I Unit normal vector valued function

II Curvature

III Function of several variables

Unit tangent vector : The unit tangent vector of the curve $R(t)$ is

$$T(t) = \frac{R'(t)}{\|R'(t)\|} \quad \text{whenever } R'(t) \neq 0.$$

Considering the expression

$$\frac{ds}{dt} = \|R'(t)\|$$

We have

$$T(t) = \frac{\frac{dR}{dt}}{\frac{ds}{dt}} \Rightarrow T(t) = \frac{dR}{dt} \times \left(\frac{ds}{dt}\right)^{-1}$$

$$= \frac{dR}{dt} \times \left(\frac{dt}{ds}\right)$$

$$= \frac{dR}{ds} \qquad (\text{By chain rule}).$$

Consequently, $\left\|\frac{dR}{ds}\right\| = \|T(t)\| = 1.$

This shows that $R(s)$ is a unit speed curve.

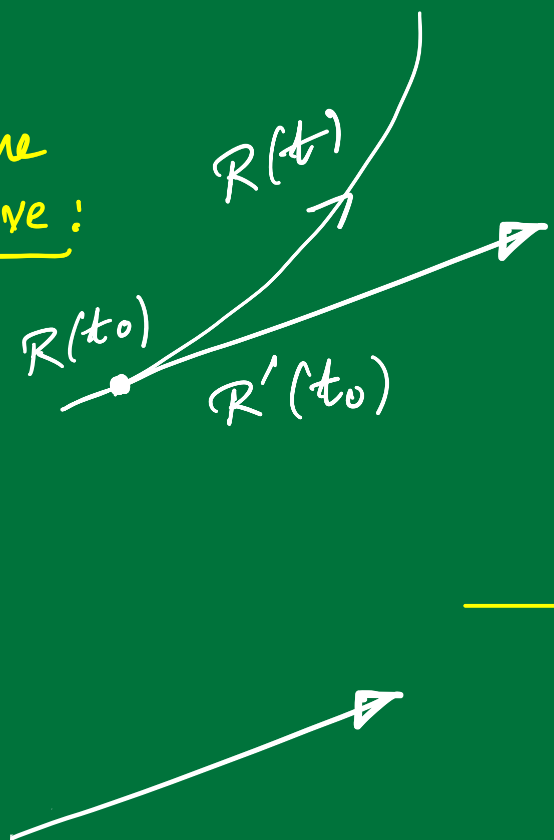
Consider differentiable vector valued function $R : I \to \mathbb{R}^3$

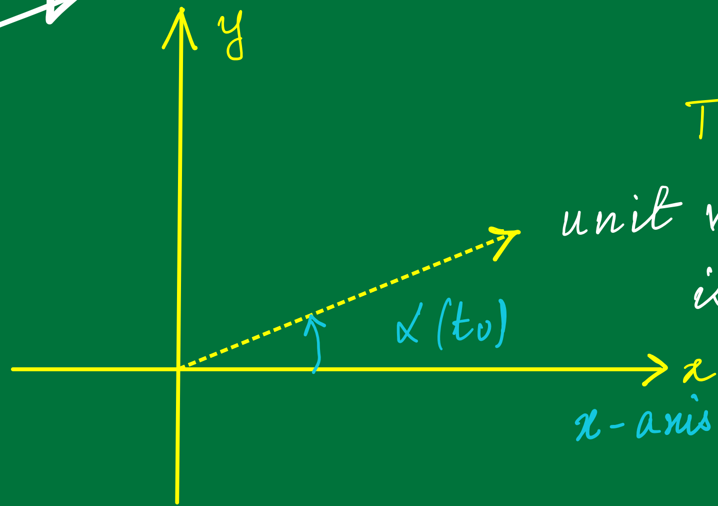
$$t \longmapsto R(t)$$

The unit tangent vector of $R(t)$

$$T(t) = \frac{R'(t)}{\|R'(t)\|}$$

Plane Curve:





$T(t_0)$

unit vector along $R'(t_0)$ is $(\cos\alpha(t_0), \sin\alpha(t_0))$

Suppose T is a differentiable vector valued function, we get

$$T' : I \to \mathbb{R}^3$$
$$t \longmapsto T'(t) \quad \overset{?}{\sim} \quad T(t)$$

How they are related ?

Perpendicular !

Note that

$$T(t) = \cos\alpha(t)\,\vec{i} + \sin\alpha(t)\,\vec{j}$$

$$\Rightarrow \quad T'(t) = -\alpha'(t)\sin\alpha(t)\,\vec{i} + \alpha'(t)\cos\alpha(t)\,\vec{j}$$

$$= \alpha'(t)\cos\left(\frac{\pi}{2} + \alpha(t)\right)\vec{i} + \alpha'(t)\sin\left(\frac{\pi}{2} + \alpha(t)\right)\vec{j}$$

$$= \alpha'(t)\,u(t) \quad \text{where the unit vector}$$

$$u(t) = \cos\left(\frac{\pi}{2} + \alpha(t)\right)\vec{i} + \cancel{\alpha'(t)}\sin\left(\frac{\pi}{2} + \alpha(t)\right)\vec{j}.$$

$\Rightarrow$ $T'(t)$ is perpendicular to $T(t)$ for all $t \in I$.

ie, $T'(t) \cdot T(t) = 0$.

In fact it is a special case of a more general case of vector valued function.

Let $F : I \to \mathbb{R}^3$ defined by $F(t) = \left(f_1(t), f_2(t), f_3(t)\right)$

and $G : I \to \mathbb{R}^3$ defined by $G(t) = \left(g_1(t), g_2(t), g_3(t)\right)$

be two vector valued functions.

Then $F.G : I \to \mathbb{R}$ is a real valued function

$(F.G)(t) = F(t) . G(t)$ dot product.

$$= f_1(t)\, g_1(t) + f_2(t)\, g_2(t) + f_3(t)\, g_3(t)$$

$$= (f_1 g_1)(t) + (f_2 g_2)(t) + (f_3 g_3)(t)$$

$$= \left(\sum_{i=1}^{3} f_i g_i\right)(t)$$

$\Rightarrow$ $F.G : I \to \mathbb{R}$

$t \mapsto F(t).G(t)$ is a real valued function.

Suppose F and G are differentiable vector valued functions then $F.G$ is a differentiable real valued function and the derivative function is given by $(F.G)'(t) = (F'.G + F.G')(t)$.

Note that

$$F'(t) = (f_1'(t), f_2'(t), f_3'(t)) \text{ and}$$

$$G'(t) = (g_1'(t), g_2'(t), g_3'(t)).$$

So, $(F'.G)(t) = F'(t).G(t) = \sum_{i=1}^{3} f_i'(t) g_i(t)$ and

$$(F.G')(t) = F(t).G'(t) = \sum_{i=1}^{3} f_i(t) g_i'(t)$$

Now $\quad (F \cdot G)'(t) = \sum_{i=1}^{3} (f_i g_i)'(t)$

$$= \sum_{i=1}^{3} f_i'(t) g_i(t) + \sum_{i=1}^{3} f_i(t) g_i'(t)$$

$$= (F' \cdot G)(t) + (F \cdot G')(t)$$

for all t

$$\Rightarrow \quad (F \cdot G)' = F' \cdot G + F \cdot G'$$

Theorem: Let I be an interval and F be a vector valued function on I such that $\|F(t)\| = \alpha$ - constant for all $t \in I$. Then $F \cdot F' = 0$ on I, that is $F'(t)$ is perpendicular to $F(t)$ for each $t \in I$.

$t \mapsto \|F(t)\|$
$\in \mathbb{R}$

<u>Proof</u>: Define $g : I \longrightarrow \mathbb{R}$ by

$$t \longmapsto \|F(t)\|^2 = \alpha^2$$

Then g is a constant function and so $g'(t) = 0$ for all $t \in I$.

Now,
$$g'(t) = (F \cdot F)'(t)$$
$$= (F' \cdot F + F \cdot F')(t)$$

$$= 2(F \cdot F')(t)$$

Therefore,

$$F \cdot F' = 0$$

For the unit tangent vector valued function T, we have $\| T(t) \| = 1$ — constant for all t.

By the previous result $T \cdot T' = 0$

or T' is perpendicular to T.

In view of we define the principle normal to the curve

$$N(t) = \frac{T'(t)}{\|T'(t)\|} \text{ whenever } \|T'(t)\| \neq 0.$$

Note that

$$T(t) = \cos\alpha(t)\,\vec{i} + \sin\alpha(t)\,\vec{j}$$

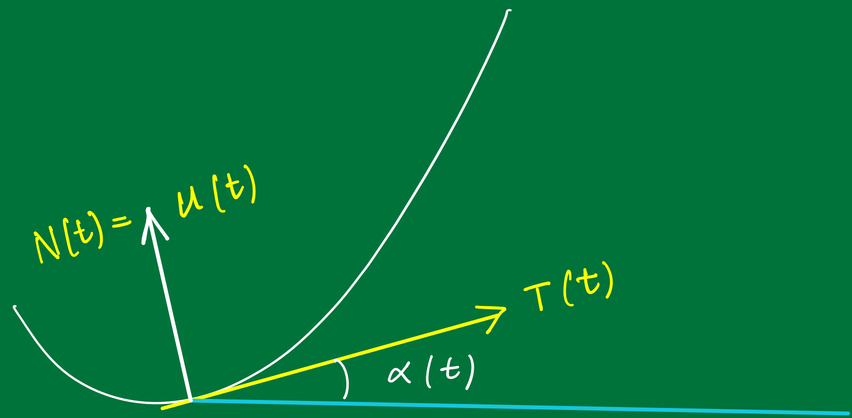
$$\Rightarrow \quad T'(t) = -\alpha'(t)\sin\alpha(t)\,\vec{i} + \alpha'(t)\cos\alpha(t)\,\vec{j}$$

$$= \alpha'(t)\cos\left(\frac{\pi}{2} + \alpha(t)\right)\vec{i} + \alpha'(t)\sin\left(\frac{\pi}{2} + \alpha(t)\right)\vec{j}$$

$= \alpha'(t)\,u(t)$ where the unit vector

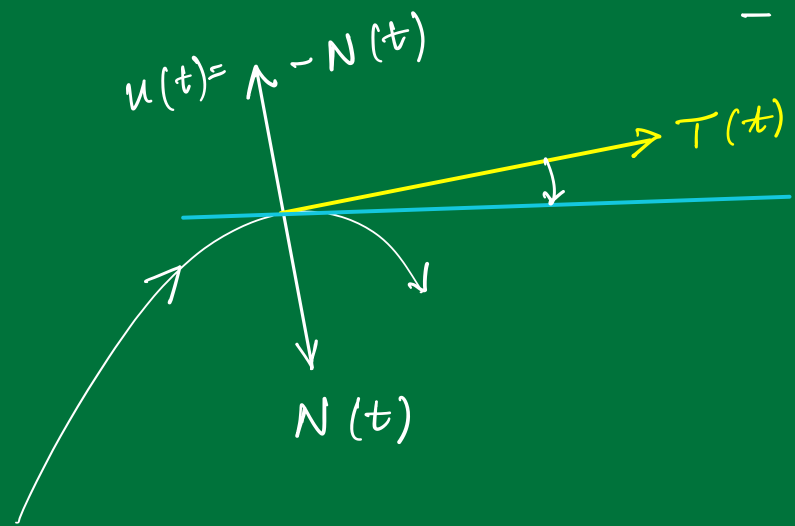
$$u(t) = \cos\left(\frac{\pi}{2} + \alpha(t)\right)\vec{i} + \alpha'(t)\sin\left(\frac{\pi}{2} + \alpha(t)\right)\vec{j}.$$

Case I: $\alpha'(t) > 0$, the angle $\alpha(t)$ is increasing and in this case $N(t) = \dfrac{\alpha'(t)\,u(t)}{|\alpha'(t)|} = u(t)$



Case II $\quad \alpha'(t) < 0$, the angle $\alpha(t)$ is decreasing

and in this case $N(t) = \dfrac{\alpha'(t)\, u(t)}{-\alpha'(t)} = -u(t)$



For a plane curve we have $\| T'(t) \| = |\alpha'(t)|$

Where $T(t) = \cos\alpha(t)\vec{i} + \sin\alpha(t)\vec{j}$

By chain rule

$$\frac{d\alpha}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt}$$

$$= \| R'(t) \| \frac{d\alpha}{ds}$$

$$\Rightarrow \quad K(t) = \frac{\| T'(t) \|}{\| R'(t) \|} = \frac{\left| \frac{d\alpha}{dt} \right|}{\| R'(t) \|}$$

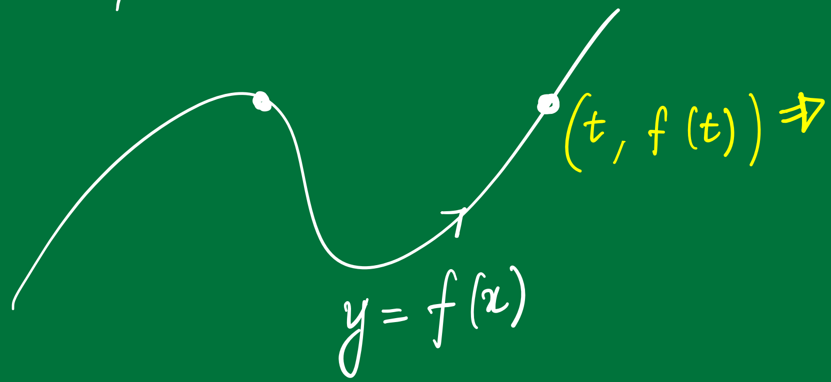
$$\Rightarrow \quad K(t) = \left| \frac{d\alpha}{ds} \right| \qquad (\| R'(t) \| \neq 0)$$

Sometimes the curvature of a plane curve is defined to be the rate of change of the direction of tangent vectors (tangent lines).

<u>Theorem</u>: Let $v(t)$ and $a(t)$ denote the velocity and the acceleration vectors of a motion of a particle on a curve defined by $R(t)$. Then

$$K(t) = \frac{\| a(t) \times v(t) \|}{\| v(t) \|^3}.$$

Application:



graph of $y = f(x)$

$$\vec{R}(t) = t\,\vec{i} + f(t)\,\vec{j}$$

$$v(t) = R'(t)$$

$$= \vec{i} + f'(t)\,\vec{j}$$

$$a(t) = \frac{dv}{dt}$$

$$= R''(t)$$

$$= f''(t)\,\vec{j}$$

$$\Rightarrow \quad v(t) \times a(t)$$

$$= f''(t)\left(\vec{i} \times \vec{j}\right)$$

$$= f''(t)\,\vec{k}$$

Now

$$K(t) = \frac{\|a(t) \times v(t)\|}{\|v(t)\|^3}$$

$$= \frac{\| f''(t)\,\vec{k} \|}{\left(\sqrt{1 + f'(t)^2}\right)^3}$$

$$= \frac{| f''(t) |}{\left(1 + f'(t)^2\right)^{3/2}}$$

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y, z) \longmapsto f(x, y, z)$$

$$(x, y, z) \longmapsto x^2 + y^2 + z^2$$

Let $X_0 = (a, b, c) \in \mathbb{R}^3$

$$X \to X_0$$

$$(x, y, z) \to (a, b, c)$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

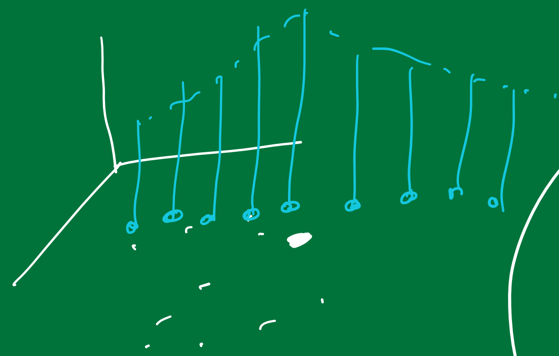
$$(x, y) \longmapsto 1$$

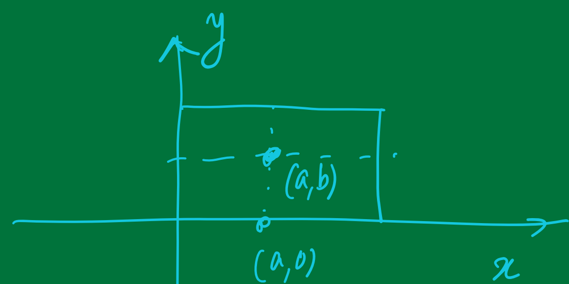
$$\{(x, y, f(x, y)) \in \mathbb{R}^3\}$$

$$= (x, y, 1) \in \mathbb{R}^3$$

$$\lim_{x \to x_0} f(x) = L$$

for $\varepsilon > 0$, there exists a $\delta > 0$





$$|f(x,y) - f(a,b)|$$

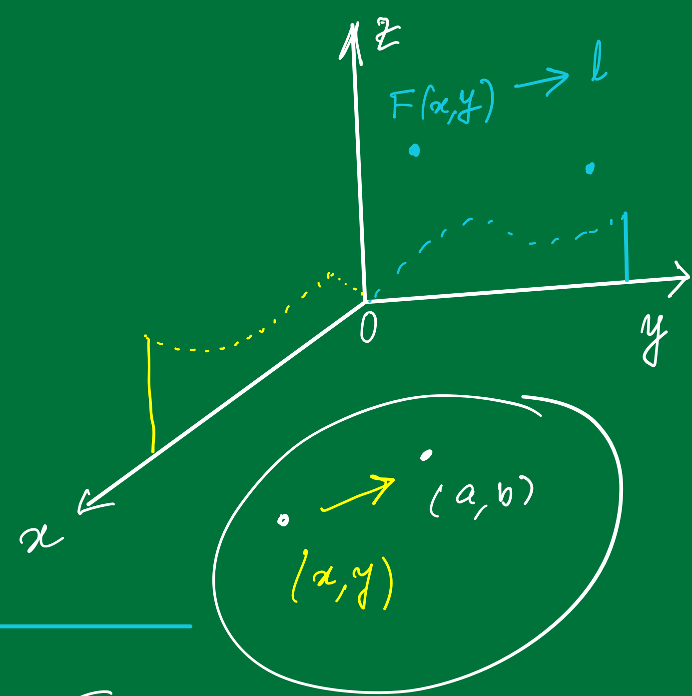
$$\|(x,y) - (a,b)\|$$

$$|f(x,y) - L|$$

Let $(a,b) \in \mathbb{R}^2$, $\qquad (x,y) \longrightarrow F(x,y) \in \mathbb{R}$

We say $F(x,y)$ approaches to ℓ as $(x,y) \to (a,b)$ and write

$$\lim_{(x,y)\to(a,b)} F(x,y) = \ell .$$



If for $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\|(x,y) - (a,b)\| < \delta$ we have $|F(x,y) - \ell| < \varepsilon$.

OR, For every sequence $(x_n, y_n) \to (a,b)$ we have $F(x_n, y_n) \to \ell$.

Example: $F(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

If we vary (x,y)
along the x-axis then $F(x,0) \to 0$

If we vary (x,y)
along the y-axis then $F(0,y) \to 0$

But for arbitrary line $y = mx$ in the xy plane,
if we vary (x,y) along the line $y = mx$, we find that

$F(x,y) = F(x, mx)$ and

$F(x, mx) \to \dfrac{2m}{1+m^2}$ – a quantity depends on the value of m (the slope of the line).

In this case, $\lim_{(x,y)\to(0,0)} F(x,y) = ?$

We want $\ell \in \mathbb{R}$ such that

$F(x,y) \to \ell$ whenever $(x,y) \to (0,0)$.

— No such ℓ exists.

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