Lecture 14 : Orthogonal Basis, Gram-Schmidt Orthogonalization & Orthogonal Projection

Recall that two vectors u & v in an inner product space is orthogonal if $\langle u, v\rangle = 0$

Orthogonal Set : A set of vectors S in an inner product space is said to be orthogonal set if each pair of distinct elements u & v of S is orthogonal i.e $\langle u, v\rangle = 0 \ \forall\, u \neq v$ in S.

Proposition: Any orthogonal set of non-zero vectors in an inner product space V is linearly independent (L.I.)

Proof. Let S be an orthogonal set of non-zero vectors in V & let $\sum_{i=1}^{n} \alpha_i u_i = 0$

$$\Rightarrow \left\langle \sum_{i=1}^{n} \alpha_i u_i , u_i \right\rangle = 0$$

$$\Rightarrow \alpha_i \langle u_i, u_i\rangle = 0 \quad \text{(since } \langle u_j, u_i\rangle = 0 \ \ j \neq i\text{)}$$

$$\Rightarrow \alpha_i \|u_i\|^2 = 0$$

$$\Rightarrow \alpha_i = 0 \quad \text{as } \|u_i\|^2 \neq 0$$

$\boxtimes$

Example: (1) $\{(1,-1),(1,1)\}$ is an orthogonal set in $\mathbb{R}^2$.

(2) In $\mathbb{R}^n$, let $e_i = (0, \ldots 0, \underset{\downarrow}{1}, 0, \ldots, 0)$
ith position

$\{e_1, e_2, \ldots, e_i, \ldots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

(3) Consider the vector space $C([0,1])$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$

Let $f(x) = 1-2x, \; 0 \le x \le \frac{1}{2}$
$\quad\quad = 0, \; \frac{1}{2} \le x \le 1$

& $g(x) = 0, \; 0 \le x \le 1$
$\quad\quad = 2x-1, \; \frac{1}{2} \le x \le 1$

$\langle f, g\rangle = 0$ (check) $\Rightarrow \{f, g\}$ is an orthogonal set in $C([0,1])$

Orthogonal Basis: A basis of an inner product space is orthogonal if the basis is an orthogonal set.

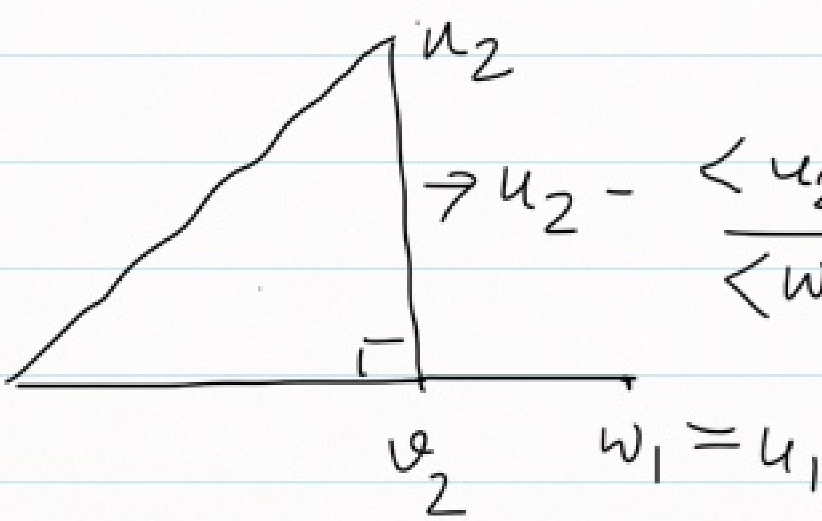
(Gram-Schmidt Orthogonalization Process)

Proposition: A finite dimensional inner product space has orthogonal basis.

Proof: Let V be an inner product space of dimension n. Let $\{u_1, \cdots, u_n\}$ be a basis of V. We will construct an orthogonal basis of V from $\{u_1, \cdots, u_n\}$.

Let $w_1 = u_1$,

Let v_2 be orthogonal projection of u_2 on $L(w_1)$ then $v_2 = \dfrac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$



$\to u_2 - \dfrac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = w_2$

Let $w_2 = u_2 - \dfrac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$\langle w_1, w_2\rangle = 0$ (check)

Also, note that $w_2 \neq 0$ (if $w_2 = 0$ then $\{u_1, u_2\}$ becomes L.D., contradiction)

Let $w_3 = u_3 - \frac{\langle u_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \frac{\langle u_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$

It is easy to check $\langle w_3, w_2 \rangle = 0 = \langle w_3, w_1 \rangle$

Note that $w_2 \in L(\{u_1, u_2\})$

If $w_3 = 0$ then $u_3 \in L(\{u_1, u_2\})$, this contradicts the fact that $\{u_1, u_2, u_3\}$, being a subset of basis, is L.I. Therefore, $w_3 \neq 0$ & $w_3 \in L(\{u_1, u_2, u_3\})$.

Proceeding in this way

Let $w_k = u_k - \sum_{r=1}^{k-1} \frac{\langle u_k, w_r \rangle}{\langle w_r, w_r \rangle} w_r$

where $k \in \{1, 2, \ldots, n\}$

Note that $w_k \in L(\{u_1, \ldots, u_k\})$, hence $w_k \neq 0$ & $\langle w_i, w_k \rangle = 0$, $1 \leq i \leq k-1$.

Thus, the set $\{w_1, \ldots, w_n\}$ is orthogonal set of non-zero vectors, hence it is L.I.

& $L(\{w_1, \ldots, w_n\}) = L(\{u_1, \ldots, u_n\}) = V$

as $\{u_1, \ldots, u_n\}$ is a basis.

So, $\{w_1, \ldots, w_n\}$ is an orthogonal basis of V. $\boxed{//}$

Orthonormal set : A set S of vectors in an inner product space V is said to be an orthonormal set if it is orthogonal set & $\|v\| = 1$ for all $v \in S$.

☆ If S is an orthogonal set of non-zero vectors, then the set $\left\{ \frac{v}{\|v\|} : v \in S \right\}$ is an orthonormal set.

Proposition : Let V be a finite dimensional inner product space. Then $\exists$ an orthonormal basis of V.

Proof: Use Gram-Schmidt orthogonalization process to get an orthogonal basis. Use ☆ to get orthonormal basis.

Proposition: Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of an inner product space V. Then every $x \in V$ can be written as $x = \langle x, u_1 \rangle u_1 + \cdots + \langle x, u_n \rangle u_n$.

Proof: Let $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$

$$\langle x, u_i \rangle = \alpha_i \langle u_i, u_i \rangle = \alpha_i.$$

Example: (1) $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$ is an orthonormal basis of $\mathbb{R}^2$.

Let $u_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $u_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Let $x = (2,3)$, $\langle x, u_1 \rangle = \frac{5}{\sqrt{2}}$, $\langle x, u_2 \rangle = -\frac{1}{\sqrt{2}}$

$$x = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2$$
$$= \frac{5}{\sqrt{2}} u_1 + \left(\frac{-1}{\sqrt{2}}\right) u_2 .$$

(2) $e_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th position}}}{1}, 0, \ldots, 0)$

$\{e_i : 1 \leq i \leq n\}$ is orthonormal basis of $\mathbb{R}^n$.

Orthogonal Projection: Let W be a subspace of an inner product space V. Let $\{u_1, u_2, \ldots, u_m\}$ be an orthonormal basis of W. Orthogonal projection of $x \in V$ on W, denoted by $P_W(x)$, is $P_W(x) := \langle x, u_1 \rangle u_1 + \cdots + \langle x, u_m \rangle u_m$

$$\langle x - P_W(x), u_i \rangle = \langle x, u_i \rangle - \langle x, u_i \rangle \langle u_i, u_i \rangle$$
$$= 0 \quad \forall\ 1 \le i \le m$$

$\Rightarrow \langle x - P_W(x), u \rangle = 0 \ \forall\ u \in W.$

$\Rightarrow$ $x - P_W(x)$ is orthogonal to W.

$\Rightarrow$ $x - P_W(x) \in W^\perp$.

We have seen $W \cap W^\perp = \{0\}$

Let $x \in W$ then we can write

$$x = P_W(x) + (x - P_W(x))$$

$\Rightarrow x \in W + W^\perp$.

So, $\underline{V = W + W^\perp}$.

(Orthogonal Projection is unique)

Let W be a subspace of a finite dimensional inner product space V. Let $P_W^1(x)$ & $P_W^2(x)$ respectively be orthogonal projections of x on W with respect to bases $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$. Then $P_W^1(x) = P_W^2(x)$.

Proof:- $P_W^1(x), P_W^2(x) \in W \Rightarrow P_W^1(x) - P_W^2(x) \in W$

$$x - P_W^1(x) \in W^\perp \ \& \ x - P_W^2(x) \in W^\perp$$

$$P_W^1(x) - P_W^2(x) = \underbrace{(x - P_W^2(x)) - (x - P_W^1(x))}_{\in W^\perp}$$

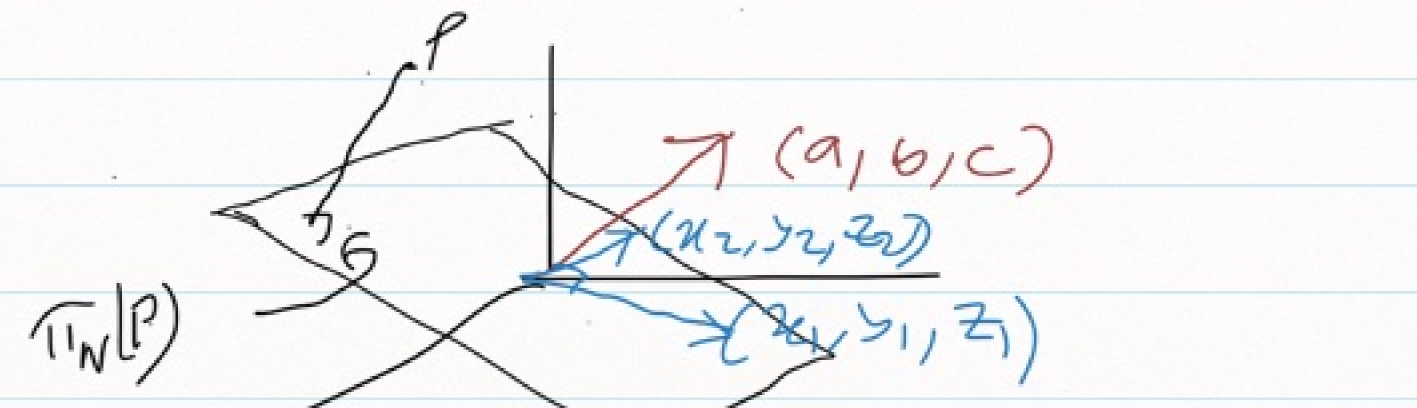
$$\Rightarrow P_W^1(x) - P_W^2(x) \in W \cap W^\perp = \{0\}$$

$$\Rightarrow P_W^1(x) = P_W^2(x). \quad \boxtimes$$

Example: Consider the subspace $W: ax + by + cz = 0$ in $\mathbb{R}^3$ $(a, b, c) \neq (0, 0, 0)$. (It is plane passing through origin)

Let $A = (x_1, y_1, z_1) \in W \Rightarrow$

$$ax_1 + by_1 + cz_1 = 0$$



Let $B = (x_2, y_2, z_2) = (cy_1 - bz_1, az_1 - cx_1, bx_1 - ay_1)$
(obtained by taking cross product $(x_1, y_1, z_1) \times (a, b, c)$)

$\{A, B\}$ is an orthogonal basis of W.

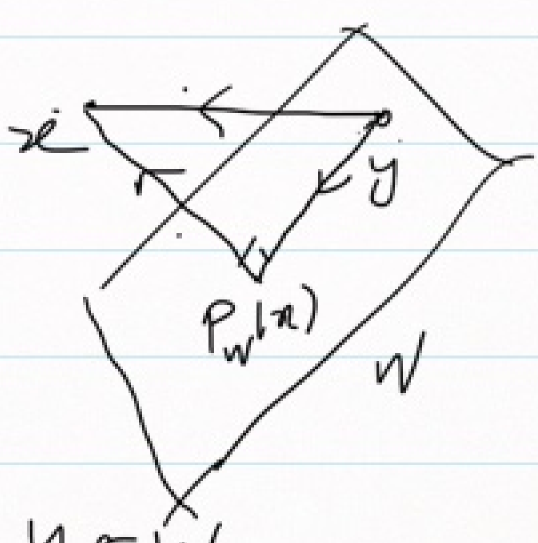
$\left\{ \frac{A}{\|A\|}, \frac{B}{\|B\|} \right\}$ is an orthonormal basis of W.

Let $P = (p, q, r)$ be any point in $\mathbb{R}^3$.

Orthogonal projection of P on W is

$$P_W(P) = \left\langle P, \frac{A}{\|A\|} \right\rangle P + \left\langle Q, \frac{B}{\|B\|} \right\rangle Q \quad \square$$

Nearest Point: Let W be a subspace of an inner product space V & $x \in V$. Then $\forall\, y \in W$, check that



$$\|x - P_W(x)\|^2 + \|P_W(x) - y\|^2$$
$$= \|x - y\|^2 \quad \text{(Pythagoras theorem)}$$

$\Rightarrow \|x - P_W(x)\| \leq \|x - y\| \quad \forall\, y \in W$

$\Rightarrow P_W(x)$ is nearest point to x in W.