Lecture 7 : Linear span of vectors, Linearly Independent & Basis vectors

Let $x = (x_1, x_2, x_3)$ & $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The linear span of x & y is

$$L(x, y) = \{\alpha x + \beta y : \alpha, \beta \in \mathbb{R}\}$$

If $x = (1, 0, 0)$ & $y = (0, 1, 0)$ then
$L(x, y) = \{(\alpha, \beta, 0) : \alpha, \beta \in \mathbb{R}\}$, it is the xy-plane.
We will define linear span more generally for vectors in a vector space V.

Let V be a vector space over K ($K = \mathbb{R}$ or $\mathbb{C}$)

Linear span: Let $S = \{v_1, \ldots, v_n\}$ be a subset of vectors in V. Linear span of S, denoted by $L(S)$, is the set

$$L(S) = \{\alpha_1 v_1 + \cdots + \alpha_n v_n : \alpha_1, \ldots, \alpha_n \in K\}$$

The expression $\alpha_1 v_1 + \cdots + \alpha_n v_n$ is called linear combination of vectors $v_1, \ldots, v_n$ by $\alpha_1, \ldots, \alpha_n$.

Proposition :- $L(S)$ is a subspace of V containing S and it is the smallest subspace of V containing.

Proof: First part is easy to verify.

$$v_i = 0 \cdot v_1 + \cdots + 1 \cdot v_i + \cdots + 0 \cdot v_n \in L(S)$$

$$\Rightarrow S \subseteq L(S)$$

Let W be a subspace of V containing S. This implies $v_i \in W \ \forall i \in \{1, \ldots, n\}$

By definition of W, any linear combination of v_i's belongs to W. Hence $L(S) \subseteq W$.

Linear Dependent (L.D.) vectors.

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ of V is said to be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that $\sum_{i=1}^{n} \alpha_i \cdot v_i = 0$

Example (i) $\{(1,1,1), (3,3,3)\}$ is L.D.

$$(-3) \cdot (1,1,1) + 1 \cdot (3,3,3) = (0, 0, 0)$$

(iii) $\{(1,0), (0,1), (2,3)\}$ is L.D. in $\mathbb{R}^2$.

$$2(1,0) + 3(0,1) + (-1)(2,3) = (0,0)$$

Linearly Independent (L.I.) set of vectors

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to linearly independent if it is not linearly dependent i.e

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \ \forall\, i \in \{1, \ldots, n\}$$

Example: $\{(1,0,0), (0,1,0)\}$ is L.I.

$$\alpha_1 (1,0,0) + \alpha_2 (0,1,0) = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, 0) = (0,0,0)$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2$$

The following properties are easy to check: Let S be a finite set of vectors in V.

☆ S is L.I. $\Rightarrow 0 \notin S$

☆ S is L.I. $\Rightarrow$ for any $S' \subseteq S$, S' is also L.I.

☆ S is L.D. $\Rightarrow$ for any S' containing S, S' is L.D.

Linear Dependence, Linear Independence for infinite set of vectors.

Let V be a vector space and S be an infinite subset of vectors in V.
S is said to L.D. if there exists a finite subset F of S such that F is L.D.
S is said to L.I. if for all subset S' of S, S' is L.I.

Example: Let $P(x)$ = set of all single variable polynomials with coeffecient from $\mathbb{R}$.
Then $P(x)$ is a vector space.
Consider the set $S = \{x^n : n \geq 1\}$
This set S is linearly independent.

Basis Vectors

A subset S of a vector space V is said to be basis of V if

(i) S is linearly independent (L.I.)

(ii) S spans V i.e. $L(S) = V$

i.e for every $v \in V$ there exists finitely many vectors $e_1, e_2, \ldots, e_n$ in S such that $v = \sum_{i=1}^{n} \alpha_i e_i$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(every element of V can be written as finite linear combination of elements from S)

Example (i) Consider $\mathbb{R}^n$ & $e_i = (0, \ldots 0, \underset{\substack{\downarrow \\ i\text{th position}}}{1}, 0, \ldots 0)$

The set $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

(ii) $\{(1,1), (1,2)\}$ is a basis of $\mathbb{R}^2$.

$$c_1(1,1) + c_2(1,2) = (0,0)$$
$$\Rightarrow (c_1 + c_2, c_1 + 2c_2) = (0,0)$$
$$\Rightarrow c_1 + c_2 = 0 \ \& \ c_1 + 2c_2 = 0$$
$$c_1 = -c_2 \quad \& \quad c_1 = -2c_2$$
$$\Rightarrow c_2 = 0 \ \& \ c_1 = 0$$

Hence $\{(1,1), (1,2)\}$ is L.I.

Let $(x,y) \in \mathbb{R}^2$, $(x,y) = \alpha(1,1) + \beta(1,2)$

$$\Rightarrow \alpha + \beta - x = 0$$
$$\alpha + 2\beta - y = 0$$

$$\frac{\alpha}{2x-y} = \frac{\beta}{y-x} = \frac{1}{1}$$

$$\Rightarrow \alpha = 2x - y, \quad \beta = y - x$$

$\therefore$ $(x,y) = (2x-y)(1,1) + (y-x)(1,2)$

So, $L(\{(1,1),(1,2)\}) = \mathbb{R}^2$.

(iii) Check that $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for the vector space $P(X)$ of polynomials.

Proposition: Let V be a vector space & S be a basis of V, then every vector $v \in V$ can be expressed uniquely as finite linear combinations of elements of S.

Proof: Suppose $v = \alpha_1 e_{i_1} + \cdots + \alpha_k e_{i_k}$

$$\& \quad v = \beta_1 e_{j_1} + \cdots + \beta_r e_{j_r}$$

$$\Rightarrow \alpha_1 e_{i_1} + \cdots + \alpha_k e_{i_k} - \beta_1 e_{j_1} \cdots - \beta_r e_{j_r} = 0$$

Note that here it may be possible that $\{i_1, i_2, \cdots i_k\} \cap \{j_1, j_2, \cdots, j_r\} \neq \emptyset$.

As $\{e_{i_1}, \cdots, e_{i_k}, e_{j_1}, e_{j_2}, \cdots, e_{j_r}\}$ is L.I.

Either $\alpha_p = 0$ or $\beta_q = 0$ or for $s \in \{i_1, \cdots, i_k\} \cap \{j_1, \cdots, j_r\}$ $\alpha_s = \beta_s$.

Hence, we have the required result. $\square$