Name: ______________________

Roll Number: ________________

Practice final

MTH302 - Set Theory and Mathematical Logic

(Odd Semester 2024/25, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.
2. This exam contains **5 + 1** questions and is worth **50%** of your grade.
3. Answer **ALL** questions.

Question 1. [5 $\times$ 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) There exists a sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ such that $\alpha_{n+1} < \alpha_n$ for every $n < \omega$.

(ii) $(\mathbb{Q}, <)$ is an elementary submodel of $(\mathbb{R}, <)$.

(iii) Every consistent first order theory has an infinite model.

(iv) $\mathsf{Th}([0,1], <)$ is ω-categorical.

(v) If $A \subseteq \omega$ is c.e., then $A + A = \{m + n : m, n \in A\}$ is c.e.

Solution

(i) False. Otherwise, $\{\alpha_n : n < \omega\}$ is a nonempty set of ordinals with no least member.

(ii) True. See HW 34.

(iii) False. Let $T = \{(\forall x)(\forall y)(x = y)\}$.

(iv) True. Showed in class.

(v) True. Showed in class.

Question 2. [10 Points]

Call a subset $E \subseteq \mathbb{R}$ mid-point free if there do not exist $x < y < z$ in E such that $x + z = 2y$.

(a) [**2 Point**] State Zorn's lemma.

(b) [**4 Points**] Show that there is a maximal mid-point free set $E \subseteq \mathbb{R}$. This means that E is mid-point free and for every $x \in \mathbb{R} \setminus E$, $E \cup \{x\}$ is not mid-point free.

(c) [**4 Points**] Show that every maximal mid-point free set has cardinality $\mathfrak{c}$.

Solution

(a) Every Let $(P, \preceq)$ be a partial ordering such that every chain in P has an upper bound in P. Then $(P, \preceq)$ has a maximal element. $\square$

(b) Let P be the set of all mid-point free $E \subseteq \mathbb{R}$ ordered by $\subseteq$. Then P is nonempty since $\emptyset \in P$.

Let $\mathcal{C}$ be a chain in P. Put $A = \bigcup \mathcal{C}$. Then for every $E \in \mathcal{C}$, $E \subseteq A$. We claim that $E \in P$. Suppose not. Then there are $a < b < c$ in E such that $2b = a + c$. Choose E_1, E_2, E_3 in $\mathcal{C}$ such that $a \in E_1$, $b \in E_2$ and $c \in E_2$. As $\mathcal{C}$ is a chain, one of the sets, say E_1 is a superset of the other two. This implies that $\{a, b, c\} \subseteq E_1$. But this contradicts the fact that E_1 is mid-point free. So $A \in P$ is an upper bound of $\mathcal{C}$.

By Zorn's lemma, P has a maximal member E. It follows that E is mid-point free and for every $x \in \mathbb{R} \setminus E$, $E \cup \{x\}$ is not mid-point free. $\square$

(c) It suffices to show that if $E \subseteq \mathbb{R}$ is mid-point free and $|E| < \mathfrak{c}$, then there exists $x \in \mathbb{R} \setminus E$ such that $E \cup \{x\}$ is mid-point free. For each pair $x \neq y$ in E, there are exactly 3 points $z \in \mathbb{R}$ such that $\{x, y, z\}$ is not mid-point free − namely, $x + y = 2z$, $y + z = 2x$ and $x + z = 2y$. Let $B_{x,y}$ be the set of all these z's. Then $B = \bigcup \{B_{x,y} : x, y \in E \wedge x \neq y\}$ is a union of $|E \times E| < \mathfrak{c}$ sets each of size 3. So $|B| < \mathfrak{c}$ and hence $|E \cup B| < \mathfrak{c}$. Choose $x \in \mathbb{R} \setminus (E \cup B)$. Then $E \cup \{x\}$ is mid-point free. $\square$

Question 3. [10 Points]

Let $\mathcal{L}$ be the empty language. For each $n \geq 2$, recall that $\exists_{\geq n}$ denotes the following $\mathcal{L}$-sentence:

$$(\exists x_1)(\exists x_2)\ldots(\exists x_n)\left(\bigwedge_{i<j\leq n} \neg(x_i = x_j)\right)$$

Define $T = \{\exists_{\geq n} : n \geq 2\}$. Prove the following.

(a) [**2 Points**] T does not have a finite model.

(b) [**2 Points**] T is consistent.

(c) [**2 Points**] T is ω-categorical.

(d) [**2 Points**] T is complete.

(e) [**2 Points**] $\{\phi : T \vdash \phi\}$ is computable.

Solution

As $\mathcal{L}$ is empty, an $\mathcal{L}$-structure is just a nonempty set M (its domain).

(a) Suppose M is finite and $|M| = n$. Then $M \models \neg(\exists_{\geq n+1})$. So M does not model T.

(b) Every infinite set M is a model of T. As T has a model, it is consistent.

(c) Let $|M| = |N| = \omega$. Then any bijection between M and N is an isomorphism because $\mathcal{L}$ is empty. So every $\mathcal{L}$-theory (and hence T) is ω-categorical.

(d) Since $\mathcal{L}$ is countable, T is ω-categorical and T has no finite models, it is complete (Slide 141).

(e) Since T is computable and complete, $\{\psi : T \vdash \psi\}$ is computable (Slide 165). $\square$

Question 4. [10 Points]

Let $T = \mathsf{Th}(\mathbb{Z}, <)$. So T is an $\mathcal{L}$-theory where $\mathcal{L} = \{<\}$.

(a) [**2 Point**] Is T a complete theory? Justify your response.

(b) [**2 Point**] State the compactness theorem for first order logic.

(c) [**6 Points**] Show that T is not ω-categorical. **Hint:** Put $\mathcal{L}_1 = \{<, c, d\}$ where c, d are constant symbols and consider the $\mathcal{L}_1$-theory $T_1 = T \cup \{\psi_n : n \geq 1\}$ where ψ_n is says "there are at least n elements between c and d".

Solution

(a) Yes. Because every $\mathcal{L}$-sentence is either true or false in $(\mathbb{Z}, <)$.

(b) Let T be a first order theory such that every finite subset of T has a model. Then T has a model.

(c) This is similar to the application of compactness theorem on Slide 140.

For each $n \geq 1$, consider the $\mathcal{L}_1$-structure $\mathcal{M}_n = (\mathbb{Z}, <, c^{\mathcal{M}} = -n, d^{\mathcal{M}} = n)$. Then $\mathcal{M}_n \models T \cup \{\psi_k : k \leq n\}$. It follows that every finite subset of T has a model. Hence T has a model and as $\mathcal{L}_1$ is countable, by the Lowenheim-Skolem theorem it must have a countable model say $\mathcal{A} = (A, <^{\mathcal{A}}, c^{\mathcal{A}}, d^{\mathcal{A}})$. Then $|A| = \omega$ since $(\mathbb{Z}, <) \models \exists_{\geq n}$ for every $n \geq 2$. Furthermore, $(A, <^{\mathcal{A}}) \models T$ and it is a linear ordering that has two elements $c^{\mathcal{A}}$ and $d^{\mathcal{A}}$ with infinitely many members in between. Since $(\mathbb{Z}, <)$ does not have such elements, it follows that $(A, <^{\mathcal{A}})$ is not isomorphic to $(\mathbb{Z}, <)$. So T is not ω-categorical. $\square$

Question 5 [10 Points]

Let $\mathcal{N} = (\omega, 0, S, +, \cdot)$ be the standard model of PA. Define

$$\mathsf{True}_{\mathcal{N}} = \{\ulcorner \psi \urcorner : \mathcal{N} \models \psi\} \text{ and } \mathsf{False}_{\mathcal{N}} = \{\ulcorner \psi \urcorner : \mathcal{N} \models \neg\psi\}$$

(a) [**3 Points**] Let $E \subseteq \omega$. What does it mean to say that E is definable in $\mathcal{N}$?

(b) [**3 Points**] State Tarski's theorem on undefinability of truth in arithmetic.

(c) [**4 Points**] Suppose $S \subseteq \omega$, $\mathsf{True}_{\mathcal{N}} \subseteq S$ and $\mathsf{False}_{\mathcal{N}} \subseteq (\omega \setminus S)$. Show that S is not computable.

Solution

(a) There is an $\mathcal{L}_{PA}$-formula $\phi(x)$ such that for every $n \in \omega$,

$$n \in E \iff \mathcal{N} \models \phi(n)$$

(b) $\mathsf{True}_{\mathcal{N}}$ is not definable in $\mathcal{N}$.

(c) Suppose S is computable. We will show that $\mathsf{True}_{\mathcal{N}}$ is computable which is impossible since every computable subset of ω is definable in $\mathcal{N}$ while $\mathsf{True}_{\mathcal{N}}$ is not definable in $\mathcal{N}$.

Consider a program P that on input n does the following. If $n = \ulcorner \psi \urcorner$ for some $\mathcal{L}_{\mathcal{PA}}$-sentence ψ, then output $1_S(n)$. Otherwise output 0. Let us check that P computes the characteristic function of $\mathsf{True}_{\mathcal{N}}$.

First suppose $n \in \mathsf{True}_{\mathcal{N}}$. Then there is an $\mathcal{L}_{\mathcal{PA}}$-sentence ψ such that $n = \ulcorner \psi \urcorner$ and $\mathcal{N} \models \psi$. So P on input n outputs $1_S(n) = 1$ (since $n \in \mathsf{True}_{\mathcal{N}} \subseteq S$).

Next suppose $n \notin \mathsf{True}_{\mathcal{N}}$. Then either n is not the Gödel number of an $\mathcal{L}_{\mathcal{PA}}$-sentence in which case P outputs 0. Or n is the Gödel number of an $\mathcal{L}_{\mathcal{PA}}$-sentence ψ and $\mathcal{N} \models \neg\psi$ which means $n \in \mathsf{False}_{\mathcal{N}}$ and so P outputs $1_S(n) = 0$ (as $\mathsf{False}_{\mathcal{N}} \subseteq (\omega \setminus S)$). $\square$

Bonus Question [5 Points]

Recall that φ_e is the eth partial computable function on ω. Define $A = \{e : e \in \mathsf{dom}(\varphi_e) \text{ and } \varphi_e(e) = 0\}$ and $B = \{e : e \in \mathsf{dom}(\varphi_e) \text{ and } \varphi_e(e) = 1\}$. Show that $A, B \subseteq \omega$ are disjoint c.e. sets and there is no computable $R \subseteq \omega$ such that $A \subseteq R$ and $B \subseteq \omega \setminus R$.

Solution

That A, B are c.e. is easy to check. Next, towards a contradiction, suppose $R \subseteq \omega$ is computable, $A \subseteq R$ and $B \subseteq \omega \setminus R$. Then $h(n) = 1_R(n)$ is computable (here 1_R is the characteristic function of R). So there must be some $e_\star < \omega$ such that $h = \varphi_{e_\star}$. In particular, $\varphi_{e_\star}$ is total. Now check that $h(e_\star) \neq \varphi_{e_\star}(e_\star)$. A contradiction. $\square$

End of Exam