lecture 15 (Frobenius Series Solution)

Recall $y'' + p(x)y' + q(x)y = 0$.

- $x_0 = 0$ is called regular singular point if $xp(x)$ & $x^2q(x)$ are analytic at $x_0 = 0$

$\Longleftrightarrow \quad p(x) = \dfrac{b(x)}{x} \qquad q(x) = \dfrac{c(x)}{x^2}$

where $b(x)$, $c(x)$ analytic at $x_0 = 0$.

$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$

- Frobenius series,

$$y(x) = x^r \sum a_n x^n$$

- Indicial equation:

$$p(r) : r(r-1) + b_0 r + c_0 = 0$$

$$r = r_1, r_2.$$

$\boxed{\text{ASSUME}}$ $r_1, r_2 \in \mathbb{R}$ $\quad r_1 \geq r_2$.

- For $r = r_1$ we have a solution

$$y_1(x) = x^{r_1} \sum a_n(r_1) x^n.$$

Frobenius

- $r_1 - r_2 \notin \mathbb{Z}$, then $\exists$ another solution

$$y_2(x) = x^{r_2} \sum a_n(r_2) x^n.$$

- $r_1 - r_2 \in \mathbb{Z}$. Find y_2 by reduction

ie $y_2 = v y_1 \qquad v' = \dfrac{1}{y_1^2} e^{-\int p(x)\,dx}.$

Example $\quad xy'' + 2y' + xy = 0$.

$$p(x) = \frac{2}{x} = \frac{b(x)}{x} \quad \Big| \quad q(x) = 1 = \frac{c(x)}{x^2}$$

- $x_0 = 0$ is a regular singular point

$$x\,p(x) = 2 \qquad x^2 q(x) = x^2$$

are analytic at $x_0 = 0$

$$\boxed{\begin{aligned} b(x) &= 2 \\ c(x) &= x^2 \\ b_0 &= 2 \\ c_0 &= 0 \end{aligned}}$$

- Indicial equation:

$\rho(r)$: $r(r-1) + b_0 r + c_0 = 0$

$$\rho(r) = r(r+1) = 0$$

$$r = 0, -1$$

So $r_1 = 0 \qquad r_2 = -1$

At least one Frobenius Series solution exist.

Find the actual solution

$$y(x) = \sum a_n x^{n+r}. \quad (a_0 \neq 0).$$

$$x^r \Big[\underbrace{\rho(r) a_0}_{=0} + \underbrace{\rho(r+1) a_1}_{=0} x + \underbrace{\sum_{n=2}^{\infty} \big(\rho(n+r) a_n + a_{n-2}\big) x^n}_{=0} \Big] = 0$$

$$\rho(r+1) a_1 = 0$$

$$\rho(n+r) a_n + a_{n-2} = 0 \qquad n \geq 2$$

$\boxed{r = r_1 = 0}$ $\quad \rho(r_1+1) a_1 = 0 \Rightarrow a_1 = 0$

$$a_n = -\frac{a_{n-2}}{n(n+1)}. \qquad n \geq 2.$$

$$\Rightarrow a_{2n+1} = 0 \qquad a_{2n} = (-1)^n \frac{1}{\lfloor 2n+1} a_0.$$

$$y_1(x) = \left(1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \frac{x^6}{\lfloor 7} + \cdots\right) = \frac{\sin x}{x}.$$

$r_1 - r_2 = 0 - (-1) = 1 \in \mathbb{Z}$.

So second solution

$$y_2(x) = v \cdot y_1.$$

$$v' = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

$$- \quad - \quad - \quad . \quad . \quad -$$

$$= \frac{1}{\sin^2 x}.$$

$$v = -\cot x.$$

$$y_2(x) = -\cot x \cdot \frac{\sin x}{x} = -\frac{\cos x}{x}.$$

More example: Prof Shaki Ghorai lecture notes.

Gamma function

For $p>0$.

$$\Gamma(p) := \int_0^\infty e^{-t} t^{p-1}\, dt$$

$$:= \lim_{T\to\infty} \int_0^T e^{-t} t^{p-1}\, dt.$$

- This limit exist (ie the improper integr. converge)

- $\Gamma(p+1) = p\,\Gamma(p).$

$$\Gamma(p+1) = \int_0^\infty e^{-t} t^{p}\, dt$$

$$= -\cancel{e^{-t}} t^p \Big|_0^\infty + p\int_0^\infty e^{-t} t^{p-1}\, dt$$

$$= p\,\Gamma(p).$$

- $\Gamma(1) = 1$

- m positive integer.

$$\Gamma(m+1) = m\,\Gamma(m) = m(m-1)\,\Gamma(m-1)$$

$$- - - -$$

$$= m(m-1) \cdots 1\, \underbrace{\Gamma(1)}_{1}$$

$$= \lfloor m.$$

- $p<0$. $-N < p < -N+1$ N, +ve integer

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} = \frac{\Gamma(p+2)}{(p)(p+1)}$$

$$- -.$$

$$= \frac{\Gamma(p+N)}{p(p+1)\cdots(p+N-1)}$$

- $\Gamma(p)$ is not defined for $p=0$ & for negative integers.

- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\boxtimes$

Bessel equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad \nu \geq 0.$$

$$p(x) = \frac{1}{x} = \frac{b(x)}{x} \qquad q(x) = \frac{x^2 - \nu^2}{x^2} = \frac{c(x)}{x^2}$$

$$\boxed{\begin{aligned} b(x) &= 1 \\ c(x) &= x^2 - \nu^2 \\ b_0 &= 1 \\ c_0 &= -\nu^2 \end{aligned}}$$

- $x_0 = 0$ regular singular point.
- Indicial equation

$$r(r-1) + b_0 r + c_0 = 0$$

$$\rho(r) = r^2 - \nu^2 = 0$$

$$r = \nu, -\nu.$$

$$r_1 = \nu \qquad r_2 = -\nu.$$

Frobenius Series solution for $r = r_1 = \nu$

$$y = x^r \sum a_n x^n$$

Substituting in the given equation

x^r: $\rho(r) = 0$

x^{r+1}: $\rho(r+1)\, a_1 = 0$

x^{n+r}: $\rho(n+r)\, a_n = -a_{n-2} \qquad n \geq 2.$

$\boxed{r = r_1 = \nu}$

$$\rho(r_1+1)\, a_1 = 0 \Rightarrow a_1 = 0$$

$$a_n = -\frac{a_{n-2}}{n(n+2\nu)}$$

$$\boxed{\begin{aligned} &\rho(n+\nu) \\ &= (n+\nu)^2 - \nu^2 \\ &= n(n+2\nu) \end{aligned}}$$

$a_{2n+1} = 0 \quad \forall n.$

$$a_{2n} = \frac{(-1)^n}{2^{2n} \lfloor n \, (\nu+1)(\nu+2) \cdots (\nu+n)}.$$

$$y_1(x) = a_0 \, x^{\nu}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} \lfloor n \, (\nu+1)(\nu+2) \cdots (\nu+n)}\right)$$

It is convenient to take

$$a_0 = \frac{1}{2^{\nu}\,\Gamma(\nu+1)}$$

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lfloor n \, \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

— Bessel function.

To find the second soln —

If $r_1 - r_2 = 2\nu \notin \mathbb{Z}$

Then second Frobenius series sol. exist for $r_2 = -\nu$. This is given by

$$J_{-\nu}(x) = \sum \frac{(-1)^n}{\lfloor n \, \Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}.$$

Properties

- $\nu = m$ positive integer. Then

$$J_{-m} = (-1)^m J_m$$

- $\left(x^{\nu} J_{\nu}(x)\right)' = x^{\nu} J_{\nu-1}(x)$

$$\left(x^{-\nu} J_{\nu}(x)\right)' = -x^{-\nu} J_{\nu+1}(x)$$

$$J_{\nu-1} - J_{\nu+1} = 2 J_{\nu}'$$

$\boxed{}$

Orthogonality relation

$J_\nu(a) = 0 = J_\nu(b)$

$$\int_0^1 x\, J_\nu(ax)\, J_\nu(bx)\, dx = \begin{cases} 0 & a \neq b \\ \frac{1}{2} J_{\nu+1}^2(a) & a = b \end{cases}$$

Proof $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$

$\rightsquigarrow$ solution $J_\nu(x)$.

$$x (x y')' + (x^2 - \nu^2) y = 0$$

Replace x by ax.

$$ax \frac{d}{d(ax)} \left(ax \frac{d}{d(ax)}(y) \right) + (a^2 x^2 - \nu^2) y = 0$$

$$x (x y')' + (a^2 x^2 - \nu^2) y = 0$$

$\rightsquigarrow$ solution $J_\nu(ax)$

$$x (x y')' + (b^2 x^2 - \nu^2) y = 0$$

$\rightsquigarrow$ solution $J_\nu(bx)$.

Multiply 1st equation by $J_\nu(bx)$ & the second equation by $J_\nu(ax)$ & subtract

$$\frac{d}{dx}\left(x J_\nu(bx) J_\nu'(ax) - x J_\nu(ax) J_\nu'(bx) \right)$$

$$+ (a^2 - b^2)\, x\, J_\nu(ax)\, J_\nu(bx) = 0$$

Integrate from $x = 0$ to 1.

$$(a^2 - b^2) \int_0^1 x\, J_\nu(ax)\, J_\nu(bx)\, dx = 0$$

$a \neq b$

$$\Rightarrow \int_0^1 x\, J_\nu(ax)\, J_\nu(bx)\, dx = 0$$

$\boxtimes$

Orthogonality for Legendre Polynomials

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0 & m \neq n \\ \dfrac{2}{2n+1} & n = n \end{cases}$$