Marking scheme: final exam(MTH 113M)

1. (a) Show that $\{1, 2+x, x+x^2\}$ is a linearly independent subset of $P_2(\mathbb{R})$.
   (b) Let $V = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}$. Extend the linearly independent subset $S = \{(1, 0, -1, 0, 1), (0, 1, 1, 3, 1)\}$ of V to a basis of V. (4+7)

Solution: (a) Let $a_0, a_1, a_2 \in \mathbb{R}$ such that $a_0 + a_1(2+x) + a_2(x+x^2) = 0$ implies

$$a_0 + 2a_1 = 0,\ \ a_1 + a_2 = 0,\ \ a_2 = 0.$$

(2)

Therefore $a_i = 0$ for all i. Hence $\{1, 2+x, x+x^2\}$ is a linearly independent subset of $P_2(\mathbb{R})$. (2)

Alternate solution: The part $a_i = 0$ for all i can also be done by substituting different values of x (2)

(b) We note that $V = \{(2x_2 - 3x_3 + x_4 - 2x_5, x_2, x_3, x_4, x_5) \mid x_2, x_3, x_4, x_5 \in \mathbb{R}\}$ has basis

$$\{(2, 1, 0, 0, 0), (-3, 0, 1, 0, 0), (1, 0, 0, 1, 0), (-2, 0, 0, 0, 1)\}.$$

(3)

Note that $(2x_2 - 3x_3 + x_4 - 2x_5, x_2, x_3, x_4, x_5) \in \text{Span}(S)$ if and only if

$$x_2 = b,\ \ x_3 = b - a,\ \ x_4 = 3b,\ \ x_5 = a + b,$$

for some a, b. (2)

Hence $(1, 0, 0, 1, 0), (-2, 0, 0, 0, 1) \in V$ but not in $\text{Span}(S)$ and are linearly independent. Therefore

$$S' = \{(1, 0, -1, 0, 1), (0, 1, 1, 3, 1), (1, 0, 0, 1, 0), (-2, 0, 0, 0, 1)\}$$

is an extension of S to a basis of V. (2)

Other method:

- Basis of V is $\{(2, 1, 0, 0, 0), (-3, 0, 1, 0, 0), (1, 0, 0, 1, 0), (-2, 0, 0, 0, 1)\}$ (3)
- $S \cup (2, 1, 0, 0, 0)$ is a L.I. subset of V. (2)
- By considering $(1, 0, 0, 1, 0)$, $S \cup \{(2, 1, 0, 0, 0), (1, 0, 0, 1, 0)\}$ is a required basis of V that extends S. (2)

2. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$. Find the rank of T.
   (b) Find the nullity of the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$. (6 + 5)

Solution: (a) (i) $R(T) = \text{Span}(\{T(e_1), T(e_2), T(e_3)\})$ (1) (ii)
$T(e_1) = (1,1,2), T(e_2) = (0,1,1), T(e_3) = (1,2,3)\})$. (2)

Since

$$(1,2,3) = (1,1,2) + (0,1,1)$$

and $\{(1,1,2),(0,1,1)\}$ is L.I. Hence $r(T) = 2$. (3)

(b) Here $3R_2 = R_1 + R_3$ and R_1, R_3 are linearly independent gives rank is two. (3)

Given matrix is 3×4, hence nullity is also two. (2)

3. (a) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that T is one-one if and only if T is onto.
   (b) Show that any finite orthogonal set of non-zero vectors in an inner product space is a linearly independent set. (5+5)

Solution: (a) T is one-one if and only if $N(T) = 0$. Hence T is one-one if and only if $n(T) = 0$. (3)

By rank nullity theorem, this is equivalent to $r(T) = n$ or T is onto. (2)

(b) Let $\{v_1, v_2, \cdots, v_n\}$ be an orthogonal set. Let $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. (2)

Then

$$0 = \langle a_1v_1 + a_2v_2 + \cdots + a_nv_n, v_i\rangle = a_i\langle v_i, v_i\rangle$$

Since $\langle v_i, v_i\rangle \neq 0$, we must have $a_i = 0$ for all i. (3)

4. Determine the value(s) of a, for which the given linear system has i) NO solution, ii) a unique solution and iii) infinite number of solutions.

$$\begin{aligned} x + 2y + 3z + w &= 4, \\ 2x + 5y + 5z + 2w &= 6, \\ 2x + (a^2 - 6)z + 2w &= a + 20, \\ 3x + 7y + 8z + 4w &= 10. \end{aligned}$$

(10)

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 2 & 5 & 5 & 2 & 6 \\ 2 & 0 & (a^2-6) & 2 & a+20 \\ 3 & 7 & 8 & 4 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & -4 & (a^2-12) & 0 & a+12 \\ 0 & 1 & -1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & (a^2-16) & 0 & a+4 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(5)

Here $\text{Rank}(A) = \begin{cases} 3, & a^2 - 16 = 0 \\ 4, & a^2 - 16 \neq 0 \end{cases}$, (2)

$\text{Rank}(A|b) = \begin{cases} 3, & a^2 - 16 = a + 4 = 0 \\ 4, & \text{otherwise} \end{cases}$ (2)

Hence we have Unique solution if $a^2 - 16 \neq 0$, No solution if $a = 4$, infinitely many solutions if $a = -4$. (1)

5. (a) Describe the orthogonal complement of $U = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = 0,\ z + w = 0\}$ (denoted $U^\perp$) with respect to the standard inner product of $\mathbb{R}^4$.

(b) Consider $V = \text{Span}\{(1, 1, 0), (-1, 1, 1)\}$ as a subspace of $\mathbb{R}^3$. Find the orthogonal projection of $(1, 0, 1)$ onto V. (6+6)

Solution: (a) (i) Basis of U is given by $\{(1, -1, 0, 0), (0, 0, 1, -1)\}$. (2)

(ii) $U^\perp = \{(x, y, z, w) \mid \langle (x, y, z, w), (1, -1, 0, 0)\rangle = \langle (x, y, z, w), (0, 0, 1, -1)\rangle = 0\}$. (2)

(iii)Therefore

$$U^\perp = \{(x, x, y, y) \mid x, y \in \mathbb{R}\}$$

(2)

(b) (i) Orthonormal basis of V is given by $\{\frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{3}}(-1, 1, 1)\}$. (3)

(ii) Orthogonal projection of $(1, 0, 1)$ onto V is

$$\langle (1, 0, 1), \frac{1}{\sqrt{2}}(1, 1, 0)\rangle \frac{1}{\sqrt{2}}(1, 1, 0) + \langle (1, 0, 1), \frac{1}{\sqrt{3}}(-1, 1, 1)\rangle \frac{1}{\sqrt{3}}(-1, 1, 1) = (\frac{1}{2}, \frac{1}{2}, 0).$$

(3)

6. Let $U = \{(x, y, z, w) \in \mathbb{R}^4 : z = 0\}$ and $V = \{(x, y, z, w) \in \mathbb{R}^4 : x + z = y + w = 0\}$.

(i) Give an example of an onto linear transformation $T : U \to V$.

(ii) Does there exists a one-one and onto linear transformation $T : U \to V$. Justify your answer.

(iii) Does there exist a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $N(T) = U$ and $R(T) = V$? Justify your answer. (4+4+4)

Solution: Basis of U is given by $\{u_1 = (1, 0, 0, 0), u_2 = (0, 1, 0, 0), u_3 = (0, 0, 0, 1)\}$ and basis of V is given by $\{v_1 = (1, 0, -1, 0), v_2 = (0, 1, 0, -1)\}$. (3)

(i) Define $T : U \to V$ such that $T(u_1) = v_1$, $T(u_2) = v_2$ and $T(u_3) = 0$. Then $T(au_1+bu_2+cu_3) = av_1+bv_2$ is an onto linear map. (3)

(ii) Since $\dim(U) = 3$ and $\dim(V) = 2$. Any linear transformation $T : U \to V$, must satisfy $n(T) + r(T) = \dim(U)$. Since $\dim(U) = 3$, $r(T) \leq 2$, we must have $n(T) \geq 1$. Hence T cannot be one-one. (3)

(iii) There does not exist such linear transformation because $\dim(U) = n(T) = 3$, $\dim(V) = r(T) = 2$ with $T : \mathbb{R}^4 \to \mathbb{R}^4$. Any such T does not satisfy $n(T) + r(T) = 4$. (3)

7. (a) Show that $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable.

(b) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$. Find a real matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. Justify your answer. (5 + 7)

Solution: (a) (i) $\det(A - \lambda I) = (3 - \lambda)(1 - \lambda)^2$. Therefore eigenvalues of A are $1, 3$. (1)

(ii) We first note that $A-3I = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ and it has rank two. Hence $N(A-3I)$ is one dimensional. (2)

(iii) Next, $A - I = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is also of rank two, hence $N(A-I)$ also has dimension one. (1)

(iv) Therefore there exists only two L.I. eigenvectors of A. This gives that A is not diagonalizable. (1)

(b) Here eigenvalues are $1, 2, 3$. (1)

We have $A - I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$ and (x, y, z) is in the kernel of $A - I$ if and only if $x = y$ and $2x + 2y + z = 0$. Hence it is the span of $(1, 1, -4)$. (2)

We also have $A - 3I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix}$ and (x, y, z) is in the kernel of $A - 3I$ if and only if $x + y = 0$ and $2x + 2y - z = 0$. Hence an eigenvector corresponding to the eigenvalue 3 is $(1, -1, 0)$. (2)

We also have $A - 2I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$ and (x, y, z) is in the kernel of $A - 2I$ if and only if $x = y = 0$. Hence an eigenvector corresponding to the eigenvalue 2 is $(0, 0, 1)$. (1) (basically 2+2+1 marks for finding three L.I. eigenvectors)

For $Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$ (1)

we have

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(1)

8. Let $A \in M_{n\times n}(\mathbb{R})$ be an invertible matrix.

   (a) Show that every eigenvalue of A is non-zero.

   (b) Using (a), show that λ^{-1} is an eigenvalue of A^{-1} if λ is an eigenvalue of A.

   (c) Show that A^{-1} is diagonalizable if A is diagonalizable.

(2+2+2)

Solution: (a) For an invertible matrix A, $Av = 0$ will imply $v = 0$. An eigenvector by definition must be non-zero. Therefore every eigenvalue of A is non-zero. (2)

Alternate solution: $\lambda_1\lambda_2\cdots\lambda_n = \det(A)$ and $\det(A) \neq 0$ implies $\lambda_i \neq 0$) (2) (b) Let $\{v_1, v_2, \cdots, v_n\}$ be a L.I. set of eigenvectors of A with eigenvalues λ_i. Then $Av_i = \lambda_i v_i$ if and only if $A^{-1}v_i = \frac{1}{\lambda_i}v_i$, here λ_i are non-zero by (a). (2)

Alternate solution: $Av_i = \lambda_i v_i$ and $\lambda_i \neq 0$ implies $\frac{1}{\lambda_i}v_i = A^{-1}v_i$)) (2)

(c) Hence $\{v_1, v_2, \cdots, v_n\}$ is a L.I. set of eiegenvectors of A^{-1}. Hence A^{-1} is diagonalizable.

Alternate solution: $QAQ^{-1} = D$ and A invertible implies $QA^{-1}Q^{-1} = D^{-1}$. Therefore A^{-1} is diagonalizable. (2)

9. Let A be an $m \times n$ matrix, that is, as a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. Let $N(A)$ be the null space of A and $\text{Row}(A)$ be the row space of A. Show that $N(A) \oplus \text{Row}(A) = \mathbb{R}^n$. (6)

Solution: We note that $v \in N(A)$ if and only if $Av = 0$. This is equivalent to say that v is orthogonal to $\text{Row}(A)$. Hence $N(A) \subseteq \text{Row}(A)^{\perp}$ (3)

We now prove that $\dim(N(A)) = \dim(\text{Row}(A)^{\perp})$.

Note that $\dim(\text{Row}(A)^{\perp}) = n - \dim(\text{Row}(A)) = n - r(A) = n(A)$. Hence $N(A) = \text{Row}(A)^{\perp}$ and $N(A) \oplus \text{Row}(A) = \mathbb{R}^n$. (3)