MTH302: *Set Theory and Mathematical Logic*
Homework: These will be periodically updated. Odd Semester, 2024/25

(1) Show that there is no set V such that every set is a member of V.

(2) Show that $(x, y) = (a, b)$ iff $x = a$ and $y = b$.

(3) Suppose R is an equivalence relation on A. For each $a \in A$, define the R-equivalence class of a by $[a] = \{b \in A : aRb\}$. Show that $\{[a] : a \in A\}$ is a partition of A. Furthermore, show that for every partition $\mathcal{F}$ of A, there is an equivalence relation S on A such that $\mathcal{F}$ is the set of all S-equivalence classes.

(4) Suppose $X \subseteq \omega$ satisfies $0 \in X$ and $(\forall y \in X)(y \cup \{y\} \in X)$. Show that $X = \omega$.

(5) Let $(L, \prec)$ be a linear ordering. Prove the following.

(a) $(L, \prec)$ is a well-ordering iff there is no sequence $\langle x_n : n < \omega \rangle$ in L such that $(\forall n < \omega)(x_{n+1} \prec x_n)$.

(b) $(L, \prec)$ is a well-ordering iff for every $A \subseteq L$, $(A, \prec)$ is isomorphic to an initial segment of $(L, \prec)$.

(6) Suppose $(X, \prec_1)$ and $(Y, \prec_2)$ are well-orderings. Then exactly one of the following holds.

(a) $(X, \prec_1) \cong (Y, \prec_2)$.

(b) For some $x \in X$, $(\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (Y, \prec_2)$.

(c) For some $y \in Y$, $(\mathsf{pred}(Y, \prec_2, y), \prec_2) \cong (X, \prec_1)$.

Furthermore, in each of the three cases, the isomorphism is unique.

(7) Suppose X is a nonempty set and F is a choice function on $\mathcal{P}(X) \setminus \{\emptyset\}$. Let $(X, \prec_1)$ and $(X, \prec_2)$ be two well-orderings satisfying $(\forall y \in X)(F(X \setminus \mathsf{pred}(X, \prec_k, y))) = y$ for $k \in \{1, 2\}$. Show that $(\forall a, b \in X)(a \prec_1 b \iff a \prec_2 b)$.

(8) Let $f : \mathcal{P}(\omega) \setminus \{\emptyset\} \to \omega$ be defined by $f(X) = \min(X)$. Call a well-ordering $(A, \prec)$ f-directed iff $A \subseteq \omega$ and for every $x \in A$,

$$f(\omega \setminus \mathsf{pred}(A, \prec, x)) = x$$

Describe all f-directed well-orderings.

(9) Show that if $\alpha < \beta$ are ordinals, then there is an ordinal γ such that $\alpha + \gamma = \beta$. (**Hint**: $\gamma = \mathsf{type}(\beta \setminus \alpha, \in)$).

(10) Suppose α, β, γ are ordinals and $\alpha + \beta = \alpha + \gamma$. Show that $\beta = \gamma$.

(11) Suppose $\alpha \cdot \alpha = \beta \cdot \beta$. Show that $\alpha = \beta$.

(12) Show that there is an uncountable chain in $(\mathcal{P}(\omega), \subseteq)$. (**Hint**: Identify ω with the set of rationals $\mathbb{Q}$ and for each real number x, consider $\{r \in \mathbb{Q} : r < x\}$).

(13) Suppose $f : \mathbb{R} \to \mathbb{R}$ is additive and $a = f(1)$.

(a) Show that $f(0) = 0$.

(b) Show that for every $x \in \mathbb{R}$, $f(-x) = -f(x)$.

(c) Show that for every $x \in \mathbb{Q}$, $f(x) = ax$.

(14) Let $H \subseteq \mathbb{R}$ be a Hamel basis.

(a) Show that every nonzero $x \in \mathbb{R}$ can be uniquely written as

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

where $x_1 < x_2 < \cdots < x_n$ are in H and $a_1, a_2, \ldots a_n$ are nonzero rational numbers. Uniqueness means the following: Suppose

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b_1 y_1 + b_2 y_2 + \cdots + a_m y_m$$

where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ are in H and $a_1, \ldots a_n, b_1, \ldots, b_m$ are nonzero rationals. Show that $m = n$ and for every $1 \leq k \leq n$, $x_k = y_k$ and $a_k = b_k$.

(b) Let $f : H \to \mathbb{R}$. Show that there is a unique additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f \subseteq g$.

(15) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.

(a) Show that either f is identically zero or range$(f) \subseteq \mathbb{R}^+$.

(b) Suppose f is continuous and not identically zero. Show that $f(x) = a^x$ for some $a > 0$.

(16) Show that there is a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

(17) Prove the following.

(a) For every ordinal α, $|\alpha| \leq \alpha$.

(b) If κ is a cardinal and $\alpha < \kappa$, then $|\alpha| < \kappa$.

(c) There is an injection from X to Y iff $|X| \leq |Y|$.

(d) There is a surjection from X to Y iff $|Y| \leq |X|$.

(e) There is a bijection from X to Y iff $|X| = |Y|$.

(18) Prove the following.

(a) $|\mathbb{R}^\omega| = \mathfrak{c}$.

(b) $|C(\mathbb{R})| = \mathfrak{c}$ where $C(\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(c) Let A be the set of all real numbers which are roots of some polynomial equation with rational coefficients. Show that $|A| = \omega$.

(19) Suppose $(P, \preceq)$ is a partial ordering in which every countable chain has an upper bound. Must P have a maximal element?

(20) Use Zorn's lemma to show that $\mathbb{R}^+ = (0, \infty)$ can be partitioned into two nonempty sets that are both closed under addition.

(21) Is there a subset $X \subseteq \mathbb{R}^2$ that meets every circle at 2 points?

(22) Show that there is a subset $X \subseteq \mathbb{R}^2$ that meets every circle at 3 points.

(23) Suppose that the set of propositional variables $\mathcal{V}ar$ is uncountable. Use Zorn's lemma to show the following: Let S be a set of propositional formulas such that every finite subset of S is satisfiable. Then S is satisfiable. **Hint**: Apply Zorn's lemma to $(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is the set of all functions h such that $\mathrm{dom}(h) \subseteq \mathcal{V}ar$, $\mathrm{range}(h) \subseteq \{0, 1\}$ and for every finite $F \subseteq S$, there exists a valuation $val : \mathcal{V}ar \to \{0, 1\}$ such that $h \subseteq val$ and every formula in F is true under val.

(24) Let ϕ be a propositional formula in which $\neg$ does not occur. Show that ϕ is satisfiable.

(25) Show that the axiom of extensionality is not a logically valid $\mathcal{L}_{ZFC}$-sentence.

(26) Show that $(\forall x)(\forall y)(x + y = y + x)$ is not a logically valid $\mathcal{L}_{PA}$-sentence.

(27) Let $\mathcal{L}$ be a first order language and T be an $\mathcal{L}$-theory. Assume $T \vdash \phi$ and $T \vdash (\phi \implies \psi)$. Show that $T \vdash \psi$.

(28) Let $\mathcal{L}$ be a first order language and T be an $\mathcal{L}$-theory. Assume $T \vdash \phi_1$ and $T \vdash \phi_2$. Show that $T \vdash (\phi_1 \wedge \phi_2)$.

(29) Let $\mathcal{L}$ be a first order language. Call a set S of $\mathcal{L}$-sentences deductively closed if S contains every logical axiom of $\mathcal{L}$ and whenever ϕ and $(\phi \implies \psi)$ are in S, ψ is also in S. Show that for any $\mathcal{L}$-theory T, the set of theorems of T is the smallest set of $\mathcal{L}$-sentences that is deductively closed.

(30) Describe all substructures of $(\mathbb{Z}, <)$ and $(\mathbb{Z}, +)$.

(31) Describe all elementary submodels of $(\mathbb{Z}, <)$ and $(\mathbb{Z}, +)$.

(32) For each $n \geq 1$ and $\mathcal{L} = \emptyset$. Give an example of a consistent $\mathcal{L}$-theory T such that for every model $\mathcal{M} \models T$, $|M| = n$.

(33) Let $\mathcal{L} = \mathcal{L}_{PA} \cup \{c\}$ where c is a new constant symbol. Let $\mathsf{Primes} = \{2, 3, 5, 7, \ldots\}$ be the set of all primes numbers. For each $p \in \mathsf{Primes}$, let "p $\mathsf{divides}$ c" denote the $\mathcal{L}$-sentence $(\exists y)(S^p(0) \cdot y = c)$. For each $X \subseteq \mathsf{Primes}$, let T_X be the $\mathcal{L}$-theory

$$T_X = TA \cup \{(p \text{ divides } c) : p \in X\} \cup \{\neg(p \text{ divides } c) : p \in \mathsf{Primes} \setminus X\}$$

where $TA = Th(\omega, 0, S, +, \cdot)$ denotes true arithmetic.

(a) Show that T_X is consistent for every $X \subseteq \mathsf{Primes}$.

(b) Show that TA has continuum many pairwise non-isomorphic countable models.

(34) Show that $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

(35) Show that every countable linear ordering $(L, <_L)$ is isomorphic to $(A, <)$ for some $A \subseteq \mathbb{Q}$.

(36) Let $\mathcal{L} = \{\prec\}$ where $\prec$ is a binary relation symbol. Let T be the $\mathcal{L}$-theory obtained by replacing Axiom (6) of DLO (Slide 142) with the following

$$(\exists y)(\forall x)(y \prec x \vee y = x)$$

Show that T is ω-categorical and therefore complete. Use this to also show that for every $\mathcal{L}$-sentence ϕ

$$T \vdash \phi \iff (\mathbb{Q}_{\geq 0}, <) \models \phi$$

where $\mathbb{Q}_{\geq 0}$ is the set of all non-negative rationals.

(37) Let $W \subseteq \omega$ be nonempty. Show that W is c.e. iff there exists a computable function $f : \omega \to \omega$ such that range$(f) = W$.

(38) Let $W \subseteq \omega$ be infinite. Show that W is c.e. iff there exists an injective computable function $f : \omega \to \omega$ such that range$(f) = W$.

(39) Let $W \subseteq \omega$ be an infinite c.e. set. Show that W has an infinite computable subset.

(40) Let $W \subseteq \omega$ be nonempty. Show that W is c.e. iff there exists a computable $A \subseteq \omega^2$ such that $W = \{n \in \omega : (\exists m)((n, m) \in A)\}$.

(41) Suppose $X \subseteq \omega$ is numeralwise representable in PA. Show that X is computable.

(42) Let $E \subseteq \omega$ be a non-computable c.e. set. Show that E is definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$ but not numeralwise representable in PA.

(43) Do the Exercise on the last slide (180) of notes.