Q12

6.4. c is char value of T. $\Rightarrow \exists$ non-zero vector $v \in V$ s.t. $Tv = cv$

one side

$\Rightarrow f(T)v = f(c)v$

$f(T) = (T^2 + T + 2)v$

$= T^2 v + Tv + 2$

other conversely.

Let c be a eigenvalues of $f(T)$

$c = f(\lambda)$, $f(T) = (T^n + \cdots\cdots +)v$, $cv \to v$

λ is a eignvalue of T.

F is algebrically closed.

So, There exist a basis $\mathcal{B}$ of V s.t.

$[T]_{\mathcal{B}}$ is a triangulizable operator. & some thing?

6.5. $\boxed{\text{if they commute, there exist one such Basis in which they all are diagonalizable}}$

1. find P s.t. $P^{-1}AP$ & $P^{-1}BP$ are diagonal,

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$

for 2×2, only such one P.

easy way.

(a) char v. of A $\lambda = 1$, $\lambda = 2$

$A - I = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $-x + 2y = 0$

$(1, 0)$ $(2, 1)$

$\Rightarrow P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$P^{-1}BP = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(3) $T \Rightarrow$ linear operator on n-dims space,
$T \Rightarrow$ has n distinct char value.
prove any linear operator commutes with T is a polynomial in T.

Sol^n:

$$f(T) = (x-c_1)\cdots(x-c_n)$$

- $UT = TU$,

$c_1 \quad c_2 \quad \cdots c_n$

$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

U be any operator on V,

T is diagonilizable.

we can say

$U(W_i) \subseteq W_i$

$\hookrightarrow W_i$ is also invariant under W_i

$w \in W_i$

$U(w) \in W_i$

$T(U(w_1)) = U(T(w_1)) = c_1 U(w_1)$

hence W_i

As w_i's are 1-dimensional

we see that U is diagonalizable

$U(W_i) \subseteq W_i$, $\{w_1 \dots w_n\}$

$U(w_i) = 0, \; U(w_2) = d_2 w_2 \; , U(w_3) = 0 \; -$

i.e. we are not sure that they are distinct values

we get back to it!

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$V \to$ real vector

Q 9. idempotent $\to$ projection $\quad E = E^2$.

Prove $(I+E) =$ invertible.

M #1

- $(I+E)V = 0 \to \quad V = -EV$

$EV = -EV \to \boxed{EV = 0}$

$\to IV = 0$

so invertible $\leftarrow V = 0$

(good)

M #2

- $V = R(E) \oplus N(E)$

$\{w_1 \text{---} w_m\} + \{w_{m+1} \text{---} w_n\}$

$\Downarrow$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}}_{I} + \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \to \begin{bmatrix} 2 & & & & & \\ & 2 & & & & 0 \\ & & 2 & & & \\ & & & 2 & & \\ & 0 & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

$\Downarrow$

$(I+E)^{-1} \to \begin{bmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{2} & & \\ & & & 0 & \\ & & & & \ddots 0 \end{bmatrix}$

A

$\boxed{\left(I - \frac{E}{2}\right)}$

$\left(I+E\right)\left(I - \frac{E}{2}\right) \to$ ~~$I\,E$~~ $I \cdot$ ~~$E + E - \frac{E}{2}$~~ $\frac{I-E}{2} = I$

$-E = I$

Ans. $I - \frac{E}{2} + E - \frac{E}{2}$

Q10.

$\boxed{\text{Trace is independent of basis}}$

$\text{Trace}(E_1 + \cdots + E_k) = \text{trace}(I) = n$

$\Downarrow$

$\boxed{\text{trace}(E_1) + \text{trace}(E_2) + \cdots \cdots \text{tr}(E_k) = n}$

Let $W_i = \text{Range}(E_i)$

As $E_1 + \cdots\cdots + E_k = I$

for any $v \in V$

$(E_1 + \cdots\cdots + E_k)v = v \Rightarrow E_1 v + E_2 v + \cdots E_k v = 0$

$\Downarrow$

$V = W_1 + \cdots\cdots + W_k$ (not direct sum yet)

$\Leftarrow$

$\boxed{\dim V \leq \dim(W_1) + \dim(W_2) \vdots + \dim(W_k)}$

$w_i \in$

$\boxed{\dim(W_i) = \text{trace}(E_i)}$

$\Downarrow$

direct sum

$V = W_1 \oplus W_2 \oplus W_3 \cdots \cdots \oplus W_k$

$\Rightarrow$ as they are independ 0 ...

$\sum_{i=1}^{n} E_i V = 0 \qquad E_i V = 0$

$E_j\, E_j \sum E_i V = E_j V \Rightarrow E_j E_1 + \cdots + E_j E_k = E_j V$

$\Rightarrow \boxed{E_j E_i = 0}$.

Q3. $f_X(x_1, x_2) = \begin{cases} 1 & 0 < |x_2| < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 = \int_{-x_1}^{x_1} 1\, dx = \underline{2x_1}.$$

201

1) Let V be a finite dim vector space over $\mathbb{C}$.
Let $T: V \to V$ be a linear op with prop
$\underline{T^r = I}$, $r \geq 2$

Prove or disprove T is diagon.

sol^n.

T - satisfies $x^r - 1$.

Let $m(T)$ be the min poly of T.

then $m(T) \,/\, x^r - 1$, as $x^r - 1$ dont have repeated root as derivative dont have same roots.

divides

$m(T)$ dont have repeated root

hence distinct values.

what?

2) Let V be a f.d. vector space over F.
$T: V \to V$ be a linear operator.
assume that T commutes with every projection on V. what can you say about T?

- Let W be any subspace of V
$V = W \oplus W'$
P be projection onto W, T commutes with P
T leaves W-invariant

Q Let V be a f.d. vector space over F.
Let T_1 and T_2 be linear commutating diagonalisable operator on V. Prove that T_1 & T_2 are simultaneously diagonalisable.

$\Rightarrow$ As T_1 is diagonalisable
we have
$V = W_1 \oplus \cdots \oplus W_r$, where $W_i = \{ v \in V \mid (T - c_i) v = 0 \}$
$c_1, \ldots c_r$ are eigen values of T_1

~~$T_1 T_2 \ldots$~~

But a T_2 commutes with T_1,
diagonalizable $\to$ T_2 leaves each W_i invariant
$T_2 | W_i$, $\underline{1 \leq i \leq r}$ (check for 3rd) online.

$(V_1^{(1)} \oplus \cdots \oplus V_{s_1}^{(1)}) \oplus (V_1^{(2)} \oplus \cdots \oplus V_{s_2}^{(2)}) \oplus \cdots (V_1^{(r)} \oplus \cdots \oplus V_{s_r}^{(r)})$

$\{ V_1^1 \ldots V_{k_1}^1, V_1^2 \ldots V_{k_2}^2 \ldots V_1^n \ldots V_{k_r}^n \}$

Q dim$(V) < \infty$
$T: V \to V$ is linear operator. T is diagonalisable.
$\underline{W \text{ is } T\text{-inv} . \ T/W \text{ is diagonal.}}$

dim $W = r$

$B = \{ w_1 \ldots w_r, w_{r+1} \ldots w_n \}$

$[T]_B = \begin{bmatrix} \boxed{\begin{matrix} a_{11} & \\ \vdots & \cdots \\ a_{r1} & \end{matrix}} & B \\ 0 & C \end{bmatrix}$

7.8 question

3. Let V be a finite dim V.S. over F. T be a linear operator on V such that $\operatorname{rank}(T)=1$. Prove that either T is diagonalizable or T is nilpotent, not both.

$\Rightarrow$ Let $\dim V = n$

Then $\dim(\ker T) = n-1$, Let $\{V_1, \dots V_{n-1}\}$ be bais of $\ker T$.

Let $B = \{V_1, \dots, V_{n-1}, V_n\}$ be a extended basis of V.

$$[T]_B = \begin{bmatrix} 0 & 0 & \cdots & a_{1n} \\ 0 & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & \vdots \\ 0 & \vdots & \cdots & \\ & & & a_{nn} \end{bmatrix} \Rightarrow \underline{(x-a_{nn})x^{n-1}}$$

case (i) $a_{nn}=0$

$\Rightarrow$ T is nilpotent its char poly is x^n has T satsfi it.

case (2) $a_{nn} \neq 0$

$\Rightarrow$ T is diagonalizable,

$V = \underline{W_0 \oplus W_{a_{nn}}}$

[This part doubt]

---

5. $\dim V < \infty$ cover $\mathbb{C}$

Let $T: V \to V$ be a linear operator. D be a digonaliable part of T. Prove that if g is any polynomial with complex coefficients the diagonalizable part of $g(T)$ is $g(D)$.

$g = x^r + C_1 x^{r-1} + \dots + C_r, \quad C_1 \dots C_r \in \mathbb{C}$

Let $T = D+N$, operator & unique decomp

$g(T) = g(D+N)$

$= (D+N)^r + C_1(D+N)^{r-1} + \text{—} \quad C_r I$

$\sum_{k=0}^{r} D^k N^{r-k} + \dots \dots + C_r I$

$D^r + C_1 D^{r-1} + \dots + C_r + \overbrace{(\quad\quad) N}^{n/p} = g(D)$ H.P.

Q6: ~~Let V/F, $\dim V < \infty$, $T: V \to V$ be a linear operator.~~

~~Let A be a fixed $n \times n$ matrix over F~~

Let $V = M_n(F)$, let A be a fixed $n \times n$ matrix

$T: V \to V$, $T(B) = AB - BA$

Prove that T is nilpotent if A is nilpotent matrix.

$\underline{T^{2k}(B)}$, go with induction

$\hookrightarrow$ General form

205 Tutorial

Assignment 201.

Q1. Let A be an $n \times n$ matrix over a field F. Prove or disprove A is similar to A^t / A^t – A transpose.

⇒ Let $\bar{F}$ be the algebrically ~~closed~~ closure of F.
It is ~~enough~~ enough to prove that A and A^t are similar, considered as matrices over $\bar{F}$.

$F \to \mathbb{R}$, $\bar{F} \to \mathbb{C}$

over $\mathbb{C}$ A similar to its Jorden Canonical form.
A is similar to matrix

$$\text{new } A = \begin{bmatrix} \begin{bmatrix} \lambda & & 0 \\ 1 & \lambda & \\ 0 & 1 & \ddots \\ 0 & 0 & \end{bmatrix} & \\ & [\ddots] \end{bmatrix}$$

$$A^t = \begin{bmatrix} \begin{bmatrix} \lambda & 1 & 0 & \cdots \\ 0 & \lambda & 1 & \cdots \\ 0 & \cdots & \cdots & \end{bmatrix} & \\ & [\ddots] \end{bmatrix}$$

to prove similar we need to prove Jordan blocks are similar

⇓

Question boils down to ⇒

$$\begin{bmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ \vdots & 1 & \lambda & \\ 0 & 0 & 1 & \ddots & \lambda \end{bmatrix}_{n \times n} \text{ is similar } \begin{bmatrix} \lambda & 1 & 0 & \cdots \\ 0 & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}_{n \times n}$$

$B = \{v_1, \dots, v_n\}$

$T(v_1) = \lambda v_1 + v_2$
$T(v_2) = \lambda v_2 + v_3$
$T(v_n) = \lambda v_n$

⟺

$B' = \{v_n, \dots, v_1\}$

$T(v_n) = \lambda v_n$
$T(v_{n-1}) = v_n + \lambda v_{n-1}$

Q: They have same C.P., M.E, but are not similar.

N 4×4 nilpotent matrix

$Z(\alpha_1, N) \oplus Z(\alpha_2, N)$
$x^2 \qquad x^2$

$\longrightarrow \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \\ & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$

$Z(\beta_1 N) \oplus Z(\beta_2 N) + Z(\beta_3 N)$
$x^2 \qquad x \qquad x$

$\longrightarrow \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & & \\ & [0] & \\ & & [0] \end{bmatrix}$

Q: How many nilpotent 7×7 matrices upto similarity with minimal polynomial x^3.

$\dim(V) = 7$

$\Rightarrow V = \underbrace{Z(\alpha_1; N)}_{x^3} \oplus \cdots \oplus (\alpha_4, N)$

$7 = 3+3+1$
$= 3+2+2$
$3+2+1+1$
$3+1+1+1+1$

} 4 matrices with minimal polynomial x^3.

classes

7.2

Q.19 Let T be a linear operator on f.d. vector s. V.

Prove T has a cyclic vector if the following's true:-

Every linear operator U which commutes with T is polynomial in T.

Let v be the cyclic vector of T. Let $n = \dim V$.

$Z(v;T) = V$.

$\{v, Tv, \cdots, T^{n-1}v\}$ is a basis of V.

$U(v) = c_1 v + c_2 T(v) + \cdots + c_n T^{n-1}(v)$

Let $f(x) \in F[x]$, $f(x) = c_n x^{n-1} + c_{n-1} x^{n-2} + \cdots + c_1$

* claim $U = f(T)$

$U(T(v)) = TU(v)$

$= T(c_1 v + c_2 T(v) + \cdots + c_n T^{n-1} v)$

$= c_1 T v + c_2 T^2 v + \cdots c_n T^n v$

$= f(T)(Tv)$ hence prove

if α exist & U & T com

U is polynom in T.

* Assume that every commuting operator U is polynomial in T need to prove that T has a cyclic vector.

Suppose :→ T does not have any cyclic vector.

$V = Z(\alpha_1;T) \oplus \cdots \oplus Z(\alpha_r;T)$ $\underline{r > 1}$

~~$TU(\alpha_1)$~~ ~~$T\alpha_1 = U(T\alpha_1)$~~

$\{\alpha_1, T(\alpha_1) \cdots T^{k_1-1}\alpha_1\}$ $U(\alpha_1) = 0$ (define)

$U(T(\alpha_1)) = TU(\alpha_1) = 0$ as commutes

Define $U(\alpha_1) = 0$, $U(\alpha_2) = \alpha_2$, ... $U(\alpha_m) = \alpha_m$,

define your U in such way.

that U commuts with T

$U(Z_1; T) = 0$, identity on other.

as U commuts with T

$U = q(T)$ but $U(\alpha_1) = 0 \Rightarrow P_1(T) / q(T)$

$U(\alpha_2) = \alpha_2$ $\quad q(T) = P_1(T) f(T)$

$q(T)\alpha_2$

$P_1(T) f(T) \alpha_2 = 0$

which is not possible. hence $r > 1$ not possible

7.2.

Q20.

Let $\dim V/F < \infty$

$T: V \to V$ be a linear operator.

we ask: when it is true that every non-zero vector in V is cyclic vector for T. Prove that this is the case iff the char. polynomial for T is irreducible over F.

$\boxed{\text{Suppose}}$ C.P $\to$ irreducible $\Longrightarrow$ any non zero vector is cyclic.

every non-zero cyclic & C.P $\to$ not irre & M.P is not C.P. $\Rightarrow$ brings contradiction that α is cyclic.

M.P. is same $\Rightarrow$ there exist β which is not cyclic

Assignment. 5/11/24

7.3.3. $A \to 5\times 5$ matrix. ch.p. $\Rightarrow (x-2)^3(x+7)^2$

min P. $\Rightarrow (x-2)^2(x+7)$

write Jordan form for A.

Primary first

$R^5 = Z(\alpha_1, P_1) \oplus \dots Z(\alpha_r, P_r)$

$P_r | P_{r-1} \dots P_1$ } divides,

$P_1 = (x-2)^2(x+7)$ P_2 $(x-2)(x-7)$

cyclic first

$V = W_1 \oplus W_2$

$W_1 = \{v \in V \mid (T-2I)^2 v = 0\}$

$W_2 = \{v \in V \mid (T+7I)v = 0\}$

$W_1 = Z(\alpha_1, T_1) \oplus Z(\alpha_2, T_2)$
$(x-2)^2$ $(x-2)$

$W_2 = Z(\beta_1 T_2) \oplus Z(\beta_2 T_2)$
$(x+7)$. $(x+7)$

$$\begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & & & \\ & [2] & & \\ & & [-7] & \\ & & & [-7] \end{bmatrix}$$ Ans.

7.3.4

(2) How many possible Jordan forms are there for a 6×6 complex matrix with char. polynomial $(x+2)^4(x-1)^2$

$V = W_1 \oplus W_2$

$W_1 = \{v \in V \mid (T+2I)^4 v = 0\}$ $W_2 = \{v \in V \mid (T-I)^2 v = 0\}$

$(x+2)(x-1)$
$(x+2)^2(x-1)$
$(x+2)^3(x-1)$
$(x+2)^4(x-1)$

$(x+2)(x-1)^2$

$(x+2)^4(x-1)^2$

• Each case have its diff form

→ any case.

$(x+2)^3(x-1)$

$\Rightarrow Z(\alpha_1, T_1) \oplus Z(\alpha_2, T_2)$
$(x+2)^3$ $(x+2)$

but if $\cdot m(T) = \cdot (x+2)$ ...

$W_1 = Z(\alpha, T) \oplus Z(\alpha_2 T) \oplus Z(\alpha_3 T)$

$(x+2)^2 \quad (x+2) \quad (x+2) \quad [2,1,1]$

or $Z(\alpha, T) \oplus Z(\alpha_2 T) \quad [2,2]$

$(x+2)^2 \quad (x+2)^2$

$\rightarrow \boxed{2\text{ cases.}}$

calculate total sum of cases

similarly calculate W_2

☆ Q. Let T be a nilpotent linear operator on $\mathbb{R}^5$. Let d_i denote the dimension of kernel of T^i. Which of the following possibility occur as a value of (d_1, d_2, d_3)

(a) $(1,2,3)$ ✓

(b) $(2,3,5)$ ✗

(c) $(2,2,4)$

(d) $(2,4,5)$ ✓

• $d_1 \Rightarrow$ dim of kernel of T

hence no. of block in Jordan form

so. if $d_1 = 1$

only 1 block $\Rightarrow$ Ch. P. = M. P $\Rightarrow x^5$.

we know. hence on every power it reduces by 1

• ~~for $d_1 = 2$~~

(b) $(2,3,5)$ ✗

$d_3 = 5$ hence minimal polynomial $\underline{x^3}$ hence two blocks

now check dim of $\leftarrow$

$4. \leftarrow d_2$

(d) ✓

$$T = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} & \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{2\times 2} \end{bmatrix}$$

(c) $(2,2,4)$

mean x^3

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} & \\ & [0] \end{bmatrix}$$

$(2,2,4)$

$\hookrightarrow 2$ ~~...~~

$\Rightarrow$ 2 block.

• 4×1 or 3×2

$\hookrightarrow$ not poss

$\Downarrow$

x^3

Q: Let A be 3×3 real matrix rational such that

$A^5 = I$ determine possibility of A.

The minimal polynomial of A dives x^5-1,

$\Rightarrow (x-1)(x^4+x^3+x^2+x+1)$ is irreducible

the minimal poly can't deivide $(x^4+x^3 \cdots +1)$

then $(x-1)$?

hence I. Ans.

8.1 12/11/2024

Q7.

$\alpha = (\alpha, \beta) \in \mathbb{C}^2$

Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be be any non-zero vector.

operator $T(e_1) = (a, b)$

$T(e_2) = (c, d)$

$T(V) = T(\alpha, \beta) = \alpha T(e_1) + \beta T(e_2)$

$= (\alpha a + \beta c, \alpha b + \beta d)$

suppose $(\alpha | T(\alpha)) = 0$,

$(V | T(V)) = 0$

$((\alpha, \beta) ; (\alpha a + \beta c, \alpha b + \beta d)) = 0$

$\Rightarrow \alpha\,\overline{(\alpha a + \beta c)} + \beta\,\overline{(\alpha b + \beta d)} = 0$

$\Rightarrow |\alpha|^2 \bar{a} + \alpha\bar{\beta}\bar{c} + \beta\bar{\alpha}\bar{b} + |\beta|^2 \bar{d} = 0$ —— (1)

$\rightarrow |\alpha|^2\bar{a} + \alpha\bar{\beta}\bar{c} + \beta\bar{\alpha}\bar{b} + |\beta|^2\bar{d} = 0$

for $\alpha = 1, \beta = 0$ $\qquad \alpha = \beta = 1$

$\boxed{\bar{a} = 0} = \boxed{a = 0}$ $\qquad \cancel{\bar{a} + \bar{c} + \bar{b} + \bar{d} = 0}$

$\alpha = 0, \beta = 1$

$\boxed{b = 0}$ $\qquad$ take ~~real~~ some complex values, and cal, it should be 0.

8.1

8. suppose $f_A(X, X) = Y^t AX$, defines as inner product

$X = (x_1, x_2)$

$f_A(x, x) > 0$ $\qquad (x_1, x_2)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > 0$

$$\begin{bmatrix} x_1 a_{11} + x_2 a_{21} \end{bmatrix} \begin{bmatrix} a_{12}x_1 + x_2 a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\cancel{x_1 a_{12} + x_2 a_{22}}$

$$x_1^2 a_{11} + x_1 x_2 a_{21} + a_{12} x_1 x_2 + x_2^2 a_{22} > 0$$

$a_{11}a_{22} - a_{21}a_{12} > 0$

$$x_1^2 a_{11} + x_2^2 a_{22} + x_1 x_2 (a_{21} + a_{12}) > 0$$

divide by x_2^2

$b_A(x, y)$

$= b_A(y, x)$

$$a_{11}\left(\frac{x_1}{x_2}\right)^2 + \frac{x_1}{x_2}(a_{21} + a_{12}) + a_{22} > 0$$

so det < 0 $\qquad b^2 - 4ac < 0$

$$(a_{21} + a_{12})^2 - 4a_{22}a_{11} < 0$$

$$a_{21}^2 + a_{12}^2 + 2a_{21}a_{12} - 4a_{22}a_{11} < 0$$

$\boxed{\text{DO IT}}$

11. see sol^n.

12.

$$\alpha = \sum_{i=1}^{n} d_i \alpha_i$$

$$(\alpha / \alpha_j) = \left(\sum_{i=1}^{n} d_i \alpha_i / \alpha_j \right) = \sum_{i=1}^{n} d_i (\alpha_i / \alpha_j)$$

$$= \sum_{i=1}^{n} d_i G_{ji}$$

$$\begin{bmatrix} G_{11} & \cdots & \cdot \\ \vdots & \ddots & \vdots \\ \cdot & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is invertible

be an arbitrary linear

$G_{11} = [\alpha_1 | \alpha_1]$ $a_{12} = [\alpha_2 | \alpha_1]$

$a_{21} = [\alpha_1 | \alpha_2]$, $a_{22} = [\alpha_2 | \alpha_2]$.

Let $(x_1, x_2) \in \mathbb{R}^2$ $(X|Y) = Y^t G X$

$0 = [(x_1, x_2) | T(x_1, x_2)]$

$= [(x_1, x_2) | (-x_2, x_1)]$

$= (-x_2, x_1) \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, (a, d)

$= bx_1^2 - bx_1^2 + (d-a) x_1 x_2 = 0$ $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$b = 0$, $a = d$.