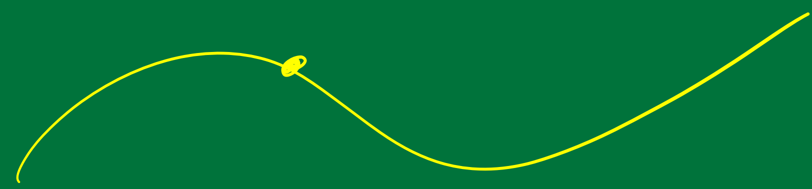
Definition: The vector $\left(\frac{\partial f}{\partial x}(X_0), \frac{\partial f}{\partial y}(X_0), \frac{\partial f}{\partial z}(X_0)\right)$ is called the gradient vector of f at X_0 and it is denoted by $\nabla f(X_0)$.



Theorem: If $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable at X_0 then all the directional derivatives $D_{X_0} f(\vec{u})$ exist for all unit vector $\vec{u}$ in $\mathbb{R}^3$. Moreover,

$(\vec{u} \in \mathbb{R}^3, \; \|\vec{u}\| = 1)$

$$D_{X_0} f(\vec{u}) = f'(X_0) \cdot \vec{u}$$
$$= \nabla f(X_0) \cdot \vec{u}$$

Note: $\quad D_{x_0} f(\vec{u}) = \nabla f(x_0) \cdot \vec{u}$

$= \|\nabla f(x_0)\| \cos\theta$ where $\theta \in [0, \pi]$

is the angle between the gradient vector $\nabla f(x_0)$ and $\vec{u}$.

$\Rightarrow \quad D_{x_0} f(\vec{u})$ is maximum when $\theta = 0$ and
is minimum when $\theta = \pi$

$\Rightarrow \quad f$ increases most rapidly around x_0 in the direction

$$\vec{u} = \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$$ and

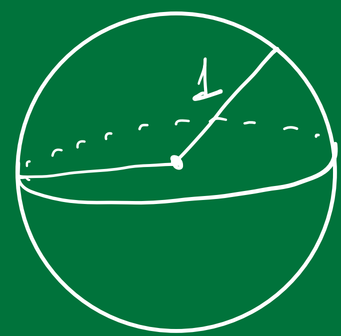
f decreases most rapidly around x_0 in the direction

$$\vec{u} = -\frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \longmapsto f(x, y, z) = x^2 + y^2 + z^2$

Then $f^{-1}(\{1\}) = \{ (x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1 \}$



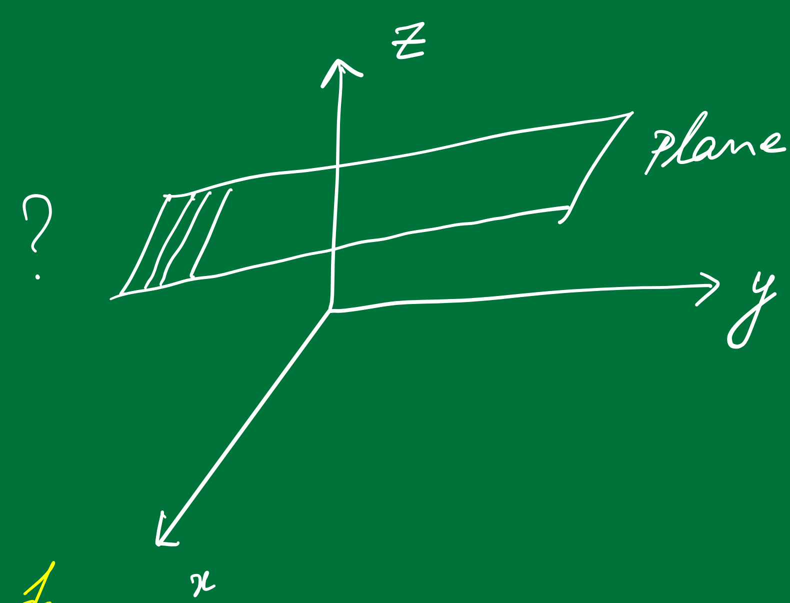
Unit Sphere

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and $c \in \mathbb{R}$.

$$(x,y,z) \longmapsto f(x,y,z)$$

Then $S = \{ (x,y,z) \in \mathbb{R}^3 \,/\, f(x,y,z) = c \} = f^{-1}(\{c\})$

is called a level surface (for the function f) at the height c.

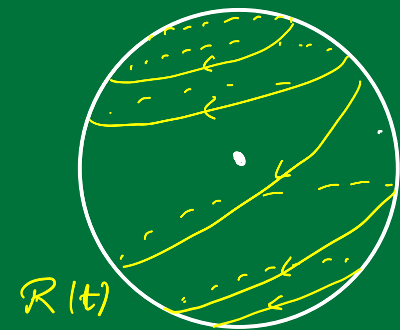


Ex. Unit Sphere is a level surface at height 1 of the function $f(x,y,z) = x^2 + y^2 + z^2$

Suppose S is a level surface and $P = (x_0, y_0, z_0) \in S$ and

$R(t) = \big(x(t),\ y(t),\ z(t)\big)$

is a differentiable curve lying on S and passing through P.

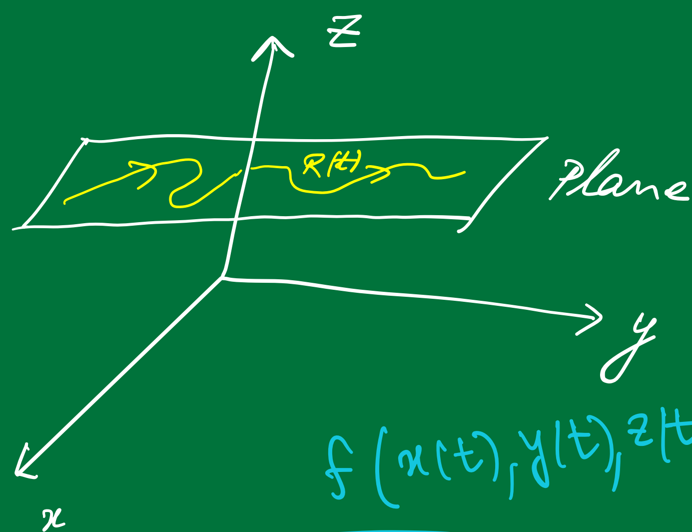


Let

$\vec{v} = R'(t_0)$

be the tangent vector to $R(t)$ at $P = (x_0, y_0, z_0) = R(t_0)$.

?



$$t \longmapsto R(t) \longrightarrow f(R(t)),$$

with the composite labelled $f\big(x(t), y(t), z(t)\big)$.

Then we have $\frac{d}{dt} f(x(t), y(t), z(t)) = 0$

or, $\frac{\partial f}{\partial x}(R(t_0)) \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(R(t_0)) \frac{dy}{dt}(t_0) + \frac{\partial f}{\partial z}(R(t_0)) \frac{dz}{dt}(t_0) = 0$

or, $\nabla f \, (R(t_0)) \cdot \left(x'(t_0), \; y'(t_0), \; z'(t_0) \right) = 0$

or, $\nabla f \cdot \vec{v} = 0$

Tangent vector

A vector $v \in \mathbb{R}^3$ is said to be a tangent vector to the level surface S at $P(x_0, y_0, z_0)$ if v is a tangent vector to a smooth curve $R(t)$ passing through the point P.

The collection of all such tangent vectors at P of the level surface constitute tangent space of The level surface S at the point P.

The tangent space of S at P :

Let $v \perp \nabla f(P)$

$\Rightarrow$ The tangent plane

$\vec{n} = \nabla f(P)$

$P = (x_0, y_0, z_0)$

Eqn: $\vec{n} \cdot P = 0$

$$\frac{\partial f}{\partial x}(p)(x - x_0) + \frac{\partial f}{\partial y}(p)(y - y_0) + \frac{\partial f}{\partial z}(p)(z - z_0) = 0$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable function;

$(x, y) \mapsto f(x, y)$

Consider the surface given by $z = f(x, y)$

i.e, $\{ (x, y, z) \in \mathbb{R}^3 \,/\, z = f(x, y) \}$

1. The partial derivative $\frac{\partial f}{\partial x}(a, b)$ is a slope of the tangent line to curve which is the intersection of the surface with the plane $y = b$.

2. $D_{x_0} f(\vec{u}) = \| \nabla f(x_0) \| \cos\theta$

where θ is the angle between $\nabla f(x_0)$ and $\vec{u}$.

3. <u>Example:</u> $S = \left\{ (x, y, z) \,\middle|\, z^2 = x^2 + y^2 \right\}$

$P = (1, 1, \sqrt{2})$

Tangent plane of S at P:

$$2(x-1) + 2(y-1) - 2\sqrt{2}(z - \sqrt{2}) = 0$$

Example: Consider $f: D \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto f(x,y)$$

$$S = \{ (x, y, f(x,y)) \,/\, (x,y) \in D \subseteq \mathbb{R}^2 \}$$

Then we take S as a level surface:

$$S = \{ (x, y, z) \,/\, F(x,y,z) = 0 \text{ where } \}$$

$$F(x,y,z) = f(x,y) - z$$

Let $X_0 = (x_0, y_0) \in D$

$z_0 = f(X_0) \in \mathbb{R}$ and $P = (x_0, y_0, z_0) \in \mathbb{R}^3$

Then the tangent plane of S is given by

$$\frac{\partial F}{\partial x}(P)(x-x_0) + \frac{\partial F}{\partial y}(P)(y-y_0) + \frac{\partial F}{\partial z}(P)(z-z_0) = 0$$

or, —

or, $z = f(X_0) + f'(X_0)\cdot(X - X_0)$

where $X = (x, y) \in \mathbb{R}^2$

Second order partial derivatives

Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto f(x,y)$$

Suppose

- f_x or $\dfrac{\partial f}{\partial x} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and f_y or $\dfrac{\partial f}{\partial y} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

  $(a,b) \longmapsto \dfrac{\partial f}{\partial x}(a,b)$ or $f_x(a,b)$ $\qquad$ $(a,b) \longmapsto \dfrac{\partial f}{\partial y}(a,b)$ or $f_y(a,b)$

exist.

Then we define

$$f_{xx} := (f_x)_x \quad \text{or} \quad \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2}$$

i.e., $$f_{xx}(a,b) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(a,b)$$

$$f_{yy} := (f_y)_y \quad \text{or} \quad \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2}$$

i.e., $$f_{yy}(a,b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(a,b)$$

$$f_{xy} := (f_x)_y \quad \text{or} \quad \frac{\partial}{\partial y}(f_x) \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x}$$

$$\text{i.e.,} \quad f_{xy}(a,b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a,b)$$

$$f_{yx} := (f_y)_x \quad \text{or} \quad \frac{\partial}{\partial x}(f_y) \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x}$$

$$\text{i.e.,} \quad f_{yx}(a,b) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b)$$

Example:

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

□. Continuous and differentiable function.

f_x, f_y ?

- $$f_x(a,b) = \begin{cases} \dfrac{4a^2b^3 + a^4b - b^5}{(a^2+b^2)^2} & \text{if } (a,b)\neq(0,0) \\ 0 & \text{if } (a,b)=(0,0) \end{cases}$$

$$f_y(a,b) = \begin{cases} \dfrac{a^5 - 4a^3b^2 - ab^4}{(a^2+b^2)^2} & \text{if } (a,b) \neq (0,0) \\ 0 & \text{if } (a,b) = (0,0) \end{cases}$$

Then

$$f_{xy}(0,0) = (f_x)_y(0,0)$$

$$= \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{-h^5/h^4 - 0}{h}$$

$$= -1$$

$$f_{yx}(0,0) = (f_y)_x(0,0)$$

$$= \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^5/h^4 - 0}{h}$$

$$= 1$$

$$\Rightarrow \quad f_{yx}(0,0) \neq f_{xy}(0,0).$$

Infact, $f_{xy}(a,b) = f_{yx}(a,b)$ for $(a,b) \neq (0,0)$

Check!

Mixed derivative theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in a neighborhood of (a,b) (for example, a rectangle containing the point (a,b)) and all are continuous functions at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Mean value theorem (MVT):

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable on the line segment joining A and B. Then there exists a point C on the line segment joining A and B such that

$$\boxed{f(B) - f(A) = (B-A)\cdot f'(C).}$$

$$\|$$

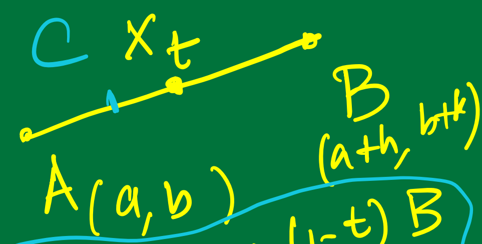
$$t \longmapsto x_t$$

$$\begin{tikzcd}
{[0,1]} \arrow[r] \arrow[d, "\varphi"'] & \mathbb{R}^2 \arrow[dl, "f"] \\
\mathbb{R} &
\end{tikzcd}$$

$\varphi(t) = f(x_t)$

$$\underset{\underset{f(B)}{\|}}{\varphi(1)} - \underset{\underset{f(A)}{\|}}{\varphi(0)} = \varphi'(\theta)$$

$\theta \in (0,1)$



$x_t = tA + (1-t)B$

Corollary: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a differentiable and

$$f_x(a,b) = 0$$

$$f_y(a,b) = 0 \quad \text{for all } (a,b) \in \mathbb{R}^2.$$

Then for $A, B \in \mathbb{R}^2$, by MVT we have

$$\begin{aligned} f(B) - f(A) &= (B-A) \cdot f'(c) \\ &= (B-A) \cdot (f_x(c), f_y(c)) \\ &= (B-A) \cdot (0,0) \\ &= 0 \end{aligned}$$

$$\Rightarrow \quad f(B) = f(A).$$

Since $A, B \in \mathbb{R}^2$ are arbitrary points, the function f is constant function.

Remark. Vector valued functions

Consider

$$g(t) = (\cos t, \sin t) \in \mathbb{R}^2$$

Check that $g : \mathbb{R} \to \mathbb{R}^2$ does not satisfy the statement of MVT.

Observe that

$$\begin{aligned} [0, 2\pi] &\xrightarrow{g} \mathbb{R}^2 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

Can there be $\alpha \in (0, 2\pi)$ such that

$$g(2\pi) - g(0) = (2\pi - 0) \cdot g'(\alpha) \quad ?$$

Extended Mean Value Theorem (EMVT):

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function.
- Let $X_0 = (x_0, y_0)$ and

$$X = X_0 + (h,k)$$
$$= (x_0+h,\ y_0+k).$$

- Suppose $f_x : \mathbb{R}^2 \to \mathbb{R}$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and they have continuous partial derivatives.

Then there exists a point C on the line segment joining X_0 and X such that $f(X) - f(X_0)$

$$= h f_x(X_0) + k f_y(X_0) + \frac{1}{2}\left[h^2 f_{xx}(C) + 2hk\, f_{xy}(C) + k^2 f_{yy}(C)\right]$$

Then there exists a point C on the line segment joining X_0 and X such that $f(X)-f(X_0)$

$$= h f_x(X_0) + k f_y(X_0) + \frac{1}{2}\left[h^2 f_{xx}(C) + 2hk\, f_{xy}(C) + k^2 f_{yy}(C)\right]$$

i.e.,

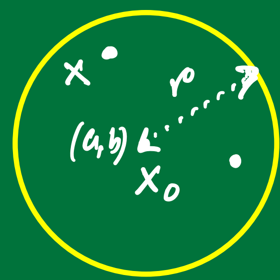
$$f(X) = f(X_0) + f'(X_0)\cdot(X-X_0)$$
$$+ \frac{1}{2}(X-X_0)\cdot\left[f''(C)(X-X_0)\right].$$

Here $f''(C) = \begin{pmatrix} f_{xx}(C) & f_{xy}(C) \\ f_{yx}(C) & f_{yy}(C) \end{pmatrix}_{2\times 2}$ and the vector $f''(C)(X-X_0)$ is given by

$$f''(C)(X-X_0) = \left(h f_{xx}(C) + k f_{xy}(C),\; h f_{yx}(C) + k f_{yy}(C)\right) \in \mathbb{R}^2.$$

Maxima and Minima for $f: \mathbb{R}^2 \to \mathbb{R}$

1. A point $(a,b) \in \mathbb{R}^2$ is called a local maximum for f if there exists $r > 0$ such that $f(x,y) \leq f(a,b)$ for all $(x,y) \in \mathbb{R}^2$ satisfying $\|(x,y) - (a,b)\| < r$.



2. A point $(a,b) \in \mathbb{R}^2$ is called a local minimum for f if there exists $r > 0$ such that $f(x,y) \geq f(a,b)$ for all $(x,y) \in \mathbb{R}^2$ satisfying $\|(x,y) - (a,b)\| < r$.

3. A point $(a,b) \in \mathbb{R}^2$ is called a <u>critical point</u> of f if

$$\nabla f(a,b) = \mathbf{0}$$

i.e., $\left(f_x(a,b),\ f_y(a,b) \right) = (0,0)$.

Proposition. (Necessary condition for local maximum and minimum).

If (a,b) is a local maximum or local minimum for a function f and $f_x(a,b)$, $f_y(a,b)$ exist then

$$\left.\begin{aligned} f_x(a,b) &= 0 \\ f_y(a,b) &= 0. \end{aligned}\right\}$$

Here, the proof follows by considering the functions

$$f_a(y) = f(a,y) \text{ and } f^b(x) = f(x,b).$$ (in one variable)

We have

$$f_a'(b) = f_y(a,b) \text{ and } \left(f^b\right)'(a) = f_x(a,b).$$

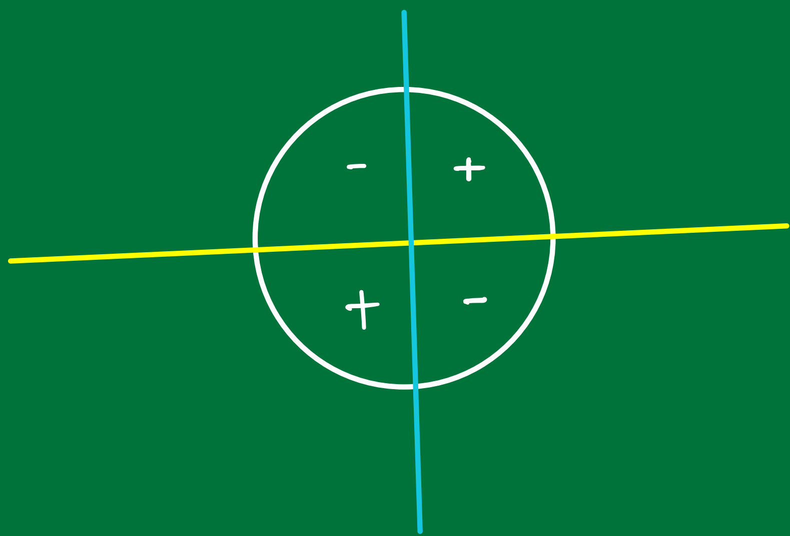
Remark:

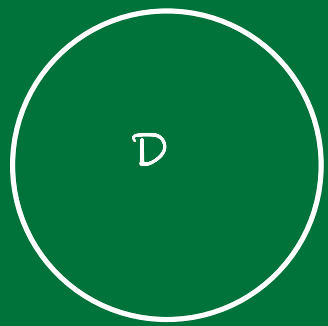
$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto f(x,y) = xy$$

$$f_x(0,0) = 0$$

$$f_y(0,0) = 0$$

But $X_0 = (0,0)$ is NOT a point of maximum or a point of minimum!





$$D = \left\{ (x, y) \in \mathbb{R}^2 \,/\, x^2 + y^2 < 1 \right\}$$

- $X_n = \left(1 - \frac{1}{n} \,,\, 0 \right) \in D$ for $n = 1, 2, 3, \dots$

- $X_n \to (1, 0)$ as $n \to \infty$.

But $(1, 0) \notin D$

Definition. A non-empty subset D of $\mathbb{R}^n$ is said to be a closed subset if a sequence (x_n) in D converges to $v \in \mathbb{R}^n$ then $v \in D$.

Example. (i) $D = \{ (x,y) \in \mathbb{R}^2 / x^2+y^2 < 1 \}$ is not a closed subset of $\mathbb{R}^2$.

(ii) $B_1 = \{ (x,y) \in \mathbb{R}^2 / x^2+y^2 \leq 1 \}$ is a closed subset of $\mathbb{R}^2$.

$N_r(x_0) = \{ x / \| x - x_0 \| < r \}$

Definition. Let $S \subseteq \mathbb{R}^n$ and $x_0 \in S$.
We say that x_0 is an interior point of S if there exists $r > 0$ such that the set $N_r(x_0) = \{ x \in \mathbb{R}^n / \| x - x_0 \| < r \}$ is contained in S.

Example. (1) $X_0 = (1,0)$ is not an interior point of B_1 but $(0,0)$ is an interior point of B_1.

(2) Every point of D is an interior point.

Open Set: A subset S of $\mathbb{R}^n$ is called an open subset if every point of S is an interior point.

Remark: ① Let $D \subseteq \mathbb{R}^n$ be a closed subset.

Then $S = \mathbb{R}^n \setminus D$ — The complement of D in $\mathbb{R}^n$

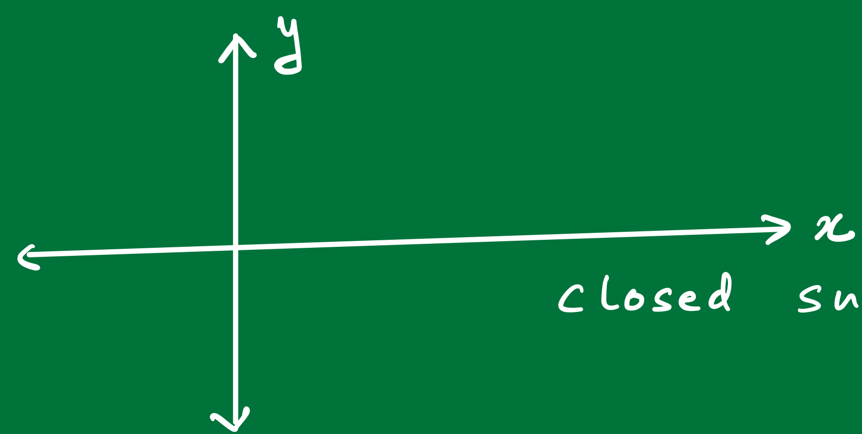
is an open subset.

②

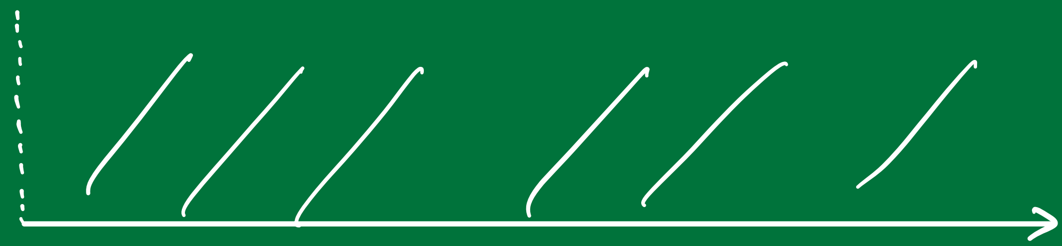
- 

  $H^+ = \{(x,y) \in \mathbb{R}^2 / x > 0, y > 0\}$

  is not a closed set.

- 

  closed subsets of $\mathbb{R}^2$



not a closed subset.

$$\{(x,y)\in\mathbb{R}^2 \,/\, x>0,\ y\geq 0\}$$