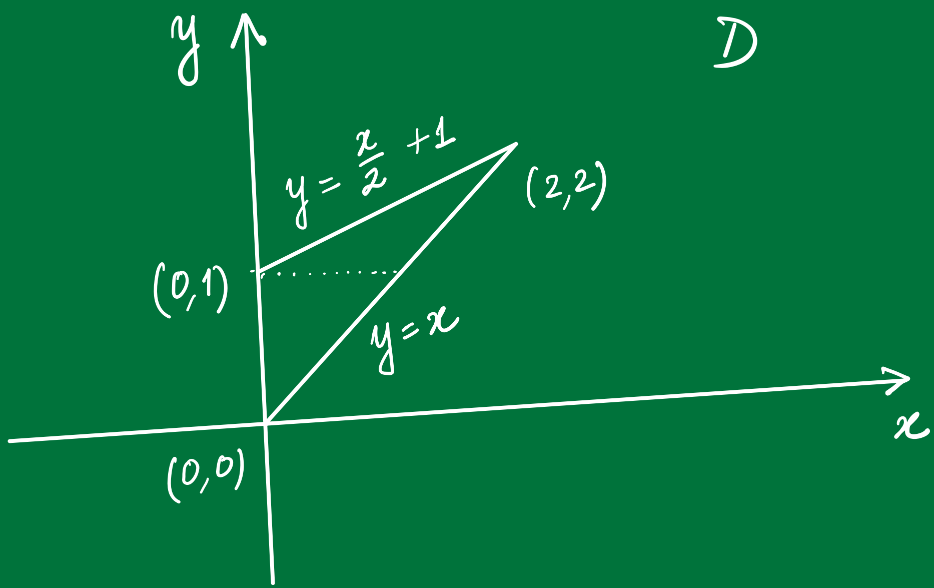
$$\iint_D f(x,y)\,dx\,dy = \iint_Q \tilde{f}(x,y)\,dx\,dy .$$

VII Compution of $\iint_D f(x,y)\,dx\,dy$

Example: Let D be the region obtained by joining $(0,0)$, $(0,1)$ and $(2,2)$ by line segments.

Evaluate $\displaystyle\iint_D (x+y)^2 \, dx\, dy$.



$$D = \left\{ (x,y) \;/\; 0 \le x \le 2 \text{ and } \underset{x}{\underset{\parallel}{f_1(x)}} \le y \le \underset{x/2+1}{\underset{\parallel}{f_2(x)}} \right\}$$

$$\iint_D (x+y)^2\,dx\,dy = \int_0^2 \left(\int_x^{x/2+1} (x+y)^2\,dy \right) dx$$

$$= \frac{1}{3} \int_0^2 \left[\left(\frac{3}{2}x + 1 \right)^3 - 8x^3 \right] dx$$

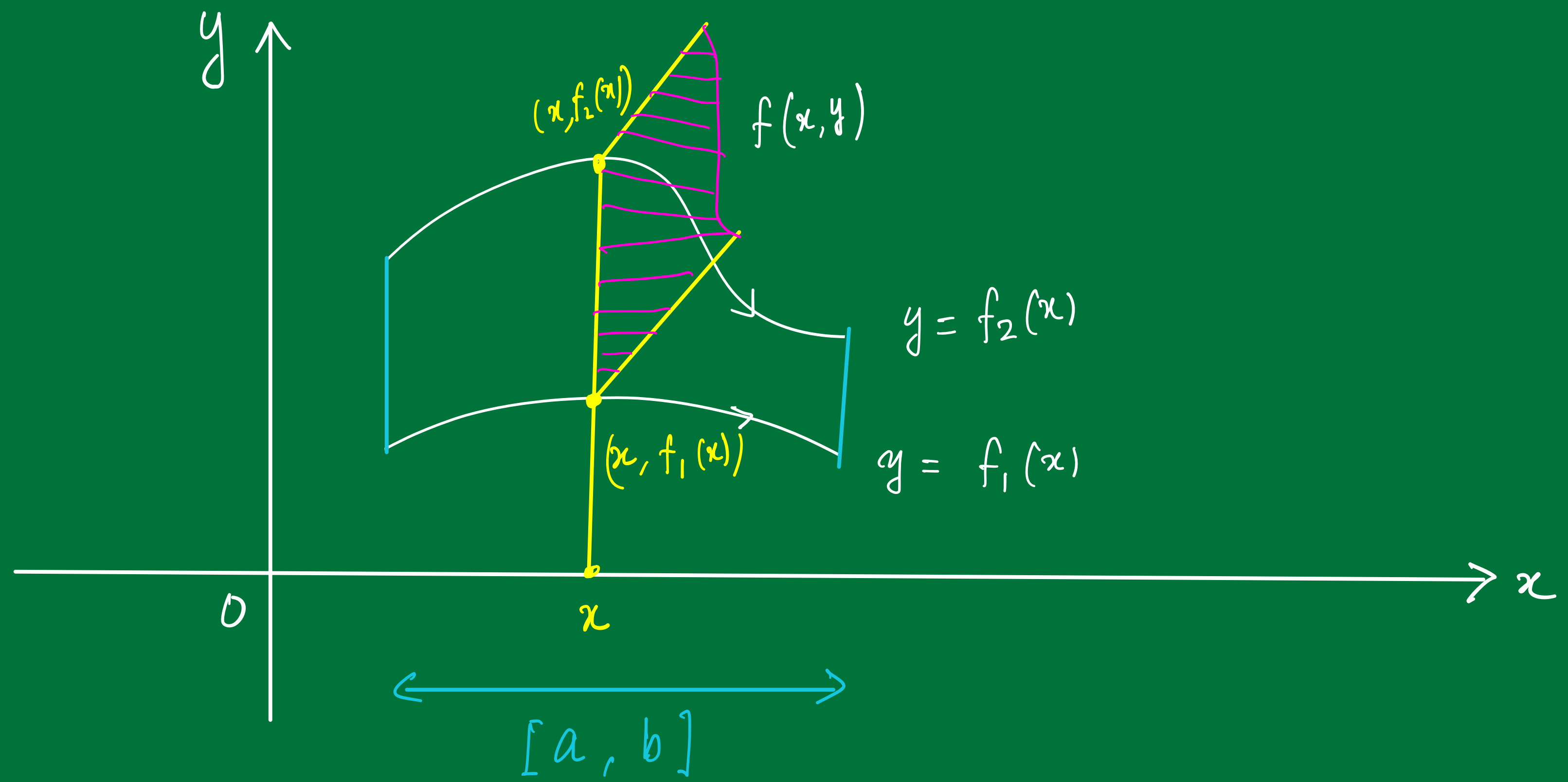
Fubini's theorem (Stronger form).

1. If $D = \{(x, y) \,/\, a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}$ for some functions $f_1, f_2 : [a, b] \longrightarrow \mathbb{R}$ then

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right) dx$$

2. If $D = \{(x, y) \,/\, c \le y \le d \text{ and } g_1(y) \le x \le g_2(y)\}$ for some functions $g_1, g_2 : [c, d] \longrightarrow \mathbb{R}$ then

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dx \right) dy \,.$$


y
O
x
x
$(x, f_2(x))$
$f(x,y)$
$y = f_2(x)$
$(x, f_1(x))$
$y = f_1(x)$
$[a, b]$

$$\iint_D (x+y)^2\,dx\,dy = \iint_{D_1} (x+y)^2\,dx\,dy + \iint_{D_2} (x+y)^2\,dx\,dy$$

where $D_1 = \{(x,y) \;/\; 0 \le y \le 1 \text{ and } 0 = g_1(y) \le x \le g_2(y) = y\}$ and

$D_2 = \{(x,y) \;/\; 1 \le y \le 2 \text{ and } 2y-1 \le x \le y\}$

$$= \int_0^1 \left(\int_0^y (x+y)^2\,dx \right) dy + \int_1^2 \left(\int_{2y-1}^y (x+y)^2\,dx \right) dy$$

Remark:

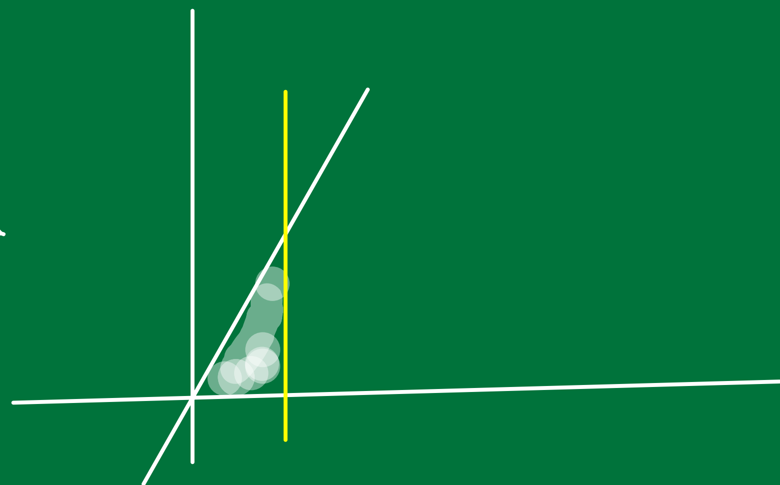
(1) $$\iint_D dx\,dy = \text{Area}(D)$$

(2) $$\iint_D f(x,y)\,dx\,dy =$$ The volume under the surface $z = f(x,y)$.

Example.

Evaluate the double integral $\iint_D e^{x^2}\,dx\,dy$

Where $D = \{(x,y) \mid 0 \le x \le 1 \text{ and } 0 \le y \le 2x\}$.



Consider

$$D' = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le y \le 2 \text{ and} \\ \frac{y}{2} \le x \le 1 \end{array} \right\}$$

We can write $\iint_D e^{x^2}\,dx\,dy = \int_0^2 \left(\int_{y/2}^1 e^{x^2}\,dx \right) dy$

We can write $\iint_D e^{x^2}\,dx\,dy = \int_0^2 \left(\int_{y/2}^1 e^{x^2}\,dx \right) dy$

$$= \int_0^1 \left(\int_{y=0}^{y=2x} e^{x^2}\,dy \right) dx$$

$$= \int_0^1 2x\, e^{x^2}\,dx$$

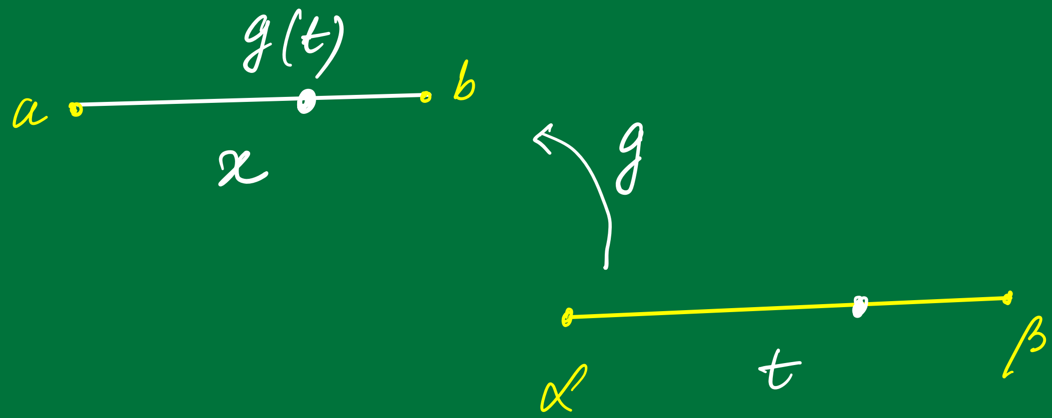
$$= e - 1$$

Change of variables.

Recall: $\displaystyle\int_a^b f(x)\,dx$

Let $x = g(t)$ and $a = g(\alpha)$
$b = g(\beta)$.

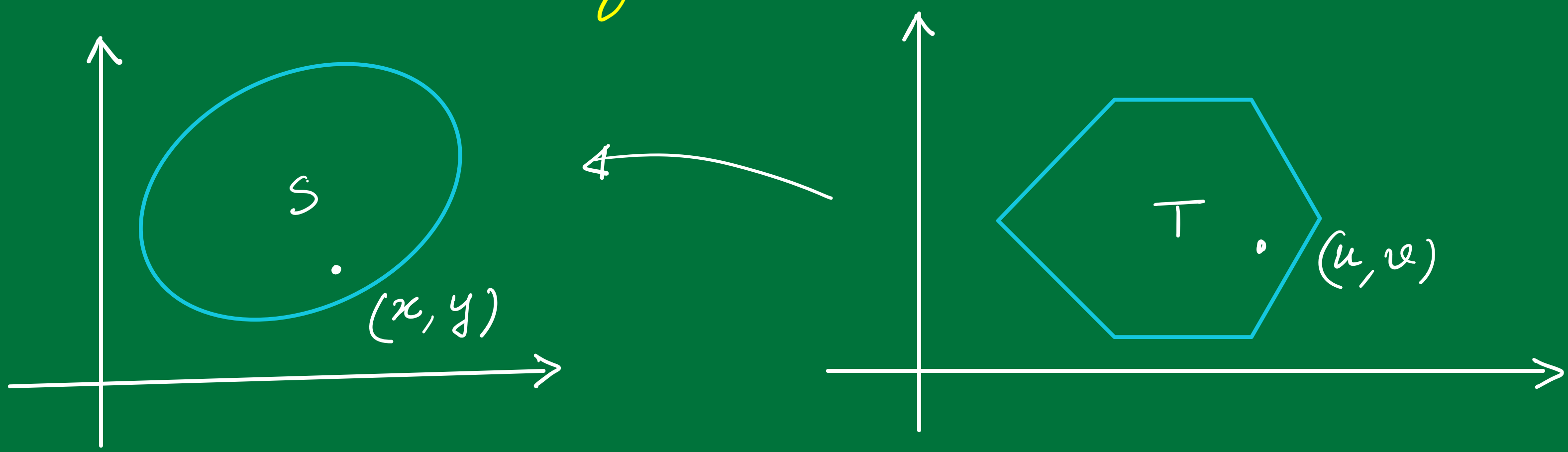
Then $$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)\,dt$$



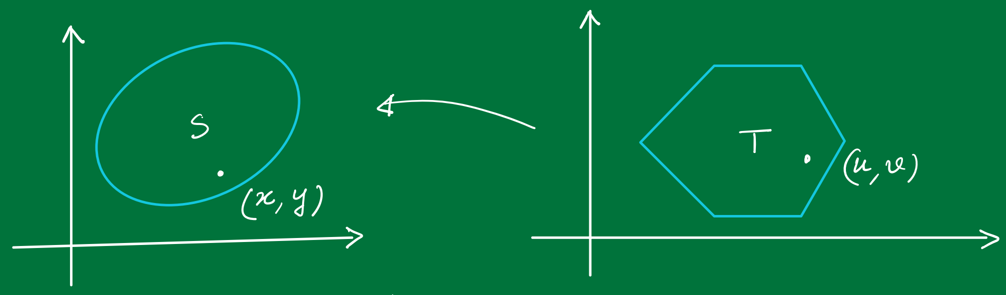
- One-to-one
- ?

A double integral defined over a region S defined in x, y variables $\iint_S f(x,y)\,dx\,dy$ can be written as

$\iint_T F(u,v)\,du\,dv$ - a double integral defined over a region T in u, v variables.



$x = g_1(u,v)$
$y = g_2(u,v)$



$x = g_1(u,v)$
$y = g_2(u,v)$

Assumptions

1. The mapping $T \to S$
$(u,v) \mapsto (g_1(u,v), g_2(u,v))$
is one-to-one. i.e, Sending distinct vectors in T to distinct vectors in S.

2. The functions g_1, g_2 are continuous and have continuous partial derivatives $\frac{\partial g_1}{\partial u}, \frac{\partial g_1}{\partial v}, \frac{\partial g_2}{\partial u}, \frac{\partial g_2}{\partial v}$.

3. The Jacobian $J(u,v) \neq 0$ where $J(u,v) = \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_2}{\partial u} \\ \frac{\partial g_1}{\partial v} & \frac{\partial g_2}{\partial v} \end{vmatrix}$.

The chage of variable formula:

$$\iint_S f(x,y)\,dx\,dy = \iint_T f\big(g_1(u,v),\, g_2(u,v)\big)\,|J(u,v)|\,du\,dv.$$

Special case: <u>Polar coordinates</u>

$$x = g_1(r,\theta) = r\cos\theta$$
$$y = g_2(r,\theta) = r\sin\theta$$

where $r > 0$ and $\theta \in [0, 2\pi]$ or $\theta \in [\theta_0, \theta_0 + 2\pi]$ for some θ_0.

Here $J(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r$$

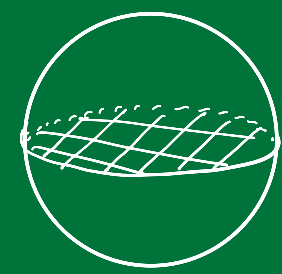
The change of variable formula is

$$\iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example: Find the volume of the sphere of radius a.

Consider the surface $z = f(x,y)$

$$= \sqrt{a^2 - x^2 - y^2}$$



over the region

$$S = \{ (x,y) \mid x^2 + y^2 \leq a^2 \}$$

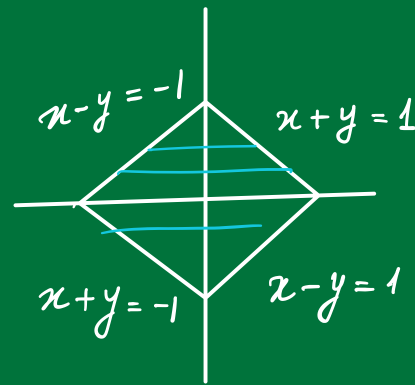
The volume $V = 2 \iint_S \sqrt{a^2 - x^2 - y^2}\, dx\, dy$

$$= 2 \iint_T \sqrt{a^2 - r^2}\; r\, dr\, d\theta$$

where $T = [0, a] \times [0, 2\pi]$

Example: Evaluate $\displaystyle\iint_D \frac{(x-y)^2}{(x+y+2)^2}\,dx\,dy$

Where D is the region



Consider $x+y=u$ and $x-y=v$

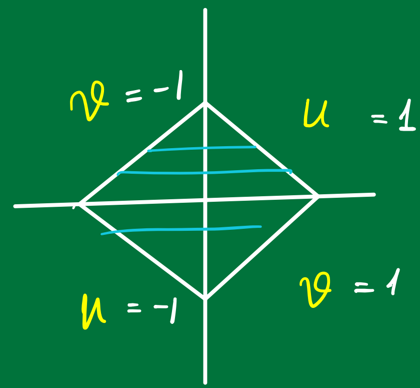
$$\Rightarrow \quad x = \frac{u+v}{2} = g_1(u,v) \quad \text{and} \quad y = \frac{u-v}{2} = g_2(u,v)$$

- $\dfrac{\partial g_1}{\partial u} = \dfrac{1}{2} = \dfrac{\partial g_1}{\partial v}$

- $\dfrac{\partial g_2}{\partial u} = \dfrac{1}{2}, \quad \dfrac{\partial g_2}{\partial v} = -\dfrac{1}{2}$

- $$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{2}$$



$$\iint_D \frac{(x-y)^2}{(x+y+2)^2}\,dx\,dy = \int_{v=-1}^{v=1}\int_{u=-1}^{u=1} \frac{v^2}{(u+2)^2}\cdot\frac{1}{2}\,du\,dv$$

$$= \frac{2}{9}$$

Recall :

$$I \xrightarrow{R} \mathbb{R}^3 \quad \text{Continuous}$$

$$t \longmapsto R(t) = \big(x(t),\ y(t),\ z(t)\big) \in \mathbb{R}^3$$

The set of points $\{ R(t) \,/\, t \in I \}$ is called a <u>parametric curve.</u>

Let $T \subseteq \mathbb{R}^2$ be a region of $\mathbb{R}^2$ and $r: T \to \mathbb{R}^3$ be given by

$$r(u,v) = \Big(X(u,v),\ Y(u,v),\ Z(u,v) \Big) \in \mathbb{R}^3$$ be a continuous function on T.

The range of r, i.e., the subset

$$S = \{ r(u,v) \in \mathbb{R}^3 \,/\, (u,v) \in T \}$$

is called a parametric surface with the parameter domain T and parameters u and v.

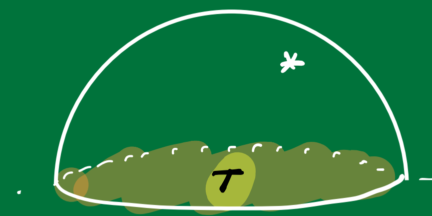
Assumption.

- Mostly, we consider the function r is one-to-one in the interior of T so that the surface does not cross itself.

A parametric surface defined by $r(u,v)$ is expressed as

$$\left.\begin{aligned} x &= X(u,v) \\ y &= Y(u,v) \\ z &= Z(u,v) \end{aligned}\right\}$$

Example: 1 S : Unit hemisphere



$$\left\{ (x, y, f(x,y)) \in \mathbb{R}^3 \right\} \xleftarrow{r} T = \left\{ (x,y) \,/\, x^2+y^2 \le 1 \right\}$$

where $f(x,y) = \sqrt{1-x^2-y^2}$

② Sphere of radius a.

$$\left\{ (x, y, z) \in \mathbb{R}^3 \;\middle/\; \begin{aligned} x &= (a \sin\phi)\cos\theta \\ y &= (a \sin\phi)\sin\theta \\ z &= a\cos\phi \end{aligned} \right\} \leftarrow \begin{aligned} T &= [0, 2\pi] \times [0, \pi] \\ &(\theta, \varphi) \end{aligned}$$

3 Cylinder

a - constant (real number)

$$\left\{ (x, y, z) \in \mathbb{R}^3 \,/\, \begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \\ z = t \end{array} \right\}$$

$T = \mathbb{R} \times [0, 2\pi]$

(t, θ)

Area of a parametric surface

Let S be a parametric surface defined on a parameter domain T. Suppose $S = r(u, v)$,

$$r_u : T \to \mathbb{R}^3$$
$$r_v : T \to \mathbb{R}^3$$

are continuous functions (vector valued)

and $r_u \times r_v : T \to \mathbb{R}^3$ is never zero.

$$\begin{tikzcd} T \arrow[r] \arrow[dr, "f"'] & \mathbb{R}^3 \arrow[d, "\|\ \|"] \\ & \mathbb{R} \end{tikzcd}$$

Then the area of S, denoted by $a(S)$ is defined as the double integral

$$a(S) = \iint_T \| r_u \times r_v \| \, du\, dv.$$

☐ $$a(S) = \iint_T \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

where $S = \{ r(x,y) = (x, y, f(x,y)) \,/\, (x,y) \in T$ and $f : T \to \mathbb{R} \}$

Hint:

$$r_x = (1, 0, f_x)$$

$$r_y = (0, 1, f_y)$$

☐ *Surface integral?*