and then equate them to zero and solve.

Osborne's method:

we will convert the problem in terms of g_0, g_1, g_2 where g_0, g_1, g_2 have defined as before.

$$\underset{n\times 2}{X}(\beta) = \begin{bmatrix} e^{\beta_1} & e^{\beta_2} \\ \vdots & \vdots \\ e^{n\beta_1} & e^{n\beta_2} \end{bmatrix}$$

$$\underset{(n-2)\times n}{G} = \begin{bmatrix} g_0 & g_1 & g_2 & 0 & - & - & 0 \\ 0 & g_0 & g_1 & g_2 & - & - & 0 \\ \vdots & & & & & & \vdots \\ 0 & - & - & - & - & - 0 & g_0 \; g_1 \; g_2 \end{bmatrix}_{(n-2)\times n}$$

here, $\text{Rank}(G) = n-2$

Note that $G^T X(\beta) = 0$

$$P_X(\beta) = X(\beta)\left[X^T(\beta)X(\beta)\right]^{-1}X^T(\beta)$$

$$\Rightarrow P_G(g) = G(g)\left(G^T(g)G(g)\right)^{-1}G^T(g)$$

so, $$S(g_0, g_1, g_2) = Y^T(I - P_{G(g)})Y$$

Note that $P_{G(g)} = P_{G^T(g)}$?

I need to maximize $Y^T(I - P_{G(g)})Y$ such that $g_0^2 + g_1^2 + g_2^2 = 1$

I need to minimize $Y^T P_{G(g)} Y$ such that $g^T g = 1$.

we write $G^T(g)$ as follows:

$$U_0 = \begin{bmatrix} 1 & 0 & - & - & 0 \\ 0 & 1 & - & - & 0 \\ \vdots & & & & \vdots \\ 0 & - & - & 1 & 0\,0 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0 & 1 & - & - & 0 \\ 0 & 0 & 1 & - & 0 \\ \vdots & & & & \vdots \\ 0 & - & - & 1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & 1 & - & - & 0 \\ 0 & 0 & 0 & 1 & - & 0 \\ \vdots & & & & & \vdots \\ 0 & & - & - & & 1 \end{bmatrix}$$

Replace g_0 by 1 else 0   Replace g_1 by 1   Replace g_2 by 1.

$$\Rightarrow G^T(g) = \left(g_0 U_0^T + g_1 U_1^T + g_2 U_2^T\right)$$

we need to minimize $$Y^T P_G Y = Y^T G(G^T G)^{-1} G^T Y$$
$$= Y^T\left(\sum_{i=0}^{2} g_i u_i\right)\left(\left(\sum_0^2 g_i u_i^T\right)\left(\sum_0^2 g_i u_i\right)\right)^{-1}\left(\sum_0^2 g_i u_i^T\right)Y$$

$$\frac{\partial}{\partial g_K}(Y^T P_G Y) = Y^T\left[\frac{\partial}{\partial g_K}\left(\sum g_i u_i\right)\right](G^T G)^{-1} G^T Y$$
$$+ Y^T G\left(\frac{\partial}{\partial g_K}(G^T G)^{-1}\right)G^T Y + Y^T G (G^T G)^{-1}\left(\frac{\partial}{\partial g_K} G^T\right)Y$$

Least square estimators:

$$y_t = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + e_t$$

$$\min \sum_{t=1}^{m} \left(y_t - \alpha_1 e^{\beta_1 t} - \alpha_2 e^{\beta_2 t} \right)^2$$

let $X = \begin{bmatrix} e^{\beta_1} & e^{\beta_2} \\ \vdots & \vdots \\ e^{m\beta_1} & e^{m\beta_2} \end{bmatrix}_{(m-2)\times m}$ $\Rightarrow$ $\min Y^T[I - P_X]Y$ where $P_X = X(\beta)(X^T(\beta)X(\beta))^{-1}X^T(\beta)$

$\Leftrightarrow \max Y^T P_X Y$ using Prony's equation.

$\Leftrightarrow \min Y^T P_G Y$ where $P_G = G(G^TG)^{-1}G^T$.

$G^T X = 0$

$[(m-2)\times m]^T (m-2)\times m$ $\quad G^T = \begin{bmatrix} g_1 & g_2 & g_3 & 0 & \cdots & 0 \\ 0 & g_1 & g_2 & g_3 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & 0 & g_1 & g_2 & g_3 \end{bmatrix}^T_{(m-2)\times m}$

Osborne's Algorithm:

$$Q(g_1, g_2, g_3) = Y^T G(G^TG)^{-1}G^TY$$

$\min Q(g_1, g_2, g_3)$ such that $g_1^2 + g_2^2 + g_3^2 = 1$.

Note that $G^T = g_1 U_1^T + g_2 U_2^T + g_3 U_3^T$ $\quad \rightarrow$ 1 where g_2 is there else 0.

$\underset{(m-2)\times m}{U_1^T} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 1 & 0 & 0 \end{bmatrix}$, $\underset{(m-2)\times m}{U_2^T} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix}$, $\underset{(m-2)\times m}{U_3^T} = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & 0 & 1 \end{bmatrix}$

$$Q(g) = Y^T \left(\sum_{i=1}^{3} g_i U_i \right) \left\{ \left(\sum g_i U_i^T \right) \left(\sum g_j U_j \right) \right\}^{-1} \left(\sum_{i=1}^{3} g_i U_i^T \right) Y$$

$$\frac{\partial Q(g)}{\partial g_k} = Y^T[U_k](G^TG)^{-1}G^TY + Y^TG\left\{ \frac{\partial}{\partial g_k}(G^TG)^{-1} \right\}G^TY + Y^TG(G^TG)^{-1}U_k^TY.$$

I want to find $\frac{\partial}{\partial g_k}\underbrace{(G^TG)}_{A}{}^{-1} \Rightarrow \frac{\partial}{\partial g_k}(A^{-1})$

$$\frac{\partial}{\partial g_k}(AA^{-1}) = \frac{\partial}{\partial g_k}(I)$$

$$\left(\frac{\partial}{\partial g_k} A \right) A^{-1} + A \left(\frac{\partial}{\partial g_k} A^{-1} \right) = 0$$

$$\left(\frac{\partial}{\partial g_k} A^{-1} \right) = - A^{-1} \left(\frac{\partial}{\partial g_k} A \right) A^{-1}$$

$$\Rightarrow \frac{\partial}{\partial g_k}(G^TG)^{-1} = -(G^TG)^{-1} \left[\frac{\partial}{\partial g_k}(G^TG) \right] (G^TG)^{-1}$$

$$\Rightarrow \quad \frac{\partial}{\partial g_K}(G^TG)^{-1} = -(G^TG)^{-1}\left[U_K^TG + G^TU_K\right](G^TG)^{-1}.$$

$\mapsto$ condition $g_1^2+g_2^2+g_3^2=1$ is already there.

so, $$\frac{\partial Q(g)}{\partial g_K} = Y^T[U_K](G^TG)^{-1}G^TY + Y^TG(G^TG)^{-1}U_K^TY - Y^TG(G^TG)^{-1}\left[U_K^TG + G^TU_K\right](G^TG)^{-1}G^TY = 0$$

$K = 1, 2, 3.$

we need to solve three equations simultaneously.

Note that the three equations can be written as follows:

$$\underset{3\times3}{B(g)}\ \underset{3\times1}{g} = 0 \qquad \text{where } g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \qquad (*)$$

The (j,K)th element of matrix $B(g)$ -

$$b_{jK} = Y^TU_K(G^TG)^{-1}U_j^TY - Y^TG(G^TG)^{-1}\left[U_K^TU_j + U_j^TU_K\right](G^TG)^{-1}G^TY + Y^TU_j(G^TG)^{-1}U_K^TY.$$

Note that $B(g)$ is symmetric matrix.

All the eigenvalues of $B(g)$ are real.

I need to solve $(*)$. if $\hat{g}$ is a solution of $(*)$ $\Rightarrow$ $\hat{g}$ is the eigenvector corresponds to 0 eigenvalue of $B(\hat{g})$.

This is known as non-linear eigenvalue problem. The following algorithm can be used to solve it.

Algorithm:

Step 1: Start with an initial guess $g^{(0)}$, normalize it.

Step 2: compute $B(g^{(0)})$ and obtain the eigenvector $g^{(1)}$ corresponds to the minimum (absolute sense) eigenvalue, normalize it.

$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Step 3: continue the process.

Suppose $\hat{g} = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$ is the solution then consider the two roots of $\hat{g}_3x^2 + \hat{g}_2x + \hat{g}_1 = 0$ & obtain $\hat{\beta}_1, \hat{\beta}_2$.

Two comments:

1) The same method works for general model: $y_t = \alpha_1e^{\beta_1t} + \alpha_2e^{\beta_2t} + \dots + \alpha_pe^{\beta_pt} + \epsilon_t$

2) The same method works for complex models also.

$$y_t = \alpha_1e^{\beta_1t} + \dots + \alpha_pe^{\beta_pt} + \epsilon_t$$

Here, $\alpha_1 \dots \alpha_p$, $\beta_1 \dots \beta_p$ can be complex numbers.

$$\alpha_1 = \alpha_{1R} + i\alpha_{1I} \ \dots \ \alpha_p = \alpha_{pR} + i\alpha_{pI}, \quad \beta_1 = \beta_{1R} + i\beta_{1I} \ \dots \ \beta_p = \beta_{pR} + i\beta_{pI}$$

$$g_1\mu_1 + \dots + g_{p+1}\mu_{p+1} = 0 \qquad \text{where } \mu_t = \alpha_1e^{\beta_1t} + \dots + \alpha_pe^{\beta_pt}.$$

$$\vdots$$

$$g_1\mu_{n-p} + \dots\ g_{p+1}\mu_n = 0$$

Another important model:

$$y_t = \sum_{k=1}^{p}\left\{A_K\cos(\omega_Kt) + B_K\sin(\omega_Kt)\right\} + \epsilon_t$$

ϵ_t's are iid $(0, \sigma^2)$

A_K's, B_K's are real numbers

$0 < \omega_K < 2\pi$

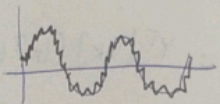
$K = 1, 2, \dots p$

The problem is to estimate $A_1, A_2, \dots A_p, B_1 \dots B_p, \omega_1 \dots \omega_p$ from $\{y_1 \dots y_n\}$.

we will consider $p=1$ for the time being:

$$y_t = A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) + \epsilon_t \quad \longleftarrow \quad y_t = 2\cos\left(\frac{\pi}{4}t\right) + 3\sin\left(\frac{\pi}{4}t\right) + \epsilon_t$$

we denote $A^\circ, B^\circ, \omega^\circ$ as the true values of the parameters.



$\rightarrow$ fluctuations due to additive noise (ϵ_t).

29/10/24 Model: $y_t = A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) + \epsilon_t$

A°, B° and ω° are the true values and we donot know these values, ϵ_t's are iid with mean 0 and variance σ^2.

We want to estimate $A^\circ, B^\circ, \omega^\circ$ based on sample $\{y_1, y_2, \ldots y_n\}$.

we would like to look at the properties of these estimators.

Properties: $\hat{A}, \hat{B}$ and $\hat{\omega}$ are estimators of A°, B° and ω°, then what will happen to $\hat{A}, \hat{B}$ and $\hat{\omega}$ as sample size increases??

Note that $\hat{A}, \hat{B}$ and $\hat{\omega}$ are all random variables.

we will be using the following facts:

(I) $$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos^2(\omega t) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin^2(\omega t) = \frac{1}{2}, \quad \omega \neq 0$$

(II) $$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin(\omega t) = 0, \quad \omega \neq 0$$

(III) $$\lim_{n\to\infty} \frac{1}{n}\sum \sin(\omega t)\cos(\omega t) = 0$$

$$y_t = A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) + \epsilon_t$$

$$Q(A,B,\omega) = \sum_{t=1}^{n} \left(y_t - A\cos(\omega t) - B\sin(\omega t)\right)^2$$

$\hat{A}, \hat{B}, \hat{\omega}$ can be obtained by minimizing $Q(A,B,\omega)$ w.r.t A, B, ω.

least square estimators of A, B, ω

Note that exactly same as before

$$Y = X(\omega^\circ)\beta^\circ + e \qquad \text{where } Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

$$Q(A,B,\omega) = (Y - X(\omega)\beta)^T (Y - X(\omega)\beta) \qquad X(\omega^\circ) = \begin{bmatrix} \cos(\omega^\circ) & \sin(\omega^\circ) \\ \vdots & \vdots \\ \cos(n\omega^\circ) & \sin(n\omega^\circ) \end{bmatrix}$$

$$\hat{\beta}(\omega) = [X^T(\omega)X(\omega)]^{-1} X^T(\omega) Y$$

$$Q(\hat{A}(\omega), \hat{B}(\omega), \omega) = Y^T\left[I - X(\omega)[X^T(\omega)X(\omega)]^{-1}X^T(\omega)\right]Y$$

$$B = \begin{bmatrix} A^\circ \\ B^\circ \end{bmatrix}, \qquad e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

we can obtain the least squares estimator of ω by minimizing

$$R(\omega) = Y^T X(\omega)[X^T(\omega)X(\omega)]^{-1}X^T(\omega)Y$$
$$= \frac{1}{n} Y^T X(\omega)\left[\frac{1}{n}X^T(\omega)X(\omega)\right]^{-1} X^T(\omega) Y$$

$$R(\omega) = \frac{1}{n} Y^T X(\omega) \left[\frac{1}{n} X^T(\omega) X(\omega) \right]^{-1} X^T(\omega) Y$$

$$X^T(\omega)X(\omega) = \begin{bmatrix} \cos(\omega) & \cdots & \cos(n\omega) \\ \sin(\omega) & \cdots & \sin(n\omega) \end{bmatrix} \begin{bmatrix} \cos(\omega) & \sin(\omega) \\ \vdots & \vdots \\ \cos(n\omega) & \sin(n\omega) \end{bmatrix}$$

$$\frac{1}{n} X^T(\omega)X(\omega) = \begin{bmatrix} \frac{1}{n}\sum_{t=1}^{n} \cos^2(\omega t) & \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t)\sin(\omega t) \\ \frac{1}{n}\sum_{t=1}^{n} \cos(\omega t)\sin(\omega t) & \frac{1}{n}\sum_{t=1}^{n} \sin^2(\omega t) \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{1}{n} X^T(\omega)X(\omega) \to \frac{1}{2} I$$

For large n, $R(\omega) \approx I(\omega) = \frac{2}{n} Y^T X(\omega) X^T(\omega) Y$

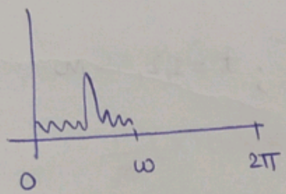
we obtain approximate least squares estimators of ω by maximizing $I(\omega) = \frac{2}{n} Y^T X(\omega) X^T(\omega) Y$

$$I(\omega) = \frac{2}{n} \left[\sum y_t \cos(\omega t) \quad \sum y_t \sin(\omega t) \right] \begin{bmatrix} \sum y_t \cos(\omega t) \\ \sum y_t \sin(\omega t) \end{bmatrix}$$

$$I(\omega) = \frac{2}{n} \left\{ \left[\sum y_t \cos(\omega t) \right]^2 + \left[\sum y_t \sin(\omega t) \right]^2 \right\}$$

↑ Periodogram function

suppose we plot ω vs $I(\omega)$



Let's look at the behavior of $I(\omega)$ for large n.

$$\cancel{y_t} = \frac{2}{n} \left[\sum_{t=1}^{n} y_t \cos(\omega t) \right]^2 + \frac{2}{n} \left[\sum_{t=1}^{n} y_t \sin(\omega t) \right]^2$$

$$= \frac{2}{n} \left\{ \left[\sum_{t=1}^{n} \left(A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) + \epsilon_t \right) \cos \omega t \right]^2 + \left[\sum \left(A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) + \epsilon_t \right) \cdot \sin(\omega t) \right]^2 \right\}$$

$$= \frac{2}{n} \left(\sum_{t=1}^{n} \left(A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) \right) \cos \omega t + \sum_{t=1}^{n} \epsilon_t \cos(\omega t) \right)^2$$

note that the approximate least squares estimators can be obtained by maximizing $I(\omega)$ $\Leftrightarrow$ maximizing $\frac{1}{n} I(\omega)$.

$$\Rightarrow \quad 2 \underbrace{\left[\frac{1}{n} \sum_{t=1}^{n} y_t \cos(\omega t) \right]^2}_{J_1(\omega)} + 2 \underbrace{\left[\frac{1}{n} \sum_{t=1}^{n} y_t \sin(\omega t) \right]^2}_{J_2(\omega)} = \frac{1}{2} \left(J_1^2(\omega) + J_2^2(\omega) \right)$$

$$\Rightarrow \frac{1}{n} \sum_{t=1}^{n} y_t \cos(\omega t) = \frac{1}{n} \sum_{t=1}^{n} \left(A^\circ \cos(\omega^\circ t) + B^\circ \sin(\omega^\circ t) + \epsilon_t \right) \cos(\omega t)$$

$$= A^\circ \frac{1}{n}\sum_{t=1}^{n} \cos(w^\circ t)\cos(wt) + \frac{B^\circ}{n}\sum \sin(w^\circ t)\cos(wt) + \frac{1}{n}\sum_{t=1}^{n} \epsilon_t \cos(wt)$$

$\downarrow$ $\cancel{J_1(w) = \frac{A^\circ}{2}}$ $\quad$ $\downarrow$ 0 $\quad$ $\downarrow$ 0 when $w = w^\circ$

$\cancel{J_1(w) = 0}$ $\quad$ 0 $\quad$ 0 when $w \neq w^\circ$

look at $E\left(\frac{1}{n}\sum_{t=1}^{n} \epsilon_t \cos(w^\circ t)\right) = 0$

$$V\left(\frac{1}{n}\sum_{t=1}^{n} \epsilon_t \cos(w^\circ t)\right) = \frac{1}{n^2} V\left(\sum_{t=1}^{n} \epsilon_t \cos(w^\circ t)\right) = \frac{\sigma^2}{n^2}\sum_{t=1}^{n} \cos^2(w^\circ t) \to 0$$

$$\Rightarrow \quad J_1(w) = \begin{cases} \left(\frac{A^\circ}{2}\right) & \text{if } w = w^\circ \\ 0 & \text{if } w \neq w^\circ \end{cases}$$

Exactly the same way, $J_2(w) = \begin{cases} \left(\frac{B^\circ}{2}\right) & \text{if } w = w^\circ \\ 0 & \text{if } w \neq w^\circ \end{cases}$

4/11/24 **Sinosoidal Model (one component):**

$$y_t = A^\circ \cos(w^\circ t) + B^\circ \sin(w^\circ t) + \epsilon_t$$

$\{y_1, y_2 \ldots y_n\}$ $\qquad$ $\epsilon_t \overset{iid}{\sim} (0, \sigma^2)$

we want to estimate $A^\circ, B^\circ, w^\circ$ based on sample $\{y_1, y_2, \ldots y_n\}$.

Multiple component Sinosoidal model:

$$y_t = \sum_{k=1}^{p} \{A_k^\circ \cos(w_k^\circ t) + B_k^\circ \sin(w_k^\circ t)\} + \epsilon_t \; ; \; t = 1, 2, \ldots n$$

$\{y_1, y_2, \ldots y_n\}$ $\quad$ $\epsilon_t \overset{iid}{\sim} (0, \sigma^2)$

we want to estimate $\{(A_k^\circ, B_k^\circ, w_k^\circ); k = 1, 2, \ldots p\}$ often in practice we need to estimate p al

we have $3p$ unknown parameters.

we will consider the most intuitive least square estimators for notational purposes we take $p=2$

$$y_t = \sum_{k=1}^{2} \{A_k^\circ \cos(w_k^\circ t) + B_k^\circ \sin(w_k^\circ t) + \epsilon_t$$

we will write it in matrix notation:

$$Y = X(w_1^\circ, w_2^\circ)\beta^\circ + \epsilon_t$$

$$Q(A_1, B_1, w_1, A_2, B_2, w_2) = [Y - X(w_1, w_2)\beta]^T [Y - X(w_1, w_2)\beta]$$

where $Y_{n\times 1} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$ $\quad \underset{\sim}{\beta} = \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix}$ $\quad \underset{n\times 4}{X(\omega_1, \omega_2)} = \begin{bmatrix} \cos(\omega_1) & \sin(\omega_1) & \cos(\omega_2) & \sin(\omega_2) \\ \cos(2\omega_1) & - & - & \sin(2\omega_2) \\ \vdots & & & \vdots \\ \cos(n\omega_1) & - & - & \sin(n\omega_2) \end{bmatrix}$

For fixed ω_1, ω_2

$$\hat{\beta}(\omega_1, \omega_2) = \left[X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) \right]^{-1} X^T(\omega_1, \omega_2) Y$$

minimizes $(*)$ w.r.t β.

Therefore the least square estimators of ω_1, ω_2 can be obtained by minimizing

$$Q(\hat{A}_1(\omega_1, \omega_2), \hat{B}_1(\omega_1, \omega_2), \omega_1, \hat{A}_2(\omega_1, \omega_2), \hat{B}_2(\omega_1, \omega_2), \omega_2) = \left[Y - X(\omega_1, \omega_2)(X^T(\omega_1, \omega_2) X(\omega_1, \omega_2))^{-1} \cdot X^T(\omega_1, \omega_2) Y \right]^T [\_\_]$$

$$I(\omega_1, \omega_2) = Y^T \left[I - P_X(\omega_1, \omega_2) \right]^T \left[I - P_X(\omega_1, \omega_2) \right] Y$$

$$= Y^T \left[I - P_X(\omega_1, \omega_2) \right] Y$$

$$P_X(\omega_1, \omega_2) = X(\omega_1, \omega_2) \left(X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) \right)^{-1} X^T(\omega_1, \omega_2)$$

First we maximize $Y^T P_X(\omega_1, \omega_2) Y$ w.r.t ω_1, ω_2 if $\hat{\omega}_1, \hat{\omega}_2$ maximize $Y^T P_X(\omega_1, \omega_2) Y$ then least squares estimators $\hat{\beta}(\hat{\omega}_1, \hat{\omega}_2)$.

In this particular case $(p=2)$ the least squares estimators can be obtained by optimizing a two dimensional problem. In general we need to solve a p-dimensional optimization problem. we also need a $2p \times 2p$ matrix inverse computation.

we observe again the matrix $X^T(\omega_1, \omega_2) X(\omega_1, \omega_2)$

$$\frac{1}{n}\left(X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) \right) = \begin{bmatrix} \frac{1}{n} \sum_{t=1}^{n} \cos^2(\omega_1 t) & \frac{1}{n} \sum \cos(\omega_1 t) \sin(\omega_1 t) & - & - \\ & & & \\ & - & - & - \\ & & & \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n} \sum_{t=1}^{n} \cos^2(\omega_1 t) & \frac{1}{n} \sum_{t=1}^{n} \cos(\omega_1 t) \sin(\omega_1 t) & \frac{1}{n} \sum_{t=1}^{n} \cos(\omega_1 t) \cos(\omega_2 t) & \frac{1}{n} \sum_{t=1}^{n} \cos(\omega_1 t) \sin(\omega_2 t) \\ & \frac{1}{n} \sum \sin^2(\omega_1 t) & & \\ & & \frac{1}{n} \sum \cos^2(\omega_2 t) & \\ & & & \frac{1}{n} \sum \sin^2(\omega_2 t) \end{bmatrix}$$

$$\lim_{n \to \infty} \frac{1}{n} \left[X^T(\omega_1, \omega_2) X(\omega_1, \omega_2) \right] = \frac{1}{2} I_4$$

for p, $\frac{1}{2} I_{2p}$.

we need to maximize

$$I(\omega_1,\omega_2) = Y^T P_X(\omega_1,\omega_2) Y$$

$$= Y^T X(\omega_1,\omega_2)\left(X^T(\omega_1,\omega_2)X(\omega_1,\omega_2)\right)^{-1} X^T(\omega_1,\omega_2) Y$$

$$(\hat{\omega}_1,\hat{\omega}_2) = \arg\max I(\omega_1,\omega_2)$$

$$= \arg\max \left\{ \frac{1}{n} Y^T X(\omega_1,\omega_2)\left(X^T(\omega_1,\omega_2)X(\omega_1,\omega_2)\right)^{-1} X^T(\omega_1,\omega_2) Y \right\}$$

$$= \arg\max \frac{1}{n} Y^T X(\omega_1,\omega_2)\left(\frac{1}{n} X^T(\omega_1,\omega_2)X(\omega_1,\omega_2)\right)^{-1} \frac{1}{n} X^T(\omega_1,\omega_2) Y.$$

so for large n

$$(\hat{\omega}_1,\hat{\omega}_2) = \arg\max \frac{1}{n^2} Y^T X(\omega_1,\omega_2) X^T(\omega_1,\omega_2) Y.$$

$$= \frac{1}{n^2}\left[\left(\sum_{t=1}^{n} y_t\cos(\omega_1 t)\right)^2 + \left(\sum y_t \sin(\omega_1 t)\right)^2 + \left(\sum_{t=1}^{n} y_t\cos(\omega_2 t)\right)^2 + \left(\sum y_t \sin(\omega_2 t)\right)^2\right]$$

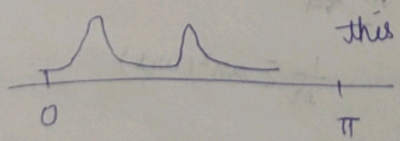
$$(\hat{\omega}_1,\hat{\omega}_2) = \arg\max \left\{\frac{1}{n^2}\left(\left(\sum y_t\cos(\omega_1 t)\right)^2 + \left(\sum y_t \sin(\omega_1 t)\right)^2\right) + \frac{1}{n^2}\left[\left(\sum y_t\cos(\omega_2 t)\right)^2 + \left(\sum y_t\sin(\omega_2 t)\right)^2\right]\right\}.$$

$$\hat{\omega}_1 = \arg\max \left\{ \frac{1}{n^2}\left[\left(\sum y_t\cos(\omega_1 t)\right)^2 + \left(\sum y_t\sin(\omega_1 t)\right)^2\right]\right\}$$

$$\hat{\omega}_2 = \arg\max \left\{ \frac{1}{n^2}\left[\left(\sum y_t\cos(\omega_2 t)\right)^2 + \left(\sum y_t\sin(\omega_2 t)\right)^2\right]\right\}$$

Therefore, instead of solving a 2-dimensional optimization problem, we need to solve two one-dimen-sional optimization problem.

$$J(\omega) = \frac{1}{n^2}\left\{\left(\sum y_t\cos(\omega t)\right)^2 + \left(\sum y_t\sin(\omega t)\right)^2\right\}$$



this $J(\omega)$ will have two local maxima.

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$$y_t = A_1^\circ \cos(\omega_1^\circ t) + A_2^\circ \cos(\omega_2^\circ t) + B_1 \sin(\omega_1^\circ t) + B_2 \sin(\omega_2^\circ t) + \epsilon(t)$$

$$\sum_{t=1}^{n} \{y_t - (\,)\}^2$$

For large n, the least square estimators of ω_1° and ω_2° can be obtained by maximizing

$$I_1(\omega_1) = \left(\frac{1}{n}\sum y_t \cos(\omega_1 t)\right)^2 + \left(\frac{1}{n}\sum y_t \sin(\omega_1 t)\right)^2$$

$$I_2(\omega_2) = \left(\frac{1}{n}\sum y_t \cos(\omega_2 t)\right)^2 + \left(\frac{1}{n}\sum y_t \sin(\omega_2 t)\right)^2$$

$$I(\omega) = \left(\frac{1}{n}\sum y_t \cos(\omega t)\right)^2 + \left(\frac{1}{n}\sum y_t \sin(\omega t)\right)^2 \geq 0 \quad (*)$$

Actually the function $I(\omega)$ has two local maximum one near ω_1° and the other one near ω_2°.

→ we will show that $\lim_{n\to\infty} I(\omega_1^\circ) > \cancel{\sigma^2}\ 0$ and $\lim_{n\to\infty} I(\omega_2^\circ) > \cancel{\sigma^2}\ 0$

$$\lim_{n\to\infty} (I(\omega)) = \cancel{\sigma^2}\ 0 \text{ if } \omega \neq \omega_1^\circ \text{ or } \omega_2^\circ .$$

Note that σ^2 is the error variance.

→ Prove:

$$\frac{1}{n}\sum y_t \cos(\omega t) = \frac{1}{n}\left[\sum\left\{\left(A_1^\circ\cos(\omega_1^\circ t) + B_1^\circ \sin(\omega_1^\circ t) + A_2^\circ \cos(\omega_2^\circ t) + B_2^\circ \sin(\omega_2^\circ t) + \epsilon_t\right\}\cos(\omega t)\right]$$

$$= A_1^\circ \frac{1}{n}\sum \cos(\omega_1^\circ t)\cos(\omega t) + B_1^\circ \frac{1}{n}\sum \sin(\omega_1^\circ t)\cos(\omega t) + A_2^\circ \frac{1}{n}\sum \cos(\omega_2^\circ t)\cos(\omega t) + B_2^\circ \frac{1}{n}\sum \sin(\omega_2^\circ t)\cos(\omega t) + \frac{1}{n}\sum \epsilon_t \cos(\omega t)$$

suppose $\omega = \omega_1^\circ$

as $n\to\infty$, $\quad \dfrac{A_1^\circ}{2} + 0 + 0 + 0 + 0 + 0$

$$E\left(\frac{1}{n}\sum \epsilon_t \cos(\omega t)\right) = 0$$

$$V\left(\frac{1}{n}\sum_{t=1}^{n} \epsilon_t \cos(\omega_1^\circ t)\right) = \frac{\sigma^2}{n^2}\sum \cos^2(\omega_1^\circ t)$$

$$\frac{1}{n}\sum \cos(\omega_1 t)\cos(\omega_2 t) = 0$$

write $\cos(\omega t) = \dfrac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2}$

$$\lim_{n\to\infty} V\left(\frac{1}{n}\sum \epsilon_t \cos(\omega_1^\circ t)\right) = 0$$

then it will become geometric series with finite summation.

so, $\dfrac{1}{n}\sum_{t=1}^{n} \epsilon_t \cos(\omega_1^\circ t) \xrightarrow{p} 0$ (chebyshev's inequality)

$$P\left[\left|\frac{1}{n}\sum \epsilon_t \cos(\omega_1^\circ t)\right| > \epsilon\right] \to 0 \ \forall \epsilon > 0$$

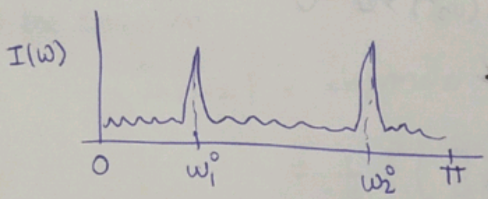
so, $\left(\dfrac{1}{n}\sum y_t \cos(\omega_1^\circ t)\right)^2 \to \left(\dfrac{A_1^{\circ 2}}{4}\right)$ and $\left(\dfrac{1}{n}\sum y_t \sin(\omega_1^\circ t)\right)^2 \to \left(\dfrac{B_1^{\circ 2}}{4}\right)$

$$I(\omega) = \left(\frac{1}{n}\sum y_t \cos(\omega t)\right)^2 + \left(\frac{1}{n}\sum y_t \sin(\omega t)\right)$$

$$I(\omega_1^\circ) \to \left(\frac{A_1^{\circ 2}}{4} + \frac{B_1^{\circ 2}}{4}\right) \quad \text{and} \quad I(\omega_2^\circ) \to \left(\frac{A_2^{\circ 2}}{4} + \frac{B_2^{\circ 2}}{4}\right)$$

$$I(\omega) \to 0 .$$

$$\lim I(\omega) = \begin{cases} \frac{1}{4}(A_1^{\circ 2} + B_1^{\circ 2}) & \text{if } \omega = \omega_1^\circ \\ \frac{1}{4}(A_2^{\circ 2} + B_2^{\circ 2}) & \text{if } \omega = \omega_2^\circ \\ 0 & \text{otherwise} \end{cases}$$



periodogram

In practice, we choose initial values at $\left(0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots \frac{n\pi}{n}\right)$.

you compute $I\left(\frac{i\pi}{n}\right)$ for $i = 0, 1, \ldots n$

choose that i for which $I\left(\frac{i\pi}{n}\right)$ is the maximum. you start your initial guess from that $\frac{i\pi}{n}$.

Problems: (1) If sample size is not very large, then we do not know how $I(\omega)$ behaves??
(we need to do simulations).

(11) If ω_1° and ω_2° are very close, there won't be two different peaks.

when p is also unknown:

Suppose I have $\quad y_t = \sum_{k=1}^{p} \left(A_k^\circ \cos(\omega_k^\circ t) + B_k^\circ \sin(\omega_k^\circ t)\right) + \epsilon_t$

one way is to check number of peaks. but what if sample size is not that large i.e. peaks have been merged?

suppose p is known(unknown), we often use sequential estimators as follows:

$$y_t = \sum_{k=1}^{2} \left(A_k^\circ \cos(\omega_k^\circ t) + B_k^\circ \sin(\omega_k^\circ t)\right) + \epsilon_t$$

looking at only one component:

$$R(A, B, \omega) = \sum_{t=1}^{n} \left(y_t - A\cos(\omega t) - B\sin(\omega t)\right)^2$$

we want to minimize $R(A, B, \omega)$ w.r.t A, B, ω.

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} \cos(\omega) & \sin(\omega) \\ \vdots & \vdots \\ \cos(n\omega) & \sin(n\omega) \end{bmatrix}, \quad \beta = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$R(A, B, \omega) = [Y - X(\omega)\beta]^T [Y - X(\omega)\beta]$$

$$\hat{\beta} = \left(X^T(\omega)X(\omega)\right)^{-1} X^T(\omega) Y .$$

$$(Y - X(\omega)\beta)^T (Y - X(\omega)\beta)$$

we can estimate ω by maximizing $Y^T X(\omega)\left(X^T(\omega) X(\omega)\right)^{-1} X^T(\omega) Y$

Multiple Sinosoidal Model :

$$Q(A,B,\omega) = \sum_{t=1}^{n} \left(y_t - A\cos(\omega t) - B\sin(\omega t) \right)^2$$

minimize $Q(A,B,\omega)$ w.r.t A, B, ω.

$$(\hat{A}, \hat{B}, \hat{\omega}) = \arg\min Q(A,B,\omega)$$
$$= \arg\min \frac{1}{n} Q(A,B,\omega)$$

$$\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega) = \frac{1}{n}\sum_{t=1}^{n}\left(y_t - A\cos(\omega t) - B\sin(\omega t) \right)^2$$

$$= \frac{1}{n}\sum_{t=1}^{n}\left(\sum_{k=1}^{p}\left(A_k^0 \cos(\omega_k^0 t) + B_k^0 \sin(\omega_k^0 t)\right) + \epsilon_t - A\cos(\omega t) - B\sin(\omega t) \right)^2$$

$$\lim \frac{1}{n} Q(A,B,\omega) = \frac{1}{n}\left\{ \left[\sum_{t=1}^{n} A_1^{0\,2} \cos^2(\omega_1^0 t) + \cdots + \sum_{t=1}^{n} A_p^{0\,2}\cos^2(\omega_p^0 t) \right] \right.$$
$$+ \left[\sum_{t=1}^{n} B_1^{0\,2} \sin^2(\omega_1^0 t) + \cdots \sum_{t=1}^{n} B_p^{0\,2} \sin^2(\omega_p^0 t) \right]$$
$$+ \left[\frac{1}{n}\sum \epsilon_t^2 + \frac{1}{n}\sum_{t=1}^{n} A^2\cos^2(\omega t) + \frac{1}{n}\sum_{t=1}^{n} B^2 \sin^2(\omega t) \right]$$

+ cross product terms .

$$\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega) = \underbrace{\frac{1}{2}\left[\sum_{k=1}^{p}\left(A_k^{0\,2} + B_k^{0\,2} \right) \right] + \frac{1}{2}(A^2+B^2) + \sigma^2}_{\text{not dependent on } \omega} + \underbrace{\text{cross product terms}}_{\to 0 \text{ on division by } n?}$$

let's check !

cross product terms :

$$\frac{1}{n}\sum_{t=1}^{n} A A_k^0 \cos(\omega_k^0 t)\cos(\omega t) = A A_k^0 \frac{1}{n}\sum_{t=1}^{n} \cos(\omega_k^0 t)\cos(\omega t) = \begin{cases} 0 & \text{if } \omega \neq \omega_k^0 \\ \frac{1}{2} A A_k^0 & \text{if } \omega = \omega_k^0 \end{cases}$$

for $\forall\, k = 1, 2, \ldots p$.

and $$\frac{1}{n}\sum_{t=1}^{n} B B_k^0 \sin(\omega_k^0 t)\sin(\omega t) = B B_k^0 \frac{1}{n}\sum_{t=1}^{n} \sin(\omega_k^0 t)\sin(\omega t) = \begin{cases} 0 & \text{if } \omega \neq \omega_k^0 \\ \frac{1}{2} B B_k^0 & \text{if } \omega = \omega_k^0 \end{cases}$$

for $\forall\, k = 1, 2, \ldots p$.

and all rest of the terms will precisely be zero .

If $\omega \neq \omega_k^0$ for $k = 1, 2, \ldots p$

$$\beta = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega) = \frac{1}{2}\left[\sum_{k=1}^{p} \left(A_k^{0\,2} + B_k^{0\,2} \right) \right] + \frac{1}{2}(A^2+B^2) + \sigma^2$$

If $\omega = \omega_k^0$ for som k.

$$\lim_{n\to\infty} \frac{1}{n} Q(A,B,\omega) = \frac{1}{2}\left[\sum_{k=1}^{p} \left(A_k^{0\,2} + B_k^{0\,2} \right) \right] + \frac{1}{2}(A^2+B^2) + \sigma^2 - \left(A A_k^0 + B B_k^0 \right)$$

without loss of generality we assume

$$(A_1^{o2} + B_1^{o2}) > (A_2^{o2} + B_2^{o2}) > \cdots > (A_p^o + B_p^o)$$

$$y_t = \sum_{k=1}^{p} \left(A_k^o \cos(w_k^o t) + B_k^o \sin(w_k^o t) \right) + \epsilon_t$$

observe that if we choose $w = w_1^o$, $A = A_1^o$, $B = B_1^o$

$$\lim_{n \to \infty} \frac{1}{n} Q(A_1^o, B_1^o, w_1^o) = \frac{1}{2} \left(\sum_{k=1}^{p} (A_k^{o2} + B_k^{o2}) \right) - \frac{1}{2}(A_1^{o2} + B_1^{o2}) + \sigma^2$$

$$\leq \lim_{n \to \infty} \frac{1}{n} Q(A, B, w)$$

claim: $\frac{1}{2}(A^2 + B^2) - (A A_1^o + B B_1^o) \geq -\frac{1}{2}(A_1^{o2} + B_1^{o2})$

$$\Leftrightarrow \cancel{\tfrac{1}{2}} (A^2 + B^2 + A_1^{o2} + B_1^{o2}) \geq 2(A A_1^o + B B_1^o)$$

$$(A - A_1^o)^2 + (B - B_1^o)^2 \geq 0$$

If we minimize $\lim_{n \to \infty} \frac{1}{n} Q(A, B, w)$, then the minimum occurs at (A_1^o, B_1^o, w_1^o).

the value of $\lim_{n \to \infty} \frac{1}{n} Q(A, B, w)$ will be $\frac{1}{2} \left[\sum_{k=1}^{p} (A_k^{o2} + B_k^{o2}) - \frac{1}{2}(A_1^{o2} + B_1^{o2}) + \sigma^2 \right]$

$$= \frac{1}{2} \left[\sum_{k=1}^{p} (A_k^{o2} + B_k^{o2}) \right] + \sigma^2$$

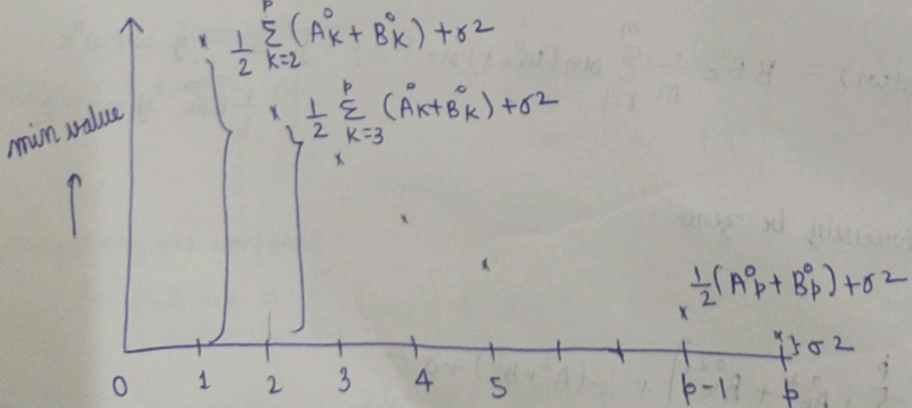
Next step will be: Once we get the estimate $(\hat{A}_1, \hat{B}_1, \hat{w}_1)$ then consider

$$\tilde{y}_t = (y_t - \hat{A}_1 \cos(\hat{w}_1 t) - \hat{B}_1 \sin(\hat{w}_1 t)); \; t = 1, 2, \ldots n$$

$$Q_2(A, B, w) = \sum_{t=1}^{n} (\tilde{y}_t - A \cos(wt) - B \sin(wt))^2$$

$\lim_{n \to \infty} \frac{1}{n} Q_2(A, B, w) - (**)$, The minimum of $(**)$ occurs at A_2^o, B_2^o, w_2^o.

The value of the minimum becomes $\frac{1}{2} \left(\sum_{k=2}^{p} (A_k^{o2} + B_k^{o2}) \right) + \sigma^2$.



$$Q_{p+1}(A, B, w) = \sum_{t=1}^{n} (\epsilon_t - A \cos(wt) - B \sin(wt))^2$$

$$= \frac{1}{n} \sum \epsilon_t^2 + A^2 \frac{1}{n} \sum \cos^2(wt) + B^2 \left(\frac{1}{n} \sin^2(wt) \right) + 0 = \frac{1}{n} \sum \epsilon_t^2 + \frac{1}{2}(A^2 + B^2)$$