lecture 4 (Existance & Uniquenss of solution)

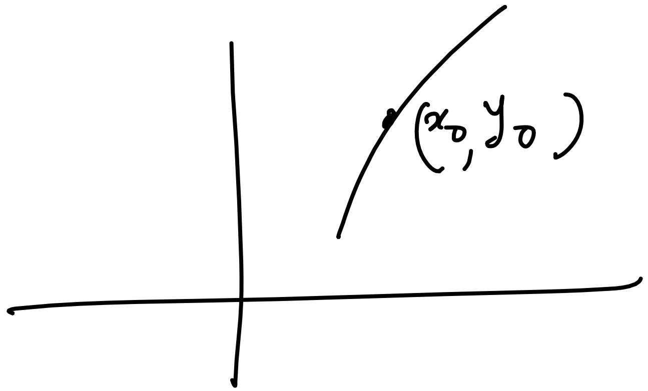
Recall IVP (Initial Value Problem)

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0 \qquad \text{—— } (*)$$

Q> When Does this IVP has a solution?

When solution is unique?

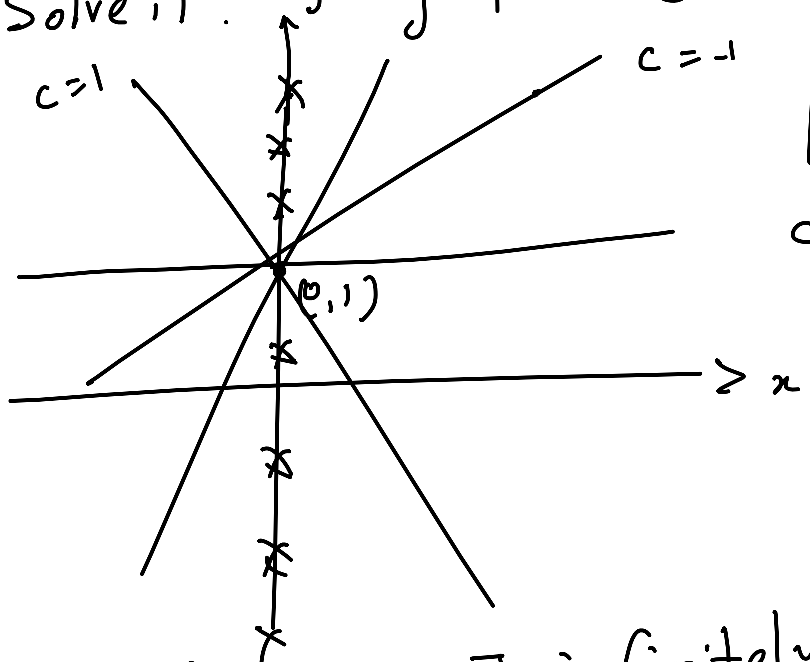
This means



Exmple 1 $\dfrac{dy}{dx} = \dfrac{y-1}{x} = f(x,y).$

f contnous except y-axis
$f_y = \dfrac{1}{x}$ cont. excep x-axis.

Solve it: $y - 1 = Cx$



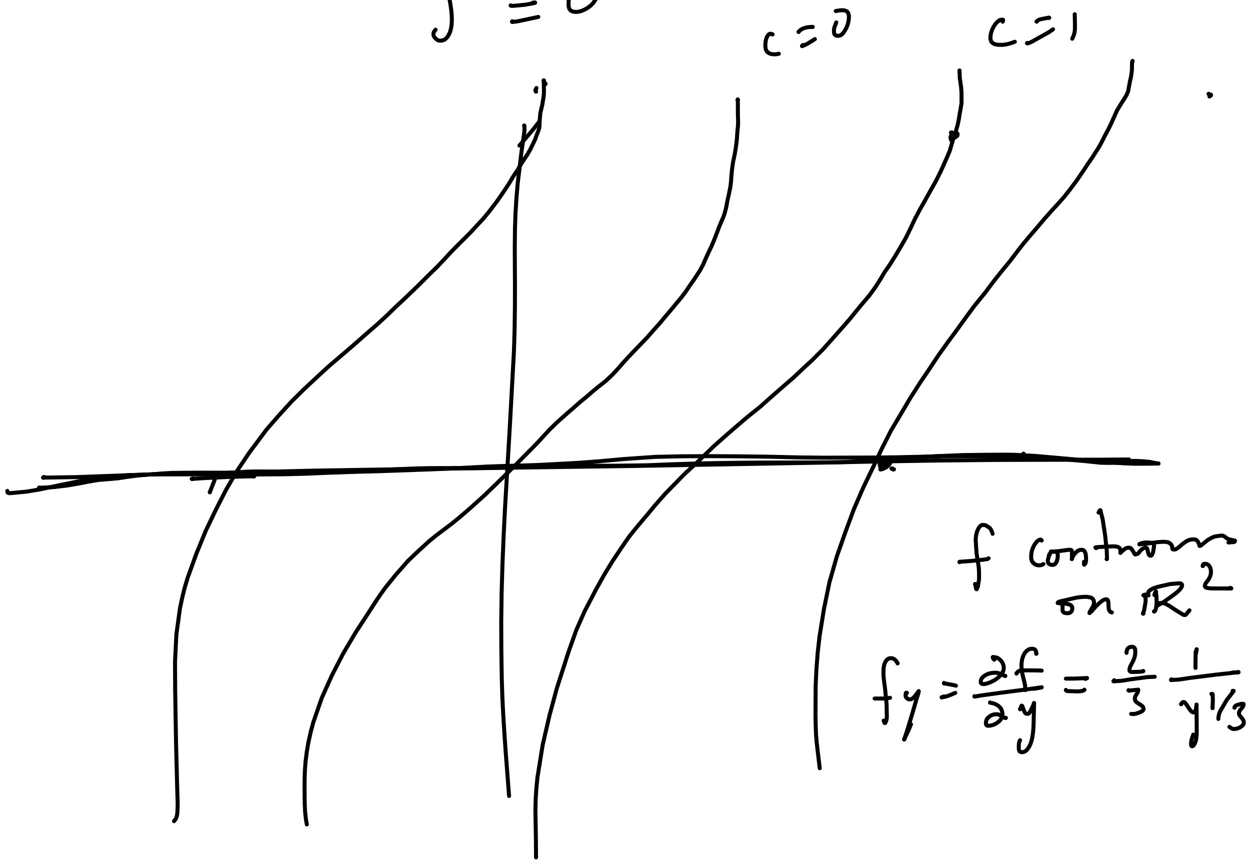
- Through $(0,1)$ $\exists$ infinitely many integral curves
- Through $(0, y_0)$ $y \neq 1$ $\nexists$ integral curve passing through it.
- $(x_0, y_0) \cdot x_0 \neq 0$, $\exists !$ integral curve passing through it.

Example 2

$$\frac{dy}{dx} = y^{2/3} = f(x,y)$$

Solving: $y = \frac{(x+c)^3}{27}$

$y \equiv 0$



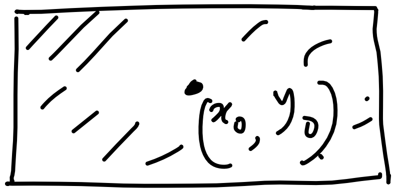
f continuous on $\mathbb{R}^2$

$$f_y = \frac{\partial f}{\partial y} = \frac{2}{3}\frac{1}{y^{1/3}}$$

- f cont: $(x_0, y_0) \in$ x-axis then $\exists$ multiple integral curves.
- f cont + f_y cont.: $(x_0, y_0) \notin$ x-axis then $\exists !$ integral curve.

Theorem (Picard Theorem)

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0 \longrightarrow (*)$$



$$R = [x_0 - a,\ x_0 + a] \times [y_0 - b,\ y_0 + b] = \left\{ (x,y) \;\middle|\; \begin{array}{l} |x - x_0| \le a \\ |y - y_0| \le b \end{array} \right\}$$

Existance If f is continuous on R, then $\exists$ a solution to the IVP $(*)$. The solution is valid for at least $(x_0 - \alpha,\ x_0 + \alpha)$ where $\alpha = \min\left\{a, \frac{b}{M}\right\}$, $M = \sup_R f(x,y)$

Uniqueness f & $f_y = \frac{\partial f}{\partial y}$ continuous on R, then $(*)$ has unique solution.

Remark

The condition of continuity of f_y can be relaxed by a weaker condition called: Lipschitz condition (on y)

$\exists\ L > 0$ s.t

$$|f(x, y_1) - f(x, y_2)| \le L\,|y_1 - y_2|$$

$\forall\ (x, y_1)\ (x, y_2) \in R$.

- f_y continuous on R $\Rightarrow$ f satisfy Lipschitz condition on R

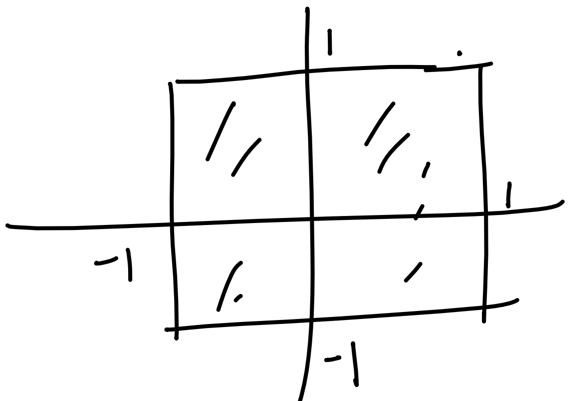
$$L = \sup_R |f_y(x,y)|$$

$$|f(x, y_1) - f(x, y_2)| = \left| (y_1 - y_2) \frac{\partial f}{\partial y}(x, \xi) \right|$$

$$\le L\,|y_1 - y_2| \qquad y_1 < \xi < y_2$$

Example $f(x,y) = x^2 |y|$

$R = [-1, 1] \times [-1, 1]$



Claim f satisfy LC on R

$$|f(x,y_1) - f(x,y_2)|$$
$$= \left| x^2 \left(|y_1| - |y_2|\right) \right|$$
$$\leq 1 \; \left| |y_1| - |y_2| \right| \leq |y_1 - y_2|$$

Claim $\frac{\partial f}{\partial y}$ is not continuous on R

$$\frac{\partial f}{\partial y}(\cancel{0}, 0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$
$$= \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

$$\frac{\partial f}{\partial y}(x, 0) = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k}$$
$$= \lim_{k \to 0} \frac{x^2|k| - 0}{k}$$
$$= x^2 \lim_{k \to 0} \frac{|k|}{k}$$

— does not exist.

$\therefore \frac{\partial f}{\partial y}$ not continuous along x-axis

$\boxtimes$

Example 2 $\quad \dfrac{dy}{dx} = 1+y^2 \qquad y(0)=0$

$x_0 = 0$

$y_0 = 0$

$$R = [-100, 100] \times [-1, 1]$$

$$a = 100, \qquad b = 1$$

Here

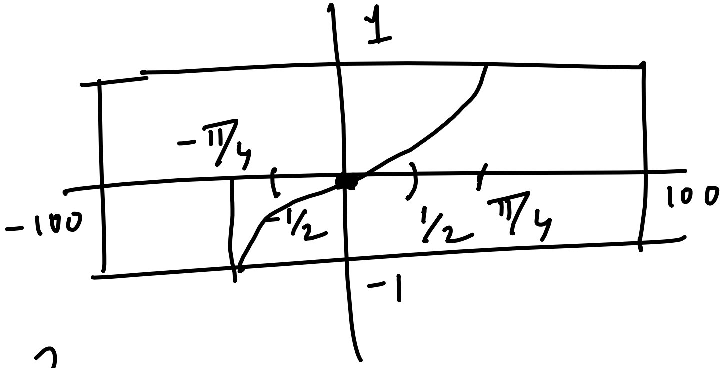
$$\left.\begin{aligned} f(x,y) &= 1+y^2 \\ f_y &= 2y \end{aligned}\right\} \text{ continuous on } R$$

By the Picard theorem, the IVP has unique solution valid for at least $(-\alpha, \alpha)$ where $\alpha = \min\left\{a, \dfrac{b}{M}\right\}$

$$M = \sup_R f(x,y)$$

$M = 2 \qquad \alpha = \min\left(100, \tfrac{1}{2}\right) = \tfrac{1}{2}$.

Q.



$$\frac{dy}{dx} = 1+y^2 \qquad y(0)=0$$

Solus $\quad y = \tan x$