Lecture 5 : Determinant, computation of inverse & Cramer's rule

The definition of determinant by permutation is not convenient for computation. Here we will give further properties of determinant which will help in computation.

Let $A = (a_{ij})$ be a square matrix of order $n > 1$.
Let A_{ij} be the submatrix of order $n-1$ obtained from A by deleting ith row and jth column of A. Determinant of A_{ij} is called ij th minor of A.

Theorem: For each $i \in \{1, 2, \ldots, n\}$, $n > 1$

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| .$$

Proof: Fix $i \in \{1, 2, \ldots, n\}$
For $j \in \{1, 2, \ldots, n\}$,

$$\text{let } \quad A_j = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & & \\ 0 & 0 & & a_{ij} & \cdots & 0 \\ \vdots & & & & & \\ a_{n1} & a_{n2} & & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

i.e. A_j is the matrix (b_{pq}) such that $b_{pq} = a_{pq}$ if $p \neq i$, $b_{iq} = 0$ if $q \neq j$ & $b_{ij} = a_{ij}$ ...

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1}+0+\cdots+0 & \cdots & 0+\cdots+a_{ij}+\cdots+0 & \cdots & 0+\cdots+0+a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & & a_{nj} & & a_{nn} \end{pmatrix}$$

Using property 4, we have

$$|A| = |A_1| + \cdots + |A_j| + \cdots + |A_n|$$

<u>Claim</u>: $$|A_j| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

$$E_{nn-1} E_{n-1n-2} \cdots E_{i+1\,i} A_j E_{jj+1} \cdots E_{n-1n}$$

$$= \begin{pmatrix} A_{ij} & \begin{matrix} a_{1j} \\ a_{2j} \\ \vdots \end{matrix} \\ 0\ 0 \cdots 0 & a_{ij} \end{pmatrix} = C \text{ (say)}$$

C is of the form $\begin{pmatrix} B & * \\ 0 & c_{nn} \end{pmatrix}$,

where B is a square matrix of order $n-1$. Let $C = (c_{ij})$

$$|C| = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, c_{1\sigma(1)} \cdots c_{n\sigma(n)}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(n)=n}} \text{sign}(\sigma)\, c_{1\sigma(1)} \cdots c_{n-1\,\sigma(n-1)}\, c_{nn}$$

(Since $c_{n\sigma(n)} = 0$ if $\sigma(n) \neq n$)

$$= c_{nn} \sum_{\sigma \in S_{n-1}} \text{sign}(\sigma)\, c_{1\sigma(1)} \cdots c_{n-1\,\sigma(n-1)}$$

$$= c_{nn}\, |B|$$

Using it we have,

$$|E_{n\,n-1} \cdots E_{i+1\,i}\, A_j\, E_{j\,j+1} \cdots E_{n-1\,n}| = a_{ij}\, |A_{ij}|$$

$$(-1)^{n-i}\, |A_j|\, (-1)^{n-j} = a_{ij}\, |A_{ij}|$$

$$\Rightarrow |A_j| = (-1)^{i+j}\, a_{ij}\, |A_{ij}|$$

$$\therefore |A| = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\, |A_{ij}| \qquad \square$$

Corollary: Let $A = (a_{ij})$ be a matrix of order n. Fix $i \in \{1, 2, \dots, n\}$, then for all $k \neq i$, we have

$$\sum_{j=1}^{n} (-1)^{i+j} a_{kj} |A_{ij}| = 0$$

Proof: Let B be the matrix obtained from A by replacing the ith row of A with kth row of A & remaining kth row as it is.

$$B = \begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow i\text{th row} \\ \\ \leftarrow k\text{th row} \\ \\ \\ \end{array}$$

By previous theorem

$$|B| = \sum_{j=1}^{n} (-1)^{i+j} a_{kj} |A_{ij}|$$

B has two identical rows, hence $|B| = 0$

$\square$

Let $B_{ij} = (-1)^{i+j} |A_{ij}|$, then

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \begin{pmatrix} B_{11} & B_{21} & \cdots & B_{n1} \\ B_{12} & B_{22} & \cdots & B_{n2} \\ \vdots & \vdots & & \vdots \\ B_{1n} & B_{2n} & \cdots & B_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & |A| \end{pmatrix} = |A| I_n$$

$$A B^T = |A| I_n$$

The element B_{ij} is called ij th cofactor of A & the matrix B^T is called (classical) adjoint (or adjugate) of the matrix A. It is denoted by $Adj(A)$. Thus,

$$\boxed{A\,(Adj(A)) = |A|\, I}$$

As $A = A^T$, we also have

$$Adj(A)\, A = |A|\, I$$

$$\boxed{A\,(\mathrm{Adj}(A)) = (\mathrm{Adj}(A))\,A = |A|\,I}$$

Thus if $|A| \neq 0$, $A^{-1} = \frac{1}{|A|}\,\mathrm{Adj}(A)$.

Cramer's rule for solving linear equations

Consider the system of linear equations:

$$AX = D, \quad \text{where } A = (a_{ij})_{n\times n}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad D = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

This system has a unique solution if $|A| \neq 0$ & the solution is given by $X = A^{-1}B$

$$\Rightarrow \quad x_i = \frac{1}{|A|}\sum_{j=1}^{n} B_{ji}\,d_j$$

$$= \frac{1}{|A|}\left(B_{1i}d_1 + B_{2i}d_2 + \dots + B_{ni}d_n\right)$$

Let B_i be the matrix obtained from A by replacing ith column by $(d_1\, d_2 \dots d_n)^T$.

Then $|B_i| = \sum_{j=1}^{n} B_{ji} d_j$ (check)

Hence, $x_i = \dfrac{|B_i|}{|A|}$, $i = 1, 2, \ldots, n$.

If $|A| = 0$, then $AX = B$ either has infinitely many solutions or no solution.

Example: $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$. $|A| = 1(3-4) - 2(9-8) + 1(6-4) = -1$

Cofactor matrix $\begin{pmatrix} -1 & -1 & 2 \\ -2 & -1 & 6 \\ 3 & 1 & -5 \end{pmatrix} = C$ (say)

$$\mathrm{Adj}(A) = C^T = \begin{pmatrix} -1 & -2 & 3 \\ -1 & -1 & 1 \\ 2 & 6 & -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \mathrm{Adj}(A)$$

$$= \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ -2 & -6 & 5 \end{pmatrix}.$$