Lecture 9 : Intersection, sum of subspaces & Quotient Space

<u>Proposition</u> : Let P and Q be two subspaces of a vector space V. Then

(i) $P \cap Q$ is a subspace of V,

(ii) $P + Q := \{a + b ; a \in P, b \in Q\}$ is a subspace of V.

<u>Proof</u>: (i) Let $x, y \in P \cap Q \Rightarrow x, y \in P \& Q$

$\Rightarrow \alpha x + \beta y \in P \& Q$ $\forall \alpha, \beta \in K$ (set of scalars)

$\Rightarrow \alpha x + \beta y \in P \cap Q$

(ii) Let $x, y \in P + Q$ then

$x = a_1 + b_1$, $y = a_2 + b_2$ for some $a_1, a_2 \in P$ & $b_1, b_2 \in Q$

$$\alpha x + \beta y = (\alpha a_1 + \beta a_2) + (\alpha b_1 + \beta b_2) \in P + Q$$

for all $\alpha, \beta \in K$.

$\boxtimes$

Examples: (1) Let $P = \{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\}$
& $Q = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 5z = 0\}$

If $(x, y, z) \in P \cap Q$, then $x = y + z$ & $x = -(2y + 5z)$

$$\Rightarrow \quad y + z = -(2y + 5z)$$
$$\Rightarrow \quad y = -2z$$
$$\& \quad x = -2z + z = -z$$

$\therefore \quad (-z, -2z, z) \in P \cap Q$.

So, $P \cap Q = \{r.(-1, -2, 1) ; r \in \mathbb{R}\}$
$= L(\{(-1, -2, 1)\})$

& $\{(-1, -2, 1)\}$ is L.I.

$\Rightarrow \{(-1, -2, 1)\}$ is a basis of $P \cap Q$
& $\dim(P \cap Q) = 1$, $P \cap Q$ represents line in $\mathbb{R}^3$.

(2) Let $P = \{(x, y, z, s, t) : x + 2y - z = 0\}$
$Q = \{(x, y, z, s, t) : y + s - t = 0\}$
$R = \{(x, y, z, s, t) : z + s + t = 0\}$

We want to find basis of $P \cap Q \cap R$.

Let $(x, y, z, s, t) \in P \cap Q \cap R$

$y = t - s$, $z = -s - t$

$x = -2y + z = -2(t - s) - s - t = s - 3t$

$$(x, y, z, s, t) = (s-3t, -s+t, -s-t, s, t)$$
$$= s(1,-1,-1,1) + t(-3,1,-1,1)$$

$\Rightarrow P \cap Q \cap R = L(\{(1,-1,-1,1), (-3,1,-1,1)\})$
& $\{(1,-1,-1,1), (-3,1,-1,1)\}$ is L.I. (check)
$\dim(P \cap Q \cap R) = 2$.

(3) Let $P = \{(x, y, z, s, t) : x - y + z = 0\}$
$Q = \{(x, y, z, s, t) : y + s + t = 0\}$

$(x, y, z, s, t) \in P \Rightarrow x = y - z$
$\Rightarrow (y-z, y, z, s, t) \in P$
$\Rightarrow y(1,1,0,0,0) + z(-1,0,1,0,0)$
$+ s(0,0,0,1,0) + t(0,0,0,0,1) \in P$

$\Rightarrow \{(1,1,0,0,0), (-1,0,1,0,0), (0,0,0,1,0),$
$(0,0,0,0,1)\}$ is a basis of P
$\Rightarrow \dim(P) = 4$
Similarly, $\{(1,0,0,0,0), (0,-1,0,1,0), (0,-1,0,0,1),$
$(0,0,0,0,1)\}$ is a basis of Q:

Let $(x, y, z, s, t) \in P \cap Q$

$x = y - z, \quad y = -s - t$

$\Rightarrow x = -s - t - z$

$\therefore (x, y, z, s, t) = (-s-t-z, -s-t, z, s, t)$

$= s(-1, -1, 0, 1, 0) + t(-1, -1, 0, 0, 1)$

$+ z(-1, 0, 1, 0, 0)$

$\Rightarrow \{(-1, -1, 0, 1, 0), (-1, -1, 0, 0, 1), (-1, 0, 1, 0, 0)\}$
is a basis of $P \cap Q$

Note that $\{(1, 1, 0, 0, 0), (-1, 0, 1, 0, 0),$
$(0, 0, 0, 1, 0), (0, 0, 0, 0, 1), (1, 0, 0, 0, 0),$
$(0, -1, 0, 1, 0), (0, -1, 0, 0, 1)\}$ spans $P + Q$.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_5 - R_1]{R_2 + R_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_7 + R_5]{R_6 + R_5} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Big\downarrow \; R_7 - R_4, \; R_6 - R_3$$

first five row vectors form a basis of $P+Q$. $\Leftarrow$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\dim(P+Q) = 5$$

Note that $\dim(P+Q) = \dim(P) + \dim(Q) - \dim(P \cap Q)$.

This fact is true in general, we will prove this using concept of quotient space

Theorem:- Let V be a finite dimensional vector space & P, Q are subspaces of V. Then

$$\dim(P+Q) = \dim(P) + \dim(Q) - \dim(P \cap Q)$$

Proof: Let $m = \dim(P \cap Q)$ & $\{e_1, \dots, e_m\}$ be a basis of $P \cap Q$.

- Extend $\{e_1, \dots, e_m\}$ to a basis $S_1 = \{e_1, \dots e_m, f_{m+1}, \dots, f_k\}$ of P
- Extend $\{e_1, \dots, e_m\}$ to a basis $S_2 = \{e_1, \dots, e_m, g_{m+1}, \dots, g_l\}$ of Q.

$S_1 \cup S_2 = \{e_1, \dots, e_m, f_{m+1}, \dots, f_k, g_{m+1}, \dots, g_l\}$

Note that $V = L(S_1 \cup S_2)$

Claim:- $S_1 \cup S_2$ is L.I.

Let $\sum_{i=1}^{m} a_i e_i + \sum_{j=1}^{k} b_j f_{m+j} + \sum_{r=1}^{l} c_r g_{m+r} = 0$ — (1)

where $a_1, \dots, a_m, b_1, \dots, b_k, c_1, \dots, c_l$ are scalars.

Let $x = \sum_{j=1}^{k} b_j f_{m+j}$ & $y = \sum_{r=1}^{l} c_r g_{m+r}$

then from (1) $x + y = -\sum_{i=1}^{m} a_i e_i \in P \cap Q$

& $x \in P$ & $y \in Q$

& $x \in P$ & $x+y \in P \cap Q \Rightarrow y \in P$

& $y \in Q$ & $x+y \in P \cap Q \Rightarrow x \in Q$

So, $x, y \in P \cap Q$

$\therefore \quad x = \sum_{j=1}^{k} b_j f_{m+j} = \sum_{i=1}^{m} \alpha_i e_i$

$\Rightarrow \quad \sum_{i=1}^{m} \alpha_i e_i - \sum_{j=1}^{k} b_j f_{m+j} = 0$

$\{e_1, -, e_m, f_{m+1} -, f_k\}$ is L.I

$\Rightarrow \quad \alpha_1 = \cdots = \alpha_m = 0 = b_1 = \cdots = b_k$

So, $b_1 = \cdots = b_k = 0$,

Similarly, $c_1 = \cdots = c_l = 0$

Now, $\sum_{i=1}^{m} a_i . e_i = 0 \Rightarrow a_1 = \cdots = a_m = 0$

as $\{e_1, \ldots, e_m\}$ is L.I.

$\therefore \quad S_1 \cup S_2$ is L.I.

Hence $S_1 \cup S_2$ is a basis of $P+Q$.

$\dim(P+Q) = |S_1 \cup S_2| = k + l - m$

$= \dim(P) + \dim(Q) - \dim(P \cap Q)$ ☒

Quotient Space: Let V be a finite dimensional vector space and W be a subspace of V (over $K = \mathbb{R}$ or $\mathbb{C}$).

Define $V/W := \{a + W ; a \in V\}$ (It is collection of subsets of V, $a + W = \{a + x : x \in W\}$

Note $a + W = b + W \iff a - b \in W$

Define addition $\oplus$ on V/W as follows:

$$(p + W) \oplus (q + W) := (p + q) + W$$

Check that $\oplus$ is well defined.

Define scalar multiplication: $\lambda \cdot (a + W) := \lambda a + W$ where $\lambda \in K$.

Under these operations, V/W becomes a vector space over K.

Let $\{u_1, \ldots, u_m\}$ be a basis of W, then it can be extended to a basis of W. Let the extended basis be $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$

Claim: $\{u_{m+1} + W, \ldots, u_n + W\}$ is basis of V/W.

Let $a + W \in V/W$, $a = \sum_{i=1}^{n} \alpha_i u_i$

$$a + W = \underbrace{(\alpha_1 u_1 + \cdots + \alpha_m u_m)}_{\in W} + (\alpha_{m+1} u_{m+1} + \cdots + \alpha_n u_n) + W.$$

$$a + W = \alpha_{m+1} u_{m+1} + \cdots + \alpha_n u_n + W$$
$$= \alpha_{m+1}(u_{m+1} + W) + \cdots + \alpha_n (u_n + W)$$

$\Rightarrow \{u_{m+1} + W, \ldots, u_n + W\}$ spans V/W.

$$\sum_{i=1}^{n} \lambda_i (u_{m+i} + W) = 0 + W \quad (\rightarrow \text{zero element of } W)$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i u_{m+i} \in W$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i u_{m+i} = \sum_{j=1}^{m} \beta_j u_j$$

$$\Rightarrow \beta_1 u_1 + \cdots + \beta_m u_m - \lambda_1 u_{m+1} - \cdots - \lambda_n u_{m+n} = 0$$

$$\Rightarrow \beta_1 = \cdots = \beta_m = \lambda_1 = \cdots = \lambda_n = 0$$

$\Rightarrow \{u_{m+1} + W, \ldots, u_n + W\}$ is also L.I. & hence it forms a basis of V/W.

$$\Rightarrow \boxed{\dim(V/W) = n - m = \dim(V) - \dim(W).}$$

Next we will prove that

$$\dim\left(\frac{P+Q}{Q}\right) = \dim\left(\frac{P}{P \cap Q}\right)$$

Where P, Q are subspaces of finite dimensional Vector space.

Define a map $\phi : {}^{P+Q}/_{Q} \to {}^{P}/_{P\cap Q}$ as follows;

$$\phi(x+y+Q) = x + P\cap Q \quad \text{where } x\in P,\, y\in Q$$

ϕ is well defined : $x_1+y_1+Q = x_2+y_2+Q$, $x_i \in P$, $y_i \in Q$, $i=1,2$

$\Rightarrow x_1+Q = x_2+Q$

$\Rightarrow x_1 - x_2 \in Q$

$\Rightarrow x_1 - x_2 \in P\cap Q$, since $x_1, x_2 \in P$

$\Rightarrow x_1 + P\cap Q = x_2 + P\cap Q$

ϕ is injective : $\phi(x_1+y_1+Q) = \phi(x_2+y_2+Q)$

$\Rightarrow x_1 + P\cap Q = x_2 + P\cap Q$

$\Rightarrow x_1 - x_2 \in P\cap Q \subseteq Q$

$\Rightarrow x_1 + Q = x_2 + Q$

$\Rightarrow x_1 + y_1 + Q = x_2 + y_2 + Q$

ϕ is surjective : Let $x + P\cap Q \in {}^{P}/_{P\cap Q}$

Let $y \in Q$ then

$\phi(x+y+Q) = x + P\cap Q$

$\therefore$ ϕ is bijective.

ϕ preserves algebraic structure i.e.

$\phi((a+Q) \oplus (b+Q)) = \phi(a+Q) \oplus \phi(b+Q)$

& $\phi(\lambda\cdot(a+Q)) = \lambda\,\phi(a+Q)$

As ϕ is bijective & preserves algebraic structure, ϕ takes basis of $P+Q/Q$ to a basis of $P/P\cap Q$

$$\Rightarrow \quad \dim\left(P+Q/Q\right) = \dim\left(P/P\cap Q\right)$$

$$\Rightarrow \dim(P+Q) - \dim(Q) = \dim(P) - \dim(P\cap Q)$$

Thus, we have proved the following theorem:

Th. $\boxed{\dim(P+Q) = \dim(P) + \dim(Q) - \dim(P\cap Q)}$ ⊠