Lecture 12 : Solvability of system of Linear Equations using rank.

Consider a system of linear equations $AX = B$ where $A = (a_{ij})$ is a matrix of order $m \times n$, $X = (x_j)$ is a matrix of order $n \times 1$ & $B = (b_j)$ is a matrix of order $n \times 1$.

The matrix A induces linear map.

$$T_A : \mathbb{R}^n \to \mathbb{R}^m$$

$$T_A(x) = AX, \quad x \in \mathbb{R}^n$$

Existence of solution

$AX = B$ has a solution say $\alpha = (\alpha_1, \dots, \alpha_n)$

$$\Leftrightarrow \quad \alpha_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \alpha_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$\Leftrightarrow \quad b \in L(\{A_1, A_2, \dots, A_n\})$

where $b = (b_1, b_2, \dots, b_n)$ & $A_j = (a_{1j}, \dots, a_{mj})$ & $j \in \{1, 2, \dots, n\}$

$\Leftrightarrow$ A & the augmented matrix $(A|B)$ have same rank.

Also note that, $AX = B$ has a solution
$\iff b = (b_1, \cdots, b_n) \in R(T_A)$

Uniqueness of Solution

The followings are equivalent:

(1) $AX = B$ has a unique solution,
(2) $AX = 0$ has only zero solution,
(3) $\text{Rank}(A) = n$

(1) $\Rightarrow$ (2) Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be unique solution of $AX = B$ & let $\beta = (\beta_1, \cdots, \beta_n)$ be a solution of $AX = 0$

$A(\alpha + \beta) = A\alpha + A\beta = B + 0 = B$
$\Rightarrow \alpha + \beta$ is also a solution of $AX = B$
As α is unique, $\alpha + \beta = \alpha \Rightarrow \beta = 0$

(2) $\Rightarrow$ (1) Let α_1, α_2 be two solutions of $AX = B \Rightarrow A\alpha_1 = B = A\alpha_2$
$\Rightarrow A(\alpha_1 - \alpha_2) = 0$
$AX = 0$ has only zero solution
$\Rightarrow \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2$.

$(2) \Leftrightarrow (3)$ $AX = 0$ has only zero solution if and only if $\ker(T_A) = \{0\}$

By Rank-Nullity theorem,

$$\dim(\ker(T_A)) + \dim(R(T_A)) = \dim(\mathbb{R}^n)$$

$$\begin{aligned}
\ker(T_A) = \{0\} &\Leftrightarrow \dim(\ker(T_A)) = 0 \\
&\Leftrightarrow 0 + \dim(R(T_A)) = n \\
&\Leftrightarrow \dim(R(T_A)) = n \\
&\Leftrightarrow \operatorname{rank}(A) = n
\end{aligned}$$

We have the following theorem:

Theorem: Consider the system of linear equations $AX = B$ where A is a matrix of order $m \times n$ and $\operatorname{rank}(A) = r$.

(i) $AX = B$ has a solution $\Leftrightarrow$ $\operatorname{rank}(A|B) = r$

(ii) If $r = m$, then $AX = B$ has a solution for every column vector $B \in \mathbb{R}^m$.

(iii) If $r = m = n$ then $AX = B$ has a unique solution for every column vector $B \in \mathbb{R}^m$.

(iv) If $r = m < n$ then $AX = B$ has infinitely many solutions for every column vector $B \in \mathbb{R}^m$.

(v) If $r < m = n$ or $r < m < n$ or $r < n < m$ and $Ax = B$ has a solution then $AX = B$ has infinitely many solutions.

(vi) If $r = n < m$ & $Ax = B$ has a solution then this solution is unique.

Proof: (i) Already shown

(ii) $r = m \Rightarrow \dim(R(T_A)) = \dim(\mathbb{R}^m)$
$$\Rightarrow R(T_A) = \mathbb{R}^m \quad (\text{as } R(T_A) \subseteq \mathbb{R}^m)$$
$\Rightarrow T_A$ is surjective, so the result follows.

(iii) Follows from (ii) & equivalent conditions proved before the theorem.

(iv) From (ii), solution of $AX = B$ exists

From Rank - Nullity theorem,
$$\dim(\ker(T_A)) + \operatorname{rank}(A) = \dim \mathbb{R}^n = n$$
$$\Rightarrow \dim(\ker(T_A)) = n - r > 0$$
$\Rightarrow$ $Ax = 0$ has infinitely many solutions

$\Rightarrow$ $AX = B$ has infinitely many solutions.

(v) $\dim(\ker(T_A)) = n - r > 0$ & $Ax = B$ has a solution implies it has infinitely many solutions.

(vi) Let α & β be two solutions of $AX = B$

$\Rightarrow \quad A(\alpha - \beta) = 0 \quad \Rightarrow \alpha - \beta \in Ker(T_A)$

$\dim(Ker(T_A)) + rank(A) = n$

$\Rightarrow \quad \dim(Ker(T_A)) = n - r = 0$

$\Rightarrow \quad Ker(T_A) = \{0\}$

$\Rightarrow \quad \alpha - \beta = 0 \Rightarrow \alpha = \beta$

Example: (1) Consider the system of linear equations

$$x_1 + x_2 - x_3 + x_4 = 2,$$
$$x_1 + 2x_2 + x_3 - x_4 = 1,$$
$$x_1 + x_2 + x_3 + 2x_4 = 2.$$

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$(A \mid B) = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 1 & 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & -2 & -1 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 3 & 3 \\ 0 & 1 & 2 & -2 & -1 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 3 & 3 \\ 0 & 1 & 2 & -2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \end{array}\right) \xrightarrow[R_2-2R_3]{R_1+3R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 9/2 & 3 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \end{array}\right)$$

$$\parallel$$

$$(C \mid D) \text{ (say)}$$

$\text{rank}(A) = \text{rank}(C) = 3$

& $\text{rank}(A|B) = \text{rank}(C|D) = 3$

$T_A : \mathbb{R}^3 \to \mathbb{R}^4$ & $\text{rank}(A) = 3 < 4$

By above theorem (i) & (iv), $AX = B$ has infinitely many solutions.

We will find the solutions of $CX = D$ (which are the solutions of $AX = B$)

$$x_1 + \frac{9}{2}x_4 = 3,$$

$$x_2 - 3x_4 = -1,$$

$$x_3 + \frac{1}{2}x_4 = 0$$

$\Rightarrow$ $x_1 = 3 - \frac{9}{2}x_4$, $x_2 = -1 + 3x_4$, $x_3 = -\frac{1}{2}x_4$

Thus, $(x_1, x_2, x_3, x_4) = \left(3 - \frac{9}{2}x_4, -1 + 3x_4, \frac{-1}{2}x_4, x_4\right)$

$= (3, -1, 0, 0) + x_4\left(-\frac{9}{2}, 3, -\frac{1}{2}, 1\right)$

The solutions are $\{(3,-1,0,0)+r(-9/2,3,-1/2,1) : r \in \mathbb{R}\}$

For any $r \in \mathbb{R}$, note that

$$A\left(r\begin{pmatrix}-9/2\\3\\-1/2\\1\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \quad \ker(T_A) = \{r(-9/2,3,-1/2,1) : r \in \mathbb{R}\}$$

Thus, each solution lie in the set

$$(3,-1,0,0) + \ker(T_A)$$

★ From (i) of previous theorem, for a system of linear equations $AX = B$ if $\operatorname{rank}(A) < \operatorname{rank}(A|B)$, then it has no solution.

Example: $AX = B$ where

$$A = \begin{pmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}, X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, B = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$x_1 - x_3 = 0$$
$$x_2 = 0$$
$$x_3 = 1$$

has no solution.

Characterisation of solutions by determinant

Proposition: Let A be a square matrix of order $n \times n$. $Ax = 0$ has only zero solution if and only if A is invertible.

Proof: Suppose $Ax = 0$ has only zero solution

$\Rightarrow \ker(A) = \{0\}$ ($A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map). By Rank-Nullity theorem,

$$\dim \ker(A) + \dim(A(\mathbb{R}^n)) = n .$$

$$\Rightarrow \quad 0 + \dim A(\mathbb{R}^n) = n \Rightarrow \dim A(\mathbb{R}^n) = n$$

$$\Rightarrow \quad A(\mathbb{R}^n) = \mathbb{R}^n \Rightarrow A \text{ is surjective}$$

Therefore, A is an isomorphism $\Rightarrow \exists$ a matrix B of order $n \times n$ such that

$$AB = BA = I_n \Rightarrow A \text{ is invertible}$$

Conversely, suppose A is invertible

$$Ax = 0 \Rightarrow A^{-1}(Ax) = 0 \Rightarrow x = 0$$

Hence, $x = 0$ is only solution.

Corollary:- A square matrix of order $n \times n$. $Ax = 0$ has only zero solution iff $\det(A) \neq 0$.

Theorem:- Suppose $Ax = b$ has a solution where A is a square matrix of order $n \times n$.
The followings are equivalent
(i) $\det(A) \neq 0$
(ii) $A\dot{x} = 0$ has only zero solution,
(iii) $Ax = b$ has unique solution
(iv) $\mathrm{rank}(A) = n$.

Corollary:- $Ax = b$ has infinitely many solutions if and only if $\det(A) = 0$.

Characterisation of Rank by determinant:-

Let A be a matrix of order $m \times n$ and
$r = \mathrm{rank}(A)$. $\exists$ r columns $c_{j_1}, \dots, c_{j_r}$ such that $\{c_{j_1}, \dots, c_{j_r}\}$ is linearly independent.
Let A_r be the submatrix
$(c_{j_1} \dots c_{j_r})$ of order $m \times r$ of A.
$r = \dim L(\{c_{j_1}, \dots, c_{j_r}\})$
Therefore, $\mathrm{rank}(A) = r$
$\Rightarrow \exists$ $R_{i_1}, \dots, R_{i_r}$ rows of A_r such that

$\{R_{i_1}, \ldots, R_{i_r}\}$ is linearly independent

Let $M = (a_{i_p j_q})$, $1 \le p \le r$, $1 \le q \le r$.
The rows of M are 'r' linearly independent row vectors of the matrix A_r
Hence M is a square matrix of order $r \times r$ and $\operatorname{rank}(M) = r$
By previous theorem, $\det(M) \neq 0$.

Let N be a submatrix of order $(r+k) \times (r+k)$ of A

Claim:- $\det(N) = 0$

If $\det(N) \neq 0$ then by previous theorem $\operatorname{rank}(N) = r+k$. Hence, all column vectors of N are L.I.

N is a submatrix of A, let $C_{l_1}, \ldots, C_{l_{r+k}}$ be columns of A coming from the matrix N. Columns of N are L.I. $\Rightarrow C_{l_1}, \ldots, C_{l_{r+k}}$ are L.I.

This contradicts any $r+k$ column vectors of A is linearly dependent.

Theorem: Let A be a matrix of order $m \times n$. $\text{Rank}(A) = r$ if and only if there exists a submatrix M of A of order $r \times r$ such that $\det(M) \neq 0$ & if N is a submatrix of order $(r+k) \times (r+k)$ then $\det(N) = 0$.

Proof: We have proved that if $\text{Rank}(A) = r$ then $\exists$ a submatrix M of A of order $r \times r$ such that $\det(M) \neq 0$ & if N is a submatrix of order $(r+k) \times (r+k)$ then $\det(N) = 0$.
Conversely, suppose the condition holds. If $\text{rank}(A) > r$ then by previous argument $\exists$ a submatrix Q of order $(r+k) \times (r+k)$, where $r+k = \text{rank}(A)$, such that $\det(Q) \neq 0$ which contradicts the hypothesis. If $\text{rank}(A) < r$ then the matrix M of order $r \times r$ has determinant 0. ($\det(M) = 0$). It contradicts $\det(M) \neq 0$. $\boxed{}$.