End-Semster Exam MTH113M/MTH102A : SET A

Date : 19/02/2024 | Time : 6:00- 7:30 pm | Will be Collected Back at 7:30 pm

NAME : SOLUTION	ROLL : END-SEM

Question A.1. Let A be a real matrix of order $n \times n$ such that $rank\,(A) = rank\,(A^2)$. Let $N(A)$ and $CS\,(A)$ denote the null space and column space of A respectively. Show that $N(A) \cap CS\,(A) = \{\mathbf{0}\}$ and $N(A) + CS\,(A) = \mathbb{R}^n$. [3+2=5 Marks]

Answer A.1.: Consider the linear transformation

$T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$T_A(X) = AX$, $X \in \mathbb{R}^n$.

$R(T_A) = CS(A)$. where $R(T_A)$ = range of T_A.

$\text{rank}(A) = \dim CS(A) = \dim R(T_A)$

Note $R(T_A^2) \subseteq R(T_A)$. — (*)

$\text{rank}(A^2) = \text{rank}(A) \Rightarrow \dim(R(T_A^2)) = \dim(R(T_A))$

From (*), $R(T_A^2) = R(T_A)$.

Note, $\ker(T_A) \subseteq \ker(T_A^2)$

By rank-nullity theorem, $n = \dim(\ker(T_A^2)) + \dim(R(T_A^2))$

$= \dim \ker(T_A^2) + \dim(R(T_A))$ — (1)

Also, $n = \dim \ker(T_A) + \dim R(T_A)$ — (2)

(2) − (1) $\Rightarrow$ $\dim \ker(T_A^2) = \dim \ker(T_A)$

$\Rightarrow \ker(T_A) = \ker(T_A^2)$,

$N(A) = \ker(T_A)$

Let $v \in N(A) \cap CS(A) \Rightarrow v \in \ker(T_A) \cap R(T_A)$

$\Rightarrow T_A(v) = 0$ & $\exists\, w \in V$ s.t $T_A(w) = v$

$T_A^2(w) = T_A(v) = 0 \Rightarrow w \in \ker(T_A^2) = \ker(T_A)$

$\Rightarrow v = T_A(w) = 0$ Thus $N(A) \cap CS(A) = \{0\}$

$\dim(N(A) + CS(A)) = \dim N(A) + \dim CS(A) - \dim(N(A) \cap CS(A))$

$= \dim \ker(T_A) + \dim R(T_A) - 0 = n$

$\Rightarrow N(A) + CS(A) = \mathbb{R}^n$.

Question A.2. (i) Prove that the map $T : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, defined by $T(A) = Trace\,(A)$, is a linear map.
(ii) Consider the subspace $V = \{A \in M_n(\mathbb{R}) : Trace\,(A) = 0\}$. Use rank-nullity theorem to prove that the dimension of V is $n^2 - 1$. [2+3=5 Marks]

Answer A.2.: (i) $T(A) = \text{Trace}(A)$.

$A = (a_{ij})$, $B = (b_{ij})$ $\quad \text{Trace}(A) = \sum_{i=1}^{n} a_{ii}$, $\text{Trace}(B) = \sum_{i=1}^{n} b_{ii}$

$$\text{Trace}(A+B) = \sum_{i=1}^{n} (a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{Trace}(A) + \text{Trace}(B).$$

$\Rightarrow T(A+B) = T(A) + T(B)$

$$\text{Trace}(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \lambda\, \text{Trace}(A).$$

(ii) $\ker T = \{A \in M_n(\mathbb{R}) : \text{Trace}(A) = 0\}$

By rank-nullity theorem,

$\dim M_n(\mathbb{R}) = \dim \ker(T) + \dim R(T)$.

$\Rightarrow n^2 = \dim \ker(T) + \dim R(T)$

$\Rightarrow \dim \ker(T) = n^2 - \dim R(T)$. — $(*)$

. The map $T : M_n(\mathbb{R}) \to \mathbb{R}$, $T(A) = \text{Trace}(A)$ is surjective.

Let $a \in \mathbb{R}$. Take $A = \begin{pmatrix} a & & 0 \\ & 0 & \\ 0 & & \ddots\ 0 \end{pmatrix}$

$T(A) = \text{Trace}(A) = a$

Thus, $R(T) = \mathbb{R}$ $\Rightarrow \dim R(T) = 1$

Thus $\dim \{A \in M_n(\mathbb{R}) : \text{Trace}(A) = 0\}$

$= \dim \ker(T) = n^2 - 1$ (from $(*)$)

Question A.3. (i) Let M_{ij} denote the $n \times n$ matrix with 1 only at (i, j)-th position and 0 elsewhere. Show that $M_{ij} = M_{ik}M_{kj} - M_{kj}M_{ik}$ for $i \neq j$ and $M_{11} - M_{jj} = M_{1j}M_{j1} - M_{j1}M_{1j}$ for $2 \leq j \leq n$.
(ii) Consider the set $S = \{AB - BA : A, B \in M_n(\mathbb{R})\}$. Let W be the linear span of S over $\mathbb{R}$ in $M_n(\mathbb{R})$. Prove that the dimension of W is $n^2 - 1$. [4+4=8 Marks]

Answer A.3.:

$$M_{ik} M_{kj} = \begin{array}{c} \\ i\text{th row} \\ \\ \end{array}\begin{pmatrix} 0 & \cdots & 0 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & - & 0 \\ 0 & \cdot & 0 & \cdot & 0 \end{pmatrix} \begin{array}{c} \\ \to k\text{th row} \\ \\ \end{array}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ k\text{th column} & j\text{th column} \end{array}$$

$$= \begin{array}{c} \\ i\text{th row} \\ \\ \end{array}\begin{pmatrix} 0 & \cdots & 0 & & 0 \\ 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = M_{ij}$$

$$\begin{array}{c} \uparrow \\ j\text{th column} \end{array}$$

$$\underset{k\text{th position}}{(0 - 1 - \cdots 0)} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = 1$$

$$M_{ki} M_{kj} = 0 \qquad \underset{i\text{th position}}{(0 \cdots 1 \cdots 0)} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{j\text{th position}} = 0 \text{ as } i \neq j$$

Thus $M_{ij} = M_{ik} M_{kj} - M_{ki} M_{kj}$ for $i \neq j$.

Note $\text{Trace}(M_{ij}) = 0$ for $i \neq j$.

$$M_{1j} M_{j1} = M_{11} \qquad M_{j1} M_{1j} = M_{jj} \Rightarrow M_{1j} M_{j1} - M_{j1} M_{1j} = M_{11} - M_{jj}$$

$\text{Trace}(M_{11} - M_{jj}) = 0$.

Let $R = \{ M_{ij} : i \neq j,\ 1 \leq i, j \leq n \} \cup \{ M_{11} - M_{jj} : j = 2, \ldots, n \}$

R contains $n^2 - 1$ elements

$LS(R) \subseteq LS(S)$, $S = \{AB - BA : A, B \in M_n(\mathbb{R})\}$

Also, $LS(S) \subseteq \{A \in M_n(\mathbb{R}) : \text{Trace}(A) = 0\} = P$ (say)

$\dim P = n^2 - 1 \Rightarrow \dim LS(R) \leq n^2 - 1$

R is linearly independent :-

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij} M_{ij} + \sum_{r=2}^{n} a_{rr} (M_{11} - M_{rr}) = 0$$

$$\Rightarrow \begin{pmatrix} \sum_{r=2}^{n} a_{rr} & a_{12} & \cdots & a_{1n} \\ a_{21} & -a_{22} & & a_{2n} \\ & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} = 0 \Rightarrow \begin{array}{l} a_{ij} = 0 \ \forall\ i \neq j \\ \& \ a_{rr} = 0 \\ \forall\ 2 \leq r \leq n \end{array}$$

Thus, $LS(R) = P \Rightarrow W = LS(S) = P \Rightarrow \dim W = n^2 - 1$.

Question A.4. Let $Q(x,y) = 8x^2 - 4xy + 5y^2$. Find a matrix A of order 2×2 such that $Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}$. Compute the eigenvalues of A and then find a matrix P such that by applying change of variable $\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$ the curve $8x^2 - 4xy + 5y^2 = 1$ changes to the form $\frac{u^2}{4} + \frac{v^2}{9} = 1$. [2+4=6 Marks]

Answer A.4.: $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$

$$\det(A - xI) = (8-x)(5-x) - 4 = x^2 - 13x + 40$$
$$= (x-4)(x-9)$$

Eigenvalues of A are roots of $\det(A - xI) = 0$

$\Rightarrow$ eigenvalues of A are $4, 9$.

Eigenvectors of A :- $A\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$ $8x - 2y = 4x$ $\Rightarrow$ $y - 2x = 0$
$-2x + 5y = 4y$

Eigenvector corresponding to 4 is $(1,2)$

$A\begin{pmatrix} x \\ y \end{pmatrix} = 9\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $8x - 2y = 9x$ $\Rightarrow x + 2y = 0$
$-2x + 5y = 9y$

Eigenvector corresponding to 9 is $(-2,1)$.

Let $v_1 = \frac{1}{\sqrt{5}}(-2,1)$, $v_2 = \frac{1}{\sqrt{5}}(1,2)$.

$\|v_i\| = 1$, $\langle v_1, v_2 \rangle = 0$ $i = 1, 2$.

Let Q be a matrix with column vectors v_1, v_2

Then Q is orthogonal & $Q^{-1} = Q^T$

$Q^{-1}AQ = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$. Note $Q = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$

$\Rightarrow A = Q\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}Q^{-1}$

$= Q^T \ (= Q^{-1})$

$= (6Q)\begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix}(6Q)^T$

$1 = (x\ y)\, A\begin{pmatrix} x \\ y \end{pmatrix} = (x\ y)\, 6Q\begin{pmatrix} 1/4 & \\ 0 & 1/9 \end{pmatrix}(6Q)^T\begin{pmatrix} x \\ y \end{pmatrix}$

Let $P = (6Q)^T$ $\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$.

$1 = (x\ y)\, A\begin{pmatrix} x \\ y \end{pmatrix} = (u\ v)\begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{u^2}{4} + \frac{v^2}{9}$.