Lecture 14 (Frobenius method)

Recall $y'' + p(x)y' + q(x)y = 0$ — $(**)$

$p(x), q(x)$ analytic at x_0

$\Rightarrow$ Solution of $(**)$ around x_0 look like

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Definition $x = x_0$ is called an **ordinary point** of $(**)$ if $p(x)$ & $q(x)$ are analytic at x_0.

x_0 - singular $:=$ x_0 - NOT ordinary point

$p(x) = \frac{1}{x}$ $q(x) = \frac{1}{x^2}$

$x_0 = 0$ singular

$p(x) = \frac{1}{x^3}$ $q(x) = x$

$x_0 = 0$ singular

$(x-x_0)\,p(x)$
$(x-x_0)^2 q(x)$] analytic

x_0 - regular singular

Irregular = NOT regular

NOT in our course

Today : Frobenius method

<u>Example</u> $x^2(x-1)^2 y'' + \sin x\, y' + (x-1)y = 0$

Find & classify the singular points.

<u>Sol</u> $p(x) = \dfrac{\sin x}{x^2(x-1)^2}$ $q(x) = \dfrac{1}{x^2(x-1)}$

Singular points: $x_0 = 0, 1$

<u>$x_0 = 0$</u> $x\,p(x) = \dfrac{\sin x}{x(x-1)^2}$ - analytic at $x_0 = 0$

$x^2 q(x) = \dfrac{1}{x-1}$ - analytic at $x_0 = 0$

$x_0 = 0$ is regular singular

<u>$x_0 = 1$</u> $(x-1)\,p(x) = \dfrac{\sin x}{x^2(x-1)}$ - NOT analytic at $x_0 = 1$

$(x-1)^2 q(x) = \dfrac{x-1}{x^2}$ - analytic at $x_0 = 1$

$x_0 = 1$ is irregular singular point.

$x = x_0$ - regular singular

$\iff p(x) = \dfrac{b(x)}{(x-x_0)}$ $q(x) = \dfrac{c(x)}{(x-x_0)^2}$

where $b(x)$ & $c(x)$ are analytic at x_0.

<u>Cauchy Euler equation</u>

$x^2 y'' + b_0 x y' + c_0 y = 0$

$y'' + \frac{b_0}{x} y' + (c_0/x^2) y = 0$

$x_0 = 0$ is a regular singular point.

$x = e^t \rightsquigarrow$ const coeff.

$y = e^{mt} = x^m$

<u>characteristic equation</u> <u>$m(m-1) + b_0 m + c_0 = 0$</u>

$m = m_1, m_2$

- $m_1 \neq m_2$ $m_1, m_2 \in \mathbb{R}$
  x^{m_1}, x^{m_2}
- $m_1 = m_2 = m$
  x^m, <u>$(\log x)\, x^m$</u>
- $m_1, m_2 = \alpha \pm i\beta$
  ....

$x = 0$ regular singular point.

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$$\Rightarrow \quad x^2 y'' + x\, b(x) y' + c(x) y = 0$$

where $b(x)$ & $c(x)$ are analytic at x_0.

Assume solution

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n \quad \Big| \; a_0 \neq 0, \ \text{w.l.o.g.}\ a_0 = 1$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$x y' = \sum_{n \geq 0} a_n (n+r) x^{n+r}$$

$$x^2 y'' = \sum_{n \geq 0} a_n (n+r)(n+r-1) x^{n+r}$$

Substituting in the given eqn

$$\sum_{n \geq 0} a_n (n+r)(n+r-1) x^{n+r} + \Big(\sum_n a_n (n+r) x^{n+r}\Big)\Big(\overbrace{\sum_n b_n x^n}^{b(x)}\Big) + \Big(\sum a_n x^{n+r}\Big)\Big(\underbrace{\sum c_n x^n}_{c(x)}\Big) = 0$$

$\boxed{\text{Coeff of } x^r}$

$$a_0 r(r-1) + a_0 r \cdot b_0 + a_0 c_0 = 0$$

$$r(r-1) + r b_0 + c_0 = 0$$

$$\rho(r) = \underbrace{r(r-1) + r b_0 + c_0}_{\text{Indicial equation}}$$

$\boxed{\text{ASSUME}}$ $\boxed{\text{both roots of } r, \text{ are real}}$.

$$r_1 \geq r_2$$

$\underline{\text{coff of } x^{n+r} \quad n \geq 1}$

$$\rho(n+r)\, a_n + \sum_{i=0}^{n-1} b_{n-i}\,(i+r)\, a_i$$
$$+ \sum_{i=0}^{n-1} C_{n-i}\, a_i = 0$$

$$a_n = \frac{(\;-\;-\;-\;--\;)}{\rho\,(n+r)}$$

$\underline{n \geq 1}$

$\boxed{r = r_1}$ $$a_n(r_1) = \frac{(\;-\;-\;-\;-\;)}{\rho\,(n+r_1)}$$

Since r_1 is the biggest root $\rho(n+r_1) \neq 0$ $\forall\, n \geq 1$

This we have a solution

$$y_1(x) = \boxed{x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)\, x^n}$$

Frobenius Series.

$\boxed{r = r_2}$

- $\underline{r_1 - r_2 \notin \mathbb{Z}}$

In this case we will have another solution

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)\, x^n$$

$$(\because \rho(n+r_2) \neq 0 \;\; \forall\, n \geq 1)$$

- $\underline{r_1 - r_2 \in \mathbb{Z}}$

To find $y_2(x)$, use the reduction method: $y_2(x) = y_1(x)\, v(x)$.

Then $v(x)$ satisfy.

$$v'(x) = \frac{1}{y_1^2} e^{-\int \frac{b(x)}{x} dx}.$$

$r_1 = r_2$

$$y_2(x) = (\log x)\, y_1 + x^{r_2} \sum a_n' x^n$$

$\mathbb{Z} \ni r_1 - r_2 \neq 0$

$$y_2(x) = c(\log x)\, y_1(x) + x^{r_2} \sum a_n' x^n$$

c - const which may zero.

<u>Summary</u>

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

- Indicial equation $\rho(r) = r(r-1) + b_0 r + c_0 = 0$

  $\rho = r_1, r_2$ <u>ASSUME</u> $r_1, r_2 \in \mathbb{R}$.

- $r_1 - r_2 \notin \mathbb{Z} \Rightarrow$ two Frobenius series solution exist

- $r_1 = r_2 \Rightarrow$ only one Frobenius series solution

- $\underbrace{r_1 - r_2}_{\in \mathbb{Z}} \neq 0 \Rightarrow$ Second Frobenius series solution may or may not exist

Example

$$2x^2 y'' + x(x+1) y' - \cos x \, y = 0$$

$x=0$ regular singular

$$b(x) = \frac{x+1}{2} \qquad c(x) = -\frac{\cos x}{2}$$

$$\left(y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \right)$$

Indicial eqtn

$$r(r-1) + b_0 r + c_0 = 0 \qquad b_0 = \frac{1}{2}$$

$$c_0 = -\frac{1}{2}$$

$$r = 1, -\frac{1}{2}.$$

$$r_1 = 1 \qquad r_2 = -\frac{1}{2}$$

$$r_1 - r_2 = 3/2 \notin \mathbb{Z}.$$

$\Rightarrow$ two Frobenius Series solutions exist. ∎