MTH302: Set Theory and Mathematical Logic

Proof of the well-ordering theorem

Well-ordering theorem

The **axiom of choice** says the following. For every family $\mathcal{E}$ of nonempty sets, there is a function F such that

(1) $\mathrm{dom}(F) = \mathcal{E}$ and

(2) for every $A \in \mathcal{E}$, $F(A) \in A$.

Any function F satisfying (1)+(2) is called a **choice function on** $\mathcal{E}$.

Theorem (Zermelo, 1904)

Suppose X is a nonempty set and F is a choice function on $\mathcal{P}(X) \setminus \{\emptyset\}$. Then there is a well-order $\prec$ on X satisfying

$$(\forall y \in X)(F(X \setminus \mathit{pred}(X, \prec, y))) = y.$$

Strategy

Fix X and a choice function F on $\mathcal{P}(X) \setminus \{\emptyset\}$. Let us assume, for a moment, that a well-order $\prec_\star$ of X has been constructed such that $(\forall y \in X)(F(X \setminus pred(X, \prec_\star, y))) = y$.

1. *Let y_0 be the $\prec_\star$-least element of X. Then $F(X \setminus pred(X, \prec_\star, y_0)) = F(X) = y_0$. So $y_0 = F(X)$.*
2. *Let y_1 be the $\prec_\star$-successor of y_0. Then $y_1 = F(X \setminus pred(X, \prec_\star, y_1)) = F(X \setminus \{y_0\})$. So $y_1 = F(X \setminus \{y_0\})$.*
3. *For any natural no. n, if $\{y_0, y_1, \cdots, y_n\} \subseteq X$ satisfies $y_k = F(X \setminus \{y_j : j < k\})$ for every $k \leq n$, then $y_0 \prec_\star y_1 \prec_\star \cdots \prec_\star y_n$ are the first $n+1$ elements of X under the order $\prec_\star$.*

If X is a finite set, then this process will exhaust X in finitely many steps and will produce the desired well order. So we will assume that X is infinite.

Proof

Definition

Define $(A, \prec)$ to be an F-directed well-ordering iff the following hold.

1. *$A \subseteq X$ and $(A, \prec)$ is a well-ordering.*
2. *For every $y \in A$, $F(X \setminus pred(A, \prec, y)) = y$.*

Claim (1)

There is a set S consisting of all F-directed well-orderings.

Proof. Put $T = \mathcal{P}(X) \times \mathcal{P}(X \times X)$. Note that every F-directed well-ordering $(A, \prec)$ must be a member of T. Now apply the axiom of comprehension to conclude that S exists since we can write

$$S = \{(A, \prec) \in T : (A, \prec) \text{ is an } F\text{-directed well-ordering}\}.$$

Is S nonempty? Yes. See the previous slide for examples of F-directed well-orderings. Note that the theorem says to show that S has a well-ordering whose domain is all of X.

Proof

Claim (2)

For any two F-directed well-orderings $(A_1, \prec_1)$ and $(A_2, \prec_2)$, exactly one of the following holds.

(1) $(A_1, \prec_1) = (A_2, \prec_2)$.

(2) *For some* $y_2 \in A_2$,

  (a) $A_1 = pred(A_2, \prec_2, y_2)$ *and*
  (b) *for every* $x, y \in A_1$, $(x \prec_1 y \iff x \prec_2 y)$.

(3) *For some* $y_1 \in A_1$,

  (a) $A_2 = pred(A_1, \prec_1, y_1)$ *and*
  (b) *for every* $x, y \in A_2$, $(x \prec_1 y \iff x \prec_2 y)$.

Proof. By a previous theorem, any two well-orderings are either isomorphic or one of them is isomorphic to an initial segment of the other. We will show that these three cases correspond to the three clauses in the Claim 2.

Proof

First suppose $(A_1, \prec_1) \cong (A_2, \prec_2)$ and fix an order isomorphism $h : A_1 \to A_2$. We will show that for every $x \in A_1$, $h(x) = x$. Since h is a bijection, it will follow that $A_1 = A_2$. Since f is an order isomorphism, it will also follow that $\prec_1 = \prec_2$.

Towards a contradiction, assume that $B = \{x \in A_1 : h(x) \neq x\}$ is nonempty and fix the $\prec_1$-least element $v \in B$. Then for every $x \in A_1$, $(x \prec_1 v \implies h(x) = x)$. Put $P_1 = \text{pred}(A_1, \prec_1, v)$ and $P_2 = \text{pred}(A_2, \prec_2, h(v))$. As h is an order isomorphism, we must have $P_2 = \{h(y) : y \in P_1\} = P_1$. Since $(A_1, \prec_1)$ and $(A_2, \prec_2)$ are F-directed, we get $v = F(X \setminus P_1 = F(X \setminus P_2) = h(v))$. A contradiction.

A similar argument works for the remaining 2 cases. Details are left to the reader. ☕

Claim (2) implies that **given finitely many F-directed well-orderings $(A_1, \prec_1), (A_2, \prec_2), \cdots, (A_n, \prec_n)$, there is a largest one among these**, i.e., there is a $k \leq n$ such that for every $i \leq n$, either $(A_i, \prec_i) = (A_k, \prec_k)$ or $(A_i, \prec_i)$ is a proper initial segment of $(A_k, \prec_k)$.

Proof

Recall that S is the set of all F-directed well-orderings. Define $Y = \bigcup\{A : (A, \prec) \in S\}$ and $\prec_\star = \bigcup\{\prec : (A, \prec) \in S\}$.

Claim (3)

$(Y, \prec_\star)$ is a linear ordering.

Proof. It is clear that $Y \subseteq X$ and $\prec_\star$ is a relation on Y. Let us first check that $\prec_\star$ is a linear order on Y.

1. (**Irreflexive**) Suppose not and fix $x \in Y$ such that $x \prec_\star x$. By the definition of $\prec_\star$, we can find $(A, \prec) \in S$ such that $(x, x) \in \prec$. This contradicts the fact that $\prec$ is irreflexive.
2. (**Transitive**) Left to the reader.
3. (**Total**) Let $x, y \in Y$ be distinct. We must show that either $x \prec_\star y$ or $y \prec_\star x$. Using the definition of Y, choose $(A_1, \prec_1) \in S$ such that $x \in A_1$. Also choose $(A_2, \prec_2) \in S$ such that $y \in A_2$. Use Claim (3) to conclude that either $\{x, y\} \subseteq A_1$ or $\{x, y\} \subseteq A_1$. The rest should be clear. ☕

Proof

Exercise (4)

Suppose $(A, \prec) \in S$, $x \in A$. *Show the following.*

1. $pred(Y, \prec_\star, x) = pred(A \prec, x)$.
2. *For every* y_1, y_2 *in* $pred(Y, \prec_\star, x)$, $(y_1 \prec_\star y_2 \iff y_1 \prec y_2)$.

Claim (5)

$(Y, \prec_\star)$ *is a well-ordering.*

Proof. Let $W \subseteq Y$ be nonempty. We will show that W has a $\prec_\star$-least element. Fix $x \in W$. Choose $(A, \prec) \in S$ such that $x \in A$. Let $Q = \text{pred}(Y, \prec_\star, x)$. By Exercise (4), $Q = \text{pred}(A \prec, x)$ and for every $y_1, y_2 \in Q$, $(y_1 \prec_\star y_2 \iff y_1 \prec y_2)$. Since $W \cap Q \subseteq A$ and $(A, \prec)$ is a well-ordering, either $W \cap Q = \emptyset$ or it has a $\prec$-least element, say z. Now check that in the former case, x is the $\prec_\star$-least element of W and in the latter case, z is the $\prec_\star$-least element of W. ☕

Proof

Claim (6)

$(Y, \prec_\star)$ *is an F-directed well-ordering.*

Proof. It is clear that $Y \subseteq X$ and we have already shown that $(Y, \prec_\star)$ is a well-ordering. Suppose $(Y, \prec_\star)$ is not F-directed. Then $B = \{x \in Y : F(X \setminus \text{pred}(Y, \prec_\star, x)) \neq x\}$ is a nonempty subset of Y. Let z be the $\prec_\star$-least element of B. Then for every $x \prec_\star z$, since $x \notin B$, we must have $F(X \setminus \text{pred}(Y, \prec_\star, x)) = x$. Choose $(A, \prec) \in S$ such that $z \in A$. Since $(A, \prec)$ is F-directed, $x = F(X \setminus \text{pred}(A, \prec, x))$. By Exercise (4), $\text{pred}(A, \prec, x) = \text{pred}(Y, \prec_\star, x)$. It follows that $x = F(X \setminus \text{pred}(X, \prec_\star, x))$ which contradicts the fact that $x \in B$. ☕

Claim (7)

$Y = X$. *Hence $\prec_\star$ is the desired well-order on X.*

Proof. Suppose not. Then $X \setminus Y \neq \emptyset$. Let $w = F(X \setminus Y)$. Define $A = Y \cup \{w\}$ and extend the well-ordering $(Y, \prec_\star)$ to a well-ordering $(A, \prec)$ by declaring w to be the largest element of A. Then it is easily checked that $(A, \prec)$ is an F-directed well-ordering. Now by its definition, $A \subseteq Y$ and hence $w \in Y$. A contradiction. ☕