- Suppose $D \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $X_0 = (x_0, y_0)$ is an interior point of D.
- Let $f_x(X_0)$ and $f_y(X_0)$ exist.
- If f has local maximum or local minimum at X_0 then

$$f_x(X_0) = 0$$
$$f_y(X_0) = 0$$

Definition. Let $D \subseteq \mathbb{R}^2$, $\quad f: D \to \mathbb{R}$.

A critical point x_0 of f is called a saddle point of f, if for every $r > 0$ we can find x_1 and x_2 in

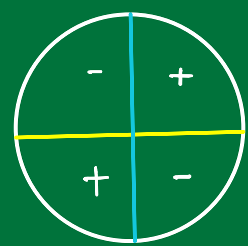
$$N_r(x_0) = \{ x \in \mathbb{R}^2 / \| x - x_0 \| < r \}$$

such that $f(x_0) > f(x_1)$ and

$$f(x_2) > f(x_0).$$

Recall:

- $f(x,y) = xy$



Second derivative test

- Let $D \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$. Suppose f has continuous partial derivatives up to order 2 i.e., $f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{yx}$ are all continuous function on D.

Let $X_0 = (x_0, y_0)$ be an interior point of D and $\nabla f(X_0) = 0$.

i.e, $\left.\begin{array}{l} f_x(X_0) = 0 \\ f_y(X_0) = 0 \end{array}\right\}$

- Suppose $\Delta = \left(f_{xx} f_{yy} - f_{xy}^2\right)(X_0) > 0$.

(i) If $f_{xx}(X_0) > 0$ then f has a local minimum at X_0.

(ii) If $f_{xx}(X_0) < 0$ then f has a local maximum at X_0.

Remark.

(1) If $\Delta < 0$ then x_0 is a saddle point.

(2) If $\Delta = 0$ the no conclusion can be drawn (and further test is required).

Example:

1. $f(x,y) = -(x^4 + y^4)$

$X_0 = (0,0)$ is a critical point of f and

$\Delta = 0$, but f has a local maximum at X_0

2. $f(x,y) = x^2 - y^4$

$X_0 = (0,0)$ is a critical point of f and $\Delta = 0$, and X_0 is a saddle point. (check).

3. $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

$X_0 = (0,0)$ is a critical point of f.

- Second derivative test ?

- $f(x,0) < f(0,0) = 0$ for sufficiently small $x \neq 0$

$$\underset{x^4 - 2x^2}{\overset{\shortparallel}{f(x,0)}}$$

- $f(x,x) = 2x^4 > f(0,0) = 0$ for $x \neq 0$

$\Rightarrow$ If we consider a circular disc of very small radius center at $X_0 = (0,0)$ then there are points X_1 inside the disc lies on x-axis and X_2 lies on the line $y = x$ such that

$$f(X_1) < f(X_0) \text{ and}$$

$$f(X_2) > f(X_0).$$

Therefore X_0 is a saddle point of the function f.

Example:

$$f : [0,1] \to \mathbb{R}$$
$$x \mapsto x$$

minimum ?
maximum ?

$f'(c) \neq 0$ for $c \in [0,1]$

Theorem. Let $D \subseteq \mathbb{R}^2$ be a closed and bounded subset of $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ be a continuous function.

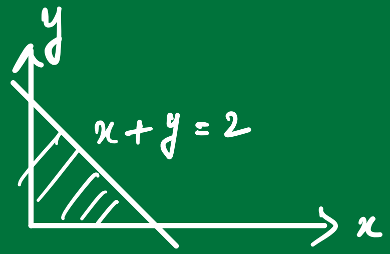
Then f has a maximum and minimum in D.

[OR, f has an absolute maximum and absolute minimum in D, i.e., there exists $x_0, y_0 \in D$ such that

$$f(x) \leq f(x_0) \text{ and}$$

$$f(x) \geq f(y_0) \text{ for all } x \in D$$].

Example: Let $D = \{ (x,y) \in \mathbb{R}^2 \,/\, x \geqslant 0,\ y \geqslant 0 \text{ and } x+y \leq 2 \}$



$f : D \to \mathbb{R}$

$f(x,y) = x^3 - xy + y^2 - x$

Find maximum and minimum of f.

Step 1. We find the critical point X_0 inside D

(means interior of D or set of interior points of D)

by solving $\nabla f\,(X_0) = 0$:

Here, $f_x = 3x^2 - y - 1$

$f_y = -x + 2y$

$\Rightarrow \quad 12y^2 - y - 1 = 0$

$\Rightarrow \quad y = 1/3\,, \quad x = 2/3$

i.e., $\quad X_0 = (2/3, 1/3) \in D$

is a critical point inside D.

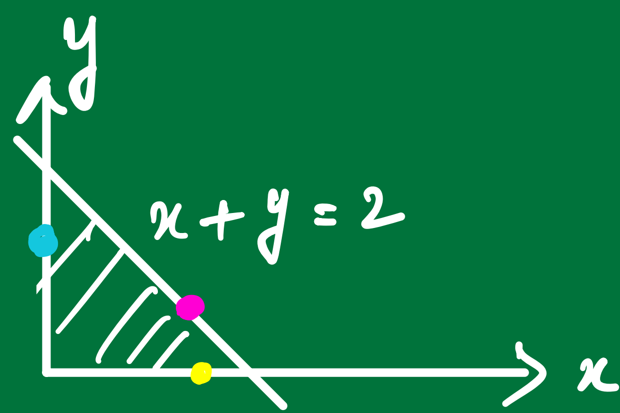
Step 2. (a) The portion of D on x-axis

i.e., $\{(x,0) \,/\, 0 \le x \le 2\}$

- $f(x,0) = x^3 - x$
- $f'(x) = 3x^2 - 1$

So, $f'(x) = 0$

$\Rightarrow \quad x = \pm \frac{1}{\sqrt{3}}$



- $(x, 2-x)$
- $(0, y)$
- $(x, 0)$

Thus the possible points of extremum are

$(0,0)$, $\left(\frac{1}{\sqrt{3}}, 0\right)$ and $(2,0)$.

2 (b) The portion of D on y-axis :

The possible points of extremum are $(0,0)$ and $(0,2)$.

2(c) The portion of D on the line $x+y=2$:

Here $f(x,y) = f(x, 2-x)$

$$= x^3 - x(2-x) + (2-x)^2 - x$$

So, $f'(x,2-x) = 3x^2 - (2-x) + x - 2(2-x) - 1$

$$= 3x^2 - 2 + 2x - 4 + 2x - 1$$

$$= 3x^2 + 4x - 7$$

$$= (3x+7)(x-1)$$

$\therefore$ $f' = 0 \Rightarrow x = 1$ or $-7/3$

Thus the possible points of extremum are

$(1,1)$, $(0,2)$ and $(2,0)$.

x_0	$(0,0)$	$(0,2)$	$(2,0)$	$(1,1)$	$(1/\sqrt{3},0)$	$(2/3,1/3)$
$f(x_0)$	0	4	6	0	$-2/3\sqrt{3}$	$-13/27$

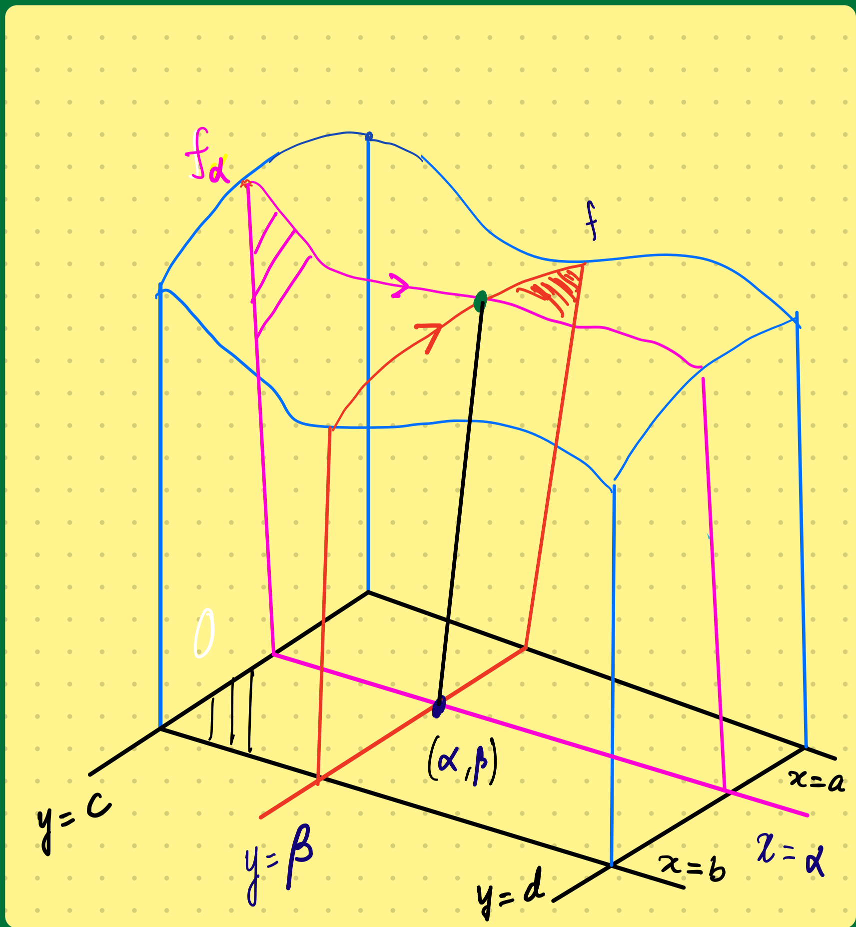
Therefore, $x_0 = (2,0)$ is a point of maximum and $y_0 = \left(\frac{2}{3}, \frac{1}{3}\right)$ is a point of minimum of the given function $f: D \to \mathbb{R}$.

Remark:

Suppose $S \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ and $f : S \to \mathbb{R}$

e.g. $\{ S = (x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1 \}$.

If x_0 is not an interior point of S, we cannot apply the tests namely The necessary condition $\nabla f(x_0) = 0$ and the second derivative test as well.



$f : \mathbb{R}^2 \to \mathbb{R}$ $\qquad Q = [a,b] \times [c,d] \subseteq \mathbb{R}^2$

$(x, y) \longmapsto f(x,y) = z$

- $(\alpha, \beta) \in \mathbb{R}^2$

Define

$$f_\alpha : [c, d] \longrightarrow \mathbb{R}$$
$$y \longmapsto f_\alpha(y) := f(\alpha, y)$$

and

$$f^\beta : [a, b] \longrightarrow \mathbb{R}$$
$$x \longmapsto f^\beta(x) := f(x, \beta)$$

Consider

$$A(\beta) = \int_a^b f(x, \beta)\, dx \quad \text{and} \quad A(\alpha) = \int_c^d f(\alpha, y)\, dy \quad \text{for } \alpha \in [a, b].$$

for $\beta \in [c, d]$

Then

V = Volume of the solid

$$= \int_c^d A(\beta)\, dy \quad \text{or} \quad \int_a^b A(\alpha)\, dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy \quad \text{or} \quad \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

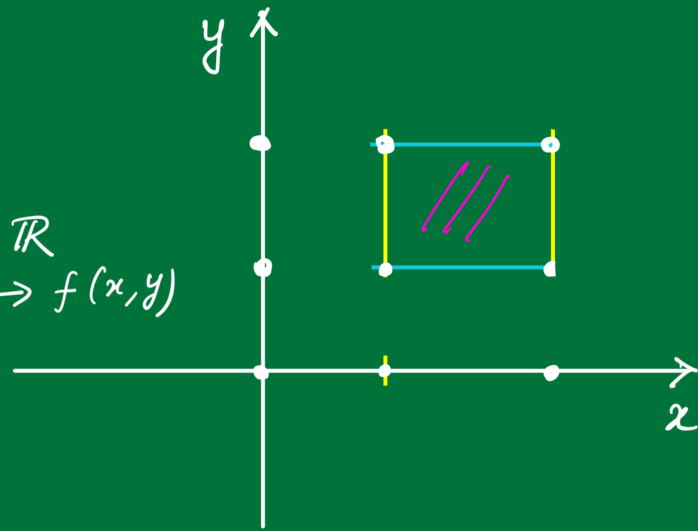
We will simply write it as

$$V = \iint_Q f(x,y)\, dx\, dy \quad \text{where} \quad Q = [a,b] \times [c,d].$$

double integral of f over Q

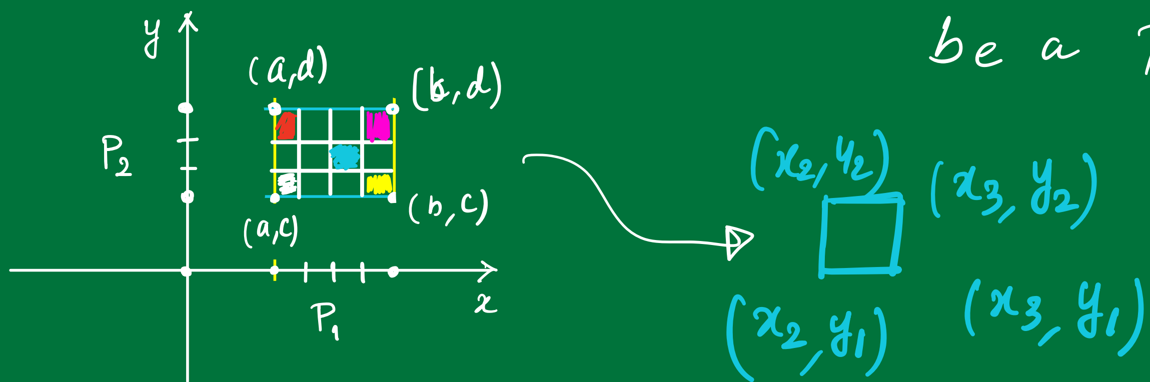
Double integral

Let $Q = [a,b] \times [c,d]$ and $f : Q \to \mathbb{R}$
$$(x,y) \mapsto f(x,y)$$
be a bounded function.



$$P_1 = \{ a = x_0 < x_1 < x_2 < \dots < x_{i-1} < x_i < \dots < x_n = b \}$$
be a partition of $[a,b]$

$$P_2 = \{ c = y_0 < y_1 < y_2 < \dots < y_{j-1} < y_j < \dots < y_m = d \}$$
be a partition of $[c,d]$.



$$\rightsquigarrow Q_{32} = [x_2, x_3] \times [y_1, y_2]$$

Then $P := P_1 \times P_2 = \{ (x_i, y_j) \,/\, 1 \le i \le n$ and $1 \le j \le m \}$ is a partition of Q, it decomposes Q into mn sub-rectangles of the form $Q_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $1 \le i \le n$ and $1 \le j \le m$.

Define

$$m_{ij} = \operatorname{Inf} \{ f(x,y) \,/\, (x,y) \in Q_{ij} \} \text{ and}$$

$$M_{ij} = \operatorname{Sup} \{ f(x,y) \,/\, (x,y) \in Q_{ij} \}, \text{ and}$$

$$L(P, f) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} m_{ij} \, \Delta x_i \, \Delta y_j, \quad U(P, f) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} M_{ij} \, \Delta x_i \, \Delta y_j$$

Lower - Riemann integral

$$\underline{\iint} f \, dx\, dy = \operatorname{Sup}\{ L(P,f) \,/\, P \text{ is a partition of } Q \}$$

Upper - Riemann integral

$$\overline{\iint} f \, dx\, dy = \operatorname{Inf}\{ U(P,f) \,/\, P \text{ is a partition of } Q \}$$

The function $f : Q \to \mathbb{R}$ is said to be Riemann integrable (or simply integrable) on Q if the lower Riemann integral is same as the upper Riemann integral.

If $f: Q \to \mathbb{R}$ is integrable, then the integral is denoted by

$$\iint_Q f(x,y)\,dx\,dy \quad \text{or} \quad \iint_Q f(x,y)\,dA .$$

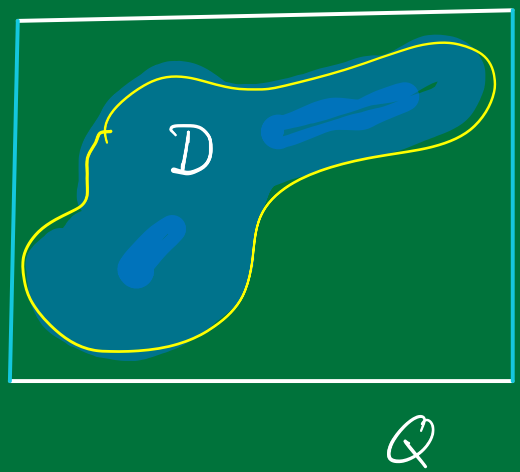
Theorem: If a function $f(x,y)$ is continuous on a rectangle $Q = [a,b] \times [c,d]$ then f is integrable on Q.

Moreover,

$$\iint_Q f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx .$$

Double integral over a bounded region $D \subseteq \mathbb{R}^2$.



Let $f : D \to \mathbb{R}$ be a bounded function on a bounded region $D \subseteq \mathbb{R}^2$.

First, we find a rectangle Q containing the region D.

Define $\tilde{f} : Q \to \mathbb{R}$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{for } (x,y) \in D \\ 0 & \text{for } (x,y) \in Q \setminus D \end{cases}$$

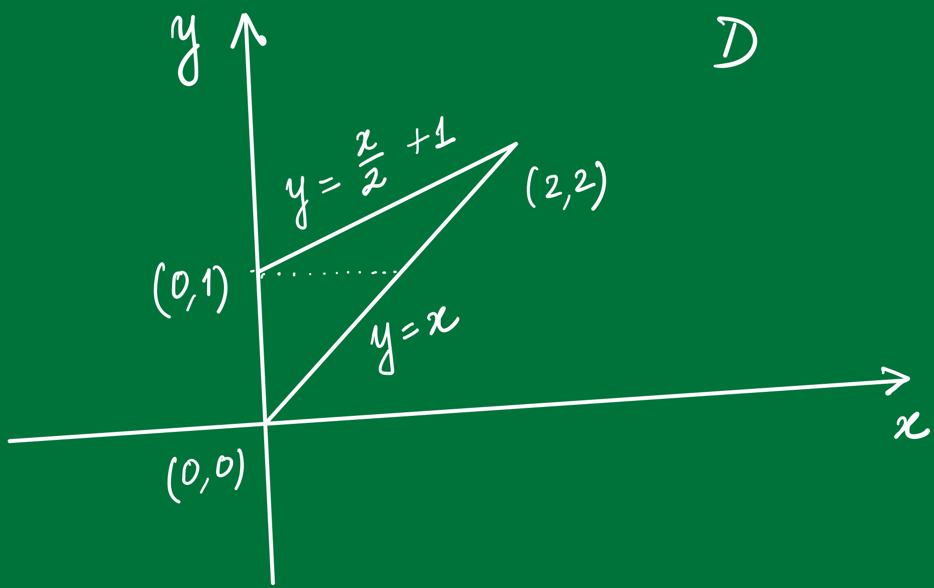
If $\tilde{f}(x,y)$ is integrable over Q then we say that $f : D \to \mathbb{R}$ is integrable over D and we define $\iint_D f(x,y)\,dx\,dy$ as

$$\iint_D f(x,y)\,dx\,dy = \iint_Q \tilde{f}(x,y)\,dx\,dy .$$

VII Compution of $\iint_D f(x,y)\,dx\,dy$

Example: Let D be the region obtained by joining $(0,0)$, $(0,1)$ and $(2,2)$ by line segments.

Evaluate $\iint\limits_D (x+y)^2 \, dx\,dy$.



$$D = \left\{ (x,y) \;/\; 0 \le x \le 2 \text{ and } \underset{\substack{\shortparallel \\ x}}{f_1(x)} \le y \le \underset{\substack{\shortparallel \\ x/2+1}}{f_2(x)} \right\}$$