Solutions to Complex Analysis, Stein & Shakarchi

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Preface

Seoul National University's "Complex Function Theory" course uses "Stein's Complex Analysis" as a textbook. However, I've seen many students struggle with their studies because it's difficult to find solutions to most problems. I'm sharing my solutions in the hopes that they will be helpful.

This solution includes all Exercises for each chapter. It doesn't contain solutions to the Problems.

This file is uploaded to https://geniuslhs.com/solutions/stein-complex-analysis.pdf. I recommend checking it regularly, as it may be updated.

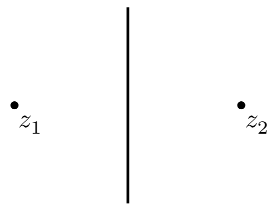
The solutions may contain mathematical errors, so we recommend reading them critically. If you find any mathematical errors or typos, please report them to qwerty12021@snu.ac.kr. Thank you for your valuable feedback.

History

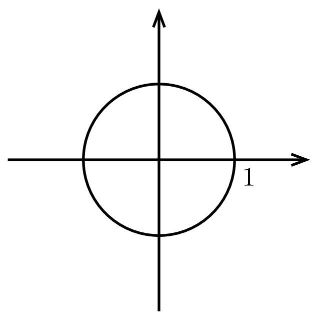
- [2025.07.29.] First release.
- [2025.10.10.] Corrected Exercise 1, Chapter 8.

Chapter 1. Preliminaries to Complex Analysis

1. (a)



(b) $1/z = \overline{z} \iff z\overline{z} = 1 \iff |z| = 1$



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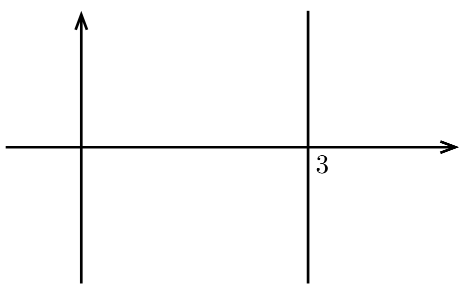

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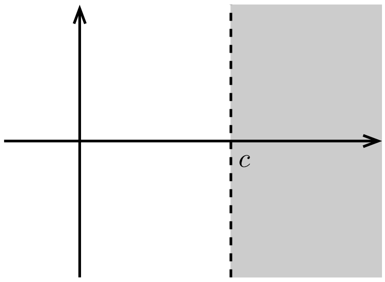
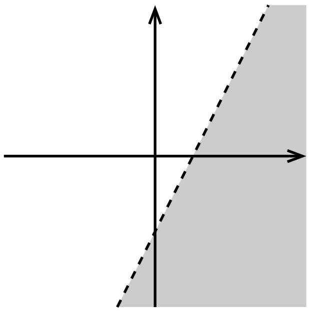
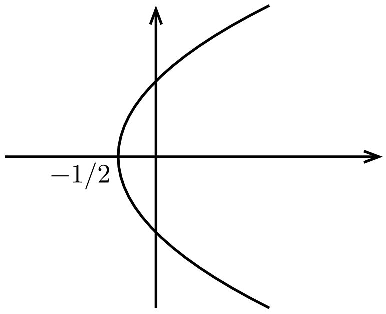


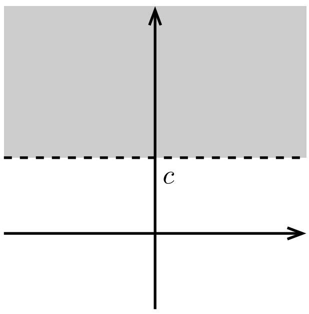
(e) The mapping $z \mapsto az$ is a linear transformation, and mapping $z \mapsto z + b$ is a translation.



(f) Let $z = x + iy$, then $|z| = \mathrm{Re}(z) + 1 \iff \sqrt{x^2 + y^2} = x + 1 \iff x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1$



(g)



2. Let $z = x_1 + iy_1, w = x_2 + iy_2$. Then

$$\langle z, w \rangle = x_1 x_2 + y_1 y_2$$

and

$$z\overline{w} = (x_1 + iy_1)(x_2 - iy_2) = x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2).$$

Hence

$$\frac{1}{2}[(z,w)+(w,z)] = \frac{1}{2}[(z,w)+(w,z)] = \mathrm{Re}(z,w) = \langle z,w\rangle.$$

3. Suppose $z = re^{i\theta}$ satisfies the equation $z^n = w$. Then

$$r^n e^{in\theta} = se^{i\varphi} \;\Rightarrow\; r^n = s \;\wedge\; e^{in\theta} = e^{i\varphi}.$$

Since $0 \le \theta < 2\pi$, We get $0 \le n\theta < 2n\pi$ and $n\theta = \varphi + 2k\pi$ $(0 \le k < n)$. Therefore there exists n solutions;

$$z = s^{\frac{1}{n}} e^{i(\frac{\varphi}{n}+2\frac{k}{n}\pi)} \; (0 \le k < n).$$

4. Suppose it is possible to define a total ordering on $\mathbb{C}$. by (i), one and only of the following is true;

$$i \succ 0, 0 \succ i, \text{or } i = 0.$$

In the first case, use property (iii) twice to get the following

$$i \cdot i \cdot i \succ 0 \cdot 0 \cdot 0$$

so that $-i \succ 0$. Use property (ii) to get $-i + i \succ 0 + i \;\Rightarrow\; 0 \succ i$, which is contradiction.

In the second case we can get a contradiction in a similar way. Indeed,

$$-i \succ 0 \;\Rightarrow\; (-i)^3 \succ 0 \;\Rightarrow\; i \succ 0 \;\Rightarrow\; 0 \succ -i.$$

The third case is obviously impossible.

5. (a) Since $z(t^*) \in \Omega = \Omega_1 \cup \Omega_2$, we get $z(t^*) \in \Omega_1$ or $z(t^*) \in \Omega_2$.

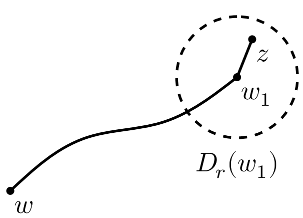
First, Suppose $z(t^*) \in \Omega_1$. Since Ω_1 is an open set, there exists $r > 0$ such that $D_r(z(t^*)) \subset \Omega_1$. z is continuous function so there exists small $\delta > 0$ such that $|z(t^* + \delta) - z(t^*)| < r$. Then we get

$$t^* = \sup_{0 \le t \le 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \le s < t\} \ge t^* + \delta,$$

which is contradiction.

Now we suppose $z(t^*) \in \Omega_2$. Since Ω_2 is an open set, there exists a $r > 0$ such that $D_r(z(t^*)) \subset \Omega_2$. Also, $t < t^* \Rightarrow z(t) \in \Omega_1$ holds by the definition of t^*. If we take $t \in [0,1]$ slightly smaller than t^* so that $|z(t) - z(t^*)| < r$, then $z(t)$ belongs to both Ω_1 and Ω_2 which is contradiction.

(b) First we prove that Ω_1 is an open set. Let $w_1 \in \Omega_1$. Since $w_1 \in \Omega$, there exists $r > 0$ such that $D_r(w_1) \subset \Omega_1$. Now we can join any $z \in D_r(w_1)$ by a curve contained in Ω, by connecting the two curves; the curve from w to w_1 which is guaranteed by the definition of Ω_1, and the straight line from w_1 to z. Hence $D_r(w_1) \subset \Omega_1$ and we conclude that Ω_1 is open.



Second we prove that Ω_2 is an open set. Similar to the argument above, we can see that the points in the neighborhood of $w_2 \in \Omega_2$ cannot be joined to w. That's because if it is possible, we can join w_2 from w by going through that point, which is contradiction.

Since we can only do one or the other, you can either connect or disconnect the line, Ω_1 and Ω_2 is disjoint and their union is Ω.

Finally, since $w \in \Omega_1$ so Ω_1 is non-empty, we get $\Omega_1 = \Omega$ and $\Omega_2 = \varnothing$.

6. (a) $\mathcal{C}_z$ is open because neighborhood of some point in $\mathcal{C}_z$ is also contained in $\mathcal{C}_z$. Also, C_z is pathwise connected, because if $w_1, w_2 \in \mathcal{C}_z$ then we can connect w_1 and w_2 by going through z. The pathwise connectedness of $\mathcal{C}_z$ implies connectedness of $\mathcal{C}_z$ by **Exercise 5**.

Now we show that $w \in \mathcal{C}_z$ defines an equivalence relation. This is quiet simple. We can join z to z, so $z \in \mathcal{C}_z$. If we can join w to z, then we can also join z to w by reversing that curve. Finally, if we can join w to z and join ζ to z, then we can join w to ζ by connecting two curves.

(b) Each component have more than one rational points. If there are uncountably many distinct connected components, then there are uncountably many rational points in Ω, which is contradiction.

(c) Since Ω is compact, there exists $M > 0$ such that $z \in \Omega \Rightarrow |z| \leq M$. Now we take $z_0 > M$ and consider $\mathcal{C}_{z_0}$. Clearly this component is unbounded. Also, another unbounded component $\mathcal{C}_{z'}$ cannot exist because if it exists, then $\mathcal{C}_{z_0}$ and $\mathcal{C}_{z'}$ cannot be disjoint.

7. (a) Observe that we can assume z is real. If not, $z = re^{i\theta}$, then

$$\left|\frac{w-z}{1-\overline{w}z}\right| = \left|\frac{e^{-i\theta}(w-z)}{1-(e^{i\theta}\overline{w})(e^{-i\theta}z)}\right| = \left|\frac{e^{-i\theta}w - r}{1-\overline{e^{-i\theta}w}\; r}\right| = \left|\frac{w'-r}{1-w'r}\right|$$

with $w' = e^{-i\theta}w$.

Now we prove $|(w-r)/(1-wr)| \leq 1$. Square both sides and expand to get

$$(r-w)(r-\overline{w}) \leq (1-rw)(1-r\overline{w}) \Leftrightarrow w\overline{w}(1-r^2) \leq 1-r^2 \Leftrightarrow (1-r^2)(1-|w|^2) \geq 0.$$

Hence equality holds when $|z| = 1$ or $|w| = 1$.

(b) (i) By **(a)**, $F(z) \leq 1$ whenever $|z| \leq 1$. Therefore F maps the unit disc to itself. Since F is product of holomorphic functions, F is also holomorphic.

(ii) $F(0) = |(w-0)/(1-\overline{w}\cdot 0)| = w$, $F(w) = |(w-w)/(1-\overline{w}\cdot w)| = 0$.

(iii) $|F(z)| = 1$ whenever $|z| = 1$ by **(a)**.

(iv) Observe that

$$F(F(z)) = \frac{w - \frac{w-z}{1-\overline{w}z}}{1-\overline{w}\frac{w-z}{1-\overline{w}z}} = \frac{w - w\overline{w}z - w + z}{1-\overline{w}z - \overline{w}w + \overline{w}z} = \frac{1-w\overline{w}}{1-w\overline{w}}z = z.$$

F is injective because $F(z) = F(w) \Rightarrow F(F(z)) = F(F(w)) \Rightarrow z = w$. F is surjective because for any $w \in \mathbb{D}$, $z = F(w)$ satisfies $F(z) = F(F(w)) = w$, so w belongs to image of F.

8. Let $f(x+iy) = a(x,y) + ib(x,y)$, $g(a+ib) = c(a,b) + id(a,b)$.

$$\begin{aligned}\frac{\partial h}{\partial z} &= \frac{1}{2}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial c}{\partial x} + i\frac{\partial d}{\partial x} - i\frac{\partial c}{\partial y} + \frac{\partial d}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial c}{\partial z}\frac{\partial a}{\partial x} + \frac{\partial c}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial d}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial d}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial x} - \frac{\partial c}{\partial a}\frac{\partial a}{\partial y} - \frac{\partial c}{\partial b}\frac{\partial b}{\partial y}\right).\end{aligned}$$

Now we calculate $\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}$.

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial a} - i\frac{\partial g}{\partial b}\right) = \frac{1}{2}\left(\frac{\partial c}{\partial a} + \frac{\partial d}{\partial b}\right) + \frac{1}{2}i\left(\frac{\partial d}{\partial a} - \frac{\partial c}{\partial b}\right),$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial d}{\partial a} - \frac{\partial c}{\partial b}\right).$$

Therefore

$$
\begin{aligned}
\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} = \frac{1}{4}\bigg(&\frac{\partial c}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial c}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial y} + \frac{\partial d}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial x} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial y} - \frac{\partial d}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial x} + \frac{\partial c}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial d}{\partial a}\frac{\partial a}{\partial y} - \frac{\partial c}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial y}\bigg) \\
&+\frac{1}{4}i\bigg(\frac{\partial c}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial x} - \frac{\partial c}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial x} - \frac{\partial d}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial y} + \frac{\partial d}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial d}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial y} - \frac{\partial c}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial x} - \frac{\partial c}{\partial b}\frac{\partial b}{\partial y}\bigg).
\end{aligned}
$$

Also,

$$
\frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial c}{\partial a} - \frac{\partial d}{\partial b}\right) + \frac{1}{2}i\left(\frac{\partial d}{\partial a} + \frac{\partial c}{\partial b}\right)
$$

$$
\frac{\partial \overline{f}}{\partial z} = \frac{1}{2}\left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) + \frac{1}{2}i\left(-\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)
$$

Therefore

$$
\begin{aligned}
\frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z} = \frac{1}{4}\bigg(&\frac{\partial c}{\partial a}\frac{\partial a}{\partial x} - \frac{\partial c}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial y} - \frac{\partial d}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial x} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial y} + \frac{\partial d}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial x} + \frac{\partial c}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial d}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial c}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial y}\bigg) \\
&+\frac{1}{4}i\bigg(-\frac{\partial c}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial x} - \frac{\partial c}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial d}{\partial b}\frac{\partial b}{\partial x} - \frac{\partial d}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial y} + \frac{\partial d}{\partial a}\frac{\partial a}{\partial x} - \frac{\partial d}{\cancel{\partial a}}\frac{\cancel{\partial b}}{\partial y} + \frac{\partial c}{\cancel{\partial b}}\frac{\cancel{\partial a}}{\partial x} - \frac{\partial c}{\partial b}\frac{\partial b}{\partial y}\bigg).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z} &= \frac{1}{4}\left(2\frac{\partial c}{\partial a}\frac{\partial a}{\partial x} + 2\frac{\partial d}{\partial b}\frac{\partial b}{\partial y} + 2\frac{\partial d}{\partial a}\frac{\partial a}{\partial y} + 2\frac{\partial c}{\partial b}\frac{\partial b}{\partial x}\right) \\
&\quad+\frac{1}{4}\left(-2\frac{\partial c}{\partial a}\frac{\partial a}{\partial y} + 2\frac{\partial d}{\partial b}\frac{\partial b}{\partial x} + 2\frac{\partial d}{\partial a}\frac{\partial a}{\partial x} - 2\frac{\partial c}{\partial b}\frac{\partial b}{\partial y}\right) \\
&= \frac{\partial h}{\partial z}.
\end{aligned}
$$

Similarly, one can get $\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}$ by calculation.

9. Let $f\big(re^{i\theta}\big) = u(r,\theta) + iv(r,\theta)$, and we calculate $f'(z)$ in two different ways.

When θ is fixed,

$$
\begin{aligned}
f'(z) &= \lim_{r'\to r}\frac{f\big(r'e^{i\theta}\big) - f\big(re^{i\theta}\big)}{r'e^{i\theta} - re^{i\theta}} \\
&= \lim_{r'\to r}\frac{u(r',\theta) + iv(r',\theta) - u(r,\theta) - iv(r,\theta)}{(r'-r)e^{i\theta}} = \frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).
\end{aligned}
$$

When r is fixed,

$$
\begin{aligned}
f'(z) &= \lim_{\theta'\to\theta}\frac{f\big(re^{i\theta'}\big) - f\big(re^{i\theta}\big)}{r\big(e^{i\theta'} - e^{i\theta}\big)} \\
&= \lim_{\theta'\to\theta}\frac{1}{r}\cdot\frac{\theta'-\theta}{e^{i\theta'} - e^{i\theta}}\cdot\frac{f\big(re^{i\theta'}\big) - f\big(re^{i\theta}\big)}{\theta'-\theta} = \frac{1}{r}\cdot\frac{1}{ie^{i\theta}}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right).
\end{aligned}
$$

These two values have to be same, that is,

$$
ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.
$$

Comparing real and imaginary parts, we get

$$
\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.
$$

Now we show that logarithm function with $u(r,\theta) = \log r, v(r,\theta) = \theta$ is holomorphic. Observe that Cauchy-Riemann equations holds in an open set because

$$r \cdot \frac{1}{r} = 1, \quad -r \cdot 0 = 0.$$

therefore $\log z$ is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

10.

$$\begin{aligned}4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} &= \left(2\frac{\partial}{\partial z}\right)\left(2\frac{\partial}{\partial \overline{z}}\right) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ &= \frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial x \partial y} + i\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta.\end{aligned}$$

Similarly, we can get $4(\partial/\partial z)(\partial/\partial z) = \Delta$.

11. Let $f(x+iy) = u(x,y) + iv(x,y)$. Since f is holomorphic in the open set Ω,

$$0 = \left(\frac{\partial f}{\partial \overline{z}}\right) = \frac{\partial u}{\partial \overline{u}} + i\frac{\partial v}{\partial \overline{z}} = 0.$$

Hence $\partial u/\partial \overline{z} = 0$ and $\partial v/\partial \overline{z} = 0$. By **Exercise 10**,

$$\begin{aligned}\Delta u &= 4\frac{\partial u}{\partial z}\frac{\partial u}{\partial \overline{z}} = 4\frac{\partial u}{\partial z} \cdot 0 = 0, \\ \Delta v &= 4\frac{\partial v}{\partial z}\frac{\partial v}{\partial \overline{z}} = 4\frac{\partial v}{\partial z} \cdot 0 = 0.\end{aligned}$$

Therefore the real and imaginary parts of f are harmonic.

12. Note that the real and imaginary parts of f are $u(x,y) = \sqrt{|x||y|}$ and $v(x,y) = 0$.

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x \to 0} \frac{\sqrt{|x| \cdot 0} - 0}{x - 0} = 0, \; \frac{\partial u}{\partial y}(0,0) = \lim_{y \to 0} \frac{\sqrt{0 \cdot |y|} - 0}{y - 0} = 0,$$

Similarly,

$$\frac{\partial v}{\partial x}(0,0) = 0, \; \frac{\partial v}{\partial y}(0,0) = 0.$$

Therefore f satisfies the Cauchy-Riemann equations at the origin.

However, we cannot apply **Theorem 2.4** because Cauchy-Riemann equations can't be satisfied in any open set containing origin. In fact, f is not holomorphic at 0 because the derivatives of f diverges near 0.

13. Let $f(x+iy) = u(x,y) + iv(x,y)$. By Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$ holds.

(a) Suppose u is constant, then $\partial u/\partial x = \partial u/\partial y = 0$. then by Cauchy-Riemann equations, $\partial v/\partial x = \partial v/\partial y = 0$. Therefore v is constant, and $f = u + iv$ is constant.

(b) Similar to argument above, if we suppose v is constant, then $\partial v/\partial x = \partial v/\partial y = 0$, hence $\partial u/\partial x = \partial u/\partial y = 0$, which means u is constant and $f = u + iv$ is constant.

(c) Since $|f|$ is constant, $u^2 + v^2 = |f|^2$ is also constant. Therefore

$$\frac{\partial(u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0, \; \frac{\partial(u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Use Cauchy-Riemann equations to get

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0,\ v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$$

and

$$(u^2+v^2)\frac{\partial u}{\partial x} = (u^2+v^2)\frac{\partial u}{\partial y} = 0.$$

If $|f(z)| = 0$ for some points in Ω, since $|f|$ is constant, $f \equiv 0$. If not, then $u^2+v^2 \neq 0$ for all points in Ω, $\partial u/\partial x = \partial u/\partial y = 0$ for all points in Ω by the formula above. Hence u is constant, and v is constant by **(a)**, also is $f = u + iv$.

14.

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n(B_n - B_{n-1}) &= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M-1}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n
\end{aligned}$$

15. Use summation by parts formula to get

$$\begin{aligned}
\sum_{n=1}^{N}(1-r^n)a_n &= (1-r^N)A_N - (1-r)A_0 - \sum_{n=1}^{N-1}((1-r^{n+1}) - (1-r^n))A_n \\
&= (1-r^N)A_N - (1-r)\sum_{n=1}^{N-1} r^n A_n.
\end{aligned}$$

Note that $\sum_{n=1}^{N} r^n a_n$ and $\sum_{n=1}^{N} r^n A_n$ converges whenever $0 < r < 1$.

Let $A = \lim_{n\to\infty} A_n$, then for every $\varepsilon > 0$, there exists $M > 0$ such that $n > M \Rightarrow |A_n - A| < \varepsilon$. Also, $(1-r)\sum_{n=1}^{\infty} r^n A_n = A$ holds because

$$\begin{aligned}
\left|(1-r)\sum_{n=1}^{\infty} r^n A_n - A\right| = (1-r)\left|\sum_{n=1}^{\infty} r^n (A_n - A)\right| &\leq (1-r)\sum_{n=1}^{\infty} r^n |A_n - A| \\
&= (1-r)\left(\sum_{n=1}^{M} r^n|A_n - A| + \sum_{n=M+1}^{\infty} r^n |A_n - A|\right) \\
&\leq (1-r)\left(\sum_{n=1}^{M} r^n|A_n - A| + \sum_{n=M+1}^{\infty} r^n \varepsilon\right) \\
&= (1-r)\sum_{n=1}^{M} r^n|A_n - A| + r^{M+1}\varepsilon \\
&< 2\varepsilon
\end{aligned}$$

for r close enough to 1. Finally we get

$$\lim_{r\to 1, r<1} \sum_{n=1}^{\infty}(1-r^n)a_n = \lim_{r\to 1, r<1} A - (1-r)\sum_{n=1}^{\infty} r^n A_n = A - A = 0$$

and

$$\lim_{r\to 1, r<1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n.$$

16. **(a)** $(\log(n+1))^2/(\log n)^2 = \left(1 + \frac{1}{\log n}\log\left(\frac{n+1}{n}\right)\right)^2 \to 1$ as $n \to \infty$, the radius of convergence is 1.

(b) $(n+1)!/n! = n+1 \to \infty$ as $n \to \infty$, the radius of convergence is 0.

(c) $\left(\frac{(n+1)^2}{4^{n+1}+3n+3}\right)/\left(\frac{n^2}{4^n+3n}\right) = \frac{4^n+3n}{4^{n+1}+3n+3} \cdot \frac{(n+1)^2}{n^2} \to 1/4$ as $n \to \infty$, the radius of convergence is 4.

(d) $\frac{((n+1)!)^3}{(3n+3)!} / \frac{(n!)^3}{(3n)!} = \left(\frac{(n+1)!}{n!}\right)^3 \cdot \frac{(3n)!}{(3n+3)!} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$ as $n \to \infty$, the radius of convergence is 27.

(e) The ratio of two consecutive terms is $\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} \to 1$ as $n \to \infty$, the radius of convergence is 1.

(f) Rewrite the series to $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \frac{1}{2^{2n}} (z^2)^n$ and treat as a function of z^2. Then the ratio of two consecutive terms is $\frac{-1}{(n+1)(n+r+1)2^2} \to 0$ as $n \to \infty$, the radius of convergence is ∞.

17. By definition, for every $\varepsilon > 0$, there exists $N > 0$ such that $n \geq N \Rightarrow \left||a_{n+1}/a_n| - L\right| < \varepsilon$, that is, $L - \varepsilon < |a_{n+1}/a_n| < L + \varepsilon$. Multiplying this inequalities from $n = N$ to $n = k-1$, we get $(L-\varepsilon)^{k-N} < |a_k/a_N| < (L+\varepsilon)^{k-N}$, and

$$(L-\varepsilon)^{1-N/k}|a_N|^{1/k} < |a_k|^{1/k} < (L+\varepsilon)^{1-N/k}|a_N|^{1/k}.$$

For sufficiently large k, we get $L - 2\varepsilon < |a_k|^{1/k} < L + 2\varepsilon$, so the proof is complete.

18. Suppose f has power series expansion

$$f(z) = \sum_{0}^{\infty} a_n z^n \quad (|z| < R).$$

Since radius of convergence of f is R,

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$$

holds. Now fix z_0 such that $|z_0| < R$. Then

$$\begin{aligned}\sum_{n=0}^{N} a_n z^n &= \sum_{n=0}^{\infty} a_n (z_0 + (z - z_0))^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z-z_0)^k \\ &= \sum_{k=0}^{N} \left(\sum_{n=k}^{N} a_n \binom{n}{k} z_0^{n-k} \right) (z-z_0)^k = \sum_{k=0}^{N} b_k(N)(z-z_0)^k,\end{aligned}$$

where we defined

$$b_k(N) = \sum_{n=k}^{N} a_n \binom{n}{k} z_0^{n-k}.$$

Observe that

$$\limsup_{n\to\infty} \left| a_n \binom{n}{k} \right|^{1/n} = \frac{1}{R},$$

and $|z_0| < R$, therefore $b_k(N)$ converges as $N \to \infty$, let's call that limit b_k.

Now we want to show that

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

for appropriate set of z. By definition, the equation above is exactly the same as the one below.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n \binom{n}{k} z_0^{n-k}(z-z_0)^k = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}(z-z_0)^k$$

Now we only need to show that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left|a_n \binom{n}{k} z_0^{n-k}(z-z_0)^k\right|\right) < \infty.$$

(See the end of Chapter 7, which discusses interchanging the order of double sums.)

Since

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left|a_n \binom{n}{k} z_0^{n-k}(z-z_0)^k\right|\right) &\leq \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}|a_n| \binom{n}{k} |z_0|^{n-k}|z-z_0|^k\right) \\ &= \sum_{n=0}^{\infty}|a_n|(|z_0| + |z - z_o|)^k,\end{aligned}$$

we can interchange the sum whenever $|z - z_0| < R - |z_0|$, hence f has power series expansion around z_0.

Note. I thought that the radius of convergence of a power series expansion centered at z_0 is exactly $R - |z_0|$. Therefore, I tried to prove the following, but was unable to do so.

$$\limsup_{k\to\infty}\left|\sum_{n=k}^{N} a_n \binom{n}{k} z_0^{n-k}\right|^{1/k} = \frac{1}{R - |z_0|}$$

19. **(a)** This power series can't converge because $|nz^n| = n \not\to 0$.

(b) This power series absolutely converges because $\left|z^n/n^2\right| = \frac{1}{n^2}$ and $\sum 1/n^2$ converges.

(c) If $z = 1$, this series obviously diverges. If not, use summation by parts formula to get

$$\sum_{n=1}^{N}\frac{1}{n}z^n = \frac{1}{N}Z_N - \sum_{n=1}^{N-1}\left(\frac{1}{n+1} - \frac{1}{n}\right)Z_n = \frac{1}{N}Z_N + \sum_{n=1}^{N-1}\frac{1}{n(n+1)}Z_n,$$

where $Z_n = \sum_{n=1}^{N} z^n = z\frac{1-z^N}{1-z}$. Since Z_n is bounded whenever $|z| = 1$, series above converges.

20. Since $a_n = \frac{1}{n!}f^{(n)}(0)$, we get

$$\begin{aligned}a_n &= \frac{m(m+1)(m+2)\cdots(m+n-1)}{n!} \\ &= \frac{(n+m-1)!}{(m-1)!n!} = \frac{1}{(m-1)!}\cdot\frac{(n+m-1)!}{n!} \sim \frac{1}{(m-1)!}n^{m-1}.\end{aligned}$$

21. Just calculate the sum of first $k+1$ terms and take a limit.

$$\sum_{n=0}^{k}\frac{z^{2^n}}{1 - z^{2^{n+1}}} = \frac{z + z^2 + \cdots + z^{2^{n+1}-1}}{1 - z^{2^{n+1}}} = \frac{z\cdot\frac{1 - z^{2^{n+1}-1}}{1-z}}{1 - z^{2^{n+1}}} = \frac{z}{1-z}\frac{1 - z^{2^{n+1}-1}}{1 - z^{2^{n+1}}} \xrightarrow{k\to\infty} \frac{z}{1-z},$$

$$\sum_{n=0}^{k-1} \frac{2^n z^{2^n}}{1+z^{2^n}} = \frac{z+z^2+\cdots+z^{2^k-1}-\left(2^k-1\right)z^{2^k}}{1-z^{2^k}} = \frac{z\cdot\frac{1-z^{2^k-1}}{1-z}-\left(2^k-1\right)z^{2^k}}{1-z^{2^k}}$$
$$= \frac{1}{1-z}\frac{1}{1-z^{2^k}}\left(z-z^{2^k}-2^k z^{2^k+1}+z^{2^k+1}\right) \xrightarrow{k\to\infty} \frac{z}{1-z}.$$

Since all but the first term are positive, any rearrangement of its terms still converges to the same value.

22. Suppose $\mathbb{N}$ can be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps. Then we can write

$$\sum_{n=1}^{\infty} z^n = \sum_{i=1}^{N}\sum_{j=0}^{\infty} z^{a_i+jd_i} \;\Rightarrow\; \frac{z}{1-z} = \sum_{i=1}^{N} \frac{z^{a_i}}{1-z^{d_i}}.$$

with $N \geq 2$ and all $d_i > 1$. Now multiply $(1-z)\left(1-z^{d_1}\right)\left(1-z^{d_2}\right)\cdots\left(1-z^{d_N}\right)$ both sides to get

$$z\left(1-z^{d_1}\right)\cdots\left(1-z^{d_N}\right)$$
$$= z^{a_1}(1-z)\left(1-z^{d_2}\right)\cdots\left(1-z^{d_N}\right)+\cdots+z^{a_N}(1-z)\left(1-z^{d_1}\right)\cdots\left(1-z^{d_{N-1}}\right).$$

Let $z_1 = e^{2\pi i/d_1}$ and consider a limit $z \to z_1$. Since all terms multiplied by $\left(1-z^{d_1}\right)$ converge to zero, the one remaining term

$$z^{a_1}(1-z)\left(1-z^{d_2}\right)\cdots\left(1-z^{d_N}\right)$$

should also converge to zero. Hence there exists a integer i_1 such that

$$z_1^{d_i} = e^{\frac{d_{i_1}}{d_1}2\pi i} = 1 \;\Rightarrow\; d_1 \mid d_{i_1}.$$

In particular, $d_i > 0$ and $d_i \neq d_j$ for any $i \neq j$, we conclude that $d_1 < d_{i_1}$. Now repeat this process N times to get

$$d_1 < d_{i_1} < d_{i_2} < \cdots < d_{i_N}.$$

Since all i_k are different, at least one i_k is equal to 1, which is a contradiction.

23. Observe that there exists a series of polynomials $\{p_n\}_{n=1}^{\infty}$ such that

$$f^{(n)}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ p_n\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} & \text{if } x > 0. \end{cases} \qquad (n \geq 0)$$

When $n = 0$, it is trivial for $p_0(x) = 1$. Now suppose above statement is true for $n = k$. Since $f^{(k)}(x)$ is differentiable for $x \neq 0$, we only have to check for $x = 0$.

$$\lim_{x\to 0^+} \frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} = \lim_{x\to 0^+} \frac{1}{x}p_k\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} = \lim_{t\to\infty} tp(t)e^{-t^2} = 0,$$

therefore $f^{(k)}$ is differentiable at $x = 0$ and $f^{(k+1)}(0) = 0$. Moreover,

$$f^{(k+1)}(x) = \frac{d}{dx}\left(p_n\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}\right) = \left(-\frac{1}{x^2}p_n'\left(\frac{1}{x}\right)+\frac{2}{x^3}p_n\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x^2}} \quad (x \geq 0),$$

hence $p_{n+1}(x) = -x^2p_n'(x)+2x^3p_n(x)$. So we can conclude that $a_n = f^{(n)}(0) = 0$ for all $n \geq 0$. Finally we conclude that f does not have a converging power series expansion $\sum a_n x^n$ for x near the origin, because if it does, then $f(x) = \sum a_n x^n = \sum 0\cdot x^n = 0$, which is contradiction.

24. By using the definition of integration along curves and making the change of variables $u = a+b-t$,

$$
\begin{aligned}
\int_{\gamma^-} f(z)dz &= \int_a^b f(z(a+b-t))(-z'(a+b-t))dt \\
&= \int_b^a f(z(u))z'(u)du = -\int_a^b f(z(u))z'(u)du \\
&= -\int_\gamma f(z)dz.
\end{aligned}
$$

25. (a) Let $z(t) = Re^{it}$ $(0 \le t < 2\pi)$ then

$$
\int_\gamma z^n dz = \int_0^{2\pi} (Re^{it})^n iRe^{it} dt = iR^{n+1} \int_0^{2\pi} e^{(n+1)it} dt = \begin{cases} iR^{n+1} 2\pi & \text{if } n = -1, \\ 0 & \text{if } n \ne -1. \end{cases}
$$

(b) Let $z(t) = A + Re^{it}$ $(0 \le t < 2\pi)$ where $|A| > |R|$.

$$
\int_\gamma z^n dz = \int_0^{2\pi} (A + Re^{it})^n iRe^{it} dt
$$

If $n \ne -1$, then this integral can be calculated as

$$
\int_0^{2\pi} (A + Re^{it})^n iRe^{it} dt = \left[\frac{1}{n+1} (A + Re^{it})^{n+1} \right]_0^{2\pi} = 0.
$$

For $n = -1$, the calculation is as follows;

$$
\begin{aligned}
\int_0^{2\pi} (A + Re^{it})^n iRe^{it} dt &= \frac{iR}{A} \int_0^{2\pi} \frac{e^{it}}{1 + \frac{R}{A} e^{it}} dt = \frac{iR}{A} \int_0^{2\pi} e^{it} \sum_{n=0}^\infty \left(-\frac{R}{A} e^{it} \right)^n dt \\
&= \frac{iR}{A} \sum_{n=0}^\infty \int_0^{2\pi} \left(-\frac{R}{A} \right)^n e^{(n+1)it} dt \\
&= \frac{iR}{A} \sum_{n=0}^\infty \left(-\frac{R}{A} \right)^n \left[\frac{1}{(n+1)i} e^{(n+1)it} \right]_0^{2\pi} \\
&= 0.
\end{aligned}
$$

Therefore the integral value is always 0.

(c) Observe that

$$
\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{1}{a-b} \left(\int_\gamma \frac{1}{z-a} dz - \int_\gamma \frac{1}{z-b} dz \right).
$$

The second integral is 0 because it was calculated in **(b)**. Now we calculate the first integral.

$$
\begin{aligned}
\int_\gamma \frac{1}{z-a} dz = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta} - a} d\theta &= i \int_0^{2\pi} \frac{1}{1 - ae^{-i\theta}} d\theta = i \int_0^{2\pi} \sum_{n=0}^\infty (ae^{-i\theta})^n d\theta \\
&= i \sum_{n=0}^\infty \int_0^{2\pi} (-a)^n e^{-in\theta} d\theta = i \cdot 2\pi = 2\pi i
\end{aligned}
$$

since the integral is not zero only when $n = 0$. Finally we get

$$
\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{1}{a-b} (2\pi i - 0) = \frac{2\pi i}{a-b}.
$$

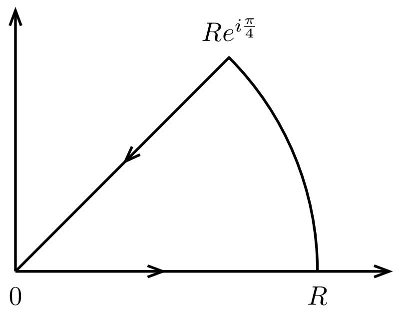
26. Suppose F, G are primitives of f, hence $F' = G' = f$. Note that Ω is connected, therefore pathwise connected. Now take any point $w \in \Omega$ and define $\gamma_{w,z}$ as the curve from w to z. Then for any $z \in \Omega$,

$$
\begin{aligned}
(F-G)(z) &= \int_{\gamma_{w,z}} (F-G)'(\zeta)\,d\zeta + (F-G)(w) \\
&= \int_{\gamma_{w,z}} 0\,d\zeta + (F-G)(w) \\
&= (F-G)(w).
\end{aligned}
$$

So $F(z), G(z)$ is differ by a constant.

Chapter 2. Cauchy's Theorem and Its Applications

1. We integrate the function e^{-z^2} over the path in following figure. This is possible and the value equals 0 because e^{-z^2} is holomorphic in $\mathbb{C}$.



By Cauchy's theorem, We get

$$
\int_0^R e^{-x^2}dx + \int_0^{\frac{\pi}{4}} e^{\left(-Re^{i\theta}\right)^2} iRe^{i\theta}d\theta + \int_R^0 e^{\left(-te^{i\frac{\pi}{4}}\right)^2} e^{i\frac{\pi}{4}}dt = 0.
$$

The first term converges to $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$. Now we estimate the second term.

$$
\left|\int_0^{\frac{\pi}{4}} e^{\left(-Re^{i\theta}\right)^2} iRe^{i\theta}d\theta\right| \le \int_0^{\frac{\pi}{4}} \left|e^{\left(-Re^{i\theta}\right)^2} iRe^{i\theta}\right|d\theta = R\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)}d\theta
$$

where the last value equals $\frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\cos x}dx$ by changing the variables $x = 2\theta$.

For large R, let $\delta = R^{-3/2} < \pi/2$. If $x \in [0, \pi/2 - \delta]$ then $\cos x \ge \cos(\pi/2 - \delta) = \sin\delta$. So

$$
\begin{aligned}
\frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\cos x}dx &= \frac{R}{2}\int_0^{\frac{\pi}{2}-\delta} e^{-R^2\cos x}dx + \frac{R}{2}\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} e^{-R^2\cos x}dx \\
&\le \frac{R}{2}\int_0^{\frac{\pi}{2}-\delta} e^{-R^2\sin\delta}dx + \frac{R}{2}\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} 1\,dx \\
&= \frac{R}{2}\Big(\frac{\pi}{2} - \delta\Big)e^{-R^2\sin\delta} + \frac{R}{2}\delta.
\end{aligned}
$$

Since $R^2\sin\delta \sim R^{1/2}$ and $R\delta \sim R^{-1/2}$ as $R \to \infty$, this sum converges to 0.

Finally, the third term is equal to

$$
\int_0^R e^{-it^2}e^{\frac{\pi}{4}i}dt = \frac{1+i}{\sqrt{2}}\left(\int_0^R \cos(t^2)dt - i\int_0^R \sin(t^2)dt\right).
$$

Now we take limit $R \to \infty$ then we get

$$\frac{\sqrt{\pi}}{2} + 0 + \frac{1+i}{\sqrt{2}}\left(\int_0^\infty \cos(t^2)dt - i\int_0^\infty \sin(t^2)dt\right) = 0$$

and

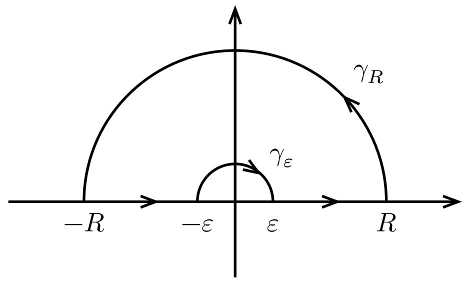
$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

2. First observe that

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}dx = \frac{1}{2i}\frac{\int_0^\infty \frac{e^{ix}-1}{x}dx - \int_0^\infty e^{-ix}-1}{x}dx) =$$

$$= \frac{1}{2i}\left(\int_0^\infty \frac{e^{ix}-1}{2}dx - \int_0^{-\infty} \frac{e^{ix}-1}{x}dx\right) = \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{2}dx.$$

Now integrate the function $(e^{iz}-1)/z$ over the indented semicircle below.



Applying Cauchy's theorem gives

$$\int_{-R}^{-\varepsilon} \frac{e^{iz}-1}{z}dz + \int_{\gamma_\varepsilon} \frac{e^{iz}-1}{z}dz + \int_\varepsilon^R \frac{e^{iz}-1}{z}dz + \int_{\gamma_R} \frac{e^{iz}-1}{z}dz = 0.$$

Since $(e^{iz}-1)/z = i + E(z)$ where $E(z) \to 0$ as $z \to 0$,

$$\left|\int_{\gamma_\varepsilon} \frac{e^{iz}-1}{z}dz\right| \leq \pi\varepsilon \cdot \sup_{z\in\gamma_\varepsilon}|i+E(z)| \xrightarrow{\varepsilon\to 0} 0.$$

Integral over γ_R is

$$\int_{\gamma_R} \frac{e^{iz}-1}{z}dz = \int_0^\pi \frac{e^{iRe^{i\theta}}-1}{Re^{i\theta}}iRe^{i\theta}d\theta = i\int_0^\pi e^{iRe^{i\theta}} - 1d\theta = -\pi i + \int_0^\pi e^{iRe^{i\theta}}d\theta.$$

For large R, let $\delta = R^{-1/2} < \pi/2$. Then

$$\begin{aligned}\left|\int_0^\pi e^{iRe^{i\theta}}d\theta\right| &\leq \int_0^\pi \left|e^{iRe^{i\theta}}\right|d\theta = \int_0^\pi e^{-R\sin\theta}d\theta \\ &= \int_0^\delta e^{-R\sin\theta}d\theta + \int_\delta^{\pi-\delta} e^{-R\sin\theta}d\theta + \int_{\pi-\delta}^\pi e^{-R\sin\theta}d\theta \\ &\leq 2\delta + (\pi-2\delta)e^{-R\sin\delta}\end{aligned}$$

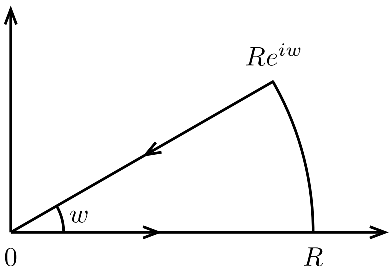
which converges to 0 as $R \to \infty$. Therefore letting $\varepsilon \to 0$ and $R \to \infty$ gives

$$\int_{-\infty}^\infty \frac{e^{ix}-1}{2}dx = \pi i$$

and

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi i}{2i} = \frac{\pi}{2}.$$

3. Integrate function e^{-Az} over the curve below.



Since e^{-Az} is holomorphic in $\mathbb{C}$, applying Cauchy's theorem to get

$$\int_0^R e^{-At} dt + \int_0^w e^{-ARe^{i\theta}} iRe^{i\theta} d\theta + \int_R^0 e^{-Ate^{iw}} e^{iw} dt = 0.$$

The first term is equal to $\frac{1}{A}(1 - e^{-AR})$, which converges to $1/A$. The absolute value of second term is

$$\begin{aligned}\left|\int_0^w e^{-ARe^{i\theta}} iRe^{i\theta} d\theta\right| &\le \int_0^w \left|e^{-ARe^{i\theta}} iRe^{i\theta}\right| d\theta \\ &\le \int_0^w Re^{-AR\cos\theta} d\theta \le \int_0^w Re^{-AR\cos w} d\theta \\ &= wRe^{-AR\cos w}\end{aligned}$$

which converges to 0 as $R \to \infty$. The third term is

$$\begin{aligned}\int_R^0 e^{-Ate^{iw}} e^{iw} dt &= e^{iw} \int_R^0 e^{-At\cos w}(\cos(At\sin w) - i\sin(At\sin w))dt \\ &= -e^{iw} \int_0^R e^{-at}(\cos(bt) - i\sin(bt))dt.\end{aligned}$$

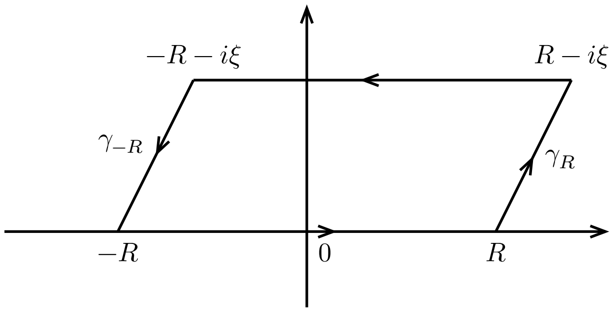
Therefore when $R \to \infty$, we get

$$\int_0^\infty e^{-at}\cos(bt)dt - \int_0^\infty e^{-at}\sin(bt)dt = \frac{1}{A}e^{-iw} = \frac{1}{A}\left(\frac{a}{A} - i\frac{b}{A}\right) = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

Hence

$$\int_0^\infty e^{-at}\cos(bt)dt = \frac{a}{a^2+b^2}, \quad \int_0^\infty e^{-at}\sin(bt)dt = \frac{b}{a^2+b^2}.$$

4. We will integrate the function $e^{-\pi z^2}$ along the curve below.



Since $e^{-\pi z^2}$ is holomorphic in $\mathbb{C}$, by Cauchy's theorem,

$$\int_{-R}^{R} e^{-\pi x^2} dx + \int_{\gamma_R} f(z) dz + \int_{R}^{-R} e^{-\pi (x-i\xi)^2} dx + \int_{\gamma_{-R}} f(z) dz = 0.$$

Integral over the real segment converges to $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$. The second and fourth term converges to 0, because the size of integrand becomes smaller proportionally to $e^{-\pi R^2}$. The third term is

$$\int_{R}^{-R} e^{-\pi (x-i\xi)^2} dx = \int_{R}^{-R} e^{-\pi x^2 + 2\pi i x\xi + \pi \xi^2} dx = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2 + 2\pi i x \xi} dx.$$

Let $R \to \infty$ and get

$$\int_{-\infty}^{\infty} e^{-\pi x^2 + 2\pi i x \xi} dx = e^{-\pi \xi^2}.$$

5. Let $f = F + iG$. Since f is holomorphic on Ω, $\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}$ and $\frac{\partial F}{\partial y} = -\frac{\partial G}{\partial x}$ hold by Cauchy-Riemann equation.

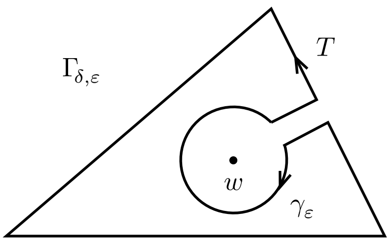
Also, by Green's theorem,

$$\int_T F\,dx - G\,dy = \int_{T^\circ} \left(-\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dxdy = \int_{T^\circ} 0\,dxdy = 0,$$
$$\int_T G\,dx + F\,dy = \int_{T^\circ} \left(\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} \right) dxdy = \int_{T^\circ} 0\,dxdy = 0.$$

Hence

$$\int_T f(z)dz = \int_T (F + iG)(dx + idy) = \int_T F\,dx - G\,dy + i \int_T G\,dx + F\,dy = 0.$$

6. Consider the keyhole $\Gamma_{\delta,\varepsilon}$, where δ is the width of the corridor, and ε is the radius of small circle centered at w.



By Cauchy's theorem,

$$\int_{\Gamma_{\delta,\varepsilon}} f(z)dz = 0.$$

Since f is continuous, letting $\delta \to 0$ gives

$$\int_T f(z)dz = \int_{\gamma_\varepsilon} f(z)dz.$$

because the integrals over the two sides of the corrider cancel out. However, $f(z)$ is bounded near w, therefore

$$\left|\int_{\gamma_\varepsilon} f(z)dz\right| \le 2\pi\varepsilon \cdot \sup_{|z-w|\le\varepsilon} |f(z)| \to 0$$

as $\varepsilon \to 0$. Hence we get

$$\int_T f(z)dz = 0.$$

7. Since $f : \mathbb{D} \to \mathbb{C}$ is holomorphic,

$$f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^2}d\zeta$$

holds for $0 < r < 1$. Also, changing the variables $\zeta \mapsto -\zeta$, then we get

$$f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} -\frac{f(-\zeta)}{\zeta^2}d\zeta.$$

Therefore

$$\begin{aligned}2|f'(0)| &= \left|\frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2}d\zeta.\right| \le \frac{1}{2\pi}\int_{|\zeta|=r} \frac{|f(\zeta)-f(-\zeta)|}{|\zeta|^2}d\zeta \\ &\le \frac{1}{2\pi}\int_{|\zeta|=r} \frac{d}{|\zeta|^2}d\zeta = \frac{1}{2\pi}\cdot 2\pi r \cdot \frac{d}{r^2} = \frac{d}{r}.\end{aligned}$$

Since $0 < r < 1$ is arbitrary,

$$2|f'(0)| \le \inf_{0<r<1} \frac{d}{r} = d.$$

8. We fix integer n and $R < 1$. First consider the case $\eta \ge 0$. For $x \ge 0$, by Cauchy inequalities,

$$\left|f^{(n)}(x)\right| \le \frac{n!}{R^n}\sup_{z\in C_R(x)} |f(z)| = \frac{n!}{R^n}A(1+x+R)^\eta.$$

Since

$$\frac{n!}{R^n}A\frac{(1+x+R)^\eta}{(1+x)^\eta} = \frac{n!}{R^n}A\left(1+\frac{R}{1+x}\right)^\eta \to \frac{n!}{R^n}A$$

as $n \to \infty$, $\frac{n!}{R^n}A\frac{(1+x+R)^\eta}{(1+x)^\eta}$ is bounded, therefore there exists constant A_n^+ such that

$$\left|f^{(n)}(x)\right| \le A_n^+(1+x)^\eta$$

for all $x \ge 0$. Similarly, for $x < 0$,

$$\left|f^{(n)}(x)\right| \le \frac{n!}{R^n}\sup_{z\in C_R(x)} |f(z)| = \frac{n!}{R^n}A(1+x-R)^\eta,$$

and

$$\frac{n!}{R^n}A\frac{(1+x-R)^\eta}{(1-x)^\eta} = \frac{n!}{R^n}\left(1-\frac{R}{x-1}\right)^\eta \to \frac{n!}{R^n}A$$

as $n \to \infty$, there exists constant A_n^- such that

$$\left|f^{(n)}(x)\right| \le A_n^+(1-x)^\eta$$

for all $x < 0$. Letting $A_n = \max\{A_n^-, A_n^+\}$ completes the proof.

In the case of $\eta < 0$, we divide $\mathbb{R}$ into $(-\infty, -R) \cup [-R, R] \cup (R, \infty)$. The case $(-\infty, -R), (R, \infty)$ is similar to argument above, we can get A_n^- and A_n^+. When $x \in [-R, R]$,

$$\left|f^{(n)}(x)\right| \le \frac{n!}{R^n}A.$$

Since the function

$$\frac{n!}{R^n}A\frac{1}{(1+|x|)^\eta}$$

is bounded in compact set $[-R, R]$, defining by

$$A_n^* = \sup_{x\in[-R,R]} \frac{n!}{R^n}A\frac{1}{(1+|x|)^\eta}$$

gives

$$\left|f^{(n)}(x)\right| \le A_n^*(1+|x|)^\eta.$$

Now letting $A_n = \max\{A_n^-, A_n^*, A_n^+\}$ completes the proof.

9. We can assume $z_0 = 0$. If not, consider the function $f : \Omega - z_0 \to \Omega - z_0$ which is defined by

$$f(z) = \varphi(z+z_0) - z_0.$$

Since $\Omega - z_0$ is also bounded open subset of $\mathbb{C}$, and $f(0) = \varphi(z_0) - z_0 = 0$ and $f'(0) = \varphi'(z_0) = 1$, we can prove by assuming a special case where $z_0 = 0$.

Let $\varphi(z) = z + a_n z^n + O(z^{n+1})$ and define $\varphi_k = \varphi \circ \cdots \circ \varphi$ (k times). Then we get

$$\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$$

by induction, because $k = 1$ case is clearly true, and equation at $k = l$ leads to

$$\begin{aligned}\varphi_{l+1}(z) = \varphi_l(\varphi(z)) &= \varphi(z) + l a_n \varphi(z)^n + O(\varphi(z)^{n+1}) \\ &= z + a_n z^n + O(z^{n+1}) + l a_n z^n + O(z^{n+1}) + O(z^{n+1}) \\ &= z + (l+1) a_n z^n + O(z^{n+1}),\end{aligned}$$

which is equation for $k = l+1$. Use Cauchy inequalities to get

$$\left|\varphi_k^{(n)}(z)\right| = k n! a_n \le \frac{n!}{R^n}\|f\|_C$$

for appropriate R. Since Ω is bounded set, $\|f\|_C$ is bounded, therefore $k \to \infty$ leads to a contradiction.

10. No. Consider the continuous function $f(z) = \mathrm{Re(z)}$. If series of polynomials $\{P_n\}$ uniformly converges to f, then

$$\int_{\partial\mathbb{D}} P_n(z)dz \to \int_{\partial\mathbb{D}} f(z)dz$$

as $n \to \infty$. Note that $\int_{\partial\mathbb{D}} P_n(z)dz = 0$ for all $n \in \mathbb{N}$. However,

$$\int_{\partial\mathbb{D}} f(z)dz = \int_0^{2\pi} \mathrm{Re}(e^{i\theta})ie^{i\theta}d\theta = i\int_0^{2\pi}(\cos^2\theta + i\cos\theta\sin\theta)d\theta$$
$$= i\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2} + i\frac{1}{2}\sin 2\theta\right)d\theta = i\cdot\frac{1}{2}\cdot 2\pi = \pi i \neq 0$$

which is contradiction.

11. (a) Note that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta,\ 0 = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - R^2/\overline{z}}d\zeta.$$

Also, by calculation,

$$\frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - R^2/\overline{z}} = \frac{\zeta}{\zeta - z} - \frac{\overline{z}}{\overline{z} - R^2/\zeta} = \frac{\zeta}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta} - \overline{z}} = \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right).$$

Therefore

$$f(z) = \frac{1}{2\pi i}\int_C\left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - R^2/\overline{z}}\right)d\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta}\left(\frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - R^2/\overline{z}}\right)d\zeta$$
$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta}\,\mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right)d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right)d\varphi.$$

(b)

$$\mathrm{Re}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r}\right) = \frac{1}{2}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} + \frac{Re^{-i\gamma} + r}{Re^{-i\gamma} - r}\right)$$
$$= \frac{1}{2}\left(\frac{2R^2 - 2r^2}{R^2 + r^2 - Rr(\cos^{i\gamma} + e^{-i\gamma})}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

12. (a) Let

$$g(z) = 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

Then g is holomorphic in $\mathbb{D}$ since it satisfies Cauchy-Riemann equations in $\mathbb{D}$, indeed,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \qquad \because \Delta u = 0,$$
$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) \qquad \because u \in \mathcal{C}^2.$$

Hence there exists holomorphic function F such that $F' = g$ by **Theorem 2.1**. Let $\mathrm{Re}(F) = U$. Then

$$F' = 2\frac{\partial U}{\partial z} \Rightarrow \frac{\partial U}{\partial z} = \frac{\partial u}{\partial z} \Rightarrow U = u + c,$$

where $c \in \mathbb{R}$ is constant. Now define $f = F - c$ then

$$\mathrm{Re}(f) = \mathrm{Re}(F - c) = U - c = u.$$

Also If f and g is a holomorphic function that satisfies the condition of problem, then

$$\mathrm{Re}(f - g) = \mathrm{Re}(f) - \mathrm{Re}(g) = u - u = 0,$$

which is constant, therefore $\mathrm{Im}(f-g)$ is also constant by **Exercise 13(a), Chapter 1**.

(b) By **(a)**, there exists holomorphic function f such that $\mathrm{Re}(f) = u$. Put $R = 1, z = e^{i\theta}$ in formula from **Exercise 11(a)** to get

$$\begin{aligned} u(z) = \mathrm{Re}(f(z)) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(f(e^{i\varphi}))\ \mathrm{Re}\left(\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right)d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\ \mathrm{Re}\left(\frac{e^{i(\varphi-\theta)}+r}{e^{i(\varphi-\theta)}-r}\right)d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(e^{i\varphi})d\varphi. \end{aligned}$$

13. Note that $c_k = 0$ means $f^{(k)}(0) = 0$ in the power series expansion below.

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Now define sequence of sets $\{A_k\}_{k=1}^{\infty}$ as follows

$$A_k = \left\{z \in \mathbb{C} :\ f^{(k)}(z) = 0\right\}.$$

Since for each $z_0 \in \mathbb{C}$, at least one coefficient is equal to 0, therefore there exists $k \in \mathbb{N}$ such that

$$z_o \in A_k.$$

Hence

$$\bigcup_{k=1}^{\infty} A_k = \mathbb{C}.$$

If every A_k are countable, then $\mathbb{C}$ is countably infinite, which is contradiction. Therefore at least one A_k is uncountable. Then there is a limit point of A_k in $\mathbb{C}$. By **Theorem 4.8**, $f^{(k)}(z) \equiv 0$. Accordingly, f is polynomial of degree up to $k-1$.

14. Suppose f has a pole of order k at z_0. By **Theorem 1.3, Chapter 3**,

$$f(z) = \left(\frac{b_{-k}}{(z-z_0)^k} + \cdots + \frac{b_{-1}}{z-z_0}\right) + G(z),$$

where $G(z)$ is a holomorphic function in a neighborhood of z_0. Since f is holomorphic in an open set containing the closed unit disc,

$$f(z) - \left(\frac{b_{-k}}{(z-z_0)^k} + \cdots + \frac{b_{-1}}{z-z_0}\right) = \sum_{n=0}^{\infty} c_n z^n \quad (|z| < 1+\delta).$$

for small enough δ. Note that $\sum c_n z^n$ converges for $|z| = 1$, therefore $c_n \to 0$ as $n \to \infty$. By definition,

$$\sum_{n=0}^{\infty} a_n z^n = \left(\frac{b_{-k}}{(z-z_0)^k} + \cdots + \frac{b_{-1}}{z-z_0}\right) + \sum_{n=0}^{\infty} c_n z^n.$$

Note that

$$\frac{1}{(z-z_0)^l} = \sum_{n=0}^{\infty} (-1)^l \binom{l+n-1}{l-1} z_0^{-n-l} z^n \quad (|z| < 1).$$

Therefore

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{l=1}^{k}\sum_{n=0}^{\infty} (-1)^l \binom{l+n-1}{l-1} z_0^{-n-l} z^n + \sum_{n=0}^{\infty} c_n z^n.$$

Since equation above holds for all $z \in \mathbb{D}$, we get

$$a_n = c_n + \sum_{l=1}^{k} (-1)^l (l+n-1.l-1) z_0^{-k-l}.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{a_{n+1}} &= \lim_{n\to\infty} \frac{c_n + \sum_{l=1}^{k} (-1)^l \binom{l+n-1}{l-1} z_0^{-n-l}}{c_{n+1} + \sum_{l=1}^{k} (-1)^l \binom{l+n}{l-1} z_0^{-n-l-1}} \\
&= \lim_{n\to\infty} \frac{c_n \cdot \frac{1}{n^{k-1}} z_0^n + \sum_{l=1}^{k} (-1)^l \frac{1}{n^{k-1}} \binom{l+n-1}{l-1} z_0^{-l}}{c_{n+1} \cdot \frac{1}{n^{k-1}} z_0^n + \sum_{l=1}^{k} (-1)^l \frac{1}{n^{k-1}} \binom{l+n}{l-1} z_0^{-l-1}} \\
&= z_0^{-l} / z_0^{-l-1} = z_0.
\end{aligned}$$

15. This proof is made up of several steps.

Step 1. Define g as

$$g(z) = \begin{cases} f(z) & |z| \le 1 \\ 1/\overline{f(1/\overline{z})} & |z| > 1 \end{cases}$$

This is well-defined because $f(z) \neq 0$ for $z \in \overline{\mathbb{D}}$, and obviously g is continuous in $\mathbb{C}$.

Step 2. g is bounded in $\mathbb{C}$. Since g is continuous on compact set $\overline{\mathbb{D}}$, there exists positive number m, M such that $m \le |g(z)| \le M$ for $z \in \overline{\mathbb{D}}$. therefore $1/M \le |g(z)| \le 1/m$ for $|z| > 1$.

Step 3. g is holomorphic in $|z| > 1$, since g satisfies Cauchy-Riemann equations in open region $\{z : |z| > 1\}$. Let $g(x+iy) = u(x.y) + iv(x,y)$ for $x + iy \in \mathbb{D}$. Note that

$$\frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta}, \ \frac{\partial u}{\partial \beta} = -\frac{\partial v}{\partial \alpha}$$

holds in $\mathbb{D}$ because g is holomorphic in $\mathbb{D}$.

For $x + iy \in \{z : |z| > 1\}$,

$$g(x+iy) = 1/\overline{g(1/\overline{x+iy})} = \frac{u(\alpha,\beta)}{u(\alpha,\beta)^2 + v(\alpha,\beta)^2} + i\frac{v(\alpha,\beta)}{u(\alpha,\beta)^2 + v(\alpha,\beta)^2},$$

where $\alpha(x,y) = \frac{x}{x^2+y^2}, \beta(x,y) = \frac{y}{x^2+y^2}$. Now compute the partial derivatives;

$$\frac{\partial}{\partial x}\left(\frac{u}{u^2+v^2}\right) = \frac{\partial u}{\partial \alpha}\left(\frac{y^2-x^2}{(x^2+y^2)^2}(v^2-u^2) - 2uv\frac{-2xy}{(x^2+y^2)^2}\right) + \frac{\partial u}{\partial \beta}\left(\frac{-2xy}{(x^2+y^2)^2}(v^2-u^2) + 2uv\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{v}{u^2+v^2}\right) = \frac{\partial v}{\partial \beta}\left(\frac{x^2-y^2}{(x^2+y^2)^2}(u^2-v^2) - 2uv\frac{-2xy}{(x^2+y^2)^2}\right) + \frac{\partial v}{\partial \alpha}\left(\frac{-2xy}{(x^2+y^2)^2}(u^2-v^2) + 2uv\frac{x^2-y^2}{(x^2+y^2)^2}\right)$$

Therefore

$$\frac{\partial}{\partial x}\left(\frac{u}{u^2+v^2}\right) = \frac{\partial}{\partial y}\left(\frac{v}{u^2+v^2}\right).$$

Also,

$$\frac{\partial}{\partial y}\left(\frac{u}{u^2+v^2}\right)=\frac{\partial u}{\partial \beta}\left(\frac{x^2-y^2}{(x^2+y^2)^2}(v^2-u^2)-2uv\frac{-2xy}{(x^2+y^2)^2}\right)+\frac{\partial u}{\partial \alpha}\left(\frac{-2xy}{(x^2+y^2)^2}(v^2-u^2)+2uv\frac{x^2-y^2}{(x^2+y^2)^2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{v}{u^2+v^2}\right)=\frac{\partial v}{\partial \alpha}\left(\frac{y^2-x^2}{(x^2+y^2)^2}(u^2-v^2)-2uv\frac{-2xy}{(x^2+y^2)^2}\right)+\frac{\partial v}{\partial \beta}\left(\frac{-2xy}{(x^2+y^2)^2}(u^2-v^2)+2uv\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

Therefore

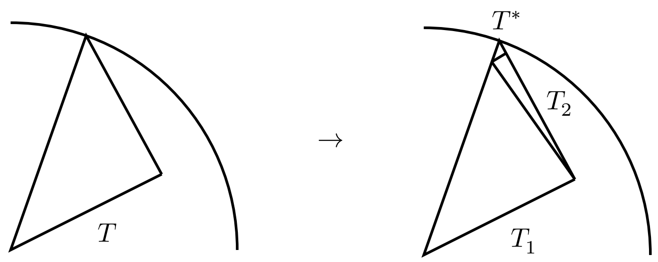
$$\frac{\partial}{\partial y}\left(\frac{u}{u^2+v^2}\right)=-\frac{\partial}{\partial x}\left(\frac{v}{u^2+v^2}\right).$$

Hence g satisfies Cauchy-Riemann equations in $\{z : |z| > 1\}$.

Step 4. T is any triangle that belongs to $\mathbb{D}$, then

$$\int_T g(z)dz = 0.$$

If $T \subset \mathbb{D}$, then result is trivial by **Theorem 1.1**. Now suppose one vertex of a triangle touches the boundary of a circle. We split a triangle into multiple triangles. Let T^* be the triangle that touches the boundary of the circle.



Integral over other triangles is simply 0, therefore

$$\int_T g(z)dz = \int_{T^*} g(z)dz.$$

Since g is bounded in compact set $\overline{\mathbb{D}}$, sending perimeter of T^* to 0 gives

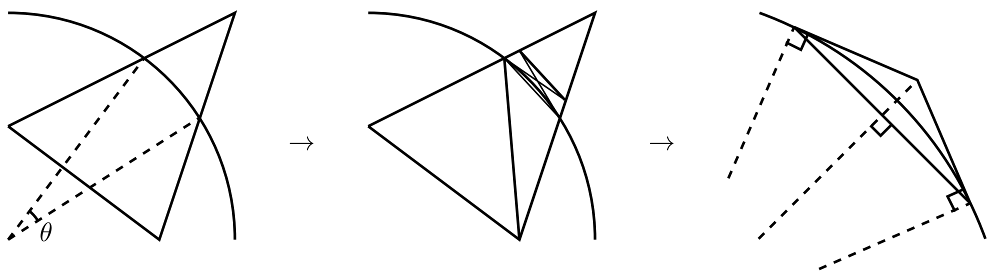
$$\int_T g(z)dz = 0.$$

If more than two vertexes of a triangle touch the boundary of circle, we can show that integral over the triangle is 0, by splitting to triangles which have only one intersection with circle.

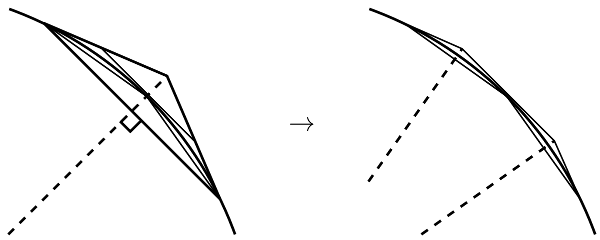
Furthermore, we can see that integral over the triangle $T \in \{z : |z| \geq 1\}$ is also zero.

Step 5. g is holomorphic in $\mathbb{C}$. We prove this by showing that $\int_T g(z)dz = 0$ for all triangles $T \in \mathbb{C}$. The first step is to convert an arbitrary triangle into an isosceles triangle of a certain shape by splitting it.

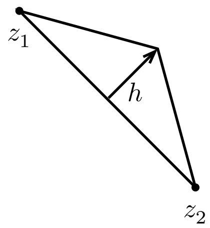
Suppose there are two intersection with $\partial\mathbb{D}$. The goal is a triangle with tangents at those intersections as sides. This is easily attained by drawing tangents at each intersection.



You can then draw a tangent line from the point where the circle meets the perpendicular bisector of the straight line connecting the intersection, so split the triangle into smaller triangles with same shape.



Repeat this n times, then there are 2^n triangles of this shape. Let $m = 2^n$, and let's call each triangle γ_k $(1 \leq k \leq m)$. Now we estimate integrals $\int_{\gamma_k} g(z)dz$.



Let h is a vector that represents height of γ_k. Note that $|h| = \sin^2(\theta/2m)\sec(\theta/2m) = O(1/m^2)$. Since f is continuous on compact set $\overline{D_2}$, f is uniformly continuous on $\overline{D_2}$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$. Then

$$\begin{aligned}
\left|\int_{\gamma_k} g(z)dz\right| &= \Bigg|\int_0^1 g(z_1 + tw)wdt \\
&\quad - \left(\int_0^{1/2} g(z_1 + t(w + 2h))wdt + \int_{1/2}^1 g(z_1 + tw + 2(1-t)h)wdt\right)\Bigg| \\
&\leq \int_0^{1/2} |g(z_1 + tw) - g(z_1 + tw + 2h)||w|dt \\
&\quad + \int_{1/2}^1 |g(z_1 + tw) - g(z_1 + tw + 2(1-t)h)||w|dt \\
&\leq \frac{1}{2}|w|\varepsilon + \frac{1}{2}|w|\varepsilon = \sin\left(\frac{\theta}{2m}\right)\varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\int_T g(z)dz\right| = \left|\sum_{k=1}^m \int_{\gamma_k} g(z)dz\right| \leq \sum_{k=1}^m \left|\int_{\gamma_k} g(z)dz\right| \leq \sum_{k=1}^m \sin\left(\frac{\theta}{2m}\right)\varepsilon \\
= m\sin\left(\frac{\theta}{2m}\right)\varepsilon < m\frac{\theta}{2m}\varepsilon = \frac{\theta}{2}\varepsilon
\end{aligned}$$

for small ε and large m. Hence $\int_T g(z)dz = 0$.

Step 6. Since g is bounded and holomorphic in $\mathbb{C}$, g is constant. $f(z) = g(z)$ for $z \in \mathbb{D}$, so f is also constant.

Chapter 3. Meromorphic Functions and the Logarithm

1. The zeros of $\sin \pi z$ are

$$\sin \pi z = 0 \Rightarrow e^{i\pi z} = e^{-i\pi z} \Rightarrow e^{2\pi i z} = 1 \Rightarrow z = n \ (n \in \mathbb{Z}).$$

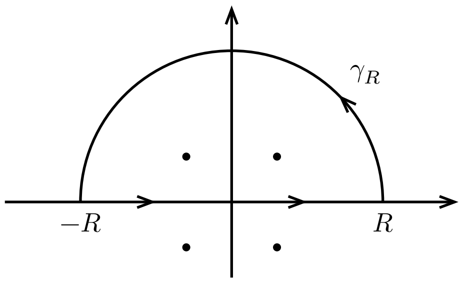
and these are each of order 1 because

$$\lim_{z \to n} \frac{\sin \pi z - \sin \pi n}{z - n} = \pi \cos \pi z|_{z=n} = (-1)^n \pi \neq 0.$$

Therefore the residue of $\frac{1}{\sin \pi z}$ at $z = n$ is

$$\operatorname*{Res}_{z=n} \frac{1}{\sin \pi z} = \lim_{z \to n} (z - n) \frac{1}{\sin \pi z - \sin \pi n} = \frac{(-1)^n}{\pi}.$$

2. Integrate the function $1/(1 + z^4)$ over the semicircle with radius R.



Note that there are two simple poles of $1/(1 + z^4)$ inside the semicircle, $z = \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}$. By residue formula,

$$\int_{-R}^{R} \frac{1}{1 + x^4} dx + \int_{\gamma_R} \frac{1}{1 + z^4} dz = 2\pi i \left(\operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{1}{1 + z^4} + \operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{1}{1 + z^4} \right).$$

The integral over the arc is less than $2\pi R \cdot \frac{1}{R^4 - 1}$, hence converges to 0. Now calculate residues.

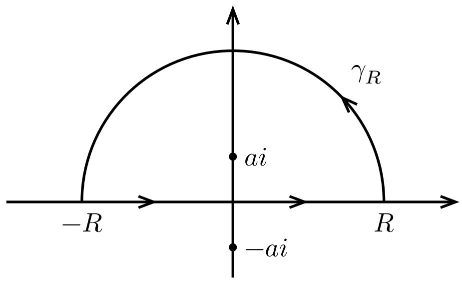
$$\begin{aligned}
\operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{1}{1 + z^4} &= \frac{1}{\left(\frac{1+i}{\sqrt{2}} - \frac{-1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}} - \frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{1}{\sqrt{2} \cdot \sqrt{2} i \cdot \sqrt{2}(1 + i)} = \frac{1}{2\sqrt{2} i (1 + i)},
\end{aligned}$$

$$\begin{aligned}
\operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{1}{1 + z^4} &= \frac{1}{\left(\frac{-1+i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}} - \frac{-1-i}{\sqrt{2}}\right)} \\
&= \frac{1}{-\sqrt{2} \cdot \sqrt{2} i \cdot \sqrt{2}(-1 + i)} = \frac{1}{2\sqrt{2} i (1 - i)}.
\end{aligned}$$

Let $R \to \infty$ to get

$$\int_{-\infty}^{\infty} \frac{1}{1 + z^4} dz = 2\pi i \cdot \frac{1}{2\sqrt{2}} \cdot \frac{1}{i} \left(\frac{1}{1 + i} + \frac{1}{1 - i} \right) = \frac{\pi}{\sqrt{2}} \left(\frac{1 + i + 1 - i}{1 - i^2} \right) = \frac{\pi}{\sqrt{2}}.$$

3. Integrate the function $e^{iz}/(z^2 + a^2)$ over the semicircle of radius R.



By residue formula, we get

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}dx + \int_{\gamma_R} \frac{e^{iz}}{z^2+a^2}dz = 2\pi i \operatorname*{Res}_{z=ai} \frac{e^{iz}}{z^2+a^2}.$$

Note that if $\mathrm{Im}(z) \geq 0$ then $\left|e^{iz}\right| = e^{-\mathrm{Im}(z)} \leq 0$. Therefore the integral over the arc is less than $2\pi R \cdot \frac{1}{R^2-a^2}$, which converges to 0. Also,

$$\operatorname*{Res}_{z=ai} \frac{e^{iz}}{z^2+a^2} = \frac{e^{i\cdot ai}}{ai+ai} = \frac{e^{-a}}{2ai}.$$

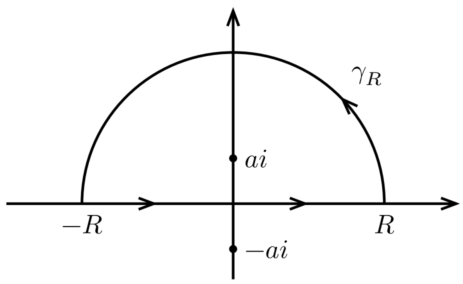
Hence let $R \to \infty$ to get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2} = 2\pi i \cdot \frac{e^{-a}}{2ai} = \pi \frac{e^{-a}}{a}$$

and

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\right) = \pi \frac{e^{-a}}{a}.$$

4. Integrate the function $ze^{iz}/(z^2+a^2)$ over the semicircle of radius R.



There is are only one simple pole $z = ai$ in the semicircle, hence by residue formula,

$$\int_{-R}^{R} \frac{xe^{ix}}{x^2+a^2}dx + \int_{\gamma_R} \frac{ze^{iz}}{z^2+a^2}dz = 2\pi i \operatorname*{Res}_{z=ai} \frac{ze^{iz}}{z^2+a^2}.$$

We now estimate the integral over the arc.

$$\begin{aligned}\left|\int_{\gamma_R} \frac{ze^{iz}}{z^2+a^2}dz\right| &= \left|\int_0^{2\pi} \frac{Re^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2} iRe^{i\theta}d\theta\right| \leq \int_0^{2\pi} \left|\frac{Re^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2} iRe^{i\theta}\right| d\theta \\ &= R^2 \int_0^{2\pi} \frac{e^{-R\sin\theta}}{|R^2e^{2i\theta}+a^2|}d\theta \leq \frac{R^2}{R^2-a^2}\int_0^{2\pi} e^{-R\sin\theta}d\theta\end{aligned}$$

For large R, let $\delta = R^{-1/2} < \pi/2$. Note that $\theta \in [\delta, \pi-\delta]$ then $\sin\theta \geq \sin\delta$.

$$\begin{aligned}\int_0^{2\pi} e^{-R\sin\theta}d\theta &= \int_0^{\delta} e^{-R\sin\theta}d\theta + \int_{\delta}^{\pi-\delta} e^{-R\sin\theta}d\theta + \int_{\pi-\delta}^{\pi} e^{-R\sin\theta}d\theta \\ &\le \delta\cdot 1 + (\pi - 2\delta)e^{-R\sin\delta} + \delta\cdot 1 \\ &= \frac{2}{\sqrt{R}} + \left(\pi - \frac{2}{\sqrt{R}}\right)e^{-R\sin\frac{1}{\sqrt{R}}} \xrightarrow{R\to\infty} 0.\end{aligned}$$

Therefore let $R \to \infty$ to get

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}dx = 2\pi i \operatorname*{Res}_{z=ai} \frac{ze^{iz}}{z^2+a^2} = 2\pi i\cdot\frac{aie^{-a}}{2ai} = i\pi e^{-a}$$

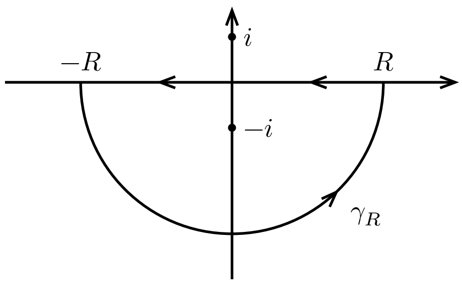
and

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}dx\right) = \pi e^{-a}.$$

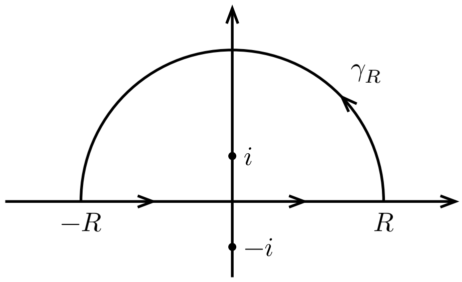
5. When $\xi = 0$, then changing the variables $x = \tan\theta$ to get

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx = \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\pi}{2}.$$

Suppose $\xi > 0$. Observe that $\operatorname{Im}(z) \le 0$ then $\left|e^{-2\pi ix\xi}\right| = e^{2\pi\xi\operatorname{Im}(z)} \le 1$. We integrate the function $e^{-2\pi iz\xi}/(1+z^2)^2$ over the lower semicircle of radius R.



contour when $\xi > 0$



contour when $\xi < 0$

Since there are pole of order 2 inside the curve, by residue formula,

$$\int_R^{-R} \frac{e^{-2\pi ix\xi}}{(1+x^2)^2}dx + \int_{\gamma_R} \frac{e^{-2\pi iz\xi}}{(1+z^2)^2}dz = 2\pi i \operatorname*{Res}_{z=-i} \frac{e^{-2\pi iz\xi}}{(1+z^2)^2}.$$

We can easily see that integral over the arc tends to 0 as $R \to \infty$. Therefore

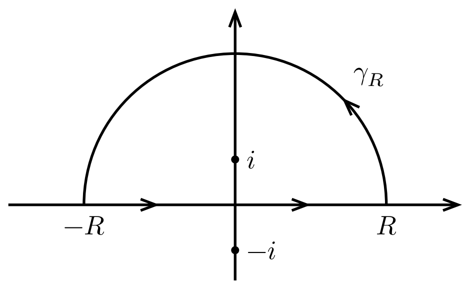
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-2\pi ix\xi}}{(1+x^2)^2}dx &= -2\pi i \operatorname*{Res}_{z=-i} \frac{e^{-2\pi iz\xi}}{(1+z^2)^2} = -2\pi i \lim_{z\to -i} \frac{d}{dz}\left(\frac{e^{-2\pi iz\xi}}{(z-i)^2}\right) \\ &= -2\pi i \lim_{z\to -i} \frac{-2\pi i\xi e^{-2\pi iz\xi}(z-i)^2 - e^{-2\pi iz\xi}\cdot 2(z-i)}{(z-i)^4} \\ &= -2\pi i\cdot\frac{(-4\pi\xi - 2)e^{-2\pi\xi}}{8i} = \frac{\pi}{2}(1+2\pi\xi)e^{-2\pi\xi}.\end{aligned}$$

Proof for case $\xi < 0$ is similar to proof for case $\xi > 0$. Integrate the function $e^{-2\pi iz\xi}/(1+z^2)^2$ over the upper semicircle of radius R, and let $R \to \infty$ to get

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} = 2\pi i \lim_{z\to -i} \frac{d}{dz}\left(\frac{e^{-2\pi i z\xi}}{(z+i)^2}\right)$$

$$= 2\pi i \cdot \frac{(4\pi\xi - 2)e^{2\pi\xi}}{-8i} = \frac{\pi}{2}(1-2\pi\xi)e^{2\pi\xi} = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}.$$

6. Integrate the function $\frac{1}{(1+z^2)^{n+1}}$ over the upper semicircle.



By residue formula,

$$\int_{-R}^{R} \frac{1}{(1+x^2)^{n+1}} dx + \int_{\gamma_R} \frac{1}{(1+z^2)^{n+1}} dz = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(1+z^2)^{n+1}}.$$

Obviously the integral over the arc goes to 0. Since $z = i$ is pole of order $n+1$,

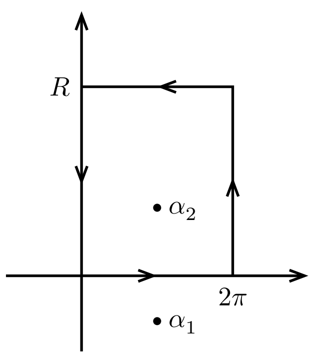
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(1+z^2)^{n+1}} = 2\pi i \lim_{z\to i} \frac{1}{n!}\left(\frac{d}{dz}\right)^n (z+i)^{-n-1}$$

$$= 2\pi i \lim_{z\to i} \frac{1}{n!}(-n-1)(-n-2)\cdots(-n-n)(z+i)^{-n-n-1}$$

$$= 2\pi i \cdot \frac{(2n)!}{(n!)^2} \cdot \frac{1}{2^{2n+1}} \cdot \frac{1}{i} = \frac{(2n)!}{(2^n n!)^2} \cdot \pi$$

$$= \frac{1 \cdot 2 \cdot \cdots \cdot (2n)}{(1 \cdot 2 \cdot \cdots \cdot n)^2} \cdot \pi = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \cdot \pi.$$

7. We integrate the function $1/(a+\cos z)^2$ over the rectangle below.



Note that

$$\frac{1}{(a+\cos z)^2} = \frac{1}{\left(a + \frac{e^{iz}-e^{-iz}}{2}\right)^2} = \frac{4e^{2iz}}{(e^{2iz}+2ae^{iz}+1)^2} = \frac{4e^{iz}}{(e^{iz}-r_1)^2(e^{iz}-r_2)^2}$$

where

$$r_1 = -a - \sqrt{a^2-1},\ r_2 = -a + \sqrt{a^2-1}.$$

Now let

$$\alpha_i = \pi - \ln(-r_i)i \quad \text{for } i = 1, 2,$$

then $1/(a+\cos z)^2$ have pole of order 2 at $z = \alpha_1, \alpha_2$. By residue formula,

$$\int_0^{2\pi} \frac{1}{(a+\cos x)^2} dx + \int_0^R \frac{1}{\left(a + \frac{e^{2\pi i + t} + e^{-2\pi i - t}}{2}\right)^2} i dt$$
$$+ \int_{2\pi}^0 \frac{1}{a + \cos(t+iR)} dt + \int_R^0 \frac{1}{\left(a + \frac{e^t + e^{-t}}{2}\right)^2} i dt = 2\pi i \operatorname*{Res}_{z=\alpha_2} \frac{1}{(a+\cos z)^2}.$$

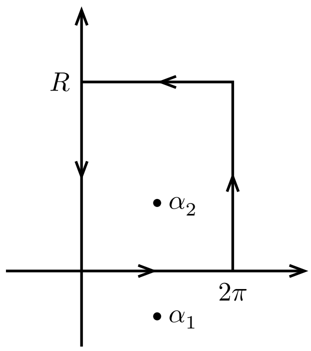
The integral over the top side of rectangles goes to 0 as $R \to \infty$. The integrals over the vertical sides cancel out. The residue is

$$\operatorname*{Res}_{z=\alpha_2} \frac{1}{(a+\cos z)^2} = \lim_{z\to\alpha_2} \frac{d}{dz}\left((z-\alpha_2)^2 \cdot \frac{4e^{2iz}}{(e^{iz}-r_1)^2(e^{iz}-r_2)^2} \right) = \frac{1}{i} \cdot \frac{a}{(a^2-1)^{3/2}},$$

hence

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

8. We integrate the function $1/(a + b\cos z)$ over the rectangle below.



Observe that

$$\frac{1}{a+b\cos z} = \frac{2e^{iz}}{b(e^{iz}-r_1)(e^{iz}-r_2)}$$

where

$$r_1 = -\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1},\ r_2 = -\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}.$$

Similar to **Exercise 7**, the integral over the top side tends to 0 as $R \to 0$, and integrals over vertical sides cancel out. Therefore the integral over the real segment equals to

$$\begin{aligned} 2\pi i \operatorname*{Res}_{z=\alpha_2} \frac{1}{a+b\cos z} &= \lim_{z\to\alpha_2} (z-\alpha_2) \frac{2e^{iz}}{b(e^{iz}-r_1)(e^{iz}-r_2)} \\ &= \lim_{z\to\alpha_2} \frac{2e^{iz}}{b(e^{iz}-r_1)} \cdot \frac{z-\alpha_2}{e^{iz}-r_2} = \frac{2r_2}{b(r_2-r_1)} \cdot \frac{1}{ir_2} \\ &= \frac{2}{ib(r_2-r_1)} = \frac{1}{i\sqrt{a^2-b^2}} \end{aligned}$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

9. Observe that

$$\sin \pi z = (1 - e^{2\pi i z}) e^{-\pi i z} \left(\frac{1}{2} i\right).$$

Since

$$-\frac{\pi}{2} < \arg(1 - e^{2\pi i z}) < \frac{\pi}{2}, -\pi < \arg(e^{-\pi i z}) < 0, \arg\left(\frac{i}{2}\right) = \frac{\pi}{2},$$

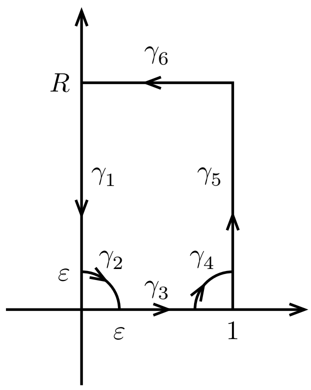
we get

$$\log(\sin \pi z) = \log(1 - e^{2\pi i z}) - \pi i z + \log\left(\frac{1}{2} i\right).$$

Therefore

$$\begin{aligned}
\int_0^1 \log(\sin \pi x) dx &= \int_0^1 \log(1 - e^{2\pi i x}) dx - \int_0^1 \pi i x dx + \int_0^1 \log\left(\frac{1}{2} i\right) dx \\
&= \int_0^1 \log(1 - e^{2\pi i x}) dx - i\frac{\pi}{2} - \log 2 + i\frac{\pi}{2} \\
&= \int_0^1 \log(1 - e^{2\pi i x}) dx - \log 2.
\end{aligned}$$

Now we integrate the function $f(z) = \log(1 - e^{2\pi i z})$ over the contour below.



By Cauchy's theorem,

$$\sum_{k=1}^{6} \int_{\gamma_k} f(z) dz = 0.$$

Integrals over the vertical sides cancel out, and $\int_{\gamma_6} f(z)dz \to 0$ as $R \to \infty$. Now we estimate the integral over γ_2.

$$\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^{\pi/2} \log\left(1 - e^{2\pi i\varepsilon e^{i\theta}}\right) i\varepsilon e^{i\theta} d\theta\right| \le \varepsilon \int_0^{\pi/2} \left|\log\left(1 - e^{2\pi i\varepsilon e^{i\theta}}\right)\right| d\theta$$
$$\le \varepsilon \int_0^{\pi/2} \sum_{n=1}^{\infty} \frac{1}{n}\left|e^{2\pi i\varepsilon e^{i\theta}}\right| d\theta = \varepsilon \int_0^{\pi/2} \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi n\varepsilon \sin\theta} d\theta$$
$$= \varepsilon \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi/2} e^{-2\pi n\varepsilon\sin\theta} d\theta$$

Let $n_\varepsilon = \left\lfloor \frac{8}{\pi^3}\frac{1}{\varepsilon^2} \right\rfloor + 1$. For $n > n_\varepsilon$, $\delta = n^{-1/3}\varepsilon^{-2/3} < \pi/2$, so

$$\int_0^{\pi/2} e^{-2\pi n\varepsilon\sin\theta} d\theta = \int_0^{\delta} e^{-2\pi n\varepsilon\sin\theta} d\theta + \int_{\delta}^{\pi/2} e^{-2\pi n\varepsilon\sin\theta} d\theta$$
$$\le \delta + \left(\frac{\pi}{2} - \delta\right) e^{-2\pi n\varepsilon\sin\delta} \le \delta + \left(\frac{\pi}{2} - \delta\right) e^{-4n\varepsilon\delta}$$
$$\le n^{-1/3}\varepsilon^{-2/3} + \frac{\pi}{2} e^{-4n^{2/3}\varepsilon^{1/3}}.$$

Hence

$$\varepsilon \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi/2} e^{-2\pi n\varepsilon\sin\theta} d\theta \le \varepsilon \sum_{n=1}^{n_\varepsilon - 1} \frac{\pi}{2n} + \varepsilon \sum_{n=n_\varepsilon}^{\infty} \frac{1}{n}\left(\frac{1}{n^{1/3}}\varepsilon^{-2/3} + \frac{\pi}{2} e^{-4n^{2/3}\varepsilon^{1/3}}\right)$$
$$\le \frac{\pi}{2}\varepsilon(\ln(n_\varepsilon - 1) + 1) + \varepsilon^{1/3}\zeta(4/3) + \frac{\pi}{2}\varepsilon \sum_{n=1}^{\infty} \frac{1}{n} e^{-4n^{2/3}\varepsilon^{1/3}}.$$

Note that

$$\sum_{n=1}^{\infty} e^{-an^{3/2}} \le \sum_{n=1}^{\infty} \frac{1}{\frac{1}{2}a^2 n^{4/3}} = \frac{2}{a^2}\zeta(4/3).$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-4n^{2/3}\varepsilon^{1/3}} \le \sum_{n=1}^{\infty} e^{-4n^{2/3}\varepsilon^{1/3}} \le \frac{2}{\left(4\varepsilon^{1/3}\right)^2}\zeta(4/3) = \frac{1}{8}\varepsilon^{-3/2}\zeta(4/3).$$

As a consequence,

$$\frac{\pi}{2}\varepsilon(\ln(n_\varepsilon - 1) + 1) + \varepsilon^{1/3}\zeta(4/3) + \frac{\pi}{2}\varepsilon \sum_{n=1}^{\infty} \frac{1}{n} e^{-4n^{2/3}\varepsilon^{1/3}}$$
$$\le \frac{\pi}{2}\varepsilon\left(1 + \ln\left(\frac{8}{\pi^3}\right) - 2\ln\varepsilon\right) + e^{1/3}\zeta(4/3) + \frac{\pi}{16}\varepsilon^{1/3}\zeta(4/3) = O\left(\varepsilon^{1/3}\right).$$

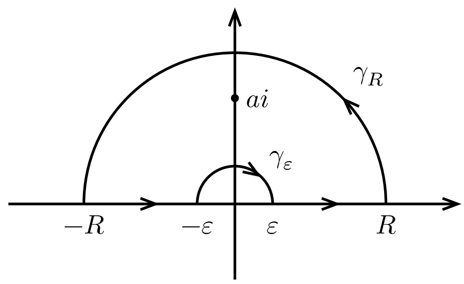
Therefore $\int_{\gamma_2} f(z)dz \to 0$ as $\varepsilon \to 0$. Similarly, $\int_{\gamma_4} f(z)dz \to 0$. So we get

$$\int_0^1 \log(1 - e^{2\pi i x})dx = 0$$

and

$$\int_0^1 \log(\sin \pi x)dx = -\log 2.$$

10. We integrate the function $f(z) = \frac{\log z}{z^2 + a^2}$ over the contour below.



By residue formula,

$$\int_{-R}^{-\varepsilon} f(x)dx + \int_{\gamma_\varepsilon} f(z)dz + \int_{\varepsilon}^{R} f(x)dx + \int_{\gamma_R} f(z)dz = 2\pi i \operatorname*{Res}_{z=ai} f(z).$$

We denote

$$I = \int_\varepsilon^R f(x)dx.$$

Then

$$\int_{-R}^{-\varepsilon} \frac{\log x}{x^2+a^2}dx = \int_R^\varepsilon \frac{\log(-t)}{t^2+a^2}(-dt) = \int_\varepsilon^R \frac{\log t + \pi i}{t^2+a^2}dt = I + \pi i \cdot \frac{\pi}{2a}.$$

The integral over the γ_R goes to 0 as $R \to \infty$ because

$$\left| \int_{\gamma_R} f(z)dz \right| \leq 2\pi R \cdot \frac{A \log R}{R^2 - a^2} \to 0.$$

Also, integral over the γ_ε goes to 0 as $\varepsilon \to 0$ because

$$\begin{aligned} \int_{\gamma_\varepsilon} \frac{\log z}{z^2+a^2}dz &= -\int_0^\pi \frac{\log\left(\varepsilon e^{i\theta}\right)}{\varepsilon^2 e^{2i\theta}+a^2} i\varepsilon e^{i\theta} d\theta = -\int_0^\pi \frac{\log \varepsilon + i\theta}{\varepsilon^2 e^{2i\theta}+a^2} i\varepsilon e^{i\theta} d\theta \\ &= -i\left(\varepsilon \log\varepsilon \int_0^\pi \frac{e^{i\theta}}{\varepsilon^2 e^{2i\theta}+a^2}d\theta + i\varepsilon \int_0^\pi \frac{\theta e^{i\theta}}{\varepsilon^2 e^{2i\theta}+a^2}d\theta \right) \to 0. \end{aligned}$$

Therefore taking the limits $R \to \infty, \varepsilon \to 0$ gives

$$2I + \frac{\pi^2}{2a}i = 2\pi i \operatorname*{Res}_{z=ai} \frac{\log z}{z^2+a^2} = 2\pi i \cdot \frac{\log a + \frac{\pi}{2}i}{2ai} = \pi \frac{\log a}{a} + \frac{\pi^2}{2a}i.$$

Hence

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{\pi}{2a}\log a.$$

11. First, we can assume $a > 0$. Otherwise, if $a = a' e^{i\varphi}$ $(a' > 0, \varphi \in (0, 2\pi))$, we can make it equal to the case $a > 0$ by doing the following;

$$\int_0^{2\pi} \log\left|1 - a' e^{i(\theta-\varphi)}\right| d\theta = \int_\varphi^{2\pi+\varphi} \log\left|1 - a' e^{i\theta}\right| d\theta = \int_0^{2\pi} \log\left|1 - a' e^{i\theta}\right| d\theta.$$

Now define

$$f(a) = \int_0^{2\pi} \log\left|1 - a e^{i\theta}\right| d\theta,$$

then

$$\begin{aligned} f(a) &= \int_0^{2\pi} \log\sqrt{(1-a\cos\theta)^2+(a\sin\theta)^2}\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\log(1-2a\cos\theta+a^2)\,d\theta = \int_0^{\pi} \log(1-2a\cos\theta+a^2)\,d\theta. \end{aligned}$$

we are going to calculate the derivative of f in respect to a.

$$\begin{aligned} f'(a) &= \int_0^{\pi} \frac{\partial}{\partial a}\log(1-2a\cos\theta+a^2)\,d\theta = \int_0^{\pi} \frac{-2\cos\theta+2a}{1-2a\cos\theta+a^2}\,d\theta \\ &= 2\left(\int_0^{\pi/2} \frac{a-\cos\theta}{1-2a\cos\theta+a^2}\,d\theta + \int_0^{\pi/2} \frac{a+\cos\theta}{1+2a\cos\theta+a^2}\,d\theta\right) \\ &= 4\int_0^{\pi/2} \frac{a+a^3-2a\cos^2\theta}{\left(1+a^2\right)^2-4a^2\cos^2\theta}\,d\theta \\ &= \frac{2(a^4-1)}{a}\int_0^{\pi/2} \frac{1}{\left(1+a^2\right)^2-4a^2\cos^2\theta}\,d\theta + 4\int_0^{\pi/2}\frac{1}{2a}\,d\theta \end{aligned}$$

By changing the variables $t=\tan\theta$,

$$\begin{aligned} \int_0^{\pi/2} \frac{1}{\left(1+a^2\right)^2-4a^2\cos^2\theta}\,d\theta &= \int_0^{\pi/2} \frac{\sec^2\theta}{\left(1+a^2\right)^2\tan^2\theta+\left(1-a^2\right)^2}\,d\theta \\ &= \int_0^{\infty} \frac{1}{\left(1+a^2\right)^2t^2+\left(1-a^2\right)^2}\,dt = \frac{1}{\left(1+a^2\right)^2}\cdot\frac{1+a^2}{|1-a^2|}\cdot\frac{\pi}{2}. \end{aligned}$$

Hence $f'(a)=0$ for all $0<a<1$. Since $f(0)=0$, $f(a)=0$ for all $0<a<1$. In the case of $a=1$,

$$\begin{aligned} \int_0^{\pi}\log(2-2\cos\theta)\,d\theta &= \int_0^{\pi}\log\left(4\sin^2\frac{\theta}{2}\right)d\theta = 2\int_0^{\pi}\log\left(2\sin\frac{\theta}{2}\right)d\theta \\ &= 2\left(\pi\log 2+\int_0^{\pi}\log\sin\frac{\theta}{2}\,d\theta\right) \\ &= 4\left(\frac{\pi\log 2}{2}+\int_0^{\pi/2}\log\sin t\,dt\right) \end{aligned}$$

which is 0 by **Exercise 9**.

12. We integrate the function $f(z)=\pi\cot\pi z/(u+z)^2$ over the circle of radius $N+1/2$, centered at 0. Observe that $z=n$ $(-N\le n\le N)$ are simple poles, and $z=-u$ is pole of order 2. By residue formula,

$$\int_0^{2\pi} \frac{\pi\cot\left(\pi R_N e^{i\theta}\right)}{\left(u+R_N e^{i\theta}\right)^2} iR_N e^{i\theta}\,d\theta = 2\pi i\left(\sum_{n=-N}^{N}\operatorname*{Res}_{z=n} f(z)+\operatorname*{Res}_{z=-u} f(z)\right).$$

First, the integral over the circle goes to 0 as $R_N\to\infty$. This is because $\cot\left(\pi R_N e^{i\theta}\right)$ is bounded for large N. Note that

$$|\cot z| = \left|\frac{\cos z}{\sin z}\right| = \left|\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right| \le \frac{e^y+e^{-y}}{|e^y-e^{-y}|}$$

where $z=x+iy$. If $\operatorname{Im}(z)\ge 1$, then

$$|\cot z| \le \frac{e^y+e^{-y}}{e^y-e^{-y}} = 1+\frac{2}{e^{2y}-1} \le 1+\frac{2}{e^2-1}.$$

Similarly, $|\cot z|$ is bounded for $\mathrm{Im}(z) \leq -1$. Now consider $|\mathrm{Im}(z)| < 1$. Since $|z| = \pi R_N$, we can choose N large enough so

$$|\mathrm{Im}(z)| < 1 \;\Rightarrow\; \pi N + \pi/4 < \mathrm{Re}(z) \leq \pi N + \pi/2.$$

This leads to

$$|\sin(z)| = \frac{1}{2}|\cos x(e^{-y} - e^{y}) + i \sin x(e^{y} + e^{-y})| \geq \frac{1}{2}|\sin x||e^{y} + e^{-y}| \geq \frac{1}{\sqrt{2}}.$$

and

$$|\cos(z)| = \frac{1}{2}(e^{y} + e^{-y}) \leq \frac{e + e^{-1}}{2},$$

so

$$|\cot z| \leq \frac{e + e^{-1}}{\sqrt{2}}.$$

Hence $\cot\left(\pi R e^{i\theta}\right)$ is bounded for all $\theta \in [0, 2\pi]$. Moreover,

$$\operatorname*{Res}_{z=n} f(z) = \lim_{z \to n} \frac{\pi}{(u+n)^2} \cdot i(e^{2\pi i n} + 1) \cdot \frac{z-n}{e^{2\pi i z} - e^{2\pi i n}} = \frac{1}{(u+n)^2},$$

$$\operatorname*{Res}_{z=-u} f(z) = \lim_{z \to -u} (\pi \cot \pi z)' = -\pi^2 \csc^2 \pi u.$$

Therefore taking the limit $N \to \infty$ gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

13. Let $g(z) = (z - z_0) f(z)$. Observe that

$$|g(z)| = |z - z_0||f(z)| \leq |z - z_0|^{\varepsilon}$$

converges to 0 when $z \to z_0$, therefore g is bounded on punctured disc $D_r(z_0) - \{z_0\}$. By Riemann's theorem on removable singularities, z_0 is a removable singularity of g. Hence g can be extended to holomorphic function on $D_r(z_0)$, which is

$$g(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots.$$

However, $g(z) \to 0$ as $z \to z_0$, we get $a_0 = 0$. So

$$f(z) = a_1 + a_2(z - z_0) + \cdots$$

near the z_0. Then f is also bounded near z_0, therefore z_0 is removable singularity of f.

14. We are going to determine the type of singularity of function $f(1/z)$ at $z = 0$.

First suppose that $f(1/z)$ has removable singularity at $z = 0$. Then $|f(1/z)|$ is bounded in $|z| < \delta$, so $|f(z)|$ is bounded in $|z| \geq 1/\delta$. Since f is holomorphic and bounded in $\mathbb{C}$, f is constant this is contradiction with the fact that f in injective.

Next, we suppose $f(1/z)$ has essential singularity. By Casorati-Weierstrass Theorem, image of $D_r(0) - \{0\}$ by $z \mapsto f(1/z)$ is dense in $\mathbb{C}$. Now fix $\varepsilon < 1/r$ and consider $f(0) \in \mathbb{C}$. Since f is open mapping, there exists $\delta > 0$ such that

$$D_\delta(f(0)) \subset f(D_\varepsilon(0)).$$

Also, there exists $|z_0| < r$ such that

$$\left|f\left(\frac{1}{z_0}\right) - f(0)\right| < \delta, \quad \text{that is,} \quad f\left(\frac{1}{z_0}\right) \in D_\delta(f(0)).$$

Hence there exists $z_1 \in D_\varepsilon(0)$ such that $f(z_1) = f(1/z_0)$, which is contradiction.

Therefore $f(1/z)$ has pole at $z = 0$, and we can write

$$f\left(\frac{1}{z}\right) = \frac{a_{-k}}{z^k} + \frac{a_{-k+1}}{z^{k-1}} + \cdots + \frac{a_{-1}}{z} + G(z),$$

where $G(z)$ is bounded near 0. Hence

$$f(z) = a_{-k}z^k + a_{-k+1}z^{k-1} + \cdots + a_{-1}z + H(z),$$

where $H(z) = G(1/z)$ is bounded as $|z| \to \infty$. Since H is holomorphic and bounded in $\mathbb{C}$, it's constant. Hence f is a polynomial of degree k.

k must be less than 2 to be injective; Otherwise, according to the fundamental theorem of algebra, it would have more than one zero and would not be a injective function. Even if f has one zero of multiplicity k, so $f(z) = (z - z_0)^k$, it cannot be injective because

$$f\left(z_0 + e^{2\pi i/k}\right) = f\left(z_0 + e^{4\pi i/k}\right).$$

Therefore $f(z) = az + b$ $(a \neq 0)$.

15. (a) By Cauchy inequalities, for each $n \geq k + 1$,

$$\left|f^{(n)}(0)\right| \leq \frac{n!}{R^n} \sup_{|z|=R} |f(z)| \leq n! \cdot \frac{AR^k + B}{R^n} \to 0$$

as $R \to \infty$. Therefore $f^{(n)}(0) = 0$ for all $n \geq k + 1$, and f is polynomial of degree $\leq k$.

(b) Let $0 < \alpha < \varphi - \theta$ and m be a minimal natural number such that $m\alpha > 2\pi$. Also, define $M = \sup_{z \in \mathbb{D}} |f(z)|$. Now define

$$g(z) = f(z)f\left(ze^{i\alpha}\right)f\left(ze^{i2\alpha}\right) \cdots f\left(ze^{im\alpha}\right).$$

Since f is converges uniformly to zero in the sector $\theta < \arg z < \varphi$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$r > \delta \;\Rightarrow\; \left|f\left(re^{it}\right)\right| < \varepsilon \;(\forall t \in (\theta, \varphi)).$$

We prove that g uniformly converges to zero as $r \to 1$. For any $z = re^{it}$, there exists k such that $\theta < t + k\alpha < \varphi$, so $\left|f\left(re^{it}e^{ik\alpha}\right)\right| < \varepsilon$. Therefore

$$|g(z)| \leq M^m \left|f\left(ze^{ik\alpha}\right)\right| < M^m \varepsilon.$$

Hence g uniformly converges to zero as $r \to 1$. By maximum modulus principle,

$$\sup_{z \in D_r} |g(z)| \leq \sup_{z \in C_r} |g(z)| \to 0 \quad \text{as} \quad r \to 1,$$

which means $g(z) = 0$.

Now we show that there is sequence $\{z_n\}$ such that $f(z_n) = 0$ and $z_n \to 0$. Since

$$g\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right)f\left(\frac{1}{n}e^{i\alpha}\right) \cdots f\left(\frac{1}{n}e^{im\alpha}\right) = 0,$$

$f\left(\frac{1}{n}e^{ik_n\alpha}\right) = 0$ for at least one k_n. Let $z_n = \frac{1}{n}e^{ik_n\alpha}$. Observe that $|z_n| = 1/n \to 0$. By **Theorem 4.8, Chapter 2**, $f(z) = 0$.

(c) Define $f(z) = (z - w_1) \cdots (z - w_n)$. We have to show that $|f(z)| \geq 1$ for at least one $z \in \partial\mathbb{D}$. By Cauchy inequalities,

$$n! = \left|f^{(n)}(0)\right| \leq \frac{n!}{1^n} \sup_{z \in \partial\mathbb{D}} |f(z)|.$$

Therefore $\sup_{z \in \partial\mathbb{D}} |f(z)| \geq 1$. Since $\partial\mathbb{D}$ is compact, $f(z)$ attains its maximum in unit circle, therefore $\max_{z \in \partial\mathbb{D}} |f(z)| \geq 1$. This completes the proof.

Also, by intermediate value theorem on $g(z) = f\left(e^{i\theta}\right)$, there exists z such that $|f(z)|$ is exactly equal to 1.

(d) Observe that $\left|e^f\right| = e^{\mathrm{Re}(f)}$. Therefore if $\mathrm{Re}(f)$ is bounded, then e^f is also bounded. Since e^f is bounded and holomorphic in $\mathbb{C}$, it is constant. Thus f is also constant.

16. (a) Note that f has unique solution $z = 0$ in $\overline{\mathbb{D}}$. Now we want to take small enough $\varepsilon > 0$ such that

$$|f(z)| > \varepsilon |g(z)|$$

for all $|z| = 1$. This is possible if we choose ε such that

$$\varepsilon < \frac{\inf_{|z|=1} |f(z)|}{\sup_{|z|=1} |g(z)|}.$$

Note that $\sup_{|z|=1} |g(z)|$ exists because $g(z)$ is bounded on compact set $\overline{\mathbb{D}}$. Also, since $\{z : |z| = 1\}$ is compact, we can say that $\inf_{|z|=1} |f(z)| = \min_{|z|=1} |f(z)| > 0$. By Rouché's theorem, $f(z) + \varepsilon g(z) = 0$ also has unique solution in $\overline{\mathbb{D}}$.

(b) Fix small enough $\varepsilon_0 > 0$. We want to prove that for every $\xi > 0$, there exists $\delta > 0$ such that

$$\text{for all } \varepsilon > 0,\ |\varepsilon - \varepsilon_0| < \delta \ \Rightarrow \ \left|z_\varepsilon - z_{\varepsilon_0}\right| < \xi.$$

Note that

$$f_\varepsilon(z) = f(z) + \varepsilon g(z) = f(z) + \varepsilon_0 g(z) + (\varepsilon - \varepsilon_0) g(z).$$

Since $\left|z_{\varepsilon_0}\right| < 1$, for every small enough $\xi > 0$, there is only one zero of $f(z) + \varepsilon_0 g(z)$ in $D_\xi\left(z_{\varepsilon_0}\right)$. Now take $\delta > 0$ such that

$$|\varepsilon - \varepsilon_0| < \delta \ \Rightarrow \ |f(z) + \varepsilon_0 g(z)| > |\varepsilon - \varepsilon_0| |g(z)| \ \text{ for } \ z \in C_\xi\left(z_{\varepsilon_0}\right).$$

This is possible similar to argument in **(a)**. By Rouché's theorem, $f(z) + \varepsilon g(z) = 0$ has unique zero in $D_\xi\left(z_{\varepsilon_0}\right)$, and that is equal to z_ε. Hence $\left|z_\varepsilon - z_{\varepsilon_0}\right| < \xi$.

17. (a) Since $|f(z)| = 1$ and $|w_0| < 1$ in unit circle, by Rouché's theorem, $f(z) = w_0$ has a root in $\mathbb{D}$ whenever $f(z) = 0$ has a root in $\mathbb{D}$. Hence it suffices to show that $f(z) = 0$ has a root.

Suppose $f(z) = 0$ has no root, so $|f(z)| > 0$ for all $z \in \mathbb{D}$. By maximum modulus principle,

$$|f(z)| \leq \sup_{z \in \mathbb{D}} |f(z)| \leq \sup_{z \in \overline{\mathbb{D}} - \mathbb{D}} |f(z)| = 1,$$

$$\left|\frac{1}{f(z)}\right| \leq \sup_{z \in \mathbb{D}} \left|\frac{1}{f(z)}\right| \leq \sup_{z \in \overline{\mathbb{D}} - \mathbb{D}} \left|\frac{1}{f(z)}\right| = 1.$$

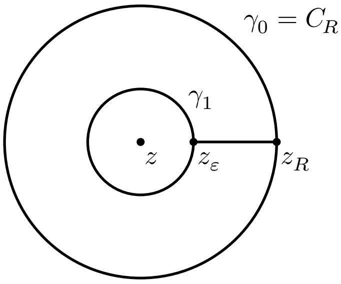
Hence $|f(z)| = 1$ for all $z \in \mathbb{D}$. By **Exercise 13, Chapter 1**, $f(z)$ is also constant, which is contradiction.

(b) Suppose there is no $z \in \mathbb{D}$ such that $f(z) = 0$. By maximum modulus principle,

$$\left|\frac{1}{f(z)}\right| \le \sup_{z\in\mathbb{D}}\left|\frac{1}{f(z)}\right| \le \sup_{z\in\overline{\mathbb{D}}-\mathbb{D}}\left|\frac{1}{f(z)}\right| \le 1.$$

So $|f(z)| \ge 1$ for all $z \in \mathbb{D}$. But $|f(z_0)| < 1$, which is contradiction.

18. Pick any two points $z_\varepsilon \in C_\varepsilon(z), z_R \in C_R(z)$ on the circle and consider the two curves. First one is the circle centered at z of radius R, that is, $\gamma_0(t) = Re^{2\pi i t}$ $(0 \le t \le 1)$. Second, $\gamma_1(t) : [0,1] \to \overline{D_R(z)}$ is equal to line segment $\overline{z_R z_\varepsilon}$ for $0 \le t < 1/3$, the circle $C_\varepsilon(z)$ for $1/3 < t \le 2/3$, the line segment $\overline{z_\varepsilon z_R}$ for $2/3 \le t \le 1$.



Since γ_0 and γ_1 is homotopic, we get

$$\int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

The integral over the line segment canceled so

$$\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Observe that $(f(\zeta) - f(z))/(\zeta - z)$ is bounded near z, hence

$$\int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{C_\varepsilon} \frac{1}{\zeta - z} d\zeta \to 0 + f(z) \cdot 2\pi i$$

as $\varepsilon \to 0$. Therefore

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

19. (a) Let f be holomorphic near z_0 with $u = \mathrm{Re}(f)$. Since f is an open mapping, $f(D_r(z_0))$ is open for small enough $r > 0$. So there exists $z_1 \in D_r(z_0)$ such that $\mathrm{Re}(f(z_1)) > \mathrm{Re}(f(z_0))$, so u cannot have local maximum at z_0.

(b) u is continuous function, so $|u(z)|$ have maximum at $\overline{\Omega}$. But u cannot have maximum in Ω by **(a)**. Hence given inequality holds.

20. (a) By mean-value property,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

Therefore

$$\pi t^2 f(z) = \int_0^t 2\pi f(z) r dr = \int_0^r \int_0^{2\pi} f(z + re^{i\theta}) r d\theta dr = \int_{D_t(z)} f(x + iy) dx dy$$

whenever f is holomorphic in a disc $D_t(z)$. Note that for every $z \in D_s(z_0)$, $D_{r-s}(z) \in D_r(z_0)$. Now we set $t = r - s$, then

$$
\begin{aligned}
|f(z)| &= \frac{1}{\pi t^2}\left|\int_{D_t(z)} f(z)dxdy\right| \le \frac{1}{\pi t^2}\sqrt{\left(\int_{D_t(z)} |f(z)|^2 dxdy\right)\left(\int_{D_t(z)} 1\, dxdy\right)} \\
&= \frac{1}{\sqrt{\pi}t}\sqrt{\int_{D_t(z)} |f(z)|^2 dxdy} \le \frac{1}{\sqrt{\pi}t}\sqrt{\int_{D_r(z_0)} |f(z)|^2 dxdy} \\
&= \frac{1}{\sqrt{\pi}t}\|f\|_{L^2(D_r(z_0))}.
\end{aligned}
$$

Hence

$$
\|f\|_{L^\infty(D_s(z_0))} = \sup_{z\in D_s(z_0)} |f(z)| \le \frac{1}{\sqrt{\pi}(r-s)}\|f\|_{L^2(D_r(z_0))}.
$$

(b) Let d be a distance between K and $U^{\complement}$. Since both K and $U^{\complement}$ are close sets, $d > 0$. Now

$$
\|f\|_{L^\infty(K)} = \frac{1}{\sqrt{\pi}(d/2)}\|f\|_{L^2(U)}
$$

holds for all holomorphic f. Therefore

$$
\|f_n - f_m\|_{L^\infty(K)} = \frac{1}{\sqrt{\pi}(d/2)}\left\|f_f - f_m\right\|_{L^2(U)} < \frac{2}{\sqrt{\pi}d}\varepsilon
$$

for large n, m. Hence $\{f_n\}$ converges to a function f. Note that $\{f_n\}$ uniformly converges to f because of inequality above. Since each f_n are holomorphic, and $f_n \rightrightarrows f$, f is a holomorphic function.

21. (a) Suppose $\gamma_0(t), \gamma_1(t)$ are two curves lying in Ω, then we can define $\gamma_s(t)$ by

$$
\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t).
$$

Since $\gamma_0(t), \gamma_1(t) \in \Omega$, by the definition of convex set, $\gamma_s(t) \in \Omega$. Hence Ω is simply connected set.

(b) Suppose $\gamma_0(t), \gamma_1(t) : [0,1]^2 \to \Omega$ are two curves lying in Ω. We define $\gamma_s(t)$ as

$$
\gamma_s(t) = \begin{cases}
\alpha\gamma_0(u) + (1-\alpha)z_0 & \text{for } t \in [0,1/2] \ \wedge \ s \in [0,1/2] \\
\alpha\gamma_0(1-u) + (1-\alpha)z_0 & \text{for } t \in (1/2,1] \ \wedge \ s \in [0,1/2] \\
\alpha\gamma_1(-u) + (1-\alpha)z_0 & \text{for } t \in [0,1/2] \ \wedge \ s \in (1/2,1] \\
\alpha\gamma_1(1+u) + (1-\alpha)z_0 & \text{for } t \in (1/2,1] \ \wedge \ s \in (1/2,1]
\end{cases}
$$

where

$$
\begin{aligned}
\alpha(s,t) &= \sqrt{\cos^2 \pi t + (1-2s)^2 \sin^2 \pi t}, \\
u(s,t) &= \frac{1}{\pi}\arcsin\left(\frac{(1-2s)\sin \pi t}{\sqrt{\cos^2 \pi t + (1-2s)^2 \sin^2 \pi t}}\right).
\end{aligned}
$$

Since Ω is star-shaped set with star center z_0, $\gamma_s(t) \in \Omega$. Hence Ω is simply connected set.

(c) A horseshoe shaped region.

22. Suppose that there is a holomorphic function f satisfying such a condition. Then for all $0 < r < 1$,

$$
\int_{\partial D_r(0)} f(z)dz = 0
$$

since f is holomorphic in $\mathbb{D}$. However,

$$\int_{\partial\mathbb{D}} f(z)dz = 2\pi i$$

since $f(z) = 1/z$ for $z \in \partial\mathbb{D}$. Taking a limit $r \to 1^-$, we get a contradiction.

Chapter 4. The Fourier Transform

1. (a) By definition of Fourier transform,

$$A(\xi) - B(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi(x-t)}dx = e^{2\pi i\xi t}\hat{f}(\xi) = 0.$$

So $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$.

(b) First we prove that $A(z)$ is holomorphic in upper half-plane. Define

$$A_n(z) = \int_{-n}^{t} f(x)e^{-2\pi iz(x-t)}dx.$$

Since $f(x)e^{-2\pi iz(x-t)}$ is holomorphic for each x and continuous in $[-n,t] \times \{z : z > 0\}$, $A_n(z)$ is holomorphic. Also, $A_n(z)$ uniformly converges to $A(z)$ in any compact subset K of $\{z : z > 0\}$ because

$$|A_n(z) - A(z)| = \left|\int_{-\infty}^{-n} f(x)e^{-2\pi iz(x-t)}dx\right| \le \int_{-\infty}^{-n} |f(x)|dx.$$

Therefore $A(z)$ is holomorphic. Similarly, $B(z)$ is holomorphic in $\{z : z < 0\}$. Since $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$, by Schwarz reflection principle, a function F defined by

$$F(z) = \begin{cases} A(z) & \text{for } \operatorname{Im}(z) \ge 0, \\ B(z) & \text{for } \operatorname{Im}(z) < 0 \end{cases}$$

is holomorphic in $\mathbb{C}$. Note that

$$|A(z)| \le \int_{-n}^{t} |f(x)|dx \le \int_{-\infty}^{t} |f(x)|dx,$$
$$|B(z)| \le \int_{t}^{n} |f(x)|dx \le \int_{t}^{\infty} |f(x)|dx.$$

So F is bounded. Since F is holomorphic and bounded, it is constant. In particular, $A(iy) \to 0$ as $y \to \infty$ thus $F(z) = 0$.

(c) For all $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} f(x)dx = F(0) = 0.$$

Hence $\int_{t_1}^{t_2} f(x)dx = 0$ for any $t_1, t_2 \in \mathbb{R}$. If $f(x)$ is not identically zero, then there exists x_0 such that $f(x_0) \ne 0$. Suppose $f(x_0) > 0$. (The case $f(x_0) < 0$ is proved similarly.) Since f is continuous, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| > \frac{1}{2}|f(x_0)| \Rightarrow f(x) > \frac{1}{2}f(x_0).$$

Then

$$\int_{x_0-\delta}^{x_0+\delta} f(x)dx > \int_{x_0-\delta}^{x_0+\delta} \frac{1}{2} f(x_0)dx = \delta f(x_0) > 0,$$

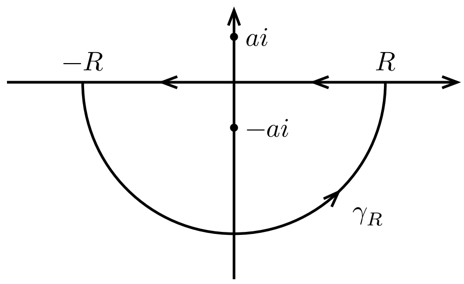
which is contradiction.

2. The case of $b = 0$ is trivial. Suppose $0 < b < a$. Since $f^{(n)}$ is holomorphic in $S_b \subset S_a$, using Cauchy inequality gives

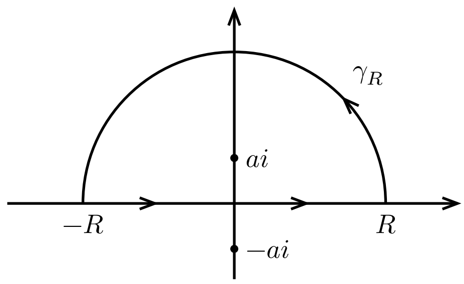
$$\left|f^{(n)}(x+iy)\right| \le \frac{n!}{R^n} \sup_{z\in C_R(z+iy)} |f(z)| \le \frac{n!}{R^n} \frac{A}{1+(|x|+R)^2} \le \frac{n!}{R^n} \frac{A'}{1+x^2}$$

for all $(x, y) \in \mathbb{R} \times (-b, b)$, where $R = (b-a)/2$.

3. The case of $\xi = 0$ is simple. If $\xi > 0$, we integrate the function $f(z) = \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$ over the lower semicircle of radius R.



contour when $\xi > 0$



contour when $\xi < 0$

By residue formula,

$$-\int_{-R}^{R} f(x)dx + \int_{\gamma_R} f(z)dz = 2\pi i \operatorname*{Res}_{z=-ai} f(z).$$

Note that $\left|e^{-2\pi i z\xi}\right| \le 0$ whenever $\mathrm{Im}(\xi) \le 0$, so integral over the semicircle goes to 0 as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} f(x)dx = -2\pi i \operatorname*{Res}_{z=-ai} f(z) = -2\pi i \cdot \frac{a}{-2ai} e^{-2\pi i(-ai)\xi} = \pi e^{-2\pi a\xi} = \pi e^{-2\pi a|\xi|}.$$

Conversely, if $\xi < 0$, then integrate the function $f(z)$ over the upper semicircle. Similarly to argument above, the integral over the semicircle tends to 0 thus

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \operatorname*{Res}_{z=ai} f(z) = 2\pi i \cdot \frac{a}{2ai} e^{-2\pi i(ai)\xi} = \pi e^{2\pi a\xi} = \pi e^{-2\pi a|\xi|}.$$

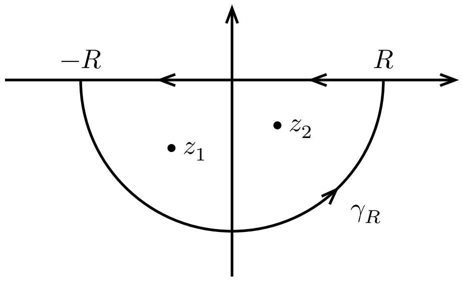
Therefore

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}.$$

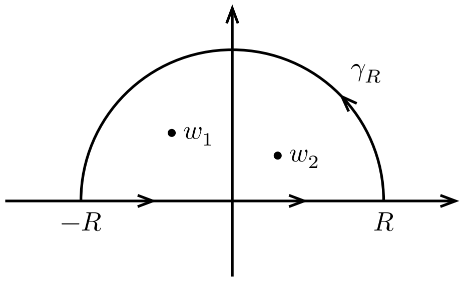
By the Fourier inversion formula, we get

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i\xi x} d\xi = \frac{1}{\pi} \frac{a}{a^2+x^2}.$$

4. Let's say $\{w_k\}$ be the roots of Q in the upper half-plane, and $\{z_k\}$ be the roots of Q in the lower half-plane. Similar to **Exercise 3**, we can integrate the function $f(z) = e^{-2\pi i z\xi}/Q(z)$ over the appropriate semicircle to obtain a integral.



contour when $\xi > 0$



contour when $\xi < 0$

For the case $\xi > 0$, integrate the function $f(z)$ over the lower semicircle and take $R \to \infty$ to get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)} dx &= -2\pi i \sum_{z_k} \operatorname*{Res}_{z=z_k} \frac{e^{-2\pi i z\xi}}{Q(z)} = -2\pi i \sum_{z_k} \lim_{z \to z_k} \frac{z - z_k}{Q(z)} e^{-2\pi i z\xi} \\ &= -2\pi i \sum_{z_k} \frac{e^{-2\pi i z_k \xi}}{Q'(z_k)}.\end{aligned}$$

Otherwise, if $\xi < 0$, integrate the function $f(z)$ over the upper semicircle and take $R \to \infty$ to get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)} dx &= 2\pi i \sum_{w_k} \operatorname*{Res}_{z=w_k} \frac{e^{-2\pi i z\xi}}{Q(z)} = 2\pi i \sum_{w_k} \lim_{z \to w_k} \frac{z - w_k}{Q(z)} e^{-2\pi i z\xi} \\ &= 2\pi i \sum_{w_k} \frac{e^{-2\pi i w_k \xi}}{Q'(w_k)}.\end{aligned}$$

Finally, if $\xi = 0$, we just get

$$\int_{-\infty}^{\infty} \frac{1}{Q(x)} dx = \widehat{1/Q}(0).$$

Even if several roots coincide, we can simply find the residue of the that poles and add it.

5. (a) Similar to **Exercise 4**, just integrate the function $f(z) = R(z)e^{-2\pi i x\xi}$ in the upper semicircle with radius R and take a limit $R \to \infty$ to get

$$\int_{-\infty}^{\infty} R(x)e^{-2\pi i x\xi} dx = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=\alpha_j} R(z)e^{-2\pi i z\xi}.$$

Let $m_j = \mathrm{mul}(\alpha_j)$, then

$$\begin{aligned}\operatorname*{Res}_{z=\alpha_j} R(z)e^{-2\pi i z\xi} &= \frac{1}{(m_j - 1)!} \lim_{z \to \alpha_j} \left(\frac{d}{dz}\right)^{m_j - 1} \left(\frac{P(z)}{Q(z)} e^{-2\pi i z\xi} (z - \alpha_j)^{m_j}\right) \\ &= P_j(\xi) e^{-2\pi i \alpha_j \xi}.\end{aligned}$$

This completes the proof.

(b) If $Q(z)$ has no zeros in the upper half-plane, then there are no residues to add up, so the integral is simply 0.

(c) Denote the zeros of $f(z)$ in the lower half-plane as $\{\beta_j\}$. Then integrating the function $f(z)$ over the upper semicircle of radius R and taking $R \to \infty$ gives

$$\int_{-\infty}^{\infty} R(x)e^{-2\pi i x\xi} dx = \sum_{z=\beta_j} Q_j(\xi) e^{-2\pi i \beta_j \xi},$$

Where Q_j is a polynomial of degree less than the multiplicity of β_j.

(d) Since $\left|e^{-2\pi i\alpha_j\xi}\right| = e^{-2\pi \mathrm{Im}(\alpha_i)|\xi|}$ for $\xi < 0$, and $\left|e^{-2\pi i\beta_j\xi}\right| = e^{-2\pi \mathrm{Im}(-\beta_j)|\xi|}$ for $\xi > 0$, a has to satisfy the inequality

$$a < 2\pi \min\left(\{\mathrm{Im}(\alpha_j) \cup \mathrm{Im}(-\beta_j)\}\right) = 2\pi \min\{|\mathrm{Im}(\gamma_j)|\}$$

where the $\{\gamma_j\}$ is the zeros of Q.

6. Define $f(z) = \frac{1}{\pi}\frac{a}{a^2+z^2}$. By **Exercise 3**, we know that $\hat{f}(\xi) = e^{-2\pi a|\xi|}$. By substituting f and $\hat{f}$ to the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

we get

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}.$$

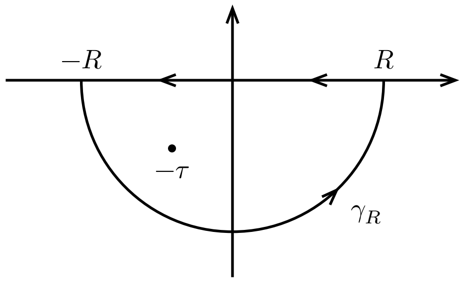
In addition,

$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = 1 + 2\sum_{n=1}^{\infty} e^{-2\pi an} = 1 + 2\frac{e^{-2\pi a}}{1-e^{-2\pi a}} = \frac{e^{2\pi a}+1}{e^{2\pi a}-1} = \coth \pi a.$$

7. (a) Define $f(z) = 1/(\tau+z)^k$. We want to calculate

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{1}{(\tau+x)^k} e^{-2\pi ix\xi} dx.$$

If $\xi > 0$, then we integrate the function $e^{-2\pi iz\xi}/(\tau+z)^k$ over the lower semicircle of radius R.



Since $\left|e^{-2\pi iz\xi}\right| \le 1$ whenever $\mathrm{Im}(z) \le 0$, the integral over the arc goes to 0 as $R \to \infty$. By residue formula,

$$\hat{f}(\xi) = -2\pi i \operatorname*{Res}_{z=-\tau} \frac{e^{-2\pi iz\xi}}{(\tau+z)^k} = -2\pi i \cdot \frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}\left(e^{-2\pi iz\xi}\right) = \frac{(-2\pi i)^k}{(k-1)!} \cdot \xi^{k-1} e^{-2\pi iz\xi}.$$

However, if $\xi < 0$, we integrate the function over the upper semicircle. Since there is no poles inside the contour, the integral is

$$\hat{f}(\xi) = 0.$$

Simply $\hat{f}(0)$ is also 0. Therefore by Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} \xi^{k-1} e^{-2\pi iz\xi}.$$

(b) Set $k = 2$ then the right hand side becomes

$$(-2\pi i)^2 \sum_{m=1}^{\infty} m e^{2\pi i m \tau}.$$

Since

$$\sum_{m=1}^{\infty} e^{2\pi i m \tau} = \frac{e^{2\pi i \tau}}{1 - e^{2\pi i \tau}},$$

differentiate both sides to obtain

$$2\pi i \sum_{m=1}^{\infty} m e^{2\pi i m \tau} = 2\pi i \frac{e^{-2\pi \tau i}}{(1 - e^{-2\pi \tau i})^2}.$$

Therefore

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau + n)^2} = -4\pi^2 \sum_{m=1}^{\infty} m e^{2\pi i m \tau} = -4\pi^2 \cdot \frac{1}{(e^{\pi \tau i} - e^{-\pi \tau i})^2} = \frac{\pi^2}{\sin^2(\pi \tau)}.$$

(c) The case $\mathrm{Im}(\tau) = 0$ is proved via taking the limit $\mathrm{Im}(\tau) \to 0$. If $\mathrm{Im}(\tau) < 0$, then $\mathrm{Im}(-\tau) > 0$ so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(-\tau + n)^2} = \frac{\pi^2}{\sin^2(-\pi \tau)}.$$

Therefore

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau + n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(\tau - n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(-\tau + n)^2} = \frac{\pi^2}{\sin^2(-\pi \tau)} = \frac{\pi^2}{\sin^2(\pi \tau)}.$$

8. By the Fourier inversion formula,

$$f(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \ \Rightarrow \ f^{(n)}(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i x \xi} (2\pi i \xi)^n d\xi.$$

Therefore

$$a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{n!} (2\pi i)^n \int_{-M}^{M} \hat{f}(\xi) \xi^n d\xi.$$

$\left|\hat{f}\right|$ is continuous in $[-M, M]$ so bounded, hence there exists $A > 0$ such that $\left|\hat{f}\right| \le A$. Then

$$n!|a_n| \le (2\pi)^n \int_{-M}^{M} \left|\hat{f}(\xi)\right| |\xi^n| d\xi \le (2\pi)^n A \int_{-M}^{M} |\xi|^n d\xi = (2\pi)^n A \cdot 2 \cdot \frac{1}{n+1} M^{n+1}$$

and

$$(n!|a_n|)^{1/n} \le 2\pi M \cdot \left(\frac{2AM}{n+1}\right)^{1/n}.$$

Therefore

$$\limsup_{n\to\infty} (n!|a_n|)^{1/n} \le 2\pi M.$$

Conversely, now suppose f be any power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with

$$\limsup_{n\to\infty} (n!|a_n|)^{1/n} \le 2\pi M.$$

Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \ \Rightarrow \ (n!|a_n|)^{1/n} \le 2\pi(M + \varepsilon).$$

First observe that $|a_n|^{1/n} \le 2\pi(M+\varepsilon)/(n!)^{1/n} \to 0$ as $n \to \infty$, so radius of convergence of f is ∞. Also,

$$\begin{aligned} |f(z)| &\le \sum_{n=0}^{N} |a_n||z|^n + \sum_{n=N+1}^{\infty} \frac{(2\pi(M+\varepsilon))^n}{n!}|z|^n \\ &\le \sum_{n=0}^{N} |a_n||z|^n + e^{2\pi(M+\varepsilon)|z|}. \end{aligned}$$

Since the rate of growth of $\sum_{n=0}^{N}|a_n||z|^n$ is no more than an polynomial of degree $\le N$, there exists $A_\varepsilon > 0$ such that

$$|f(z)| \le A_\varepsilon e^{2\pi(M+\varepsilon)|z|}.$$

9. **(a)** This is special case of **(b)** with $\beta = 1$.

(b) We define

$$F_\varepsilon(z) = F(z)e^{-\varepsilon z^\gamma},$$

where $\alpha < \gamma < \beta$. Note that z^γ means

$$z = re^{i\theta} \;\Rightarrow\; z^\gamma = r^\gamma e^{i\gamma\theta}.$$

Since $-\pi/2\beta < \theta < \pi/2\beta$ obtains

$$-\frac{\pi}{2} < -\frac{\pi}{2}\frac{\gamma}{\beta} < \gamma\theta < \frac{\pi}{2}\frac{\gamma}{\beta} < \frac{\pi}{2},$$

$\cos(\gamma\theta) > 0$ for all $z = re^{i\theta} \in S$.

Observe that

$$|F_\varepsilon(z)| = |F(z)|\left|e^{-\varepsilon z^\gamma}\right| \le Ce^{c|z|^\alpha} \cdot e^{-\varepsilon|z|^\gamma \cos^\gamma(\pi/(2\beta))} \to 0 \quad \text{as} \quad |z| \to \infty.$$

Therefore F_ε is bounded. Let

$$M = \sup_{z\in\overline{S}} |F_\varepsilon(z)|.$$

Suppose F is not identically zero, let $\{w_j\}$ be a sequence of points such that $\left|F_\varepsilon(w_j)\right| \to M$. Since $M \ne 0$ and F_ε converges to 0 as $|z| \to \infty$, $\{w_j\}$ is bounded. Hence $w_j \to w \in \overline{S}$. By the maximum principle, w cannot be interior point of S. So w is on the boundary of S. Since $|F(z)| \le 1$ for z on the boundary of S,

$$|F_\varepsilon(w)| = |F(w)|\left|e^{-\varepsilon w^\gamma}\right| \le 1 \;\Rightarrow\; M \le 1.$$

Finally, taking a limit $\varepsilon \to 0$ concludes the proof.

10. By shifting the contour of integration, we obtain

$$\hat{f}(\xi + i\eta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x(\xi+i\eta)}dx = \int_{-\infty}^{\infty} f(x-iy)e^{-2\pi i(x-iy)(\xi+i\eta)}dx.$$

Therefore

$$
\begin{aligned}
\left|\hat{f}(\xi+i\eta)\right| &= \left|\int_{-\infty}^{\infty} f(x-iy)e^{-2\pi i(x-iy)(\xi+i\eta)}dx\right| \\
&= \left|\int_{-\infty}^{\infty} f(x-iy)e^{-2\pi ix\xi-2\pi y\xi+2\pi x\eta-2\pi iy\eta}dx\right| \\
&= e^{-2\pi y\xi}\int_{-\infty}^{\infty}|f(x-iy)|e^{2\pi x\eta}dx \le e^{-2\pi y\xi}\int_{-\infty}^{\infty} ce^{-ax^2+by^2}e^{2\pi x\eta}dx \\
&= ce^{-2\pi y\xi+by^2}\int_{-\infty}^{\infty} e^{-ax^2+2\pi\eta x}dx = ce^{-2\pi y\xi+by^2+\frac{\pi^2}{a}\eta^2}\int_{-\infty}^{\infty} e^{-a(x-\pi\eta/a)^2}dx.
\end{aligned}
$$

Now letting $y = \frac{\pi}{b}\xi$ gives

$$\left|\hat{f}(\xi+i\eta)\right| \le \left(C\int_{-\infty}^{\infty} e^{-ax^2}dx\right)e^{-\frac{\pi^2}{b}\xi^2+\frac{\pi^2}{a}\eta^2}.$$

Hence $C' = \sqrt{\pi/a}C, a' = \pi^2/b, b' = \pi^2/a$ satisfies the inequality.

11. If $x^2 \le y^2$, then

$$|f(z)| \le Ce^{c_1|z|^2} = Ce^{c_1x^2+c_2y^2} \le Ce^{-c_1x^2+3c_1y^2}$$

so $|f(z)| = O\left(e^{-ax^2+by^2}\right)$. Now suppose $x^2 \ge y^2$. Without loss of generality, we can consider the case $\arg z \in [0, \pi/4]$. We denote $r = |z|$. Note that $|f(z)| \le C_2e^{-c_2r^2}$ when $\arg z = 0$, and $|f(z)| \le C_1e^{c_1r^2}$ when $\arg z = \pi/4$. Define

$$F(z) = f(z)e^{(c_2+ic_1)z^2}.$$

Then

$$
\begin{aligned}
\arg z = 0: \ |F(z)| &\le C_2e^{-c_2r^2}e^{c_2r^2} = C_2, \\
\arg z = \pi/4: \ |F(z)| &\le C_1e^{c_1r^2}e^{-c_1r^2} = C_1.
\end{aligned}
$$

By Phragmén–Lindelöf principle, we can get

$$|F(z)| \le C, \quad \arg z \in [0, \pi/4].$$

Hence

$$
\begin{aligned}
|f(z)| &\le C\left|e^{(-c_2-ic_1)z^2}\right| = C\left|e^{(-c_2-ic_1)(x+iy)^2}\right| = Ce^{-c_2(x^2-y^2)+2c_1xy} \\
&= Ce^{-c_2x^2+c_2y^2+2c_1xy} \le Ce^{-\frac{1}{2}c_1x^2+\left(c_2+\frac{2c_1^2}{c_2}\right)y^2}.
\end{aligned}
$$

In the last inequality, we used arithmetic mean-geometric mean inequality

$$2c_1xy \le \frac{c_2}{2}x^2 + \frac{2c_1^2}{c_2}y^2.$$

12. (a) Let $\xi = \sigma + i\tau$. Since $f(x) = O\left(e^{-\pi x^2}\right)$,

$$
\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx\right| = \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi ix\sigma+2\pi x\tau}dx\right| \le \int_{-\infty}^{\infty}|f(x)|e^{2\pi x\tau}dx \\
&\le C\int_{-\infty}^{\infty} e^{-\pi x^2+2\pi x\tau}dx = C\int_{-\infty}^{\infty} e^{-\pi(x-\tau)^2+\pi\tau^2}dx = Ce^{\pi\tau^2}.
\end{aligned}
$$

Therefore a function f_n defined by $f_n(\xi) = \int_{-n}^{n} f(x)e^{-2\pi i x\xi}dx$ is holomorphic and uniformly converges to $\hat{f}$ in all compact subset of $\mathbb{C}$. Hence $\hat{f}(\xi)$ is holomorphic. $\hat{f}$ is even because

$$\hat{f}(-\xi) = \int_{-\infty}^{\infty} f(x)e^{2\pi i x(-\xi)}dx = \int_{\infty}^{-\infty} f(-t)e^{2\pi i t\xi}(-dt) = \int_{-\infty}^{\infty} f(t)e^{2\pi i t\xi}dt = \hat{f}(\xi).$$

Now define $g(z) = \hat{f}\left(z^{1/2}\right)$. Then $|g(x)| \le ce^{-\pi x}$ since $\hat{f}(\xi) = O\left(e^{-\pi\xi^2}\right)$. Moreover,

$$|g(z)| = \left|\hat{f}\left(z^{1/2}\right)\right| \le ce^{\pi y^2} \le ce^{\pi\left(\sqrt{R}\sin(\theta/2)\right)^2} = ce^{\pi R\sin^2(\theta/2)} \le ce^{\pi|z|}.$$

(b) Now define $F(z) = g(z)e^{\gamma z}$, where $\gamma = i\pi\frac{e^{-i\pi/(2\beta)}}{\sin\pi/(2\beta)} = \pi + i\pi\cot\pi/(2\beta)$. Observe that

$$\begin{aligned} \arg z = 0: \ & |F(x)| = |g(x)||e^{\gamma x}| \le ce^{-\pi x}|e^{\gamma x}| = ce^{-\pi x}e^{\pi x} = c, \\ \arg z = \pi/4: \ & |F(z)| = |g(z)||e^{\gamma z}| \le ce^{\pi|z|}e^{-\pi|z|} = c. \end{aligned}$$

By Phragmén–Lindelöf principle, we get

$$|F(z)| \le c.$$

Take a limit $\beta \to 1^+$ then $\gamma \to \pi$, so $e^{\pi z}g(z)$ is bounded on closed upper half-plane. Similarly, same result holds in lower half-plane. Since $e^{\pi z}g(z)$ is holomorphic and bounded in $\mathbb{C}$, $e^{\pi z}g(z)$ is constant.

(c) If f is odd, then $\hat{f}$ is also odd so $\hat{f}(0) = 0$. Then $\hat{f}(z)/z$ is entire. Note that $g(z) = \hat{f}\left(z^{1/2}\right)/z^{1/2}$ also satisfies the conclusion of **(a)**, since $g(z)$ is bounded in $|z| \le 1$ and $g(z)$ is even smaller than $\hat{f}\left(z^{1/2}\right)$ when $|z| > 1$.

Now we white $f(z) = f_{\text{even}}(z) + f_{\text{odd}}(z)$, where

$$f_{\text{even}}(z) = \frac{f(z) + f(-z)}{2}, \quad f_{\text{odd}}(z) = \frac{f(z) - f(-z)}{2}.$$

Since $f(-x) = O\left(e^{-\pi(-x)^2}\right) = O\left(e^{-\pi x^2}\right)$, f_{even} and f_{odd} is also $O\left(e^{-\pi x^2}\right)$. Therefore we can apply the above argument and deduce that $f = \hat{f} = 0$.

Chapter 5. Entire Functions

1. Observe that if f_1 and f_2 satisfies the hypotheses and conclusion, then the product $f_1 f_2$ also satisfies the hypothesis and conclusion. Let $\{z_1, \cdots, z_N\}$ be the zeros of f inside $\mathbb{D}$. Then $g(z) = f(z)/\left(\psi_{z_1}\psi_{z_2}\cdots\psi_{z_N}\right)$ is bounded near each z_j, so each z_j is removable singularity of g. Also, g nowhere vanishes in $\mathbb{D}$ since $1/|\overline{z_k}| > 1$. We write

$$f(z) = \psi_{z_1}(z)\,\psi_{z_2}(z)\cdots\psi_{z_N}(z)\,g(z).$$

For g, proof is same with step 3 of **Theorem 1.1**. We prove that Jensen formula holds for each ψ_w, that is,

$$\log|w| = \log\left(\frac{|w|}{1}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{w - e^{i\theta}}{1 - \overline{w}e^{i\theta}}\right|d\theta.$$

Since

$$\int_0^{2\pi}\log\left|\frac{w - e^{i\theta}}{1 - \overline{w}e^{i\theta}}\right|d\theta = \int_0^{2\pi}\log\left|w - e^{i\theta}\right|d\theta - \int_0^{2\pi}\log\left|1 - \overline{w}e^{i\theta}\right|d\theta = 0 - 0 = 0,$$

the equation holds.

2. **(a)** For every integer $m > 0$, there exists $\varepsilon > 0$ such that

$$|z|^m \le Ae^{B|z|^\varepsilon} \qquad \text{for all } z \in \mathbb{C}.$$

Hence the order of growth is 0.

(b) The order of growth is n.

(c) There is no ρ such that

$$\left|e^{e^z}\right| \le Ae^{B|z|^\rho} \qquad \text{for all } z \in \mathbb{C},$$

because for real $z = x$, taking a logarithm on the inequality above gives

$$e^x \le \log A + B|x|^\rho \qquad \text{for all } x \in \mathbb{R}$$

which is false for large x. Thus the order of growth is $\inf \varnothing = \infty$.

3. Observe that

$$|\Theta(z|\tau)| \le \sum_{n=-\infty}^{\infty} \left|e^{\pi i n^2 \tau} e^{2\pi i n z}\right| = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \left|e^{2\pi i n z}\right| \le \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{2\pi i n |z|}.$$

The last power series converges because

$$\begin{aligned} e^{-\pi n^2 t + 2\pi n|z|} &\le e^{-\pi n^2 t} && \text{when } n \to -\infty, \\ e^{-\pi n^2 t + 2\pi n|z|} &\le e^{-\pi n^2 t/2} && \text{when } n \to \infty. \end{aligned}$$

Hence

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{2\pi i n|z|} = \sum_{n=-\infty}^{\infty} e^{-\pi t(n-|z|/t)^2 + \pi|z|^2/t} = \left(\sum_{n=-\infty}^{\infty} e^{-\pi t(n-|z|/t)^2}\right) e^{\pi|z|^2/t}$$

which has order of growth 2.

Note. I proved that $\Theta(z|\tau)$ has an order of growth ≤ 2, but i don't know how to show that order of growth is exactly equal to 2.

4. (a) We write

$$F_1(z) = \prod_{n=1}^{N} (1 - e^{-2\pi n t} e^{2\pi i z}) \quad \text{and} \quad F_2(z) = \prod_{n=N+1}^{\infty} (1 - e^{-2\pi n t} e^{2\pi i z})$$

where $N = \left\lceil \frac{|z|}{t} - 1 \right\rceil$. Since $(N+1)T \ge |z|$,

$$\sum_{n=N+1}^{\infty} e^{-2\pi n t} e^{2\pi|z|} \le \frac{1}{1 - e^{-2\pi t}}.$$

Hence

$$\begin{aligned} |F_2(z)| = \prod_{n=N+1}^{\infty} \left|1 - e^{-2\pi n t + 2\pi i z}\right| &= \exp\left(\sum_{n=N+1}^{\infty} \log\left|1 - e^{-2\pi n t + 2\pi i z}\right|\right) \\ &\le \exp\left(\sum_{n=N+1}^{\infty} e^{-2\pi n t + 2\pi|z|}\right) \le \exp(1/(1 - e^{-2\pi t})), \end{aligned}$$

where we used the inequality $\log|1 - z| \le |1 - z| + 1 \le |z|$. However,

$$\left|1 - e^{-2\pi n t} e^{2\pi i z}\right| \le 1 + e^{2\pi|z|} \le 2e^{2\pi|z|}.$$

Thus

$$|F_1(z)| \le 2^N e^{2\pi N|z|} \le 2^{|z|/t} e^{2\pi|z|^2/t}.$$

Therefore $F(z) = F_1(z)F_2(z)$ has order of growth ≤ 2.

(b) Since the order of growth of F is 2, by **Theorem 2.1 (ii)**,

$$\sum \frac{1}{|z_n|^{2+\varepsilon}} < \infty.$$

for every positive number ε. Note that

$$\sum_{m\in\mathbb{Z}} \frac{1}{m^2+a^2} \ge \sum_{m=1}^{\infty} \frac{1}{m^2+a^2} \ge \int_1^{\infty} \frac{1}{x^2+a^2}dx = \frac{\pi}{2a} - \frac{1}{a}\arctan\left(\frac{1}{a}\right).$$

Hence

$$\sum \frac{1}{|z_n|^2} = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{n^2t^2+m^2} \ge \sum_{n=1}^{\infty} \frac{\pi}{2nt} - \frac{1}{nt}\arctan\left(\frac{1}{nt}\right) = \infty.$$

5. Note that following inequality holds;

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \le c|z|^{\alpha/(\alpha-1)}$$

because if $|t|^{\alpha-1} \le 4\pi|z|$, then

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \le 2\pi|z||t| \le 2\pi(4\pi)^{1/(\alpha-1)}|z|^{\alpha/(\alpha-1)},$$

otherwise, $|t|^{\alpha-1} \le 4\pi|z|$ implies

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| = \frac{1}{2}|t|(4\pi|z| - |t|^{\alpha-1}) \le 0 \le |z|^{\alpha/(\alpha-1)}.$$

Now we can get

$$|F_\alpha(z)| \le \int_{-\infty}^{\infty} \left|e^{-|t|^\alpha + 2\pi izt}\right| dt \le \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi i|z||t|} dt \le \left(\int_{-\infty}^{\infty} e^{-|t|^\alpha/2} dt\right) e^{c|z|^{\alpha/(\alpha-1)}}$$

Hence F_α has order of growth $\le \alpha/(\alpha-1)$.

6. By the product formula for $\sin z$,

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Put $z = 1/2$ to get

$$1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) \;\Rightarrow\; \frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty} \frac{2n\cdot 2n}{(2n-1)(2n+1)}.$$

7. (a) Note that $|a_n| \to 0$ as $n \to \infty$ because $\sum |a_n|^2$ converges.

First we suppose that $\sum a_n$ converges. Note that

$$\log(1+a_n) = a_n - \frac{1}{2}a_n^2 + \frac{1}{3}a_n^3 - \frac{1}{4}a_n^4 + \cdots = a_n - \frac{1}{2}a_n^2 + k_n a_n^2$$

where $k_n = \frac{1}{3}a_n - \frac{1}{4}a_n^2 + \cdots \to 0$ as $n \to \infty$. Since $1/2 - k_n$ is bounded,

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2}-k_n\Big)a_n^2$$

converges. Hence

$$\begin{aligned}\prod_{n=1}^{N}(1+a_n) &= \prod_{n=1}^{N} e^{\log(1+a_n)} = \exp\left(\sum_{n=1}^{N}\log(1+a_n)\right) \\ &= \exp\left(\sum_{n=1}^{N}a_n - \sum_{n=1}^{N}\Big(\frac{1}{2}-k_n\Big)a_n^2\right) \to \exp\left(\sum_{n=1}^{\infty}a_n - \sum_{n=1}^{\infty}\Big(\frac{1}{2}-k_n\Big)a_n^2\right).\end{aligned}$$

Therefore $\prod(1+a_n)$ converges to a non-zero limit.

Conversely, suppose $\prod(1+a_n)$ converges to a non-zero limit. Similar to argument above, we get

$$\exp\left(\sum_{n=1}^{N}a_n\right) = \prod_{n=1}^{N}(1+a_n)\cdot\exp\left(\sum_{n=1}^{N}\Big(\frac{1}{2}-k_n\Big)a_n^2\right).$$

Since right hand side converges as $N\to\infty$ and $\lim_{n\to\infty}a_n = 0$, $\sum a_n$ also converges.

(b) Let $a_n = (-1)^n/\sqrt{n}$. Then $\sum a_n$ converges because it is alternating series. Note that $\sum a_n^2 = \sum 1/n$ diverges. Now let

$$l_n = \frac{1}{3} - \frac{1}{4}a_n + \frac{1}{5}a_n^2 - \frac{1}{6}a_n^3 + \cdots$$

then $l_n \to 1/3$ as $n\to\infty$ and $\sum l_n a_n^3$ converges. Similar to argument above,

$$\prod_{n=1}^{N}(1+a_n) = \exp\left(\sum_{n=1}^{N}a_n\right)\cdot\exp\left(-\frac{1}{2}\sum_{n=1}^{N}a_n^2\right)\cdot\exp\left(\sum_{n=1}^{N}l_n a_n^3\right).$$

If $\prod(1+a_n)$ converges, then $\sum a_n^2$ also converges, which is contradiction. Hence $\prod(1+a_n)$ diverges.

(c) $a_n = (-1)^n$ then $\prod(1+a_n) = 0$ converges to 0, while $\sum a_n = \sum(-1)^n$ diverges.

8. Use the fact that $\sin 2z = 2\sin z\cos z$.

$$\prod_{k=1}^{N}\cos\Big(\frac{z}{2^k}\Big) = \prod_{k=1}^{N}\left(\frac{\sin(z/2^{k-1})}{2\sin(z/2^k)}\right) = \frac{1}{2^N}\cdot\frac{\sin(z)}{\sin(z/2^N)} = \frac{z/2^N}{\sin(z/2^N)}\cdot\frac{\sin z}{z} \to \frac{\sin z}{z}$$

as $N\to\infty$, since $z/2^N \to 0$ for all $z\in\mathbb{C}$.

9. Use the fact that $(1-z^n)(1+z^n) = 1-z^{2n}$. Since

$$(1-z)\prod_{k=0}^{N}\Big(1+z^{2^k}\Big) = 1 - z^{2^{N+1}},$$

we get

$$\prod_{k=0}^{N}\Big(1+z^{2^k}\Big) = \frac{1-z^{2^{N+1}}}{1-z} \to \frac{1}{1-z}$$

as $N\to\infty$ for every $|z|<1$.

10. (a) Since $e^z - 1$ has zeros at $2\pi n i$ $(n\in\mathbb{Z})$ and has order of growth 1, by Hadamard's factorization theorem,

$$\begin{aligned} e^z - 1 &= e^{az+b} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{2\pi n i}\right) e^{z/(2\pi n i)} \left(1 + \frac{z}{2\pi n i}\right) e^{-z/(2\pi n i)} \\ &= e^{az+b} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right). \end{aligned}$$

Divide both sides by z and take $z \to 0$ to get $e^b = 1$. Furthermore, $(e^z - 1)/e^{az}$ is odd, hence $a = 1/2$. Therefore

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

(b) Since $\cos \pi z$ has zeros at $z = n + 1/2$ $(n \in \mathbb{Z})$ and has order of growth 0, by Hadamard's factorization theorem,

$$\cos \pi z = e^c \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right).$$

Put $z = 0$ to get $e^c = 1$. Hence $\cos \pi z = \prod_{n=1}^{\infty} (1 - 4z^2/(2n+1)^2)$.

11. Suppose f misses a, b. Since f is of finite order, there exists a polynomial p such that

$$f(z) - a = e^{p(z)} \quad \text{for all } z \in \mathbb{C}.$$

There is no z such that $f(z) = b$, so there is no z such that $p(z) = \log(b - a)$. Hence p must be constant, then f is also constant.

12. Since f is entire and never vanishes, and of finite order, there exists polynomial p such that $f(z) = e^{p(z)}$ for all $z \in \mathbb{C}$. Hence $f'(z) = p'(z)e^{p(z)}$. f' also never vanishes, thus $p'(z)$ must be constant. Therefore we get $p(z) = az + b$ and $f(z) = e^{az+b}$.

13. Note that the order of growth of $e^z - z$ is 1. If equation $e^z - z = 0$ does not have infinity zeros, then

$$e^z - z = e^{Az+B} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

by Hadamard's factorization theorem. Putting $z = 0$ gives $e^B = 1$. Observe that letting $C = A + \sum_{n=1}^{N} 1/a_n$ gives

$$e^z - z = e^{Cz} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right).$$

Considering the rate of growth, it must be $C = 1$. Dividing both sides by e^z to get

$$1 - \frac{z}{e^z} = \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right).$$

Taking $z \to \infty$ for real z gives $1 = \pm\infty$, which is contradiction.

14. Suppose F does not have infinity zeros. By Hadamard's factorization theorem,

$$F(z) = e^{p(z)} z^m \prod_{n=1}^{N} E_k(z/a_n) = e^{q(z)} z^m \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right)$$

where $q(z) = p(z) + \sum_{n=1}^{N} \left(z + z^2/2 + \cdots + z^{\lfloor\rho\rfloor}/\lfloor\rho\rfloor\right)$ is a polynomial of degree $\leq \lfloor\rho\rfloor$. Observe that order of growth of left-hand side is ρ, while the right-hand side is $\lfloor\rho\rfloor$. Since ρ is not an integer, $\rho \neq \lfloor\rho\rfloor$, we get contradiction.

15. The any meromorphic function f in $\mathbb{C}$, by definition, is holomorphic in $\mathbb{C} - \{z_0, z_1, \cdots\}$ and has poles at the points $\{z_0, z_1, \cdots\}$, where $\{z_0, z_1, \cdots\}$ has no limit points. By **Theorem 4.1**, There exists a function g such that has zeros at each $z = z_k$, and there are no other zeros. Then $h = gf$ has no pole, therefore entire. Now we can write $f = h/g$.

Suppose $\{a_n\}$ and $\{b_n\}$ are disjoint sequences having no finite limit points. Then there exists function f and g such that f has zeros exactly at $\{a_n\}$ and g has zeros exactly at $\{b_n\}$. Now $h = f/g$ is a meromorphic function that vanishes exactly at $\{a_n\}$ and has poles exactly at $\{b_n\}$.

16. Since $\{a_n\}$ has no limit points, $\lim_{n\to\infty}|a_n| = \infty$. We can assume that $a_n \neq 0$. For any compact subset K of $\mathbb{C}$, there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |a_n| > \sup_{z\in K}|z|.$$

Observe that each $\frac{1}{z-a_n}$ is holomorphic in disc $|z| < |a_n|$ for $n \geq 1$. By Runge's approximation theorem, $Q_n\left(\frac{1}{z-a_n}\right)$ can be uniformly approximated by polynomials $P_n(z)$, thus

$$\left|Q_n\left(\frac{1}{z-a_n}\right) - P_n(z)\right| \leq \frac{1}{2^n}, \quad \text{in disc } |z| < |a_n|$$

for all $n \geq 1$. Now consider the function

$$f(z) = \sum_{n=1}^{\infty}\left(Q_n\left(\frac{1}{z-a_n}\right) - P_n(z)\right).$$

Since $z \in K \Rightarrow |z| < |a_n|$,

$$\sum_{n=N+1}^{\infty}\left(Q_n\left(\frac{1}{z-a_n}\right) - P_n(z)\right)$$

is holomorphic in K. Moreover,

$$\sum_{n=1}^{N}\left(Q_n\left(\frac{1}{z-a_n}\right) - P_n(z)\right)$$

is meromorphic function which has poles at a_n $(1 \leq n \leq N)$ with principal parts $Q_n\left(\frac{1}{z-a_n}\right)$. By the arbitrariness of K, $f(z)$ is meromorphic function which has poles at a_n $(n \geq 1)$ and principal parts $Q_n\left(\frac{1}{z-a_n}\right)$ at each poles.

17. (a) Define

$$P(z) = \sum_{i=1}^{n} \frac{(z-a_1)\cdots(z-a_{i-1})(z-a_{i+1})\cdots(z-a_n)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} b_i$$

then $P(a_i) = b_i$ holds for all $1 \leq i \leq n$.

(b) For any compact subset K of $\mathbb{C}$, there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |a_n| > \sup_{z\in K}|z|$$

because $\lim_{k\to\infty}|a_k| = \infty$. Note that $a_n \notin K$ for all $n > N$. Since $E(z)/(z-a_n)$ is bounded in K for each n, there exists $M_n > 0$ such that

$$M_n = \sup_{z\in K}\left|\frac{E(z)}{z-a_n}\right|.$$

Moreover, $|E(z)|$ is also bounded in K,

$$\lim_{n\to\infty}\left|\frac{E(z)}{z-a_n}\right| \le \lim_{n\to\infty}\frac{|E(z)|}{a_n-|z|} = 0, \quad \forall z \in K.$$

Thus $\lim_{n\to\infty} M_n = 0$. Therefore $M = \sup_{n\in\mathbb{N}} M_n$ satisfies

$$\left|\frac{E(z)}{z-a_n}\right| \le M, \quad \forall z \in K, \forall n > N.$$

Let $F_N(z)$ be the partial sum of F up to $k = N$. There exists $q \in (0,1)$ such that $|z/a_n| < q$ for $\forall z \in K, \forall n > N$. Hence

$$|F(z) - F_N(z)| = \left|\sum_{k=N+1}^{\infty}\frac{b_k}{E'(a_k)}\frac{E(z)}{z-a_k}\left(\frac{z}{a_k}\right)^{m_k}\right| \le M\sum_{k=N+1}^{\infty}\left|\frac{b_k}{E'(a_k)}\right|q^{m_k} \le \frac{M}{2^N}$$

where we choose m_k large enough so $|b_k/E'(a_k)|q^{m_k} < 1/2^k$ for all $k \ge 1$. Therefore F_n uniformly converges to F in every compact subset K of $\mathbb{C}$. Since $E(z)/z, E(z)/(z-a_k)$ are holomorphic, F is also holomorphic. Note that

$$\lim_{a\to a_l}\frac{b_k}{E'(a_k)}\frac{E(z)}{z-a_k} = \lim_{z\to a_l}\frac{b_k}{E'(a_k)}\frac{E(z)-E(a_k)}{z-a_k} = \begin{cases} b_k & (l=k) \\ 0 & (l\ne k)\end{cases}.$$

Accordingly,

$$\begin{aligned} F(a_k) &= \lim_{z\to a_k} F(z) = \lim_{z\to a_k}\lim_{N\to\infty} F_N(z) \\ &= \lim_{N\to\infty}\lim_{z\to a_k} F_N(z) = \lim_{N\to\infty} F_N(a_k) = \lim_{N\to\infty}\left(\begin{cases} b_k & (N\ge k) \\ 0 & (N<k)\end{cases}\right) = b_k \end{aligned}$$

for all $k \ge 0$.

Note. The interpolation formulas to both problems have similar forms.

$$\begin{aligned} &(\mathrm{a}): \quad F(z) = \sum_{k=0}^{n}\frac{b_k}{E'(a_k)}\frac{E(z)}{z-a_k} && \text{where } E(z) = \prod_{k=0}^{n}(z-a_k), \\ &(\mathrm{b}): \quad F(z) = \sum_{k=0}^{n}\frac{b_k}{E'(a_k)}\frac{E(z)}{z-a_k}\left(\frac{z}{a_k}\right)^{m_k} && \text{where } (z/0)^0 := 1. \end{aligned}$$

The term $(z/a_k)^{m_k}$ is added in order to guarantee the uniform convergence of series.

Chapter 6. The Gamma and Zeta Functions

1. Note that $1/\Gamma(s)$ satisfies the following equation:

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)e^{-s/k}.$$

Therefore

$$\Gamma(s) = \lim_{n\to\infty} e^{-\gamma s}\frac{1}{s}\prod_{k=1}^{N}\frac{k}{s+k}e^{s/k} = \lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}e^{s(1+1/2+\cdots+1/n-\log n-\gamma)}$$
$$= \lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}.$$

2. Simple calculation leads to

$$\prod_{k=1}^{n}\frac{k(k+a+b)}{(k+a)(k+b)} = \frac{(a+b+1)\cdots(a+b+n)\cdot n!}{(a+1)\cdots(a+n)\cdot(b+1)\cdots(b+n)}$$
$$= \left(\frac{n^{a+1}n!}{(a+1)\cdots(a+1+n)}\right)\left(\frac{n^{b+1}n!}{(b+1)\cdots(b+1+n)}\right)\cdot$$
$$\left(\frac{(a+b+1)\cdots(a+b+1+n)}{n^{a+b+1}n!}\right)\left(\frac{(a+1+n)(b+1+n)}{n(a+b+1+n)}\right)$$
$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

3. Note that the general term of Wallis's product is

$$\left(\frac{2\cdot 2}{1\cdot 3}\right)\cdot\left(\frac{4\cdot 4}{3\cdot 5}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right) = 2^{2n}\left(\frac{1\cdot 1}{1\cdot 3}\right)\cdot\left(\frac{2\cdot 2}{3\cdot 5}\right)\cdots\left(\frac{n\cdot n}{(2n-1)(2n+1)}\right)$$
$$= 2^{2n}\frac{(n!)^2}{((2n+1)!!)^2}(2n+1)$$
$$= 2^{4n}\frac{(n!)^4}{((2n+1)!)^2}(2n+1).$$

Hence

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} 2^{2n}\frac{(n!)^2}{(2n+1)!}\sqrt{2n+1}.$$

As a result,

$$\frac{\Gamma(s)\Gamma(s+1/2)}{\Gamma(2s)} = \lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}\cdot\frac{n^{s+\frac{1}{2}}n!}{\left(s+\frac{1}{2}\right)\left(s+\frac{3}{2}\right)\cdots\left(s+\frac{1}{2}+n\right)}\cdot\frac{2s(2s+1)\cdots(2s+n)}{n^{2s}n!}$$
$$= \lim_{n\to\infty}\frac{2^{2n+2}n^{\frac{1}{2}}n!}{(2s+n+1)\cdots(2s+2n+1)}$$
$$= \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}(2n+1)^{\frac{1}{2}}\cdot 2^2\sqrt{\frac{n}{2n+1}}\cdot\frac{(n+1)\cdots(2n+1)}{(2s+n+1)\cdots(2s+2n+1)}.$$

Since letting $a_n = \frac{(n+1)\cdots(2n+1)}{(2s+n+1)\cdots(2s+2n+1)}$ gives

$$-\log(a_n) = \sum_{k=1}^{n+1}\log\left(1+\frac{2s}{n+k}\right) = \sum_{k=1}^{n+1}\left(\frac{2s}{n+k}+O\left(\frac{1}{n^2}\right)\right)$$
$$= \sum_{k=1}^{n+1}\frac{2s}{1+\frac{k}{n}}\cdot\frac{1}{n}+O\left(\frac{1}{n}\right)\xrightarrow{n\to\infty}\int_0^1\frac{2s}{1+x}dx = 2s\ln 2,$$

we get $a_n \to 2^{-2s}$ and

$$\frac{\Gamma(s)\Gamma(s+1/2)}{\Gamma(2s)} = \sqrt{\frac{\pi}{2}}\cdot\frac{4}{\sqrt{2}}\cdot 2^{-2s} = 2^{1-2s}\sqrt{\pi}.$$

4. Since

$$f^{(n)}(z) = (-\alpha)(-\alpha-1)\cdots(-\alpha-(n-1))(1-z)^{-\alpha-n}\cdot(-1)^n,$$

we get

$$a_n(\alpha) = \frac{1}{n!}f^{(n)}(0) = \frac{1}{n!}\alpha(\alpha+1)\cdots(\alpha+(n-1)) = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n^\alpha n!}\cdot\frac{n^\alpha}{n+\alpha}.$$

Therefore

$$\frac{a_n(\alpha)}{n^{\alpha-1}/\Gamma(\alpha)} = \Gamma(\alpha)\cdot\frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n^\alpha n!}\cdot\frac{n}{n+\alpha}\to 1$$

as $n\to\infty$.

5. We first prove that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$. This is because

$$\Gamma(\bar{s}) = \lim_{n\to\infty}\frac{n^{\bar{s}}n!}{\bar{s}(\bar{s}+1)\cdots(\bar{s}+n)} = \lim_{n\to\infty}\overline{\left(\frac{n^s n!}{s(s+1)\cdots(s+n)}\right)} = \overline{\Gamma(s)}.$$

Hence

$$|\Gamma(1/2+it)| = \sqrt{\Gamma(1/2+it)\Gamma(1/2-it)} = \sqrt{\frac{\pi}{\sin\pi(1/2+it)}} = \sqrt{\frac{\pi}{\cos i\pi t}} = \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}.$$

6. Let's denote $H_n = 1+1/2+\cdots+1/n$.

$$\begin{aligned}1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}-\frac{1}{2}\log n &= \left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{1}{2}\log n\\ &= H_{2n}-\frac{1}{2}H_n-\frac{1}{2}\log n\\ &= (H_{2n}-\log(2n))-\frac{1}{2}(H_n-\log n)+\log 2\\ &\to \gamma-\frac{1}{2}\gamma+\log 2 = \frac{\gamma}{2}+\log 2.\end{aligned}$$

7. (a) By definition,

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-t-s}dtds.$$

Now make the change of variables $s = ur, t = u(1-r)$. Note that

$$\left|\frac{\partial(s,t)}{\partial(u,r)}\right| = \left|\begin{pmatrix} r & u\\ 1-r & -u\end{pmatrix}\right| = u$$

and $dsdt = u\,dudr$. Thus

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^1 (u(1-r))^{\alpha-1}(ur)^{\beta-1}e^{-u}udrdu \\
&= \int_0^\infty \int_0^1 u^{\alpha+\beta-1}(1-r)^{\alpha-1}r^{\beta-1}e^{-u}drdu \\
&= \left(\int_0^\infty u^{\alpha+\beta-1}e^{-u}du\right)\left(\int_0^1 (1-r)^{\alpha-1}r^{\beta-1}dr\right) \\
&= \Gamma(\alpha+\beta)B(\alpha,\beta).
\end{aligned}$$

(b) Simply change the variable $t = 1/(1+u)$ to get

$$\begin{aligned}
B(\alpha,\beta) &= \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt = \int_\infty^0 \left(\frac{u}{1+u}\right)^{\alpha-1}\left(\frac{1}{1+u}\right)^{\beta-1}\frac{-1}{(1+u)^2}du \\
&= \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du.
\end{aligned}$$

8. By the properties of beta function,

$$\begin{aligned}
\int_{-1}^1 e^{ixt}(1-t^2)^{\nu-\frac{1}{2}}dt &= \int_{-1}^1 \sum_{n=1}^\infty \frac{(ixt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}} = \sum_{n=0}^\infty \frac{i^n x^n}{n!}\int_{-1}^1 t^n(1-t^2)^{\nu-\frac{1}{2}}dt \\
&= \sum_{m=0}^\infty \frac{i^{2m}x^{2m}}{(2m)!}\cdot 2\cdot \int_0^1 t^{2m}(1-t^2)^{\nu-\frac{1}{2}}dt \\
&= \sum_{m=0}^\infty \frac{(-1)^m x^{2m}}{(2m)!}\int_0^1 u^{m-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}du = \sum_{m=0}^\infty \frac{(-1)^m x^{2m}}{(2m)!}B\left(\nu+\frac{1}{2}, m+\frac{1}{2}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\int_{-1}^1 e^{ixt}(1-t^2)^{\nu-\frac{1}{2}}dt &= \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\sum_{m=0}^\infty \frac{(-1)^m x^{2m}}{(2m)!}B\left(\nu+\frac{1}{2}, m+\frac{1}{2}\right) \\
&= \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\sum_{m=1}^\infty \frac{(-1)^m x^{2m}}{\Gamma(2m+1)}\frac{\Gamma(\nu+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(\nu+m+1)} \\
&= \left(\frac{x}{2}\right)^\nu \sum_{m=1}^\infty \frac{(-1)^m x^{2m}}{\Gamma(\nu+m+1)}\frac{1}{m!2^{2m}} = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^\infty \frac{(-1)^m (x^2/4)^m}{m!\Gamma(\nu+m+1)}
\end{aligned}$$

whenever $x > 0$.

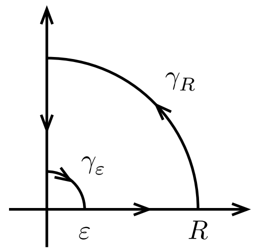
9. Note that

$$(1-zt)^{-\alpha} = \sum_{n=0}^\infty \alpha(\alpha+1)\cdots(\alpha+(n-1))(zt)^n.$$

Hence

$$
\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(\sum_{n=0}^{\infty}\alpha(\alpha+1)\cdots(\alpha+(n-1))(zt)^n\right)dt \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\alpha(\alpha+1)\cdots(\alpha+(n-1))z^n\int_0^1 t^{n+\beta-1}(1-t)^{\gamma-\beta-1}dt \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\alpha(\alpha+1)\cdots(\alpha+(n-1))\frac{\Gamma(\gamma-\beta)\Gamma(n+\beta)}{\Gamma(\gamma+n)}z^n \\
&= 1+\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+(n-1))\beta(\beta+1)\cdots(\beta+(n-1))}{\gamma(\gamma+1)\cdots(\gamma+(n-1))}z^n.
\end{aligned}
$$

10. (a) Integrate the function $f(w)=e^{-w}w^{z-1}$ around the contour below.



$f(w)$ does not have pole in that contour, by Cauchy's theorem,

$$\int_\varepsilon^R f(x)dx+\int_{\gamma_\varepsilon}f(w)dw+\int_R^\varepsilon f(it)idt+\int_{\gamma_R}f(w)dw=0.$$

First, we show that the integrals over the quadrant converges to 0.

$$
\begin{aligned}
\left|\int_{\gamma_\varepsilon}f(w)dw\right| &\le \frac{\pi}{2}\varepsilon\cdot\sup_{w\in\gamma_\varepsilon}\left|e^{-w}w^{z-1}\right| = \frac{\pi}{2}\varepsilon\cdot\sup_{w=\varepsilon e^{i\theta}\in\gamma_\varepsilon} e^{-\varepsilon\cos\theta}\varepsilon^{\mathrm{Re}(z)-1}e^{-\theta\mathrm{Im}(z-1)} \\
&\le \frac{\pi}{2}\varepsilon\cdot A\varepsilon^{\mathrm{Re}(z)-1} = \frac{\pi}{2}A\varepsilon^{\mathrm{Re}(z)} \xrightarrow{\varepsilon\to 0^+} 0.
\end{aligned}
$$

Moreover, $\delta=R^{-(1+\mathrm{Re}(z))/2}<\pi/2$ to get

$$
\begin{aligned}
\left|\int_{\gamma_R}f(w)dw\right| &\le \int_0^{\frac{\pi}{2}} e^{-R\cos\theta}R^{\mathrm{Re}(z)}e^{-\theta\mathrm{Im}(z-1)}d\theta \\
&= \int_0^{\frac{\pi}{2}-\delta} CR^{\mathrm{Re}(z)}e^{-R\sin\delta}d\theta+\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} CR^{\mathrm{Re}(z)}d\theta \\
&= \left(\frac{\pi}{2}-\delta\right)Ce^{-R\sin\delta}R^{\mathrm{Re}(z)}+C\delta R^{\mathrm{Re}(z)} \xrightarrow{R\to\infty} 0.
\end{aligned}
$$

Therefore we get

$$\int_0^\infty e^{-x}x^{z-1}dx=\int_0^\infty e^{-it}(it)^{z-1}dt.$$

Left hand side is equal to $\Gamma(z)$, and the right hand side is

$$
\begin{aligned}
\int_0^\infty e^{-it}(it)^{z-1}dt &= \int_0^\infty (\cos t - i\sin t)i(it)^{z-1}dt \\
&= i^{z-1}\int_0^\infty \sin(t)t^{z-1}dt + i^z\int_0^\infty \cos(t)t^{z-1}dt \\
&= \left(\sin\left(\pi\frac{z}{2}\right) - i\cos\left(\pi\frac{z}{2}\right)\right)\mathcal{M}(\sin)(z) + \left(\cos\left(\pi\frac{z}{2}\right) + i\sin\left(\pi\frac{z}{2}\right)\right)\mathcal{M}(\cos)(z).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sin\left(\pi\frac{z}{2}\right)\mathcal{M}(\sin)(z) + \cos\left(\pi\frac{z}{2}\right)\mathcal{M}(\cos)(z) = \Gamma(z) \\
\cos\left(\pi\frac{z}{2}\right)\mathcal{M}(\sin)(z) + \sin\left(\pi\frac{z}{2}\right)\mathcal{M}(\cos)(z) = 0,
\end{aligned}
$$

and

$$
\mathcal{M}(\cos)(z) = \Gamma(z)\cos\left(\pi\frac{z}{2}\right), \quad \mathcal{M}(\sin)(z) = \Gamma(z)\sin\left(\pi\frac{z}{2}\right).
$$

(b) Since $\left|\sin(t)t^{z-1}\right| \sim t^{\mathrm{Re}(z)}$ near $t=0$, the integral $\int_0^\infty \sin(t)t^{z-1}dt$ converges for $-1 < \mathrm{Re}(z) < 1$. Second of the above identities is valid in the larger strip $-1 < \mathrm{Re}(z) < 1$ by **Corollary 4.9, Chapter 2**. Finally, taking $z \to 0$ and $z = -1/2$ gives

$$
\begin{aligned}
\int_0^\infty \frac{\sin x}{x}dx &= \lim_{z\to 0}\Gamma(z)\sin\left(\pi\frac{z}{2}\right) = \lim_{z\to 0} z\Gamma(z)\frac{\sin(\pi\frac{z}{2})}{z} = \frac{\pi}{2}, \\
\int_0^\infty \frac{\sin x}{x^{3/2}}dx &= \Gamma\left(-\frac{1}{2}\right)\sin\left(\pi\frac{-1/2}{2}\right) = -2\sqrt{\pi}\cdot\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2\pi}.
\end{aligned}
$$

11. First observe that

$$
f(x+iy) = e^{a(x+iy)}e^{-e^{x+iy}} = e^{a(x+iy)}e^{-e^x(\cos y + i\sin y)},
$$

so

$$
|f(x+iy)| = e^{ax - e^x\cos y}.
$$

Since $a > 0$ and $\cos y > 0$, this function exponentially decreases as $|x| \to \infty$. Also,

$$
\hat{f}(\xi) = \int_{-\infty}^\infty f(x)e^{-2\pi i x\xi}dx = \int_{-\infty}^\infty e^{ax}e^{-e^x}e^{-2\pi i x\xi}dx = \int_0^\infty e^{-t}t^{a-2\pi i\xi-1}dt = \Gamma(a - 2\pi i\xi)
$$

Where we changed the variable $t = e^x$.

12. (a) Since $\Gamma(s+1) = s\Gamma(s)$,

$$
\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(-1-\frac{1}{2}\right)\cdots\left(-k-\frac{1}{2}\right)\Gamma\left(-k-\frac{1}{2}\right).
$$

Hence

$$
\frac{1}{\left|\Gamma\left(-k-\frac{1}{2}\right)\right|} = \frac{1}{\Gamma(\frac{1}{2})}\left|\left(-\frac{1}{2}\right)\left(-1-\frac{1}{2}\right)\cdots\left(-k-\frac{1}{2}\right)\right| \geq \frac{1}{2\Gamma(\frac{1}{2})}k! = \frac{k!}{2\sqrt{\pi}}.
$$

Therefore $1/|\Gamma(s)|$ is not $O\left(e^{c|s|}\right)$ for any $c > 0$.

(b) Since the order of growth of F is 1, by Hadamard's factorization theorem,

$$
F(s) = e^{As+B}s\prod_{n=1}^\infty\left(1+\frac{s}{n}\right)e^{-s/n} = e^{(A-\gamma)s+B}\cdot e^{\gamma s}s\prod_{n=1}^\infty\left(1+\frac{s}{n}\right)e^{-s/n} = \frac{e^{(A-\gamma)s+B}}{\Gamma(s)}.
$$

If $F(s) = O(e^{c|s|})$, then $1/\Gamma(s) = F(s) \cdot e^{-(A-\gamma)s-B}$ is also $O(e^{c|s|})$, which is contradiction.

13. Note that $\Gamma(s) = \prod_{n=1}^{\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}$. Hence

$$\begin{aligned}\log\Gamma(s) &= \lim_{n\to\infty} s\log n - \log s + \sum_{k=1}^{n}(\log(s+k) - \log(k)) \\ &= \lim_{n\to\infty} s(\log n - H_n) - \log s + \sum_{k=1}^{n}\left(\frac{s}{k} - \log\left(1+\frac{s}{k}\right)\right) \\ &= -s\gamma - \log s + \sum_{k=1}^{n}\left(\frac{s}{k} - \log\left(1+\frac{s}{k}\right)\right).\end{aligned}$$

Since the series $\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+s}\right) = \sum_{k=1}^{\infty}\frac{s}{k(k+s)}$ uniformly converges in any compact subset of $\mathbb{C}$,

$$\frac{d\log\Gamma(s)}{ds} = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+s}\right).$$

Also, $\sum_{k=1}^{\infty}\frac{1}{(k+s)^2}$ uniformly converges in any compact subset of $\mathbb{C}$, thus

$$\frac{d^2\log\Gamma(s)}{ds^2} = \frac{1}{s^2} + \sum_{k=1}^{\infty}\frac{1}{(k+s)^2} = \sum_{n=1}^{\infty}\frac{1}{(s+n)^2}$$

whenever s is positive number. Observe that $\Gamma'(s)/\Gamma(s)$ is well defined for $s \neq 0, -1, -2, \cdots$, and the formula

$$\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)' = \sum_{n=1}^{\infty}\frac{1}{(s+n)^2}$$

holds for all $s > 0$, it also holds for all complex numbers with $s \neq 0, -1, -2, \cdots$ by **Corollary 4.9, Chapter 2**.

14. (a) Let $f(x) = \log\Gamma(x)$, and $F(x)$ be a primitive function of $f(x)$. Then

$$\begin{aligned}\frac{d}{dx}\int_x^{x+1} f(t)dt &= \frac{d}{dx}(F(x+1) - F(x)) = f(x+1) - f(x) = \log\Gamma(x+1) - \log\Gamma(x) \\ &= \log(x\Gamma(x)) - \log\Gamma(x) = \log x.\end{aligned}$$

Therefore, we integrate both sides to get

$$\int_x^{x+1} f(t)dt = x\log x - \log x + c.$$

(b) Note that $\Gamma(x)$ is monotonically increasing for all large x. So

$$\log\Gamma(x) \leq \int_x^{x+1}\log\Gamma(t)dt \leq \log\Gamma(x+1),$$

that is,

$$\log\Gamma(x) \leq x\log x - x + c \leq \log\Gamma(x+1).$$

Therefore

$$(x-1)\log(x-1) - (x-1) + c \leq \log\Gamma(x) \leq x\log x - x + c$$

and

$$\frac{(x-1)\log(x-1)-(x-1)+c}{x\log x - x} \le \frac{\log\Gamma(x)}{x\log x - x} \le \frac{x\log x - x + c}{x\log x - x},$$

which follows by

$$\lim_{x\to\infty} \frac{\log\Gamma(x)}{x\log x - x} = 1.$$

In fact, $\log\Gamma(n) \sim n\log n + O(n)$ since

$$\lim_{x\to\infty} \frac{\log\Gamma(x)}{x\log x + O(x)} = \lim_{x\to\infty} \frac{\log\Gamma(x)}{x\log x - x} \cdot \frac{x\log x - x}{x\log x + O(x)} = 1.$$

15. Since $1/(e^x-1) = \sum_{n=1}^\infty e^{-nx}$,

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx &= \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx = \sum_{n=1}^\infty \int_o^\infty e^{-nx}x^{s-1}dx \\ &= \sum_{n=1}^\infty \int_0^\infty e^{-t}\left(\frac{t}{n}\right)^{s-1}\frac{1}{n}dt = \left(\sum_{n=1}^\infty \frac{1}{n^s}\right)\left(\int_0^\infty e^{-t}t^{s-1}dt\right) = \zeta(s)\Gamma(s).\end{aligned}$$

Note that we changed the variables $t = nx$.

16. Write

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{e^x-1}dx + \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-1}}{e^x-1}dx.$$

The second integral defines an entire function, while

$$\int_0^1 \frac{x^{s-1}}{e^x-1}dx = \sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)},$$

where B_m denotes the m^{th} Bernoulli number defined by

$$\frac{x}{e^x-1} = \sum_{m=0}^\infty \frac{B_m}{m!}x^m.$$

Then $B_0 = 1$, and since $z/(e^z-1)$ is holomorphic for $|z| < 2\pi$, we have $\limsup_{n\to\infty} |B_m/m!|^{1/m} = \frac{1}{2\pi}$. Therefore

$$\limsup_{n\to\infty} \left|\frac{B_m}{m!(s+m-1)}\right|^{1/m} = \frac{1}{2\pi},$$

which means that the series $\sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)}$ converges absolutely. Since

$$1/\Gamma(s) \text{ has simple poles at } s = 0,-1,-2,\cdots, \quad \text{and}$$
$$\sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)} \text{ has simple poles at } s = 1,0,-1,-2,\cdots,$$

$\zeta(s)$ is continuable in the complex plane with only singularity a simple pole at $s = 1$.

17. (a) We prove that

$$I(s) = \frac{(-1)^k}{\Gamma(s+k)}\int_0^\infty f^{(k)}(x)x^{s+k-1}dx, \quad \forall k \ge 0$$

by induction. The case $k=0$ is trivial by definition. Now suppose the equation holds when $k=k_0 \geq 0$. Then

$$\begin{aligned}
I(s) &= \frac{(-1)^{k_0}}{\Gamma(s+k_0)}\int_0^\infty f^{(k_0)}(x)x^{s+k_0-1}dx \\
&= \frac{(-1)^{k_0}}{\Gamma(s+k_0)}\left(\left[f^{(k_0)}(x)\frac{1}{s+k_0}x^{s+k_0}\right]_0^\infty - \int_0^\infty f^{(k_0+1)}(x)\frac{1}{s+k_o}x^{s+k_0}dx\right) \\
&= \frac{(-1)^{k_0+1}}{(s+k_0)\Gamma(s+k_0)}\int_0^\infty f^{(k_0+1)}(x)x^{s+k_0}dx \\
&= \frac{(-1)^{k_0+1}}{\Gamma(s+k_0+1)}\int_0^\infty f^{(k_0+1)}(x)x^{s+k_0}dx.
\end{aligned}$$

Therefore the equation also holds for $k=k_0+1$. Observe that $1/\Gamma(s+k)$ is holomorphic in $\mathrm{Re}(s) > -k$, and $\int_0^\infty f^{(k)}(x)x^{s+k-1}dx$ is holomorphic in $\mathbb{C}$. Hence $I(s)$ is holomorphic in $\mathrm{Re}(s) > -k$. Since $k \geq 0$ is arbitrary, $I(s)$ has an analytic continuation as an entire function in $\mathbb{C}$.

(b) Put $s=-n, k=n+1$ to get

$$I(-n) = \frac{(-1)^{n+1}}{\Gamma(1)}\int_0^\infty f^{(n+1)}(x)dx = (-1)^{n+1}\big(-f^{(n)}(0)\big) = (-1)^n f^{(n)}(0).$$

In particular, $I(0)=f(0)$ when $n=0$.

Chapter 7. The Zeta Function and Prime Number Theorem

1. Use summation by parts to get

$$\sum_{n=1}^N a_n n^{-s} = N^{-s}a_N + \sum_{n=1}^{N-1} A_n(n^{-s}-(n+1)^{-s}).$$

By the mean value theorem,

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| = \left|\frac{-s}{x^{s+1}}\right| \leq \left|\frac{s}{n^{s+1}}\right| = \frac{|s|}{n^{\sigma+1}},$$

where $s=\sigma+it$. Hence the series $\sum_{n=1}^\infty a_n/n^s$ converges in $\mathrm{Re}(s)>0$. Note that A_n is bounded so $|a_N| = |A_N - A_{N-1}| \leq 2A$. Now we show that this series converge uniformly on every compact subset K of the half plane $\mathrm{Re}(s)>0$. There exist $\sigma_0>0$ and $M>0$ such that $s\in K \Rightarrow \mathrm{Re}(s)\geq\sigma_0, |s| \leq M$. Hence

$$\begin{aligned}
\left|\sum_{n=N+1}^\infty\right| &= \left|-N^{-s}a_N + \sum_{n=N}^\infty A_n(n^{-s}-(n+1)^{-s})\right| \leq N^{-\sigma}|a_N| + \sum_{n=N}^\infty |A_n|\frac{|s|}{n^{\sigma+1}} \\
&\leq N^{-\sigma_0}\cdot 2A + MA\sum_{n=N}^\infty \frac{1}{n^{\sigma_0+1}} \to 0
\end{aligned}$$

as $N\to\infty$. Therefore this series converges absolutely in K. Finally, $\sum_{n=1}^\infty a_n/n^s$ defines a holomorphic function in half plane by **Theorem 5.2, Chapter 2**.

2. (a) Since $\{a_m\}$ and $\{b_k\}$ are bounded, $\sum a_m/m^s$ and $\sum b_k/k^s$ converge absolutely when $\mathrm{Re}(s)>1$. Moreover, $\sum c_n/n^s$ converges because

$$\sum_{n=1}^{N}\frac{c_n}{n^s}=\sum_{n=1}^{N}\sum_{mk=n}\frac{a_m}{m^s}\cdot\frac{b_k}{k^s}=\sum_{m=1}^{N}\left(\frac{a_m}{m^s}\sum_{k=1}^{\lfloor N/m\rfloor}\frac{b_k}{k^s}\right)\to\left(\sum_{m=1}^{\infty}\frac{a_m}{m^s}\right)\left(\sum_{k=1}^{\infty}\frac{b_k}{k^s}\right)$$

as $N\to\infty$. Let $|a_m|\le A$, $|b_k|\le B$. Then

$$\begin{aligned}\sum_{n=1}^{N}\left|\frac{c_n}{n^s}\right|&=\sum_{n=1}^{N}\left|\sum_{mk=n}\frac{a_m}{m^s}\cdot\frac{b_k}{k^s}\right|\le\sum_{m=1}^{N}\left(\left|\frac{a_m}{m^s}\right|\sum_{k=1}^{\lfloor N/m\rfloor}\left|\frac{b_k}{k^s}\right|\right)\\&=\sum_{m=1}^{N}\left(\frac{|a_m|}{m^\sigma}\sum_{k=1}^{\lfloor N/m\rfloor}\frac{|b_k|}{k^\sigma}\right)\le AB\left(\sum_{l=1}^{\infty}\frac{1}{l^\sigma}\right)^2.\end{aligned}$$

Hence above series converge absolutely when $\mathrm{Re}(s)>1$.

(b) By **(a)**,

$$(\zeta(s))^2=\left(\sum_{m=1}^{\infty}\frac{1}{m^s}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k^s}\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{mk=n}1\cdot1\right)=\sum_{n=1}^{\infty}\frac{d(n)}{n^s}$$

for $\mathrm{Re}(s)>1$ and

$$\zeta(s)\zeta(s-a)=\left(\sum_{m=1}^{\infty}\frac{1}{m^s}\right)\left(\sum_{k=1}^{\infty}\frac{k^a}{k^s}\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{mk=n}1\cdot k^a\right)=\sum_{n=1}^{\infty}\frac{\sigma_{a(n)}}{n^s}$$

for $\mathrm{Re}(s-a)>1$.

3. **(a)** Using the Euler product formula,

$$\frac{1}{\zeta(s)}=\prod_p(1-p^{-s}).$$

Since every natural numbers less than N is the product of primes less than N,

$$\left|\prod_{p<N}(1-p^{-s})-\sum_{n=1}^{N-1}\frac{\mu(n)}{n^s}\right|<\sum_{n=N}^{\infty}\frac{1}{n^s}.$$

Hence $N\to\infty$ to get

$$\frac{1}{\zeta(s)}=\prod_p(1-p^{-s})=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}.$$

(b) Define $a_n=\sum_{k|n}\mu(k)$. Observe that $|a_n|\le n$. Since $\zeta(s)\cdot\frac{1}{\zeta(s)}=1$, by **Exercise 2**,

$$1=\left(\sum_{m=1}^{\infty}\frac{1}{m^s}\right)\left(\sum_{k=1}^{\infty}\frac{\mu(k)}{k^s}\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{k|n}\mu(k)\right)=a_1+\frac{a_2}{2^s}+\frac{a_3}{3^s}+\cdots.$$

The equation above holds for all $\mathrm{Re}(s)>1$. Since $|a_n/n^s|\le1/n^{\sigma-1}$ for $s=\sigma+it$, taking $s\to\infty$ gives $a_1=1$. Similarly, multiply both sides by 2^s and taking $s\to\infty$ gives $a_2=0$. Repeat this process to get

$$\sum_{k|n}\mu(k)=\begin{cases}1&\text{if }n=1\\0&\text{otherwise.}\end{cases}$$

4. Since $1/(e^{qx}-1)=\sum_{n=1}^{\infty}e^{-nqx}$,

$$\int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx = \int_0^\infty Q(x)x^{s-1}\sum_{n=1}^\infty e^{-nqx}dx = \sum_{n=1}^\infty \int_0^\infty \sum_{m=0}^{q-1} a_{q-m}e^{mx}x^{s-1}e^{-nqx}dx$$
$$= \sum_{n=1}^\infty \int_0^\infty \sum_{m=0}^{q-1} a_{q-m}x^{s-1}e^{(m-nq)x}dx$$
$$= \sum_{n=1}^\infty \int_0^\infty \sum_{m=0}^{q-1} a_{q-m}\left(\frac{t}{nq-m}\right)^{s-1} e^{-t}\frac{dt}{nq-m}$$
$$= \sum_{n=1}^\infty \left(\int_0^\infty t^{s-1}e^{-t}dt\right)\sum_{m=0}^{q-1}\frac{a_{nq-m}}{(nq-m)^s} = \Gamma(s)L(s).$$

Hence

$$L(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx, \quad \mathrm{Re}(s) > 1.$$

Now write

$$L(s) = \frac{1}{\Gamma(s)}\int_0^{1/q} \frac{Q(x)x^{s-1}}{e^{qx}-1}dx + \frac{1}{\Gamma(s)}\int_{1/q}^\infty \frac{Q(x)x^{s-1}}{e^{qx}-1}dx.$$

The second integral defines an entire function, while

$$\int_0^{1/q} \frac{Q(x)x^{s-1}}{e^{qx}-1}dx = \sum_{m=0}^{q-1} a_{q-m}\int_0^{1/q}\frac{e^{mx}x^{s-1}}{e^{qx}-1}dx = \sum_{m=0}^{q-1} a_{q-m}\int_0^1 \frac{e^{\frac{m}{q}u}\left(\frac{u}{q}\right)^{s-1}}{e^u-1}\frac{dt}{q}$$
$$= \frac{1}{q^s}\sum_{m=0}^{q-1} a_{q-m}\int_0^1 \frac{e^{\frac{m}{q}u}u^{s-1}}{e^u-1}du.$$

Moreover,

$$\int_0^1 \frac{e^{\frac{m}{q}u}u^{s-1}}{e^u-1}du = \int_0^1 e^{\frac{m}{q}u}\sum_{k=0}^\infty \frac{B_k}{k!}u^{s+k-2}du = \sum_{k=0}^\infty \frac{B_k}{k!}\int_0^1 \sum_{l=0}^\infty \frac{1}{l!}\left(\frac{m}{q}u\right)^l u^{s+k-2}du$$
$$= \sum_{k=0}^\infty \frac{B_k}{k!}\sum_{l=0}^\infty \frac{1}{l!}\frac{m^l}{q^l}\frac{1}{s+k+l-1}.$$

where B_k denotes the k^{th} Bernoulli number. (see **Exercise 16, Chapter 6**.) Therefore

$$\int_0^{1/q} \frac{Q(x)x^{s-1}}{e^{qx}-1}dx$$
$$= \frac{1}{q^s}\left(\sum_{m=0}^{q-1} a_{q-m}\right)\frac{1}{s-1} + \frac{1}{q^s}\left(\sum_{m=0}^{q-1} a_{q-m}\left(\frac{B_1}{1!} + \frac{B_0}{0!}\frac{m}{q}\right)\right)\frac{1}{s}$$
$$+\frac{1}{q^s}\left(\sum_{m=0}^{q-1} a_{q-m}\left(\frac{B_2}{2!} + \frac{B_1}{1!}\frac{m}{q} + \frac{B_0}{0!}\frac{m^2}{q^2}\right)\right)\frac{1}{s+1} + \cdots.$$

If this series have simple poles at each of $s = 0, -1, -2, \cdots$, then it is cancelled with simple zeros of $1/\Gamma(s)$. However, there is no zeros of $1/\Gamma(s)$ such that simple pole at $s = 1$ can be cancelled with. Hence $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s = 1$. In addition, by series expansion above, $L(s)$ is regular at $s = 1$ if and only if $\sum_{m=0}^{q-1} a_{q-m} = 0$.

5. (a) Since $\sum_{n=1}^N (-1)^n$ is bounded, by **Exercise 1**, the series defining $\tilde{\zeta}(s)$ converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in that half-plane.

(b) Define $\tilde{\zeta}_N(s) = \sum_{n=1}^{N} (-1)^{n+1}/n^s$. Then

$$\begin{aligned}\tilde{\zeta}_N(s) &= \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{1-(-1)^{n+1}}{n^s} \\ &= \sum_{n=1}^{N} \frac{1}{n^s} - 2\sum_{k=1}^{\lfloor N/2 \rfloor} \frac{1}{(2k)^s} = \zeta_N(s) - 2^{1-s}\zeta_{\lfloor N/2 \rfloor}(s).\end{aligned}$$

Taking $N \to \infty$ gives $\tilde{\zeta}(s) = (1-2^{1-s})\zeta(s)$.

(c) Let $s = \sigma + it$. Since both $\tilde{\zeta}(s)$ and $(1-2^{1-s})\zeta(s)$ are holomorphic in $\mathrm{Re}(s) > 0$ and coincides for $s > 1$, we know that $\tilde{\zeta}(s) = (1-2^{1-s})\zeta(s)$ for all $\mathrm{Re}(s) > 0$. Moreover, $\tilde{\zeta}(s)$ is given as an alternating series, so $\tilde{\zeta}(s) \neq 0$ for segment $0 < \sigma < 1$. Hence $\zeta(s) \neq 0$ for $0 < \sigma < 1$.

By **Exercise 4**, $\tilde{\zeta}(s)$ can be expressed by

$$\tilde{\zeta}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{(e^x - 1)x^{s-1}}{e^{2x} - 1} dx = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1} dx$$

and $\tilde{\zeta}(s)$ is entire function. Integration by parts to get

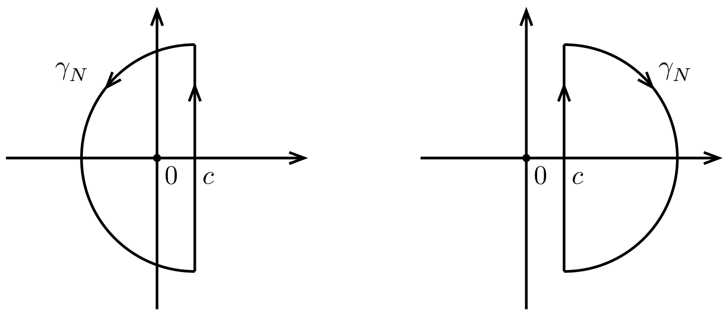
$$\tilde{\zeta}(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s e^x}{(e^x+1)^2} dx.$$

In particular,

$$\tilde{\zeta}(0) = \int_0^\infty \frac{e^x}{(e^x+1)^2} dx = \frac{1}{2}.$$

Hence $\zeta(0) = -1/2$ and we can extend last assertion to $\sigma = 0$.

6. We will integrate the function $f(s) = a^s/s$ over the appropriate semicircle.



contour when $a > 1$ contour when $0 \le a < 1$

First consider the case $a > 1$. We integrate the function $f(s) = a^s/s$ over the left semicircle of radius N centered at c. Since $f(s)$ has simple pole at $s = 0$ and $\mathrm{Res}_{s=0}\, a^s/s = 1$,

$$\int_{c-iN}^{c+iN} \frac{a^s}{s} ds + \int_{\gamma_N} \frac{a^s}{s} = 2\pi i.$$

Now we estimate the integral over the semicircle.

$$\begin{aligned}\left| \int_{\gamma_N} \frac{a^s}{s} ds \right| &= \left| \int_{\pi/2}^{3\pi/2} \frac{a^{c+Ne^{i\theta}}}{c+Ne^{i\theta}} iNe^{i\theta} d\theta \right| \le \int_{\pi/2}^{3\pi/2} \frac{N}{N-c} a^c \left| a^{Me^{i\theta}} \right| d\theta \\ &= \frac{Na^c}{N-c} \int_{\pi/2}^{3\pi/2} a^{N\cos\theta} d\theta.\end{aligned}$$

Let $\delta = N^{-1/2} < \pi/2$ for large N, then

$$\int_{\pi/2}^{3\pi/2} a^{N\cos\theta} d\theta = \int_{\pi/2}^{\pi/2+\delta} a^{N\cos\theta} d\theta + \int_{\pi/2+\delta}^{3\pi/2-\delta} a^{N\cos\theta} d\theta + \int_{3\pi/2-\delta}^{3\pi/2} a^{N\cos\theta} d\theta$$
$$\leq 2\delta + (\pi - 2\delta) a^{N\sin\delta} \to 0$$

as $N \to \infty$. Hence $\lim_{N\to\infty} \int_{c-iN}^{c+iN} \frac{a^s}{s} ds = 2\pi i$.

If $0 \leq a < 1$, integrate the function $f(s)$ over the right semicircle of radius N centered at c. Since there is no poles of $f(s)$,

$$\int_{c-iN}^{c+iN} \frac{a^s}{s} ds + \int_{\gamma_N} \frac{a^s}{s} = 0.$$

Similar to the case $a < 1$, the integral over the semicircle converges to 0 because

$$\left| \int_{\gamma_N} \frac{a^s}{s} ds \right| = \left| \int_{\pi/2}^{-\pi/2} \frac{a^{c+Ne^{i\theta}}}{c + Ne^{i\theta}} iNe^{i\theta} d\theta \right|$$
$$\leq \frac{Na^c}{N-c} \int_{-\pi/2}^{\pi/2} a^{N\cos\theta} d\theta = \frac{Na^c}{N-c} \int_{\pi/2}^{3\pi/2} \left(\frac{1}{a}\right)^{N\cos\theta} d\theta$$

and $1/a > 1$. Thus $\lim_{N\to\infty} \int_{c-iN}^{c+iN} \frac{a^s}{s} ds = 0$.

Finally, consider the case $a = 1$.

$$\int_{c-iN}^{c+iN} \frac{a^s}{s} ds = \int_{c-iN}^{c+iN} \frac{1}{s} ds = \log(c+iN) - \log(c-iN) = i(\theta_2 - \theta_1),$$

where $\theta_1 = \arg(c - iN), \theta_2 = \arg(c + iN)$. Therefore

$$\lim_{N\to\infty} \int_{c-iN}^{c+iN} \frac{a^s}{s} ds = \lim_{N\to\infty} \frac{i(\theta_2 - \theta_1)}{2\pi i} = \frac{\pi i}{2\pi i} = \frac{1}{2}.$$

7. Recall the formula

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u) du,$$

where $\psi(u) = \frac{\vartheta(u)-1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 u}$. Obviously $\xi(s)$ is real when s is real. Now suppose $\mathrm{Re}(s) = 1/2$, that is, $s = 1/2 + it$. Then

$$\xi(s) = \xi(1/2 + it) = \frac{1}{-\frac{1}{2} + it} + \frac{1}{-\frac{1}{2} - it} + \int_1^\infty \left(u^{-\frac{3}{4}-i\frac{t}{2}} + u^{-\frac{3}{4}+i\frac{t}{2}}\right)\psi(u) du.$$

Since $\overline{\xi(s)} = \xi(s)$, we conclude that $\xi(s)$ is real when $\mathrm{Re}(s) = 1/2$.

8. (a) Let

$$F(s) = \frac{1}{\left(\frac{1}{2} + s\right)\left(\frac{1}{2} - s\right)} \xi\left(\frac{1}{2} + s\right).$$

Since $\xi(1/2 + s) = \xi(1/2 - s)$, $F(s)$ is an even function of s. Define

$$G(s) = F\left(\pm s^{\frac{1}{2}}\right).$$

G is well-defined because F is even. Hence $G(s^2) = F(s)$.

(b) Recall the functional equation $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u)du, \quad \psi(u) = \frac{\vartheta(u)-1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 u}.$$

Then

$$|\xi(s)| \le \left|\frac{1}{s-1}\right| + \left|\frac{1}{s}\right| + \int_1^\infty \left|u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right|\psi(u)du.$$

Moreover,

$$\begin{aligned}\int_1^\infty \left|u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right|\psi(u)du &\le \int_1^\infty 2u^{\frac{|\sigma|}{2}+1}\sum_{n=1}^\infty e^{-\pi n^2 u}du \le 2\sum_{n=1}^\infty \int_0^\infty u^{\frac{|\sigma|}{2}+1}e^{-\pi n^2 u}du \\ &\le 2\sum_{n=1}^\infty \int_0^\infty \frac{x^{|\sigma|/2+1}}{(\pi n^2)^{|\sigma|/2+1}}e^{-x}\frac{dx}{\pi n^2} = 2\Gamma\left(\frac{|\sigma|}{2}+2\right)\sum_{n=1}^\infty \frac{1}{(\pi n^2)^{|\sigma|/2+2}} \\ &\le c\Gamma\left(\frac{|\sigma|}{2}+2\right)\end{aligned}$$

for $s = \sigma + it$. Let $N \in \mathbb{N}$ be $N - 1 < |\sigma|/2 + 2 \le N$, then

$$\begin{aligned}\Gamma\left(\frac{|\sigma|}{2}+2\right) &\le \Gamma(N) = (N-1)! \le (N-1)^{N-1} = e^{(N-1)\log(N-1)} \le e^{\left(\frac{|\sigma|}{2}+2\right)\log\left(\frac{|\sigma|}{2}+2\right)} \\ &\le e^{\left(\frac{|s|}{2}+2\right)\log\left(\frac{|s|}{2}+2\right)} \le e^{c_1|s|\log|s|}\end{aligned}$$

for large $|s|$. Hence $|\xi(s)| \le c_2 e^{c_3|s|\log|s|}$ for large $|s|$. Additionally $|1/\Gamma(s)| \le c_4 e^{c_5|s|\log|s|}$, we know

$$(s-1)\zeta(s) = \pi^{s/2}\frac{1}{\Gamma(s/2)}(s-1)\xi(s)$$

is entire and has order of growth 1. Therefore

$$\left(s^2 - \frac{1}{4}\right)F(s) = -\xi\left(\frac{1}{2}+s\right)$$

has order of growth 1, and

$$\left(s - \frac{1}{4}\right)G(s)$$

has order of growth $1/2$.

(c) Since G has order of growth that is non-integral, by **Exercise 14, Chapter 5**, there is infinitely many zeros of G. By definition, $F(s)$ has infinitely many zeros, and $\xi(s)$ also has infinitely many zeros.

In the case of $\mathrm{Re}(s) \ge 1$, then both $\Gamma(s/2)$ and $\zeta(s)$ have no zeros, so $\xi(s)$ is zero-free. If $\mathrm{Re}(s) < 0$, then $\zeta(s)$ has zeros at $s = -2, -4, -6, \cdots$, but these are cancelled with simple poles of $\Gamma(s/2)$ at $s = 0, -2, -4, \cdots$. Hence $\xi(s)$ have infinitely many zeros in $0 < \mathrm{Re}(s) < 1$. Since $\Gamma(s/2)$ has no zeros in that strip, $\zeta(s)$ has infinitely many zeros in the critical strip.

9. (a) By **Proposition 2.5, Corollary 2.6 of Chapter 6**,

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} + \sum_{n=1}^{\mathbb{O}} \delta_{n(s)} = \frac{1}{s-1} + \sum_{n=0}^{N-1} \delta_n(s) + \sum_{n=N}^{\infty} \delta_n(s) \\ &= \frac{1}{s-1} + \sum_{n=1}^{N-1} \frac{1}{n^s} - \int_1^N \frac{1}{x^s} dx + \sum_{n=N}^{\infty} \delta_n(s) \\ &= \sum_{n=1}^{N-1} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \sum_{n=N}^{\infty} \delta_n(s).\end{aligned}$$

Hence

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \sum_{n=N}^{\infty} \delta_n(s)$$

for $N \geq 2, \mathrm{Re}(s) > 0$. Now let $s = 1 + it$ and choose $N = \lfloor |t| + 1 \rfloor$ to get

$$\begin{aligned}|\zeta(1+it)| &\leq \sum_{n=1}^{N-1} \frac{1}{n} + \frac{1}{|t|} + \sum_{n=N}^{\infty} \frac{\sqrt{t^2+1}}{n^2} \leq 1 + \int_1^{N-1} \frac{1}{x} dx + \frac{1}{|t|} + \frac{\sqrt{5}}{2}|t| \int_{N-1}^{\infty} \frac{1}{x^2} dx \\ &\leq 1 + \log(N-1) + \frac{1}{|t|} + \frac{\sqrt{5}}{2}|t| \frac{1}{N-1} \leq 1 + \log|t| + \frac{1}{|t|} + \frac{\sqrt{5}}{2} \frac{|t|}{|t|-1} \\ &\leq A \log|t|.\end{aligned}$$

(b) Differentiate both sides of functional equation to get

$$\zeta'(s) = \sum_{n=1}^{N-1} \frac{-\log n}{n^s} + \frac{N^{1-s}(1-s)\log N + N^{1-s}}{(1-s)^2} + \sum_{n=N}^{\infty} \delta_n'(s),$$

where

$$|\delta_n'(s)| \leq \int_n^{n+1} \left| \frac{\log n}{n^s} - \frac{\log x}{x^s} \right| dx \leq \frac{1 + |s| \log n}{n^{\sigma+1}}$$

for large n. Since there exists $k > 0$ such that $|s| \leq k|t|$ for $|t| \geq 2$, and $N = \lfloor |t| + 1 \rfloor$,

$$\begin{aligned}|\zeta'(1+it)| &\leq \sum_{n=1}^{N-1} \frac{\log n}{n} + \frac{|t| \log N + 1}{|t|^2} + \sum_{n=N}^{\infty} \frac{1 + k|t| \log n}{n^2} \\ &\leq \frac{\log 2}{2} + \frac{\log 3}{3} + \int_3^{N-1} \frac{\log x}{x} dx + \frac{\log N}{|t|} + \frac{1}{|t|^2} + \frac{\pi^2}{6} + k|t| \int_{N-1}^{\infty} \frac{\log x}{x^2} dx \\ &= C + \frac{1}{2}(\log|t|)^2 + \frac{\log(|t|+1)}{|t|} + \frac{1}{|t|^2} + \frac{k|t|}{|t|-1} - k\log(|t|-1) \\ &\leq A(\log|t|)^2.\end{aligned}$$

(c) In fact, the estimates of zeta function in **(a), (b)** is also hold for $1 \leq \sigma \leq 2 \wedge |t| \geq 2$. For example,

$$\begin{aligned}|\zeta(\sigma+it)| &\leq \sum_{n=1}^{N-1} \frac{1}{n^\sigma} + \frac{N^{1-\sigma}}{|t|} + k \sum_{n=N}^{\infty} \frac{|t|}{n^2} \\ &\leq \sum_{n=1}^{N-1} \frac{1}{n} + \frac{1}{|t|} + k \sum_{n=N}^{\infty} \frac{|t|}{n^2} \leq A \log|t|,\end{aligned}$$

So $|\zeta(\sigma+it)| \leq A \log|t|$ for $1 \leq \sigma \leq 2$ and $|t| \geq 2$. Note that the condition $\sigma \leq 2$ is needed to guarantee the existence of constant k such that $|s| \leq k|t|$ for all $|t| \geq 2$. Similarly, $|\zeta'(\sigma+it)| \leq A(\log|t|)^2$ for $|t| \geq 2$.

Now we estimate $1/\zeta$. Since

$$\left|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)\right| \geq 1, \quad \text{whenever } \sigma \geq 1,$$

we find that

$$\left|\zeta^4(\sigma+it)\right| \geq c\left|\zeta^{-3}(\sigma)\right|(\log|t|)^{-1} \geq c'(\sigma-1)^3(\log|t|)^{-1}$$

for all $1 \leq \sigma \leq 2$ and $|t| \geq 2$. Thus

$$|\zeta(\sigma+it)| \geq c'(\sigma-1)^{3/4}(\log|t|)^{-1/4}.$$

We will consider two separate cases. If $\sigma - 1 \geq A(\log|t|)^{-9}$, then

$$|\zeta(\sigma+it)| \geq A'(\log|t|)^{-27/4}(\log|t|)^{-1/4} = A'(\log|t|)^{-9}$$

so the inequality is proved. If, however, $\sigma - 1 < A(\log|t|)^{-9}$, then we can select $\sigma' > \sigma$ with $\sigma' - 1 = A(\log|t|)^{-9}$. The triangle inequality then implies

$$|\zeta(\sigma+it)| \geq |\zeta(\sigma'+it)| - |\zeta(\sigma'+it) - \zeta(\sigma+it)|,$$

and an application of the mean value theorem,

$$|\zeta(\sigma'+it) - \zeta(\sigma+it)| \leq c''|\sigma'-\sigma|(\log|t|)^2 \leq c''|\sigma'-1|(\log|t|)^2.$$

Hence

$$|\zeta(\sigma+it)| \geq c'(\sigma'-1)^{3/4}(\log|t|)^{-1/4} - c''(\sigma'-1)(\log|t|)^2.$$

We can choose $A = (c'/(2c''))^4$ to conclude

$$|\zeta(\sigma+it)| \geq A''|\log(t)|^{-9}.$$

10. (a) Integrate by parts to get

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}dt = \left[\frac{t}{\log t}\right]_2^x + \int_2^x \frac{1}{(\log t)^2}dt.$$

Now we show that $\int_2^x \frac{1}{(\log t)^2}dt = O\big(\frac{x}{(\log x)^2}\big)$. Indeed,

$$\begin{aligned}\int_2^x \frac{1}{(\log t)^2}dt &= \int_2^{\sqrt{x}} \frac{1}{(\log t)^2}dt + \int_{\sqrt{x}}^x \frac{1}{(\log t)^2}dt \\ &\leq \left(\sqrt{x}-2\right)\frac{1}{(\log 2)^2} + \left(x-\sqrt{x}\right)\frac{1}{\left(\log\sqrt{x}\right)^2} = O\left(\frac{x}{(\log x)^2}\right).\end{aligned}$$

Therefore

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{\log x^2}\right)$$

and

$$\pi(x) \sim \frac{x}{\log x} \sim \mathrm{Li}(x).$$

(b) Using integration by parts N times,

$$\begin{aligned}\mathrm{Li}(x) = {}&\left[\frac{t}{\log t}\right]_2^x + 1!\left[\frac{t}{(\log t)^2}\right]_2^x + 2!\left[\frac{t}{(\log t)^3}\right]_2^x + \cdots + (N-1)!\left[\frac{t}{(\log t)^N}\right]_2^x \\ &+N!\int_2^x \frac{1}{(\log t)^{N+1}}dt.\end{aligned}$$

In addition,

$$\int_2^x \frac{1}{(\log t)^{N+1}}dt = \int_2^{\sqrt{x}} \frac{1}{(\log t)^{N+1}}dt + \int_{\sqrt{x}}^x \frac{1}{(\log t)^{N+1}}dt$$
$$\leq (\sqrt{x}-2)\frac{1}{(\log 2)^{N+1}} + (x-\sqrt{x})\frac{2^{N+1}}{(\log x)^{N+1}} = O\left(\frac{x}{(\log x)^{N+1}}\right).$$

Therefore the given asymptotic expansion holds for every integer $N > 0$.

11. We already proved that (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii) in Chapter 7.

(i) $\Rightarrow$ (ii): Note that

$$\frac{\varphi(x)}{x} \leq \frac{\pi(x)\log x}{x} \leq \frac{1}{\alpha}\left(\frac{\varphi(x)}{x} + \frac{\alpha\pi(x^\alpha)\log x}{x}\right)$$

for arbitrary $\alpha \in (0,1)$, thus we get

$$1 \leq \liminf_{x\to\infty}\frac{\pi(x)\log x}{x} \leq \limsup_{x\to\infty}\frac{\pi(x)\log x}{x} \leq \frac{1}{\alpha}.$$

Since α is arbitrary, $\pi(x) \sim x/\log x$.

(ii) $\Rightarrow$ (i): Since

$$\frac{\alpha\log x(\pi(x)-\pi(x^\alpha))}{x} \leq \frac{\varphi(x)}{x} \leq \frac{\pi(x)\log x}{x}$$

for arbitrary $\alpha \in (0,1)$, we get

$$\alpha \leq \liminf_{x\to\infty}\frac{\varphi(x)}{x} \leq \limsup_{x\to\infty}\frac{\varphi(x)}{x} \leq 1$$

and $\varphi(x) \sim x$.

(ii) $\Rightarrow$ (iii): Since

$$\frac{\alpha\log x(\pi(x)-\pi(x^\alpha))}{x} \leq \frac{\psi(x)}{x} \leq \frac{\pi(x)\log x}{x}$$

for arbitrary $\alpha \in (0,1)$, we get

$$\alpha \leq \liminf_{x\to\infty}\frac{\psi(x)}{x} \leq \limsup_{x\to\infty}\frac{\psi(x)}{x} \leq 1$$

and $\psi(x) \sim x$.

(iii) $\Rightarrow$ (iv): Since $\varphi(x) \sim x$, there exists $x_0 > 0$ such that

$$x > x_0 \Rightarrow \left|\frac{\psi(x)}{x} - 1\right| < \varepsilon \Rightarrow (1-\varepsilon)x < \psi(x) < (1+\varepsilon)x.$$

Since

$$\psi_1(x) = \int_1^x \psi(u)du = A_0 + \int_{x_0}^x \psi(u)du, \quad \text{where } A_0 = \int_1^{x_0}\psi(u)du,$$

we get

$$A_0 + (1-\varepsilon)\left(\frac{x^2}{2} - \frac{x_0^2}{x}\right) \leq \psi_1(x) \leq A_0 + (1+\varepsilon)\left(\frac{x^2}{2} - \frac{x_0^2}{x}\right)$$

and

$$\frac{A_0}{x^2/2} - \frac{x_0^2}{x^2} - \varepsilon\left(1 - \frac{x_0^2}{x^2}\right) \leq \frac{\psi_1(x)}{x^2/2} - 1 \leq \frac{A_0}{x^2/2} - \frac{x_0^2}{x^2} + \varepsilon\left(1 - \frac{x_0^2}{x^2}\right).$$

Now taking x large enough implies

$$-2\varepsilon \leq \frac{\psi_1(x)}{x^2/2} - 1 \leq 2\varepsilon.$$

Hence $\psi_1(x) \sim x^2/2$.

12. **(a)** Since $\pi(x) \sim x/\log x$,

$$\lim_{x\to\infty} \frac{\log \pi(x) + \log\log x}{\log x} = \lim_{x\to\infty} 1 + \frac{1}{\log(x)} \log\left(\frac{\pi(x)}{x/\log x}\right) = 1.$$

Therefore $\log \pi(x) + \log\log x \sim \log x$.

(b) As a consequence, $\log \pi(x) \sim \log x$ since

$$\lim_{x\to\infty} \frac{\log \pi(x)}{\log x} = \lim_{x\to\infty} \frac{\log \pi(x) + \log\log x}{\log x} - \frac{\log\log x}{\log x} = 1.$$

Thus

$$\lim_{x\to\infty} \frac{x}{\pi(x)\log \pi(x)} = \lim_{x\to\infty} \frac{x/\log x}{\pi(x)} \cdot \frac{\log x}{\log \pi(x)} = 1.$$

Put $x = p_n$ and $\pi(p_n) = n$ gives

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1,$$

hence $p(n) \sim n \log n$.

Chapter 8. Conformal Mappings

1. First suppose holomorphic map $f : U \to V$ is local bijection on U. For any $z_0 \in U$, there exists an open disc $D \subset U$ centered at z_0 such that $f : D \to f(D)$ is bijection. Applying **Proposition 1.1** to $f : D \to f(D)$ gives $f'(z) \neq 0$ for all $z \in D$. In particular, $f'(z_0) \neq 0$.

Now suppose $f'(z) \neq 0$ for all $z \in U$. Choose any $z_0 \in U$ and write

$$f(z) - f(z_0) = a(z - z_0) + G(z) \quad \text{for all } z \text{ near } z_0,$$

with $a \neq 0$, and G vanishing to order ≥ 2 at z_0. Consider the small disc $D_r(z_0)$ such that

$$|a(z - z_0)| > |G(z)| \quad \text{for all } z \in \partial D_r(z_0).$$

Since $a(z - z_0)$ has exactly one zero in $D_r(z_0)$, $f(z) - f(z_0) = a(z - z_0) + G(z)$ also has exactly one zero in $D_r(z_0)$ by Rouché's theorem. Choose any $r' < r$ and define

$$\mu = \inf_{|z-z_0|=r'} |f(z) - f(z_0)|.$$

Note that $\mu > 0$. Otherwise, there exists z^* such that $|z^* - z_0| = r'$ and $f(z^*) - f(z_0) = 0$ by **Theorem 4.8, Chapter 2**. But this is contradiction with the fact that the only zero of $f(z) - f(z_0)$ with $|z - z_0| < r$ is $z = z_0$.

If w satisfies $|f(z_0) - w| < \mu$, then

$$|f(z) - f(z_0)| \geq \mu > |f(z_0) - w| \quad \text{for all } z \in \partial D_{r'}(z_0).$$

By Rouché's theorem, $f(z) - w$ and $f(z) - f(z_0)$ have the same number of zeros in $D_{r'}(z_0)$, namely one. Now set $r'' = \min\{r, \mu/(2|a|)\}$. For every $z \in D_{r''}(z_0)$, we have

$$|f(z) - f(z_0)| \leq |a(z - z_0)| + |G(z)| < 2|a(z - z_0)| < \mu,$$

hence $f(D_{r''}(z_0)) \subset D_\mu(f(z_0))$. Letting $D = D_{r''}(z_0)$, we conclude that $f : D \to f(D)$ is a bijection.

2. We can write $F(z) = (g(z))^2$ because $F(z_0) = F'(z_0) = 0$. Since $g'(z_0) = (F''(z_0))^{1/2} \neq 0$, g is bijective near z_0. Now consider two curves

$$\Gamma_1 : [-\delta, \delta] \to \mathbb{C},\ t \mapsto g^{-1}(t),$$
$$\Gamma_2 : [-\delta, \delta] \to \mathbb{C},\ t \mapsto g^{-1}(it)$$

for small enough δ. Then

$$F|_{\Gamma_1} = F(g^{-1}(t)) = t^2, \quad F|_{\Gamma_2} = F(g^{-1}(it)) = -t^2,$$

therefore F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has maximum at z_0. Moreover,

$$(g^{-1})'(0) = 1/g'(z_0), \quad \left.\frac{d(g^{-1}(it))}{dt}\right|_{t=0} = i/g'(z_0),$$

hence two curves are orthogonal at z_0.

3. Consider any two curves $\gamma_0, \gamma_1 : [a, b] \to V$ such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. Since U and V are conformally equivalent, there exists conformal map $f : U \to V$. Then two curves defined by

$$\gamma_0'(t) = f^{-1}(\gamma_0(t)), \quad \gamma_1'(t) = f^{-1}(\gamma_1(t))$$

lies in U and have common end-points. Since U is simply connected, γ_0' and γ_1' is homotopic so there exists $\gamma'(s, t) : [0, 1] \times [a, b] \to U$ such that

$$\gamma'(0, t) = \gamma_0'(t), \quad \gamma'(1, t) = \gamma_1'(t), \quad \gamma'(s, a) = \gamma_0'(a), \quad \gamma'(s, b) = \gamma_0'(b)$$

and γ' is jointly continuous in $s \in [0, 1]$ and $t \in [a, b]$. Now let $\gamma(s, t) = f(\gamma'(s, t))$, then γ is homotopy of two curves γ_0 and γ_1 because f is holomorphic. Hence V is simply connected.

4. Define $f : \mathbb{D} \to \mathbb{C}$ as

$$f(w) = (G(w) - i)^2, \quad \text{where } G(w) = i\frac{1 - w}{1 + w}.$$

Note that $G : \mathbb{D} \to \mathbb{H}$ is conformal map. We prove that f is a holomorphic surjection. First suppose $z \in \mathbb{R}_{\geq 0}$. Since $\sqrt{z} + i \in \mathbb{H}$, there exists $w \in \mathbb{D}$ such that $G(w) = \sqrt{z} + i$ thus $z = (G(w) - i)^2 = f(w)$. Otherwise, if $z \in \mathbb{C} - \mathbb{R}_{\geq 0}$, then we can define $z^{1/2}$ as

$$z^{1/2} = r^{1/2} e^{i\theta/2} \quad \text{where} \quad z = re^{i\theta},\ 0 < \theta < 2\pi.$$

Since $0 < \arg(z^{1/2}) < \pi$, we have $z^{1/2} + i \in \mathbb{H}$, hence there exists $w \in \mathbb{D}$ such that $G(w) = z^{1/2} + i$ and $z = (G(w) - i)^2 = f(w)$.

5. Obviously, f is holomorphic in upper half-disc. f is injective, because $f(z_1) = f(z_2)$ implies

$$z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2} \ \Rightarrow \ (z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right) = 0 \ \Rightarrow \ z_1 = z_2.$$

In addition, f is surjective. To see this, choose any $w \in \mathbb{H}$. The equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$. This equation has two distinct roots in $\mathbb{C}$ because $w \in \mathbb{H}$. The product of two zeros is 1, so we call $re^{i\theta}$, $\frac{1}{r}e^{-i\theta}$ for $r > 0$. Then

$$\left(r - \frac{1}{r}\right)\sin\theta = \mathrm{Im}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = \mathrm{Im}(-2w) < 0,$$

hence $r < 1$. So there exists $z \in \{z : |z| < 1, \mathrm{Im}(z) > 0\}$ such that $f(z) = w$. Finally, f is a conformal map by definition.

6. Let $F(x,y) = \alpha(x,y) + i\beta(x,y)$. By Cauchy-Riemann equations,

$$\frac{\partial\alpha}{\partial x} = \frac{\partial\beta}{\partial y}, \quad \frac{\partial\alpha}{\partial y} - \frac{\partial\beta}{\partial x}$$

and $\Delta\alpha = \Delta\beta = 0$. Since

$$\frac{\partial^2}{\partial x^2}(u \circ F) = \left(\frac{\partial^2 u}{\partial\alpha^2}\frac{\partial\alpha}{\partial x} + \frac{\partial^2 u}{\partial\beta\partial\alpha}\frac{\partial\beta}{\partial x}\right)\frac{\partial\alpha}{\partial x} + \frac{\partial u}{\partial\alpha}\frac{\partial^2\alpha}{\partial x^2} + \left(\frac{\partial^2 u}{\partial\alpha\partial\beta}\frac{\partial\alpha}{\partial x} + \frac{\partial^2 u}{\partial\beta^2}\frac{\partial\beta}{\partial x}\right)\frac{\partial\beta}{\partial x} + \frac{\partial u}{\partial\beta}\frac{\partial^2\beta}{\partial x^2},$$

$$\frac{\partial^2}{\partial y^2}(u \circ F) = \left(\frac{\partial^2 u}{\partial\alpha^2}\frac{\partial\alpha}{\partial y} + \frac{\partial^2 u}{\partial\beta\partial\alpha}\frac{\partial\beta}{\partial y}\right)\frac{\partial\alpha}{\partial y} + \frac{\partial u}{\partial\alpha}\frac{\partial^2\alpha}{\partial y^2} + \left(\frac{\partial^2 u}{\partial\alpha\partial\beta}\frac{\partial\alpha}{\partial y} + \frac{\partial^2 u}{\partial\beta^2}\frac{\partial\beta}{\partial y}\right)\frac{\partial\beta}{\partial y} + \frac{\partial u}{\partial\beta}\frac{\partial^2\beta}{\partial y^2},$$

we get

$$\begin{aligned}\Delta(u \circ F) &= \Delta u \cdot \left(\left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2\right) + 2\frac{\partial^2 u}{\partial\alpha\partial\beta}\left(\frac{\partial\beta}{\partial x}\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y}\frac{\partial\alpha}{\partial y}\right) \\ &\quad + \left(\frac{\partial u}{\partial\alpha}\right)\Delta\alpha + \left(\frac{\partial u}{\partial\beta}\right)\Delta\beta \\ &= 0.\end{aligned}$$

7. (a) If $re^{i\theta} = G(iy)$, then

$$\begin{aligned}re^{i\theta} = G(iy) &= \frac{i - e^{\pi iy}}{i + e^{\pi iy}} = \frac{i - \cos\pi y - i\sin\pi y}{i + \cos\pi y + i\sin\pi y} \\ &= \frac{(i - \cos\pi y - i\sin\pi y)(-i + \cos\pi y - i\sin\pi y)}{(\cos\pi y)^2 + (1 + \sin\pi y)^2} = i\frac{\cos\pi y}{1 + \sin\pi y}.\end{aligned}$$

Furthermore, $\left(e^{i\theta}\right)^2 = e^{2i\theta} = -1$ implies that

$$r^2 = e^{-2i\theta} \cdot i^2 \frac{\cos^2\pi y}{(1 + \sin\pi y)^2} = \frac{1 - \sin^2\pi y}{(1 + \sin\pi y)^2} = \frac{1 - \sin\pi y}{1 + \sin\pi y},$$

and

$$P_r(\theta - \varphi) = \frac{1 - r^2}{1 - 2r\sin\varphi + r^2} = \frac{2\sin\pi y}{2 - 2(1 + \sin\pi y)r\sin\varphi} = \frac{\sin\pi y}{1 - \cos\pi y\sin\varphi} \quad \text{for } \theta = \pi/2,$$

$$P_r(\theta - \psi) = \frac{1 - r^2}{1 + 2r\sin\varphi + r^2} = \frac{2\sin\pi y}{2 + 2(1 + \sin\pi y)r\sin\varphi} = \frac{\sin\pi y}{1 - \cos\pi y\sin\varphi} \quad \text{for } \theta = -\pi/2.$$

(b) Since

$$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}} = -\frac{(e^{\pi t} - i)^2}{e^{2\pi t} + 1} \quad \Rightarrow \quad \sin\varphi = \operatorname{Im}(e^{i\varphi}) = \operatorname{sech}\pi t,$$

$\cos\varphi = \pm\tanh\pi t$. Observe that

$$\begin{aligned} 0 \le \varphi \le \pi/2: &\quad \cos\varphi \ge 0, t \le 0, \tanh\pi t \le 0, \\ \pi/2 < \varphi \le \pi: &\quad \cos\varphi < 0, t > 0, \tanh\pi t > 0. \end{aligned}$$

Therefore $\cos\varphi = -\tanh\pi t$ and $d\varphi/dt = \pi\operatorname{sech}\pi t$. Hence

$$\begin{aligned} \frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\tilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin\pi y}{1 - \cos\pi y\sin\varphi}\tilde{f}_0(\varphi)d\varphi \\ &= \frac{\sin\pi y}{2\pi}\int_{-\infty}^{\infty}\frac{f_0(t)}{1 - \cos\pi y\operatorname{sech}\pi t}\pi\operatorname{sech}\pi t dt \\ &= \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_0(t)}{\cosh\pi t - \cos\pi y}dt. \end{aligned}$$

(c) Now substitute $t = F(e^{i\varphi}) - i$, then $\sin\varphi = -\operatorname{sech}\pi t$ and $d\varphi/dt = -\pi\cosh\pi t$. Therefore

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^0 P_r(\theta - \varphi)\tilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_{-\pi}^0 \frac{\sin\pi y}{1 - \cos\pi y\sin\varphi}\tilde{f}_1(\varphi)d\varphi \\ &= \frac{\sin\pi y}{2\pi}\int_{\infty}^{-\infty}\frac{f_1(t)}{1 - \cos\pi y(-\operatorname{sech}\pi t)}(-\pi\operatorname{sech}\pi t)dt \\ &= \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_1(t)}{\cosh\pi t + \cos\pi y}dt. \end{aligned}$$

8. The conformal maps indicated in Figure 11 are

$$F_1(z) = \frac{z-1}{z+1},\ F_2(z) = \log z,\ F_3(z) = \frac{1}{i}z - \frac{\pi}{2},\ F_4(z) = \sin z,\ F_5(z) = z - 1.$$

Observe that $\arg(z)/\pi$ is harmonic on the upper half-plane, equals 0 on the positive real axis, and 1 on the negative real axis. Thus the harmonic function

$$u = F_1^{-1} \circ F_2^{-1} \circ F_3^{-1} \circ F_4^{-1} \circ F_5^{-1} \circ \left(z \mapsto \frac{1}{\pi}\arg(z)\right)$$

satisfies the given condition.

9. Since $f(z) = (i+z)/(i-z)$ is holomorphic in $\mathbb{D}$, $u = \operatorname{Re}(f)$ is harmonic in unit disc. Furthermore,

$$u(x,y) = \operatorname{Re}\left(\frac{i+z}{i-z}\right) = \operatorname{Re}\left(\frac{x + i(1+y)}{-x + i(1-y)}\right) = \frac{1 - x^2 - y^2}{x^2 + (1-y)^2},$$

thus u vanishes on its boundary.

10. Recall the conformal map between $\mathbb{D}$ and $\mathbb{H}$, $F^*(z) = \frac{i-z}{i+z}$ and $G(w) = i\frac{1-w}{1+w}$. Define function

$$f: \mathbb{D} \to \mathbb{C}, \quad f(w) = F(G(w)) = F\left(i\frac{1-w}{1+w}\right).$$

Since $|f(w)| \le 1$ for all $w \in \mathbb{D}$ and $f(0) = F(i) = 0$, by Schwarz lemma,

$$|f(w)| \le |w| \quad \text{for all } w \in \mathbb{D}.$$

Now substitute $w = F^*(z)$ to get

$$|F(z)| = |f(F^*(z))| \le |F^*(z)| = \left|\frac{z-i}{z+i}\right| \quad \text{for all } z \in \mathbb{H}.$$

11. Define $g : \mathbb{D} \to \overline{\mathbb{D}}$ and $h : \mathbb{D} \to \overline{\mathbb{D}}$ as

$$g(w) = \frac{1}{M} f(Rw), \quad h(w) = \frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}.$$

Since $h(0) = 0$, by Schwarz lemma, $|h(w)| \le |w|$ for all $w \in \mathbb{D}$. Letting $z = Rw$ gives

$$\left|\frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}\right| = \left|\frac{\frac{1}{M}f(Rw) - \frac{1}{M}f(0)}{1 - \frac{1}{M}\overline{f(0)}\frac{1}{M}f(Rw)}\right| = M\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \le \frac{|z|}{R}.$$

Therefore given inequality holds.

12. **(a)** Define $g : \mathbb{D} \to \mathbb{D}$ as $g = \psi_\alpha^{-1} \circ f \circ \psi_\alpha$. Let $\gamma = \psi_\alpha^{-1}(\beta) \in \mathbb{D} - \{0\}$, then

$$\begin{aligned} g(0) &= \psi_\alpha^{-1}(f(\psi_\alpha(0))) = \psi_\alpha^{-1}(f(\alpha)) = \psi_\alpha^{-1}(\alpha) = 0, \\ g(\gamma) &= \psi_\alpha^{-1}(f(\psi_\alpha(\gamma))) = \psi_\alpha^{-1}(f(\beta)) = \psi_\alpha^{-1}(\beta) = \gamma. \end{aligned}$$

Hence g is identity by Schwarz lemma. Therefore $f = \psi_\alpha \circ g \circ \psi_\alpha^{-1} = \psi_\alpha \circ \mathrm{id}_{\mathbb{D}} \circ \psi_\alpha^{-1} = \mathrm{id}_{\mathbb{D}}$.

(b) No. Consider the function $f : \mathbb{D} \to \mathbb{D}$ defined as

$$f = F \circ (z \mapsto z + 1) \circ G,$$

where $F(z) = \frac{i-z}{i+z}, G(w) = \frac{i(1-w)}{1+w}$ is conformal map between $\mathbb{D}$ and $\mathbb{H}$. Then f does not have fixed point. Otherwise, it has fixed point w, then

$$F(G(w) + 1) = w \;\Rightarrow\; G(w) + 1 = G(w),$$

which is contradiction.

13. **(a)** Consider the function $g = \psi_{f(w)} \circ f \circ \psi_w^{-1}$. Since $g : \mathbb{D} \to \mathbb{D}$ and $g(0) = 0$, $|g(z)| \le |z|$ for all $z \in \mathbb{D}$ by Schwarz lemma. Hence

$$\begin{aligned} \left|\left(\psi_{f(w)} \circ f \circ \psi_w^{-1}\right)(z)\right| \le |z| \;&\Rightarrow\; \left|\left(\psi_{f(w)} \circ f\right)(z)\right| \le |\psi_w(z)| \\ &\Rightarrow\; \rho(f(z), f(w)) \le \rho(z, w) \quad \text{for all } z, w \in \mathbb{D}. \end{aligned}$$

(b) Since

$$\left|\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\right| \le \left|\frac{z - w}{1 - \overline{w}z}\right| \;\Rightarrow\; \left|\frac{\frac{f(z)-f(w)}{z-w}}{1 - \overline{f(w)}f(z)}\right| \le \frac{1}{|1 - \overline{w}z|},$$

taking the limit $w \to z$ gives

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

14. Note that $F(z) = \frac{i-z}{i+z}$ and $G(w) = i\frac{1-w}{1+w}$ is conformal map between $\mathbb{D}$ and $\mathbb{H}$. Suppose f is conformal map from $\mathbb{H}$ to $\mathbb{D}$. Then $w \mapsto f(G(w))$ is conformal map from $\mathbb{D}$ to $\mathbb{D}$. Hence there exists $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that

$$f(G(w)) = e^{i\theta}\frac{\alpha - w}{1 - \overline{\alpha}w}.$$

Now choose $z = G(w)$ then

$$f(z) = e^{i\theta}\frac{\alpha - F(z)}{1 - \overline{\alpha}F(z)} = e^{i\theta}\frac{\alpha - \frac{i-z}{i+z}}{1 - \overline{\alpha}\frac{i-z}{i+z}} = e^{i\theta}\frac{(\alpha+1)z + i(\alpha-1)}{(\overline{\alpha}+1)z + i(1-\overline{\alpha})} = e^{i\theta}\frac{z-\beta}{z-\overline{\beta}},$$

where $\beta = i\frac{1-\alpha}{1+\alpha} = G(\alpha) \in \mathbb{H}$.

15. (a) Since Φ is an automorphism of $\mathbb{H}$, there exists a, b, c, d such that $\Phi(z) = (az+b)/(cz+d)$. Then the equation

$$\frac{az+b}{cz+d} = z \ \Rightarrow\ cz^2 + (d-a)z - b = 0$$

has three distinct real roots, so $c = d - a = b = 0$. Therefore $\Phi(z) = az/d = z$, which is identity.

(b) Observe that the equation $y = \frac{ax+b}{cx+d} \Leftrightarrow cyx - ax + dy - b = 0$ is hyperbola. Since this function passes through the (x_j, y_j), this equation is equivalent to

$$\left|\begin{pmatrix} xy & x & y & 1 \\ x_1y_1 & x_1 & y_1 & 1 \\ x_2y_2 & x_2 & y_2 & 1 \\ x_3y_3 & x_3 & y_3 & 1 \end{pmatrix}\right| = 0.$$

by means of a Laplace expansion along the first row, we get

$$\begin{aligned}
a &= x_1y_1(y_2 - y_3) + x_2y_2(y_3 - y_1) + x_3y_3(y_1 - y_2), \\
b &= x_1y_1(x_2y_3 - x_3y_2) + x_2y_2(x_3y_1 - x_1y_3) + x_3y_3(x_1y_2 - x_2y_1), \\
c &= y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1), \\
d &= x_1y_1(x_2 - x_3) + x_2y_2(x_3 - x_1) + x_3y_3(x_1 - x_2).
\end{aligned}$$

Note that $ad - bc = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)(y_1 - y_3)(y_2 - y_3) > 0$. Therefore

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{\sqrt{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies the condition $a'd' - b'c' = 1$, so there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j$.

Even if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_2$,

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)(y_1 - y_3)(y_2 - y_3) > 0,$$

therefore the conclusion is the same.

16. (a) Given $\theta \in \mathbb{R}$,

$$\begin{aligned}
f^{-1}(e^{i\theta}f(z)) = i\frac{1 - e^{i\theta}\frac{i-z}{i+z}}{1 + e^{i\theta}\frac{i-z}{i+z}} &= \frac{i(1+e^{i\theta})z - (1 - e^{i\theta})}{(1-e^{i\theta})z + i(1+e^{i\theta})} \\
&= \frac{\frac{e^{i\theta/2}+e^{-i\theta/2}}{2}z + \frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}z}{-\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}z + \frac{e^{i\theta/2}+e^{-i\theta/2}}{2}} = \frac{\cos(\theta/2)z + \sin(\theta/2)}{-\sin(\theta/2)z + \cos(\theta/2)}.
\end{aligned}$$

Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

(b) Given $\alpha = r + is \in \mathbb{D}$, then

$$\psi_\alpha(f(z)) = \frac{\alpha - \frac{i-z}{i+z}}{1-\overline{\alpha}\frac{i-z}{i+z}} = \frac{(\alpha+1)z+(\alpha-1)i}{(1+\overline{\alpha})z+(1-\overline{\alpha})i}$$

and

$$f^{-1}(\psi_\alpha(f(z))) = i\frac{1-\frac{(\alpha+1)z+(\alpha-1)i}{(1+\overline{\alpha})z+(1-\overline{\alpha})i}}{1+\frac{(\alpha+1)z+(\alpha-1)i}{(1+\overline{\alpha})z+(1-\overline{\alpha})i}} = i\frac{(\overline{\alpha}-\alpha)z+(2-\alpha-\overline{\alpha})i}{(2+\alpha+\overline{\alpha})z+(\alpha-\overline{\alpha})i} = \frac{sz-(1-r)}{(1+r)z-s}.$$

Therefore

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{1-|\alpha|^2}}\begin{pmatrix} s & -(1-r) \\ 1+r & -s \end{pmatrix}.$$

(c) By **Theorem 2.2**, there exist $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$ such that $g(z) = e^{i\theta}\psi_\alpha(z)$. Apply **(a)**, **(b)**, then

$$\begin{aligned} f^{-1}\circ g \circ f &= f^{-1}\circ\left(z\mapsto e^{i\theta}z\right)\circ\psi_\alpha\circ f = \left(f^{-1}\circ\left(z\mapsto e^{i\theta}z\right)\circ f\right)\circ\left(f^{-1}\circ\psi_\alpha\circ f\right) \\ &= \left(z\mapsto\frac{a_1z+b_1}{c_1z+d_1}\right)\circ\left(z\mapsto\frac{a_2z+b_2}{c_2z+d_2}\right) = \left(z\mapsto\frac{az+b}{cz+d}\right) \end{aligned}$$

where $ad-bc=1$.

17. We change the variable $w=\psi_\alpha(z)$. Since the determinant of the Jacobian is simply $|\psi_\alpha'|^2$,

$$\frac{1}{\pi}\iint_{\mathbb{D}}|\psi_\alpha'|^2dxdy = \frac{1}{\pi}\iint_{\psi_\alpha(\mathbb{D})}dxdy = \frac{1}{\pi}\iint_{\mathbb{D}}dxdy = 1.$$

Since

$$\psi_\alpha'(z) = -\frac{1-|\alpha|^2}{(1-\overline{\alpha}z)^2} \Rightarrow |\psi_\alpha'(z)| = \frac{1-|\alpha|^2}{(1-\overline{\alpha}z)^2},$$

the second integral is

$$\frac{1}{\pi}\iint_{\mathbb{D}}|\psi_\alpha'|dxdy = \frac{1-|\alpha|^2}{\pi}\int_0^1\int_0^{2\pi}\frac{1}{(1-\overline{\alpha}re^{i\theta})(1-\alpha re^{-i\theta})}rd\theta dr.$$

Using residue formula for $f(z) = 1/(1-\overline{\alpha}rz)(1-\alpha z)$,

$$\begin{aligned} \int_0^1\int_0^{2\pi}\frac{1}{(1-\overline{\alpha}re^{i\theta})(1-\alpha re^{-i\theta})}rd\theta dr &= \int_0^1 r\left(\int_0^{2\pi}\frac{1}{(1-\overline{\alpha}re^{i\theta})(1-\alpha re^{-i\theta})}d\theta\right)dr \\ &= \int_0^1 r\left(\int_{\partial\mathbb{D}}\frac{1}{i(1-\overline{\alpha}rz)(z-\alpha r)}dz\right)dr \\ &= \int_0^1 r\left(2\pi i\cdot\frac{1}{i}\cdot\frac{1}{1-\overline{\alpha}r\cdot\alpha r}\right)dr = \int_0^1\frac{2\pi r}{1-|\alpha|^2r^2}dt \\ &= \frac{\pi}{|\alpha|^2}\log\frac{1}{(1-|\alpha|)^2}. \end{aligned}$$

Hence

$$\frac{1}{\pi}\iint_{\mathbb{D}}|\psi_\alpha'|dxdy = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{(1-|\alpha|)^2}.$$

18. Remember that in the proof of **Theorem 4.2**, the key geometric property of the unit disc and polygonal region is that if z_0 belongs to the boundary of Ω, and C is any small circle centered at z_0, then $C \cap \Omega$ consists of an arc. Note that piecewise-smooth closed curve γ also has this property because

$$z(t) = z(t_0) + z'(t_0)(t - t_0) + o(|t - t_0|)$$

at any point $z_0 = z(t_0)$ on γ. Therefore we can generalize **Theorem 4.2** to the piecewise-smooth closed curve.

19. Suppose any two curves $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ lying in $\Omega = \mathbb{C} - \cup_{k=1}^{n} \{A_k + iy : y \le 0\}$ and write $\gamma_j(z) = r_j(z) + is_j(z)$. First, we show that for every small $\varepsilon > 0$, there exists A_j such that

$$s_j(t) + A_j t(1-t) < 0 \ \Rightarrow\ \left|r_j(t) - r(0)\right| < \varepsilon \text{ or } \left|r_j(t) - r(1)\right| < \varepsilon.$$

Since r_j is continuous, there exists $\delta > 0$ such that

$$t < \delta \text{ or } t > 1 - \delta \ \Rightarrow\ \left|r_j(t) - r(0)\right| < \varepsilon \text{ or } \left|r_j(t) - r(1)\right| < \varepsilon.$$

Now let $A_j = \sup\{-s_j(t)\}/(\delta(1-\delta))$. Then

$$\delta \le t \le 1 - \delta \ \Rightarrow\ -s_j(t) \le A_j \delta(1-\delta) \le A_j t(1-t) \ \Rightarrow\ s_j(t) + A_j t(1-t) \ge 0.$$

Hence the desired proof is obtained.

Let $A = \max\{A_0, A_1\}$ and define $\gamma_j^*(t) = \gamma_j(t) + iAt(1-t)$ $(j = 0, 1)$. Note that γ_0 and γ_0^* are homotopic, γ_1 and γ_1^* are homotopic. Since both γ_0^* and γ_1^* are contained in $\Omega' = \mathbb{H} \cup \{z \in \mathbb{C} : |\text{Re}(z) - r(0)| < \varepsilon \text{ or } |\text{Re}(z) - r(1)| < \varepsilon\}$, and Ω' is simply connected by **Problem 4, Chapter 3**, γ_0^* and γ_1^* are homotopic. Therefore γ_0 and γ_1 are homotopic, and conclude that Ω is simply connected.

20. (a) If $\lambda \ne 0, 1$, then the integral is equal to

$$\int_0^z \frac{d\zeta}{\zeta^{1/2}(\zeta - 1)^{1/2}(\zeta - \lambda)^{1/2}},$$

which is Schwarz-Christoffel integral with $\beta_1 = \beta_2 = \beta_3 = 1/2$. Since $\sum \beta_k = 3/2$, By **Proposition 4.1**, this function maps the upper-half plane conformally to a rectangle. Moreover, the angle of the vertices is $\alpha_1\pi = \alpha_2\pi = \alpha_3\pi = \alpha_4\pi = \pi/2$, so the image is rectangle.

Note. In the case of $\lambda = 0$, the integral diverges and the definition does not seem to be valid.

(b) Since the image is a rectangle, the lengths of the two opposite sides are equal. The lengths of the two adjacent sides are

$$\int_0^1 \frac{1}{\sqrt{x(1-x^2)}}\,dx \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{u(1-u^2)}}\,du,$$

respectively, and we can see that they are equal through changing the variables $u = 1/x$. Hence the image is a square. Furthermore, changing the variables $x^2 = t$, the side length is

$$\begin{aligned}\int_0^1 x^{-1/2}(1-x^2)^{-1/2}dx &= \int_0^1 t^{-1/4}(1-t)^{-1/2}\frac{dt}{2t^{1/2}} = \frac{1}{2}\int_0^1 t^{-3/4}(1-t)^{-1/2}dt \\ &= \frac{1}{2}B\Big(\frac{1}{4}, \frac{1}{2}\Big) = \frac{1}{2}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{1}{2}\frac{\Gamma^2(1/4)}{\sqrt{2\pi}} = \frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}.\end{aligned}$$

21. (a) This is Schwarz-Christoffel integral with $1 < \beta_1 + \beta_2 < 2$, it maps $\mathbb{H}$ to a triangle whose vertices are the images of $0, 1, \infty$, and with angles $\alpha_1\pi, \alpha_2\pi, \alpha_3\pi$ with $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) The images of the two intervals $(-\infty, 0], [1, \infty)$ become parallel.

(c) The images of the two intervals $(-\infty, 0], [1, \infty]$ diverge in opposite directions.

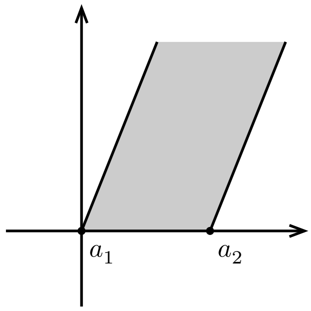


Image when $\beta_1 + \beta_2 = 1$

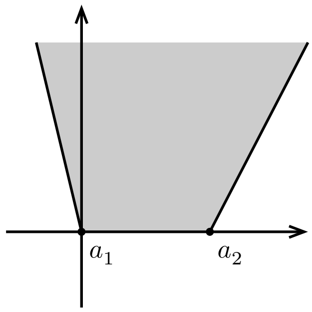


Image when $0 < \beta_1 + \beta_2 < 1$

(d) Let l_j be the length of the side of the triangle opposite angle $\alpha_j \pi$. Then

$$\begin{aligned} l_3 &= \int_0^1 x^{-\beta_1}(1-x)^{1-\beta_2}dx = B(1-\beta_1, 1-\beta_2) = B(\alpha_1, \alpha_2) \\ &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(1-\alpha_3)} = \frac{\sin(\alpha_3\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3), \end{aligned}$$

$$\begin{aligned} l_1 &= \int_1^\infty x^{-\beta_1}(x-1)^{-\beta_2}dx = \int_0^1 t^{\beta_1+\beta_2-2}(1-t)^{-\beta_2}dt = B(\beta_1+\beta_2-1, 1-\beta_2) \\ &= B(\alpha_3, \alpha_2) = \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)} = \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(1-\alpha_1)} = \frac{\sin(\alpha_1\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) \quad (t = 1/x), \end{aligned}$$

$$\begin{aligned} l_2 &= \int_{-\infty}^0 (-x)^{-\beta_1}(1-x)^{-\beta_2}dx = \int_0^1 t^{\beta_1+\beta_2-2}(1-t)^{-\beta_1}dt = B(\beta_1+\beta_2-1, 1-\beta_1) \\ &= B(\alpha_3, \alpha_1) = \frac{\Gamma(\alpha_3)\Gamma(\alpha_1)}{\Gamma(\alpha_3+\alpha_1)} = \frac{\Gamma(\alpha_3)\Gamma(\alpha_1)}{\Gamma(1-\alpha_2)} = \frac{\sin(\alpha_2\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) \quad \left(t = -\frac{1}{x-1}\right). \end{aligned}$$

22. Let $\mathbb{F}(z) = \frac{i-z}{i+z}$ and $\mathbb{G}(w) = i\frac{1-w}{1+w}$, which are conformal map between $\mathbb{H}$ and $\mathbb{D}$. Then $G = F \circ \mathbb{F}$ is conformal from $\mathbb{H}$ to P. We first suppose that F did not map the point at $z = -1$ to a vertex of P. Then by **Theorem 4.6**, we can represent G as

$$G(z) = c_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\beta - A_n)^{\beta_n}} + c_2.$$

Therefore change the variables $\xi = \mathbb{F}(\zeta)$ to get

$$\begin{aligned} F(z) = G(\mathbb{G}(z)) &= c_1 \int_0^{i\frac{1-z}{1+z}} \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\beta - A_n)^{\beta_n}} + c_2 \\ &= c_1 \int_1^z \frac{-2i\frac{1}{(1+\xi)^2}d\xi}{\left(i\frac{1-\xi}{1+\xi} - A_1\right)^{\beta_1} \cdots \left(i\frac{1-\xi}{1+\xi} - A_n\right)^{\beta_n}} + c_2 = c_1' \int_1^z \frac{d\xi}{(\xi - B_1)^{\beta_1} \cdots (\xi - B_n)^{\beta_n}} + c_2, \end{aligned}$$

where $B_j = \mathbb{F}(A_j)$.

If there is a vertex of P such that corresponding to $-1 \in \mathbb{D}$ with F, then this point corresponds to ∞ for G. Hence by **Theorem 4.7**, we can represent G as

$$G(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\beta - A_{n-1})^{\beta_{n-1}}} + C_2.$$

Similar to above,

$$
\begin{aligned}
F(z) = G(\mathbb{G}(z)) &= C_1 \int_0^{i\frac{1-z}{1+z}} \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\beta - A_{n-1})^{\beta_{n-1}}} + C_2 \\
&= C_1' \int_1^z \frac{d\zeta}{(\xi - B_1)^{\beta_1} \cdots (\xi - B_n)^{\beta_n}} + C_2
\end{aligned}
$$

where $B_j = \mathbb{F}(A_j)$ and $A_n = \infty$.

23. By **Exercise 22**, F maps the unit disc conformally onto the interior of a polydon with n sides. The side lengths are

$$
\begin{aligned}
\left| \int_{e^{2\pi k i/n}}^{e^{2\pi(k+1)i/n}} \frac{d\zeta}{(1-\zeta^n)^{2/n}} \right| &= \left| \int_{2\pi k/n}^{2\pi(k+1)/n} \frac{ie^{i\theta}}{(1-e^{in\theta})^{2/n}} d\theta \right| = \left| \int_0^{2\pi/n} \frac{ie^{i(\varphi + 2\pi k/n)}}{\left(1-e^{in(\varphi+2\pi k/n)}\right)^{2/n}} d\varphi \right| \\
&= \left| ie^{2\pi k i/n} \int_0^{2\pi/n} \frac{e^{i\varphi}}{(1-e^{in\varphi})^{2/n}} d\varphi \right|,
\end{aligned}
$$

respectively, and they are all the same. Therefore the image is regular polygon. The perimeter is

$$
\begin{aligned}
n \left| \int_0^{2\pi/n} \frac{e^{i\varphi}}{(1-e^{in\varphi})^{2/n}} d\varphi \right| = n \left| \int_0^{\pi} \frac{e^{2i\theta/n}}{1-e^{(2i\theta)^{2/n}}} \frac{2}{n} d\theta \right| &= 2 \left| \left(\frac{i}{2}\right)^{2/n} \int_0^{\pi} \frac{1}{\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^{2/n}} d\theta \right| \\
&= 2^{\frac{n-2}{n}} \int_0^{\pi} (\sin\theta)^{-2/n} d\theta.
\end{aligned}
$$

24. (a) We change variables $x = \left(1 - \tilde{k}^2 y^2\right)^{-1/2}$ in the integral defining $K'(k)$, then

$$
\begin{aligned}
K'(k) &= \int_0^{1/k} \frac{1}{\sqrt{(x^2-1)(1-k^2x^2)}} dx = \int_0^1 \frac{1}{\sqrt{\frac{\tilde{k}^2 y^2}{1-\tilde{k}^2y^2} \cdot \tilde{k}^2 \frac{1-y^2}{1-\tilde{k}^2y^2}}} \frac{\tilde{k}^2 y}{\sqrt{1-\tilde{k}^2y^2}^3} dy \\
&= \int_0^1 \frac{1}{\sqrt{(1-y^2)\left(1-\tilde{k}^2y^2\right)}} dy = K\big(\tilde{k}\big).
\end{aligned}
$$

(b) Change variables $x = 2t/\left(1+\tilde{k}+\left(1-\tilde{k}\right)t^2\right)$, then

$$
\begin{aligned}
K(k) &= \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} dx \\
&= \int_0^1 \frac{1}{\frac{\sqrt{(1-t^2)\left(\left(1+\tilde{k}\right)^2 - \left(1-\tilde{k}\right)^2 t^2\right)}}{\left(1+\tilde{k}\right)+\left(1-\tilde{k}\right)t^2} \cdot \frac{\left(1+\tilde{k}\right)-\left(1-\tilde{k}\right)t^2}{\left(1+\tilde{k}\right)+\left(1-\tilde{k}\right)t^2}} \cdot \frac{2\left(\left(1+\tilde{k}\right)-\left(1-\tilde{k}\right)t^2\right)}{\left(\left(1+\tilde{k}\right)+\left(1-\tilde{k}\right)t^2\right)^2} dt \\
&= 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)\left(\left(1+\tilde{k}\right)^2 - \left(1-\tilde{k}\right)^2 t^2\right)}} \\
&= \frac{2}{1+\tilde{k}} \int_0^1 \frac{dt}{\sqrt{(1-t^2)\left(1 - \left(\frac{1-\tilde{k}}{1+\tilde{k}}\right)^2 t^2\right)}} = \frac{2}{1+\tilde{k}} K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right).
\end{aligned}
$$

(c) Using the integral representation for F given in **Exercise 9, Chapter 6**,

$$
\begin{aligned}
K(k) &= \int_0^1 (1-x^2)^{-1/2}(1-k^2x^2)^{-1/2}dx = \int_0^1 (1-t)^{-1/2}(1-k^2t)^{-1/2}\frac{dt}{2\sqrt{t}} \\
&= \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-k^2t)^{-1/2}dt = \frac{1}{2}\left(\frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1-1/2)}\right)^{-1} F(1/2,1/2,1;k^2) \\
&= \frac{\pi}{2}F(1/2,1/2,1;k^2).
\end{aligned}
$$

Chapter 9. An Introduction to Elliptic Functions

1. (a) First suppose that f is periodic with the simple period $w_0 = \frac{1}{q}w_1$. Since $w_1 = qw_0$ and $w_2 = \frac{p}{q}w_1 = pw_0$ where p, q are integers, f has two periods w_1 and w_2.

Now suppose f has two periods w_1 and w_2 where $w_2/w_1 = p/q$. Then there exists integers m and n such that $mq + np = 1$. Hence

$$f(z + w_0) = f(z + (mq + np)w_0) = f(z + mw_1 + nw_2) = f(z),$$

which means that f is periodic with simple period $w_0 = \frac{1}{q}w_1$.

(b) Let $\tau = w_2/w_1$. Note that $\{m - n\tau\}_{m,n\in\mathbb{Z}}$ is dense in $\mathbb{R}$ since τ is irrational. Hence

$$\{z \in \mathbb{C} : f(z) = f(0)\}$$

has subsequence of distinct points with limit point 0. By **Corollary 4.9, Chapter 2**, $f(z) = 0$ for all $z \in \mathbb{C}$.

2. Suppose that the boundary of the parallelogram contains no zeros or poles. Using the periodicity of f,

$$
\begin{aligned}
\int_{\partial P_0} \frac{zf'(z)}{f(z)}dz &= \int_0^{w_1} \frac{zf'(z)}{f(z)}dz + \int_{w_1}^{w_1+w_2} \frac{zf'(z)}{f(z)}dz + \int_{w_1+w_2}^{w_2} \frac{zf'(z)}{f(z)}dz + \int_{w_2}^{0} \frac{zf'(z)}{f(z)}dz \\
&= \int_0^{w_1} \frac{zf'(z)}{f(z)}dz + \int_0^{w_2} \frac{(z+w_1)f'(z)}{f(z)}dz + \int_{w_1}^{0} \frac{(z+w_2)f'(z)}{f(z)}dz + \int_{w_2}^{0} \frac{zf'(z)}{f(z)}dz \\
&= w_1\int_0^{w_2} \frac{f'(z)}{f(z)}dz - w_2\int_0^{w_1} \frac{f'(z)}{f(z)}dz.
\end{aligned}
$$

Observe that if f has zero of order n at z_0, then

$$\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + G(z) \;\Rightarrow\; \frac{zf'(z)}{f(z)} = \frac{nz}{z-z_0} + zG(z),$$

otherwise f has pole of order n at z_0, then

$$\frac{f'(z)}{f(z)} = -\frac{n}{z-z_0} + H(z) \;\Rightarrow\; \frac{zf'(z)}{f(z)} = -\frac{nz}{z-z_0} + zH(z).$$

Therefore

$$\int_{\partial P_0} \frac{zf'(z)}{f(z)}dz = 2\pi i\left(\sum_{j=0}^{r} a_j - \sum_{j=0}^{r} b_j\right).$$

Meanwhile, the integral of $f'(z)/f(z)$ over a side is an integer multiple of 2π. To see this, write $f(z) = r(z)e^{i\theta(z)}$ for $r(z) > 0$ and continuous $\theta(z)$. Then

$$\int_0^w \frac{f'(z)}{f(z)}dz = \int_0^w \frac{r'e^{i\theta}+ire^{i\theta}}{re^{i\theta}}dz = \int_0^w \left(\frac{r'}{r}+i\theta'\right)dz = [\log r(z)]_0^w + i[\theta]_0^w = i(\theta(w)-\theta(0)),$$

where $f(0)=f(w)$ implies $e^{i(\theta(w)-\theta(0))}=1$ and $i(\theta(w)-\theta(0))=2\pi i\cdot m$ for $m\in\mathbb{Z}$. Therefore

$$2\pi i\left(\sum_{j=0}^{r} a_j - \sum_{j=0}^{r} b_j\right) = \int_{\partial P_0}\frac{zf'(z)}{f(z)}dz = 2\pi i(nw_1+mw_2), \quad \text{where } n,m\in\mathbb{Z}.$$

If there are zeros or poles on the side of the parallelogram, we can translate it by a small amount to reduce the problem to the first case.

3. Since $|n+m\tau|^2 \approx (|n|+|m|)^2 \approx n^2+m^2$, it's enough to prove $\sum 1/(n^2+m^2)=\infty$. Note that

$$\sum_{1\le n^2+m^2\le R^2}\frac{1}{n^2+m^2} = 4\sum_{1\le n\le R}\frac{1}{n^2} + 4\sum_{\substack{n^2+m^2\le R^2\\ 1\le n,m}}\frac{1}{n^2+m^2}.$$

First term is obviously $O(1)$. Moreover,

$$\sum_{\substack{n^2+m^2\le R^2\\ 1\le n,m}}\frac{1}{n^2+m^2} \le \int_0^{\pi/2}\int_0^R \frac{1}{r^2}r\,drd\theta = \frac{\pi}{2}\log R,$$

$$\begin{aligned}\sum_{\substack{n^2+m^2\le R^2\\ 1\le n,m}}\frac{1}{n^2+m^2} &\ge \int_0^{\pi/2}\int_0^R \frac{1}{(r\cos\theta-1)^2+(r\sin\theta-1)^2}r\,drd\theta \\ &\ge \int_0^{\pi/2}\int_0^R \frac{1}{r^2+2}r\,drd\theta \\ &= \frac{\pi}{2}\log\sqrt{R^2+2} - \frac{\pi}{4}\log 2 \ge \frac{\pi}{2}\log R + O(1).\end{aligned}$$

Hence

$$\sum_{1\le n^2+m^2\le R^2} 1/(n^2+m^2) = 2\pi\log R + O(1) \quad \text{as } R\to\infty$$

and the given series does not converge.

4. For R sufficiently large,

$$\wp(z)-\wp^R(z) = \sum_{|w|\ge R}\left[\frac{1}{(z+w)^2}-\frac{1}{w^2}\right] = \sum_{|w|\ge R} O\left(\frac{1}{|w|^3}\right) = O\left(\int_R^\infty \frac{1}{r^3}r dr\right) = O\left(\frac{1}{R}\right).$$

Hence $\wp(z)=\wp^R(z)+O(1/R)$. Next,

$$\begin{aligned}\wp^R(z+1)-\wp^R(z) &= \frac{1}{(z+1)^2} + \sum_{0<|w|<R}\left[\frac{1}{(z+1+w)^2}-\frac{1}{w^2}\right] - \frac{1}{z^2} - \sum_{0<|w|<R}\left[\frac{1}{(z+w)^2}-\frac{1}{w^2}\right] \\ &= \sum_{0\le|w-1|\le R}\frac{1}{(z+w)^2} - \sum_{0\le|w|\le R}\frac{1}{(z+w)^2} = O\left(\sum_{R-1\le|w|\le R+1}\frac{1}{|w|^2}\right) \\ &= O\left(R\cdot\frac{1}{R^2}\right) = O\left(\frac{1}{R}\right).\end{aligned}$$

Similarly, $\wp^R(z+\tau)=\wp^R(z)+O(1/R)$. Therefore for any $w\in\Lambda$,

$$\wp(z+w)=\wp(z)+O(1/R).$$

Taking the limit $R \to \infty$ gives $\wp(z+w) = \wp(z)$.

5. (a) First prove that $\sigma(z)$ is the entire function of order 2. That is, we will show that

$$\left|\prod_{j=1}^{\infty} E_2(z/\tau_j)\right| \leq e^{c|z|^s}$$

for any s with $2 < s < 3$. This proof is similar to the proof of **Lemma 5.3, Chapter 5**. We write

$$\prod_{j=1}^{\infty} E_2(z/\tau_j) = \prod_{|\tau_j| \leq 2|z|} E_2(z/\tau_j) \prod_{|\tau_j| > 2|z|} E_2(z/\tau_j).$$

For the second product,

$$\left|\prod_{|\tau_j|>2|z|} E_2(z/\tau_j)\right| = \prod_{|\tau_j|>2|z|} |E_2(z/\tau_j)| \leq \prod_{|\tau_j|>2|z|} e^{c|z/\tau_j|^3} = e^{c|z|^3 \sum_{|\tau_j|>2|z|} |\tau_j|^{-3}}.$$

But $|\tau_j| > 2|z|$ and $s < 3$, so we have

$$|\tau_j|^{-3} = |\tau_j|^{-s} |\tau_j|^{s-3} \leq C|\tau_j|^{-s} |z|^{s-3}.$$

Therefore, the fact that $\sum |\tau_n|^{-s}$ converges implies that

$$\left|\prod_{|\tau_j|>2|z|} E_2(z/\tau_j)\right| \leq e^{c|z|^s}.$$

Now estimate the first product.

$$\left|\prod_{|\tau_j|\leq 2|z|} E_2(z/\tau_j)\right| = \prod_{|\tau_j|\leq 2|z|} \left|1 - \frac{z}{\tau_j}\right| \prod_{|\tau_j|\leq 2|z|} e^{c|z/\tau_j|^2}.$$

Since $|\tau_j|^{-2} = |\tau_j|^{-s}|\tau_j|^{s-2} \leq C|\tau_j|^{-s}|z|^{s-2}$, therefore

$$\prod_{|\tau_j|\leq 2|z|} e^{c|z/\tau_j|^2} = e^{c|z|^2 \sum_{|\tau_j|\leq 2|z|} |\tau_j|^{-2}} \leq e^{c|z|^s}.$$

Moreover, Letting $\tau' = \min\{1, |\tau|\}$ gives

$$\begin{aligned}\prod_{|\tau_j|\leq 2|z|} \left|1 - \frac{z}{\tau_j}\right| &\leq \prod_{|\tau_j|\leq 2|z|} \left(1 + \left|\frac{z}{\tau_j}\right|\right) \leq \left(1 + \frac{|z|}{\tau'}\right)^{\mathfrak{n}(2|z|)} \\ &= e^{\mathfrak{n}(2|z|)\log(1+|z|/\tau')} \leq e^{c|z|^\rho \log(1+|z|/\tau')} \leq e^{c|z|^s},\end{aligned}$$

where $2 < \rho < s$. Hence $\sigma(z)$ is entire function of order 2. By Hadamard's factorization theorem, any function f of order 2 and has simples zeros at $\{n + m\tau\}$ is equal to $e^{P(z)}\sigma(z)$, where $P(z)$ is polynomial of degree ≤ 2. Therefore $\sigma(z)$ also have simple zeros at all the periods $n + m\tau$, and vanishes nowhere else.

(b) By **Lemma 4.2, Chapter 5**,

$$\begin{aligned}\frac{\sigma'(z)}{\sigma(z)} &= \frac{1}{z} + \sum_{(n,m)\neq(0,0)} \frac{1}{\tau_j}\frac{E_2'(z/\tau_j)}{E_2(z/\tau_j)} = \frac{1}{z} + \sum_{(n,m)\neq(0,0)} \left[\frac{1}{z-\tau_j} + \frac{1}{\tau_j} + \frac{z}{\tau_j^2}\right] \\ &= \frac{1}{z} + \sum_{(n,m)\neq(0,0)} \left[\frac{1}{z-n-m\tau} + \frac{1}{n+m\tau} + \frac{z}{(n+m\tau)^2}\right].\end{aligned}$$

(c) Also by **Lemma 4.2, Chapter 5**,

$$L'(z) = -\left(\frac{\sigma'(z)}{\sigma(z)}\right)' = \frac{(\sigma'(z))^2 - \sigma(z)\sigma''(z)}{(\sigma(z))^2}$$
$$= \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)}\left[\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2}\right] = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] = \wp(z).$$

6. Recall that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, where $g_2 = 60E_4$ and $g_3 = 140E_6$. Differentiate both sides then

$$2\wp'\wp'' = 12\wp^2\wp' - g_2\wp' \;\Rightarrow\; \wp'' = 6\wp^2 - g_2/2.$$

Hence $\wp''$ is a quadratic polynomial in $\wp$.

7. Setting $\tau = 1/2$ in the expression

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}$$

gives

$$\sum_{m\in\mathbb{Z}}\frac{1}{(m+1/2)^2} = \pi^2 \;\Rightarrow\; \sum_{m\in\mathbb{Z}}\frac{1}{(2m+1)^2} = \frac{\pi^2}{4} \;\Rightarrow\; \sum_{m\geq 1, m\text{ odd}}\frac{1}{m^2} = \frac{\pi^2}{8}.$$

Since

$$\sum_{m\geq 1, m\text{ odd}}\frac{1}{m} = \sum_{m\geq 1}\frac{1}{m^2} - \sum_{m\geq 1}\frac{1}{(2m)^2} = \frac{3}{4}\sum_{m\geq 1}\frac{1}{m^2},$$

deduce that

$$\sum_{m\geq 1}\frac{1}{m^2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6} = \zeta(2).$$

Moreover, differentiating both sides of expression above twice then

$$6\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^4} = 2\pi^4\csc^2(\pi\tau)(1+3\tan^2(\pi\tau)).$$

Set $\tau = 1/2$ to get

$$\sum_{m\in\mathbb{Z}}\frac{1}{(m+1/2)^4} = \frac{\pi^4}{3} \;\Rightarrow\; \sum_{m\in\mathbb{Z}}\frac{1}{(2m+1)^4} = \frac{\pi^4}{48} \;\Rightarrow\; \sum_{m\geq 1, m\text{ odd}}\frac{1}{m^4} = \frac{\pi^4}{96}$$

and

$$\sum_{m\geq 1}\frac{1}{m^4} = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90} = \zeta(4).$$

8. (a) Putting $k = 4$ to **Theorem 2.5**,

$$E_4(\tau) = 2\zeta(4) + \frac{2(2\pi)^4}{6}\sum_{r=1}^{\infty}\sigma_3(r)e^{2\pi i\tau r}$$

where $2\zeta(4) = \pi^4/45$. Let $\tau = x + it$, then

$$\left|\sum_{r=1}^{\infty}\sigma_3(r)e^{2\pi i\tau r}\right| \le \sum_{r=1}^{\infty} r^4 e^{-2\pi rt} \le \sum_{r=1}^{\infty} e^{\frac{4}{3}\pi r}e^{-2\pi rt} = \frac{e^{\pi(4/3-2t)}}{1-e^{\pi(4/3-2t)}} \to 0$$

as $t\to\infty$. We used the identity

$$r \le \frac{\pi}{3}r \le \frac{\pi}{3}r + 1 \le e^{\frac{\pi}{3}r}, \quad r > 0.$$

(b) By **(a)**,

$$\begin{aligned}\left|E_4(\tau) - \frac{\pi^4}{45}\right| &= \frac{16}{3}\pi^4\left|\sum_{r=1}^{\infty}\sigma_3(r)e^{2\pi i\tau r}\right| \le \frac{16}{3}\pi^4\frac{e^{4\pi/3}}{1-e^{-\pi(2t-4/3)}}e^{-2\pi t}\\ &\le \frac{16}{3}\pi^4\frac{e^{4\pi/3}}{1-e^{-2\pi/3}}e^{-2\pi t} = ce^{-2\pi t},\end{aligned}$$

whenever $t \ge 1$.

(c) Since $E_k(\tau) = \tau^{-k}E_k(-1/\tau)$ (**Theorem 2.1**) and $\mathrm{Im}(-1/\tau) = 1/t \ge 1$,

$$\left|E_4(\tau) - \tau^{-4}\frac{\pi^4}{45}\right| = \left|\tau^{-4}E_4(-1/\tau) - \tau^{-4}\frac{\pi^4}{45}\right| = t^{-4}\left|E_4(-1/\tau) - \frac{\pi^4}{45}\right| \le t^{-4}ce^{-2\pi/t}.$$

The last inequality is held by **(b)**.

Chapter 10. Applications of Theta Functions

1. Observe that $F(z) = \left((\Theta')^2 - \Theta\Theta''\right)/\Theta^2$ and $\wp_\tau(z-1/2-\tau/2)$ have same property; they are both elliptic function of order 2 with periods 1 and τ, and with a double pole at $z = 1/2+\tau/2+n+m\tau$. Moreover, the principal part of two functions are same as $1/(z-z_0)^2$ at each pole $z_0 = 1/2+\tau/2+n+m\tau$. By Liouville's theorem, $F(z) - \wp_\tau(z-1/2-\tau/2)$ is constant, thus

$$F(z) = \wp_\tau(z-1/2-\tau/2) + c_\tau.$$

Now we calculate c_τ. Note that

$$\wp(z-z_0) = \frac{1}{(z-z_0)^2} + 0 + 3E_4(z-z_0)^2 + \cdots.$$

Since $\Theta(z|\tau) = \Theta'(z_0|\tau)(z-z_0) + \frac{1}{2}\Theta''(z_0|\tau)(z-z_0)^2 + \frac{1}{6}\Theta'''(z_0|\tau)(z-z_0)^3 + \cdots$, we get

$$\begin{aligned}\frac{\Theta'(z|\tau)}{\Theta(z|\tau)} &= \frac{1}{z-z_0}\frac{\Theta_0' + \Theta_0''(z-z_0) + \frac{1}{2}\Theta_0'''(z-z_0)^2 + \cdots}{\Theta_0' + \frac{1}{2}\Theta_0''(z-z_0) + \frac{1}{6}\Theta_0'''(z-z_0)^2 + \cdots}\\ &= \frac{1}{z-z_0}\left(1 + \frac{\Theta_0''}{2\Theta_0'}(z-z_0) + \left(\frac{1}{3}\frac{\Theta_0'''}{\Theta_0'} - \frac{1}{4}\left(\frac{\Theta_0''}{\Theta_0'}\right)^2\right)(z-z_0)^2 + \cdots\right),\end{aligned}$$

where $\Theta_0^{(k)} = \Theta^{(k)}(z_0|\tau)$. Hence

$$F(z) = -\left(\frac{\Theta'(z|\tau)}{\Theta(z|\tau)}\right)' = \frac{1}{(z-z_0)^2} - \left(\frac{1}{3}\frac{\Theta_0'''}{\Theta_0'} - \frac{1}{4}\left(\frac{\Theta_0''}{\Theta_0'}\right)^2\right) + \cdots$$

and

$$c_\tau = -\frac{1}{3}\frac{\Theta_0'''}{\Theta_0'} + \frac{1}{4}\left(\frac{\Theta_0''}{\Theta_0'}\right)^2.$$

Note. It is unclear whether c_τ can be expressed in terms of the first two derivatives of $\Theta(z|\tau)$.

2. **(a)** Since $F_n \le 2^n$ for all $n \ge 0$, $F(x)$ converges absolutely near 0. Observe that

$$\begin{aligned}
&F(x) - x^2F(x) - xF(x) - x \\
&= (F_0 + F_1x) - (F_0x) - x + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})x^n = 0.
\end{aligned}$$

Hence $F(x) = x^2F(x) + xF(x) + x$ for all x in a neighborhood of 0.

(b) $q(x) = 1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$ where $\alpha, \beta = \left(1 \pm \sqrt{5}\right)/2$.

(c) By simple calculation,

$$F(x) = \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha - \beta}\frac{1}{1-\alpha x} + \frac{1}{\beta - \alpha}\frac{1}{1 - \beta x}.$$

(d) In the neighborhood of 0,

$$\sum_{n=0}^{\infty} F_n x^n = F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \sum_{n=0}^{\infty}(A\alpha^n + B\beta^n)x^n.$$

Hence $F_n = A\alpha^n + B\beta^n$ for all $n \ge 0$.

3. We will briefly touch on the solution for the case where $\alpha = \beta$. If so,

$$U(x) = \frac{u_0 + (u_1 - au_0)x}{(1-\alpha x)^2} = \frac{A}{1-\alpha x} + \frac{B\alpha}{(1-\alpha x)^2} = A\sum_{n=0}^{\infty}\alpha^n x^n + B\sum_{n=0}^{\infty}(n+1)\alpha^{n+1}x^n.$$

Therefore

$$u_n = A\alpha^n + B(n+1)\alpha^{n+1} = A'\alpha^n + B'n\alpha^n$$

for all $n \ge 0$.

4. Note that

$$\begin{aligned}
p(n) &= \sum_{0<k(3k+1)/2\le n} (-1)^{k+1} p\left(n - \frac{k(3k+1)}{2}\right) \\
\Leftrightarrow \quad 0 &= \sum_{0\le k(3k+1)/2\le n} (-1)^{k+1} p\left(n - \frac{k(3k+1)}{2}\right).
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} p(n)z^n = \prod_{k=1}^{\infty}\frac{1}{1-x^k}, \quad \prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}},$$

we have

$$\begin{aligned}
1 &= \left(\prod_{j=1}^{\infty}\frac{1}{1-x^j}\right)\left(\prod_{n=1}^{\infty}(1-x^n)\right) = \sum_{k=-\infty}^{\infty}(-1)^k\left(\prod_{j=1}^{\infty}\frac{1}{1-x^j}\right)x^{\frac{k(3k+1)}{2}} \\
&= \sum_{k=-\infty}^{\infty}(-1)^k\left(\sum_{n=k(3k+1)/2}^{\infty} p\left(n - \frac{k(3k+1)}{2}\right)x^{n-\frac{k(3k+1)}{2}}\right)x^{\frac{k(3k+1)}{2}} \\
&= \sum_{n=0}^{\infty}\left(\sum_{0\le k(3k+1)/2\le n}(-1)^k p\left(n - \frac{k(3k+1)}{2}\right)\right)x^n,
\end{aligned}$$

whenever $|x| < 1$. Therefore

$$\sum_{0\le k(3k+1)/2\le n} (-1)^k p\left(n-\frac{k(3k+1)}{2}\right)=0$$

for all $n \ge 1$.

5. Observe that

$$\log F(x)=\sum_{n=1}^{\infty}\log\left(\frac{1}{1-x^n}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}x^{nm}=\sum_{m=1}^{\infty}\frac{1}{m}\frac{x^m}{1-x^m}.$$

Since $mx^{m-1}(1-x)<1-x^m<m(1-x)$ for $0<x<1$,

$$\frac{1}{1-x}\sum_{m=1}^{\infty}\frac{x^m}{m^2}\le \log F(x)\le \frac{\pi^2}{6}\frac{x}{1-x}.$$

Applying Abel's theorem,

$$\lim_{\substack{x\to 1\\ 0<x<1}}\sum_{m=1}^{\infty}\frac{1}{m^2}x^m=\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}.$$

So by the sandwich theorem

$$\lim_{\substack{x\to 1\\ 0<x<1}}\frac{\log F(x)}{\pi^2/(6(1-x))}=1.$$

6. Since $\log F(x)\sim c/(1-x)$ as $x\to 1$, we have $\log F(e^{-y})\sim c/(1-e^{-y})$ as $y\to 0$. Also, $cy/(1-e^{-y})$ is bounded near $y=0$, so we know

$$\frac{cy}{1-e^{-y}}\le A \;\Rightarrow\; \frac{c}{1-e^{-y}}\le\frac{A}{y}.$$

Hence We get $F(e^{-y})=\sum p(n)e^{-ny}\le Ce^{c/y}$, and $p(n)e^{-ny}\le ce^{c/y}$. Take $y=1/n^{1/2}$ to get $p(n)\le c'e^{c'n^{1/2}}$. In the opposite direction, first see that $cy/(1-e^{-y})\ge A'>0$ near $y=0$, so

$$\frac{cy}{1-e^{-y}}\ge A' \;\Rightarrow\; \frac{c}{1-e^{-y}}\ge\frac{A'}{y}.$$

This leads to inequality

$$Ce^{c/y}\le F(e^{-y})=\sum_{n=0}^{\infty}p(n)e^{-ny}\le\sum_{n=0}^{m}p(n)e^{-ny}+C\sum_{n=m+1}^{\infty}e^{cn^{1/2}}e^{-ny}.$$

Take $y=Am^{-1/2}$ where A is a large constant. we have $cn^{1/2}-Am^{-1/2}n\le-\frac{1}{2}Am^{-1/2}n$ for $n\ge m+1$, thus

$$\sum_{n=m+1}^{\infty}e^{cn^{1/2}}e^{-Am^{-1/2}n}\le\sum_{n=m+1}^{\infty}e^{-\frac{1}{2}Am^{-1/2}n}=\frac{e^{-\frac{1}{2}Am^{-1/2}(m+1)}}{1-e^{-\frac{1}{2}Am^{-1/2}}}=O(1).$$

Moreover, the sequence $p(m)$ is increasing,

$$\sum_{n=0}^{m}p(n)e^{-ny}\le p(m)\sum_{n=0}^{m}e^{-ny}\le p(m)\frac{A'}{y}=\frac{A'}{A}\sqrt{m}p(m).$$

Summarizing all the inequalities so far, we finally have

$$p(m)\ge\frac{K}{\sqrt{m}}e^{\frac{c}{A}\sqrt{m}}+O(1)\ge e^{c_1m^{1/2}}.$$

7. (a) Let $x = e^{2\pi i u}, q = e^{\pi i u}, z = u/2$. Then

$$\prod_{n=0}^{\infty}(1+x^n)(1-x^{2n+2}) = \prod_{n=1}^{\infty}(1-x^n)(1+x^n)(1+x^{n-1})$$
$$= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}).$$

By **Theorem 1.3** the product equals

$$\sum_{n=-\infty}^{\infty} e^{\pi i n^2 u} e^{2\pi i n(u/2)} = \sum_{n=-\infty}^{\infty} \left(e^{2\pi i u}\right)^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2}.$$

(b) Let $x = e^{2\pi i u}, q = e^{5\pi i u}, z = 1/2 + 3u/2$. Then

$$\prod_{n=0}^{\infty}(1-x^{5n+1})(1-x^{5n+4})(1-x^{5n+5}) = \prod_{n=1}^{\infty}(1-x^{5n})(1-x^{5n-4})(1-x^{5n-1})$$
$$= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}).$$

By **Theorem 1.3** the product equals

$$\sum_{n=-\infty}^{\infty} e^{\pi i n^2 5u} e^{2\pi i n(1/2+3u/2)} = \sum_{n=-\infty}^{\infty} (-1)^n \left(e^{2\pi i u}\right)^{n(5n+3)/2} = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(5n+3)/2}.$$

8. (a) If both a and b are even, then it is contradiction with the fact that a and b have no common factors. If both a and b are odd, then $c^2 \equiv 2 \pmod 4$, which is impossible.

(b) Assume a is odd and b even, and write $b^2 = c^2 - a^2$. Then $c-a$ and $c+a$ are both even, so

$$\left(\frac{b}{2}\right)^2 = \left(\frac{c-a}{2}\right)\left(\frac{c+a}{2}\right).$$

Note that $(c-a)/2$ and $(c+a)/2$ are coprime. Otherwise, $g > 1$ divides $(c-a)/2$ and $(c+a)/2$, then g divides $c = \frac{c+a}{2} + \frac{c-a}{2}$ and $a = \frac{c+a}{2} - \frac{c-a}{2}$, which is contradiction. Therefore there exists integer n, m such that $(c-a)/2 = n^2, (c+a)/2 = m^2$. then

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

(c) If $c = m^2 + n^2$ for some integers m, n, then letting $a = m^2 - n^2$ and $b = 2mn$ to get $a^2 + b^2 = c^2$.

9. (a) Since $d_1(p) = 2$ and $d_3(p) = 0$, $r_2(p) = 4(2-0) = 8$.

(b) q^a has $a+1$ divisors, which are $1, q, q^2, \cdots, q^a$, and the remainders modulo 4 is $1, 3, 1, 3, \cdots$. Hence

$$r_2(q^a) = 4\left(\left(\frac{a}{2}+1\right) - \frac{a}{2}\right) > 0, \qquad \text{when } a \text{ is even, and}$$
$$r_2(q^a) = 4\left(\frac{a+1}{2} - \frac{a+1}{2}\right) = 0, \qquad \text{when } a \text{ is odd.}$$

(c) Note that if q is prime of the form $4k+3$ and $q \mid a^2 + b^2$ then $q \mid a$ and $q \mid b$. Otherwise, if $q \nmid a$, then $1 + \left(a^{-1}b\right)^2 \equiv 0 \pmod q$ and $1 = \left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}} = -1$, which is contradiction.

Suppose n can be represented as the sum of two squares, $n = a^2 + b^2$. If there is prime $q \mid n$ of the form $4k+3$, then $q \mid a, b$ implies that $q^2 \mid a^2 + b^2 = n$. Therefore $(n/q)^2 = (a/q)^2 + (b/q)^2$. Repeating this process, we can see that every prime $q = 4k+3$ occur with even exponents.

Now suppose all the primes of the form $4k+3$ that arise in the prime decomposition of n occur with even exponents. Observe that

$$
\begin{aligned}
2 &= 1^2 + 1^2, \\
p &= a^2 + b^2, \quad \text{where } p \text{ is prime of the form } 4k+1 \text{ (by (a))}, \\
p^2 &= p^2 + 0^2, \quad \text{where } p \text{ is prime of the form } 4k+3.
\end{aligned}
$$

Since n is product of numbers above, and $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$, n is also sum of two squares.

10. (a) If q is prime of the form $4k+3$ and a is odd, then $r_2(q^a) = 0$. Meanwhile, $n = 5^k$ then $r_2(n) = 4(d_1(5^k) - d_3(5^k)) = 4(k+1-0) = 4k+4$, hence $\limsup_{n\to\infty} r_2(n) = \infty$.

(b) If $n = 2^k$, then $r_4(n) = 8\sigma^*(n) = 8(1+2) = 24$. Now define $n_k = (p_1 p_2 \cdots p_k)^k$, where p_j denotes j-th prime. Then

$$
\frac{\sigma^*(n_k)}{n_k} = \frac{\sigma(n_k)}{n_k} = \left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_1^k}\right) \cdots \left(1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^k}\right) \geq \sum_{i=1}^{k} \frac{1}{i},
$$

hence $\limsup_{n\to\infty} r_4(n)/n = \infty$.

11. Since $z/(1-z) = \sum_{n=1}^{\infty} z^n$,

$$
\sum_{n=1}^{\infty} \frac{n^l z^n}{1-z^n} = \sum_{n=1}^{\infty} n^l \sum_{m=1}^{\infty} z^{nm} = \sum_{k=1}^{\infty} \sum_{n|k} n^l z^k = \sum_{k=1}^{\infty} \sigma_l(k) z^k.
$$

12. (a) For $|q| < 1$,

$$
\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_1(n) q^n = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m q^{nm} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}.
$$

(b) Note that

$$
\sigma_1^*(n) = \begin{cases} \sigma_1(n) & \text{if } n \text{ is not divisible by 4}, \\ \sigma_1(n) - 4\sigma_1(n/4) & \text{if } n \text{ is divisible by 4}. \end{cases}
$$

Therefore

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{4nq^n}{1-q^{4n}} &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 4\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} \\
&= \sum_{n=1}^{\infty} \sigma_1(n) q^n - 4 \sum_{4|n} \sigma_1\left(\frac{n}{4}\right) q^n \\
&= \sum_{n=1}^{\infty} \sigma_1^*(n) q^n.
\end{aligned}
$$

(c) Left hand side is equal to $\sum_{n=0}^{\infty} r_4(n) q^n$. Observe that right hand side is

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^n}{(1+(-1)^n q^n)^2} &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} + \sum_{n=1}^{\infty} \left(\frac{q^{2n}}{(1+q^{2n})^2} - \frac{q^{2n}}{(1-q^{2n})^2} \right) \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 4\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} \\
&= \sum_{n=1}^{\infty} \sigma_1^*(n) q^n.
\end{aligned}
$$

Hence the identity is equivalent to

$$\sum_{n=0}^{\infty} r_4(n)q^n = 1 + 8\sum_{n=1}^{\infty} \sigma_1^*(n)q^n,$$

which is also equivalent to four-squares theorem $r_4(n) = 8\sigma_1^*(n)\ (n \geq 1)$.