MTH 114M ODE

Ordinary Differential Equations

Instructor: DEBASIS SEN

References

(i) Prof. Shakti Ghorai lecture notes

(ii) Engineering Mathematics
— Kreyszig

(iii) Differential Equations with historical notes
— Simmons.

ODE

$y = y(x)$ – function of a single variable x

$\frac{dy}{dx} = y'$ $\quad \frac{d^2y}{dx^2} = y''$ - -

Example

	Order	linear
$y' = \cos x$	1	✓
$y'' + 9y = 0$	2	✓
$y'y''' - \frac{3}{2}y'^2 = 0$	3	X
$y'' + \cos y = 0$	2	X

Definition An ODE is a relation that involves a function of single variable y & its derivatives $y', y'', \dots y^{(n)}$.

$$F(x, y, y', y'', \dots y^{(n)}) = 0$$

Definition The order of an ODE is the order of the heighest derivative of y appearing in the relation.

Definition (linear ODE).

An ODE is called linear if y & its derivaties $y', y'' \dots y^{(n)}$ appear linearly in the equation. ie the ODE looks like.

$$\underbrace{a_0(x)\, y + a_1(x)\, y' + a_2(x)\, y'' + \dots + a_n(x)\, y^{(n)}}_{\shortparallel \atop L(y)} = b(x)$$

Remark

Then $L(y)$ is a linear map

ie $L(y_1 + y_2) = L(y_1) + L(y_2)$

$L(cy) = c\,L(y)$ $\quad \underbrace{c \in \mathbb{R}}_{\text{real numbers}}$

If $b(x)=0$, then the linear equation is called <u>homogeneous</u>.

Given an ODE, we would like to solve it.

Q) (i) What is a solution?

(ii) Does an ODE always admit a solution.
If yes, then how many?

(iii) If solution exist can we find an explicit formula of the solution!
If <u>not</u> can we get a feel of the solution?

<u>Definition</u> Given an ODE

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

A function $\varphi(x)$ is called a solution on an interval I $((a, b))$ if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \dots \varphi^{(n)}(x)) = 0$$

$\forall x \in I.$

Exmple

1st order ODE

$$\frac{dy}{dx} = f(x, y)$$

① solve $\frac{dy}{dx} = y^2$ $\quad y(0) = 1.$

$$\frac{dy}{y^2} = dx$$

Integrating $\quad -\frac{1}{y} = x + C$

c-const.

$-1 = 0 + C \quad C = -1$

$-\frac{1}{y} = x - 1$

$y = \frac{1}{1-x} \quad x \in (-\infty\ 1)$

② Verify $xy + \log y = 1$, is a solution of

$$(xy+1)\, y' + y^2 = 0$$

Sol Differentiate the ~~given~~ given equation $xy + \log y = 1$

$$x y' + y + \frac{1}{y} y' = 0$$

Simplyfing

$$(xy+1)\, y' + y^2 = 0$$

Remk In this example, the solution $y(x)$ is given implicitely as a function of x. This kind of solution is called implicit soln.

Graphical method

$$\frac{dy}{dx} = f(x,y)$$

Example $\quad \dfrac{dy}{dx} = -\dfrac{x}{y} = f(x,y)$

Let us first draw the curve,

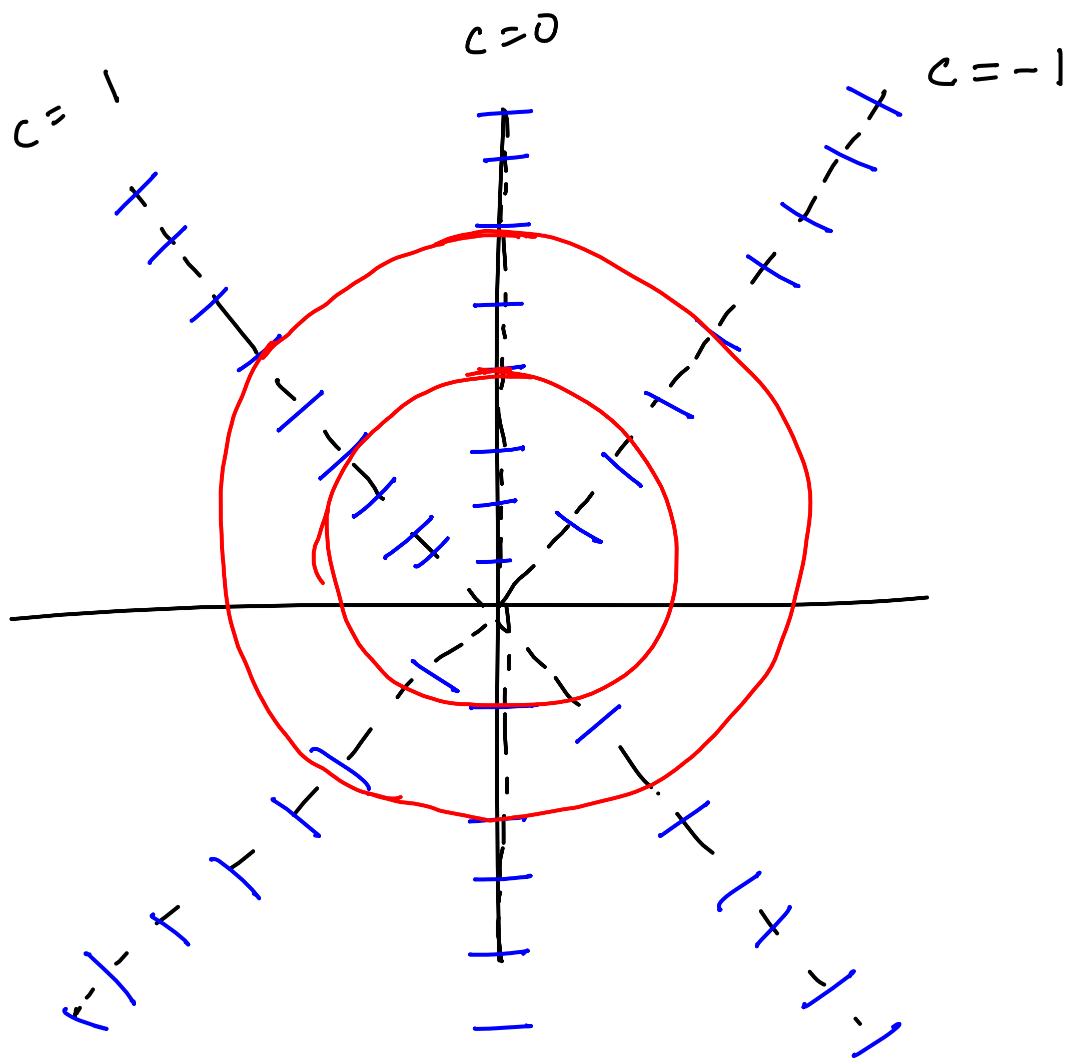
$f(x,y) = c$

$c=0$: $x = 0$

$c=1$: $y = -x$

$c=-1$: $y = x$

- $f(x,y) = C$: isoclines
- Blue small lines: slope field/direction field
- Red curves: approximate solc.



② $\dfrac{dy}{dx} = 1 + x - y = f(x, y)$

Step 1 Draw the Isocline,

$1 + x - y = c$

$c = 0$: $1 + x - y = 0$

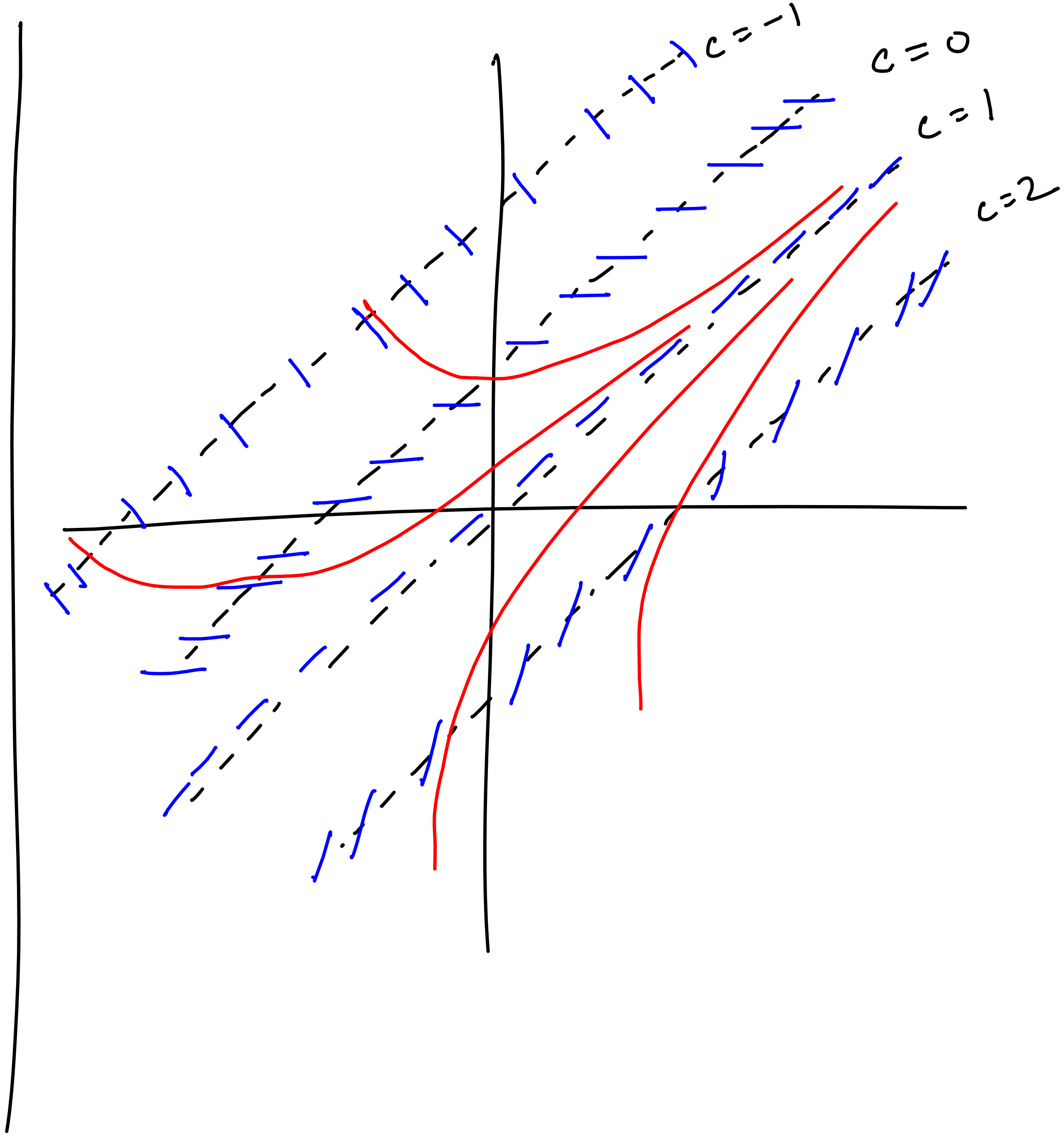
$c = 1$: $x - y = 0$

$c = -1$: $x - y = -2$

$c = 2$ $x - y = 1$

Step 2 Draw the slope field/direction field.
(blue lines)

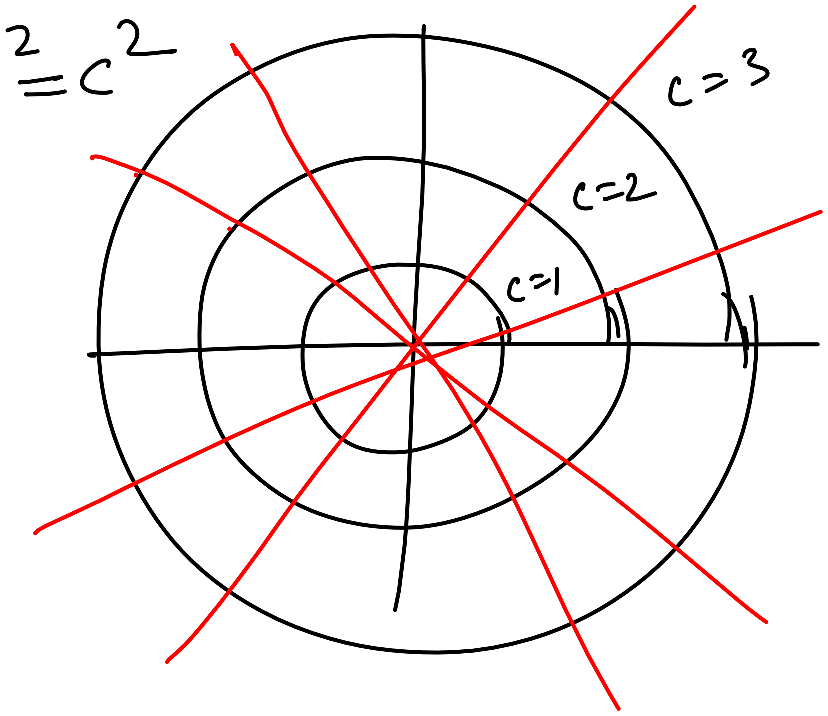
Step 3 Draw the approximate solution
(Red curves)



Orthogonal family of curves

$x^2+y^2=c^2$

$y=mx$



Definition Two family of curves are called orthogonal if they intersect orthogonally each other.

Suppose we are given a family of curves $\varphi(x,y,c)=0$

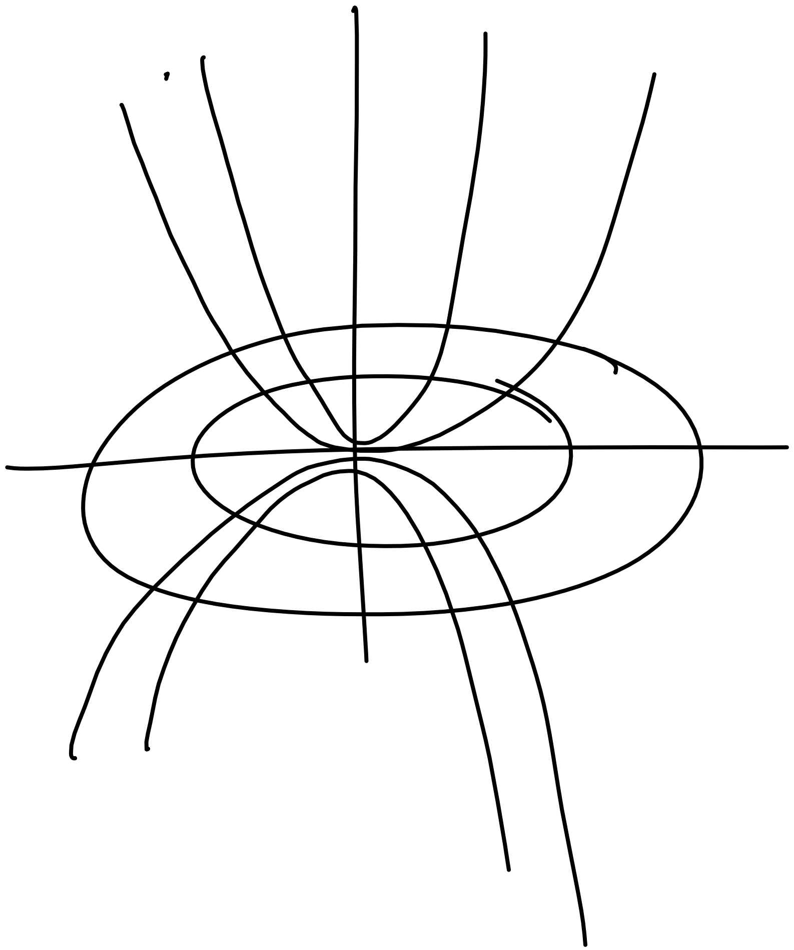
To find a orthogonal family to $\varphi(x,y,c)=0$

- First eliminate c to find the DE $\dfrac{dy}{dx}=f(x,y)$
- The slope of the orthogonal family $-\dfrac{1}{f(x,y)}$
- So the DE of the orthogonal family is $\dfrac{dy}{dx}=-\dfrac{1}{f(x,y)}$

Exmple

Find orlhogonl famly to $y = cx^2$

Sol



$$y = cx^2$$

Diff $\quad \frac{dy}{dx} = c \cdot 2x$

$$= \frac{y}{x^2} \cdot 2x$$

$$= 2\frac{y}{x} = \frac{2y}{x}$$

So the DE of the orlhogonal famly

$$\frac{dy}{dx} = -\frac{x}{2y}$$

$$2y\,dy = -x\,dx$$

$$y^2 + \frac{x^2}{2} = C$$

$$x^2 + 2y^2 = 2C.$$

Example Find the family of curves that intersect $x^2+y^2=r^2$ at an angle $\pi/4$.

Sol $x^2+y^2=r^2$

Diff $2x+2y\frac{dy}{dx}=0$

$$\frac{dy}{dx}=-\frac{x}{y}$$

slope of new curves

$$\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$\tan(\pi/4)=\frac{-\frac{x}{y}-\frac{dy}{dx}}{1+\left(-\frac{x}{y}\right)\frac{dy}{dx}}$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

$$x\,dy-y\,dy=x\,dx+y\,dx$$

$$\frac{x\,dx+y\,dy}{x^2+y^2}+\frac{y\,dx-x\,dy}{x^2+y^2}=0$$

$$d\left(\frac{1}{2}\log(x^2+y^2)\right)+d\left(\tan^{-1}\frac{y}{x}\right)=0$$

Integrating -

$$\frac{1}{2}\log(x^2+y^2)+\tan^{-1}\frac{y}{x}=c$$

In polar coordinate

$$\frac{1}{2}\log r^2+\theta=c \qquad r=c_1e^{-\theta}$$