Under certain restriction on the partial derivatives of a given function f, we get the continuity of f.

Let

$$S = \left\{ (x,y) \in \mathbb{R}^2 \;/\; \begin{array}{l} a < x < b \text{ and} \\ c < y < d \end{array} \right\}.$$

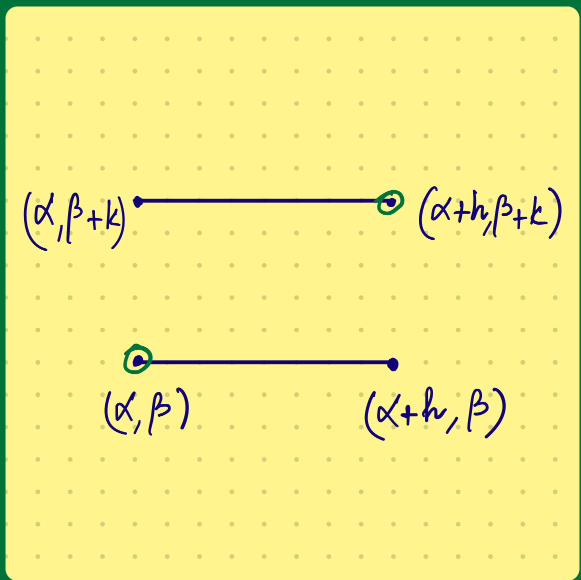
Suppose $f : S \to \mathbb{R}$ such that both partial derivative

$$\frac{\partial f}{\partial x} : S \to \mathbb{R} \qquad \text{and} \qquad \frac{\partial f}{\partial y} : S \to \mathbb{R}$$

$$x_0 \mapsto \frac{\partial f}{\partial x}(x_0) \qquad\qquad x_0 \mapsto \frac{\partial f}{\partial y}(x_0)$$

are bounded functions. Then $f : S \to \mathbb{R}$ is a continuous function.

Let $\left|\frac{\partial f}{\partial x}(\alpha,\beta)\right| \le M$ and $\left|\frac{\partial f}{\partial y}(\alpha,\beta)\right| \le M$



$$f(\alpha+h,\beta+k) - f(\alpha,\beta)$$

$$= f(\alpha+h,\beta+k) - f(\alpha+h,\beta) + f(\alpha+h,\beta) - f(\alpha,\beta)$$

$$= \left\{ f_{\alpha+h}(\beta+k) - f_{\alpha+h}(\beta) \right\} + \left\{ f^{\beta}(\alpha+h) - f^{\beta}(\alpha) \right\}$$

$$= k\, f'_{\alpha+h}(\beta+\theta_1 k) + h\left(f^{\beta}\right)'(\alpha+\theta_2 h)$$

Apply MVT on $f_{\alpha+h} : [\beta, \beta+k] \to \mathbb{R}$ and

$f^{\beta} : [\alpha, \alpha+h] \to \mathbb{R}$

$$= k\, f'_{\alpha+h}(\beta+\theta_1 k) + h\,(f^{\beta})'(\alpha+\theta_2 h) \quad \text{for some } \theta_1, \theta_2 \in (0,1).$$

$$= k\,\frac{\partial f}{\partial y}(\alpha+h, \beta+\theta_1 k) + h\,\frac{\partial f}{\partial x}(\alpha+\theta_2 h, \beta)$$

$$\Rightarrow \left| f(\alpha+h, \beta+k) - f(\alpha,\beta) \right| = \left| k\,\frac{\partial f}{\partial y}(\alpha+h, \beta+\theta_1 k) + h\,\frac{\partial f}{\partial x}(\alpha+\theta_2 h, \beta) \right|$$

$$\leq M\,(|k| + |h|)$$

$$\leq 2M\sqrt{h^2+k^2}$$

$$\Rightarrow |f(\alpha+h, \beta+k) - f(\alpha,\beta)| < \varepsilon \quad \text{whenever}$$

$$\sqrt{h^2+k^2} < \delta = \left(\varepsilon/2M\right) \quad \Rightarrow f \text{ is continuous at } (\alpha,\beta) \in S.$$

$$\Rightarrow \underbrace{\lim_{(x,y)\to(\alpha,\beta)} f(x,y) = f(\alpha,\beta)}_{\text{How?}}$$

<u>Need</u>:

$$|f(x,y) - f(\alpha,\beta)| < \varepsilon \quad \text{whenever}$$

$$\|(x,y) - (\alpha,\beta)\| < \delta.$$

Derivative: For $f: \mathbb{R} \to \mathbb{R}$ or $f: I \to \mathbb{R}$ the function f is differentiable at $c \in I$ with derivative $f'(c) \Rightarrow$ $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

$$\Leftrightarrow \lim_{h \to 0} \frac{f(c+h) - f(c) - h f'(c)}{h} = 0$$

$$\Leftrightarrow \lim_{h \to 0} \frac{|f(c+h) - f(c) - h f'(c)|}{|h|} = 0$$

Let $f: \mathbb{R}^3 \to \mathbb{R}$ and $X_0 = (a,b,c) \in \mathbb{R}^3$. We say that f is differentiable at X_0 if there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that the "error function" $E(H) \to 0$ as $H \to 0$

$$E(H) = \frac{f(X_0+H) - f(X_0) - \alpha . H}{\| H \|} \to 0$$

as $H \to 0$

$$\Leftrightarrow \lim_{\| H \| \to 0} \frac{f(X_0+H) - f(X_0) - \alpha . H}{\| H \|} = 0$$

In this case, the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is called the derivative of f at X_0 and we write it as

$$F'(X_0) = (\alpha_1, \alpha_2, \alpha_3).$$

Example:

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x + y$$

$$X_0 = (a, b) \quad , \quad \alpha = f'(a, b) = ?$$

For $H = (h, k)$

$$E(H) = \frac{f(X_0 + H) - f(X_0) - \alpha . H}{\| H \|}$$

$$= \frac{f(a+h, b+k) - f(a, b) - (\alpha_1, \alpha_2).(h, k)}{\sqrt{h^2 + k^2}}$$

$$= \frac{(a+h) + (b+k) - (a+b) - (\alpha_1 h + \alpha_2 k)}{\sqrt{h^2 + k^2}}$$

Then for $\alpha = (1,1)$, we have $E(H) \to 0$ as $\|H\| \to 0$

Therefore $\alpha = f'(a,b) = (1,1)$.

Example.

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto xy$$

$$\alpha = f'(a,b) = ?$$

□ Directional derivative Vs derivative.

Remark:

If $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $X_0 \in \mathbb{R}^3$ then f is continuous at X_0.

Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable at X_0 and $\alpha = (\alpha_1, \alpha_2, \alpha_3) = f'(X_0)$.

Now

$$E(H) = \frac{f(X_0 + H) - f(X_0) - \alpha . H}{\| H \|} \to 0 \text{ as } \|H\| \to 0$$

$$\Rightarrow \left| f(X_0 + H) - f(X_0) - \alpha . H \right| = \|H\| \, |E(H)|$$

and $E(H) \to 0$ as $\|H\| \to 0$.

$$\left| f(X_0 + H) - f(X_0) \right| - |\alpha . H| < \| H \| \, | E(H) |$$

$$|a-b|$$

$$|a| = |(a-b) + (b)|$$

$$|b| + |a-b| > |a|$$

$$|a-b| + |b| > |a|$$

$$|a-b| > |a| - |b|$$

$$\left| f(X_0 + H) - f(X_0) \right| < \|H\| \left(\|\alpha\| \quad \|E(H)\| \right)$$

As $\|H\| \to 0$ we get $f(X_0+H) - f(X_0) \to 0$.

$\Rightarrow$ $f(X_0 + H) \to f(X_0)$ as $\|H\| \to 0$.

$\Rightarrow$ f is continuous at x_0.

Example:

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Check if the function f is differentiable at origin.

* We can check that f is not continuous at $(0,0)$, So f is not differentiable at $(0,0)$. Because if f is differentiable at a point (a,b) then f is Continuous at (a,b) aswell.

Example:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Q. Is f is differentiable at $X_0 = (0,0)$?

I The given function $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous everywhere.
— is a continuous function.

II $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous at $X_0 \neq (0,0)$.

I f is differentiable at $X_0 \neq (0,0)$.

We checked before that the function f is continuous everywhere.

II. For $(a,b) \neq (0,0)$, We can check that the partial derivatives $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ exist

and the function $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous.

Thus $f(x,y)$ is differentiable at $(a,b) \neq (0,0)$.

Here $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$

At the point $(0,0)$;

For $H = (h,k)$ and $\alpha = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right)$

the error function $E(H) = \frac{f(h,k)}{\sqrt{h^2+k^2}}$

$$= \frac{hk}{h^2+k^2}$$

As $H \to (0,0)$ (or $(h,k) \to (0,0)$)

the function $E(H) \not\to 0$!

Therefore f is not differentiable at $(0,0)$.

Theorem. Suppose a given function $f(x,y,z)$ of three variables is differentiable at $X_0 \in \mathbb{R}^3$.

Then The partial derivatives $\frac{\partial f}{\partial x}(X_0)$, $\frac{\partial f}{\partial y}(X_0)$ and $\frac{\partial f}{\partial z}(X_0)$ exists at X_0 and the derivative

$$\boxed{f'(X_0) = \left(\frac{\partial f}{\partial x}(X_0), \frac{\partial f}{\partial y}(X_0), \frac{\partial f}{\partial z}(X_0) \right).}$$