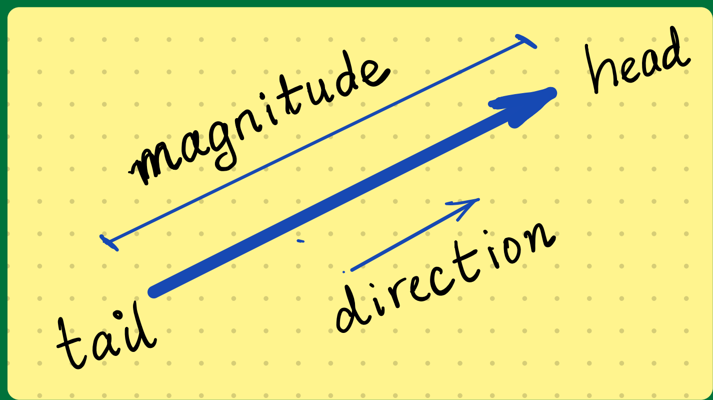
I Introduction to vector, vector operations

II Lines and planes

III Sequence in space

Vector:

A vector is an object that has both a magnitude and a direction. Geometrically, we draw a vector as a directed line segment, whose length is the magnitude of the vector and an arrow indicating the direction. The direction of a vector is from its tail to its head.



Remark.

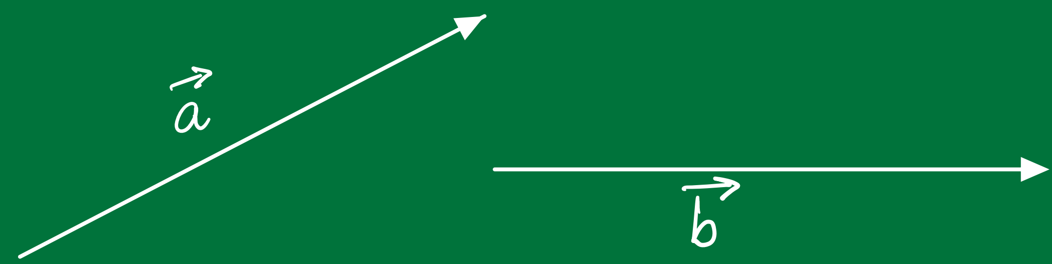
Two vectors are the same if they have the same magnitude and direction. This means that if we take a vector and translate it to a new position (without rotating it), then the vector we obtain at the end of this process is the same vector we had in the beginning.

Examples: Vectors represent

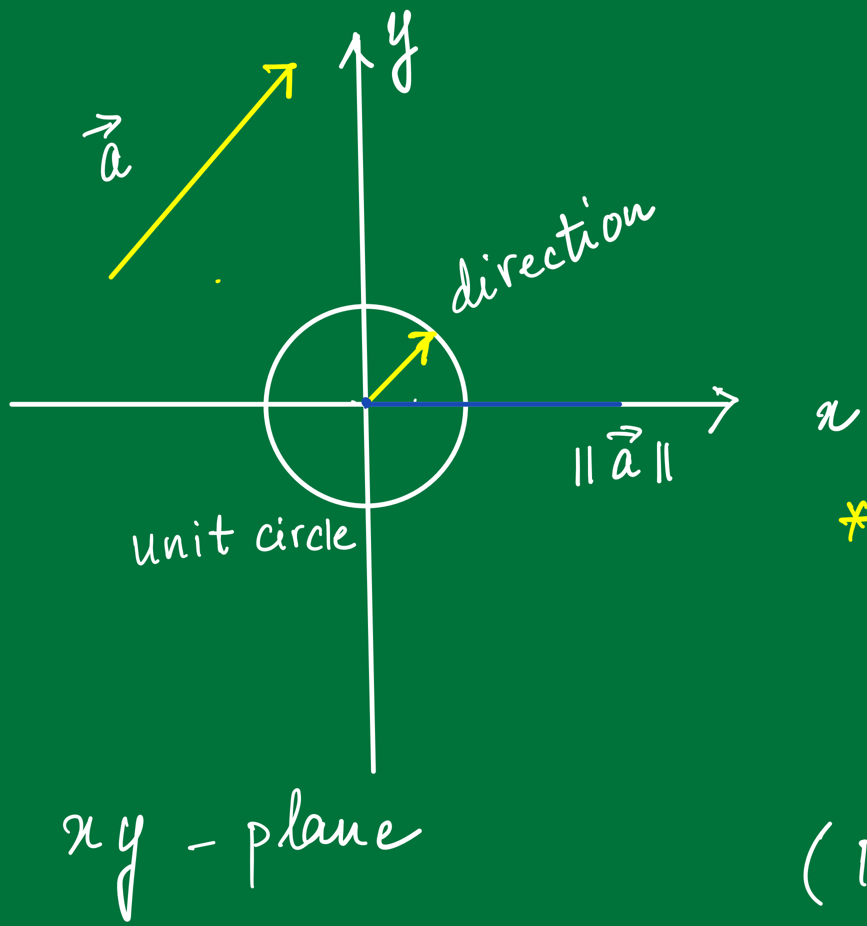
- Force and
- Velocity

– The magnitude of the vector determine the strength of the force or the speed associated to the velocity

<u>Notation</u>: $\mathbf{a}$, $\mathbf{b}$, ... or $\vec{a}$, $\vec{b}$, ...



and the magnitude of The vector $\vec{a}$ by $\|a\|$ or simply by a.



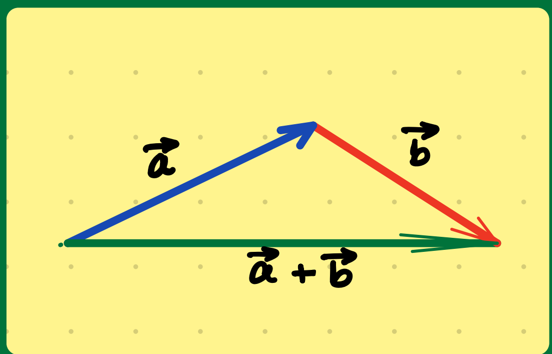
* Exception :

If $\vec{a}$ is the zero vector : $\mathbf{0}$
(The only vector with zero magnitude),
for the direction is not defined ;
it has no length , it is not pointing in
any particular direction.

Vector operations / Operations on vectors

- addition, subtraction, multiplication by scalar
- multiply two vectors - the dot product and the cross product

Addition of vectors.



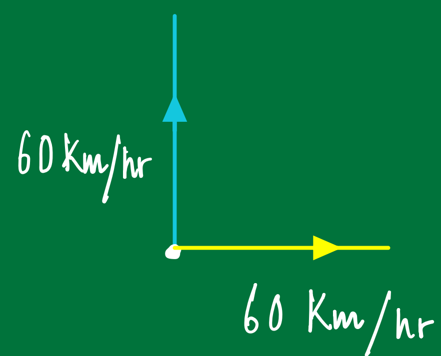
Given two vectors $\vec{a}$ and $\vec{b}$, we form their sum $\vec{a}+\vec{b}$ as follows. We translate the vector $\vec{b}$ until its tail coincides with the head of $\vec{a}$. (note that it does not change the vector!)

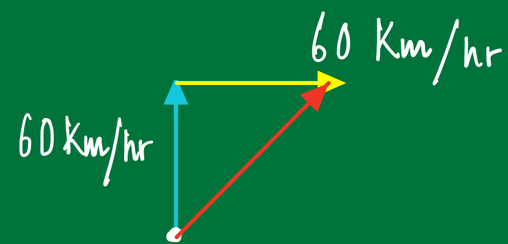
Then the directed line segment from the tail of $\vec{a}$ to the head of $\vec{b}$ is the vector $\vec{a}+\vec{b}$.

Example.

A car is travelling toward north at 60 Km/hr and a child in the back seat behind throws a ball at 60 Km/hr toward his sibling who is sitting due east of him, then the velocity of the ball (relative to the ground) will be in north-east direction. The velocity vectors form a right angle triangle and the total velocity is the hypotenuse. Therefore the total speed of the ball, - the magnitude of the velocity vector is

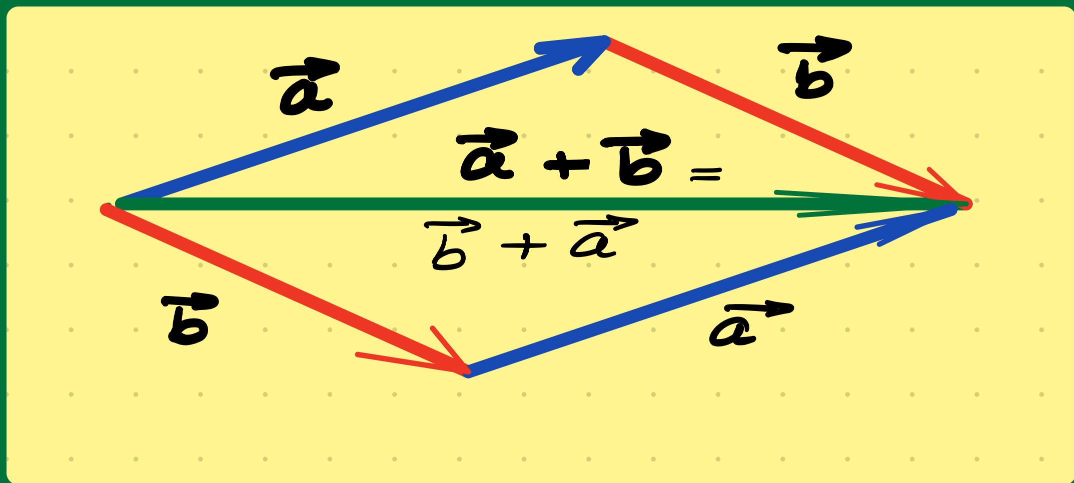
$\sqrt{60^2 + 60^2} = 60\sqrt{2}$ Km/hr relative to the ground.





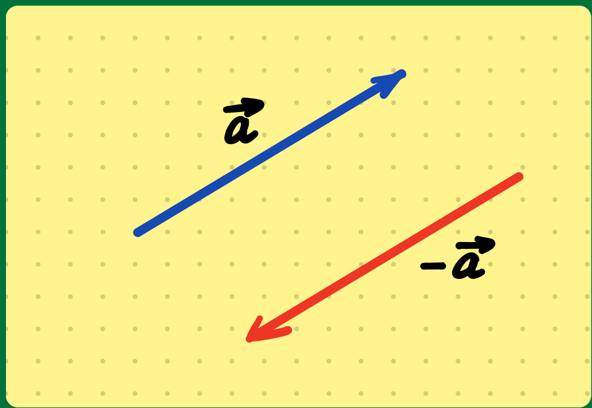
Addition of vectors satisfies two important properties.

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ The parallelogram law.



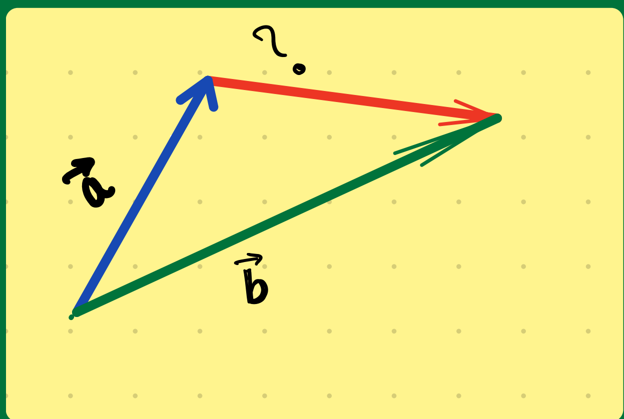
2. The associative law, which states that the sum of three vectors does not depend on which pair of vectors is added first: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Vector subtraction



For a vector $\vec{a}$,
We define $-\vec{a}$, which is the opposite of $\vec{a}$. The vector $-\vec{a}$ is the vector with same magnitude $\|\vec{a}\|$ but that is pointed in the opposite direction.

We define, $\underbrace{\vec{b} - \vec{a}}_{\text{subtraction}} = \underbrace{\vec{b} + (-\vec{a})}_{\text{addition with the opposite vector.}}$



Scalar multiplication

Given a vector $\vec{a}$ and a real number (scalar) λ, we form the vector $\lambda \vec{a}$ as follows:

- If $\lambda > 0$ then $\lambda \vec{a}$ is the vector whose direction is same as direction of $\vec{a}$ and magnitude is λ times that of $\vec{a}$.

  ie, $\|\lambda \vec{a}\| = \lambda \|\vec{a}\|$.

- If $\lambda < 0$ then $\lambda \vec{a}$ is the vector whose direction is opposite to the direction of $\vec{a}$ and magnitude is $|\lambda|$ times that of $\vec{a}$.

  ie, $\|\lambda \vec{a}\| = |\lambda| \|\vec{a}\|$.

Remark: In both cases, we have $\|\lambda \vec{a}\| = |\lambda| \|\vec{a}\|$.

Properties satisfied by scalar multiplication.

1. $\lambda(\vec{a}+\vec{b}) = \lambda\vec{a} + \lambda\vec{b}$ (Distributive law. 1)

2. $(\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$ (Distributive law. 2)

3. $1\,\vec{a} = \vec{a}$

4. $(-1)\,\vec{a} = -\vec{a}$

5. $\underbrace{0}_{\text{number zero}}\,\vec{a} = \underbrace{\mathbf{0}}_{\text{vector zero.}}$

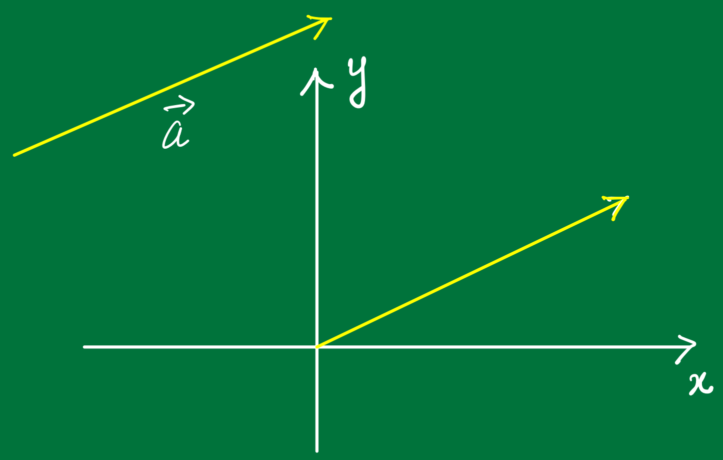
Remark. If $\vec{a}, \vec{b}$ are two vectors with $\vec{b} = \lambda\vec{a}$ for some scalar λ then we say that the vectors $\vec{a}$ and $\vec{b}$ are parallel.

Note: Our discussion was independent of any specific coordinate system in which the vectors live.

We express a vector in a coordinate system, we identify a vector with a tuple of numbers called coordinates or components, that specify the geometry of the vector in terms of the coordinate system

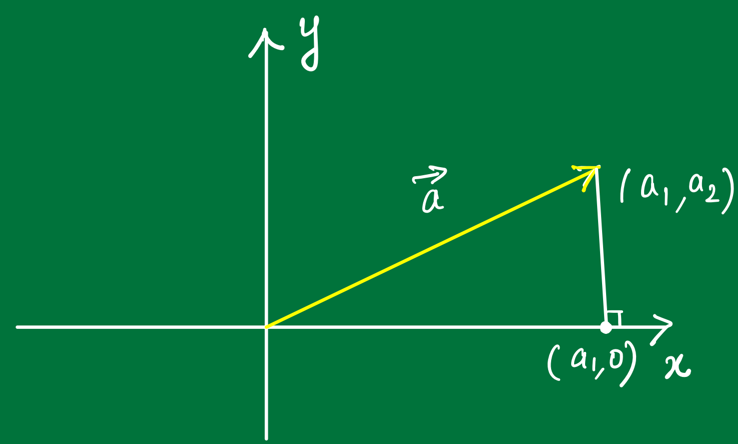
First we discuss the standard Cartesian coordinate systems in the plane and in three-dimensional space.

Vectors in the plane



Step 1. translate the vector, so that its tail is at origin of the coordinate system. Then the head of the vector will be at some point (a_1, a_2) in the plane.

step 2 We call (a_1, a_2) the coordinates or components of the vector $\vec{a}$.



We write $\vec{a} \in \mathbb{R}^2$ and

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

Example: 1.

$\vec{a}$: Line segment from $(2,3)$ to $(5,7)$.

- (a_1, a_2) ?
- $\|\vec{a}\|$?

Step 1. To find the coordinates (a_1, a_2):

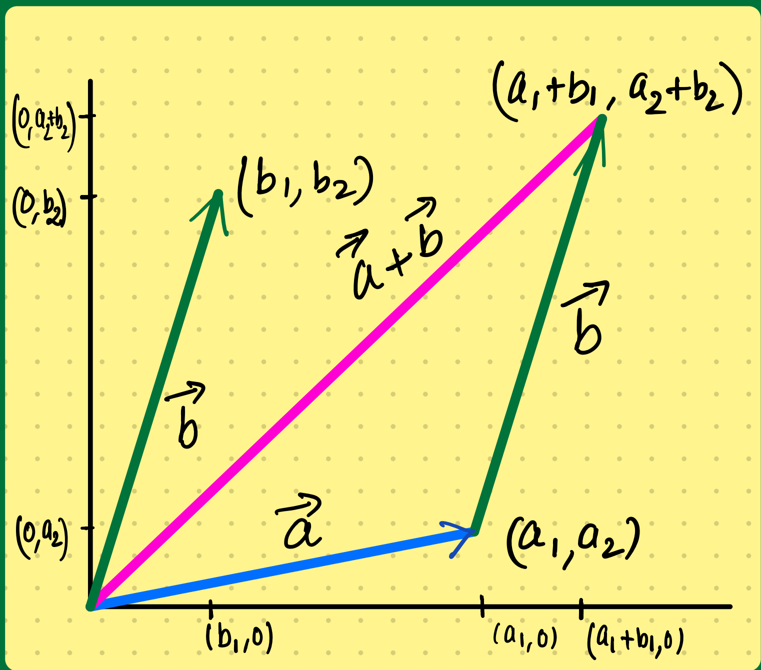
Translate the line segment 2 units left and 3 units down. The line segment begins at the origin $(0,0)$ and ends at $(5-2, 7-3) = (3,4)$

Step 2: Therefore $\vec{a} = (2, 3)$ and

$\|\vec{a}\| = \sqrt{3^2 + 4^2} = 5.$

□ If $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$

then $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)$.



Check that;

$$\vec{b} - \vec{a} = (b_1 - a_1, b_2 - a_2)$$

and $\lambda \vec{a} = (\lambda a_1, \lambda a_2)$

for any scalar λ.

Note: A point (α, β) can also be represented by a vector whose tail is fixed at the origin.

Another way to denote vectors by using standard unit vectors $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$:

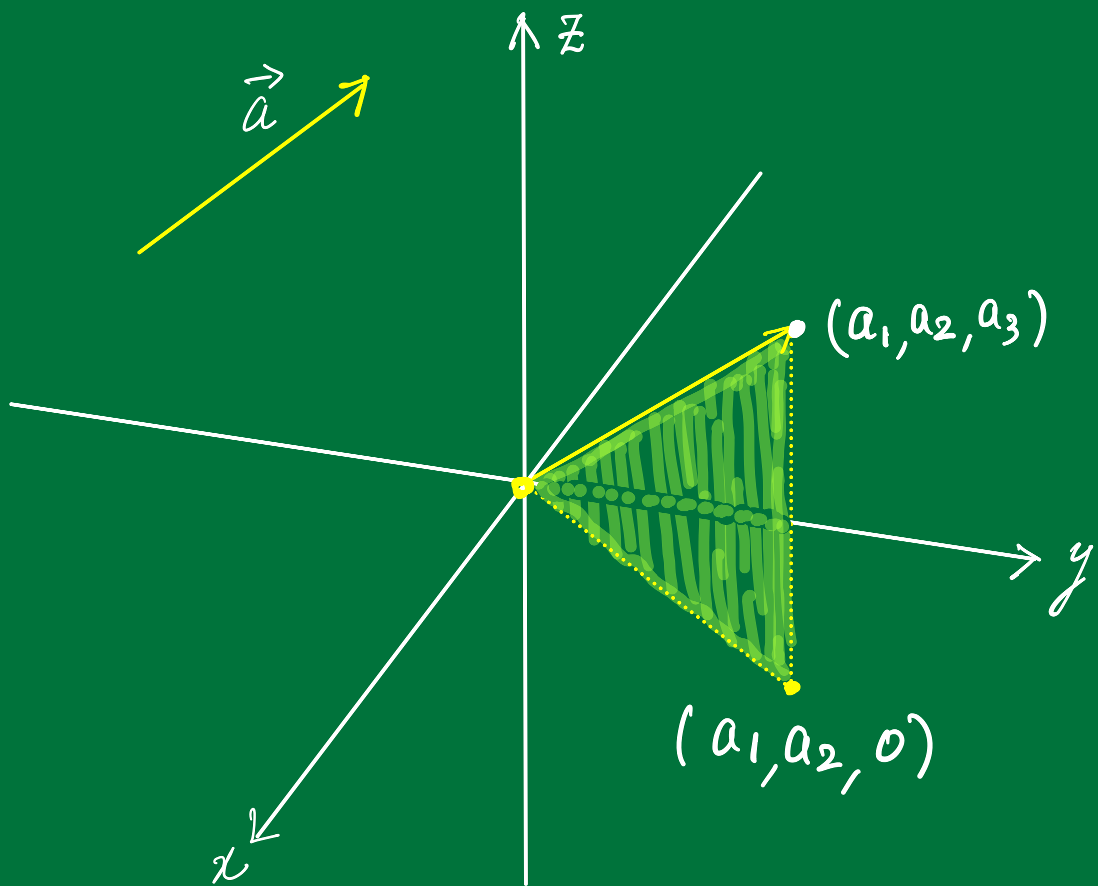
If $\vec{a} = (a_1, a_2)$ then $\vec{a} = a_1\vec{i} + a_2\vec{j}$.

Vectors in three-dimensional space

We assign coordinates of a vector $\vec{a}$ by translating its tail to the origin and finding the coordinates of the point of its head. In this way, we can write a vector as $\vec{a} = (a_1, a_2, a_3)$. We often write $\vec{a} \in \mathbb{R}^3$ to denote that the vector $\vec{a}$ can be described by three real coordinates.

If $\vec{a} = (a_1, a_2, a_3)$,
$\vec{b} = (b_1, b_2, b_3)$

then
$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
$$\vec{b} - \vec{a} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$$
and
$$\lambda \vec{a} = (\lambda a_1, \lambda a_2, \lambda a_3)$$
for any scalar λ.



Note :

1. In terms of standard unit vectors $\vec{i}, \vec{j}, \vec{k}$ we write $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$.

2. The Length of a vector $\vec{a} = (a_1, a_2, a_3)$

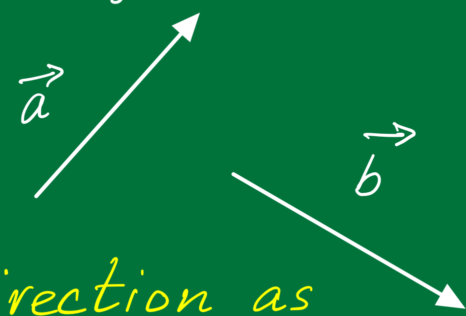
is $\sqrt{a_1^2 + a_2^2 + a_3^2}$

$$
\left[\;
\begin{aligned}
* \quad \|\vec{a}\|^2 &= \|(a_1, a_2, a_3)\|^2 \\
&= \|(a_1, a_2, 0)\|^2 + \|(0, 0, a_3)\|^2 \\
&= a_1^2 + a_2^2 + a_3^2
\end{aligned}
\;\right]
$$

Going beyond 3-dimension!

The projection of one vector onto another.

Let $\vec{a}$, $\vec{b}$ be two given vectors.

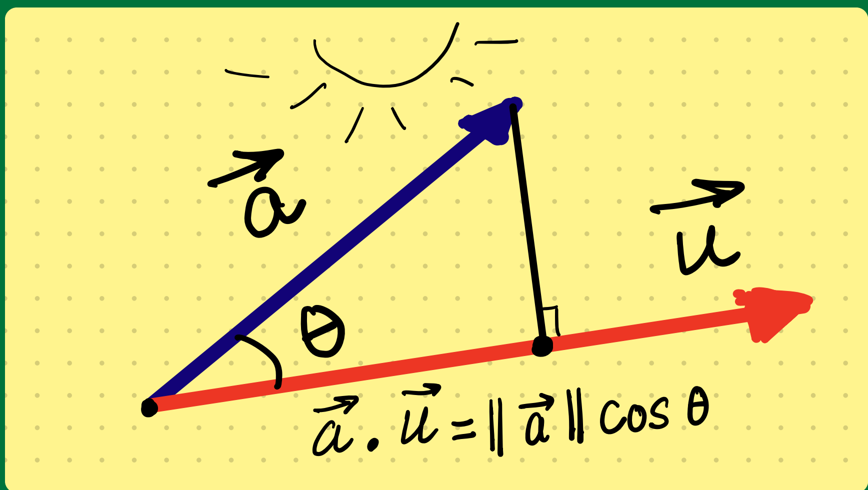


Q. How much of $\vec{a}$ is pointing in the same direction as the vector $\vec{b}$?

— does not have anything to do with the magnitude of $\vec{b}$, it is based only on the direction of $\vec{b}$ (unit vector along $\vec{b}$).

Let us replace $\vec{b}$ with the unit vector $\vec{u}$ that points in the same direction as $\vec{b}$, where $\vec{u}$ is defined as $\vec{u} = \frac{b}{\|b\|}$

The dot product of $\vec{a}$ with unit vector $\vec{u}$, denoted by $\vec{a}.\vec{u}$ is defined to be the projection of $\vec{a}$ in the direction of $\vec{u}$, or the amount $\vec{a}$ is pointing in the same direction as unit vector $\vec{u}$.



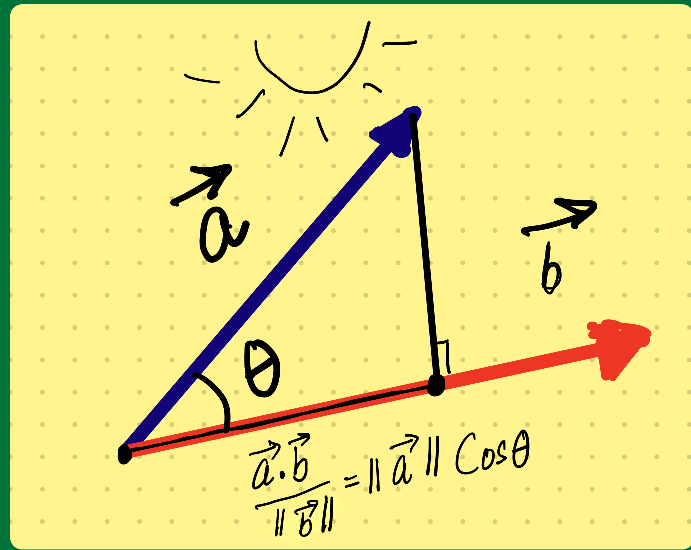
$\vec{a}.\vec{u}$:

Length of the shadow of $\vec{a}$ onto $\vec{u}$ if their tails were together and the sun was shining from a direction perpendicular to $\vec{u}$.

For the vector $\vec{b}$ with magnitude different than one, we now define the dot product:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta ,$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.



[The geometric definition of the dot product.]

* This dot product depends on magnitude of both vectors.

Note: (1). $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (2). $\vec{i}\cdot\vec{j} = 0$ (3) $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

$\vec{j}\cdot\vec{k} = 0$

$\vec{k}\cdot\vec{i} = 0$

(4) For any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, and any scalar λ,

- $(\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$
- $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + b\cdot\vec{c}$

(5) For $\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and

$\vec{b} = (b_1, b_2, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

We get $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ [Component formula for The dot product]

Example: Let $\vec{a} = (1, 2, 3)$ and $\vec{b} = (4, 5, -6)$.
1.

Do the vector form an acute angle, right angle or obtuse angle?

Using the component formula for the dot product,

$$\vec{a} \cdot \vec{b} = 1.4 + 2.5 + 3(-6)$$
$$= -4$$

Since $\vec{a} \cdot \vec{b}$ is negetive, we can infer from the geometric definition of dot product that the vectors form an obtuse angle.

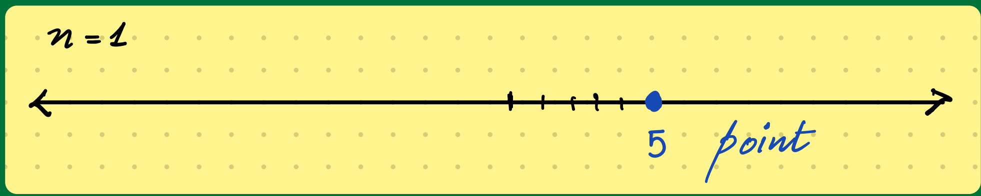
2. Let $\vec{a} = (7, 8)$ and $\vec{b}$ $(9, 10)$.

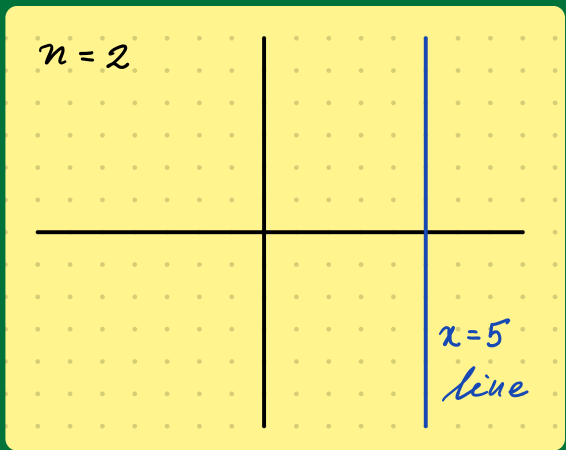
Do the vector form an acute angle, right angle or obtuse angle?

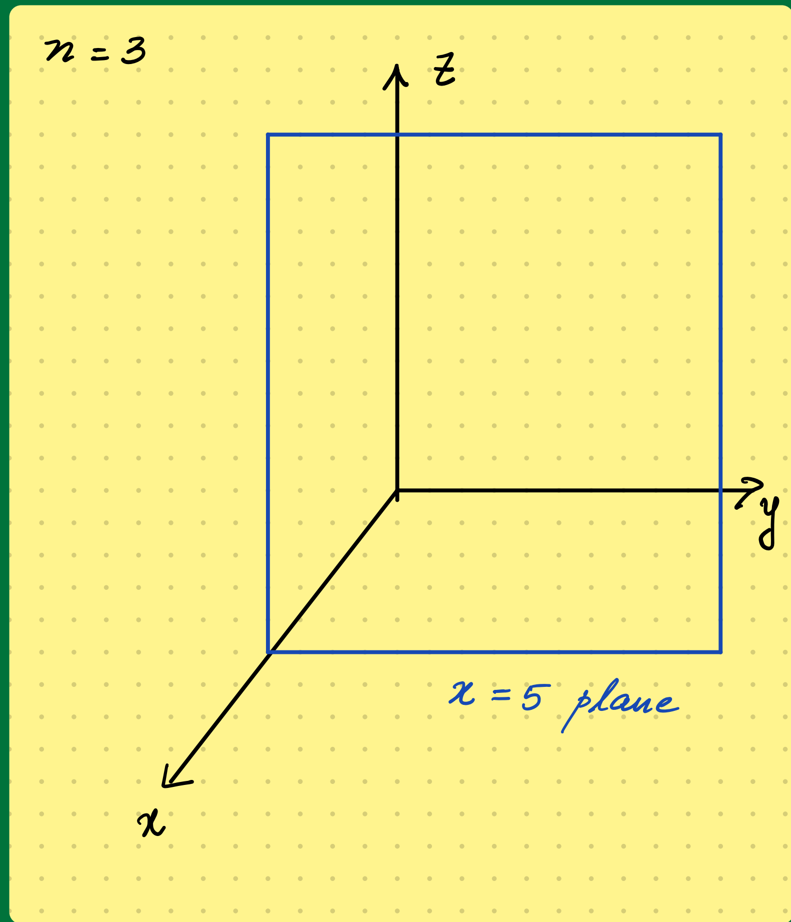
Since $\vec{a} \cdot \vec{b}$ is positive, we can infer from the geometric definition of dot product that the vectors form an acute angle.

Q. Is the graph of

$x = 5$ the graph of a line or a plane or a point?

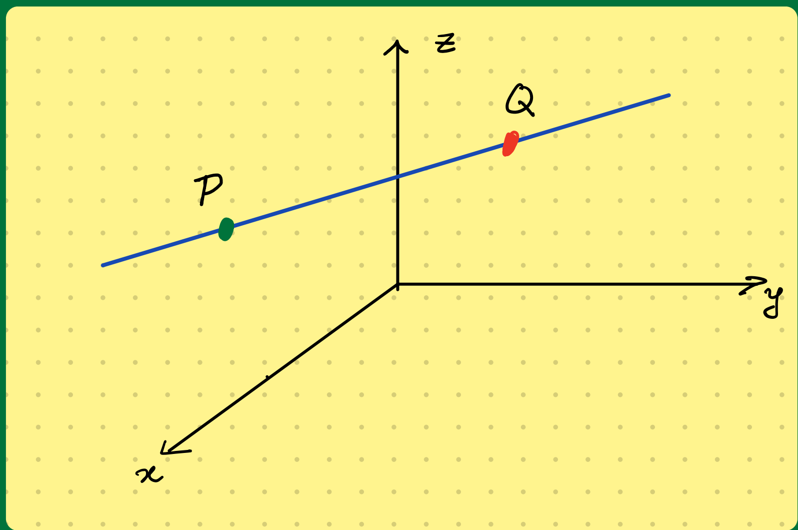

n = 1
5
point

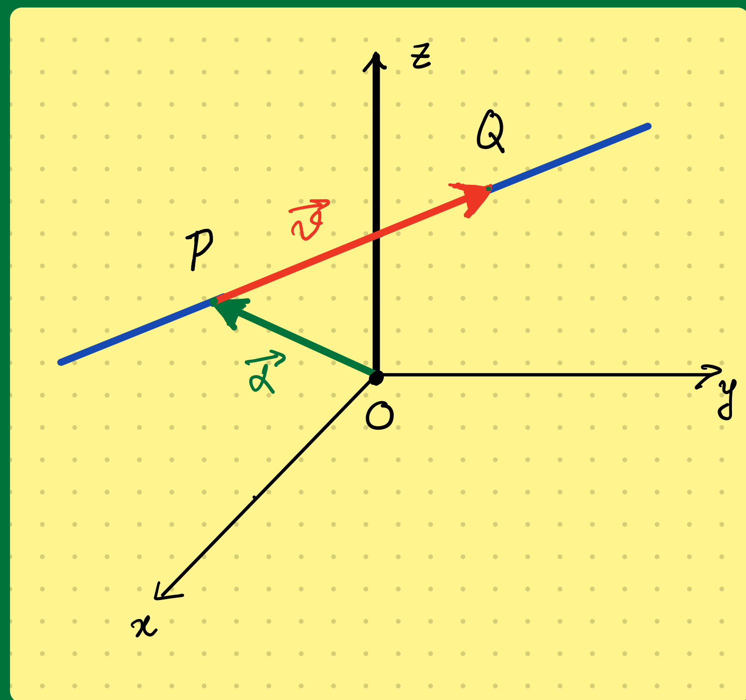

n = 2
x = 5
line


n = 3
z
y
x
x = 5 plane

Parametrization of a line.

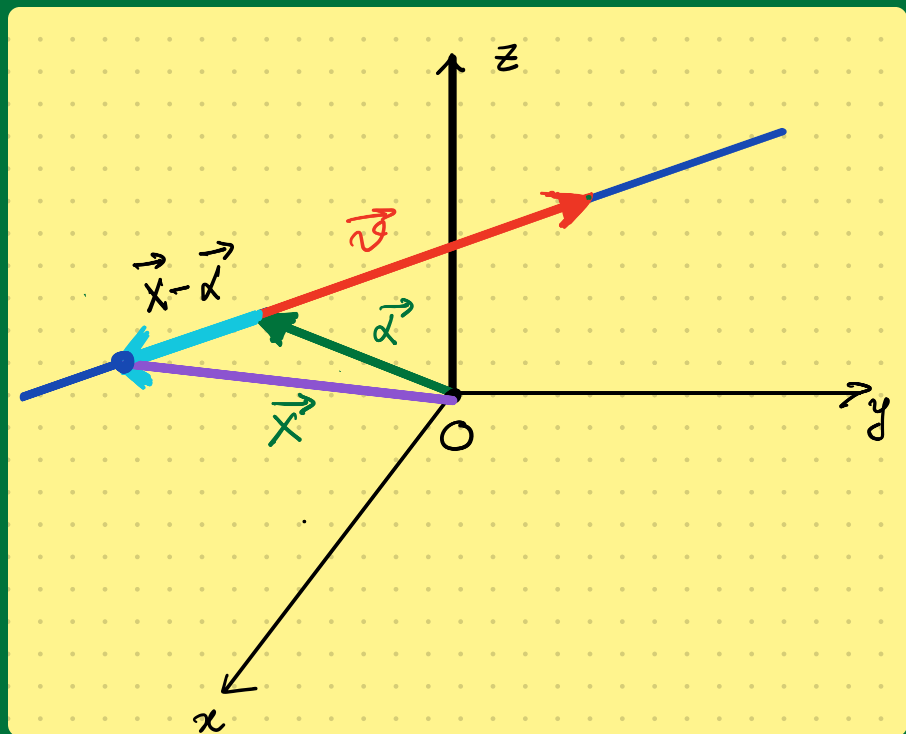
- A line is determined by two points P and Q.





□ We now consider a line by a point P and a vector $\vec{v}$ parallel to the line. ($\vec{v}$ can be taken to be the vector from the point P to the point Q).

Now think the point P as the head of a vector $\vec{\alpha}$ with tail at the origin.



A point X on the line determined by two vectors. The point X is on the line (passing through the point $\vec{\alpha}$ and parallel to the vector $\vec{v}$), the vector $\vec{v}$ is always $\underbrace{\text{parallel}}$ to the vector $\vec{X} - \vec{\alpha}$.

So, $\vec{X} - \vec{\alpha} = t\vec{v}$ for some $t \in \mathbb{R}$.

or, $\vec{X} = t\vec{v} + \vec{\alpha}$

Two vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if

$$\vec{a} = \lambda \vec{b}$$

for some scalar λ.

This equation is called the parametrization of the line.

As the parameter t sweeps through all real numbers, the point X sweeps out the line.

- For $\vec{v} = (a, b, c)$, $\vec{\alpha} = (x_0, y_0, z_0)$ we get the point

$$X = (x, y, z) = (ta + x_0,\ tb + y_0,\ tc + z_0)$$

or,

$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\} \text{ or, } \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

—— / ——

In case $a, b, c \neq 0$, the equation of the line is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

* If $a = 0$, $b, c \neq 0$, then The line is represented as

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example. Find a parametrization of a line through the points $(1,2,4)$ and $(7,0,5)$.

The line is parallel to the vector

$$\vec{v} = (7,0,5) - (1,2,4)$$
$$= (6,-2,1).$$

Hence a parametrization for the line is

$$X = (1,2,4) + t\,(6,-2,1)$$

for $-\infty < t < \infty$.

Equivalently, $X = (1+6t, 2-2t, 4+t)$

for $t \in \mathbb{R}$.

Or, if we write $X = (x, y, z) \in \mathbb{R}^3$, we can write the parametric equation in component form as

$$x = 1 + 6t$$
$$y = 2 - 2t$$
$$z = 4 + t$$

for $t \in \mathbb{R}$.

or, $$\frac{x-1}{6} = \frac{y-2}{-2} = \frac{z-4}{1}.$$