lecture 17 (Laplace Transform)

Recall $f:[0\ \infty) \longrightarrow \mathbb{R}$

- $\mathcal{L}(f)(s) = F(s) = \displaystyle\int_0^\infty f(t)\, e^{-st}\, dt$

- $\mathcal{L}(u_a) = \dfrac{e^{-as}}{s}$, $s>0$ $\qquad u_a(t) = \begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$

- $\mathcal{L}(t^n) = \dfrac{\lfloor n}{s^{n+1}}$, $s>0$

  $\mathcal{L}(t^a) = \dfrac{\Gamma(a+1)}{s^{a+1}}$ $(a>-1)$

  $\mathcal{L}\left(\dfrac{1}{\sqrt{t}}\right) = \dfrac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\dfrac{\pi}{s}}$. $s>0$

- $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ $\quad s>a$

- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$ $\quad s>0$

- $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$ $\quad s>0$

- $\mathcal{L}(f) = F(s)$ then $F(s) \to 0$ as $s \to \infty$

- f piecewise continuous + f exponential order $|f(t)| \leq M e^{ct}$ $\Rightarrow$ $F(s)$ exist.

- **First shift formula**

  $\mathcal{L}(e^{at} f(t)) = F(s-a)$ $\qquad \mathcal{L}(t^n e^{at}) = \dfrac{\lfloor n}{(s-a)^{n+1}}$

More properties

- Scaling $\mathcal{L}\left(f(ct)\right) = \frac{1}{c} F\left(\frac{s}{c}\right) \quad c>0$

$$\mathcal{L}\left(f(ct)\right) = \int_{t=0}^{\infty} e^{-st} f(ct)\, dt$$

$ct = u$

$$= \int_{u=0}^{\infty} e^{-\frac{s}{c}u} f(u) \frac{du}{c}$$

$$= \frac{1}{c} F\left(\frac{s}{c}\right)$$

- $\mathcal{L}\left(t f(t)\right) = -F'(s)$.

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$F'(s) = \int_0^{\infty} \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt = -\int_0^{\infty} e^{-st}\, t f(t)\, dt$$

note $\mathcal{L}\left(t^n f(t)\right) = (-1)^n F^{(n)}(s) \quad s>0$

$n \in \mathbb{N}$

- $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_{p=s}^{\infty} F(p)\, dp$

$$F(p) = \int_0^{\infty} f(t)\, e^{-pt}\, dt$$

$$\Rightarrow \int_{p=s}^{\infty} F(p)\, dp = \int_{p=s}^{\infty} \int_{t=0}^{\infty} f(t)\, e^{-pt}\, dt\, dp$$

$$= \int_{t=0}^{\infty} \left(f(t) \int_{p=s}^{\infty} e^{-pt}\, dp \right) dt$$

$$= \int_{t=0}^{\infty} f(t) \left. \frac{e^{-pt}}{-t} \right|_{p=s}^{\infty} dt = \int_0^{\infty} \frac{f(t)}{t} e^{-st}\, dt$$

Example

① $\mathcal{L}\left(t e^{-t}\right) = -\frac{d}{ds}\left(\mathcal{L}(e^{-t})\right)$

$= -\frac{d}{ds}\left(\frac{1}{s+1}\right) \quad s > -1$

$= \frac{1}{(s+1)^2} \quad s > -1$

② $\mathcal{L}\left(t \cos at\right) = -\frac{d}{ds}\left(\mathcal{L}(\cos at)\right)$

$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$

$= \frac{s^2-a^2}{(s^2+a^2)^2}$

③ $F(s) = \log\left(\frac{s-a}{s-b}\right)$

Find f such that $\mathcal{L}(f) = F$

(ie find inverse Laplace transform)

Sol

$F(s) = \log(s-a) - \log(s-b)$

$F'(s) = \frac{1}{s-a} - \frac{1}{s-b}$

$-\mathcal{L}(t f(t)) = \mathcal{L}(e^{at}) - \mathcal{L}(e^{bt})$

$f(t) = \frac{e^{bt} - e^{at}}{t}$

④ Find $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$

Sol $\frac{1}{s(s^2+a^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+a^2}$

Find A, B, C.

$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = A + B\cos at + \frac{C}{a}\sin(at)$

• $\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$

$= \int_0^\infty f'(t)\, e^{-st}\, dt$

$= f(t)\, e^{-st}\Big|_{t=0}^{\infty} - \int_0^\infty f(t)\, (-s e^{-st})\, dt$

$= -f(0) + s\mathcal{L}(f)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

$\mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0)$

Solve

① $y'' - y' - 2y = 0$ $\quad y(0) = 1$

$y'(0) = 0$

Apply Laplace Transformation on both side

$Y = \mathcal{L}(y)$.

$$\left(s^2 Y - s y(0) - y'(0)\right) - (sY - y(0)) - 2Y = 0$$

$$Y(s^2 - s - 2) = s - 1$$

$$Y = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}$$

$$Y = \frac{A}{s-2} + \frac{B}{s+1}$$

$$y = A e^{2t} + B e^{-t}$$

$A = \frac{1}{3}$

$B = 2/3$

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$$

(2) Solve
$$y'' - 2y' + 5y = 8\sin t - 4\cos t$$
$$y(0) = 1$$
$$y'(0) = 3.$$

Apply L.T. $\quad Y = \mathcal{L}(y).$

$$\left(s^2 Y - s\,y(0) - y'(0)\right) - 2\left(sY - y(0)\right)$$
$$+5Y = 8\frac{1}{1+s^2} - \frac{4s}{1+s^2}$$

$$Y = \frac{s^3 + s^2 - 3s + 9}{(s^2 - 2s + 5)(s^2+1)}$$

$$= \frac{Cs + D}{s^2 - 2s + 5} + \frac{As + B}{s^2+1}.$$

$$y = A\cos t + B\sin t$$
$$+ C' e^{t} \cos 2t$$
$$+ D' e^{t} \frac{\sin 2t}{2}$$

$$\frac{Cs + D}{s^2 - 2s + 5}$$

$$= \frac{Cs + D}{(s-1)^2 + 4}$$

$$= \frac{C'(s-1) + D'}{(s-1)^2 + 4}$$

$A = 0$

$B = 2$

$C' = 1$

$D' = 0$

$\boxtimes$

Theorem. $f:[0, \infty) \longrightarrow \mathbb{R}$

- f piecewise continuous
- f exponential order $\exists\, M, c$ s.t

$$|f(t)| \le M e^{ct} \quad \forall\, t \ge t_0$$

$$\Rightarrow \mathcal{L}(f) \text{ exist. } \forall\, s \ge s_0$$

(• Polynomials are exponential order.

$$e^{at} = \sum \frac{t^n a^n}{n!} \ge \frac{t^n a^n}{n!} \qquad t \ge 0.$$

$$\Rightarrow t^n \le \underbrace{\boxed{\frac{n!}{a^n}}}_{M} e^{at}$$

- e^{t^2} is NOT of exponential order)

Proof

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \underbrace{\int_0^A f(t)\, e^{-st}\, dt}_{\text{exist because } f\text{-piecewise continuous}} + \underbrace{\int_A^\infty f(t)\, e^{-st}\, dt}_{I_2}$$

$$\left|f(t)\, e^{-st}\right| \le M e^{ct} e^{-st} = M e^{(c-s)t}$$

For large s $\displaystyle\int_A^\infty e^{(c-s)t}\, dt$ exist.

$\Rightarrow$ L.H.S integral exist.

$$\left[\int_A^\infty e^{\alpha t} = \left.\frac{e^{\alpha t}}{\alpha}\right|_{t=A}^{\infty}\right.$$