lecture 7

Second order linear equations

(*) $$\begin{cases} \dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)\,y = r(x) \\ p(x),\ q(x),\ r(x) \text{ are continuous functions on some interval } I : \longrightarrow \mathbb{R} \\ y(x_0) = \alpha_1 \qquad y'(x_0) = \alpha_2 \qquad x_0 \in I \end{cases}$$

Theorem (Existence & Unique)
The IVP (*) has unique solution over the interval I.

Proof This follows Picard's Theorem.

Take $$\begin{cases} y'(x) = v(x) \\ v'(x) = r(x) - p(x)\,v - q(x)\,y \end{cases}$$
$v(x_0) = \alpha_2 \quad y(x_0) = \alpha_1$

This is a system of 1st order IVP's

Hence by Picard's Theorem the result follows.

Homogenous 2nd order liner eqution

$$(**)\quad \left\{ \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0 \right.$$

- $y_1(x)$ & $y_2(x)$ are two solutions of $(**)$ then $c_1 y_1(x) + c_2 y_2(x)$ also a soln where $c_1, c_2 \in \mathbb{R}$.

- $y \equiv 0$ is always a solution of $(**)$.

Thus the set of solutions of $(**)$ is a real vector space.

Recall

- Let $f, g : I \longrightarrow \mathbb{R}$ are two functions. They are called Linearly dependent (LD) over I if $\exists$ $(c_1, c_2) \neq (0, 0)$ s.t.
$$c_1 f(x) + c_2 g(x) = 0 \quad \forall\, x \in I.$$

- f, g called linearly independent (LI) if they are not LD.

Suppose f, g are LD over I.
$$c_1 f(x) + c_2 g(x) = 0 \quad \forall\, x \in I$$
If f, g differentiable,
$$c_1 f'(x) + c_2 g'(x) = 0 \quad \forall\, x \in I$$
There are two linear equations with unknowns c_1, c_2. and it has non-trivial solution.

($Ax = 0$ has non-trivial solution
$\Leftrightarrow |A| = 0$)

Thus $\underbrace{\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}}_{W(f,g)(x)} = 0 \quad \forall x \in I.$

$W(f,g)(x)$ = Wronskian of the functions f, g.

Thus we have proved

Lemma 1: f, g LD over I, f, g differentiable
$\Rightarrow W(f,g)(x) = 0 \quad \forall x \in I.$

($W(f,g)(x_0) \neq 0$ for some $x_0 \in I$
$\Rightarrow f, g$ are LI over I)

Example

$f(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$ $\quad g(x) = x^2$

$I = [-1, 1]$

- $W(f,g)(x) = 0 \quad \forall x.$

- f, g are L.I

($\because \nexists\, c$ s.t $f(x) = c\, g(x) \quad \forall x \in [-1, 1]$)

Lemma

Lemma 2

Suppose $y_1(x)$, $y_2(x)$ s.t

$W(y_1, y_2)(x) = 0$ over I

& y_1, & y_2 are solutions of same

ODE (**).

$\Rightarrow$ y_1 & y_2 are LD. over I.

Proof Choose $x_0 \in I$

$$W(Y_1, Y_2)(x_0) = \begin{vmatrix} Y_1(x_0) & Y_2(x_0) \\ Y_1'(x_0) & Y_2'(x_0) \end{vmatrix} = 0$$

$\Rightarrow$ $\exists (c_1, c_2) \neq (0, 0)$ s.t

$$c_1 Y_1(x_0) + c_2 Y_2(x_0) = 0$$

$$c_1 Y_1'(x_0) + c_2 Y_2'(x_0) = 0$$

Take $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

$\Rightarrow$ y is also solution of (**).

$y(x_0) = 0$, $y'(x_0) = 0$

But $0(x) = 0$ $\forall x$ is also a solution of (**). satisfying same initial condition $0(x_0) = 0$

$0'(x_0) = 0$

So by uniqueness of solution, we get $y(x) = 0$ $\forall x \in I$.

i.e. $c_1 Y_1(x) + c_2 Y_2(x) = 0$

i.e. Y_1, Y_2 are LD. ∎

Summary. Assume y_1, y_2 are solutions of Same (**).

Then

y_1 & y_2 are LD over I $\Longleftrightarrow$ $W(y_1, y_2)(x) = 0$ $\forall x \in I$.

y_1 & y_2 are LI $\Longleftrightarrow$ $\exists\, x_0 \in I$ s.t. $W(f,g)(x_0) \neq 0$

$\xLeftrightarrow{\text{lemma 3}}$ $W(f,g)(x) \neq 0$ $\forall x \in I$

Example (1) $y_1(x) = x|x|$ $\quad y_2(x) = x^2$

$I = [-1\ \ 1]$

Then y_1 & y_2 are NOT solution of Same (**) over $[-1\ \ 1]$.

(2) $y(x) = x^3$ is NOT a solution of any (**) over $[-1\ \ 1]$.

Sol $\quad y'' + p(x)\, y' + q(x)\, y = 0$ over $[-1, 1]$

Putting $y = x^3$

$$6x + p(x)\, 3x^2 + q(x)\, x^3 = 0 \quad \forall x$$

Dividing by x^2

$$\frac{6}{x} + 3p(x) + q(x)\, x = 0 \quad \forall x \neq 0$$

$$3p(x) + q(x)\, x = -\frac{6}{x} \quad \forall x \neq 0.$$

Take limit as $x \longrightarrow 0$

L.H.S $\longrightarrow 3p(0) + q(0)\cdot 0 = 3p(0)$

R.H.S $\longrightarrow$ does not exist

This is a contradiction. So x^3 is not solution of (**) over $[-1\ \ 1]$.

$y(x) = x^3$

$y' = 3x^2$

$y'' = 6x$

$x^2 y'' = 6x^3 = 6y.$

$x^2 y'' - 6y = 0$

$y'' - \frac{6}{x^2} y = 0$ $\qquad p(x) = 0$

$q(x) = -\frac{6}{x^2}$

Example $W(2x^2, x^4) = 4x^5$

Thus $2x^2, x^4$ can not solution of same $(**)$ over $[-1 \quad 1]$.

lemma 3. Suppose y_1 & y_2 are solutions of same $(**)$.

Then $W(y_1, y_2)$ either always zero or never zero.

Proof $\quad y_1'' + p(x) y_1' + q(x) y_1 = 0$ — ①

$y_2'' + p(x) y_2' + q(x) y_2 = 0$ — ②

① $\times y_2$ — ② $\times y_1$

$(y_2 y_1'' - y_2'' y_1) + p(x)(y_1' y_2 - y_2' y_1) = 0$

$\frac{dW}{dx} - p(x) W = 0.$

Solve: $W(x) = c\, e^{\int p(x)\,dx}$

Thus $W \equiv 0$ if $c = 0$ & W never zero if $c \neq 0$. ∎