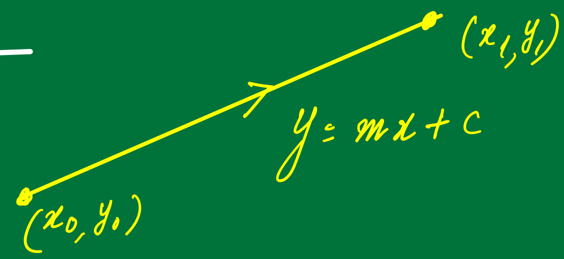
Second lecture

I. Length of a plane curve

II. Volume of a solid by slicing

I. Length of a plane curve (Smooth curve)



Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' - the derivative function is continuous.

Such a function f is said to be smooth function and its graph

$$\{ (x, f(x)) \in \mathbb{R}\times\mathbb{R} / x \in [a, b] \} \subseteq \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$$

is said to be a smooth curve.

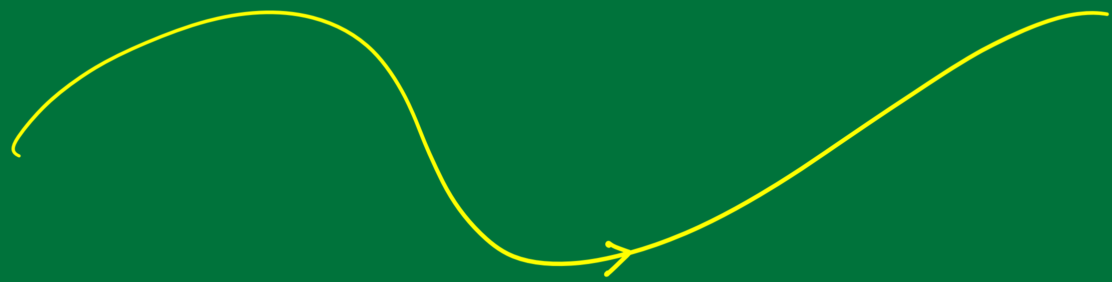
Example: $f : [0,4] \to \mathbb{R}$ defined as $f(x) = x^{3/2}$ for all $x \in [0,4]$.

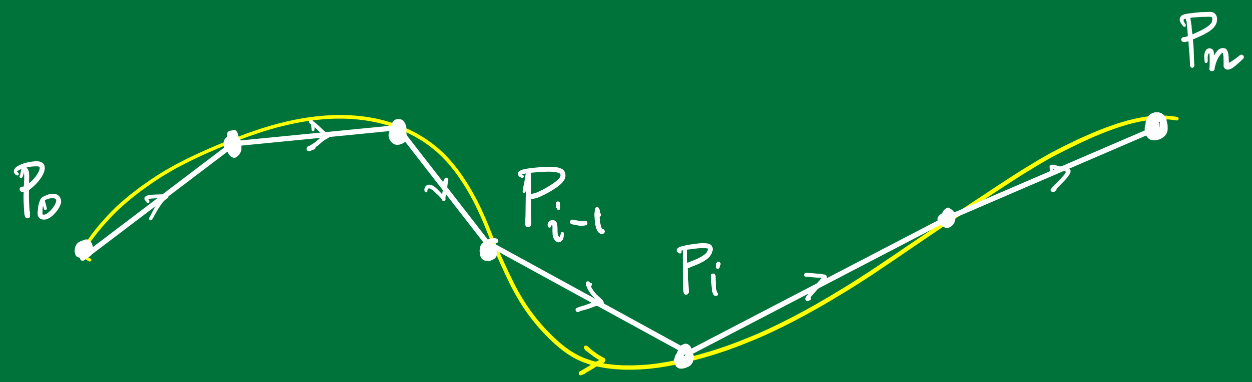
Here, the derivative function $f'(x) = \frac{3}{2} x^{1/2}$ is continuous.

(— Riemann integrable)

- We will describe the length of smooth curve in terms of an integral expression;

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

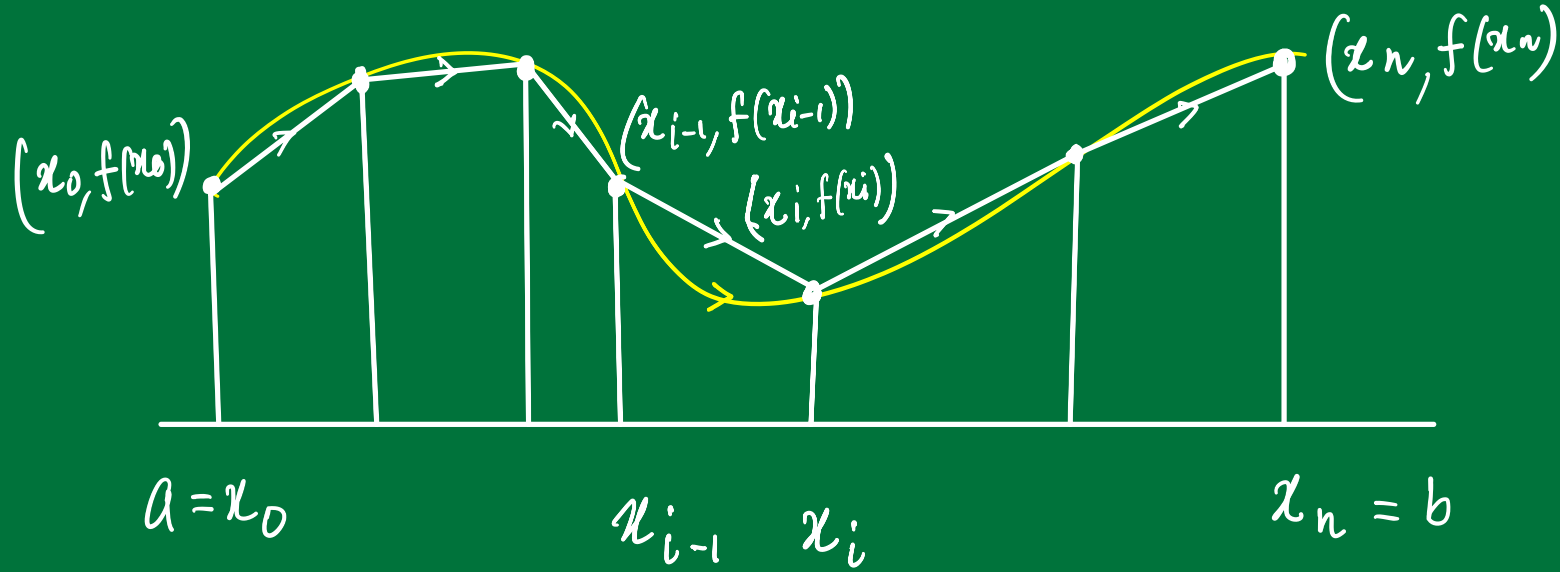




$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + \left(f(x_i) - f(x_{i-1})\right)^2}$$

is an approximation

$$\sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$



Let $P: a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$ be a partition of $[a, b]$.

Join $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ by a straight line.

Then we define the length L of the curve to be

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} |P_{i-1} P_i|$$

$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + f'(c_i)^2 \Delta x_i^2}$$

$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} \underbrace{\left(\sqrt{1 + f'(c_i)^2}\right) \Delta x_i}$$

[By MVT:

$$\Delta y_i = f(x_i) - f(x_{i-1})$$
$$= f'(c_i)(x_i - x_{i-1})$$

for some $c_i \in (x_{i-1}, x_i)$.

[$S(P, F)$;

$$F(x) = \sqrt{1 + f'(x)^2}$$

$$L = \int_a^b \sqrt{1+f'(x)^2}\, dx$$

Example (Continued): $f : [0,4] \to \mathbb{R}$ defined as $f(x) = x^{3/2}$ for all $x \in [0,4]$.

$$F(x) = \sqrt{1+f'(x)^2}$$

$$= \sqrt{1+\frac{9}{4}x}$$

is Riemann integrable on $[0,4]$

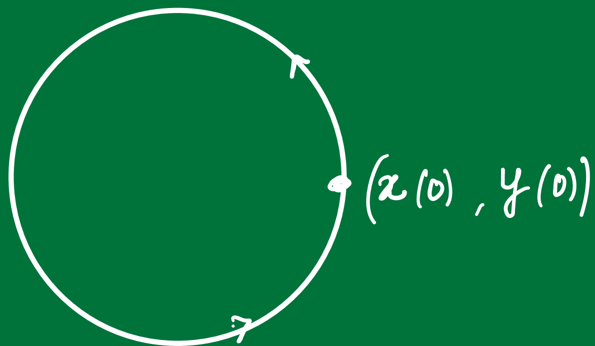
$$L = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx$$

$$= \frac{8}{27}\left(10^{3/2}-1\right)$$

II Parametric form :

$$[a,b] \to \mathbb{R}^2$$
$$t \mapsto \big(x(t), y(t)\big)$$

$$x : [a,b] \to \mathbb{R}$$
$$y : [a,b] \to \mathbb{R}$$



Example : $[0, 2\pi] \to \mathbb{R}^2$
$$t \mapsto (\cos t, \sin t)$$

$$\{ (x(t), y(t)) : t \in [0, 2\pi] \}$$

x, y are continuous functions.

Suppose a plane curve is given by the set of points $\left\{ (x(t), y(t)) \,/\, t \in [a,b] \text{ where } x, y : [a,b] \to \mathbb{R} \text{ are continuous functions} \right\}$ is called a parametric curve.

Smooth : Moreover, if the derivatives x', y' are continuous functions then such a curve is said to be smooth parametric curve (Plane curve).

For smooth parametric curve $(x(t), y(t))$

the length L is defined as

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

III Polar form

If a curve is given in the polar form:

$$\begin{cases} r = f(\theta) \text{ where } f(\theta) \geq 0 \\ \text{and } \alpha \leq \theta \leq \beta, \end{cases}$$

then the curve can also be expressed as the parametric form

$$\{ (x(\theta), y(\theta)) / \theta \in [\alpha, \beta] \}$$

In this case, the length

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

Here,

$$x(\theta) = r(\theta)\cos\theta = f(\theta)\cos\theta$$

$$y(\theta) = r(\theta)\sin\theta = f(\theta)\sin\theta$$

So,

$$x'(\theta) = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$y'(\theta) = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

and

$$\left(x'(\theta)\right)^2 + \left(y'(\theta)\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2.$$

Example: Find length of the curve given by the equation $x^{2/3} + y^{2/3} = 1$.

We consider the given curve in parametric form $\left\{ \left(\cos^3(t), \sin^3(t) \right) / t \in [0, 2\pi] \right\}$.

It is a smooth parametric curve:

$$x'(t) = -3\cos^2 t \sin t$$

$$y'(t) = 3\cos t \sin^2 t$$

are continuous functions

$$\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{9\cos^2 t \sin^2 t}$$

is Riemann integrable on $[0, \pi/2]$.

The length $L = 4 \times \int_0^{\pi/2} \sqrt{9 \sin^2 t \cos^2 t} \, dt$

$$= 6 \left[\frac{\cos 2t}{2} \right]_{\pi/2}^{0}$$

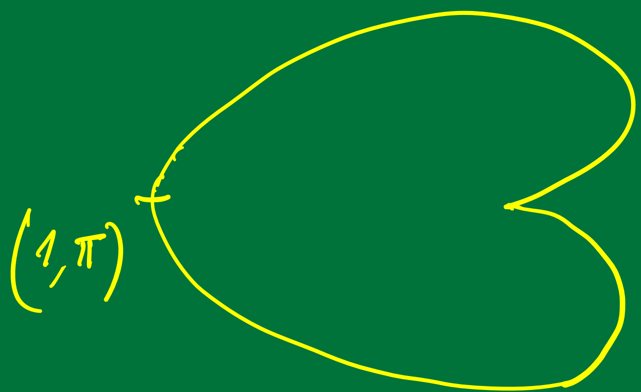
$$= 6$$

Example (2) Compute the length of the curve

$$r = \sin^2(\theta/2) = \frac{1}{2}(1-\cos\theta)$$

— Cardioid.

Hint:



$$\frac{dr}{d\theta} = \frac{1}{2}\sin\theta$$

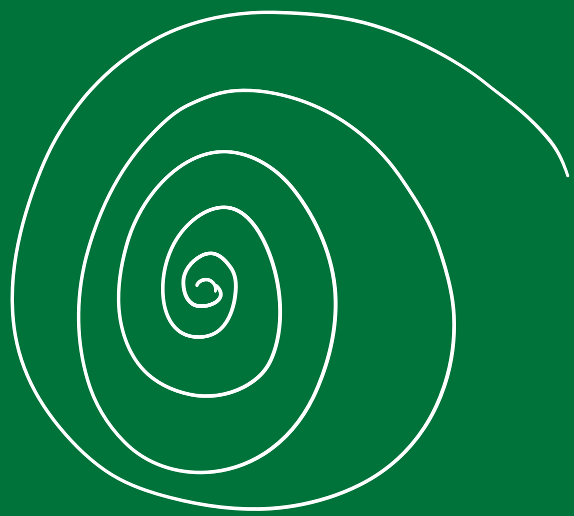
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \ldots = \sin^2(\theta/2)$$

$$L = 2\int_0^\pi \sin\frac{\theta}{2}\, d\theta$$

$$= \ldots$$

$$= 4.$$

Example 3. $r = e^{-\theta}$



$$\frac{dr}{d\theta} = -e^{-\theta}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2e^{-2\theta}$$

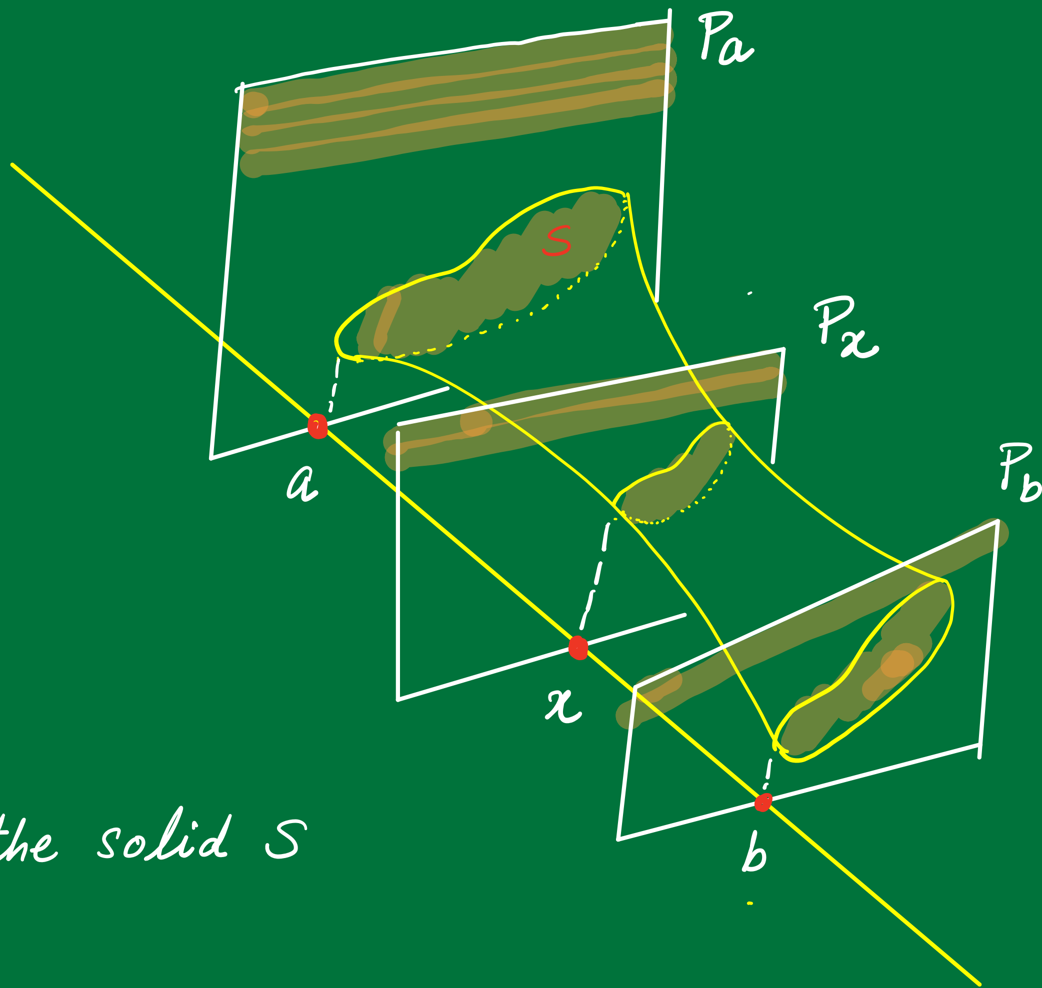
$$L = \int_0^\infty \sqrt{2}\, e^{-\theta}\, d\theta$$

$$= \sqrt{2}$$

Check !

Volume of solid bodies by integral expressions.

Let S be a solid, is bounded by two parallel plane P_a and P_b perpendicular to x-axis at $x = a$ and $x = b$ respectively.



We define volume of the solid S to be

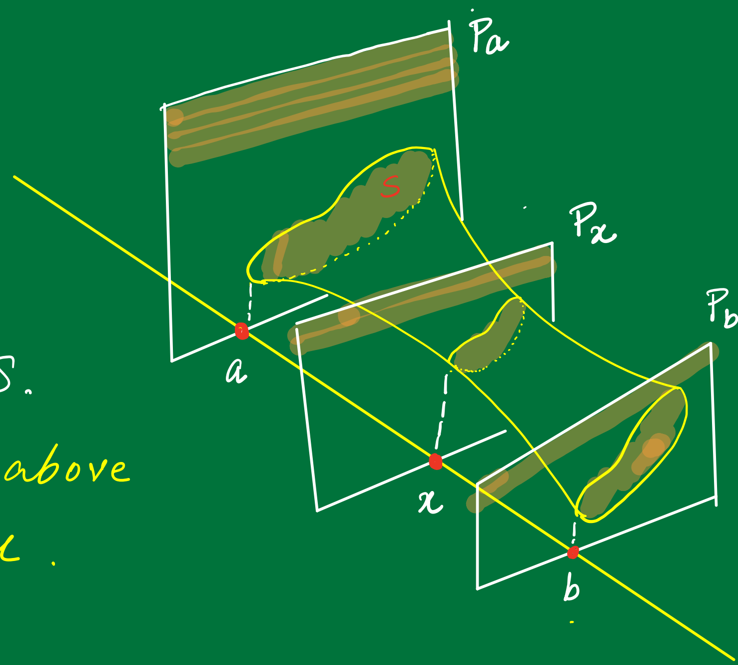
$$V = \int_a^b A(x)\,dx \;;$$

— by slicing method.

Let $P = \{ a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b \}$ be a partition of $[a, b]$ and P_{x_i} be the plane perpendicular to x-axis at x_i. Suppose S_{x_i} be the cross-section of S, which is the planer region formed by intersecting S with P_{x_i}.

Let $A(x_i)$ be the area of the region S_{x_i}.

Then volume of the cylindrical solid between $P_{x_{i-1}}$ and P_{x_i} is approximately $A(x_i) \times \Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$.

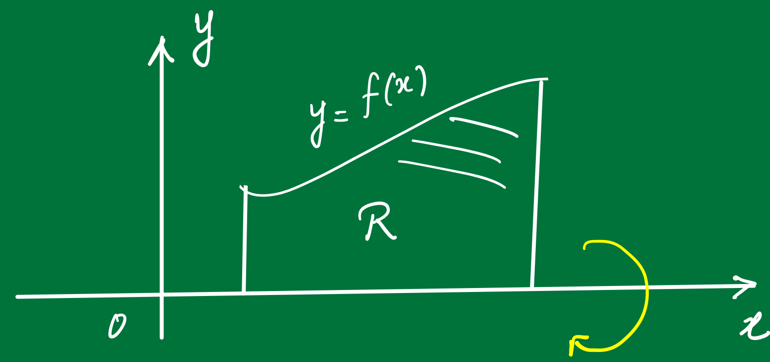


Consider $\sum_{i=1}^{n} A(x_i) \Delta x_i$ as an approximation of the volume of S.

If $A(x)$ is continuous on $[a, b]$ then the above Riemann sum converges to $\int_a^b A(x)\,dx$.

II Volumes of solids of revolution

S : solid in $\mathbb{R}^3$ generated by revolving a planer region R about an axis (e.g. x-axis).

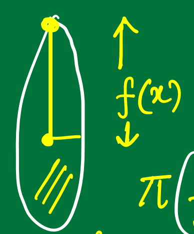


The volume of the solid of revolution, by slicing method, is

$$V = \int_a^b A(x)\,dx$$

$$= \int_a^b \pi \big(f(x)\big)^2\,dx$$

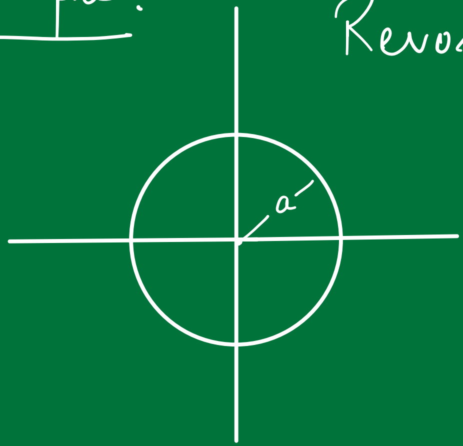
Consider $f : [a,b] \to \mathbb{R}$ a continuous function, $f(x) \geq 0$.



$\pi\big(f(x)\big)^2 = A(x)$

a slice — is a disc of radius $f(x)$

Example:

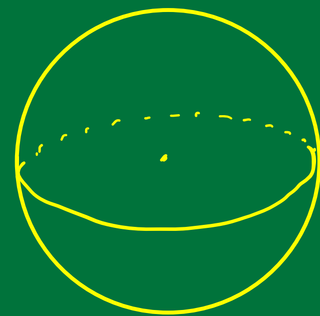


Revolve the circular disc

$$\{ (x,y) \mid x^2 + y^2 \le a^2 \}$$

about x-axis

and generate the sphere

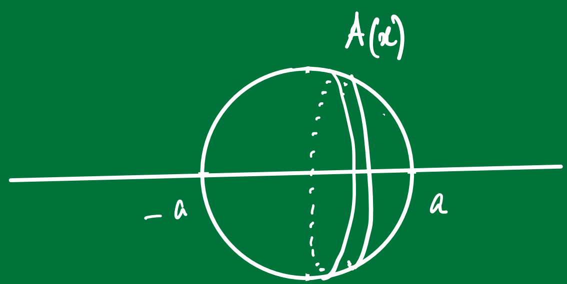


The volume is

$$V = \int_{-a}^{a} \pi \left(f(x)\right)^2 \, dx$$

$$= \int_{-a}^{a} \pi \left(a^2 - x^2\right) dx$$

$$= \frac{4}{3} \pi a^3 .$$

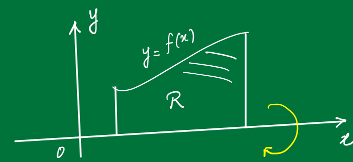


$\cdot (x, f(x))$

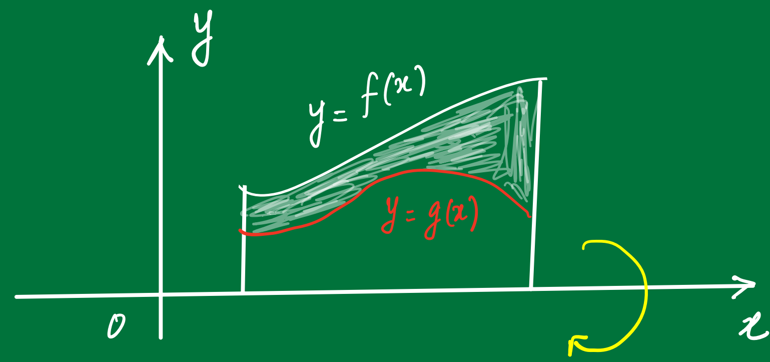
$x^2 + (f(x))^2 = a^2$

$(\dot{x}, 0)$

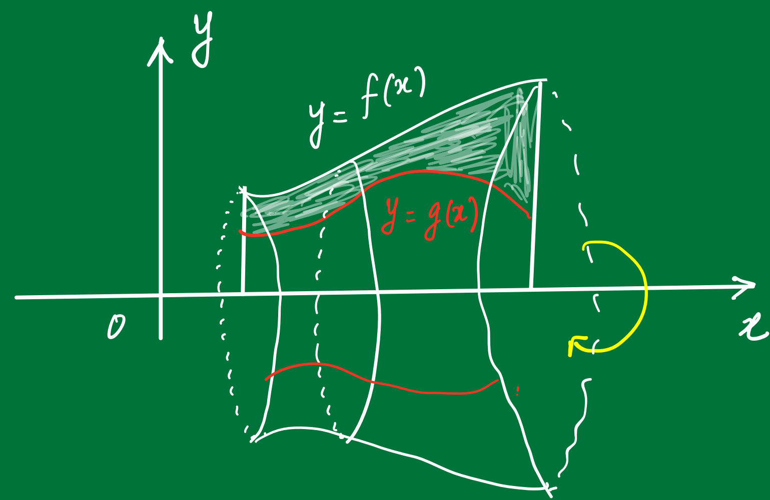
II (a) Washer method



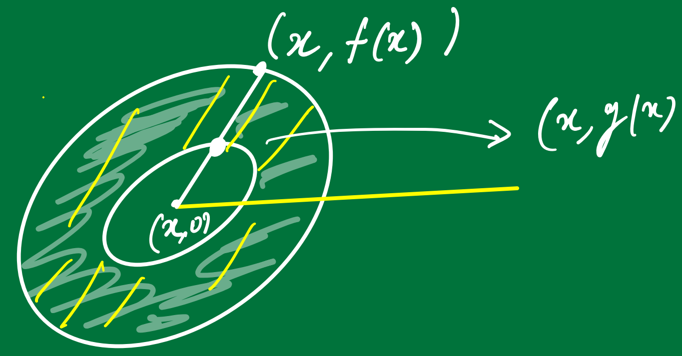
Let $f, g : [a,b] \to \mathbb{R}$
continuous functions
with $f(x) \geq g(x) \geq 0$



Consider the region R between the graphs and the solid S obtained by revolving the region R about x-axis.



The cross-sections perpendicular to the x-axis are now washer of inner radius $g(x)$ and outer radius $f(x)$ respectively.

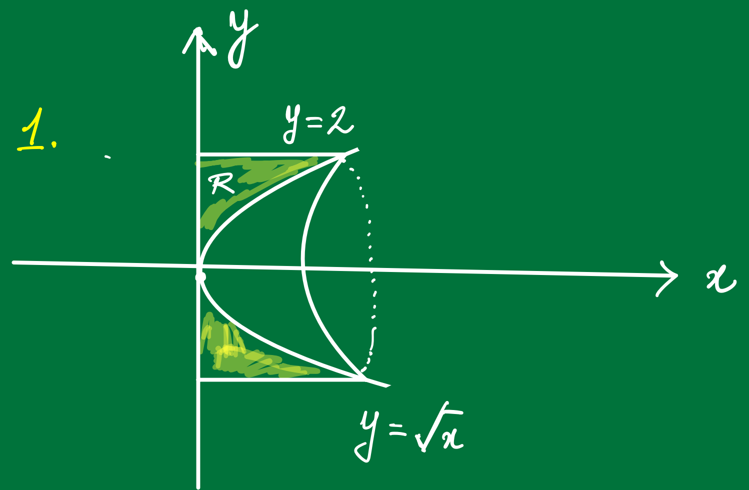


The slices perpendicular to the axis of revolution look like washers

$$V = \int_a^b A(x)\, dx$$

$$= \int_a^b \pi \left(f(x)^2 - g(x)^2 \right) dx$$

Example. 1.



Compute the volume of the solid S obtained by revolving the region R about x-axis.

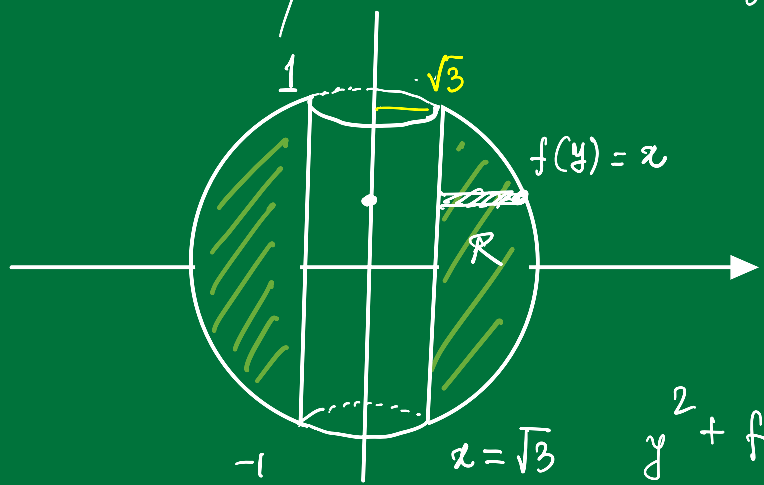
Here $\quad A(x) = f(x)^2 - g(x)^2$

$$= 4 - x$$

and $\quad V = \int_0^4 \pi (4-x)\,dx$

Example 2.

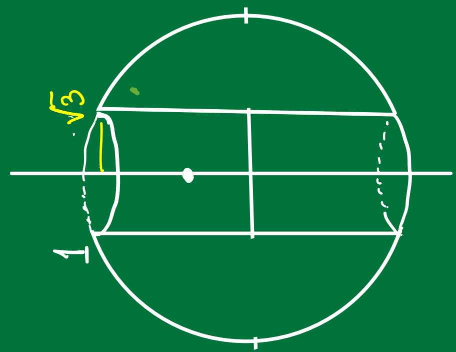
Consider a solid sphere of radius 2 cm and remove a portion by making a cylindrical hole of radius $\sqrt{3}$ cm. Using Washer method compute the volume of the remaining portion of the sphere.

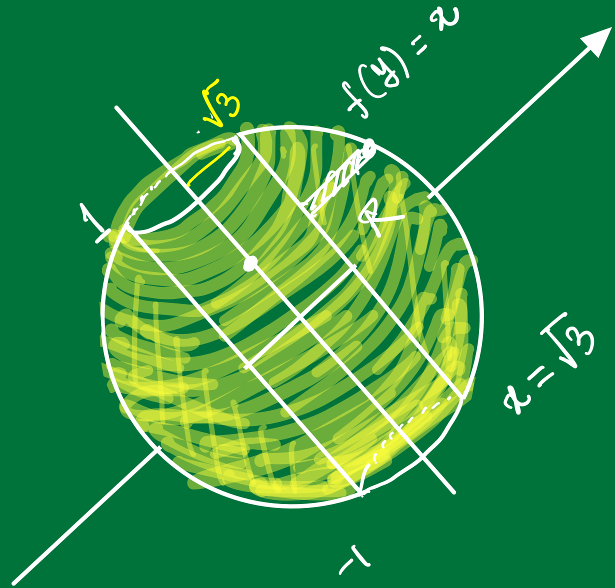


$y^2 + f(y)^2 = 4$

- $A(y) = \pi\left(f(y)^2 - (\sqrt{3})^2\right)$

$$= \pi(4 - y^2 - 3)$$





The region R is revolved about the y-axis

$$V = \int_{-1}^{1} A(y)\,dy = \int_{-1}^{1} \pi\left(4 - y^2 - 3\right)dy$$

$$= \int_{-1}^{1} \pi\left(1 - y^2\right)dy$$

$$= \frac{4\pi}{3}$$