MSO205 PRACTICE PROBLEMS SET 9

Question 1. Let $X = (X_1, X_2)$ be a discrete random vector with joint p.m.f.

$$f_X(x_1, x_2) = \begin{cases} \alpha(2x_1 + x_2), & \text{if } x_1, x_2 \in \{1, 2\}, \\ 0, & \text{otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. Find the value of α and identify the marginal p.m.f.s of X_1 and X_2. Are X_1, X_2 independent? If not independent, find the conditional p.m.f. of X_2 given $X_1 = x_1 \in \{1, 2\}$.

Question 2. Let $X = (X_1, X_2, X_3)$ be a continuous random vector with joint p.d.f.

$$f_X(x_1, x_2, x_3) = \begin{cases} \frac{\alpha}{x_1 x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1, \\ 0, & \text{otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. Find the value of α and identify the marginal p.d.f.s of X_1, X_2 and X_3. Are X_1, X_2, X_3 independent? If not independent, find the conditional DF and conditional p.d.f. of X_2 given $(X_1, X_3) = (x_1, x_3)$ with $0 < x_3 < x_1 < 1$.

Question 3. Let $X = (X_1, X_2)$ be a bivariate continuous random vector with joint p.d.f. given by

$$f_X(x_1, x_2) = \begin{cases} 1, & \text{if } 0 < |x_2| \le x_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal p.d.f.s of X_1 and X_2 and show that X_1, X_2 are not independent.

Question 4. Let $X \sim Exponential(\lambda)$ for some $\lambda > 0$. For $r, s > 0$, show that

$$\mathbb{P}(X > r + s \mid X > r) = \mathbb{P}(X > s).$$

Note: This property is usually referred to as the 'no memory' property of the Exponential distribution.

Question 5. Let $X_i \sim Gamma(\alpha_i, \beta), i = 1, 2, \cdots, n$ be independent RVs, with $\alpha_i > 0, \forall i$ and $\beta > 0$. Show that $X_1 + X_2 + \cdots + X_n \sim Gamma(\sum_{i=1}^n \alpha_i, \beta)$.

Question 6. Let $X \sim Gamma(\alpha_1, \beta), Y \sim Gamma(\alpha_2, \beta)$ be independent RVs, for some $\alpha_1, \alpha_2, \beta > 0$. Identify the distribution of $\frac{X}{X+Y}$.

Question 7. If $X_1, X_2, \cdots, X_n$ are independent RVs with $X_i \sim N(\mu_i, \sigma_i^2)$, then find the distribution of $X_1 + X_2 + \cdots + X_n$.

Question 8. Let X and Y be i.i.d. $N(0,1)$ RVs. Fix $a \neq 0, b \neq 0$ and set $U := aX + bY, V := bX - aY$. Find the joint p.d.f. of U, V. Are U and V independent?

Question 9. Let $X_i \sim Poisson(\lambda_i), i = 1, 2, \cdots, n$ be independent RVs, with $\lambda_i > 0, \forall i$. Show that $X_1 + X_2 + \cdots + X_n \sim Poisson(\sum_{i=1}^n \lambda_i)$.
(Note: A special case of this result is the following: If $X_1, X_2, \cdots, X_n$ be a random sample from $Poisson(\lambda)$ distribution, then $X_1 + X_2 + \cdots + X_n \sim Poisson(n\lambda)$.)

Question 10. Let $X \sim Poisson(\lambda), Y \sim Poisson(\mu)$ be independent RVs. Find the conditional distribution of X given $X + Y = k$ for $k = 0, 1, \cdots$.

Question 11. Consider the set

$$A := \left\{ t = (t_1, t_2, \ldots, t_p) \in \mathbb{R}^p : \mathbb{E}\left(e^{\sum_{i=1}^p t_i X_i} \right) < \infty \right\}$$

for a given random vector $X = (X_1, X_2, \cdots, X_p)$ and look at $\Psi_X(t) := \ln M_X(t), t \in A$. Verify that

$$\left[\frac{\partial^2}{\partial t_i \partial t_j} \Psi_X(t) \right]_{(t_1, t_2 \ldots t_p) = (0, \ldots, 0)} = Cov(X_i, X_j).$$