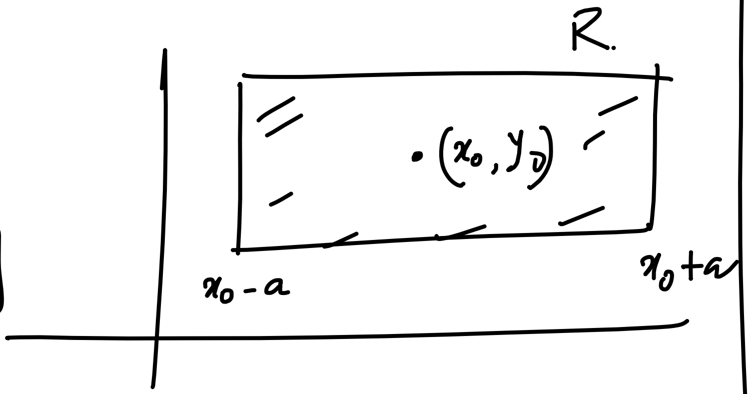
lecture 19

Recall (Picard Theorem) (Local)

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0 \quad \text{———} \quad (*)$$

$R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$



IF

- f is continuous on R
- f satisfy LC on R

$\Rightarrow$ IVP $(*)$ has unique solution.

The solution $y(x)$ is valid for $|x - x_0| < h$ where $h = \min\left\{a, \frac{b}{M}\right\}$

$$M = \sup_R |f(x,y)|$$

Picard's iterate,

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$$

$y(x) = \lim y_n(x)$ is the solution.

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt$$

$$= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

— defined for $|x - x_0| < a$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(\underbrace{t, y_1(t)})\, dt$$

To this to be well-defined we must have $|y_1(t) - y_0| \leq b$.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_0)\, dt \right|$$

$$\leq M\, |x - x_0| < b$$

$$|x - x_0| < \frac{b}{M}.$$

Thus for $|x - x_0| < \frac{b}{M}$, we have

$$|y_1(x) - y_0| < b.$$

Hence for $|x - x_0| < \frac{b}{M}$ & $|x - x_0| < a$

$y_2(x)$ is well defined.

in $|x - x_0| < h$ for $h = \min\left\{a, \frac{b}{M}\right\}$.

- $\boxed{\text{For } |x - x_0| < h.}$
  $\boxed{|y_1(x) - y_0| < b.}$

Also $|y_2(x) - y_0| < b$

. .

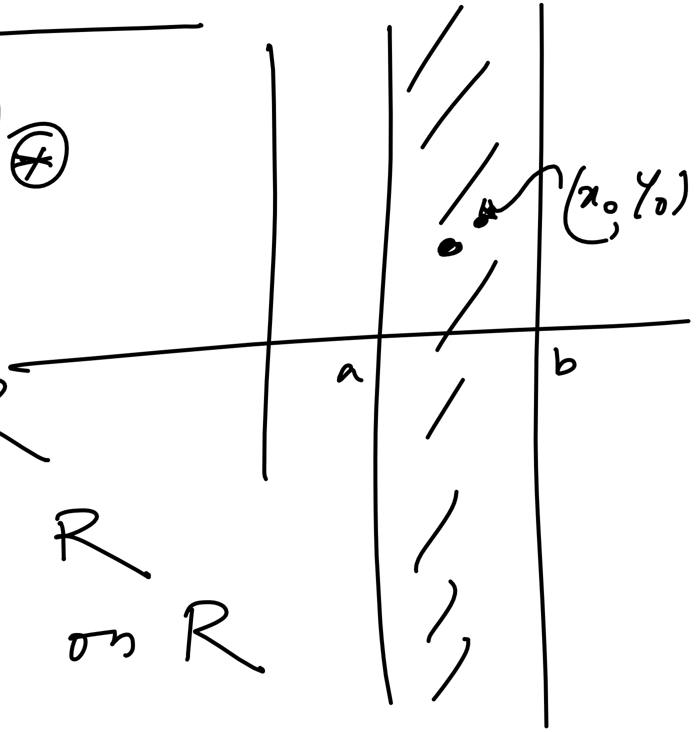
$$|y_{n-1}(x) - y_0| < b$$

$$|y_2(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_1(t))\, dt \right|$$

$$\leq M\, |x - x_0| < M \cdot h \leq b$$

Global Picard Theorem

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}\right] (*)$$



$R = [a\ b] \times \mathbb{R}$

IF
- f is continuous R
- f satisfy LC on R

$\Rightarrow \exists!$ Solution of the IVP $*$
- the solution is valid for $[a\ b]$

Remark: The proof is similar. Only observe that $y_n(x)$ are defined for $[a\ b]$.

Example Linear equation

$$\frac{dy}{dx} = y\, p(x) + r(x)$$

where $p(x)$, $r(x)$ are continuous on $[a\ b]$.

- $f(x,y) = y\, p(x) + r(x)$

$R = [a\ b] \times \mathbb{R}$

- $|f(x,y_1) - f(x,y_2)|$

$$= |p(x)|\, |y_1 - y_2|$$

$$\leq L\, |y_1 - y_2|$$

$\blacksquare$

Remark Local & global Picard's Theorem can be generalized for a system of equations.

$$y_1'(x) = f_1(x, y_1, y_2, y_3) \qquad y_1(x_0) =$$

$$y_2'(x) = f_2(x, y_1, y_2, y_3) \qquad y_2(x_0) =$$

$$y_3'(x) = f_3(x, y_1, y_2, y_3) \qquad y_3(x_0) =$$

$$\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\underline{Y}' = \underline{f}(x, \underline{Y})$$

Second order linear equation

Theorem

$$y'' + p(x)\, y' + q(x)\, y = 0$$

$p(x)$, $q(x)$ are continuous functions over $[a, b) = I$

$$y(x_0) = \alpha \qquad y'(x_0) = \beta.$$

$\Rightarrow$ This has ! solution over $[a, b)$.

Proof

$$y' = v.$$
$$v' = -p(x)\, v - q(x)\, y.$$

This is a system of 1st order linear equations.

$$v(x_0) = \beta$$
$$y(x_0) = \alpha.$$

Hence by global Picard Theorem $\exists!$ solution over $[a, b)$. $\blacksquare$

Analytic functions (Real)

- $f(x)$ is called analytic $I \subseteq \mathbb{R}$. if $f(x)$ is analytic at x_0 $\forall\, x_0 \in I$
- f is called at a point x_0 if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \Big| \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

Examples: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\forall\, x \in \mathbb{R}$.

This is analytic on $\mathbb{R}$.

- Polynomials / $\sin x$ / $\cos x$ are analytic on $\mathbb{R}$.

- $\frac{1}{1-x}$ is analytic on $\mathbb{R} \setminus \{1\}$.

$g(x)$ is analytic at x_0 + $g(x_0) \neq 0$ $\Rightarrow$ $\frac{1}{g}$ is analytic at x_0

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

What is the power series around $x_0 = 5$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} a_n (x-5)^n \qquad a_n = \frac{f^{(n)}(5)}{n!}$$

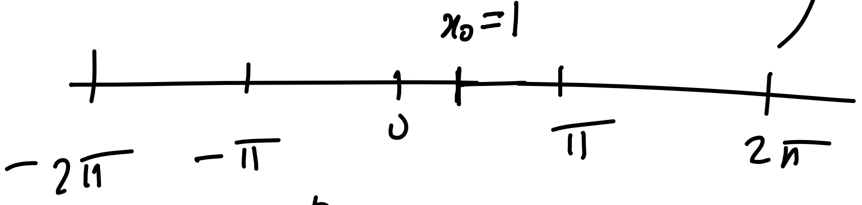
$$\frac{1}{1-x} = \frac{1}{-4-(x-5)} = \frac{1}{-4\left(1+\frac{x-5}{4}\right)}$$

$$= \frac{1}{-4}\left(1 - \frac{x-5}{4} + \left(\frac{x-5}{4}\right)^2 - \left(\frac{x-5}{4}\right)^3 + \cdots\right)$$

$$\left|\frac{x-5}{4}\right| < 1 \qquad 1 < x < 9$$

Example $\frac{1}{\sin x} = f(x)$

analytic for $\mathbb{R} \setminus \{ n\pi \}$



$$\frac{1}{\sin x} = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$a_n = \frac{f^{n}(1)}{n!}$$

- Radius of convergence of this power series

$= R =$ distance from $x_0 (= 1)$ to the nearest singularity (0)

$= 1.$ ∎