MTH 114: ODE: Assignment-5

1. Solve: (i) $x^2y'' + 2xy' - 12y = 0$ (ii)(**T**) $x^2y'' + 5xy' + 13y = 0$ (iii) $x^2y'' - xy' + y = 0$

2. (i) Let $y_1(x), y_2(x)$ are two linearly independent solutions of $y''+p(x)y'+q(x)y = 0$. Show that $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions if and only if $\alpha\delta \neq \beta\gamma$.

   (ii) Show that the zeros of the functions $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$.

3. (**T**) Show that any nontrivial solution $u(x)$ of $u'' + q(x)u = 0,\ q(x) < 0$ for all x, has at most one zero.

4. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

5. Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

6. Let J_p be any non-trivial solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

   Show that J_p has infinitely many positive zeros.

7. (**T**) Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) > m^2$ for all $t \in I$. Show any non trivial solution $u(x)$ has infinitely many zeros and distance between two consecutive zeros is at most π/m.

8. Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) < m^2$ for all $t \in I$. Show that distance between two consecutive zeros is at least π/m.

9. (**T**) Let J_p be any non-trivial solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

   Show that (i) If $0 \leq p < 1/2$, then every interval of length π has at least contains at least one zero of J_p.

   (ii) If $p = 1/2$ then distance between consecutive zeros of J_p is exactly π.

   (iii) If $p > 1/2$ then every interval of length π contains at most one zero of J_p.

10. Let $y(x)$ be a non-trivial solution of $y'' + q(x)y = 0$. Prove that if $q(x) > k/x^2$ for some $k > 1/4$ then y has infinitely many positive zeros. If $q(x) < \frac{1}{4x^2}$ then y has only finitely many positive zeros.

11. Find the eigen values and eigen functions of the following Sturm-Liouville problems:

(i) (**T**) $y'' + \lambda y = 0, \quad y(0) = y'(1) + y(1) = 0$

(ii) $(e^{2x}y')' + (\lambda + 1)e^{2x}y = 0, \quad y(0) = y(\pi) = 0. \qquad$ [Substitute $y = e^{-x}u$]

[Recall: (Sturm-Liouville Boundary Value Problem (SL-BVP)) With the notation

$$L[y] = \frac{d}{dx}(p(x)\frac{dy}{dx}) + q(x)y$$

consider the Sturm-Liouville equation

$$L[y] + \lambda r(x)y = 0$$

where $p > 0$, $r \geq 0$, and p, q, r are continuous functions on interval [a, b]; along with the boundary conditions $a_1y(a) + a_2p(a)y'(a) = 0, \;\; b_1y(b) + b_2p(b)y'(b) = 0$ where $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$. The problem of finding a values of λ if any, such that the BVP has a non-trivial solution is called a Sturm-Liouville Eigen Value Problem (SL-EVP). Such a value of λ is called an eigenvalue and the corresponding non-trivial solutions are called eigenfunctions.]

12. If $p(x), q(x), r(x)$ are all greater than zero on (a, b), then prove that the eigen values of the Sturm-Liouville problem, $(p(x)y')' + q(x)y + \lambda r(x)y = 0$, are positive with any of the boundary conditions: (i) $p(a) = p(b) = 0$, (ii) $y(a) - ky'(a) = y(b) + my'(b) = 0, k, m > 0$, (iii) $p(a) = p(b)$ with $y(b) = y(a), \; y'(b) = y'(a)$.

13. (**T**) Consider the Strum-Liouville problem

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$

with $p(x) > 0$ on $[a, b]$ and $y(a) \neq y(b)$, $y'(a) \neq y'(b)$. Show that every eigen function is unique except for a constant factor.