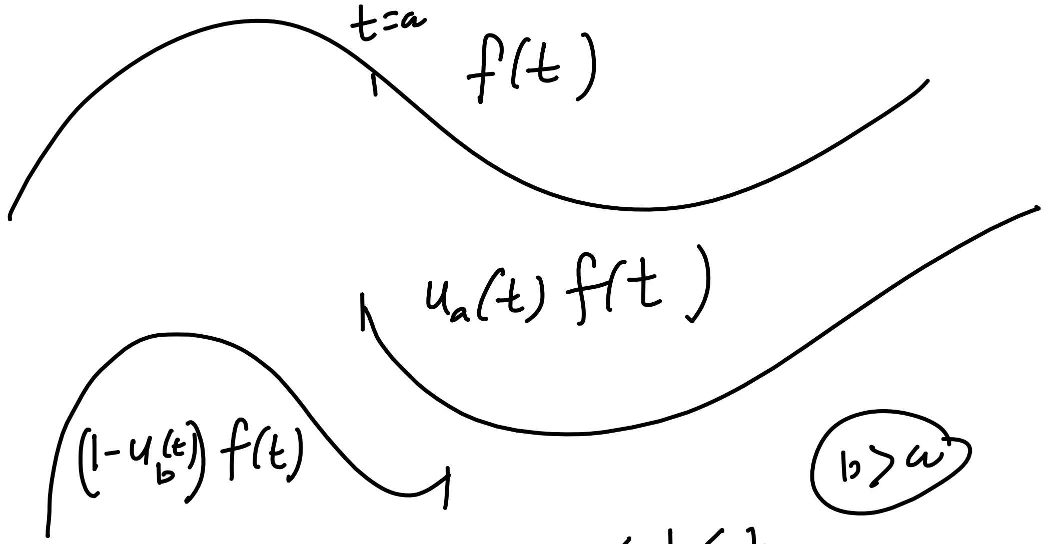
lecture 18 (Laplace Transform)

Recoll

$$u_a(t) = \begin{cases} 0 & t \le a \\ 1 & t > a \end{cases}$$

$$\mathcal{L}(u_a) = \frac{e^{-as}}{s}, \quad s > 0.$$



- $u_a(t) - u_b(t) = \begin{cases} 1 & a \le t < b \\ 0 & \text{else} \end{cases}$

First shift formula

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

Second shift formula

$$\mathcal{L}\left(u_a(t)\, f(t-a)\right) = e^{-as} F(s).$$

Prof

$$\text{L.H.S} = \int_{t=0}^{\infty} e^{-st} u_a(t) f(t-a)\, dt$$

$$= \int_{t=a}^{\infty} e^{-st} f(t-a)\, dt \qquad t-a = v$$

$$= \int_{v=0}^{\infty} e^{-s(a+v)} f(v)\, dv$$

$$= e^{-as} \int_{v=0}^{\infty} e^{-sv} f(v)\, dv = e^{-as} F(s)$$

$\blacksquare$

$$\mathcal{L}\left(u_a(t) f(t-a)\right) = e^{-as} F(s)$$

Exmple $f(t) = \begin{cases} t^2 & 0 \le t < 1 \\ \sin 2t & 1 \le t < \pi \\ \cos t & t > \pi . \end{cases}$

Find $\mathcal{L}(f)$.

Sol $f(t) = \underline{(u_0 - u_1)\, t^2} + \underline{(u_1 - u_\pi) \sin 2t}$
$+ \underline{u_\pi \cdot \cos t}$

- $(u_0 - u_1) t^2 = u_0 (t-0)^2 - u_1 (t-1+1)^2$

$= u_0 (t-0)^2 - u_1 (t-1)^2$
$- 2u_1 (t-1) - u_1$

$$\mathcal{L}\left((u_0 - u_1) t^2\right) = e^{-0.s} \frac{2}{s^3} - e^{-s} \frac{2}{s^3} - 2e^{-s} \frac{1}{s^2} - e^{-s} \frac{1}{s}$$

- $(u_1 - u_\pi) \sin 2t$

$= u_1 \sin 2t - u_\pi \sin 2t$

$= u_1 \sin\left(2(t-1) + 2\right) - u_\pi \sin\left(2(t-\pi) + 2\pi\right)$

$= u_1 \left\{ \sin\left(2(t-1)\right) \underline{\cos 2} + \cos\left(2(t-1)\right) \sin 2 \right\}$

$- u_\pi \sin\left(2(t-\pi)\right)$

$$\mathcal{L}\left((u_1 - u_\pi) \sin 2t\right)$$

$= \cos 2 \; e^{-s} \frac{2}{s^2+4} + \sin 2 \cdot e^{-s} \frac{s}{s^2+4}$

$- e^{-\pi s} \cdot \frac{2}{s^2+4}$

- $\mathcal{L}(u_\pi \cos t) = -\mathcal{L}\left(u_\pi \cos(t-\pi)\right) = -e^{-\pi s} \frac{s}{s^2+1}$

Convolution

$f, g : [0\ \infty) \longrightarrow \mathbb{R}.$

$$(f*g)(t) := \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$$

• $f*g = g*f$

Prf
$$(f*g)(t) = \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau \qquad t-\tau = v$$

$$= \int_{v=t}^{0} f(t-v)\, g(v)\, (-dv)$$

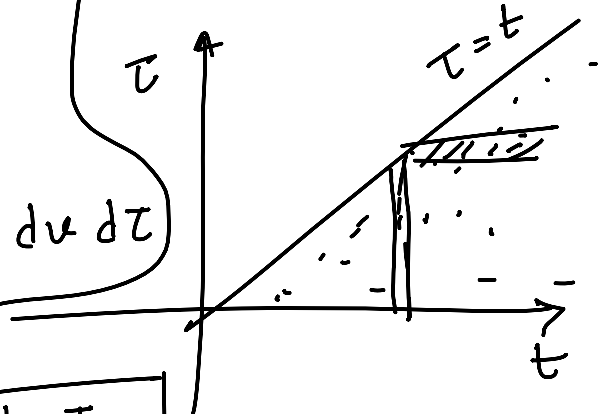
$$= \int_{v=0}^{t} g(v)\, f(t-v)\, dv = (g*f)(t). \blacksquare$$

• $\mathcal{L}(f*g) = \mathcal{L}(f)\,\mathcal{L}(g) = F(s)\, G(s)$

Prf
$$\text{L.H.S} = \int_{t=0}^{\infty} e^{-st} (f*g)(t)\, dt$$

$$= \int_{t=0}^{\infty} e^{-st} \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau\, dt$$

$$= \int_{\tau=0}^{\infty} f(\tau) \int_{t=\tau}^{\infty} e^{-st} g(t-\tau)\, dt\, d\tau$$



$$= \int_{\tau=0}^{\infty} f(\tau) \int_{v=0}^{\infty} e^{-s(\tau+v)} g(v)\, dv\, d\tau$$

$$\boxed{t-\tau = v}$$

$$= F(s) \cdot G(s). \blacksquare$$

Exmple ① Find $\mathcal{L}^{-1}\left(\frac{1}{s(s+1)^2}\right)$

$$\frac{1}{s(s+1)^2} = \underbrace{\frac{1}{s}}_{F(s)} \cdot \underbrace{\frac{1}{(s+1)^2}}_{G(s)}$$

$$F(s) = \mathcal{L}(f) \qquad G(s) = \mathcal{L}(g)$$

$$f(t) = 1 \qquad g(t) = t e^{-t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+1)^2}\right) = (f * g)(t)$$

$$= (g * f)(t)$$

$$= \int_0^t g(\tau) f(t-\tau)\, d\tau$$

$$= \int_0^t \tau e^{-\tau}\, d\tau$$

$$= - - - - \cdot$$

$$= 1 - (t+1) e^{-t}.$$

② Find $\mathcal{L}^{-1} \frac{1}{(s^2+a^2)^2}$

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}$$

$$\quad F \qquad = G$$

$$F(s) = \mathcal{L}(f) \qquad G = \mathcal{L}(g)$$

$$f(t) = \frac{\sin(at)}{a} = g(t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = (f * g)(t)$$

$$= \frac{1}{a^2}\int_0^t \sin(a\tau)\sin(a(t-\tau))\, d\tau$$

$$\cdot - - - - - - \cdot$$

$$= \frac{1}{2a^3}\left(\sin(at) - at\cos(at)\right)$$

③ Solve

$$y' + \int_0^t y(t-\tau)\, e^{-2\tau}\, d\tau = 1$$

$$y(0) = 1.$$

Apply L.T. $Y = \mathcal{L}(y)$.

$$sY - y(0) + Y \frac{1}{s+2} = \frac{1}{s}$$

$$Y = \frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}$$

$$y = 2 - e^{-t}$$

Linear Independence & Wronskian

$f, g : I \longrightarrow \mathbb{R}$. $\quad I = [a\ b)$

- f, g are called LI over $I = [a\ b]$
  if $c_1 f(t) + c_2 g(t) = 0 \quad \forall\ t \in I$

  $\Rightarrow c_1 = c_2 = 0$

- $W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

  $W(f, g)(t_0) \neq 0$ for some $t_0 \in I$

  $\Rightarrow f, g$ are L.I.

- y_1, y_2 are solution of (**)
  $$y'' + p(x)\, y' + q(x)\, y = 0$$
  p, q continuous on I.

Either
$$\underbrace{W(Y_1, Y_2) \equiv 0}_{LD} \quad \text{or} \quad \underbrace{W(Y_1, Y_2) \text{ never zero.}}_{L.I.}$$

Q) y_1, y_2 are two L.I functions.
Are they solution of (**) ?

$W(Y_1, Y_2)$ never zero

$\Rightarrow y_1, y_2$ independent solution of (**).

N

$y = c_1 y_1 + c_2 y_2$

$y' = c_1 y_1' + c_2 y_2'$

$y'' = c_1 y_1'' + c_2 y_2''$

Elimity c_1, c_2

$$\begin{vmatrix} y'' & y_1'' & y_2'' \\ y' & y_1' & y_2' \\ y & y_1 & y_2 \end{vmatrix} = 0$$

$$y''\left(W(y_1, y_2)\right) + y'(\ ) + y(\ ) = 0$$

$$y'' + y' \underbrace{\frac{(\ )}{W(y_1, y_2)}}_{p} + y \underbrace{\frac{(\ )}{W(y_1, y_2)}}_{q} = 0$$

Example

$y_1 = x^2$, $y_2 = x^3$

$I = [\ ]$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= 3x^4 - 2x^4 = x^4.$$

$I = (0 \quad \infty)$