MTH 201 2024-2025 (1)

Tutorial 4

27/08/2024

1. Let V be a finite dimensional vector space over F. Let $T : V \to V$ be linear map with the property that $\text{rank}(T) = \text{rank}(T^2)$. Prove that $R \cap N = \{0\}$, where R and N denote the range space and the null space of T respectively.

2. Let V be a finite dimensional vector space over F and let S and T be linear operators on V. Prove that there exist ordered bases B and B' of V such that $[S]_B = [T]_{B'}$ iff there is an invertible operator U on V such that $S = U^{-1}TU$.

3. Let F be a field of characteristic zero and let V be a finite dimensional vector space over F. Let $\alpha_1, \ldots, \alpha_n$ are finitely many non zero vectors of V. Prove that there is a linear functional f on V such that $f(\alpha_i) \neq 0$ for $i = 1, \ldots, m$.

4. Let A be a $m \times n$ matrix with entries in $\mathbb{R}$ and let $b_0 \in \mathbb{R}^{m\times 1}$. Suppose the system of linear equation $AX = b_0$ has a unique solution. Which of the following statement/s is/are true:
   (a) $AX = b$ has a solution for every $b \in \mathbb{R}^m$.
   (b) If $AX = b$ has a solution, then its unique.
   (c) $AX = 0$ has a unique solution.
   (d) A has rank m.

5. Let T be a linear operator on $\mathbb{R}^3$ defined by
$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$
   (a) What is the matrix of T in the standard basis of $\mathbb{R}^3$.
   (b) What is the matrix of T in the ordered basis $\{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$?
   (c) Prove that T is invertible and give a rule of T^{-1} like the one which defines T.

6. Let V be an n-dimensional vector space over F, and let $\{\alpha_1, \ldots, \alpha_n\}$ be ordered basis for V.
   (a) There is a unique linear operator T on V such that $T(\alpha_j) = \alpha_{j+1}$ for $j = 1, \ldots, n-1$ and $T(\alpha_n) = 0$. What is the matrix A of T relative the ordered basis $\{\alpha_1, \ldots, \alpha_n\}$.
   (b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there exists an ordered basis B' of V such that the matrix of S relative to B' is the matrix A of part (a).

(d) Prove that if M and N are $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.