# Lecture I

Introduction, concept of solutions, application

**Definition 1.** *A differential equation (DE) is a relation that contains a finite set of functions and their derivatives with respect to one or more independent variables.*

**Definition 2.** *An ordinary differential equation (ODE) is a relation that contains derivatives of functions of only one variable.*

If the number of functions is one, then it is called simple or scalar ODE. Otherwise, we have a system of ODEs.

**Note:** ODEs are distinguished from partial differential equations (PDEs), which contain partial derivatives of functions of more than one variable.

From now on, we shall mostly deal with simple or scalar ODE.

**Definition 3.** *The highest derivative that appear in a ODE is the order of that ODE.*

**Notations:** A general simple ODE of order $n$ can be written as

$$F\left(x, y(x), y^{(1)}(x), y^{(2)}(x), \cdots, y^{(n)}(x)\right) = 0. \tag{1}$$

If the order of the ODE is small, then we shall use $'$ for the derivative. For example, a second order ODE is written as

$$F(x, y, y', y'') = 0.$$

**Definition 4.** *If the ODE (1) can be written as*

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y^{(1)}(x) + a_n(x)y(x) = F(x), \tag{2}$$

*then the given ODE is linear. If such representation is not possible, then we say that the given ODE is nonlinear.*

If the function $F(x)$ is zero, then (2) is called linear homogeneous ODE. If the functions $a_0(x), a_1(x), \cdots, a_n(x)$ are constants, then (2) is called linear ODE with constant coefficients. Similarly, if the functions $a_0(x), a_1(x), \cdots, a_n(x)$ are constants and $F(x)$ is zero, then (2) is called linear homogeneous ODE with constant coefficients.

**Example 1.** *Both* $y'' + y^2 = 2$ *and* $y'' + \cos y = 0$ *are nonlinear, whereas both* $y'' + e^x y = x^2$ *and* $e^x = y'/(\cos x + y'')$ *are linear.*

**Example 2.** *A pendulam is released from rest when the supporting string is at angle* $\alpha$ *to the vertical. If the air resistence is negligible, find the ODE describing the motion of the pendulam.*

**Solution**: Let $s$ be the arc length measured from the lowest point as shown in Figure 1. If $l$ is the length of the string, then $s = l\theta$. Now the velocity $v$ and accleration $a$ is given by

$$v = \frac{ds}{dt} = l\frac{d\theta}{dt}, \qquad a = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

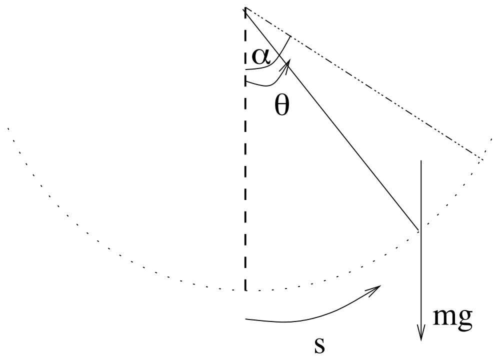


Figure 1: Motion of a pendulam.

Resolving the gravity force, we find

$$\frac{d^2\theta}{dt^2} + \kappa \sin\theta = 0, \tag{3}$$

where $\kappa = g/l$. Note that (3) is of second order and nonlinear. Usually, the nonlinear ODEs are very difficult to solve analytically.

**Example 3.** *Consider a pond in which only two types of fishes, A and B, live. Fish A feeds on phytoplankton whereas fish B survives by eating A. We are interested to model the populations growths of both the fishes. A model for this kind of Predator-Prey interaction is popularly known as the Lotka–Volterra system.*

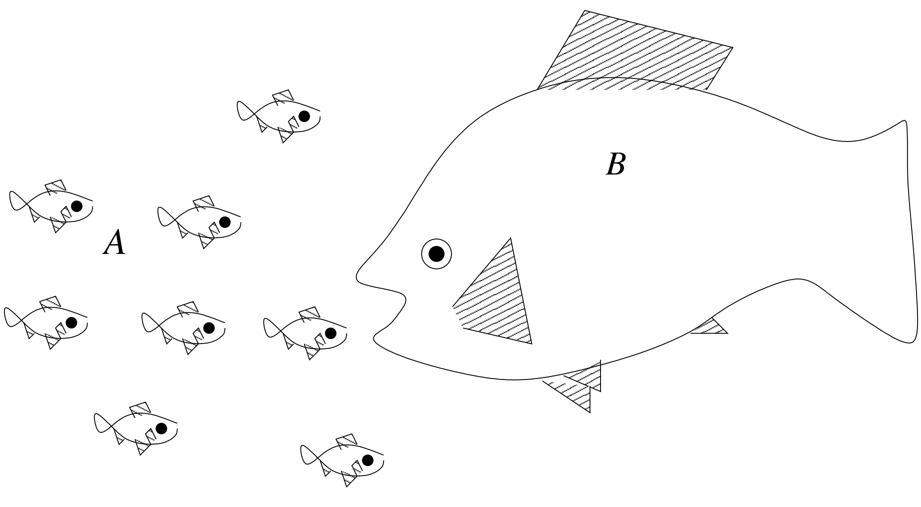


Figure 2: Predator-prey interactions

**Solution**: In absence of B, fish population A will increase and the rate of increase is proportional to number density of A. Similarly, in absence of A, fish population B will decrease and the rate of decrease is proportional to the number density of B. When both are present, $A$ will decrease and $B$ will increase. This rate of change, as a first approximation, can be taken as proportional to the product of number density of A and B. Thus, we get the following system of ODEs:

$$\begin{aligned}\frac{dA}{dt} &= \alpha A - \beta AB\\ \frac{dB}{dt} &= -\gamma B + \delta AB,\end{aligned}$$

where $\alpha, \beta, \gamma$ and $\delta$ are positive constants.

**Definition 5.** *A functon $\phi$ is said to be the solution of the ODE (1) in an interval $\mathcal{I}$ if $\phi \in \mathcal{C}^n(\mathcal{I})$ and*

$$F\Big(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \cdots, \phi^{(n)}(x)\Big) = 0, \qquad \forall x \in \mathcal{I}.$$

[$\mathcal{C}^n(\mathcal{I})$ denotes the set of functions having derivatives up to the order $n$ continuous in the interval $\mathcal{I}$]

**Example 4.** *$y = x^2 + c,\ x \in (-\infty, \infty)$ (c is an arbitray constant) is a solution to $y' = 2x$. On the other hand $y = 1/(1-x),\ x \in (-\infty, 1)$ or $x \in (1, \infty)$ is a solution to $y' = y^2$. Note that the last solution is not valid in any interval which contains $x = 1$.*

In general, we are interested to know whether a given ODE, under certain circumstances, has a unique solution or not. Usually, this question can be answered with some extra conditions such as initial conditions. Thus, we consider the so called *initial value problem* (IVP) which takes the following form for a first order ODE:

$$\left.\begin{aligned} &y'(x) = f\Big(x, y(x)\Big), \quad x \in \mathcal{I},\\ &y(x_0) = y_0, \quad x_0 \in \mathcal{I},\ y_0 \in \mathcal{J}, \end{aligned}\right\} \tag{4}$$

where $\mathcal{I}$ and $\mathcal{J}$ are some intervals.

For higher order ODE, the initial conditions are also applied on the lower derivatives of the function. For example, the IVP for Example 2 can be written as

$$\left.\begin{aligned} &\frac{d^2\theta}{dt^2} + \kappa \sin\theta = 0,\\ &\theta(0) = \alpha, \quad \theta'(0) = 0. \end{aligned}\right\} \tag{5}$$

Note that here, the notation $\theta(0)$ means $\theta(t = 0)$ and similar is the case for $\theta'(0)$.

**Definition 6.** *A* general solution *of an ODE of order $n$ is a solution which is given in terms of $n$ independent parameters. A* particular solution *of an ODE is a solution which is obtained from the general solution by choosing specific values of independent parameters.*

**Example 5.** *(i) Consider $y' = 2x$. Clearly $y = C + x^2$ is the general solution. If we choose $C = 2$, then $y = 2 + x^2$ is a particular solution. This particular solution can be thought of as the solution of the IVP $y' = 2x,\ y(0) = 2$ **OR** $y' = 2x,\ y(1) = 3$ **OR** $\cdots$. Here the particular solution is valid for all $x$.*

*(ii) Consider $y'' - 4y' + 4y = 0$. For this, $y(x) = (C_1 + C_2 x)e^{2x}$ is the general solution, whereas $y_p(x) = xe^{2x}$ is a particular solution obtained by choosing $C_1 = 0, C_2 = 1$. The conditions on $C_1$ and $C_2$ are equivalent to initial conditions $y(0) = 0, y'(0) = 1$. This particular solution is valid for all $x$.*

*(iii) Clearly $y = 1/(C - x)$ is a genreal solution to $y' = y^2$. Now $y = 1/(1 - x)$ is a particular solution to the IVP $y' = y^2, y(0) = 1$, which is valid in $(-\infty, 1)$. But $y = -1/(1 + x)$ is a particular solution to the IVP $y' = y^2, y(0) = -1$, which is valid in $(-1, \infty)$.*

*Why the solution $y = -1/(1 + x)$ of $y' = y^2, y(0) = -1$ is not valid in $(-\infty, -1)$?*

**Comment**: *An ODE may sometimes have an additional solution that can not be obtained from the general solution. Such a solution is called singular solution.*

**Example 3**: $y'^2 - xy' + y = 0$ has the general solution $y = cx - c^2$ (verify!). It also has a solution $y_s(x) = x^2/4$ that cannot be obtained from the general solution by choosing specific values of $c$. Hence, the later solution is a singular solution.

**Comment**: The solutions $y(x)$ of an ODE often are given *implicitly*. For example, $xy + \ln y = 1$ is an implicit solution of the ODE $(xy + 1)y' + y^2 = 0$.

**Application**: A stone of mass $m$ is dropped from a height $H$ above the ground. Suppose the air resistence is proportional to the speed Derive the ODE that describe the equations of motion.

**Solution**: Suppose the distance to the ball is measured from the ground. Now the velocity $v = dy/dt$ is positive upward. The air resistence acts in the opposite direction to the motion. Hence,

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}, \qquad y(0) = H, y'(0) = 0. \tag{6}$$

Note that the gravity and the air resistence both oppose the motion if the stone is traveling up. On the other hand, gavity aids and resistence opposes if the stone is traveling down. (What will be the form of the above IVP if the distance $y$ is measured from height $H$ in the downward direction?)

Clearly, (6) is a linear second order equation with constant coefficients. This can be solved using the method to be developed later. Using $v = dy/dt$, (6) becomes a first

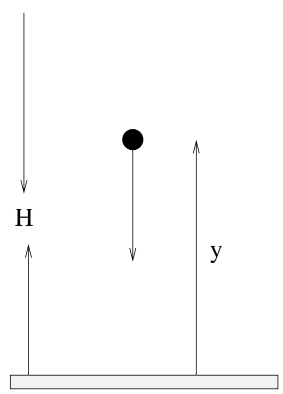


Figure 3: Motion of a falling stone.

order ODE:

$$\frac{dv}{dt} + \alpha v = -g, \qquad v(0) = 0, \tag{7}$$

where $\alpha = k/m$. This equation can be solved easily for $v$. Once, $v$ is found, $y$ is obtained from

$$\frac{dy}{dt} = v, \qquad y(0) = H.$$

Even without solving, we can derive some important informations. For example, the stone will attain a terminal velocity $v_t = -g/\alpha$ (why negative sign?) provided $H$ is large enough. (Is it possible for a stone to attain terminal velocity when it is moving up?)

**Comment**: Care must be taken if the air resistence is proportional to square of the speed. Using the same coordinate system, the governing equation becomes

$$m\frac{d^2y}{dt^2} = -mg + k\left(\frac{dy}{dt}\right)^2, \qquad y(0) = H, y'(0) = 0. \tag{8}$$

The terminal velocity in this case becomes $v_t = -\sqrt{g/\alpha}$.

# Lecture II

## Geomterical interpretation

Here we concentrate on a first order ODE of the form

$$y' = f(x, y). \tag{1}$$

From calculas, we know that $y'$ is the slope of the curve $y(x)$ at $x$. Hence, if (1) has a solution curve passing through the point $(x_0, y_0)$, then the slope of that curve at $(x_0, y_0)$ is $f(x_0, y_0)$. Thus, the value of $f(x, y)$ at each point gives the slope of the solution curve passing through that point.

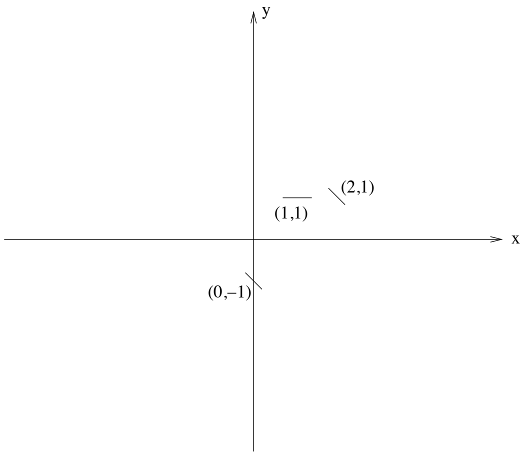


Figure 1: Some lineal elements for $y' = y - x$.

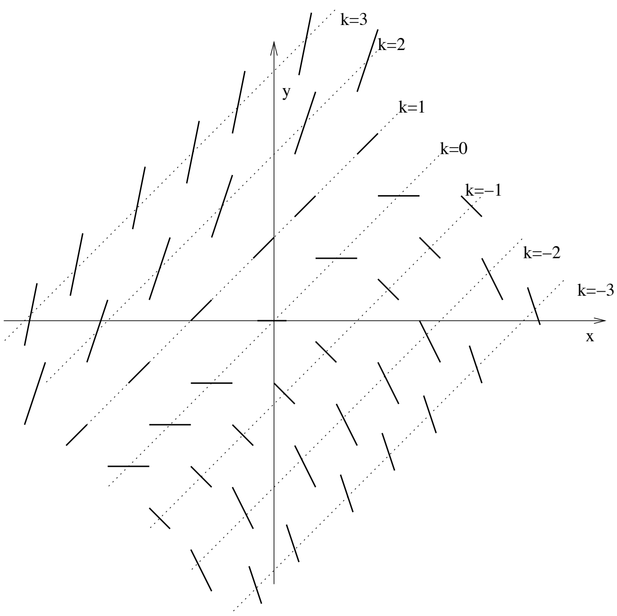


Figure 2: Different isoclines for $y' = y - x$.

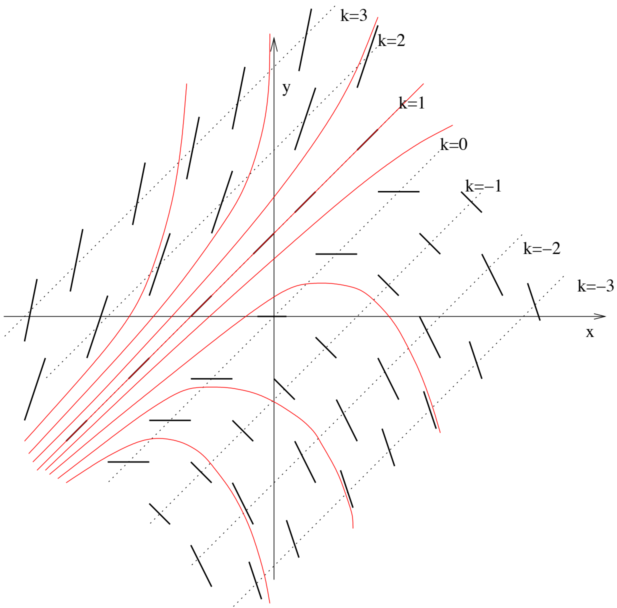


Figure 3: Approximate solution curves (in red) for $y' = y - x$.

Given the differential equation (1), we indicate the slope $f(x, y)$ by a short line segments called lineal elements. For example, lineal elements at three points, for $y' = y - x$, are shown in Figure 1. The lineal elements are also called direction or slope fields.

To draw approximate solution curve through arbitrary points, we need to cover the whole domian with lineal elements. This process is very cumbersome and time consuming. Hence, we use the following method. On the domian we trace the curve $f(x, y) = k$. The curve $f(x, y) = k$, on which the slope $y'$ is constant and equals to $k$, is called *isoclines*. If $k = 0$, then it has a special name, *nullclines*. Using isoclines, we can draw lineal elements in the domain of $f(x, y)$. These can be seen in Figure 2 for $y' = y - x$.

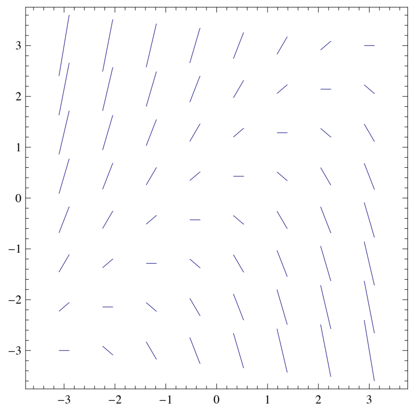


Figure 4: Lineal elements using *Mathematica* for $y' = y - x$.

Since $f(x, y)$ is continuous, we can add more arrows (if needed) between two null clines. To draw solution curve, we follow the lineal elements since these are the directions of

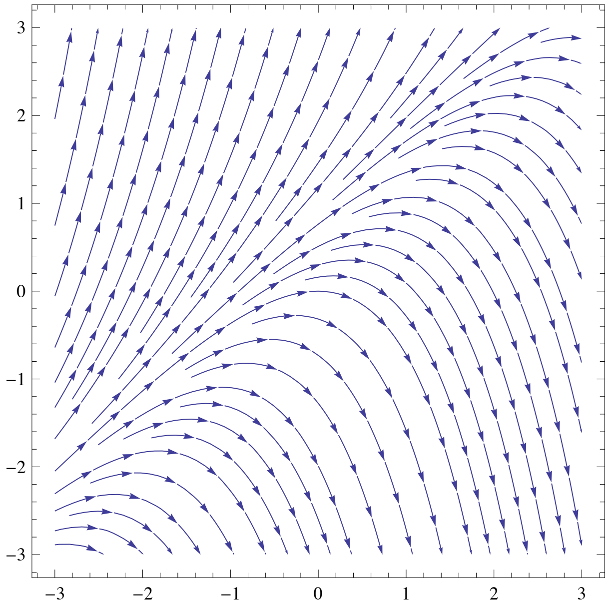


Figure 5: Solution curves using *Mathematica* for $y' = y - x$.

tangents to the curve. It is clear that $y = x + 1$ is a solution curve for $y' = y - x$ (see Figure 3). Other approximate solution curves are show in Figure 3.

Most of the commercial software packages can draw direction field as well as solution curves. For example, the *Mathematica* command

```
VectorPlot[{1, y - x}, {x, -3, 3}, {y, -3, 3}, VectorPoints -> 8,
VectorStyle -> Arrowheads[0]]
```

produces Figure 4. Similarly, the *Mathematica* command

```
StreamPlot[{1, y - x}, {x, -3, 3}, {y, -3, 3}]
```

produces Figure 5.

**Excercise** Draw the direction field of the following differential equations:
(i) $y' = 2x$ (ii) $xy' = 2y$

Lecture III

Solution of first order equations

**Theorem 1.** *abd*

**Proposition 1.** *abd*

# 1 Separable equations

These are equations of the form

$$y' = f(x)g(y)$$

Assuing $g$ is nonzero, we divide by $g$ and integrate to find

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

What happens if $g(y)$ becomes zero at a point $y = y_0$?

**Example 1.** $xy' = y + y^2$

**Solution**: We write this as

$$\int \frac{dy}{y+y^2} = \int \frac{dx}{x} + C \Rightarrow \int \frac{dy}{y} - \int \frac{dy}{1+y} = \ln x + C \Rightarrow \ln y - \ln(1+y) = \ln x + C$$

**Note**: Strictly speaking, we should write the above solution as

$$\ln|y| - \ln|1+y| = \ln|x| + C$$

When we wrote the solution without the modulas sign, it was (implicitly) assumed that $x > 0, y > 0$. This is acceptable for problems in which the solution domain is not given explicitly. But for some problems, the modulas sign is necessary. For example, consider the following IVP:

$$xy' = y + y^2, \quad y(-1) = -2.$$

Try to solve this.

# 2 Reduction to separable form

## 2.1 Substitution method

Let the ODE be

$$y' = F(ax + by + c)$$

Suppose $b \neq 0$. Substituting $ax + by + c = v$ reduces the equation to a separable form. If $b = 0$, then it is already in separable form.

**Example 2.** $y' = (x + y)^2$

**Solution**: Let $v = x + y$. Then we find

$$v' = v^2 + 1 \Rightarrow \tan^{-1} v = x + C \Rightarrow x + y = \tan(x + C)$$

## 2.2 Homogeneous form

Let the ODE be of the form

$$y' = f(y/x)$$

In this case, substitution of $v = y/x$ reduces the above ODE to a seprable ODE.

**Comment 1**: Sometimes, substitution reduces an ODE to the homogeneous form. For example, if $ae \neq bd$, then $h$ and $k$ can be chosen so that $x = u + h$ and $y = v + k$ reduces the following ODE

$$y' = F\left(\frac{ax + by + c}{dx + ey + f}\right)$$

to a homeogeneous ODE. What happens if $ae = bd$?

**Comment 2**: Also, an ODE of the form

$$y' = y/x + g(x)h(y/x)$$

can be reduced to the separable form by substituting $v = y/x$.

**Example 3.** $xyy' = y^2 + 2x^2, \qquad y(1) = 2$

**Solution**: Substituting $v = y/x$ we find

$$v + xv' = v + 2/v \Rightarrow y^2 = 2x^2(C + \ln x^2)$$

Using $y(1) = 2$, we find $C = 2$. Hence, $y = 2x^2(1 + \ln x^2)$

# 3 Exact equation

A first order ODE of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is exact if there exits a function $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

Then the above ODE can be written as $du = 0$ and hence the solution becomes $u = C$.

**Theorem 2.** *Let $M$ and $N$ be defined and continuously differentiable on a rectangle rectangle $R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\}$. Then (1) is exact if and only if $\partial M/\partial y = \partial N/\partial x$ for all $(x, y) \in R$.*

**Proof**: We shall only prove the necessary part. Assume that (1) is exact. Then there exits a function $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

Since $M$ and $N$ have continuous first partial derivatives, we have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Now continuity of 2nd partial derivative implies $\partial M/\partial y = \partial N/\partial x$.

**Example 4.** *Solve* $(2x + \sin x \tan y)dx - \cos x \sec^2 y\, dy = 0$

**Solution**: Here $M = 2x + \sin x \tan y$ and $N = -\cos x \sec^2 y$. Hence, $M_y = N_x$. Hence, the solution is $u = C$, where $u = x^2 - \cos x \tan y$

## 4 Reduction to exact equation: integrating factor

An integrating factor $\mu(x, y)$ is a function such that

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2}$$

becomes exact on multiplying it by $\mu$. Thus,

$$\mu M\, dx + \mu N\, dy = 0$$

is exact. Hence

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}.$$

**Comment**: If an equation has an integrating factor, then it has infinitely many integrating factors.

**Proof**: Let $\mu$ be an integrating factor. Then

$$\mu M\, dx + \mu N\, dy = du$$

Let $g(u)$ be any continuous function of $u$. Now multiplying by $\mu g(u)$, we find

$$\mu g(u) M\, dx + \mu g(u) N\, dy = g(u) du \Rightarrow \mu g(u) M\, dx + \mu g(u) N\, dy = d\left(\int^u g(u)\, du\right)$$

Thus,

$$\mu g(u) M\, dx + \mu g(u) N\, dy = dv, \quad \text{whare} \quad v = \int^u g(u)\, du$$

Hence, $\mu g(u)$ is an integrating factor. Since, $g$ is arbitrary, there exists an infinite number of integrating factors.

**Example 5.** $xdy - ydx = 0$.

**Solution**: Clearly $1/x^2$ is an integrating factor since

$$\frac{xdy - ydx}{x^2} = 0 \Rightarrow d(y/x) = 0$$

Also, $1/xy$ is an integrating factor since

$$\frac{xdy - ydx}{xy} = 0 \Rightarrow d\ln(y/x) = 0$$

Similarly it can be shown that $1/y^2$, $1/(x^2 + y^2)$ etc. are integrating factors.

## 4.1 How to find intgrating factor

**Theorem 3.** *If (2) is such that*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

*is a function of $x$ alone, say $F(x)$, then*

$$\mu = e^{\int F\,dx}$$

*is a function of $x$ only and is an integrating factor for (2).*

**Example 6.** $(xy-1)dx + (x^2 - xy)dy = 0$

**Solution**: Here $M = xy - 1$ and $N = x^2 - xy$. Also,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x}$$

Hence, $1/x$ is an integrting factor. Multiplying by $1/x$ we find

$$\frac{(xy-1)dx + (x^2 - xy)dy}{x} = 0 \Rightarrow xy - \ln x - y^2/2 = C$$

**Theorem 4.** *If (2) is such that*

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

*is a function of $y$ alone, say $G(y)$, then*

$$\mu = e^{\int G\,dy}$$

*is a function of $y$ only and is an integrating factor for (2).*

**Example 7.** $y^3dx + (xy^2 - 1)dy = 0$

**Solution**: Here $M = y^3$ and $N = xy^2 - 1$. Also,

$$-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{y}$$

Hence, $1/y^2$ is an integrting factor. Multiplying by $1/y^2$ we find

$$\frac{y^3dx + (xy^2 - 1)dy}{y^2} = 0 \Rightarrow xy + \frac{1}{y} = C$$

**Comment**: Sometimes it may be possible to find integrating factor by inspection. For this, some known differential formulas are useful. Few of these are given below:

$$\begin{aligned}
d\left(\frac{x}{y}\right) &= \frac{ydx - xdy}{y^2}\\
d\left(\frac{y}{x}\right) &= \frac{xdy - ydx}{x^2}\\
d(xy) &= xdy + ydx\\
d\left(\ln\frac{x}{y}\right) &= \frac{ydx - xdy}{xy}
\end{aligned}$$

**Example 8.** $(2x^2y + y)dx + xdy = 0$

Obviously, we can write this as

$$2x^2ydx + (ydx + xdy) = 0 \Rightarrow 2x^2ydx + d(xy) = 0$$

Now if we divide this by $xy$, then the last term remains differential and the first term also becomes differential:

$$2xdx + \frac{d(xy)}{xy} = 0 \Rightarrow d\left(x^2 + \ln(xy)\right) = 0 \Rightarrow x^2 + \ln(xy) = C$$

Lecture IV

Linear equations, Bernoulli equations, Orthogonal trajectories, Oblique trajectories

# 1 Linear equations

A first order linear equations is of the form

$$y' + p(x)y = r(x) \tag{1}$$

This can be written as

$$(p(x)y - r(x))dx + dy = 0.$$

Here $M = p(x)y - r(x)$ and $N = 1$. Now

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$$

Hence,

$$\mu(x) = e^{\int p(x)\,dx}$$

is an integrating factor. Multiplying (1) by $\mu(x)$ we get

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right) = r(x)e^{\int p(x)\,dx}$$

Integrating we get

$$e^{\int p(x)\,dx}y = \int^x r(s)e^{\int p(s)\,ds}\,ds + C$$

which on simplification gives

$$y = e^{-\int p(x)\,dx}\left(C + \int^x r(s)e^{\int p(s)\,ds}\,ds\right)$$

**Example 1.** *Solve* $y' + 2xy = 2x$

**Solution**: An integrating factor is $e^{x^2}$. Hence,

$$ye^{x^2} = \int^x 2te^{t^2}\,dt + C \Rightarrow y = 1 + Ce^{-x^2}$$

**Comment**: The usual notation $dy/dx$ implies that $x$ is the independent variable and $y$ is the dependent variable. In trying to solve first order ODE, it is sometimes helpful to reverse the role of $x$ and $y$, and work on the resulting equations. Hence, the resulting equation

$$\frac{dx}{dy} + p(y)x = r(y)$$

is also a linear equation.

**Example 2.** *Solve* $(4y^3 - 2xy)y' = y^2, \quad y(2) = 1$

**Solution**: We write this as

$$\frac{dx}{dy} + \frac{2}{y}x = 4y$$

Clearly, $y^2$ is an integrating factor. Hence,

$$xy^2 = \int^y 4y^3\,dy + C \Rightarrow xy^2 = y^4 + C$$

Using initial condition, we find $xy^2 = y^4 + 1$.

# 2 Bernoulli's equation

This is of the form

$$y' + p(x)y = r(x)y^\lambda, \tag{2}$$

where $\lambda$ is a real number. Equation (2) is linear for $\lambda = 0$ or 1. Otherwise, it is nonlinear and can be reduced to a linear form by substituting $z = y^{1-\lambda}$

**Example 3.** *Solve* $y' - y/x = y^3$

**Solution**: We write this as

$$y^{-3}y' - y^{-2}/x = 1$$

Substitute $y^{-2} = z \Rightarrow -2y^{-3}y' = z'$. This leads to

$$z' + 2z/x = -2$$

This is a linear equation whose solution is

$$zx^2 = -2x^3/3 + C$$

Replacing $z$ we find

$$3\frac{x^2}{y^2} + 2x^3 = C$$

# 3 Reducible second order ODE

A general 2nd order ODE is of the form

$$F(x, y, y', y'') = 0$$

In some cases, by making substitution, we can reduce this 2nd order ODE to a 1st order ODE. Few cases are described below

**Case I**: If the *independent variable is missing*, then we have $F(y, y', y'') = 0$. If we substitute $w = y'$, then $y'' = w\frac{dw}{dy}$. Hence, the ODE becomes $F(y, w, w\frac{dw}{dy}) = 0$, which is a 1st order ODE.

**Example 4.** *Solve* $2y'' - y'^2 - 4 = 0$

**Solution**: With $w = y'$, the above equation becomes

$$2w\frac{dw}{dy} - w^2 - 4 = 0 \Rightarrow \ln[(w^2+4)/C] = y \Rightarrow w = \pm\sqrt{Ce^y - 4}$$

SInce $w = y'$, we find

$$\frac{dy}{\sqrt{Ce^y - 4}} = \pm x + D$$

The integral on the LHS can be evaluated by substitution.

**Case II**: If the *dependent variable is missing*, then we have $F(x, y', y'') = 0$. If we substitute $w = y'$, then $y'' = w'$. Hence, the ODE becomes $F(x, w, w') = 0$, which is a 1st order ODE.

**Example 5.** *Solve* $xy'' + 2y' = 0$

**Solution**: Substitute $w = y'$, then we find

$$\frac{dw}{dx} + \frac{2}{x}w = 0 \Rightarrow w = Cx^{-2}$$

Since $w = y'$, we further get

$$y' = C/x^2 \Rightarrow y = -C/x + D$$

# 4 Orthogonal trajectories

**Definition 1.** *Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical, they we say that the family is self-orthogonal.*

**Comment**: Orthogonal trajectories has important applications in the field of physics. For example, the equipotential lines and the streamlines in an irratational 2D flow are orthogonal.

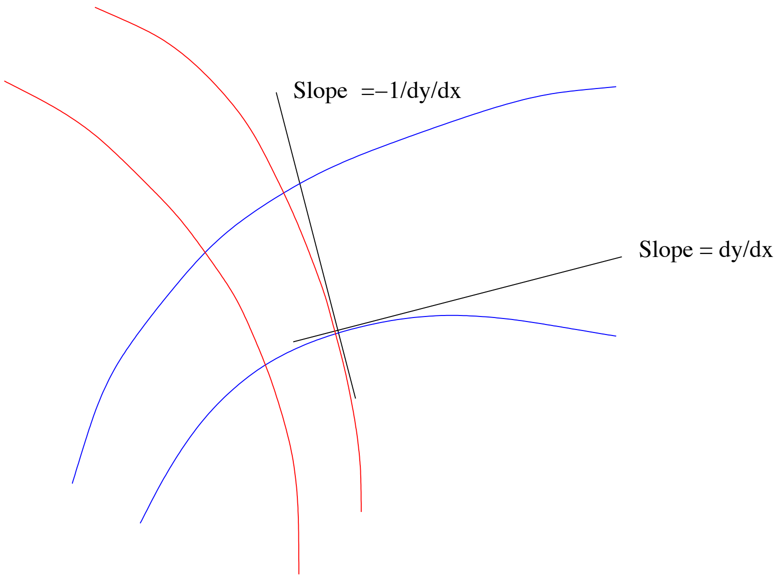


Figure 1: Orthogonal trajectories.

## 4.1 How to find orthonal trajectories

Suppose the first familiy

$$F(x, y, c) = 0. \tag{3}$$

To find the orthogonal trajectories of this family we proceed as follows. First, differentiate (3) w.r.t. $x$ to find

$$G(x, y, y', c) = 0. \tag{4}$$

Now eliminate $c$ between (3) and (4) to find the differential equation

$$H(x, y, y') = 0 \tag{5}$$

corresponding to the first family. As seen in Figure 1, the differential equation for the other family is obtained by replacing $y'$ by $-1/y'$. Hence, the differetial equation of the orthogonal trajectories is

$$H(x, y, -1/y') = 0 \tag{6}$$

General solution of (6) gives the required orthogonal trajectories.

**Example 6.** *Find the orthogonal trajectories of familiy of straight lines through the origin.*

**Solution**: The familiy of straight lines through the origin is given by

$$y = mx$$

The ODE for this familiy is

$$xy' - y = 0$$

The ODE for the orthogonal family is

$$x + yy' = 0$$

Integrating we find

$$x^2 + y^2 = C,$$

which are family of circles with centre at the origin.

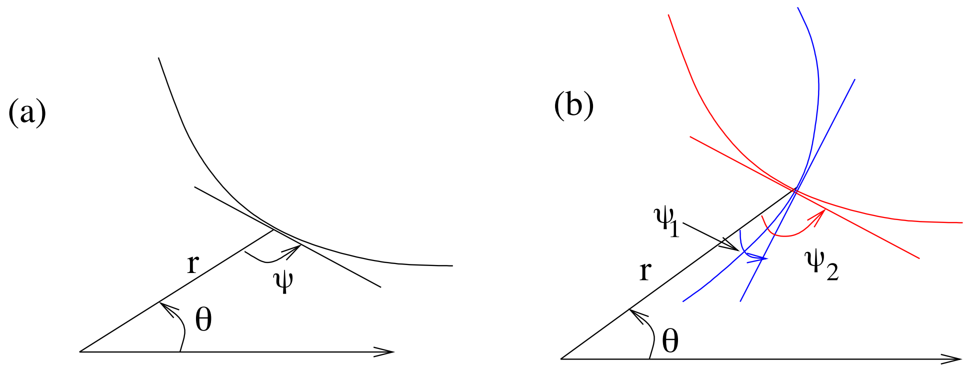


Figure 2: Orthogonal trajectories.

## 4.2 *Orthogonal trjactories in polar coordinates

Consider a curve in polar cordinate. The angle $\psi$ between the radial and tangent directions is given by

$$\tan\psi = \frac{r\,d\theta}{dr}$$

Consider the curve with angle $\psi_1$. The curve that intersects it orthogonally has angle $\psi_2 = \psi_1 + \pi/2$. Now

$$\tan \psi_2 = -\frac{1}{\tan \psi_1}$$

Thus, at the point of orthogonal intersection, the value of

$$\frac{r\,d\theta}{dr} \tag{7}$$

for the second famility should be negative raciprocal of the value of (7) of the first family. To illustrate, consider the differential equation for the first family:

$$Pdr + Qd\theta = 0.$$

Thus we find $r\,d\theta/dr = -Pr/Q$. Hence, the differential equation of the orthogonal family is given by

$$\frac{r\,d\theta}{dr} = \frac{Q}{Pr}$$

or

$$Q\,dr - r^2\,P\,d\theta = 0$$

General solution of the last equation gives the orthogonal trajectories.

**Example 7.** *Find the orthogonal trajectories of familiy of straight lines through the origin.*

**Solution**: The familiy of straight lines through the origin is given by

$$\theta = A$$

The ODE for this familiy is

$$d\theta = 0$$

The ODE for the orthogonal family is

$$dr = 0$$

Integrating we find

$$r = C,$$

which are family of circles with centre at the origin.

## 4.3 Oblique trajectories

Here the two families of curves intersect at an arbitrary angle $\alpha \neq \pi/2$. Suppose the first familiy

$$F(x, y, c) = 0. \tag{8}$$

To find the oblique trajectories of this family we proceed as follows. First, differentiate (8) w.r.t. $x$ to find

$$G(x, y, y', c) = 0. \tag{9}$$

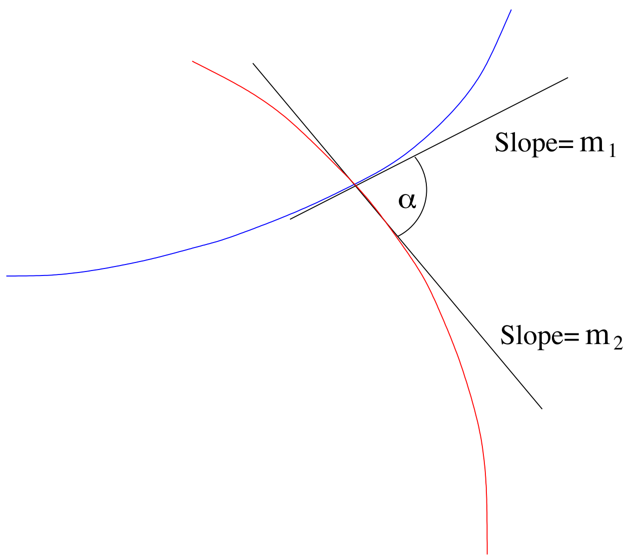


Figure 3: Oblique trajectories.

Now eliminate $c$ between (8) and (9) to find the differential equation

$$H(x, y, y') = 0. \tag{10}$$

Now if $m_1$ is the slope of this family, then we write (10) as

$$H(x, y, m_1) = 0, \tag{11}$$

Let $m_2$ be the slope of the second famililly. Then

$$\pm \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Thus, we find

$$m_1 = \frac{m_2 \pm \tan\alpha}{1 \mp m_2 \tan\alpha}$$

Hence, from (11), the ODE for the second family satisfies

$$H\left(x, y, \frac{m_2 \pm \tan\alpha}{1 \mp m_2 \tan\alpha}\right) = 0,$$

Replacing $m_2$ by $y'$, the ODE for the second family is written as

$$H\left(x, y, \frac{y' \pm \tan\alpha}{1 \mp y' \tan\alpha}\right) = 0. \tag{12}$$

General solution of (12) gives the required oblique trajectories.
**Note**: If we let $\alpha \to \pi/2$, we obtained ODE for the orthogonal trajectories.

**Example 8.** *Find the oblique trajectories that intersects the familiy $y = x + A$ at an angle of* $60^o$

**Solution**: The ODE for the given family is

$$y' = 1$$

For the oblique trajectories, we replace

$$y' \quad \text{by} \quad \frac{y' \pm \tan(\pi/3)}{1 \mp y' \tan(\pi/3)} = \frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'}$$

Thus, the ODE for the oblique trajectories is given by

$$\frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'} = 1$$

Simplifying we obtain

$$y' = \frac{1-\sqrt{3}}{1+\sqrt{3}} \quad \text{OR} \quad y' = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Hence, the oblique trajectories are either

$$y = \frac{1-\sqrt{3}}{1+\sqrt{3}}x + C_1$$

Or

$$y = \frac{1+\sqrt{3}}{1-\sqrt{3}}x + C_2$$

Lecture V

Picard's existence and uniquness theorem, Picard's iteration

# 1 Existence and uniqueness theorem

Here we concentrate on the solution of the first order IVP

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

We are interested in the following questions:
1. Under what conditions, there exists a solution to (1).
2. Under what conditions, there exists a unique solution to (1).

**Comment**: An ODE may have no solution, unique solution or infinitely many solutions. For, example $y'^2 + y^2 + 1 = 0$, $y(0) = 1$ has no solution. The ODE $y' = 2x$, $y(0) = 1$ has unique solution $y = 1+x^2$, whereas the ODE $xy' = y-1$, $y(0) = 1$ has infinitely many solutions $y = 1 + \alpha x$, $\alpha$ is any real number.

(We only state the theorems. For proof, one may see 'An introduction to ordinary differential equation' by E A Coddington.)

**Theorem 1.** (Existence theorem)*: Suppose that $f(x, y)$ is continuous function in some region*

$$R = \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\}, \quad (a, b > 0).$$

*Since $f$ is continuous in a closed and bounded domain, it is necessarily bounded in $R$, i.e., there exists $K > 0$ such that $|f(x, y)| \le K \quad \forall (x, y) \in R$. Then the IVP (1) has atleast one solution $y = y(x)$ defined in the interval $|x - x_0| \le \alpha$ where*

$$\alpha = \min\left\{a, \frac{b}{K}\right\}$$

(Note that the solution exists possibly in a ~~smaller~~ larger interval)

**Theorem 2.** (Uniquness theorem)*: Suppose that $f$ and $\partial f/\partial y$ are continuous function in $R$* (defined in the existence theorem)*. Hence, both the $f$ and $\partial f/\partial y$ are bounded in $R$, i.e.,*

$$\text{(a) } |f(x, y)| \le K \quad \textit{and} \quad \text{(b) } \left|\frac{\partial f}{\partial y}\right| \le L \qquad \forall (x, y) \in R$$

*Then the IVP (1) has atmost one solution $y = y(x)$ defined in the interval $|x - x_0| \le \alpha$ where*

$$\alpha = \min\left\{a, \frac{b}{K}\right\}.$$

*Combining with existence thereom, the IVP (1) has unique solution $y = y(x)$ defined in the interval $|x - x_0| \le \alpha$.*

**Comment**: Condition (b) can be replaced by a *weaker* condition which is known as Lipschitz condition. Thus, instead of continuity of $\partial f/\partial y$, we require

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2| \qquad \forall (x, y_i) \in R.$$

If $\partial f/\partial y$ exists and is bounded, then it necessarily saitisfies Lipschitz condition. On the other hand, a function $f(x,y)$ may be Lipschitz continuous but $\partial f/\partial y$ may not exists. For example $f(x,y) = x^2|y|,\ |x| \le 1, |y| \le 1$ is Lipschitz continous in $y$ but $\partial f/\partial y$ does not exist at $(x,0)$ (prove it!).

***Note 1**: The existence and uniqueness theorems stated above are *local* in nature since the interval, $|x-x_0| \le \alpha$, where solution exists may be smaller than the original interval, $|x-x_0| \le a$, where $f(x,y)$ is defined. However, in some cases, this restrictions can be removed. Consider the linear equation

$$y' + p(x)y = r(x), \tag{2}$$

where $p(x)$ and $r(x)$ are defined and continuous in a the interval $a \le x \le b$. Here $f(x,y) = -p(x)y + r(x)$. If $L = \max_{a \le x \le b} |p(x)|$, then

$$|f(x,y_1) - f(x,y_2)| = |-p(x)(y_1 - y_2)| \le L|y_1 - y_2|$$

Thus, $f$ is Lipschitz continuous in $y$ in the infinite vertical strip $a \le x \le b$ and $-\infty < y < \infty$. In this case, the IVP (2) has a unique solution in the *original* interval $a \le x \le b$.

***Note 2**: Though the theorems are stated in terms of interior point $x_0$, the point $x_0$ could be left/right end point.

**Comment**: The conditions of the existence and uniqueness theorem are sufficeint but not necessary. For example, consider

$$y' = \sqrt{y} + 1, \quad y(0) = 0, \quad x \in [0,1]$$

Clearly $f$ does not satisfy Lipschitz condition near origin. But still it has unique solution. Can you prove this? [Hint: Let $y_1$ and $y_2$ be two solutions and consider $z(x) = \left(\sqrt{y_1(x)} - \sqrt{y_2(x)}\right)^2$.]

**Comment**: The existence and uniqueness theorem are also valid for certain system of first order equations. These theorems are also applicable to a certain higher order ODE since a higher order ODE can be reduced to a system of first order ODE.

**Example 1.** *Consider the ODE*

$$y' = xy - \sin y, \qquad y(0) = 2.$$

*Here $f$ and $\partial f/\partial y$ are continuous in a closed rectangle about $x_0 = 0$ and $y_0 = 2$. Hence, there exists unique solution in the neighbourhood of $(0,2)$.*

**Example 2.** *Consider the ODE*

$$y' = 1 + y^2, \qquad y(0) = 0.$$

*Consider the rectangle*

$$S = \{(x,y) : |x| \le 100, |y| \le 1\}.$$

*Clearly $f$ and $\partial f/\partial y$ are continuous in $S$. Hence, there exists unique solution in the neighbourhood of $(0,0)$. Now $f = 1 + y^2$ and $|f| \le 2$ in $S$. Now $\alpha = min\{100, 1/2\} =$*

1/2. *Hence, the theorems guarantee existence of unique solution in* $|x| \leq 1/2$, *which is much smaller than the original interval* $|x| \leq 100$.
*Since, the above equation is separable, we can solve it exactly and find* $y(x) = \tan(x)$. *This solution is valid only in* $(-\pi/2, \pi/2)$ *which is also much smaller than* $[-100, 100]$ *but nevertheless bigger than that predicted by the existence and uniqueness theorems.*

**Example 3.** *Consider the IVP*

$$y' = x|y|, \quad y(1) = 0.$$

*Since* $f$ *is continuous and satisfy Lipschitz condition in the neighbourhood of the* $(1, 0)$, *it has unique solution around* $x = 1$.

**Example 4.** *Consider the IVP*

$$y' = y^{1/3} + x, \quad y(1) = 0$$

*Now*

$$|f(x, y_1) - f(x, y_2)| = |y_1^{1/3} - y_2^{1/3}| = \frac{|y_1 - y_2|}{|y_1^{2/3} + y_1^{1/3}y_2^{1/3} + y_2^{1/3}|}$$

*Suppose we take* $y_2 = 0$. *Then*

$$|f(x, y_1) - f(x, 0)| = \frac{|y_1 - 0|}{|y_1^{2/3}|}$$

*Now we can take* $y_1$ *very close to zero. Then* $1/|y_1^{2/3}|$ *becomes unbounded. Hence, the relation*

$$|f(x, y_1) - f(x, 0)| \leq L|y_1 - 0|$$

*does not always hold around a region about* $(1, 0)$.
*Since* $f$ *does not satisfy Lipschitz condition, we can not say whether unique solution exits or does not exist (remember the existence and uniqueness conditios are sufficient but not necessary).*
*On the other hand*

$$y' = y^{1/3} + x, \quad y(1) = 1$$

*has unique solution around* $(1, 1)$.

**Example 5.** *Discuss the existence and unique solution for the IVP*

$$y' = \frac{2y}{x}, \quad y(x_0) = y_0$$

**Solution**: Here $f(x, y) = 2y/x$ and $\partial f/\partial y = 2/x$. Clearly both of these exist and bounded around $(x_0, y_0)$ if $x_0 \neq 0$. Hence, unique solution exists in a interval about $x_0$ for $x_0 \neq 0$.
For $x_0 = 0$, nothing can be said from the existence and uniqness theorem. Fortunately, we can solve the actual problem and find $y = Ax^2$ to be the general solution. When $x_0 = 0$, there exists no solution when $y_0 \neq 0$. If $y_0 = 0$, then we have infinite number of solutions $y = \alpha x^2$ ($\alpha$ any real number) that satisfy the IVP $y' = 2y/x, \quad y(0) = 0$.

# 2 Picard iteration for IVP

This method gives approximate solution to the IVP (1). Note that the IVP (1) is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \tag{3}$$

A rough approximation to the solution $y(x)$ is given by the function $y_0(x) = y_0$, which is simply a horizontal line through $(x_0, y_0)$. (don't confuse function $y_0(x)$ with constant $y_0$). We insert this to the RHS of (3) in order to obatin a (perhaps) better approximate solution, say $y_1(x)$. Thus,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

The nest step is to use this $y_1(x)$ to genereate another (perhaps even better) approximate solution $y_2(x)$:

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

At the $n$-th stage we find

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

**Theorem 3.** *If the function $f(x, y)$ satisfy the existence and uniqueness theorem for IVP (1), then the succesive approximation $y_n(x)$ converges to the unique solution $y(x)$ of the IVP (1).*

**Example 6.** *Apply Picard iteration for the IVP*

$$y' = 2x(1 - y), \quad y(0) = 2.$$

**Solution**: Here $y_0(x) = 2$. Now

$$y_1(x) = 2 + \int_0^x 2t(1-2)\,dt = 2 - x^2$$

$$y_2(x) = 2 + \int_0^x 2t(t^2 - 1)\,dt = 2 - x^2 + \frac{x^4}{2}$$

$$y_3(x) = 2 + \int_0^x 2t\left(t^2 - \frac{t^4}{2} - 1\right)\,dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$$

$$y_4(x) = 2 + \int_0^x 2t\left(\frac{t^6}{3!} - \frac{t^4}{2} + t^2 - 1\right)\,dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{8!}$$

By induction, it can be shown that

$$y_n(x) = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!}$$

Hence, $y_n(x) \to 1 + e^{-x^2}$ as $n \to \infty$. Now $y(x) = 1 + e^{-x^2}$ is the exact solution of the given IVP. Thus, the Picard iterates converge to the unique solution of the given IVP.

**Comment**: Picard iteration has more theoretical value than practical value. It is used in the proof of existence and uniqueness theorem. On the other hand, finding approximate solution using this method is almost impractical for complicated function $f(x, y)$.

# Lecture VI

Numerical methods: Euler's method, improved Euler's method

In most real situations, it is impossible to find analytical solution to the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \tag{1}$$

Thus, we need to solve (1) numerically. Following two methods, described below, are the simplest of the numerous numerical methods that are used to solve (1). They can be derived different in many ways.

Note the (1) is equivalent to

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2}$$

Suppose $x_0 = a$ and we need to find the solution at $x = b > a$. We divide the interval $[a = x_0, b]$ in to $N$ intervals of size $h = (b-a)/N$. Thus, the uniform mesh points are $a = x_0 < x_1 < x_2 < \cdots < x_N$.

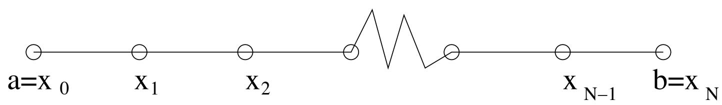


Figure 1: Mesh points

Consider arbitrary interval $[x_n, x_{n+1}]$. Using (2), we find

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))\, dt. \tag{3}$$

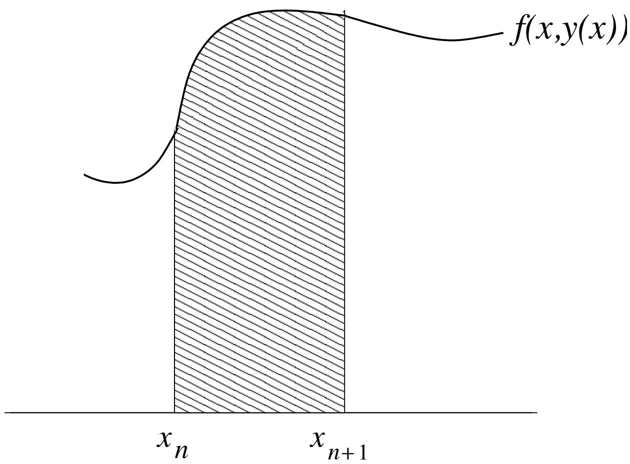


Figure 2:

The integral on the RHS is the shaded area shown in Figure 2. But since $y(t)$ is unknown, we have to approximate the integral in the RHS of (3).

Now from the mean value theorem of integral calculus, we write (3) as

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n)f(\zeta, y(\zeta)), \quad \zeta \in (x_n, x_{n+1})$$

or equivalently

$$y(x_{n+1}) = y(x_n) + hf(\zeta, y(\zeta)), \quad \zeta \in (x_n, x_{n+1}) \tag{4}$$

Now $f(\zeta, y(\zeta))$ is the slope of the solution curve $y = y(x)$ at the unknown point $x = \zeta$. Further $y$ is also itself unknown. We need to approximate the slope using numerical methods.

# 1 Euler's method

We assume that the slope $f$ varies so little in $[x_n, x_{n+1}]$ that we can approximate it by its value at $x = x_n$, i.e. $f(\zeta, y(\zeta)) \approx f(x_n, y(x_n))$. Hence,

$$y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n))$$

But the exact value $y(x_n)$ is not known except $y(x_0) = y_0$. Hence, if we assume $y(x_n) \approx y_n$, then $f(x_n, y(x_n)) \approx f(x_n, y_n)$. Finally the Euler's method becomes

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0, 1, 2, N - 1. \tag{5}$$

**Geometrical interpretation**

The curve between $x_0$ and $x_1$ is approximated by the straight line that passes through $(x_0, y_0)$ having slope $f(x_0, y_0)$. Similarly, the curve between $x_1$ and $x_2$ is approximated by the straight line that passes through $(x_1, y_1)$ having slope $f(x_1, y_1)$. Obviously, at each step we are making error (shown by the red segment in Figure3). The error between the exact and computed solution at the $n$-th step is $|y(x_n) - y_n|$.

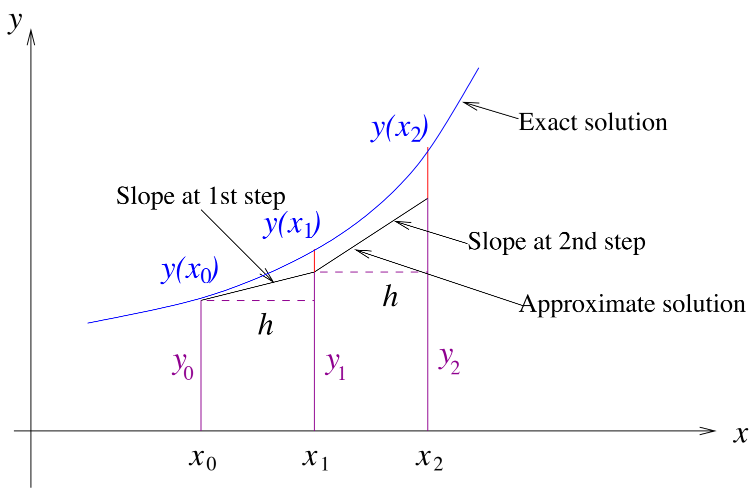


Figure 3: Euler's method

## 2 Improved Euler's method

A better approximate for $f(\zeta, y(\zeta))$ in (4) is the average of the slopes at $x_n$ and $x_{n+1}$, i.e.

$$f(\zeta, y(\zeta)) \approx \frac{f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))}{2}.$$

Hence, we find

$$y(x_{n+1}) \approx y(x_n) + \frac{h}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right] \tag{6}$$

As before $y(x_n) \approx y_n$. Now the unknown $y(x_{n+1})$ also occurs in the RHS of (6). This $y(x_{n+1})$ in the RHS of (6) is approximated using the value from the Euler's method. Thus, we substitute $y(x_{n+1}) \approx y_n + hf(x_n, y_n) = y^*_{n+1}$ in the RHS of (6). Hence, finally, the improved Euler's method becomes

$$\left.\begin{aligned} y^*_{n+1} &= y_n + hf(x_n, y_n) \\ y_{n+1} &= y_n + \tfrac{h}{2}\Big[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\Big] \end{aligned}\right\}, \; n = 0, 1, 2, N-1. \tag{7}$$

**Geometrical interpretation**

First the Euler's method is used to predict the value $y^*_1$ at $x = x_1$. Slope at $x = x_1$ is calculated from $f(x_1, y^*_1)$. Then we obtained the average slope from the slopes at $x = x_0$ and at $x = x_1$. Now the approximate solution is the straight line that passes throgh $(x_0, y_0)$ with slope equal to the average value of the slopes. This process is repeated for the next step $[x_1, x_2]$ and so on.

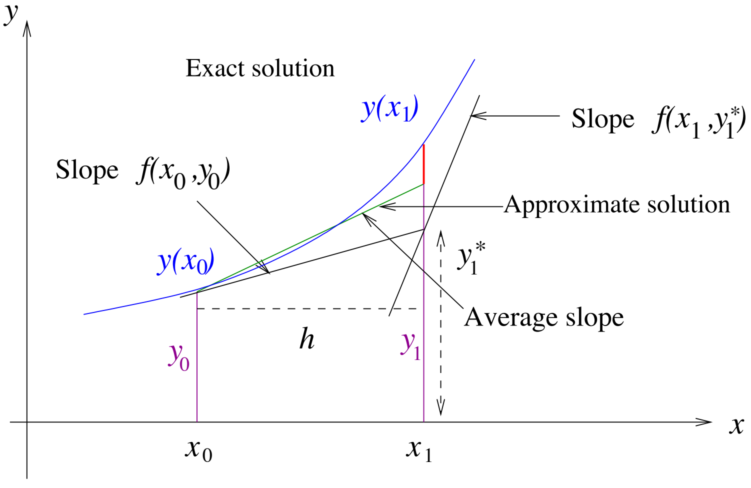


Figure 4: Improved Euler's method

**Comment 1**: Improved Euler's method has many different names such as Modified Euler's method, Heun method and Runge-Kutta method of order 2. It also belongs to the category of predictor-corrector method.

**Comment 2**: It can be proved that the accuracy of Euler's method is proportional to $h$ and that of improved Euler's method to $h^2$, where $h$ is the step size. Hence, Improved Euler's method has better accuracy than that of Euler's method.

**Example 1.** *Apply (i) Euler's method and (ii) improved Euler's method to compute $y(x)$ at $x = 0.4$ with stepsize $h = 0.2$ for the initial value problem:*

$$y' = 2x(1 - y), \ y(0) = 2.$$

*Compare the errors $e_n = |y(x_n) - y_n|$ in each step with the exact solution $y(x) = 1 + e^{-x^2}$.*

**Solution**: The values are rounded to three places of decimal in the Tables.

| $n$ | $x_n$ | $y_n$ | $y(x_n)$ | $e_n$ | $f(x_n, y_n)$ |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 0 | 0 |
| 1 | 0.2 | 2 | 1.961 | 0.039 | -0.4 |
| 2 | 0.4 | 1.92 | 1.852 | 0.068 | |

Table 1: Euler's method

| $n$ | $x_n$ | $y_n$ | $y(x_n)$ | $e_n$ | $f(x_n, y_n)$ | $y^*_{n+1}$ | $f(x_{n+1}, y^*_{n+1})$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 0 | 0 | 2 | $-0.4$ |
| 1 | 0.2 | 1.96 | 1.961 | 0.001 | $-0.384$ | 1.883 | $-0.706$ |
| 2 | 0.4 | 1.845 | 1.852 | 0.007 | | | |

Table 2: Improved Euler's method

It is clear that the improved Euler's method has better accuracy than the Euler's method which is expected. Also, decreasing the step size $h$ improves the accuracy of both of the methods.

# Lecture VII

Second order linear ODE, fundamental solutions, reduction of order

A second order linear ODE can be written as

$$y'' + p(x)y' + q(x)y = r(x), \qquad x \in \mathcal{I}, \tag{1}$$

where $\mathcal{I}$ is an interval. If $r(x) = 0,\ \forall\, x \in I$, then (1) is a homogeneous 2nd order linear ODE, otherwise it is non-homogeneous. We shall assume the following existence and uniqueness theorem for (1).

**Theorem 1.** *Let $p(x), q(x)$ and $r(x)$ be continuous in $\mathcal{I}$. If $x_0 \in \mathcal{I}$ and $K_0, K_1$ are two arbitrary real numbers, then (1) has unique solution $y(x)$ on $\mathcal{I}$ such that $y(x_0) = K_0$ and $y'(x_0) = K_1$.*

We shall also consider the homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = 0, \qquad x \in \mathcal{I}. \tag{2}$$

**Theorem 2.** *Let $y_1(x)$ and $y_2(x)$ be two solutions of (2). Then $y(x) = c_1y_1(x)+c_2y_2(x)$ ($c_1, c_2$ arbitrary constants) is also a solution of (2).*

**Proof**: Trivial

**Definition 1.** *Two function $f$ and $g$ are defined in $\mathcal{I}$. If there exists constant $a, b$, not both zero such that*

$$af(x) + bg(x) = 0 \quad \forall x \in \mathcal{I},$$

*then $f$ and $g$ are linearly dependent (LD) in $\mathcal{I}$, otherwise they are linearly independent (LI) in $\mathcal{I}$.*

**Example 1.**
*(i)* $\sin x, \cos x,\ x \in (-\infty, \infty)$ *are LI.*
*(ii)* $x|x|, x^2,\ x \in (-1, 1)$ *are LI.*
*(iii)* $x|x|, x^2,\ x \in (0, 1)$ *are LD*

**Definition 2.** *Let $f$ and $g$ be two differentiable functions. Then*

$$W(f, g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

*is called the Wronskian of $f$ and $g$*

**Note** : Let $f$ and $g$ be differentiable. If $f$ and $g$ are LD in an interval $\mathcal{I}$, then $W(f, g) = 0, \forall x \in \mathcal{I}$. Hence, if two differentiable functions $f$ and $g$ are such that $W(f, g) \neq 0$ at a point $x_0 \in \mathcal{I}$, then $f$ and $g$ are LI.
*But the converse is not true.* If $W(f, g) = 0, \forall x \in \mathcal{I}$, then $f$ and $g$ may not be LD. For example, consider $f(x) = x|x|, g(x) = x^2,\ x \in (-\infty, \infty)$. Here $W(f, g) = 0, \forall x$ but still $f$ and $g$ are LI.

**Example 2.** *For $f(x) = x, g(x) = \sin x$, we find $W(f, g) = x \cos x - \sin x$ which is nonzero, for example, at $x = \pi$. Hence, $x$ and $\sin x$ are LI. Note that $W(f, g)$ may be zero at some point such as $x = 0$.*

**Theorem 3.** *Two solutions $y_1, y_2$ of (2) are LD* iff *$W(y_1, y_2) = 0$ at certain point $x_0 \in \mathcal{I}$.*

**Proof**: Let $y_1, y_2$ be LD. Thus, there exists $a, b$ not both zero such that

$$ay_1(x) + by_2(x) = 0 \tag{3}$$

We can differentiate (3) once and obtain

$$ay_1'(x) + by_2'(x) = 0 \tag{4}$$

Now (3) and (4) can be viewed as linear homogeneous equations in two unknowns $a$ and $b$. Since the solution is nontrivial, the determinant must be zero. Thus $W(y_1, y_2) = 0$, $\forall x \in \mathcal{I}$. Hence, $W(y_1, y_2)$ must be zero at $x_0 \in \mathcal{I}$.
Conversely, suppose $W(y_1, y_2) = 0$ at $x_0 \in \mathcal{I}$. Now consider

$$ay_1(x_0) + by_2(x_0) = 0 \tag{5}$$

and

$$ay_1'(x_0) + by_2'(x_0) = 0 \tag{6}$$

Now the determinant of the system of linear equations (in unknowns $a, b$) of (5) and (6) is the Wronskian $W(y_1, y_2)$ at $x_0$. Since, this is zero, we can find nontrivial solution for $a$ and $b$. Take these nontrivial $a$ and $b$ and form

$$y(x) = ay_1(x) + by_2(x)$$

By (5) and (6), we find $y(x_0) = y'(x_0) = 0$. Hence, by uniqueness theorem $y(x) \equiv 0$, i.e. for nontrivial $a$ and $b$

$$ay_1(x) + by_2(x) = 0, \qquad x \in \mathcal{I}$$

Hence $y_1, y_2$ are LD.
**Comment**: This theorem says that if $f$ and $g$ are solutions of (2) and $W(f, g) = 0$ at $x_0 \in \mathcal{I}$, then $f$ and $g$ must be LD. But in Example 2, $W(f, g) = 0$ at $x = 0$ but still $f$ and $g$ are LI. Do you find any contradiction in it?

**Corollary 1.** *Let $y_1, y_2$ be solutions of (2). If the Wronskian $W(y_1, y_2) = 0$ at $x_0 \in I$, then $W(y_1, y_2) = 0$ $\forall x \in \mathcal{I}$.*

**Proof**: We proceed as in the converse part of the previous theorem to prove that $y_1$ and $y_2$ are LD. Now proceed as in the first part of the same theorem to prove that $W(y_1, y_2) = 0$, $\forall x \in \mathcal{I}$.
**Aliter**: Since $y_1$ and $y_2$ are solutions of (2), we obtain

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \tag{7}$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{8}$$

Multiply (7) by $y_2$ and (8) by $y_1$ and subtract. This leads to

$$\frac{dW}{dx} + p(x)W = 0,$$

where we have used the short notation $W$ for $W(y_1, y_2)$. Integrating, we find

$$W(x) = Ce^{-\int p(x)\,dx}$$

Since $W(x_0) = 0$, this gives $C = 0$ and hence $W \equiv 0$.

**Theorem 4.** *Let $y_1, y_2$ be solutions of (2). If there exists a point $x_0 \in \mathcal{I}$ such that $W(y_1, y_2) \neq 0$ at $x_0$, then $y_1$ and $y_2$ are LI and forms a basis solution for (2).*

**Proof**: If $y_1$ are $y_2$ are LD, then $W(y_1, y_2) \equiv 0$ which contradicts $W(y_1, y_2) \neq 0$ at $x_0$. Hence, $y_1$ and $y_2$ are LI.

Now we shall show that $y_1$ and $y_2$ spans the solution space for (2). Let $y$ be any solution with $y(x_0) = K_0$ and $y'(x_0) = K_1$. Now, the system

$$\begin{aligned} ay_1(x_0) + by_2(x_0) &= K_0 \\ ay_1'(x_0) + by_2'(x_0) &= K_1 \end{aligned}$$

has unique solution $a = c_1$ and $b = c_2$, since the determinant is nonzero. Let $\zeta(x) = c_1y_1(x) + c_2y_2(x)$. Then, $\zeta(x_0) = K_0$, $\zeta'(x_0) = K_1$. But by the existence and uniqueness theorem, we have $y(x) \equiv \zeta(x)$ and thus

$$y(x) = c_1y_1(x) + c_2y_2(x), \quad \forall x \in \mathcal{I}$$

Hence, $y_1$ and $y_2$ spans the solution space. Thus, $y_1$ and $y_2$ form a basis of solution for (2). Thus, a *general solution* $y(x)$ of (2) can be written as

$$y(x) = Ay_1(x) + By_2(x),$$

where $A$ and $B$ are arbitrary constants. For an IVP, these constants take particular values to satisfy the initial condition.

**Existence of basis**: By the existence and uniqueness theorem, there exists a solution $y_1(x)$ of (2) with $y_1(x_0) = 1, y_1'(x_0) = 0$. Similarly, there exists a solution $y_2(x)$ of (2) with $y_2(x_0) = 0, y_2'(x_0) = 1$. Hence, $W(y_1, y_2) = 1 \neq 0$ at $x_0$. By the previous, theorem $y_1$ and $y_2$ form a basis solution for (2).

**Example 3.** $y_1(x) = \sin x$ *and* $y_2(x) = \cos x$ *satisfy* $y'' + y = 0$ *and* $W(y_1, y_2) = -1 \neq 0$. *Hence,* $\sin x$ *and* $\cos x$ *form a basis of solution for* $y'' + y = 0$. *Thus, a general solution of* $y'' + y = 0$ *is* $y(x) = C_1 \sin x + C_2 \cos x$.

**Reduction of order**: Consider the homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = 0. \tag{9}$$

If we know one nonzero solution $y_1(x)$ (by any method) of (9), then it is easy to find the second solution $y_2(x)$ which is independent of $y_1$. Thus, $y_1$ and $y_2$ will form a basis of solution.

We assume that $y_2(x) = v(x)y_1(x)$, where $v(x)$ is an unknown function. Since, $y_2$ is a solution, we substitute $y_2(x) = v(x)y_1(x)$ into (9). Taking into account the fact that $y_1$ is also a solution of (9), we find

$$y_1v'' + (2y_1' + py_1)\,v' = 0.$$

Dividing this by $y_1$ and writing $U$ for $v'$, we get

$$U' + \left(\frac{2y_1'}{y_1} + p\right) U = 0$$

Since this is linear equation, it has general solution

$$U = \frac{C}{y_1^2} e^{-\int p\,dx},$$

where $C$ is a constant of integration. Thus, we find

$$v(x) = C \int \frac{1}{y_1^2} e^{-\int p\,dx} + D,$$

where $D$ is another constant of integration. Finally, multiply $v$ by $y_1$ to find $y_2$:

$$y_2(x) = Cy_1(x) \int \frac{1}{y_1^2} e^{-\int p\,dx} + Dy_1(x).$$

Since, we are looking for a solution independent of $y_1$, this can be taken with $C = 1$ and $D = 0$. Thus

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{-\int p\,dx}.$$

To show that they are LI, note that

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = y_1^2 v' = y_1^2 U = e^{-\int p\,dx} \neq 0.$$

Thus, $y_1$ and $y_2$ form a basis of solution.

**Example 4.** *Solve* $xy'' + (2x+1)y' + (x+1)y = 0$

**Solution**: Since at $x = 0$, the equation becomes singular, we solve the above for $x \neq 0$. WLOG, we assume that $x > 0$. Clearly, $y_1(x) = e^{-x}$ is a solution. We write this equation as

$$y'' + \left(2 + \frac{1}{x}\right) y' + \frac{x+1}{x} y = 0.$$

Hence, $p(x) = 2 + 1/x$. Substituting $y_2(x) = v(x)y_1(x)$ and solving we find

$$v(x) = \int \frac{1}{e^{-2x}} \exp\left(-\int (2 + 1/x)\,dx\right) = \ln x$$

Hence, $y_2(x) = e^{-x} \ln x$. Thus, the general solution is $y(x) = e^{-x}(C_1 + C_2 \ln x), \quad x > 0$. What is the general solution for $x < 0$?

Lecture VIII

Homogeneous linear ODE with constant coefficients

# 1 Homogeneous 2nd order linear equation with constant coefficients

If the ODE is of the form

$$ay'' + by' + cy = 0, \qquad x \in \mathcal{I}, \tag{1}$$

where $a, b, c$ are constants, then two independent solutions (i.e. basis) depend on the quadratic equation

$$am^2 + bm + c = 0. \tag{2}$$

Equation (2) is called *characteristic equation* for (1).

**Theorem 1.** *(i) If the roots of (2) are real and distinct, say $m_1$ and $m_2$, then two linearly independent (LI) solutions of (1) are $e^{m_1x}$ and $e^{m_2x}$. Thus, the general solution to (1) is*

$$y = C_1 e^{m_1x} + C_2 e^{m_2x}.$$

*(ii) If the roots of (2) are real and equal, say $m_1 = m_2 = m$, then two LI solutions of (1) are $e^{mx}$ and $xe^{mx}$. Thus, the general solution to (1) is*

$$y = (C_1 + C_2 x)e^{mx}.$$

*(iii) If the roots of (2) are complex conjugate, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two real LI solutions of (1) are $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$. Thus, the general solution to (1) is*

$$y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

**Proof**: For convenience (specially for higher order ODE) (1) is written in the operator form $L(y) = 0$, where

$$L \equiv a\frac{d^2}{dx^2} + b\frac{d}{dx} + c.$$

We also sometimes write $L$ as

$$L \equiv aD^2 + bD + c,$$

where $D = d/dx$. Now

$$L(e^{mx}) = (am^2 + bm + c)e^{mx} = p(m)e^{mx}, \tag{3}$$

where $p(m) = am^2 + bm + c$. Thus, $e^{mx}$ is a solution of (1) if $p(m) = 0$.

(i) If $p(m) = 0$ has two distinct real roots $m_1, m_2$, then both $e^{m_1x}$ and $e^{m_2x}$ are solutions of (1). Since, $m_1 \neq m_2$, they are also LI. Thus, the general solution to (1) is

$$y = C_1 e^{m_1x} + C_2 e^{m_2x}.$$

**Example 1.** *Solve* $y'' - y' = 0$

**Solution**: The characteristic equation is $m^2 - m = 0 \Rightarrow m = 0, 1$. The general solution is $y = C_1 + C_2 e^x$

(ii) If $p(m) = 0$ has real equal roots $m_1 = m_2 = m$, then $e^{mx}$ is a solution of (1). To find the other solution, note that if $m$ is repeated root, then $p(m) = p'(m) = 0$. This suggests differentiating (3) w.r.t. $m$. Since $L$ consists of differentiation w.r.t. $x$ only,

$$\frac{\partial}{\partial m}\Big(L(e^{mx})\Big) = L\left(\frac{\partial}{\partial m}e^{mx}\right) = L(xe^{mx}).$$

$$L(xe^{mx}) = p(m)xe^{mx} + p'(m)e^{mx},$$

where $'$ represents the derivative. Since, $m$ is a repeated root, the RHS is zero. Thus, $xe^{mx}$ is also a solution to (1) and it is independent of $e^{mx}$. Hence, the general solution to (1) is

$$y = (C_1 + C_2 x)e^{mx}.$$

**Example 2.** *Solve* $y'' - 2y' + y = 0$

**Solution**: The characteristic equation is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$. The general solution is $y = (C_1 + C_2 x)e^x$

(iii) If the roots of (2) are complex conjugate, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two LI solutions are $Y_1 = e^{(\alpha+i\beta)x}$ and $Y_2 = e^{(\alpha-i\beta)x}$. But these are complex valued. Note that if $Y_1, Y_2$ are LI, then so does $y_1 = (Y_1 + Y_2)/2$ and $y_2 = (Y_1 - Y_2)/2i$. Hence, two real LI solutions of (1) are $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$. Thus, the general solution to (1) is

$$y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

**Example 3.** *Solve* $y'' - 2y' + 5y = 0$

**Solution**: The characteristic equation is $m^2 - 2m + 5 = 0 \Rightarrow m = 1 \pm 2i$. The general solution is $y = e^x(C_1 \cos 2x + C_2 \sin 2x)$

# 2 Homogeneous $n$-th order linear equation with constant coefficients

Now the ODE is of the form

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + a_2 y^{(n-2)}(x) + \cdots + a_{n-1} y^{(1)}(x) + a_n y(x) = 0, \qquad x \in \mathcal{I}, \quad (4)$$

where the superscript $(i)$ denotes the $i$-th derivative and all $a_i$'s are constants. As in the case of 2nd order linear equation, the LI solutions of (4) depends on the characteristic equations

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \qquad (5)$$

Obviously, this equation has $n$ roots. As in the case of 2nd order equation, the following can be proved.

**Theorem 2.** *The fundamental set of solutions $\mathcal{B}$ for (4) is obtained using the following two rules:*

**Rule 1**: *If a root $m$ of (5) is real and repeated $k$ times, then this root gives $k$ number of LI solutions $e^{mx}, xe^{mx}, x^2e^{mx}, \cdots, x^{k-1}e^{mx}$ to $\mathcal{B}$.*

**Rule 2**: *If the roots $m = \alpha \pm i\beta$ of (5) is complex conjugate ($\beta \neq 0$) and are repeated $k$ times each, then they contribute $2k$ number of LI solutions $e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)$, $xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), x^2e^{\alpha x}\cos(\beta x), x^2e^{\alpha x}\sin(\beta x), \cdots\cdots, x^{k-1}e^{\alpha x}\cos(\beta x)$ and $x^{k-1}e^{\alpha x}\sin(\beta x)$ to $\mathcal{B}$.*

**Example 4.** *Solve* $y^{(5)}(x) + y^{(4)}(x) - 2y^{(3)}(x) - 2y^{(2)}(x) + y^{(1)}(x) + y = 0$

**Solution**: The characteristic equation is $m^5 + m^4 - 2m^3 - 2m^2 + m + 1 = 0 \Rightarrow (m+1)^3(m-1)^2 = 0 \Rightarrow m = -1, -1, -1, 1, 1$. The general solution is $y = e^{-x}(C_1 + C_2x + C_3x^2) + e^x(C_4 + C_5x)$

**Example 5.** *Solve* $y^{(6)}(x)+8y^{(5)}(x)+25y^{(4)}(x)+32y^{(3)}(x)-y^{(2)}(x)-40y^{(1)}(x)-25y = 0$

The characteristic equation is $m^6 + 8m^5 + 25m^4 + 32m^3 - m^2 - 40m - 25 = 0 \Rightarrow (m+1)(m-1)(m^2+4m+5)^2 = 0 \Rightarrow m = -1, 1, -2 \pm i, -2 \pm i$. The general solution is $y = C_1e^{-x} + C_2e^x + e^{-2x}\Big((C_3 + C_4x)\cos x + (C_5 + C_6x)\sin x\Big)$

Lecture IX

Non-homogeneous linear ODE, method of undetermined coefficients

# 1 Non-homogeneous linear equation

We shall mainly consider 2nd order equations. Extension to the $n$-th order is straight forward.

Consider a 2nd order linear ODE of the form

$$y'' + p(x)y' + q(x)y = r(x), \qquad x \in \mathcal{I}, \tag{1}$$

where $p, q$ are continuous functions. Let $y_p(x)$ is a (particular) solution to (1). Then

$$y_p'' + p(x)y_p' + q(x)y_p = r(x).$$

Let $y$ be any solution to (1). Now consider $Y = y - y_p$ satisfies

$$Y'' + p(x)Y' + q(x)Y = 0.$$

Thus, $Y$ satisfies a homogeneous linear 2nd order ODE. Hence, we express $Y$ as linear combination of two LI solutions $y_1$ and $y_2$. This gives

$$Y = C_1y_1 + C_2y_2$$

or

$$y = C_1y_1 + C_2y_2 + y_p \tag{2}$$

Thus, the general solution to (1) is given by (2). We have seen how to find $y_1$ and $y_2$. Here, we concentrate on methods to find $y_p$.

## 1.1 Method of undetermined coefficients

This method works for the following nonhomogeneous linear equation:

$$ay'' + by' + cy = r(x), \qquad x \in \mathcal{I}, \tag{3}$$

where $a, b, c$ are constants and $r(x)$ is a finite linear combination of products formed from the polynomial, exponential and sines or cosines functions. Thus, $r(x)$ is a finite linear combination of functions of the following form:

$$e^{\alpha x}x^m \left\{ \begin{array}{l} \sin\beta x \\ \cos\beta x \end{array} \right. ,$$

where $m$ is a nonnegative integer.

Suppose

$$r(x) = r_1(x) + r_2(x) + \cdots + r_n(x).$$

If $y_{pi}(x),\ 1 \le i \le n$ is a particular solution to

$$ay'' + by' + cy = r_i(x),$$

then it is trivial to prove that

$$y_p(x) = y_{p1}(x) + y_{p2}(x) + \cdots + y_{pn}(x)$$

is a particular solution to

$$ay'' + by' + cy = r(x).$$

Hence, we shall consider the case when $r(x)$ is one of the $r_i(x)$. Thus, we choose $r(x)$ to be of the following form:

$$e^{\alpha x} x^m \begin{cases} \sin \beta x \\ \cos \beta x \end{cases}.$$

**Rule 1**: If none of the terms in $r(x)$ is a solution of the homogeneous problem, then for $y_p$, choose a linear combination of $r(x)$ and all its derivatives that form a finite set of linearly independent functions.

**Example 1.** *Consider*

$$y'' - 2y' + 2y = x \sin x.$$

**Solution**: The LI solutions of the homogeneous part are $e^x \cos x$ and $e^x \sin x$. Clearly, neither $x$ nor $\sin x$ is a solution of the homogeneous part. Hence, we choose

$$y_p(x) = ax \sin(x) + bx \cos x + c \cos x + d \sin x.$$

Now substituting into the governing equation, we get

$$(a+2b)x \sin x + (-2a+b)x \cos x + (2a-2b-2c+d) \cos x + (-2a-2b+c+2d) \sin x = x \sin x.$$

Hence

$$a + 2b = 1, \quad -2a + b = 0, \quad (2a - 2b - 2c + d) = 0, \quad (-2a - 2b + c + 2d) = 0.$$

Solving, we get

$$a = \frac{1}{5},\ b = \frac{2}{5},\ c = \frac{2}{25},\ d = \frac{14}{25}$$

Hence, the general solution is

$$y = e^x(C_1 \cos x + C_2 \sin x) + \frac{1}{5} x \sin(x) + \frac{2}{5} x \cos x + \frac{2}{25} \cos x + \frac{24}{25} \sin x$$

**Aliter:** (*Annihilator method*) Writing $D \equiv d/dx$, we write

$$(D^2 - 2D + 2)y_p = x \sin x.$$

Note that $(D^2 + 1)^2 x \sin x = 0$. Hence, operating $(D^2 + 1)^2$ on both sides, we find

$$(D^2 + 1)^2 (D^2 - 2D + 2) y_p = 0.$$

The characteristic roots are found from $(m^2 + 1)^2 (m^2 - 2m + 2) = 0$. Thus, $m = -1 \pm i$ and $m = \pm i, \pm i$. Now solution to this homogeneous linear ODE with constant coefficient is

$$y_p = e^x(c_1 \cos x + c_2 \sin x) + (c_3 \cos x + c_4 \sin x) + x(c_5 \cos x + c_6 \sin x)$$

Since, the first two terms are the solution of the original homogeneous part and hence contribute nothing. Thus, the form for $y_p$ must be

$$y_p = (c_3 \cos x + c_4 \sin x) + x(c_5 \cos x + c_6 \sin x),$$

which conforms with previous form.

**Rule 2**: If $r(x)$ contains terms that are solution of the homogeneous linear part, then to choose the trial form of $y_p$ follow the following steps. First, choose a linear combination of $r(x)$ and its derivatives which are LI. Second, this linear combination is multiplied by a power of $x$, say $x^k$, where $k$ is the smallest nonnegative integer that makes all the new terms not to be solutions of the homogeneous problem.

**Example 2.** *Consider*

$$y'' - 2y' - 3y = xe^{-x}.$$

**Solution**: The LI solutions of the homogeneous part are $e^{-x}$ and $e^{3x}$. Clearly, $e^{-x}$ is a solution of the homogeneous part. Hence, we choose $y_p(x) = x(axe^{-x} + be^{-x})$. Substituting, we find

$$e^{-x}(-4b + 2a - 8ax) = xe^{-x}$$

This, gives $-4b + 2a = 0, -8a = 1$ and thus $a = -1/8, b = -1/16$. Thus, the general solution is

$$y = C_1 e^{-x} + C_2 e^{3x} - \frac{xe^{-x}}{16}(2x + 1)$$

**Aliter:** (*Annihilator method*) Writing $D \equiv d/dx$, we write

$$(D^2 - 2D - 3)y_p = xe^{-x}.$$

Since $(D + 1)^2 xe^{-x} = 0$, operating $(D + 1)^2$ on both sides we find

$$(D + 1)^2(D^2 - 2D - 3)y_p = 0$$

The characteristic roots are found from $(m + 1)^2(m^2 - 2m - 3) = 0$. Thus, $m = -1, -1, -1, 3$. Now solution to this homogeneous linear ODE with constant coefficient is

$$y_p = c_1 e^{3x} + e^{-x}(c_2 + c_3 x + c_4 x^2)$$

Since, the first two terms are the solution of the original homogeneous part and hence contribute nothing. Thus, the form for $y_p$ must be

$$y_p = e^{-x}(c_3 x + c_4 x^2),$$

which conforms with the previous form.

**Example 3.** *Consider*

$$y'' - 2y' + y = 6xe^x$$

**Solution**: The LI solutions of the homogeneous part are $e^x$ and $xe^x$. Clearly, both $e^x, xe^x$ are solutions of the homogeneous part. Hence, we choose $y_p(x) = x^2(axe^x + be^x)$. Substituting, we find

$$e^x(2b + 3ax) = 6xe^x$$

This, gives $a = 1, b = 0$. Thus, the general solution is

$$y = e^x(C_1 + C_2 x + x^3)$$

**Example 4.** *Consider*

$$y''' - 3y'' + 2y' = 10 + 4xe^{2x}.$$

**Solution**: The LI solutions of the homogeneous part related to the characteristic equation

$$m^3 - 3m^2 + 2m = 0 \quad \Rightarrow \quad m = 0, 1, 2.$$

Thus the LI solutions are $1, e^x$ and $e^{2x}$. Clearly, $e^{2x}$ and 10 are solutions of the homogeneous part. Hence, we choose $y_p(x) = ax + x(bxe^{2x} + ce^{2x})$. Substituting, we find

$$2a + (6b + 2c)e^{2x} + 4bxe^{2x} = 10 + 4xe^{2x}$$

This, gives $a = 5, b = 1$ and $c = -3$ Thus, the general solution is

$$y = C_1 + C_2e^x + C_3e^{2x} + 5x + e^{2x}(x^2 - 3x)$$

**Aliter:** (*Annihilator method*) Writing $D \equiv d/dx$, we write

$$(D^3 - 3D + 2D)y_p = 10 + 4xe^{2x}.$$

To annihilate 10 we apply $D$ and to annihilate $xe^{2x}$, we apply $(D - 2)^2$. Thus,

$$D(D - 2)^2(D^3 - 3D + 2D)y_p = 0$$

The characteristic roots are found from $m(m - 2)^2(m^3 - 3m + 2m) = 0$. Thus, $m = 0, 0, 1, 2, 2, 2$. Now solution to this homogeneous linear ODE with constant coefficient is

$$y_p = c_1 + c_2x + c_3e^x + e^{2x}(c_4 + c_5x + c_6x^2)$$

The terms with $c_1, c_3$ and $c_4$ are the solution of the original homogeneous part and hence contribute nothing. Thus, the form for $y_p$ must be

$$y_p = c_2x + e^{2x}(c_5x + c_6x^2).$$

which conforms with the previous form.

# Lecture X

Non-homegeneous linear ODE, method of variation of parameters

## 0.1 Method of variation of parameters

Again we concentrate on 2nd order equation but it can be applied to higher order ODE. This has much more applicability than the method of undetermined coefficeints. First, the ODE need not be with constant coefficeints. Second, the nonhomogeneos part $r(x)$ can be a much more general function.

**Theorem 1.** *A particular solution $y_p$ to the linear ODE*

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

*is given by*

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(y_1,y_2)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(y_1,y_2)}dx, \tag{2}$$

*where $y_1$ and $y_2$ are basis solutions for the homogeneous counterpart*

$$y'' + p(x)y' + q(x)y = 0.$$

**Important note**: *The (leading) coefficeint of $y''$ in (1) must be unity. If it is not unity, then make it unity by dividing the ODE by the leading coefficeint.*

**Proof**: We try for $y_p$ of the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$

where $u(x)$ and $v(x)$ are unknown functins. Now $y_p$ should satisfy (1). First, we find $y_p'(x) = u'(x)y_1(x) + v'(x)y_2(x) + u(x)y_1'(x) + v(x)y_2'(x)$. Now to make calculations easier (!), we take

$$u'(x)y_1(x) + v'(x)y_2(x) = 0. \tag{3}$$

Next we find $y_p''(x) = u'(x)y_1'(x) + v'(x)y_2'(x) + u(x)y_1''(x) + v(x)y_2''(x)$. Subtituting $y_p(x), y_p'(x)$ and $y_p''(x)$ into (1) (and using the fact that $y_1$ and $y_2$ are solution of the homogeneos part), we get

$$u'(x)y_1'(x) + v'(x)y_2'(x) = r(x). \tag{4}$$

We solve $u', v'$ from (3) and (4) as follows (Cramer's rule):

$$u' = -\frac{r(x)y_2(x)}{W(y_1,y_2)}, \quad v' = \frac{r(x)y_1(x)}{W(y_1,y_2)}$$

Integrating we find

$$u = -\int \frac{y_2(x)r(x)}{W(y_1,y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1,y_2)}dx.$$

Substituting $u$ and $v$ in $y_p(x) = y_1(x)u(x) + y_2(x)v(x)$, we find the required form of $y_p$ given in (2).

**Note**: *We don't write constant of integration in the expression of $u$ and $v$, since these can be absorbed with the constants of the genral solution of the homogeneous part.*

**Example 1.** *Consider*

$$y'' - 2y' - 3y = xe^{-x}.$$

(This has been solved before by the method of undetermined coefficeints.) The LI solutions of the homogenous part are $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$. Hence,

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Now $W(y_1, y_2) = 4e^{2x}$. Hence,

$$u(x) = -\int \frac{x}{4}dx = -\frac{x^2}{8}$$

$$v(x) = \int \frac{xe^{-4x}}{4}dx = -\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}$$

Thus,

$$y_p(x) = -\frac{x^2}{8}e^{-x} + e^{3x}\left(-\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}\right)$$

Hence, the general solution is

$$y(x) = C_1e^{-x} + C_2e^{3x} + y_p(x)$$

Since, the last term of $y_p$ can be absorbed with the constant $C_1$, we get the same solution as obatined before.

**Example 2.** *Consider*

$$y'' + y = \tan x.$$

**Solution**: (This can not be solved by the method of undetermined coefficeint.) The LI solutions of the homogenous part are $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Hence,

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Now $W(y_1, y_2) = 1$. Hence,

$$u(x) = -\int \sin x \tan x\, dx = -\ln|\sec x + \tan x| + \sin x$$

$$v(x) = \int \sin x\, dx = -\cos x$$

Thus,

$$y_p(x) = -\cos x \ln|\sec x + \tan x|$$

Hence, the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x + y_p(x)$$

**Example 3.** *Consider*

$$y'' + y = |x|, \qquad x \in (-1, 1)$$

**Solution**: You can find the general solution using either the method of undetermined coefficients (tricky!) OR method of variation of parameters. Try yourself.

**Example 4.** *Consider*

$$xy'' - (1+x)y' + y = x^2 e^{2x}, \qquad x > 0$$

This is linear but the coefficients are not constants. Note that $y_1(x) = e^x$ is a solution (by inspection!) of

$$xy'' - (1+x)y' + y = 0.$$

Let us first divide by the leading coefficient to find

$$y'' - \frac{(1+x)}{x} y' + \frac{1}{x} y = 0$$

Using reduction of order, we find

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{\int (1+x)/x \, dx} \, dx = (x+1)$$

The LI solutions of the homogenous part are $y_1(x) = e^x$ and $y_2(x) = (1+x)$. Divinding by the leading coefficient leads to

$$y'' - \frac{(1+x)}{x} y' + \frac{1}{x} y = xe^{2x}$$

Hence, $r(x) = xe^{2x}$. Now

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)} dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)} dx.$$

Here $W(y_1, y_2) = -xe^x$. Hence,

$$u(x) = \int (x+1)e^x \, dx = xe^x$$

$$v(x) = -\int e^{2x} \, dx = -\frac{1}{2} e^{2x}$$

Thus,

$$y_p(x) = \frac{(x-1)}{2} e^{2x}$$

Hence, the general solution is

$$y(x) = C_1 e^x + C_2(x+1) + y_p(x)$$

## 0.2 Method of variation of parameters: extension to higher order

We illustrate the method for the third order ODE

$$y''' + a(x)y'' + b(x)y' + c(x)y = r(x). \tag{5}$$

Note that the leading coefficeint is again unity. Suppose the three LI solutions to (5) are $y_1, y_2$ and $y_3$. As before let

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) + w(x)y_3(x). \tag{6}$$

We find

$$y_p'(x) = u'(x)y_1(x) + v'(x)y_2(x) + w'(x)y_3(x) + u(x)y_1'(x) + v(x)y_2'(x) + w(x)y_3'(x)$$

As before for the ease of computation (!) we set

$$u'(x)y_1(x) + v'(x)y_2(x) + w'(x)y_3(x) = 0 \tag{7}$$

Now

$$y_p''(x) = u'(x)y_1'(x) + v'(x)y_2'(x) + w'(x)y_3'(x) + u(x)y_1''(x) + v(x)y_2''(x) + w(x)y_3''(x)$$

Again for the ease of computation (!!), we set

$$u'(x)y_1'(x) + v'(x)y_2'(x) + w'(x)y_3'(x) = 0 \tag{8}$$

Further

$$y_p''' = u'(x)y_1''(x) + v'(x)y_2''(x) + w'(x)y_3''(x) + u(x)y_1'''(x) + v(x)y_2'''(x) + w(x)y_3'''(x)$$

Subtituting $y_p(x), y_p'(x), y_p''(x)$ and $y_p'''(x)$ into (5) (and using the fact that $y_1, y_2$ and $y_3$ are solutions of the homogeneos part), we get

$$u'(x)y_1''(x) + v'(x)y_2''(x) + w'(x)y_3'(x) = r(x). \tag{9}$$

Now we find $u', v', w'$ from (7),(8) and (7) by Cramer's rule. Let

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

be the Wronskian of three LI solutions. (Wronskian can similarly defined for $n$ LI solutions). Then

$$u' = \frac{W_1}{W(y_1, y_2, y_3)},\ v' = \frac{W_2}{W(y_1, y_2, y_3)}\ w' = \frac{W_3}{W(y_1, y_2, y_3)}.$$

Here $W_i$ is the determinate obtained from $W(y_1, y_2, y_3)$ by replacing the $i$-th column by the column vector $(0, 0, r(x))^T$. Hence,

$$u = \int \frac{W_1}{W(y_1, y_2, y_3)}dx,\ v = \int \frac{W_2}{W(y_1, y_2, y_3)}dx,\ w = \int \frac{W_3}{W(y_1, y_2, y_3)}dx.$$

These $u, v, w$ are then substituted into (6) to get $y_p$.

Lecture XI
Euler-Cauchy Equation

# 1 Homogeneous Euler-Cauchy equation

If the ODE is of the form

$$ax^2y'' + bxy'' + cy = 0, \tag{1}$$

where $a, b$ and $c$ are constants; then (1) is called homogeneous Euler-Cauchy equation. Two linearly independent solutions (i.e. basis) depend on the quadratic equation

$$am^2 + (b-a)m + c = 0. \tag{2}$$

Equation (2) is called *characteristic equation* for (1). The ODE (1) is singular at $x = 0$. Hence, we solve (1) for $x \neq 0$. We consider the case when $x > 0$.

**Theorem 1.** *(i) If the roots of (2) are real and distinct, say $m_1$ and $m_2$, then two linearly independent (LI) solutions of (1) are $x^{m_1}$ and $x^{m_2}$. Thus, the general solution to (1) is*

$$y = C_1x^{m_1} + C_2x^{m_2}.$$

*(ii) If the roots of (2) are real and equal, say $m_1 = m_2 = m$, then two LI solutions of (1) are $x^m$ and $x^m \ln x$. Thus, the general solution to (1) is*

$$y = (C_1 + C_2 \ln x)x^m.$$

*(iii) If the roots of (2) are complex conjugate, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two real LI solutions of (1) are $x^\alpha \cos(\beta \ln x)$ and $x^\alpha \sin(\beta \ln x)$. Thus, the general solution to (1) is*

$$y = x^\alpha \Big(C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\Big).$$

**Proof**: We have seen that the trial solution for a constant coefficient equation is $e^{mx}$. Now since power of $x^m$ is reduced by 1 by a differentiation, let us take $x^m$ as trial solution for (1).

For convenience, (1) is written in the operator form $L(y) = 0$, where

$$L \equiv ax^2\frac{d^2}{dx^2} + bx\frac{d}{dx} + c.$$

We also sometimes write $L$ as

$$L \equiv ax^2D^2 + bxD + c,$$

where $D = d/dx$. Now

$$L(x^m) = \Big(am(m-1) + bm + c\Big)x^m = p(m)x^m, \tag{3}$$

where $p(m) = am^2 + (b-a)m + c$. Thus, $x^m$ is a solution of (1) if $p(m) = 0$.

(i) If $p(m) = 0$ has two distinct real roots $m_1, m_2$, then both $x^{m_1}$ and $x^{m_2}$ are solutions of (1). Since, $m_1 \neq m_2$, they are also LI. Thus, the general solution to (1) is

$$y = C_1x^{m_1} + C_2x^{m_2}.$$

**Example 1.** *Solve* $x^2y'' - xy' - 3y = 0$

**Solution**: The characteristic equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3$. The general solution is $y = C_1/x + C_2x^3$

(ii) If $p(m) = 0$ has real equal roots $m_1 = m_2 = m$, then $x^m$ is a solution of (1). To find the other solution, note that if $m$ is repeated root, then $p(m) = p'(m) = 0$. This suggests differentiating (3) w.r.t. $m$. Since $L$ consists of differentiation w.r.t. $x$ only,

$$\frac{\partial}{\partial m}\Big(L(x^m)\Big) = L\left(\frac{\partial}{\partial m}x^m\right) = L(x^m \ln x).$$

Now

$$L(x^m \ln x) = \Big(p'(m) + p(m)\ln x\Big)x^m,$$

where $'$ represents the derivative. Since, $m$ is a repeated root, the RHS is zero. Thus, $x^m \ln x$ is also a solution to (1) and it is independent of $x^m$. Hence, the general solution to (1) is

$$y = (C_1 + C_2 \ln x)x^m.$$

**Example 2.** *Solve* $x^2y'' - 3xy' + 4y = 0$

**Solution**: The characteristic equation is $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$. The general solution is $y = (C_1 + C_2 \ln x)x^2$.

(iii) If $p(m) = 0$ has complex conjugate roots, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two LI solutions are

$$Y_1 = x^{(\alpha+i\beta)} = x^\alpha e^{i\beta \ln x}, \quad \text{and} \quad Y_2 = x^\alpha e^{-i\beta \ln x}.$$

But these are complex valued. Note that if $Y_1, Y_2$ are LI, then so are $y_1 = (Y_1 + Y_2)/2$ and $y_2 = (Y_1 - Y_2)/2i$. Hence, two real LI solutions of (1) are $y_1 = x^\alpha \cos(\beta \ln x)$ and $y_2 = x^\alpha \sin(\beta \ln x)$. Thus, the general solution to (1) is

$$y = x^\alpha\Big(C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\Big).$$

**Example 3.** *Solve* $x^2y'' - 3xy' + 5y = 0$

**Solution**: The characteristic equation is $m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i$. The general solution is $y = x^2\Big(C_1 \cos(\ln x) + C_2 \sin(\ln x)\Big)$

**Comment 1**: The solution for $x < 0$ can be obtained from that of $x > 0$ by replacing $x$ by $-x$ everywhere.

**Comment 2**: Homogeneous Euler-Cauchy equation can be transformed to linear constant coefficient homogeneous equation by changing the independent variable to $t = \ln x$ for $x > 0$.

**Comment 3**: This can be generalized to equations of the form

$$a(\gamma x + \delta)^2 y'' + b(\gamma x + \delta)y' + cy = 0.$$

In this case we consider $(\gamma x + \delta)^m$ as the trial solution.

# 2 Nonhomogeneous Euler-Cauchy equation

If the ODE is of the form

$$ax^2y'' + bxy'' + cy = \tilde{r}(x), \tag{4}$$

where $a, b$ and $c$ are constants; then (4) is called nonhomogeneous Euler-Cauchy equation. We can use the method of variation of parameters as follows. First divide (4) by $ax^2$ so that the coefficient of $y''$ becomes unity:

$$y'' + \frac{b}{ax}y'' + \frac{c}{ax^2}y = r(x), \tag{5}$$

where $r(x) = \tilde{r}(x)/ax^2$. Now we already know two LI solutions $y_1, y_2$ of the homogeneous part. Hence, the particular solution to
(4) is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Thus, the general solution to (4) is

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x).$$

**Example 4.**

**Comment:** In few cases, it can be solved also using method of undetermined coefficients. For this, we first convert it to constant coefficient liner ODE by $t = \ln x$. If the the transformed RHS is of special form then the method of undetermined coefficients is applicable.

**Example 5.** *Consider*

$$x^2y'' - xy' - 3y = \frac{\ln x}{x}, \qquad x > 0.$$

The characteristic equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3$. Hence $y_1 = 1/x$ and $y_2 = x^3$. Hence,

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Now $W(y_1, y_2) = 4x$ and $\boxed{r(x) = \ln x/x^3}$! Hence,

$$u(x) = -\int \frac{\ln x}{4x}dx = -\frac{(\ln x)^2}{8}$$

$$v(x) = \int \frac{\ln x}{4x^5}dx = -\frac{\ln x}{16x^4} - \frac{1}{64x^4}$$

Hence,

$$y_p(x) = -\frac{(\ln x)^2}{8x} - \frac{\ln x}{16x} - \frac{1}{64x}$$

Hence the general solution is $y = c_1y_1 + c_2y_2 + y_p$, i.e.

$$y(x) = \frac{A}{x} + Bx^3 - \frac{(\ln x)^2}{8x} - \frac{\ln x}{16x}.$$

Note that last term of $y_p$ is absorbed with $y_1$.

**Aliter**: Let us make the transformation $t = \ln x$. Then the given transformed to

$$\ddot{y} - 2\dot{y} - 3y = te^{-t},$$

where $\dot{} = d/dt$. This is the same problem we have solved in lecture 9 using method of undetermined coefficients. The solution is (see lecture 9)

$$y(t) = C_1e^{-t} + C_2e^{3t} - \frac{te^{-t}}{16}(2t + 1),$$

which in terms of original $x$ variable becomes

$$y(x) = \frac{C_1}{x} + C_2x^3 - \frac{\ln x}{16x}(2\ln x + 1),$$

Lecture XII

Power Series Solutions: Ordinary points

# 1 Analytic function

**Definition 1.** *Let $f$ be a function defined on an interval $\mathcal{I}$. We say $f$ is analytic at point $x_0 \in \mathcal{I}$ if $f$ can be expanded in a power series about $x_0$ which has a positive radius of convergence.*

Thus $f$ is analytic at $x = x_0$ if $f$ has the representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n. \tag{1}$$

Here $c_n$ are constant and (1) converges for $|x - x_0| < R$ where $R > 0$. Radius of convergence $R$ can be found from ratio test/root test.

If $f$ has power series representation (1), then its derivative exits in $|x-x_0| < R$. These derivatives are obtained by differentiating the RHS of (1) term by term. Thus,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1} \equiv \sum_{n=0}^{\infty} (n+1) c_{n+1} (x - x_0)^n, \tag{2}$$

and

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x - x_0)^{n-2} \equiv \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} (x - x_0)^n. \tag{3}$$

# 2 Ordinary points

Consider a linear 2nd order homogeneous ODE of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0,$$

where $a_0, a_1$ and $a_2$ are continuous in an interval $\mathcal{I}$. The points where $a_0(x) = 0$ are called *singular points*. If $a_0(x) \neq 0, \forall x \in \mathcal{I}$, then the above ODE can be written as (by dividing by $a_0(x)$)

$$y'' + p(x)y' + q(x)y = 0. \tag{4}$$

**Definition 2.** *A point $x_0 \in \mathcal{I}$ is called an ordinary point for (4) if $p(x)$ and $q(x)$ are analytic at $x = x_0$.*

**Theorem 1.** *Let $x_0$ be an ordinary point for (4). Then there exists a unique solution $y = y(x)$ of (4) which is also analytic at $x_0$ and satisfies $y(x_0) = K_0, y'(x_0) = K_1$ ($K_0, K_1$ are arbitrary constants). Further, if $p$ and $q$ have convergent power series expansion in $|x - x_0| < R$, ($R > 0$), then the power series expansion of $y$ is also convergent in $|x - x_0| < R$.*

**Example 1.** *Find power series solution around $x_0 = 0$ for*

$$(1 + x^2)y'' + 2xy' - 2y = 0.$$

**Solution**: $\Big($This can be solved by reduction of order technique since $Y_1 = x$ is a solution. The other solution is given by

$$Y_2(x) = Y_1(x)\int \frac{1}{x^2} e^{-\int 2x/(1+x^2)\,dx} dx = x\int\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx = -(1 + x\tan^{-1} x)$$

Thus, two LI solutions are $Y_1 = x$ and $Y_2 = 1 + x\tan^{-1} x\Big)$

Here $p(x) = 2x/(1 + x^2)$ and $q(x) = -2/(1 + x^2)$ are analytic at $x = 0$ with common radius of convergence $R = 1$. Let

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Now using (3), we get

$$(1 + x^2)y''(x) = \sum_{n=0}^{\infty}(n + 2)(n + 1)c_{n+2}x^n + \sum_{n=2}^{\infty} n(n - 1)c_n x^n.$$

Note that the summation in the last term can be taken from $n = 0$ since the contributions due to $n = 0$ and $n = 1$ vanish. Thus

$$(1 + x^2)y''(x) = \sum_{n=0}^{\infty}\Big[(n + 2)(n + 1)c_{n+2} + n(n - 1)c_n\Big]x^n.$$

Similarly

$$2xy'(x) = \sum_{n=0}^{\infty} 2nc_n x^n.$$

Substuting into the given ODE we find

$$\sum_{n=0}^{\infty}\Big[(n + 2)(n + 1)c_{n+2} + n(n - 1)c_n + 2nc_n - 2c_n\Big]x^n = 0.$$

Now all the coefficients of powers of $x$ must be zero. Hence,

$$(n+2)(n+1)c_{n+2} = -\Big(n(n-1)c_n+2nc_n-2c_n\Big) \Rightarrow c_{n+2} = -\frac{n-1}{n+1}c_n, \qquad n = 0, 1, 2, \cdots.$$

This enables us to find $c_n$ in terms $c_0$ or $c_1$. For $n = 0$ get

$$c_2 = c_0,$$

and for $n = 1$ we obtain

$$c_3 = 0.$$

Similarly, letting $n = 2, 3, 4, \cdots$ we find that $c_n = 0$, $n = 5, 7, 9, \cdots$, and

$$c_4 = -\frac{1}{3}c_2 = -\frac{1}{3}c_0, \quad c_6 = -\frac{3}{5}c_4 = \frac{1}{5}c_0, \cdots.$$

By induction we find that for $m = 1, 2, 3, \cdots$,

$$c_{2m} = (-1)^{m-1} \frac{1}{2m-1} c_0,$$

and

$$c_{2m+1} = 0.$$

Now we write

$$y(x) = c_0 y_1(x) + c_1 y_2(x),$$

where

$$y_1(x) = 1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots$$

OR

$$y_1(x) = 1 + x \sum_{m=0}^{\infty} (-1)^m \frac{1}{2m+1} x^{2m+1}$$

and

$$y_2(x) = x.$$

Here $c_0$ and $c_1$ are arbitrary. Thus, $y_1$ is a solution corresponding to $c_0 = 1, c_1 = 0$ and $y_2$ is a solution corresponding to $c_0 = 0, c_1 = 1$. They form a basis of solutions. Obviously $y_2$ being polynomial has radius of convergence $R = \infty$ and $y_1$ has $R = 1$. Thus, the power series solution is valid at least in $|x| < 1$. We can identify $y_1$ with $1 + x \tan^{-1} x$ obtained earlier.

**Comment:** In the above problem, it was possible to write the series (after substitution of $y = \sum_{n=0}^{\infty} c_n x^n$) in the form

$$\sum_{n=0}^{\infty} b_n x^n = 0,$$

which ultimately gives $b_n = 0,\ n = 0, 1, 2, \cdots$. Sometimes, we need to leave few terms outside of the summation OR define few new terms inside the summation. For example, consider

$$(1 + x^2) y'' + x^2 y = 0.$$

If we substitute $y = \sum_{n=0}^{\infty} c_n x^n$, then we find

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0. \tag{5}$$

This can be arranged in two different ways:

(A) Here we write (5) as

$$2c_2 + 3 \cdot 2 c_3 x + \sum_{n=2}^{\infty} \Big[ (n+2)(n+1) c_{n+2} + n(n-1) c_n + c_{n-2} \Big] x^n = 0$$

Hence $c_2 = 0,\ c_3 = 0,\ \ (n+2)(n+1) c_{n+2} + n(n-1) c_n + c_{n-2} = 0,\ n \geq 2$

(B) Here we write (5) as

$$\sum_{n=0}^{\infty} \Big[ (n+2)(n+1) c_{n+2} + n(n-1) c_n + c_{n-2} \Big] x^n = 0, \qquad c_{-2} = c_{-1} = 0.$$

Thus, $(n+2)(n+1) c_{n+2} + n(n-1) c_n + c_{n-2} = 0,\ \ n \geq 0, c_{-2} = c_{-1} = 0$

Lecture XIII

Legendre Equation, Legendre Polynomial

# 1 Legendre equation

This equation arises in many problems in physics, specially in boundary value problems in spheres:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{1}$$

where $\alpha$ is a constant.

We write this equation as

$$y'' + p(x)y' + q(x)y = 0,$$

where

$$p(x) = \frac{-2x}{1-x^2} \quad \text{and} \quad q(x) = \frac{\alpha(\alpha+1)}{1-x^2}.$$

Clearly $p(x)$ and $q(x)$ are analytic at the origin and have radius of convergence $R = 1$. Hence $x = 0$ is an ordinary point for (1). Assume

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Proceeding as in the case of example 1 in lecture note XII, we find

$$c_{n+2} = -\frac{(\alpha+n+1)(\alpha-n)}{(n+2)(n+1)} c_n, \qquad n = 0, 1, 2, \cdots$$

Taking $n = 0, 1, 2$ and $3$ we find

$$c_2 = -\frac{(\alpha+1)\alpha}{1\cdot 2} c_0, \quad c_3 = -\frac{(\alpha+2)(\alpha-1)}{1\cdot 2\cdot 3} c_1, \quad c_4 = \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{1\cdot 2\cdot 3\cdot 4} c_0,$$

and

$$c_5 = \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5} c_1.$$

By induction, we can prove that for $m = 1, 2, 3, \cdots$

$$c_{2m} = (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} c_0$$

$$c_{2m+1} = (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} c_1.$$

Thus, we can write

$$y(x) = c_0 y_1(x) + c_1 y_1(x),$$

where

$$y_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} x^{2m}, \tag{2}$$

and

$$y_2(x) = x+\sum_{m=1}^{\infty}(-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}. \tag{3}$$

Taking $c_0 = 1, c_1 = 0$ and $c_0 = 0, c_1 = 0$, we find that $y_1$ and $y_2$ are solutions of Legendre equation. Also, these are LI, since their Wronskian is nonzero at $x = 0$. The series expansion for $y_1$ and $y_2$ may terminate (in that case the corresponding solution has $R = \infty$), otherwise they have radius of convergence $R = 1$.

# 2 Legendre polynomial

We note that if $\alpha$ in (1) is a nonnegative integer, then either $y_1$ given in (2) or $y_2$ given in (3) terminates. Thus, $y_1$ terminates when $\alpha = 2m$ $(m = 0, 1, 2, \cdots)$ is nonnegative even integer:

$$\begin{array}{rcll} y_1(x) &=& 1, & (\alpha = 0), \\ y_1(x) &=& 1-3x^2, & (\alpha = 2), \\ y_1(x) &=& 1-10x^2+\frac{35}{3}x^4, & (\alpha = 4). \end{array}$$

Note that $y_2$ does not terminate when $\alpha$ is a nonnegative even integer.

Similarly, $y_2$ terminates (but $y_1$ does not terminate) when $\alpha = 2m+1$ $(m = 0, 1, 2, \cdots)$ is nonnegative odd integer:

$$\begin{array}{rcll} y_2(x) &=& x, & (\alpha = 1), \\ y_2(x) &=& x-\frac{5}{3}x^3, & (\alpha = 5), \\ y_2(x) &=& x-\frac{14}{3}x^2+\frac{21}{5}x^5, & (\alpha = 5). \end{array}$$

Notice that the polynomial solution of

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \tag{4}$$

where $n$ is nonnegative integer, is polynomial of degree $n$. Equation (4) is the same as (1) with $n$ replacing $\alpha$.

**Definition 1.** *The polynomial solution, denoted by $P_n(x)$, of degree $n$ of (4) which satisfies $P_n(1) = 1$ is called the Legendre polynomial of degree $n$.*

Let $\psi$ be a polynomial of degree $n$ defined by

$$\psi(x) = \frac{d^n}{dx^n}(x^2-1)^n. \tag{5}$$

Then $\psi$ is a solution of (4). To prove it, we proceed as follows: Assume $u(x) = (x^1-1)^n$. Then

$$(x^2-1)u^{(1)} = 2nxu. \tag{6}$$

Now we take $(n+1)$-th derivative of both sides of (6):

$$\left((x^2-1)u^{(1)}\right)^{(n+1)} = 2n(xu)^{(n+1)}. \tag{7}$$

Now we use Leibniz rule for the derivative of product two functions $f$ and $g$:

$$(f \cdot g)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k)},$$

which can be proved easily by induction.
Thus from (7) we get

$$(x^2-1)u^{(n+2)} + 2x(n+1)u^{(n+1)} + (n+1)nu^{(n)} = 2n\Big(xu^{(n+1)} + (n+1)u^{(n)}\Big).$$

Simplifying this and noting that $\psi = u^{(n)}$, we get

$$(1-x^2)\psi'' - 2x\psi' + n(n+1)\psi = 0.$$

Thus, $\psi$ satisfies (4). Note that we can write

$$\psi(x) = \Big((x+1)^n(x-1)^n\Big)^{(n)} = (x+1)^n n! + (x-1)s(x),$$

where $s(x)$ is a polynomial. Thus, $\psi(1) = 2^n n!$. Hence,

$$P_n(x) = \frac{1}{2^n n!}\psi(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{8}$$

# 3 Properties of Legendre polynomials

a. *Generating function*: The function $G(t,x)$ given by

$$G(t,x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

is called the generating function of the Legendre polynomials. It can be shown that for small $t$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

b. *Orthogonality*: The following property holds for Legendre polynomials:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1}, & \text{if } m = n. \end{cases}$$

c. *Fourier-Legendre series*: By using the orthogonality of Legendre polynomials, any piecewise continuous function in $-1 \le x \le 1$ can be expresses in terms of Legendre polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x),$$

where

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx.$$

Now

$$\sum_{n=0}^{\infty} c_n P_n(x) = \begin{cases} f(x), & \text{where } f \text{ is continuous} \\ \dfrac{f(x^-)+f(x^+)}{2}, & \text{where } f \text{ is discontinuous} \end{cases}$$

Lecture XIV

Frobenius series: Regular singular points

# 1 Singular points

Consider the second order linear homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \qquad x \in \mathcal{I} \tag{1}$$

Suppose that $a_0, a_1$ and $a_2$ are analytic at $x_0 \in \mathcal{I}$. If $a_0(x_0) = 0$, then $x_0$ is a *singular* point for (1).

**Definition 1.** *A point $x_0 \in \mathcal{I}$ is a regular singular point for (1) if (1) can be written as*

$$b_0(x)(x - x_0)^2 y'' + b_1(x)(x - x_0)y' + b_2(x)y = 0, \tag{2}$$

*where $b_0(x_0) \neq 0$ and $b_0, b_1, b_2$ are analytic at $x_0$.*

**Comment 1:** Since $b_0(x_0) \neq 0$, we get an equivalent definition of regular singular point by dividing (2) by $b_0(x)$. Thus, a point $x_0 \in \mathcal{I}$ is a regular singular point for (1) if (1) can be written as

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0, \tag{3}$$

where $p$ and $q$ are analytic at $x_0$.

**Comment 2:** Any singular point of (1) which is not regular is called irregular singular point.

**Example 1.** *Consider*

$$x^3 y'' - (1 - \cos x)y' + xy = 0$$

*The singular point $x_0 = 0$ is regular.*

**Example 2.** *Consider*

$$x^2(x - 1)^2 y'' + (\sin x)y' + (x - 1)y = 0$$

*The singular point $x_0 = 0$ is regular whereas $x_0 = 1$ is irregular.*

**Example 3.** *Euler-Cauchy equation:*

$$ax^2 y'' + bxy' + cy = 0,$$

*where $a, b, c$ are constants. Here $x_0 = 0$ is a regular singular point.*

For simplicity, we consider a second order linear ODE with a regular singular point at $x_0 = 0$. If $x_0 \neq 0$, it is easy to convert the given ODE to an equivalent ODE with regular singular point at $x_0 = 0$. For this, we substitute $t = x - x_0$ and let $z(t) = y(x_0 + t)$. Then (3) becomes

$$t^2 \ddot{z} + t\tilde{p}(t)\dot{z} + \tilde{q}(t)z = 0,$$

where $\dot{} = d/dt$. Thus, we consider following second order homogeneous linear ODE

$$x^2y'' + xp(x)y' + q(x)y = 0, \tag{4}$$

where $p, q$ are analytic at the origin.

**Ordinary point vs. regular singular point**: This can explained by taking two examples. Consider

$$y'' + y = 0,$$

which has 0 as the ordinary point. Note that the general solution is $y = c_1 \cos x + c_2 \sin x$. At the ordinary point $x_0 = 0$, we can find unique $c_1, c_2$ for a given $K_0, K_1$ such that $y(0) = K_0, y'(0) = K_1$. Thus, unique solution exists for initial conditions specified at the ordinary point.

Now consider the Euler-Cauchy equation

$$x^2y'' - 2xy' + 2y = 0,$$

for which $x_0 = 0$ is a regular singular point. The general solution is $y = c_1x + c_2x^2$. Now it is not possible to find unique values of $c_1, c_2$ for a given $K_0, K_1$ such that $y(0) = K_0, y'(0) = K_1$. Note that solution does not exist for $K_0 \neq 0$ since $y(0) = 0$.

## 2 Frobenius method

We would like to find two linearly independent solutions of (4) so that these form a basis solution for $x \neq 0$. *We find the basis solution for* $x > 0$. For $x < 0$, we substitute $t = -x$ and carry out similar procedure for $t > 0$.

If $p$ and $q$ in (4) are constants, then a solution of (4) is of the form $x^r$. But since $p$ and $q$ are power series, we assume that a solution of (4) can be represented by an extended power series

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \tag{5}$$

which is a product of $x^r$ and a power series. We also assume that $a_0 \neq 0$. We formally substitute (5) into (4) and find $r$ and $a_1, a_2, \cdots$ in terms of $a_0$ and $r$. Once we find (5), we next check the convergence of the series. If it converges, then (5) becomes solution for (4).

Now from (5), we find

$$x^2y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}, \quad xy'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}.$$

Since $p$ and $q$ are analytic, we write

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Substituting into (4), we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0.$$

OR

$$x^r \sum_{n=0}^{\infty} \left[ (r+n)(r+n-1)a_n + \sum_{k=0}^{n} \Big( (r+k)p_{n-k} + q_{n-k} \Big) a_k \right] x^n = 0.$$

Since $x > 0$, this becomes

$$\sum_{n=0}^{\infty} \left[ (r+n)(r+n-1)a_n + \sum_{k=0}^{n} \Big( (r+k)p_{n-k} + q_{n-k} \Big) a_k \right] x^n = 0. \tag{6}$$

Thus, we must have

$$[(r+n)(r+n-1)+(n+r)p_0+q_0]a_n + \sum_{k=0}^{n-1} [(r+k)p_{n-k}+q_{n-k}]a_k = 0, \quad n = 0, 1, 2, \cdots \tag{7}$$

Now (7) gives $a_n$ in terms of $a_0, a_1, \cdots, a_{n-1}$ and $r$.
For $n = 0$, we find

$$r(r-1) + p_0 r + q_0 = 0, \tag{8}$$

since $a_0 \neq 0$. Equation (8) is called indicial equation for (4). The form of the linearly independent solutions of (4) depends on the roots of (8).
Let $\rho(r) = r(r-1) + p_0 r + q_0$. Then for $n = 1, 2, \cdots$, we find

$$\rho(r+n)a_n + b_n = 0,$$

where

$$b_n = \sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}]a_k.$$

Notice that $b_n$ is a linear combination of $a_0, a_1, \cdots, a_{n-1}$. Thus, we can find $a_n$ uniquely in terms of $r$ and $a_0$ if $\rho(r+n) \neq 0$. If $\rho(r+n) = 0$, then it is possible to find value of $a_n$ in certain cases.
Let $r_1, r_2$ be the roots of the indicial equation (8). We assume that the roots are real and $r_1 \geq r_2$. For $r_1$, clearly $\rho(r_1+n) \neq 0$ for $n = 1, 2, \cdots$. Thus, we can determine $a_1, a_2, a_3, \cdots$ corresponding to $r_1$. Clearly, one Frobenius series (extended power series) solution $y_1$ corresponding to the larger root $r_1$ always exists. Suppose $a_0 = 1$, then

$$y_1(x) = x^{r_1} \Big( 1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \Big). \tag{9}$$

Now for $r = r_2$, three cases may appear. These are as follows:

A. $r_1 - r_2$ **is not a nonnegative integer**: Then $r_2 + n \neq r_1$ for any integer $n \geq 1$ and as a result $\rho(r_2+n) \neq 0$ for any $n \geq 1$.Thus, we can determine $a_1, a_2, a_3, \cdots$ corresponding to $r_2$. Clearly, another Frobenius series solution $y_2$ corresponding to the smaller root exists. Suppose $a_0 = 1$, then

$$y_2(x) = x^{r_2} \Big( 1 + \sum_{n=1}^{\infty} a_n(r_2) x^n \Big). \tag{10}$$

B. $r_1 = r_2$**, double root**: Clearly a second extended power series (Frobenius series) solution does not exist.

C. $r_1 - r_2 = m$**,** $m \geq 1$ **is a positive integer**: In this case $\rho(r_2 + m) = \rho(r_1) = 0$. Thus, we can find $a_1, a_2, \cdots, a_{m-1}$. But for $a_m$, we have

$$\rho(r_2 + m)a_m = -b_m.$$

Since $\rho(r) = (r - r_1)(r - r_2)$, we have

$$\rho(r + m) = (r + m - r_1)(r + m - r_2) = (r - r_2)(r + m - r_2).$$

Clearly two cases may arise here:

C.i $b_m$ has a factor $r - r_2$, i.e. $b_m(r_2) = 0$. In this case, we cancel factor $r - r_2$ from both sides and find $a_m(r_2)$ as a finite number. Then we can continue calculating remaining coefficients $a_{m+1}, a_{m+2}, \cdots$. Hence, a second Frobenius series solution exists.

C.ii On the other hand, if $b_m(r_2) \neq 0$, then it is not possible to continue the calculations of $a_n$ for $n \geq m$. Hence, a second Frobenius series solution does not exist.

To find the form of the solution in the case of B and C described above, we use the reduction of order technique. We know that $y_1(x)$ (corresponding the larger root) always exists. Let $y_2(x) = v(x)y_1(x)$. Then

$$\begin{aligned} v' &= \frac{1}{y_1^2} e^{-\int p(x)/x\, dx} \\ &= \frac{1}{x^{2r_1}\Big(1 + a_1(r_1)x + a_2(r_1)x^2 + \cdots\Big)^2} e^{-p_0 \ln x - p_1 x - \cdots} \\ &= \frac{1}{x^{2r_1+p_0}\Big(1 + a_1(r_1)x + a_2(r_1)x^2 + \cdots\Big)^2} e^{-p_1 x - \cdots} \\ &= \frac{1}{x^{2r_1+p_0}} g(x), \end{aligned}$$

where $g(x)$ is analytic at $x = 0$ and $g(0) = 1$. Since $g(x)$ is analytic at $x = 0$ with $g(0) = 1$, we must have $g(x) = 1 + \sum_{n=1}^{\infty} g_n x^n$. Since $r_1, r_2$ are roots of (8), we must have

$$r_1 + r_2 = 1 - p_0 \Rightarrow 2r_1 + p_0 = m + 1.$$

Hence,

$$v' = \frac{1}{x^{m+1}} + \frac{g_1}{x^m} + \cdots + \frac{g_{m-1}}{x^2} + \frac{g_m}{x} + g_{m+1} + \cdots,$$

OR

$$v(x) = \frac{x^{-m}}{-m} + \frac{g_1 x^{-m+1}}{-m+1} + \cdots + \frac{g_{m-1}x^{-1}}{-1} + g_m \ln x + g_{m+1}x + \cdots. \tag{11}$$

Thus,

$$\begin{aligned} y_2(x) &= y_1(x)\left[\frac{x^{-m}}{-m} + \frac{g_1 x^{-m+1}}{-m+1} + \cdots + \frac{g_{m-1}x^{-1}}{-1} + g_m \ln x + g_{m+1}x + \cdots\right] \\ &= g_m y_1(x)\ln x + x^{r_1}\Big(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\Big)\left[\frac{x^{-m}}{-m} + \frac{g_1 x^{-m+1}}{-m+1} + \cdots + \frac{g_{m-1}x^{-1}}{-1} + g_{m+1}x + \cdots\right] \end{aligned}$$

Now we take the factor $x^{-m}$ from the series inside the third bracket. Since $r_1 - m = r_2$, we finally find

$$y_2(x) = cy_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} c_n x^n, \tag{12}$$

where we put $g_m = c$.

Now for $r_1 = r_2$, we have $m = 0$ and hence $g_m = g_0 = g(0) = 1 = c$. Thus, $\ln x$ term is definitely present in the second solution. Also in this case, the series in (11) starts with $g_0 \ln x$ and the next term is $g_1 x$. Hence, for $r_1 = r_2$, we must have $c_0 = 0$ in (12). In certain cases, $g_m = c$ becomes zero (case C.ii) for $m \geq 1$. Then the second solution is also a Frobenius series solution; otherwise, the second Frobenius series solution does not exist.

# 3 Summary

The results derived in the previous section can be summarized as follows. Consider

$$x^2 y'' + xp(x)y' + q(x)y = 0, \tag{13}$$

where $p$ and $q$ have convergent power series expansion in $|x| < R,\ R > 0$. Let $r_1, r_2$ ($r_1 \geq r_2$) be the roots of the indicial equation:

$$r^2 + (p(0) - 1)r + q(0) = 0 \tag{14}$$

For $x > 0$ we have the following theorems:

**Theorem 1.** *If $r_1 - r_2$ is not zero or a positive integer, then there are two linearly independent solutions $y_1$ and $y_2$ of (13) of the form*

$$y_1(x) = x^{r_1}\sigma_1(x), \quad y_2(x) = x^{r_2}\sigma_2(x), \tag{15}$$

*where $\sigma_1, \sigma_2$ are analytic at $x = 0$ with radius of convergence $R$ and $\sigma_1(0) \neq 0$ and $\sigma_2(0) \neq 0$.*

**Theorem 2.** *If $r_1 = r_2$, then there are two linearly independent solutions $y_1$ and $y_2$ of (13) of the form*

$$y_1(x) = x^{r_1}\sigma_1(x), \quad y_2(x) = (\ln x)y_1(x) + x^{r_2+1}\sigma_2(x), \tag{16}$$

*where $\sigma_1, \sigma_2$ are analytic at $x = 0$ with radius of convergence $R$ and $\sigma_1(0) \neq 0$.*

**Theorem 3.** *If $r_1 - r_2$ is a positive integer, then there are two linearly independent solutions $y_1$ and $y_2$ of (13) of the form*

$$y_1(x) = x^{r_1}\sigma_1(x), \quad y_2(x) = c(\ln x)y_1(x) + x^{r_2}\sigma_2(x), \tag{17}$$

*where $\sigma_1, \sigma_2$ are analytic at $x = 0$ with radius of convergence $R$ and $\sigma_1(0) \neq 0$ and $\sigma_2(0) \neq 0$. It may happen that $c = 0$.*

**Example 4.** *Discuss whether two Frobenius series solutions exist or do not exist for the following equations:*
*(i)* $2x^2y'' + x(x+1)y' - (\cos x)y = 0$,
*(ii)* $x^4y'' - (x^2 \sin x)y' + 2(1 - \cos x)y = 0$.

**Solution**: (i) We can write this as

$$x^2y'' + \frac{(x+1)}{2}xy' - \frac{\cos x}{2}y = 0.$$

Hence $p(x) = (x+1)/2$ and $q(x) = -\cos x/2$. Thus, $p(0) = 1/2$ and $q(0) = -1/2$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow 2r^2 - r - 1 = 0 \Rightarrow r_1 = 1, r_2 = -1/2.$$

Since $r_1 - r_2 = 3/2$, which is not zero or a positive integer, two Frobenius series solutions exist.
(ii) We can write this as

$$x^2y'' - \frac{\sin x}{x}xy' + 2\frac{1 - \cos x}{x^2}y = 0.$$

Hence $p(x) = -\sin x/x$ and $q(x) = 2(1 - \cos x)/x^2$. Thus, $p(0) = -1$ and $q(0) = 1$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r_1 = 1 = r_2.$$

Since $r_1 = r_2$, only one Frobenius series solutions exists.

**Example 5. (Case A)** *Find two independent solutions around $x = 0$ for*

$$2xy'' + (x+1)y' + 3y = 0$$

**Solution**: We write this as

$$x^2y'' + \frac{(x+1)}{2}xy' + (3x/2)y = 0.$$

Hence $p(x) = (x+1)/2$ and $q(x) = 3x/2$. Thus, $p(0) = 1/2, q(0) = 0$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow 2r^2 - r = 0 \Rightarrow r_1 = 1/2, r_2 = 0.$$

Since $r_1 - r_2 = 1/2$, is not zero or a positive integer, two independent Frobenius series solution exist.
Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling $x^r$) we find

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n + \sum_{n=1}^{\infty} \Big((n+r-1)+3\Big)a_{n-1}x^n = 0,$$

where $\rho(r) = r(2r-1)$. Rearranging the above, we get

$$\rho(r)a_0 + \sum_{n=1}^{\infty} \Big[\rho(n+r)a_n + (n+r+2)a_{n-1}\Big] x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(n+r)a_n + (n+r+2)a_{n-1} = 0 \ \text{ for } n \geq 1.$$

From the first relation we find roots of the indicial equation $r_1 = 1/2, r_2 = 0$. Now with the larger root $r = r_1 = 1/2$, we find

$$a_n = -\frac{(2n+5)a_{n-1}}{2n(2n+1)}, \quad n \geq 1.$$

Iterating we find

$$a_1 = -\frac{7}{6}a_0, \quad a_2 = \frac{21}{40}a_0, \cdots$$

Hence, by induction

$$a_n = (-1)^n \frac{(2n+5)(2n+3)}{15 \cdot 2^n n!} a_0, \quad n \geq 1 \qquad \text{(Check!)}$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = x^{1/2}\left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \cdots\right)$$

Now with $r = r_2 = 0$, we find

$$a_n = -\frac{(n+2)a_{n-1}}{n(2n-1)}, \quad n \geq 1.$$

Iterating we find

$$a_1 = -3a_0, \quad a_2 = 2a_0, \cdots$$

Hence, by induction

$$a_n = (-1)^n \left(\frac{5}{2n-1} - \frac{2}{n}\right)\left(\frac{5}{2n-3} - \frac{2}{n-1}\right)\cdots\left(\frac{5}{1} - \frac{2}{1}\right)a_0, \quad n \geq 1 \qquad \text{(Check!)}$$

Thus, taking $a_0 = 1$, we find

$$y_2(x) = \left(1 - 3x + 2x^2 - \cdots\right)$$

**Example 6. (Case B)** *Find the general solution in the neighbourhood of origin for*

$$4x^2y'' - 8x^2y' + (4x^2+1)y = 0$$

**Solution**: We write this as

$$x^2y'' - (2x)xy' + (x^2 + 1/4)y = 0.$$

Hence $p(x) = -2x$ and $q(x) = x^2 + 1/4$. Thus, $p(0) = 0, q(0) = 1/4$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 - r + 1/4 = 0 \Rightarrow r_1 = r_2 = 1/2.$$

Since the indicial equation has a double root, only one Frobenius series solution exists. Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling $x^r$) we find

$$\sum_{n=0}^{\infty} \rho(n+r) a_n x^n - \sum_{n=1}^{\infty} 8(n+r-1) a_{n-1} x^n + \sum_{n=2}^{\infty} 4a_{n-2} x^n = 0,$$

where $\rho(r) = (2r-1)^2$. Rearranging the above, we get

$$\rho(r)a_0 + \Big(\rho(r+1)a_1 - 8ra_0\Big)x + \sum_{n=2}^{\infty} \Big[\rho(n+r)a_n - 8(n+r-1)a_{n-1} + 4a_{n-2}\Big]x^n = 0.$$

Now with $r = 1/2$, we find

$$a_1 = a_0, \ a_n = \frac{(2n-1)a_{n-1}}{n^2} - \frac{a_{n-2}}{n^2}, \quad n \geq 2.$$

Iterating we find

$$a_2 = \frac{1}{2!}a_0, \quad a_3 = \frac{1}{3!}a_0, \quad a_4 = \frac{1}{4!}a_0, \cdots$$

Hence, by induction

$$a_n = \frac{1}{n!}a_0, \quad n \geq 1.$$

$\Big\{$Induction: Claim $a_k = a_0/k!$. True for $k = 1, 2$. Assume it is true for $k = m$. Now for $k = m+1$,

$$a_{k+1} = \frac{(2k+1)a_k}{(k+1)^2}a_0 - \frac{a_{k-1}}{(k+1)^2}a_0 = \frac{1}{(k-1)!(k+1)^2}\frac{k+1}{k}a_0 = \frac{a_0}{(k+1)!} \quad \Big\}$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = x^{1/2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x^{1/2}e^x.$$

For the general solution, we need to find another solution $y_2$. For this we use reduction of order. Let $y_2(x) = y_1(x)v(x)$. Then

$$v = \int \frac{1}{y_1^2} e^{-\int p dx}\, dx,$$

where $p(x) = -2$. Hence

$$v(x) = \int \frac{1}{x} dx = \ln x$$

and $y_2 = (\ln x)x^{1/2}e^x$. Thus, the general solution is

$$y(x) = x^{1/2}e^x(c_1 + c_2 \ln x)$$

**Example 7. (Case C.i)** *Find two independent solutions around $x = 0$ for*

$$xy'' + 2y' + xy = 0$$

**Solution**: We write this as

$$x^2y'' + 2xy' + x^2y = 0.$$

Hence $p(x) = 2$ and $q(x) = x^2$. Thus, $p(0) = 2, q(0) = 0$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 + r = 0 \Rightarrow r_1 = 0, r_2 = -1.$$

A Frobenius series solution exists for the larger root $r_1 = 0$. Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling $x^r$) we find

$$\sum_{n=0}^{\infty} \rho(n + r)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0,$$

where $\rho(r) = r(r + 1)$. Rearranging the above, we get

$$\rho(r)a_0 + \rho(r + 1)a_1 x + \sum_{n=2}^{\infty} \Big[\rho(n + r)a_n + a_{n-2}\Big]x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(r + 1)a_1 = 0, \quad \rho(n + r)a_n + a_{n-2} = 0 \quad \text{for } n \geq 2.$$

From the first relation we find roots of the indicial equation $r_1 = 0, r_2 = -1$. Now with the larger root $r = r_1$, we find

$$a_1 = 0, a_n = -\frac{a_{n-2}}{n(n + 1)}, \quad n \geq 2.$$

Iterating we find

$$a_2 = -\frac{1}{3!}a_0, \quad a_3 = 0, \quad a_4 = \frac{1}{5!}a_0, \cdots$$

Hence, by induction

$$a_{2n} = (-1)^n \frac{1}{(2n + 1)!}a_0, \quad a_{2n+1} = 0.$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = \frac{\sin x}{x}$$

Since $r_1 - r_2 = 1$, a positive integer, the second Frobenius series solution may or may not exist. Hence, to be sure, we need to compute it. With $r = r_2 = -1$, we find

$$0 \cdot a_1 = 0, \; a_n = -\frac{a_{n-2}}{n(n - 1)}, \quad n \geq 2.$$

Now the first relation can be satisfied by taking any value of $a_1$. For simplicity, we choose $a_1 = 0$. Iterating we find

$$a_2 = -\frac{1}{2!}a_0, \quad a_3 = 0, \quad a_4 = \frac{1}{4!}a_0, \cdots$$

Hence, by induction

$$a_{2n} = (-1)^n \frac{1}{(2n)!}a_0, \quad a_{2n+1} = 0.$$

Thus, indeed a second Frobenius series solution exists and taking $a_0 = 1$, we get

$$y_2(x) = x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = \frac{\cos x}{x}.$$

**Comment:** The second solution could have been obtained using reduction of order also. Suppose $y_2 = vy_1$, then

$$v = \int \frac{x^2}{\sin^2 x} e^{-\int 2/x\, dx} dx = \int \operatorname{cosec}^2 x\, dx = -\cot x.$$

Hence $y_2(x) = \cos x/x$ (disregarding minus sign)

**Example 8. (Case C.ii)** *Find general solution around $x = 0$ for*

$$(x^2 - x)y'' - xy' + y = 0$$

**Solution**: We write this as

$$x^2 y'' - \frac{x}{x-1}xy' + \frac{x}{x-1}y = 0.$$

Hence $p(x) = -x/(x-1)$ and $q(x) = x/(x-1)$. Thus, $p(0) = q(0) = 0$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 - r = 0 \Rightarrow r_1 = 1, r_2 = 0.$$

Since $r_1 - r_2 = 1$, a positive integer, two independent Frobenius series solution may or may not exist.
Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling $x^r$) we find

$$(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^n - x\sum_{n=0}^{\infty}(n+r)a_n x^n + x\sum_{n=0}^{\infty} a_n x^n = 0.$$

Rearranging the above, we get

$$(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^n - x\sum_{n=0}^{\infty}\Big((n+r) - 1\Big)a_n x^n = 0.$$

OR

$$x\sum_{n=0}^{\infty}(n+r-1)^2 a_n x^n - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^n = 0.$$

OR

$$\sum_{n=1}^{\infty}(n+r-2)^2 a_{n-1} x^n - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^n = 0.$$

OR

$$r(r-1)a_0 + \sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n - (n+r-2)^2 a_{n-1}\Big]x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(n+r)a_n - (n+r-2)^2 a_{n-1} = 0 \text{for } n \geq 1,$$

where $\rho(r) = r(r-1)$. From the first relation we find roots of the indicial equation $r_1 = 1, r_2 = 0$. Now with the larger root $r = r_1 = 1$, we find

$$a_n = \frac{(n-1)a_{n-1}}{n(n+1)}, \quad n \geq 1.$$

Iterating we find

$$a_n = 0, \qquad n \geq 1.$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = x$$

Now with $r = r_2 = 0$, we find

$$n(n-1)a_n = (n-2)^2 a_{n-1}, \quad n \geq 1.$$

Now for $n = 1$, we find $0 = a_0$ which is a contradiction. Hence, second Frobenius series solution does not exist. To find the second independent solution, we use reduction of order technique. Let $y_2(x) = v(x)y_1(x)$. Then

$$v(x) = \int \frac{1}{y_1^2} e^{-\int p dx} dx,$$

where $p(x) = -x/(x^2 - x) = -1/(x-1)$. Hence,

$$v(x) = \int \frac{1}{x^2} e^{\ln(1-x)}\, dx = \int \left(\frac{1}{x^2} - \frac{1}{x}\right) dx = -\left(\frac{1}{x} + \ln x\right).$$

(Why I wrote $\ln(1-x)$ NOT $\ln(x-1)$?) Hence, $y_2(x) = (1 + x\ln x)$ (disregarding the minus sign, since the ODE is homogeneous and linear). Thus, the general solution is given by

$$y(x) = c_1 x + c_2(1 + x\ln x).$$

Lecture XV

Bessel's equation, Bessel's function

# 1 Gamma function

*Gamma function* is defined by

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt, \qquad p > 0. \tag{1}$$

The integral in (1) is convergent that can be proved easily. Some special properties of the gamma function are the following:

i. It is readily seen that $\Gamma(p+1) = p\Gamma(p)$, since

$$\begin{aligned}\Gamma(p+1) &= \lim_{T\to\infty} \int_0^T e^{-t} t^p\, dt \\ &= \lim_{T\to\infty} \left[ -e^{-t} t^p \Big|_0^T + p \int_0^T e^{-t} t^{p-1}\, dt \right] \\ &= p \int_0^\infty e^{-t} t^{p-1}\, dt = p\Gamma(p).\end{aligned}$$

ii. $\Gamma(1) = 1$ (trivial proof)

iii. If $p = m$, a positive integer, then $\Gamma(m+1) = m!$ (use i. repeatedly)

iv. $\Gamma(1/2) = \sqrt{\pi}$. This can be proved as follows:

$$I = \Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2}\, dt = 2 \int_0^\infty e^{-u^2}\, du.$$

Hence

$$I^2 = 4 \int_0^\infty e^{-u^2}\, du \int_0^\infty e^{-v^2}\, dv = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy.$$

Using polar coordinates $\rho, \theta$, the above becomes

$$I^2 = 4 \int_0^\infty \int_0^{\pi/2} e^{-\rho^2} \rho\, d\rho\, d\theta \Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi}$$

v. Using relation in i., we can extend the definition of $\Gamma(p)$ for $p < 0$. Suppose $N$ is a positive integer and $-N < p < -N + 1$. Now using relation of i., we find

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} = \frac{\Gamma(p+2)}{p(p+1)} = \cdots = \frac{\Gamma(p+N)}{p(p+1)\cdots(p+N-1)}.$$

Since $p + N > 0$, the above relation is well defined.

vi. $\Gamma(p)$ is not defined when $p$ is zero or a negative integer. For small positive $\epsilon$,

$$\Gamma(\pm\epsilon) = \frac{\Gamma(1 \pm \epsilon)}{\pm\epsilon} \approx \frac{1}{\pm\epsilon} \to \pm\infty \quad \text{as} \quad \epsilon \to 0.$$

Since $\Gamma(0)$ is undefined, $\Gamma(p)$ is also undefined when $p$ is a negative integer.

# 2 Bessel's equation

Bessel's equation of order $\nu$ ($\nu \geq 0$) is given by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{2}$$

Obviously, $x = 0$ is regular singular point. Since $p(0) = 1, q(0) = -\nu^2$, the indicial equation is given by

$$r^2 - \nu^2 = 0.$$

Hence, $r_1 = \nu, r_2 = -\nu$ and $r_1 - r_2 = 2\nu$. A Frobenius series solution exists for the larger root $r = r_1 = \nu$. To find this series, we substitute

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \qquad x > 0$$

into (2) and (after some manipulation) find

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

where $\rho(r) = r^2 - \nu^2$. This equation is rearranged as

$$\rho(r)a_0 + \rho(r+1)a_1 x + \sum_{n=2}^{\infty} \Big(\rho(n+r)a_n + a_{n-2}\Big)x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \; \rho(r+1)a_1 = 0, \; \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

From the first relation, we get $r_1 = \nu, r_2 = -\nu$. Now with the larger root $r = r_1$ we find

$$a_1 = 0, \quad a_n = -\frac{a_{n-2}}{n(n+2\nu)}, \quad n \geq 2.$$

Iterating we find (by induction),

$$a_{2n+1} = 0, \quad a_{2n} = (-1)^n \frac{1}{2^{2n} n!(\nu+1)(\nu+2)\cdots(\nu+n)} a_0, \quad n \geq 1.$$

Hence

$$y_1(x) = a_0 x^{\nu} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(\nu+1)(\nu+2)\cdots(\nu+n)}\right). \tag{3}$$

Here it is usual to choose (*instead of* $a_0 = 1$ *as was done in lecture 14*)

$$a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}.$$

Then the Frobenius series solution (3) is called the Bessel function of order $\nu$ of the first kind and is denoted by $J_\nu(x)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{4}$$

To find the second independent solution, we consider the following three cases:

A. $r_1 - r_2 = 2\nu$ **is not a nonnegative integer:** We know that a second Frobenius series solution for $r_2 = -\nu$ exist. We do similar calculation as in the case of $r_1$ and it turns out that the resulting series is given by (4) with $\nu$ replaced by $-\nu$. Hence, the second solution is given by

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{5}$$

B. $r_1 = r_2$**:** Obviously this corresponds to $\nu = 0$ and a second Frobenius series solution does not exist.

C. $r_1 - r_2 = 2\nu$ **is a positive integer:** Now there are two cases. We discuss them separately.

C.i $\nu$ **is not a positive integer:** Clearly $\nu = (2k+1)/2$, where $k \in \{0, 1, 2, \cdots\}$. Now we have found earlier that (since $a_0 \neq 0$)

$$\rho(r) = 0,\ \rho(r+1)a_1 = 0,\ \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

With $r = r_2 = -\nu$, we get

$$\rho(r) = 0;\ 1 \cdot \Big(1 - (2k+1)\Big)a_1 = 0;\ n \cdot \Big(n - (2k+1)\Big)a_n = -a_{n-2}, \quad n \geq 2.$$

It is clear that the even terms $a_{2n}$ can be determined uniquely. For odd terms, $a_1 = a_3 = \cdots = a_{2k-1} = 0$ but for $a_{2k+1}$ we must have

$$n \cdot 0 \cdot a_{2k+1} = -a_{2k-1} \Rightarrow 0 \cdot a_{2k+1} = 0.$$

This can be satisfied by taking any value of $a_{2k+1}$ and for simplicity, we can take $a_{2k+1} = 0$. Rest of the odd terms thus also vanish. Hence, the second solution in this case is also given by (5), i.e.

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{6}$$

C.ii $\nu$ **is a positive integer:** Clearly $\nu = k$, where $k \in \{1, 2, 3, \cdots\}$. Now we find (since $a_0 \neq 0$)

$$\rho(r) = 0,\ \rho(r+1)a_1 = 0,\ \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

With $r = r_2 = -\nu$, we get

$$\rho(r) = 0;\ 1 \cdot \Big(1 - 2k\Big)a_1 = 0;\ n \cdot \Big(n - 2k\Big)a_n = -a_{n-2}, \quad n \geq 2.$$

It is clear that all the odd terms $a_{2n+1}$ vanish. For even terms, $a_2, a_4, \cdots, a_{2k-2}$ each is nonzero. For $a_{2k}$ we must have

$$n \cdot 0 \cdot a_{2k} = -a_{2k-2} \Rightarrow 0 \cdot a_{2k} \neq 0,$$

which is a contradiction. Thus, a second Frobenius series solution does not exist in this case.

**Summary of solutions for Bessel's equation**: The Bessel's equation of order $\nu$ ($\nu \geq 0$)

$$x^2y'' + xy' + (x^2 - \nu^)y = 0,$$

has two independent Frobenius series solutions $J_\nu$ and $J_{-\nu}$ when $\nu$ is not an (nonnegative) integer:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}, \qquad J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}.$$

Thus the general solution, when $\nu$ is not an (nonnegative) integer, is

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

When $\nu$ is a (nonnegative) integer, a second solution, which is independent of $J_\nu$, can be found. This solution is called Bessel function of second kind and is denoted by $Y_\nu$. Hence, the general solution, when $\nu$ is an (nonnegative) integer, is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).$$

# 3 Linear dependence of $J_m$ and $J_{-m}$, m is a +ve integer

When $\nu = m$ is a positive integer, then

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m},$$

since $\Gamma(n+m+1) = (n+m)!$.
Since $\Gamma(\pm 0) = \pm\infty$, we define $1/\Gamma(k)$ to be zero when $k$ is nonpositive integer. Now

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}.$$

Now each term in the sum corresponding to $n = 0$ to $n = m-1$ is zero since $1/\Gamma(k)$ is zero when $k$ is nonpositive integer. Hence, we write the sum starting from $n = m$:

$$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}.$$

Substituting $n - m = k$, we find

$$\begin{aligned} J_{-m}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+m}}{(m+k)!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2(m+k)-m} \\ &= (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \\ &= (-1)^m J_m(x). \end{aligned}$$

Hence $J_m$ and $J_{-m}$ becomes linearly dependent when $m$ is a positive integer.

# 4 Properties of Bessel function

Few important relationships are very useful in application. These are described here.

A. From the expression for $J_\nu$ given in (4), we find

$$x^\nu J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+\nu+1)} x^{2n+2\nu}$$

Taking derivative with respect to $x$ we find

$$\Big(x^\nu J_\nu(x)\Big)' = \sum_{n=0}^{\infty} \frac{(-1)^n (n+\nu)}{2^{2n+\nu-1} n! \Gamma(n+\nu+1)} x^{2n+2\nu-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu-1} n! \Gamma(n+\nu)} x^{2n+2\nu-1},$$

where we have used $\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu)$. We can write the above relation as

$$\Big(x^\nu J_\nu(x)\Big)' = x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma\Big(n+(\nu-1)+1\Big)} \left(\frac{x}{2}\right)^{2n+\nu-1}.$$

Hence,

$$\Big(x^\nu J_\nu(x)\Big)' = x^\nu J_{\nu-1}(x). \tag{7}$$

B. From (4), we find

$$x^{-\nu} J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+\nu+1)} x^{2n}.$$

Taking derivative with respect to $x$ we find

$$\Big(x^{-\nu} J_\nu(x)\Big)' = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+\nu-1} (n-1)! \Gamma(n+\nu+1)} x^{2n-1}.$$

Note that the sum runs from $n = 1$ (in contrast to that in A). Let $k = n - 1$, then we obtain

$$\begin{aligned} \Big(x^{-\nu} J_\nu(x)\Big)' &= x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k! \Gamma\Big(k+(\nu+1)+1\Big)} \left(\frac{x}{2}\right)^{2k+\nu+1} \\ &= -x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k! \Gamma\Big(k+(\nu+1)+1\Big)} \left(\frac{x}{2}\right)^{2k+\nu+1}. \end{aligned}$$

Hence,

$$\Big(x^{-\nu} J_\nu(x)\Big)' = -x^{-\nu} J_{\nu+1}(x). \tag{8}$$

**Note:** In the first relation A, while taking derivative, we keep the sum running from $n = 0$. This is true only when $\nu > 0$. In the second relation B, we only need $\nu \geq 0$. Taking $\nu = 0$ in B, we find $J_0' = -J_1$. If we put $\nu = 0$ in A, then we find $J_0' = J_{-1}$. But $J_{-1} = -J_1$ and hence we find the same relation as that in B. Hence, the first relation is also valid for $\nu \geq 0$.

C. From A and B, we get

$$\begin{aligned} J_\nu' + \frac{\nu}{x} J_\nu &= J_{\nu-1} \\ J_\nu' - \frac{\nu}{x} J_\nu &= -J_{\nu+1} \end{aligned}$$

Adding and subtracting we find

$$J_{\nu-1} - J_{\nu+1} = 2J_\nu' \tag{9}$$

and

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu. \tag{10}$$

# 5 Orthogonality of Bessel function $J_\nu$

It will be shown in Lecture XVI that there exist infinitely many orthogonal sets of Bessel's function, such as one set for $J_0$, one set for $J_{1/2}$ etc. Each set is orthogonal with respect to weight function $x$ in the interval $0 \le x \le R$, where $R$ is a fixed positive real number. The elements of orthogonal set for $J_\nu$ are $J_\nu(k_{\nu p} x)$, $p = 1, 2, 3, \cdots$, where $\nu \ge 0$ is fixed and $J_\nu(k_{\nu p} R) = 0$, $p = 1, 2, 3, \cdots$.

**Proposition 1. (Orthogonality)** *The Bessel functions $J_\nu$ satisfy*

$$\int_0^R x J_\nu(k_{\nu p} x) J_\nu(k_{\nu q} x)\, dx = \frac{R^2}{2} J_{\nu+1}^2(k_{\nu p} R) \delta_{pq}, \tag{11}$$

*where $\delta_{pq} = 1$ for $p = q$ and $\delta_{pq} = 0$ for $p \ne q$.*

**Proof:** See Lecture XVI

**Theorem 1. (Fourier-Bessel series)** *Suppose a function $f$ is defined in the interval $0 \le x \le R$ and that it has a Fourier-Bessel series expansion:*

$$f(x) \sim \sum_{p=1}^{\infty} c_p J_\nu(k_{\nu p} x),$$

*where $J_\nu(k_{\nu p} R) = 0$, $p = 1, 2, 3, \cdots$. Using orthogonality, we find*

$$c_p = \frac{2}{R^2\, J_{\nu+1}^2(k_{\nu p} R)} \int_0^R x f(x) J_\nu(k_{\nu p} x)\, dx.$$

*Suppose that $f$ and $f'$ are piecewise continuous on the interval $0 \le x \le R$. Then for $0 < x < R$,*

$$\sum_{p=1}^{\infty} c_p J_\nu(k_{\nu p} x) = \begin{cases} f(x), & \text{where } f \text{ is continuous} \\ \dfrac{f(x^-) + f(x^+)}{2}, & \text{where } f \text{ is discontinuous} \end{cases}$$

*At $x = 0$, it converges to zero for $\nu > 0$ and to $f(0+)$ for $\nu = 0$. On the other hand, it converges to zero at $x = R$.*

# Lecture XVI

Strum comparison theorem, Strum-Liouville Problems, Orthogonal functions

## 1 Normal form of second order homogeneous linear ODE

Consider a second order linear ODE in the standard form

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

By a change of dependent variable, (1) can be written as

$$u'' + Q(x)u = 0, \tag{2}$$

which is called the normal form of (1).
To find the transformation, let us put $y(x) = u(x)v(x)$. When this is substituted in (1), we get

$$vu'' + (2v' + pv)u' + (v'' + pv' + qv)u = 0.$$

Now we set the coefficient of $u'$ to zero. This gives

$$2v' + pv = 0 \Rightarrow v = e^{-\int p/2\,dx}.$$

Now coefficient of $u$ becomes

$$\left(q(x) - \frac{1}{4}p^2 - \frac{1}{2}p'\right) v = Q(x)v.$$

Since $v$ is nonzero, cancelling $v$ we get the required normal form. Also, since $v$ never vanishes, $u$ vanishes if and only if $y$ vanishes. Thus, the above transformation has no effect on the zeros of solution.

**Example 1.** *Consider the Bessel equation of order $\nu \geq 0$:*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0.$$

**Solution**: Here $v = e^{-\int x/2\,dx} = 1/\sqrt{x}$. Now

$$Q(x) = 1 - \frac{\nu^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2} = 1 + \frac{1/4 - \nu^2}{x^2}.$$

Thus, Bessel equation in normal form becomes

$$u'' + \left(1 + \frac{1/4 - \nu^2}{x^2}\right) u = 0. \tag{3}$$

**Theorem 1. (Strum comparison theorem)** *Let $\phi$ and $\psi$ be nontrivial solutions of*

$$y'' + p(x)y = 0, \qquad x \in \mathcal{I},$$

*and*

$$y'' + q(x)y = 0, \qquad x \in \mathcal{I},$$

*where $p$ and $q$ are continuous and $p \leq q$ on $\mathcal{I}$. Then between any two consecutive zeros $x_1$ and $x_2$ of $\phi$, there exists at least one zero of $\psi$ unless $p \equiv q$ on $(x_1, x_2)$.*

**Proof:** Consider $x_1$ and $x_2$ with $x_1 < x_2$. WLOG, assume that $\phi > 0$ in $(x_1, x_2)$. Then $\phi'(x_1) > 0$ and $\phi'(x_2) < 0$. Further, suppose on the contrary that $\psi$ has no zero on $(x_1, x_2)$. Assume that $\psi > 0$ in $(x_1, x_2)$. Since $\phi$ and $\psi$ are solutions of the above equations, we must have

$$\phi'' + p(x)\phi = 0,$$
$$\psi'' + q(x)\psi = 0.$$

Now multiply first of these by $\psi$ and second by $\phi$ and subtracting we find

$$\frac{dW}{dx} = (q-p)\phi\psi,$$

where $W = \phi'\psi - \psi'\phi'$ is the Wronskian of $\phi$ and $\psi$. Integrating between $x_1$ and $x_2$, we find

$$W(x_2) - W(x_1) = \int_{x_1}^{x_2} (q-p)\phi\psi \, dx.$$

Now $W(x_2) \leq 0$ and $W(x_1) \geq 0$. Hence, the left hand side $W(x_2) - W(x_1) \leq 0$. On the other hand, right hand side is strictly greater than zero unless $p \equiv q$ on $(x_1, x_2)$. This contradiction proves that between any two consecutive zeros $x_1$ and $x_2$ of $\phi$, there exists at least one zero of $\psi$ unless $p \equiv q$ on $(x_1, x_2)$.

**Proposition 1.** *Bessel function of first kind $J_v$ ($\nu \geq 0$) has infinitely number of positive zeros.*

**Proof:** The number of zeros $J_\nu$ is the same as that of nontrivial $u$ that satisfies (3), i.e.

$$u'' + \left(1 + \frac{1/4 - \nu^2}{x^2}\right) u = 0. \tag{4}$$

Now for large enough $x$, say $x_0$, we have

$$\left(1 + \frac{1/4 - \nu^2}{x^2}\right) > \frac{1}{4}, \qquad x \in (x_0, \infty). \tag{5}$$

Now compare (4) with

$$v'' + \frac{1}{4}v = 0. \tag{6}$$

Due to (5), between any two zeros of a nontrivial solution of (6) in $(x_0, \infty)$, there exists at least one zero of nontrivial solution of (4). We know that $v = \sin(x/2)$ is a nontrivial solution of (6), which has infinite number of zeros in $(x_0, \infty)$. Hence, any nontrivial solution of (4) has infinite number of zeros in $(x_0, \infty)$. Thus, $J_\nu$ has infinite number of zeros in $(x_0, \infty)$, i.e. $J_\nu$ has infinitely number of positive zeros. We label the positive zeros of $J_\nu$ by $\lambda_n$, thus $J_\nu(\lambda_n) = 0$ for $n = 1, 2, 3, \cdots$.

# 2 Strum-Liouville problems

Here we consider boundary value problems (BVP) where we solve an ordinary differential equation subject to a set of boundary conditions. For example, consider the BVP

$$y'' + p(x)y' + q(x)y = r(x), \qquad x \in I = (a, b),$$

where $p, q$ and $r$ are continuous functions in $I$. The above equation is subjected to the boundary conditions of the following form

$$\begin{aligned} k_1 y(a) + k_2 y'(a) &= b_1, \\ l_1 y(b) + l_2 y'(b) &= b_2, \end{aligned}$$

where $k_1^2 + k_2^2 \neq 0$ and $l_1^2 + l_2^2 \neq 0$. The set of boundary conditions can be replaced with periodic boundary conditions too.

Compared to initial value problem, BVP may have no solution or infinitely many solutions. For example, consider the ODE

$$y'' + y = 0, \qquad 0 < x < \pi.$$

Now $y = \cos x + \sin x$ is the unique solution with the boundary conditions $y(0) = 1, y'(\pi) = 1$. On the other the number of solutions with boundary conditions $y'(0) = 1, y'(\pi) = -1$ is infinite. Further, there exists no solution with boundary conditions $y'(0) = 1, y'(\pi) = 1$.

We now consider a special kind of BVP which is called Strum-Liouville problem. It consists of the Strum-Liouville equation

$$\Big(p(x)y'\Big)' + \Big(q(x) + \lambda r(x)\Big)y = 0, \qquad a < x < b, \tag{7}$$

involving a parameter $\lambda$.

- A Sturm Liouville (S-L) problem is said to be regular if $p(x) > 0, r(x) > 0$ and $p, p', q, r$ are continuous functions over the finite interval $[a, b]$, and has separated boundary conditions of the form

$$\left.\begin{aligned} k_1 y(a) + k_2 y'(a) = 0, \\ l_1 y(b) + l_2 y'(b) = 0, \end{aligned}\right\} \tag{8}$$

  where $k_1^2 + k_2^2 \neq 0$ and $l_1^2 + l_2^2 \neq 0$.

- A Sturm Liouville (S-L) problem is said to be singular if $p(x)$ or $r(x)$ is zero for some $x$ or the interval is infinite.

- A Sturm Liouville (S-L) problem is said to be periodic if $p(a) = p(b), p(x) > 0, r(x) > 0$ and $p, p', q, r$ are continuous functions over the finite interval $[a, b]$ coupled with the boundary conditions $y(a) = y(b)$and $y'(a) = y'(b)$.

**Example 2.** *Consider the Legendre equation:*

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \qquad x \in (-1, 1)$$

*This can be written as Strum-Liouville equation*

$$\Big((1 - x^2)y'\Big)' + \lambda y = 0, \qquad x \in (-1, 1).$$

*Here* $p = (1 - x^2)$, $q = 0$, $r(x) = 1$ *and* $\lambda = n(n+1)$.

**Example 3.** *Consider the Bessel's equation:*

$$\tilde{x}^2\ddot{\tilde{y}} + \tilde{x}\dot{\tilde{y}} + (\tilde{x}^2 - \nu^2)\tilde{y} = 0, \qquad \tilde{x} \in (0,\infty), \qquad \dot{} = \frac{d}{d\tilde{x}},$$

*where $\nu \geq 0$ is a parameter. In applications, it convenient to have another parameter in addition to $\nu$. This can be accomplished by introducing $\tilde{x} = kx$ and let $\tilde{y}(\tilde{x}) = \tilde{y}(kx) = y(x)$. If $' = d/dx$, then $y' = k\dot{\tilde{y}}$ and $y'' = k^2\ddot{\tilde{y}}$. Hence, we find*

$$x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0,$$

*which can be written as Strum-Liouville equation*

$$(xy')' + \left(-\frac{\nu^2}{x} + k^2x\right)y = 0, \qquad x \in (0,\infty). \tag{9}$$

*Here $p = x$, $q = -\nu^2/x$, $r = x$ and $\lambda = k^2$.*

## 2.1 Eigenvalues and Eigenfunctions

Note that $y(x) \equiv 0$ is a solution of (7). We want non-trivial solution $y(x)$ of (7) which satisfy the given boundary boundary conditions. This non-trivial solution is called eigenfunction and the corresponding $\lambda$ is called an eigenvalue.

**Example 4.** *Find the eigenvalues and eigenfunctions for the Strum-Liouville problem*

$$y'' + \lambda y = 0, \qquad y(0) = y'(\pi) = 0.$$

*First consider the case when $\lambda$ is negative. Then $\lambda = -p^2$ with $p \neq 0$. Solution becomes $y(x) = Ae^{px} + Be^{-px}$. Now $y(0) = 0$ and $y'(\pi) = 0$ give $A + B = 0$ and $Ape^{p\pi} - Bpe^{-p\pi} = 0$. Since $p \neq 0$, the last relation gives $Ae^{2p\pi} - B = 0$. Using $B = -A$, we get $A(e^{2p\pi} + 1) = 0 \Rightarrow A = 0 = B$. Hence $y(x) \equiv 0$ which is not an eigenfunction.*

*Next consider the case of $\lambda = 0$. This gives $y(x) = A + Bx$. Again application of $y(0) = 0$ and $y'(\pi) = 0$ give $A = B = 0$ and hence $y(x) \equiv 0$. Hence, again eigenfunction does not exist for $\lambda = 0$.*

*Finally, let $\lambda = p^2 > 0$. Then $y(x) = A\cos px + B\sin px$. Now $y(0) = 0 \Rightarrow A = 0$. Then $y'(\pi) = 0$ gives $Bp\cos p\pi = 0$. Since $p \neq 0$, this reduces to $B\cos p\pi = 0$. If $B = 0$ then $y(x) \equiv 0$ which is not an eigenfuncion. Hence, we impose*

$$\cos p\pi = 0 \Rightarrow p\pi = (2n+1)\frac{\pi}{2} \quad \textit{for} \quad n = 0, \pm1, \pm2, \pm3, \cdots.$$

*This gives $p = (2n+1)/2$ for $n = 0, \pm1, \pm2, \pm3, \cdots$. Hence, the eigenvalues are*

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2, \quad n = 0, \pm1, \pm2, \pm3, \cdots.$$

*and the corresponding eigenfunctions are*

$$y_n(x) = \sin\left(\frac{2n+1}{2}x\right), \quad n = 0, \pm1, \pm2, \pm3, \cdots.$$

*Note that eigenvalues corresponding* $n = -1, -2, -3, \cdots$ *concides with that of* $n = 0, 1, 2, 3, \cdots$ *respectively. This is true for eigenfunctions too. Hence, the eigenvalues and eigenfunctions are given by*

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2, \quad y_n(x) = \sin\left(\frac{2n+1}{2}x\right), \qquad n = 0, 1, 2, 3, \cdots.$$

*The lowest eigenvalue and the corresponding eigenfunction are of great interest in application. They are given by*

$$\lambda_0 = \frac{1}{4}, \quad y_0(x) = \sin\left(\frac{x}{2}\right).$$

## 2.2 Orthogonality of Eigenfunctions

**Theorem 2.** *Suppose that* $p, p', q, r$ *in Strum-Liouville equation (7) are real valued and continuous in* $[a, b]$ *and* $r(x) > 0$ *for all* $x \in (a, b)$. *Let* $y_m(x)$ *and* $y_n(x)$ *respectively be two eigenfunctions corresponding to two distinct eigenvalues* $\lambda_m$ *and* $\lambda_n$ *of the Strum-Liouville problem given by (7) and (8). Then* $y_m$ *and* $y_n$ *are orthogonal in the interval* $[a, b]$ *with respect to the weight function* $r(x)$, *i.e.,*

$$\int_a^b r(x) y_m(x) y_n(x)\, dx = 0, \qquad m \neq n.$$

*If* $p(a) = 0$ *[resp.* $p(b) = 0$*], then first (resp. second) boundary condition in (8) can be dropped. If* $p(a) = p(b) = 0$, *then both boundary conditions given in (8) can be dropped. If* $p(a) = p(b)$, *then the boundary conditions given in (8) can be replaced by periodic boundary condtion* $y(a) = y(b)$ *and* $y'(a) = y'(b)$.

**Remark:** We require that $y$ and $y'$ remain bounded at the end point $x = a$ if $p(a) = 0$ and the same hold at $x = b$ if $p(b) = 0$.

**Proof:** By assumptions, $y_m$ and $y_n$ satisfy

$$(py_m')' + (q + \lambda_m r) y_m = 0 \quad \text{and} \quad (py_n')' + (q + \lambda_n r) y_n = 0.$$

Multiplying first by $y_n$ and second by $y_m$ and subtracting them, we get

$$(\lambda_m - \lambda_n) r y_m y_n = y_m (py_n')' - y_n (py_m')' = \Big(p(y_m y_n' - y_n y_m')\Big)'$$

Integrating over the interval from $x = a$ to $x = b$, we find

$$(\lambda_m - \lambda_n) \int_a^b r(x) y_m(x) y_n(x)\, dx = \Big[p(x)(y_m(x) y_n'(x) - y_n(x) y_m'(x))\Big]_a^b$$

Hence,

$$(\lambda_m - \lambda_n) \int_a^b r(x) y_m(x) y_n(x)\, dx = p(b)\Big(y_m(b) y_n'(b) - y_n(b) y_m'(b)\Big) - p(a)\Big(y_m(a) y_n'(a) - y_n(a) y_m'(a)\Big). \tag{10}$$

Since $\lambda_m \neq \lambda_n$, the required orthogonality follows if the right hand side (rhs) of (10) is zero.

**Case 1.** $p(a) = p(b) = 0$. Then the rhs of (10) is zero and we don't need (8).
**Case 2.** $p(a) = 0, p(b) \neq 0$. Then the rhs of (10) becomes

$$p(b)\Big(y_m(b)y_n'(b) - y_n(b)y_m'(b)\Big)$$

and we don't need the boundary condition at $x = a$. Now $y_m$ and $y_n$ satisfies

$$\begin{aligned} l_1 y_m(b) + l_2 y_m'(b) &= 0, \\ l_1 y_n(b) + l_2 y_n'(b) &= 0, \end{aligned}$$

where at least one of $l_1$ and $l_2$ is nonzero. Hence, the determinant of the coefficient matrix vanishes. This leads to $y_m(b)y_n'(b) - y_n(b)y_m'(b) = 0$ and hence the rhs of (10) is zero.
**Case 3.** $p(a) \neq 0, p(b) = 0$. This case is similar to Case 2 and we don't need boundary condition at $x = b$.
**Case 4.** $p(a) \neq 0, p(b) \neq 0$. We use both the boundary conditions and proceed as in Cases 2 and 3.
**Case 5.** $p(a) = p(b)$. Using periodic boundary condition we see that the rhs of (10) is zero.

**Example 5. Orthogonality of Legendre polynomials:** *We have seen that the Legendre equation can be written as*

$$\Big((1-x^2)y'\Big)' + \lambda y = 0, \qquad x \in (-1, 1).$$

*Here* $\lambda = n(n+1)$. *We know that for* $n = 0, 1, 2, \cdots$, *i.e.,* $\lambda_n = 0, 2, 6, \cdots$, *this equation has Legendre polynomial* $P_n(x)$ *as a solution. Here* $r(x) = 1$ *and* $p(-1) = p(1) = 0$. *Hence, we don't need any boundary condition and the orthogonality relation is*

$$\int_{-1}^{1} P_m(x)P_n(x)\, dx = 0, \qquad\qquad (m \neq n).$$

*We have also seen that*

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

**Example 6. Orthogonality of Bessel's functions** $J_\nu(x)$**:** *We have seen that the Bessel's equation*

$$\tilde{x}^2\ddot{\tilde{y}} + \tilde{x}\dot{\tilde{y}} + (\tilde{x}^2 - \nu^2)\tilde{y} = 0, \qquad \tilde{x} \in (0, \infty), \qquad\qquad {}^{\cdot} = \frac{d}{d\tilde{x}},$$

*which can be written as Strum-Liouville equation*

$$(xy')' + \left(-\frac{n^2}{x} + k^2 x\right) y = 0, \qquad x \in (0, \infty),$$

*where* $\tilde{x} = kx$ *and* $\tilde{y}(\tilde{x}) = \tilde{y}(kx) = y(x)$. *For a given* $\nu \geq 0$, *a solution of the former is* $J_\nu(\tilde{x}) = J_\nu(kx) = y_\nu(x)$. *We consider the interval* $[0, R]$ *such that*

$$y_\nu(R) = 0 \Rightarrow J_\nu(kR) = 0.$$

*Now $J_\nu(x)$ has infinitely many positive zeros $\alpha_{\nu 1} < \alpha_{\nu 2} < \alpha_{\nu 3} < \cdots$. Hence, we choose $k_{\nu p} = \alpha_{\nu p}/R$, $p = 1, 2, 3, \cdots$. The eigenfunctions corresponding to $k_{\nu p}$ is $y_{\nu p}(x) = J_\nu(k_{\nu p} x)$. Hence, using orthogonality, we find*

$$\int_0^R x\, y_{\nu p}(x) y_{\nu q}(x) dx = 0, \qquad (p \neq q), \tag{11}$$

*which can be written as*

$$\int_0^R x\, J_\nu(k_{\nu p} x) J_\nu(k_{\nu q} x) dx = 0, \qquad (p \neq q).$$

*Note that (11) follows from*

$$\left(k_{\nu p}^2 - k_{\nu q}^2\right) \int_0^R x\, y_{\nu p}(x) y_{\nu q}(x) dx = R\left(y_{\nu p}(R) y'_{\nu q}(R) - y'_{\nu p}(R) y_{\nu q}(R)\right)$$

*Noting that $y'_\nu(x) = kJ'_\nu(kx)$, we can write this equation as [writing $k = k_{\nu p}, k' = k_{\nu q}$]*

$$\int_0^R x J_\nu(kx) J_\nu(k'x) dx = \frac{R\Big(k' J_\nu(kR) J'_\nu(k'R) - k J'_\nu(kR) J_\nu(k'R)\Big)}{k^2 - k'^2}$$

*Then [using l'Hospital's rule and $J_\nu(kR) = 0$]*

$$\begin{aligned}
\int_0^R x\, J_\nu^2(kx) dx &= -\lim_{k' \to k} \frac{R\Big(k' J_\nu(kR) J'_\nu(k'R) - k J'_\nu(kR) J_\nu(k'R)\Big)}{k'^2 - k^2} \\
&= -\lim_{k' \to k} \frac{R\Big(J_\nu(kR) J'_\nu(k'R) + k' R J_\nu(kR) J''_\nu(k'R) - kR J'_\nu(kR) J'_\nu(k'R)\Big)}{2k'} \\
&= \frac{R^2}{2} \left(J'_\nu(kR)\right)^2
\end{aligned}$$

*Using the relation $J'_\nu = -J_{\nu+1} + \nu J_\nu/x$ we get $J'_\nu(kR) = -J_{\nu+1}(kR)$ and hence*

$$\int_0^R x\, J_\nu^2(k_{\nu p} x) dx = \frac{R^2}{2} J_{\nu+1}^2(k_{\nu p} R).$$

*Hence, we have the following*

$$\int_0^R x J_\nu(k_{\nu p} x) J_\nu(k_{\nu q} x)\, dx = \frac{R^2}{2} J_{\nu+1}^2(k_{\nu p} R) \delta_{pq}, \tag{12}$$

*where $\delta_{pq} = 1$ for $p = q$ and $\delta_{pq} = 0$ for $p \neq q$.*

Lecture XVII

Laplace Transform, inverse Laplace Transform, Existence and Properties of Laplace Transform

# 1 Introduction

Differential equations, whether ordinary or partial, describe the ways certain quantities of interest vary over time. These equations are generally coupled with initial conditions at time $t = 0$ and boundary conditions.

Laplace transform is a powerful technique to solve differential equations. It transforms an IVP in ODE to algebraic equations. The solution of the algebraic equations is than back-transformed to the original problem. In case of PDE, it can be applied to any independent variable $x, y, z$ and $t$ that varies from 0 to $\infty$. After applying Laplace transform, the original PDE becomes a new PDE with one less independent variable or an ODE. The resulting problem can be solved by other method (such as separation of variables, another transform) and the solution is again back-transformed to the original problem.

# 2 Laplace transform

Let a function $f$ be defined for $t \geq 0$. We define the Laplace transform of $f$, denoted by $F(s)$ or $\mathcal{L}\Big(f(t)\Big)$, as

$$F(s) = \mathcal{L}\Big(f(t)\Big) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

for those $s$ for which the integral in (1) exists. We also refer $f(t)$ as the inverse Laplace transform of $F(s)$ and we write

$$f(t) = \mathcal{L}^{-1}\Big(F(s)\Big).$$

**Comment 1:** Laplace transform is defined for complex valued function $f(t)$ and the parameter $s$ can also be complex. But we restrict our discussion only for the case in which $f(t)$ is real valued and $s$ is real.

**Comment 2:** Since the integral in (1) is an improper integral, existence of Laplace transform implies that the following limit exists:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{A \to \infty} \int_0^A e^{-st} f(t)\, dt.$$

**Example 1.** *Consider the function defined by*

$$f(t) = \begin{cases} 0, & 0 \leq t < a, \\ 1, & t > a. \end{cases}$$

*Now*

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_a^\infty e^{-st}\, dt \\ &= \frac{e^{-as}}{s}, \qquad s > 0. \end{aligned}$$

**Example 2.** *Consider $f(t) = t^a,\ a > -1$. Now*

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} t^a\, dt \\ &= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^{a+1-1}\, du \\ &= \frac{\Gamma(a+1)}{s^{a+1}}, \qquad s > 0. \end{aligned}$$

*Hence $F(1) = 1/s$; $F(t) = 1/s^2$; $F(t^n) = n!/s^{n+1}$, where $n$ is nonnegative integer.*

**Example 3.** *Consider $f(t) = e^{at}$. Now*

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at}\, dt \\ &= \int_0^\infty e^{-(s-a)t}\, dt \\ &= \frac{1}{s-a}, \qquad s > a. \end{aligned}$$

**Example 4.** *Consider $f(t) = \cos(\omega t)$. Using*

$$\int \cos(at) e^{bt}\, dt = \frac{e^{bt}}{a^2+b^2}\Big(b\cos(at) + a\sin(at)\Big)$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \cos(\omega t)\, dt \\ &= \frac{s}{s^2+\omega^2}, \qquad s > 0. \end{aligned}$$

**Example 5.** *Consider $f(t) = \sin(\omega t)$. Using*

$$\int \sin(at) e^{bt}\, dt = \frac{e^{bt}}{a^2+b^2}\Big(b\sin(at) - a\cos(at)\Big)$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sin(\omega t)\, dt \\ &= \frac{\omega}{s^2+\omega^2}, \qquad s > 0. \end{aligned}$$

## 2.1 Existence of Laplace transform

We give sufficient condition for the existence of LT. We need the concept of piecewise continuous function.

**Definition 1.** (**Piecewise continuous function**) *A function $f$ is piecewise continuous on the interval $[a, b]$ if*
*(i) The interval $[a, b]$ can be broken into a finite number of subintervals $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, such that $f$ is continuous in each subinterval $(t_i, t_{i+1})$, for $i = 0, 1, 2, \cdots, n-1$*
*(ii) The function $f$ has jump discontinuity at $t_i$, thus*

$$\left|\lim_{t\to t_i^+} f(t)\right| < \infty, \quad i = 0, 1, 2, \cdots, n-1; \qquad \left|\lim_{t\to t_i^-} f(t)\right| < \infty, \quad i = 1, 2, 3, \cdots, n.$$

**Note:** A function is piecewise continuous on $[0, \infty)$ if it is piecewise continuous in $[0, A]$ for all $A > 0$

**Example 6.** *The function defined by*

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ 3-t, & 1 < t \le 2, \\ t+1, & 2 < t \le 3, \end{cases}$$

*is piecewise continuous on* $[0, 3]$

**Example 7.** *The function defined by*

$$f(t) = \begin{cases} \dfrac{1}{2-t}, & 0 \le t < 2, \\ t+1, & 2 \le t \le 3, \end{cases}$$

*is NOT piecewise continuous on* $[0, 3]$

**Definition 2.** (**Exponential order**) *A function $f$ is said to be of exponential order if there exist constants $M$ and $c$ such that*

$$|f(t)| \le Me^{ct} \qquad \textit{for sufficiently large } t.$$

**Example 8.** *Any polynomial is of exponential order. This is clear from the fact that*

$$e^{at} = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} \ge \frac{t^n a^n}{n!} \implies t^n \le \frac{n!}{a^n} e^{at}$$

*But $f(t) = e^{t^2}$ is not of exponential order.*

**Sufficient condition for the existence of Laplace transform:** Let $f$ be a piecewise continuous function in $[0, \infty)$ and is of exponential order. Then Laplace transform $F(s)$ of $f$ exists for $s > c$, where $c$ is a real number that depends on $f$.

**Proof:** Since $f$ is of exponential order, there exists $A, M, c$ such that

$$|f(t)| \le Me^{ct} \qquad \text{for } t \ge A.$$

Now we write

$$I = \int_0^\infty f(t)e^{-st}\,dt = I_1 + I_2,$$

where

$$I_1 = \int_0^A f(t)e^{-st}\,dt \quad \text{and} \quad I_2 = \int_A^\infty f(t)e^{-st}.$$

Since $f$ is piecewise continuous, $I_1$ exists. For the second integral $I_2$, we note that for $t \ge A$

$$|e^{-st}f(t)| \le Me^{-(s-c)t}.$$

Thus

$$\int_A^\infty |f(t)e^{-st}|\,dt \le M\int_A^\infty e^{-(s-c)t}\,dt \le M\int_0^\infty e^{-(s-c)t}\,dt = \frac{M}{s-c}, \quad s > c.$$

Since the integral in $I_2$ converges absolutely for $s > c$, $I_2$ converges for $s > c$. Thus, both $I_1$ and $I_2$ exist and hence $I$ exists for $s > c$.

**Comment** The above condition is not necessary. For example, consider $f(t) = 1/\sqrt{t}$, which is not piecewise continuous in $[0, \infty)$. But

$$\int_0^\infty \frac{e^{-st}}{\sqrt{t}} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}u^{1/2-1}\,du = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

# 3 Basic properties of Laplace transform

**Theorem 1.** *(**Uniqueness of Laplace transform**) Let $f(t)$ and $g(t)$ be two functions such that $F(s) = G(s)$ for all $s > k$. Then $f(t) = g(t)$ at all $t$ where both are continuous.*

**Proposition 1.** *(**Linearity**) Suppose $F_1(s) = \mathcal{L}\Big(f_1(t)\Big)$ exists for $s > a_1$ and $F_2(s) = \mathcal{L}\Big(f_2(t)\Big)$ exists for $s > a_2$. Then*

$$\mathcal{L}\Big(c_1 f_1(t) + c_2 f_2(t)\Big) = c_1 F_1(s) + c_2 F_2(s),$$

*for $s > a$, where $a = max\{a_1, a_2\}$.*

**Proof:** Trivial

**Example 9.** *Consider $f(t) = \cosh(\omega t)$. Then*

$$\mathcal{L}\Big(\cosh(\omega t)\Big) = \frac{1}{2}\Big(\mathcal{L}(e^{\omega t}) + \mathcal{L}(e^{-\omega t})\Big) = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}.$$

**Example 10.** *Consider $f(t) = \sinh(\omega t)$. Proceeding as above, we find*

$$F(s) = \omega/(s^2 - \omega^2)$$

**Theorem 2.** (**First shifting theorem**) *If* $\mathcal{L}\Big(f(t)\Big) = F(s)$, *then*

$$\mathcal{L}\Big(e^{at}f(t)\Big) = F(s-a), \qquad \textit{and} \qquad e^{at}f(t) = \mathcal{L}^{-1}\Big(F(s-a)\Big).$$

**Proof:** Suppose $\mathcal{L}\Big(f(t)\Big) = F(s)$ holds for $s > k$. Now

$$\mathcal{L}\Big(e^{at}f(t)\Big) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a), \qquad s-a > k.$$

**Example 11.** *Consider* $f(t) = e^{-5t}\cos(4t)$. *Since*

$$\mathcal{L}\Big(\cos(4t)\Big) = \frac{s}{s^2+16} \implies \mathcal{L}\Big(e^{-5t}\cos(4t)\Big) = \frac{s+5}{(s+5)^2+16}$$

**Proposition 2.** *If* $\mathcal{L}\Big(f(t)\Big) = F(s)$, *then* $F(s) \to 0$ *as* $s \to \infty$.

**Proof:** We prove this for a piecewise continuous function which is of exponential order. But the result is valid for any function for which Laplace transform exists. Now

$$I = \int_0^\infty e^{-st}f(t)\,dt \implies |I| \le \int_0^\infty e^{-st}|f(t)|\,dt.$$

Now since the function is exponential order, there exists $M, \alpha, A$ such that $|f(t)| \le M_1e^{\alpha t}$ for $t \ge A$. Also, since the function is piecewise continuous in $[0, A]$, we must have $|f(t)| \le M_2e^{\beta t}$ for $0 \le t \le A$ except possibly at some finite number of points where $f(t)$ is not defined. Now we take $M = \max\{M_1, M_2\}$ and $\gamma = \max\{\alpha, \beta\}$. Then we have

$$|F(s)| = |I| \le \int_0^\infty e^{-st}|f(t)|\,dt \le M\int_0^\infty e^{-(s-\gamma)t}\,dt = \frac{M}{s-\gamma}, \quad s > \gamma.$$

Thus, $F(s) \to 0$ as $s \to \infty$.
**Comment:** Any function $F(s)$ without this behaviour can not be Laplace transform of a certain function. For example, $s/(s-1), \sin s, s^2/(1+s^2)$ are not Laplace transform of any function.

# Lecture XVIII

Unit step function, Laplace Transform of Derivatives and Integration, Derivative and Integration of Laplace Transforms

## 1 Unit step function $u_a(t)$

**Definition 1.** *The unit step function (or Heaviside function) $u_a(t)$ is defined*

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a. \end{cases}$$

This function acts as a mathematical 'on-off' switch as can be seen from the Figure 1. It has been shown in Example 1 of Lecture Note 17 that for $a > 0$, $\mathcal{L}\Big(u_a(t)\Big) = e^{-as}/s$.

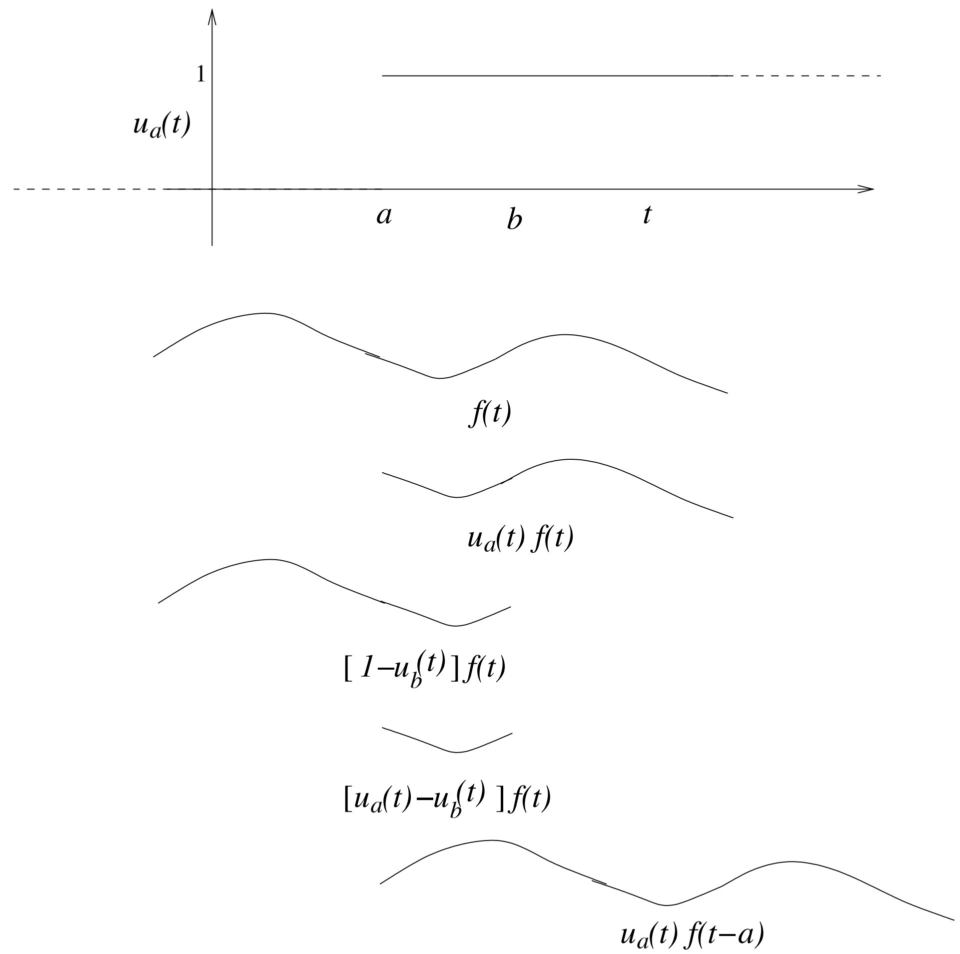


Figure 1: Effects of unit step function on a function $f(t)$. Here $b > a$.

**Example 1.** *Consider the function*

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ \sin 2t, & 1 < t \le \pi, \\ \cos t, & t > \pi \end{cases}$$

*Now let us consider a function $g$ defined by*

$$g(t) = \Big(u_0(t) - u_1(t)\Big)t^2 + \Big(u_1(t) - u_\pi(t)\Big)\sin 2t + u_\pi(t)\cos t.$$

*Now $f(t)$ is piecewise continuous function. Hence, Laplace transform of $f$ exists. Clearly $f(t) = g(t)$ at all $t$ except possibly at a finite number points $t = 0, 1, \pi$ where $f(t)$ possibly has jump discontinuity. Hence, using Uniqueness Theorem of Laplace Transform (see Lecture Note 17), we conclude that $\mathcal{L}\Big(f(t)\Big) = \mathcal{L}\Big(g(t)\Big)$.*

**Theorem 1.** *(***Second shifting theorem***) If $\mathcal{L}\Big(f(t)\Big) = F(s)$, then*

$$\mathcal{L}\Big(u_a(t)f(t-a)\Big) = e^{-as}F(s).$$

*Conversely,*

$$\mathcal{L}^{-1}\Big(e^{-as}F(s)\Big) = u_a(t)f(t-a).$$

**Proof:** From the definition of Laplace transform

$$\begin{aligned}
\mathcal{L}\Big(u_a(t)f(t-a)\Big) &= \int_0^\infty e^{-st}u_a(t)f(t-a)\,dt \\
&= \int_a^\infty e^{-st}f(t-a)\,dt \\
&= e^{-as}\int_0^\infty e^{-su}f(u)\,du, \qquad t-a=u \\
&= e^{-as}F(s).
\end{aligned}$$

**Example 2.** *Find the Laplace transform of*

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ \sin 2t, & 1 < t \le \pi, \\ \cos t, & t > \pi \end{cases}$$

**Solution**: We know that if

$$g(t) = \Big(u_0(t) - u_1(t)\Big)t^2 + \Big(u_1(t) - u_\pi(t)\Big)\sin 2t + u_\pi(t)\cos t,$$

then $F(s) = G(s)$. Now we write $g(t)$ in such a way that second shifting theorem (see Theorem 1) can be applied. Hence, we manipulate $g(t)$ in the following way:

$$\begin{aligned}
g(t) &= u_0(t)t^2 - u_1(t)(t-1+1)^2 + u_1(t)\sin[2(t-1)+2] - u_\pi(t)\sin[2(t-\pi)] - u_\pi(t)\cos(t-\pi) \\
&= u_0(t)t^2 - u_1(t)(t-1)^2 - 2u_1(t)(t-1) - u_1(t) + \cos(2)u_1(t)\sin[2(t-1)] \\
&+ \sin(2)u_1(t)\cos[2(t-1)] + u_\pi(t)\sin[2(t-\pi)] - u_\pi(t)\cos(t-\pi)
\end{aligned}$$

Now every term is of the form $u_a(t)h(t-a)$. For example

$$u_0(t)t^2 \equiv u_0(t)(t-0)^2 \quad \text{and} \quad u_1(t) \equiv u_1(t)h(t-1) \quad \text{where } h(t) = 1.$$

Now we know that

$$\mathcal{L}\Big(1\Big) = \frac{1}{s},\ \mathcal{L}\Big(t\Big) = \frac{1}{s^2},\ \mathcal{L}\Big(t^2\Big) = \frac{2}{s^3},\ \mathcal{L}\Big(\cos t\Big) = \frac{s}{s^2+1},\ \mathcal{L}\Big(\sin t\Big) = \frac{1}{s^2+1}$$

and

$$\mathcal{L}\Big(\cos 2t\Big) = \frac{s}{s^2+4},\ \mathcal{L}\Big(\sin 2t\Big) = \frac{2}{s^2+4}$$

Hence,

$$F(s) = \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} + \frac{2e^{-s}\cos 2}{s^2+4} + \frac{s\sin 2e^{-s}}{s^2+4} + \frac{2e^{-\pi s}}{s^2+4} - \frac{se^{-\pi s}}{s^2+1}$$

## 2 Laplace transform of derivatives and integrals

**Theorem 2.** *Let $f(t)$ be continuous for $t \geq 0$ and is of exponential order. Further suppose that $f$ is differentiable with $f'$ piecewise continuous in $[0,\infty)$. Then $\mathcal{L}\Big(f'\Big)$ exists and is given by*

$$\mathcal{L}\Big(f'\Big) = s\mathcal{L}\Big(f\Big) - f(0). \tag{1}$$

**Proof:** Since $f'$ is piecewise continuous in $[0,\infty)$, $f'$ is piecewise continuous in $[0,R]$ for any $R > 0$. Let $x_i$, $i = 0, 1, 2, \cdots, n$ are the possible points of jump discontinuity where $x_0 = 0$ and $x_n = R$. Now

$$\begin{aligned}
\int_0^R e^{-st} f'(t) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} e^{-st} f'(t)\, dt \\
&= \sum_{i=0}^{n-1} e^{-st} f(t)\Big|_{x_i}^{x_{i+1}} + s\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} e^{-st} f(t)\, dt \\
&= e^{-sR} f(R) - f(0) + s\int_0^R e^{-st} f(t)\, dt
\end{aligned}$$

Since $f$ is of exponential order, $|f(R)| \leq Me^{cR}$. This implies

$$|e^{-sR} f(R)| \leq Me^{-(s-c)R} \to 0 \quad \text{as} \quad R \to \infty \quad \text{for} \quad s > c.$$

Hence taking $R \to \infty$, we find

$$\mathcal{L}\Big(f'\Big) = s\mathcal{L}\Big(f\Big) - f(0). \tag{2}$$

**Corollary 1.** *Let $f$ and its derivatives $f^{(1)}, f^{(2)}, \cdots, f^{(n-1)}$ be continuous for $t \geq 0$ and are of exponential order. Further suppose that $f^{(n)}$ is piecewise continuous in $[0,\infty)$. Then Laplace transform of $f^{(n)}$ exists and is given by*

$$\mathcal{L}\Big(f^{(n)}\Big) = s^n\mathcal{L}\Big(f\Big) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

*In particular for $n = 2$, we get*

$$\mathcal{L}\Big(f''\Big) = s^2\mathcal{L}\Big(f\Big) - sf(0) - f'(0). \tag{4}$$

**Proof:** for $n = 2$, use (2) twice to find

$$\begin{aligned}\mathcal{L}\Big(f''\Big) &= s\mathcal{L}\Big(f'\Big) - f'(0)\\ &= s\left(s\mathcal{L}\Big(f\Big) - f(0)\right) - f'(0)\\ &= s^2\mathcal{L}\Big(f\Big) - sf(0) - f'(0).\end{aligned}$$

For general $n$, prove by induction.

**Example 3.** *Find Laplace transform of*

$$t\cos(\omega t).$$

**Solution**: Since $f(t) = t\cos(\omega t)$, we find

$$f'(t) = -\omega t\sin(\omega t) + \cos(\omega t)$$

and

$$f''(t) = -\omega^2 f(t) - 2\omega\sin(\omega t).$$

Hence taking Laplace transform on both sides, we find

$$\mathcal{L}\Big(f''\Big) = -\omega^2\mathcal{L}\Big(f\Big) - 2\omega\mathcal{L}\Big(\sin(\omega t)\Big)$$

Hence,

$$s^2\mathcal{L}\Big(f\Big) - sf(0) - f'(0) = -\omega^2\mathcal{L}\Big(f\Big) - 2\omega\frac{\omega}{s^2+\omega^2}.$$

Now $f(0) = 0, f'(0) = 1$. Simplifying, we find

$$\mathcal{L}\Big(f\Big) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

**Theorem 3.** *Let $F(s)$ be the Laplace transform of $f$. If $f$ is piecewise continuous in $[0,\infty)$ and is of exponential order, then*

$$\mathcal{L}\Big(\int_0^t f(\tau)\,d\tau\Big) = \frac{F(s)}{s}. \tag{5}$$

**Proof:** Since $f$ is piecewise continuous,

$$g(t) = \int_0^t f(\tau)\,d\tau$$

is continuous. Since $f(t)$ is piecewise continuous, $|f(t)| \le Me^{kt}$ for all $t \ge 0$ except possibly at finite number of points where $f$ has jump discontinuities. Hence,

$$|g(t)| \le M\int_0^t e^{k\tau}\,d\tau = \frac{M}{k}(e^{kt}-1) \le \frac{M}{k}e^{kt}.$$

Thus, $g$ is continuous and is of exponential order. Hence, Laplace transform of $g$ exists. Further $g'(t) = f(t)$ and $g(0) = 0$. Using (2), we find

$$\mathcal{L}\Big(g'\Big) = s\mathcal{L}\Big(g\Big) - g(0) \implies \mathcal{L}\Big(g'\Big) = s\mathcal{L}\Big(g\Big) \implies G(s) = \frac{F(s)}{s}.$$

**Example 4.** *Find the inverse Laplace transform of* $1/s(s+1)^2$.

**Solution**: Since

$$\mathcal{L}\Big(t\Big) = \frac{1}{s^2} \implies \mathcal{L}\Big(te^{-t}\Big) = \frac{1}{(s+1)^2}$$

Hence for $f(t) = te^{-t}$, we have $F(s) = 1/(s+1)^2$. Thus,

$$\frac{1}{s(s+1)^2} = \frac{F(s)}{s} \implies \mathcal{L}^{-1}\left(\frac{1}{s(s+1)^2}\right) = \int_0^t \tau e^{-\tau}\, d\tau = 1-(t+1)e^{-t}$$

## 3 Derivative and integration of the Laplace transform

**Theorem 4.** *If $F(s)$ is the Laplace transform of $f$, then*

$$\mathcal{L}\Big(-tf(t)\Big) = F'(s), \quad \text{and} \quad \mathcal{L}^{-1}\Big(F'(s)\Big) = -tf(t). \tag{6}$$

**Comment:** The derivative formula for $F(s)$ can be derived by differentiating under the integral sign, i.e.

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}f(t)\right)\,dt \\ &= \int_0^\infty e^{-st}(-tf(t))\,dt \\ &= \mathcal{L}\Big(-tf(t)\Big). \end{aligned}$$

**Example 5.** *Consider the same problem as in Example 3, i.e. Laplace transform of $t\cos(\omega t)$. Let $f(t) = \cos(\omega t)$. Then*

$$F(s) = \frac{s}{s^2+\omega^2} \implies F'(s) = \frac{\omega^2 - s^2}{(s^2+\omega^2)^2}.$$

*Hence using (6), we find*

$$\mathcal{L}\Big(-t\cos(\omega t)\Big) = \frac{\omega^2-s^2}{(s^2+\omega^2)^2} \implies \mathcal{L}\Big(t\cos(\omega t)\Big) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

**Example 6.** *Find the inverse Laplace transform of*

$$F(s) = \ln\left(\frac{s-a}{s-b}\right)$$

**Solution**: If $\mathcal{L}\Big(f(t)\Big) = F(s)$, then $\mathcal{L}\Big(tf(t)\Big) = -F'(s)$. Hence

$$\mathcal{L}\Big(tf(t)\Big) = \frac{1}{s-b} - \frac{1}{s-a} = \mathcal{L}\Big(e^{bt} - e^{at}\Big) \implies f(t) = \frac{e^{bt}-e^{at}}{t}.$$

**Theorem 5.** *If $F(s)$ is the Laplace transform of $f$ and the limit of $f(t)/t$ exists as $t \to 0^+$, then*

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(p)\,dp, \qquad \text{and} \qquad \mathcal{L}^{-1}\Big(\int_s^\infty F(p)\,dp\Big) = \frac{f(t)}{t}. \tag{7}$$

**Proof:** Let

$$g(t) = f(t)/t, \qquad \text{and} \qquad g(0) = \lim_{t\to 0^+} \frac{f(t)}{t}.$$

Now

$$F(s) = \mathcal{L}\Big(f(t)\Big) \implies F(s) = \mathcal{L}\Big(tg(t)\Big) = -G'(s), \qquad \text{[using (6)]}$$

Hence,

$$G(s) = \int_s^A F(p)\,dp.$$

Since $G(s) \to 0$ as $s \to \infty$, we must have

$$0 = \int_\infty^A F(p)\,dp$$

Thus,

$$G(s) = \int_s^A F(p)\,dp - \int_\infty^A F(p)\,dp \implies G(s) = \int_s^\infty F(p)\,dp \implies \mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(p)\,dp.$$

**Example 7.** *Find the Laplace transform of*

$$\frac{\sin\omega t}{t}.$$

**Solution**: Let $f(t) = \sin\omega t$. Using the formula (7), we find

$$\mathcal{L}\left(\frac{\sin\omega t}{t}\right) = \int_s^\infty \frac{\omega}{p^2+\omega^2}dp = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right).$$

**Example 8.** *Consider the same problem as in Example 6, i.e. inverse Laplace transform of*

$$F(s) = \ln\left(\frac{s-a}{s-b}\right)$$

**Solution***: Note that*

$$\mathcal{L}\Big(f(t)\Big) = \ln\left(\frac{s-a}{s-b}\right) = \int_s^\infty \frac{1}{s-b}dp - \int_s^\infty \frac{1}{s-a}dp = \mathcal{L}\left(\frac{e^{bt}}{t}\right) - \mathcal{L}\left(\frac{e^{at}}{t}\right)$$

*Hence,*

$$\mathcal{L}\Big(f(t)\Big) = \mathcal{L}\left(\frac{e^{bt}-e^{at}}{t}\right) \implies f(t) = \frac{e^{bt}-e^{at}}{t}.$$

Lecture XIX

Laplace Transform of Periodic Functions, Convolution, Applications

# 1 Laplace transform of periodic function

**Theorem 1.** *Suppose that $f : [0,\infty) \to \mathbb{R}$ is a periodic function of period $T > 0$, i.e. $f(t+T) = f(t)$ for all $t \geq 0$. If the Laplace transform of $f$ exists, then*

$$F(s) = \frac{\displaystyle\int_0^T f(t)e^{-st}\,dt}{1 - e^{-sT}}. \tag{1}$$

**Proof:** We have

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\
&= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt \\
&= \sum_{n=0}^{\infty} \int_0^T f(u+nT)e^{-su-snT}\,du \qquad u = t - nT \\
&= \sum_{n=0}^{\infty} e^{-snT} \int_0^T f(u)e^{-su}\,du \\
&= \left(\int_0^T f(u)e^{-su}\,du\right) \sum_{n=0}^{\infty} e^{-snT} \\
&= \frac{\displaystyle\int_0^T f(u)e^{-su}\,du}{1 - e^{-sT}}.
\end{aligned}$$

The last line follows from the fact that

$$\sum_{n=0}^{\infty} e^{-snT}$$

is a geometric series with common ration $e^{-sT} < 1$ for $s > 0$.

**Example 1.** *Consider $f(t) = \sin(\omega t)$, which is a periodic function of period $2\pi/\omega$.*

**Solution**: Using (1), we find

$$F(s) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \sin(\omega t)\,dt = \frac{\omega}{s^2+\omega^2} \frac{1 - e^{-2\pi s/\omega}}{1 - e^{-2\pi s/\omega}} = \frac{\omega}{s^2+\omega^2}$$

**Example 2.** *Consider a* saw-tooth *function (see Figure 1)*

$$f(t) = \begin{cases} t, & 0 \leq t < 1, \\ f(t-1), & t \geq 1. \end{cases}$$

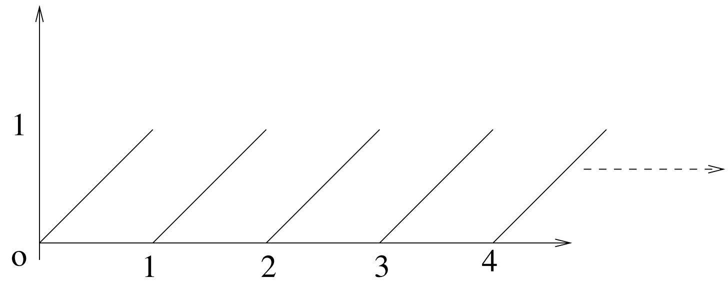


Figure 1: A saw-tooth function.

**Solution**: Here period $T = 1$. Using (1), we find

$$F(s) = \frac{1}{1-e^{-s}} \int_0^1 te^{-st}\, dt = \frac{1-e^{-s}(1+s)}{s^2(1-e^{-s}} = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$$

**Example 3.** *Consider the following function (see Figure 2)*

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \\ f(t-2), & t \ge 2. \end{cases}$$

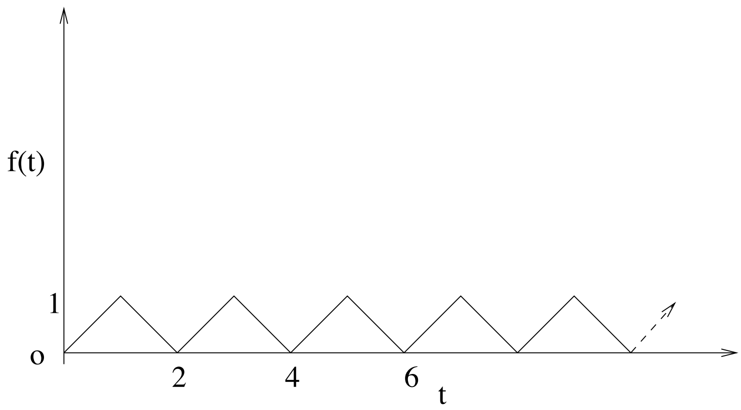


Figure 2: A saw-tooth function.

**Solution**: Here $f(t)$ is a periodic function of period $T = 2$. Hence, using (1), we find

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st}\, dt = \frac{1}{1-e^{-2s}} \left( \int_0^1 te^{-st}\, dt + \int_1^2 (2-t)e^{-st}\, dt \right)$$

Simplifying the RHS, we find

$$F(s) = \frac{(1-e^{-s})^2}{s^2(1-e^{-2s})} = \frac{1-e^{-s}}{s^2(1+e^{-s})} = \frac{1}{s^2}\tanh(s/2)$$

**Aliter:** Note that

$$f'(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \\ f(t-2), & t > 2. \end{cases}$$

Since $f'$ is piecewise continuous and is of exponential order, its Laplace transform exist. Also, $f'$ is periodic with period $T = 2$. Hence,

$$\mathcal{L}\Big(f'\Big) = \frac{1}{1-e^{-2s}}\int_0^2 f'(t)e^{-st}\,dt = \frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\,dt + \int_1^2 -e^{-st}\,dt\right) = \frac{(1-e^{-s})^2}{s(1-e^{-2s})}.$$

Hence,

$$\mathcal{L}\Big(f'\Big) = \frac{1}{s}\tanh(s/2) \implies sF(s) - f(0) = \frac{1}{s}\tanh(s/2) \implies F(s) = \frac{1}{s^2}\tanh(s/2)$$

**Comment:** Is it possible to do similar calculations (like in aliter) in Example 2? If not, why not?

# 2 Convolution

Suppose we know that a Laplace transform $H(s)$ can be written as $H(s) = F(s)G(s)$, where $\mathcal{L}\Big(f(t)\Big) = F(s)$ and $\mathcal{L}\Big(g(t)\Big) = G(s)$. We need to know the relation of $h(t) = \mathcal{L}^{-1}\Big(H(s)\Big)$ to $f(t)$ and $g(t)$.

**Definition 1.** (**Convolution**) *Let $f$ and $g$ be two functions defined in $[0,\infty)$. Then the convolution of $f$ and $g$, denoted by $f * g$, is defined by*

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \tag{2}$$

**Note:** It can be shown (easily) that $f * g = g * f$. Hence,

$$(f*g)(t) = \int_0^t g(\tau)f(t-\tau)\,d\tau \tag{3}$$

We use either (2) or (3) depending on which is easier to evaluate.

**Theorem 2.** (**Convolution theorem**) *The convolution $f * g$ has the Laplace transform property*

$$\mathcal{L}\Big((f*g)(t)\Big) = F(s)G(s). \tag{4}$$

*OR conversely*

$$\mathcal{L}^{-1}\Big(F(s)G(s)\Big) = (f*g)(t)$$

**Proof:** Using definition, we find

$$\begin{aligned}\mathcal{L}\Big((f*g)(t)\Big) &= \int_0^\infty (f*g)(t)e^{-st}\,dt \\ &= \int_0^\infty \left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)e^{-st}\,dt\end{aligned}$$

The region of integration is the area in the first quadrant bounded by the $t$-axis and

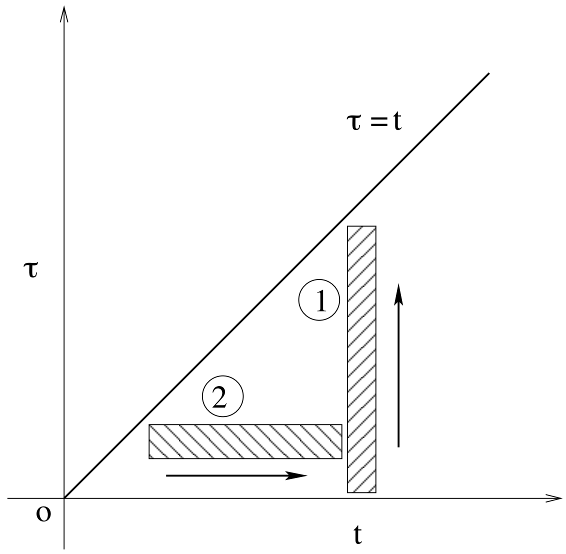


Figure 3: Effects of unit step function on a function $f(t)$. Here $b > a$.

the line $\tau = t$. The variable limit of integration is applied on $\tau$ which varies from $\tau = 0$ to $\tau = t$.

Let us change the order of integration, thus apply variable limit on $t$. Then $t$ would vary from $t = \tau$ to $t = \infty$ and $\tau$ would vary from $\tau = 0$ to $\tau = \infty$. Hence, we have

$$\begin{aligned}
\mathcal{L}\Big((f*g)(t)\Big) &= \int_0^\infty \left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right) f(\tau)\,d\tau \\
&= \int_0^\infty \left(\int_0^\infty e^{-su} g(u)\,du\right) f(\tau)e^{-s\tau}\,d\tau, \qquad t-\tau = u \\
&= \left(\int_0^\infty e^{-su} g(u)\,du\right)\left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right) \\
&= F(s)G(s)
\end{aligned}$$

**Example 4.** *Consider the same problem as given in Example 4 of Lecture Note 18, i.e. find inverse Laplace transform of* $1/s(s+1)^2$.

**Solution**: We write $H(s) = F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s+1)^2$. Thus $f(t) = 1$ and $g(t) = te^{-t}$. Hence, using convolution theorem, we find

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t \tau e^{-\tau}\,d\tau = 1 - (t+1)e^{-t}.$$

**Note:** We have used $f(t-\tau)g(\tau)$ in the convolution formula since $f(t) = 1$. This helps a little bit in the evaluation of the integration.

**Example 5.** *Find inverse Laplace transform of* $1/(s^2+\omega^2)^2$.

**Solution**: Let $H(s) = F(s)G(s)$, where $F(s) = 1/(s^2+\omega^2)$ and $G(s) = 1/(s^2+\omega^2)$.

Thus, $f(t) = \sin(\omega t)/\omega = g(t)$. Hence,

$$\begin{aligned} h(t) &= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin(\omega(t-\tau))d\tau \\ &= \frac{1}{2\omega^3}\Big(\sin(\omega t) - \omega t\cos(\omega t)\Big). \end{aligned}$$

# 3 Applications

**Example 6.** *(Differential equation) Solve the IVP*

$$y'' + y = t, \quad y(0) = 0, y'(0) = 2$$

**Solution**: Take Laplace transform on both sides. This gives

$$s^2Y - 2 + Y = \frac{1}{s^2} \implies Y = \frac{1}{s^2(s^2+1)} + \frac{2}{s^2+1}$$

Using partial fraction, we find

$$Y = \frac{1}{s^2} + \frac{1}{s^2+1} \implies y(t) = t + \sin t$$

**Aliter:** In the method above, we evaluated Laplace transform of the nonhomogeneous term in the right hand side. Now here we don't evaluate it. Let $g(t)$ be nonhomogeneous term (in this case $g(t) = t$). Let $G(s)$ be the Laplace transform of $g$. Now Take Laplace transform on both sides. This gives

$$s^2Y - 2 + Y = G(s) \implies Y = \frac{G(s)}{s^2+1} + \frac{2}{s^2+1}$$

Taking inverse transform and convolution, we find

$$y(t) = \int_0^t g(t-\tau)\sin(\tau)\,dt + 2\sin t \implies y(t) = \int_0^t (t-\tau)\sin(\tau)\,dt + 2\sin t$$

OR (using integration by parts)

$$y(t) = t + \sin t$$

**Example 7.** *(Differential equation) Solve the IVP*

$$y'' + 9y = \begin{cases} 8\sin t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases} \qquad y(0) = 0, y'(0) = 4.$$

**Solution**: Consider $g(t) = 8\Big(u_0(t) - u_\pi(t)\Big)\sin t$. Then Laplace transform of the nonhomogeneous term is the same as that of $g(t)$. Now we write $g(t)$ as

$$g(t) = 8u_0(t)\sin t + +8u_\pi(t)\sin(t-\pi).$$

Now taking Laplace transform of the ODE, we get

$$s^2Y-4+9Y=\frac{8}{s^2+1}+8\frac{e^{-\pi s}}{s^2+1} \implies Y=\frac{4}{s^2+9}+8\frac{1}{(s^2+1)(s^2+9)}+8e^{-\pi s}\frac{1}{(s^2+1)(s^2+9)}$$

Using partial fraction, we get

$$Y=\frac{4}{s^2+9}+\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)+e^{-\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right).$$

Now

$$\mathcal{L}\left(\sin t-\frac{1}{3}\sin 3t\right)=\frac{1}{s^2+1}-\frac{1}{s^2+9}.$$

Hence, using shifting theorem and inverse transform, we find

$$y(t)=\frac{4\sin 3t}{3}+\sin t-\frac{1}{3}\sin 3t+u_\pi(t)\left(\sin(t-\pi)-\frac{1}{3}\sin 3(t-\pi)\right)$$

Further, this can be break up as

$$y(t)=\begin{cases}\sin 3t+\sin t, & 0\le t\le\pi,\\ \dfrac{4}{3}\sin 3t, & t\ge\pi,\end{cases}$$

**Example 8.** *(Differential equation) (Variable coefficient) Solve the IVP*

$$y''-2xy'+4y=0,\quad y(0)=1, y'(0)=0$$

**Solution**: Take Laplace Transform on both sides, we find

$$s^2Y-sy(0)-y'(0)+2\frac{d}{ds}\Big(\mathcal{L}(y')\Big)+4Y=0,$$

OR

$$s^2Y-s+2\frac{d}{ds}\Big(sY-y(0)\Big)+4Y=0 \implies 2sY'+(s^2+6)Y=s \implies Y'+\left(\frac{s}{2}+\frac{3}{s}\right)Y=\frac{1}{2}$$

This is linear equation. Hence,

$$Ys^3e^{s^2/4}=\frac{1}{2}\int s^3e^{s^2/4}\,ds+C$$

OR

$$Y=\frac{s^2-4}{s^3}+C\frac{e^{-s^2/4}}{s^3}$$

OR

$$Y=\frac{1}{s}-\frac{4}{s^3}+C\frac{e^{-s^2/4}}{s^3}.$$

Now it can be shown by Bromwich integral method (*not in the syllabus*) that

$$\mathcal{L}\left(\frac{x^2}{2}-\frac{1}{4}\right)=\frac{e^{-s^2/4}}{s^3}$$

Hence, we find

$$y(t) = (1 - 2x^2) + C\left(\frac{x^2}{2} - \frac{1}{4}\right).$$

OR

$$y(t) = (1 - C/4) + (C/2 - 2)x^2$$

Now $y(0) = 1 \implies C = 4$. Hence

$$y(x) = (1 - 2x^2)$$

**Comment:** If we expand $e^{-s^2/4}/s^3$ then we find

$$\frac{e^{-s^2/4}}{s^3} = \frac{1}{s^3} - \frac{1}{4s} + \text{non-negative power of } s.$$

If we assume $\mathcal{L}^{-1}(s^k) = 0,\ k = 0, 1, 2, \cdots$, then we find

$$\mathcal{L}\left(\frac{x^2}{2} - \frac{1}{4}\right) = \frac{e^{-s^2/4}}{s^3}$$

**Example 9.** *(Integral equation) Solve*

$$y' + \int_0^t y(t-\tau)e^{-2\tau}d\tau = 1, \quad y(0) = 1.$$

**Solution**: Take Laplace Transform on both sides, we find

$$sY - y(0) + \frac{Y}{s+2} = \frac{1}{s} \implies Y = \frac{s+2}{s(s+1)} \implies Y = \frac{2}{s} - \frac{1}{s+1}$$

Hence,

$$y(t) = 2 - e^{-t}$$