Lecture 17 : Diagonalization of Matrices, Examples and application

Recall that a square matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. The following theorem characterizes when a matrix is diagonalizable.

Theorem: Let A be a square matrix of order n. A is diagonalizable if and only if it has n L.I. eigen vectors.

Proof: Suppose A is diagonalizable then $\exists$ an invertible matrix P of order n such that $P^{-1}AP = D$, where D is a diagonal matrix. Let c_i be column vectors of P, $1 \le i \le n$. & $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

Now $AP = PD \Rightarrow A(c_1 \cdots c_i \cdots c_n) = (c_1 \cdots c_i \cdots c_n)D$

$\Rightarrow \quad Ac_i = \lambda_i c_i \quad 1 \le i \le n.$

$\Rightarrow$ c_i is an eigen vector.

P is invertible $\Rightarrow$ rank$(P) = n$

$= \{c_1, c_2, \ldots, c_n\}$ is L.I.

Conversely, suppose A has n L.I. eigen vectors $c_1, c_2, \ldots, c_n$. Then $\exists$ $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$Ac_i = \lambda_i c_i, \quad 1 \le i \le n. \quad - (\star)$$

Let $P = (c_1\ c_2 \ldots c_n)$

$c_1, c_2, \ldots, c_n$ are L.I. set of vectors

$\Rightarrow$ P is invertible.

$\star$ implies $\quad AP = PD, \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}$

$\Rightarrow \quad P^{-1}AP = D.$

Theorem: Let A be a square matrix of order n having distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let c_i be a non-zero eigen vector corresponding to eigen value λ_i. Then $c_1, c_2, \ldots, c_n$ are L.I.

<u>Proof</u>: $\quad A c_i = \lambda_i c_i, \quad 1 \le i \le n$

If possible, let $c_1, c_2, \ldots, c_n$ be L.D. . There exists $1 \le p < n$ such that $c_1, c_2, \ldots, c_p$ is L.I. and p is maximal $\Rightarrow c_1, c_2, \ldots, c_p, c_{p+1}$ are L.D $\Rightarrow \exists\ \alpha_i$'s, $1 \le i \le p+1$ not all zero such that

$$\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_p c_p + \alpha_{p+1} c_{p+1} = 0 \quad - \star$$

Note that $\alpha_{p+1} \ne 0$ & at least one of $\alpha_1, \ldots \alpha_p$ is not zero.

Now, $A(\alpha_1 c_1 + \cdots + \alpha_{p+1} c_{p+1}) = 0$

$$\Rightarrow \alpha_1 A c_1 + \cdots + \alpha_{p+1} A c_{p+1} = 0$$

$$\Rightarrow \alpha_1 \lambda_1 c_1 + \cdots + \alpha_p \lambda_p c_p + \alpha_{p+1} \lambda_{p+1} c_{p+1} = 0 \quad - \star\star$$

$$\lambda_{p+1} \times \star - \star\star \Rightarrow$$

$$\alpha_1 (\lambda_{p+1} - \lambda_1) c_1 + \cdots + \alpha_p (\lambda_{p+1} - \lambda_p) c_p = 0$$

$\{c_1, c_2, \ldots, c_p\}$ is L.I $\Rightarrow \alpha_i (\lambda_{p+1} - \lambda_i) = 0$

$\exists\ i$ such that $\alpha_i \ne 0 \Rightarrow \lambda_i = \lambda_{p+1}$

This contradicts the fact that $\lambda_1, \ldots, \lambda_n$ are distinct.

Therefore, $c_1, c_2, \ldots, c_n$ are L.I. $\boxtimes$

Corollary: If a square matrix has distinct eigen values then it is diagonalizable.

Application of Diagonalization:

Fibonacci Numbers: Consider the sequence $\{F_n\}$ of numbers defined recursively as: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

Thus the sequence is

$0, 1, 1, 2, 3, 5, 8, 13, \ldots$

Binet Formula:

Consider $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$,

$F_{n+1} = F_{n+1}$.

$$\Rightarrow \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

$$u_{n+1} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix}, \quad u_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

$$u_{n+1} = A u_n, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Characteristic equation of A is

$$x^2 - x - 1 = 0$$

Its roots are $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

A has distinct eigen values implies A is diagonalizable.

Let $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$

Let X_1 be non-zero eigen vector corresponding to λ_1 & let $X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$AX_1 = \lambda_1 X_1 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_1 x_2 \end{pmatrix}$$

$$\Rightarrow \quad x_1 + x_2 = \lambda_1 x_1, \quad x_1 = \lambda_1 x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to λ_1.

Similarly, $\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to λ_2.

Let $P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$, then

$$P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow P^{-1} A^n P = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \quad \forall\, n \geq 1$$

$$\Rightarrow A^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}$$

$$u_{n+1} = A u_n = A^2 u_{n-1} = \cdots = A^{n+1} u_0$$

So,

$$u_{n+1} = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} u_0 \quad \& \quad u_0 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Check $u_n = \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}$

& $u_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$

$\Rightarrow F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$

So, $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$