ASSIGNMENT 4

(1) Let $\{w_1, w_2, ..., w_n\}$ be a basis of a finite dimensional vector space V. Let v be a non zero vector in V. Show that there exists w_i such that if we replace w_i by v in the basis it still remains a basis of V.

(2) Find the dimension of the following vector spaces :
(i) $\{A : A \text{ is } n \times n \text{ real upper triangular matrices}\}$,
(ii) $\{A : A \text{ is } n \times n \text{ real symmetric matrices}\}$,
(iii) $\{A : A \text{ is } n \times n \text{ real skew symmetric matrices}\}$,
(iv) $\{A : A \text{ is } n \times n \text{ real matrices with } Tr(A) = 0\}$

(3) Let $\mathcal{P}(X, \mathbb{R})$ be vector space of all single variable polynomials with real coeffecients and $\mathcal{P}_n(X, \mathbb{R})$ be the subspace of all poynomials with degree less or equal to n. Find a basis of $\mathcal{P}_n(X, \mathbb{R})$. Prove that $S = \{X + 1, X^2 - X + 1, X^2 + X - 1\}$ is a basis of $\mathcal{P}_2(X, \mathbb{R})$. Hence, determine the coordinates of following elements: $2X - 1, 1 + X^2, X^2 + 5X - 1$.

(4) Let W be a subspace of a finite dimensional vector space V
(i) Show that there is a subspace U of V such that $V = W + U$ and $W \cap U = \{0\}$,
(ii) Show that there is no subspace U of V such that $W \cap U = \{0\}$ and $dim(W) + dim(U) > dim(V)$.

(5) Let $W_1 = L(\{(1, 0, -1), (1, 0, 1)\})$ and $W_2 = L(\{(0, 1, 2), (0, 1, -1)\})$ be two subspaces of $\mathbb{R}^3$. Prove that $W_1 + W_2 = \mathbb{R}^3$. Given an example $v \in \mathbb{R}^3$ such that v can be written in two different ways of the form $v = w_1 + w_2$ where $w_1 \in W_1, w_2 \in W_2$.

(6) *** Decide which of the followings are linear transformation:
(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + 2y, z, |x|)$,
(ii) Let $M_n(\mathbb{R})$ be set of all $n \times n$ real matrices. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by
(a) $T(A) = A^T$,
(b) $T(A) = I + A$, where I is identity matrix of order n,
(c) $T(A) = BAB^{-1}$, where $B \in M_n(\mathbb{R})$ is an invertible matrix.

(7) *** Let $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = \overline{z}$, is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

(8) *** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1, 0, 0) = (1, 0, 0), T(1, 1, 0) = (1, 1, 1), T(1, 1, 1) = (1, 1, 0)$. Find $T(x, y, z)$, $Ker(T)$, $R(T)$ (Range of T). Prove that $T^3 = T$.

(9) *** Find all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$.

(10) *** Let V and W be two finite dimensional vector spaces over k, where $k = \mathbb{R}$ or $\mathbb{C}$. Prove that set $L(V, W)$ of all linear transformations from V to W is a vector space over k of dimension $dim(V).dim(W)$.