MTH 114: ODE: Assignment-5

1. Solve: (i) $x^2y'' + 2xy' - 12y = 0$ (ii)(**T**) $x^2y'' + 5xy' + 13y = 0$ (iii) $x^2y'' - xy' + y = 0$

Solution: [Recall: The ODE of the form $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$, where a, b are constants, is called the Cauchy-Euler equation. Under the transformation $x = e^t$ (when $x > 0$) for the independent variable, the above reduces to $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$, which is an equation with constant coefficients.]

(i) Using the substitution $x = e^t$, the given equation reduces to

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 12u = 0 \implies m^2 + m - 12 = 0 \implies m = -4, 3 \implies u(t) = Ae^{-4t} + Be^{3t} = y(e^t).$$

The general solution is thus

$$y(x) = \frac{A}{x^4} + Bx^3.$$

(ii) Using the substitution $x = e^t$, the given equation reduces to,

$$\frac{d^2u}{dt^2} + 4\frac{du}{dt} + 13u = 0 \implies m^2 + 4m + 13 = 0 \implies m = -2 \pm 3i.$$

Thus

$$u(t) = e^{-2t}(A\cos 3t + B\sin 3t) = y(e^t).$$

The general solution is

$$y(x) = \frac{1}{x^2}\big(A\cos(3\ln x) + B\sin(3\ln x)\big).$$

(iii) Using the substitution $x = e^t$, the given equation reduces to

$$\frac{d^2u}{dt^2} - 2\frac{du}{dt} + u = 0 \implies m^2 - 2m + 1 = 0 \implies m = 1, 1 \implies u(t) = e^t(A + Bt) = y(e^t)$$

The general solution is thus

$$y(x) = e^x(A + B\ln x).$$

2. (i) Let $y_1(x), y_2(x)$ are two linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$. Show that $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions if and only if $\alpha\delta \neq \beta\gamma$.

(ii) Show that the zeros of the functions $a\sin x + b\cos x$ and $c\sin x + d\cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$.

Solution:

(i) We have $W(\phi, \psi) = (\alpha\delta - \beta\gamma)W(y_1, y_2)$. Since y_1, y_2 are fundamental solutions, $W(y_1, y_2) \neq 0$. If $\alpha\delta \neq \beta\gamma$, then $W(\phi, \psi) \neq 0$. Conversely if $W(\phi, \psi) \neq 0$, then $\alpha\delta \neq \beta\gamma$.

(ii) We know $\sin x, \cos x$ are independent solutions of $y'' + y = 0$. So by part (i) $a\sin x + b\cos x$ and $c\sin x + d\cos x$ are independent solutions whenever $ad - bc \neq 0$. Hence the result follows from Sturm Separation theorem (Simmons, page 190, Theorem A).

3. (**T**) Show that any nontrivial solution $u(x)$ of $u'' + q(x)u = 0,\ q(x) < 0$ for all x, has at most one zero.

   Solution:

   Consider the equation $z'' = 0$. Then $z = 1$ is a solution of the equation. By Strum comparison theorem, between two zeros of $u(x)$ there must be at least one zero of $z(x)$. But $z = 1$ has no zero. Hence $u(x)$ can have at most one zero.

4. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

   Solution:

   Consider

   $$v'' + v = 0, \qquad u'' + \big(1 + q(x)\big)u = 0$$

   Now $v = \sin x$ is a nontrivial solution of $v'' + v = 0$. Since $1 + q(x) > 1$, by Strum comparison theorem, u must vanish between two zeros of $\sin x$. Since, $\sin x$ has infinitely many zeros, u also has infinitely may zeros.

5. Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

   Solution:

   Suppose, on the contrary, $u(x)$ has infinite number of zeros in $[a, b]$. It follows that there exists $x_0 \in [a, b]$ and a sequence of zeros $x_n \neq x_0$ such that $x_n \to x_0$. Since $u(x)$ is continuous and differentiable at x_0, we have

   $$u(x_0) = \lim_{x_n \to x_0} u(x_n) = 0, \qquad u'(x_0) = \lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0$$

   By uniqueness theorem, $u \equiv 0$ which contradicts the fact that u is nontrivial.

6. Let J_p be any non-trivial solution of the Bessel equation

   $$x^2 y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

   Show that J_p has infinitely many positive zeros.

   Solution:

   The normal form of Bessel equation is

   $$u'' + (1 + \frac{1/4 - p^2}{x^2})u = 0.$$

   Given $p \geq 0$, we can choose x_0 large enough such that $1+\frac{1/4-p^2}{x^2} > 1/4$ for all $x \in (x_0, \infty)$. Compare J_p with $\sin(x/2)$ which is solution of $v''+\frac{1}{4}v = 0$ in (x_0, ∞). Clearly $\sin(x/2)$ has infinitely many zeros in (x_0, ∞). By Sturm comparison theorem, between two consecutive zeros of $\sin(x/2)$ there is a zero of J_p. Hence J_p has infinitely many zero in (x_0, ∞).

7. (**T**) Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) > m^2$ for all $t \in I$. Show any non trivial solution $u(x)$ has infinitely many zeros and distance between two consecutive zeros is at most π/m.

   Solution: Compare $u(x)$ with $\sin mx$ which is a solution of $v'' + m^2 v = 0$. By Sturm comparison theorem, between two consecutive zeros of $v(x) = \sin(mx)$ there is a zero of $u(x)$. Hence $u(x)$ has infinitely many zero in (x_0, ∞).

   Let $u(a) = 0$. We will show that $u(x)$ has a zero in $(a, a + \pi/m]$. Consider $v(x) = \sin(mx - ma)$ which is a solution of $v'' + m^2 v = 0$. Clearly $v(a) = v(a + \pi/m) = 0$. Hence by Sturm comparison theorem, there exists at least one zero of $u(x)$ in $(a, a + \pi/m)$. Hence distance between two consecutive zeros of $u(x)$ is at most π/m.

8. Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) < m^2$ for all $t \in I$. Show that distance between two consecutive zeros is at least π/m.

   Solution:

   Suppose $u(a) = 0$ and $u(b)$ be two consecutive zeros. Consider $v(x) = \sin(mx - ma)$ which is a solution of $v'' + m^2 v = 0$. By Sturm comparison theorem, there exists a zero of $v(x)$ in (a, b). But we know that $v(a) = 0$ and next zero of v is at $a + \pi/m$. So $b > a + \pi/m$.

9. (**T**) Let J_p be any non-trivial solution of the Bessel equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

   Show that (i) If $0 \le p < 1/2$, then every interval of length π has at least contains at least one zero of J_p.

   (ii) If $p = 1/2$ then distance between consecutive zeros of J_p is exactly π.

   (iii) If $p > 1/2$ then every interval of length π contains at most one zero of J_p.

   Solution: The normal form of Bessel equation is

$$u'' + \left(1 + \frac{1/4 - p^2}{x^2}\right)u = 0.$$

   The zeros of J_p and $u(x)$ are same.

   (i)Apply exercise 7 with $m = 1$.

   (ii) Clear from normal form.

   (iii) Apply exercise 8 with $m = 1$.

10. Let $y(x)$ be a non-trivial solution of $y'' + q(x)y = 0$. Prove that if $q(x) > k/x^2$ for some $k > 1/4$ then y has infinitely many positive zeros. If $q(x) < \frac{1}{4x^2}$ then y has only finitely many positive zeros.

    Solution:

Consider the Cauchy-Euler equation $y'' + \frac{ky}{x^2} = 0$. With $x = e^t$, it transforms into $y'' - y' + ky = 0$. So characteristic equation $m^2 - m + k = 0$. So $1 - 4k = 0$ implies two equal real roots and so the solution has finitely many zeros. If $1 - 4k < 0$ then complex conjugate roots and solution look like $x^m \sin(\beta x)$ and it has infinitely many zeros. Rest follows from Sturm comparison theorem.

11. Find the eigen values and eigen functions of the following Sturm-Liouville problems:

    (i) (**T**) $y'' + \lambda y = 0, \quad y(0) = y'(1) + y(1) = 0$

    (ii) $(e^{2x}y')' + (\lambda + 1)e^{2x}y = 0, \quad y(0) = y(\pi) = 0. \qquad$ [Substitute $y = e^{-x}u$]

    [Recall: (Sturm-Liouville Boundary Value Problem (SL-BVP)) With the notation

    $$L[y] = \frac{d}{dx}(p(x)\frac{dy}{dx}) + q(x)y$$

    consider the Sturm-Liouville equation

    $$L[y] + \lambda r(x)y = 0$$

    where $p > 0$, $r \geq 0$, and p, q, r are continuous functions on interval [a, b]; along with the boundary conditions $a_1y(a) + a_2p(a)y'(a) = 0, \; b_1y(b) + b_2p(b)y'(b) = 0$ where $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$. The problem of finding a values of λ if any, such that the BVP has a non-trivial solution is called a Sturm-Liouville Eigen Value Problem (SL-EVP). Such a value of λ is called an eigenvalue and the corresponding non-trivial solutions are called eigenfunctions.]

    Solution:

    (i) $\lambda \leq 0$ leads to trivial solution. Thus, let $\lambda = p^2 > 0$. Then $y = c_1 \cos px + c_2 \sin px$. Using the boundary conditions $c_1 = 0$ and $\sin p + p\cos p = 0$ or $p + \tan p = 0$. This has infinite number of roots (plot the curves $y = -x$ and $y = \tan x$). Thus, the eigen values are the roots of the above equation and the eigen functions are $y_p = \sin px$.

    (ii) Using the given transformations, we get $u'' + \lambda u = 0$. Again for $\lambda \leq 0$ trivial solution. Thus, $\lambda = p^2 > 0$ and $y = c_1 \cos px + c_2 \sin px$. The transformed BCs are $u(0) = u(\pi) = 0$ and thus $c_1 = 0$ and $p = n, \; n = 1, 2, 3, \cdots$. Thus, $\lambda_n = n^2$ and $y_n = e^{-x} \sin nx$.

12. If $p(x), q(x), r(x)$ are all greater than zero on (a, b), then prove that the eigen values of the Sturm-Liouville problem, $(p(x)y')' + q(x)y + \lambda r(x)y = 0$, are positive with any of the boundary conditions: (i) $p(a) = p(b) = 0$, (ii) $y(a) - ky'(a) = y(b) + my'(b) = 0, k, m > 0$, (iii) $p(a) = p(b)$ with $y(b) = y(a), \; y'(b) = y'(a)$.

    Solution:

    Multiplying by y and using integration by parts, we get

    $$\lambda \int_a^b ry^2\, dx = \int_a^b qy^2\, dx + \int_a^b py'^2\, dx - [pyy']_a^b$$

(i) $p(a) = p(b) = 0 \implies [pyy']_a^b = 0$ (ii) $y(a) - ky'(a) = y(b) + my'(b) = 0, k, m > 0, \implies [pyy']_a^b = -mp(b)y'(b)^2 - kp(a)y'(a)^2$ (iii) $p(a) = p(b)$ with $y(b) = y(a),\ y'(b) = y'(a) \implies [pyy']_a^b = 0$

Thus, in (i) & (iii) $[pyy']_a^b = 0$ and in (ii) $[pyy']_a^b \leq 0$. Thus, λ is positive.

13. (**T**) Consider the Strum-Liouville problem

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$

with $p(x) > 0$ on $[a, b]$ and $y(a) \neq y(b)$, $y'(a) \neq y'(b)$. Show that every eigen function is unique except for a constant factor.

Solution:

The boundary conditions $y(a) \neq y(b) \implies$ either $y(a) \neq 0$ or $y(b) \neq 0$ and $y'(a) \neq y'(b) \implies$ either $y'(a) \neq 0$ or $y'(b) \neq 0$. Also $y(a) = y'(a) = 0$ is not possible since then we get trivial solution only. Similarly $y(b) = y'(b) = 0$ is not possible. Thus, we can write the BCs as

$$c_1 y(a) + c_2 y'(a) = 0 \quad \text{and} \quad d_1 y(b) + d_2 y'(b) = 0$$

where c_1 or c_2 not equal to zero and d_1 or d_2 not equal to zero.

Let u and v are eigen functions corrsponding to an eigen value λ. Then $(pu')' + qu + \lambda ru = 0$ and $(pv')' + qv + \lambda rv = 0$. Multiplying the 1st by v and the second by u and subtracting we get $[pW(u, v)]' = 0$ where W is the Wronskian. Since u and v satisfy the above BCs, $W(u, v) = 0$ at $x = a$ and $x = b$. Thus, $pW(u, v) \equiv 0$ or $W(u, v) \equiv 0$. Hence u and v are linealy dependent.