MTH 112M

I. Area between two curves

II. Polar Coordinates
and
their relation to
Cartesian Coordinates.

III. Area of the region bounded by the curves
$r = f(\theta)$ and by the rays $\theta = \alpha$ and
$\theta = \beta$.

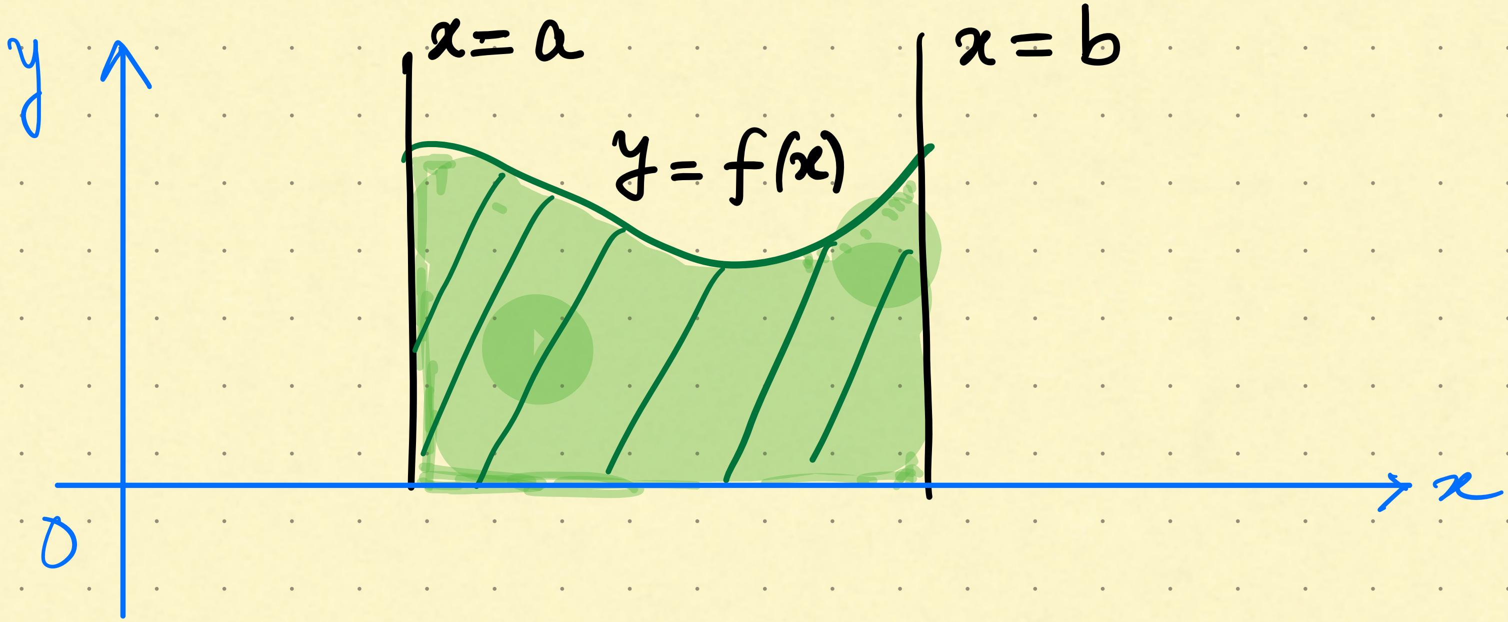
I. Area between two curves

We have seen before the computation of the area of the region under the graph of a function $f : [a,b] \to \mathbb{R}$ and x-axis.

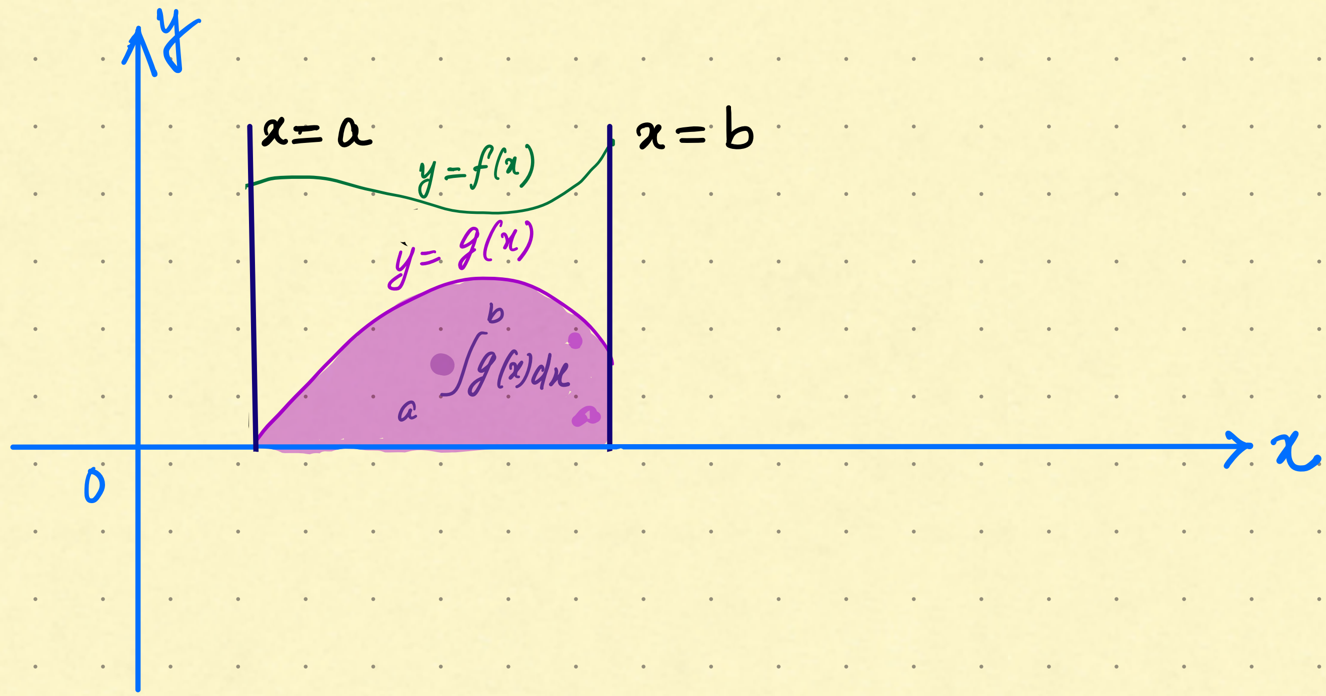
Let f be continuous function and $f(x) \geqslant 0$.

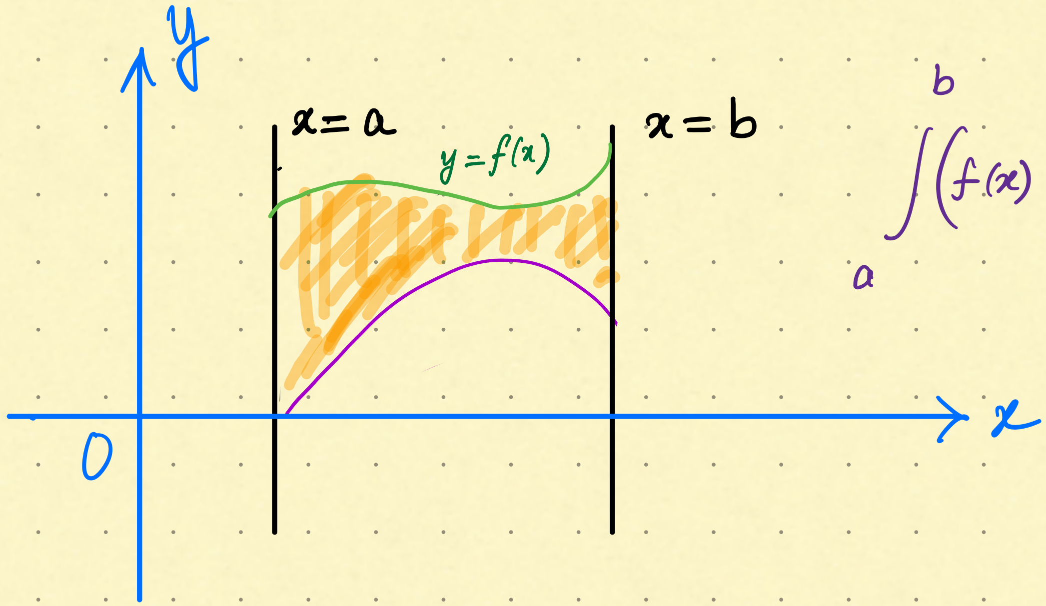
then by Riemann integral

$$\int_a^b f(x)\,dx = \text{Area under the given curve } y = f(x) \text{ enclosed by the lines } y=0,\ x=a \text{ and } x=b.$$



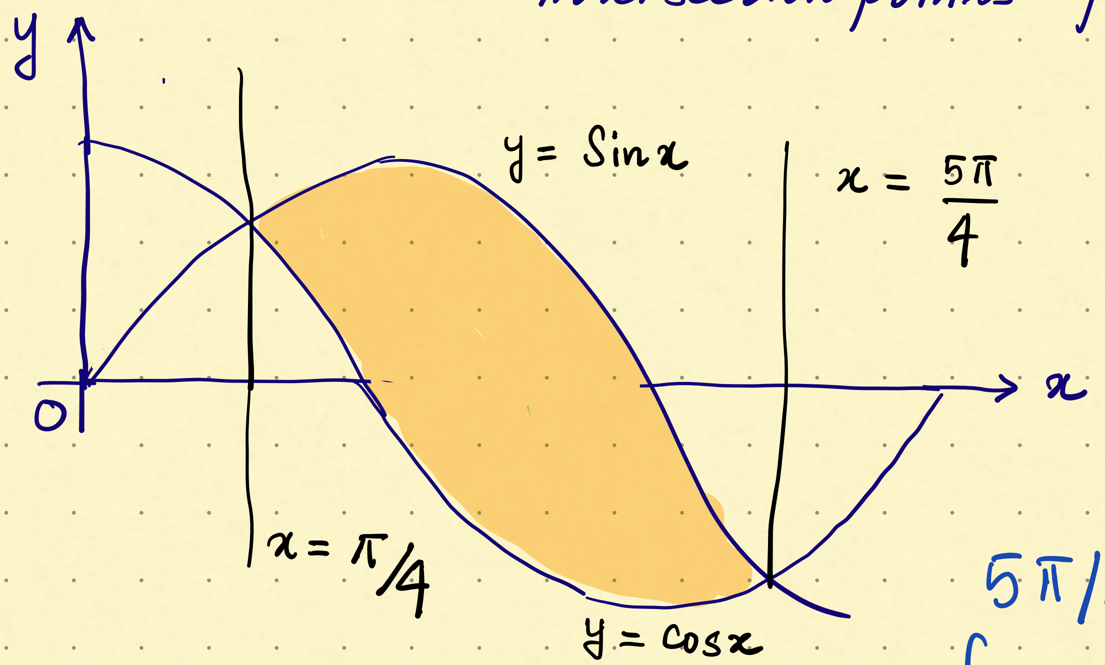
Consider a new region by replacing the x-axis (i.e., the curve $y=0$) by $y = g(x)$ where $g:[a,b] \to \mathbb{R}$ be a continuous function such that $g(x) \le f(x)$ for $x \in [a,b]$.





$$\int_a^b \left(f(x) - g(x)\right) dx =: \text{The area of the region bounded by the graphs } y = f(x),\ y = g(x),\ x = a \text{ and } x = b.$$

Example 1. Area of the region enclosed between consecutive intersection points of the curve $y = \sin x$ and $y = \cos x$ is $2\sqrt{2}$.



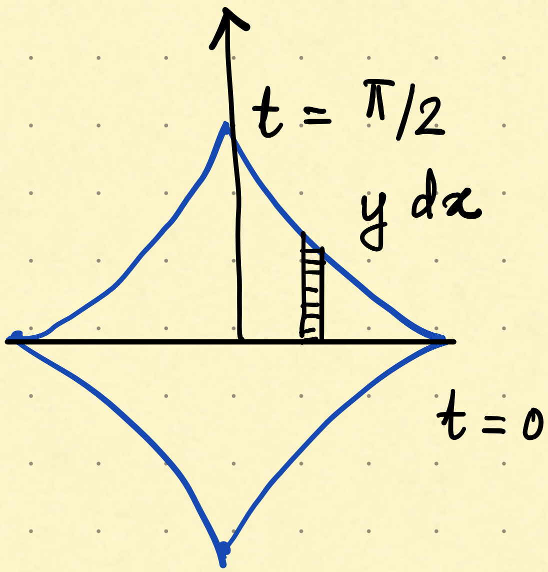
Area $\Rightarrow$ $$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = 2\sqrt{2}$$

Example 2. Consider the equation $x^{2/3} + y^{2/3} = 1$.

Compute the area of the enclosed region bounded by the solution curves of the given equation.

Hint: $t \mapsto (x(t), y(t))$; $x(t)^{2/3} + y(t)^{2/3} = 1$

$$\Rightarrow \quad x(t) = \cos^3 t$$
$$y(t) = \sin^3(t)$$



$$\int_0^{\pi/2} y\,dx = \text{Area of the enclosed region in the first quadrant}$$

$$= \int_0^{\pi/2} \left(-3 \sin^4 t \cos^2 t\right) dt$$

$$= 3\pi/32$$

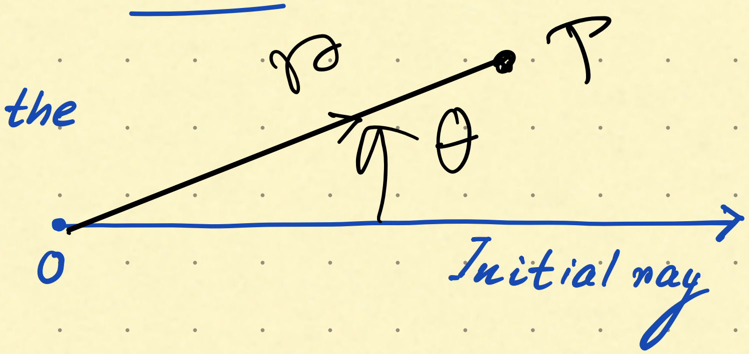
II Polar Coordinates and their relation to Cartesian Coordinates.

- function $f : [a,b] \to \mathbb{R}$
- Curve: $\{(x, f(x)) / x \in [a,b]\}$ or $y = f(x)$ (graph of the function)

Why?

- Relate a function with a curve which is graph of the function
- Express a given curve by a function or an equation.
- The use of rectangular coordinates may not be always easy.
- Sometimes the polar coordinate system is better suited for the representation of a curve given geometrically.
- Polar coordinates are very useful for computation of multiple integrals.

Fix a point in the plane called an origin O (or the pole) and an initial ray from O.

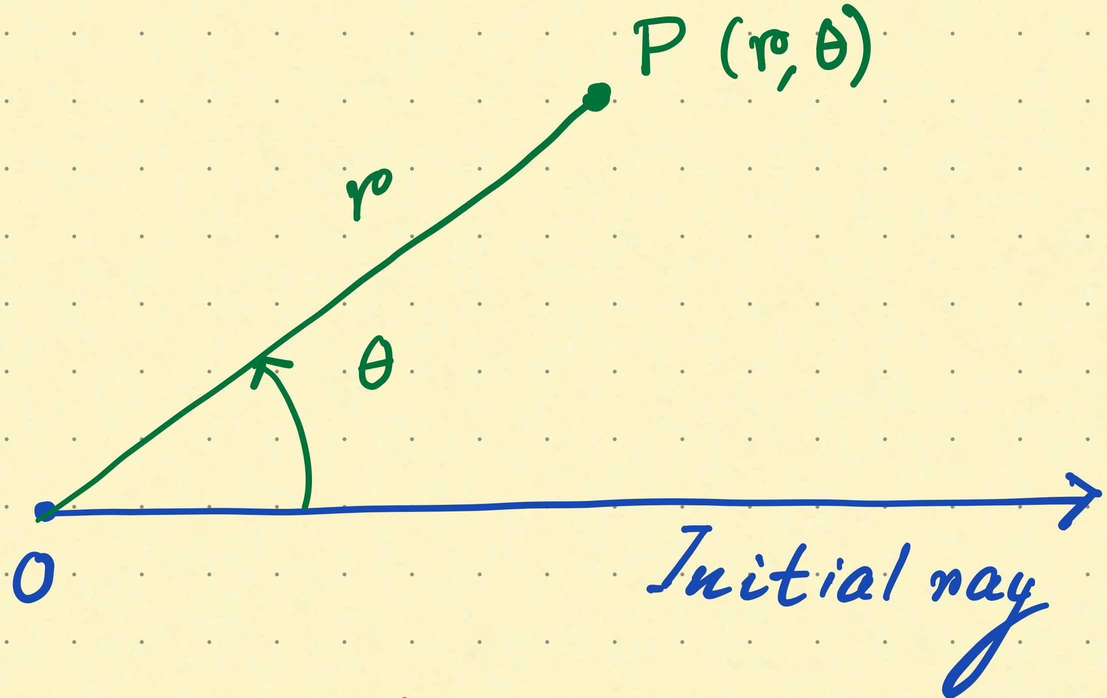


Usually the positive x-axis is chosen as the initial ray. Each point P in the plane can be determined by assigning to it a polar coordinate

$$(r, \theta)$$

where r is the directed distance from O to P and θ is the directed angle from the initial ray to the line segment OP.

Here the directed angle θ is positive when measured counterclockwise and negetive when measured clockwise.



* The same point P can be represented by infinitely many pairs of polar coordinates (but P has a unique cartesian coordinate).

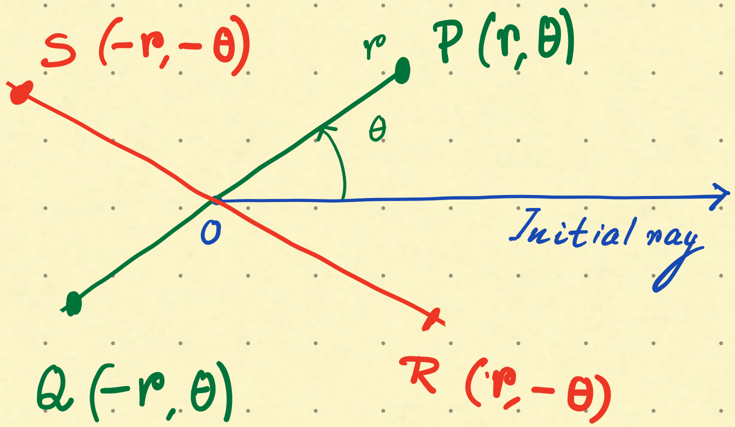
$P = (5, 30^\circ)$ $\qquad\qquad$ $P = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right) \in \mathbb{R}^2$

$= (-5, -150^\circ)$

$= (-5, 210^\circ)$

$= (5, 390^\circ)$

The directed distance can be understood as going forward r units from the origin O or going backward r units respectively.



- For a given value of θ, we find a ray and we allow r to be negetive and interpret this to mean measuring brackward from the origin.

In general, we consider $r \geqslant 0$, but for plotting polar equation we allow negetive values of r as well.

Change of coordinates.

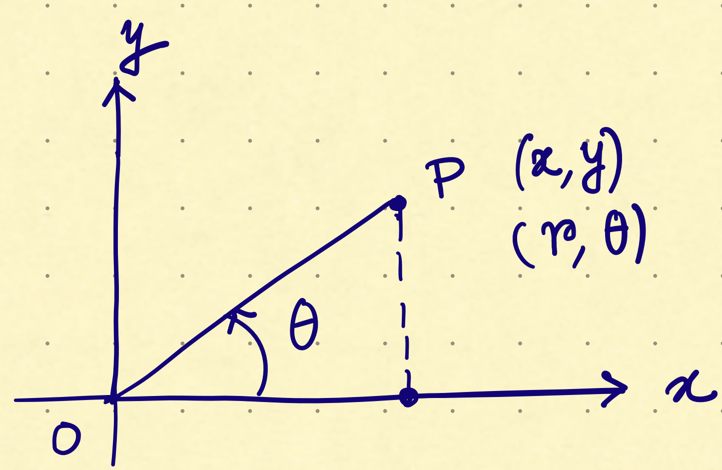
For common origin O and the initial ray as the positive x-axis, the polar coordinates (r, θ) of a point $P = (x, y)$ in rectangular coordinate are related by the equations:

$x = r\cos\theta$,

$y = r\sin\theta$

or, $x^2 + y^2 = r^2$,

$\dfrac{y}{x} = \tan\theta$



These equations are used to find Cartesian equations equivalent to polar equation and polar equations from given cartesian equations.

Examples.

(1) The curve given by the polar equation

$$r^2 = 3r\sin\theta$$

is equivalent to $x^2+y^2=3y$, which is equation of a circle.

(2). The curve given by the polar equation

$$r\cos\theta = -4$$

is equivalent to $x=-4$ which is equation of a line parallel to y-axis.

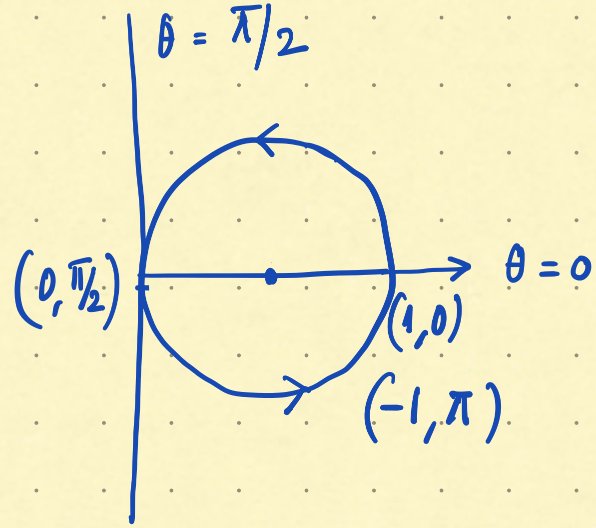
(3). Let us consider the polar equation $r = \cos\theta$.

Then $r^2 = \cos^2\theta = r\cos\theta$

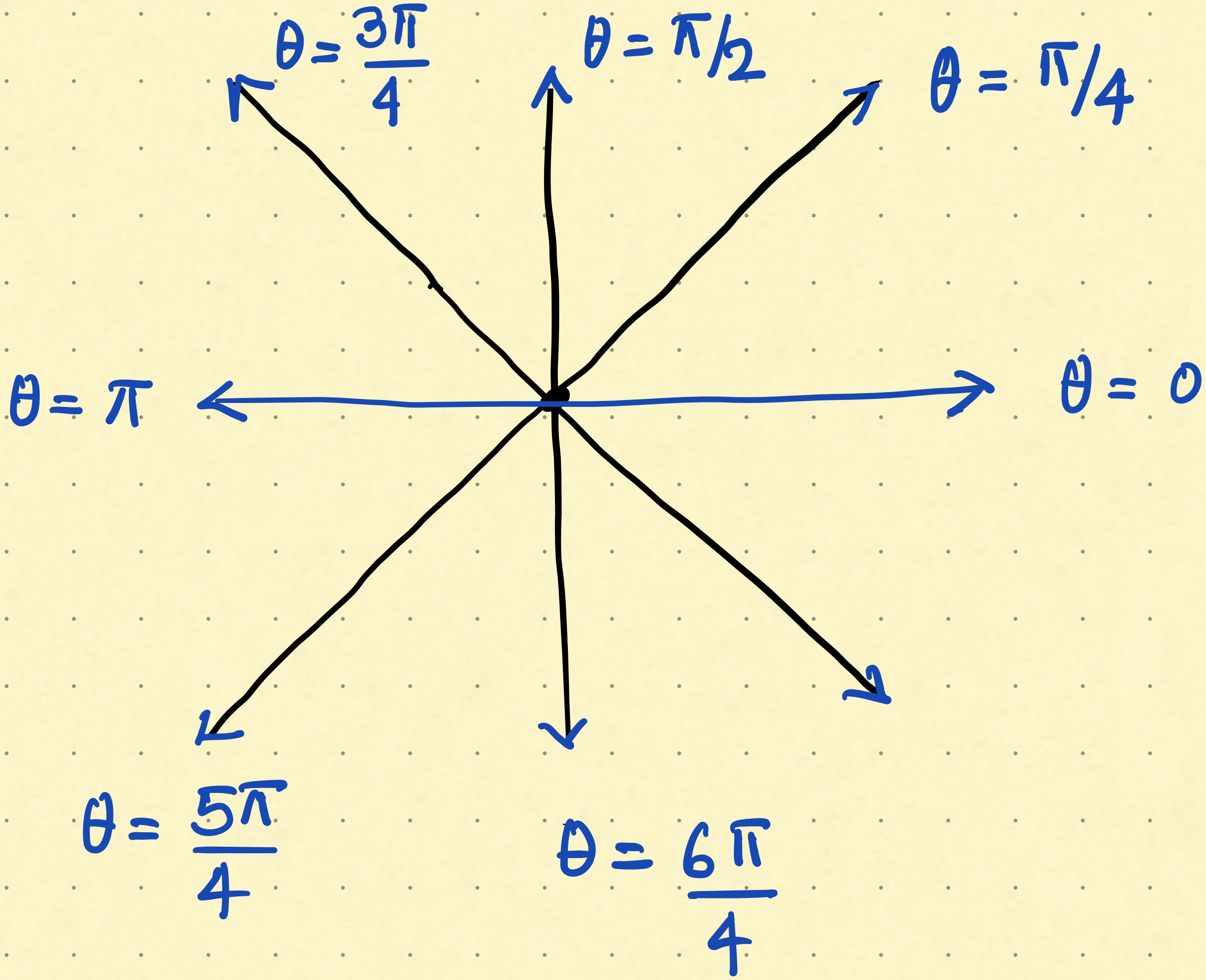
or, $x^2 + y^2 = x$

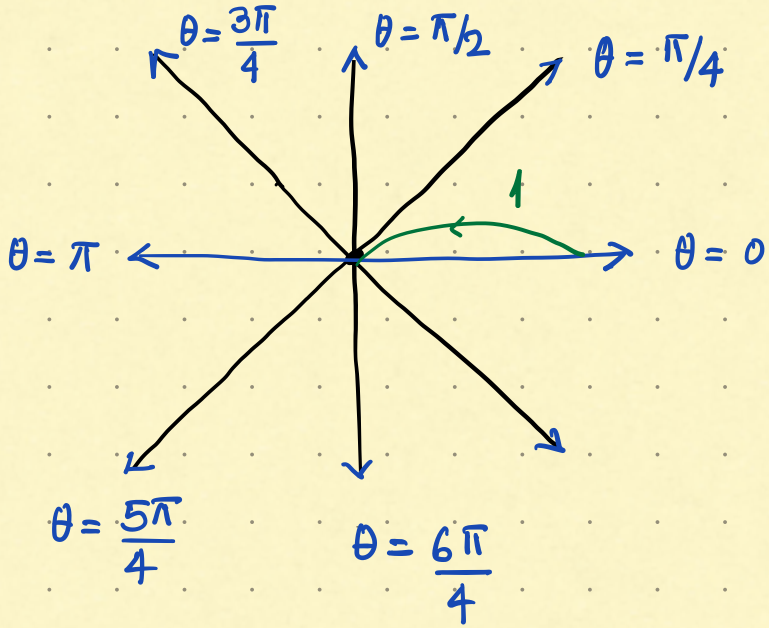
or, $\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$

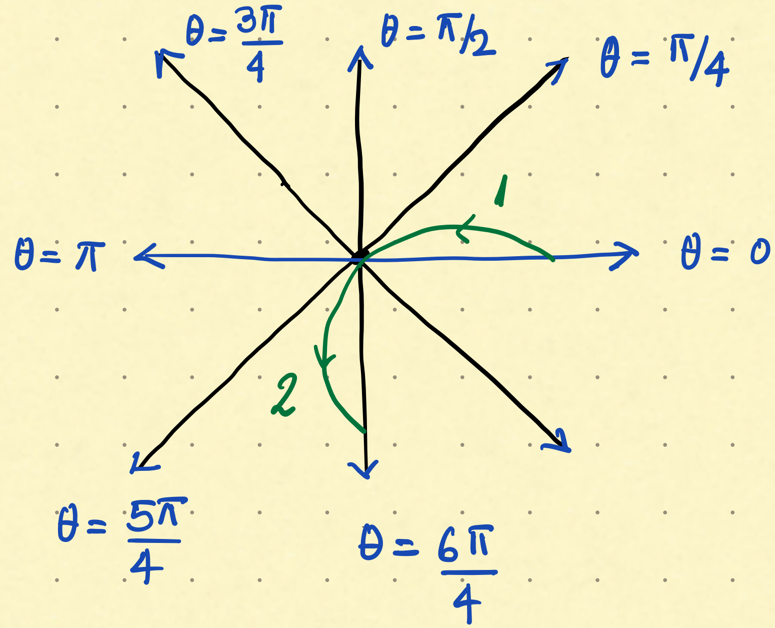
represent a circle.

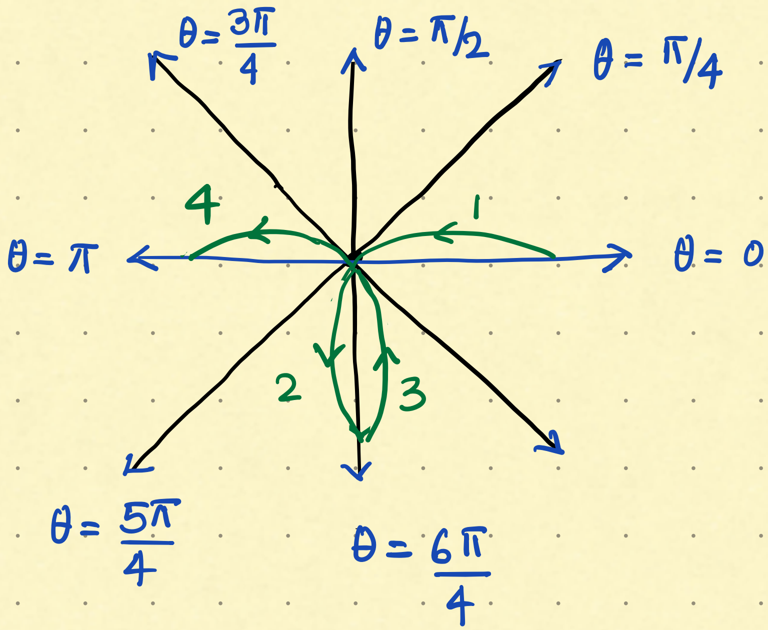


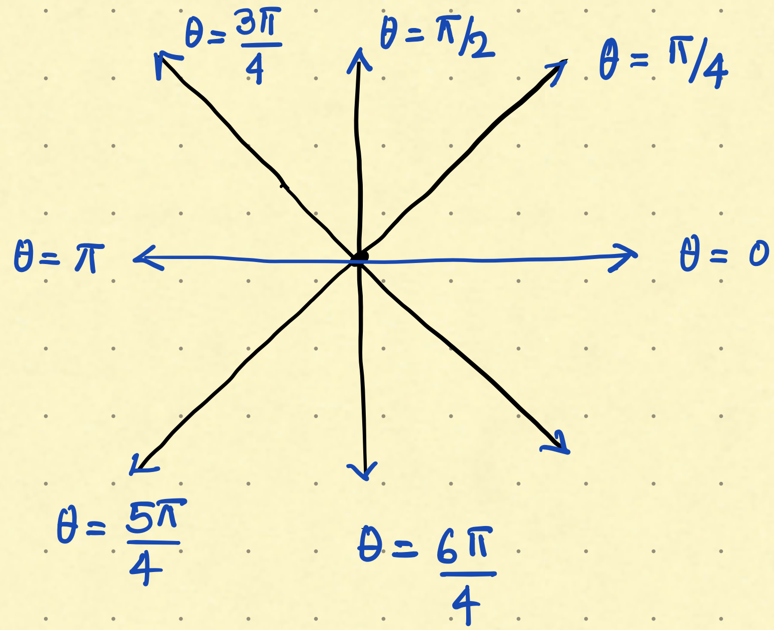
4. Draw the curve given by $r = \cos 2\theta$ and $r = \sin 3\theta$.

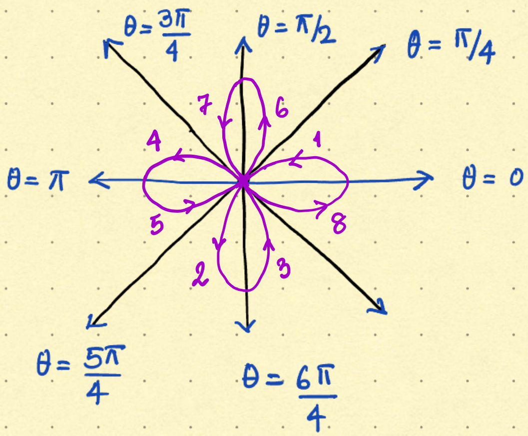


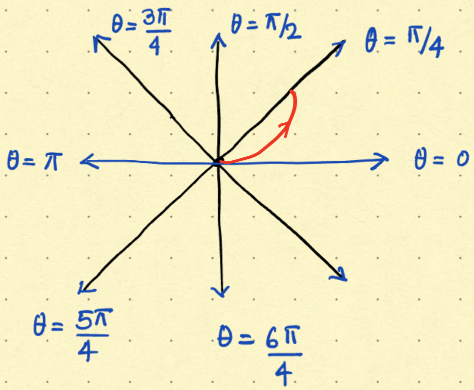

θ = 3π/4
θ = π/2
θ = π/4
1
θ = π
θ = 0
θ = 5π/4
θ = 6π/4

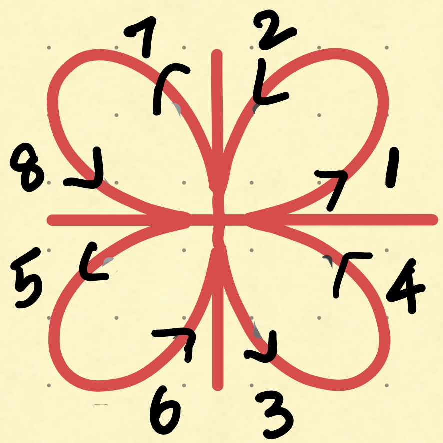

θ = 3π/4
θ = π/2
θ = π/4
1
θ = π
θ = 0
2
θ = 5π/4
θ = 6π/4


θ = 3π/4
θ = π/2
θ = π/4
4
1
θ = π
θ = 0
2
3
θ = 5π/4
θ = 6π/4


θ = 3π/4
θ = π/2
θ = π/4
θ = π
θ = 0
θ = 5π/4
θ = 6π/4


θ = 3π/4
θ = π/2
θ = π/4
θ = π
θ = 0
θ = 5π/4
θ = 6π/4
1
2
3
4
5
6
7
8

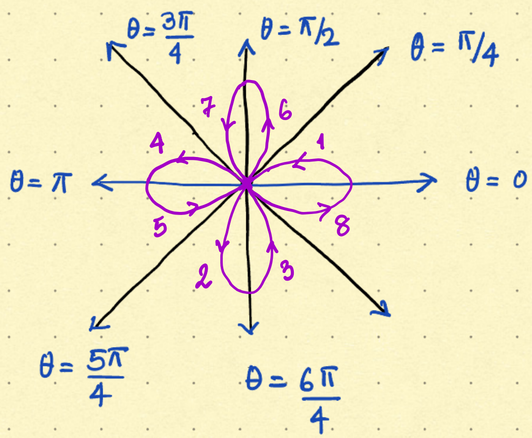

θ = 3π/4
θ = π/2
θ = π/4
θ = π
θ = 0
θ = 5π/4
θ = 6π/4


7
2
8
1
5
4
6
3

Note:

To plot a curve, we first determine few points on the curve corresponding to correct choice of the θ from the equation $r = f(\theta)$.

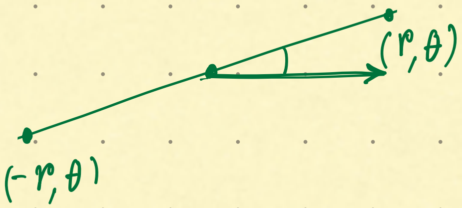
To get the actual shape of the curve, it is desirable to consider θ's for which $f(\theta) = r$ is maximum or minimum. As in the Cartesian system it is also desirable to consider the symmetry about the origin and the axes.



Symmetric about the origin:

If the equation is unchanged (i) when (r, θ) is replaced by $(-r, \theta)$ or

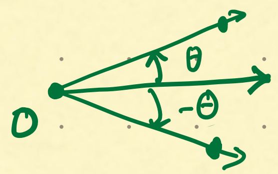
(ii) when (r, θ) is replaced by $(-r, \pi + \theta)$.

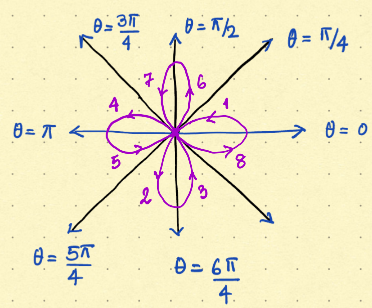


Symmetric about the **x-axis**:

If the equation is unchanged (i) when (r, θ) is replaced by $(r, -\theta)$ or

(ii) when (r, θ) is replaced by $(-r, \pi - \theta)$.



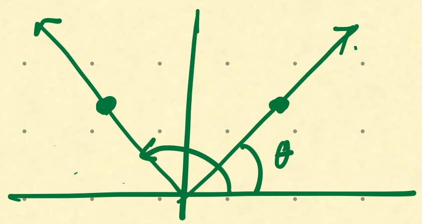


Symmetric about origin, x-axis and y-axis?

Symmetric about the **y- axis:**

If the equation is unchanged (i) when (r, θ) is replaced by $(r, \pi - \theta)$ or

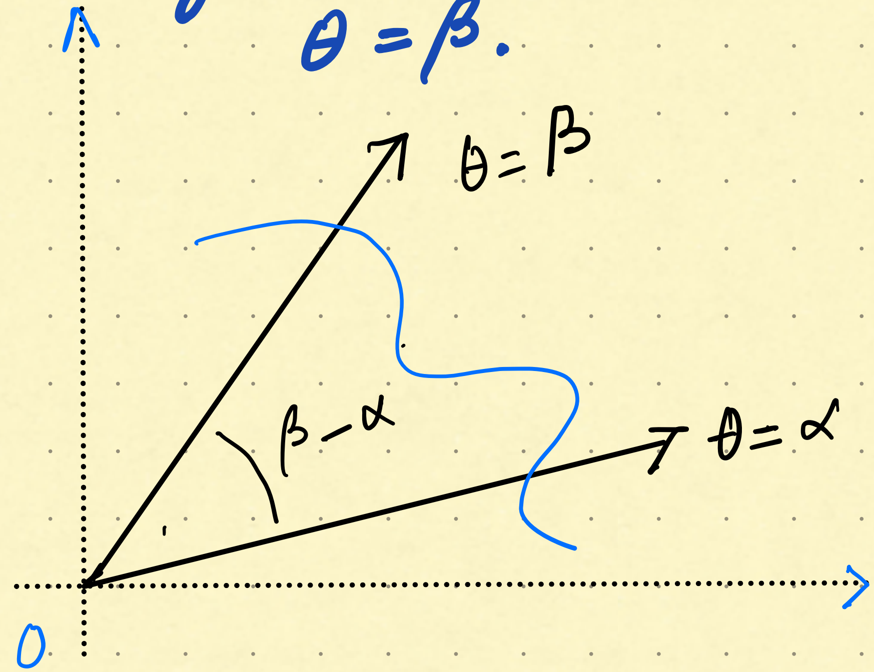
(ii) when (r, θ) is replaced by $(-r, -\theta)$.



Q. Write five examples of curves in each of the cases of symmetry.

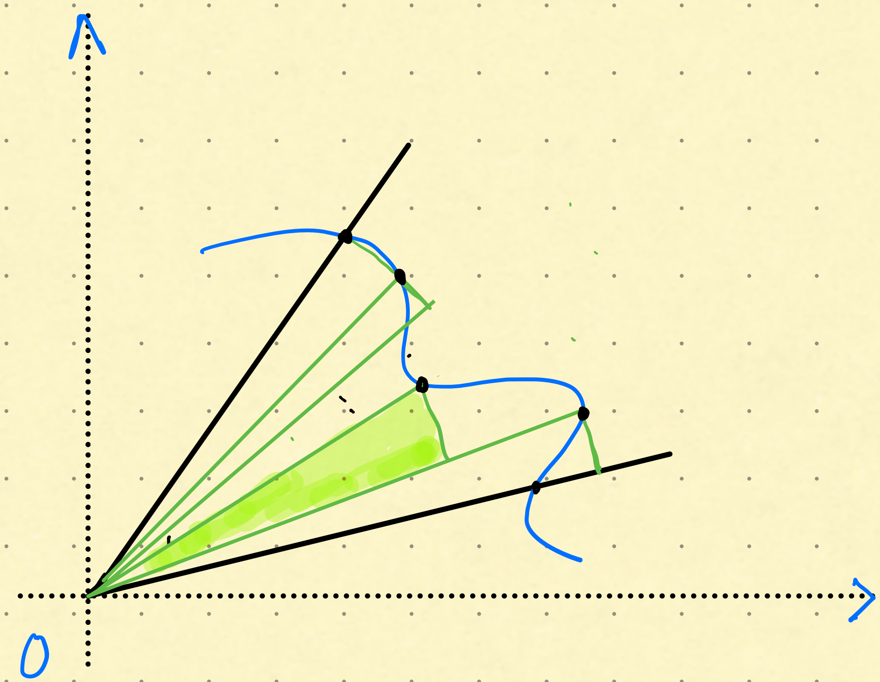
III. Area of the region bounded by the curves $r = f(\theta)$ and by the rays $\theta = \alpha$ and $\theta = \beta$.

Suppose $r = f(\theta)$ is a continuous function for $\alpha \leq \theta \leq \beta$. Assume that $(\beta - \alpha) \leq 2\pi$.



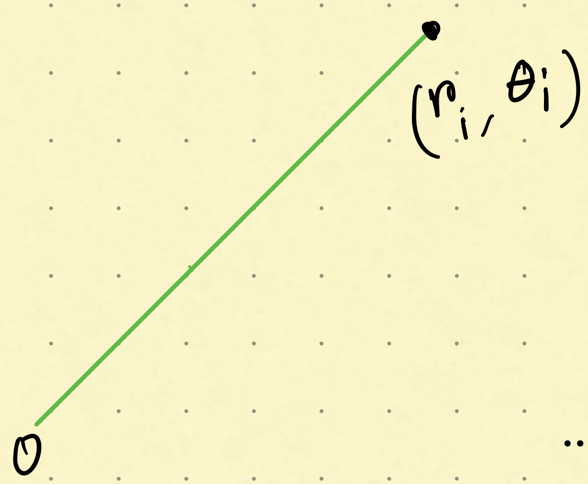
We consider a partition

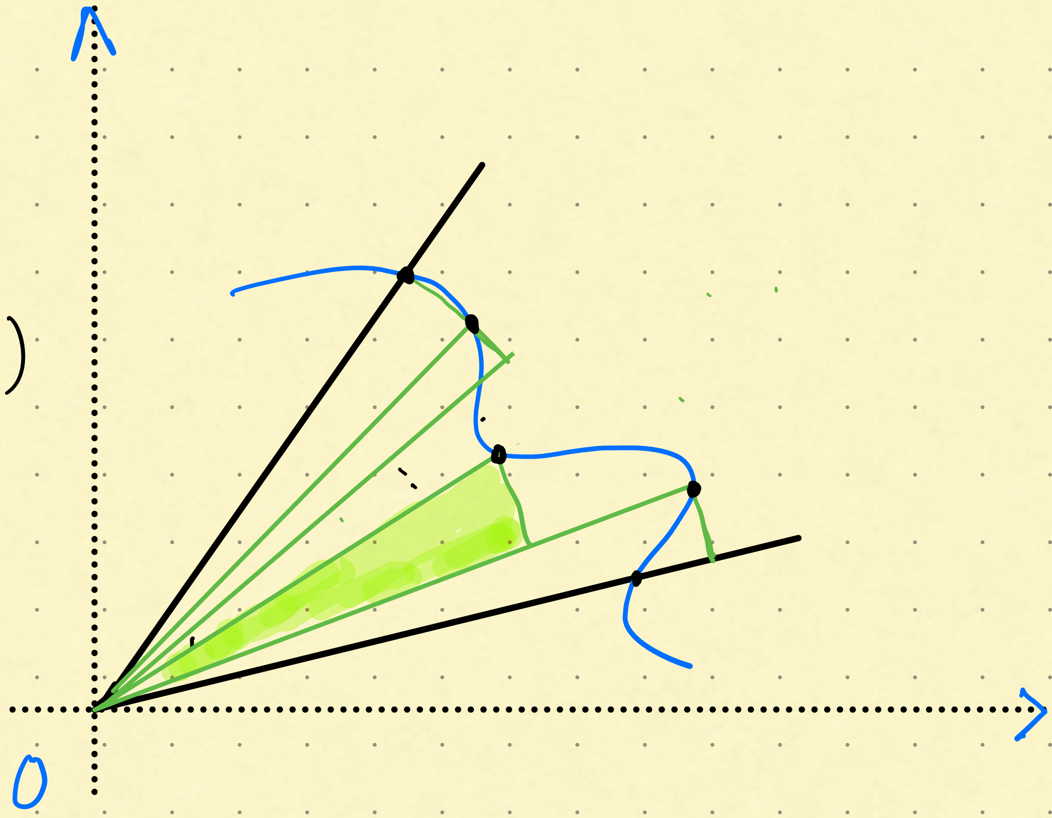
$$P = \{ \alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{i-1} < \theta_i < \cdots < \theta_n = \beta \}$$



The i-th circular sector of radius $r_i = f(\theta_i)$ and with a central angle $\Delta\theta_i = (\theta_i - \theta_{i-1})$ radian measure. Its area

$$A_i = \frac{\Delta\theta_i}{2\pi} \pi f(\theta_i)^2 = \frac{1}{2} \Delta\theta_i f(\theta_i)^2$$





Sum of the areas of all circular sectors given by the partition P is

$$= \sum_{i=1}^{n} \frac{1}{2} \Delta \theta_i \, f(\theta_i)^2 = S(P, f)$$

Theorem. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable function. Then

$$\lim_{\|P\| \to 0} S(P,f) = \int_a^b f(\theta)\, d\theta$$

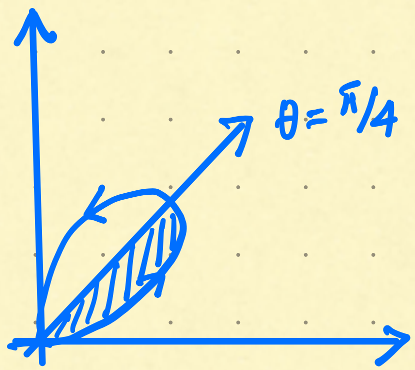
Since $f : [\alpha, \beta] \to \mathbb{R}$ is continuous, it is

$$\theta \longmapsto f(\theta)$$

Riemann integrable, and the Riemann sum converges to

$$\frac{1}{2} \int_\alpha^\beta f(\theta)^2\, d\theta$$

Example. Area bounded by the curve $r = \sin 2\theta$ is



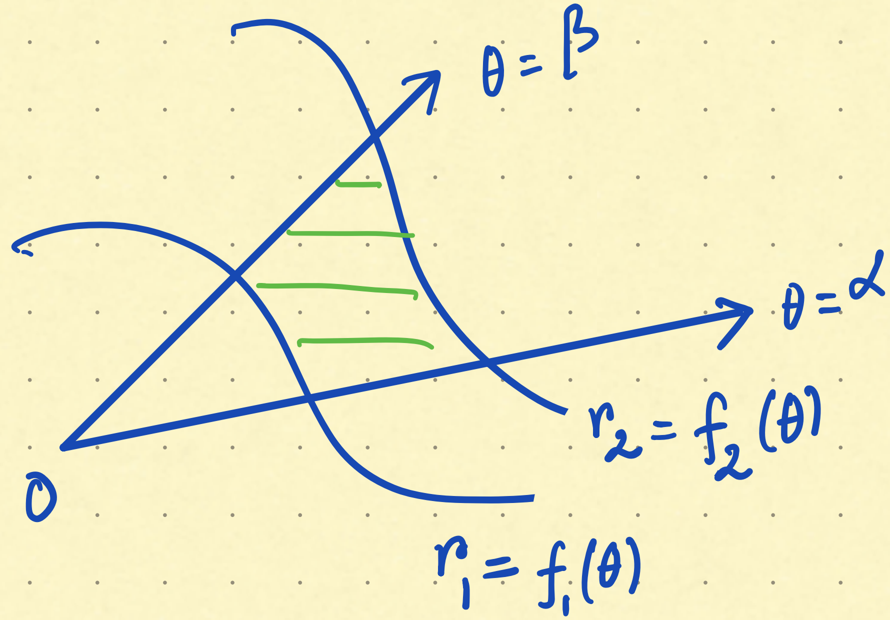
8 x Area of the shaded portion in the first quadrant

$$= 8 \times \frac{1}{2} \int_0^{\pi/4} (\sin 2\theta)^2 \, d\theta$$

$= ?$

Remark:

A = Area of the enclosed region between the curves $r_1 = f_1(\theta)$, $r_2 = f_2(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$



$$= \frac{1}{2}\int_{\alpha}^{\beta} f_2(\theta)^2\,d\theta - \frac{1}{2}\int_{\alpha}^{\beta} f_1(\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta}\left(f_2^2 - f_1^2\right)d\theta \quad \text{or} \quad \frac{1}{2}\int_{\alpha}^{\beta}\left(r_2^2 - r_1^2\right)d\theta$$