1. Consider the differential equation:

$$\dot{x}(t) = |x(t)|^{3/4}.$$

(a) Suppose $x(0) = 0$. Find a **nontrivial** solution. with proper justification. that is defined on all of $\mathbb{R}$. [5 marks]

(b) Now suppose $x(0) = 1$. Show that the local solution in this case could be extended to all of $\mathbb{R}$ in more than one ways. [5 marks]

Name (Capital):   Roll No.:

Consider

$①\cdots \dot{x}(t) = |x(t)|^{3/4}, \quad x(0) = x_0 > 0.$

Consider now

$F(y) = \int_{x_0}^{y} \frac{1}{f(s)}\, ds, \quad y \in (0,\infty),$

where $f(s) = |s|^{3/4}$. Therefore,

$F(y) = \int_{x_0}^{y} \frac{1}{s^{3/4}}\, ds = 4(y^{1/4} - x_0^{1/4});$
$\qquad\qquad\qquad y \in (0,\infty).$

$\text{Range } F = \left(\lim_{y \to 0} F(y), \lim_{y \to \infty} F(y)\right)$

$\qquad\quad = \left(-4x_0^{1/4}, \infty\right).$

Note that F can be extended at $y = 0$, by putting $F(0) = -4x_0^{1/4}$.

Case a. $x_0 = 0$.

In this case, a solution of $①$ on $\mathbb{R}_+$ is given by the inverse of F, namely

$x(t) = \dfrac{t^4}{4^4}.$

There are two ways of extending it to all of $\mathbb{R}$, namely

$x_1(t) = \begin{cases} t^4/4^4, & t \geq 0 \\ 0 & t \leq 0 \end{cases} \qquad \text{or,}$

Observe that in this case, if $x(t)$ is a solution on $t \geq 0$, then $-(x(-t))$ is a solution of $①$ on $t \leq 0$. Therefore

$x_2(t) = \begin{cases} t^4/4^4, & t \geq 0 \\ -t^4/4^4 & t \leq 0 \end{cases}$

is also a solution of $①$ on $\mathbb{R}$.

Case b: $x_0 = 1$.

In this case, on $\mathbb{R}_+$, the inverse of F, defined on $[-4,\infty)$, by

$x(t) = \left(\dfrac{t}{4} + 1\right)^4$

is a solution.

One way of extending this to all of $\mathbb{R}$, is to ~~put~~ extend it by defining

$x_1(t) = \begin{cases} (t/4+1)^4, & [-4,\infty), \\ 0, & (-\infty, -4]. \end{cases}$

Another way to extend it is by observing that
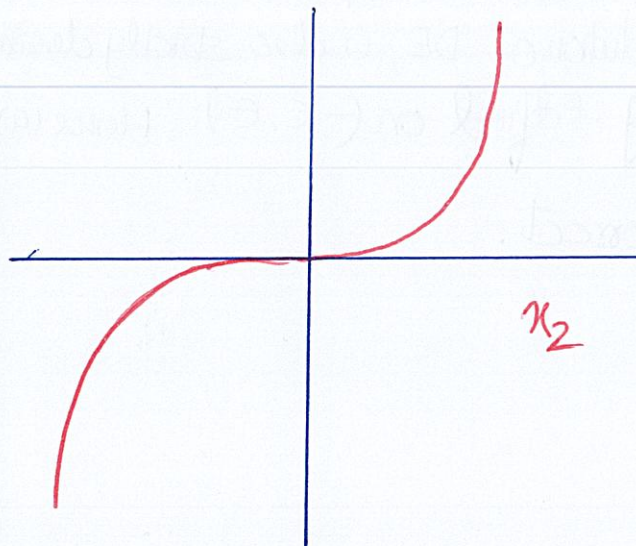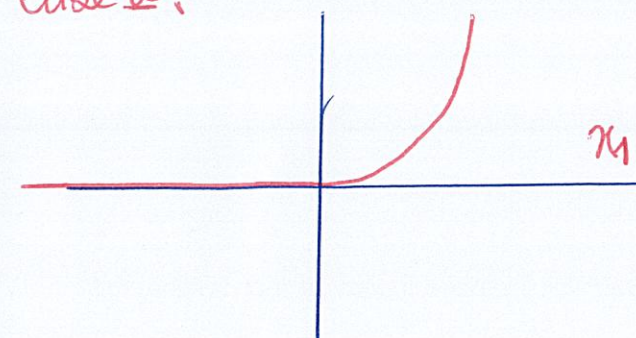
$-\dfrac{(t+4)^4}{4}$ is a solution of

$\dot{x}(t) = |x(t)|^{3/4}$ on $t \leq -4$ and

satisfying $x(-4) = 0$. Hence

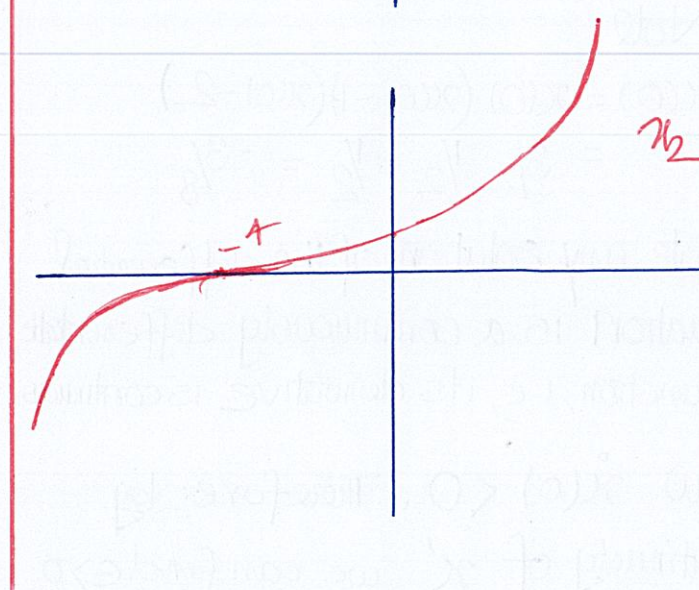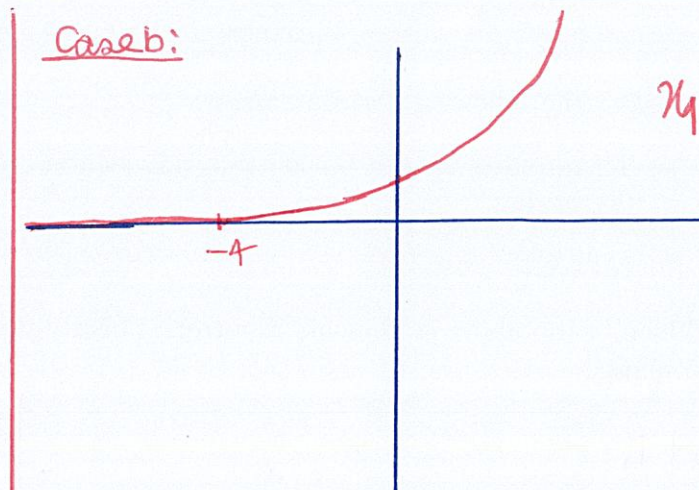$x_2(t) = \begin{cases} (t/4+1)^4, & [-4,\infty); \\ -(t/4+1)^4, & (-\infty, -4). \end{cases}$

Case a:

Case b:

ROUGH WORK

2. Consider the nonautonomous differential equation:

$$\dot{x}(t) = x(t)(x(t) - 1)(x(t) - 2). \quad x(0) = 3/2.$$

Consider the following two statements:

(a) There exists an $\epsilon > 0$ such that any solution of the above differential equation on $(-\epsilon, \epsilon)$ is strictly decreasing.

(b) No such ϵ exists.

Which of the above statements is correct? Justify your answer (no marks without proper reasoning).
[5 marks]

Sol 1.

Note

$\dot{x}(0) = x(0) \, (x(0) - 1)(x(0) - 2)$

$= \frac{3}{2} \, \frac{1}{2} - \frac{1}{2} = -\frac{3}{8}$.

Note any solution of the differential equation is a continuously differentiable function, i.e, its derivative is continuous

Now $\dot{x}(0) < 0$. Therefore, by continuity of x', we can find $\epsilon > 0$ such that

$\dot{x}(t) < 0$ on $(-\epsilon, \epsilon)$.

Hence x is strictly decreasing.

Sol-2 Consider the function

$f(s) = s(s-1)(s-2)$.

Its graph looks like

Note $f < 0$ on $(1, 2)$.

Hence the function

$F(y) = \int_{\frac{3}{2}}^{y} \frac{1}{f(s)} ds$ is strictly decreasing on $(1, 2)$.

Note $F(3/2) = 0$. Clearly Range(F) $\supset$ $(-\epsilon, \epsilon)$ for some $\epsilon > 0$.

Hence the inverse of F, which is a solution of DE, is also strictly decreasing defined on $(-\epsilon, \epsilon)$. Hence (a) is correct.

QUIZ 1(B): MTH421

Time: 2:55 P.M., Date: 22/08/2025, Duration: 30M, Maximum Marks: 10.

Name (Capital):   Roll No.:

1. Consider the nonautonomous differential equation:

$$\dot{x}(t) = x(t)(x(t) - 1)(x(t) - 2), \quad x(0) = 1/2.$$

Consider the following two statements:

(a) There exists an $\epsilon > 0$ such that any solution of the above differential equation on $(-\epsilon, \epsilon)$ is strictly increasing.

(b) No such ϵ exists.

Which of the above statements is correct? Justify your answer (no marks without proper reasoning).
[5 marks]

Solution: Solution for this is exactly same as Problem-2 in Version A of the Quiz 1.

ROUGH WORK

2. Consider the differential equation:
$$\dot{x}(t) = |x(t)|^{4/5}.$$

(a) Suppose $x(0) = 0$. Find a **nontrivial** solution. with proper justification. that is defined on all of $\mathbb{R}$.
[5 marks]

(b) Now suppose $x(0) = 1$. Show that the local solution in this case could be extended to all of $\mathbb{R}$ in more than one ways.
[5 marks]

Consider
$$\dot{x}(t) = |x(t)|^{4/5}, \quad x(0) = x_0 > 0. \cdots ①$$

In this case $f(s) = |s|^{4/5}$ remains positive on $(0, \infty)$. Consider the function

$$F(y) = \int_{x_0}^{y} \frac{1}{f(s)} ds, \quad y \in (0, \infty),$$

$$= \int_{x_0}^{y} \frac{1}{s^{4/5}} ds = 5(y^{1/5} - x_0^{1/5})$$

$$y \in (0, \infty).$$

Range $F = (-5x_0^{1/5}, \infty)$.

Note that F can be extended continuously at $y=0$ by setting
$$F(0) = -5x_0^{1/5}.$$

<u>Case a:</u> $x_0 = 0$.   nontrivial

In this case on $\mathbb{R}_+$, a solution of ①
is given by the inverse of F, namely

$$x(t) = (t/5)^5. \text{ Now}$$

you can extend it to get

$$x_1(t) = \begin{cases} (t/5)^5, & t \geq 0; \\ 0, & t \leq 0. \end{cases}$$

Another way to extend it is to observe
that <u>in this case</u> if $x(t)$ is a solution
for $t \geq 0$, then $-x(-t)$ is a solution on
$t \leq 0$. Note

$$-x(-t) = -\left(\frac{-t}{5}\right)^5 = \left(\frac{t}{5}\right)^5. \text{ Hence}$$

$$x_2(t) = (t/5)^5 \text{ is another nontrivial}$$
solution of ① with $x_0 = 0$.

<u>Case b:</u> $x_0 = 1$

In this case, on $\mathbb{R}_+$, the inverse of
F defined on $[-5, \infty)$ and given by

$$x(t) = \left(\frac{t}{5} + 1\right)^5 \text{ is a solution. One}$$
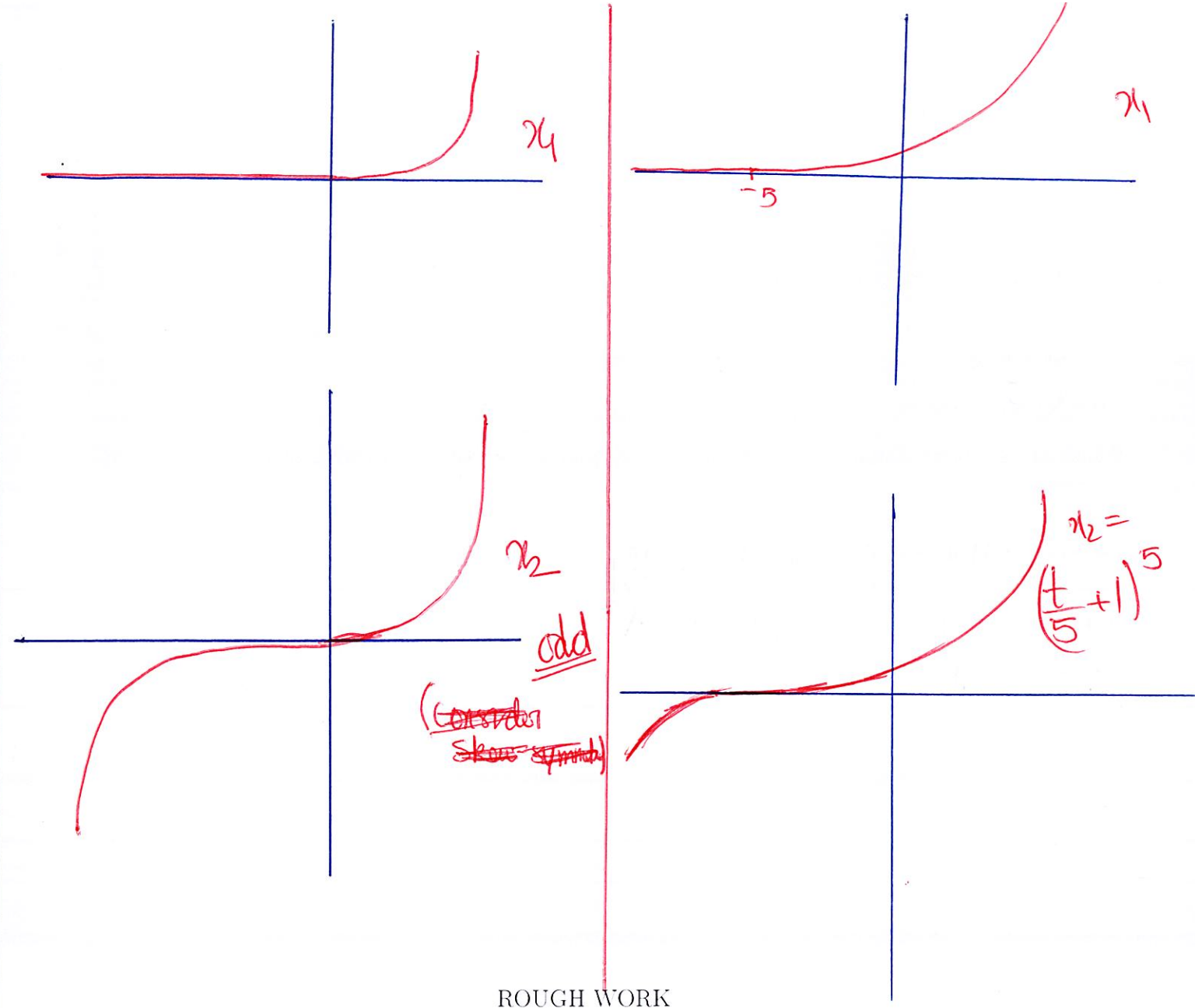
way of extending it is:

$$x_1(t) = \begin{cases} (t/5+1)^5, & t \geq -5 \\ 0, & t \leq -5 \end{cases}$$

Another way of extending is to consider

$$\left(\frac{t - (-5)}{5}\right)^{\cancel{5}} \text{ on } t \leq -5. \text{ Therefore}$$

$$x_2(t) = \cancel{\int}$$

$$x_2(t) = \left(\frac{t}{5} + 1\right)^5 \text{ is a solution on } \mathbb{R}.$$

x_1

x_1

-5

x_2

$x_2 = \left(\frac{t}{5} + 1\right)^5$

<u>odd</u>

(~~Consider~~
~~skew symmetry~~)

ROUGH WORK