$F(x,y) = F(x, mx)$ and

$F(x, mx) \to \dfrac{2m}{1+m^2}$ – a quantity depends on the value of m (the slope of the line).

In this case, $\lim\limits_{(x,y)\to(0,0)} F(x,y) = ?$

We want $\ell \in \mathbb{R}$ such that

$F(x,y) \to \ell$ whenever $(x,y) \to (0,0)$.

— No such ℓ exists.

② Q. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$?

Here,

$$|F(x,y) - 0| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \left(\frac{x^2+y^2}{2}\right) \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{\sqrt{x^2+y^2}}{2}$$

For $\varepsilon > 0$, if we take $\delta = 2\varepsilon$, then whenever

$\|(x,y) - (0,0)\| < \delta = 2\varepsilon$

we get $|F(x,y) - 0| \leq \frac{1}{2}\sqrt{x^2+y^2}$

$< \varepsilon$

$\Rightarrow \lim_{(x,y)\to(0,0)} F(x,y) = 0.$

③ Compute $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$.

For $y = mx$, $F(x, mx) \to ?$

Now $F(x, mx) = \dfrac{mx^3}{x^4+m^2x^2} = \dfrac{mx}{x^2+m^2}$ – depends on both the values of x and m!

For, $y = x^2$

we have $F(x, x^2) \to 1/2$ | limit does not exists.

Example 4 Check that $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{2x^2+3y^2} = 0$

Here, $|F(x,y) - 0| = \left|\dfrac{x^2 y}{2(x^2+y^2)+y^2}\right| \leq \dfrac{|x^2 y|}{2(x^2+y^2)}$

$$\leq \frac{|x| \times \frac{1}{2}(x^2+y^2)}{2(x^2+y^2)}$$

$$= \frac{|x|}{4}$$

and

$\|(x,y) - (0,0)\| = \sqrt{x^2+y^2}$, $\quad \dfrac{|x|}{4} < \dfrac{\sqrt{x^2+y^2}}{4}$

$\Rightarrow |F(x,y) - 0| < \varepsilon \quad$ if $\quad \|(x,y) - (0,0)\| < \underset{4\varepsilon}{\underset{\shortparallel}{\delta}}$

For any $\varepsilon > 0$, if we take $\delta = 4\varepsilon$ then

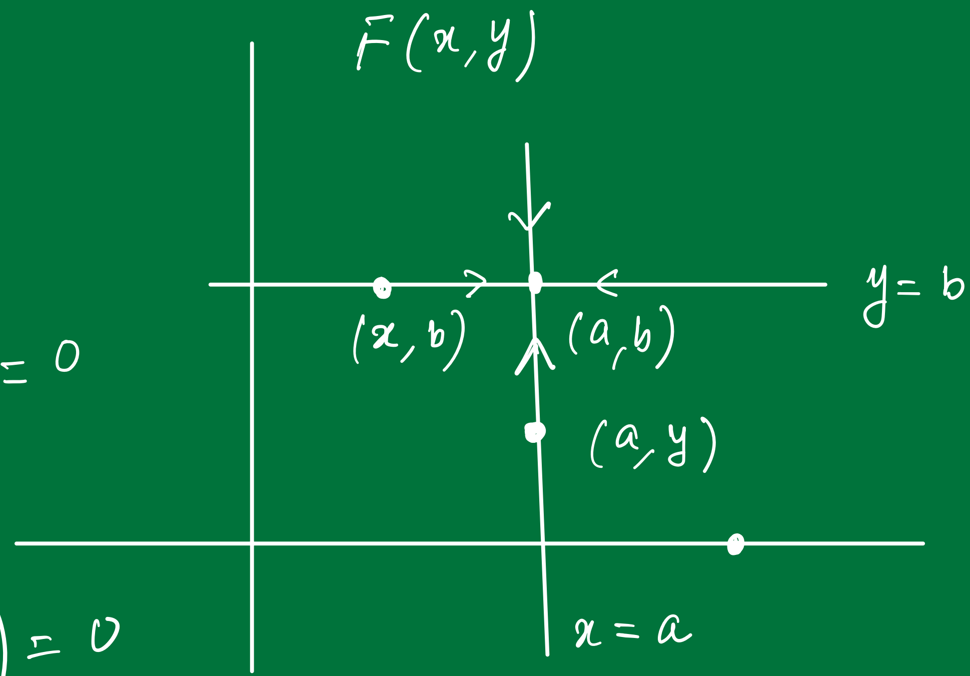
$|F(x,y) - 0| < \varepsilon$ whenever $\|(x,y) - (0,0)\| < \delta$. Consequently, $\lim_{(x,y)\to(0,0)} F(x,y) = 0$.

$$(x, y) \to (a, b)$$

If

$$\lim_{y \to b} \left(\lim_{x \to a} F(x, y) \right) = 0$$

$$\lim_{x \to a} \left(\lim_{y \to b} F(x, y) \right) = 0$$



Can we say that $\displaystyle\lim_{(x,y) \to (a,b)} F(x, y) = 0$?

Example: 5

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} & (x,y) \neq (0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

For

$y = x$ line: $\quad f(x,x) \to 1.$

Example

$$\lim_{(x,y)\to(0,0)} x^4 \sin \frac{1}{x^2+|y|} = 0 \quad ?$$

Example 6

$$\lim_{(x,y)\to(0,0)} \underbrace{x^4 \sin\left(\frac{1}{x^2+|y|}\right)} = 0 \quad ?$$

Let

$$F(x,y) = \begin{cases} x^4 \sin\left(\frac{1}{x^2+|y|}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Now $|F(x,y) - 0|$

$= \left| x^4 \sin\left(\frac{1}{x^2+|y|}\right) \right| \leq |x|^4$ and $|x|^4 < \varepsilon$ if $|x| < \delta = \varepsilon^{1/4}$

$\Rightarrow$ For any $\varepsilon > 0$ taking $\delta = \varepsilon^{1/4}$ we have

$|F(x,y) - 0| < \varepsilon$ whenever $\|(x,y) - (0,0)\| < \delta$.

Note: $\|(x,y) - (0,0)\| < \delta$

$\Rightarrow |x| \leq \sqrt{x^2+y^2} < \delta$

$\Rightarrow |x| < \delta$

A function $f: \mathbb{R}^2 \to \mathbb{R}$ is said to be continuous at $x_0 \in \mathbb{R}^2$

if $\underbrace{\lim_{x \to x_0} f(x)}_{}$ $= f(x_0)$ (in other words

$f(x) \to f(x_0)$ as $x \to x_0$)

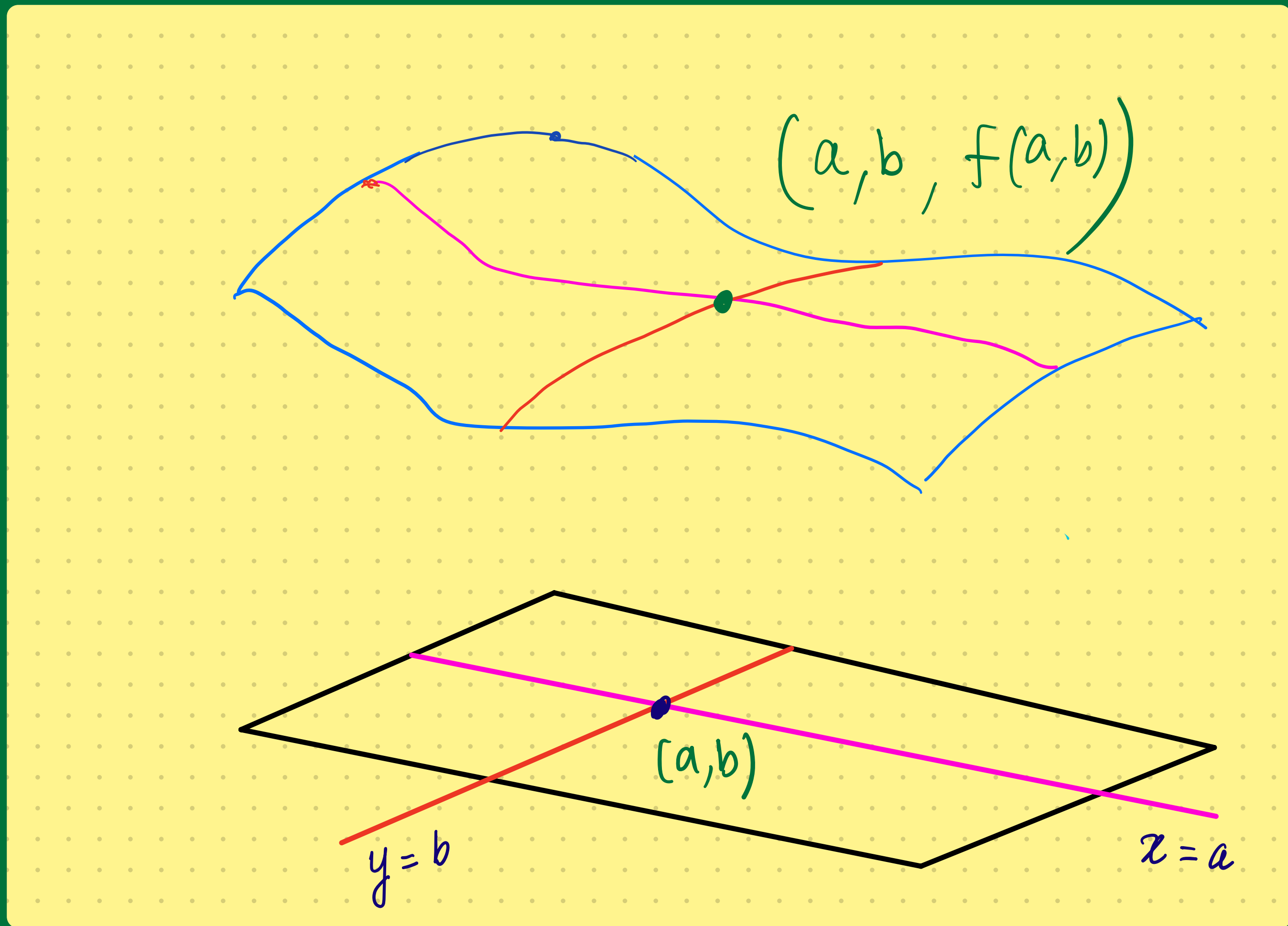
* limiting value is the value of f

A function $f: \mathbb{R}^2 \to \mathbb{R}$ is said to continuous if it is continuous at every $x_0 \in \mathbb{R}^2$.

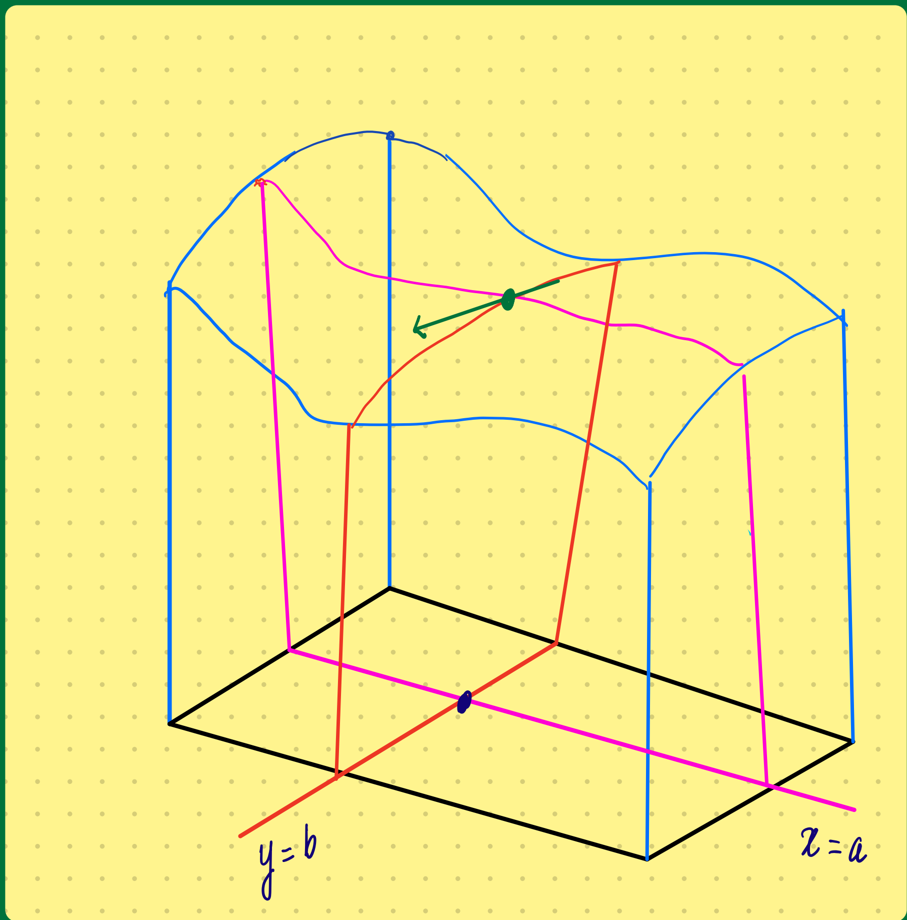
□ Continuous function $f: \mathbb{R}^2 \to \mathbb{R}$

$f : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto f(x,y)$



$$\lim_{(x,y)\to(0,0)} f(x,y)$$



f_a and f^b both are functions of one variable ;

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \longmapsto f(x,y)$$

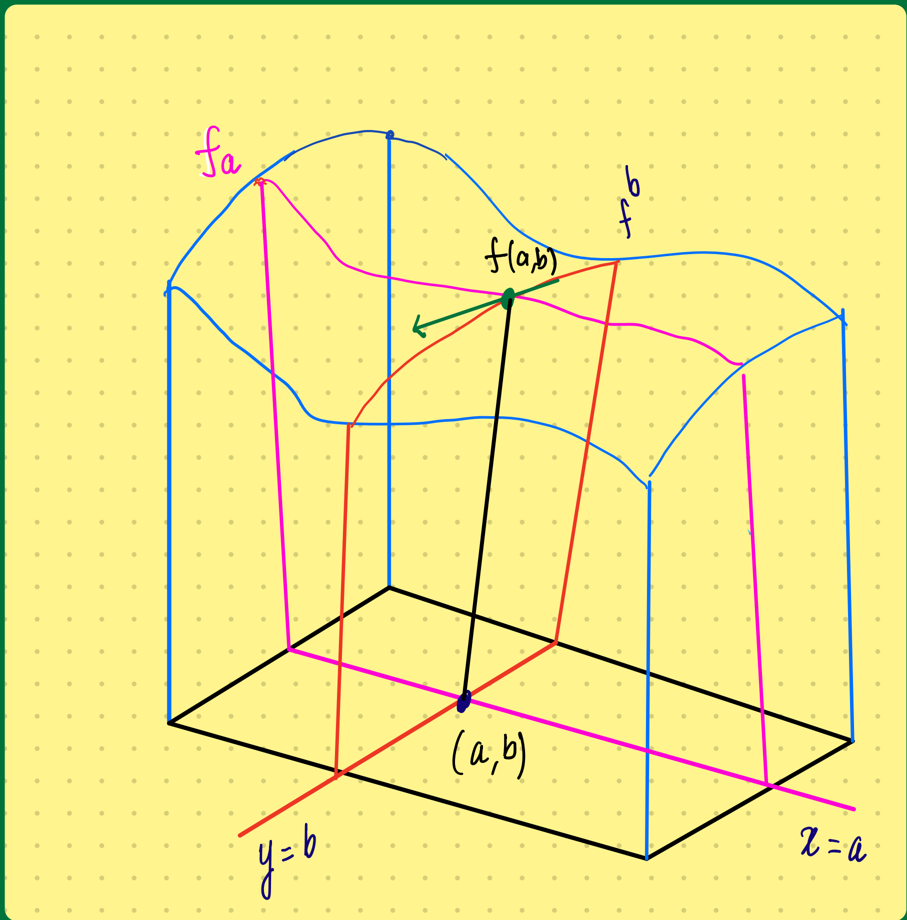
Let $(a,b) \in \mathbb{R}^2$;

$$f_a : \mathbb{R} \to \mathbb{R}$$
$$y \longmapsto f_a(y) = f(a,y)$$

$$f^b : \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto f^b(x) = f(x,b)$$

$f_a'(b)$ – derivative of f_a at b

$(f^b)'(a)$ – derivative of f^b at a



$$f_a'(b) = \lim_{h\to 0} \frac{f_a(b+h) - f_a(b)}{h}$$

$$= \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

$$:= \frac{\partial f}{\partial y}(a,b)$$

Similarly,

$$(f^b)'(a) = \lim_{h\to 0} \frac{f^b(a+h) - f^b(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$:= \frac{\partial f}{\partial x}(a,b)$$

Partial derivative of $f: \mathbb{R}^2 \to \mathbb{R}$ at $X_0 = (a, b)$ with respect to the 1st variable,

$$\frac{\partial f}{\partial x}(X_0) = \lim_{h \to 0} \frac{f(X_0 + h(1,0)) - f(X_0)}{h}$$

$$= \lim_{t \to 0} \frac{f(X_0 + t\vec{i}) - f(X_0)}{t} \quad , \ \vec{i} = (1,0)$$

and $\dfrac{\partial f}{\partial y}(X_0)$

$$= \lim_{t \to 0} \frac{f(X_0 + t\vec{j}) - f(X_0)}{t}$$

Consider a unit vector $\vec{u} \in \mathbb{R}^2$ where $\vec{u} = (v_1, v_2)$
(or a direction) $= v_1 \vec{i} + v_2 \vec{j}$.

We define the directional derivative of f at x_0 in the direction of $\vec{u}$ is

$$D_{x_0} f(\vec{u}) = \lim_{t \to 0} \frac{f(x_0 + t\vec{u}) - f(x_0)}{t}$$

If $f : \mathbb{R}^3 \to \mathbb{R}$, then

$$D_{X_0} f(\vec{i}) = \lim_{t \to 0} \frac{f(X_0 + t\vec{i}) - f(X_0)}{t} = \frac{\partial f}{\partial x}(X_0) = f_x(X_0)$$

$\vec{i} = (1, 0, 0)$

$$D_{X_0} f(\vec{j}) = \lim_{t \to 0} \frac{f(X_0 + t\vec{j}) - f(X_0)}{t} = \frac{\partial f}{\partial y}(X_0) = f_y(X_0)$$

$\vec{j} = (0, 1, 0)$

$$D_{X_0} f(\vec{k}) = \lim_{t \to 0} \frac{f(X_0 + t\vec{k}) - f(X_0)}{t} = \frac{\partial f}{\partial z}(X_0)$$

$$= f_z(X_0).$$

$\vec{k} = (0, 0, 1)$

Q. Suppose all the directional derivatives $D_{x_0} f(\vec{u})$ exists for $f: \mathbb{R}^2 \to \mathbb{R}$ and $x_0 \in \mathbb{R}^2$. What can we say about continuity of f at x_0?

$f: \mathbb{R}^2 \to \mathbb{R}$ is continuous?

Example: $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

The function f is NOT continuous at $(0,0)$.

Let $\vec{u} = (v_1, v_2)$ be a unit vector in $\mathbb{R}^2$.

Then
$$D_{(0,0)} f(\vec{u}) = \lim_{t \to 0} \frac{f((0,0) + t\vec{u}) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(tv_1, tv_2) - 0}{t}$$

$$= \lim_{t \to 0} \frac{t^3 v_1^2 v_2}{(t^4 v_1^4 + t^2 v_2^2)\, t}$$

$$= \lim_{t \to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2}$$

$$= \begin{cases} v_1^2 / v_2 & \text{if } v_2 \neq 0 \\ 0 & \text{if } v_2 = 0 \end{cases} \qquad \blacksquare$$

Under certain restriction on the partial derivatives of a given function f, we get the continuity of f.

Let $S = \left\{ (x,y) \in \mathbb{R}^2 \,/\, \begin{array}{l} a < x < b \text{ and} \\ c < y < d \end{array} \right\}$.

Suppose $f : S \to \mathbb{R}$ such that both partial derivative

$$\frac{\partial f}{\partial x} : S \to \mathbb{R}, \quad x_0 \mapsto \frac{\partial f}{\partial x}(x_0) \quad \text{and} \quad \frac{\partial f}{\partial y} : S \to \mathbb{R}, \quad x_0 \mapsto \frac{\partial f}{\partial y}(x_0)$$

are bounded functions. Then $f : S \to \mathbb{R}$ is a continuous function.