lecture 13 (Power Series & ODE)

Recall:

Power series : $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

Analytic function : • $f(x)$ is analytic at x_0 if $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for x near x_0

• f analytic on $I \subseteq \mathbb{R}$ if f analytic at x_0 $\forall$ $x_0 \in I$.

Examples (i) $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

— polynomials are analytic on $\mathbb{R}$

(ii) $\sin x$ / $\cos x$ / e^x — — — — .

(iii) $\frac{1}{1-x}$ analytic on $\mathbb{R} - \{1\}$

(iv) f analytic at x_0 + $f(x_0) \neq 0$

$\Rightarrow$ $\frac{1}{f}$ analytic at x_0.

(v) $\frac{1}{1-x^2}$ analytic on $\mathbb{R} \setminus \{\pm 1\}$

$\boxed{\text{Theorem}}$

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad (**)$$

- If $p(x)$ & $q(x)$ are analytic at x_0, then above $(**)$ admits two independent solutions of the form $\sum_n a_n (x-x_0)^n$

- If the power series of p, q are conv for $|x-x_0| < R$, then so is the solution $\sum a_n (x-x_0)^n$.

Example $\quad y'' + y = 0$

$p(x) = 0 \qquad q(x) = 1.$

- analytic on $\mathbb{R}$, in particular at $x_0 = 0$

Assume solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y'(x) = \sum_{n=1}^{\infty} a_n\, n\, x^{n-1}$$

$$= \sum_{n=0}^{\infty} a_{n+1} (n+1)\, x^n$$

$$y''(x) = \sum_{n=2}^{\infty} a_n\, n(n-1)\, x^{n-2}$$

$$= \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)\, x^n$$

Substitute in the given equation

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Compare the coff of x^n on both side

$$(n+2)(n+1) a_{n+2} + a_n = 0 \qquad n = 0, 1, 2, \ldots$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

$\underline{n=0}$ $\quad a_2 = -\dfrac{a_0}{\lfloor 2}$

$\underline{n=1}$ $\quad a_3 = -\dfrac{a_1}{\lfloor 3}$

$\underline{n=2}$ $\quad a_4 = -\dfrac{a_2}{4 \cdot 3} = \dfrac{a_0}{\lfloor 4}$

$\underline{n=3}$ $\quad a_5 = \dfrac{a_1}{\lfloor 5}$

$$a_6 = -\frac{a_0}{\lfloor 6}$$

$$a_7 = -\frac{a_1}{\lfloor 7}$$

So solution

$$y(x) = a_0 \underbrace{\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right)}$$

$$+ a_1 \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

② $(1-x^2)\,y'' - 2x\,y' + \alpha(\alpha+1)\,y = 0$

(Legendre eqution) $\alpha \in \mathbb{R}$ - const.

$$p(x) = -\frac{2x}{1-x^2} \qquad q(x) = \frac{\alpha(\alpha+1)}{1-x^2}$$

$= -2x(1+x^2+x^4+x^6---)$ $|x|<1$

These functions are analytic on $\mathbb{R} \setminus \{\pm 1\}$.

In particular analytic at $x_0 = 0$.

So assume solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n$$

$$y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n.$$

Substituting in the given equation.

$$(1-x^2)\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n - 2x \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n + \alpha(\alpha+1)\sum a_n x^n = 0$$

Comparing the coeff of x^n on both side

$$(n+1)(n+2)\, a_{n+2} - (n-1)\cdot n\, a_n - 2n\, a_n + \alpha(\alpha+1)\, a_n = 0$$

$n = 0, 1, 2 ---$

$$\Rightarrow a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)}\, a_n.$$

$$a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)}\, a_n.$$

$$y = \sum a_n x^n$$

- $\underline{n=0}$ $\qquad a_2 = -\dfrac{\alpha(\alpha+1)}{2!}\, a_0$
- $\underline{n=1}$ $\qquad a_3 = -\dfrac{(\alpha-1)(\alpha+2)}{3!}\, a_1$

$\underline{n=2}$ $\qquad a_4 = \dfrac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}\, a_0$

$\underline{n=3}$ $\qquad a_5 = -\dfrac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}\, a_1$

$\underline{n=4}$ $\qquad a_6 = -\dfrac{\alpha(\alpha-2)(\alpha-4)(\alpha+1)(\alpha+3)(\alpha+5)}{6!}\, a_0$

Solution

$$y(x) = a_0 \underbrace{\left(1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 - \frac{[\alpha(\alpha-2)(\alpha-4)][(\alpha+1)(\alpha+3)(\alpha+5)]}{6!}x^6 - - - \right)}_{y_1}$$

$$+\; a_1 \underbrace{\left(x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{[(\alpha-1)(\alpha-3)][(\alpha+2)(\alpha+4)]}{5!}x^5 - - - \right)}_{y_2}$$

$y_1(0) = 1 \qquad y_1'(0) = 0$

$y_2(0) = 0 \qquad y_2'(0) = 1 \qquad W(y_1, y_2)(0) = 1 \neq 0$

- $y_1(x)$ & $y_2(x)$ convg for $|x|<1$.

Observe.

- If α = even positive integer $= 2m$
$\Rightarrow y_1(x)$ polynomial of degree $2m$

$$y_1 = \begin{cases} 1 & \alpha = 0 \\ 1-3x^2 & \alpha = 2 \\ 1-10x^2+\frac{35}{3}x^4 & \alpha = 4 \end{cases}$$

- If α = odd positive integer $= 2m+1$
$\Rightarrow y_2(x)$ polynomial of degree $2m+1$

$$y_2 = \begin{cases} x & \alpha = 1 \\ x-\frac{5}{3}x^3 & \alpha = 3 \\ x-\frac{14}{3}x^3+\frac{21}{5}x^5 & \alpha = 5 \end{cases}$$

Summary. Given a positive integer n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (**)$$

has a polynomial solution of degree n.

Definition (Legendre Polynomials)
The polynomial solution $P_n(x)$ of $(**)$ with $P_n(1)=1$ are called the Legendre polynomials.

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$ - -

Theorem

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}\left((x^2-1)^n\right) = \frac{1}{2^n \lfloor n} V(x)$$

Proof R.H.S = polynomial of degree n.

To show

- R.H.S - satisfy Legendre equation
- R.H.S - satisfy the normalisation condition ie

$$\text{R.H.S}(1) = 1.$$

$$u(x) = (x^2-1)^n$$

$$u'(x) = n(x^2-1)^{n-1}\, 2x$$

$$(1-x^2)u' + 2nx\,u = 0$$

Diff this once more.

$$(1-x^2)u'' + 2x(n-1)u' + 2nu = 0$$

Diff this n-times & simplify.

$$(1-x^2)u^{(n+2)} - 2x\,u^{(n+1)} + n(n+1)u^{(n)} = 0$$

$$\boxed{(A\cdot B)^{(n)} = A^{(n)}B + {}^nC_1 A^{(n-1)}B^{(1)} + {}^nC_2 A^{(n-2)}B^{(2)} + \dots + B^{(n)}}$$

$$V(x) = u^{(n)}(x)$$

V satisfy Legendre equation (**).

$$\Rightarrow V(1) = \lfloor n\, 2^n$$

$$V = \frac{d^n}{dx^n}(x^2-1)^n = \frac{d^n}{dx^n}\Big[\underbrace{(x-1)^n}_{A}\underbrace{(x+1)^n}_{B}\Big] = \lfloor n\,(x+1)^n + (\dots\dots)(x-1)$$

Then $\frac{V(x)}{\lfloor n\, 2^n}$ satisfy Legendre equation & also the normalisation condition.