Midsemester Exam: MTH421

Time: 13:00–15:00, Date: 19/09/2025, Duration: 2 Hours, Final score: min{30, marks scored}

1. Consider the (IVP) $z'(t) = A^2 + z(t)^2$, $z(0) = 1$, where $A > 0$ is a fixed parameter. Denote by $[0, T_A)$ the *maximal positive interval* on which the solution exists.

(a) Solve the differential equation explicitly, and determine the blow-up time T_A in terms of A. [7]

(b) Show that if $B > A$, then $T_B < T_A$. [3]

(a) $f(z) = A^2 + z^2$. Note $f > 0$ on $\mathbb{R}$.

Consider now

$$F(y) = \int_1^y \frac{1}{A^2+s^2}\,ds = \left[\frac{1}{A}\tan^{-1}\frac{s}{A}\right]_1^y$$

$$= \frac{1}{A}\left(\tan^{-1}\frac{y}{A} - \tan^{-1}\frac{1}{A}\right).$$

Note that F is strictly increasing & the solution is the inverse of F. The inverse of F is given by:

$$g(t) = A\tan\left(At + \tan^{-1}(1/A)\right).$$

The blow-up time T_A is given by

$A T_A + \tan^{-1}(1/A) = \pi/2$. Therefore

$$T_A = \frac{\pi/2 - \tan^{-1}(1/A)}{A} = \frac{\tan^{-1}A}{A}.$$

(b) Consider the function

$$g(x) = \frac{\tan^{-1}x}{x}, \quad x \neq 0.$$

$$g'(x) = \frac{1}{x(1+x^2)} - \frac{\tan^{-1}x}{x^2}$$

$$= \frac{1}{x^2}\left(\frac{x}{1+x^2} - \tan^{-1}x\right).$$

Now let $f(x) = \frac{x}{1+x^2} - \tan^{-1}(x)$

Note $f(0) = 0$.

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} - \frac{1}{1+x^2}$$

$$= \frac{1-x^2-1-x^2}{(1+x^2)^2}$$

$$= \frac{-2x^2}{(1+x^2)^2} \leq 0.$$

Hence $f \leq 0$. Hence $g'(x) \leq 0$. Hence g is strictly decreasing.

Name (Capital):

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2. Let $A \in M_n(\mathbb{C})$. Which one of the following statements are true? Justify your answers.

(a) Let (λ, v) be an eigenpair of A. Then $(\exp(\lambda), v)$ is an eigenpair of $\exp(A)$. [2]

(b) There exists an A such that $\exp(A)$ is not an invertible matrix. [2]

(c) Suppose $\exp(A) = \mathbb{I}_n$, then each eigenvalue of A is of the form $2\pi i k$ for some $k \in \mathbb{Z}$. [2]

(a) Note if (λ, v) is an eigenpair then

$Av = \lambda v$, $v \neq 0$.

$\Rightarrow A^k v = \lambda^k v$. Therefore, for

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}, \text{ we get } e^A v = e^\lambda v. \quad \text{TRUE.}$$

(b) FALSE:

One simple way: since A & $-A$ ~~compute~~ commute we have

$$e^A \cdot e^{-A} = e^{-A}\cdot e^{A} = e^{-A+A} = e^0 = \mathbb{I}.$$

Alternative one can use part (a) to argue that if e^A is not invertible then one of its eigenvalue is zero. & Now use part (a) that this cannot be the case.

(c) True:

Each eigenvalue of $\exp(A)$ is of the form e^λ where λ is an eigenvalue of A. Clearly

$$e^\lambda = 1 \Leftrightarrow \lambda = 2\pi i k \text{ for some } k \in \mathbb{Z}.$$

3. Consider the (IVP) $z'(t) = t^2 + z(t)^2$, $z(0) = 1$.

(a) Using the method of upper/lower solution, show that if $[0, T)$ is the maximal interval on which the solution ϕ exists then $T < 1$ (Hint: show that $\phi(t) \geq 1/(1-t)$). [5]

(b) Once again, use the method of upper/lower solution, to show that $T > \pi/4$. [5]

(a)

Let $f(t,z) = t^2 + z^2$.

If ψ is a solution of (IVP)

$z'(t) = (z(t))^2$, $z(0) = 1$, then

$\psi'(t) - f(t,\psi(t)) = (\psi(t))^2 - t^2 - (\psi(t))^2$

$= -t^2$

$< \varphi'(t) - f(t,\varphi(t)) \equiv 0$,

here φ is the solution of the given (IVP).

Note $\varphi(0) = \psi(0) = 1$. Therefore by the first theorem (comparison theorem) done in the class we get

$\varphi(t) \geq \psi(t)$.

By solving explicitely, we get $\psi(t) = \frac{1}{1-t}$, therefore

$\varphi(t) \geq \frac{1}{1-t}$, wherever the solution exists. Clearly if $[0,T)$ is the maximal positive interval on which the sol. exists then

$T < 1$.

(b) Fix $S < T$ & consider the interval $[0, S]$. Again, let ψ be the solution of the IVP

$z'(t) = S^2 + (z(t))^2$, $z(0) = 1$.

Then

$\psi'(t) - f(t,\psi(t)) = S^2 + (\psi(t))^2 - t^2 - (\psi(t))^2$

$= S^2 - t^2 \geq 0$ on $[0, S]$

Therefore

$\psi'(t) - f(t,\psi(t)) \geq \varphi'(t) - f(t,\varphi(t))$

$\psi(0) = \varphi(0) = 1$.

Again by the aforementioned theorem

$\varphi(t) \leq \psi(t)$ whenever the solution exist.

Note by Problem 1 or solving directly, we see that

$\psi(t) = S\tan(St + \tan^{-1}(1/S))$

which blows up at $\frac{\tan^{-1}(S)}{S}$.

Since

$S < T < 1$

$\Rightarrow \frac{\tan^{-1} S}{S} > \frac{\tan^{-1}(T)}{T} > \frac{\pi}{4}$.

Hence ψ is bounded at $\pi/4$, and hence φ is bounded at $\pi/4$. This clearly implies $T > \pi/4$.

Note if we take $S = 1$ then $\psi(t) = \tan(t + \pi/4)$ blows up at $\pi/4$. However, we don't get any information about φ at $\pi/4$ (it may blow, it may not). Therefore considering $S = 1$ is not helpful.

4. Consider the (IVP) $y'(t) = f(t, y(t))$, $y(0) = 1$, where

$$f(t,y) = |y|(1 + \sin t) + \frac{\sqrt{1+|t|}}{1+|y|}.$$

(a) Prove that f is *locally Lipschitz in y, uniformly with respect to t*. Hence the (IVP) has a unique local solution through $(0, 1)$. [5]

(b) Show that every solution is global (exists for all $t \in \mathbb{R}$). [5]

a)

$f(t,y) = |y|(1+\sin t) + \frac{\sqrt{1+|t|}}{1+|y|}$, $t, y \in \mathbb{R}$

Note, in general, f is not differential. Therefore we cannot appeal to the derivative test.

Note:

$|f(t,y_1) - f(t,y_2)| = ||y_1| - |y_2|| \times \left|(1+\sin t) - \frac{\sqrt{1+|t|}}{(1+|y_1|)(1+|y_2|)}\right|$ ...①

$\leq |y_1 - y_2| \left(|1+\sin t| + \frac{\sqrt{1+|t|}}{(1+|y_1|)(1+|y_2|)}\right)$ ...②

$\leq |y_1 - y_2| \left(2 + \sqrt{1+|t|}\right)$ ...③

All the steps are important here!

Note given a compact set $K \subset \mathbb{R}^2$ if $(t,y_1), (t,y_2) \in K$ then clearly $\exists\, T(K)$ s.t.

$|t| \leq T(K)$, $T(K) = \sup\{|t| : (t,y) \in K\}$.

Hence, for any compact set $K \subset \mathbb{R}^2$ we have

$\sup_{(t,y_1) \neq (t,y_2) \in K} \frac{|f(t,y_1) - f(t,y_2)|}{|y_1 - y_2|} \leq 2 + \sqrt{1 + T(K)} = L(K)$

Hence f is locally Lipschitz in y, uniformly with respect to t.

(b) Consider a slab $(-T, T) \times \mathbb{R}$, $T > 0$.

Note, on this slab

$|f(t,y)| = \left||y|(1+\sin t) + \frac{\sqrt{1+|t|}}{1+|y|}\right|$

$\leq 2|y| + \frac{\sqrt{1+|t|}}{1+|y|}$

$\leq 2|y| + \sqrt{1+|t|}$

$\leq 2|y| + \sqrt{1+T}$

In other words, f satisfies the linear growth condition

$|f(t,y)| \leq L(T) + M(T)|y|$ on $(-T, T) \times \mathbb{R}$

where $L(T) = \sqrt{1+T}$ & $M(T) = 2$.

Hence, by a result done in the class we see that each solution is a global solution.