Lecture 8 : Dimension of vector space

Here we will deal with vector spaces having finitely many element in a basis. Natural examples are $\mathbb{R}^n$ or $\mathbb{C}^n$. The number 'n' here will correspond to 'dimension' of $\mathbb{R}^n$ or $\mathbb{C}^n$. The dimension of a vector space is the number of elements in a basis. But we need to check that this definition is well defined that is any two basis of a vector space contains same number of elements.

Theorem: Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of a vector space V. Then for any finite set $S = \{v_1, v_2, \ldots, v_m\}$ of vectors with $m > n$, the set S is L.D.

Proof: As $\{e_1, e_2, \ldots, e_n\}$ is a basis

$$v_j = \sum_{i=1}^{n} a_{ij} e_i \quad \text{for some scalars } a_{ij}\text{'s}$$
$$\& \ \forall j \in \{1, 2, \ldots, m\}$$

$$\sum_{j=1}^{m} x_j v_j = 0 \Rightarrow \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} a_{ij} e_j \right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_j a_{ij} \right) e_j = 0$$

As $\{e_1, e_2, \dots, e_n\}$ is L.I.,

$$\sum_{j=1}^{m} a_{ij} x_j = 0 \quad \forall\, i = 1, 2, \dots, n \quad \star$$

Let $A = (a_{ij})_{n \times m}$ & $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

$\star$ reduces to following homogenous system of linear equations:

$$AX = 0 \quad \cdots \quad (1)$$

FACT: As number of unknown variables is more than the number of equations, using 'Rank-Nullity' theorem, we will prove later that it has a non-zero solution.

Using this fact, (1) has a non-zero solution say $(\alpha_1, \alpha_2, \dots \alpha_m)$ & α_j's satisfy

$$\sum_{j=1}^{m} \alpha_j v_j = 0 \Rightarrow \{v_1, \dots v_m\} \text{ is L.D.} \quad \boxtimes$$

Corollary: Let V be a vector space with basis $\{e_1, \dots, e_n\}$. If $\{f_1, \dots f_m\}$ is also a basis of V then $m = n$.

Proof: If $m < n$ then as $\{f_1, \dots f_m\}$ is a basis of V, from the previous theorem it implies that $\{e_1, \dots, e_n\}$ is L.D., which is a contradiction. So $m \geq n$. Similarly, $n \geq m$. So $m = n$.

Dimension of a vector space

☆ A vector space is called finite dimensional if it contains a basis consisting of finitely many elements. Otherwise it is called infinite dimensional vector space.

☆ Dimension of a finite dimensional vector space is said to be 'n' if it contains a basis consisting of n-vectors.

(From the corollary, every basis of the vector space contains 'n' many elements)

Example (i) Dimension of $\mathbb{R}^n$ is n.

(ii) Dimension of $P(x)$ is infinite.

Notation: Dimension of a vector space V is denoted by $\dim(V)$.

Theorem: Let V be a vector space of $\dim(V) = n$. Then any set $\{v_1, v_2, \dots, v_n\}$ of n-L.I. vectors is a basis of V.

Proof: We need only to prove that

$$L(\{v_1, \dots, v_n\}) = V.$$

Let $v \in V$ s.t $v \neq v_i$, $i = 1, \dots, n$.

As $\dim(V) = n$, $\{v_1, v_2, \dots, v_n, v\}$ is a L.D. set (by previous theorem).

Thus $\exists$ α_i's & α (not all zero) such that

$$\alpha_1 v_1 + \dots + \alpha_n v_n + \alpha v = 0$$

$\alpha = 0 \Rightarrow \sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow \alpha_1 = \dots = \alpha_n = 0$ (Contradiction)
(as $\{v_1, \dots, v_n\}$ is L.I.

Therefore, $\alpha \neq 0 \Rightarrow v = \left(\frac{-\alpha_1}{\alpha}\right) v_1 + \dots + \left(\frac{-\alpha_n}{\alpha}\right) v_n$

$\in L(\{v_1, \dots, v_n\})$ $\boxtimes$

Example (1) Let $S = \{(1,1,1,1), (-1,1,-1,1), (0,1,0,1), (1,1,1,0)\}$

We want to find a basis of Linear span $L(S)$. We will use row reduction method to find a basis.

Consider the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_4 - R_1]{R_2 + R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow[(-1)R_4]{R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_{34}]{\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Note that each row vector of B is linear combination of elements of S. Hence

$L(\{(1,1,1,1), (0,1,0,1), (0,0,0,1), (0,0,0,0)\}) = L(S)$.

Consider the $U = \{(1,1,1,1), (0,1,0,1), (0,0,0,1)\}$

$c_1(1,1,1,1) + c_2(0,1,0,1) + c_3(0,0,0,1) = \bar{0}$

$\Rightarrow (c_1, c_1+c_2, c_1, c_1+c_2+c_3) = (0,0,0,0)$

$\Rightarrow c_1 = c_2 = c_3 = 0$

Hence U is L.I. & $L(U) = L(S)$

$\Rightarrow U$ is a basis.

(2) In the above example, let us take the set $U = \{(1,1,1,1), (0,1,0,1), (0,0,0,1)\}$

We have seen $\dim(L(U)) = 3$.

$L(U)$ is strictly contained in $\mathbb{R}^4$.

Let $x = (x_1, x_2, x_3, x_4) \notin L(U)$

Consider $Q = U \cup \{x\}$, then Q is L.I.

$\dim(L(Q)) \geq 4$

$L(Q) \subseteq \mathbb{R}^4 \Rightarrow \dim(L(Q)) \leq 4$

So, $\dim(L(Q)) = 4$ & $L(Q) \subseteq \mathbb{R}^4$

$\Rightarrow L(Q) = \mathbb{R}^4$.

Using this idea, we can easily extend a L.I. set of vectors to basis of a vector space.

Theorem: Let V be a finite dimensional vector space with $\dim(V) = n$. Let $\{u_1, \dots, u_m\}$ be a L.I. set in V, then $\{u_1, \dots, u_m\}$ can be extended to basis of V.

Proof If $L(\{u_1, \dots, u_m\}) = V$ (i.e $n=m$) then $\{u_1, \dots, u_m\}$ is a basis of V. Otherwise $\exists\, x_1 \in V$ s.t. $x_1 \notin L(\{u_1, \dots, u_m\})$

$\Rightarrow$ $\{u_1, \dots, u_m, x_1\}$ is L.I.

Let $u_{m+1} = x_1$, if $L(\{u_1, \dots, u_{m+1}\}) = V$ then $\{u_1, \dots, u_{m+1}\}$ is a basis of V. Otherwise we continue this process & it will stop at $n-m$ steps. $\square$

Algorithm to find a basis of a linear span in finite dimensional vector space:

Let V be a vector space with $\dim(V) = n$ & $\{e_1, \dots, e_n\}$ be a basis of V.

Let $S = \{v_1, \dots, v_m\}$ be a set of distinct vectors in V.

We want to find a basis of $L(S)$:

For all $i \in \{1, .., m\}$, $v_i = \sum_{j=1}^{n} a_{ij} e_j$ for some scalars a_{ij}'s.

a_{ij}'s are called coordinates of v_i with respect to basis $e_1, .., e_n$.

v_i can be thought of as vector $(a_{i1}, .., a_{in})$.

Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The row vectors of A correspond to vectors $v_1, .., v_m$.

$L(S) = L(\text{row vectors of } A)$

Observe that $L(\{v_i, v_j\}) = L(\{v_i + c v_j, v_j\})$ for $c \neq 0$

& $L(\{v_i, v_j\}) = L(\{c v_i, v_j\})$ for $c \neq 0$.

Therefore, for an elementary matrix E with $B = EA$

$L(\text{row vectors of } B) = L(\text{row vectors of } A)$

Now apply row operations on A, so that it gets in RREF.

There exists E product of elementary matrices such that

$$B = EA \quad \& \quad B \text{ is in RREF}$$

★ $L(\text{row vectors of } B) = L(\text{row vectors of } A)$
$= L(S)$

★ Check that non-zero row vectors of B is L.I to each other.

Hence, non-zero row vectors of B forms a basis of $L(S)$. ∎