$$\text{I}\quad \left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}\overset{x_n}{\wedge} = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right.$$

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

$$\text{II}\quad \left\{\begin{array}{l} d_{11}x_1 + \cdots + d_{1n}x_n = e_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ d_{k1}x_1 + \cdots + d_{kn}x_n = e_k \end{array}\right.$$

$$\left[\begin{array}{ccc|c} d_{11} & \cdots & d_{1n} & e_1 \\ \vdots & & & \vdots \\ d_{k1} & \cdots & d_{kn} & e_k \end{array}\right]$$

Assume

$r = 1, \ldots, k$

$$\left.\begin{array}{l} d_{r1}x_1 + \cdots + d_{rn}x_n = \underline{c_{r1}}(a_{11}x_1 + \cdots + a_{1n}x_n) + c_{r2}(\qquad\qquad) + \cdots \\ \qquad\qquad\qquad \cdots + c_{rm}(a_{m1}x_1 + \cdots + a_{mn}x_n) \\ e_r = c_{r1}b_1 + \cdots + c_{rm}b_m \end{array}\right\}$$

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ & \vdots & & \\ c_{r1} & c_{r2} & \cdots & c_{rm} \\ \vdots & & & \\ c_{k1} & \cdots & & c_{km} \end{bmatrix}}_{C} \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] = \left[\begin{array}{ccc|c} d_{11} & \cdots & d_{1n} & e_1 \\ \vdots & & & \vdots \\ d_{k1} & \cdots & d_{kn} & e_k \end{array}\right]$$

$$\underbrace{}_{A} \quad \underset{b}{\overset{\shortparallel}{}} \qquad \underbrace{}_{D} \quad \underbrace{}_{e}$$

PROP:- Suppose $k = m$ & $\exists$ C - $m \times m$ / $\mathbb{F}$ & C is inv. s.t $C[A|b] = [D|e]$

$\Leftrightarrow$ $[A|b] = C^{-1}[D|e]$

PROOF:- Let $(y_1, \ldots, y_n)$ be a soln of $Ax = b$ $\Rightarrow$ $Ay = b$, where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ -

$\Rightarrow$ $CAy = Cb \Rightarrow Dy = e$

$\therefore$ $(y_1, \ldots, y_n)$ is soln of $[D|e]$

$(z_1, \ldots, z_n)$ $\quad$ $Dx = e$

$$\Rightarrow \quad Dz = e, \qquad z = \begin{bmatrix} z_1 \\ \cdot \\ \vdots \\ z_n \end{bmatrix}$$

$$\Rightarrow \quad C^{-1}(Dz) = C^{-1}e$$

$$\Rightarrow \quad (C^{-1}D)z = C^{-1}e$$

$$\Rightarrow \quad Az = b$$

$\therefore$ $Ax = b$ & $Dx = e$ have same sol.

COR: Applying elementary row op., the solutions remain exactly same.

INVERTIBLE MATRIX :- A - $n \times n$ matrix $/ \mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ is said to be inv. if $\exists$ $n \times n$ matrix $\underline{\underline{B}}$ s.t. $AB = BA = I_n$

REMARK: If B_1 & B_2 are matrices s.t. $\begin{cases} AB_1 = B_1A = I_n \\ AB_2 = B_2A = I_n \end{cases}$ then $B_1 = B_2$.

EXERCISE

B is called the inverse of A, denoted by A^{-1}.

Example :-

1. I_n
2. Elementary matrices
3. A, B - $n \times n / \mathbb{F}$ are inv. $\Rightarrow$ AB is inv & $(AB)^{-1} = B^{-1}A^{-1}$. (More gen. any finite product of inv. matrices)

$$\begin{aligned} & (AB)(B^{-1}A^{-1}) \\ &= A(B(B^{-1}A^{-1})) \\ &= A((BB^{-1})A^{-1}) \\ &= A\cdot(I_n \cdot A^{-1}) \\ &= A\cdot A^{-1} \\ &= I_n \end{aligned}$$

$(B^{-1}A^{-1})(AB) = -$

TYPE - I

$$\begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & c & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & \ddots \ 1 \end{bmatrix} \qquad \begin{bmatrix} \quad & 1 & \quad \end{bmatrix}$$

Such a matrix is inv. + inverse is also an ele. matrix of the same type.

TYPE II

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & c & \\ & & & & 1 \ \ddots \ 1 \end{bmatrix}$$

$c \neq 0.$

,,

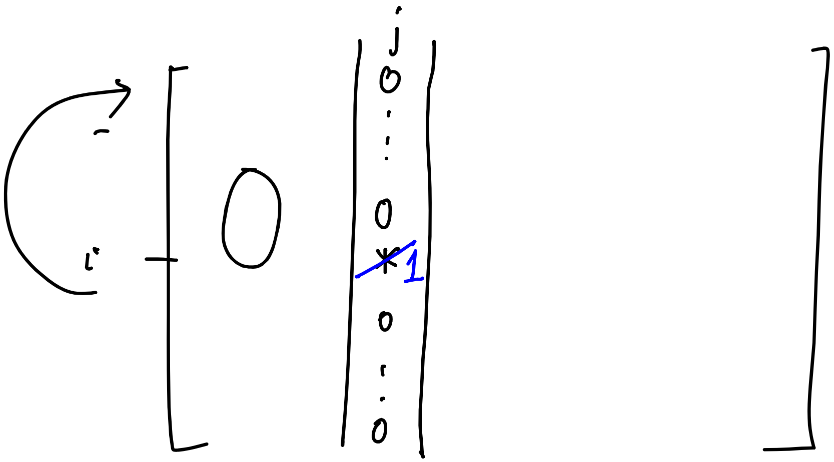
TYPE III

,,

(3') If $E_1, E_2, \dots, E_k$ are elementary matrices then $E_1 \cdots E_K$ is inv.

THEOREM :- Let A - $m \times n$ matrix / $\mathbb{F}$. Then there exist elementary matrices $E_1, \dots, E_k$ s.t. $E_k \cdots E_1 A$ is an RREF, i.e., A can be brought to an RREF by applying elementary row op.

PROOF :-



STEP 1 :- Choose the first column from left that has a $\neq 0$ entry

STEP 2: Now choose the first $\neq 0$ entry in the j-th col.

8: Make it 1

4. Clear out everything else

5. Bring it to the first row

$$\left[\begin{array}{c|c|c} & 1 & * \ \ * \ \cdots \ * \\ \cline{2-3} & 0 & \\ 0 & \vdots & B \\ & 0 & \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} & 1 & \cancel{*} \ \ * \ \cdot \ \cancel{\cdots} \ \ * \\ \hline 0 & 0 & R \end{array}\right]$$