Eigen Value Problem

Eigenvalue Problem: The Power Method

Power method is normally used to determine the largest eigenvalue (in magnitude) and the corresponding eigenvector of the system

$$Ax = \lambda x.$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

and further assume that the corresponding eigenvectors $v_1, v_2, \cdots, v_n$ forms a basis for $\mathbb{R}^n$. Therefore, any vector $v \in \mathbb{R}^n$ can be written as

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

Premultiplying by A and substituting $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, $i = 1, \cdots, n$, we get

$$\begin{aligned} A\boldsymbol{v} &= c_1\lambda_1\boldsymbol{v}_1 + \cdots + c_n\lambda_n\boldsymbol{v}_n \\ &= \lambda_1 \left(c_1\boldsymbol{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right) \boldsymbol{v}_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right) \boldsymbol{v}_n \right) \end{aligned}$$

Premultiplying by A again and simplying, we get

$$\begin{aligned} A^2\boldsymbol{v} &= \lambda_1^2 \left(c_1\boldsymbol{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^2 \boldsymbol{v}_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^2 \boldsymbol{v}_n \right) \\ &\cdots \\ &\cdots \\ &\cdots \\ A^k\boldsymbol{v} &= \lambda_1^k \left(c_1\boldsymbol{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \boldsymbol{v}_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k \boldsymbol{v}_n \right) \end{aligned}$$

Using the assumption $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$ we can see that

$$\left|\frac{\lambda_k}{\lambda_1}\right| < 1, \quad k = 2, \cdots, n.$$

Therefore, we have

$$\lim_{k\to\infty} \frac{A^k\boldsymbol{v}}{\lambda_1^k} = c_1\boldsymbol{v}_1.$$

For $c_1 \neq 0$, the RHS is a scalar multiple of the eigenvector.

Also, from the above expression for $A^k\boldsymbol{v}$, we get

$$\lim_{k\to\infty} \frac{(A^{k+1}\boldsymbol{v})_i}{(A^k\boldsymbol{v})_i} = \lambda_1,$$

where i denotes a component of the corresponding vectors.

Eigenvalue Problem: The Power Method

Algorithm Choose an arbitrary initial guess $x^{(0)}$. For $k = 1, 2, \cdots$

Step 1 Compute $y^{(k)} = Ax^{(k-1)}$

Step 2 Take $\mu_k = y_i^{(k)}$, where $\|\boldsymbol{y}^{(k)}\|_\infty = |y_i^{(k)}|$,

Step 3 Set $x^{(k)} = \dfrac{\boldsymbol{y}^{(k)}}{\mu_k}$.

Step 4 If $\|x^{(k-1)} - x^{(k)}\|_\infty > \epsilon$, go to step 1.

For some pre-assigned positive quantity ϵ.

Example Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda_1 = 3$, $\lambda_2 = 1$ and $\lambda_3 = 0$. The corresponding eigen vectors are $\boldsymbol{X}_1 = (1, 0, 2)^T$, $\boldsymbol{X}_2 = (0, 2, -5)^T$ and $\boldsymbol{X}_3 = (0, 1, -3)^T$.

Initial Guess 1: Let us take $x_0 = (1, 0.5, 0.25)^T$. The power method gives the following:
Iteration No: 1

$$\begin{aligned} \boldsymbol{y}_1 &= A\boldsymbol{x}_0 = (3.000000, -0.500000, 7.250000,)^T \\ \mu_1 &= 7.250000 \\ \boldsymbol{x}_1 &= \frac{\boldsymbol{y}_1}{\mu_1} = (0.413793, -0.068966, 1.000000,)^T \end{aligned}$$

Iteration No: 2

$$
\begin{aligned}
\boldsymbol{y}_2 &= A\boldsymbol{x}_1 = (1.241379, -0.068966, 2.655172,)^T \\
\mu_2 &= 2.655172 \\
\boldsymbol{x}_2 &= \frac{\boldsymbol{y}_2}{\mu_2} = (0.467532, -0.025974, 1.000000,)^T
\end{aligned}
$$

Iteration No: 3

$$
\begin{aligned}
\boldsymbol{y}_3 &= A\boldsymbol{x}_2 = (1.402597, -0.025974, 2.870130,)^T \\
\mu_3 &= 2.870130 \\
\boldsymbol{x}_3 &= \frac{\boldsymbol{y}_3}{\mu_3} = (0.488688, -0.009050, 1.000000,)^T
\end{aligned}
$$

Iteration No: 4

$$
\begin{aligned}
\boldsymbol{y}_4 &= A\boldsymbol{x}_3 = (1.466063, -0.009050, 2.954751,)^T \\
\mu_4 &= 2.954751 \\
\boldsymbol{x}_4 &= \frac{\boldsymbol{y}_4}{\mu_4} = (0.496172, -0.003063, 1.000000,)^T
\end{aligned}
$$

Iteration No: 5

$$
\begin{aligned}
\boldsymbol{y}_5 &= A\boldsymbol{x}_4 = (1.488515, -0.003063, 2.984686,)^T \\
\mu_5 &= 2.984686 \\
\boldsymbol{x}_5 &= \frac{\boldsymbol{y}_5}{\mu_5} = (0.498717, -0.001026, 1.000000,)^T
\end{aligned}
$$

Iteration No: 6

$$
\begin{aligned}
\boldsymbol{y}_6 &= A\boldsymbol{x}_5 = (1.496152, -0.001026, 2.994869,)^T \\
\mu_6 &= 2.994869 \\
\boldsymbol{x}_6 &= \frac{\boldsymbol{y}_6}{\mu_6} = (0.499572, -0.000343, 1.000000,)^T
\end{aligned}
$$

Iteration No: 7

$$
\begin{aligned}
\boldsymbol{y}_7 &= A\boldsymbol{x}_6 = (1.498715, -0.000343, 2.998287,)^T \\
\mu_7 &= 2.998287 \\
\boldsymbol{x}_7 &= \frac{\boldsymbol{y}_7}{\mu_7} = (0.499857, -0.000114, 1.000000,)^T
\end{aligned}
$$

Iteration No: 8

$$
\begin{aligned}
\boldsymbol{y}_8 &= A\boldsymbol{x}_7 = (1.499571, -0.000114, 2.999429,)^T \\
\mu_8 &= 2.999429 \\
\boldsymbol{x}_8 &= \frac{\boldsymbol{y}_8}{\mu_8} = (0.499952, -0.000038, 1.000000,)^T
\end{aligned}
$$

Iteration No: 9

$$
\begin{aligned}
\boldsymbol{y}_9 &= A\boldsymbol{x}_8 = (1.499857, -0.000038, 2.999809,)^T \\
\mu_9 &= 2.999809 \\
\boldsymbol{x}_9 &= \frac{\boldsymbol{y}_9}{\mu_9} = (0.499984, -0.000013, 1.000000,)^T
\end{aligned}
$$

Iteration No: 10

$$y_10 = Ax_9 = (1.499952, -0.000013, 2.999936,)^T$$
$$\mu_10 = 2.999936$$
$$x_10 = \frac{y_10}{\mu_10} = (0.499995, -0.000004, 1.000000,)^T$$

Initial Guess 2: Let us take $x_0 = (0, 0.5, 0.25)^T$. The power method gives the following:
Iteration No: 1

$$y_1 = Ax_0 = (0.000000, 3.500000, -8.750000,)^T$$
$$\mu_1 = 8.750000$$
$$x_1 = \frac{y_1}{\mu_1} = (0.000000, 0.400000, -1.000000,)^T$$

Iteration No: 2

$$y_2 = Ax_1 = (0.000000, 0.400000, -1.000000,)^T$$
$$\mu_2 = 1.000000$$
$$x_2 = \frac{y_2}{\mu_2} = (0.000000, 0.400000, -1.000000,)^T$$

Iteration No: 3

$$y_3 = Ax_2 = (0.000000, 0.400000, -1.000000,)^T$$
$$\mu_3 = 1.000000$$
$$x_3 = \frac{y_3}{\mu_3} = (0.000000, 0.400000, -1.000000,)^T$$

Iteration No: 4

$$y_4 = Ax_3 = (0.000000, 0.400000, -1.000000,)^T$$
$$\mu_4 = 1.000000$$
$$x_4 = \frac{y_4}{\mu_4} = (0.000000, 0.400000, -1.000000,)^T$$

Note that in the second initial guess, the first coordinate is zero and therefore, c_1 in the power method is zero. This makes the iteration to converge to λ_2, which is the next dominant eigenvalue. □

Eigenvalue Problem: The Power Method

Algorithm Choose an arbitrary initial guess $x^{(0)}$. For $k = 1, 2, \cdots$

Step 1 Compute $y^{(k)} = Ax^{(k-1)}$

Step 2 Take $\mu_k = y_i^{(k)}$, where $\|\boldsymbol{y}^{(k)}\|_\infty = |y_i^{(k)}|$,

Step 3 Set $x^{(k)} = \dfrac{\boldsymbol{y}^{(k)}}{\mu_k}$.

Step 4 If $\|x^{(k-1)} - x^{(k)}\|_\infty > \epsilon$, go to step 1.

For some pre-assigned positive quantity ϵ.

Let us now study the convergence of this method.

Theorem 3.21 (Power method).
Let A be an non-singular $n \times n$ matrix with the following conditions:

I. A has n linearly independent eigenvectors, $\boldsymbol{v}_i$, $i = 1, \cdots, n$.
II. The eigenvalues λ_i satisfy

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

III. The vector $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ is such that

$$\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} c_j \boldsymbol{v}_j, \quad c_1 \neq 0.$$

Then the power method converges in the sense that there exists constants C_1 and C_2 such that

$$\|\boldsymbol{x}^{(k)} - K\boldsymbol{v}_1\| \leq C_1 \left|\frac{\lambda_2}{\lambda_1}\right|^k, \quad \textit{for some } K \neq 0$$

and

$$|\lambda_1 - \mu_k| \leq C_1 \left|\frac{\lambda_2}{\lambda_1}\right|^k.$$

Proof. From the definition of $\boldsymbol{x}^{(k)}$, we have

$$\boldsymbol{x}^{(k)} = \frac{A\boldsymbol{x}^{(k-1)}}{\mu_k} = \frac{A\boldsymbol{y}^{(k-1)}}{\mu_k\mu_{k-1}} = \frac{AA\boldsymbol{x}^{(k-2)}}{\mu_k\mu_{k-1}} = \frac{A^2\boldsymbol{x}^{(k-2)}}{\mu_k\mu_{k-1}} = \cdots = \frac{A^k\boldsymbol{x}^{(0)}}{\mu_k\mu_{k-1}\cdots\mu_1}.$$

Therefore, we have

$$\boldsymbol{x}^{(k)} = m_k A^k \boldsymbol{x}^{(0)},$$

where $m_k = 1/(\mu_1\mu_2\cdots\mu_k)$. But, $\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} c_j\boldsymbol{v}_j, \quad c_1 \neq 0$. Therefore

$$\boldsymbol{x}^{(k)} = m_k\lambda_1^k\left(c_1\boldsymbol{v}_1 + \sum_{j=2}^{n} c_j\left(\frac{\lambda_j}{\lambda_1}\right)^k\boldsymbol{v}_j\right).$$

Taking maximum norm on both sides and noting that $\|\boldsymbol{x}^{(k)}\|_\infty = 1$, we get

$$1 = |m_k\lambda_1^k|\left\|c_1\boldsymbol{v}_1 + \sum_{j=2}^{n} c_j\left(\frac{\lambda_j}{\lambda_1}\right)^k\boldsymbol{v}_j\right\|_\infty.$$

This implies on taking limit,

$$|\lim_{k\to\infty} m_k\lambda_1^k| = \frac{1}{|c_1|\|\boldsymbol{v}_1\|_\infty} < \infty.$$

This is equivalent to

$$\lim_{k\to\infty} m_k\lambda_1^k = \pm\frac{1}{c_1\|\boldsymbol{v}_1\|_\infty} < \infty.$$

Finally,

$$\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \lim_{k\to\infty} m_k\lambda_1^k . c_1\boldsymbol{v}_1 = K\boldsymbol{v}_1$$

Moreover,

$$\|\boldsymbol{x}^{(k)} - K_k\boldsymbol{v}_1\|_\infty = \left\|m_k\lambda_1^k\sum_{j=2}^{n} c_j\left(\frac{\lambda_j}{\lambda_1}\right)^k\boldsymbol{v}_j\right\|_\infty \leq C\left|\frac{\lambda_2}{\lambda_1}\right|^k.$$

For eigenvalue,

$$\mu_k \boldsymbol{x}^{(k)} = \boldsymbol{y}^{(k)}.$$

Therefore,

$$\mu_k = \frac{\boldsymbol{y}_i^{(k)}}{\boldsymbol{x}_i^{(k)}} = \frac{(A\boldsymbol{x}^{(k-1)})_i}{(\boldsymbol{x}^{(k)})_i}$$

Taking limit, we have

$$\lim_{k\to\infty} \mu_k = \frac{A(K\boldsymbol{v}_1)_i}{K(\boldsymbol{v}_1)_i} = \frac{\lambda(\boldsymbol{v}_1)_i}{(\boldsymbol{v}_1)_i} = \lambda_1.$$

which gives the desired result.