)^n}{n+1}\frac{1}{2^n}$

$= \sum \frac{(-1)^n}{n+1}$ - convg

$x-3 = -\frac{1}{2}$ $\quad \sum \frac{(-2)^n}{n+1}\left(\frac{-1}{2}\right)^n$

$= \sum \frac{1}{n+1}$ - divg.

∎

Theorem (I) (Power Series can be differentiated term by term)

$$f(x) = \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \quad \boxed{|x| < R}$$

Then $f(x)$ is differentiable. with

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad |x| < R$$

($\Rightarrow$ f is infinitely differentiable
$a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = \frac{f''(0)}{\lfloor 2}$
$a_n = f^{(n)}(0) / \lfloor n$.)

(II) Termwise integration

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

$|x| < R$.

Example (1)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$|x| < 1$

diff

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$|x| < 1$.

integrating

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$|x| < 1$.

Q) Which function can be written as a power series?

Example

(i) $|x| \overset{??}{=} \sum a_n x^n$ — NO!

(not diff)

(ii) $x|x| \overset{??}{=} \sum a_n x^n$ — NO!

(not twice dif)

(iii) $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Ex

$$f^{(n)}(0) = 0$$

$$f(x) \overset{??}{=} \sum a_n x^n$$

$$a_n = \frac{f^{(n)}(0)}{\lfloor n} = 0$$

(Infinitely diff $\not\Longrightarrow$ may not be represented as power series)

Definition (Analytic function)

A function $f(x)$ is called analytic at point x_0, iff f can be written as power series around x_0.

$$f(x) = \sum a_n (x - x_0)^n \text{ near } x_0.$$

Example

(i) polynomial,

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

— anlytic on $\mathbb{R}$.

(ii) $\sin x / \cos x / e^x$ — anlytic on $\mathbb{R}$

(iii) $\dfrac{1}{1-x}$ anlytic for $\mathbb{R} - \{1\}$

(iv) $\dfrac{1}{1-x^2}$ anlytic for $\mathbb{R} - \{\pm 1\}$

(v) $\dfrac{1}{\sin x}$ anlytic on $\mathbb{R} - \{n\pi\}$

(vi) $\dfrac{\sin x}{x} = \dfrac{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots}{x} = 1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \cdots$

anlytic on $\mathbb{R}$.

• f, g are anlytic at x_0

$\Rightarrow$ $f+g$ anlytic at x_0

$f \cdot g$

$\left(f = \sum a_n (x-x_0)^n \quad g = \sum b_n (x-x_0)^n \right.$

$\left. f+g = \sum (a_n + b_n)(x-x_0)^n \right)$

• f, g are anlytic at x_0 + $g(x_0) \neq 0$

$\Rightarrow$ $\dfrac{f}{g}$ is anlytic at x_0.

$g(x_0) = b_0 \neq 0$

$$\frac{f}{g} = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$\sum a_n (x-x_0)^n = \left(\sum c_n (x-x_0)^n \right) \left(\sum b_n (x-x_0)^n \right)$$

$a_0 = c_0 b_0$ $\qquad a_1 = c_0 b_1 + c_1 b_0$

$c_0 = a_0 / b_0$ $\qquad c_1 = \dfrac{a_1 - c_0 b_1}{b_0}$

• $\frac{1}{1-x}$ analytic on $\mathbb{R}-\{1\}$.

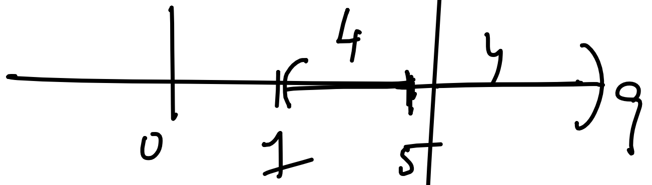
$x_0=0$

$$\frac{1}{1-x} = 1+x+x^2+\cdots \qquad |x|<1$$

Take $x_0=5$ what is the power series around this!

$$\frac{1}{1-x}$$

$$= \frac{1}{-4-(x-5)}$$



$$= -\frac{1}{4}\,\frac{1}{\left(1+\frac{x-5}{4}\right)} \qquad \left|\frac{x-5}{4}\right|<1 \Rightarrow 1<x<9$$

$$= -\frac{1}{4}\left(1-\left(\frac{x-5}{4}\right)+\left(\frac{x-5}{4}\right)^2-\left(\frac{x-5}{4}\right)^3+\cdots\right)$$