MTH302: *Set Theory and Mathematical Logic*
Midterm Exam Odd Semester 2024/25

Instructions

- This is a **closed notes exam**.
- The exam contains **4 + 1** questions and is worth **30%** of your grade.
- You have **2 hours**.

Question 1 [10 Points]

For each of the following statements, determine whether it is **true or false**. No justification required. [**5 × 2 Points**]

(a) There is an uncountable chain in $(\mathcal{P}(\mathbb{Q}), \subseteq)$.

(b) There is a subset of plane that intersects every line at exactly 7 points.

(c) If ϕ and ψ are satisfiable propositional formulas, then $(\phi \wedge \psi)$ is satisfiable.

(d) There exist infinite ordinals $\alpha \neq \beta$ such that $\alpha + \alpha + \alpha = \beta + \beta + \beta$.

(e) ZFC has finitely many axioms.

Question 2 [7 Points])

(a) [**1 Points**] State the Schröder-Bernstein theorem.

(b) [**2 Points**] Suppose X is an uncountable set and Y is a countable subset of X. Show that $|X \setminus Y| = |X|$.

(c) [**4 Points**] Let $\mathcal{F}$ be the set of all bijections from $\mathbb{R}$ to $\mathbb{R}$. Show that $|\mathcal{F}| > |\mathbb{R}|$.

Question 3 [5 Points]

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$,

$$f(x + y) = f(x) + f(y) - f(x)f(y).$$

(a) [**1 Point**] Define $h = 1 - f$. Show that $h(x + y) = h(x)h(y)$.

(b) [**4 Points**] Suppose f is continuous. Show that **either** f is identically 1 **or** $f(x) = 1 - a^x$ for some constant $a > 0$.

Question 4 [8 Points]

(a) [**2 Points**] State Zorn's lemma.

(b) Consider the partial ordering $(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is the family of all $X \subseteq \mathbb{R}^2$ that satisfy: For every circle $C \subseteq \mathbb{R}^2$,

$$|X \cap C| \leq 3.$$

(i) [**2 Points**] Show that $(\mathcal{F}, \subseteq)$ has a maximal element.

(ii) [**4 Points**] Show that there exists $M \subseteq \mathbb{R}^2$ such that for every circle $C \subseteq \mathbb{R}^2$,

$$|M \cap C| = 3.$$

Bonus problem [5 Points]

Prove or disprove the following statement. There is an uncountable family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ such that for every $A, B \in \mathcal{F}$,

$$A \neq B \implies A \cap B \text{ is finite.}$$