lecture 3

Recall $\dfrac{dy}{dx} = f(x,y)$

We have seen:

(i) graphical method of drawing approx. sol.

(ii) Seperation of variables

(iii) Homogeneous equations (ODE)

(iv) Exact equation, Integrating factor.

Today
- Linear 1st order equations
- Bernoullie Equations.

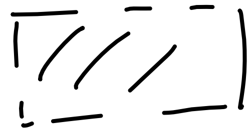
Recall

$M\,dx + N\,dy$ $\underline{\text{is called}}$ **exact** $(*)$
if $\exists\ u(x,y)$ s.t

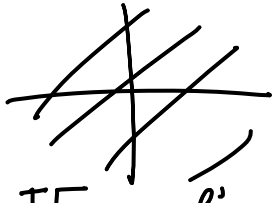
$$M\,dx + N\,dy = du.$$

Theorm

$(*)$ is exact on



$$\iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$



Integrating Factor (IF) $\mu(x,y)$ is called IF of $(*)$
if $\mu M\,dx + \mu N\,dy$ is exact.

Theorem If

(i) $\frac{1}{N}\left(M_y - N_x\right) = f(x)$

then $e^{\int f(x)\,dx}$ is an IF for (*)

(ii) $\frac{1}{M}\left(M_y - N_x\right) = g(y)$, then

$e^{-\int g(y)\,dy}$ is an IF for (*).

Proof (i) $\mu(x,y) = e^{\int f(x)\,dx}$

To show μ is an IF for (*), we have to show that

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\frac{\partial}{\partial y}(\mu M) = \cancel{\frac{\partial \mu}{\partial y}}_{0} M + \mu \frac{\partial M}{\partial y}$$

$$= \mu \frac{\partial M}{\partial y}.$$

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$= \mu f(x) N + \mu \frac{\partial N}{\partial x}$$

$$= \mu \frac{1}{\cancel{N}}(M_y - N_x)\cancel{N} + \mu N_x$$

$$= \mu M_y.$$

$\blacksquare$

$$\mu = e^{\int f(x)\,dx}$$

$$\log \mu = \int f(x)\,dx$$

$$\frac{1}{\mu}\frac{\partial \mu}{\partial x} = f(x).$$

$$\frac{\partial \mu}{\partial x} = \mu f(x)$$

Remark

IF is not unique. In fact if μ is an IF for $M\,dx + N\,dy$ (i.e. $\underline{\mu M\,dx + \mu N\,dy} = du$), then $g(u)\mu$ is also an IF for $M\,dx + N\,dy$, where $g(u)$ is any continuous function.

$$v = \int g(u)\,du$$

$$\frac{dv}{du} = g(u) \quad \text{i.e.}$$

i.e. $dv = g(u)\,du = g(u)\mu\, M\,dx + g(u)\mu\, N\,dy.$ ☐

1st order linear equation

$$\frac{dy}{dx} + p(x)\,y = r(x). \quad (**)$$

where $p(x)$ & $r(x)$ are continuous functions.

$$\underbrace{\left(p(x)y - r(x)\right)}_{M} dx + \underbrace{1}_{N} \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = p(x) \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{1}{N}\left(M_y - N_x\right) = \frac{1}{1}\cdot p(x) = p(x)$$

So $e^{\int p(x)\,dx}$ is an IF for $(**)$

$$\underbrace{e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y} = r(x)e^{\int p(x)dx}$$

$$\frac{d}{dx}\left(\underline{y e^{\int p(x)dx}}\right) = r(x)e^{\int p(x)dx}$$

integrate

$$y e^{\int p(x)dx} = \int r(x)e^{\int p(x)dx}dx + C$$

Example

(1) Solve $y' + 2xy = 2x$

This is 1st order linear equation

with $p(x) = 2x$ $\quad r(x) = 2x$

$$IF = e^{\int p(x)dx} = e^{x^2}$$

Multiplying by it

$$\underbrace{e^{x^2}y' + e^{x^2}\cdot 2xy} = e^{x^2}\cdot 2x.$$

$$\frac{d}{dx}\left(ye^{x^2}\right) = 2xe^{x^2}$$

integrating $\quad ye^{x^2} = e^{x^2} + C$

$$y = 1 + Ce^{-x^2}$$

Exmple 2 Solve.

$$(4y^3 - 2xy)\,y' = y^2 \qquad y(2) = 1$$

— This is 1st order non-linear in y

But $\dfrac{4y^3 - 2xy}{y^2} = \dfrac{dx}{dy}$.

$$\frac{dx}{dy} + x \cdot \frac{2}{y} = 4y.$$

— linear in x

$$IF = e^{\int \frac{2}{y} dy} = e^{2\log y} = y^2$$

Multiplying by y^2

$$\frac{d}{dy}\left(y^2 \cdot x\right) = 4y^3$$

integrating

$$y^2 x = y^4 + C$$

Using the initial condition $y(2) = 1$

$$1 \cdot 2 = 1 + C \qquad C = 1$$

Solution $y^2 x = y^4 + 1$ ☒

Berroullie equation

$$\frac{dy}{dx} + p(x) y = r(x) y^n \quad n \in \mathbb{R}.$$

This is linear only for $n = 0, 1$

How to solve it for $n \neq 0, 1$.

$$y^{-n} \frac{dy}{dx} + p(x) y^{1-n} = r(x).$$

Put $\boxed{v = y^{1-n}.}$

$$\frac{dv}{dx} = (1-n) y^{-n} \frac{dy}{dx}.$$

Substituting in the given equation

$$\frac{1}{1-n} \frac{dv}{dx} + p(x) v = r(x)$$

$$\frac{dv}{dx} + (1-n) p(x) v = (1-n) r(x).$$

— linear 1st order.

Example $\frac{dy}{dx} + y = x y^3$

This is Berroullie equation with $n = 3$

$$v = y^{1-3} = \frac{1}{y^2}$$

$$\frac{dv}{dx} = -\frac{2}{y^3} \frac{dy}{dx}$$

Putting in the given equatn.

$$\frac{dv}{dx} - 2v = -2x.$$

$$IF = e^{\int -2\,dx} = e^{-2x}$$

Multiplying by it

$$\frac{d}{dx}\left(e^{-2x}\cdot v\right) = -2x e^{-2x}.$$

Integrate.

$$e^{-2x} v = e^{-2x}\left(x + \frac{1}{2}\right) + C$$

$$\frac{1}{y^2} = v = \left(x + \frac{1}{2}\right) + C e^{2x}.$$

$\blacksquare$

Remark

$$f'(y)\frac{dy}{dx} + f(y)\,p(x) = r(x)$$

$\left(f(y) = y^{1-n}\right.$, then we get Bernoullic form$\left.\right)$

In this case

$$v = f(y)$$

$$\frac{dv}{dx} = \underline{f'(y)\frac{dy}{dx}}$$

$$\frac{dv}{dx} + v\,p(x) = r(x)$$

— linear $\boxed{}$

Example

$$\underbrace{\cos y}_{f'(y)} \frac{dy}{dx} + \underbrace{\sin y}_{f(y)} \frac{1}{x} = 1$$

$$v = \sin y$$

$$\frac{dv}{dx} = \cos y \frac{dy}{dx}$$

Substituting in the given eq.

$$\frac{dv}{dx} + v\frac{1}{x} = 1$$

$$IF = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Multiplying by IF

$$\frac{d}{dx}(x \cdot v) = 1 \cdot x = x$$

Integrate:

$$x \sin y = \frac{x^2}{2} + C$$

Reducible second order ODE

$$F(x, y, y', y'') = 0 \quad (***)$$

Case 1 x is missing (***) ie

$$F(y, y', y'') = 0$$

Example $2y'' - y'^2 - 4 = 0$

- 2nd order non linear.
- x is missing.

$$\boxed{y' = w}$$

$$y'' = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = \frac{dw}{dy} w$$

Putting in the given eqn.

$$w\frac{dw}{dy} - w^2 - 4 = 0$$

- Variables can be separated.

$$\frac{2w}{w^2+4}\,dw = dy.$$

Integrating.

$$\log\left(\frac{w^2+4}{c}\right) = y.$$

$$w = \pm\sqrt{ce^y - 4.} = \frac{dy}{dx}.$$

$$\pm\, dx = \frac{dy}{\sqrt{ce^y - 4}}$$

Integrate . . . . . .

Case 2 y is missing in $***$

ie $F(x, y', y'') = 0.$

Example $xy'' + 2y' = 0.$

— 2nd order, 'y is missing'

$$y' = w$$

$$y'' = \frac{dw}{dx}.$$

Putting in the given eq

$$x\frac{dw}{dx} + 2w = 0$$

— 1st order

$$\frac{dw}{w} + \frac{2dx}{x} = 0.$$

Integrating $wx^2 = c$. Again integrate

$$\frac{dy}{dx} = \frac{c}{x^2}. \quad y = -c\,\frac{1}{x} + D.$$