

**Problem Set - 2**  
**MTH-204, MTH-204A**  
**Abstract Algebra**

1. Let  $G$  be a group such that the intersection of all its subgroups which are different from  $\{e\}$  is a subgroup different from  $\{e\}$ . Prove that every element in  $G$  has finite order.
2. If  $G$  has no nontrivial subgroups, show that  $G$  must be finite of prime order.
3. If  $H$  is a subgroup of  $G$ , and  $a \in G$ , let  $aHa^{-1} = \{aha^{-1} : h \in H\}$ . Show that  $aHa^{-1}$  is a subgroup of  $G$ . If  $H$  is finite, what is  $o(aHa^{-1})$ ?
4. Suppose that  $H$  is a subgroup of  $G$  such that whenever  $Ha \neq Hb$  then  $aH \neq bH$ . Prove that  $gHg^{-1} \subseteq H$  for all  $g \in G$ .
5. For  $m, n \in \mathbb{Z}$ , compute  $m\mathbb{Z} \cap n\mathbb{Z}$ .
6. Let  $G$  be an abelian group and suppose that  $G$  has elements of orders  $m$  and  $n$ , respectively. Prove that  $G$  has an element whose order is the least common multiple of  $m$  and  $n$ .
7. Prove that every subgroup of a cyclic group is cyclic.
8. Let  $G$  be a cyclic group of order  $n$ , then prove that for each  $d$  dividing  $n$ ,  $G$  has a unique subgroup of order  $d$ .
9. Let  $G$  be a cyclic group of order  $n$ . Prove that  $G$  has  $\phi(n)$  generators.
10. Let  $G$  be a cyclic group of order  $n$ . If  $d$  divides  $n$ , show that the number of elements of order  $d$  in  $G$  is  $\phi(d)$ . It is 0 otherwise.
11. Show that  $U_9, U_{17}, U_{18}$  are cyclic groups whereas  $U_8, U_{20}$  are not cyclic.
12. If  $p$  is a prime, prove that  $\phi(p^a) = p^a - p^{a-1}$ .
13. If  $\gcd(m, n) = 1$ , prove that  $\phi(mn) = \phi(m)\phi(n)$ .

4. <https://math.stackexchange.com/questions/181928/h-is-normal-whenever-ha-not-hb-implies-ah-not-bh>

6. <https://math.stackexchange.com/a/10618>

7. <https://math.stackexchange.com/questions/295564/a-subgroup-of-a-cyclic-group-is-cyclic-understanding-proof>

9. <https://sharmaeklavya2.github.io/theoremdp/nodes/abstract-algebra/groups/cyclic-groups/cyclic-group-has-phi-generators.html>

12. [https://math.stackexchange.com/questions/3201004/prove-that-varphi\(pk-pk-1-for-prime-p](https://math.stackexchange.com/questions/3201004/prove-that-varphi(pk-pk-1-for-prime-p)